Analysis of certain integrable models via random matrices

Ronan Memin

▶ **To cite this version:**

Ronan Memin. Analysis of certain integrable models via random matrices. Mathematical Physics [math-ph]. Ecole normale supérieure de lyon - ENS LYON, 2023. English. NNT : 2023ENSL0054 . tel-04236939

HAL Id: tel-04236939
https://theses.hal.science/tel-04236939

Submitted on 11 Oct 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THESE

en vue de l'obtention du grade de Docteur, délivré par

l'ECOLE NORMALE SUPERIEURE DE LYON

Ecole Doctorale N° 512
École Doctorale en Informatique et Mathématiques de Lyon

Discipline : **Mathématiques**

Soutenue publiquement le 22/09/2023, par :

Ronan MEMIN

Analyse de certains modèles intégrables grâce aux matrices aléatoires

---

Devant le jury composé de :

CHAFAÏ, Djalil	Prof. Univ.	Université Paris-Dauphine	Rapporteur
LAMBERT, Gaultier	Ass. Prof.	KTH Stockholm	Rapporteur
GUIONNET, Alice	Dir. Rech.	ENS Lyon	Directrice de thèse
GRAVA, Tamara	Prof. Univ.	Bristol University	Examinatrice
KOZLOWSKI, Karol	Ch. Rech.	ENS Lyon	Examinateur
MAÏDA, Mylène	Prof. Univ.	Université de Lille	Examinatrice
PAIN, Michel	Ch. Rech.	Université Paul Sabatier	Examinateur
SPOHN, Herbert	Prof. Univ.	Technische Universitaet Muenchen	Examinateur

Remerciements-Aknowledgements

Ces remerciements seront adressés dans un franglais de circonstance : les personnes auxquelles ceux-ci sont adressés n'étant pas toutes francophones.

Mes premiers remerciements vont à Alice Guionnet, qui m'a patiemment encadré et fait confiance tout au long de cette thèse. Ses conseils m'ont toujours été d'une aide précieuse. J'ai découvert à son contact les plaisirs de la recherche mathématique, et le magnifique sujet des matrices aléatoires.

Je remercie très chaleureusement Djalil Chafaï et Gaultier Lambert de me faire l'honneur de rapporter ma thèse. I am also extremely grateful to Tamara Grava, Karol Kozlowski, Mylène Maïda, Michel Pain, and Herbert Spohn for being part of my jury. It is an honor to benefit from the expertise of the reporters and of the jury. I had the chance to meet all of you during the course of my PhD: it always led to fruitful discussions.

I had the pleasure to collaborate with Alice Guionnet, Guido Mazzuca and Charlie Dworaczek Guera. Thank you for your inspiring views and your constant enthousiasm: with you I discovered the fascinating subjects of random matrices and integrable systems.

Je remercie Magalie, Virginia, Laure et Jessica pour leur aide constante : rien ne serait possible sans elles.

Mes remerciements vont aussi à Grégory Miermont, Eric Dumas et Emmanuel Jacob, pour la confiance qu'ils mont accordé en me confiant la charge de travaux dirigés dans les matières qu'ils enseignaient.

Merci à mes frères et soeur de thèse : Jonathan, Félix, Charlie, Vanessa et Thomas. Je suis chanceux d'être tombé sur des personnes aussi intentionnées que vous.
Merci à tous.tes mes camarades quotidiens : Arthur - mon co-bureau initial - les membres du bureau des légendes - Benoît, Thuy, Charlie, Corentin et Jules - Jeanne, Hugues, Antoine, Basile, William, Aymane, Léo, Héloïse, Valentine, Simon, Justin, Vianney, Raphaël, Raphaël, Clément, Denis, Riccardo, Mickaël et les autres. I also have a thought for my friends Rouven, Dylan, Adam, Sarah, Harriet, Issa and the others, with who I had the pleasure to share conferences or seminars. To all of you : thank you for everything. It was a pleasure to meet you during this unique adventure!

Merci à mes compères mathématiques élargis : Lucille, Quentin, Alice, Enguerrand, Grégoire, François, Mériadec, Louis, Louis, Juliette, Camille, Thomas, Antoine, Antoine, Fabrice, Adrian, Roméo et tous.tes les autres.

Merci tout simplement à tous.tes mes ami.es. Vous êtes exceptionnel.les. J'ai vu du pays grâce à vous : Rennes, Lyon, Quimper, Dijon, Saint Cast, Paris, Lille, Marseille, Arles, Strasbourg, Crozon ...

Merci à MC Marty.

Merci enfin à Maman, Papa, Sarah, Ninnog, Maya, Laure, Antonin, Vovonne. On a de la chance.

Résumé de la thèse

Cette thèse vise à établir de manière mathématiquement rigoureuse certains aspects d'un lien entre systèmes intégrables et matrices aléatoires. Motivé par la compréhension de l'échelle hydrodynamique des systèmes intégrables, Herbert Spohn a récemment remarqué qu'il est possible d'étudier les propriétés statistiques de ces systèmes en les comparant à des ensembles de matrices pour lesquelles on a accès à la loi du spectre : les β-ensembles.

Les systèmes que nous considérons sont des systèmes de particules qui évoluent dans le temps selon une interaction aux plus proches voisins. Le qualificatif "intégrable" signifie qu'un grand nombre de quantités indépendantes sont conservées le long de la dynamique. Cette propriété très spéciale permet d'accéder, en principe, à des formules permettant d'analyser ces systèmes. Cependant, ces formules peuvent être compliquées et il est commode de relier leur étude à celle de matrices aléatoires. Notre exemple clé est celui de la chaîne de Toda, un système introduit dans les années 1960, qui est un système de particules interagissant selon un potentiel non linéaire. Une manière directe de démontrer son intégrabilité est d'établir l'existence d'une paire de Lax : la dynamique est encodée par l'évolution de deux matrices B et L, appelées matrices de Lax, et qui satisfont une équation impliquant que les valeurs propres de L sont conservées dans le temps. Dans cette thèse, par "système intégrable", on entend "système possédant une paire de Lax".

Pour comprendre l'échelle hydrodynamique de ces systèmes — c'est-à-dire une échelle où on ne s'intéresse plus au comportement individuel des particules, mais aux changements macroscopiques du système — il est nécessaire d'adopter un point de vue statistique. Plutôt qu'essayer de comprendre le mouvement de chaque particule, on fait l'hypothèse que localement, le système est à l'équilibre : aléatoire, distribué selon une loi de probabilité bien précise, appelée Ensemble de Gibbs Généralisé (Generalized Gibbs Ensemble), qui est invariante dans le temps. Sous cette hypothèse, il s'agit alors de comprendre la distribution de la matrice de Lax L lorsque le système est distribué selon les mesures invariantes de Gibbs. Dans ce contexte, la matrice de Lax devient une matrice aléatoire dont on vise à comprendre les valeurs propres : lorsque sa taille tend vers l'infini (ce qui correspond à considérer un nombre de particules dans le système qui tend vers l'infini), a-t-on convergence du spectre ? Si oui, peut-on accéder à une caractérisation de la limite ? Que peut-on dire des fluctuations autour de cette limite ?

Les matrices de Lax des systèmes considérés ont la particularité d'être à bande, en raison de la structure d'interactions aux plus proches voisins. De plus, lorsque l'on considère une mesure de Gibbs particulière, on observe que celles-ci ressemblent à des matrices bien connues, en tant que représentation matricielle des β-ensembles. Par exemple, dans le cas de la chaîne de Toda, la matrice de Lax ressemble à la représentation tridiagonale du β-ensemble réel, découverte par Dumitriu et Edelman.

Le β-ensemble réel est une loi de probabilité sur $\mathbb{R}^N$, dépendant d'une température inverse $\beta > 0$ et d'un potentiel extérieur V, qui peut être interprétée comme une loi de probabilité sur un ensemble de particules vivant sur l'axe réel, se repoussant selon un potentiel logarithmique, et confinées par le potentiel extérieur V. En particulier, on peut voir ce modèle comme un gaz de Coulomb restreint à la droite réelle. On peut également voir le β-ensemble réel comme la loi jointe du spectre de matrices aléatoires : les plus fameux représentants de ce fait sont les cas du GOE (Gaussian orthogonal ensemble) et du GUE (Gaussian unitary ensemble) dans les cas particuliers d'une température inverse $\beta = 1$ ou 2.

L'existence d'une représentation matricielle pour tout β a été établie par Dumitriu et Edelman, grâce à un modèle de matrices tridiagonales. Le β-ensemble est donc l'un des (très rares) cas où il est possible de

donner explicitement la loi du spectre d'une matrice aléatoire. Pour ces raisons, les propriétés statistiques du β-ensemble sont très étudiées. Il est également possible de définir des β-ensembles sur d'autres espaces que $\mathbb{R}^N$: d'autres exemples sont donnés par le β-ensemble circulaire, où les particules sont sur le cercle unité, ou encore le β-ensemble antisymétrique, avec des particules sur l'axe imaginaire pur. Dans tous ces cas, on possède une représentation matricielle du type de celle donnée par Dumitriu et Edelman.

Pour comparer systèmes intégrables et β-ensembles, on se place dans les β-ensembles à grande température, c'est à dire dans le régime où β est de l'ordre de $1/N$, N étant le nombre de particules. Ce régime a beaucoup intéressé les communautés de physique mathématique et de matrices aléatoires récemment : la limite du spectre, ses fluctations et ses statistiques locales ont entre autres été étudiées.

Dans ce contexte, cette thèse établit mathématiquement la correspondance entre systèmes intégrables et β-ensembles sous deux points de vue : celui des grandes déviations et de la mesure limite, et celui des fluctuations autour de la limite.

Ce manuscrit se divise en cinq chapitres :

1. Le premier chapitre introduit le problème et son contexte, en présentant les principales idées permettant de comprendre le lien entre ensembles de matrices et systèmes intégrables.

2. Le deuxième chapitre, basé sur [GM22], en collaboration avec Alice Guionnet, établit ce lien d'un point de vue des grandes déviations pour les mesures empiriques des matrices d'intérêt, permettant d'obtenir une caractérisation de la mesure limite de la matrice de Lax pour la chaîne de Toda en termes de la mesure limite dans le β-ensemble réel à grande température.

3. Le troisième chapitre, basé sur [MM23b], en collaboration avec Guido Mazzuca, établit ce lien d'un point de vue des grandes déviations, cette fois pour le système d'Ablowitz-Ladik, une discrétisation intégrable de l'équation de Schrödinger avec non-linéarité cubique, relié lui au β-ensemble circulaire à grande température.

4. Le quatrième chapitre, basé sur [DGM23], en collaboration avec Charlie Dworaczek Guera, établit un théorème central limite pour la mesure empirique du β-ensemble réel à grande température, testée contre des fonctions test suffisamment régulières.

5. Le cinquième chapitre, basé sur [MM23a], en collaboration avec Guido Mazzuca, établit pour divers systèmes intégrables le lien avec les β-ensembles d'un point de vue des fluctuations : on y établit un théorème central limite pour la mesure empirique de la matrice de Lax, et on le compare au théorème central limite dans le β-ensemble correspondant.

Contents

Chapter 1

Introduction

1.1 Random matrices: A first example

Random matrix theory focuses on understanding the spectrum of large matrices with random entries. Its use has proven to be extremely fruitful in various domains of mathematics and physics: Since its first use in Statistics by Wishart in 1928 [Wis28], it has been extensively used in quantum mechanics, as pioneered by Wigner in the 50's who modeled the Hamiltonian of heavy nuclei by large random matrices. Other important fields of application include statistical mechanics, number theory, and integrable models. This manuscript aims at presenting recent developments and contributions to this last field, more precisely to the theory of **integrable systems**, which are very particular systems evolving in time - in some sense that we will describe precisely in the course of this first chapter. In this introduction, we try to give some context to the problem we are concerned with, and some of the most important ideas of this topic. It is divided in three main parts:

- Sections 1.1-1.4 are dedicated to the presentation of several random matrix models and some of their statistical properties.

- Sections 1.5-1.8 present the notion of integrable system and a way to analyse them by linking them to random matrix ensembles. Our key example is the one of the *Toda chain*.

- In Section 1.9, we present related works, and in Section 1.10 we give a summary of the main results of this thesis.

Most of the time, it is not possible to work out directly the law of the spectrum of a given random matrix, and one is led to search for asymptotic results about the spectrum. In this setting, of particular interest is the *empirical measure of eigenvalues*, given by

$$\hat{\mu}_N(A) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(A)} \,,$$

where A is a $N \times N$ matrix, and $\lambda_1(A), \dots, \lambda_N(A)$ are its eigenvalues. When A is random, *i.e.* when its entries are random variables, $\hat{\mu}(A)$ defines a random probability measure which one aims to understand. In particular, a natural question is the existence (and computation when possible) of a limit for this measure when the size N of the matrix goes to infinity. A fundamental example of such an asymptotic result is given by Wigner's semicircle law, which was discovered by Wigner in [Wig55], motivated by the statistical modelization of atoms with heavy nuclei:

Definition 1.1.1 (Wigner matrix)**.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*
*Let $(a_{i,j})_{1\leqslant i<j\leqslant N}$ be a family of i.i.d random variables with values in $\mathbb{K}$ with $\mathbb{E}[a_{i,j}] = 0$, $\mathbb{E}[|a_{i,j}|^2] = 1$, and let $(d_i)_{i\geqslant 1}$ be i.i.d real random variables with $\mathbb{E}[d_i] = 0$, $\mathbb{E}[d_i^2] = \sigma^2 > 0$. We say that the hermitian matrix X is a **Wigner matrix** if it is given by*

$$X_{i,j} = \begin{cases} \frac{1}{\sqrt{N}} a_{i,j} & \text{if } i<j, \\ \frac{1}{\sqrt{N}} d_i & \text{if } i=j, \\ \frac{1}{\sqrt{N}} \overline{a_{j,i}} & \text{otherwise.} \end{cases}$$

Theorem 1.1.2 (Wigner's semicircle law)**.** *Let $X^{(N)}$ be a sequence of $N \times N$ Wigner matrices.*

Then, the sequence of empirical measures $\hat{\mu}_N := \hat{\mu}(X^{(N)})$ converges almost surely in distribution towards the semicircle law sc *given by*

$$d\mathsf{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}(x)dx\,. \tag{1.1.1}$$

Remark 1.1.3. *The convergence of Theorem 1.1.2 says that on a set of probability one, the measures $\hat{\mu}_N$ converge in distribution towards* sc*, that is, for any $f : \mathbb{R} \to \mathbb{R}$ bounded continuous,*

$$\int_{\mathbb{R}} f d\hat{\mu}_N \to \int_{[-2,2]} f d\mathsf{sc}\,.$$

Because of the normalization $\frac{1}{\sqrt{N}}$ in the definition of a Wigner matrix, and by the law of large numbers, the euclidean norm of the rows of $X^{(N)}$ are asymptotically of order one, which helps understand why we find a non trivial limit for the empirical measure of eigenvalues.

This result can be proven for example via a moments method, which consists in establishing that the moments of the measure $\hat{\mu}_N$

$$\int x^n d\hat{\mu}_N = \frac{1}{N}\mathrm{Tr}X^n$$

converge towards the moments of the semicircle law.

Given a family of random matrices whose empirical measure of eigenvalues converge, one can ask more refined questions, among which:

- Can one compute the limiting measure ?
- Are there large deviations for this sequence of measures ?
- What is the rate of the convergence ? Is there an associated Central Limit Theorem ?

In the present work, we will be interested in those questions for particular models of random matrices.

1.2 GOE,GUE, β-ensembles

A symmetric $N \times N$ matrix is said to be in the Gaussian orthogonal ensemble GOE(N) if its entries are independent (up to the symmetry constraint), standard real Gaussians outside the diagonal and real Gaussians with mean zero and variance 2 on the diagonal, *i.e.*

$$A_{k,l} = x_{k,l} \text{ for } k < l, \quad A_{k,k} = \sqrt{2}x_{k,k}, \quad \text{and } A_{k,l} = A_{l,k},$$

where $(x_{k,l})_{1\leqslant k\leqslant l\leqslant N}$ is an independent family of standard Gaussians.

The Gaussian unitary ensemble, formed of Hermitian matrices, is defined similarly in terms of complex Gaussians, namely the Hermitian matrix A is in the Gaussian unitary ensemble GUE(N) if one has

$$A_{k,l} = \frac{x_{k,l} + iy_{k,l}}{\sqrt{2}}, \text{ for } k < l, \quad A_{k,k} = x_{k,k}, \quad \text{and } A_{l,k} = \overline{A_{k,l}}$$

for an independent family of real standard Gaussians $(x_{k,l}, y_{k,l})_{k \leqslant l}$.

Notice that those matrices are Wigner matrices. In particular, their empirical measures converge towards the semi circle law. However, a lot more can be said about those particular cases: both models are highly symmetric, in the sense that the GOE is invariant under orthogonal conjugation, and the GUE is invariant under unitary conjugation. In symbols, for the case of the GOE, this means that if $A \in \mathrm{GOE}(N)$ and O is a non random $N \times N$ orthogonal matrix, then OAO^T is also in $\mathrm{GOE}(N)$.

As a consequence of this symmetry, one is able to extract the joint law of eigenvalues of those ensembles. The following Theorem can be found in [AGZ10][Theorem 2.5.2].

Theorem 1.2.1 (Law of the unordered spectrum of GOE/GUE)**.** *Let $A \in GOE(N)$ or $A \in GUE(N)$. Let $\beta = 1$ in the first case, and $\beta = 2$ in the second case. Then, the law of the unordered spectrum of A is given by the probability measure on $\mathbb{R}^N$*

$$d\mathbb{P}_N^\beta(x_1, \dots, x_N) = \frac{1}{Z_N^\beta} \prod_{i<j} |x_i - x_j|^\beta e^{-\beta \sum_{i=1}^N x_i^2/4} dx_1 \dots dx_N \,. \tag{1.2.1}$$

Remark 1.2.2. *One can also construct an analogue of the previous models having eigenvalue distribution* (1.2.1) *with $\beta = 4$ using quaternions. For details, see [AGZ10][Section 4.1]. Furthermore, the normalizing constant Z_N^β in* (1.2.1)*, called the partition function of the model, can be computed by the use of Selberg's integral formula, [AGZ10][Section 2.5.3].*

Equation (1.2.1) is to be compared with the so-called Coulomb gas, defined as the probability measure on $(\mathbb{R}^d)^N$

$$d\mathbb{P}_{\mathrm{Coulomb},N}(\mathbf{x}_1, \dots, \mathbf{x}_N) = \frac{1}{Z_{\mathrm{Coulomb},N}} \exp\left\{ -\beta \left(\sum_{i=1}^N V(\mathbf{x}_i) - \sum_{1 \leqslant i < j \leqslant N} g(\mathbf{x}_i - \mathbf{x}_j) \right) \right\} ,$$

where g is the fundamental solution of the Laplacian, *i.e.* satisfies

$$\Delta g = \delta_0$$

in the sense of distributions; and V is a function going to infinity at infinity, having the effect of *confining* the particles of the gas. In the case $d = 2$, the function g is given by $g(\mathbf{x}) = \log|\mathbf{x}|$, hence one can see the probability measure (1.2.1) as a two dimensional Coulomb gas constrained to the real line. This comparison encourages to apply the statistical physics formalism to the GOE and the GUE, resulting in a good understanding of the repartition of their eigenvalues. A natural generalization of these particular ensembles of matrices is the β-ensemble on the real line.

Definition 1.2.3 (β-ensemble on the real line)**.** *Let $\beta \geqslant 0$ and $N \geqslant 2$ integer. Let $V : \mathbb{R} \to \mathbb{R}$ continuous and such that for some $\delta > 0$ and $c \in \mathbb{R}$;*

$$V(x) \geqslant (\beta N + 1 + \delta) \log|x| + c \quad \text{for all } x \in \mathbb{R} \,. \tag{1.2.2}$$

The β-ensemble (on the real line) of size N and potential V is the probability measure on $\mathbb{R}^N$ given by

$$d\mathbb{P}_N^{V,\beta}(x_1, \dots, x_N) = \frac{1}{Z_N^{V,\beta}} \prod_{1 \leqslant i < j \leqslant N} |x_i - x_j|^\beta e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N \,. \tag{1.2.3}$$

The $N \times N$ random matrix A is said to be in the β-ensemble with potential V if its law of unordered eigenvalues is given by (1.2.3)*.*

Remark 1.2.4. *A few comments about this definition.*

i) Condition (1.2.2) *ensures that the partition function*

$$Z_N^{V,\beta} = \int_{\mathbb{R}^N} \exp\left(\beta \sum_{1\leqslant i<j\leqslant N} \log|x_i - x_j| - \sum_{i=1}^N V(x_i)\right) dx_1 \dots dx_N$$

converges. This can be checked using that for $x, y \in \mathbb{R}$,

$$\log|x-y| \leqslant \log(1+|x|) + \log(1+|y|)\,.$$

ii) The parameter β *can be interpreted as an* ***inverse temperature*** $1/T$ *as is classical in statistical physics. Roughly speaking, at fixed* N*, the smaller* β *is, the more the term* $\sum_{i=1}^N V(x_i)$ *is prominent in front of* $\beta\sum_{i,j}\log|x_i - x_j|$ *(ignoring possible issues that may arise from the singularity of* log *at zero). The density of* $\mathbb{P}_N^{V,\beta}$ *then almost factorizes into the law of independent variables, which intuitively corresponds to a disordered system, that is a system at high temperature. On the contrary, increasing* β *has the effect of introducing correlations between the* x_i*'s, intuitively lowering the temperature of the system.*

iii) The law $\mathbb{P}_N^\beta$ *of Theorem 1.2.1 corresponds to* $\mathbb{P}_N^{\frac{\beta x^2}{4},\beta}$ *in the notations of the previous definition.*

By Theorem 1.2.1, GOE(N) is then contained in the 1-ensemble of size N with potential $x^2/4$ and GUE(N) is contained in the 2-ensemble of size N and potential $x^2/2$. We can then adress the question of the existence of a matrix representation for the β-ensembles with general β:

For $\beta = 1, 2, 4$ the β-ensembles with quadratic potentials have matrix representations with independent entries. Does such a matrix representation exist for any $\beta \geqslant 0$?

This question was positively answered by Dumitriu and Edelman in [DE02], giving a **tridiagonal representation** of the β ensemble with quadratic potential. Before stating their theorem, recall that for $a > 0$ the χ_a distribution is the probability measure on $\mathbb{R}$ with density

$$f_a(x) = \frac{2^{1-a/2}}{\Gamma(a/2)} x^{a-1} e^{-x^2/2}\,, \tag{1.2.4}$$

where Γ is the classical Gamma function and by convention the χ_0 distribution is constant to zero. For integer $a \geqslant 0$, χ_a is the law of the euclidean norm of a standard a-dimensional Gaussian vector:

$$\chi_a \overset{(d)}{=} \sqrt{g_1^2 + \dots + g_a^2}\,,$$

where the g_i's are independents Gaussians $N(0,1)$.

Theorem 1.2.5 (Tridiagonal representation of the β-ensemble)**.** *Let* $\beta > 0$ *and* $N \geqslant 2$ *integer. Let* $(g_i, b_j)_{1\leqslant i\leqslant N, 1\leqslant j\leqslant N-1}$ *be an idependent family with* $g_i \sim N(0,1)$ *and* $b_j \sim \chi_{(N-j)\beta}$*. The tridiagonal matrix*

$$T = \begin{pmatrix} g_1 & \frac{b_1}{\sqrt{2}} & & & \\ \frac{b_1}{\sqrt{2}} & g_2 & \frac{b_2}{\sqrt{2}} & & \\ & \frac{b_2}{\sqrt{2}} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{b_{N-1}}{\sqrt{2}} \\ & & & \frac{b_{N-1}}{\sqrt{2}} & g_N \end{pmatrix}$$

is in the β*-ensemble of size* N *and potential* $x^2/2$*.*

Let us say a few words about the particular case of $\beta = 1$ which is obtained via a tridiagonalization procedure: Take A the $N \times N$ symmetric matrix with independent (up to the symmetry) centered Gaussian entries, having variance $1/2$ outside the diagonal and variance 1 on the diagonal (notice the normalization by $1/\sqrt{2}$ of the GOE, a matrix of the GOE being in the 1-ensemble with potential $x^2/4$). In other words, $A \in \frac{1}{\sqrt{2}}\mathrm{GOE}(N)$. Write A as

$$A = \begin{pmatrix} a_{11} & x^T \\ x & B \end{pmatrix},$$

where a_{11} is a standard Gaussian, x is a vector of size $N-1$ having independent Gaussian entries $N(0, 1/2)$ and B is in $\frac{1}{\sqrt{2}}\mathrm{GOE}(N-1)$. Let then $H \in \mathcal{O}(N-1)$ such that $Hx = \|x\|e_1$, with $e_1^T = (1, 0, \dots, 0)$. We have

$$\begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} a_{11} & x^T \\ x & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & H^T \end{pmatrix} = \begin{pmatrix} a_{11} & \|x\|e_1^T \\ \|x\|e_1 & HBH^T \end{pmatrix},$$

and because $B \in \frac{1}{\sqrt{2}}\mathrm{GOE}(N-1)$, HBH^T also is. We now notice that $\|x\|$ is $\frac{1}{\sqrt{2}}\chi_{N-1}$ distributed, and we conclude by induction.

Theorem 1.2.5 is particularly useful when tackling problems related with the quadratic β-ensembles, allowing one to study independent entries of a matrix instead of directly working with the correlated probability measure (1.2.3). For instance, to investigate the moments of the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ of $\mathbf{x} = (x_1, \dots, x_N)$ distributed with respect to $\mathbb{P}_N^{\frac{x^2}{2},\beta}$, it suffices to study the sequence of traces $\frac{1}{N}\mathrm{Tr}T^n$ where the matrix T is given by Theorem 1.2.5. This substitution is then convenient because of the independence of the coefficients of T.

In this thesis, we are interested in the regime where β scales as $\frac{1}{N}$, namely $\beta = \dfrac{2P}{N}$, $P \geqslant 0$ independent of N.

Definition 1.2.6 (β ensemble at high temperature)**.** *Let $P \geqslant 0$ and $V : \mathbb{R} \to \mathbb{R}$ continuous be such that for some $\alpha > 2P + 1$, there exists a constant c such that for all $x \in \mathbb{R}$,*

$$V(x) \geqslant \alpha \log|x| + c\,.$$

The β-ensemble at high temperature with parameter $2P \geqslant 0$ and potential $V : \mathbb{R} \to \mathbb{R}$ is the probability distribution on $\mathbb{R}^N$

$$d\mathbb{P}_N^{V,\frac{2P}{N}}(x_1, \dots, x_N) = \frac{1}{Z_N^{V,\frac{2P}{N}}} \prod_{1 \leqslant i,j \leqslant N} |x_i - x_j|^{2P/N} e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N\,. \tag{1.2.5}$$

This regime has attracted the attention of the mathematical physics and random matrices community lately. It was first considered in [CL97]. In [ABG12], the authors derived the limiting measure of the eigenvalues in the quadratic case. It was considered among others in [GZ19] from the large deviations point of view, in [NT18], [HL21], [NTT23], where the authors studied the fluctuations of the empirical measure around its limit, in [BGP15], [NT18] [Lam21] where the authors established local statistics in those ensembles.

In this case, and with the quadratic potential $V(x) = x^2/2$, considering the $N \times N$ matrix T of Theorem 1.2.5, the entries of T are of order 1:

$$T(i,i) \sim N(0,1) \text{ and } T_{i,i+1} \sim \chi_{2P(N-i)/N}\,.$$

Let us consider the k-th moment of T,

$$\frac{1}{N}\mathrm{Tr}T^k = \frac{1}{N} \sum_{\substack{1 \leqslant i_1, \dots, i_k \leqslant N, \\ |i_{j+1} - i_j| \leqslant 1}} T_{i_1,i_2} T_{i_2,i_3} \dots T_{i_k,i_1}\,,$$

where we used the tridiagonal structure of T. This last sum has $O(N)$ terms which are all of order 1, therefore $\frac{1}{N}\mathrm{Tr}T^k$ should converge towards some deterministic $c_k \in \mathbb{R}$. With this heuristics, we guess that the empirical measure $\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ of $\mathbf{x} \sim \mathbb{P}_N^{x^2/2,\frac{2P}{N}}$ should converge towards some deterministic probability measure on $\mathbb{R}$. To make this statement more precise and in order to extend it to more general potentials V, we next give some facts about large deviations, and some consequences for the β-ensembles, and give a glimpse of Central Limit Theorems for such ensembles.

1.3 Large deviations, equilibrium measure, fluctuations

We give here the definition (in the case of a metric space) of a large deviation principle, and explain how those can be applied to random matrix theory. We defer to Appendix A for some general facts about large deviations.

Let (E,d) be a metric space and let $(X_N)_{N\in\mathbb{N}}$ be a sequence of E-valued random variables defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$.
Let $I : E \to \mathbb{R}_+ \cup +\infty$ be a *lower semi continuous* function, meaning that for any $a \geqslant 0$, the level set $I^{-1}([0,a])$ is closed.

Definition 1.3.1 (Large deviation principle)**.**
*We say that $(X_N)_N$ satisfies a **Large deviation principle** (LDP) with rate function I and at scale N if for any borelian $A \in \mathcal{B}(E)$, one has*

$$-\inf_{x\in\mathring{A}} I(x) \leqslant \liminf_N \frac{1}{N}\log\mathbb{P}\left(X_N \in A\right) \leqslant \limsup_N \frac{1}{N}\log\mathbb{P}\left(X_N\in A\right) \leqslant -\inf_{x\in\overline{A}} I(x)\,, \tag{1.3.1}$$

with $\mathring{A}$ the interior of A and $\overline{A}$ its closure.

Remark 1.3.2. *We assume I to be lower semi continuous to ensure uniqueness of the rate function: the chain of inequalities* (1.3.1) *uniquely determines I among the set of lower semicontinuous functions on E.*

Roughly speaking, $(X_N)_N$ satisfies the previous LDP if for small ε and any $x \in E$, we have for large N (up to subexponential factors)

$$\mathbb{P}(X_N \in B_x(\varepsilon)) \approx e^{-NI(x)}\,,$$

where $B_x(\varepsilon)$ is the open ball centered in x with radius ε.

In our applications, the space E will always be the set of probability measures $\mathcal{P}(X)$ on a complete separable (i.e. polish) metric space X such as $\mathbb{R}$ or the unit circle $\mathbb{T}$, endowed with a metric compatible with the topology of weak convergence. An example we will be using in the sequel of such a metric is given by

$$d(\mu,\nu) = \sup_{\|f\|_{\mathrm{BV}}\leqslant 1, \|f\|_{\mathrm{Lip}}\leqslant 1} \left|\int_X f d\mu - \int_X f d\nu\right|\,, \tag{1.3.2}$$

with $\|f\|_{\mathrm{BV}}$ the bounded variation of f defined as

$$\|f\|_{\mathrm{BV}} = \sup_n \sum_{x_1<\ldots<x_n} |f(x_{i+1}) - f(x_i)|$$

where the supremum runs over the subdivisions of X and $\|f\|_{\mathrm{Lip}}$ the Lipschitz norm of f.

Large deviations are a powerful tool to prove almost sure convergences. The following proposition is a mere application of the Borel-Cantelli lemma.

Proposition 1.3.3. *Let $(X_N)_N$ satisfy a LDP at scale N and with rate function I.*
*Suppose that I is **good**, meaning that for any $a \geqslant 0$, the set $I^{-1}([0,a])$ is compact. Furthermore, assume that I admits a unique minimizer x_0.*

Then X_N converges almost surely towards x_0.

Remark 1.3.4. *If x_0 is a minimizer of the rate function I, then $I(x_0) = 0$. This can be seen by taking $A = E$ in* (1.3.1).

By Sanov's Theorem, see Theorem A.3.2, if (x_i) is an *i.i.d* sequence of random variables, then its sequence of empirical measures $\hat{\mu}_N$ satisfies a large deviation principle at scale N, with good rate function $H(\cdot|\mu)$ given by

$$H(\nu|\mu) = \begin{cases} \int_X f \log f d\mu \text{ if } \nu \text{ is absolutely continuous with respect to } \mu \text{ with density } f = \frac{d\nu}{d\mu} \\ +\infty \text{ otherwise.} \end{cases}$$

Let us come back to the β-ensemble at high temperature with general potential V. The density of $\mathbb{P}_N^{V,\frac{2P}{N}}$ can be written in terms of the empirical measure $\hat{\mu}_N$ of $\mathbf{x} = (x_1, \dots, x_N)$ by

$$d\mathbb{P}_N^{V,\frac{2P}{N}}(x_1, \dots, x_N) = \frac{1}{Z_N^{V,\frac{2P}{N}}} \exp\left(-N \left[-P \iint_{\mathbb{R}^2 \setminus \Delta} \log|x-y| d\hat{\mu}_N(x) d\hat{\mu}_N(y) + \int_{\mathbb{R}} V(x) d\hat{\mu}_N \right] \right),$$

where $\Delta = \{(x,x);\, x \in \mathbb{R}\} \subset \mathbb{R}^2$. Let us ignore for a moment that the function log has a singularity at zero. Then, as a consequence of Varadhan-Bryc Lemmas: Lemmas A.2.1 and A.2.2; and of Sanov's theorem, we deduce that the sequence of empirical measures satisfies a LDP at scale N and rate function

$$J_P^V(\mu) = \begin{cases} \mathcal{E}_P^V(\mu) - \inf_{\nu \in \mathcal{P}(\mathbb{R})} \mathcal{E}_P^V(\nu) \text{ if } \mu \text{ is absolutely continuous with respect to Lebesgue measure} \\ +\infty \text{ otherwise,} \end{cases} \tag{1.3.3}$$

where, setting ρ for the density of μ with respect to Lebesgue measure, $\mathcal{E}_P^V$ is given by

$$\mathcal{E}_P^V(\mu) = -P \iint_{\mathbb{R}^2} \log|x-y| d\mu(x) d\mu(y) + \int_{\mathbb{R}} \rho(x) \log \rho(x) dx + \int_{\mathbb{R}} V(x) d\mu(x)\,.$$

Of course, the singularity at zero of the log has to be properly handled. This is done in greater generality in [GZ19]. As a consequence we have:

Theorem 1.3.5 (LDP for the empirical measure at high temperature)**.**
Assume $\mathbf{x}$ *to be distributed with respect to* $\mathbb{P}_N^{V,\frac{2P}{N}}$*. The empirical measure*

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$$

satisfies a large deviation principle at scale N and with good, strictly convex rate function J_P^V given by (1.3.3).

This result is in the continuity of the large deviations for the empirical measure in the regime where $\beta > 0$ is **fixed**, that was established by Ben Arous and Guionnet in [BAG97]. As a consequence of their result, one deduces that the limiting measure of the β ensemble at fixed β is the semicircle distribution (1.1.1).

Using Proposition 1.3.3, we deduce that under $\mathbb{P}_N^{V,\frac{2P}{N}}$ the sequence of empirical measures converges almost surely in distribution towards the minimizer of the functional $\mathcal{E}_P^V$. The following result can be found in Lemma 2.3.2 and Proposition 4.2.2.

Proposition 1.3.6 (Equilibrium measure for $\mathbb{P}_N^{V,\frac{2P}{N}}$)**.** *Let μ_P^V be the minimizer of the functional $\mathcal{E}_P^V$, denote by ρ_P^V its density with respect to Lebesgue measure. Then, there exists some $\lambda_P^V \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,*

$$V(x) - 2P \int_{\mathbb{R}} \log|x-y| d\mu_P^V(y) + \log \rho_P^V = \lambda_P^V\,. \tag{1.3.4}$$

As a consequence, there exists a constant C_P^V such that for all $x \in \mathbb{R}$,

$$0 < \rho_P^V(x) \leqslant C_P^V(1+|x|)^{2P} e^{-V(x)}\,.$$

In contrast with the fixed β case, the equilibrium measure in the high temperature regime has unbounded support. This can be informally explained by the fact that the correlations are less prominent in the high temperature regime than in the fixed temperature one, where the compactness of the limiting measure is a manifestation of such correlations. Another way of illustrating this difference in both regimes in via Central Limit Theorems. As was shown first in [Joh98] for $\beta = 2$ and quadratic potential, one has the following convergence in distribution: let $\mathbf{x}$ be distributed according to $\mathbb{P}_N^2$ (1.2.1). Then, as N goes to infinity, and for smooth enough functions $f : \mathbb{R} \to \mathbb{R}$, the random variables

$$N\left(\frac{1}{N}\sum_{j=1}^{N} f(x_j) - \int_{\mathbb{R}} f(x) d\mathrm{sc}(x)\right)$$

converge towards a Gaussian variable whose mean and variance depend on f. The fluctuations of the empirical measure around its limit are thus of order $1/N$. This theorem was then generalized and further developed in the regime where β is fixed in [Shc13], [BG13a], [BG13b],[BLS18],[LLW19], [BMP22]. In contrast with the fixed β case, as was established by [NT18] in the high temperature regime with quadratic potential $V(x) = \frac{x^2}{2}$, the fluctuations of the empirical measure are of order $1/\sqrt{N}$:

$$\sqrt{N}\left(\frac{1}{N}\sum_{j=1}^{N} f(x_j) - \int_{\mathbb{R}} f(x) d\mu_P^{x^2/2}\right) \to N(0, \sigma^2_{x^2/2,P}(f))\,.$$

Extending this result to more general potentials is the purpose of Chapter 4. As we will see in the next section, the fluctuations are also of order $1/\sqrt{N}$ in the **circular** β ensemble in the high temperature regime. Consequently, the fluctuations in the high temperature regime are of bigger order than the fluctuations in the fixed temperature regime, giving evidence of more disorder in the high temperature case.

We close this section by mentioning the local statistics of the β ensemble in the high temperature regime, see [BGP15], [NT18], [Lam21], from which the local statistics of the β-ensemble can be seen to be described by Poisson processes. More precisely, consider $\mathbf{x}$ to be distributed according to the high temperature β-ensemble. Then, the sequence of random measures

$$\sum_{j=1}^{N} \delta_{Nx_j}$$

converges (in an appropriate sense) towards a Poisson point process on $\mathbb{R}$. This process has the property that given two disjoint sets A and B, the number of particles that fall in A is independent of the number of particles falling in B.

Again, this convergence can be put in contrast with the fixed temperature case, in which case, the limiting process is **number rigid** as was shown in [DHLM21], meaning that under this point process, the number of particles inside a compact set is a deterministic function of the configuration outside this compact.

1.4 The circular β-ensemble at high temperature

Similarly to the β-ensemble on the real line, one can consider the **circular β-ensemble** at inverse temperature β. It is the probability measure on the set $\mathbb{T}^N$, where $\mathbb{T} = [-\pi, \pi)$ is the torus, given by

$$d\mathbb{P}_{C,N}^{V,\beta}(\theta_1, \ldots, \theta_N) = \frac{1}{Z_{C,N}^{V,\beta}} \prod_{1 \leqslant j \leqslant \ell \leqslant N} |e^{i\theta_j} - e^{i\theta_\ell}|^\beta e^{-\sum_{j=1}^N V(\theta_j)}\,, \tag{1.4.1}$$

where $V : \mathbb{T} \to \mathbb{R}$ is a continuous potential. The definition of this probability measure is very similar to the one of the β-ensemble on the real line, and both ensembles share important common features. We list in this section important results about the circular case. Those results always have (or are expected to) have a counterpart in the β-ensemble on the real line.

The circular β-ensemble has a matrix representation, this time via unitary matrices. Before stating the matrix representation, we introduce the law Θ_ν on the unit disk $\overline{\mathbb{D}}$, where $\nu > 1$, by:
The random variable X is Θ_ν distributed if for any bounded measurable $f : \overline{\mathbb{D}} \to \mathbb{R}$

$$\mathbb{E}[f(X)] = \frac{\nu - 1}{2\pi} \int_{\overline{\mathbb{D}}} f(z)(1 - |z|^2)^{\frac{\nu-3}{2}} d^2 z \,. \tag{1.4.2}$$

For integer $\nu \geqslant 2$, Θ_ν has the following geometrical interpretation: if $\mathbf{u} = (u_1, \dots, u_{\nu+1})$ is sampled according to the surface measure on the unit sphere S^ν in $\mathbb{R}^{\nu+1}$, then $u_1 + iu_2$ is Θ_ν distributed [KN07]. We then define Θ_1 as the uniform measure on the unit circle.

Next theorem can be found in [KN04][Theorem 1.2].

Theorem 1.4.1 (Matrix representation of the circular β-ensemble with null potential)**.** *Consider the block diagonal $N \times N$ matrices*

$$L = diag\,(\Xi_1, \Xi_3, \Xi_5 \dots,) \quad \text{and} \quad M = diag\,(\Xi_0, \Xi_2, \Xi_4, \dots) \,, \tag{1.4.3}$$

where the blocks Ξ_j, $j = 1, \dots, N-1$, take the form

$$\Xi_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix} , \quad \rho_j = \sqrt{1 - |\alpha_j|^2}, \tag{1.4.4}$$

while $\Xi_0 = (1)$ and $\Xi_N = (\overline{\alpha}_N)$ are 1×1 matrices. Define the $N \times N$ sparse matrix

$$E = LM, \tag{1.4.5}$$

and suppose that the entries α_j are independent complex random variables with $\alpha_j \sim \Theta_{\beta(N-j)+1}$ for $1 \leqslant j \leqslant N-1$ and α_N is uniformly distributed on the unit circle. Then the eigenvalues of E are distributed according to the Circular Ensemble (1.4.1) *at temperature β^{-1} and potential $V = 0$.*

Remark 1.4.2.

- *The matrix E is called a CMV matrix (after Cantero, Moral and Velazquez [CMV05]).*
- *Because the Ξ_j's are unitary matrices, so are L, M and E, therefore the eigenvalues of E indeed land on the unit circle.*

We will be interested in the high temperature regime, that is in the case where $\beta = 2\alpha/N$ for some $\alpha > 0$. As in the real case, the circular β ensemble at high temperature displays large deviations for its empirical measure

$$\hat{\mu}_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{e^{i\theta_j}} \,,$$

as can be deduced from [Ber18], [GZ19].

Theorem 1.4.3. *Let $\beta = \frac{2\alpha}{N}$, with fixed $\alpha > 0$ and assume $V : \mathbb{T} \to \mathbb{R}$ to be continuous. Define for any $\mu \in \mathcal{P}(\mathbb{T})$ absolutely continuous with respect to the Lebesgue measure the functional*

$$\begin{aligned} f_\alpha^V(\mu) = -\alpha \int_{\mathbb{T}\times\mathbb{T}} \log\left(|e^{i\theta} - e^{i\varphi}|\right) \mu(d\theta)\mu(d\varphi) + \alpha \log(2) + \int_{\mathbb{T}} V(\theta)\mu(d\theta) + \\ \int_{\mathbb{T}} \log\left(\frac{d\mu}{d\theta}(\theta)\right) \mu(d\theta) + \log(2\pi) \,, \end{aligned} \tag{1.4.6}$$

then

i. the functional $f_\alpha^V(\mu)$ is strictly convex and achieves its minimal value at a unique probability measure μ_α^V, absolutely continuous with respect to the Lebesgue measure;

ii. the sequence $(\hat{\mu}_N)$ satisfies a large deviation principle in $\mathcal{P}(\mathbb{T})$ equipped with the weak topology at speed αN with rate function defined for absolutely continuous $\mu \in \mathcal{P}(\mathbb{T})$ with respect to Lebesgue measure by $I_\alpha^V(\mu) = f_\alpha^V(\mu) - f_\alpha^V(\mu_\alpha^V)$, and $I_\alpha^V(\mu) = +\infty$ otherwise. In particular

$$\mu_N \xrightarrow[N\to\infty]{a.s.} \mu_\alpha^V \,. \tag{1.4.7}$$

In particular, this last theorem implies the following law of large numbers: Let $\boldsymbol{\theta}$ be distributed according to $\mathbb{P}_{C,N}^{V,\frac{2\alpha}{N}}$. If f is bounded measurable, we have the almost sure convergence

$$\frac{1}{N}\sum_{j=1}^{N} f(\theta_j) \to \int_{\mathbb{T}} f(\theta) d\mu_\alpha^V(\theta)\,.$$

A natural question following this fact is the one of the fluctuations around the limit. Fix a *reasonable* (smooth enough) function $\psi : \mathbb{T} \to \mathbb{R}$. Can one find $\eta > 0$ such that the random variables

$$N^\eta \left(\frac{1}{N}\sum_{j=1}^{N} \psi(\theta_j) - \int_{\mathbb{T}} \psi(\theta) d\mu_\alpha^V(\theta) \right)$$

converge in distribution towards a non trivial random variable ? In [HL21], Hardy and Lambert solve this problem with $\eta = 1/2$ and the limiting law being Gaussian with zero mean and variance depending of the test function f.

Theorem 1.4.4 (CLT for the circular β-ensemble at high temperature, [HL21])**.** *Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)$ be distributed with respect to* (1.4.1)*, with $\beta = \frac{2\alpha}{N}$ for some $\alpha > 0$. Let ψ be $\mathcal{C}^{2\gamma+1}(\mathbb{T})$ for some integer $\gamma \geqslant 2$. Then, we have the convergence in distribution*

$$\sqrt{N}\left(\frac{1}{N}\sum_{j=1}^{N} f(\theta_j) - \int_{\mathbb{T}} f(\theta) d\mu_\alpha^V(\theta) \right) \to N(0, \sigma_{\alpha,V}^2(\psi))\,,$$

where $\sigma^2(\psi) = \langle \psi, \mathcal{L}^{-1}\psi \rangle_{\mathrm{H}}$, $\mathcal{L}$ being a linear operator on some Hilbert space H*.*

Remark 1.4.5. *We do not explicit here the form of the operator $\mathcal{L}$. We will come back to it (more precisely, we will come back to its real line counterpart) when presenting the CLT in the β-ensemble on the real line at high temperature. For the moment, we refer to [HL21] for detail. Let us also mention that the proof of Hardy and Lambert is based on a normal approximation method, that was first established in [LLW19] for the β-ensemble on the real line* ***in the fixed $\beta > 0$ regime****. Using this approach, they are able to derive a Berry-Esseen bound, that is a speed of convergence towards the limiting Gaussian.*

1.5 Integrable systems: the example of the Toda chain

In this section we present a notion of *integrability* for systems of interacting particles, evolving with time. We give the main ideas by illustrating them on the practical example of the *Toda chain*, also called the *Toda lattice*. It was introduced in [Tod67] as a an example of one dimensional, continuous time and discrete space system with nonlinear interaction, which has soliton solutions, *i.e.* traveling solutions whose shape is conserved along time.

The (periodic) Toda chain is a system of N particles evolving on the real line, labeled $1 \leqslant j \leqslant N$, with positions $\mathbf{q}(t) = (q_j(t))_{1\leqslant j\leqslant N}$, momenta $\mathbf{p}(t) = (p_j(t))_{1\leqslant j\leqslant N}$, and governed by the system of ordinary differential equations

$$\frac{dq_j}{dt} = p_j, \quad \frac{dp_j}{dt} = e^{-r_{j-1}} - e^{-r_j} \quad 1 \leqslant j \leqslant N\,, \tag{1.5.1}$$

where $r_j = q_{j+1} - q_j$ is the *stretch* between particles $j+1$ and j, and where we assume the periodicity boundary condition

$$q_{N+1} = q_1 + \ell \tag{1.5.2}$$

for some $\ell \in \mathbb{R}$.

The particles are not supposed to be ordered, thus the stretches can take negative values. The set of equations (1.5.1) is a Hamiltonian system with Hamiltonian

$$H = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + e^{-r_j} ,$$

i.e. (1.5.1) can be written

$$\frac{dq_j}{dt} = \frac{\partial H(\mathbf{q},\mathbf{p})}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H(\mathbf{q},\mathbf{p})}{\partial q_j} \quad 1 \leqslant j \leqslant N .$$

An important feature of the Toda chain is its nearest neighbors type of interactions: during an infinitesimal time, particle j only interacts with particles $j+1$ and $j-1$. We emphasize that we do not suppose any ordering on the positions of the particles, and the expression "nearest neighbors" has to be taken in the sense of "nearest labels". An equivalent way of seeing the Toda dynamic is as a disretization of a continuous field $(x,t) \mapsto q(x,t)$. With this point of view, neighboring sites interact during an infinitesimal time.

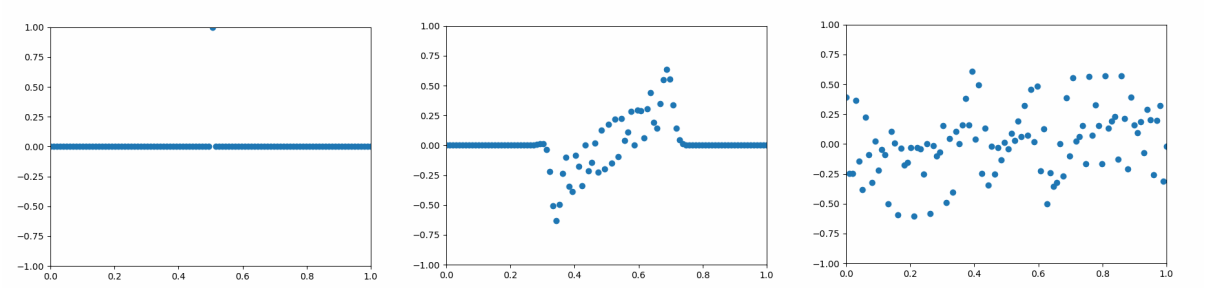

Figure 1.1: Simulation of the Toda chain with total stretch $\ell = 0$ and 'Dirac' initial condition $q_{N/2} = 1$, $q_j = 0$ for $j \neq N/2$, and $p_j = 0$ for all $1 \leqslant j \leqslant N$, at three different times. The sites are represented on the x-axis, and on the y-axis we display the value of q at the corresponding site.
Left: 'Dirac' initial condition.
Middle: After a short time. The impulsion propagates through the nearest neighbors interaction of the Toda chain.
Right: After some time. The systems almost looks random.

Aside from the obvious conserved quantity

$$\sum_{j=1}^{N} r_j = \ell,$$

the N-particles Toda chain was suspected via computer simulations, see [FST73], to have N nontrivial conserved quantities. This fact was established by Hénon [Hen74], who gave formulae for those constants of motion. Approximately in the same time, Flaschka [Fla74b] and Manakov [Man74] gave independently an alternative proof of the existence of those conserved quantities. We use here the notations of Flaschka: Introducing the variable $a_j = e^{-r_j/2}$, the dynamic (1.5.1) is equivalent to

$$\frac{da_j}{dt} = \frac{1}{2} a_j (p_j - p_{j+1}), \quad \frac{dp_j}{dt} = a_{j-1}^2 - a_j^2, \quad 1 \leqslant j \leqslant N . \tag{1.5.3}$$

Introducing the $N \times N$ matrices

$$L_N = \begin{pmatrix} p_1 & a_1 & 0 & \dots & a_N \\ a_1 & p_2 & a_2 & \ddots & 0 \\ 0 & a_2 & p_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ a_N & 0 & \dots & a_{N-1} & p_N \end{pmatrix}, \qquad B_N = \frac{1}{2} \begin{pmatrix} 0 & -a_1 & 0 & \dots & a_N \\ a_1 & 0 & -a_2 & \ddots & 0 \\ 0 & a_2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -a_{N-1} \\ -a_N & 0 & \dots & a_{N-1} & 0 \end{pmatrix}, \tag{1.5.4}$$

the set of equations (1.5.3) can then be rewritten as

$$\frac{dL_N}{dt} = L_N B_N - B_N L_N \,.$$

The couple (L_N, B_N) is called a **Lax pair**.

In this manuscript, by "integrable system", we mean "system possessing a Lax pair".

Definition 1.5.1 (Integrable system)**.** *A system of N ordinary differential equations is said the be integrable if it possesses a Lax pair, i.e if the dynamic is equivalent to*

$$\frac{dL}{dt} = [L, B] = LB - BL \,, \tag{1.5.5}$$

where L and B are $N \times N$ matrices.

Remark 1.5.2. *In general, a system of N particles evolving with time is said to be integrable if it possesses N independent conserved quantities. As we will now see, having a Lax pair guarantees this property.*

Having a Lax pair is a very special property which has an immediate consequence about the existence of conserved quantities. Indeed, with (L, B) of the previous definition, let S be the solution of the ODE

$$\frac{dS}{dt} = -B(t)S(t), \quad S(0) = I \,. \tag{1.5.6}$$

Then, considering its determinant one sees that S is invertible at any time and that the matrix $S(t)L(0)S(t)^{-1}$ satisfies the Lax equation (1.5.5) and is equal to $L(0)$ at time zero, thus

$$L(t) = S(t)L(0)S(t)^{-1} \,. \tag{1.5.7}$$

This ensures that the eigenvalues of L are conserved quantities. Furthermore, up to computing the matrix $S(t)$ and recovering the data of interest from L, the system is solved, hence the integrability. It is worth to mention that finding a formula for $S(t)$ is a delicate task in general.

Others examples of N-particles integrable systems (in the Lax pair sense) are given by the **Ablowitz-Ladik lattice** [AL75] and the **Schur flow** [Gol06] (which we introduce in Chapter 3), the **exponential Toda lattice** [GGGM23], the **Volterra lattice** and the **Itoh-Narita-Bogoyavleskii lattices** [Bog91] (we introduce the latter three systems in Chapter 5). An important feature that those systems share is, as in the Toda lattice case, the locality/nearest neighbors structure of the interactions. As a consequence, the Lax matrices of all those systems are periodic band matrices with **band size independent of N**, meaning that for each of these systems, if (L, B) is the associated Lax pair, there is some $K \in \mathbb{N}$ such that

$$L_{i,i+a} = 0 \text{ and } B_{i,i+a} = 0 \text{ whenever } |a| > K \,,$$

where the indices are considered modulo N.

We now introduce the so-called Generalized Gibbs Ensemble (GGE) of the Toda chain.

1.6 Generalized Gibbs Ensemble, and a link with β-ensembles

Let us start by recalling the following geometrical fact for Hamiltonian systems. We defer to the book [Gio22] for an introduction to the topic of Hamiltonian dynamics.

Theorem 1.6.1 (Preservation of Lebesgue measure)**.** *Let $(\mathbf{q},\mathbf{p}) = (q_j,p_j)_{1\leqslant j\leqslant N} \in \mathbb{R}^N \times \mathbb{R}^N$ be a Hamiltonian system with Hamiltonian $H(\mathbf{q},\mathbf{p})$, i.e. the equations of motions are given by*

$$\frac{dq_j}{dt} = \frac{\partial H(\mathbf{q},\mathbf{p})}{\partial p_j}\,, \quad \frac{dp_j}{dt} = -\frac{\partial H(\mathbf{q},\mathbf{p})}{\partial q_j}\,. \tag{1.6.1}$$

Then, the Lebesgue measure λ_{2N} on $\mathbb{R}^N \times \mathbb{R}^N$ is preserved by the flow of (1.6.1), that is, denoting the flow Φ_t defined by $(\mathbf{q}(t),\mathbf{p}(t)) = \Phi_t(\mathbf{q}(0),\mathbf{p}(0))$, one has for any Borelian set $A \subset \mathbb{R}^N \times \mathbb{R}^N$

$$\lambda_{2N}\left(\Phi_t^{-1}A\right) = \lambda_{2N}(A)\,.$$

As we saw in the previous section, if $V : \mathbb{R} \to \mathbb{R}$ is any function, $\mathrm{Tr}V(L)$ and $\sum_{j=1}^N r_j$ are conserved quantities of the Toda chain. In combinations with Theorem 1.6.1, the following measure, introduced by Spohn in [Spo20c] is invariant under the Toda flow.

Definition 1.6.2 (GGE for the Toda lattice)**.** *Let $V : \mathbb{R} \to \mathbb{R}$ such that there exist c, C such that*

$$V(x) \geqslant cx^2 + C \quad \text{for all } x \in \mathbb{R}\,. \tag{1.6.2}$$

The Generalized Gibbs Ensemble of the Toda chain with potential V and pressure $P > 0$ is the probability measure on $\mathbb{R}^N \times \mathbb{R}^N$ given by

$$d\mathbb{T}_N^{P,V}(p,r) = \frac{1}{\mathbb{Z}_{N,\mathbb{T}}^{V,P}} \exp\left\{-\mathrm{Tr}(V(L_N))\right\} \prod_{i=1}^N e^{-Pr_i} dr_i dp_i\,. \tag{1.6.3}$$

Remark 1.6.3.

i) Invariance of $\mathbb{T}_N^{V,P}$ means that if the initial condition $(\mathbf{q}(0),\mathbf{p}(0))$ is sampled with respect to $\mathbb{T}_N^{V,P}$, then the law of $(\mathbf{q}(t),\mathbf{p}(t))$ is $\mathbb{T}_N^{V,P}$ at any time t.

ii) Condition (1.6.2) is enough to ensure that the partition function $\mathbb{Z}_{N,\mathbb{T}}^{V,P}$ converges.

The idea behind the introduction of the GGE (which can be introduced also for the other previously mentioned integrable systems) is that when trying to understand the large N behavior of those systems, and despite of the possible formulas one would be able to extract via equations (1.5.6),(1.5.7) (which would be too complicated to analyse directly), a statistical approach is needed. In the framework of statistical mechanics, the observables of the Toda chain are expected to tend, as time goes to infinity, to be distributed according to $\mathbb{T}_N^{V,P}$. More precisely, one expects for reasonable $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and for generic initial condition $(\mathbf{q}(0),\mathbf{p}(0))$ the convergence

$$\frac{1}{T}\int_0^T f(\mathbf{q}(t),\mathbf{p}(t))dt \underset{T\to\infty}{\longrightarrow} \mathbb{E}_{N,\mathbb{T}}^{V,P}\left[f(\mathbf{q},\mathbf{p})\right]\,, \tag{1.6.4}$$

where the last expectation is taken with respect to $\mathbb{T}_N^{V,P}$ for some potential V and pressure parameter $P > 0$, depending on the initial condition and on the total stretch ℓ (1.5.2).

By invariance of the GGE, if the initial state is sampled according to $\mathbb{T}_N^{V,P}$, the law of the Lax matrix L_N is independent of time. Let us give here the fundamental observation made by Spohn in [Spo20c], allowing to relate the study of the Toda chain with the one of the β-ensemble at high temperature.

Consider the GGE for the Toda lattice with quadratic potential $\mathbb{T}_N^{x^2/2,P}$. With this choice of potential, we can compute the trace in (1.6.3) and find, in the notations of (1.5.4),

$$\frac{1}{2}\mathrm{Tr}L_N^2 = \frac{1}{2}\sum_{i=1}^N p_i^2 + \sum_{i=1}^N a_i^2\,,$$

and so the density of $\mathbb{T}_N^{x^2/2,P}$ factorizes into the law of independent random variables. More precisely, we find

$$p_i \sim N(0,1) \text{ and } \frac{a_i}{\sqrt{2}} \sim \chi_{2P}\,,$$

where we recall (1.2.4), i.e.

$$L \overset{(d)}{=} \begin{pmatrix} N(0,1) & \frac{\chi_{2P}}{\sqrt{2}} & & & \frac{\chi_{2P}}{\sqrt{2}} \\ \frac{\chi_{2P}}{\sqrt{2}} & N(0,1) & \frac{\chi_{2P}}{\sqrt{2}} & & \\ & \frac{\chi_{2P}}{\sqrt{2}} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{\chi_{2P}}{\sqrt{2}} \\ \frac{\chi_{2P}}{\sqrt{2}} & & & \frac{b_{N-1}}{\sqrt{2}} & N(0,1) \end{pmatrix}, \tag{1.6.5}$$

where the entries are independent up to the symmetry. This matrix model is reminiscent of the tridiagonal representation of the β-ensemble given by Theorem 1.2.5. We notice two differences in the structure of the matrix (1.6.5):

- Here, L_N is *periodic* tridiagonal, meaning that the entries $L_N(1,N)$ and $L_N(N,1)$ are nonzero.
- The law of the off diagonal variables is independent of their position: $L_N(j,j+1) \overset{(d)}{=} L_N(1,2)$, whereas in Theorem 1.2.5 the parameter defining the law of the off diagonal entries varies linearly with their position.

The first point is not going to play a big role in our analysis. Indeed, up to a perturbation of rank 2, L is a *true* tridiagonal matrix. We take advantage of the second point as follows:

Denote by $L_N^{(P)}$ the matrix of (1.6.5). Take $\beta = \frac{2P}{N}$ in the representation of Theorem 1.2.5. Then, in the top left corner of T, the off diagonal entries are χ-distributed, with parameter approximately equal to $2P$, almost matching the top corner of the Lax matrix $L_N^{(P)}$. Now, fix some $M \geqslant 1$ and consider the euclidean division of N by M: $N = kM + r$ for some $k \geqslant 1$ and $0 \leqslant r \leqslant M-1$. Assume here for simplicity that $r = 0$. Then, consider a sequence of $k \times k$ Lax matrices with varying pressure parameter $L_k^{(P(1-j/M))}$, $0 \leqslant j \leqslant M-1$, and form the block diagonal matrix

$$S = \mathrm{diag}\left(L_k^{(P)}, L_k^{(P(1-1/M))}, \dots, L_k^{(P/M)}\right).$$

Last, replace the entries $S(jk+1,(j+1)k)$ and $S((j+1)k,jk+1)$ for $0 \leqslant j \leqslant M-1$ by zero and denote by the resulting matrix by S'. Morally, replacing S by S' should not produce too big of a mistake because it can be written as a small rank perturbation. This statement can be made more precise:

Lemma 1.6.4. *Let A and B be $N \times N$ Hermitian matrices, and consider the distance d defined in* (1.3.2). *Then we have*

$$d(\hat{\mu}_N(A), \hat{\mu}_N(B)) \leqslant \min\left\{\frac{\mathrm{rank}(A-B)}{N}, \frac{1}{N}\sum_{1\leqslant i,j\leqslant N}|A_{i,j}-B_{i,j}|\right\}.$$

Now, both matrices S' and T are tridiagonal, their diagonal entries have same law and their off diagonal entries are *almost* the same in law: the difference between the parameters defining their distributions is

bounded by a $O(1/M)$. We can now compare the empirical measures: because S' is defined by blocks, its empirical measure reads

$$\hat{\mu}_N(S') = \frac{1}{M}\sum_{j=0}^{M-1} \hat{\mu}_k(B_j),$$

where the B_j's are the blocks of S'. Using Lemma 1.6.4,

$$\frac{1}{M}\sum_{j=0}^{M-1} \hat{\mu}_k(L_k(P(1-j/M))) \approx \frac{1}{M}\sum_{j=0}^{M-1} \hat{\mu}_k(B_j) \approx \hat{\mu}_N(T)\,,$$

thus when taking the limits $k \to \infty$ then $M \to \infty$, and assuming that for any $Q > 0$ the empirical measure $\hat{\mu}_k(L_k^{(Q)})$ converges towards some measure ν_Q as $k \to +\infty$, it is reasonable to expect the equality between measures

$$\int_0^1 \nu_{sP} ds = \mu_P^{x^2/2}\,, \tag{1.6.6}$$

where $\mu_P^{x^2/2}$ is the limiting measure of the high temperature β ensemble given by Proposition 1.3.6 and where this equality means that for any f bounded continuous one has

$$\int_0^1 \int f d\nu_{sP} ds = \int f d\mu_P^{x^2/2}\,.$$

At this point, natural questions arise:

i- Consider the measure $\mathbb{T}_N^{V,P}$ with general V. Does the empirical measure of the Lax matrix of the Toda chain converge ?

ii- If yes, does equation (1.6.6) hold ?

iii- Can this link between the Toda chain and the high temperature β-ensemble be pushed further, in other words, can we expect an analogue of (1.6.6) for different objects than empirical measures, for example free energies, limiting variances, ... ?

iv- Does a similar picture hold for other integrable systems/β-ensembles ?

In Chapter 2, we tackle questions *i* and *ii* by establishing a large deviation principle for the empirical measure of the Toda chain, and comparing it with the large deviations for the empirical measure of the high temperature β-ensemble.

In Chapter 3, we show a similar Large deviation principle for the empirical measure of the Ablowitz-Ladik lattice and for the Schur flow, and link them respectively to the **circular** β-ensemble and the **Jacobi** β-ensemble at high temperature: this goes in the direction of question *iv*, showing that this picture is valid for other integrable systems than the Toda chain.

Those chapters are based on two publications of the present author, respectively [GM22] in collaboration with Alice Guionnet, and [MM23b] in collaboration with Guido Mazzuca.

We now present the idea and stategy of derivation of a hydrodynamic equation for integrable systems, following [Spo21], in the case of the Toda lattice. This will further motivate the results of Chapters 2 and 3, and motivate the results of Chapters 4 and 5.

1.7 Hydrodynamics, currents and fluctuations

An important - and open in most cases - question is the existence of a **hydrodynamic limit** of particle systems, that is, informally, an equation over the density (number of particles per space unit) of particles after a suitable rescaling:

- a scaling in space so that the distance between two particles is of order $1/N$, where N is the number of particles in the system,
- a scaling in time such that after rescaling, in a time window of order one, one observes a macrospopic evolution of the system.

Being able to derive such an equation on the density of particles is crucial to study the system from a global perspective, rather than looking at the microscopic scale of single particles. However, a mathematically rigorous proof of such a limit is very often far from reach.

As explained in [Spo21], the question of a hydrodynamic limit for integrable systems has been tackled recently through the introduction of Generalized Gibbs Ensembles (as the one of Definition 1.6.3), based on the idea that those systems should locally behave, in a suitable time scale, as if they where distributed following a GGE. Before being more precise on this last point, let us mention that the structure of the equation is expected to be common for a wide class of integrable systems, see for example [Spo22b] for the Ablowitz-Ladik case, [Spo21][Sections 5 and 6] for the hard rods system and the for Toda chain respectively.

In the example of the N particles periodic Toda chain, the space scaling is $1/N$ so that the distance between two neighboring sites is of order $1/N$. Let Φ_j be a **local quantity**, that is: there is some K independent of N such that $\Phi_j(t)$ only depends on particles $j-K, j-K+1, \dots, j, \dots, j+K$ at time t. Then it is expected that:

For any $t > 0$ and $x \in [0,1]$, as $j/N \to x$, there is a potential-pressure couple $(V_{x,t}, P_{x,t})$ such that

$$\frac{1}{Nt}\int_0^{Nt} \Phi_j(s)ds \underset{N\to\infty}{\longrightarrow} \lim_N \mathbb{E}_N^{V_{x,t},P_{x,t}}\left[\Phi_j(0)\right] . \tag{1.7.1}$$

This last equation is called a **local equilibrium** property at speed N for the system. The hydrodynamic scaling in time is then $t \to Nt$. The local equilibrium assumption is at the basis of the derivation of the hydrodynamic limit, and it is the biggest difficulty in a mathematical proof of such a limit. However, based on this assumption, Spohn is able to derive a candidate hydrodynamic equation. The strategy is based on the fact that the conserved quantities are local. Following the notations of [Spo21], we define for $n \geqslant 1$

$$Q_j^{[n],N} = L_N^n(j,j)$$

so that the conserved quantity $\mathrm{Tr}L_N^n$ can be written

$$\mathrm{Tr}L_N^n = \sum_{j=1}^N Q_j^{[n],N}$$

where $Q_j^{[n],N}$ only depends on a_{j+i}, p_{j+i}, $-n \leqslant i \leqslant n$ due to the tridiagonal structure of the matrix L_N. For $n = 0$, we denote

$$Q_j^{[0],N} = r_j ,$$

and the associated conserved quantity is the total stretch $\ell = \sum_{j=1}^N r_j$. We emphasize that while $\mathrm{Tr}L_N^n$ is invariant, each $Q_j^{[n],N}$ depends on time. Now, define the **currents** $J_j^{[n],N}$ (also depending on time) via the conservation law

$$\frac{d}{dt}Q_j^{[n],N} + \partial_j J_j^{[n],N} = 0, \quad 1 \leqslant j \leqslant N ,$$

where ∂_j is the difference operator $\partial_j f = f(j+1) - f(j)$.

Because of the local equilibrium assumption (1.7.1), it is enough to compute, for arbitrary pressure-potential couple (P, V) the limiting means

$$\lim_N \mathbb{E}_N^{V,P}\left[Q_j^{[n],N}(0)\right] \text{ and } \lim_N \mathbb{E}_N^{V,P}\left[J_j^{[n],N}(0)\right] . \tag{1.7.2}$$

Computing those means for each $n \geqslant 1$, Spohn is able to derive the hydrodynamic limit for the Toda lattice [Spo21][Section 6] (establishing it in the weak sense, tested against all $x \mapsto x^n$, $n \geqslant 0$):

With the local equilibrium assumption (1.7.1), denote around the macroscopic point x at time t the average stretch

$$\nu(x,t) = \lim_N \mathbb{E}_N^{V_{x,t},P_{x,t}}[r_1]$$

and the limiting measure

$$\mu(x,t;v) = \rho(x,t;v)dv, \quad \mu(x,t) = \lim_N \mathbb{E}_N^{V_{x,t},P_{x,t}}\left[\frac{1}{N}\sum_{j=1}^N \delta_{\lambda_j(L_N)}\right].$$

Set $\rho_{\mathsf{P}}(x,t;v) = \nu(x,t)\rho(x,t;v)$ and define implicitly the *effective velocity* through

$$v_{\text{eff}}(x,t;v) = v + 2\int_{\mathbb{R}} \log(|v-w|\rho_{\mathsf{P}}(x,t;w))(v_{\text{eff}}(x,t;w) - v_{\text{eff}}(x,t;v))dw\,.$$

Finally, defining

$$q_1(x,t) = \nu(x,t)\int_{\mathbb{R}} v\rho_{\mathsf{P}}(x,t;v)dv\,,$$

the hydrodynamic equation reads

$$\begin{aligned}&\partial_t\nu(x,t) - \partial_x q_1(x,t) = 0\\ &\partial_t\big(\nu(x,t)\rho_{\mathsf{P}}(x,t;v)\big) + \partial_x\big([v_{\text{eff}}(x,t;v) - q_1(x,t)]\rho_{\mathsf{P}}(x,t;v)\big) = 0\,.\end{aligned}$$

We now give the scheme for the derivation of limits (1.7.2), using the heuristic link between the Toda chain and the high temperature β-ensemble explained earlier.

- By translation invariance of the law of the entries of the Lax matrix L_N under $\mathbb{T}_N^{V,P}$, we have

$$\mathbb{E}_N^{V,P}\left[Q_j^{[n],N}(0)\right] = \frac{1}{N}\mathbb{E}_N^{V,P}\left[\mathrm{Tr}L_N^n\right] = \mathbb{E}_N^{V,P}\left[\int_{\mathbb{R}} x^n d\hat{\mu}_N\right],$$

and computing this quantity when N goes to infinity is thus associated with the question of the limiting empirical measure of eigenvalues under the GGE $\mathbb{T}_N^{V,P}$

$$\lim_N \hat{\mu}_N =: \nu_P^V\,.$$

By the heuristics leading to (1.6.6), and assuming that it is valid under a general potential V, we get the formula

$$\nu_P^V = \partial_P\left(P\mu_P^V\right), \tag{1.7.3}$$

and because of the characterization of μ_P^V given by Proposition 1.3.6, we get a characterization of the limiting density of ν_P^V.

- By the Lax pair formulation of the Toda chain

$$\frac{dL_N}{dt} = [L_N, B_N]\,,$$

we can compute the evolution equation for such quantities as

$$\frac{d}{dt}Q_j^{[n],N} = (BL^n - L^nB) = b_{j-1}L^n_{j,j-1} - b_jL^n_{j+1,j}\,. \tag{1.7.4}$$

Defining $J_j^{[n],N} = b_{j-1}L^n_{j,j-1}$, we have

$$\frac{d}{dt}Q_j^{[n],N} = J_j^{[n],N} - J_{j+1}^{[n],N}\,.$$

$J_j^{[n],N}$ is then the current of the local conserved field $Q_j^{[n],N}$. In particular, defining the matrix $L_N^{\downarrow}$ as

$$L_N^{\downarrow}(i,j) = \begin{cases} L_N(i,j) & \text{if } j < i \text{ or } i = 1, j = N \\ 0 & \text{otherwise} \end{cases} \tag{1.7.5}$$

we can recast the previous definition as

$$J_j^{[n],N} = (L_N^n L_N^{\downarrow})(j,j)\,, \tag{1.7.6}$$

giving a formula for the currents in terms of the Lax matrix L_N. A priori, the derivation of the limiting current

$$\lim_N \mathbb{E}_N^{V,P}[J_j^{[n],N}] = \lim_N \frac{1}{N}\mathbb{E}_N^{V,P}[\mathrm{Tr}L_N^n L_N^{\downarrow}]$$

is more intricate than the derivation of the limiting average empirical measure, this expression not being expressible only in terms of the eigenvalues of L_N. However, as explained in [Spo21], it can be expressed via the associated *fluctuations*, by showing that taking derivatives with respect to P, and modulo the *decay of correlations* between local observables,

$$\partial_P \lim_N \mathbb{E}_N^{V,P}[J_j^{[n],N}] = \lim_N \frac{1}{N}\mathrm{Cov}_N^{V,P}(\mathrm{Tr}L_N, \mathrm{Tr}L_N^n)\,, \tag{1.7.7}$$

where the last covariance is taken with respect to $\mathbb{T}_N^{V,P}$. From this perspective, it would then be sufficient to show that the spectral statistics of the Toda Lax matrix

$$\int_{\mathbb{R}} x^n d\hat{\mu}_N(x)$$

satisfy a Central Limit Theorem at speed $\sqrt{N}$, i.e. that for any $n \geqslant 1$, there is some $\sigma^2_{\mathrm{Toda}}(V,P,n)$ such that we have as $N \to \infty$ the convergence in distribution towards a Gaussian random variable

$$\sqrt{N}\left(\int_{\mathbb{R}} x^n d\hat{\mu}_N(x) - \int_{\mathbb{R}} x^n d\nu_P^V(x)\right) \xrightarrow{(d)} N(0, \sigma^2_{\mathrm{Toda}}(V,P,n))\,.$$

To compute the currents, it would then suffice to compute the limiting variance $\sigma^2_{\mathrm{Toda}}(V,P,n)$. As an analogue of equation (1.6.6), **provided there is a central limit theorem for the empirical measure in the β ensemble at high temperature** with limiting variance σ^2_{HT}, we might expect the following equation between the limiting variances

$$\int_0^1 \sigma^2_{\mathrm{Toda}}(V, sP, n)ds = \sigma^2_{\mathrm{HT}}(V,P,n), \quad \textit{i.e.} \quad \sigma^2_{\mathrm{Toda}}(V,P,n) = \partial_P\left(P\sigma^2_{\mathrm{HT}}(V,P,n)\right)\,, \tag{1.7.8}$$

therefore allowing to compute the limiting average currents in terms of the limiting variance in the CLT for the high temperature β-ensemble. **Chapter 4** is devoted to the Central Limit Theorem for the β-ensemble at high temperature, based on the joint work with Charlie Dworaczek-Guera [DGM23]. **Chapter 5** is devoted to the CLT in integrable systems and to their link with high temperature β-ensembles, allowing to justify mathematically equation (1.7.8) not only for the couple Toda chain/real β-ensemble at high temperature but for several other couples integrable systems/high temperature β-ensemble. It is based on the joint work with Guido Mazzuca [MM23a].

We now introduce the Ablowitz-Ladik lattice, which is an integrable system, similar to the Toda lattice in many points: in particular, it can be compared to the high temperature circular β-ensemble.

1.8 The Ablowitz-Ladik lattice

The (periodic) **Ablowitz-Ladik lattice** is the system of ODEs

$$i\frac{da_j}{dt} = -(a_{j+1} + a_{j-1} - 2a_j) + |a_j|^2(a_{j-1} + a_{j+1})\,, \tag{1.8.1}$$

that describe the evolution of the complex functions $a_j(t)$, $j \in \mathbb{Z}$ and $t \in \mathbb{R}$. We assume N-periodic boundary conditions $a_{j+N} = a_j$, for all $j \in \mathbb{Z}$.

This system was introduced by Ablowitz and Ladik in [AL75, AL76] as a spatial discretization of the defocusing **cubic Nonlinear Schrödinger Equation**

$$i\partial_t \psi(x,t) = -\frac{1}{2}\partial_x^2 \psi(x,t) + |\psi(x,t)|^2 \psi(x,t). \tag{1.8.2}$$

This equation is known to be integrable, in the sense that it possesses a Lax pair (the elements of the pair being infinite dimensional operators), see [ZS72]. The Ablowitz-Ladik lattice is a discretization of (1.8.2) that is itself integrable, as was shown by Ablowitz and Ladik in [AKN74, AL75] by discretizing the Lax pair of the Cubic nonlinear Schrödinger equation.

In [Nen05, Sim05], Nenciu and Simon constructed another Lax pair for the Ablowitz-Ladik lattice as follows. We take here N to be even for simplicity.

As a preliminary remark, we notice by a direct computation that the quantity

$$K^{(0)} := \prod_{j=1}^{N} \left(1 - |a_j|^2\right) \tag{1.8.3}$$

is a constant of motion for the AL lattice. This implies that if $|a_j(0)| < 1$ for all $j = 1, \dots, N$, then $|a_j(t)| < 1$ for all times. Therefore, considering an initial condition having coordinates in the unit disk, $\boldsymbol{a} = (a_1, \dots, a_N) \in \overline{\mathbb{D}}^N$, we can form the 2×2 unitary matrices Ξ_j (well defined at any time because of our preliminary remark)

$$\Xi_j = \begin{pmatrix} \overline{a}_j & \rho_j \\ \rho_j & -a_j \end{pmatrix}, \quad j = 1, \dots, 2N, \quad \rho_j = \sqrt{1 - |a_j|^2}. \tag{1.8.4}$$

Then, we define the $2N \times 2N$ matrices $\mathcal{L}$ and $\mathcal{M}$ as

$$\mathcal{L} = \begin{pmatrix} \Xi_1 & & & \\ & \Xi_3 & & \\ & & \ddots & \\ & & & \Xi_{2N-1} \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} -a_{2N} & & & \rho_{2N} \\ & \Xi_2 & & \\ & & \ddots & \\ & & \Xi_{2N-2} & \\ \rho_{2N} & & & \overline{a}_{2N} \end{pmatrix}. \tag{1.8.5}$$

Finally, define $\mathcal{E}$ as

$$\mathcal{E} = \mathcal{L}\mathcal{M} \tag{1.8.6}$$

and $\mathcal{E}^+$ as

$$\mathcal{E}^+_{j,k} = \begin{cases} \frac{1}{2}\mathcal{E}_{j,j} & j = k \\ \mathcal{E}_{j,k} & k = j+1 \bmod N \text{ or } k = j+2 \bmod N \\ 0 & \text{otherwise.} \end{cases} \tag{1.8.7}$$

Then, the equations (1.8.1) are equivalent to the Lax equation

$$\frac{d\mathcal{E}}{dt} = \left[\mathcal{E}, i\left(\mathcal{E}^+ + (\mathcal{E}^+)^\dagger\right)\right]$$

where † stands for the hermitian conjugate. We recognize a construction similar to the one of Theorem 1.4.1: the Lax matrix $\mathcal{E}$ is a **periodic CMV matrix**. In the spirit of Definition 1.6.2, and as in [Spo22b, GM23], we introduce the Generalized Gibbs ensemble for the Ablowitz-Ladik lattice as

Definition 1.8.1 (GGE for the Ablowitz-Ladik lattice)**.** *Let $\alpha > 0$ and let $V : \mathbb{T} \to \mathbb{R}$ be a continuous function on the torus. Denote by $\mathbb{D}$ the unit disk.*

The Generalized Gibbs Ensemble (GGE) with parameters α, V for the Ablowitz-Ladik lattice is the probability measure on $\mathbb{D}^N$ given by

$$d\mathbb{P}^{V,\alpha}_{AL,N}(a_1,\dots,a_N) = \frac{1}{Z^{V,\alpha}_{AL,N}} \prod_{j=1}^{N} (1-|a_j|^2)^{\alpha-1} \mathbb{1}_{a_j\in\mathbb{D}} e^{-\mathrm{Tr}V(\mathcal{E})} \,. \tag{1.8.8}$$

Remark 1.8.2. *Because the state space $\mathbb{D}^N$ is compact, any choice of continuous V ensures that the partition function*

$$Z^{V,\alpha}_{AL,N} = \int_{\mathbb{D}^N} \prod_{j=1}^{N} (1-|a_j|^2)^{\alpha-1} \mathbb{1}_{a_j\in\mathbb{D}} e^{-\mathrm{Tr}V(\mathcal{E})}$$

converges.

We now make the fundamental observation that when taking the special potential $V(x) = 0$ (instead of $V(x) = x^2/2$ in the Toda case), the a_i's are independent and identically distributed with distribution $\Theta_{2\alpha+1}$, where we recall that the Θ distribution is given by (1.4.2). As in the Toda case, we note that with this specific choice of potential, the Lax matrix $\mathcal{E}$ resembles the matrix model for the circular β-ensemble given by Theorem 1.4.1. As before, the main difference between both models is that in the Ablowitz-Ladik case, the law of the entries does not depend on their location in the matrix, whereas the parameter in the law of the coefficients of the Killip and Nenciu representation depend linearly on their position. The link between Ablowitz-Ladik lattice and circular β-ensemble therefore seems to be of the same nature than the one between Toda and β ensemble on the real line. We thus expect the ideas we presented in the previous section to apply in this case also. In particular, Spohn derived the hydrodynamics of the Ablowitz-Ladik lattice in [Spo22a] following those ideas.

We close this section by emphasizing that those ideas are expected to be fairly general, as the Toda chain and the Ablowitz-Ladik lattice are not the only examples of systems displaying a link with a β ensemble in the high temperature regime. In [GGGM23], the authors draw this link for several integrable systems/β-ensembles, which are listed in the following table.

Integrable System	**β-ensemble at high-temperature**
Toda lattice	Real
Defocusing Ablowitz-Ladik lattice	Circular
Exponential Toda lattice	Laguerre
Defocusing Schur flow	Jacobi
Volterra lattice	Antisymmetric Gaussian

Table 1.1: Integrable systems and random matrix ensembles

Beyond the question of hydrodynamics, the correspondance between integrable systems and random matrix ensembles could help to have a broader understanding of both subjects. It would also be fascinating to understand when an integrable system can be mapped to a β-ensemble and vice-versa.

1.9 Some related works

We list here some works in the context of the previously described ideas. We do not re-cite all of the papers we already mentioned, but those are of course to be considered as related to our works.

The idea of analysing the Lax matrix of integrable systems from a random matrix perspective appeared in [BGS09, BGS11]. This approach then regained attention through the work of Spohn [Spo20c, Spo20a, Spo20b, Spo21, Spo22a]. The density of states (*i.e.* limiting empirical measure) of Laguerre, Jacobi and Real β-ensembles at high temperature, and the relation of the latter with the Toda chain, were investigated in [Maz22] in the case of a quadratic potential. In [GM23], the authors proved the analogue of (1.7.3)

for the Ablowtiz-Ladik lattice/circular β-ensemble at high temperature for polynomial potentials, using a transfer operator approach. The link between integrable systems and random matrix ensembles was investigated for the systems of Table 1.8 in [GGGM23]. Hydrodynamics and fluctuations for the hard rods system are considered in [DS17], [OF22]. The generalized hydrodynamics of the Sinh-Gordon model are considered in [BDWY18].

The real β-ensemble at high temperature was first considered [CL97]. In [ABG12], the authors derived the limiting density of eigenvalues for the quadratic potential. The large deviations for high temperature Gibbs measures were established in [GZ19], with a direct application for the β-ensembles at high temperature. The fluctuations around the equilibrium measure, and the local statistics for the high temperature β-ensembles were considered, among others, in [BGP15, NT18, Pak18, NT20, HL21, NTT23, Lam21].

1.10 Presentation of the results

1.10.1 Main results of Chapter 2

This chapter is based on the collaboration with Alice Guionnet [GM22]. It is motivated by the justification of formula (1.7.3) under the Generalized Gibbs Ensemble (1.6.3) with general potential V.

One way to tackle this problem is noticing that when the potential V is a polynomial, the interactions between the matrix entries are local. As noticed by Spohn in [Spo20c], this allows to use a transfer operator approach. This idea was used in [GM23] to prove the analogue of equation (1.7.3) in the case of the Ablowitz-Ladik lattice. We use this approach in Chapter 5.

Here, we adopt a large deviations strategy. In this chapter, we change the definition of $\mathbb{T}_N^{P,V}$ (1.6.3) by shifting the potential V by $x^2/2$:

$$d\mathbb{T}_N^{V,P}(p,r) = \frac{1}{\mathbb{Z}_{N,\mathbb{T}}^{V,P}} \exp\{-\mathrm{Tr}(V(L_N)) - \frac{1}{2}\mathrm{Tr}(L_N^2)\} \prod_{i=1}^{N} e^{-Pr_i} dr_i dp_i \,. \tag{1.10.1}$$

Notice that in particular, the definition of the normalizing constant $\mathbb{Z}_{N,\mathbb{T}}^{V,P}$ differs from the one of (1.6.3). This definition is convenient in the present setup as we want to compare general potentials to the quadratic one. The approach of large deviations allows us to tackle the case of potentials with polynomial growth. More precisely, we prove the following.

Theorem 1.10.1. *Let $P > 0$ and assume that V is continuous. Assume that either V is uniformly bounded or there exists $k \in \mathbb{N}^*$ such that*

$$\lim_{|x|\to\infty} \frac{V(x)}{x^{2k}} = a \,, \tag{1.10.2}$$

with $a > 0$ if $k > 1$ and $a > -1/2$ if $k = 1$. Then:

1. *The law of $\hat{\mu}_{L_N}$ under $\mathbb{T}_N^{V,P}$ satisfies a large deviation principle in the scale N with a good rate function, denoted T_P^V.*
2. *T_P^V achieves its minimal value at a unique probability measure ν_P^V.*
3. *As a consequence $\hat{\mu}_{L_N}$ converges almost surely and in L^1 towards ν_P^V.*

Our proof is made of two independent parts: the case of a potential of the form $\frac{x^2}{2} + V$ with V bounded continuous, which can be seen as a perturbative case of the quadratic potential, and the case of a general potential with polynomial growth. The second part applies to the first case, but the first approach has the advantage of setting the basic ideas.

Here also, we shift the potential by $x^2/2$ in the definition of the β ensemble. In this section, and in the corresponding chapter, we set

$$\mathrm{d}\mathbb{P}_N^{V,\beta}(x_1,\cdots,x_N) = \frac{1}{Z_{N,C}^{V,\beta}} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i=1}^N (\frac{1}{2}x_i^2 + V(x_i))} \mathrm{d}x_1 \cdots \mathrm{d}x_N \,, \tag{1.10.3}$$

and as before take $\beta = 2P/N$. **When needed, we add the subscript C (for Coulomb) and write $\mathbb{P}_{N,C}^{V,P}$** to avoid possible confusions between the β ensemble measure and the Toda GGE. **The measure $\mathbb{P}_{N,C}^{V,P}$ is not to be confused with the circular β-ensemble, which does not appear in this chapter.** In those notations, and with the ones of Definition 1.2.6,

$$\mathbb{P}_{N,C}^{V,\frac{2P}{N}} = \mathbb{P}_N^{V+\frac{x^2}{2},\frac{2P}{N}} \,.$$

Recall that as a consequence of [GZ19], the empirical measure of $\mathbf{x} = (x_1,\ldots,x_N)$ following the β ensemble $\mathbb{P}_N^{V,2P/N}$ satisfies a large deviation principle at scale N with rate function I_P^V (which is deduced from J_P^V of Theorem 1.3.5 by shifting V by $x^2/2$), given by

$$I_P^V(\mu) = \begin{cases} f_P^V(\mu) - \inf_{\nu\in\mathcal{P}(\mathbb{R})} f_P^V(\nu) \text{ if } \mu \text{ is absolutely continuous with respect to Lebesgue measure} \\ +\infty \text{ otherwise,} \end{cases} \tag{1.10.4}$$

where, setting ρ for the density of μ with respect to Lebesgue measure, f_P^V is given by

$$f_P^V(\mu) = -P \iint_{\mathbb{R}^2} \log|x-y| d\mu(x) d\mu(y) + \int_{\mathbb{R}} \rho(x)\log\rho(x)dx + \int_{\mathbb{R}} \left(V(x) + x^2/2\right) d\mu(x) \,.$$

The main point is that we are able to compare the minimizers of the Toda rate function T_P^V to the ones of J_P^V. This is done by comparing the *free energies* of both models $F_{\mathbb{T}}^{V,P} = \lim_{N\to\infty} \frac{1}{N}\mathbb{Z}_{N,\mathbb{T}}^{V,P}$ and $F_C^{V,P} = \lim_{N\to\infty} \frac{1}{N} Z_{N,C}^{V,\frac{2P}{N}}$, showing that the map $P \mapsto F_C^{V,P}$ is differentiable and

$$F_{\mathbb{T}}^{V,P} = \partial_P(PF_C^{V,P}) \,. \tag{1.10.5}$$

This identity is proven in the bounded case by expressing the rate function of the high temperature β-ensemble I_P^V in terms of the Toda rate function:

Theorem 1.10.2. *For any continuous function V such that*

$$\limsup_{|x|\to\infty} \frac{|V(x)|}{x^2} = 0, \tag{1.10.6}$$

the law of the empirical measure $\hat{\mu}_N$ under $\mathbb{P}_N^{V,2P/N}$ satisfies a large deviation principle in the scale N and with good rate function $I_P^V(\mu) = f_P^V(\mu) - \inf f_P^V$ where

$$f_P^V(\mu) = \lim_{\delta\to 0} \liminf_M \inf_{\substack{\nu_{P/M},\cdots,\nu_P s.t. \\ \frac{1}{M}\sum_i \nu_{iP/M} \in B_\mu(\delta)}} \left\{ \frac{1}{M} \sum_{i=1}^M (T_{iP/M}(\nu_{iP/M}) + \int V d\nu_{iP/M}) \right\}. \tag{1.10.7}$$

In the general case, equation (1.10.5) is proven directly, in Lemma 2.4.5.

Because of Varadhan's lemma, those free energies are related with the minimizers of the rate functions, and we deduce formula (1.7.3):

Theorem 1.10.3. *Let P be a positive real number. Then, denoting ν_P^V the minimizer of T_P^V and μ_P^V the minimizer of I_P^V, we have:*

for any bounded continuous function f on the real line,

$$\int f(x)d\nu_P^V(x) = \partial_P(P\int f(x)d\mu_P^V(x))\,.$$

Note that as a byproduct of this approach, we prove a large deviation principle for the empirical measure of general tridiagonal matrices with independent coefficients, see Theorem 2.2.4.

1.10.2 Main results of Chapter 3

Chapter 3 is based on the collaboration with Guido Mazzuca [MM23b]. It is motivated by the justification of the equivalent of equation (1.7.3) for the Ablowitz-Ladik lattice and the circular β-ensemble at high temperature. As noticed at the begining of Paragraph 1.10.1 for the Toda chain, one can approach this problem by a transfer operator approach: this was done in [GM23]. This approach allows the authors to conclude for polynomial potentials.

We tackle the problem using large deviations, in the spirit of the results of Chapter 2. We therefore consider the empirical measure of the matrix $\mathcal{E}$ introduced in equation (1.8.6)

$$\hat{\mu}_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^{N}\delta_{e^{i\theta_j(\mathcal{E})}}\,,$$

where the matrix $\mathcal{E}$ is distributed with respect to

$$d\mathbb{P}^{V,\alpha}_{\mathrm{AL},N}(a_1,\dots,a_N) = \frac{1}{Z^{V,\alpha}_{\mathrm{AL},N}}\prod_{j=1}^{N}(1-|a_j|^2)^{\alpha-1}\mathbb{1}_{a_j\in\mathbb{D}}e^{-\mathrm{Tr}V(\mathcal{E})}\,.$$

We then show

Theorem 1.10.4. *Let $\alpha > 0$. For any continuous function $V : \mathbb{T} \to \mathbb{R}$ the following holds:*

a. the sequence $\hat{\mu}_N(\mathcal{E})$ under the law $\mathbb{P}^{V,\alpha}_{AL,N}$ satisfies a large deviations principle at speed N with a good rate function J_α^V,

b. J_α^V achieves its minimum at a unique probability measure ν_α^V,

c. $\hat{\mu}_N(\mathcal{E})$ converges almost surely and in $L^1(\mathbb{T})$ towards ν_α^V.

To identify the limiting measure ν_α^V, we compare the Ablowitz-Ladik lattice to the circular β ensemble at high temperature $\mathbb{P}^{V,2\alpha/N}_{C,N}$(equation (1.4.1)), as previously motivated.

Recall that by Theorem 1.4.3, the empirical measure of the circular β-ensemble at high temperature satisfies a large deviation principle with rate function given for μ absolutely continuous with respect to Lebesgue measure by

$$I_\alpha^V(\mu) = f_\alpha^V(\mu) - \inf f_\alpha^V\,,$$

$I_\alpha^V(\mu) = +\infty$ otherwise, where

$$f_\alpha^V(\mu) = -\alpha\int_{\mathbb{T}\times\mathbb{T}}\log\left(|e^{i\theta}-e^{i\varphi}|\right)\mu(d\theta)\mu(d\varphi) + \alpha\log(2) + \int_{\mathbb{T}}V(\theta)\mu(d\theta) + \int_{\mathbb{T}}\log\left(\frac{d\mu}{d\theta}(\theta)\right)\mu(d\theta) + \log(2\pi)\,.$$

The rate function I_α^V is stricly convex. Denoting its unique minimizer μ_α^V, we then show the following equation linking ν_α^V to μ_α^V, for any choice of continuous $V : \mathbb{T} \to \mathbb{R}$.

Theorem 1.10.5. *For any continuous $V, f : \mathbb{T} \to \mathbb{R}$*

$$\int_{\mathbb{T}} f(\theta) d\nu_\alpha^V(\theta) = \partial_\alpha \left(\alpha \int_{\mathbb{T}} f(\theta) d\mu_\alpha^V(\theta) \right) . \tag{1.10.8}$$

The advantage of this setup is the compactness of the torus: as before, the proof is based on the comparison of the measure $\mathbb{P}_{\mathrm{AL},N}^{V,\alpha}$ to its version where $V = 0$, namely $\mathbb{P}_{\mathrm{AL},N}^{0,\alpha}$, for which the entries of the matrix $\mathcal{E}$ are easier to handle. As before, we are able to relate the free energies of both models by showing

$$F_{\mathrm{AL}}^{V,\alpha} = \partial_\alpha(\alpha F_C^{V,\alpha}). \tag{1.10.9}$$

This is done by establishing the reformulation of the circular β-ensemble rate function in terms of the Ablowtiz-Ladik one.

Theorem 1.10.6. *Let $\alpha > 0$, and $V : \mathbb{T} \to \mathbb{R}$ continuous. The law of the empirical measures $\hat{\mu}_N(E)$ under $d\mathbb{P}_{C,N}^{V,\frac{2\alpha}{N}}$ satisfies a large deviations principle at speed N, with a good rate function $I_\alpha^V(\mu) = f_\alpha^V(\mu) - \inf_{\nu \in \mathcal{P}(\mathbb{T})} f_\alpha^V(\nu)$, where*

$$f_\alpha^V(\mu) = \lim_{\delta \to 0} \liminf_{q \to \infty} \inf_{\substack{\nu_{\alpha/q}, \dots, \nu_\alpha \\ \frac{1}{q}\sum_i \nu_{i\alpha/q} \in B_\mu(\delta)}} \left\{ \frac{1}{q} \sum_{i=1}^{q} \left(J_{i\alpha/q}(\nu_{i\alpha/q}) + \int_{\mathbb{T}} V d\nu_{i\alpha/M} \right) \right\} . \tag{1.10.10}$$

As a byproduct of this approach, we prove a large deviation principle for (periodic or non periodic) CMV matrices (that is of the form of $\mathcal{E}$ given by (1.8.6) or of the matrix representation E of Theorem 1.4.1), in the case where the a_i's appearing in its construction are *i.i.d.*

1.10.3 Main results of Chapter 4

Chapter 4 is based on the collaboration with Charlie Dworaczek Guera [DGM23]. It is dedicated to the establishment of the Central Limit Theorem for the empirical measure in the β-ensemble on the real line at high temperature. More precisely, we establish that for *smooth enough, decaying at infinity* test functions $\phi : \mathbb{R} \to \mathbb{R}$, and for a class of *confining* potentials $V : \mathbb{R} \to \mathbb{R}$, with $\mathbf{x} = (x_1, \dots, x_N)$ distributed under the measure

$$\mathbb{P}_N^{V,2P/N}(x_1 \dots, x_N) = \frac{1}{Z_N^{V,2P/N}} \prod_{1 \leqslant i < j \leqslant N} |x_i - x_j|^{\frac{2P}{N}} e^{-\sum_{j=1}^N V(x_j)} ,$$

we have the convergence in distribution, as $N \to \infty$

$$\sqrt{N} \left(\frac{1}{N} \sum_{j=1}^{N} \phi(x_j) - \int_{\mathbb{R}} \phi(x) d\mu_P^V(x) \right) \to N\Big(0, (\sigma_P^V)^2(\phi)\Big) .$$

In the previous equation, the measure μ_P^V is the limit given by Proposition 1.3.6 of the empirical measures $\hat{\mu}_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j}$. It is characterized by the equation

$$V(x) - 2P \int_{\mathbb{R}} \log|x - y| d\mu_P^V(y) + \log \rho_P^V = \lambda_P^V . \tag{1.10.11}$$

The limiting variance σ_P^V is expressed in terms of a certain operator acting on test functions (the counterpart of $\mathcal{L}$ in Theorem 1.4.4).

In the context of Section 1.7, assuming that:

- The empirical measure of the Toda chain, tested against a test function f, statisfies a central limit theorem with limiting variance $\sigma_{Toda}^2(P, V, \phi)$,

- We have the relation between the limiting variances

$$\int_0^1 \sigma^2_{Toda}(V, sP, \phi)ds = (\sigma_P^V(\phi))^2, \quad i.e. \quad \sigma^2_{\text{Toda}}(V, P, \phi) = \partial_P(P(\sigma_P^V(\phi))^2\,,$$

Then we are able to access the limiting variance of the Toda chain, therefore giving a way to compute the currents of the conserved quantities.

Before stating the theorem, let us give a heuristic that, when made precise, leads to the proof:

Take $\phi : \mathbb{R} \to \mathbb{R}$ smooth, vanishing fast enough at infinity, and do the change of variables in $Z_N^{V,P}$, $x_i = y_i + \frac{t}{\sqrt{N}}\phi(y_i)$, $1 \leqslant i \leqslant N$, to get

$$Z_N^{V,P} = \int_{\mathbb{R}^N} \prod_{i<j} \left| y_i - y_j + \frac{t}{\sqrt{N}}(\phi(y_i) - \phi(y_j)) \right|^{2P/N} e^{-\sum_{i=1}^N V\left(y_i + \frac{t}{\sqrt{N}}\phi(y_i)\right)} \prod_{i=1}^N \left(1 + \frac{t}{\sqrt{N}}\phi'(y_i)\right) d^N\mathbf{y}\,.$$

Expanding the different terms in this integral, one gets

$$Z_N^{V,P} = \int_{\mathbb{R}^N} \prod_{i<j} |y_i - y_j|^{\frac{2P}{N}} e^{-\sum_{i=1}^N V(y_i)} e^{\frac{t}{\sqrt{N}}\left[\frac{2P}{N}\sum_{i<j}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j} + \sum_{i=1}^N\left(\phi'(y_i) - V'(y_i)\phi(y_i)\right)\right]} e^{-\frac{t^2}{2}\sigma_N^2(\phi)} d^N\mathbf{y}\,,$$

where the term $\sigma_N^2(\phi)$ converges towards a limiting variance $\sigma^2(\phi)$ depending on ϕ, P and V. After dividing both parts of the equation by $Z_N^{V,P}$, and because of equation (1.10.11) one deduces the convergence of the Laplace transform

$$\mathbb{E}\left[e^{t\sqrt{N}(\nu_N(\Xi\phi) + \text{error term})}\right] \underset{N\to\infty}{\longrightarrow} \exp\left(\frac{t^2}{2}\sigma^2(\phi)\right),$$

where we denoted

$$\nu_N(f) = \frac{1}{N}\sum_{j=1}^N f(x_j)d\hat{\mu}_N - \int_{\mathbb{R}} f(x)d\mu_P^V\,,$$

and where Ξ is a linear operator acting on test functions and defined by

$$(\Xi\phi)(x) = 2P\int_{\mathbb{R}} \frac{\phi(x) - \phi(y)}{x - y} d\mu_P^V(y) + \phi'(x) - V'(x)\phi(x)\,. \tag{1.10.12}$$

Once the error term is taken care of, this shows the central limit theorem for test functions of the form $\Xi\phi$, with an explicit variance. Following [HL21], the operator $\mathcal{L}$ given by

$$\mathcal{L}\phi = \Xi\phi' \tag{1.10.13}$$

can be analyzed using Hilbert space techniques. In particular, the operator $\mathcal{L}$, seen as an unbounded operator of the Hilbert space

$$\mathsf{H} = \left\{ u \in L^2(\mu_P^V) \;\middle|\; u' \in L^2(\mu_P^V), \int_{\mathbb{R}} u\rho_P^V dx = 0 \right\}, \qquad \langle u, v\rangle_{\mathsf{H}} = \left\langle u', v'\right\rangle_{L^2(\rho_P^V)}\,,$$

can be decomposed as

$$-\mathcal{L} = \mathcal{A} + 2P\mathcal{W}\,,$$

where $\mathcal{A}$ is a positive Sturm-Liouville operator and $\mathcal{W}$ is positive and self-adjoint. Such a writing allows us to show that $-\mathcal{L}$ is invertible, see Section 4.6. With the previous heuristics, assuming we can choose $\mathcal{L}^{-1}\phi$ as a reasonable test function, we get

$$\mathbb{E}\left[e^{t\sqrt{N}(\nu_N(\phi) + \text{error term}}\right] \to \exp\left(-\frac{t^2}{2}\sigma^2\left((\mathcal{L}^{-1}\phi)'\right)\right). \tag{1.10.14}$$

We refer to Chapter 4 for the precise assumptions made on the potential V. See Assumptions 4.1.1 and 4.1.2.

To fix the ideas, confining potentials of the form

$$V = V_{\text{conv}} + \varphi\,,$$

where V_{conv} is convex such that $|V'_{\text{conv}}| \to +\infty$ at infinity and where φ is $\mathcal{C}^3$ with bounded derivatives, satisfy our assumptions.

We then have

Theorem 1.10.7. *Assume that V satisfies Assumptions 4.1.1 and Assumption 4.1.2. Then for ϕ verifying the following conditions:*

- $\phi \in \mathcal{C}^1(\mathbb{R})$
- *there exists $\varepsilon > 0$, $\phi(x) = \underset{|x|\to\infty}{O}(x^{-\frac{1}{2}-\varepsilon})$ and $\phi'(x) = \underset{|x|\to\infty}{O}(x^{\frac{1}{2}-\varepsilon})$ at infinity*
- $\displaystyle\int_{\mathbb{R}} \phi(x) d\mu_P^V(x) = 0$

we have the convergence in law

$$\sqrt{N}\nu_N(\phi) \to N\left(0, (\sigma_P^V)^2(\phi)\right) \tag{1.10.15}$$

where the limiting variance $(\sigma_P^V)^2(\phi)$ is given by

$$(\sigma_P^V)^2(\phi) = \langle \phi, \mathcal{L}^{-1}\phi\rangle_{\mathsf{H}} = \int_{\mathbb{R}} \left(\left(\mathcal{L}^{-1}\phi\right)''(x)^2 + V''(x)\left(\mathcal{L}^{-1}\phi\right)'(x)^2 \right) d\mu_P^V(x) + P \iint\limits_{\mathbb{R}^2} \left(\frac{\left(\mathcal{L}^{-1}\phi\right)'(x) - \left(\mathcal{L}^{-1}\phi\right)'(y)}{x-y} \right)^2 d\mu_P^V(x) d\mu_P^V(y)\,. \tag{1.10.16}$$

As a consequence, we recover the equivalent of the formula giving the limiting variance in Theorem 1.4.4. The decay conditions on ϕ appear to ensure that the error term in (1.10.14) is well behaved.

As a tool for the proof of the CLT, we establish a concentration inequality for the empirical measure. It is stated in terms of the following distance over the set of probability distributions $\mathcal{P}(\mathbb{R})$.

For $\mu, \mu' \in \mathcal{P}(\mathbb{R})$, define the distance

$$d(\mu, \mu') = \sup_{\substack{\|f\|_{\text{Lip}} \leqslant 1 \\ \|f\|_{1/2} \leqslant 1}} \left\{ \left| \int f d\mu - \int f d\mu' \right| \right\}\,, \tag{1.10.17}$$

where $\|f\|_{\text{Lip}}$ denotes the Lipschitz constant of f, and $\|f\|_{1/2}^2 = \displaystyle\int_{\mathbb{R}} |t|\, |\mathcal{F}[f](t)|^2\, dt$, where $\mathcal{F}$ denotes the Fourier transform.

We then have

Theorem 1.10.8. *There exists $K \in \mathbb{R}$ (depending on P and on V), such that for any $N \geqslant 1$ and $r > 0$,*

$$\mathbb{P}_N^{V,\frac{2P}{N}}\left(d(\hat{\mu}_N, \mu_P^V) > r\right) \leqslant e^{-Nr^2\frac{P\pi^2}{2} + 5P\log N + K}\,. \tag{1.10.18}$$

This result is the analog of [HL21, Theorem 1.4], and its proof is inspired by an idea of Maïda and Maurel-Segala [MMS14], consisiting in a regularization of the empirical measure (making it absolutely continuous). We also mention [CHM18], where the authors were able to establish a similar concentration bound for Coulomb gases in dimension $d \geqslant 2$.

1.10.4 Main results of Chapter 5

This chapter is based on the collaboration with Guido Mazzuca [MM23a], where we consider the question of the fluctuations of the empirical measure in integrable systems and in high temperature β ensembles.

More precisely, motivated by the ideas of Section 1.7, we establish the existence of a central limit theorem for the fluctuations of the empirical measure of the Lax matrix of several integrable systems, including Toda and Ablowitz-Ladik, and we show the analogue of equation (1.7.8) for those systems and their β-ensemble counterpart. The integrable systems and β ensembles are described in the following table.

Integrable System	**β-ensemble at high-temperature**
Toda lattice	Real
Defocusing Ablowitz-Ladik lattice	Circular
Exponential Toda lattice	Laguerre
Defocusing Schur flow	Jacobi
Volterra lattice	Antisymmetric Gaussian

Table 1.2: Integrable systems and random matrix ensembles

The global picture is the one we described before:

- Each of the integrable systems at stake have a **periodic, band** (with fixed band size) Lax matrix, which can be described by a set of coordinates $x_1, \dots, x_N \in X$, where X is a subset of $\mathbb{R}$ or $\mathbb{C}$. For instance, in the case of the Toda lattice, recalling (1.5.4), $X = \mathbb{R}^2$ and the matrix L is described by $x_i = (a_i, p_i)$, $1 \leqslant i \leqslant N$. In the Ablowitz-Ladik case, $X = \mathbb{D}$ and the matrix $\mathcal{E}$ is described by $x_i = a_i$, $1 \leqslant i \leqslant N$, see equations (1.8.5), (1.8.6). **We label those matrix models as "type 1".**

- The β ensembles at hand have a **non periodic** matrix representation (such as the Dumitriu-Edelman representation, Theorem 1.2.5 or the Killip-Nenciu one, Theorem 1.4.1), which resembles the Lax matrix of the corresponding integrable system in Table 1.10.4. **We label those non periodic representations as "type 2".**

We refer to Chapter 5 for a description of the matrix models.

We model the Generalized Gibbs Ensembles of those integrable systems by the probability measure on X^N

$$\mu_N^{(1)} = \frac{1}{Z_N^{(1)}(\alpha, G)} \left(\prod_{j=1}^{N} F(x_j, \alpha) \right) e^{-\mathrm{Tr}G(L)} .$$

Here the matrix L is of **type 1**. The parameter α plays the same role as P in (1.6.3) and as α in (1.8.8), and G is the potential.

The β ensemble counterpart of those ensembles is modeled by the probability measure

$$\mu_N^{(2)} = \frac{1}{Z_N^{(2)}(\alpha, G)} \left(\prod_{j=1}^{N-1} F\left(x_j, \alpha \left(1 - \frac{j}{N} \right) \right) \right) R(x_N) e^{-\mathrm{Tr}G(L)} d\mathbf{x} .$$

Here the matrix L is of **type 2**. The function R is added because of the non periodicity of those matrices.

We consider **circular** potentials, in the sense that there exists some $k \geqslant 1$ such that, writing the

euclidean division of N by k: $N = kM + \ell$, we have

$$\mathrm{Tr}G(L) = \begin{cases} \sum_{j=1}^{M-1} W(\mathbf{x}_j, \mathbf{x}_{j+1}) + W_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1) & \text{for type 1} \\ \sum_{j=1}^{M-1} W(\mathbf{x}_j, \mathbf{x}_{j+1}) + \widetilde{W}_1(\mathbf{x}_1) + W(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0) & \\ + \widetilde{W}_2(x_{kM+1}, \dots, x_{kM+\ell})) & \text{for type 2} \end{cases}, \tag{1.10.19}$$

here $\mathbf{x}_j$ is the **block of coordinates** $(x_{(j-1)k+1}, \dots, x_{jk})$. The function W is called the **seed** of G. The functions $\widetilde{W}_1$ and $\widetilde{W}_2$ are to be seen like remainders and can be neglected for simplicity. The previous decomposition is typically valid when the potential G is a polynomial; in which case the number k is related to its degree and to the size of the band of the matrices at stake.

For example, in the Toda/real β ensemble at high temperature case, L is tridiagonal (periodic if of type 1, non periodic if of type 2): if $G(x) = x^d$, we have

$$\mathrm{Tr}G(L) = \mathrm{Tr}L^d = \sum_{\substack{1 \leqslant i_1, \dots, i_d \leqslant N, \\ |i_{j+1} - i_j| \leqslant 1}} L_{i_1,i_2} L_{i_2,i_3} \dots L_{i_d,i_1} ,$$

and we can set k to be equal to d. This way, only neighboring blocks of coordinates interact in both probability measures $\mu_N^{(1)}$ and $\mu_N^{(2)}$.

We also consider a test function H which has the same cyclicity property as G, *i.e.* the same decomposition for $\mathrm{Tr}H(L)$ as in (1.10.19).

We then show the following theorem. For readers convenience, we do not give all the assumptions, see Theorem 5.1.3 for the full statement.

Theorem 1.10.9. *Let L be a matrix of type 1 or 2, and let G, H be cyclic. Let*

$$S_N = \frac{1}{N} \mathrm{Tr}H(L) .$$

Then, there exist two continuous functions

$$A(x) : \mathbb{R}_+ \longrightarrow \mathbb{R} , \tag{1.10.20}$$

$$\sigma^2(x) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ , \tag{1.10.21}$$

such that under $\mu_N^{(1)}$ (5.1.12)

$$(S_N - NA(\alpha))/\sqrt{N}$$

converges to a Gaussian distribution $N(0, \sigma^2(\alpha))$ as N tends to infinity, and under $\mu_N^{(2)}$ (5.1.13),

$$\left(S_N - N \int_0^1 A(\alpha x) dx \right) / \sqrt{N}$$

converges to a Gaussian distribution $N(0, \int_0^1 \sigma^2(\alpha x) dx)$ as N tends to infinity. Furthermore, defining the free energies $\mathcal{F}_N^{(1)}(\alpha, G), \mathcal{F}_N^{(2)}(\alpha, G)$ as

$$\mathcal{F}_N^{(1)}(\alpha, G) = -\lim_{N \to \infty} \frac{1}{N} \log \left(Z_N^{(1)}(\alpha, G) \right) , \tag{1.10.22}$$

$$\mathcal{F}_N^{(2)}(\alpha, G) = -\lim_{N \to \infty} \frac{1}{N} \log \left(Z_N^{(2)}(\alpha, G) \right) , \tag{1.10.23}$$

then

i. $\mathcal{F}^{(1)}(\alpha, G) = \partial_\alpha \left(\alpha \mathcal{F}^{(2)}(\alpha, G)\right)$

ii. $A(\alpha) = i\partial_t \mathcal{F}^{(1)}(\alpha, G + itH)_{|t=0}$

iii. $\int_0^1 A(\alpha x) = i\partial_t \mathcal{F}^{(2)}(\alpha, G + itH)_{|t=0}$

iv. $\sigma^2(\alpha) = \partial_t^2 \mathcal{F}^{(1)}(\alpha, G + itH)_{|t=0}$

v. $\int_0^1 \sigma^2(\alpha x)dx = \partial_t^2 \mathcal{F}^{(2)}(\alpha, G + itH)_{|t=0}$

This result therefore allows us to justify the equivalent of equation (1.7.8) for the couples integrable model/β ensemble of Table 1.10.4. Furthermore, the limiting variances are expressed as the second derivatives of the free energies.

As mentioned previously, we prove this result by a **transfer operator approach.** The idea is that under our assumptions, because of the locality of the interactions in the both measures $\mu_N^{(1)}$ and $\mu_N^{(2)}$, the expectations in both models can be written in the approximate forms, for $t \in \mathbb{R}$, (see Theorems 5.2.1 and 5.2.2)

$$\mathbb{E}_1\left[e^{-it\mathrm{Tr}H}\right] = c_{k,\ell}(\alpha, t)\lambda(\alpha, t)^{M-2}\left(1 + o_M(1)\right), \tag{1.10.24}$$

and

$$\mathbb{E}_2\left[e^{-it\mathrm{Tr}H}\right] = c_{k,\ell,M}(\alpha, t)\prod_{j=1}^{M-2}\lambda\left(\alpha\frac{j}{M}, t\right)\left(1 + o_M(1)\right). \tag{1.10.25}$$

In the two previous formulas, $\lambda(\alpha, t)$ is the spectral radius of some operator $\mathcal{L}$ defined on $L^2(X^k)$, hence the name of the approach. This approach was used in [GM23] to prove the relation (1.10.9) in the case of a polynomial potential.

As a consequence of this approach, we are also able to prove the **decay of correlations** for local functions (in the sense that they only depend on a fixed number, independent from N, of consecutive variables) in those integrable systems, see Theorem 5.1.7 for a full statement.

Theorem 1.10.10 (Decay of correlations)**.** *Let G be cyclic, and let $I, J : X^k \to \mathbb{R}$ two (local) functions. Write $N = kM + \ell$, and let $j \in \{1, \dots, M\}$. Then there exists some $0 < \mu < 1$ such that*

$$\mathbb{E}_1\left[I(\mathbf{x}_1)J(\mathbf{x}_j)\right] - \mathbb{E}_1\left[I(\mathbf{x}_1)\right]\mathbb{E}_1\left[J(\mathbf{x}_j)\right] = O(\mu^{M-j} + \mu^j).$$

This results allows us to justify equation (1.7.7), derived in [Spo21], linking the computation of the currents of the conserved quantities to the fluctuations of the empirical measure of the Lax matrix.

Chapter 2

Large deviations for Gibbs ensembles of the classical Toda chain

Abstract We prove large deviation principles for the distribution of the empirical measure of the eigenvalues of Lax matrices following the Generalized Gibbs ensembles of the classical Toda chain introduced in [Spo20c]. We deduce the almost sure convergence of this empirical measure towards a limit which we describe in terms of the limiting empirical measure of Beta-ensembles. Our results apply to general smooth potentials.

2.1 Introduction

In a breakthrough paper [Spo20c], Herbert Spohn introduced the generalized Gibbs ensembles of the classical Toda chain as invariant measures of the dynamics of the classical Toda lattice. He analyzes them by comparing the Toda Lax matrices for these Generalized Gibbs ensembles with Dumitriu-Edelman tri-diagonal representations of β-ensembles. Thanks to this beautiful comparison, [Spo20c] showed that the empirical measure of the eigenvalues of Toda Lax matrices converges towards a probability measure related with the equilibrium measure for β-ensembles. One of the key tools of Herbert Spohn analysis is the use of transfer matrices, which are restricted to polynomial potentials. We refer the interested reader to subsequent developments in [Spo20a, Spo20b, Spo21] and [Maz22] where the transfer matrix approach is used in the similar context of the so-called Ablowitz-Ladik lattice.

The main goal of this article is to generalize some of the results of [Spo20c] by using large deviations theory, which allows to consider more general potentials. More precisely, we will show the convergence of the free energy and of the empirical measure of the eigenvalues of Toda Lax matrices following these Generalized Gibbs ensembles. Moreover, we will express the limits in terms of the well known β-ensembles. Indeed, a key tool is again to compare the Toda Lax matrices with Dumitriu-Edelman tri-diagonal representations of β-ensembles. Moreover, we will derive large deviation principles for the empirical measure of the eigenvalues of tri-diagonal matrices with more general coefficients. However, in this generality, the rate functions and the limits will not be explicit as the comparison with β-ensembles is not possible.

More precisely, the Hamiltonian of the Toda chain on sites $j = 1, \dots, N$ is given by

$$H = \sum_{j=1}^{N} (\frac{1}{2} p_j^2 + e^{-r_j}), \quad r_j = q_{j+1} - q_j$$

with the periodic conditions $q_{N+j} = q_j + cN$ for some real constant c. The equations of motion are then given by

$$\frac{d}{dt} q_j = p_j, \qquad \frac{d}{dt} p_j = e^{-r_{j-1}} - e^{-r_j} . \tag{2.1.1}$$

Let L_N be the Lax matrix given by the $N \times N$ tri-diagonal matrix with entries

$$(L_N)_{j,j} = p_j \text{ and } (L_N)_{j,j+1} = (L_N)_{j+1,j} = e^{-r_j/2} \tag{2.1.2}$$

with periodic boundary conditions $(L_N)_{1,N} = (L_N)_{N+1,N}$ and $(L_N)_{N,1} = (L_N)_{N,N+1}$, then for all integer number p,

$$Q_N^p = \mathrm{Tr}(L_N^p)$$

is conserved by the dynamics (2.1.1) as well as $\sum_{i=1}^N r_i$. It is therefore natural to consider that the finite N Toda chain is distributed according to the Gibbs measure with density $e^{-\mathrm{Tr}(W(L_N))-P\sum r_i}$ with respect to Lebesgue measure. Here, $P > 0$ controls the pressure of the chain and W is a potential to be chosen later, which can be a polynomial or a general measurable function from $\mathbb{R}$ into $\mathbb{R}$. We will assume it goes to infinity faster than x^2: namely there exists $c > 0$ and a finite constant C such that for all $x \in \mathbb{R}$

$$W(x) \geqslant cx^2 + C\,. \tag{2.1.3}$$

This assumption is used to compare our distribution to the case where $W(x) = cx^2$ in which case the entries of the Lax matrix L_N are independent. We can without loss of generality assume $c = \frac{1}{2}$ up to rescaling and therefore put

$$W(x) = \frac{1}{2}x^2 + V(x). \tag{2.1.4}$$

In the following we will denote

$$d\mathbb{T}_N^{V,P}(p,r) = \frac{1}{\mathbb{Z}_{N,\mathbb{T}}^{V,P}} \exp\{-\mathrm{Tr}(V(L_N)) - \frac{1}{2}\mathrm{Tr}(L_N^2)\} \prod_{i=1}^N e^{-Pr_i} dr_i dp_i \tag{2.1.5}$$

where $\mathbb{Z}_{N,\mathbb{T}}^{V,P}$ is the partition function of the Toda Gibbs measure :

$$\mathbb{Z}_{N,\mathbb{T}}^{V,P} = \int \exp\{-\mathrm{Tr}(V(L_N)) - \frac{1}{2}\mathrm{Tr}(L_N^2)\} \prod_{i=1}^N e^{-Pr_i} dr_i dp_i\,. \tag{2.1.6}$$

We denote in short $\mathbb{T}_N^P$ for $\mathbb{T}_N^{0,P}$. Our goal in this article is to study the empirical measure of the eigenvalues $\lambda_N \leqslant \cdots \leqslant \lambda_1$ of L_N denoted by

$$\hat{\mu}_{L_N} = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}\,.$$

We shall call $\hat{\mu}_{L_N}$ the empirical measure of L_N, or the empirical density of states of the Lax matrix following [Spo20c]. Our main result is a large deviations principle for the distribution of $\hat{\mu}_{L_N}$ under $d\mathbb{T}_N^{V,P}$, from which we deduce the almost sure convergence of $\hat{\mu}_{L_N}$ under $d\mathbb{T}_N^{V,P}$.

Theorem 2.1.1. *Let $P > 0$ and assume that V is continuous. Assume that either V is uniformly bounded or there exists $k \in \mathbb{N}^*$ such that*

$$\lim_{|x|\to\infty} \frac{V(x)}{x^{2k}} = a\,, \tag{2.1.7}$$

with $a > 0$ if $k > 1$ and $a > -1/2$ if $k = 1$. Then:

1. *The law of $\hat{\mu}_{L_N}$ under $\mathbb{T}_N^{V,P}$ satisfies a large deviation principle in the scale N with good rate function T_P^V.*
2. *T_P^V achieves its minimal value at a unique probability measure ν_P^V.*
3. *As a consequence $\hat{\mu}_{L_N}$ converges almost surely and in L^1 towards ν_P^V.*

ν_P^V corresponds to the density of states of the Lax matrix in [Spo20c]. Moreover, following [Spo20c], we can identify the equilibrium measure ν_P^V using the equilibrium measure for Coulomb gases in dimension one at temperature of order of the number of particles. More precisely, for a probability measure μ on the real line, we define the function f_P^V by

$$f_P^V(\mu) = \frac{1}{2}\int \left(\frac{1}{2}(x^2+y^2) + V(x) + V(y) - 2P\log|x-y|\right) d\mu(x)d\mu(y) + \int \log\frac{d\mu}{dx} d\mu(x)$$

if $\mu \ll dx$, whereas f_P^V is infinite otherwise. f_P^V achieves its minimal value at a unique probability measure $\mu_P^V \ll dx$ which satisfies the non-linear equation

$$\frac{1}{2}x^2 + V(x) - 2P\int \log|x-y| d\mu_P^V(y) + \log\frac{d\mu_P^V}{dx} = \lambda_P^V \qquad a.s \tag{2.1.8}$$

where λ_P^V is a finite constant. We show in section 2.3 that μ_P^V is absolutely continuous with respect to Lebesgue measure and that its depends smoothly on the parameter P. In Lemma 2.3.6, we show it is in fact differentiable in P. We then show that

Theorem 2.1.2. *Let P be a positive real number. Then, for any bounded continuous function f on the real line,*

$$\int f(x) d\nu_P^V(x) = \partial_P\left(P\int f(x) d\mu_P^V(x)\right)$$

This result was already shown in [Spo20c] when V is a polynomial. Our strategy is to prove first a large deviation principle in the case when V vanishes: then, L_N has independent entries (modulo the symmetry constraint) under $\mathbb{T}_N^P$. We then derive large deviation principles for more general bounded continuous potentials by using Varadhan's Lemma, see section 2.2.
Indeed, in the case where V vanishes, the random variables $(p_j, r_j)_{1\leqslant j\leqslant N}$ are independent, $(L_N)_{j,j}$ are standard Gaussian $N(0,1)$ variables and $\sqrt{2}(L_N)_{j,j+1}$ follows a χ_{2P} distribution with density with respect to Lebesgue measure given by

$$\chi_{2P}(x) = \frac{2^{1-P}x^{2P-1}e^{-x^2/2}}{\Gamma(P)}\mathbf{1}_{x>0}. \tag{2.1.9}$$

The central observation is that we can compare this matrix to the tri-diagonal matrix C_N^β introduced by Dumitriu and Edelman [DE02]. This is the symmetric matrix with independent (up to symmetry) entries whose diagonal elements are independent standard Gaussians variables, and off diagonal elements so that $\sqrt{2}C_N^\beta(j,j+1)$ follow a χ distribution with parameter $\beta(N-j)$. When $\beta = 2P/N$, the matrix is therefore similar to L_N except that the parameters of the off-diagonal entries vary linearly. The key point is that the law of the eigenvalues of C_N^β is explicit and given by the β-ensemble, see Section 2.3. This comparison allows to compare the free energy, the rate function and the equilibrium measure of the Toda chain with those of Coulomb gases in section 2.3. In section 2.4, we study the case of general potentials. The proof is nearly independent from the quadratic case, but requires additional arguments in particular because the eigenvalues of the Toda matrix are not simple functions of the empirical measure of the entries. Note that the proof given in section 2.4 also applies to the case where V is bounded. We nevertheless choose to give a separate proof, dedicated to this case: the computations being simpler, the core of the proof seems more accessible and introduces ideas we re-use in the case where V is unbounded.

Moreover, our result allows to derive large deviation principles for the empirical measure of the tri-diagonal matrices with independent standard Gaussian entries on the diagonal and independent chi distributed variables with general parameters profile on the off-diagonal. Namely let L_N^σ be a tri-diagonal symmetric matrix with independent Gaussian variables on the diagonal and independent variables $\sqrt{2}L_N^\sigma(j,j+1)$ chi distributed with parameter $\sigma(\frac{i}{N}), 1\leqslant i\leqslant N$. Let $\mathbb{T}_N^{V,\sigma}$ be the distribution with density $e^{-\mathrm{Tr}(V(L_N^\sigma))}/Z$ with respect to the distribution of L_N^σ.

Theorem 2.1.3. *Assume that V is continuous and satisfies* (2.1.7)*. Then, if σ is bounded continuous,*

1. *the law of* $\hat{\mu}_{L_N^\sigma}$ *under* $\mathbb{T}_N^{V,\sigma}$ *satisfies a large deviation principle in the scale* N *with good rate function* T_σ^V,

2. T_σ^V *achieves its minimal value at a unique probability measure* $\nu_\sigma^V = \int_0^1 \nu_{\sigma(P)}^V dP$,

3. *As a consequence,* $\hat{\mu}_{L_N^\sigma}$ *converges almost surely and in* L^1 *towards* ν_σ^V.

Acknowledgments : We are very grateful to Herbert Spohn for asking us to investigate the convergence of the density of states for general potentials V and many fruitful discussions that followed. We would also like to thank David García-Zelada for showing us how to derive Theorem 2.3.1 from [GZ19]. We thank an anonymous referee for helping us to improve the presentation of our results.

2.2 Large deviation principles for tri-diagonal matrices

In this section, we consider a tri-diagonal matrix M_N with entries

$$(M_N)_{j,j} = a_j \text{ and } (M_N)_{j,j+1} = (M_N)_{j+1,j} = b_j \tag{2.2.1}$$

with periodic boundary conditions, the random variables $(a_i, b_i)_{1\leqslant i\leqslant N}$ being iid, with (a_1, b_1) with law $Q_a \otimes Q_b$ on $\mathbb{R}^2$. We denote by $\hat{\mu}_{M_N}$ the empirical measure of the eigenvalues of M_N and prove the existence of a large deviation principle for the distribution of $\hat{\mu}_{M_N}$. In [Zha17, Theorem 4.2], the author proves a large deviation principle for the empirical moments $\hat{\mu}_{M_N}(x^k)$ by noticing that

$$\hat{\mu}_{M_N}(x^k) = \frac{1}{N}\sum_{i=1}^{N} f_k(a_j, b_j, |i-j| \leqslant k)$$

where $f_k(a_j, b_j, |i-j| \leqslant k) = (M_N^k)_{ii}$ is an homogeneous polynomial of degree k in the entries $a_j, b_j, |i-j| \leqslant k$. Noting that f_k does not depend on i, one can use the large deviation principle for Markov chains (or k-dependent large deviation principle), see e.g [DZ10, Theorem 3.1.2 or Section 6.5.2], as well as the contraction principle, to deduce a large deviation principle for the distribution of the empirical moments $\{\hat{\mu}_{M_N}(x^k), k \leqslant p\}$. This could be used to deduce the existence of a large deviation principle for $\hat{\mu}_{M_N}$ for the weak topology after approximations, but the rate function would not be particularly explicit. We prefer to develop a more straightforward sub-additivity argument and prove separately the existence of a weak large deviation principle and exponential tightness, see e.g [DZ10, Lemma 1.2.18].

2.2.1 Exponential tightness

In this section we assume that

Assumption 2.2.1. *There exists* $\gamma > 0$ *such that*

$$D_\gamma := \int e^{\gamma x^2} dQ_a(x) \times \int e^{\gamma y^2} dQ_b(y) < \infty\,.$$

We equip the set of probability measures on the real line $\mathcal{P}(\mathbb{R})$ with the weak topology. We then show that

Lemma 2.2.2. *If* $(a_j, b_j)_{1\leqslant j\leqslant N}$ *are iid with law* $Q_a \otimes Q_b$ *satisfying Assumption 2.2.1, the sequence* $(\hat{\mu}_{M_N})_{N\geqslant 0}$ *is exponentially tight, namely for each* $L \geqslant 0$ *there exists a compact set* K_L *(*$K_L = \{\mu \in \mathcal{P}(\mathbb{R}) : \int x^2 d\mu(x) \leqslant \frac{2}{\gamma}(L + \log D_\gamma)\}$ *with* γ *as in Assumption 2.2.1) such that*

$$\limsup_N \frac{1}{N}\log \mathbb{P}(\hat{\mu}_{M_N} \in K_L^c) < -L. \tag{2.2.2}$$

Proof. For $N \geqslant 1$, notice that

$$\begin{aligned} \int x^2 \mathrm{d}\hat{\mu}_{M_N}(x) &= \frac{1}{N}\mathrm{Tr}(M_N^2) \\ &= \frac{1}{N}\sum_{j=1}^{N}((M_N)_{j,j})^2 + \frac{1}{N}\sum_{j=1}^{N}\left(\sqrt{2}(M_N)_{j,j+1}\right)^2. \end{aligned} \tag{2.2.3}$$

As a consequence, Tchebychev's inequality implies that, for any $\gamma > 0$,

$$\mathbb{P}\left(\int x^2 \mathrm{d}\hat{\mu}_{M_N}(x) > K\right) \leqslant e^{-\frac{1}{2}\gamma N K}\mathbb{E}\big[e^{\frac{1}{2}N\gamma\int x^2 \mathrm{d}\hat{\mu}_{M_N}(x)}\big] \leqslant e^{-\frac{1}{2}\gamma N K} D_\gamma^N .$$

The conclusion follows by taking $K = \frac{2}{\gamma}(L + \log D_\gamma)$.

□

2.2.2 Weak large deviation principle

We next establish a weak large deviation principle, based on the general ideas developed in [DZ10, Lemma 6.1.7]. To this end, we use the following distance on $\mathcal{P}(\mathbb{R})$:

$$d(\mu,\nu) = \sup_{\|f\|_{\mathrm{BV}}\leqslant 1, |f|_{\mathrm{Lip}}\leqslant 1}\left\{\left|\int_{\mathbb{R}} f(x)d\mu(x) - \int_{\mathbb{R}} f(x)d\nu(x)\right|\right\}, \tag{2.2.4}$$

where $\|f\|_{\mathrm{BV}}$ is the total variation norm of f given by

$$\|f\|_{\mathrm{BV}} = \sup\sum_{k\in\mathbb{N}} |f(x_{k+1}) - f(x_k)|,$$

where the supremum holds over all increasing sequences $(x_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. $\|f\|_L$ is the Lipschitz norm of f. If f is continuously differentiable and we put without loss of generality $f(0) = 0$, $\|f\|_{BV} = \int_{-\infty}^{+\infty} |f'(y)|dy$ and $\|f\|_L = \|f'\|_\infty$. The distance d is smaller than the Wasserstein distance where one takes the supremum over all functions whose L^∞ and Lipschitz norms are bounded by one, and is easily seen to be as well compatible with the weak topology. Then, we shall prove that if $B_\mu(\delta) = \{\nu \in \mathcal{P}(\mathbb{R}) : d(\mu,\nu) < \delta\}$ denotes the open ball with radius δ centered at μ, we have :

Lemma 2.2.3. *For any μ in $\mathcal{P}(\mathbb{R})$, there exists a limit*

$$\lim_{\delta\to 0}\liminf_N \frac{1}{N}\log\mathbb{P}\left(\hat{\mu}_{M_N} \in B_\mu(\delta)\right) = \lim_{\delta\to 0}\limsup_N \frac{1}{N}\log\mathbb{P}\left(\hat{\mu}_{M_N} \in B_\mu(\delta)\right). \tag{2.2.5}$$

We denote this limit by $-J_M(\mu)$.

Proof. The advantage of the distance d is the following control: For any symmetric $N \times N$ matrices A and B with empirical measures of eigenvalues $\hat{\mu}_A$ and $\hat{\mu}_B$, we have:

$$d(\hat{\mu}_A, \hat{\mu}_B) \leqslant \min\left\{\frac{\mathrm{rank}(A-B)}{N}, \frac{1}{N}\sum_{i,j}|A(i,j) - B(i,j)|\right\}. \tag{2.2.6}$$

Indeed, for any function f with bounded variation we have thanks to Weyl interlacing property, see e.g. [Gui09, (1.17)],

$$\left|\int f d\hat{\mu}_A - \int f d\hat{\mu}_B\right| \leqslant \frac{1}{N}\mathrm{rank}(A-B). \tag{2.2.7}$$

Moreover, one can check that, if f is continuously differentiable, we have

$$\begin{aligned}\int f d\hat{\mu}_A - \int f d\hat{\mu}_B &= \int_0^1 \frac{1}{N}\mathrm{Tr}\left((A-B)f'(\alpha A + (1-\alpha)B)\right) d\alpha \\ &= \int_0^1 \left(\frac{1}{N}\sum_{i,j=1}^N (A-B)_{ij} f'(\alpha A + (1-\alpha)B)_{ji}\right) d\alpha\end{aligned}$$

which implies since for all indices i, j, $|f'(\alpha A + (1-\alpha)B)_{ji}| \leqslant \|f'\|_\infty$ that

$$\left|\int f d\hat{\mu}_A - \int f d\hat{\mu}_B\right| \leqslant \|f'\|_\infty \frac{1}{N}\sum_{i,j=1}^N |(A-B)_{ij}|\,. \tag{2.2.8}$$

Since continuously differentiable functions with bounded L^∞ norm are dense in Lipschitz functions, we deduce (2.2.6) from (2.2.7) and (2.2.8). We are now ready to prove Lemma 2.2.3. To this end, we shall approximate our matrix M_N by a diagonal block matrix with independent blocks. Let $q \geqslant 1$. For $N \geqslant 1$ we decompose $N = k_N q + r_N$ with $r_N \in \{0, \ldots, q-1\}$ and set $M_N = M_N^q + R_N^q$, where M_N^q is the diagonal block matrix

$$M_N^q = \begin{bmatrix} M_q^1 & & & \\ & \ddots & & \\ & & M_q^{k_N} & \\ & & & B \end{bmatrix}. \tag{2.2.9}$$

Here, for all $i \in \{1, \ldots, k_N\}$, M_q^i has the same distribution than M_q and B the same distribution than M_{r_N}. The matrices $M_q^i, 1 \leqslant i \leqslant k_N$, are independent, and are independent from B. R_N^q is the self-adjoint matrix with null entries except $R_N^q(1,N) = R_N^q(N,1) = b_N$, $R_N^q(k_N q+1, N) = R_N^q(N, k_N q+1) = -b_N$, and those given, for $k \in \{1, \ldots, k_N\}$, by $R_N^q(kq+1, kq) = R_N^q(kq, kq+1) = b_{kq}$, $R_N^q((k-1)q+1, kq)) = R_N^q(kq, (k-1)q+1) = -b_{kq}$. Therefore $\mathrm{rank}(R_N^q) \leqslant 2k_N + 2 \leqslant 4k_N$. By (2.2.6), we deduce that

$$d(\hat{\mu}_{M_N}, \hat{\mu}_{M_N^q}) \leqslant \frac{4}{q}. \tag{2.2.10}$$

Moreover, we can write $\hat{\mu}_{M_N^q}$ as the sum

$$\hat{\mu}_{M_N^q} = \sum_{i=1}^{k_N} \frac{q}{N}\hat{\mu}_{M_q^i} + \frac{r_N}{N}\hat{\mu}_B\,.$$

Therefore, for any $\mu \in \mathcal{P}(\mathbb{R})$ and $\delta > 0$, we have

$$\begin{aligned}\mathbb{P}\left(\hat{\mu}_{M_q^1} \in B_\mu(\delta)\right)^{k_N} \mathbb{P}\left(\hat{\mu}_{M_{r_N}} \in B_\mu(\delta)\right) &= \mathbb{P}\left(\forall\ i \in \{1, \ldots, k_N\},\ \hat{\mu}_{M_q^i} \in B_\mu(\delta),\ \hat{\mu}_B \in B_\mu(\delta)\right) \\ &\leqslant \mathbb{P}\left(\hat{\mu}_{M_N^q} \in B_\mu(\delta)\right) \\ &\leqslant \mathbb{P}\left(\hat{\mu}_{M_N} \in B_\mu(\delta + \frac{4}{q})\right),\end{aligned}$$

where we used the convexity of balls and (2.2.10). As a consequence,

$$u_N(\delta) := -\log \mathbb{P}\left(\hat{\mu}_{M_N} \in B_\mu(\delta)\right)$$

satisfies

$$u_N(\delta + 4/q) \leqslant k_N u_q(\delta) + u_{r_N}(\delta).$$

It is easy (and classical) to deduce the convergence of $u_N(\delta)/N$ when N goes to infinity, and then δ goes to zero. Indeed let $\delta > 0$ be given and choose q large enough so that $\frac{4}{q} < \delta$. Then, since $\delta \to u_N(\delta)$ is decreasing and non-negative, we have:

$$\frac{u_N(2\delta)}{N} \leqslant \frac{u_N(\delta + 4/q)}{N} \leqslant \frac{u_q(\delta)}{q} + \frac{u_{r_N}(\delta)}{N} . \tag{2.2.11}$$

Since $\frac{u_{r_N}(\delta)}{N} \leqslant \frac{\max_{1 \leqslant i \leqslant q-1} u_i(\delta)}{N}$ goes to zero when $N \to \infty$, we conclude that

$$\limsup_N \frac{u_N(2\delta)}{N} \leqslant \frac{u_q(\delta)}{q} .$$

Since this is true for all q large enough, we get

$$\limsup_N \frac{u_N(2\delta)}{N} \leqslant \liminf_N \frac{u_N(\delta)}{N} .$$

Since the left and right hand sides decrease as δ goes to zero, we conclude that

$$\lim_{\delta \to 0} \limsup_{N \to \infty} -\frac{1}{N} \log \mathbb{P}\left(\hat{\mu}_{M_N} \in B_\mu(\delta)\right) \leqslant \lim_{\delta \to 0} \liminf_{N \to \infty} -\frac{1}{N} \log \mathbb{P}\left(\hat{\mu}_{M_N} \in B_\mu(\delta)\right) ,$$

and the conclusion follows. □

2.2.3 Full large deviation principle

As a consequence of Lemmas 2.2.2 and 2.2.3, we have by [DZ10, Theorem 1.2.18] the following large deviation principle.

Theorem 2.2.4. *Under Assumption 2.2.1, the law of $\hat{\mu}_M$ satisfies a large deviation principle in the scale N with a good rate function J_M. Moreover, J_M is convex. In other words,*

- *$J_M : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ has compact level sets $\{\mu : J_M(\mu) \leqslant L\}$ for all $L \geqslant 0$. Moreover, J_M is convex.*
- *For any closed set $F \subset \mathcal{P}(\mathbb{R})$,*

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\hat{\mu}_{M_N} \in F) \leqslant -\inf_F J_M ,$$

whereas for any open set $O \subset \mathcal{P}(\mathbb{R})$

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\hat{\mu}_{M_N} \in O) \geqslant -\inf_O J_M .$$

Proof. J_M exists and is defined by Lemma 2.2.3. The lower semi-continuity of J_M follows from [DZ10, Theorem 4.1.11]. We then deduce that the level sets of J_M are compact by the exponential tightness, see [DZ10, Lemma 1.2.18 (b)].

In the spirit of [DZ10, Lemma 4.1.21], we show that J_M is convex. Let μ_1, $\mu_2 \in \mathcal{P}(\mathbb{R})$. Since $\hat{\mu}_{M_{2N}}$ can be decomposed as the independent sum of $\hat{\mu}_{M_N}$ divided by 2 plus an error term of smaller than $4/N$ by (2.2.7), we have for all $\delta_1, \delta_2 > 0$

$$\mathbb{P}\left(d(\hat{\mu}_{M_N}, \mu_1) < \delta_1\right) \mathbb{P}\left(d(\hat{\mu}_{M_N}, \mu_2) < \delta_2\right) \leqslant \mathbb{P}\left(d(\hat{\mu}_{M_{2N}}, \frac{\mu_1 + \mu_2}{2}) < \delta_3\right). \tag{2.2.12}$$

for any $\delta_3 \geqslant \frac{1}{2}(\delta_1 + \delta_2) + \frac{4}{N}$. Taking the logarithm, dividing by $2N$ and letting N go to infinity, δ_1, δ_2 and then δ_3 to zero, we conclude that

$$J_M\left(\frac{\mu_1 + \mu_2}{2}\right) \leqslant \frac{1}{2}\Big(J_M(\mu_1) + J_M(\mu_2)\Big), \tag{2.2.13}$$

from which we deduce the convexity of J_M as in [DZ10, Lemma 4.1.21].
The second point, namely that a weak large deviation principle and exponential tightness implies a full large deviation principle, is classical, see [DZ10, Lemma 1.2.18]. □

2.2.4 Large deviation principle for the Toda-Chain with quadratic potential

Recall that we denoted by Q_a and Q_b respectively the laws of the a_i's and b_i's, see (2.2.1). In the case of the Toda chain with Gaussian potential, that is $V = 0$, with entries following $\mathbb{T}_N^P$, we take Q_a to be the standard Gaussian law and Q_b to be the chi distribution $\sqrt{2}^{-1}\chi_{2P}$ given in (2.1.9). We let $L_N(P)$ be the tridiagonal matrix whose entries follow $\mathbb{T}_N^P$. These entries clearly satisfy Assumption 2.2.1 and therefore we have

Corollary 2.2.5. *For any $P > 0$, the law of $\hat{\mu}_{L_N(P)}$ satisfies a large deviation principle in the scale N with good, convex, rate function denoted by T_P.*

For further use, we show that

Lemma 2.2.6. *For each $\mu \in \mathcal{P}(\mathbb{R})$, the map $P \in (0, +\infty) \mapsto T_P(\mu)$ is lower semi-continuous.*

Proof. Let P, h be positive real numbers. We first couple the matrices $(L_N(P), L_N(P+h))_N$, where $L_N(u)$ follows $\mathbb{T}_N^u$ for $u = P$ and $u = P + h$, in such a way that there exists a finite constant c so that

$$\mathbb{P}\left(d(\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(P+h)}) > \delta\right) \leqslant e^{N(c - \sqrt{-\log(h)}\delta/2)}. \tag{2.2.14}$$

This coupling is done as follows:
• The diagonal coefficients are the same set of standard independent Gaussian variables
• The coefficient below and above the diagonal X_u^i, follow a $\sqrt{2}^{-1}\chi_{2u}$ for $u = P$, $u = h$ and $P + h$. By definition of the χ distribution we can construct these variables so that almost surely

$$X_{P+h}^i = \sqrt{(X_P^i)^2 + (X_h^i)^2}. \tag{2.2.15}$$

This coupling allows by (2.2.6) to write

$$d(\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(P+h)})) \leqslant \frac{2}{N}\sum_{i=1}^{N} |X_{P+h}^i - X_P^i| = \frac{2}{N}\sum_{i=1}^{N} (X_{P+h}^i - X_P^i) \leqslant \frac{2}{N}\sum_{i=1}^{N} X_h^i,$$

where we ultimately used that, for all $i \in \{1, \dots, N\}$, $X_{P+h}^i \leqslant X_h^i + X_P^i$ because $X_h^i X_P^i$ is non-negative and (2.2.15) holds. Equation (2.2.14) follows by Tchebychev inequality since $\mathbb{E}[\exp\{\sqrt{\log h^{-1}} X_h^i\}]$ is finite, see (2.3.14). (2.2.14) implies that $(\hat{\mu}_{L_N(P+h)})_{N\geqslant 0}$ is an exponential approximation of $(\hat{\mu}_{L_N(P)})_{N\geqslant 0}$ when h goes to zero. By [DZ10, Theorem 4.2.16] , we deduce that for any $\mu \in \mathcal{P}(\mathbb{R})$, we have

$$T_P(\mu) = \lim_{\delta\to 0} \liminf_{h\to 0} \inf_{B_\mu(\delta)} T_{P+h}.$$

By monotonicity of the right hand side and the lower semi-continuity of T_{P+h} we deduce that, see [DZ10, (4.1.2)],

$$\lim_{\delta\to 0} \inf_{B_\mu(\delta)} T_{P+h} = T_{P+h}(\mu),$$

and therefore

$$T_P(\mu) = \lim_{\delta\to 0} \liminf_{h\to 0} \inf_{B_\mu(\delta)} T_{P+h} \leqslant \liminf_{h\to 0} T_{P+h}(\mu),$$

and so $P \mapsto T_P(\mu)$ is lower semi-continuous.

□

We shall also use later that Corollary 2.2.5 gives a large deviation principle for the empirical measure of the Toda chain with general bounded continuous potential.

Corollary 2.2.7. *Let V be a bounded continuous function on the real line and P be a positive real number. Let $L_N(P)$ be the tridiagonal matrix whose entries follow $\mathbb{T}_N^{V,P}$. Then:*

- *The law of $\hat{\mu}_{L_N(P)}$ satisfies a large deviation principle in the scale N with convex good rate function given, for any $\mu \in \mathcal{P}(\mathbb{R})$,*

$$T_P^V(\mu) = T_P(\mu) + \int V d\mu - \inf_\nu \{T_P(\nu) + \int V d\nu\}\,.$$

- *The set M_P^V where T_P^V achieves its minimum value is a compact convex subset of $\mathcal{P}(\mathbb{R})$. It is continuous in the sense that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $\delta < \delta_\varepsilon$, any $P, Q > 0$ such that for $|P - Q| \leqslant \delta$,*

$$M_P^V \subset (M_Q^V)^\varepsilon$$

where $A^\varepsilon = \{\mu : d(\mu, A) \leqslant \varepsilon\}$.

Proof. The first point is a direct consequence of Varadhan's lemma since when V is bounded continuous, $\mu \to \int V(x) d\mu(x)$ is also continuous. We hence need only to prove the second point, that is the continuity of $P \in (0, +\infty) \mapsto M_P^V$. Note that since T_P^V is a good rate function, M_P^V is compact for all positive real number P. We let $\mathbb{T}_N$ be the coupling of $L_N(P)$ and $L_N(Q)$ introduced in Lemma 2.2.6. By definition, for $R = P$ and Q, B a measurable subset of $\mathcal{P}(\mathbb{R})$, we have

$$\mathbb{T}_N^{V,R}(\hat{\mu}_{L_N} \in B) = \frac{1}{Z_{N,\mathbb{T}}^{V,R}} \int 1_{\{\hat{\mu}_{L_N(R)} \in B\}} e^{-N \int V(x) d\hat{\mu}_{L_N(R)}(x)} d\mathbb{T}_N,$$

where we used the notation

$$Z_{N,\mathbb{T}}^{V,R} = \int e^{-N \int V(x) d\hat{\mu}_{L_N(R)}(x)} d\mathbb{T}_N^R.$$

Therefore, since $((M_Q^V)^\varepsilon)^c$ is open, we can use the large deviation principle for the empirical measure of $L_N(P)$, Corollary 2.2.5, to state that for any $\kappa > 0$

$$\begin{aligned} -\inf_{((M_Q^V)^\varepsilon)^c} T_P^V &\leqslant \limsup_{N\to\infty} \frac{1}{N} \log \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \int_{\{d(\hat{\mu}_{L_N(P)}, M_Q^V) > \varepsilon\}} e^{-N \int V(x) d\hat{\mu}_{L_N(P)}(x)} d\mathbb{T}_N \\ &\leqslant \max\{\limsup_{N\to\infty} \frac{1}{N} \log \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \int_{\{d(\hat{\mu}_{L_N(P)}, M_Q^V) > \varepsilon\} \cap \{d(\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(Q)}) \leqslant \kappa\}} e^{-N \int V(x) d\hat{\mu}_{L_N(P)}(x)} d\mathbb{T}_N, \\ &\qquad 2\|V\|_\infty + c - \sqrt{-\log|P-Q|}\kappa/2\} \end{aligned} \tag{2.2.16}$$

where we used (2.2.14) and $Z_{N,\mathbb{T}}^{V,P} \geqslant e^{-N\|V\|_\infty}$. We next remark that by Lemma 2.2.2, there exists a positive constant c and a finite constant C such that uniformly on P in a compact set, if we denote by $K_L = \{\int x^2 d\mu(x) \leqslant L\}$,

$$\mathbb{T}_N^P(\hat{\mu}_{L_N} \in K_L^c) \leqslant e^{-(cL+C)N}\,.$$

Hence, fixing some $L > 0$, (2.2.16) implies

$$\begin{aligned} -\inf_{((M_Q^V)^\varepsilon)^c} T_P^V \leqslant \max\Big\{ & 2\|V\|_\infty + c - \sqrt{-\log|P-Q|}\kappa/2, 2\|V\|_\infty - cL - C, \\ & \limsup_{N\to\infty} \frac{1}{N} \log \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \int \mathbf{1}_{d(\hat{\mu}_{L_N(P)}, M_Q^V) > \varepsilon} \mathbf{1}_{d(\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(Q)}) \leqslant \kappa} \mathbf{1}_{\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(Q)} \in K_L} e^{-N \int V(x) d\hat{\mu}_{L_N(P)}(x)} d\mathbb{T}_N \Big\}. \end{aligned} \tag{2.2.17}$$

We next notice that $\int V(d\mu - d\nu)$ is bounded by some $\varepsilon_V^L(\kappa)$ going to zero as κ does uniformly on $\{d(\mu,\nu) \leqslant \kappa\}$ and μ, ν in the compact set K_L. Indeed, this is obvious if V has bounded variation and

Lipschitz norms, with $\varepsilon_V(\kappa) = \max\{\|V\|_{\mathrm{BV}}, \|V\|_{\mathrm{L}}\}\kappa$. If V is bounded continuous, we let $\eta > 0$ and choose $M = \sqrt{2\eta^{-1}L}$ so that $\mu([-M,M]^c) + \nu([-M,M]^c) \leqslant \eta$ for $\mu, \nu \in K_L$, and then V_η with finite bounded variation and Lipschitz norm so that

$$\sup_{x\in[-M,M]} |V(x) - V_\eta(x)| \leqslant \eta\,.$$

We then check that

$$\left|\int V(d\mu - d\nu)\right| \leqslant \|V\|_\infty \eta + 2\eta + \left|\int V_\eta(d\mu - d\nu)\right| \leqslant (\|V\|_\infty + 2)\eta + \max\{\|V_\eta\|_{\mathrm{BV}}, \|V_\eta\|_{\mathrm{L}}\}\kappa\,.$$

We finally choose $\eta = \eta(\kappa)$ going to zero slowly enough with κ so that the above right hand side goes to zero. Hence, we can bound the third term in the right hand side of (2.2.17) to find that

$$\begin{aligned}\frac{1}{Z^{V,P}_{N,\mathbb{T}}} \int 1_{d(\hat\mu_{L_N(P)}, M_Q^V)>\varepsilon} 1_{d(\hat\mu_{L_N(P)},\hat\mu_{L_N(Q)})\leqslant\kappa} \mathbf{1}_{\hat\mu_{L_N(P)},\hat\mu_{L_N(Q)}\in K_L} e^{-N\int V(x)d\hat\mu_{L_N(P)}(x)} d\mathbb{T}_N \\ \leqslant e^{N\varepsilon_V^L(\kappa)} \frac{Z^{V,Q}_{N,\mathbb{T}}}{Z^{V,P}_{N,\mathbb{T}}} \frac{1}{Z^{V,Q}_{N,\mathbb{T}}} \int 1_{\{d(\hat\mu_{L_N(Q)}, M_Q^V)\geqslant\varepsilon-\kappa\}} e^{-N\int V(x)d\hat\mu_{L_N(Q)}(x)} d\mathbb{T}_N\,.\end{aligned}$$

Similarly, we find that

$$\begin{aligned} Z^{V,Q}_{N,\mathbb{T}} &\leqslant \int e^{-N\int V d\hat\mu_{L_N(Q)}} \mathbf{1}_{\hat\mu_{L_N(P)},\hat\mu_{L_N(Q)}\in K_L} 1_{\{d((\hat\mu_{L_N(P)},\hat\mu_{L_N(Q)})\leqslant\kappa\}} d\mathbb{T}_N \\ &\quad + e^{(\|V\|_\infty + c - \sqrt{-\log|P-Q|}\kappa/2)N} + 2e^{(\|V\|_\infty - cL - C)N} \\ &\leqslant Z^{V,P}_{N,\mathbb{T}}(e^{N\varepsilon_V^L(\kappa)} + e^{(2\|V\|_\infty + c - \sqrt{-\log|P-Q|}\kappa/2)N} + 2e^{(2\|V\|_\infty - cL - C)N}) \end{aligned}$$

where we used that the partition function is bounded from below by $e^{-\|V\|_\infty N}$. Moreover the previous large deviation principle implies if $\kappa \leqslant \varepsilon/2$ that

$$\limsup_{N\to\infty} \frac{1}{N} \log \frac{1}{Z^{V,Q}_{N,\mathbb{T}}} \int_{\{d(\hat\mu_{L_N(Q)}, M_Q^V)\geqslant\varepsilon-\kappa\}} e^{-N\int V(x)d\hat\mu_{L_N(Q)}(x)} d\mathbb{T}_N \leqslant -\inf_{d(\mu, M_Q^V)\geqslant\varepsilon/2} \{T_Q^V\}\,.$$

Hence, we find that if L is big enough, $P - Q$ small enough so that $\varepsilon_V^L(\kappa) > \max\{2\|V\|_\infty + c - \sqrt{-\log|P-Q|}\kappa, 2\|V\|_\infty - cL - C\}$, (2.2.17) yields

$$-\inf_{((M_Q^V)^\varepsilon)^c} T_P^V \leqslant 2\varepsilon_V^L(\kappa) - \inf_{d(\mu, M_Q^V)\geqslant\varepsilon/2} \{T_Q^V\}$$

We then conclude that the right hand side is negative for such choices of parameters if κ is small enough and therefore $\inf_{((M_Q^V)^\varepsilon)^c} T_P^V > 0$ so that $((M_Q^V)^\varepsilon)^c \subset (M_P^V)^c$ which yields the result. □

2.3 β-ensembles

2.3.1 Large deviation principles for β-ensembles

In this section we consider the β-ensembles and collect already known results about their large deviation principles. We then relate these large deviation principles with the previous ones thanks to Dumitriu-Edelman tri-diagonal representation, as pioneered in [Spo20c]. Coulomb gases on the real line are given by the following β-ensembles distribution:

$$\mathrm{d}\mathbb{P}_N^{V,\beta}(x_1, \cdots, x_N) = \frac{1}{Z^{V,\beta}_{N,C}} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i=1}^N (\frac{1}{2}x_i^2 + V(x_i))} \mathrm{d}x_1 \cdots \mathrm{d}x_N. \tag{2.3.1}$$

V will be a continuous potential. When $V = 0$ and $\beta = 1$, it is well known [AGZ10, Section 2.5.2] that $d\mathbb{P}_N^{0,1}$ is the law of the eigenvalues of the Gaussian orthogonal ensemble of random matrices with standard Gaussian entries. Hereafter, we keep the potential to be under the form of a quadratic potential plus a general potential only to have simpler notations later on. In this article, we are however interested in the scaling where $\beta = \frac{2P}{N}$. The large deviation principles for the empirical measure $\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ have been derived in [GZ19] and yields the following result.

Theorem 2.3.1. *[GZ19] Let P_N be a sequence of positive real numbers converging towards $P > 0$. Let $W(x) = \frac{1}{2}x^2 + V(x)$ be a continuous function such that for some $P' > P + 1$ there exists a finite constant C_V such that for all x*

$$W(x) \geqslant P' \log(|x|^2 + 1) + C_V \tag{2.3.2}$$

Then the law of $\hat{\mu}_N$ under $\mathbb{P}_N^{V,\frac{2P_N}{N}}$ satisfies a large deviation principle in the scale N and with good rate function $I_P^V(\mu) = f_P^V(\mu) - \inf f_P^V$ where

$$f_P^V(\mu) = \frac{1}{2}\int (W(x) + W(y) - 2P\log|x-y|)d\mu(x)d\mu(y) + \int \log\frac{d\mu}{dx}d\mu(x)$$

if $\mu \ll dx$ and $\log\frac{d\mu}{dx}$ is μ-integrable, whereas f_P^V is infinite otherwise.

In fact, neglecting the singularity of the logarithm, this result would be a direct consequence of Sanov's theorem and Varadhan's lemma. Dealing with this singularity requires extra-care, a difficulty which was addressed in [GZ19]. Indeed, [GZ19, Theorem 1.1] can be applied, as was kindly shown to us by David García-Zelada. For $\frac{1}{2P'} < \alpha < 1 - \frac{P}{P'}$, we can rewrite

$$d\mathbb{P}_N^{V,\frac{2P_N}{N}}(x_1,\dots,x_N) = \frac{1}{\tilde{Z}_N^{V,P_N}} e^{-2P_N N H_N(x_1,\dots,x_N)} d\pi(x_1)\dots d\pi(x_N),$$

where, if $\gamma(N) = (1 - N^{-1})\frac{1}{2P'} + \frac{\alpha - 1}{2P_N}$, we set

$$H_N(x_1,\dots,x_N) = \frac{1}{N^2}\sum_{1\leqslant i<j\leqslant N}\left(\frac{W(x_i)}{2P'} + \frac{W(x_j)}{2P'} - \log|x_i - x_j|\right) - \frac{\gamma(N)}{N}\sum_{i=1}^N W(x_i)$$

and π is the probability measure given by

$$d\pi(x) = \frac{1}{Z}e^{-\alpha W(x)}dx.$$

The sequence $(H_N)_{N\geqslant 0}$ is (up to considering N large enough) uniformly bounded from below by (2.3.2). Moreover, letting $\gamma(\infty) = \frac{1}{2P'} + \frac{\alpha - 1}{2P}$, we set for $\mu \in \mathcal{P}(\mathbb{R})$,

$$H(\mu) := \frac{1}{2}\int\left(\frac{W(x)}{2P'} + \frac{W(y)}{2P'} - \log|x-y|\right)d\mu(x)d\mu(y) - \gamma(\infty)\int W(x)d\mu(x),$$

we find [GZ19, Lemma 2.1] that the couple $(\{H_N\}_{N\geqslant 0}, H)$ fulfills the assumptions of [GZ19, Theorem 1.1]. Thus the law of $\hat{\mu}_N$ satisfies a large deviation principle at speed N with rate function $I_P^V = f_P^V - \inf f_P^V$, where

$$f_P^V(\mu) = \begin{cases} 2PH_V(\mu) + \int \log\frac{d\mu}{d\pi}d\mu & \text{if } \mu \ll \pi \text{ and } \log\frac{d\mu}{dx} \text{ is } \mu\text{-integrable} \\ +\infty & \text{otherwise.} \end{cases}$$

It is not hard to see that

Lemma 2.3.2. *For any continuously differentiable function W, any $P' > P + 1$ such that (2.3.2) holds,*

- $\mu \mapsto I_P^V(\mu)$ *is strictly convex,*
- I_P^V *achieves its minimal value at a unique probability measure* $\mu_P^V(dx) \ll dx$ *which satisfies the non-linear equation*

$$W(x) - 2P \int \log|x-y| d\mu_P^V(y) + \log \frac{d\mu_P^V}{dx} = \lambda_P^V \qquad a.s \tag{2.3.3}$$

where λ_P^V *is a finite constant. Furthermore the support of* μ_P^V *is the whole real line and the density of* $\frac{d\mu_P^V}{dx}$ *is bounded from above by* $C_P(|x|+1)^{2(P-P')}$ *where* C_P *is a constant which is uniformly bounded on compact subsets of* $(0, P'-1)$.

- *Let* D *be the distance on* $\mathcal{P}(\mathbb{R})$ *given by*

$$\begin{aligned} D(\mu,\mu') &= \left(-\int \log|x-y| d(\mu-\mu')(x) d(\mu-\mu')(y)\right)^{1/2} \\ &= \left(\int_0^\infty \frac{1}{t}\left|\int e^{itx} d(\mu-\mu')(x)\right|^2 dt\right)^{1/2} \end{aligned} \tag{2.3.4}$$

Then $P \mapsto \mu_P^V$ *is locally 1/2-Hölder for the distance* D*: For any* $\delta > 0$ *such that* $[P-\delta, P+\delta] \subset (0, P'-1)$*, there exists a constant* $D > 0$ *such that for all* $P - \delta \leqslant R \leqslant P + \delta$*, we have*

$$D(\mu_P^V, \mu_R^V) \leqslant D\sqrt{|P-R|}.$$

We will see later that in fact $P : (0, P'-1) \to \mu_P^V$ is differentiable, see Lemma 2.3.6. Observe that if f is in L^2 with derivative in L^2, we can set $\|f\|_{\frac{1}{2}} = (\int_0^\infty t|\hat{f}_t|^2 dt)^{1/2}$. Then, for any measure ν with zero mass,

$$\int f(x) d\nu(x) = \int_{-\infty}^{\infty} \hat{f}_t \hat{\nu}_t dt = \int_{-\infty}^{\infty} \sqrt{t}\hat{f}_t \frac{1}{\sqrt{t}} \hat{\nu}_t dt$$

so that by Cauchy-Schwartz inequality, we get,

$$\left|\int f(x) d\nu(x)\right|^2 \leqslant \int_{-\infty}^{\infty} |t\hat{f}_t|^2 dt \int_{-\infty}^{\infty} \frac{1}{|t|} |\hat{\nu}_t|^2 dt = 4\|f\|_{1/2}^2 D(\nu, 0)^2 \tag{2.3.5}$$

In particular, the last point in the theorem shows that for any f with finite $\|f\|_{1/2}$, $P \to \int f d\mu_P^V$ is Hölder $1/2$.

Proof. For $P'' > 1$, we denote by $\lambda_{P''}$ the probability measure on the real line given by $\lambda_{P''}(dx) := Z_{P''}^{-1}(|x^2|+1)^{-P''/2} dx$ and rewrite f_P^V (up to a constant $\log Z_{P''}$) as

$$f_P^V(\mu) = \frac{1}{2}\int (\bar{W}(x) + \bar{W}(y) - 2P\log|x-y|) d\mu(x) d\mu(y) + \int \log \frac{d\mu(x)}{d\lambda_{P''}(x)} d\mu(x)$$

where $\bar{W}(y) := W(y) - \frac{1}{2}P''\log(|y|^2+1)$. Because $\lambda_{P''}$ is a probability measure so that, for every probability measure μ,

$$\int \log \frac{d\mu}{d\lambda_{P''}}(x) d\mu(x) \geqslant 0$$

by Jensen's inequality since $x \mapsto x\log x$ is convex.

The first point of the lemma is clear as $\mu \mapsto \int (\bar{W}(x) + \bar{W}(y) - 2P\log|x-y|) d\mu(x) d\mu(y)$ is strictly convex [AGZ10, Lemma 2.6.2] whereas the relative entropy $\mu \mapsto \int \log \frac{d\mu}{d\lambda_{P''}}(y) d\mu(y)$ is well known to be convex. Since f_P^V is a good rate function, it achieves its minimal value at a unique probability measure μ_P^V. Writing that for any measure ν with mass zero such that $\mu_P^V + \varepsilon\nu$ is a probability measure for small

enough ε, $I_P^V(\mu_P^V + \varepsilon\nu) \geqslant I_P^V(\mu_P^V)$, we get that (2.3.3) holds μ_P^V almost surely and that the left hand side in (2.3.3) is greater or equal than the right hand side outside of the support of μ_P^V. Since the left hand side equals $-\infty$ when the density vanishes, we conclude that the support is the whole real line. We finally show the boundedness of the density. Note that (2.3.3) implies that

$$\frac{d\mu_P^V}{dx}(x) = e^{\lambda_P^V} e^{-W(x)+2P\int \log|x-y| d\mu_P^V(y)} \tag{2.3.6}$$

We get from (2.3.2), and the fact that $\log|x-y| \leqslant \frac{1}{2}\log(|x|^2+1) + \frac{1}{2}\log(|y|^2+1)$ the bound

$$-W(x) + 2P\int \log|x-y| d\mu_P^V(y) \leqslant -(P'-P)\log(|x|^2+1) + C_V + P\int \log(|x|^2+1) d\mu_P^V\,.$$

We thus only need to bound $\int \log(|x|^2+1) d\mu_P^V$ and λ_P^V from above. We first notice that $P \mapsto \inf f_P^V$ is concave since it is the limit of the free energy $-N^{-1}\log Z_N^{V,\frac{2P}{N}}$. This is enough to guarantee that this quantity is uniformly bounded on compact sets (as it is at any given point). We denote by C such a bound for a fixed compact set. As in [AGZ10, Lemma 2.6.2 (b)], since the relative entropy is non-negative we find that

$$\int (\bar{W}(x) - P\log(|x|^2+1)) d\mu_P^V(x) \leqslant f_P^V(\mu_P^V) \leqslant C\,.$$

This implies by our hypothesis (2.3.2) that

$$(P'-P''-P)\int \log(|x|^2+1) d\mu_P^V(x) \leqslant C - C_V$$

and therefore plugging this estimate in the infimum of f_P^V gives if $P'-P-P''>0$ (which is always possible as we assumed $P'-P>1$)

$$\int W(x) d\mu_P^V(x) \geqslant C + \frac{C-C_V}{2(P'-P-P'')}$$

Moreover, again because the relative entropy is non-negative,

$$\begin{aligned} -P\Sigma(\mu_P^V) &:= -P\int \log|x-y| d\mu_P^V(x) d\mu_P^V(y) \\ &\leqslant C - \int \bar{W}(x) d\mu_P^V(x) \leqslant C - 2(P'-P'')\int \log(|x|^2+1) d\mu_P^V(x) - C_V \end{aligned}$$

is uniformly bounded. Finally, from (2.3.3) we have after integration under μ_P^V

$$\lambda_P^V = \inf f_P^V - P\int \log|x-y| d\mu_P^V(x) d\mu_P^V(y) \tag{2.3.7}$$

is thus uniformly bounded from above. This completes the proof of the upper bound of the density: $\frac{d\mu_P^V}{dx}$ is bounded by $C_P(|x|+1)^{2(P-P')}$ where C_P is uniformly bounded on compacts so that $P'-P-1 \geqslant \varepsilon > 0$ for some fixed ε.

We next study the regularity of the equilibrium measure μ_P^V in the parameter P. Let $\delta > 0$ be such that $[P-\delta, P+\delta] \subset (0, P'-1)$, and let $P-\delta \leqslant R \leqslant P+\delta$. If $\Delta\mu = \mu_P^V - \mu_R^V$, since μ_P^V minimizes f_P^V,

we have

$$\begin{aligned}
0 &\geqslant f_P^V(\mu_P^V) - f_P^V(\mu_R^V) \\
&= \int W(x)d\Delta\mu(x) - 2P\int \log|x-y|d\mu_R^V(x)d\Delta\mu(y) - P\int \log|x-y|d\Delta\mu(x)d\Delta\mu(y) \\
&\quad + \int \log\frac{d\mu_P^V}{dx}d\mu_P^V - \int \log\frac{d\mu_R^V}{dx}d\mu_R^V \\
&= \int (2R\int \log|x-y|d\mu_R^V(y) - \log\frac{d\mu_R^V}{dx})(x)d\Delta\mu(x) - 2P\int \log|x-y|d\mu_R^V(x)d\Delta\mu(y) \\
&\quad -P\int \log|x-y|d\Delta\mu(x)d\Delta\mu(y) + \int \log\frac{d\mu_P^V}{dx}d\mu_P^V - \int \log\frac{d\mu_R^V}{dx}d\mu_R^V \\
&= 2(R-P)\int \log|x-y|d\mu_R^V(x)d\Delta\mu(y) - P\int\int \log|x-y|d\Delta\mu(x)d\Delta\mu(y) + \int \log\frac{d\mu_P^V}{d\mu_R^V}d\mu_P^V
\end{aligned}$$

where in the second line we used (2.3.3) and the fact that $\Delta\mu(1) = 0$. By using the Fourier transform of the logarithm, the centering of $\Delta\mu$ and the definition (2.3.4) we deduce

$$\int \log\frac{d\mu_P^V}{d\mu_R^V}d\mu_P^V + PD(\mu_P^V,\mu_R^V)^2 \leqslant 2(P-R)\int\int \log|x-y|d\mu_R^V(x)d\Delta\mu(y)\,. \tag{2.3.8}$$

We can assume without loss of generality that $R < P$. We now show that the integral of the right hand side is bounded independently of $R \in [P-\delta, P]$. We have $\frac{d\mu_R^V}{dx} \leqslant \frac{C_R}{(1+|x|)^{2(P'-R)}}$, where $R \mapsto C_R$ is bounded on any compact of $(0, P'-1)$, and in particular on $[P-\delta, P+\delta]$. Thus there exists $C > 0$ such that $\frac{d\mu_R^V}{dx} \leqslant \frac{C}{(1+|x|)^2}$, and the same bound holds for μ_P^V. Using that for any x, y with $x \neq y$ we have $\log(|x-y|) \leqslant \log(1+|x|) + \log(1+|y|)$ and the previous bound on the density of μ_R^V, we conclude that $\int\int \log|x-y|d\mu_R^V(x)d\Delta\mu(y)$ is uniformly bounded in $R \in [P-\delta, P+\delta]$. Since $\int \log\frac{d\mu_P^V}{d\mu_R^V}d\mu_P^V \geqslant 0$ by Jensen's inequality equation (2.3.8) gives the existence of a finite constant D such that

$$D(\mu_P^V,\mu_R^V) \leqslant D\sqrt{|P-R|}\,.$$

□

2.3.2 Relation with the large deviation principle for Toda matrices with quadratic potential

When $V = 0$, for any $\beta > 0$, Dumitriu and Edelman [DE02, Theorem 2.12] have shown that $\mathbb{P}_N^{0,\beta}$ is the law of the eigenvalues of a $N \times N$ tri-diagonal matrix C_N^β such that $\left((C_N^\beta)_{j,j}\right)_{1\leqslant j\leqslant N}$ are independent standard normal variables, independent from the off diagonal entries $(C_N^\beta)_{j,j+1} = (C_N^\beta)_{j+1,j}$ which are independent and such that $\sqrt{2}C_N^\beta(j,j+1)$ follows a $\chi_{(N-j)\beta}$ distribution. As in the case of the Toda measure we hereafter identify $\mathbb{P}_N^{0,\beta}$ with $\mathbb{P}_N^\beta$. We are now going to give an alternate large deviation principle for the empirical measure under $\mathbb{P}_N^{2P/N}$ based on this representation, this will allow to relate the rate function $I_P = I_P^0$ of the Coulomb Gas in terms of the large deviation rate function $T_s, s \leqslant P$ for Toda matrices.

Lemma 2.3.3. *The law of the empirical measure $\hat{\mu}_N$ under $\mathbb{P}_N^{2P/N}$ satisfies a large deviation principle in the scale N and with good rate function*

$$I_P(\mu) = \lim_{\delta\to 0}\liminf_{M\to\infty} \inf_{\substack{\nu_{P/M},\cdots,\nu_P\, s.t. \\ \frac{1}{M}\sum_i \nu_{iP/M}\in B_\mu(\delta)}} \left\{\frac{1}{M}\sum_{i=1}^M T_{iP/M}(\nu_{iP/M})\right\}. \tag{2.3.9}$$

Observe for later purpose that we must have $I_P = I_P^0$ where I_P^V is defined just above Lemma 2.3.2.

Proof. We shall proceed by exponential approximation. We write $N = k_N M + r_N$, $0 \leqslant r_N \leqslant M-1$, and consider the matrices

$$S_N^M = \begin{pmatrix} L_{k_N}^1 & & & \\ & \ddots & & \\ & & L_{k_N}^M & \\ & & & 0 \end{pmatrix},$$

with $(L_{k_N}^i)_{1\leqslant i\leqslant M}$ a family of independent square matrices with size k_N distributed according to $\mathbb{T}_{k_N}^{\left(P\frac{N-ik_N}{N}\right)}$, and a block with null entries of size $r_N \times r_N$. We shall prove that they provide good exponential approximation for the matrix $C_N^{\frac{2P}{N}}$ following the distribution $\mathbb{P}_N^{2P/N}$, see [DZ10, Definition 4.2.14]. More precisely, we show that for any positive real number δ :

$$\lim_{M\to+\infty} \limsup_N \frac{1}{N} \log \mathbb{P}(d(\hat{\mu}_{C_N^{\frac{2P}{N}}}, \hat{\mu}_{S_N^M}) > \delta) = -\infty\,. \tag{2.3.10}$$

The lemma is then a direct application of [DZ10, Theorem 4.2.16 and Exercise 4.2.7]. We first approximate S_N^M by the following matrix

$$U_N^M = \begin{pmatrix} \boxed{C_1} & * & & & \\ * & \ddots & & & \\ & & * & & \\ & * & \boxed{C_M} & * & \\ & & * & \boxed{R_N^M} & \end{pmatrix},$$

where the symbols $*$ denote entries following the law of a matrix distributed according to $\mathbb{P}_N^{2P/N}$:

$$U_N(ik_N, ik_N+1) = U_N(ik_N+1, ik_N) \sim \frac{1}{\sqrt{2}}\chi_{2P\frac{N-ik_N}{N}},\ 1 \leqslant i \leqslant M\ ;$$

R_N^M has same distribution as the $r_N \times r_N$-bottom-right corner of a $\mathbb{P}_N^{2P/N}$- distributed matrix. C_i has the same coefficients as $L_{k_N}^i$ except for the top-right and bottom-left corner entries which are put to zero :

$$C_i = \begin{pmatrix} g_{(i-1)k_N+1} & \ddots & & 0 \\ \ddots & \ddots & \frac{1}{\sqrt{2}}c_j^i & \\ & \frac{1}{\sqrt{2}}c_j^i & \ddots & \ddots \\ 0 & & \ddots & g_{ik_N} \end{pmatrix}.$$

The $(c_j^i)_{1\leqslant j\leqslant k_N-1}$ are distributed according to $\chi_{2P\frac{N-ik_N}{N}}$.
For $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant k_N - 1$, let $b_j^i = \sqrt{(c_j^i)^2 + \chi_{i,j}^2}$, where $(\chi_{i,j})_{1\leqslant i\leqslant M, 1\leqslant j\leqslant k_N}$ is an independent

family of χ variables with parameter $2P\frac{k_N-j}{N}$, independent from U_M^N.
We set, for $1 \leqslant i \leqslant M$, B_i to be the matrix

$$B_i = \begin{pmatrix} g_{(i-1)k_N+1} & \ddots & & 0 \\ \ddots & \ddots & \frac{1}{\sqrt{2}}b_j^i & \\ & \frac{1}{\sqrt{2}}b_j^i & \ddots & \ddots \\ 0 & & \ddots & g_{ik_N} \end{pmatrix}.$$

The matrix

$$C_N^{2P/N} = \begin{pmatrix} \boxed{B_1} & * & & & \\ * & \ddots & & & \\ & & * & & \\ & * & \boxed{B_M} & * & \\ & & * & \boxed{R_N^M} & \end{pmatrix}$$

is distributed according to $\mathbb{P}_N^{2P/N}$, where the symbols $*$ denote the same coefficients as those of U_N^M. Because the rank of $S_N^M - U_N^M$ is bounded by $2M + r_N \leqslant 3M$, by (2.2.6) we have

$$d(\hat{\mu}_{U_N^M}, \hat{\mu}_{S_N^M}) \leqslant \frac{3M}{N} = \frac{3}{k_N}. \tag{2.3.11}$$

Let δ be a positive real number. Then for N large enough so that k_N verifies $\frac{3}{k_N} \leqslant \delta/2$,

$$\begin{aligned} \mathbb{P}\left(d(\hat{\mu}_{C_N^{2P/N}}, \hat{\mu}_{S_N^M}) > \delta\right) &\leqslant \mathbb{P}\left(d(\hat{\mu}_{C_N^{2P/N}}, \hat{\mu}_{U_N^M}) + d(\hat{\mu}_{U_N^M}, \hat{\mu}_{S_N^M}) > \delta\right) \\ &\leqslant \mathbb{P}\left(d(\hat{\mu}_{C_N^{2P/N}}, \hat{\mu}_{U_N^M}) > \delta/2\right). \end{aligned}$$

Moreover (2.2.6) yields

$$d(\hat{\mu}_{U_N^M}, \hat{\mu}_{C_N^{2P/N}}) \leqslant \frac{2}{N}\sum_{i=1}^{N} |Y_i|, \tag{2.3.12}$$

where Y_i is the ith coefficient above or below the (i,i) the coefficient of $C_N^{2P/N} - U_N^M$. Applying the inequality $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ for $a, b \geqslant 0$ and $a = c_j^i$ and $b = \chi_{i,j}$, we deduce

$$d(\hat{\mu}_{U_N^M}, \hat{\mu}_{C_N^{2P/N}}) \leqslant \frac{\sqrt{2}}{k_N M}\sum_{i=1}^{k_N M} \chi_{2P/M}^i, \tag{2.3.13}$$

where the last sum denotes the sum of iid variables with law $\chi_{2P/M}$ (and we used that there exists a coupling between a $\chi_{2P\frac{k_N-j}{N}}$ and a $\chi_{2P/M}$ variable such that the first is always bounded above by the second).

Thus for all $\delta > 0$, for any integer numbers N such that $\frac{3}{k_N} \leqslant \delta/2$ (i.e for N larger than some N_0 depending on M) and for any non-negative function $A : M \mapsto A(M)$

$$\begin{aligned}\mathbb{P}\left(d(\hat{\mu}_{S_N^M}, \hat{\mu}_{C_N^{2P/N}}) > \delta\right) &\leqslant \mathbb{P}\left(\sum_{i=1}^{k_N M} \chi_{2P/M} > \frac{k_N M \delta}{2\sqrt{2}}\right) \\ &\leqslant e^{-A(M)k_N M\delta/(2\sqrt{2})}\mathbb{E}\left[e^{A(M)\chi_{2P/M}}\right]^{k_N M}.\end{aligned}$$

It is not hard to see that with $A(M) = \sqrt{\log(M)}$, there exists a finite constant K such that

$$\sup_{M\geqslant 0} \mathbb{E}\int e^{A(M)x} d\chi_{1/M}(x) \leqslant K \tag{2.3.14}$$

insuring that

$$\frac{1}{N}\log \mathbb{P}(d(\hat{\mu}_{C_N^{2P/N}}, \hat{\mu}_{S_N^M} > \delta) \leqslant -A(M)\frac{\delta}{2\sqrt{2}} + K,$$

which yields the result.

□

We shall use the previous lemma to study the case with a non trivial potential. Indeed, as a direct consequence of Lemma 2.3.3 and Varadhan's lemma, we deduce the following Theorem.

Theorem 2.3.4. *For any continuous function V such that*

$$\limsup_{|x|\to\infty} \frac{|V(x)|}{x^2} = 0, \tag{2.3.15}$$

the law of the empirical measure $\hat{\mu}_N$ under $\mathbb{P}_N^{V,2P/N}$ satisfies a large deviation principle in the scale N and with good rate function $I_P^V(\mu) = f_P^V(\mu) - \inf f_P^V$ where

$$f_P^V(\mu) = \lim_{\delta\to 0}\liminf_M \inf_{\substack{\nu_{P/M},\cdots,\nu_P s.t. \\ \frac{1}{M}\sum_i \nu_{iP/M}\in B_\mu(\delta)}} \left\{\frac{1}{M}\sum_{i=1}^M (T_{iP/M}(\nu_{iP/M}) + \int V d\nu_{iP/M})\right\}. \tag{2.3.16}$$

Remark 2.3.5. *Varadhan's lemma gives the result for bounded continuous function V. However, we can approximate V by $V(x)(1+\varepsilon x^2)^{-1}$ with overwhelming probability thanks to Lemma 2.2.2, which allows to conclude for any potential V satisfying* (2.3.15)

We shall use this relation to give a better description of the rate function T_P. In fact we first consider the free energy

$$F_{\mathbb{T}}^{V,P} = \lim_{N\to\infty}\frac{1}{N}\log Z_{N,\mathbb{T}}^{V,P},\ F_C^{V,P} = \lim_{N\to\infty}\frac{1}{N}\log Z_{N,C}^{V,P} = -\inf f_P^V.$$

Lemma 2.3.6. *For any continuous function V satisfying* (2.3.15),

- *$P \mapsto F_C^{V,P} = -\inf f_P^V$ is continuously differentiable on $(0,+\infty)$. Moreover, for any $P > 0$*

$$F_{\mathbb{T}}^{V,P} = \partial_P(PF_C^{V,P})$$

- *For any bounded continuous function f, the map $P \in (0,+\infty) \mapsto P\mu_P^V(f)$ is continuously differentiable. Moreover, there exists a unique minimizer ν_P^V of $\mu \mapsto T_P(\mu) + \int V d\mu(x)$, which satisfies, for any bounded continuous function f,*

$$\nu_P^V(f) = \partial_P(P\mu_P^V(f)).$$

Therefore, we have

$$\nu_P^V = \partial_P(P\mu_P^V). \tag{2.3.17}$$

- *For any probability measure μ,*

$$T_P(\mu) = -\inf_{V\in\mathcal{C}_b^0}\left\{\int_{\mathbb{R}} V d\mu + F_{\mathbb{T}}^{V,P}\right\}. \tag{2.3.18}$$

Proof. First notice that, for any probability measure μ, Lemma 2.3.3 implies that

$$\begin{aligned} f_P^V(\mu) = I_P(\mu) + \int_{\mathbb{R}} V\mathrm{d}\mu &\geqslant \liminf_M \frac{1}{M}\sum_{i=1}^M \inf_\nu \left\{T_{iP/M}(\nu) + \int_{\mathbb{R}} V d\nu\right\} \\ &= \int_0^1 \inf_\nu \left\{T_{sP}(\nu) + \int_{\mathbb{R}} V\mathrm{d}\nu\right\}\mathrm{d}s = -\int_0^1 F_{\mathbb{T}}^{V,sP} ds\,. \end{aligned} \tag{2.3.19}$$

In the equality between the lim inf and the integral, we used the fact that $s\in(0,1)\mapsto F_{\mathbb{T}}^{V,sP}$ is convex and therefore continuous. We claim that this lower bound is achieved. For $s\in[0,1]$, let ν_{sP}^* be a minimizer of $\mu\mapsto T_{sP}(\mu)+\int V\mathrm{d}\mu$. By Corollary 2.2.7, we can choose ν_{sP}^* such that $s\mapsto\nu_{sP}^*$ is continuous. Hence, $\mu_P^* := \int_0^1 \nu_{sP}^*\mathrm{d}s$ makes sense and is a probability measure on $\mathbb{R}$. We claim it minimizes f_P^V. Indeed, by Lemma 2.3.3, we have

$$\begin{aligned} f_P^V(\mu_P^*) &= \lim_{\delta\to 0}\liminf_M \inf_{\frac{1}{M}\sum_{i=1}^M \nu_{iP/M}\in B_{\mu_P^*}(\delta)} \left\{\frac{1}{M}\sum_{i=1}^M T_{iP/M}(\nu_{iP/M}) + \int_{\mathbb{R}} V d\nu_{iP/M}\right\} \\ &\leqslant \liminf_M \frac{1}{M}\sum_{i=1}^M \left\{T_{iP/M}(\nu_{iP/M}^*) + \int_{\mathbb{R}} V d\nu_{iP/M}^*\right\} \\ &= \liminf_M \frac{1}{M}\sum_{i=1}^M \inf_\nu \left\{T_{iP/M}(\nu) + \int_{\mathbb{R}} V d\nu\right\} \\ &= \int_0^1 \inf_\nu \left\{T_{sP}(\nu) + \int_{\mathbb{R}} V\mathrm{d}\nu\right\}\mathrm{d}s = -\int_0^1 F_{\mathbb{T}}^{V,Ps} ds. \end{aligned}$$

With (2.3.19), we deduce that the above inequality is an equality and that f_P^V achieves its minimal value at μ_P^*. By Lemma 2.3.2, this minimizer is unique and therefore $\mu_P^* = \mu_P^V$ for any choices of paths $\nu_\cdot^*$ and any positive real number P. Hence, we find that

$$-F_C^{V,P} = \inf f_P^V = I_P(\mu_P^V) + \int_{\mathbb{R}} V\mathrm{d}\mu_P^V = -\int_0^1 F_{\mathbb{T}}^{V,Ps} ds\,.$$

By a change of variable we deduce

$$PF_C^{V,P} = \int_0^P F_{\mathbb{T}}^{V,s} ds\,.$$

Since $s\mapsto F_{\mathbb{T}}^{V,s}$ is convex, it is continuous. This shows that $P\mapsto PF_C^{V,P}$ is continuously differentiable, and that for all $P>0$,

$$F_{\mathbb{T}}^{V,P} = \partial_P(PF_C^{V,P})\,.$$

Moreover, we have seen that for any choice of continuous minimizing path $\nu_\cdot^*$ of $\mu\mapsto T_\cdot(\mu)+\int V d\mu$ and any positive real number P,

$$\mu_P^V = \int_0^1 \nu_{sP}^* ds = \frac{1}{P}\int_0^P \nu_s^*\, ds\,.$$

Integrating the last equality against f bounded continuous we have

$$\mu_P^V(f) = \frac{1}{P}\int_0^P \nu_s^*(f) ds\,.$$

By continuity of $s \mapsto \nu_s^*(f)$, we deduce that $P \mapsto \mu_P^V(f)$ is continuously differentiable and that

$$\nu_P^*(f) = \partial_P(P\mu_P^V(f)).$$

But Corollary 2.3.2 implies that any probability which minimizes T_P^V can be seen as the endpoint of a continuous path $s \in (0, P] \mapsto \nu_s^*$ where each ν_s^* minimizes T_s^V. By the latter, such a measure is then equal to $\partial_P(P\mu_P^V(f))$, showing the uniqueness of the minimizer ν_P^V of T_P^V and the equality

$$\nu_P^V = \partial_P(P\mu_P^V).$$

The last point of the Lemma is a direct consequence of [DZ10, Theorem 4.5.10] since T_P^V is convex for all bounded continuous function V.

□

By Lemma 2.3.2, ν_P^V is a probability measure which satisfies almost surely

$$d\nu_P^V(x) = (C_P^V + 2P \int \log|x-y| d\nu_P^V(y)) d\mu_P^V(x)$$

with C_P^V a constant such that

$$C_P^V + 2P \int \log|x-y| d\nu_P^V(y) d\mu_P^V(x) = 1$$

Furthermore we must have $C_P^V + 2P \int \log|x-y| d\nu_P^V(y) \geqslant 0$ for all x.

2.4 Large deviations for Toda Gibbs measure with general potentials

We now consider the measures $\mathbb{T}_N^{V,P}$ given by (2.1.5), with potential given by $W : x \in \mathbb{R} \mapsto ax^{2k} + U(x)$, $k \geqslant 2$, with $U(x)/x^{2k}$ going to zero at infinity. We show that under these laws, the law of the empirical measures $(\hat{\mu}_{L_N})_{N \geqslant 1}$ still fulfills a large deviation principle, by extending the subadditivity argument previously used. We then identify the rate function as before. By Varadhan's Lemma, it is enough to consider the case where $U(x) = \frac{1}{2}x^2$ (we detail this in Section 2.5). We hereafter continue to use the notation (2.1.5) with now $V(x) = ax^{2k}$.

2.4.1 Exponential tightness

In this section we prove that if $W(x) = ax^{2k} + \frac{1}{2}x^2$, *i.e* $V(x) = ax^{2k}$ with $k \geqslant 2$ and $a > 0$, then the law of the empirical measure of the eigenvalues is exponentially tight under $\mathbb{T}_N^{V,P}$. More precisely, we let $\mathcal{K}_L = \{\mu \in \mathcal{P}(\mathbb{R}) \mid \int V(x) d\mu(x) \leqslant L\}$ which is a compact of $\mathcal{P}(\mathbb{R})$. Then we shall prove

Lemma 2.4.1. *There exists a finite constant c_W such that*

$$\mathbb{T}_N^{V,P}(\hat{\mu}_N \in \mathcal{K}_L^c) \leqslant e^{-(L - c_W)N}.$$

Proof. We first bound from below the free energy by Jensen's inequality

$$Z_{N,\mathbb{T}}^{V,P} = \int_{\mathbb{R}^{2N}} e^{-N \int_{\mathbb{R}} V d\hat{\mu}_N} d\mathbb{T}_N^P \geqslant \exp\{-N \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}} V d\hat{\mu}_N d\mathbb{T}_N^P\} \geqslant \exp\{-c_V N\}. \tag{2.4.1}$$

From here we deduce exponential tightness for $(\hat{\mu}_N)_N$ under $\mathbb{T}_N^{V,P}$: for $L > 0$,

$$\mathbb{T}_N^{V,P}\left(\int_{\mathbb{R}} V d\hat{\mu}_N \geqslant L\right) = \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \int_{\mathbb{R}^{2N}} \mathbf{1}_{\{\int_{\mathbb{R}} V d\hat{\mu}_N \geqslant L\}} e^{-N\int_{\mathbb{R}} V d\hat{\mu}_N} d\mathbb{T}_N^P$$
$$\leqslant e^{N(c_V - L)}. \tag{2.4.2}$$

□

For later purpose we prove the following result showing that the off diagonal entries $b_i = e^{-r_i/2}, 1 \leqslant i \leqslant N$ of the Lax matrix L_N do not become too small :

Lemma 2.4.2. *For any $P > 0$*

$$\limsup_L \limsup_N \frac{1}{N} \log \mathbb{T}_N^{V,P}\left(\frac{1}{N}\sum_{i=1}^N \log b_i \leqslant -L\right) = -\infty.$$

Proof. Since V is bounded from below and we have bounded from below the partition function (2.4.1), it enough to prove this estimate when $V = 0$. But, in this case the entries are independent and so we only need to prove it for independent chi distributed variables. But then, for any $0 < \delta < P$, with $\mathbb{Z}_{N,\mathbb{T}}^P = \mathbb{Z}_{N,\mathbb{T}}^{0,P}$ the partition function in (2.1.6), we find

$$\mathbb{T}_N^P\left(\frac{1}{N}\sum_{i=1}^N \log b_i \leqslant -L\right) \leqslant e^{-\delta L N} \frac{\mathbb{Z}_{N,\mathbb{T}}^{P-\delta/2}}{\mathbb{Z}_{N,\mathbb{T}}^P} = e^{-\delta L N}\left(\frac{\Gamma(P-\delta/2)}{2^{\delta/2}\Gamma(P)}\right)^N$$

from which the result follows by taking for instance $\delta = P/2$.

□

2.4.2 Weak LDP

In this section, we prove that $\hat{\mu}_{L_N}$ satisfies a weak large deviation principle, namely Lemma 2.2.3. In this more general setup, we follow again a subadditivity argument, which is however more sophisticated since the entries of L_N are not independent anymore. We will restrict ourselves to the case where $V(x) = ax^{2k}$, $a > 0$, the case of a more general potential with the same asymptotic behavior being again a consequence of Varadhan's Lemma. We first show that the large deviation principles is the same if we remove the entries (equal to b_N) in the corners $(N, 1)$ and $(1, N)$ in the Toda matrix. Namely, let $\tilde{L}_N$ be the tridiagonal matrix with entries equal to those of L_N except for the entries $(1, N)$ and $(N, 1)$ which vanish and consider the following modification of $\mathbb{T}_N^{V,P}$ given by

$$d\tilde{\mathbb{T}}_N^{V,P} = \frac{1}{\tilde{Z}_N^{V,P}} e^{-\mathrm{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P\,. \tag{2.4.3}$$

Lemma 2.4.3. *For any probability measure μ, we have*

$$\lim_{\delta\to 0}\liminf_{N\to\infty} \frac{1}{N}\log\int 1_{d(\hat{\mu}_{L_N},\mu)<\delta} e^{-\mathrm{Tr}V(L_N)} d\mathbb{T}_N^P = \lim_{\delta\to 0}\liminf_{N\to\infty} \frac{1}{N}\log\int 1_{d(\hat{\mu}_{\tilde{L}_N},\mu)<\delta} e^{-\mathrm{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P$$

Moreover,

$$\liminf_{N\to\infty} \frac{1}{N}\log\int e^{-\mathrm{Tr}V(L_N)} d\mathbb{T}_N^P = \liminf_{N\to\infty} \frac{1}{N}\log\int e^{-\mathrm{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P\,.$$

The same results hold if we replace all the liminf by limsup.

Proof. To simplify the notations we take $a = 1$ in the proof. First notice that $V(L_N) - V(\tilde{L}_N)$ is an homogeneous polynomial of degree $2k$ in L_N and $\Delta L_N = L_N - \tilde{L}_N$, with degree at least one in the latter. Observe that ΔL_N only depends on b_N. Therefore, there exists a finite constant C_k such that on $B_N^{K,M} := \{b_N \leqslant K\} \cap \{\frac{1}{N}\mathrm{Tr}(L_N^{2k}) \leqslant M\}$ (or $\tilde{B}_N^{M,K} := \{b_N \leqslant K\} \cap \{\frac{1}{N}\mathrm{Tr}(\tilde{L}_N^{2k}) \leqslant M\}$), Hölder's inequality implies

$$\begin{aligned}\left|\frac{1}{N}\mathrm{Tr}\left(V(L_N) - V(\tilde{L}_N)\right)\right| &\leqslant C_k \sum_{l=1}^{2k} \left(\frac{1}{N}\mathrm{Tr}\left((\Delta L_N)^{2k}\right)\right)^{l/2k} \left(\frac{1}{N}\mathrm{Tr}\left(L_N^{2k}\right)\right)^{\frac{2k-l}{2k}} \\ &\leqslant C(M,K)N^{-\frac{1}{2k}}\end{aligned}$$

where $C(M,K)$ is a finite constant depending only on M, K, k. Note that in the above right hand side $\mathrm{Tr}(L_N^{2k})$ can be replaced by $\mathrm{Tr}(\tilde{L}_N^{2k})$ as they play a symmetric role. Moreover, by (2.2.7), $d(\hat{\mu}_{L_N}, \hat{\mu}_{\tilde{L}_N}) \leqslant 2/N$ since ΔL_N has rank at most two. We fix a probability measure μ and first prove that

$$\liminf_{N\to\infty} \frac{1}{N}\log \int 1_{d(\hat{\mu}_{L_N},\mu)<\delta} e^{-\mathrm{Tr}V(L_N)} d\mathbb{T}_N^P \geqslant \liminf_{N\to\infty} \frac{1}{N}\log \int 1_{d(\hat{\mu}_{L_N},\mu)<\delta} e^{-\mathrm{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \,. \tag{2.4.4}$$

We can assume without loss of generality that the right hand side does not equal $-\infty$. Then, we have by the previous remark

$$\begin{aligned}\int 1_{d(\hat{\mu}_{L_N},\mu)<\delta} e^{-\mathrm{Tr}V(L_N)} d\mathbb{T}_N^P &\geqslant e^{-C(M,K)N^{\frac{2k-1}{2k}}} \int 1_{\tilde{B}_N^{M,K}\cap\{d(\hat{\mu}_{\tilde{L}_N},\mu)<\delta-\frac{2}{N}\}} e^{-\mathrm{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \\ &\geqslant C' e^{-C(M,K)N^{\frac{2k-1}{2k}}} \int 1_{\{\mathrm{Tr}V(\tilde{L}_N)\leqslant NM\}\cap\{d(\hat{\mu}_{\tilde{L}_N},\mu)<\delta-\frac{2}{N}\}} e^{-\mathrm{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \\ &\geqslant C' e^{-C(M,K)N^{\frac{2k-1}{2k}}} \left\{\int 1_{\{d(\hat{\mu}_{\tilde{L}_N},\mu)<\delta-\frac{2}{N}\}} e^{-\mathrm{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P - e^{-NM}\right\}\end{aligned}$$

where in the second line we integrated over $b_N \leqslant K$ and in the last line we used that

$$\int 1_{\{\mathrm{Tr}V(\tilde{L}_N)\geqslant NM\}} e^{-\mathrm{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \leqslant e^{-NM}\,.$$

We next choose M so that this term is smaller than the first term (which we assumed bounded below by e^{-NC} for some finite C). We deduce that (2.4.4) holds. To prove the converse inequality, we notice that there exists one b_i bounded by K with probability greater than $1 - e^{-a(K)N}$ under $\mathbb{T}_N^P$, with $a(K) = -\log P(b \geqslant K) > 0$ which goes to + infinity when K does. By symmetry with respect to the order of the indices, we may assume it is b_N. Therefore, because V is bounded below by some finite constant C, setting $a'(K) = a(K) - C$, and using Lemma 2.4.1, we find

$$\begin{aligned}\int 1_{d(\hat{\mu}_{L_N},\mu)<\delta} e^{-\mathrm{Tr}V(L_N)} d\mathbb{T}_N^P &\leqslant e^{-Na'(K)} + N\int 1_{\{b_N\leqslant K\}\cap\{d(\hat{\mu}_{L_N},\mu)<\delta\}} e^{-\mathrm{Tr}V(L_N)} d\mathbb{T}_N^P \\ &\leqslant e^{-Na'(K)} + Ne^{-N(M-c_V)} + Ne^{C(M,K)N^{\frac{2k-1}{2k}}} \int 1_{B_N^{M,K}\cap\{d(\hat{\mu}_{\tilde{L}_N},\mu)<\delta+\frac{2}{N}\}} e^{-\mathrm{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \\ &\leqslant e^{-Na'(K)} + Ne^{-N(M-c_V)} + Ne^{C(M,K)N^{\frac{2k-1}{2k}}} \int 1_{\{d(\hat{\mu}_{\tilde{L}_N},\mu)<\delta+\frac{2}{N}\}} e^{-\mathrm{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P\end{aligned}$$

which gives the converse bound, letting N going to infinity, provided K and M are large enough. The same arguments also hold when there is no indicator function, giving the same estimates for the free energy. □

Lemma 2.4.4. *Let $V(x) = ax^{2k}$ and $P > 0$. For any μ in $\mathcal{P}(\mathbb{R})$, there exists a limit*

$$\lim_{\delta\to 0}\liminf_N \frac{1}{N}\log \mathbb{T}_N^{V,P}\left(\hat{\mu}_{L_N} \in B_\mu(\delta)\right) = \lim_{\delta\to 0}\limsup_N \frac{1}{N}\log \mathbb{T}_N^{V,P}\left(\hat{\mu}_{L_N} \in B_\mu(\delta)\right). \tag{2.4.5}$$

We denote this limit by $-T_P^V(\mu)$. Then, $\mu \mapsto T_P^V(\mu)$ is convex.

Proof. We use the notations of Lemma 2.2.3. Let $q \geqslant 1$ be fixed. For $N \geqslant 1$ we write $N = k_N q + r_N$, $0 \leqslant r_N \leqslant q-1$, and define L_N^q by removing the off diagonal entries $b_{\ell q} = L_N(\ell q, \ell q + 1), L_N(\ell q+1, \ell q), 1 \leqslant \ell \leqslant k_N$, as well as the entries $L_N(1, N), L_N(N, 1)$, from L_N. We set $R_N^q = L_N - L_N^q$. Let $Z_N^V = Z_{N,\mathbb{T}}^{V,P}$ denote in short the partition function for the Toda Gibbs measure with potential V and set

$$Z_{N,q}^V = \mathbb{E}_{\mathbb{T}_N^P}\left[e^{-\mathrm{Tr}V(L_N^q)}\right] = \int e^{-\mathrm{Tr}V(L_N^q)} d\mathbb{T}_N^P.$$

We first show that there is some constant C_k (independent of N) such that for all $N \geqslant 1$,

$$\frac{1}{N}\log\frac{Z_{N,q}^V}{Z_N^V} \geqslant -\frac{C_k}{q^{1/2k}}. \tag{2.4.6}$$

By Jensen's inequality we have

$$\frac{1}{N}\log\frac{Z_{N,q}^V}{Z_N^V} = \frac{1}{N}\log\mathbb{E}_{\mathbb{T}_N^{V,P}}\left[e^{\mathrm{Tr}(V(L_N)-V(L_N^q))}\right] \geqslant \frac{1}{N}\mathbb{E}_{\mathbb{T}_N^{V,P}}\left[\mathrm{Tr}(V(L_N)-V(L_N^q))\right]. \tag{2.4.7}$$

As in the proof of Lemma 2.4.1, we bound the right hand side by first noticing that $V(L_N) - V(L_N^q)$ is an homogeneous polynomial of degree $2k$ in L_N and $L_N - L_N^q$, with degree at least one in the latter. Therefore, Hölder's inequality implies that there exists a finite constant C depending only on k such that

$$\left|\frac{1}{N}\mathbb{E}_{\mathbb{T}_N^{V,P}}\left[\mathrm{Tr}(V(L_N)-V(L_N^q))\right]\right| \leqslant C\sum_{l=1}^{2k}\mathbb{E}_{\mathbb{T}_N^{V,P}}\left[\frac{1}{N}\mathrm{Tr}\left((L_N-L_N^q)^{2k}\right)\right]^{l/2k}\mathbb{E}_{\mathbb{T}_N^{V,P}}\left[\frac{1}{N}\mathrm{Tr}(L_N^{2k})\right]^{\frac{2k-l}{2k}}$$

Now, $R_N^q = L_N - L_N^q$ has non zero entries only at the sites $(i, i+1)$ and $(i+1, i)$, $i \in J = \{\ell q, 1 \leqslant \ell \leqslant k_N\}$, as well as $(N, 1)$ and $(1, N)$. We can assume without loss of generality that $q > 2k$ so that $\mathrm{Tr}(R_N^q)^{2k}$ simply depends on the $2k$th power of the its non-vanishing entries. Thus, there exists a finite constant C_k which only depends on k such that

$$\mathrm{Tr}\left((R_N^q)^{2k}\right) \leqslant C_k\sum_{i\in J} L_N(i, i+1)^{2k} + C_k L_N(N, 1)^{2k}.$$

Next notice that

$$L_N(i, i+1)^2 \leqslant L_N(i, i)^2 + L_N(i, i+1)^2 + L_N(i, i-1)^2 = L_N^2(i, i).$$

Moreover, diagonalizing $L_N = \sum \lambda_j v_j v_j^T$, we find by Hölder's inequality (since $\sum v_j(i)^2 = 1$ for all $i \in \{1, \ldots, N\}$) that

$$L_N^2(i, i)^k = \left(\sum \lambda_j^2 v_j(i)^2\right)^k \leqslant \sum \lambda_j^{2k} v_j(i)^2 = L_N^{2k}(i, i).$$

Thus,

$$L_N(i, i+1)^{2k} \leqslant L_N^2(i, i)^k \leqslant L_N^{2k}(i, i).$$

Because L_N has periodic boundary conditions, the distribution of the entries of L_N are invariant under the shift $\theta : i \to i + 1$, so that under $\mathbb{T}_N^{V,P}$, $L_N(i, i+1)$ has the same law than $L_N(i+1, i+2)$, and $L_N(i, i)$ has the same law than $L_N(i+1, i+1)$. As a consequence, we have

$$\mathbb{E}_{\mathbb{T}_N^{V,P}}\left[\frac{1}{N}\mathrm{Tr}\left((L_N-L_N^q)^{2k}\right)\right] \leqslant \frac{1}{N}C_k\sum_{i\in J}\mathbb{E}_{\mathbb{T}_N^{V,P}}\left[L_N^{2k}(i, i)\right] = C_k\frac{k_N}{N}\mathbb{E}_{\mathbb{T}_N^{V,P}}\left[\frac{1}{N}\mathrm{Tr}(L_N^{2k})\right].$$

But (2.4.2) implies that $\mathbb{E}_{\mathbb{T}_N^{V,P}}\left[\frac{1}{N}\mathrm{Tr}(L_N^{2k})\right]$ is bounded by some finite constant independent of N. We therefore deduce (2.4.6) from (2.4.7).

We next prove the subadditivity property. Let $\delta > 0$ and $L > 0$ be given. Let $\mathcal{K}_L = \{\hat{\mu}_{L_N}(V) \leqslant L\}$. As in equation (2.2.10), we have for q big enough,

$$\mathbb{T}_N^{V,P}\left(\{\hat{\mu}_{L_N} \in B_\mu(\delta)\} \cap \mathcal{K}_L\right) \geqslant \frac{Z_{N,q}^V}{Z_N^V}\frac{1}{Z_{N,q}^V}\int_{\mathcal{K}_L \cap K_A} \mathbf{1}_{\hat{\mu}_{L_N^q} \in B_\mu(\delta - 4/q)} e^{-\mathrm{Tr}(V(L_N))} d\mathbb{T}_N^P, \tag{2.4.8}$$

where we set $K_A = K_{A,N} = \cap_{i\in J}\{\ b_i^{2k} \leqslant A\} \cap \{b_N^{2k} \leqslant A\}$. As before, noticing that $V(L_N) - V(L_N^q)$ is a polynomial in L_N^q and $L_N - L_N^q$, we find a finite constant C such that, on $\mathcal{K}_L \cap K_A$, for N large enough,

$$\frac{1}{N}\left|\mathrm{Tr}(V(L_N) - V(L_N^q))\right| \leqslant C\left(\frac{k_N}{N}C_k A\right)^{1/2k} L^{\frac{2k-1}{2k}}.$$

Therefore if we set $\mathcal{K}_L^q = \{\hat{\mu}_{L_N^q}(V) \leqslant L\}$, we deduce that $K_A \cap \mathcal{K}_L$ contains $K_A \cap \mathcal{K}_{L-\varepsilon(q)}^q$ for some $\varepsilon(q)$ going to zero as q goes to infinity. We deduce from (2.4.6) and (2.4.8) that there exists a finite constant C independent of q (but dependent on L and k) such that

$$\mathbb{T}_N^{V,P}\left(\{\hat{\mu}_{L_N} \in B_\mu(\delta)\} \cap \mathcal{K}_L\right) \geqslant \frac{e^{-NCq^{-1/2k}}}{Z_{N,q}^V}\int_{K_A \cap \mathcal{K}_{L-\varepsilon(q)}^q} \mathbf{1}_{\hat{\mu}_{L_N^q} \in B_\mu(\delta - 4/q)} e^{-\mathrm{Tr}(V(L_N^q))} d\mathbb{T}_N^P, \tag{2.4.9}$$

Since L_N^q is independent of the entries $b_i, i \in J$ and therefore of K_A, we see that we can integrate the indicator function of K_A yielding a contribution $C_A^{k_N}$ for some positive constant C_A depending only on A. We observe as well that L_N^q is a block diagonal matrix $\mathrm{diag}(L_q^1, \ldots, L_q^{k_N}, B)$ where L_q^i, $1 \leqslant i \leqslant k_N$, are independent and independent from B, L_q^i following $\tilde{\mathbb{T}}_q^P$ defined in (2.4.3) and B following $\tilde{\mathbb{T}}_{r_N}^P$. Finally, we notice that $\mathcal{K}_{L-\varepsilon(q)}^q$ contains $\cap_{1\leqslant i\leqslant k_N}\{\frac{1}{q}\mathrm{Tr}((L_q^i)^{2k}) \leqslant L - \varepsilon(q)\} \cap \{\frac{1}{N-k_N q}\mathrm{Tr}(B^{2k}) \leqslant L - \varepsilon(q)\}$ since the trace of $(L_N^q)^{2k}$ is a linear combination of the latter traces. Thus by independence of the matrices $L_q^1, \ldots, L_q^{k_N}$ under $\frac{1}{Z_{N,q}^V}e^{-\mathrm{Tr}V(L_N^q)}d\mathbb{T}_N^P$ and convexity of balls, we deduce by taking the logarithm that if we set $u_N(\delta, L) = -\log \mathbb{T}_N^{V,P}(\{\hat{\mu}_{M_N} \in B_\mu(\delta)\} \cap \mathcal{K}_L)$ and $v_N(\delta, L) = -\log \tilde{\mathbb{T}}_N^{V,P}(\{\hat{\mu}_{\tilde{L}_N} \in B_\mu(\delta)\} \cap \{\mathrm{Tr}(\tilde{L}_N)^{2k}) \leqslant LN\})$, then we have

$$u_N(\delta + 4/q, L + \varepsilon(q)) \leqslant N(Cq^{-1/2k} + \log(C_A)/q) + k_N v_q(\delta, L) + v_{r_N}(\delta, L). \tag{2.4.10}$$

We conclude as in Lemma 2.2.3 that

$$\limsup_N \frac{u_N(\delta + 4/q, L + \varepsilon(q))}{N} \leqslant \frac{v_q(\delta, L)}{q} + Cq^{-1/2k} + \frac{\log(C_A)}{q}. \tag{2.4.11}$$

We then notice that for all N, δ, $u_N(\delta, L) \geqslant u_N(\delta, \infty)$ and $v_N(\delta, L) \leqslant v_N(\delta, \infty) + \log 2$ for L large enough by Lemma 2.2.2 (for $\tilde{L}_N$). If therefore we choose a subsequence q going to infinity along which the liminf is taken, we deduce by Lemma 2.4.3 that

$$\limsup_N \frac{u_N(2\delta, \infty)}{N} \leqslant \liminf_{q\to\infty} \frac{v_q(\delta, \infty)}{q} = \liminf_{q\to\infty} \frac{u_q(\delta, \infty)}{q}$$

If there is no such subsequence then both sides go to infinity and there is nothing to say. Otherwise we conclude as in Lemma 2.2.3.

We see that we can adapt in the same fashion the proof of Theorem 2.2.4 (which stands for quadratic V) to our setting and get that $\mu \mapsto T_P^V(\mu)$ is convex, which concludes the proof. □

2.4.3 Convergence of the free energy and large deviation principle

In the case where $V(x) = ax^{2k}, a > 0$, Lemmas 2.4.1 and 2.4.4 of the previous two sections showed that a large deviation principle holds for the empirical measure of the eigenvalues of L_N under $\mathbb{T}_N^{V,P}$ with good, convex rate function which, using [DZ10, Theorem 4.5.10], can be represented as

$$T_P^V(\mu) = -\inf_{W\in C_b^0}\{\int W d\mu + F_{\mathbb{T}}^{V+W,P} - F_{\mathbb{T}}^{V,P}\} \tag{2.4.12}$$

where

$$F_{\mathbb{T}}^{V,P} = \lim_{N\to\infty} \frac{1}{N} \log \int e^{-\mathrm{Tr}V(L_N)} d\mathbb{T}_N^P .$$

To identify T_P^V and its minimizer, our goal is to show that

Lemma 2.4.5. *For $a > 0$ and $V(x) = ax^{2k} + U(x)$ with $U \in C_b^0(\mathbb{R})$, for every $P > 0$, we have*

$$\int_0^1 F_{\mathbb{T}}^{V,sP} ds = F_C^{V,P} . \tag{2.4.13}$$

As a consequence, the unique minimizer of T_P^V is given by $\nu_P^V = \partial_P(P\mu_P^V)$ with μ_P^V the equilibrium measure for the β-ensemble with parameter $\beta = 2P/N$.

Proof. We first prove (2.4.13). Clearly, for all bounded continuous functions U, U', uniformly in P,

$$|F_{\mathbb{T}}^{ax^{2k}+U,P} - F_{\mathbb{T}}^{ax^{2k}+U',P}| \leqslant \|U - U'\|_\infty \text{ and } |F_C^{ax^{2k}+U,P} - F_C^{ax^{2k}+U',P}| \leqslant \|U - U'\|_\infty .$$

Therefore it is enough to prove (2.4.13) for $U \in C_b^1(\mathbb{R})$ by density. We prove that for $U \in C_b^1(\mathbb{R})$,

$$F_{\mathbb{T}}^{V,P} = \partial_P(PF_C^{V,P}) . \tag{2.4.14}$$

Let us consider the tridiagonal matrix C_P^N of the Coulomb model with distribution $\mathbb{P}_N^{\frac{2P}{N}}$. We decompose, for $\varepsilon > 0$, this matrix as

$$C_P^N = \begin{pmatrix} M_P^{\lfloor \varepsilon N \rfloor} & R_N \\ R_N^T & C_{P_N^\varepsilon}^{N_\varepsilon} \end{pmatrix}$$

where $M_P^{\lfloor \varepsilon N \rfloor}$ is a $\lfloor N\varepsilon \rfloor \times \lfloor N\varepsilon \rfloor$ tri-diagonal symmetric matrix with standard independent Gaussian variables on the diagonal and chi distributed variables above the diagonal with parameters $2\frac{i}{N}P, N - \lfloor \varepsilon N \rfloor \leqslant i \leqslant N-1$, $C_{P_N^\varepsilon}^{N_\varepsilon}$ is a $N_\varepsilon = N - \lfloor \varepsilon N \rfloor$ square tridiagonal Coulomb matrix with parameter $2P_N^\varepsilon/N$ with $P_N^\varepsilon = N_\varepsilon N^{-1} P = (1 - \lfloor \varepsilon N \rfloor / N)P$, and R_N has only one non-zero entry r at position $(\lfloor \varepsilon N \rfloor, \lfloor \varepsilon N \rfloor + 1)$. Our first goal is to show that, with $V(x) = ax^{2k} + U(x)$, we have

$$\lim_{N\to\infty} \frac{1}{\varepsilon N} \log \mathbb{E}[e^{-\mathrm{Tr}V(M_P^{\lfloor \varepsilon N \rfloor})}] = \frac{1}{\varepsilon}(F_C^{V,P} - F_C^{V,P-\varepsilon}) + F_C^{V,P-\varepsilon} . \tag{2.4.15}$$

We will then complete the argument by showing that

$$\lim_{\varepsilon\downarrow 0} \lim_{N\to\infty} \frac{1}{\varepsilon N} \log \mathbb{E}[e^{-\mathrm{Tr}V(M_P^{\lfloor \varepsilon N \rfloor})}] = F_{\mathbb{T}}^{V,P} \tag{2.4.16}$$

We next turn to the proof of (2.4.15). Let us denote

$$\tilde{C}_P^N = \begin{pmatrix} M_P^{\lfloor \varepsilon N \rfloor} & 0 \\ 0 & C_{P_N^\varepsilon}^{N_\varepsilon} \end{pmatrix} .$$

We now show that

$$\mathrm{Tr}((C_P^N)^{2k}) \geqslant \mathrm{Tr}((\tilde{C}_P^N)^{2k}) . \tag{2.4.17}$$

Indeed, by Klein's lemma [AGZ10, Lemma 4.4.12], $B \mapsto \mathrm{Tr}(B^{2k})$ is convex on the set of symmetric matrices. Moreover $\nabla \mathrm{Tr}(B^{2k}) = (2kB^{2k-1})_{ij}$. As a consequence, for any symmetric matrices A, B

$$\mathrm{Tr}((A+B)^{2k}) - \mathrm{Tr}(B^{2k}) \geqslant \mathrm{Tr}(2kB^{2k-1}A) .$$

We apply the above inequality with $A = C_P^N - \tilde{C}_P^N$ and $B = \tilde{C}_P^N$ and notice that the entry $\lfloor \varepsilon N \rfloor, \lfloor \varepsilon N \rfloor + 1$ of $(\tilde{C}_P^N)^{2k-1}$ vanishes so that $\mathrm{Tr}((\tilde{C}_P^N)^{2k-1}(C_P^N - \tilde{C}_P^N)) = 0$, proving (2.4.17).
Moreover, if U is C_b^1,

$$|\mathrm{Tr}(U(C_P^N)) - \mathrm{Tr}(U(\tilde{C}_P^N))| \leqslant \int_0^1 |\mathrm{Tr}(U'(\alpha C_P^N + (1-\alpha)\tilde{C}_P^N)(C_N^P - \tilde{C}_N^P))| d\alpha \leqslant \|U'\|_\infty |r| \tag{2.4.18}$$

Consequently, using the independence of r and $\tilde{C}_P^N$ and the fact that $C_U = \mathbb{E}\big[e^{+\|U'\|_\infty |r|}\big]$ is finite since r has sub-Gaussian distribution, we deduce from (2.4.17) that

$$\mathbb{E}\big[e^{-\mathrm{Tr}(V(C_P^N))}\big] \leqslant \mathbb{E}\big[e^{-\mathrm{Tr}(V(\tilde{C}_P^N))+\|U'\|_\infty |r|}\big] \leqslant C_U \mathbb{E}\big[e^{-\mathrm{Tr}(V(\tilde{C}_P^N))}\big]\,. \tag{2.4.19}$$

As a consequence

$$\mathbb{E}\big[e^{-\mathrm{Tr}(V(C_P^N))}\big] \leqslant C_U \mathbb{E}\big[e^{-\mathrm{Tr}V(M_P^{\lfloor \varepsilon N \rfloor})}\big]\mathbb{E}\big[e^{-\mathrm{Tr}(V(C_{P_N^\varepsilon}^{N_\varepsilon}))}\big]$$

which gives the desired lower bound:

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}\big[e^{-\mathrm{Tr}V(M_P^{\lfloor \varepsilon N \rfloor})}\big] \geqslant F_C^{P,V} - (1-\varepsilon) F_C^{P(1-\varepsilon),V} \tag{2.4.20}$$

where we used that Theorem 2.3.1 is valid for $P_N^\varepsilon \to (1-\varepsilon)P$.

To get the complementary lower bound we restrict ourselves to

$$\{|r| \leqslant \frac{1}{N}\} \cap \{\frac{1}{N}\mathrm{Tr}((\tilde{C}_P^N)^{2k}) \leqslant M\}$$

Because of (2.4.18) and applying Hölder's inequality as in the proof of Lemma 2.4.3, we see that on this set $\mathrm{Tr}(V(C_P^N)) - \mathrm{Tr}(V(\tilde{C}_P^N))$ goes to zero uniformly for all M. On the other hand the probability of the set $\{|r| \leqslant \frac{1}{N}\}$ is of order $1/N$. Again by independence we deduce that

$$\begin{aligned}
\mathbb{E}\big[e^{-\mathrm{Tr}(V(C_P^N))}\big] &\geqslant e^{o(1)}\mathbb{E}\big[1_{\{|r|\leqslant \frac{1}{N}\}\cap\{\frac{1}{N}\mathrm{Tr}((\tilde{C}_P^N)^{2k})\leqslant M\}} e^{-\mathrm{Tr}(V(\tilde{C}_P^N))}\big] \\
&\geqslant e^{o(1)}\left(\mathbb{E}\big[e^{-\mathrm{Tr}(V(\tilde{C}_P^N))}\big] - \mathbb{E}\big[1_{\{\frac{1}{N}\mathrm{Tr}((\tilde{C}_P^N)^{2k})\geqslant M\}} e^{-\mathrm{Tr}(V(\tilde{C}_P^N))}\big]\right).
\end{aligned} \tag{2.4.21}$$

But we can show exactly as in the proof of Lemma 2.4.1 that for M large enough

$$\limsup_{N\to\infty} \frac{\mathbb{E}\big[1_{\{\mathrm{Tr}((\tilde{C}_P^N)^{2k})\geqslant MN\}} e^{-\mathrm{Tr}(V(\tilde{C}_P^N))}\big]}{\mathbb{E}\big[e^{-\mathrm{Tr}(V(\tilde{C}_P^N))}\big]} \leqslant \frac{1}{2}\,,$$

yielding the desired lower bound and therefore (2.4.15).

To prove (2.4.16), we proceed by approximation. We notice that if we denote by D_T^ε the density of the distribution of $M_P^{\lfloor \varepsilon N \rfloor}$ with respect to the distribution of a Toda matrix $\tilde{L}_{\lfloor \varepsilon N \rfloor}$ with parameter P to which we removed the extreme entries at $(1, \lfloor \varepsilon N \rfloor)$ and $(\lfloor \varepsilon N \rfloor, 1)$, then we get

$$D_T^\varepsilon = \prod_{i=1}^{N\varepsilon} b_i^{-2P(\frac{i}{N})}\,.$$

Therefore

$$\begin{aligned}
\mathbb{E}\big[e^{-\mathrm{Tr}V(M_P^{\lfloor \varepsilon N \rfloor})}\big] &\geqslant e^{-\varepsilon^2 NM}\mathbb{E}\big[e^{-\mathrm{Tr}V(\tilde{L}_{\lfloor \varepsilon N \rfloor})} 1_{-2P\sum_{i=1}^{\varepsilon N} \frac{i}{N}\log b_i \geqslant -\varepsilon^2 NM}\big] \\
&= e^{-\varepsilon^2 NM}\mathbb{E}\big[e^{-\mathrm{Tr}V(\tilde{L}_{\lfloor \varepsilon N \rfloor})}\big](1 - \tilde{\mathbb{T}}^{V,P}_{\lfloor N\varepsilon \rfloor}(-2P\sum_{i=1}^{\varepsilon N}\frac{i}{N}\log b_i \leqslant -\varepsilon^2 NM))
\end{aligned}$$

On the other hand

$$\{2P\sum_{i=1}^{\varepsilon N}\frac{i}{N}\log b_i \geqslant \varepsilon^2 NM\} \subset \{P\frac{1}{N\varepsilon}\sum_{i=1}^{\varepsilon N} b_i^2 \geqslant M\} \subset \{\frac{1}{N\varepsilon}\mathrm{Tr}((\tilde{L}_{\lfloor N\varepsilon \rfloor})^2) \geqslant M/P\}$$

has exponentially small probability under $\tilde{\mathbb{T}}^{V,P}_{\lfloor N\varepsilon\rfloor}$ for ge enough. This shows, using Lemma 2.4.3, that there exists a finite constant M such that

$$\liminf_{N\to\infty} \frac{1}{N\varepsilon} \log \mathbb{E}[e^{-\mathrm{Tr}V(M_P^{\lfloor \varepsilon N\rfloor})}] \geqslant F_{\mathbb{T}}^{V,P} + M\varepsilon$$

Similarly, we can see that the density $\tilde{D}^{\varepsilon}_T = \prod_{i=1}^{N\varepsilon} b_i^{2P(\frac{i}{N}-\varepsilon)}$ of the law a Toda matrix $\tilde{L}_{\lfloor \varepsilon N\rfloor}$ with respect to $M_P^{\lfloor \varepsilon N\rfloor}$ is bounded below by $-\varepsilon^2 NM$ on $\{\sum_{i=1}^{\varepsilon N}(\varepsilon - \frac{i}{N})\log b_i \leqslant \varepsilon^2 NM\}$ so that we get similarly a finite constant M' such that

$$\limsup_{N\to\infty} \frac{1}{N\varepsilon} \log \mathbb{E}[e^{-\mathrm{Tr}V(M_P^{\lfloor \varepsilon N\rfloor})}] \leqslant F_{\mathbb{T}}^{V,P(1-\varepsilon)} + M'\varepsilon \tag{2.4.22}$$

We hence conclude by the continuity of $\varepsilon \to F_{\mathbb{T}}^{V,P(1-\varepsilon)}$ (which is due to its convexity) Equality (2.4.14) follows then from (2.4.22).

We finally show that (2.4.13) implies that T_P^V achieves its minimum value at $\partial_P(P\mu_P^V)$. Indeed, by (2.4.12), for any bounded continuous U, any probability measure ν, we have

$$T_P^V(\nu) \geqslant -\left(\int U d\nu + F_{\mathbb{T}}^{V+U,P} - F_{\mathbb{T}}^{V,P}\right)$$

We integrate this inequality at $\nu = \nu_{sP}$ a measurable probability measure valued process such that $\mu = \int_0^1 \nu_{sP} ds$ to deduce from (2.4.13) that

$$\int_0^1 T_P^V(\nu_{sP}) ds \geqslant -\left(\int U d\mu + F_C^{V+U,P} - F_C^{V,P}\right).$$

We finally optimize over U to conclude that

$$\int_0^1 T_P^V(\nu_{sP}) ds \geqslant -\inf_U \left(\int U d\mu + F_C^{V+U,P} - F_C^{V,P}\right) = I_P^V(\mu).$$

Since I_P^V vanishes only at μ_P^V we deduce that any measurable minimizing path $(\nu_{sP})_{0\leqslant s\leqslant 1}$ must satisfy $\int_0^1 \nu_{sP} ds = \mu_P^V$. If we can consider a continuous $s \mapsto \nu_{sP}$, we conclude that $\partial_P(P\mu_P^V)$ makes sense and that it is equal to ν_P. We therefore now show that such a path can be chosen to be continuous. But we can follow arguments similar to those of Corollary 2.2.7 to show that the set M_P^V where T_P^V achieves its minimum value is a compact convex subset of $\mathcal{P}(\mathbb{R})$ and is continuous in the sense that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $\delta < \delta_\varepsilon$, any $P, Q > 0$ such that for $|P - Q| \leqslant \delta$

$$M_Q^V \subset (M_P^V)^{\varepsilon}.$$

Indeed, even if we do not have the coupling of Corollary 2.2.7, we easily see that the density of $\mathbb{T}_N^{V,Q}$ with respect to $\mathbb{T}_N^{V,P}$ is bounded by $e^{MN|P-Q|}$ with probability greater than $1 - e^{-c(M)N}$ with $c(M)$ going to infinity when M goes to infinity. Indeed, the density equals $(P-Q)\sum \log b_i$ from which the remark follows from Lemma 2.4.2. This implies that

$$-\inf_{((M_P^V)^{\varepsilon})^c} T_Q^V \leqslant \max\{M|Q-P| - \inf_{((M_P^V)^{\varepsilon})^c} T_P^V, -c(M)N\}$$

which implies that for any $\varepsilon > 0$, for M large enough and $|Q-P|$ small enough $\inf_{((M_P^V)^{\varepsilon})^c} T_Q^V > 0$, from which the continuity follows. □

2.5 Proof of Theorem 2.1.1 and 2.1.3

Lemma 2.4.4 combined with the exponential tightness of Lemma 2.4.1 proves a large deviation principle for the potential $V(x) = ax^{2k}$. If now we consider the case where $V(x)/x^{2k}$ goes to $a > 0$ at infinity, we can always write $V(x) = ax^{2k} + U(x)$ where $U(x)/x^{2k}$ goes to zero at infinity. We have seen by Lemma 2.4.1 that under $\mathbb{T}_N^{P,V}$, the event $\{\frac{1}{N}\mathrm{Tr}(L_N^{2k}) > M\}$ has exponentially small probability. Let for $\varepsilon > 0$, $V_\varepsilon(x) = ax^{2k} + (1+\varepsilon x^{2k})^{-1}U(x)$. Then, the large deviation principle for the distribution of $\hat{\mu}_{L_N}$ under $\mathbb{T}_N^{V_\varepsilon,P}$ follows from Varadhan's lemma. Moreover, on $\{\mathrm{Tr}(L_N^{2k}) \leqslant MN\}$, if $|U(x)| \leqslant \delta x^{2k}$ on $|x| \geqslant L$,

$$\begin{aligned}\left|\frac{1}{N}\mathrm{Tr}V(L_N) - \frac{1}{N}\mathrm{Tr}V_\varepsilon(L_N)\right| &\leqslant \frac{\varepsilon L^{2k}}{1+\varepsilon L^{2k}} \max_{|x|\leqslant L}|U(x)| + \delta\varepsilon\frac{1}{N}\mathrm{Tr}\left(\frac{L_N^{4k}}{1+\varepsilon L_N^{2k}}\right) \\ &\leqslant \frac{\varepsilon L^{2k}}{1+\varepsilon L^{2k}} \max_{|x|\leqslant L}|U(x)| + M\delta\end{aligned}$$

which is as small as wished if M is fixed, L taken large so that δ is small, provided ε is taken small enough. This shows that we can approximate $\mathbb{T}_N^{V,P}$ by $\mathbb{T}_N^{V_\varepsilon,P}$ in the exponential scale from which the result follows.

The proof of Theorem 2.1.3 follows the same arguments than those developed in the last section: we approximate the general variance profile by a stepwise constant profile, remove a negligible number of off diagonal entries and then use the large deviation principle for the Toda matrices. We leave the details to the reader.

Chapter 3

Large Deviations for Ablowitz-Ladik lattice, and the Schur flow

Abstract We consider the Generalized Gibbs ensembles of the Ablowitz-Ladik lattice and of the Schur flow. We derive large deviations principles for the distribution of the empirical measures for these ensembles. As a consequence, we deduce their almost sure convergence. Moreover, we are able to characterize their limit in terms of the equilibrium measure of the Circular, and the Jacobi beta ensemble respectively.

3.1 Introduction

The defocusing Ablowitz-Ladik (AL) lattice is the system of ODEs

$$i\dot{\alpha}_j = -(\alpha_{j+1} + \alpha_{j-1} - 2\alpha_j) + |\alpha_j|^2(\alpha_{j-1} + \alpha_{j+1}), \tag{3.1.1}$$

that describe the evolution of the complex functions $\alpha_j(t)$, $j \in \mathbb{Z}$ and $t \in \mathbb{R}$, here $\dot{\alpha}_j = \dfrac{d\alpha_j}{dt}$. We assume N-periodic boundary conditions $\alpha_{j+N} = \alpha_j$, for all $j \in \mathbb{Z}$. For simplicity, we consider the case N even, and, when not mentioned, the limits as $N \to \infty$ is taken along N even. This system was introduced by Ablowitz and Ladik [AL75, AL76] as a spatial discretization of the defocusing Nonlinear Schrödinger Equation (NLS)

$$i\partial_t\psi(x,t) = -\frac{1}{2}\partial_x^2\psi(x,t) + |\psi(x,t)|^2\psi(x,t). \tag{3.1.2}$$

The NLS is a well-known integrable model [ZS72], and the Ablowitz-Ladik lattice is one of the several discretizations that preserve integrability [QNCVDL84].

It is straightforward to verify that the two quantities

$$K^{(0)} := \prod_{j=1}^{N}\left(1 - |\alpha_j|^2\right), \quad K^{(1)} := -\sum_{j=1}^{N}\alpha_j\overline{\alpha}_{j+1}, \tag{3.1.3}$$

are constants of motion for the AL lattice. Since $K^{(0)}$ is conserved along the flow, it implies that if $|\alpha_j(0)| < 1$ for all $j = 1, \dots, N$, then $|\alpha_j(t)| < 1$ for all times. Thus, we can consider $\mathbb{D}^N$ as our phase space, where $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$.

On this phase space we consider the symplectic form ω [EL06, GN09, Nen05, Sim05]

$$\omega = i\sum_{j=1}^{N}\frac{1}{\rho_j^2}d\alpha_j \wedge d\overline{\alpha}_j, \quad \rho_j = \sqrt{1 - |\alpha_j|^2}. \tag{3.1.4}$$

The corresponding Poisson bracket is defined for functions $f, g \in \mathcal{C}^\infty(\mathbb{D}^N)$ as

$$\{f, g\} = i \sum_{j=1}^{N} \rho_j^2 \left(\frac{\partial f}{\partial \overline{\alpha}_j} \frac{\partial g}{\partial \alpha_j} - \frac{\partial f}{\partial \alpha_j} \frac{\partial g}{\partial \overline{\alpha}_j} \right) . \tag{3.1.5}$$

Using this Poisson bracket, it is possible to rewrite the equations of motion (3.1.1) of the AL lattice in Hamiltonian form as

$$\dot{\alpha}_j = \{\alpha_j, H_{AL}\}, \quad H_{AL}(\boldsymbol{\alpha}) = -2\log(K^{(0)}) + K^{(1)} + \overline{K^{(1)}}\,, \tag{3.1.6}$$

here $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$.

Conserved quantities. As we already mentioned, the AL lattice is an integrable model: this was proved by Ablowitz and Ladik [AKN74, AL75]. Specifically, they were able to obtain a Lax pair for the Ablowitz-Ladik lattice by discretizing the 2×2 Zakharov-Shabat Lax pair of the cubic nonlinear Schrödinger equation.

Nenciu and Simon in [Nen05, Sim05] constructed a new Lax pair for the Ablowitz-Ladik lattice, exploiting the connection of this system to the orthogonal polynomials on the unit circle. This link is the analogue of the well-known link between the Toda lattice and orthogonal polynomials on the real line (see e.g. [Dei99]). This connection was also generalized to the non-commutative case [Caf09].

Following [Nen05, Sim05], we construct the Lax matrix as follows. Consider the 2×2 unitary matrices

$$\Xi(\alpha_j) = \Xi_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix} , \quad \rho_j = \sqrt{1 - |\alpha_j|^2}, \quad j = 1, \ldots, N\,, \tag{3.1.7}$$

and the $N \times N$ matrices

$$\mathcal{M} = \begin{pmatrix} -\alpha_N & & & & & \rho_N \\ & \Xi_2 & & & & \\ & & \Xi_4 & & & \\ & & & \ddots & & \\ & & & & \Xi_{N-2} & \\ \rho_N & & & & & \overline{\alpha}_N \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} \Xi_1 & & & \\ & \Xi_3 & & \\ & & \ddots & \\ & & & \Xi_{N-1} \end{pmatrix}. \tag{3.1.8}$$

Now let us define the Lax matrix

$$\mathcal{E} = \mathcal{L}\mathcal{M}\,, \tag{3.1.9}$$

which has the following structure

$$\mathcal{E} = \begin{pmatrix} * & * & * & & & & & & & * \\ * & * & * & & & & & & & * \\ & * & * & * & * & & & & & \\ & * & * & * & * & & & & & \\ & & & & & \ddots & \ddots & & & \\ & & & & & & * & * & * & * \\ & & & & & & * & * & * & * \\ * & & & & & & & * & * & * \\ * & & & & & & & * & * & * \end{pmatrix} .$$

The matrix $\mathcal{E}$ is a periodic CMV matrix (after Cantero, Moral and Velazquez [CMV05]). It is straightforward to verify that the equations of motions (3.1.1) are equivalent to the following Lax equation for the matrix $\mathcal{E}$:

$$\dot{\mathcal{E}} = i \left[\mathcal{E}, \mathcal{E}^+ + (\mathcal{E}^+)^\dagger \right] , \tag{3.1.10}$$

where † stands for hermitian conjugate, and

$$\mathcal{E}^+_{j,k} = \begin{cases} \frac{1}{2}\mathcal{E}_{j,j} & j = k \\ \mathcal{E}_{j,k} & k = j+1 \bmod N \text{ or } k = j+2 \bmod N \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.11}$$

Remark 3.1.1. *We notice that since all the Ξ_j are unitary, then also $\mathcal{E}$ is unitary, this implies that all the eigenvalues λ_j lie on the unit circle, and they can be written in terms of their argument, namely for all $j = 1, \dots, N$ there exists a $\theta_j \in \mathbb{T} := [-\pi, \pi)$ such that*

$$\lambda_j = e^{i\theta_j} .$$

In view of this identification, and in order to simplify the notations, for any function $f(z) : \partial\mathbb{D} \to \mathbb{R}$, we write $f(\theta)$ in place of $f(e^{i\theta})$ when it is convenient. Further, we will write indifferently $\int_{\mathbb{T}} f(\theta)d\mu(\theta)$ or $\int_{\partial\mathbb{D}} f(z)d\mu(z)$ for any probability measure μ having support on the circle.

Remark 3.1.2. *We notice that $(\mathcal{E}^+)^\dagger + (\mathcal{E}^\dagger)^+ = \mathcal{E}^\dagger$ and $[\mathcal{E}, \mathcal{E}^\dagger] = 0$ since $\mathcal{E}$ is unitary. Therefore, the Lax pair* (3.1.10) *can be rewritten in the equivalent form*

$$\dot{\mathcal{E}} = i\left[\mathcal{E}, \mathcal{E}^+ - (\mathcal{E}^\dagger)^+\right] . \tag{3.1.12}$$

The formulation (3.1.10) implies that the quantities

$$K^{(\ell)} = \mathrm{Tr}\left(\mathcal{E}^\ell\right), \quad \ell = 1, \dots, N-1, \tag{3.1.13}$$

are constants of motion for the defocusing AL system (3.1.1).

As in [Spo22b, GM23], we introduce the Generalized Gibbs ensemble for the Ablowitz-Ladik lattice, namely the following probability measure on the phase space $\mathbb{D}^N$

$$d\mathbb{P}^{V,\beta}_{AL,N}(\alpha_1, \dots, \alpha_N) = \frac{1}{Z^{AL}_N(V,\beta)} \prod_{j=1}^N (1-|\alpha_j|^2)^{\beta-1} \mathbf{1}_{\{\alpha_j \in \mathbb{D}\}} \exp(-\mathrm{Tr}(V(\mathcal{E})))d^2\boldsymbol{\alpha}, \tag{3.1.14}$$

where $V(e^{i\theta}) : \mathbb{T} \to \mathbb{R}$ is a continuous function, $\mathbf{1}_A$ is the indicator function of the set A, and $Z^{AL}_N(V,\beta)$ is the partition function of the system

$$Z^{AL}_N(V,\beta) = \int_{\mathbb{D}^N} \prod_{j=1}^N (1-|\alpha_j|^2)^{\beta-1} \exp(-\mathrm{Tr}(V(\mathcal{E})))d^2\boldsymbol{\alpha}.$$

Furthermore, we consider the empirical measure $\mu_N(\mathcal{E})$ of the eigenvalues $e^{i\theta_1}, \dots, e^{i\theta_N}$ of the matrix $\mathcal{E}$ (3.1.9), namely

$$\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^N \delta_{e^{i\theta_j(\mathcal{E})}} , \tag{3.1.15}$$

here δ_x is the delta function centred at x, furthermore, we notice that we can just consider the arguments $\theta_1, \dots, \theta_N$ of the eigenvalues since the matrix $\mathcal{E}$ is unitary, see Remark 3.1.1.

Our main result is a large deviations principle (LDP) with good rate function for the sequence $(\mu_N(\mathcal{E}))$ under the law $\mathbb{P}^{V,\beta}_{AL,N}$ (3.1.14). Namely, denoting by $\mathcal{P}(\mathbb{T})$ the set of probability measures on the Torus $\mathbb{T}$ endowed with the topology of weak convergence, there exists a function $J^V_\beta : \mathcal{P}(\mathbb{T}) \to [0, +\infty]$ such that:

1. it is *lower semicontinuous/good*, namely for any $a \geqslant 0$, $\{\mu \in \mathcal{P}(\mathbb{T}) \mid J^V_\beta(\mu) \leqslant a\} \subset \mathcal{P}(\mathbb{T})$ is compact,

2. it satisfies a *large deviations lower bound*, namely for all $O \subset \mathcal{P}(\mathbb{T})$ open,

$$-\inf_{O} J \leqslant \liminf_{N \text{ even}} \frac{1}{N} \log \mathbb{P}(\mu_N(\mathcal{E}) \in O), \tag{3.1.16}$$

3. it satisfies a *large deviations upper bound*, namely for all $F \subset \mathcal{P}(\mathbb{T})$ closed,

$$\limsup_{N \text{ even}} \frac{1}{N} \log \mathbb{P}(\mu_N(\mathcal{E}) \in F) \leqslant -\inf_{F} J. \tag{3.1.17}$$

We refer to [DZ10] for a general introduction to large deviations.

Remark 3.1.3. *We notice that, by compactness of $\mathcal{P}(\mathbb{T})$, it is sufficient to prove a **weak** large deviations principle, see [DZ10, Section 1.2], which is the same as a full large deviations principle, except that the large deviation upper bound (point 3) holds only for **compact** subsets of $\mathcal{P}(\mathbb{T})$.*

From this large deviations principle we are able to deduce that $\mu_N(\mathcal{E})$ converges almost surely as N goes to infinity.

Theorem 3.1.4. *Let $\beta > 0$. For any continuous function $V : \mathbb{T} \to \mathbb{R}$ the following holds:*

a. the sequence $\mu_N(\mathcal{E})$ under the law $\mathbb{P}^{V,\beta}_{AL,N}$ satisfies a large deviations principle at speed N with a good rate function J^V_β,

b. J^V_β achieves its minimum at a unique probability measure ν^V_β,

c. $\mu_N(\mathcal{E})$ converges almost surely and in $L^1(\mathbb{T})$ towards ν^V_β.

Moreover, following [GM23, Spo22b, GM22], we are able to characterize the measure ν^V_β in terms of the equilibrium measure of the Circular beta ensemble at high temperature [KN04, HL21]. More precisely, consider the functional $\mu \mapsto f^V_\beta(\mu)$ given, for any $\mu \in \mathcal{P}(\mathbb{T})$ absolutely continuous with respect to Lebesgue measure and with density $\frac{d\mu}{d\theta}$, by

$$\begin{aligned} f^V_\beta(\mu) = &-\beta \int_{\mathbb{T}\times\mathbb{T}} \log\left(|e^{i\theta} - e^{i\varphi}|\right) \mu(d\theta)\mu(d\varphi) + \beta \log(2) + \int_{\mathbb{T}} V(\theta)\mu(d\theta) \\ &+ \int_{\mathbb{T}} \log\left(\frac{d\mu}{d\theta}(\theta)\right) \mu(d\theta) + \log(2\pi)\,. \end{aligned} \tag{3.1.18}$$

It is shown in [HL21], that the previous functional reaches its minimum for a unique absolutely continuous probability measure μ^V_β. Moreover, in [GM23] it is proved that this measure is almost surely differentiable with respect to β. Exploiting this result, and Theorem 3.1.4 we are able to show that

Theorem 3.1.5. *For any continuous $V, f : \mathbb{T} \to \mathbb{R}$*

$$\int_{\mathbb{T}} f(\theta) d\nu^V_\beta(\theta) = \partial_\beta \left(\beta \int_{\mathbb{T}} f(\theta) d\mu^V_\beta(\theta)\right). \tag{3.1.19}$$

Thus, we obtain a unique characterization of the measure ν^V_β.

In [Spo22b, GM23], the authors considered the GGE (3.1.14) with polynomial potential, and they were able to prove Theorem 3.1.5 for this particular class of potentials using a transfer operator technique. In this sense, we generalize their result, extending it to the class of continuous and bounded potentials.

In the last part of the manuscript, we consider another integrable model related to the Ablowitz-Ladik lattice, namely the so-called Schur flow [Gol06]. Also for this system, the Lax matrix is $\mathcal{E}$ (3.1.9). Following the same construction as in the Ablowitz-Ladik lattice case, we define a GGE for this model.

We are able to show analogous results to Theorem 3.1.4 and Theorem 3.1.5 for the Schur flow. The main difference is that in place of the Circular beta ensemble, we have the Jacobi one.

To give a wider overview of the relevant literature, we mention that, H. Spohn in [Spo22b], applying the theory of generalized hydrodynamics [Doy20], argues that the correlation functions of the Ablowitz-Ladik lattice with respect to the GGE (3.1.14) show a ballistic behaviour. As we already mentioned, in [GM23] the authors rigorously proved Theorem 3.1.5 for polynomial potential $V(z)$. Moreover, they computed explicitly the density of states in the case $V(z) = \eta(z + \overline{z})$, which corresponds to the classical Gibbs ensemble.

It is worth to mention that this link between random matrix and integrable system was first noticed in [Spo20c]. In this paper the author considered the GGE for the Toda lattice, and he was able to study this ensemble, comparing it with the Gaussian beta ensemble [DE06]. We refer to [Spo20a, Spo20b, Spo21, Maz22, GM22, GM23] for subsequent developments.

In particular, our work was inspired by the recent paper [GM22]. In this paper, the authors obtained a large deviations principle for the Toda lattice, and obtain an analogous result to Theorem 3.1.5, where in place of the Circular beta ensemble, they had the Gaussian one.

Moreover, we want to underline that in [KS09, NT20, Lam21] the authors were able to describe the local statistics at high temperature of respectively the Circular β ensemble, the Gaussian β ensemble, and Coulomb and Riesz gases, including the case of the Gaussian β-ensemble with general potential. They discovered that they are described by a Poisson point process.

Finally, we notice that in [BNR09] the authors introduced the Circular Jacobi Ensemble, which has a matrix representation in terms of a Hessenberg matrix. It would be interesting to understand if this ensemble admits a matrix representation in terms of a CMV matrix as the Circular β ensemble. In this way, it would be possible to apply our result also to this other matrix ensemble.

The structure of the paper is the following. In Section 3.2, we prove the first point of Theorem 3.1.4. In Section 3.3, we collect some known results related to the Circular beta ensemble in the high-temperature regime. Moreover, we reformulate the already known large deviations principle for this ensemble in terms of the AL lattice. In Section 3.4, we conclude the proof of Theorem 3.1.4, and we prove Theorem 3.1.5. Section 3.5, is dedicated to the Schur flow, where we prove the analogue of Theorem 3.1.4 and Theorem 3.1.5 for this integrable model. Finally, we defer to the appendix the most technical results of our manuscript.

3.2 Existence of a Large deviations principle for the empirical measure of the Ablowitz-Ladik lattice

The aim of this section is to prove the first point of Theorem 3.1.4, namely to show that, for $\mathbb{N} \ni N \geqslant 2$ and even, the sequence of empirical measures $\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^{N} \delta_{e^{i\theta_j(\mathcal{E})}}$ satisfies a large deviations principle. The strategy of proof is the following. First, we show that if $\mathcal{E}$ is distributed according to $\mathbb{P}^{\beta}_{AL,N} := \mathbb{P}^{0,\beta}_{AL,N}$ defined in (3.1.14), then the sequence of random probability measures $(\mu_N(\mathcal{E}))_{N \text{ even}}$ satisfies a large deviations principle in $\mathcal{P}(\mathbb{T})$, the space of probability measures on $\mathbb{T}$, endowed with the topology of weak convergence. Since according to this toppology $\mathcal{P}(\mathbb{T})$ is compact, it suffices to show that the sequence $(\mu_N(\mathcal{E}))_{N \text{ even}}$ satisfies a **weak** large deviations principle, see Remark 3.1.3. Then, applying Varadhan's Lemma [DE97, Theorem 1.2.1], we obtain the existence of a large deviations principle for arbitrary continuous V, *i.e.* the first point of Theorem 3.1.4.

We also notice that when $V = 0$ in (3.1.14) the α_i's are independent and identically distributed (*i.i.d*) with distribution $\Theta_{2\beta+1}$, where Θ_ν is defined for $\nu > 1$ as the random variable such that for $f : \mathbb{C} \to \mathbb{R}$ bounded and measurable

$$\mathbb{E}[f(X)] = \frac{\nu - 1}{2\pi} \int_{\overline{\mathbb{D}}} f(z)(1 - |z|^2)^{\frac{\nu-3}{2}} d^2 z. \tag{3.2.1}$$

Remark 3.2.1. *We recall that for integer $\nu \geqslant 2$, such measure has the following geometrical interpretation: if $\mathbf{u} = (u_1, \dots, u_{\nu+1})$ is chosen at random according to the surface measure on the unit sphere S^ν*

in $\mathbb{R}^{\nu+1}$*, then* $u_1 + iu_2$ *is* Θ_ν *distributed [KN07].*

To show that the sequence $(\mu_N(\mathcal{E}))_{N \text{ even}}$ satisfies a weak large deviations principle according to the law $\mathbb{P}^{\beta}_{AL,N}$, we only need the α_i's to be *i.i.d* according to some law σ with $\text{supp}(\sigma) \subseteq \overline{\mathbb{D}}$. Thus, we just assume the latter hypothesis, and we prove the result in more generality.

3.2.1 Large Deviations Principle for periodic CMV matrix

Let d be the distance on $\mathcal{P}(\mathbb{T})$ defined by

$$d(\mu,\nu) = \sup_{\|f\|_{\text{Lip}}\leqslant 1, \|f\|_{\text{BV}}\leqslant 1} \left\{ \left| \int f d\mu - \int f d\nu \right| \right\}, \tag{3.2.2}$$

where the Lipschitz and the bounded variation norms are defined on the space of functions $f : \mathbb{T} \to \mathbb{R}$ as

$$\|f\|_{\text{Lip}} = \sup_{\substack{\theta_1,\theta_2\in\mathbb{T} \\ \theta_1\neq\theta_2}} \frac{|f\left(e^{i\theta_1}\right) - f\left(e^{i\theta_2}\right)|}{|e^{i\theta_1} - e^{i\theta_2}|}, \tag{3.2.3}$$

$$\|f\|_{\text{BV}} = \sup_{n\geqslant 1, 0=\theta_1<\theta_2<\ldots<\theta_n=2\pi} \sum_{k=1}^{n-1} \left| f\left(e^{i\theta_{k+1}}\right) - f\left(e^{i\theta_k}\right) \right|. \tag{3.2.4}$$

The distance d is compatible with the weak convergence of probability measures [GM22]. We recall that for a $N \times N$ matrix A, its empirical measure of eigenvalues is defined by

$$\mu(A) = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(A)},$$

where $\lambda_j(A)$, $j = 1, \ldots, N$, are the eigenvalues of A. The following Lemma, whose proof can be found in Appendix 3.6, gives an upper bound on the distance of the empirical measures of two unitary matrices.

Lemma 3.2.2. *For any* A*,* B *unitary matrices of size* $N \times N$*,*

- *For* f *with bounded variation,*

$$\left| \int f d\mu(A) - \int f d\mu(B) \right| \leqslant \|f\|_{BV} \frac{rank(A-B)}{N},$$

- *For* f *Lipschitz,*

$$\left| \int f d\mu(A) - \int f d\mu(B) \right| \leqslant \|f\|_{Lip} \frac{1}{N} \sum_{i,j=1}^{N} |(A-B)_{i,j}|.$$

As a consequence,

$$d(\mu(A),\mu(B)) \leqslant \min \left\{ \frac{rank(A-B)}{N}, \frac{1}{N} \sum_{i,j=1}^{N} |(A-B)_{i,j}| \right\}. \tag{3.2.5}$$

We are now in position to prove that the sequence $\mu_N(\mathcal{E})$ with $(\alpha_i)_{i\geqslant 1}$ *i.i.d* with law σ, such that $\text{supp}(\sigma) \subseteq \overline{\mathbb{D}}$, satisfies a large deviations principle. The proof of the following Lemma follows the same line as the corresponding one in [GM22].

Lemma 3.2.3. *Let* $(\alpha_i)_{i\geqslant 1}$ *be an i.i.d sequence of law* σ *with* $supp(\sigma) \subseteq \overline{\mathbb{D}}$*, and* $\mathcal{E}$ *be the associated matrix defined in* (3.1.9)*. Then the sequence of empirical measures* $(\mu_N(\mathcal{E}))_{N \text{ even}}$ *satisfies a large deviations principle in* $\mathcal{P}(\mathbb{T})$ *endowed with the topology of weak convergence.*

Proof. We use a subadditivity argument to show that for any fixed $\mu \in \mathcal{P}(\mathbb{T})$ the following holds

$$\lim_{\delta\to 0} \liminf_{N \text{ even}} \frac{1}{N} \log \mathbb{P}\Big(\mu_N(\mathcal{E}) \in B_\mu(\delta)\Big) = \lim_{\delta\to 0} \limsup_{N \text{ even}} \frac{1}{N} \log \mathbb{P}\Big(\mu_N(\mathcal{E}) \in B_\mu(\delta)\Big), \tag{3.2.6}$$

where $B_\mu(\delta) := \{\nu \in \mathcal{P}(\mathbb{T}) \mid d(\mu,\nu) < \delta\}$. Then, applying [DZ10, Theorem 4.1.11], along with the fact that in our setting a weak LDP is equivalent to a full LDP, due to the compactness of $\mathcal{P}(\mathbb{T})$, see remark 3.1.3, we conclude.
The first step to prove the result is to approximate the matrix $\mathcal{E}$ (whose law we denote by $\mathcal{E}^{(N)}$) by a diagonal block matrix of independent blocks. To this end, fix $q \in \mathbb{N}$ even such that $q \leqslant N$, write the euclidean division of N by q, $N = kq + r$ with $0 \leqslant r < q$. We consider $\mathcal{M}$ given by (3.1.8),

$$\mathcal{M} = \begin{pmatrix} -\alpha_N & & & & & & \rho_N \\ & \Xi_2 & & & & & \\ & & \Xi_4 & & & & \\ & & & \ddots & & & \\ & & & & \Xi_{N-2} & & \\ \rho_N & & & & & & \overline{\alpha}_N \end{pmatrix},$$

and approximate it the following way.

Let $\widetilde{\mathcal{M}} = \operatorname{diag}(\mathcal{M}_1, \dots, \mathcal{M}_k, R)$, where $\mathcal{M}_i$ is the block diagonal matrix given by

$$\mathcal{M}_i = \begin{pmatrix} -\widetilde{\alpha}_{(i-1)q} & & & & & & \widetilde{\rho}_{(i-1)q} \\ & \Xi_{(i-1)q+2} & & & & & \\ & & \Xi_{(i-1)q+4} & & & & \\ & & & \ddots & & & \\ & & & & \Xi_{iq-2} & & \\ \widetilde{\rho}_{(i-1)q} & & & & & & \overline{\widetilde{\alpha}}_{(i-1)q} \end{pmatrix},$$

where $(\widetilde{\alpha}_{(i-1)q})_{1\leqslant i\leqslant k}$ are *i.i.d* of law σ, independent of the α_i's, $\widetilde{\rho}_i = \sqrt{1-|\widetilde{\alpha}_i|^2}$, and the remaining block (of size $r \times r$) R is defined similarly:

$$R = \begin{pmatrix} -\widetilde{\alpha}_{kq} & & & & & \widetilde{\rho}_k \\ & \Xi_{kq+2} & & & & \\ & & \Xi_{kq+4} & & & \\ & & & \ddots & & \\ & & & & \Xi_{N-2} & \\ \widetilde{\rho}_{kq+1} & & & & & \overline{\widetilde{\alpha}_{kq}} \end{pmatrix}.$$

Following the same decomposition of $N = kq + r$ we write $\mathcal{L} = \operatorname{diag}(\mathcal{L}_1, \dots, \mathcal{L}_k, \mathcal{L}_{k+1})$, with $\mathcal{L}_i$ of size q for $1 \leqslant i \leqslant k$ and $\mathcal{L}_{k+1}$ of size r.
Notice that by construction, we have

$$\operatorname{rank}(\mathcal{M} - \widetilde{\mathcal{M}}) \leqslant 2(k+1). \tag{3.2.7}$$

Now, defining $\widetilde{\mathcal{E}} = \mathcal{L}\widetilde{\mathcal{M}}$, $\widetilde{\mathcal{E}}$ is a block diagonal matrix $\operatorname{diag}(\mathcal{E}_1, \dots, \mathcal{E}_k, \mathcal{E}_{k+1})$.Then, the blocks $\mathcal{E}_i$, $1 \leqslant i \leqslant k+1$ are independent, each $\mathcal{E}_i$, $i = i, \dots, k$, has law $\mathcal{E}^{(q)}$, and $\mathcal{E}_{k+1}$ has law $\mathcal{E}^{(r)}$.
Furthermore, using that $\operatorname{rank}(AB) \leqslant \min\{\operatorname{rank}(A); \operatorname{rank}(B)\}$ for A, B two square matrices, and (3.2.7) we get

$$\operatorname{rank}(\mathcal{E} - \tilde{\mathcal{E}}) = \operatorname{rank}(\mathcal{L}(\mathcal{M} - \widetilde{\mathcal{M}})) \leqslant 4(k+1).$$

By the first point of Lemma 3.2.2 we deduce

$$d(\mu_N(\mathcal{E}), \mu_N(\widetilde{\mathcal{E}})) \leqslant \frac{4(k+1)}{N} \leqslant \frac{8}{q}.$$

Moreover, we can rewrite $\mu_N(\widetilde{\mathcal{E}})$ as

$$\mu_N(\widetilde{\mathcal{E}}) = \frac{q}{N}\sum_{\ell=1}^{k}\mu_q(\mathcal{E}_\ell) + \frac{r}{N}\mu_r(\mathcal{E}_{k+1})\,. \tag{3.2.8}$$

Using the independence of the blocks of $\widetilde{\mathcal{E}}$, we deduce that

$$\begin{aligned}
\mathbb{P}\Big(\mu_q(\mathcal{E}_1)\in B_\mu(\delta)\Big)^k\mathbb{P}\Big(\mu_r(\mathcal{E}_{k+1})\in B_\mu(\delta)\Big) &= \mathbb{P}\Big(\mu_q(\mathcal{E}_1),\dots,\mu_q(\mathcal{E}_k),\mu_r(\mathcal{E}_{k+1})\in B_\mu(\delta)\Big)\\
&\leqslant \mathbb{P}\Big(\frac{q}{N}\sum_{l=1}^{k}\mu_q(\mathcal{E}_l)+\frac{r}{N}\mu_r(\mathcal{E}_{k+1})\in B_\mu(\delta)\Big)\\
&= \mathbb{P}\Big(\mu_N(\tilde{\mathcal{E}})\in B_\mu(\delta)\Big)\\
&\leqslant \mathbb{P}\Big(\mu_N(\mathcal{E})\in B_\mu\left(\delta+\frac{8}{q}\right)\Big),
\end{aligned}$$

Where we used the convexity of balls in the first inequality.
This implies that, setting

$$u_N(\delta) = -\log\left(\mathbb{P}(\mu_N\in B_\mu(\delta))\right)\,, \tag{3.2.9}$$

we have

$$u_N\left(\delta+\frac{8}{q}\right)\leqslant k u_q(\delta)+u_r(\delta). \tag{3.2.10}$$

We now conclude as in [GM22, Lemma 2.3]. Let $\delta>0$ and choose q in such a way that $\frac{8}{q}\leqslant\delta$, so we deduce that

$$\frac{u_N(2\delta)}{N}\leqslant\frac{u_N\left(\delta+\frac{8}{q}\right)}{N}\leqslant\frac{u_q(\delta)}{q}+\frac{u_r(\delta)}{N}\,, \tag{3.2.11}$$

since $\frac{u_r(\delta)}{N}\to 0$ as $N\to\infty$, we deduce that

$$\limsup_{N\to\infty}\frac{u_N(2\delta)}{N}\leqslant\frac{u_q(\delta)}{q}\,. \tag{3.2.12}$$

The previous inequality holds true for all q big enough, so we conclude that

$$\limsup_{N\to\infty}\frac{u_N(2\delta)}{N}\leqslant\liminf_{N\to\infty}\frac{u_N(\delta)}{N}\,. \tag{3.2.13}$$

From this last inequality we deduce that

$$\lim_{\delta\to0}\limsup_{N\to\infty}\frac{u_N(\delta)}{N}\leqslant\lim_{\delta\to0}\liminf_{N\to\infty}\frac{u_N(\delta)}{N}\,, \tag{3.2.14}$$

thus we obtain (3.2.6), and the conclusion follows applying [DZ10, Theorem 4.1.11]. □

Since $(\mathcal{P}(\mathbb{T}),d)$ is compact, Lemma 3.2.3 automatically implies the existence of a **strong** large deviations principle. Furthermore, the corresponding rate function J, which depends on the distribution σ of the entries of $\mathcal{L}$ and $\mathcal{M}$, can be seen to be convex. We collect these results in the following proposition.

Proposition 3.2.4. *In the same hypothesis and notations as in Lemma 3.2.3, the sequence of empirical measures $(\mu_N(\mathcal{E}))_{N\ even}$ satisfies a large deviations principle with good, convex rate function $J:\mathcal{P}(\mathbb{T})\to[0,+\infty]$, i.e.*

- *The function J is convex and its level sets $J^{-1}([0,a])$, $a\geqslant 0$, are compact,*

- *For all $O \subset \mathcal{P}(\mathbb{T})$ open,*
$$-\inf_O J \leqslant \liminf_{N \text{ even}} \frac{1}{N} \log \mathbb{P}(\mu_N(\mathcal{E}) \in O),$$
- *For all $F \subset \mathcal{P}(\mathbb{T})$ closed,*
$$\limsup_{N \text{ even}} \frac{1}{N} \log \mathbb{P}(\mu_N(\mathcal{E}) \in F) \leqslant -\inf_F J.$$

Proof. We already established all the claims except the fact that the function J is convex and that the level sets $J^{-1}([0,a])$ are compact. The latter comes from the fact that these sets are closed, see [DZ10, Theorem 4.1.11]. To prove the convexity of J, we follow the same argument as [GM22, Theorem 2.4].

Let $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{T})$. Since $\mu_{2N}(\mathcal{E})$ can be approximated by the sum of two independent $\mu_N(\mathcal{E})$'s up to a mistake smaller than $\frac{4}{N}$ by the first point of Lemma 3.2.2, for $\delta > 0$ the following holds

$$\mathbb{P}\left(\mu_N(\mathcal{E}) \in B_{\mu_1}(\delta)\right) \mathbb{P}\left(\mu_N(\mathcal{E}) \in B_{\mu_2}(\delta)\right) \leqslant \mathbb{P}\left(\mu_{2N}(\mathcal{E}) \in B_{\frac{\mu_1+\mu_2}{2}}\left(\delta + \frac{4}{N}\right)\right), \tag{3.2.15}$$

taking minus the logarithm of both sides, dividing by $2N$, taking the limit for N going to infinity and then for δ to zero, we deduce that:

$$J\left(\frac{\mu_1+\mu_2}{2}\right) \leqslant \frac{1}{2}\left(J(\mu_1) + J(\mu_2)\right), \tag{3.2.16}$$

which, together with the lower semi-continuity of J, implies the convexity of J, see [DZ10, Lemma 4.1.21] . □

3.2.2 Large deviations principle for the Ablowitz-Ladik lattice

Taking $\sigma = \Theta_{2\beta+1}$ given by equation (3.2.1), Proposition 3.2.4 applies to $(\mu_N(\mathcal{E}))_{N \text{ even}}$, where $\mathcal{E}$ follows $\mathbb{P}^{\beta}_{AL,N}$ defined in (3.1.14). Thus, $(\mu_N(\mathcal{E}))_{N \text{ even}}$ with law $\mathbb{P}^{\beta}_{AL,N}$ satisfies a large deviations principle, with a good convex rate function, that we denote by J_β.

We can now state the existence of a large deviations principle for $(\mu_N(\mathcal{E}))_{N \text{ even}}$ under $\mathbb{P}^{V,\beta}_{AL,N}$ for V continuous.

Corollary 3.2.5. *Let $\beta > 0$, and $V : \mathbb{T} \to \mathbb{R}$ be continuous. Under $\mathbb{P}^{V,\beta}_{AL,N}$ the sequence $(\mu_N(\mathcal{E}))_{N \text{ even}}$ fulfils a large deviations principle with good, convex rate function $J^V_\beta(\mu) = g^V_\beta(\mu) - \inf_{\nu \in \mathcal{P}(\mathbb{T})} g^V_\beta(\nu)$, where $g^V_\beta(\mu)$ is given for $\mu \in \mathcal{P}(\mathbb{T})$ by*

$$g^V_\beta(\mu) = J_\beta(\mu) + \int_{\mathbb{T}} V d\mu\,. \tag{3.2.17}$$

Proof. Let us write

$$d\mathbb{P}^{V,\beta}_{AL,N} = \frac{Z^{AL}_N(0,\beta)}{Z^{AL}_N(V,\beta)} e^{-N\int_{\mathbb{T}} V d\mu_N} d\mathbb{P}^{\beta}_{AL,N} = \frac{1}{\mathcal{Z}^{AL,V}_N} e^{-N\int_{\mathbb{T}} V d\mu_N} d\mathbb{P}^{\beta}_{AL,N}.$$

The function $\mu \mapsto \int_{\mathbb{T}} V d\mu$ being bounded continuous, by the large deviations principle under $\mathbb{P}^{\beta}_{AL,N}$ and Varadhan's Lemma, [DE97, Theorem 1.2.1], we see that for any bounded continuous $f : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ we have

$$\lim_N \frac{1}{N} \log \int_{\mathbb{D}^N} e^{Nf(\mu_N)} d\mathbb{P}^{V,\beta}_{AL,N} = \sup_{\mu \in \mathcal{P}(\mathbb{T})} \left\{ f(\mu) - \left(J_\beta(\mu) + \int V d\mu - \inf_{\nu \in \mathcal{P}(\mathbb{T})} \left\{ J_\beta(\nu) + \int V d\nu \right\} \right) \right\},$$

which ensures by [DE97, Theorem 1.2.3] that (μ_N) satisfies a large deviations principle under $\mathbb{P}^{V,\beta}_{AL,N}$ with the announced rate function. Since the function J^V_β is an affine perturbation of J_β, which is convex, J^V_β is also convex. □

The first point of Theorem 3.1.4 is proven.

3.3 Circular β ensemble at high temperature

In this section, we consider the Circular β ensemble, and we collect some known results that we exploit in our treatment. The aim of this section is to prove an alternative formulation of the large deviations principle for the Circular beta ensemble in the high-temperature regime, see Theorem 3.3.11 below. Our formulation allows us to relate the large deviations principle of the Coulomb gas with the one of Ablowitz-Ladik, proved in the previous section.

3.3.1 Large deviations principle for Circular β ensemble

Coulomb gas on the torus $\mathbb{T} = [-\pi, \pi)$ at temperature $\widetilde{\beta}^{-1}$ are described by

$$d\mathbb{P}_{C,N}^{V,\widetilde{\beta}} = \frac{1}{Z_N^C(V,\widetilde{\beta})} \prod_{j\leqslant \ell \leqslant N} |e^{i\theta_j} - e^{i\theta_\ell}|^{\widetilde{\beta}} e^{\sum_{j=1}^N V(\theta_j)} d\boldsymbol{\theta}\,, \tag{3.3.1}$$

here $V : \mathbb{T} \to \mathbb{R}$ is a continuous potential, and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)$. When $V = 0$, Killip and Nenciu showed that $d\mathbb{P}_{C,N}^{0,\widetilde{\beta}}$ is the law of the eigenvalues of a CMV matrix [KN04], see Theorem 3.3.4. In this manuscript, we are interested in the so-called *high-temperature regime* of this ensemble, namely the limit of number of particles N going to infinity with the constraint that $\widetilde{\beta}N \to 2\beta > 0$. From [GZ19], [Ber18], one deduces the following large deviations principle for the measure $\mu_N = \frac{1}{N}\sum_{j=1}^N \delta_{e^{i\theta_j}}$, where the θ_j are distributed according to (3.3.1).

Theorem 3.3.1. *Let $\widetilde{\beta} = \frac{2\beta}{N}$, $\beta > 0$ and assume $V : \mathbb{T} \to \mathbb{R}$ to be continuous. Define for any $\mu \in \mathcal{P}(\mathbb{T})$ absolutely continuous with respect to the Lebesgue measure the functional*

$$\begin{aligned} f_\beta^V(\mu) = -\beta \int_{\mathbb{T}\times\mathbb{T}} \log\left(|e^{i\theta} - e^{i\varphi}|\right) \mu(d\theta)\mu(d\varphi) + \beta\log(2) + \int_{\mathbb{T}} V(\theta)\mu(d\theta) + \\ \int_{\mathbb{T}} \log\left(\frac{d\mu}{d\theta}(\theta)\right)\mu(d\theta) + \log(2\pi)\,, \end{aligned} \tag{3.3.2}$$

then

i. the functional $f_\beta^V(\mu)$ is strictly convex and achieves its minimal value at the unique probability measure μ_β^V absolutely continuous with respect to the Lebesgue measure;

ii. the sequence (μ_N) satisfies a large deviations principle in $\mathcal{P}(\mathbb{T})$ equipped with the weak topology at speed βN with rate function defined for absolutely continuous $\mu \in \mathcal{P}(\mathbb{T})$ with respect to Lebesgue measure by $I_\beta^V(\mu) = f_\beta^V(\mu) - f_\beta^V(\mu_\beta^V)$, and $I_\beta^V(\mu) = +\infty$ otherwise. In particular

$$\mu_N \xrightarrow[N\to\infty]{a.s.} \mu_\beta^V\,. \tag{3.3.3}$$

Exploiting this result, in [GM23] the authors deduced several useful properties of the minimizer μ_β^V, specifically they proved the following.

Lemma 3.3.2 (cf. [GM23] Lemma 3.5)**.** *Let $\beta > 0$, consider a continuous potential $V : \mathbb{T} \to \mathbb{R}$, then the following holds*

i. The map $\beta \to \inf\left(f_\beta^V(\mu)\right)$ is Lipschitz;

ii. Let D be the distance on $\mathcal{P}(\mathbb{T})$ given by

$$\begin{aligned} D(\mu,\mu') &= \left(-\int_{\mathbb{T}\times\mathbb{T}} \log\left|\sin\left(\frac{\theta-\phi}{2}\right)\right| (\mu-\mu')(d\theta)(\mu-\mu')(d\phi)\right)^{1/2} \\ &= \sqrt{\sum_{k\geqslant 1} \frac{1}{k}\,|\widehat{\mu}_k - \widehat{\mu}'_k|^2}\,, \end{aligned} \tag{3.3.4}$$

where $\widehat{\mu}_k = \int_{\mathbb{T}} e^{ik\theta}\mu(d\theta)$. *Then for any* $\varepsilon > 0$ *there exists a finite constant* C_ε *such that for all* $\beta, \beta' > \varepsilon$

$$D(\mu_\beta^V, \mu_{\beta'}^V) \leqslant C_\varepsilon \left|\beta - \beta'\right| . \tag{3.3.5}$$

Remark 3.3.3. *We observe that if* $f \in L^2(\mathbb{T})$ *with derivative in* $L^2(\mathbb{T})$, *we can set* $||f||_{\frac{1}{2}} = \sqrt{\sum_{k\geqslant 1} k|\widehat{f}_k|^2}$. *So, for any measure* ν *with zero mass we obtain the following*

$$\int_{\mathbb{T}} f(\theta)\nu(d\theta) = \sum_{k\neq 0} \widehat{f}_k\overline{\widehat{\nu}_k} = \sum_{k\neq 0} \sqrt{|k|}\widehat{f}_k \frac{\overline{\widehat{\nu}_k}}{\sqrt{|k|}} . \tag{3.3.6}$$

Then, by Cauchy-Schwartz inequality, we deduce the following inequality

$$\left|\int_{\mathbb{T}} f(\theta)\nu(d\theta)\right|^2 \leqslant \left|\sum_{k\neq 0} \sqrt{|k|}\widehat{f}_k \sum_{k\neq 0} \frac{\overline{\widehat{\nu}_k}}{\sqrt{|k|}}\right| \leqslant 4||f||_{\frac{1}{2}}^2 D(\nu, 0)^2. \tag{3.3.7}$$

Combining (3.3.5) *and* (3.3.7), *we deduce that for any function* f *with finite* $||f||_{\frac{1}{2}}$ *norm, the map* $\beta \to \int_{\mathbb{T}} f d\mu_\beta^V(\theta)$ *is Lipschitz for* $\beta > 0$.

3.3.2 Relation with the large deviations principle of the Ablowitz-Ladik lattice

In the case $V = 0$, for any $\widetilde{\beta} > 0$, Killip and Nenciu in [KN04] showed that the law $\mathbb{P}_{C,N}^{0,\widetilde{\beta}}$ (3.3.1) coincides with the distribution of the eigenvalues of a certain CMV matrix. Specifically they proved the following:

Theorem 3.3.4 (cf. [KN04] Theorem 1.2)**.** *Consider the block diagonal* $N \times N$ *matrices*

$$L = diag\,(\Xi_1, \Xi_3, \Xi_5 \ldots,) \quad \text{and} \quad M = diag\,(\Xi_0, \Xi_2, \Xi_4, \ldots) , \tag{3.3.8}$$

where the block Ξ_j, $j = 1, \ldots, N-1$, *takes the form*

$$\Xi_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix} , \quad \rho_j = \sqrt{1 - |\alpha_j|^2}, \tag{3.3.9}$$

while $\Xi_0 = (1)$ *and* $\Xi_N = (\overline{\alpha}_N)$ *are* 1×1 *matrices. Define the* $N \times N$ *sparse matrix*

$$E = LM, \tag{3.3.10}$$

and suppose that the entries α_j *are independent complex random variables with* $\alpha_j \sim \Theta_{\widetilde{\beta}(N-j)+1}$ *for* $1 \leqslant j \leqslant N-1$ *and* α_N *is uniformly distributed on the unit circle. Then the eigenvalues of* E *are distributed according to the Circular Ensemble* (3.3.1) *at temperature* $\widetilde{\beta}^{-1}$.

To simplify the notation, we will denote by $\mathbb{P}_{C,N}^{\widetilde{\beta}}$ the law $\mathbb{P}_{C,N}^{0,\widetilde{\beta}}$. We give an alternative formulation of the large deviations principle for the empirical measure under the law $\mathbb{P}_{C,N}^{\frac{2\beta}{N}}$ based on the Killip-Nenciu matrix representation. This alternative formulation allows us to relate the rate function of the Coulomb gas I_β in terms of the rate function J_β of the Ablowitz-Ladik lattice. Finally, applying Varadhan's Lemma [DE97, Theorem 1.2.1] we obtain an alternative formulation of the large deviations principle for the Circular beta ensemble at high temperature with continuous potential, see Theorem 3.3.11 below.

To achieve our goal, we need several technical results regarding the distribution Θ_ν (3.2.1), and the CMV matrix E (3.3.10). First, in the next Lemma, we give a representation of Θ_ν in terms of Gaussian, and Chi distributions.

Lemma 3.3.5. *Let $\nu > 1$. Let X_1, X_2, Y_ν be independent, X_1, X_2 standard Gaussian variables and Y_ν be $\chi_{\nu-1}$ distributed, i.e. with density*

$$\chi_{\nu-1}(x) = \frac{2^{\frac{3-\nu}{2}}}{\Gamma(\frac{\nu-1}{2})} x^{\nu-2} e^{-x^2/2} \mathbf{1}_{x>0}\,,$$

here $\Gamma(x)$ is the classical Gamma function [DLMF, §5]

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt\,, \tag{3.3.11}$$

Then, $Z := \frac{X_1+iX_2}{(X_1^2+X_2^2+Y_\nu^2)^{1/2}}$ follows the law Θ_ν.

Remark 3.3.6. *If $\nu \geqslant 2$ is an integer, Y_ν^2 has the law of $\sum_{i=1}^{\nu-1} N_i^2$ where the N_i's are i.i.d. standard gaussians random variables, thus Z is equal in distribution to $\frac{X_1+iX_2}{(X_1^2+\cdots+X_{\nu+1}^2)^{\frac{1}{2}}}$, which follows the law Θ_ν by Remark 3.2.1.*

Proof. We identify $\mathbb{C}$ with $\mathbb{R}^2$ and check that for any $f : \mathbb{D} \to \mathbb{R}$ bounded and measurable,

$$\mathbb{E}[f(Z)] = \frac{\nu-1}{2\pi} \int_{\mathbb{D}} f(z)(1-|z|^2)^{\frac{\nu-3}{2}} d^2z,$$

i.e. that for some constant c,

$$\begin{aligned}\int_{\mathbb{R}^2\times\mathbb{R}_+^*} f\left(\frac{x_1}{(x_1^2+x_2^2+y^2)^{1/2}}, \frac{x_2}{(x_1^2+x_2^2+y^2)^{1/2}}\right) e^{-\frac{x_1^2+x_2^2}{2}} e^{-\frac{y^2}{2}} y^{\nu-2} dx_1 dx_2 dy \\ = c\int_{\mathbb{D}} f(u,v)(1-(u^2+v^2))^{\frac{\nu-3}{2}} du dv\,.\end{aligned} \tag{3.3.12}$$

For fixed $y > 0$, we perform the diffeomorphic change of variables $(u, v) = \frac{1}{(x_1^2+x_2^2+y^2)^{1/2}}(x_1, x_2)$. Its inverse $(x_1, x_2) = \frac{y}{(1-(u^2+v^2))^{1/2}}(u, v)$ has Jacobian equal to $y^2(1-(u^2+v^2))^{-2}$. The integral becomes

$$\int_{\mathbb{D}} \frac{f(u,v)}{(1-(u^2+v^2))^2} \int_{\mathbb{R}_+^*} e^{-\frac{y^2}{2(1-(u^2+v^2))}} y^\nu dy du dv = \sqrt{\frac{\pi}{2}} \int_{\mathbb{D}} \frac{f(u,v)}{(1-(u^2+v^2))^{3/2}} \mathbb{E}[|X_{u,v}|^\nu] du dv\,, \tag{3.3.13}$$

where $X_{u,v}$ denotes a Gaussian variable $N(0, 1-(u^2+v^2))$. By [Win14],

$$\mathbb{E}[|X_{u,v}|^\nu] = c_\nu (1-(u^2+v^2))^{\frac{\nu}{2}}$$

for some constant c_ν independent of u, v. Substituting this last equality in the previous integral, we conclude.

□

To obtain our main result, we need some technical lemmas. Since they are based on standard techniques, we just state them here, and we defer their proofs to the appendix 3.6.

The first one gives an estimate that we use combined with Lemma 3.2.2.

Lemma 3.3.7. *Let $N = 2k$ be even and A be a $N \times N$ matrix. Then,*

- $\sum_{i,j=1}^N |(\mathcal{L}A)_{i,j}| \leqslant 2\sum_{i,j=1}^N |A_{i,j}|$,
- $\sum_{i,j=1}^N |(A\mathcal{M})_{i,j}| \leqslant 2\sum_{i,j=1}^N |A_{i,j}|$,

where $\mathcal{M}$, and $\mathcal{L}$ are defined in (3.1.8).

We now give an explicit coupling between Θ_ν (3.2.1) and $\Theta_{\nu+h}$ for $\nu > 1$, $h > 0$.
Let X_1, X_2 be $N(0,1)$ independent variables, and let $Y_{\nu-1} \sim \chi_{\nu-1}$, $Y_h \sim \chi_h$ be independent, and independent of X_1, X_2 (notice that $(Y_h^2 + Y_{\nu-1}^2)^{\frac{1}{2}}$ is $\chi_{\nu+h-1}$ distributed). Let

$$\alpha_\nu = \frac{X_1 + iX_2}{(X_1^2 + X_2^2 + Y_{\nu-1}^2)^{\frac{1}{2}}}, \qquad \alpha_{\nu+h} = \frac{X_1 + iX_2}{(X_1^2 + X_2^2 + Y_{\nu-1}^2 + Y_h^2)^{\frac{1}{2}}}. \tag{3.3.14}$$

By Lemma 3.3.5, $\alpha_\nu \sim \Theta_\nu$ and $\alpha_{\nu+h} \sim \Theta_{\nu+h}$.

Exploiting this coupling, we bound the differences $|\alpha_\nu - \alpha_{\nu+h}|$ and $|\rho_\nu - \rho_{\nu+h}|$ by a random variable Z_h, where $\rho_\nu = \sqrt{1 - |\alpha_\nu|^2}$, and $\rho_{\nu+h} = \sqrt{1 - |\alpha_{\nu+h}|^2}$. Moreover, we find an upper bound for the exponential moments of Z_h.

Lemma 3.3.8. *Let α_ν and $\alpha_{\nu+h}$ defined by equation (3.3.14). Define $\rho_\nu = \sqrt{1 - |\alpha_\nu|^2}$, and $\rho_{\nu+h} = \sqrt{1 - |\alpha_{\nu+h}|^2}$, then the following holds*

i.

$$\begin{aligned} |\alpha_\nu - \alpha_{\nu+h}| &\leqslant \frac{Y_h}{(X_1^2 + X_2^2 + Y_h^2)^{\frac{1}{2}}} \quad \text{almost surely}, \\ |\rho_\nu - \rho_{\nu+h}| &\leqslant \frac{Y_h}{(X_1^2 + X_2^2 + Y_h^2)^{\frac{1}{2}}} \quad \text{almost surely}, \end{aligned} \tag{3.3.15}$$

where $X_1, X_2 \sim N(0,1)$, $Y_h \sim \chi_h$ are all independent.

ii. define $Z_h = \frac{Y_h}{(X_1^2+X_2^2+Y_h^2)^{\frac{1}{2}}}$, and $a(h) = -\frac{1}{2}\log(h) + 1$, then there exists a constant K independent of h such that

$$\sup_{0<h<1} \mathbb{E}\left[\exp(a(h)Z_h)\right] \leqslant K\,. \tag{3.3.16}$$

Remark 3.3.9. *Let $h < h'$, and let $Z_h, Z_{h'}$ be given by*

$$Z_h = \frac{Y_h}{(X_1^2 + X_2^2 + Y_h^2)^{\frac{1}{2}}}, \qquad Z_{h'} = \frac{Y_{h'}}{(X_1^2 + X_2^2 + Y_{h'}^2)^{\frac{1}{2}}}\,,$$

where $Y_h \sim \chi_h$ and $Y_{h'} \sim \chi_{h'}$ are χ variables coupled by

$$Y_{h'} = \sqrt{Y_h^2 + Z^2},$$

Z being a $\chi_{h'-h}$ variable independent of Y_h. Then, because of the monotonicity of the function $x \mapsto \frac{x}{\sqrt{a+x^2}}$ for $a > 0$, we have almost surely $Z_h \leqslant Z_{h'}$.

We are now in position to give an alternate formulation of the large deviations principle for the sequence of measures $(\mu_N(E))$ under the law $\mathbb{P}_{C,N}^{\frac{2\beta}{N}}$, given by Theorem 3.3.1.

Lemma 3.3.10. *Let $\beta > 0$. The law of the empirical measure $(\mu_N(E))_{N\,even}$ under $\mathbb{P}_{C,N}^{\frac{2\beta}{N}}$ satisfies a large deviations principle at speed N and with a good rate function*

$$I_\beta(\mu) = \lim_{\delta\to 0} \liminf_{q\to\infty} \inf_{\substack{\nu_{\beta/q},\dots,\nu_\beta \\ \frac{1}{q}\sum_i \nu_{i\beta/q} \in B_\mu(\delta)}} \left\{ \frac{1}{q} \sum_{i=1}^{q} J_{i\beta/q}(\nu_{i\beta/q}) \right\}, \tag{3.3.17}$$

where J_η is the rate function of Proposition 3.2.4 applied to $\sigma = \Theta_{2\eta+1}$.

Proof. Following the same line as in [GM22, Lemma 3.3], we proceed by exponential approximation. Let $q \geqslant 1$ be an integer, since N is even, we can write $N = kq + r$, with k even, and with $0 \leqslant r < 2q - 2$. Consider the following family of matrices $\mathcal{L}^{(i)}, \mathcal{M}^{(i)}$, $i = 1, \dots q$ defined as

$$\mathcal{M}_k^{(i)} = \begin{pmatrix} -\alpha_{ik} & & & & & \rho_{ik} \\ & \Xi_2^{(i)} & & & & \\ & & \Xi_4^{(i)} & & & \\ & & & \ddots & & \\ & & & & \Xi_{k-2}^{(i)} & \\ \rho_{ik} & & & & & \overline{\alpha}_{ik} \end{pmatrix}, \qquad \mathcal{L}_k^{(i)} = \begin{pmatrix} \Xi_1^{(i)} & & & & \\ & \Xi_3^{(i)} & & & \\ & & \ddots & & \\ & & & \Xi_{k-1}^{(i)} & \end{pmatrix}, \tag{3.3.18}$$

where $\Xi_\ell^{(i)}$ are defined as

$$\begin{aligned} &\Xi_\ell^{(i)}(\alpha_{i(k-1)+\ell}) = \Xi_\ell^{(i)} = \begin{pmatrix} \overline{\alpha}_{i(k-1)+\ell} & \rho_{i(k-1)+\ell} \\ \rho_{i(k-1)+\ell} & -\alpha_{i(k-1)+\ell} \end{pmatrix}, \quad \ell = 1, \dots, k-1\,, \\ &\rho_{i(k-1)+\ell} = \sqrt{1 - |\alpha_{i(k-1)+\ell}|^2}\,, \end{aligned} \tag{3.3.19}$$

and $(\alpha_{i(k-1)+\ell})_{1 \leqslant i \leqslant q, 1 \leqslant \ell \leqslant k}$ is a family of independent random variables such that

$$\alpha_{i(k-1)+\ell} \sim \Theta_{2\beta\frac{N-ik}{N}+1},\ \ell = 1, \dots k,\ i = 1, \dots, q.$$

From these two families of matrices, we can define a third one, namely $\mathcal{E}_k^{(i)} = \mathcal{L}_k^{(i)} \mathcal{M}_k^{(i)}$, $i = 1, \dots, q$. We notice that $\mathcal{E}_k^{(i)}$ is distributed according to $\mathbb{P}_{AL,k}^{\beta\frac{N-ik}{N}}$, and that the $\mathcal{E}_k^{(i)}$, $1 \leqslant i \leqslant q$, are independent.

Our aim is to prove that the empirical measure of the matrix C_N^q

$$C_N^q = \begin{pmatrix} \mathcal{E}_k^{(1)} & & & & \\ & \mathcal{E}_k^{(2)} & & & \\ & & \ddots & & \\ & & & \mathcal{E}_k^{(q)} & \\ & & & & 0_{r\times r} \end{pmatrix}, \tag{3.3.20}$$

where $0_{r\times r}$ is a null block of size $r \times r$, is an exponential approximation (see [DZ10, Definition 4.2.14]) of the empirical measure of $E \sim \mathbb{P}_{C;N}^{\frac{2\beta}{N}}$ (3.3.10), that is, for any positive real number δ:

$$\lim_{q\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log\left(\mathbb{P}\left(d\left(\mu_N(E), \mu_N(C_N^q)\right)\right) > \delta\right) = -\infty\,, \tag{3.3.21}$$

where $\mathbb{P}$ denotes the coupling introduced in equation (3.3.14). In this way, we obtain the claim as an application of [DZ10, Theorem 4.2.16]. The strategy of proof is the following. First we approximate C_N^q and E by two block diagonal matrices $\widetilde{C}_N^q, \widetilde{E}_N^q$ respectively. Finally, we will prove that both $\widetilde{C}_N^q$, and $\widetilde{E}_N^q$ approximate a third matrix B_N^q.

Consider another family of matrices $(\widetilde{\mathcal{M}}_k^{(i)})_{1\leqslant i\leqslant q}$ of size $k \times k$, defined as

$$\widetilde{\mathcal{M}}_k^{(i)} = \operatorname{diag}\left(\widetilde{\Xi}_0^{(i)}, \Xi_2^{(i)}, \Xi_4^{(i)}, \dots, \widetilde{\Xi}_k^{(i)}\right), \tag{3.3.22}$$

where the matrices $\Xi_\ell^{(i)}$ are defined in (3.3.19), while $\widetilde{\Xi}_0^{(i)} = (1)$ and $\widetilde{\Xi}_k^{(i)} = (\overline{\alpha}_{ik})$ are 1×1 matrices, where the α_{ik} are independent, uniformly distributed on the unit circle for all $i = 1, \dots, q$, and independent of $(\alpha_{i(k-1)+l})_{1\leqslant i\leqslant q, 1\leqslant \ell\leqslant k-1}$. Define the $k \times k$ family of CMV matrices

$$\widetilde{\mathcal{E}}_k^{(i)} = \mathcal{L}_k^{(i)} \widetilde{\mathcal{M}}_k^{(i)} \quad i = 1, \dots, q\,.$$

From the family of matrices $(\widetilde{\mathcal{E}}_k^{(i)})_{1\leqslant i\leqslant q}$, we define the block diagonal matrix:

$$\widetilde{C}_N^q = \begin{pmatrix} \widetilde{\mathcal{E}}_k^{(1)} & & & & \\ & \widetilde{\mathcal{E}}_k^{(2)} & & & \\ & & \ddots & & \\ & & & \widetilde{\mathcal{E}}_k^{(q)} & \\ & & & & 0_{r\times r} \end{pmatrix}. \tag{3.3.23}$$

We claim that $\widetilde{C}_N^q$ is such that

$$\operatorname{rank}(C_N^q - \widetilde{C}_N^q) \leqslant 2q\,. \tag{3.3.24}$$

Indeed, we take the same $\alpha_j^{(i)}$ in the construction of $\widetilde{\mathcal{E}}_k^i$ and of $\mathcal{E}_k^i$, except for the entries of the corners of $\mathcal{M}_k^{(i)}$, where $\mathcal{M}_k^{(i)}(1,1)$ is replaced by 1, $\mathcal{M}_k^{(i)}(k,k)$ is replaced by a uniform variable on the circle, and both entries $\mathcal{M}_k^{(i)}(1,k)$ and $\mathcal{M}_k^{(i)}(k,1)$ are replaced by 0. This shows that

$$\operatorname{rank}(\mathcal{M}_k^{(i)} - \widetilde{\mathcal{M}}_k^{(i)}) \leqslant 2,$$

and

$$\operatorname{rank}(\mathcal{E}_k^{(i)} - \widetilde{\mathcal{E}}_k^{(i)}) = \operatorname{rank}(\mathcal{L}_k^{(i)}(\mathcal{M}_k^{(i)} - \widetilde{\mathcal{M}}_k^{(i)})) \leqslant \operatorname{rank}(\mathcal{M}_k^{(i)} - \widetilde{\mathcal{M}}_k^{(i)}) \leqslant 2,$$

and we deduce (3.3.24). From (3.3.24) and Lemma 3.2.2, we deduce that

$$d(\mu_N(C_N^q), \mu_N(\widetilde{C}_N^q)) \leqslant \frac{2}{k}\,, \tag{3.3.25}$$

and for any $\delta > 0$ and sufficiently large N, we can take k such that $\frac{2}{k} \leqslant \frac{\delta}{4}$.

Consider now another two families of matrices $(\mathfrak{L}_k^{(i)})_{1\leqslant i\leqslant q}$, and $(\mathfrak{M}_k^{(i)})_{1\leqslant i\leqslant q}$, constructed in the same way as $(\mathcal{L}_k^{(i)})_{1\leqslant i\leqslant q}$, and $(\widetilde{\mathcal{M}}_k^{(i)})_{1\leqslant i\leqslant q}$ by means of independent variables $\widetilde{\alpha}_{(i-1)k+j}$, where each $\widetilde{\alpha}_{(i-1)k+j} \sim \Theta_{2\beta\frac{N-(i-1)k-j}{N}}$ is coupled to $\alpha_{(i-1)k+j}$ by equation (3.3.14), for all $j = 0,\dots,k-1$, and $i = 1,\dots,q$, and where $\widetilde{\alpha}_{ik} = \alpha_{ik}$ for $i = 1,\dots,q$ is uniformly distributed on the unit circle. Define the family of CMV matrices $(\mathfrak{E}_k^{(i)})_{1\leqslant i\leqslant q}$ as

$$\mathfrak{E}_k^{(i)} = \mathfrak{L}_k^{(i)}\mathfrak{M}_k^{(i)}\,, \quad i = 1,\dots,q\,. \tag{3.3.26}$$

Define the block diagonal matrix E_N^q as:

$$E_N^q = \begin{pmatrix} \mathfrak{E}_k^{(1)} & & & & \\ & \mathfrak{E}_k^{(2)} & & & \\ & & \ddots & & \\ & & & \mathfrak{E}_k^{(q)} & \\ & & & & 0_{r\times r}\,. \end{pmatrix} \tag{3.3.27}$$

From the definition of $\mathbb{P}_{C,N}^{\frac{2\beta}{N}}$ and E_N^q, we conclude that for some $E \sim \mathbb{P}_{C,N}^{\frac{2\beta}{N}}$, we have

$$\operatorname{rank}(E - E_N^q) \leqslant 2q + r\,. \tag{3.3.28}$$

As before, from the previous inequality we deduce that

$$d(\mu_N(E), \mu_N(\widetilde{E}_N^q)) \leqslant \frac{4}{k}\,. \tag{3.3.29}$$

Finally, we define the matrix B_N^q as

$$B_N^q = \begin{pmatrix} B_k^{(1)} & & & & \\ & B_k^{(2)} & & & \\ & & \ddots & & \\ & & & B_k^{(q)} & \\ & & & & 0_{r\times r}\,, \end{pmatrix} \tag{3.3.30}$$

where $B_k^{(i)} = \mathfrak{l}_k^{(i)} \widetilde{\mathcal{M}}_k^{(i)}$.

Let $\delta > 0$, for N large enough such that $\frac{4}{k} \leqslant \frac{\delta}{4}$, we have almost surely

$$d(\mu_N(C_N^q), \mu_N(\widetilde{C}_N^q)) + d(\mu_N(E), \mu_N(E_N^q)) > \frac{\delta}{2}.$$

As a consequence,

$$\begin{aligned} &\mathbb{P}\left(d(\mu_N(C_N^q), \mu_N(E)) > \delta\right) \\ &\qquad \leqslant \mathbb{P}\Big(d(\mu_N(C_N^q), \mu_N(\widetilde{C}_N^q)) + d(\mu_N(\widetilde{C}_N^q), \mu_N(B_N^q)) \\ &\qquad + d(\mu_N(B_N^q), \mu_N(E_N^q)) + d(\mu_N(E_N^q), \mu_N(E)) > \delta\Big) \\ &\qquad \leqslant \mathbb{P}\left(d(\mu_N(\widetilde{C}_N^q), \mu_N(B_N^q)) + d(\mu_N(B_N^q), \mu_N(E_N^q)) > \frac{\delta}{2}\right). \end{aligned} \tag{3.3.31}$$

Moreover, combining Lemma 3.2.2 and Lemma 3.3.7 we deduce that

$$\begin{aligned} d(\mu_N(\widetilde{C}_N^q), \mu_N(B_N^q)) &\leqslant \frac{2}{N} \sum_{i=1}^{q} \sum_{1\leqslant \ell,j\leqslant k} \left|\mathfrak{l}_k^{(i)}(\ell,j) - \mathcal{L}_k^{(i)}(\ell,j)\right|, \\ d(\mu_N(B_N^q), \mu_N(\widetilde{E}_N^q)) &\leqslant \frac{2}{N} \sum_{i=1}^{q} \sum_{1\leqslant \ell,j\leqslant k} \left|\mathfrak{M}_k^{(i)}(\ell,j) - \widetilde{\mathcal{M}}_k^{(i)}(\ell,j)\right|. \end{aligned} \tag{3.3.32}$$

Applying Lemma 3.3.8 point $i.$, we deduce that

$$d(\mu_N(\widetilde{C}_N^q), \mu_N(B_N^q)) + d(\mu_N(B_N^q), \mu_N(\widetilde{E}_N^q)) \leqslant \frac{8}{N} \sum_{i=1}^{q} \sum_{j=0}^{k-1} Z^{(i)}_{\frac{k-j}{N}}, \tag{3.3.33}$$

where the last sum denotes the sum of independent random variables with law $Z_{\frac{k-j}{N}}$, defined in Lemma 3.3.8.

Thus, for N large enough such that $\frac{4}{k} \leqslant \frac{\delta}{4}$, we deduce that for any non-negative function $a(q^{-1})$:

$$\begin{aligned} \mathbb{P}\left(d(\mu_N(C_N^q), \mu_N(E)) > \delta\right) &\leqslant \mathbb{P}\left(\sum_{i=1}^{q} \sum_{j=0}^{k-1} Z^{(i)}_{\frac{k-j}{N}} > \frac{N\delta}{16}\right) \\ &\leqslant e^{-a(q^{-1})N\delta/16} \left(\sup_{0<h<1} \mathbb{E}\left[\exp(a(h)Z_h)\right]\right)^{qk}, \end{aligned} \tag{3.3.34}$$

Where in the last inequality we used Remark 3.3.9, namely, since $\frac{k-j}{N} \leqslant \frac{1}{q}$, we have

$$\mathbb{E}[\exp(a(q^{-1})Z_{\frac{k-j}{N}}] \leqslant \mathbb{E}[\exp(a(q^{-1})Z_{\frac{1}{q}}].$$

Setting $a(h) = -\frac{1}{2}\log(h) + 1$ and applying Lemma 3.3.8 point $ii.$, we deduce that there exist constants $\widetilde{K}$ and $c > 0$, independent of q, such that

$$\frac{1}{N}\log\left(\mathbb{P}\left(d(\mu_N(C_N^q),\mu_N(E))>\delta\right)\right)\leqslant -c\log(q)\delta+\tilde{K}\,, \tag{3.3.35}$$

And we obtain the claim. □

We can apply the previous Lemma to study the case of continuous potential, indeed as a consequence of Varadhan's Lemma we obtain the main result of this section, namely

Theorem 3.3.11. *In the same notation as before. Let $\beta>0$, and $V:\mathbb{T}\to\mathbb{R}$ continuous. The law of the empirical measures $\mu_N(E)$ under $d\mathbb{P}_{C,N}^{V,\frac{2\beta}{N}}$ satisfies a large deviations principle at speed N, with a good rate function $I_\beta^V(\mu)=f_\beta^V(\mu)-\inf_{\nu\in\mathcal{P}(\mathbb{T})}f_\beta^V(\nu)$, where*

$$f_\beta^V(\mu)=\lim_{\delta\to 0}\liminf_{q\to\infty}\inf_{\substack{\nu_{\beta/q},\dots,\nu_\beta\\ \frac{1}{q}\sum_i\nu_{i\beta/q}\in B_\mu(\delta)}}\left\{\frac{1}{q}\sum_{i=1}^{q}\left(J_{i\beta/q}(\nu_{i\beta/q})+\int_{\mathbb{T}}V d\nu_{i\beta/M}\right)\right\}. \tag{3.3.36}$$

3.4 Proof of the main results

In this section, we conclude the proof of Theorem 3.1.4 and prove Theorem 3.1.5. The main tool to prove these theorems is the uniqueness of the minimizer of the rate function for the β ensemble.

Define the free energies of the Ablowtiz-Ladik lattice and the Circular beta ensemble at high temperature as

$$F_{AL}(V,\beta)=\inf_{\nu\in\mathcal{P}(\mathbb{T})}g_\beta^V(\nu)\,,\quad F_C(V,\beta)=\inf_{\nu\in\mathcal{P}(\mathbb{T})}f_\beta^V(\nu)\,, \tag{3.4.1}$$

where g_β^V, and f_β^V are given by (3.2.17) and (3.3.36). We claim that

Lemma 3.4.1. *Let $\beta>0$, and $V:\mathbb{T}\to\mathbb{R}$ continuous, then the following holds:*

a. the map $\beta\to F_C(V,\beta)$ is continuously differentiable on $\mathbb{R}_+^*$. Moreover:

$$F_{AL}(V,\beta)=\partial_\beta(\beta F_C(V,\beta))\,; \tag{3.4.2}$$

b. for almost all $\beta>0$ there exists a unique minimizer ν_β^V of the functional $J_\beta^V(\mu)$, see Corollary 3.2.5, given by

$$\nu_\beta^V=\partial_\beta(\beta\mu_\beta^V)\,, \tag{3.4.3}$$

i.e for continuous $f:\mathbb{T}\to\mathbb{R}$,

$$\int_{\mathbb{T}}f d\nu_\beta^V=\partial_\beta\left(\beta\int_{\mathbb{T}}f d\mu_\beta^V\right). \tag{3.4.4}$$

we recall that the measure μ_β^V is defined as the unique minimizer of the functional I_β^V in Theorem 3.3.11.

Remark 3.4.2. *Our definition of Free Energy is different from the one used in [GM23, Spo22b]. Indeed, in virtue of Varadhan's Lemma [DE97, Theorem 1.2.1], we have*

$$\begin{aligned}F_{AL}(V,\beta)&=\inf_{\nu\in\mathcal{P}(\mathbb{T})}g_\beta^V(\nu)=-\lim_N\frac{1}{N}\log\mathbb{E}_{AL,N}^\beta\left[e^{-\mathrm{Tr}V(\mathcal{E})}\right],\\ F_C(V,\beta)&=\inf_{\nu\in\mathcal{P}(\mathbb{T})}f_\beta^V(\nu)=-\lim_N\frac{1}{N}\log\mathbb{E}_{C,N}^\beta\left[e^{-\mathrm{Tr}V(E)}\right],\end{aligned} \tag{3.4.5}$$

instead in [GM23, Spo22b], the authors defined the free energies as

$$\begin{aligned}\widetilde{F}_{AL}(V,\beta) &= -\lim_{N\to\infty}\frac{1}{N}\log(Z_N^{AL}(V,\beta))\,,\\ \widetilde{F}_{C}(V,\beta) &= -\lim_{N\to\infty}\frac{1}{N}\log(Z_N^{C}(V,\beta))\,.\end{aligned}\tag{3.4.6}$$

We notice that it is possible to recover one expression from the other since

$$\begin{aligned}F_{AL}(V,\beta) &= \widetilde{F}_{AL}(V,\beta)-\widetilde{F}_{AL}(0,\beta)\,,\\ F_{C}(V,\beta) &= \widetilde{F}_{C}(V,\beta)-\widetilde{F}_{C}(0,\beta)\,.\end{aligned}\tag{3.4.7}$$

To prove uniqueness of the minimizer ν_β^V, we need to consider a continuous family $(\mu_s^*)_{0<s<\beta}$, where each μ_s^* minimizes J_s^V, see Corollary 3.2.5. We address the existence of such a family in the next Lemma, which we prove in the appendix 3.6.

Lemma 3.4.3. *Let $M_\beta^V = (J_\beta^V)^{-1}(\{0\})$ be the set of minimizers of J_β^V, defined in Corollary 3.2.5. Then, $\beta \mapsto M_\beta^V$ is continuous in the sense that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 \leqslant h \leqslant \delta$, $M_{\beta+h}^V \subset (M_\beta^V)^\varepsilon$, where for $A \subset \mathcal{P}(\mathbb{T})$ we denote $A^\varepsilon = \{\mu \in \mathcal{P}(\mathbb{T}) \mid d(\mu, A) \leqslant \varepsilon\}$.*

Proof of Lemma 3.4.1. First, we notice that for any probability measure $\mu \in \mathcal{P}(\mathbb{T})$, Theorem 3.3.11 implies

$$\begin{aligned}f_\beta^V(\mu) &\geqslant \liminf_{q\to\infty}\inf_{\nu\in\mathcal{P}(\mathbb{T})}\left\{\frac{1}{q}\sum_{i=1}^{q}\left(J_{i\beta/q}(\nu)+\int_{\mathbb{T}}V d\nu\right)\right\}\\ &= \int_0^1 \inf_{\nu\in\mathcal{P}(\mathbb{T})} g_{s\beta}^V(\nu) = \int_0^1 F_{AL}(V,s\beta)ds\,,\end{aligned}\tag{3.4.8}$$

Where we noticed that the Riemann sums indeed converge towards the integral since $s \mapsto F_{\mathrm{AL}}(V, s\beta)$ is concave, this can be seen by applying Hölder inequality to equation (3.4.5).

To prove the first part of the claim, we show that the lower bound is achieved. For $s \in [0,1]$, let $\nu_{s\beta}^*$ be a minimizer of $\inf_{\nu\in\mathcal{P}(\mathbb{T})} g_{s\beta}^V(\nu)$. From Lemma 3.4.3, we can choose $\nu_{s\beta}^*$ such that the map $s \to \nu_{s\beta}^*$ is continuous. This implies that $\mu_\beta^* = \int_0^1 \nu_{s\beta}^* ds$ is a well-defined probability measure on $\mathbb{T}$. We claim that this measure minimizes f_β^V (3.3.36), and so I_β^V. Indeed, from Theorem 3.3.11, we deduce that

$$\begin{aligned}f_\beta^V(\mu_\beta^*) &= \lim_{\delta\to 0}\liminf_{q\to\infty}\inf_{\substack{\nu_{\beta/q},\dots,\nu_\beta\\ \frac{1}{q}\sum_i \nu_{i\beta/q}\in B_{\mu_\beta^*}(\delta)}}\left\{\frac{1}{q}\sum_{i=1}^{q}\left(J_{i\beta/q}(\nu_{i\beta/q})+\int_{\mathbb{T}}V d\nu_{i\beta/q}\right)\right\}\\ &\leqslant \liminf_{q\to\infty}\left\{\frac{1}{q}\sum_{i=1}^{q}\left(J_{i\beta/q}(\nu_{i\beta/q}^*)+\int_{\mathbb{T}}V d\nu_{i\beta/q}^*\right)\right\}\\ &= \liminf_{q\to\infty}\left\{\frac{1}{q}\sum_{i=1}^{q}\inf_{\nu\in\mathcal{P}(\mathbb{T})}\left(J_{i\beta/q}(\nu)+\int_{\mathbb{T}}V d\nu\right)\right\}\\ &= \int_0^1 \inf_{\nu\in\mathcal{P}(\mathbb{T})} g_{s\beta}^V(\nu) = \int_0^1 F_{AL}(V,s\beta)ds\,.\end{aligned}\tag{3.4.9}$$

Combining (3.4.8)-(3.4.9), and performing the change of coordinates $s\beta = t$ we deduce that:

$$\beta F_C(V,\beta) = \int_0^\beta F_{AL}(V,t)dt\,.\tag{3.4.10}$$

Moreover, from Lemma 3.3.2 we deduce that the map $\beta \to F_C(V,\beta)$ is Lipschitz in β, and so almost surely differentiable. This implies that for almost all $\beta > 0$

$$F_{AL}(V,\beta) = \partial_\beta(\beta F_C(V,\beta))\,. \tag{3.4.11}$$

Furthermore, we have just shown that $I_\beta^V(\mu) = f_\beta^V(\mu) - \inf_{\nu\in\mathcal{P}(\mathbb{T})} f_\beta^V(\nu)$ reaches its minimum at $\int_0^1 \nu_{s\beta}^* ds$. By uniqueness of the minimizer of $I_\beta^V(\mu)$, Theorem 3.3.1, we deduce that we have the equality between probability measures $\mu_\beta^V = \int_0^1 \nu_{s\beta}^* ds$. Taking $f : \mathbb{T} \to \mathbb{R}$ continuous we get

$$\beta \int_{\mathbb{T}} f d\mu_\beta^V = \int_0^\beta \int_{\mathbb{T}} f d\nu_s^* ds.$$

Note that the function $s \mapsto \int_{\mathbb{T}} f d\nu_s^*$ is continuous, therefore by differentiating this equality, we get that ν_β^* is the unique minimizer of J_β^V, which we denote by ν_β^V, and satisfies for f continuous

$$\int_{\mathbb{T}} f\nu_\beta^V = \partial_\beta\left(\beta \int_{\mathbb{T}} f\mu_\beta^V\right)\,, \tag{3.4.12}$$

proving point b.

□

Remark 3.4.4. *As a corollary of the previous Lemma we obtain Theorem 3.1.4.*

3.5 The Schur Flow

In this section, we consider another integrable model, namely the Schur flow. Our goal is to show that is possible to obtain a similar result to the one that we presented for the Ablowitz-ladik lattice. Namely, we prove the existence of a large deviations principle for the Schur flow, and we relate its density of state to the one of the Jacobi beta ensemble in the high temperature regime.

3.5.1 Generalized Gibbs Ensemble

The Schur flow is the system of ODEs [Gol06]

$$\dot{\alpha_j} = \rho_j^2(\alpha_{j+1} - \alpha_{j-1})\,, \quad \rho_j = \sqrt{1-|\alpha_j|^2}\,, \tag{3.5.1}$$

and, as before, we consider periodic boundary conditions, namely $\alpha_j = \alpha_{j+N}$ for all $j \in \mathbb{Z}$.

In [AL76], it is argued that the continuum limit of (3.5.1) is the modified Korteweg-de Vries equation:

$$\partial_t u = \partial_x^3 u - 6u^2\partial_x u\,. \tag{3.5.2}$$

We notice that, if one chooses an initial data such that $\alpha_j(0) \in \mathbb{R}$ for all $j = 1,\dots,N$, then $\alpha_j(t) \in \mathbb{R}$ for all times. Moreover, it is straightforward to verify that $K_0 = \prod_{j=1}^N \left(1-|\alpha_j|^2\right)$ is conserved along the Schur flow. This implies that we can choose as phase space for the Schur flow the N-cube $\mathbb{I}^N$, where $\mathbb{I} := (-1,1)$.

On this phase space, we consider the Poisson braket (3.1.5), so we can rewrite the Schur flow (3.5.1) in Hamiltonian form as

$$\dot{\alpha}_j = \{\alpha_j, H_S\}, \quad H_S = -i\sum_{j=1}^N \left(\alpha_j\overline{\alpha}_{j+1} - \overline{\alpha}_j\alpha_{j+1}\right)\,. \tag{3.5.3}$$

It is well known that the Schur flow admits as Lax matrix the same one as the AL [Gol06], namely $\mathcal{E}$ (3.1.9) is the Lax matrix of the Schur flow. This implies that the Ablowitz-Ladik's constants of motion are conserved also along the Schur flow (3.5.1).

Following the same construction made for the Ablowitz-Ladik lattice, on $\mathbb{I}^N$ we define the finite volume limit GGE as

$$d\mathbb{P}_{S,N}^{V,\beta}(\alpha_1,\dots,\alpha_N) = \frac{1}{Z_N^S(V,\beta)}\prod_{j=1}^{N}(1-\alpha_j^2)^{\beta-1}\mathbf{1}_{\{\alpha_j\in\mathbb{I}\}}\exp(-\mathrm{Tr}(V(\mathcal{E})))d\boldsymbol{\alpha}, \tag{3.5.4}$$

where $Z_N^S(V,\beta)$ is the partition function of the system

$$Z_N^S(V,\beta) = \int_{\mathbb{I}^N}\prod_{j=1}^{N}(1-\alpha_j^2)^{\beta-1}\exp(-\mathrm{Tr}(V(\mathcal{E})))d\boldsymbol{\alpha}.$$

Since according to the measure (3.5.4) the matrix $\mathcal{E}$ is real, its eigenvalues come in pairs [Sim05], meaning that if $e^{i\theta_j}$ is an eigenvalue, then its conjugate $e^{-i\theta_j}$ is also an eigenvalue. This implies that for a system of size N even, there are just $n = N/2$ independent eigenvalues. Following the same idea as in [KN04], it is more convenient to restrict the argument of the eigenvalues in $[0,\pi)$ and then consider $x_j = \cos(\theta_j)$, $j = 1,\dots,n$. In these variables, the empirical spectral measure $\mu_n(\mathcal{E})$ reads:

$$\mu_n(\mathcal{E}) = \frac{1}{n}\sum_{j=1}^{n}\delta_{x_j}\,, \quad x_j\in\mathbb{I}\,. \tag{3.5.5}$$

As a corollary of Lemma 3.2.3 and Proposition 3.2.4, we obtain the existence of a large deviations principle for the sequence $(\mu_n(\mathcal{E}))$, namely:

Corollary 3.5.1. *Let $V : \mathbb{I}\to\mathbb{R}$ be continuous. Under $\mathbb{P}_{S,n}^{V,\beta}$ the sequence $(\mu_n(\mathcal{E}))_{n\geqslant 1}$ fulfils a large deviations principle with good, convex rate function $S_\beta^V(\mu) = h_\beta^V(\mu) - \inf_{\nu\in\mathcal{P}(\mathbb{I})} h_\beta^V(\nu)$, where*

$$h_\beta^V(\nu) = K_\beta(\nu) + \int_{\mathbb{I}} V d\nu\,, \tag{3.5.6}$$

where $K_\beta(\nu)$ is the rate function of μ_n under the law $\mathbb{P}_{S,n}^{0,\beta}$.

3.5.2 Jacobi beta ensemble in the high temperature regime

The Jacobi beta ensemble refers to the distribution of charges constrained to the segment $\mathbb{I}$, and subjected to an external potential $W(x) = -a\log(1-x) - b\log(1+x) + V(x)$, here $a,b > -1$ and $W(x)\in C^0(\mathbb{I})$. Specifically the joint distribution of these particles is

$$d\mathbb{P}_{J,n}^{(V,\tilde\beta)} = \frac{1}{Z_N^J(V,\widetilde\beta)}\prod_{i<j}|x_i-x_j|^{\tilde\beta}\prod_{j=1}^{n}(1-x_j)^a(1+x_j)^b e^{-V(x_j)}dx_j\,. \tag{3.5.7}$$

In [KN04], Killip and Nenciu were able to show that the distribution (3.5.7) can be realized as the eigenvalues distribution of a particular CMV matrix, specifically they proved the following

Theorem 3.5.2 (cf. [KN04] Proposition 5.3)**.** *Let $N = 2n$, consider the CMV matrix E in* (3.3.10) *with parameters $\alpha_1,\dots,\alpha_{2n-1}\in\mathbb{I}$ distributed according to*

$$d\mathbb{B}_n^{(V,\tilde\beta)} = \frac{1}{\mathfrak{Z}_n(V,\widetilde\beta)}\prod_{j=1}^{2n-1}(1-\alpha_j^2)^{\tilde\beta(2n-j)/4-1}\prod_{j=1}^{2n-1}(1-\alpha_j)^{a+1-\tilde\beta/4}(1+(-1)^{j+1}\alpha_j)^{b+1-\tilde\beta/4}e^{-\mathrm{Tr}V(E)}d\alpha_j\,, \tag{3.5.8}$$

and $\alpha_{2n} = -1$, here $\mathfrak{Z}_N(V,\beta)$ is the normalization constant. Then all the eigenvalues of E come in pairs, meaning that if $e^{i\theta_j}$ is an eigenvalue, then also $e^{-i\theta_j}$ is one. Moreover, under the change of variables $\cos(\theta_j) = x_j$, the x_js are distributed according to (3.5.7)*.*

Remark 3.5.3. *We notice that the previous proposition is not stated in this way in [KN04], but this equivalent formulation is more useful for our purpose.*

Also in this case, we are interested in the high temperature regime for this ensemble. Specifically we consider the situation $\widetilde{\beta} = \frac{4\beta}{N} = \frac{2\beta}{n}$, and $a = b = -1 + \frac{\tilde{\beta}}{4}$, in this regime $d\mathbb{P}_{J,n}^{\left(V,\frac{\beta}{n}\right)}$ reads

$$d\mathbb{P}_{J,n}^{\left(V,\frac{2\beta}{n}\right)} = \frac{1}{Z_N^J\left(V,\frac{\beta}{n}\right)} \prod_{i<j} |x_i - x_j|^{\frac{2\beta}{n}} \prod_{j=1}^{n} (1-x_j)^{-1+\frac{\beta}{2n}} (1+x_j)^{-1+\frac{\beta}{2n}} e^{-V(x_j)} dx_j \,, \tag{3.5.9}$$

and $d\mathbb{B}_n^{\left(V,\frac{\beta}{n}\right)}$ becomes

$$d\mathbb{B}_n^{\left(V,\frac{\beta}{n}\right)} = \frac{1}{\mathfrak{Z}_n\left(V,\frac{\beta}{n}\right)} \prod_{j=1}^{2n-1} (1-\alpha_j^2)^{\beta\left(1-\frac{j}{2n}\right)-1} \prod_{j=1}^{2n-1} e^{-\mathrm{Tr}V(E)} d\alpha_j \,. \tag{3.5.10}$$

We mention that this particular regime was considered in [FM21, TT21]. In these papers the authors computed the density of states for this ensemble in the case $V = 0$.

We can apply [GZ19, Corollary 1.3] to (3.5.9) to obtain a large deviations principle for the empirical measure $\mu_n(E) = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_j}$. Specifically, we deduce that

Proposition 3.5.4. *For any continuous $V : \mathbb{I} \to \mathbb{R}$. The law of the empirical measures $\mu_n(E)$ under $d\mathbb{P}_{J,n}^{\left(V,\frac{2\beta}{n}\right)}$ satisfies a large deviations principle at speed n in the space $\mathcal{P}(\mathbb{I})$, with a good rate function $\mu \mapsto Q_\beta^V(\mu)$ given for μ absolutely continuous with respect to Lebesgue measure, and with density $\frac{d\mu}{dx}$, by $Q_\beta^V(\mu) = q_\beta^V(\mu) - \inf_{\nu\in\mathcal{P}(\mathbb{I})} q_\beta^V(\nu)$, where*

$$q_\beta^V(\mu) = \int_{\mathbb{I}} (V(x) + \log(1+x) + \log(1-x)) d\mu(x) - 2\beta \int_{\mathbb{I}\times\mathbb{I}} \log(|x-y|) d\mu(x) d\mu(y) + \int_{\mathbb{I}} \log\left(\frac{d\mu}{dx}(x)\right) d\mu(x) \,, \tag{3.5.11}$$

and $Q_\beta^V(\mu) = +\infty$ otherwise.

We notice that the arguments in Section 3.3 and 3.4 can be applied also in this context with $d\mathbb{P}_{J,n}^{\left(V,\frac{2\beta}{n}\right)}$ in place of $d\mathbb{P}_{C,N}^{\left(V,\frac{2\beta}{N}\right)}$, and $d\mathbb{P}_{S,N}^{V,\beta}$ in place of $d\mathbb{P}_{AL,N}^{V,\beta}$. Hence, we deduce the following result

Theorem 3.5.5. *Consider the sequence of measures $\mu_n(\mathcal{E})$ (3.5.5) under the law $d\mathbb{P}_{S,2n}^{V,\beta}$ (3.5.4), then*

$$\mu_n(\mathcal{E}) \xrightarrow{a.s.} \nu_\beta^V \,. \tag{3.5.12}$$

Moreover, ν_β^V is absolutely continuous with respect to the Lebesgue measure, and it reads

$$\nu_\beta^V = \partial_\beta(\beta\mu_\beta^V) \,, \tag{3.5.13}$$

where μ_β^V is the unique minimizer of the functional q_β^V (3.5.11).

Finally, it is worth to mention that in the case $V(x) = 0$, it is possible to compute explicitly the densities of states for both the Jacobi beta ensemble at high temperature and for the Schur flow [Maz22, FM21].

3.6 Appendix: Technical Results

In this appendix we collect the proof of all the technical results that we exploit along the proof of the main theorem. For reader convenience, we report here the statement of Lemmas.

Proof of Lemma 3.2.2

Lemma 3.6.1. *For any A, B unitary matrices of size $N \times N$, we have*

- *For f with bounded variation,*
$$\left| \int f d\mu - \int f d\nu \right| \leqslant \|f\|_{BV} \frac{rank(A-B)}{N},$$
- *For f Lipschitz,*
$$\left| \int f d\mu - \int f d\nu \right| \leqslant \|f\|_{Lip} \frac{1}{N} \sum_{i,j} |(A-B)_{i,j}|.$$

As a consequence,
$$d(\mu(A), \mu(B)) \leqslant \min \left\{ \frac{rank(A-B)}{N}, \frac{1}{N} \sum_{i,j} |(A-B)_{i,j}| \right\}. \tag{3.6.1}$$

Proof. The first point is a consequence of the fact that the eigenvalues of A and B interlace on the unit circle.

First, we order the eigenvalues $\lambda_1(A), \dots, \lambda_N(A), \lambda_1(B), \dots, \lambda_N(B)$ of A, B in such a way that
$$-\pi \leqslant \arg(\lambda_1(A)) \leqslant \dots \leqslant \arg(\lambda_N(A)) < \pi\,, \tag{3.6.2}$$
and analogously for B.

Write $B = (I_N + (B-A)A^{-1})A$ and set $U := I_N + (B-A)A^{-1}$. One checks that U is unitary, $B = UA$, and that $\operatorname{rank}(U-I) = \operatorname{rank}(B-A) =: r$. By [AG88, section 6, equation (85)], we deduce that for $1 \leqslant j \leqslant N$
$$\arg(\lambda_{j-r}(A)) \leqslant \arg(\lambda_j(B)) \leqslant \arg(\lambda_{j+r}(A))\,. \tag{3.6.3}$$

This means that $\lambda_j(B)$ lies on the anticlockwise arc $(\arg(\lambda_{j-r}(A)), \arg(\lambda_{j+r}(A)))$ of the circle. If $j - r \leqslant 0$ we identify λ_{j-r} with λ_{j-r+N}, and analogously for the case $j + r > N$.
It is a classical result (see [AGZ10]) to deduce from (3.6.3) that
$$\left| \int f d\mu_N(A) - \int f d\mu_N(B) \right| \leqslant \|f\|_{\mathrm{BV}} \frac{r}{N} = \frac{r}{N}\,,$$
for any $f : \mathbb{T} \to \mathbb{R}$ such that $||f||_{BV} \leqslant 1$. As a consequence, we obtain the first point.
The proof of the second point is the same as in the symmetric case, see [GM22, (16)]. Indeed, we only use the fact that a normal matrix is unitarily diagonalizable. □

Proof of Lemma 3.3.7

Lemma 3.6.2. *Let $N = 2k$ be even and A be a $N \times N$ matrix. Then,*

- $\sum_{i,j} |(\mathcal{L}A)_{i,j}| \leqslant 2 \sum_{i,j} |A_{i,j}|$,
- $\sum_{i,j} |(A\mathcal{M})_{i,j}| \leqslant 2 \sum_{i,j} |A_{i,j}|$,

where $\mathcal{M}$, and $\mathcal{L}$ are defined in (3.1.8).

Proof. We will just prove the first point, since the proof of the second one follows the same lines.

For $0 \leqslant l \leqslant k-1$ and $1 \leqslant j \leqslant N$, consider
$$(\mathcal{L}A)_{2l+1,j} = \overline{\alpha}_{2l+1} A_{2l+1,j} + \rho_{2l+1} A_{2l+2,j} \quad \text{and} \quad (\mathcal{L}A)_{2l+2,j} = \rho_{2l+1} A_{2l+1,j} - \alpha_{2l+2} A_{2l+2,j}.$$
Summing over i, j,
$$\sum_{i,j} |(\mathcal{L}A)_{i,j}| = \sum_{l=0}^{k-1} \sum_{j=1}^{N} |(\mathcal{L}A)_{2l+1,j}| + |(\mathcal{L}A)_{2l+2,j}| \leqslant 2 \sum_{l=0}^{k-1} \sum_{j=1}^{N} |A_{2l+1,j}| + |A_{2l+2,j}| = 2 \sum_{i,j} |A_{i,j}|,$$
where we used that $|\alpha_i|, \rho_i \leqslant 1$. □

Proof of Lemma 3.3.8

Lemma 3.6.3. *Let α_ν and $\alpha_{\nu+h}$ defined by equation (3.3.14). Define $\rho_\nu = \sqrt{1-|\alpha_\nu|^2}$, and $\rho_{\nu+h} = \sqrt{1-|\alpha_{\nu+h}|^2}$, then the following hold*

i.

$$\begin{aligned} |\alpha_\nu - \alpha_{\nu+h}| &\leqslant \frac{Y_h}{(X_1^2+X_2^2+Y_h^2)^{\frac{1}{2}}}\,, \text{ almost surely}, \\ |\rho_\nu - \rho_{\nu+h}| &\leqslant \frac{Y_h}{(X_1^2+X_2^2+Y_h^2)^{\frac{1}{2}}}\,, \text{ almost surely}, \end{aligned} \tag{3.6.4}$$

where $X_1, X_2 \sim N(0,1)$, $Y_h \sim \chi_h$ are all independent.

ii. define $Z_h = \frac{Y_h}{(X_1^2+X_2^2+Y_h^2)^{\frac{1}{2}}}$, and $a(h) = -\frac{1}{2}\log(h)+1$, then there exists a constant K independent of h such that

$$\sup_{0<h<1} \mathbb{E}\left[\exp(a(h)Z_h)\right] \leqslant K\,. \tag{3.6.5}$$

Proof. First, we focus on claim *i.*.

We recall that $\alpha_\nu, \alpha_{\nu+h}$ are defined by

$$\alpha_\nu = \frac{X_1+iX_2}{(X_1^2+X_2^2+Y_{\nu-1}^2)}\,, \quad \alpha_{\nu+h} = \frac{X_1+iX_2}{(X_1^2+X_2^2+Y_{\nu-1}^2+Y_h^2)}\,. \tag{3.6.6}$$

From the previous equation, we deduce that

$$\begin{aligned} |\alpha_\nu - \alpha_{\nu+h}| &= \frac{|X_1+iX_2|}{(X_1^2+X_2^2+Y_{\nu-1}^2)^{\frac{1}{2}}}\left(1-\left(\frac{X_1^2+X_2^2+Y_{\nu-1}^2}{X_1^2+X_2^2+Y_{\nu-1}^2+Y_h^2}\right)^{\frac{1}{2}}\right) \\ &= \frac{|X_1+iX_2|}{(X_1^2+X_2^2+Y_{\nu-1}^2)^{\frac{1}{2}}}\left(1-\left(1-\frac{Y_h^2}{X_1^2+X_2^2+Y_{\nu-1}^2+Y_h^2}\right)^{\frac{1}{2}}\right) \\ &\leqslant \left(\frac{X_1^2+X_2^2}{X_1^2+X_2^2+Y_{\nu-1}^2}\right)^{\frac{1}{2}} \frac{Y_h}{(X_1^2+X_2^2+Y_{\nu-1}^2+Y_h^2)^{\frac{1}{2}}}, \end{aligned}$$

where we used in the previous line that for $0 \leqslant a \leqslant b$ we have

$$\sqrt{b} \leqslant \sqrt{b-a} + \sqrt{a}\,, \tag{3.6.7}$$

and we took $a = \frac{Y_h^2}{X_1^2+X_2^2+Y_{\nu-1}^2+Y_h^2}$, $b=1$. The last term is bounded by the announced bound.

One can proceed analogously for $|\rho_\nu - \rho_{\nu+h}|$ obtaining that

$$\begin{aligned} |\rho(\alpha_{\nu+h}) - \rho(\alpha_\nu)| &= \sqrt{1-|\alpha_{\nu+h}|^2} - \sqrt{1-|\alpha_\nu|^2} \leqslant \sqrt{|\alpha_\nu|^2 - |\alpha_{\nu+h}|^2} \\ &= \sqrt{\frac{X_1^2+X_2^2}{X_1^2+X_2^2+Y_{\nu-1}^2}}\sqrt{1-\frac{X_1^2+X_2^2+Y_{\nu-1}^2}{X_1^2+X_2^2+Y_{\nu-1}^2+Y_h^2}} \leqslant \frac{Y_h}{(X_1^2+X_2^2+Y_{\nu-1}^2+Y_h^2)^{\frac{1}{2}}}\,, \end{aligned} \tag{3.6.8}$$

where we used again equation (3.6.7) with $a = 1-|\alpha_\nu|^2$ and $b = 1-|\alpha_{\nu+h}|^2$. Thus, point *i.* is proved.

To prove point *ii.*, we find explicitly the law of Z_h. Thus, we consider a continuous function $f : (0,1) \to \mathbb{R}$, and we compute:

$$\int_{\mathbb{R}^2\times\mathbb{R}_+} f\left(\frac{y}{(x_1^2+x_2^2+y^2)^{\frac{1}{2}}}\right) e^{-\frac{x_1^2+x_2^2+y^2}{2}} y^{h-1} dx_1 dx_2 dy\,. \tag{3.6.9}$$

Performing the change of coordinates $(u,v) = \frac{1}{(x_1^2+x_2^2+y^2)^{1/2}}(x_1,x_2)$, which is the same one that we performed in Lemma 3.3.5, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbb{R}_+} f\left(\frac{y}{(x_1^2+x_2^2+y^2)^{\frac{1}{2}}}\right) e^{-\frac{x_1^2+x_2^2+y^2}{2}} y^{h-1} dx_1 dx_2 dy \\
&= \int_{\mathbb{D}\times\mathbb{R}_+} \frac{f\left(\sqrt{1-u^2-v^2}\right)}{(1-u^2-v^2)^2} e^{-\frac{y^2}{2(1-u^2-v^2)}} y^{h+1} du dv dy \\
&\overset{\sqrt{1-u^2-v^2}t=y}{=} \int_{\mathbb{D}\times\mathbb{R}_+} f\left(\sqrt{1-u^2-v^2}\right)\left(1-u^2-v^2\right)^{\frac{h}{2}-1} e^{-\frac{t^2}{2}} t^{h+1} du dv dt\,.
\end{aligned} \tag{3.6.10}$$

We can now explicitly compute the integral in t. Moreover, we can express the remaining part of the integral in polar coordinates; namely, we apply the change of variables $u = \rho\cos(\theta), v = \rho\sin(\theta)$, obtaining that:

$$\begin{aligned}
&\int_{\mathbb{D}\times\mathbb{R}_+} f\left(\sqrt{1-u^2-v^2}\right)\left(1-u^2-v^2\right)^{\frac{h}{2}-1} e^{-\frac{t^2}{2}} t^{h+1} du dv dt \\
&= 2\pi 2^{\frac{h}{2}}\Gamma\left(\frac{h}{2}+1\right)\int_0^1 \rho f(\sqrt{1-\rho^2})\left(1-\rho^2\right)^{\frac{h}{2}-1} d\rho \\
&\overset{\sqrt{1-\rho^2}=w}{=} 2\pi 2^{\frac{h}{2}}\Gamma\left(\frac{h}{2}+1\right)\int_0^1 f(w) w^{h-1} dw\,,
\end{aligned} \tag{3.6.11}$$

here $\Gamma(x)$ is the gamma function (3.3.11). Thus, in order to obtain the estimate (3.6.5), we have to deduce an upper bound for

$$\sup_{0<h<1} \frac{\int_0^1 e^{a(h)w} w^{h-1} dw}{\int_0^1 w^{h-1} dw}\,. \tag{3.6.12}$$

For any $0 < h < 1$, we can explicitly compute the denominator as

$$\int_0^1 w^{h-1} dw = \frac{1}{h}\,. \tag{3.6.13}$$

Moreover, we can give an upper bound on the numerator as

$$\begin{aligned}
\int_0^1 e^{a(h)w} w^{h-1} dw &\overset{a(h)w=r}{=} \frac{1}{a(h)^h}\int_0^{a(h)} e^r r^{h-1} dr = \frac{1}{a(h)^h}\left(\int_0^1 e^r r^{h-1} dr + \int_1^{a(h)} e^r r^{h-1} dr\right) \\
&\leqslant \frac{1}{a(h)^h}\left(e\int_0^1 r^{h-1} dr + \int_1^{a(h)} e^r dr\right) \leqslant \frac{e}{a(h)^h h} + \frac{e^{a(h)}}{a(h)^h}\,.
\end{aligned} \tag{3.6.14}$$

Combining (3.6.13)-(3.6.14), with our choice of $a(h) = -\frac{1}{2}\log(h) + 1$, we deduce that there exists a constant K independent of h such that (3.6.5) holds.

□

Proof of Lemma 3.4.3

Lemma 3.6.4. *Let $M_\beta^V = (J_\beta^V)^{-1}(\{0\})$ be the set of minimizers of J_β^V. Then, $\beta \mapsto M_\beta^V$ is continuous in the sense that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 \leqslant h \leqslant \delta$, $M_{\beta+h}^V \subset (M_\beta^V)^\varepsilon$, where for $A \subset \mathcal{P}(\mathbb{T})$ we denote $A^\varepsilon = \{\mu \in \mathcal{P}(\mathbb{T}) \mid d(\mu, A) \leqslant \varepsilon\}$.*

Proof. Let $\varepsilon > 0$. We are going to show that for $h > 0$ small enough, we have

$$-\inf_{\left[(M_\beta^V)^\varepsilon\right]^c} J_{\beta+h}^V < 0,$$

which will ensure that $J_{\beta+h}^V > 0$ on $\left[(M_\beta^V)^\varepsilon\right]^c$, thus $\left[(M_\beta^V)^\varepsilon\right]^c \subset \left[(M_{\beta+h}^V)\right]^c$, and hence the conclusion.

By the large deviations principle for $(\mu_N)_{N \text{ even}}$ under $\mathbb{P}_{AL,N}^{V,\beta}$, Corollary 3.2.5, since $\left[(M_\beta^V)^\varepsilon\right]^c$ is open, we have

$$\begin{aligned}
-\inf_{\left[(M_\beta^V)^\varepsilon\right]^c} J_{\beta+h}^V &\leqslant \liminf_{N \text{ even}} \frac{1}{N} \log \mathbb{P}_{AL,N}^{V,\beta+h}\left(\mu_N(\mathcal{E}) \in \left[(M_\beta^V)^\varepsilon\right]^c\right) \\
&= \liminf_{N \text{ even}} \frac{1}{N} \log \mathbb{P}_{AL,N}^{V,\beta+h}\left(d(\mu_N(\mathcal{E}), M_\beta^V) > \varepsilon\right) \\
&\leqslant \limsup_{N \text{ even}} \frac{1}{N} \log \mathbb{P}_{AL,N}^{V,\beta+h}\left(d(\mu_N(\mathcal{E}), M_\beta^V) \geqslant \varepsilon\right) .
\end{aligned}$$

Since for any positive h and $\boldsymbol{\alpha} \in \mathbb{D}^N$ $\prod_{j=1}^N (1-|\alpha_j|^2)^h \leqslant 1$,we deduce that for any $A \subseteq \mathbb{D}^N$

$$\frac{1}{N} \log\left(\mathbb{P}_{AL,N}^{V,\beta+h}(A)\right) \leqslant \frac{1}{N}\left(\log\left(\frac{Z_N^{AL}(V,\beta)}{Z_N^{AL}(V,\beta+h)}\right) + \log\left(\mathbb{P}_{AL,N}^{V,\beta}(A)\right)\right) , \tag{3.6.15}$$

we recall that $\mathbb{P}_{AL,N}^{V,\beta}$ is defined in (3.1.14).

Applying the previous inequality in the case $A = \{d(\mu_N(\mathcal{E}), M_\beta^V) \geqslant \varepsilon\}$, we conclude that

$$-\inf_{\left[(M_\beta^V)^\varepsilon\right]^c} J_{\beta+h}^V \leqslant \limsup_{N\to\infty} \frac{1}{N}\left(\log\left(\frac{Z_N^{AL}(V,\beta)}{Z_N^{AL}(V,\beta+h)}\right) + \log\left(\mathbb{P}_{AL,N}^{V,\beta}(d(\mu_N(\mathcal{E}), M_\beta^V) \geqslant \varepsilon)\right)\right) . \tag{3.6.16}$$

From Corollary 3.2.5, we deduce that there exists a positive constant c, independent of h, such that

$$\limsup_{N\to\infty} \frac{1}{N} \mathbb{P}_{AL,N}^{V,\beta}(d(\mu_N(\mathcal{E}), M_\beta^V) \geqslant \varepsilon)) \leqslant -\inf_{\left[(M_\beta^V)^\varepsilon\right]^c} J_\beta^V < -c . \tag{3.6.17}$$

Thus, to conclude we have just to prove that the function $g(\beta) = \lim_{N\to\infty} \frac{1}{N} \log\left(Z_N^{AL}(V,\beta)\right)$ is continuous in β. Actually, we prove that this function is convex in β. Let $1/p + 1/q = 1$, and $\beta_1, \beta_2 \in \mathbb{R}_+$ then

$$\begin{aligned}
Z_N^{AL}\left(V, \frac{\beta_1}{p} + \frac{\beta_2}{q}\right) &= \int_{\mathbb{D}^N} \prod_{j=1}^N (1-|\alpha_j|^2)^{\frac{\beta_1}{p} + \frac{\beta_2}{q} - 1} \exp(-\mathrm{Tr}(V(\mathcal{E}))) d^2\boldsymbol{\alpha} \\
&= \int_{\mathbb{D}^N} \prod_{j=1}^N (1-|\alpha_j|^2)^{\frac{\beta_1 - 1}{p} + \frac{\beta_2 - 1}{q}} \exp\left(-\left(\frac{1}{p} + \frac{1}{q}\right)\mathrm{Tr}(V(\mathcal{E}))\right) d^2\boldsymbol{\alpha} \\
&\leqslant Z_N^{AL}(V,\beta_1)^{\frac{1}{p}} Z_N^{AL}(V,\beta_2)^{\frac{1}{q}} ,
\end{aligned} \tag{3.6.18}$$

where in the last inequality we used Hölder inequality. This implies that

$$g\left(\frac{\beta_1}{p} + \frac{\beta_2}{q}\right) \leqslant \frac{1}{p} g(\beta_1) + \frac{1}{q} g(\beta_2) , \tag{3.6.19}$$

thus $g(\beta)$ is convex, and so continuous, for $\beta > 0$. We can now choose h is such a way that

$$\left| \limsup_{N\to\infty} \frac{1}{N} \log\left(\frac{Z_N^{AL}(V,\beta)}{Z_N^{AL}(V,\beta+h)} \right) \right| < c\,,$$

so we obtain that

$$\inf_{\left[(M_\beta^V)^\varepsilon\right]^c} J_{\beta+h}^V > 0\,. \tag{3.6.20}$$

□

Acknowledgements
This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while G.M. participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 20-21 semester "Universality and Integrability in Random Matrix Theory and Interacting Particle Systems". G.M. has received funding from the European Union's H2020 research and innovation programme under the Marie Skłodowska–Curie grant No. 778010 *IPaDEGAN*.

R.M. has received funding from the European Research Council (ERC) under the European Union Horizon 2020 research and innovation program (grant agreement No. 884584).

We thank the two anonymous referees for their comments and suggestions.

Chapter 4

CLT for real β-ensembles at high temperature

Abstract We establish a central limit theorem for the fluctuations of the empirical measure in the β-ensemble of dimension N at a temperature proportional to N and with confining smooth potential. The space of test functions for which the CLT holds includes $\mathcal{C}^1$, vanishing functions at infinity. It is obtained by the inversion of an operator which is a pertubation of a Sturm-Liouville operator. The method that we use is based on a change of variables introduced in [BFG15] and in [Shc14b].

4.1 Introduction and main result

The β-ensemble of dimension $N \geqslant 1$ with parameter $\beta > 0$ and potential V is the probability measure on $\mathbb{R}^N$ given by

$$d\mathbb{P}_N^{V,\beta}(x_1, \dots, x_N) = \frac{1}{Z_N^{V,\beta}} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N \,. \tag{4.1.1}$$

The potential V has to be chosen so that the partition function

$$Z_N^{V,\beta} = \int_{\mathbb{R}^N} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N$$

is finite. This is the case for example if for some $\beta' > \max(1, \beta)$,

$$\liminf_{|x|\to\infty} \frac{V(x)}{N\beta' \log|x|} > 1 \,, \tag{4.1.2}$$

see [AGZ10, equation (2.6.2)]. The parameter β, which is allowed to depend on N, is the so-called inverse temperature.

Under the special choice of $V_G(x) = \dfrac{x^2}{2}$, the measure (4.1.1) can be seen as the joint law of the (unordered) eigenvalues of certain matrix models:

- For $\beta = 1$ (resp. $\beta = 2$), it is the law of the eigenvalues of the Gaussian orthogonal ensemble (resp. Gaussian unitary ensemble), [AGZ10][Theorem 2.5.2].
- For general $\beta > 0$, potentially depending on N, it is the law of the spectrum of certain tri-diagonal random matrices as shown by Dumitriu and Edelman in [DE02].

We consider here the *high temperature regime* where β scales as $1/N$, and write $\beta = \frac{2P}{N}$ for some $P > 0$. The corresponding measure is therefore

$$d\mathbb{P}_N^{V,\frac{2P}{N}}(x_1,\dots,x_N) = \frac{1}{Z_N^{V,\frac{2P}{N}}} \prod_{i<j} |x_i - x_j|^{\frac{2P}{N}} e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N \,, \tag{4.1.3}$$

with partition function

$$Z_N^{V,\frac{2P}{N}} = \int_{\mathbb{R}^N} \prod_{i<j} |x_i - x_j|^{\frac{2P}{N}} e^{-\sum_{i=1}^N V(x_i)} dx_1 \dots dx_N \,. \tag{4.1.4}$$

As a consequence of [GZ19], under $\mathbb{P}_N^{V,\frac{2P}{N}}$, the sequence of empirical measures

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$$

satisfies a large deviation principle at speed N with strictly convex, good rate function. As a consequence, $\hat{\mu}_N$ converges almost surely in distribution towards a deterministic measure μ_P^V as N goes to infinity, meaning that almost surely, for every bounded continuous $f : \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} f d\hat{\mu}_N \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} f d\mu_P^V \,.$$

The limiting measure μ_P^V can be seen to have a density ρ_P^V which satisfies for almost every $x \in \mathbb{R}$

$$V(x) - 2P \int_{\mathbb{R}} \log|x-y| \rho_P^V(y) dy + \log \rho_P^V(x) = \lambda_P^V \,, \tag{4.1.5}$$

where λ_P^V is constant (see [GM22, Lemma 3.2] for example).

The β-ensemble in the regime $\beta N \underset{N\to\infty}{\longrightarrow} 2P > 0$ has drawn a lot of attention from the random matrix and statistical physics communities lately. This regime was first considered by [CL97] with the study of Dyson Brownian motion with vanishing repulsive coefficient scaled like $\frac{1}{N}$. Gases of vortices were also studied with temperature proportional to N in [BG99]. The limiting density was then described in the case of the quadratic potential in [ABG12], as a crossover between the Wigner semicircle law (fixed $\beta > 0$ case) and the Gaussian density (case $\beta = 0$). The fluctuations of the eigenvalues in the bulk and at the edge of a configuration were studied for example in [BGP15],[NT18],[NT20],[Pak18], [Lam21]. These fluctuations were shown to be described by Poisson statistics in this regime. Recently, Spohn uncovered in [Spo20c] a link between the study of the Classical Toda chain and the β-ensemble in the high temperature regime, showing that the limiting density of states of the classical Toda chain, distributed according to the generalized Gibbs ensemble with polynomial potential, can be computed by means of the limiting empirical measure of the β-ensemble at high temperature. In [Maz22], the author established this relation using the matrix representation of the β-ensemble and a moment method, and in [GM22] the authors proved a large deviation principle for the empirical measure of the Toda chain, establishing the previous result for potentials with polynomial growth. See also [Spo22b],[GM23],[MM23b] for a similar link between the Ablowitz-Ladik lattice and the circular β-ensemble at high temperature. This relation can be further pushed to compute the limiting *currents* of the Toda chain through the central limit theorem for the empirical measure in the β ensemble. The computation of these currents is a crucial step to the derivation of a hydrodynamic equation for the Toda chain, and to the analysis of the correlations of the locally conserved quantities at equilibrium through *linearized hydodynamics*, see [Spo21].

The Central Limit Theorem for the fluctuations of the linear statistics of β-ensembles was first established by [Joh98] for $\beta = 2$ and polynomial potential, then generalized and further developed in the

regime where β is fixed in [Shc13], [BG13a], [BG13b],[BLS18],[LLW19]. Also an optimal local law was found in this regime in [BMP22]. The CLT was obtained in the high-temperature regime $\beta N \to 2P > 0$ by Nakano and Trinh in [NT18, Theorem 4.9] for quadratic V, relying on the tridiagonal representation for the β-ensemble with quadratic potential in [DE02]. In [HL21], the authors prove the CLT in the case of the *circular* β-ensemble at high temperature with general potential, using a normal approximation method involving the spectral analysis of an operator associated to the limiting covariance structure. Their method allowed them to derive a Berry-Esseen bound, *i.e.* a speed of convergence of the fluctuations towards a Gaussian variable.

In this paper, we adapt part of the arguments of [HL21] to our setup. More precisely, we show that for a class of regular, convex potentials V satisfying a growth condition of the type

$$\lim_{|x|\to\infty} \frac{V''(x)}{V'(x)^2} = 0\,,$$

denoting $\nu_N = \hat{\mu}_N - \mu_P^V$ and considering test functions f belonging to the range of a certain integro-differential operator, the scaled fluctuations of $\hat{\mu}_N$, defined by

$$\sqrt{N}\nu_N(f) := \sqrt{N}\left(\int_{\mathbb{R}} f d\mu_N - \int_{\mathbb{R}} f d\mu_P^V\right),$$

converge in law towards centered Gaussian law with variance depending on f.

When considering the fixed temperature regime, *i.e.* β fixed, one has to renormalize the x_i's by $\sqrt{N}$. It is shown in [AGZ10][Theorem 2.6.1] that the measure

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i/\sqrt{N}}$$

satisfies a large deviation principle, and the limiting measure is characterized in [AGZ10][Lemma 2.6.2] by an equation similar to (4.1.5). In fact, the term $\log \rho_P^V$ in the left-hand side of (4.1.5) is the only difference in the equation characterizing the limiting measure in the fixed β case. We point out the very similar characterization of the equilibrium measure corresponding to the minimization problem arising in [BGK16]. There again, the limiting measure is compactly supported. The term $\log \rho_P^V$ is of prime importance because its presence implies that the support of ρ_P^V is the whole real line. It leads to technicalities to deal with the behavior at infinity of most of the associated objects, namely dealing with weighted Lebesgue spaces $L^2(\mu_P^V)$ and the corresponding Sobolev spaces $H^k(\mu_P^V)$.

Our strategy is based on a change of variables in the partition function $Z_N^{V,\frac{2P}{N}}$ (4.1.4), used for the β-ensemble at fixed temperature introduced in [BFG15] and [Shc14b], and used in [Gui19] and in [BGK16] to derive the loop equations and in [BLS18] to derive a CLT in the β-ensemble with β fixed. The outline of the argument goes as follows: Take $\phi : \mathbb{R} \to \mathbb{R}$ smooth, vanishing fast enough at infinity, and do the change of variables in $Z_N^{V\frac{2P}{N}}$, $x_i = y_i + \frac{t}{\sqrt{N}}\phi(y_i)$, $1 \leqslant i \leqslant N$, to get

$$Z_N^{V,\frac{2P}{N}} = \int_{\mathbb{R}^N} \prod_{i<j} \left| y_i - y_j + \frac{t}{\sqrt{N}}(\phi(y_i) - \phi(y_j)) \right|^{2P/N} e^{-\sum_{i=1}^N V\left(y_i + \frac{t}{\sqrt{N}}\phi(y_i)\right)} \prod_{i=1}^N \left(1 + \frac{t}{\sqrt{N}}\phi'(y_i)\right) d^N\mathbf{y}\,.$$

Expanding the different terms in this integral, one gets

$$Z_N^{V,\frac{2P}{N}} = \int_{\mathbb{R}^N} \prod_{i<j} |y_i - y_j|^{\frac{2P}{N}} e^{-\sum_{i=1}^N V(y_i)} e^{\frac{t}{\sqrt{N}}\left[\frac{2P}{N}\sum_{i<j}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j} + \sum_{i=1}^N \left(\phi'(y_i) - V'(y_i)\phi(y_i)\right)\right]} e^{-\frac{t^2}{2}\sigma_N^2(\phi)} d^N\mathbf{y}\,,$$

where the term $\sigma_N^2(\phi)$ converges towards a limiting variance $\sigma^2(\phi)$ depending on ϕ, P and V. After dividing both parts of the equation by $Z_N^{V,P}$, and because of equation (4.1.5) characterizing μ_P^V, one can deduce from the last equation the convergence of the Laplace transform

$$\mathbb{E}\left[e^{t\sqrt{N}(\nu_N(\Xi\phi) + \text{error term})}\right] \underset{N\to\infty}{\longrightarrow} \exp\left(\frac{t^2}{2}\sigma^2(\phi)\right), \tag{4.1.6}$$

where Ξ is a linear operator acting on test functions and defined by

$$(\Xi\phi)(x) = 2P\int_{\mathbb{R}} \frac{\phi(x)-\phi(y)}{x-y} d\mu_P^V(y) + \phi'(x) - V'(x)\phi(x)\,. \tag{4.1.7}$$

Once the error term is taken care of, (4.1.6) shows the central limit theorem for test functions of the form $\Xi\phi$. Following [HL21], the operator $\mathcal{L}$ given by

$$\mathcal{L}\phi = \Xi\phi' \tag{4.1.8}$$

can be analyzed using Hilbert space techniques. In particular, the operator $\mathcal{L}$, seen as an unbounded operator of the Hilbert space

$$\mathsf{H} = \left\{ u \in L^2(\mu_P^V) \;\middle|\; u' \in L^2(\mu_P^V), \int_{\mathbb{R}} u\rho_P^V dx = 0 \right\}, \qquad \langle u, v\rangle_{\mathsf{H}} = \left\langle u', v'\right\rangle_{L^2(\rho_P^V)}\,,$$

can be decomposed as

$$-\mathcal{L} = \mathcal{A} + 2P\mathcal{W}\,,$$

where $\mathcal{A}$ is a positive Sturm-Liouville operator and $\mathcal{W}$ is positive and self-adjoint. Such a writing allows us to show that $-\mathcal{L}$ is invertible, see Theorem 4.6.7.

We now state the assumptions we make on the potential V. Recall that a probability measure μ supported on $\mathbb{R}$ satisfies the Poincaré inequality if there exists $C > 0$ such that for all $f \in \mathcal{C}^1(\mathbb{R})$ with compact support:

$$\mathrm{Var}_\mu(f) := \int_{\mathbb{R}} \left(f - \int_{\mathbb{R}} f d\mu \right)^2 d\mu \leqslant C \int f'^2 d\mu\,. \tag{4.1.9}$$

Assumptions 4.1.1. *The potential V satisfies:*

i) $V \in \mathcal{C}^3(\mathbb{R})$, $V(x) \underset{|x|\to+\infty}{\longrightarrow} +\infty$, $|V'(x)| \underset{|x|\to+\infty}{\longrightarrow} +\infty$ *and is such that μ_P^V satisfies the Poincaré inequality* (4.1.9).

ii) For all polynomial $Q \in \mathbb{R}[X]$ and $\alpha > 0$, $Q\Big(V'(x)\Big)e^{-V(x)} = \underset{|x|\to\infty}{o}(x^{-\alpha})$.

iii) Furthermore, for any sequence x_N such that $|x_N|$ goes to infinity, and for all real $a < b$, we have, as N goes to infinity,

$$\frac{1}{V'(x_N)^2} \sup_{a\leqslant x\leqslant b} |V''(x_N + x)| \underset{N\to\infty}{\longrightarrow} 0\,.$$

iv) The function $\dfrac{1}{V'^2}$ is integrable at infinity. $\dfrac{V''(x)}{V'(x)} = \underset{|x|\to\infty}{O}(1)$ and $\dfrac{V^{(3)}(x)}{V'(x)} = \underset{|x|\to\infty}{O}(1)$.

Taking $V = V_{\mathrm{conv}} + \phi$ with V_{conv}, $\phi \in \mathcal{C}^3(\mathbb{R})$ such that $\phi^{(k)}$ is bounded for $k = 0, \dots, 3$, V_{conv} is convex with $|V'_{\mathrm{conv}}| \to +\infty$ at infinity, satisfying hypotheses *ii)*, *iii)* and *iv)* such that there exists $\varepsilon > 0$ such that $V_{\mathrm{conv}} - 2Pf_\varepsilon$ is convex (see Lemma 4.2.4), then V satisfies Assumptions 4.1.1.

Because *i)* implies that V goes to infinity faster than linearly, we will see that it ensures exponential decay at infinity of ρ_P^V. Recalling the sufficient condition for $\mathbb{P}_N^{V,\frac{2P}{N}}$ of equation (4.1.2) to be defined, this first assumption implies that there exists $\alpha > 0$ such that $\liminf_{|x|\to\infty} \frac{V(x)}{|x|} > \alpha$. This guarantees in particular that the β-ensemble (4.1.3) is well-defined for all $N \geqslant 1$ and $P \geqslant 0$. We will use the fact that μ_P^V satisfies the Poincaré inequality to ensure that H endowed with $\langle\cdot,\cdot\rangle_{\mathsf{H}}$ is a Hilbert space.

The second assumption ensures that any power of V' and V'' is in $L^2(\mu_P^V)$ and ρ_P^V, which behaves like e^{-V} up to a sub-exponential factor, belongs to the Sobolev space $H^2(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R})$. Indeed, for $k \leqslant 2$, using *iv)*, ${\rho_P^V}^{(k)}$ behaves at infinity like $(V')^k\rho_P^V$ as shown in Lemma 4.2.2 which is in $L^2(\mathbb{R})$ by assumption *ii)*.

Assumption *iii)* will be used to localize the minimum/maximum point of a typical configuration $(x_1, \dots, x_N)$ following the law $\mathbb{P}_N^{V,\frac{2P}{N}}$: this will be done in Corollary 4.4.2, which comes as a consequence of [Lam21][Theorem 3.4]. More precisely, Corollary 4.4.2 establishes that for sequences $(\alpha_N^+)_N, (\alpha_N^-)_N$ going to infinity, the random variables

$$\alpha_N^+ \left(\max_{1 \leqslant j \leqslant N} x_j - E_N^+ \right) \qquad \text{and} \qquad \alpha_N^- \left(\max_{1 \leqslant j \leqslant N} x_j - E_N^- \right)$$

converge in distribution. For large N, the scalars E_N^+ and E_N^- can thus be seen as the edges of a typical configuration. Furthermore,

$$V(E_N^\pm) \sim \log N \,. \tag{4.1.10}$$

We refer to Section 4.4 for detailed statements. The final step in the proof of Theorem 4.1.3 consists in lifting the result of Proposition 4.5.1 from compactly supported functions to more general functions.

We use Assumption *iv)* to control integral remainders in the proof of Theorem 4.7.1, ensuring that $\mathcal{L}^{-1}$ is regular enough *i.e.* that for sufficiently smooth functions f, $\left(\mathcal{L}^{-1} f\right)' \in H^2(\mathbb{R})$.

We will need another technical assumption to ensure that Taylor remainders arising in the proof of Theorem 4.5.2 are negligible.

Assumption 4.1.2. *With the notations of Theorem 4.4.1, we have*

$$\sup_{d(x, I_N) \leqslant 1} \left| V^{(3)}(x) \right| = o(N^{1/2}) \,,$$

where $I_N = \left[E_N^- - 2; E_N^+ + 2\right]$.

Again taking $V = V_{\text{conv}} + \phi$ with $V_{\text{conv}}, \phi \in \mathcal{C}^3(\mathbb{R})$ such that $\phi^{(k)}$ is bounded for $k = 0, \dots, 3$, V_{conv} is convex with $|V'_{\text{conv}}| \to +\infty$ at infinity, satisfying hypotheses *ii)*, *iii)*, *iv)* and Assumption 4.1.2, such that there exists $\varepsilon > 0$ such that $V_{\text{conv}} - 2P f_\varepsilon$ is convex (see Lemma 4.2.4), then V satisfies Assumptions 4.1.1.

This last hypothesis is satisfied whenever V_{conv} is sufficiently convex in a compact centered at 0 and is made so that V_{conv} compensate the *small* lack of concavity in the bulk of a function behaving like $-\log|x|$ at infinity (note that assuming that $V'' \geqslant \alpha$ for some $\alpha > 0$ is sufficient). This is the reason why we introduce the function f_ε, which are functions with required growth at infinity and with second derivative as small as desired. The main point that needs to be checked is that the measure μ_P^V satisfies the Poincaré inequality, this will be done in Proposition 4.2.6.

The type of potential V_{conv} that one can consider is typically the convex polynomials or $\cosh(\alpha x)$. On the other hand a *scaled* potentials like e^{x^2} which have a faster growing derivative at infinity and so doesn't satisfy assumptions *iii)* and *iv)*.

We are now able to state the main result, *ie* the central limit theorem for functions belonging to the image of the operator $\mathcal{L}$ introduced in (4.1.8).

Theorem 4.1.3. *Assume that V satisfies Assumptions 4.1.1 and Assumption 4.1.2. Then for ϕ verifying the following conditions:*

- $\phi \in \mathcal{C}^1(\mathbb{R})$
- *there exists* $\varepsilon > 0$, $\phi(x) = \underset{|x| \to \infty}{O}(x^{-\frac{1}{2}-\varepsilon})$ *and* $\phi'(x) = \underset{|x| \to \infty}{O}(x^{\frac{1}{2}-\varepsilon})$ *at infinity*
- $\displaystyle\int_{\mathbb{R}} \phi(x) d\mu_P^V(x) = 0$

we have the convergence in law

$$\sqrt{N} \nu_N(\phi) \to N\left(0, (\sigma_P^V)^2(\phi)\right) \tag{4.1.11}$$

where the limiting variance $(\sigma_P^V)^2(\phi)$ *is given by*

$$(\sigma_P^V)^2(\phi) = \langle \phi, \mathcal{L}^{-1}\phi \rangle_{\mathsf{H}} = \int_{\mathbb{R}} \left(\big(\mathcal{L}^{-1}\phi\big)''(x)^2 + V''(x)\big(\mathcal{L}^{-1}\phi\big)'(x)^2 \right) d\mu_P^V(x) + P \iint\limits_{\mathbb{R}^2} \left(\frac{\big(\mathcal{L}^{-1}\phi\big)'(x) - \big(\mathcal{L}^{-1}\phi\big)'(y)}{x-y} \right)^2 d\mu_P^V(x) d\mu_P^V(y)\,. \quad (4.1.12)$$

Remark 4.1.4. *Since* $\nu_N(\phi + c) = \nu_N(\phi)$ *for all constant* $c \in \mathbb{R}$*, the assumption* $\int_{\mathbb{R}} \phi(x) d\mu_P^V = 0$ *can be dropped by replacing* ϕ *by* $\phi - \int_{\mathbb{R}} \phi(x) d\mu_P^V$ *in the expression of the limiting variance.*

As a tool to deal with the error term of equation (4.1.6), we establish a concentration inequality for the empirical measure. This inequality is stated in terms of the following distance over the set of probability distributions $\mathcal{P}(\mathbb{R})$.

For $\mu, \mu' \in \mathcal{P}(\mathbb{R})$ we define the distance

$$d(\mu, \mu') = \sup_{\substack{\|f\|_{\mathrm{Lip}} \leqslant 1 \\ \|f\|_{1/2} \leqslant 1}} \left\{ \left| \int f d\mu - \int f d\mu' \right| \right\}, \quad (4.1.13)$$

where $\|f\|_{\mathrm{Lip}}$ denotes the Lipschitz constant of f, and $\|f\|_{1/2}^2 = \int_{\mathbb{R}} |t| \, |\mathcal{F}[f](t)|^2 \, dt$, where $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R})$ which takes the following expression $\mathcal{F}[f](t) = \int_{\mathbb{R}} f(x) e^{-\mathrm{i}tx} dx$ for f in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

We then have

Theorem 4.1.5. *There exists* $K \in \mathbb{R}$ *(depending on* P *and on* V*), such that for any* $N \geqslant 1$ *and* $r > 0$*,*

$$\mathbb{P}_N^{V, \frac{2P}{N}} \left(d(\hat{\mu}_N, \mu_P^V) > r \right) \leqslant e^{-Nr^2 \frac{P\pi^2}{2} + 5P \log N + K}. \quad (4.1.14)$$

This result is the analog of [HL21, Theorem 1.4].

The paper is organized as follows. In Section 4.2 we discuss the regularity of the equilibrium density ρ_P^V under Assumption 4.1.1. In Section 4.3 we prove Theorem 4.1.5. Section 4.4 is dedicated to the localization of the edge of a typical configuration, mentioned in the discussion preceding the statement of Assumption 4.1.2. We next prove in Section 4.5 the convergence of the Laplace transform of $\sqrt{N}\nu_N(\mathcal{L}\phi)$ for general functions ϕ which establishes Theorem 4.1.3 for functions of the form $\mathcal{L}\phi$. Section 4.6 is dedicated to the diagonalization and inversion of $\mathcal{L}$ given by (4.1.8). In Section 4.7, we show regularity properties of $\mathcal{L}^{-1}$ to establish Theorem 4.1.3. We detail in Appendix 4.8 elements of proof for the spectral theory of Schrödinger operators, used in Section 4.6.

Acknowledgments The authors wish to thank Alice Guionnet and Karol Kozlowski for their helpful suggestions. We also thank Arnaud Debussche for pointing out the link with Schrödinger operators theory and Gaultier Lambert for pointing out [Lam21]. We would also like to thank Jeanne Boursier, Corentin Le Bihan and Jules Pitcho for their intuition about the regularity of the inverse operator. We would like to thank Jean-Christophe Mourrat for telling us about a more general framework for Poincaré inequalities.

4.2 Regularity of the equilibrium measure and Hilbert transform

In this section, we discuss the regularity properties of the equilibrium density ρ_P^V, namely its decay at infinity and its smoothness, and give formulas for its two first derivatives.

The Hilbert transform, whose definition we recall, plays a central role in the analysis of the equilibrium measure. It is first defined on the Schwartz class through $\forall \phi \in \mathcal{S}(\mathbb{R})$, $\forall x \in \mathbb{R}$,

$$\mathcal{H}[\phi](x) := \fint_{\mathbb{R}} \frac{\phi(t)}{t-x} dt = \lim_{\varepsilon \downarrow 0} \int_{|t-x|>\varepsilon} \frac{\phi(t)}{t-x} dt = \int_0^{+\infty} \frac{\phi(x+t)-\phi(x-t)}{t} dt, \tag{4.2.1}$$

where $\fint$ denotes the Cauchy principal value integral, and then extended to $L^2(\mathbb{R})$ thanks to property *ii)* of Lemma 4.2.1: $\|f\|_{L^2(dx)} = \frac{1}{\pi}\|\mathcal{H}[f]\|_{L^2(dx)}$. The last expression in (4.2.1) is a definition where the integral converges in the classical sense. We also recall the definition of the logarithmic potential U^f of a density of probability $f : \mathbb{R} \to \mathbb{R}$, given for $x \in \mathbb{R}$ by

$$U^f(x) = -\int_{\mathbb{R}} \log|x-y| f(y) dy\,. \tag{4.2.2}$$

Because we assume $f \in L^1(\mathbb{R})$ to be nonnegative, U^f takes values in $[-\infty, +\infty)$. If f integrates the function log, *i.e* $\int_{\mathbb{R}} \log|x| f(x) dx < +\infty$, then U^f takes real values. Additionally, one can check that the logarithmic potential and the Hilbert transform of f are linked through the distributional identity $\left(U^f\right)' = \mathcal{H}[f]$.

We recall in the next lemma some properties of the Hilbert transform that we will use in the rest of the paper.

Lemma 4.2.1 (Properties of the Hilbert transform)**.**

i) Fourier transform: For all $\phi \in L^2(\mathbb{R})$, $\mathcal{F}\Big[\mathcal{H}[\phi]\Big](\omega) = \mathrm{i}\pi sgn(\omega)\mathcal{F}[\phi](\omega)$ for all $\omega \in \mathbb{R}$.

ii) As a consequence, $\frac{1}{\pi}\mathcal{H}$ is an isometry of $L^2(\mathbb{R})$, and $\mathcal{H}$ satisfies on $L^2(\mathbb{R})$ the identity $\mathcal{H}^2 = -\pi^2 I$.

iii) Derivative: For any $f \in H^1(\mathbb{R})$, $\mathcal{H}[f]$ is also $H^1(\mathbb{R})$ and $\mathcal{H}[f]' = \mathcal{H}[f']$.

iv) For all $p > 1$, the Hilbert transform can be extended as a bounded operator $\mathcal{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$.

v) Skew-self adjointness: For any $f, g \in L^2(\mathbb{R})$, $\langle \mathcal{H}[f], g\rangle_{L^2(\mathbb{R})} = -\langle f, \mathcal{H}[g]\rangle_{L^2(\mathbb{R})}$.

Proof. We refer to [Kin09] for the proofs of these properties. □

As a consequence of [GZ19], $\hat{\mu}_N$ converges almost surely under $\mathbb{P}_N^{V,\frac{2P}{N}}$ towards the unique minimizer of the energy-functional $\mathcal{E}_P^V$, defined for $\mu \in \mathcal{P}(\mathbb{R})$ by

$$\mathcal{E}_P^V(\mu) = \begin{cases} \displaystyle\int_{\mathbb{R}} \Big[V + \log\Big(\frac{d\mu}{dx}\Big)\Big] d\mu - P \iint_{\mathbb{R}^2} \log|x-y| d\mu(x) d\mu(y) \text{ if } \mu \ll dx \\ +\infty \text{ otherwise} \end{cases}. \tag{4.2.3}$$

(Here we wrote $\mu \ll dx$ for "μ is absolutely continuous with respect to Lebesgue measure")

Consequently, following [GM22, Lemma 3.2], the density ρ_P^V of μ_P^V satisfies equation (4.1.5), which we rewrite here for convenience.

$$V(x) - 2P \int_{\mathbb{R}} \log|x-y| \rho_P^V(y) dy + \log \rho_P^V(x) = \lambda_P^V \,, \tag{4.2.4}$$

where λ_P^V is a constant (depending on V and P). Using this equation, we will show in the next lemma that ρ_P^V decays exponentially and is twice continuously differentiable via the representation:

$$\forall x \in \mathbb{R},\ \rho_P^V(x) = \exp\Big(- V(x) - 2PU^{\rho_P^V}(x) - \lambda_P^V \Big)$$

In the Gaussian potential case *ie* $V_G(x) = \dfrac{x^2}{2}$, an explicit formula has been found [ABG12]:

$$\rho_P^{V_G}(x) = \frac{\Gamma(P)}{P\sqrt{2\pi}} \frac{\exp\Big(-\dfrac{x^2}{2}\Big)}{\displaystyle\int_0^{+\infty} t^{P-1} e^{-\frac{t^2}{2} + ixt} dt}$$

It has been established in [BGP15] that $\sqrt{P+1}\rho_P^{V_G}(\sqrt{P+1}x)$ converges to the Gaussian distribution when P goes to zero and the semi-circle law when P goes to infinity. So in the Gaussian case, μ_P can be seen as an interpolation between the Gaussian distribution and the semi-circular one. **We now drop the superscript of ρ_P^V and μ_P^V and denote it ρ_P and μ_P for convenience.** In the next lemma, we prove that ρ_P has the same regularity as V.

Lemma 4.2.2. *Under Assumption 4.1.1,*

- *The support of μ_P is $\mathbb{R}$ and there exists a constant C_P^V such that for all $x \in \mathbb{R}$,*

$$\rho_P(x) \leqslant C_P^V (1+|x|)^{2P} e^{-V(x)} \,.$$

- *The density ρ_P is in $\mathcal{C}^3(\mathbb{R})$ and we have*

$$\rho_P' = -\Big(V' + 2P\mathcal{H}[\rho_P]\Big)\rho_P \tag{4.2.5}$$

and

$$\rho_P'' = \Big(- 2P\mathcal{H}[\rho_P]' - V'' + V'^2 + 4P^2\mathcal{H}[\rho_P]^2 + 4PV'\mathcal{H}[\rho_P]\Big)\rho_P \,. \tag{4.2.6}$$

Proof. For the first point, [GM22, Lemma 3.2] establishes that the support of μ_P is the whole real axis, and that under the first condition of Assumptions 4.1.1, we have the bound, valid for all $x \in \mathbb{R}$

$$\rho_P(x) \leqslant \frac{K_P^V}{(1+|x|)^2} \,, \tag{4.2.7}$$

with K_P^V a positive constant. Using (4.2.4) and the fact that

$$\log|x-y| \leqslant \log\big(1+|x|\big) + \log\big(1+|y|\big) \,,$$

we see that for all $x \in \mathbb{R}$,

$$\rho_P(x) \leqslant C_P^V \exp\Big(- V(x) + 2P\log(1+|x|)\Big) \,, \tag{4.2.8}$$

with

$$C_P^V = \exp\Big(2P\int_{\mathbb{R}} \log(1+|y|)\rho_P(y)dy + \lambda_P^V\Big)$$

which is indeed finite by (4.2.7).

For the second point, we use that $\big(U^{\rho_P}\big)' = \mathcal{H}[\rho_P]$ weakly and equation (4.2.4) to conclude on the distributional identity

$$\rho_P' = \Big(- V' - 2P\mathcal{H}[\rho_P]\Big)\rho_P\,.$$

By the second point of Assumption 4.1.1, $V'(x)e^{-V(x)+2P\log(1+|x|)} = o(x^{-1})$ as $|x| \to \infty$, thus by (4.2.8), $V'\rho_P \in L^2(\mathbb{R})$. Also since ρ_P is $L^2(\mathbb{R})$ and bounded, we deduce, by using that $\mathcal{H}\big[L^2(\mathbb{R})\big] = L^2(\mathbb{R})$, that $\mathcal{H}[\rho_P]\rho_P \in L^2(\mathbb{R})$. Adding up these terms we get $\rho_P \in H^1(\mathbb{R})$. Because $\mathcal{H}[\rho_P]' = \mathcal{H}[\rho_P']$ in a weak sense by Lemma 4.2.1, $\mathcal{H}[\rho_P] \in H^1(\mathbb{R})$. By the classical fact that $H^1(\mathbb{R})$ is contained in the set of 1/2-Hölder functions $\mathcal{C}^{1/2}(\mathbb{R})$, we have $\mathcal{H}[\rho_P] \in \mathcal{C}^{1/2}(\mathbb{R})$ and so $U^{\rho_P} \in \mathcal{C}^{1,1/2}(\mathbb{R})$, the set of functions in $\mathcal{C}^1(\mathbb{R})$ with derivative of class 1/2-Hölder.
Using the fact that V is continuously differentiable, the previous equation for the weak derivative of ρ_P then ensures that $\rho_P \in \mathcal{C}^1(\mathbb{R})$ and equation (4.2.5) holds in the strong sense.

Differentiating (in a weak sense) equation (4.2.5) we obtain

$$\rho_P'' = \Big(- 2P\mathcal{H}[\rho_P]' - V'' + V'^2 + 4P^2\mathcal{H}[\rho_P]^2 + 4PV'\mathcal{H}[\rho_P]\Big)\rho_P\,.$$

The three first terms belong to $L^2(\mathbb{R})$ for the same reasons as before. Since $\rho_P \in H^1(\mathbb{R})$n by Lemma 4.2.1*iii)* so is $\mathcal{H}[\rho_P] \in H^1(\mathbb{R})$, it is then bounded over $\mathbb{R}$ hence the two last term are in $L^2(\mathbb{R})$ when multiplied by ρ_P. Finally, we can conclude that $\rho_P \in H^2(\mathbb{R})$ and so that $\mathcal{H}[\rho_P] \in H^2(\mathbb{R})$ with $\mathcal{H}[\rho_P]'' = \mathcal{H}[\rho_P'']$ (in a weak sense). As before, we conclude that $\rho_P \in \mathcal{C}^2(\mathbb{R})$ and that equation (4.2.6) holds in a strong sense. By the exact same method, we can show that $\rho_P \in \mathcal{C}^3(\mathbb{R})$. □

We next show that the Hilbert transform of ρ_P is continuous and decays at infinity.

Lemma 4.2.3. *Let $u \in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} u(t)dt$ exists and $f : t \mapsto tu(t) \in H^1(\mathbb{R})$ then*

$$\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} u(t)dt}{x}.$$

Moreover if $\displaystyle\int_{\mathbb{R}} u(t)dt = 0$, $\int_{\mathbb{R}} f(t)dt$ exists and $g : t \mapsto t^2u(t) \in H^1(\mathbb{R})$, then

$$\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} tu(t)dt}{x^2}.$$

As a consequence, we obtain that $\mathcal{H}[\rho_P](x) \underset{|x|\to\infty}{\sim} x^{-1}$ and the logarithmic potential U^{ρ_P} is Lipschitz bounded, with bounded derivative $\mathcal{H}[\rho_P]$.

Proof. Let $u \in L^2(\mathbb{R})$, such that $\int_{\mathbb{R}} u(t)dt$ exists and $f : t \mapsto tu(t) \in H^1(\mathbb{R})$. Then

$$x\mathcal{H}[u](x) + \int_{\mathbb{R}} u(t)dt = \int_{\mathbb{R}} \Big[\frac{xu(x+t) - xu(x-t)}{2t} + \frac{u(x+t)}{2} + \frac{u(x-t)}{2}\Big]dt = \mathcal{H}[f](x).$$

Since $f \in H^1(\mathbb{R})$, so is $\mathcal{H}[f]$, proving that it goes to zero at infinity. Hence

$$\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} u(t)dt}{x}$$

Moreover if $\displaystyle\int_{\mathbb{R}} u(t)dt = 0$, $\int_{\mathbb{R}} f(t)dt$ exists and $g : t \mapsto t^2u(t) \in H^1(\mathbb{R})$, then by the same argument:

$$x^2\mathcal{H}[u](x) = x\mathcal{H}[f](x) = \mathcal{H}[g](x) - \int_{\mathbb{R}} f(t)dt$$

where $g(t) = t^2u(t)$. We deduce that $\mathcal{H}[u](x) \underset{|x|\to\infty}{\sim} \frac{-\int_{\mathbb{R}} tu(t)dt}{x^2}$ since $\mathcal{H}[g]$ goes to zero at infinity. □

Lemma 4.2.4 (Asymptotic of the logarithmic potential)**.** *We have the following asymptotic expansion at infinity* $U^{\rho_P} = \log|x| + \underset{|x|\to\infty}{O}(1)$.

Proof. Since $\mathcal{H}[\rho_P](x) = x^{-1} + \underset{|x|\to\infty}{O}(x^{-2})$, and recalling that $U^{\rho_P^V}$ (defined by (4.2.2)) satisfies $(U^{\rho_P^V})' = \mathcal{H}[\rho_P^V]$, we deduce the result by integrating $t \mapsto \mathcal{H}[\rho_P^V](t) - 1/t$ in a neighborhood of infinity. □

We conclude this section by stating the Poincaré inequality for the measure μ_P under the assumption that V is a bounded perturbation of a strictly convex potential V_{conv}.

Lemma 4.2.5. *Let* $\varepsilon > 0$*, there exists a function* $f \in \mathcal{C}^2(\mathbb{R})$ *such that* $f_\varepsilon(x) - \log|x| = \underset{|x|\to\infty}{O}(1)$*, and* $\|f_\varepsilon''\|_\infty \leqslant \varepsilon$.

Proposition 4.2.6. *Assume that* $V = V_{\text{conv}} + \phi$*, where* $V_{\text{conv}} \in \mathcal{C}^3(\mathbb{R})$ *with* V_{conv} *convex, such that there exists* $\varepsilon > 0$ *such that* ϕ *is bounded and* $V_{\text{conv}} - 2Pf_\varepsilon$ *is convex (*f_ε *being given by Lemma 4.2.5). Then, the measure* μ_P *satisfies the Poincaré inequality: there exists a constant* $C > 0$ *such that for all* $f \in \mathcal{C}_c^1(\mathbb{R})$*,*

$$\mathrm{Var}_{\mu_P}(f) \leqslant C \int_{\mathbb{R}} |f'|^2 d\mu_P \,. \tag{4.2.9}$$

Proof. We use the fact that if μ_1, μ_2 are two absolutely continuous probability measures supported on $\mathbb{R}$ such that $\dfrac{1}{C} \leqslant \dfrac{d\mu_1}{d\mu_2} \leqslant C$ for some $C > 0$ and μ_1 satisfies Poincaré inequality with constant C_1 then so does μ_2 for some other constant. Indeed, in that case let $f \in \mathcal{C}_c^1(\mathbb{R})$, we have

$$\mathrm{Var}_{\mu_2}(f) = \inf_a \int_{\mathbb{R}} (f-a)^2 \, d\mu_2 \leqslant C \,\mathrm{Var}_{\mu_1}(f) \leqslant C^2 C_1 \int_{\mathbb{R}} f' d\mu_2.$$

Here we take $d\mu_2(x) := \rho_P^V(x)dx$ and we want to compare it to a measure μ_1 supported on $\mathbb{R}$ defined by $d\mu_1(x) = \dfrac{1}{Z}\exp\big(-W(x)\big)dx$ for some convex function W. The measure μ_1 then clearly verifies the Poincaré inequality. This fact comes as a direct consequence of [BBCG08][Corollary 1.9], which states that if a probability measure μ has a log-concave density on $\mathbb{R}$, then it satisfies (4.2.9). With the definition $W := V_{\text{conv}} - 2Pf_\varepsilon$ with $\varepsilon > 0$ such that $V_{\text{conv}} - 2Pf_\varepsilon$ is convex, $W - V - 2PU^{\rho_P}$ is bounded on $\mathbb{R}$. It is then not hard to see that $\dfrac{1}{C} \leqslant \dfrac{d\mu_1}{d\mu_P} \leqslant C$ for some $C > 0$ which allows to conclude that μ_P satisfies the Poincaré inequality.

□

Remark 4.2.7. *We will apply later inequality* (4.2.9) *to more general functions than* $\mathcal{C}_c^1(\mathbb{R})$*, namely functions of the weighted Sobolev space* $H^1(\rho_P)$*, defined in Section 4.6; which can be seen as the completion of* $\mathcal{C}_c^\infty(\mathbb{R})$ *with respect to the norm* $\|u\|_{L^2(\rho_P)} + \|u'\|_{L^2(\rho_P)}$.

4.3 Concentration inequality, proof of Theorem 4.1.5

We prove in this section the concentration Theorem 4.1.5. Its proof is a direct adaptation of Theorem 1.4 of [HL21], which shows the analogous estimate in the circular setup. It is inspired by [MMS14] and based on a comparison between a configuration $\mathbf{x}_N = (x_1, \dots, x_N)$ sampled with $\mathbb{P}_N^{V, \frac{2P}{N}}$ and a regularized version $\mathbf{y}_N = (y_1, \dots, y_N)$, which we describe here.

Definition 4.3.1. $y_1 := x_1$*, and for* $0 \leqslant k \leqslant N-1$*,* $y_{k+1} := y_k + \max\{x_{k+1} - x_k, N^{-3}\}$.

Note that the configuration $\mathbf{y}_N$ given by the previous definition satisfies $y_{k+1} - y_k \geqslant N^{-3}$, and $\mathbf{y}_N$ is close to $\mathbf{x}_N$ in the sense that

$$\sum_{k=1}^{N} |x_k - y_k| \leqslant \frac{1}{2N}\,. \tag{4.3.1}$$

Indeed, by construction we have $|x_k - y_k| = y_k - x_k \leqslant (k-1)N^{-3}$, and we get the bound by summing these inequalities.
The key point of the proof of Theorem 4.1.5 is comparing the empirical measure $\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$, where $\mathbf{x}_N$ follows $\mathbb{P}_N^{V,\frac{2P}{N}}$, to the regularized measure

$$\widetilde{\mu}_N := \lambda_{N^{-5}} * \frac{1}{N}\sum_{i=1}^{N} \delta_{y_i}, \tag{4.3.2}$$

ie the convolution of $\lambda_{N^{-5}}$ and the empirical measure, where $\lambda_{N^{-5}}$ is the uniform measure on $[0, N^{-5}]$. The interest of introducing the measure $\widetilde{\mu}_N$ is that it is close to $\hat{\mu}_N$, while having a finite energy $\mathcal{E}_P^V(\widetilde{\mu}_N)$, given by (4.2.3). Finally, notice that the empirical measure doesn't change when reordering $x_1, \ldots, x_N$, and thus we do not lose in generality for our purposes in assuming that $x_1 \leqslant \ldots \leqslant x_N$ in definition 4.3.1.

We now introduce a distance on $\mathcal{P}(\mathbb{R})$ which is well-suited to our context.

Definition 4.3.2. *For $\mu, \mu' \in \mathcal{P}(\mathbb{R})$ we define the distance (possibly infinite) $D(\mu, \mu')$ by*

$$D(\mu,\mu') := \left(-\int \log|x-y| d(\mu-\mu')(x) d(\mu-\mu')(y)\right)^{1/2} \tag{4.3.3}$$

$$= \left(\int_0^{+\infty} \frac{1}{t}\left|\mathcal{F}[\mu-\mu'](t)\right|^2 dt\right)^{1/2}. \tag{4.3.4}$$

where the Fourier transform of a signed measure ν is defined by $\mathcal{F}[\nu](x) := \int_{\mathbb{R}} e^{-\mathrm{i}tx} d(\mu-\mu')(x)$

Let $f : \mathbb{R} \to \mathbb{R}$ with finite 1/2 norm $\|f\|_{1/2} := \left(\int_{\mathbb{R}} |t|\,|\mathcal{F}[f](t)|^2\,dt\right)^{1/2}$. By Plancherel theorem and Hölder inequality, for any $\mu, \mu' \in \mathcal{P}(\mathbb{R})$, setting $\nu = \mu - \mu'$,

$$\left|\int_{\mathbb{R}} f d\mu - \int_{\mathbb{R}} f d\mu'\right|^2 = \left|\frac{1}{2\pi}\int_{\mathbb{R}} |t|^{1/2}\mathcal{F}[f](t)\frac{\overline{\mathcal{F}[\nu](t)}}{|t|^{1/2}} dt\right|^2 \leqslant \frac{1}{2\pi^2}\|f\|_{1/2}^2 D^2(\mu,\mu').$$

Therefore the metric d defined in (4.1.13) is dominated by D:

$$d(\mu,\mu') \leqslant \frac{1}{\sqrt{2\pi}} D(\mu,\mu'). \tag{4.3.5}$$

The following lemma shows how the distance D is related to the energy-functional $\mathcal{E}_P^V$ defined in (4.2.3), we will write $\mathcal{E}_P$ for simplicity.

Lemma 4.3.3. *We have for any absolutely continuous $\mu \in \mathcal{P}(\mathbb{R})$ with finite energy $\mathcal{E}_P^V(\mu)$,*

$$\mathcal{E}_P(\mu) - \mathcal{E}_P(\mu_P) = PD^2(\mu,\mu_P) + \int \log\left(\frac{d\mu}{d\mu_P}\right) d\mu\,. \tag{4.3.6}$$

Proof of Lemma 4.3.3. Subtracting $\mathcal{E}_P(\mu) - \mathcal{E}_P(\mu_P)$ we find

$$\mathcal{E}_P(\mu) - \mathcal{E}_P(\mu_P) = \int V d(\mu-\mu_P) + \int \log\frac{d\mu}{dx} d\mu - \int \log\rho_P d\mu_P - P\iint \log|x-y| d\mu(x) d\mu(y) + P\iint \log|x-y| d\mu_P(x) d\mu_P(y)\,. \tag{4.3.7}$$

Now, if ν is a signed measure of mass zero, integrating (4.2.4) we get

$$\int V(x)d\nu(x) - 2P\iint \log|x-y|d\nu(x)d\mu_P(y) + \int \log(\rho_P)(x)d\nu(x) = 0\,.$$

We take $\nu = \mu - \mu_P$, and get

$$\int V(x)d(\mu-\mu_P)(x) = 2P\iint \log|x-y|d\mu(x)d\mu_P(y) - 2P\iint \log|x-y|d\mu_P(x)d\mu_P(y) \\ - \int \log(\rho_P)(x)d\mu(x) + \int \log(\rho_P)(x)d\mu_P(x)\,.$$

Plugging this last identity in (4.3.7), we find

$$\mathcal{E}_P(\mu) - \mathcal{E}_P(\mu_P) = -P\iint \log|x-y|d\nu(x)d\nu(y) + \int \log\left(\frac{d\mu}{d\mu_P}\right)(x)d\mu(x)$$

which establishes the result. □

Proof of Theorem 4.1.5. We first give a lower bound for the partition function $Z_N^{V,\frac{2P}{N}}$ (4.1.4) of $\mathbb{P}_N^{V,\frac{2P}{N}}$. We rewrite it as

$$Z_N^{V,\frac{2P}{N}} = \int_{\mathbb{R}^N} \exp\left(\frac{2P}{N}\sum_{i<j}\log|x_i-x_j| - \sum_{i=1}^N\Big[V(x_i)+\log\rho_P(x_i)\Big]\right)d\rho_P(x_1)\dots d\rho_P(x_N)\,,$$

and apply Jensen inequality to obtain:

$$\begin{aligned}\log Z_N^{V,\frac{2P}{N}} &\geqslant \int_{\mathbb{R}^N}\left(\frac{2P}{N}\sum_{i<j}\log|x_i-x_j| - \sum_{i=1}^N\Big[V(x_i)+\log\rho_P(x_i)\Big]\right)d\rho_P(x_1)\dots d\rho_P(x_N)\\ &\geqslant P(N-1)\iint\log|x-y|d\rho_P(x)d\rho_P(y) - N\int_{\mathbb{R}}\Big[V+\log\rho_P\Big]d\rho_P\\ &\geqslant -N\mathcal{E}_P^V\big[\mu_P\big] - P\iint\log|x-y|d\rho_P(x)d\rho_P(y).\end{aligned}$$

Using this estimate and the fact that for $1\leqslant i,j\leqslant N$ we have $|x_i-x_j|\leqslant|y_i-y_j|$, with $\mathbf{y}_N=(y_1,\dots,y_N)$ of definition 4.3.1, we deduce the bound on the density of probability

$$\frac{d\mathbb{P}_N^{V,\frac{2P}{N}}}{d\mathbf{x}}(x_1,\dots,x_N) \leqslant e^{N\mathcal{E}_P(\mu_P)+P\iint\log|x-y|d\mu_P(x)d\mu_P(y)+\frac{P}{N}\sum_{i\neq j}\log|y_i-y_j|-\sum_{i=1}^N V(x_i)}\,. \tag{4.3.8}$$

Recalling (4.3.2), we now show the following estimate

$$\sum_{i\neq j}\log|y_i-y_j| \leqslant 2 + N^2\iint\log|x-y|d\widetilde{\mu}_N(x)d\widetilde{\mu}_N(y) + 5N\log N + \frac{3}{2}N\,. \tag{4.3.9}$$

Let $i\neq j$ and $u,v\in[0,N^{-5}]$. Since for $x\neq 0$ and $|h|\leqslant\frac{|x|}{2}$, we have $\big|\log|x+h|-\log|x|\big|\leqslant\frac{2|h|}{|x|}$, we deduce

$$\big|\log|y_i-y_j+u-v|-\log|y_i-y_j|\big| \leqslant \frac{2|u-v|}{|y_i-y_j|} \leqslant \frac{2N^{-5}}{N^{-3}} = \frac{2}{N^2}.$$

Thus, summing over $i\neq j$ and integrating with respect to u and v, we get

$$\begin{aligned}\sum_{i\neq j}\log|y_i-y_j| &\leqslant 2 + \sum_{i\neq j}\iint\log|y_i-y_j+u-v|d\lambda_{N^{-5}}(u)d\lambda_{N^{-5}}(v)\\ &= 2 + N^2\iint\log|x-y|d\widetilde{\mu}_N(x)d\widetilde{\mu}_N(y) - N\iint\log|u-v|d\lambda_{N^{-5}}(u)d\lambda_{N^{-5}}(v)\,.\end{aligned}$$

The last integral is equal to $-\frac{3}{2} - 5\log N$, so we deduce (4.3.9). We now combine (4.3.8) and (4.3.9). Recall (4.2.3) and set

$$c_N = P\left(\iint \log|x-y| d\mu_P(x) d\mu_P(y) + 3/2 + 2/N\right).$$

Then we get

$$\begin{aligned}\frac{d\mathbb{P}_N^{V,\frac{2P}{N}}}{d\mathbf{x}}(x_1,\dots,x_N) &\leqslant e^{c_N+5P\log N}\exp\left[N\left\{\mathcal{E}_P(\mu_P)-\mathcal{E}_P(\tilde{\mu}_N)+\int\left(V+\log\frac{d\tilde{\mu}_N}{dx}\right)d\tilde{\mu}_N\right\}-\sum_{i=1}^N V(x_i)\right]\\ &= e^{c_N+5P\log N}\exp\left[-NPD^2(\tilde{\mu}_N,\mu_P)+N\int(V+\log\rho_P)\,d\tilde{\mu}_N-\sum_{i=1}^N V(x_i)\right]\end{aligned}$$

where we used equation (4.3.6) in the last equality. Using again equation (4.2.4) we then see that the density $\frac{d\mathbb{P}_N^{V,\frac{2P}{N}}}{d\mathbf{x}}(x_1,\dots,x_N)$ is bounded by

$$e^{c_N+5P\log N}\exp\left[-NPD^2(\tilde{\mu}_N,\mu_P)+2PN\iint\log|x-y|d(\tilde{\mu}_N-\hat{\mu}_N)(x)d\mu_P(y)\right]\prod_{i=1}^N\rho_P(x_i)\,.$$

Recalling (4.2.2), we used that $\iint\log|x-y|d(\tilde{\mu}_N-\hat{\mu}_N)(x)d\mu_P(y) = -\int U^{\rho_P}d(\tilde{\mu}_N-\hat{\mu}_N)$. As a consequence of the bound on the density $\frac{d\mathbb{P}_N^{V,\frac{2P}{N}}}{d\mathbf{x}}(x_1,\dots,x_N)$ we established, we have for all $r>0$

$$\mathbb{P}_N^{V,\frac{2P}{N}}\left(D^2(\tilde{\mu}_N,\mu_P)>r\right)\leqslant e^{-NPr+c_N+5P\log N}\int_{\mathbb{R}^N}\exp\left\{-2PN\int U^{\rho_P}d(\tilde{\mu}_N-\hat{\mu}_N)\right\}\prod_{i=1}^N\rho_P(x_i)dx_i\,. \tag{4.3.10}$$

Next, we show that $-N\int U^{\rho_P}d(\tilde{\mu}_N-\hat{\mu}_N)$ is bounded. By Lemma 4.2.3, U^{ρ_P} is differentiable with bounded derivative $\mathcal{H}[\rho_P]$ on $\mathbb{R}$. As a consequence,

$$\begin{aligned}\left|N\int U^{\rho_P}d(\tilde{\mu}_N-\hat{\mu}_N)\right| &\leqslant \sum_{i=1}^N\int|U^{\rho_P}(y_i+u)-U^{\rho_P}(x_i)|\,d\lambda_{N^{-5}}(u)\\ &\leqslant \|\mathcal{H}[\rho_P]\|_\infty\left(\sum_{i=1}^N|y_i-x_i|+N\int u d\lambda_{N^{-5}}(u)\right)\\ &\leqslant \|\mathcal{H}[\rho_P]\|_\infty\Big(\frac{1}{2N}+N^{-4}/2\Big),\end{aligned}$$

where we used (4.3.1) in the last inequality. Therefore, we deduce from (4.3.10)

$$\mathbb{P}_N^{V,\frac{2P}{N}}\left(D^2(\tilde{\mu}_N,\mu_P)>r\right)\leqslant e^{-NPr+c_N+5P\log N+\frac{2P}{N}\|\mathcal{H}[\rho_P]\|_\infty}=e^{-NPr+5P\log N+K_N} \tag{4.3.11}$$

with $K_N := c_N + \frac{2P}{N}\|\mathcal{H}[\rho_P]\|_\infty$. Since $(c_N)_N$ is bounded, so is $(K_N)_N$.
Finally, let f be a Lipschitz bounded function with $\|f\|_{\mathrm{Lip}}\leqslant 1$, then, we have (as we did for U^{ρ_P})

$$\left|\int f d\hat{\mu}_N-\int f d\tilde{\mu}_N\right|\leqslant N^{-2}\,.$$

Thus by (4.3.5)

$$d(\hat{\mu}_N,\mu_P)\leqslant d(\hat{\mu}_N,\tilde{\mu}_N)+d(\tilde{\mu}_N,\mu_P)\leqslant N^{-2}+\frac{1}{\sqrt{2\pi}}D(\tilde{\mu}_N,\mu_P)\,,$$

and for any N such that $r - N^{-2} \geqslant r/2$ (in particular $r - N^{-2} > 0$) we get

$$\mathbb{P}_N^{V,\frac{2P}{N}}\left(d(\hat{\mu}_N,\mu_P) > r\right) \leqslant \mathbb{P}_N^{V,\frac{2P}{N}}\left(\frac{1}{2\pi^2}D^2(\tilde{\mu}_N,\mu_P) > (r-N^{-2})^2\right) \leqslant \mathbb{P}_N^{V,\frac{2P}{N}}\left(\frac{1}{2\pi^2}D^2(\tilde{\mu}_N,\mu_P) > r^2/4\right),$$

and the last term is bounded by $e^{-Nr^2\frac{P\pi^2}{2}+5P\log N+K}$ for some K large enough, which concludes the proof. □

As a consequence of Theorem 4.1.5, we are able to control the quantities

$$\zeta_N(\phi) := \iint_{\mathbb{R}^2} \frac{\phi(x)-\phi(y)}{x-y} d(\hat{\mu}_N - \mu_P)(x) d(\hat{\mu}_N - \mu_P)(y) \tag{4.3.12}$$

for a certain class of test functions ϕ.

Corollary 4.3.4. *There exist $C, K > 0$ such that for all $\phi \in \mathcal{C}^2(\mathbb{R}) \cap H^2(\mathbb{R})$ with bounded second derivative, we have for $\varepsilon > 0$ and N large enough,*

$$\mathbb{P}_N^{V,\frac{2P}{N}}\left(\sqrt{N}|\zeta_N(\phi)| \leqslant N^{-1/2+\varepsilon}\right) \geqslant 1 - \exp\left\{-\frac{PN^\varepsilon}{2C\|\psi\|_{H^2(\mathbb{R})}} + 5P\log N + K\right\}$$

with $N_2(\phi) = \|\phi'\|_{L^2(dx)} + \|\phi''\|_{L^2(dx)}$.

Proof. We follow the proof given in [Gui19][Cor. 4.16] and adapt it to our setting. Let us denote by $\widetilde{\zeta_N}(\phi)$ the quantity

$$\iint_{\mathbb{R}^2} \frac{\phi(x)-\phi(y)}{x-y} d(\tilde{\mu}_N - \mu_P)(x) d(\tilde{\mu}_N - \mu_P)(y).$$

We have the almost sure inequality, by a Taylor estimate

$$|\zeta_N(\phi) - \widetilde{\zeta_N}(\phi)| \leqslant 2N^{-2}\|\phi''\|_\infty. \tag{4.3.13}$$

Thus, for any $\delta > 0$,

$$\begin{aligned}\mathbb{P}_N^{V,\frac{2P}{N}}\left(|\zeta_N(\phi)| > \delta\right) &\leqslant \mathbb{P}_N^{V,\frac{2P}{N}}\left(|\zeta_N(\phi) - \widetilde{\zeta_N}(\phi)| > \delta/2\right) + \mathbb{P}_N^{V,\frac{2P}{N}}\left(|\widetilde{\zeta_N}(\phi)| > \delta/2\right)\\ &\leqslant \mathbb{P}_N^{V,\frac{2P}{N}}\left(2N^{-2}\|\phi''\|_\infty > \delta/2\right) + \mathbb{P}_N^{V,\frac{2P}{N}}\left(|\widetilde{\zeta_N}(\phi)| > \delta/2\right),\end{aligned}$$

where the first term of the right-hand side is either 0 or 1. With $\delta = N^{-1+\varepsilon}$, $\varepsilon > 0$, it is zero for N large enough. For such a choice of δ, and for N large enough,

$$\mathbb{P}_N^{V,\frac{2P}{N}}\left(|\zeta_N(\phi)| > N^{-1+\varepsilon}\right) \leqslant \mathbb{P}_N^{V,\frac{2P}{N}}\left(|\widetilde{\zeta_N}(\phi)| > \frac{1}{2}N^{-1+\varepsilon}\right).$$

We next show that, for some $C > 0$ independent of ϕ, we have

$$|\widetilde{\zeta_N}(\phi)| \leqslant CD^2(\tilde{\mu}_N,\mu_P)\|\phi\|_{H^2(\mathbb{R})}. \tag{4.3.14}$$

We begin by showing this inequality for $\psi \in \mathcal{S}(\mathbb{R})$. By using the inverse Fourier transform we have

$$\begin{aligned}\tilde{\zeta}_N(\psi) &= \frac{1}{2\pi}\iint \frac{\int dt\mathcal{F}[\psi](t)e^{\mathrm{i}tx} - \int dt\mathcal{F}[\psi](t)e^{\mathrm{i}ty}}{x-y} d\big(\tilde{\mu}_N-\mu_P\big)(x) d\big(\tilde{\mu}_N-\mu_P\big)(y)\\ &= \frac{1}{2\pi}\int dt\mathrm{i}t\mathcal{F}[\psi](t)\iint e^{\mathrm{i}ty}\frac{e^{\mathrm{i}t(x-y)}-1}{\mathrm{i}t(x-y)} d\big(\tilde{\mu}_N-\mu_P\big)(x) d\big(\tilde{\mu}_N-\mu_P\big)(y)\\ &= \frac{1}{2\pi}\int dt\mathrm{i}t\mathcal{F}[\psi](t)\iint e^{\mathrm{i}ty}\int_0^1 d\alpha e^{\mathrm{i}\alpha t(x-y)} d\big(\tilde{\mu}_N-\mu_P\big)(x) d\big(\tilde{\mu}_N-\mu_P\big)(y)\\ &= \frac{1}{2\pi}\int dt\mathrm{i}t\mathcal{F}[\psi](t)\int_0^1 d\alpha\int e^{\mathrm{i}\alpha tx} d\big(\tilde{\mu}_N-\mu_P\big)(x)\int e^{\mathrm{i}(1-\alpha)ty} d\big(\tilde{\mu}_N-\mu_P\big)(y)\end{aligned}$$

We then apply in order the triangular inequality, Cauchy-Schwarz inequality, a change of variable and the fact that $\left|\mathcal{F}\left[\widetilde{\mu}_N - \mu_P\right]\right|^2$ is an even function.

$$
\begin{aligned}
|\widetilde{\zeta}_N(\psi)| &\leqslant \frac{1}{2\pi}\int_{\mathbb{R}} dt\, |t\mathcal{F}[\psi](t)| \int_0^1 d\alpha\, \left|\mathcal{F}\left[\widetilde{\mu}_N - \mu_P\right](\alpha t)\right| . \left|\mathcal{F}\left[\widetilde{\mu}_N - \mu_P\right]\big((1-\alpha)t\big)\right| \\
&\leqslant \frac{1}{2\pi}\int_{\mathbb{R}} dt\, |t\mathcal{F}[\psi](t)| \Big(\int_0^1 d\alpha\, \left|\mathcal{F}\left[\widetilde{\mu}_N - \mu_P\right](\alpha t)\right|^2\Big)^{\frac{1}{2}} \Big(\int_0^1 d\alpha\, \left|\mathcal{F}\left[\widetilde{\mu}_N - \mu_P\right]\big((1-\alpha)t\big)\right|^2\Big)^{\frac{1}{2}} \\
&\leqslant \frac{1}{2\pi}\int_{\mathbb{R}} dt\, |t\mathcal{F}[\psi](t)| \int_0^1 d\alpha\, \left|\mathcal{F}\left[\widetilde{\mu}_N - \mu_P\right](\alpha t)\right|^2 \\
&\leqslant \frac{1}{2\pi}\int_0^{+\infty} dt\, |t\mathcal{F}[\psi](t)| \int_0^1 \frac{td\alpha}{t\alpha} \left|\mathcal{F}\left[\widetilde{\mu}_N - \mu_P\right](\alpha t)\right|^2 + \frac{1}{2\pi}\int_{-\infty}^0 dt\, |t\mathcal{F}[\phi](t)| \int_0^1 \frac{-td\alpha}{-t\alpha} \left|\mathcal{F}\left[\widetilde{\mu}_N - \mu_P\right](\alpha t)\right|^2 \\
&\leqslant \frac{1}{2\pi}\int_{\mathbb{R}} dt\, |t\mathcal{F}[\psi](t)|\, D^2(\widetilde{\mu}_N, \mu_P) \\
&\leqslant \frac{1}{2\pi}\Big(\int_{\mathbb{R}} dt\, |t\mathcal{F}[\psi](t)|^2 (1+t^2)\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} \frac{dt}{1+t^2}\Big)^{\frac{1}{2}} D^2(\widetilde{\mu}_N, \mu_P) \\
&\leqslant \frac{1}{2\sqrt{\pi}} D^2(\widetilde{\mu}_N, \mu_P) N_2(\psi) \\
&\leqslant \frac{1}{2\sqrt{\pi}} D^2(\widetilde{\mu}_N, \mu_P) \|\psi\|_{H^2(\mathbb{R})}
\end{aligned}
$$

By density of $\mathcal{S}(\mathbb{R})$ in $H^2(\mathbb{R})$, and since $\widetilde{\zeta}_N : \Big(H^2(\mathbb{R}), \|\cdot\|_{H^2(\mathbb{R})}\Big) \to \mathbb{R}$ is continuous, the inequality still holds for ϕ. Thus, using equation (4.3.11),

$$
\mathbb{P}_N^{V,\frac{2P}{N}}\left(|\widetilde{\zeta_N}(\phi)| > \frac{1}{2}N^{-1+\varepsilon}\right) \leqslant \mathbb{P}_N^{V,\frac{2P}{N}}\left(D^2(\widetilde{\mu}_N, \mu_P) > \frac{N^{-1+\varepsilon}}{2C\|\phi\|_{H^2(\mathbb{R})}}\right) \leqslant \exp\left\{-P\frac{N^{\varepsilon}}{2C\|\phi\|_{H^2(\mathbb{R})}} + 5P\log N + K\right\},
$$

which concludes the proof. □

4.4 Localization of the edge of a configuration

In [Lam21][Theorem 1.8, Theorem 3.4], Lambert was able to control the edge (*i.e* the minimum and the maximum) of a typical configuration $(x_1, \ldots, x_N)$ distributed according to $\mathbb{P}_N^{V,\frac{2P}{N}}$, by showing that the random measure

$$
\Xi_N := \sum_{j=1}^N \delta_{\varphi_N^{-1}(x_j)}
$$

converges in distribution towards a Poisson point process for a function φ_N which takes the form

$$
\varphi_N(x) := E_N + \alpha_N^{-1} x\,.
$$

Before being more precise on the construction of $(E_N)_N$ and $(\alpha_N)_N$, we explain, following [Lam21], how one can use this convergence to localize the edge of a typical configuration $(x_1, \ldots, x_N)$. Let us assume for a moment that Ξ_N converges towards a Poisson point process with intensity $\theta(x) = e^{-x}$, with $E_N \to +\infty$. In particular, the random variable

$$
\Xi_N(t, +\infty)
$$

converges in distribution towards a Poisson random variable with mean $\int_t^{+\infty} e^{-x}dx$. Combined with the equalities

$$\begin{aligned}\mathbb{P}_N^{V,\frac{2P}{N}}\left(\Xi_N(t,+\infty)=0\right) &= \mathbb{P}_N^{V,\frac{2P}{N}}\left(\forall\ 1\leqslant j\leqslant N,\ \varphi_N^{-1}(x_j)=\alpha_N(x_j-E_N)\leqslant t\right)\\ &= \mathbb{P}_N^{V,\frac{2P}{N}}\left(\alpha_N\left(\max_{1\leqslant j\leqslant N}x_j-E_N\right)\leqslant t\right),\end{aligned}$$

we deduce that for all $t\in\mathbb{R}$

$$\mathbb{P}_N^{V,\frac{2P}{N}}\left(\alpha_N\left(\max_{1\leqslant j\leqslant N}x_j-E_N\right)\leqslant t\right)\underset{N\to\infty}{\longrightarrow}\exp(-e^{-t})\,.$$

Therefore, the random variable

$$\alpha_N\left(\max_{1\leqslant j\leqslant N}x_j-E_N\right)$$

converges in distribution to the Gumbel law, showing that the maximum of a configuration is of order E_N. Furthermore, as will be clear from the construction of α_N and E_N, α_N is positive, and goes to infinity as N goes to infinity.

Replacing in the previous analysis $\theta(x)=e^x$ and $E_N\to-\infty$, we would have deduced in the same fashion that

$$\alpha_N\left(\min_{1\leqslant j\leqslant N}x_j-E_N\right)$$

converges in law.

With the above notations, we can apply [Lam21][Theorem 3.4] to our context.

Theorem 4.4.1. *Let $v=\pm$. There exists $(E_N^v)_N$, $(\alpha_N^v)_N$ sequences of real numbers with $|E_N^v|\to+\infty$, $\alpha_N^v>0$ for large enough N, satisfying $V'(E_N^v)=\alpha_N^v v$, such that:*

a) $\dfrac{Ne^{-V(E_N^v)+2P\log|E_N^v|+\lambda_V^P}}{\alpha_N^v}\underset{N\to\infty}{\longrightarrow}1$ *(recall λ_P^V is defined through equation (4.1.5)),*

b) $\frac{\log(\alpha_N^v)}{N}\underset{N\to\infty}{\longrightarrow}0$ *and* $\alpha_N^v|E_N^v|\underset{N\to\infty}{\longrightarrow}+\infty$ *,*

c) For all compact $K\subset\mathbb{R}$,

$$(\alpha_N^v)^{-2}\sup_{x\in K}\left|V''(\varphi_N(x))\right|\underset{N\to\infty}{\longrightarrow}0\,.$$

As a consequence, the random measure Ξ_N converges in distribution as $N\to\infty$ to a Poisson point process with intensity $\theta(x)=e^{-vx}$.

Proof. We prove it in the case $v=+$, the case where $v=-$ being similar. We show that there exists a sequence $(E_N^+)_N$ going to $+\infty$ satisfying $f(E_N^+)=-\log N$, where we defined the function f by

$$f(x)=-V(x)+2P\log|x|+\lambda_P^V-\log|V'(x)|\,.$$

Recalling Assumption 4.1.1 *i)*, $|V'|$ goes to infinity at infinity, thus $\alpha_N^+=V'(E_N^+)\to+\infty$ (in the case $v=-1$ we would have looked for a sequence $(E_N^-)_N$ going to $-\infty$ and $\alpha_N^-=-V'(E_N^-)$).
As a consequence of Assumptions 4.1.1,*ii)*, one shows that $\log|V'|$ is negligible with respect to V at infinity. Therefore, because $\dfrac{\log|x|}{V(x)}\underset{|x|\to\infty}{\longrightarrow}0$,

$$f(x)=-V(x)+\underset{x\to+\infty}{o}(V(x))\,. \tag{4.4.1}$$

Because $f(x) \underset{x\to+\infty}{\longrightarrow} -\infty$ there exists $(E_N^+)_N$ going to infinity such that for all $N \geqslant 1$, $f(E_N^+) = -\log N$. Setting $x = E_N^+$ in (4.4.1), we obtain that $-V(E_N^+) \sim f(E_N^+) = -\log N$. Property c) follows from Assumptions 4.1.1, point *ii)*, along with the fact that α_N^{-1} stays bounded.

It remains to show that $\dfrac{\log(\alpha_N^+)}{N} = \dfrac{\log|V'(E_N^+)|}{N} \underset{N\to\infty}{\longrightarrow} 0$. By construction, we have

$$\frac{\log|V'(E_N^+)|}{N} = \frac{\log\left(Ne^{-V(E_N^+)+2P\log N+\lambda_P}\right)}{N} = -\frac{V(E_N^+)}{N} + o(1)\,.$$

Using that $V(E_N^+) \sim \log N$, we can conclude that $\log|V'(E_N^+)| = o(N)$ which concludes the proof. □

By the discussion preceding Theorem 4.4.1, we deduce

Corollary 4.4.2 (Edge of a configuration)**.** *Let $E_N^\pm$, $\alpha_N^\pm := |V'(E_N^\pm)|$ be the sequences of Theorem 4.4.1 associated with $v = \pm 1$. Then, both random variables*

$$\alpha_N^+\left(\max_{1\leqslant j\leqslant N} x_j - E_N^+\right)$$

and

$$\alpha_N^-\left(\min_{1\leqslant j\leqslant N} x_j - E_N^-\right)$$

converge to a Gumbel law, whose distribution function is given for $t \geqslant 0$ by $\mathcal{G}([0,t]) = \exp(e^{-t})$. Furthermore, $V(E_N^\pm) \sim \log N$ and $\alpha_N^\pm \underset{N\to\infty}{\longrightarrow} \pm\infty$.

Remark 4.4.3. *Note that[Lam21][Theorem 3.4] applies for V of class $\mathcal{C}^2$ outside of a compact set, allowing to take $V(x) = |x|^a$ for $a > 1$. In this case, we find $E_N^\pm \sim \pm(\log N)^{1/a}$. If $V(x) = \cosh(x)$, we find $E_N^+ \sim -E_N^- \sim \operatorname{arg\,cosh}(\log N) \sim \log\log N$.*

The next lemma will be convenient in the proof of Theorem 4.5.2 when dealing with error terms.

Lemma 4.4.4. *With the notations of Corollary 4.4.2, we have*

$$\mu_P([E_N^-, E_N^+]^c) = o(N^{-1/2})\,.$$

Proof. Let $0 < \delta < 1$, to be specified later. We have

$$\int_{E_N^+}^{+\infty} \rho_P dx = \int_{E_N^+}^{+\infty} (\rho_P)^\delta (\rho_P)^{1-\delta} dx \leqslant \int_{\mathbb{R}} (\rho_P)^\delta dx \sup_{[E_N^+,+\infty[} (\rho_P)^{1-\delta}\,.$$

By the first inequality of Lemma 4.2.2, the integral is finite. Also from the same inequality, we have for some constant C' and x big enough $\rho_P(x) \leqslant C'e^{-\frac{3}{4}V(x)}$. Because V is increasing in a neighborhood of $+\infty$, we get for N large enough

$$\sup_{[E_N^+,+\infty[} (\rho_P)^{1-\delta} \leqslant C'^{1-\delta} e^{-(1-\delta)\frac{3}{4}V(E_N^+)}\,.$$

Taking $\delta > 0$ such that $\frac{1}{2} - (1-\delta)\frac{3}{4} =: -\gamma < 0$ and using that $V(E_N^+) = \log N + o(\log N)$ (established in the proof of Theorem 4.4.1),

$$\sqrt{N}\int_{E_N^+}^{+\infty} \rho_P dx \leqslant Ke^{-\gamma\log N + (1-\delta)\frac{3}{4}o(\log N)}\,,$$

and the right-hand side goes to zero as N goes to infinity. We deal with the integral $\int_{-\infty}^{E_N^-} \rho_P dx$ in the same way.

□

Remark 4.4.5. *We could improve the proof to show that $\mu_P([E_N^-, E_N^+]^c) \sim \dfrac{1}{N}$ but showing that it is $o(N^{\frac{1}{2}})$ is sufficient for what we need and requires less carefulness.*

4.5 Laplace transform for smooth test functions, proof of Theorem 4.1.3

Section 4.3 allows us to justify in Proposition 4.5.1 the heuristics we gave in equation (4.1.6) for ϕ having compact support. We will then extend in Theorem 4.5.2 this result to a more general set of functions, by an approximation by compactly supported functions, using Corollary 4.4.2.

Proposition 4.5.1. *For $\phi \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ with compact support, we have for any real t, as N goes to infinity,*

$$\mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{t\sqrt{N}\nu_N(\Xi\phi)}\right] \to \exp\left\{\frac{t^2}{2}q_P(\phi)\right\}, \tag{4.5.1}$$

where $\Xi\phi$ is given by equation (4.1.7), and $q_P(\phi)$ is given by

$$q_P(\phi) := \int_{\mathbb{R}}\Big(\phi'(x)^2 + V''(x)\phi(x)^2\Big)d\mu_P(x) + P\iint\limits_{\mathbb{R}^2}\Big(\frac{\phi(x)-\phi(y)}{x-y}\Big)^2 d\mu_P(x)d\mu_P(y)\,. \tag{4.5.2}$$

Proof. Let $\phi \in \mathcal{C}_c^1(\mathbb{R},\mathbb{R})$, and let $t \in \mathbb{R}$. We perform in equation (4.1.4) the change of variables $x_i = y_i + \frac{t}{\sqrt{N}}\phi(y_i)$, $1 \leqslant i \leqslant N$, which is a diffeomorphism for N big enough. We thus have

$$Z_N^{V,\frac{2P}{N}} = \int \prod_{1\leqslant i<j\leqslant N}\left|y_i - y_j + \frac{t}{\sqrt{N}}\big(\phi(y_i)-\phi(y_j)\big)\right|^{2P/N} .e^{-\sum_{i=1}^N V\left(y_i+\frac{t}{\sqrt{N}}\phi(y_i)\right)}.\prod_{i=1}^N\left(1+\frac{t}{\sqrt{N}}\phi'(y_i)\right)d^N\mathbf{y}, \tag{4.5.3}$$

and we develop separately the different terms of this integral. The first term can be written as:

$$\prod_{i<j}|y_i-y_j|^{2P/N}\prod_{i<j}\left|1+\frac{t}{\sqrt{N}}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j}\right|^{2P/N},$$

The second product above, setting $\Delta\phi_{i,j} := \frac{\phi(y_i)-\phi(y_j)}{y_i-y_j}$ and using Taylor-Lagrange theorem, equals

$$\exp\left(\frac{2P}{N}\sum_{i<j}\log\left|1+\frac{t}{\sqrt{N}}\frac{\phi(y_i)-\phi(y_j)}{y_i-y_j}\right|\right) = \exp\left(\frac{2P}{N}\sum_{i<j}\left(\frac{t}{\sqrt{N}}\Delta\phi_{i,j} - \frac{t^2}{2N}(\Delta\phi_{i,j})^2 + R_{N,1}(i,j)\right)\right),$$

where we noticed that $1+\frac{t}{\sqrt{N}}\Delta\phi_{i,j} \geqslant 1-\frac{t}{\sqrt{N}}\|\phi'\|_\infty > 0$ if N is big enough, and where

$$|R_{N,1}(i,j)| \leqslant \frac{|t|^3}{3N^{3/2}}\|\phi'\|_\infty^3.$$

Again by Taylor-Lagrange theorem, the second term in (4.5.3) equals

$$\exp\left(-\sum_{i=1}^N\left(V(y_i)+\frac{t}{\sqrt{N}}V'(y_i)\phi(y_i)+\frac{t^2}{2N}V''(y_i)\phi(y_i)^2+R_{N,2}(i)\right)\right)$$

where $R_{N,2}(i) = \frac{t^3}{6N^{3/2}}V^{(3)}\left(y_i+\frac{t\theta_i}{\sqrt{N}}\phi(y_i)\right)\phi(y_i)^3$ for some $\theta_i \in [0,1]$, thus for N large enough

$$|R_{N,2}(i)| \leqslant \frac{|t|^3}{6N^{3/2}}\|\phi\|_\infty^3 \sup_{d(x,\text{supp }\phi)\leqslant 1}|V^{(3)}(x)|.$$

The last term reads

$$\prod_{i=1}^N\left(1+\frac{t}{\sqrt{N}}\phi'(y_i)\right) = \exp\left(\sum_{i=1}^N\left(\frac{t}{\sqrt{N}}\phi'(y_i) - \frac{t^2}{2N}\phi'(y_i)^2 + R_{N,3}(i)\right)\right),$$

with $|R_{N,3}(i)| \leqslant \frac{t^3}{3N^{3/2}}\|\phi'\|_\infty^3$. Dividing both sides of equation (4.5.3) by $Z_N^{V,\frac{2P}{N}}$ we get

$$\mathbb{E}_N^{V,\frac{2P}{N}}\Bigg[\exp\left\{t\sqrt{N}\Big(P\iint\limits_{\mathbb{R}^2}\frac{\phi(x)-\phi(y)}{x-y}d\hat{\mu}_N(x)d\hat{\mu}_N(y)+\int_{\mathbb{R}}(\phi'-V'\phi)d\hat{\mu}_N\Big)\right\}\times\exp\{K_N(t,\phi)\}$$
$$\times\exp\left\{\frac{t^2}{2}\left(-P\iint\limits_{\mathbb{R}^2}\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 d\hat{\mu}_N(x)d\hat{\mu}_N(y)-\int_{\mathbb{R}}(V''\phi^2+\phi'^2)d\hat{\mu}_N\right)\right\}\Bigg]=1,$$

with $|K_N(t,\phi)| \leqslant \frac{c(t,\phi)}{\sqrt{N}}$ where $c(t,\phi)\geqslant 0$ is independent of N. This bound shows that taking the limit $N\to\infty$ we can get rid of K_N:

$$\lim_{N\to\infty}\mathbb{E}_N^{V,\frac{2P}{N}}\Bigg[\exp\left\{t\sqrt{N}\Big(P\iint\limits_{\mathbb{R}^2}\frac{\phi(x)-\phi(y)}{x-y}d\hat{\mu}_N(x)d\hat{\mu}_N(y)+\int_{\mathbb{R}}(\phi'-V'\phi)d\hat{\mu}_N\Big)\right\}$$
$$\times\exp\left\{\frac{t^2}{2}\left(-P\iint\limits_{\mathbb{R}^2}\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 d\hat{\mu}_N(x)d\hat{\mu}_N(y)-\int_{\mathbb{R}}(V''\phi^2+\phi'^2)d\hat{\mu}_N\right)\right\}\Bigg]=1.$$

Using Fubini's theorem (the function $(x,y)\mapsto\frac{\phi(x)-\phi(y)}{x-y}$ being bounded continuous on $\mathbb{R}^2$), the first line in the expectation value can be rewritten as $e^{t\sqrt{N}\Lambda_N}$ with

$$\Lambda_N := 2P\iint\limits_{\mathbb{R}^2}\frac{\phi(x)-\phi(y)}{x-y}d\mu_P(x)d(\hat{\mu}_N-\mu_P)(y)+\int_{\mathbb{R}}(\phi'-V'\phi)d(\hat{\mu}_N-\mu_P)+P\zeta_N(\phi) \tag{4.5.4}$$

where we used equation (4.1.5) and $\zeta_N(\phi)$ is given by (4.3.12). Let $F:\mathcal{P}(\mathbb{R})\to\mathbb{R}$ be defined by

$$F(\mu) = -P\iint\limits_{\mathbb{R}^2}\left(\frac{\phi(x)-\phi(y)}{x-y}\right)^2 d\mu(x)d\mu(y)-\int_{\mathbb{R}}(V''\phi^2+\phi'^2)d\mu\,. \tag{4.5.5}$$

It is continuous for the topology of weak convergence since all the functions in the integrals are bounded continuous. So far we have established that

$$\lim_{N\to\infty}\mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{t\sqrt{N}\Lambda_N+\frac{t^2}{2}F(\hat{\mu}_N)}\right]=1,$$

with Λ_N given by (4.5.4). We now replace in the latter equation the term $F(\hat{\mu}_N)$ by its limiting expression, $F(\mu_P)$. Fix a metric that is compatible with the weak convergence of probability measures on $\mathbb{R}$. For example,

$$d_{\mathrm{Lip}}(\mu,\nu)=\sup\left|\int f d\mu-\int f d\nu\right|, \tag{4.5.6}$$

where the supremum runs over $f:\mathbb{R}\to\mathbb{R}$ bounded and Lipschitz with $\|f\|_\infty\leqslant 1$ and Lipschitz constant $|f|_{\mathrm{Lip}}\leqslant 1$. By the large deviations principle for $(\hat{\mu}_N)_N$ under the probability (4.1.3) established by [GZ19, Theorem 1.1], for all $\delta>0$ the event $\{d_{\mathrm{Lip}}(\hat{\mu}_N,\mu_P)>\delta\}$ has (for N big enough) probability smaller than e^{-Nc_δ} where $c_\delta>0$. Hence,

$$\lim_{N\to\infty}\mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{t\sqrt{N}\Lambda_N+\frac{t^2}{2}F(\hat{\mu}_N)}\right]=\lim_{N\to\infty}\mathbb{E}_N^{V,\frac{2P}{N}}\left[\mathbf{1}_{\{d_{\mathrm{Lip}}(\hat{\mu}_N,\mu_P)\leqslant\delta\}}e^{t\sqrt{N}\Lambda_N+\frac{t^2}{2}F(\hat{\mu}_N)}\right].$$

By continuity of F there is some $\varepsilon(\delta)$ which goes to 0 as $\delta\to 0$ such that, for $d_{\mathrm{Lip}}(\nu,\mu_P)\leqslant\delta$, we have $|F(\nu)-F(\mu_P)|\leqslant\varepsilon(\delta)$. Taking the (decreasing) limit as δ goes to zero we deduce

$$\lim_{N\to\infty}\mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{t\sqrt{N}\Lambda_N+\frac{t^2}{2}F(\hat{\mu}_N)}\right]=\lim_{\delta\to 0}\lim_{N\to\infty}\mathbb{E}_N^{V,\frac{2P}{N}}\left[\mathbf{1}_{\{d_{\mathrm{Lip}}(\hat{\mu}_N,\mu_P)\leqslant\delta\}}e^{t\sqrt{N}\Lambda_N}\right]e^{\frac{t^2}{2}F(\mu_P)}.$$

But the same large deviations argument shows that

$$\lim_{\delta\to 0}\lim_{N\to\infty}\mathbb{E}_N^{V,\frac{2P}{N}}\left[\mathbf{1}_{\{d_{\mathrm{Lip}}(\hat{\mu}_N,\mu_P)\leqslant\delta\}}e^{t\sqrt{N}\Lambda_N}\right]=\lim_{N\to\infty}\mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{t\sqrt{N}\Lambda_N}\right].$$

Thus, we have shown that

$$\lim_{N\to\infty}\mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{t\sqrt{N}\left(2P\iint_{\mathbb{R}^2}\frac{\phi(x)-\phi(y)}{x-y}d\mu_P(x)d(\hat{\mu}_N-\mu_P)(y)+\int_{\mathbb{R}}(\phi'-V'\phi)d(\hat{\mu}_N-\mu_P)+P\zeta_N(\phi)\right)}\right]=e^{-\frac{t^2}{2}F(\mu_P)}\,,\quad(4.5.7)$$

Which establishes that $\sqrt{N}\Lambda_N=\sqrt{N}\Big(\nu_N(\Xi\phi)+P\zeta_N(\phi)\Big)$ converges in law towards a centered Gaussian random variable with announced variance. We finally get rid of the remaining term $\zeta_N(\phi)$, using Corollary 4.3.4: taking $\varepsilon=1/4$ for example, we see in particular that $\sqrt{N}\zeta_N(\phi)$ converges in probability towards zero. The conclusion follows from Slutsky's lemma. □

We now extend the result of Proposition 4.5.1 to a more general set of functions. With the notations of Proposition 4.5.1, we have

Theorem 4.5.2. *Let $\phi\in H^2(\mathbb{R})\cap\mathcal{C}^2(\mathbb{R})$ such that ϕ'' is bounded. Additionally, suppose that $V^{(3)}\phi^2$, $V''\phi\phi'$, $V''\phi^2$ and $V'\phi$ are bounded. Then, recalling* (4.5.2) *we have the convergence in distribution as N goes to infinity*

$$\sqrt{N}\nu_N(\Xi\phi)\to N(0,q_P(\phi))\,.$$

Proof. For $N\geqslant 1$, let E_N^-,E_N^+ be given by Corollary 4.4.2. Let $\chi_N:\mathbb{R}\to[0,1]$ be $\mathcal{C}^2$ with compact support such that

$$\chi_N(x)=1\text{ for }x\in[E_N^--1,E_N^++1]\text{ and }\chi_N(x)=0\text{ for }x\in[E_N^--2,E_N^++2]^c$$

and such that, denoting $\phi_N=\phi\chi_N$, $\sup_N\|\phi_N^{(k)}\|_\infty+\|\phi_N^{(k)}\|_{L^2(\mathbb{R})}<+\infty$ for $k=0,1,2$ (we assumed $\phi\in H^2(\mathbb{R})$, in particular ϕ and ϕ' is bounded and such a χ_N exists). The point of cutting ϕ outside the set $[E_N^--1,E_N^++1]$ is that with high probability, the empirical measure $\hat{\mu}_N$ doesn't see the difference between ϕ and ϕ_N.

The support of ϕ_N is then contained in $[E_N^--2,E_N^++2]$, and we now argue that the proof of Proposition 4.5.1 can be adapted so that

$$\sqrt{N}\nu_N(\Xi\phi_N)\to N(0,q_P(\phi))\,.\quad(4.5.8)$$

Similarly as in Proposition 4.5.1, we perform in $Z_N^{V,P}$ the change of variables $x_i=y_i+\frac{t}{\sqrt{N}}\phi_N(y_i)$, $1\leqslant i\leqslant N$, which is the same as before, but with ϕ replaced by ϕ_N. First, with $I_N:=[E_N^--2,E_N^++2]$, the error term

$$K_N(t,\phi_N)\leqslant 2\frac{t^3}{3N^{1/2}}\|\phi_N'\|_\infty^3+\frac{t^3}{6N^{1/2}}\|\phi_N\|_\infty\sup_{d(x,I_N)\leqslant 1}|V^{(3)}(x)|$$

of the proof of Proposition 4.5.1 is still going to zero, because of our choice of χ_N and Assumption 4.1.2. As previously, we then have

$$\lim_{N\to\infty}\mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{t\sqrt{N}\Lambda_N(\phi_N)+\frac{t^2}{2}F_N(\hat{\mu}_N)}\right]=1\quad(4.5.9)$$

with

$$\Lambda_N(\phi_N):=2P\iint_{\mathbb{R}^2}\frac{\phi_N(x)-\phi_N(y)}{x-y}d\mu_P(x)d(\hat{\mu}_N-\mu_P)(y)+\int_{\mathbb{R}}(\phi_N'-V'\phi_N)d(\hat{\mu}_N-\mu_P)+P\zeta_N(\phi_N)\,,$$

where ζ_N is given by (4.3.12), and

$$F_N(\hat{\mu}_N)=-P\iint_{\mathbb{R}^2}\left(\frac{\phi_N(x)-\phi_N(y)}{x-y}\right)^2d\hat{\mu}_N(x)d\hat{\mu}_N(y)-\int_{\mathbb{R}}(V''\phi_N^2+\phi_N'^2)d\hat{\mu}_N\,.$$

Taking again the distance d_{Lip} defined in (4.5.6), one can check that for μ, ν probability measures over $\mathbb{R}$,

$$|F_N(\mu) - F_N(\nu)| \leqslant C_N d_{\mathrm{Lip}}(\mu,\nu)\,,$$

where C_N is a term depending on the norms $\|\phi_N'\|_\infty, \|\phi_N''\|_\infty$, $\|V''\phi_N^2\|_\infty$ and $\|(V''\phi_N^2)'\|_\infty$. The choice of χ_N and the fact that ϕ is chosen so that $V^{(3)}\phi^2$ and $V''\phi\phi'$ are bounded guarantee that $\|(V''\phi_N^2)'\|_\infty$ is bounded in N. The other norms are easily bounded by hypothesis. Therefore C_N can be seen to be uniformly bounded in N, and we find some $C \geqslant 0$ independent of N such that

$$|F_N(\mu) - F_N(\nu)| \leqslant C d_{\mathrm{Lip}}(\mu,\nu)\,.$$

As in proposition 4.5.1, we use the large deviation principle for $(\hat{\mu}_N)$ to deduce

$$\lim_{N\to+\infty} \mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{t\sqrt{N}\Lambda_N(\phi_N)+\frac{t^2}{2}F_N(\hat{\mu}_N)}\right] = \lim_{N\to+\infty} \mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{t\sqrt{N}\Lambda_N(\phi_N)}\right] e^{\frac{t^2}{2}F_N(\mu_P)}\,.$$

By dominated convergence, $F_N(\mu_P)$ converges to $F(\mu_P)$, the function F being given by (4.5.5). This shows the convergence as N goes to infinity

$$\lim_{N\to+\infty} \mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{t\sqrt{N}\Lambda_N(\phi_N)}\right] = e^{-\frac{t^2}{2}F(\mu_P)}\,,$$

and $\sqrt{N}\Big(\nu_N(\Xi\phi_N)+P\zeta_N(\phi_N)\Big)$ converges towards a centered Gaussian variable with variance $-F(\mu_P) = q_P(\phi)$. Because $\sup_N \|\phi_N\|_{H^2(\mathbb{R})}$ is finite, we can apply again Corollary 4.3.4 to deduce the convergence in law (4.5.8). We now have the ingredients to conclude, by showing that the characteristic function

$$\mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{\mathrm{i}t\sqrt{N}\nu_N(\Xi\phi)}\right] = \mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{\mathrm{i}t\sqrt{N}\int\Xi\phi d\hat{\mu}_N}\right] e^{-\mathrm{i}t\sqrt{N}\int\Xi\phi d\mu_P}$$

converges to the characteristic of a Gaussian variable with appropriate variance. By Corollary 4.4.2, the probability under $\mathbb{P}_N^{V,\frac{2P}{N}}$ of the event $\mathcal{E}_N = \Big\{x_1,\dots,x_N \in [E_N^- - 1, E_N^+ + 1]\Big\}$ converges to 1. Along with the convergence (4.5.8), we deduce

$$e^{-\frac{t^2}{2}q_P(\phi)} = \lim_N \mathbb{E}_N^{V,\frac{2P}{N}}\left[e^{\mathrm{i}t\sqrt{N}\int\Xi\phi_N d\hat{\mu}_N}\right] e^{-\mathrm{i}t\sqrt{N}\int\Xi\phi_N d\mu_P} = \lim_N \mathbb{E}_N^{V,\frac{2P}{N}}\left[\mathbf{1}_{\mathcal{E}_N} e^{\mathrm{i}t\sqrt{N}\int\Xi\phi_N d\hat{\mu}_N}\right] e^{-\mathrm{i}t\sqrt{N}\int\Xi\phi_N d\mu_P}\,,$$

Where we used

$$\left|\mathbb{E}_N^{V,\frac{2P}{N}}\left[\mathbf{1}_{\mathcal{E}_N^c} e^{\mathrm{i}t\sqrt{N}\int\Xi\phi_N d\hat{\mu}_N}\right] e^{-\mathrm{i}t\sqrt{N}\int\Xi\phi_N d\mu_P}\right| \leqslant \mathbb{P}_N^{V,\frac{2P}{N}}(\mathcal{E}_N^c) \xrightarrow[N\to+\infty]{} 0\,.$$

Using that $\phi_N = \phi$ on $J_N = [E_N^- - 1, E_N^+ + 1]$,

$$\begin{aligned}
\int \Xi\phi_N d\mu_P &= 2P\iint \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_P(x)d\mu_P(y) + \int(\phi_N' - V'\phi_N)d\mu_P \\
&= 2P\iint_{J_N^2} \frac{\phi(x)-\phi(y)}{x-y} d\mu_P(x)d\mu_P(y) + 2P\iint_{(J_N^2)^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_P(x)d\mu_P(y) \\
&+ \int_{J_N}(\phi' - V'\phi)d\mu_P + \int_{J_N^c}(\phi\chi_N' + \phi'\chi_N - V'\phi\chi_N)d\mu_P\,.
\end{aligned}$$

By boundedness of $(\|\phi_N'\|_\infty)_N$, the second term is bounded by

$$C_P \iint_{(J_N^2)^c} d\mu_P d\mu_P \leqslant 2C_P\mu_P(J_N^c) = o(N^{-1/2})\,,$$

where we used the union bound and Lemma 4.4.4. By the same estimate and the fact that χ_N can be chosen so that $(\|\chi_N'\|_\infty)_N$ is bounded, and because ϕ', $V'\phi$ are bounded, the last term is also $o(N^{-1/2})$. By the previous arguments, we also conclude that

$$2P \iint\limits_{(J_N^2)^c} \frac{\phi(x)-\phi(y)}{x-y} d\mu_P(x) d\mu_P(y) + \int_{J_N^c} (\phi' - V'\phi) d\mu_P = o(N^{-1/2}),$$

thus

$$\int \Xi \phi_N d\mu_P = \int \Xi \phi d\mu_P + o(N^{-1/2}),$$

and so far we have

$$e^{-\frac{t^2}{2} q_P(\phi)} = \lim_N \mathbb{E}_N^{V, \frac{2P}{N}} \left[\mathbf{1}_{\mathcal{E}_N} e^{\mathrm{i}t\sqrt{N} \int \Xi \phi_N d\hat{\mu}_N} \right] e^{-\mathrm{i}t\sqrt{N} \int \Xi \phi d\mu_P}.$$

Finally, on $\mathcal{E}_N$, using $\phi_N = \phi$ and that $\hat{\mu}_N$ is supported in J_N,

$$\begin{aligned}\int \Xi \phi_N d\hat{\mu}_N &= 2P \iint\limits_{J_N^2} \frac{\phi(x)-\phi(y)}{x-y} d\mu_P(x) d\hat{\mu}_N(y) + 2P \iint\limits_{(J_N^2)^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_P(x) d\hat{\mu}_N(y) + \int_{J_N} (\phi' - V'\phi) d\hat{\mu}_N \\ &= 2P \iint \frac{\phi(x)-\phi(y)}{x-y} d\mu_P(x) d\hat{\mu}_N(y) + \int (\phi' - V'\phi) d\hat{\mu}_N + o(N^{-1/2}),\end{aligned}$$

Where in the second line we used, using Lemma 4.4.4 again, that

$$\iint\limits_{(J_N^2)^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_P(x) d\hat{\mu}_N(y) = \iint\limits_{J_N \times J_N^c} \frac{\phi_N(x)-\phi_N(y)}{x-y} d\mu_P(x) d\hat{\mu}_N(y) = o(N^{-1/2}),$$

and the same estimate holds for ϕ_N replaced by ϕ. Therefore,

$$e^{-\frac{t^2}{2} q_P(\phi)} = \lim_N \mathbb{E}_N^{V, \frac{2P}{N}} \left[\mathbf{1}_{\mathcal{E}_N} e^{\mathrm{i}t\sqrt{N} \int \Xi \phi d\hat{\mu}_N} \right] e^{-\mathrm{i}t\sqrt{N} \int \Xi \phi d\mu_P}.$$

This establishes that

$$\lim_N \mathbb{E}_N^{V, \frac{2P}{N}} \left[e^{\mathrm{i}t\sqrt{N} \int \Xi \phi d\hat{\nu}_N} \right] = e^{-\frac{t^2}{2} q_P(\phi)},$$

which concludes the proof. □

Remark 4.5.3. *Taking ϕ such that ϕ' satisfies the conditions of Theorem 4.5.2, we then have*

$$\mathbb{E}_N^{V, \frac{2P}{N}} \left[e^{t\sqrt{N} \nu_N(\mathcal{L}\phi)} \right] \underset{N \to \infty}{\longrightarrow} \exp\left\{ \frac{t^2}{2} q_P(\phi') \right\}, \tag{4.5.10}$$

where the operator $\mathcal{L}$ is defined as $\mathcal{L}\phi := \Xi\phi'$, ie

$$\mathcal{L}\phi = 2P \int_{\mathbb{R}} \frac{\phi'(x) - \phi'(y)}{x-y} d\mu_P(y) + \phi''(x) - V'(x)\phi'(x). \tag{4.5.11}$$

Note that $q_P^V(\phi') = \left(\sigma_P^V\right)^2(\mathcal{L}\phi)$ where σ_P^V is defined in (4.1.12). By Theorem 4.7.1, the class of functions in $\mathcal{L}^{-1}(\mathcal{T})$ where

$$\mathcal{T} := \left\{ f \in \mathcal{C}^1(\mathbb{R}), \exists \varepsilon > 0,\ f(x) = \underset{|x| \to \infty}{O}\left(x^{-\frac{1}{2}-\varepsilon}\right),\ f'(x) = \underset{|x| \to \infty}{O}\left(x^{-\frac{1}{2}-\varepsilon}\right), \int_{\mathbb{R}} f \rho_P = 0 \right\}$$

satisfies (4.5.10). This proves Theorem 4.1.3.

We now prove a more compact formula for the variance such as the one appearing in [HL21].

Lemma 4.5.4. *The following equality holds for all $\phi \in \mathcal{T}$*

$$\langle \mathcal{L}^{-1}\phi, \phi\rangle_{\mathsf{H}} = \big(\sigma_P^V\big)^2(\phi) := \int_{\mathbb{R}} \Big(\big(\mathcal{L}^{-1}\phi\big)''(x)^2 + V''(x)\big(\mathcal{L}^{-1}\phi\big)'(x)^2 \Big) d\mu_P(x) + P \iint\limits_{\mathbb{R}^2} \left(\frac{\big(\mathcal{L}^{-1}\phi\big)'(x) - \big(\mathcal{L}^{-1}\phi\big)'(y)}{x-y} \right)^2 d\mu_P(x) d\mu_P(y) \quad (4.5.12)$$

Proof. It suffices to show that $\big(\sigma_P^V\big)^2(\mathcal{L}\phi) = \langle \mathcal{L}\phi, \phi\rangle_{\mathsf{H}}$ for all ϕ, such that $\phi' \in H^2(\mathbb{R})$.

$$\langle \mathcal{L}\phi, \phi\rangle_{\mathsf{H}} = -\int_{\mathbb{R}} \Big(\frac{(\phi'\rho_P)'}{\rho_P} \Big)' \phi' \rho_P - 2P \int_{\mathbb{R}} \mathcal{H}[\phi'\rho_P]' \phi' \rho_P$$

Proceeding to integration by parts in the first integral leads to

$$-\int_{\mathbb{R}} \Big(\frac{(\phi'\rho_P)'}{\rho_P} \Big)' \phi' \rho_P = \int_{\mathbb{R}} \Big(\frac{(\phi'\rho_P)'}{\rho_P} \Big)^2 \rho_P = \int_{\mathbb{R}} \phi''^2 \rho_P + 2\phi'\phi''\rho_P' + \phi'^2 \big(\frac{\rho_P'}{\rho_P}\big)^2 \rho_P = \int_{\mathbb{R}} \phi''^2 \rho_P - \phi'^2 \frac{\rho_P''}{\rho_P} \rho_P + \phi'^2 \big(\frac{\rho_P'}{\rho_P}\big)^2 \rho_P$$

Since

$$\frac{\rho_P''}{\rho_P} = \Big(-V'' - 2P\mathcal{H}[\rho_P]' + V'^2 + 4P^2\mathcal{H}[\rho_P]^2 + 4PV'\mathcal{H}[\rho_P] \Big) = -V'' - 2P\mathcal{H}[\rho_P]' + \Big(\frac{\rho_P'}{\rho_P}\Big)^2$$

we obtain

$$\langle \mathcal{L}\phi, \phi\rangle_{\mathsf{H}} = \int_{\mathbb{R}} \phi''^2 \rho_P + V''\phi^2\rho_P - 2P \int_{\mathbb{R}} \mathcal{H}[\phi'\rho_P]'\phi'\rho_P + 2P \int_{\mathbb{R}} \mathcal{H}[\rho_P]'\phi'^2\rho_P$$

To conclude, we just have to show that

$$\iint\limits_{\mathbb{R}^2} \Big(\frac{\phi'(x) - \phi'(y)}{x-y} \Big)^2 d\mu_P(x) d\mu_P(y) = 2 \int_{\mathbb{R}} \mathcal{H}[\rho_P]'\phi'^2\rho_P - \mathcal{H}[\phi'\rho_P]'\phi'\rho_P$$

First

$$\int_{\mathbb{R}} \mathcal{H}[\phi'\rho_P]'\phi'\rho_P = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\phi'(x)\phi'(y)}{(y-x)^2} d\mu_P(x) d\mu_P(y)$$

Secondly

$$\int_{\mathbb{R}} \mathcal{H}[\rho_P]'\phi'^2\rho_P = \frac{1}{2} \iint\limits_{\mathbb{R}^2} \frac{\phi'^2(x) + \phi'^2(y)}{(y-x)^2} d\mu_P(x) d\mu_P(y)$$

which allows to conclude that $\big(\sigma_P^V\big)^2(\mathcal{L}\phi) = \langle \mathcal{L}\phi, \phi\rangle_{\mathsf{H}}$. □

4.6 Inversion of $\mathcal{L}$

This section is dedicated to the definition of $\mathcal{L}$ given by (4.1.8) and its domain and then we focus on its inversion. We rely heavily on results of Appendix 4.8: the diagonalization of the operator $\mathcal{A}$ by the use of the theory of Schrödinger operators.

Let $P > 0$ be fixed. We introduce the operators $\mathcal{A}$ and $\mathcal{W}$, acting on sufficiently smooth functions of $L^2(\rho_P)$, by

$$\mathcal{A}\phi = -\frac{\left(\phi'\rho_P\right)'}{\rho_P} = -\left(\phi'' + \frac{\rho_P'}{\rho_P}\phi'\right) \quad \text{and} \quad \mathcal{W}\phi = -\mathcal{H}\big[\phi'\rho_P\big]\,. \tag{4.6.1}$$

One can show that the operator $\mathcal{A}$ corresponds to the operator verifying:

$$\langle \phi, \psi \rangle_{\mathsf{H}} = \int_{\mathbb{R}} \phi'\psi' d\mu_P = \int_{\mathbb{R}} \phi \mathcal{A}\psi d\mu_P = \langle \phi, \mathcal{A}\psi \rangle_{L^2(\mu_P)}$$

We first show the following decomposition of $\mathcal{L}$.

Lemma 4.6.1. *For ϕ twice differentiable we have the following pointwise identity*

$$-\mathcal{L}\phi = \mathcal{A}\phi + 2P\mathcal{W}\phi\,. \tag{4.6.2}$$

Proof. We write for $x \in \mathbb{R}$

$$2P\int_{\mathbb{R}} \frac{\phi'(x) - \phi'(y)}{x - y}\rho_P(y)dy = -2P\phi'(x)\mathcal{H}[\rho_P](x) + 2P\mathcal{H}[\phi'\rho_P](x)\,. \tag{4.6.3}$$

Then,

$$\mathcal{L}\phi = \phi'' - V'\phi' - 2P\phi'\mathcal{H}[\rho_P] + 2P\mathcal{H}\big[\phi'\rho_P\big]\,.$$

By (4.2.5) we have $-V' - 2P\mathcal{H}[\rho_P] = \frac{\rho_P'}{\rho_P}$, which concludes the proof. □

In order to state the next theorem, whose proof we detail in the Appendix, we introduce the following Sobolev-type spaces. Let

$$H^1_{V'}(\mathbb{R}) := \left\{u \in H^1(\mathbb{R}),\, uV' \in L^2(\mathbb{R})\right\}.$$

We now define

$$\mathcal{D}(\mathcal{S}) = \left\{u \in H^1_{V'}(\mathbb{R}), -u'' + (w_P + \alpha)u \in L^2(\mathbb{R})\right\}$$

and

$$\mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}) := \rho_P^{-1/2}\mathcal{D}(\mathcal{S})$$

and its homogeneous counterpart

$$\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) := \left\{u \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}),\, \int_{\mathbb{R}} u\rho_P dx = 0\right\}.$$

Finally, we let $L^2_0(\rho_P)$ be the subset of $L^2(\rho_P)$ of zero mean functions with respect to ρ_P.

We detail the proof of the following theorem in Appendix 4.8 which is based on Schrödinger operators theory.

Theorem 4.6.2 (Diagonalization of $\mathcal{A}$ in $L^2_0(\rho_P)$)**.** *There exists a sequence $0 < \lambda_1 < \lambda_2 < \dots$ going to infinity, and a complete orthonormal set $(\phi_n)_{n\geqslant 1}$ of $L^2_0(\rho_P)$ of associated eigenfunctions for $\mathcal{A}$, meaning that*

- $\mathrm{Span}\{\phi_n,\, n \geqslant 1\}$ *is dense in* $L^2_0(\rho_P)$,
- *For all* i, j, $\langle \phi_i, \phi_j \rangle_{L^2(\rho_P)} = \delta_{i,j}$,
- *For all* $n \geqslant 1$, $\mathcal{A}\phi_n = \lambda_n\phi_n$.

Furthermore, each ϕ_n is in $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, $\mathcal{A} : \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \to L^2_0(\rho_P)$ is bijective, and we have the writing, for $u \in L^2_0(\rho_P)$

$$\mathcal{A}^{-1}u = \sum_{n\geqslant 1} \lambda_n^{-1} \langle u, \phi_n \rangle_{L^2(\rho_P)}\, \phi_n\,.$$

We see the operators $\mathcal{A}$ and $\mathcal{W}$ as unbounded operators on the space

$$\mathsf{H} = \Big\{ u \in H^1(\rho_P) \,|\, \int_{\mathbb{R}} u \rho_P dx = 0 \Big\}$$

endowed with the inner product $\langle u, v \rangle_{\mathsf{H}} = \langle u', v' \rangle_{L^2(\rho_P)}$. This defines an inner product on H and makes it a complete space: it can be seen that $H^1(\rho_P)$ is the completion of $\mathcal{C}_c^\infty(\mathbb{R})$ with respect to the inner product $\langle u, v \rangle_{L^2(\rho_P)} + \langle u', v' \rangle_{L^2(\rho_P)}$. The space H is then the kernel of the bounded (with respect to $\|\cdot\|_{\mathsf{H}}$) linear form, $\langle \widetilde{1}, \cdot \rangle_{L^2(\rho_P)}$ on $H^1(\rho_P)$, and both inner products are equivalent on H because of the Poincaré inequality, Proposition 4.2.6. The use of H is motivated by the fact that both $\mathcal{A}$ and $\mathcal{W}$ are self-adjoint positive on this space as we show in Lemma 4.6.4.

In the next proposition, we deduce from Theorem 4.6.2 the diagonalization of $\mathcal{A}$ in H.

Proposition 4.6.3 (Diagonalization of $\mathcal{A}$ in H)**.** *With the same eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$ as in Theorem 4.6.2, there exists a complete orthonormal set $(\psi_n)_{n \geqslant 1}$ of H formed by eigenfunctions of $\mathcal{A}$.*

Proof. With $(\phi_n)_{n \geqslant 1}$ of Theorem 4.6.2,

$$\begin{aligned} \delta_{i,j} = \langle \phi_i, \phi_j \rangle_{L^2(\rho_P)} &= \frac{1}{\lambda_j} \langle \phi_i, \mathcal{A}\phi_j \rangle_{L^2(\rho_P)} \\ &= \frac{1}{\lambda_j} \langle \phi_i', \phi_j' \rangle_{L^2(\rho_P)} \\ &= \frac{1}{\lambda_j} \langle \phi_i, \phi_j \rangle_{\mathsf{H}}. \end{aligned}$$

With $\psi_n = \frac{1}{\sqrt{\lambda_n}} \phi_n$, $(\psi_n)_{n \geqslant 1}$ is then orthonormal with respect to the inner product of H. To show that $\mathrm{Span}\{\psi_n, n \geqslant 1\}$ is dense in H, let $u \in \mathsf{H}$ be such that for all $j \geqslant 1$, $\langle u, \phi_j \rangle_{\mathsf{H}} = 0$. In the last series of equalities, replace ϕ_i by u: we see that u is orthogonal to each ϕ_j in $L^2(\rho_P)$, thus u is a constant as shown in the proof of Lemma 4.8.10, and because $u \in \mathsf{H}$ it has zero mean against ρ_P. This shows that $u = 0$. □

We set for what follows $\mathcal{D}(\mathcal{A}) := \big\{ u \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \,|\, \mathcal{A}u \in \mathsf{H} \big\}$ and $\mathcal{D}(\mathcal{W}) := \{ u \in \mathsf{H} \,|\, \mathcal{W}u \in \mathsf{H} \}$.

Lemma 4.6.4. *The following properties hold:*

- *The operator $\mathcal{W} : \mathcal{D}(\mathcal{W}) \to \mathsf{H}$ is positive: for all $\phi \in \mathcal{D}(\mathcal{W})$,*

$$\langle \mathcal{W}\phi, \phi \rangle_{\mathsf{H}} = \frac{1}{2} \|\phi' \rho_P\|_{1/2}^2 \geqslant 0 \,,$$

  with equality only for $\phi = 0$, where the $1/2$-norm of u is given by

$$\|u\|_{1/2}^2 = \int_{\mathbb{R}} |x| . \, |\mathcal{F}[u](x)|^2 \, dx \,.$$

- *Both $\mathcal{A}$ and $\mathcal{W}$ are self-adjoint for the inner product of H.*

Proof. To prove the first point, let $\phi \in \mathcal{D}(\mathcal{W})$. Then,

$$\begin{aligned} 2\pi \langle \mathcal{W}\phi, \phi \rangle_{\mathsf{H}} = -2\pi \big\langle \mathcal{H}[\phi' \rho_P]', \phi' \rho_P \big\rangle_{L^2(dx)} &= -\big\langle ix \mathcal{F}\big[\mathcal{H}[\phi' \rho_P]\big], \mathcal{F}[\phi' \rho_P] \big\rangle_{L^2(dx)} \\ &= \pi \big\langle \,|\, x \,|\, \mathcal{F}[\phi' \rho_P], \mathcal{F}[\phi' \rho_P] \big\rangle_{L^2(dx)} = \pi \|\phi' \rho_P\|_{1/2}^2 \geqslant 0 \,, \end{aligned}$$

and because ϕ is in H, this last quantity is zero if and only if ϕ vanishes.

For the second point, let $u, v \in \mathcal{D}(\mathcal{W})$. Using Plancherel's isometry and *i)* of Lemma 4.2.1,

$$\langle \mathcal{W}u, v\rangle_{\mathsf{H}} = \big\langle (\mathcal{W}u)', v'\rho_P \big\rangle_{L^2(dx)} = \frac{1}{2}\big\langle \,|\, x \,|\, \mathcal{F}[u'\rho_P], \mathcal{F}[v'\rho_P]\big\rangle_{L^2(dx)} \,,$$

and this last expression is symmetric in (u, v). The proof of the self-adjointness of $\mathcal{A}$ follows from integration by parts. □

Definition 4.6.5 (Quadratic form associated to $-\mathcal{L}$)**.** *We define for all $u, v \in \mathsf{H} \cap \mathcal{C}_c^\infty(\mathbb{R})$ the quadratic form associated to $-\mathcal{L}$ by*

$$q_{-\mathcal{L}}(u, v) = \langle \mathcal{A}u, \mathcal{A}v\rangle_{L^2(\rho_P)} + 2P\langle \mathcal{F}[u'\rho_P], \mathcal{F}[v'\rho_P]\rangle_{L^2(|x|dx)}$$

Note that for all $u, v \in \mathsf{H} \cap \mathcal{C}_c^\infty(\mathbb{R})$, $q_{-\mathcal{L}}(u, v) = \langle -\mathcal{L}u, v\rangle_{\mathsf{H}}$ and that whenever $u \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{W})$,

$$q_{-\mathcal{L}}(u, u) = \langle \mathcal{A}u, u\rangle_{\mathsf{H}} + 2P\langle \mathcal{W}u, u\rangle_{\mathsf{H}} \geqslant \lambda_1(\mathcal{A})\|u\|_{\mathsf{H}}^2 \tag{4.6.4}$$

by Proposition 4.6.3 and Lemma 4.6.4. After extending the $q_{-\mathcal{L}}$ to its form domain $Q(\mathcal{L})$ which is equal to $\Big\{u \in \mathsf{H}, \mathcal{A}u \in L^2(\rho_P),\, \mathcal{F}[u'\rho_P] \in L^2(|x|dx)\Big\} = \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$. The equality comes from the fact that $\mathcal{A}^{-1}\Big(L_0^2(\rho_P)\Big) = \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, that $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \subset \mathsf{H}$ and that $\mathcal{F}[u'\rho_P] \in L^2(x^2dx)$ whenever $u \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, indeed $u'\rho_P \in H^1(\mathbb{R})$ because $(u'\rho_P)' = -\rho_P\mathcal{A}u \in L^2(\mathbb{R})$. We now define $\mathcal{D}(\mathcal{L})$ the domain of definition of $-\mathcal{L}$ by:

$$\mathcal{D}(\mathcal{L}) := \Big\{u \in Q(\mathcal{L}), v \mapsto q_{-\mathcal{L}}(u, v) \text{ can be extended to a continuous linear form on } \mathsf{H}\Big\}$$

Proposition 4.6.6. *$\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{W})$.*

Proof. Let $u \in \mathcal{D}(\mathcal{L})$, by Riesz's theorem there exists $f_u \in \mathsf{H}$, such that $q_{-\mathcal{L}}(u, v) = \langle f_u, v\rangle_{\mathsf{H}}$ for all $v \in \mathsf{H}$, we set $-\mathcal{L}u := f_u$, it is called the Friedrichs extension of $-\mathcal{L}$. Then for all $v \in \mathsf{H} \cap \mathcal{C}_c^\infty(\mathbb{R})$, by integration by part we get:

$$\langle -\mathcal{L}u, v\rangle_{\mathsf{H}} = q_{-\mathcal{L}}(u, v) = \langle u, \mathcal{A}v\rangle_{\mathsf{H}} + 2P\langle u, \mathcal{W}v\rangle_{\mathsf{H}}\,,$$

hence we deduce the distributional identity $-\mathcal{L}u = \mathcal{A}u + 2P\mathcal{W}u$. Since $u \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, $\mathcal{W}u \in H^1(\rho_P)$ implying that $\mathcal{A}u \in \mathsf{H}$ and then that $\mathcal{W}u \in \mathsf{H}$. The converse is trivially true. □

We are now ready to state the main theorem of this section, that is the inversion of $\mathcal{L}$ on $\mathcal{D}(\mathcal{L})$.

Theorem 4.6.7 (Inversion of $\mathcal{L}$)**.** *$-\mathcal{L} : \mathcal{D}(\mathcal{L}) \longrightarrow \mathsf{H}$ is bijective. Furthermore, $(-\mathcal{L})^{-1}$ is continuous from $(\mathsf{H}, \|.\|_{\mathsf{H}})$ to $(\mathcal{D}(\mathcal{L}), q_{-\mathcal{L}})$.*

Proof. Let $f \in \mathsf{H}$, since $\langle f, .\rangle_{\mathsf{H}}$ is a linear form on $Q(\mathcal{L}) = \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$ which is, by (4.6.4), continuous with respect to $q_{-\mathcal{L}}$, one can applies Riesz's theorem so there exists a unique $u_f \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$, such that for all $v \in \mathsf{H}$, $\langle f, v\rangle_{\mathsf{H}} = q_{-\mathcal{L}}(u_f, v)$. Since, u_f is clearly in $\mathcal{D}(\mathcal{L})$ by definition of the Friedrichs extension of $-\mathcal{L}$, we have $-\mathcal{L}u = f$. □

Remark 4.6.8. *We can diagonalize $\mathcal{L}$ by the same argument we used in Appendix 4.8 to diagonalize $\mathcal{A}$ in $L_0^2(\rho_P)$.*

We now state a result that could allow one to extract more regularity for $\mathcal{L}^{-1}$ by the help of an explicit form that uses Fredholm determinant theory for Hilbert-Schmidt operators, the reader can refer to [GGK12].

Definition 4.6.9 (Fredholm determinant)**.** *Let $\mathcal{U}$ be a self-adjoint Hilbert-Schmidt operator, we denote the Fredholm determinant by* $\det(I + \mathcal{U})$.

Theorem 4.6.10 (Determinant formula for $\mathcal{L}^{-1}$)**.** *For all $u \in \mathsf{H}$, such that $x \mapsto \dfrac{1}{\rho_P(x)} \displaystyle\int_x^{+\infty} u(t)\rho_P(t)dt$ is integrable at $+\infty$, we have:*

$$\mathcal{L}^{-1}u = \mathcal{A}^{-1}u - \rho_P^{-1/2}\mathcal{R}\big[\rho_P^{1/2}\mathcal{A}^{-1}u\big] \tag{4.6.5}$$

where $\mathcal{R}$ is the kernel operator defined for all $v \in L^2(\mathbb{R})$ by:

$$\mathcal{R}[v](x) = \int_{\mathbb{R}} R(x,y)v(y)dy$$

where

$$R(x,y) = \frac{1}{\det(I+\mathcal{K})}\sum_{n\geqslant 0}\frac{1}{n!}\int_{\mathbb{R}^n}\det_{n+1}\begin{bmatrix} K(x,y) & K(x,\lambda_b) \\ K(\lambda_a,y) & K(\lambda_a,\lambda_b)\end{bmatrix}_{a,b=1\dots n} d\lambda_1\dots d\lambda_n$$

where $\mathcal{K}$ is the kernel operator defined for all $w \in L^2(\rho_P)$ by:

$$\mathcal{K}[w](x) = \int_{\mathbb{R}} K(x,y)w(y)dy \tag{4.6.6}$$

with

$$K(x,y) = -2P\rho_P(x)\rho_P(y)\log\Big|1-\frac{y}{x}\Big|. \tag{4.6.7}$$

Proof. Let $f \in \mathsf{H}$, there exists a unique $u \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}u = f$. Since $(u'\rho_P)' = \rho_P\mathcal{A}u \in L^2(\mathbb{R})$, hence $u'\rho_P \in H^1(\mathbb{R})$ so $u'(x)\rho_P(x) \underset{|x|\to+\infty}{\longrightarrow} 0$. By definition, $-\dfrac{(u'\rho_P)'}{\rho_P} = f$ hence

$$(\mathcal{A}^{-1}f)'(x)\rho_P(x) = u'(x)\rho_P(x) = \int_x^{+\infty} f(t)\rho_P(t)dt. \tag{4.6.8}$$

Using the fact that $\int_{\mathbb{R}} u(x)\rho_P(x)dx = 0$, integrating again we get:

$$u(x) = -\int_x^{+\infty}\frac{ds}{\rho_P(s)}\int_s^{+\infty} f(t)\rho_P(t)dt + C$$

where $C = \displaystyle\int_{\mathbb{R}}\rho_P(x)dx\int_x^{+\infty}\frac{ds}{\rho_P(s)}\int_s^{+\infty} f(t)\rho_P(t)dt$. Now let $g \in \mathsf{H}$, there exists a unique $v \in \mathcal{D}(\mathcal{L})$, such that $-\mathcal{L}v = \mathcal{A}v + 2P\mathcal{W}v = g$ and then $v + 2P\mathcal{W}\mathcal{A}^{-1}v = \mathcal{A}^{-1}g$. using (4.6.8), we get:

$$\mathcal{W}\mathcal{A}^{-1}v(x) = \fint_{\mathbb{R}}\frac{ds}{s-x}\int_s^{+\infty} dtv(t)\rho_P(t)$$

By Sokhotski-Plejmel formula, we have:

$$\fint_{\mathbb{R}}\frac{ds}{s-x}\int_s^{+\infty} dtv(t)\rho_P(t) = \lim_{M\to+\infty}\lim_{\varepsilon\to 0}\int_{-M}^{M}\frac{ds}{2}\Big\{\frac{1}{x-s+\mathrm{i}\varepsilon}+\frac{1}{x-s-\mathrm{i}\varepsilon}\Big\}\int_s^{+\infty} dtv(t)\rho_P(t)$$

We then proceed to an integration by part:

$$\fint_{\mathbb{R}}\frac{ds}{s-x}\int_s^{+\infty} dtv(t)\rho_P(t) = \lim_{M\to+\infty}\lim_{\varepsilon\to 0}\Big[-\frac{\log\big((x-s)^2+\varepsilon^2\big)}{2}\int_s^{+\infty} dtv(t)\rho_P(t)\Big]_{-M}^{M} - \int_{\mathbb{R}} ds\log|x-s|v(s)\rho_P(s)$$

To conclude that $\mathcal{W}\mathcal{A}^{-1}v(x) = -\int_{\mathbb{R}} ds\log|x-s|v(s)\rho_P(s)$, we just need to show that

$$\log(x)\int_x^{+\infty} dtv(t)\rho_P(t) \underset{|x|\to\infty}{\longrightarrow} 0$$

which can be seen by Cauchy-Schwarz inequality:

$$\Big|\log(x)\int_x^{+\infty} dtv(t)\rho_P(t)\Big| \leqslant |\log(x)|\|v\|_{L^2(\rho_P)}.\rho_P(x)^{1/4}\Big(\int_{\mathbb{R}}\rho_P(t)^{1/2}dt\Big)^{1/2}.$$

In this inequality, we used that ρ_P is decreasing in a neighborhood of $+\infty$, hence

$$\log|x|\int_s^{+\infty} dtv(t)\rho_P(t) \underset{x\to+\infty}{\longrightarrow} 0$$

the exact same argument allows us to conclude when x goes to $-\infty$. Using the fact that $\int_{\mathbb{R}} v(t)\rho_P(t)dt = 0$, we obtain the following equality:

$$v - 2P\int_{\mathbb{R}} ds\log|x-s|v(s)\rho_P(s) = \mathcal{A}^{-1}g := h.$$

Now setting $\tilde{v}(t) = \rho_P^{1/2}(t)v(t)$ and $\tilde{h} = \rho_P^{1/2}(t)h(t)$, we obtain $\tilde{v} + \mathcal{K}[\tilde{v}] = \tilde{h}$ where $\mathcal{K}$ is defined in (4.6.6). Since its kernel (defined in (4.6.7)) belongs to $L^2(\mathbb{R}^2)$, $\mathcal{K}$ is Hilbert-Schmidt. Hence by Fredholm determinant theory:

$$\tilde{v} = \tilde{h} - \mathcal{R}[\tilde{h}]$$

or $\mathcal{L}^{-1}g = \mathcal{A}^{-1}g - \rho_P^{-1/2}\mathcal{R}\big[\rho_P^{1/2}\mathcal{A}^{-1}g\big]$ as expected. □

4.7 Regularity of the inverse of $\mathcal{L}$ and completion of the proof of Theorem 4.1.3

Since we have proven the central limit theorem for functions of the type $\mathcal{L}\phi$ with ϕ regular enough and satisfying vanishing asymptotic conditions at infinity, we exhibit a class of functions f for which $\mathcal{L}^{-1}f$ is regular enough to satisfy conditions of Theorem 4.5.2. We define $\mathcal{T}$ the subset of H defined by

$$\mathcal{T} := \left\{ f\in\mathcal{C}^1(\mathbb{R}), \exists\varepsilon>0,\, f(x) = \underset{|x|\to\infty}{O}\left(x^{-\frac{1}{2}-\varepsilon}\right),\ f'(x) = \underset{|x|\to\infty}{O}\left(x^{-\frac{1}{2}-\varepsilon}\right), \int_{\mathbb{R}} f\rho_P = 0\right\}$$

Theorem 4.7.1. *For all $f\in\mathcal{T}$, there exists a unique $u\in\mathcal{C}^3(\mathbb{R})$ such that $u'\in H^2(\mathbb{R})$ with $u^{(3)}$ bounded wich verifies:*

- $u'(x) = \underset{|x|\to\infty}{O}\left(\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)}\right)$
- $u''(x) = \underset{|x|\to\infty}{O}\left(\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)}\right)$
- $u^{(3)}(x) = \underset{|x|\to\infty}{O}\left(\frac{1}{x^{\frac{1}{2}+\varepsilon}}\right)$

such that $f = \mathcal{L}u$.

Proof. Let $f\in\mathcal{T}\subset\mathsf{H}$, then since $-\mathcal{L}$ is bijective from $\mathcal{D}(\mathcal{L})\to\mathsf{H}$, there exists a unique $u\in\mathcal{D}(\mathcal{L})$ such that $-\mathcal{L}u = f$ *ie*:

$$-u'' - \frac{\rho_P'}{\rho_P}u' - 2P\mathcal{H}[u'\rho_P] = f \tag{4.7.1}$$

Hence we have

$$-(u'\rho_P)' = \rho_P\Big(f + 2P\mathcal{H}[u'\rho_P]\Big). \tag{4.7.2}$$

Since $u \in \mathcal{D}(\mathcal{L}) \subset \{u \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}), \mathcal{A}u \in \mathsf{H}\}$, the functions $u'\rho_P$ and its distributional derivatives $(u'\rho_P)' = -\rho_P \mathcal{A}u$ and $(u'\rho_P)'' = -\frac{\rho_P'}{\rho_P}\rho_P^{1/2}.\Big(\rho_P^{1/2}\mathcal{A}u\Big) - \rho_P\Big(\mathcal{A}u\Big)'$ are in $L^2(\mathbb{R})$. In particular $u'\rho_P$ goes to zero at infinity, and $\mathcal{H}[u'\rho_P] \in H^2(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R})$. So we can integrate (4.7.2) on $[x, +\infty[$, since by Lemma 4.2.3, the right-hand side behaves like a $\underset{|x|\to\infty}{O}\left(\frac{\rho_P(x)}{|x|^{\frac{1}{2}+\varepsilon}}\right)$, to get the following expression

$$u'(x)\rho_P(x) = \int_x^{+\infty} \frac{\rho_P(t)}{\rho_P'(t)}(f + 2P\mathcal{H}[u'\rho_P]).\rho_P'(t)dt \tag{4.7.3}$$

From this expression, we can see that $u' \in \mathcal{C}^2(\mathbb{R})$ so we just have to check the integrability condition at infinity and the fact that $u^{(3)}$ is bounded. After proceeding to an integration by parts, which is permitted by the previous argument, we obtain:

$$u'(x) = -\frac{\rho_P(x)}{\rho_P'(x)}\Big(f(x) + 2P\mathcal{H}[u'\rho_P](x)\Big) - \frac{1}{\rho_P(x)}\int_x^{+\infty}\left(\frac{\rho_P(t)}{\rho_P'(t)}(f + 2P\mathcal{H}[u'\rho_P])\right)'\rho_P(t)dt \tag{4.7.4}$$

and we define $R_1(x) := \frac{1}{\rho_P(x)}\int_x^{+\infty}\left(\frac{\rho_P(t)}{\rho_P'(t)}(f + 2P\mathcal{H}[u'\rho_P])\right)'\rho_P(t)dt$, we will need to show that R_1 is a remainder of order $\underset{x\to+\infty}{O}\left(\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\right)$ at infinity. In this case, we will have $u'(x) = \underset{x\to+\infty}{O}\left(\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)}\right)$ which will be useful for the following. If we reinject (4.7.4) in (4.7.1), we find:

$$u'' = -(f + 2P\mathcal{H}[u'\rho_P]) - \frac{\rho_P'}{\rho_P}\Big(-\frac{\rho_P}{\rho_P'}\Big(f + 2P\mathcal{H}[u'\rho_P]\Big) - R_1\Big) = \frac{\rho_P'}{\rho_P}R_1 \tag{4.7.5}$$

Hence

$$u''(x) = \frac{\rho_P'}{\rho_P^2}(x)\int_x^{+\infty}\rho_P(t)dt\Bigg\{\underbrace{\left(\frac{\rho_P}{\rho_P'}\right)'(t)}_{=\underset{t\to+\infty}{O}\left(\frac{V''(t)}{V'(t)^2}\right)}\underbrace{\big[f + 2P\mathcal{H}[u'\rho_P]\big](t)}_{=\underset{t\to+\infty}{O}\left(t^{-\frac{1}{2}-\varepsilon}\right)} + \underbrace{\frac{\rho_P}{\rho_P'}(t)}_{=\underset{t\to+\infty}{O}\left(\frac{1}{V'(t)}\right)}\underbrace{\big[f' - 2P\mathcal{H}[\rho_P\mathcal{A}u]\big](t)}_{=\underset{t\to+\infty}{O}\left(t^{-\frac{1}{2}-\varepsilon}\right)}\Bigg\}.$$

The fact that $\mathcal{H}[\rho_P\mathcal{A}u](t) = \underset{t\to+\infty}{O}(t^{-2})$ comes again from lemma 4.2.3. Finally we have that,

$$\begin{aligned} u^{(3)}(x) &= \left(\frac{\rho_P'}{\rho_P^2}\right)'(x)\rho_P(x)R_1(x) - \left(\frac{\rho_P'}{\rho_P^2}\right)(x)\left(\frac{\rho_P}{\rho_P'}(f + 2P\mathcal{H}[u'\rho_P])\right)'(x)\rho_P(x) \\ &= \underbrace{\left(\frac{\rho_P''}{\rho_P} - 2\frac{\rho_P'^2}{\rho_P^2}\right)(x)}_{=\underset{x\to+\infty}{O}\left(V'(x)^2\right)}R_1(x) - \underbrace{\left(\frac{\rho_P'}{\rho_P}\right)(x)\left(\frac{\rho_P}{\rho_P'}(f + 2P\mathcal{H}[u'\rho_P])\right)'(x)}_{=\underset{x\to+\infty}{O}\left(\frac{V''(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)} + x^{-\frac{1}{2}-\varepsilon}\right)} \end{aligned}$$

The second term is $\underset{x\to+\infty}{O}\left(x^{-\frac{1}{2}-\varepsilon}\right)$ by the assumption that $\frac{V''}{V'}(x) = \underset{|x|\to\infty}{O}(1)$. Hence, we just have to check that $R_1(x) = \underset{x\to+\infty}{O}\left(\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\right)$ to establish that u', u'', $u^{(3)}$ are in $L^2(\mathbb{R})$. By a comparison argument, we control R_1 by controlling

$$I_1(x) := \int_x^{+\infty}\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)}dt$$

By integration by parts:

$$\begin{aligned} I_1(x) :=& \underbrace{-\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)}\frac{\rho_P}{\rho_P'}(x)}_{=\underset{x\to+\infty}{O}\left(\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\right)} - \int_x^{+\infty}\rho_P(t)\Big(\frac{1}{t^{\frac{1}{2}+\varepsilon}V'}\frac{\rho_P}{\rho_P'}\Big)'(t)dt \\ =& \underset{x\to+\infty}{O}\Big(\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\Big) - \int_x^{+\infty}\rho_P(t)dt\left\{\frac{-\frac{1}{2}-\varepsilon}{t^{\frac{3}{2}+\varepsilon}V'(t)}\frac{\rho_P}{\rho_P'}(t) - \frac{V''(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^2}\frac{\rho_P}{\rho_P'} + \frac{1}{t^{\frac{1}{2}+\varepsilon}V'(t)}\Big(\frac{\rho_P}{\rho_P'}\Big)'(t)\right\} \end{aligned} \tag{4.7.6}$$

By the same argument as before, the last integral is of the form $\int_x^{+\infty}\underset{t\to+\infty}{O}\Big(\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^2}\Big)dt$ so if $I_2(x) := \int_x^{+\infty}\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^2}dt = \underset{x\to+\infty}{O}\Big(\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\Big)$ then so is I_1. By integration by parts, we obtain:

$$\begin{aligned} I_2(x) = \rho_P(x)\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\frac{\rho_P}{\rho_P'}(x) - \int_x^{+\infty}\rho_P(t)dt\left\{\Big(\frac{\rho_P}{\rho_P'}\Big)'(t)\frac{1}{t^{\frac{1}{2}+\varepsilon}V'(t)^2} - \frac{\rho_P}{\rho_P'}(t)\Big(\frac{1}{t^{\frac{3}{2}+\varepsilon}V'(t)^2} + \frac{2V''(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^3}\Big)\right\} \\ = \underset{x\to+\infty}{O}\Big(\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\Big) + \int_x^{+\infty}\underset{t\to+\infty}{O}\Big(\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^3}\Big)dt \end{aligned}$$

The last integral is a $\underset{x\to+\infty}{O}\Big(\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\Big)$ because, again, by integration by part:

$$\int_x^{+\infty}\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^3}dt = \rho_P(x)\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)^3}\frac{\rho_P}{\rho_P'}(x) - \int_x^{+\infty}\underset{t\to+\infty}{O}\Big(\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^4}\Big)$$

and finally

$$\int_x^{+\infty}\frac{\rho_P(t)}{t^{\frac{1}{2}+\varepsilon}V'(t)^4}dt \leqslant \frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\int_x^{+\infty}\frac{dt}{V'(t)^2} = \underset{x\to+\infty}{O}\Big(\frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\Big)$$

In the final step, we used the fact that $x \mapsto \frac{\rho_P(x)}{x^{\frac{1}{2}+\varepsilon}V'(x)}$ is decreasing in a neighborhood of $+\infty$ (which can be checked by differentiating) and that $x \mapsto \frac{1}{V'(x)^2}$ is integrable at ∞ by assumption *iv)*. Hence $R_1(x) = \underset{x\to+\infty}{O}\Big(\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)^2}\Big)$ (the exact same result can be shown at $-\infty$), which leads to the fact

$$u'(x) = \underset{|x|\to+\infty}{O}\Big(\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)}\Big), \qquad u''(x) = \underset{|x|\to+\infty}{O}\Big(\frac{1}{x^{\frac{1}{2}+\varepsilon}V'(x)}\Big) \quad \text{and} \quad u^{(3)}(x) = \underset{|x|\to+\infty}{O}\Big(\frac{1}{x^{\frac{1}{2}+\varepsilon}}\Big) \tag{4.7.7}$$

which establishes that these functions are in L^2 in a neighborhood of ∞. Since we already showed that $u \in \mathcal{C}^3(\mathbb{R}) \subset H^3_{\text{loc}}(\mathbb{R})$, it establishes that $u \in H^3(\mathbb{R}) \cap \mathcal{C}^3(\mathbb{R})$ with $u^{(3)}$ bounded. To complete the proof we just have to show that $(u')^2V^{(3)}$, $u'u''V''$, $(u')^2V''$ and $u'V'$ are bounded which is easily checked by (4.7.7) and Assumption 4.1.1 *iv)*. □

Remark 4.7.2. *We choose here the functions that vanishes at infinity at worst like $|x|^{-1/2-\varepsilon}$, but functions like $x \mapsto |x|^{-1/2}\log^{-1/2-\varepsilon}|x|$ or $x \mapsto |x|^{-1/2}\log^{-1/2}|x|\log^{-1/2-\varepsilon}\log|x|$ also work, the proof being the same. The only hypotheses that we use is that $f \in H^1(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$, that $f' = \underset{|x|\to+\infty}{O}(f(x))$ and that f is decreasing (resp. increasing) in a neighborhood of + (resp.-) ∞.*

4.8 Appendix: proof of Theorem 4.6.2

In order to analyze $\mathcal{A}$, we let, for $u \in L^2(\mathbb{R})$,

$$\mathcal{S}u := \rho_P^{1/2}\mathcal{A}\rho_P^{-1/2}u\,.$$

Note that $u \in \big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big) \mapsto \rho_P^{-1/2}u \in \big(L^2(\rho_P), \|.\|_{L^2(\rho_P)}\big)$ is an isometry. It turns out that it will be easier to study first the operator $\mathcal{S}$ in order to get the spectral properties of $\mathcal{A}$.

Proposition 4.8.1. *The operator $\mathcal{S}$ is a Schrödinger operator: it admits the following expression for all $u \in C_c^2(\mathbb{R})$: $\mathcal{S}u = -u'' + w_P u$ with*

$$w_P = \frac{1}{2}\left(\frac{1}{2}V'^2 - V'' + 2PV'\mathcal{H}[\rho_P] - 2P\mathcal{H}[\rho_P'] + 2P^2\mathcal{H}[\rho_P]^2\right) = \frac{1}{2}\Big[(\log\rho_P)'' + \frac{1}{2}(\log\rho_P)'^2\Big]\,.$$

Furthermore, w_P is continuous and we have $w_P(x) \underset{\infty}{\sim} \dfrac{V'(x)^2}{4} \underset{|x|\to\infty}{\longrightarrow} +\infty$.

Proof. We compute directly

$$\begin{aligned}
\frac{\Big(\rho_P\big(\rho_P^{-1/2}u\big)'\Big)'}{\rho_P} &= \big(\rho_P^{-1/2}u\big)'' + \frac{\rho_P'}{\rho_P}\big(\rho_P^{-1/2}u\big)' \\
&= \big(\rho_P^{-1/2}u' - \frac{1}{2}\rho_P^{-3/2}\rho_P'u\big)' + \rho_P'\rho_P^{-3/2}u' - \frac{1}{2}\rho_P^{-5/2}\big(\rho_P'\big)^2u \\
&= \rho_P^{-1/2}u'' + \frac{1}{4}\rho_P^{-5/2}\big(\rho_P'\big)^2u - \frac{1}{2}\rho_P^{-3/2}\rho_P''u \\
&= \rho_P^{-1/2}\Big[u'' + \frac{1}{4}\rho_P^{-2}\big(\rho_P'\big)^2u - \frac{1}{2}\rho_P^{-1}\rho_P''u\Big] \\
&= \rho_P^{-1/2}\left(u'' - \frac{1}{2}\Big[\big(\frac{\rho_P''}{\rho_P}\big) - \frac{1}{2}\big(\frac{\rho_P'}{\rho_P}\big)^2\Big]u\right) \\
&= \rho_P^{-1/2}\left(u'' - \frac{1}{2}\Big[(\log\rho_P)'' + \frac{1}{2}(\log\rho_P)'^2\Big]u\right) = \rho_P^{-1/2}\Big(u'' - w_Pu\Big)\,.
\end{aligned}$$

Now, using Lemma 4.2.2, we have

$$w_P = \frac{1}{2}\left(\frac{1}{2}V'^2 - V'' + 2PV'\mathcal{H}[\rho_P] - 2P\mathcal{H}[\rho_P'] + 2P^2\mathcal{H}[\rho_P]^2\right)\,.$$

Notice that $\mathcal{H}[\rho_P']$ and $\mathcal{H}[\rho_P]$ are bounded since they belong to $H^1(\mathbb{R})$, as we showed in Lemma 4.2.2 that ρ_P is $H^2(\mathbb{R})$. Along with Assumption 4.1.1 *iii)* and Lemma 4.2.3, we deduce $w_P(x) \underset{\infty}{\sim} \dfrac{1}{4}V'^2(x)$. □

Remark 4.8.2. *Note that the function w_P need not be positive on $\mathbb{R}$. In fact, neglecting the terms involving the Hilbert transforms of ρ_P and ρ_P', w_P would only be positive outside of a compact set. However, using the positivity of $\mathcal{A}$, which will be shown further in the article, we can show that the operator $-u'' + w_Pu$ is itself positive on $L^2(\mathbb{R})$. It can also be checked that, by integration by parts, $\mathcal{S}$ is self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R})$ with the inner product of $L^2(\mathbb{R})$.*

We now introduce an extension of $\mathcal{S}$ by defining its associated bilinear form.

Definition 4.8.3 (Quadratic form associated to $\mathcal{S}$)**.**
Let $\alpha \geqslant 0$ such that $w_P + \alpha \geqslant 1$. We define the quadratic form associated to $\mathcal{S} + \alpha I$, defined for all $u \in \mathcal{C}_c^\infty(\mathbb{R})$ by

$$q_\alpha(u,u) := \int_{\mathbb{R}} u'^2dx + \int_{\mathbb{R}} u^2(w_P + \alpha)dx$$

This quadratic form can be extended to a larger domain denoted by $Q(\mathcal{S}+\alpha I)$, called the *form domain* of the operator $\mathcal{S}+\alpha I$. By the theory of Schrödinger operators, it is well-known (see [Dav96][Theorem 8.2.1]) that such a domain is given by

$$Q(\mathcal{S}+\alpha I) = \Big\{u \in H^1(\mathbb{R}), u(w_P+\alpha)^{1/2} \in L^2(\mathbb{R})\Big\} = \big\{u \in H^1(\mathbb{R}), uV' \in L^2(\mathbb{R})\big\} =: H^1_{V'}(\mathbb{R})\,.$$

The space $H^1_{V'}(\mathbb{R})$ can be seen to be the completion of $\mathcal{C}^\infty_c(\mathbb{R})$ under the norm q_α. Now that the quadratic form associated to $\mathcal{S}+\alpha I$ has been extended to its form domain, it is possible to go back to the operator and extend it by its Friedrichs extension.

Theorem 4.8.4 (Friedrichs extension of $\mathcal{S}+\alpha I$)**.**
There exists an extension $(S+\alpha I)_F$ of the operator $\mathcal{S}+\alpha I$, called the Friedrichs extension of $\mathcal{S}+\alpha I$ defined on $\mathcal{D}\Big((S+\alpha I)_F\Big) = \Big\{u \in H^1_{V'}(\mathbb{R}), -u'' + (w_P+\alpha)u \in L^2(\mathbb{R})\Big\}$.

Proof. We denote

$$\mathcal{D}\Big((S+\alpha I)_F\Big)$$
$$:= \Big\{v \in H^1_{V'}(\mathbb{R}), u \in H^1_{V'}(\mathbb{R}) \longmapsto q_\alpha(u,v) \text{ can be extended to a continuous linear form on } L^2(\mathbb{R})\Big\}$$

If $v \in \mathcal{D}\Big((S+\alpha I)_F\Big)$, by Riesz's theorem there exists a unique $f_v \in L^2(\mathbb{R})$ such that $q_\alpha(u,v) = \langle u, f_v\rangle_{L^2(dx)}$ holds for all $u \in L^2(\mathbb{R})$ and we can set $(S+\alpha I)_F v := f_v$. Note that it is indeed a way of extending $\mathcal{S}+\alpha I$ since for all $u,v \in \mathcal{C}^\infty_c(\mathbb{R})$, $q_\alpha(u,v) = \langle u, (\mathcal{S}+\alpha I)v\rangle_{L^2(dx)}$.

We want to show that $\mathcal{D}\Big((S+\alpha I)_F\Big) = \Big\{u \in H^1_{V'}(\mathbb{R}), -u''+(w_P+\alpha)u \in L^2(\mathbb{R})\Big\}$. Let $f \in \mathcal{D}\Big((S+\alpha I)_F\Big)$ and $g := (S+\alpha I)_F f \in L^2(\mathbb{R})$. By definition of q_α, for all $u \in \mathcal{C}^\infty_c(\mathbb{R})$:

$$\int_{\mathbb{R}} gu dx = \int_{\mathbb{R}} f'u' dx + \int_{\mathbb{R}} (w_P+\alpha) fu dx = -\int_{\mathbb{R}} fu'' dx + \int_{\mathbb{R}} (w_P+\alpha) fu dx$$

Therefore in the sense of distributions, we get $-f'' + (w_P+\alpha) = g$ which is a function in $L^2(\mathbb{R})$, hence $f \in \Big\{u \in H^1_{V'}(\mathbb{R}), -u'' + (w_P+\alpha)u \in L^2(\mathbb{R})\Big\}$. Conversely, if $f \in H^1_{V'}(\mathbb{R})$ such that $-f'' + (w_P+\alpha)f \in L^2(\mathbb{R})$, it is possible to extend $u \mapsto q_\alpha(f,u)$ to a continuous linear form on $L^2(\mathbb{R})$ by

$$u \mapsto \int_{\mathbb{R}} u\Big(-f'' + f(w_P+\alpha)\Big)dx$$

which concludes the fact that $\mathcal{D}\Big((S+\alpha I)_F\Big) = \Big\{u \in H^1_{V'}(\mathbb{R}), -u'' + (w_P+\alpha)u \in L^2(\mathbb{R})\Big\}$. □

In the following, we will deal only with $(\mathcal{S}+\alpha I)_F : \mathcal{D}\Big((S+\alpha I)_F\Big) \longrightarrow L^2(\mathbb{R})$ and denote it $\mathcal{S}+\alpha I : \mathcal{D}(\mathcal{S}+\alpha I)$.

Remark 4.8.5. *Note that in the previous proof, the application of Riesz's theorem doesn't allow to say that $(\mathcal{S}+\alpha I) : v \in \Big(\mathcal{D}(S+\alpha I), \|.\|_{q_\alpha}\Big) \mapsto f_v \in \Big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\Big)$, where $\|.\|_{q_\alpha}$ stands for the norm associated to the bilinear positive definite form q_α, is continuous. It can be seen by the fact that $v \in \Big(\mathcal{D}(S+\alpha I), \|.\|_{q_\alpha}\Big) \mapsto q(.,v) \in \Big(L^2(\mathbb{R})', \|.\|_{L^2(dx)'}\Big)$, where $L^2(\mathbb{R})'$ stands for the topological dual of $L^2(\mathbb{R})$ equipped with its usual norm, is not continuous. Indeed the $\|.\|_{q_\alpha}$ norm doesn't control the second derivative of v and hence doesn't provide any module of continuity for the $L^2(\mathbb{R})$-extended linear form $q(.,v)$.*

Also note that, even though it would be convenient that $\mathcal{D}\Big((S+\alpha I)_F\Big) = L^2(\mathbb{R}, (w_P+\alpha)^2 dx) \cap H^2(\mathbb{R})$ it is not true without more properties on w_P. Such a result holds, for example when w_P belongs to B_2, the class of reverse Hölder weights, see [ABA07][Theorem 1.1].

Theorem 4.8.6 (Inversion of $\mathcal{S}+\alpha I$)**.**
For every $f \in L^2(\mathbb{R})$, there exists a unique $u \in \mathcal{D}\Big((S+\alpha I)_F\Big)$ such that $(\mathcal{S}+\alpha I)u = f$. Furthermore, the map $(S+\alpha I)^{-1}$ is continuous from $\big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$ to $\big(\mathcal{D}(S+\alpha I), \|.\|_{q_\alpha}\big)$.

Proof. Let $f \in L^2(\mathbb{R})$, the map $u \longmapsto \langle u, f\rangle_{L^2(dx)}$ is continuous on $\big(H^1_{V'}(\mathbb{R}), \|.\|_{q_\alpha}\big)$ which is a Hilbert space. Therefore by Riesz's theorem, there exists a unique $v_f \in H^1_{V'}(\mathbb{R})$ such that for all $u \in H^1_{V'}(\mathbb{R})$, $\langle f, u\rangle_{L^2(dx)} = q_\alpha(v_f, u)$ from which we deduce that, in the sense of distributions, $f = -v_f'' + (w_P + \alpha)v_f$ which implies that $v_f \in \mathcal{D}(S+\alpha I)$. Since $v_f \in \mathcal{D}(S+\alpha I)$, we have then for all $u \in L^2(\mathbb{R})$, $\langle f, u\rangle_{L^2(dx)} = q_\alpha(v_f, u) = \langle (\mathcal{S}+\alpha)v_f, u\rangle_{L^2(dx)}$, hence $(\mathcal{S}+\alpha I)v_f = f$. Finally, by Riesz's theorem, $f \in L^2(\mathbb{R}) \mapsto v_f \in H^1_{V'}(\mathbb{R})$ is continuous hence so is $(\mathcal{S}+\alpha I)^{-1}$. □

Remark 4.8.7. *It would be tempting to use Banach's isomorphism theorem to say that since $(\mathcal{S}+\alpha I)^{-1}$ is bijective and continuous, so must be $\mathcal{S}+\alpha I$. But since $\big(\mathcal{D}(S+\alpha I), \|.\|_{q_\alpha}\big)$ is not a Banach space (it's not closed in $H^1_{V'}(\mathbb{R})$) we can't apply it.*

We are now able to diagonalize the resolvent of $\mathcal{S}$.

Theorem 4.8.8 (Diagonalization of $(\mathcal{S}+\alpha I)^{-1}$)**.**
There exists a complete orthonormal set $(\psi_n)_{n\geqslant 0}$ of $L^2(\mathbb{R})$ (meaning that

$$\overline{\mathrm{Span}\{\psi_n,\ n \geqslant 0\}}^{\|.\|_{L^2(dx)}} = L^2(\mathbb{R})$$

and $\langle \psi_i, \psi_j\rangle_{L^2(dx)} = \delta_{i,j}$), where each $\psi_n \in \mathcal{D}(S+\alpha I)$ and $\big(\mu_n(\alpha)\big)_{n\geqslant 0} \in [0,1]^{\mathbb{N}}$ with $\mu_n(\alpha) \underset{N\to\infty}{\longrightarrow} 0$ such that $(\mathcal{S}+\alpha I)^{-1}\psi_n = \mu_n(\alpha)\psi_n$ for all $n \geqslant 0$. We also have $\Big|\Big|\Big|\big(\mathcal{S}+\alpha I\big)^{-1}\Big|\Big|\Big|_{\mathcal{L}\left(L^2(dx)\right)} \leqslant 1$.

Proof. By Proposition 4.8.1, $w_P + \alpha$ is continuous and goes to infinity at infinity. By Rellich criterion [RS78][see Theorem XIII.65], the unit ball of $\mathcal{D}(S+\alpha I)$, *ie* the set

$$\Big\{u \in \mathcal{D}(S+\alpha I),\ \int_{\mathbb{R}} u'^2 + \int_{\mathbb{R}} (w_P+\alpha)u^2 \leqslant 1\Big\}$$

considered as a subset of $L^2(\mathbb{R})$ is relatively compact in $\big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$. Hence, we can conclude that the injection $\iota : \big(\mathcal{D}(S+\alpha I), \|.\|_{q_\alpha}\big) \longrightarrow \big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$ is a compact operator. Since $(S+\alpha I)^{-1} : \big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big) \longrightarrow \big(\mathcal{D}(S+\alpha I), \|.\|_{q_\alpha}\big)$ is continuous then $(\mathcal{S}+\alpha I)^{-1}$ is compact from $\big(L^2(\mathbb{R}), \|.\|_{L^2(dx)}\big)$ to itself. The fact that $(\mathcal{S}+\alpha I)^{-1}$ is self-adjoint and positive allows us to apply the spectral theorem to obtain $\big(\mu_n(\alpha)\big)_{n\geqslant 0}$ positive eigenvalues verifying $\mu_n(\alpha) \underset{N\to\infty}{\longrightarrow} 0$ by compactness and a Hilbertian basis $(\psi_n)_{n\geqslant 0} \in L^2(\mathbb{R})^{\mathbb{N}}$, such that for all $n \geqslant 0$, $(\mathcal{S}+\alpha I)^{-1}\psi_n = \mu_n(\alpha)\psi_n$. It is then easy to see that for all n, $\psi_n \in \mathcal{D}(S+\alpha I)$ since they belong to the range of $(\mathcal{S}+\alpha I)^{-1}$. Finally, since for all $\phi \in L^2(\mathbb{R})$, $\langle (\mathcal{S}+\alpha I)\phi, \phi\rangle_{L^2(dx)} \geqslant \|\phi\|^2_{L^2(dx)}$, the spectrum of $(\mathcal{S}+\alpha I)^{-1}$ is contained in $[0,1]$. It allows us to conclude that $\big|\big|\big|(\mathcal{S}+\alpha I)^{-1}\big|\big|\big|_{L^2(dx)} \leqslant 1$. □

Since for all $u \in H^1_{V'}(\mathbb{R})$, $(S+\alpha)u \in L^2(\mathbb{R})$ *iff* $\mathcal{S}u \in L^2(\mathbb{R})$, if we define $\mathcal{D}(\mathcal{S})$ in the same manner that we did before, $\mathcal{D}(\mathcal{S}) = \mathcal{D}(S+\alpha I)$. It is now straightforward to see how to extend $\mathcal{A} = \rho_P^{-1/2}\mathcal{S}\rho_P^{1/2}$ on $\mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}) := \rho_P^{-1/2}\mathcal{D}(\mathcal{S})$ equipped with the norm $\|.\|_{q_\alpha,\rho_P}$ to $\big(L^2(\rho_P), \|.\|_{L^2(\rho_P)}\big)$. The norm $\|.\|_{q_\alpha,\rho_P}$ is defined for all $u \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$ by

$$\|u\|_{q_\alpha,\rho_P} = \int_{\mathbb{R}} u'^2\rho_P dx + \int_{\mathbb{R}} u^2(w_P+\alpha)\rho_P dx\,.$$

It is easy to see that $(\mathcal{A}+\alpha I)^{-1}$ is continuous.

Remark 4.8.9. *The kernel of $\mathcal{A}$ is generated by the function $\widetilde{1}$. Indeed if $\phi \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$ is in the kernel of $\mathcal{A}$ then*

$$0 = -\frac{\left(\phi'\rho_P\right)'}{\rho_P} \Rightarrow \exists c \in \mathbb{R},\ \phi' = \frac{c}{\rho_P}$$

But since ϕ' is in $L^2(\rho_P)$ then $c = 0$ which implies that ϕ is constant. We must restrict $\mathcal{A}$ to the orthogonal of $Ker\mathcal{A}$ with respect to the inner product of $L^2(\rho_P)$, ie

$$\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) := \left\{ u \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A}) \,|\, \int_{\mathbb{R}} u\rho_P = 0 \right\}.$$

Doing so makes $\mathcal{A}$ injective.

Before inverting $\mathcal{A}$, we need the following lemma:

Lemma 4.8.10. *The following equality holds*

$$(\mathcal{A} + \alpha I)\Big(\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})\Big) = L^2_0(\rho_P) := \left\{ u \in L^2(\rho_P),\ \int_{\mathbb{R}} u\rho_P dx = 0 \right\}$$

Proof. Let $\phi = \widetilde{c}$ for $c \in \mathbb{R}$, $(\mathcal{A} + \alpha I)\phi = \widetilde{\alpha c}$ then $(\mathcal{A} + \alpha I)(\mathbb{R}.\widetilde{1}) = \mathbb{R}\widetilde{1}$. Hence since $\mathcal{A} + \alpha I$ is self-adjoint with respect to the inner product of $L^2(\rho_P)$ and that $\mathbb{R}\widetilde{1}$ is stable by $\mathcal{A} + \alpha I$, then $(\mathcal{A} + \alpha I)\Big((\mathbb{R}.\widetilde{1})^\perp \cap \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})\Big) \subset (\mathbb{R}.\widetilde{1})^\perp$. For the converse, let $u \in (\mathbb{R}.\widetilde{1})^\perp$, since $\mathcal{A}+\alpha I$ is bijective, there exists $v \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$ such that $u = (\mathcal{A} + \alpha I)v$. For all $w \in \mathbb{R}.\widetilde{1}$,

$$0 = \langle u, w\rangle_{L^2(\rho_P)} = \langle (\mathcal{A} + \alpha I)v, w\rangle_{L^2(\rho_P)} = \langle v, (\mathcal{A} + \alpha I)w\rangle_{L^2(\rho_P)}$$

Hence $v \in \left[(\mathcal{A} + \alpha I)(\mathbb{R}\widetilde{1})\right]^\perp = \mathbb{R}\widetilde{1}^\perp$ and so $(\mathbb{R}.\widetilde{1})^\perp \subset (\mathcal{A} + \alpha I)\Big((\mathbb{R}.\widetilde{1})^\perp\Big)$. □

It is easy to see that $L^2_0(\rho_P)$ is a closed subset of $L^2(\rho_P)$ as it is the kernel of the linear form $\phi \in L^2(\rho_P) \mapsto \left\langle \phi, \widetilde{1} \right\rangle_{L^2(\rho_P)}$, making it a Hilbert space.

Proposition 4.8.11 (Diagonalization and invertibility of $\mathcal{A}$)**.** *There exists a complete orthonormal set of $\Big(L^2_0(\rho_P), \langle ., . \rangle_{L^2(\rho_P)}\Big)$, $(\phi_n)_{n\in\mathbb{N}} \in \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})^{\mathbb{N}}$ such that $\mathcal{A}\phi_n = \lambda_n\phi_n$ (meaning that*

$$\overline{\mathrm{Span}\{\phi_n,\ n \geqslant 0\}}^{\|.\|_{L^2(\rho_P)}} = L^2_0(\rho_P)$$

and $\langle \phi_i, \phi_j\rangle_{L^2(\rho_P)} = \delta_{i,j}$). Furthermore, $\mathcal{A} : \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \longrightarrow L^2_0(\rho_P) := \left\{ u \in L^2(\rho_P),\ \int_{\mathbb{R}} u\rho_P dx = 0 \right\}$ is bijective, $\mathcal{A}^{-1}$ is continuous when considered as an operator of $L^2_0(\rho_P)$.

Proof. Since $(\mathcal{S} + \alpha I)^{-1}$ considered as an operator of $L^2(\mathbb{R})$, is compact so is $(\mathcal{A} + \alpha I)^{-1}$ on $L^2(\rho_P)$ and since $\mathcal{A}$ is self-adjoint, by the spectral theorem, $(\mathcal{A} + \alpha I)^{-1}$ is diagonalizable. With the notations of Theorem 4.8.8, $(\mathcal{A}+\alpha I)^{-1}$ has eigenvalues $\big(\mu_n(\alpha)\big)_{n\geqslant 0}$ and corresponding eigenfunctions $\phi_n = \rho_P^{-1/2}\psi_n \in \mathcal{D}_{L^2(\mathbb{R})}(\mathcal{A})$. Hence for all $n \in \mathbb{N}$, $\mathcal{A}\phi_n = \lambda_n\phi_n$ with $\lambda_n := \big(\frac{1}{\mu_n(\alpha)} - \alpha\big)$. Now,

$$\lambda_n\|\phi_n\|_{L^2(\rho_P)} = \int_{\mathbb{R}} (\mathcal{A}\phi_n)\phi_n\rho_P dx = -\int_{\mathbb{R}} (\rho_P\phi_n')'\phi_n = \int_{\mathbb{R}} \phi_n'^2\rho_P \geqslant 0\,.$$

Furthermore, the kernel of $\mathcal{A}$ is $\mathbb{R}.\widetilde{1}$, thus the spectrum of $\mathcal{A}$ restricted to $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$ is positive. But since $(\mathcal{A} + \alpha I)^{-1}$ is a compact operator of $L^2(\rho_P)$ and that $(\mathcal{A} + \alpha I)$ maps $\mathbb{R}.\widetilde{1}^\perp$ to $\mathbb{R}.\widetilde{1}^\perp$ with respect

to the inner product of $L^2(\rho_P)$ (see lemma 4.8.10), then $\big(\mathcal{A}+\alpha I\big)^{-1}$ is compact as an operator from $L^2_0(\rho_P)$ to itself. By Fredholm alternative, for every $\lambda \in \mathbb{R}\ \lambda \neq 0$, either $(\mathcal{A}+\alpha I)^{-1} - \lambda I$ is bijective either $\lambda \in Sp\big((\mathcal{A}+\alpha I)^{-1}\big)$. These conditions are equivalent to: either $\mathcal{A} + (\alpha - \frac{1}{\lambda})I$ is bijective as an operator from $\mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A})$ to $L^2_0(\rho_P)$, either $-\alpha + \frac{1}{\lambda} \in Sp\big(\mathcal{A}\big)$. If we set $\lambda = \frac{1}{\alpha}$ then either $\mathcal{A}$ is bijective either $0 \in Sp(\mathcal{A})$, since the latter is wrong then $\mathcal{A} : \mathcal{D}_{L^2(\mathbb{R}),0}(\mathcal{A}) \to L^2_0(\rho_P)$ is bijective. The spectrum of $\mathcal{A}$ is $\left(\frac{1}{\mu_n(\alpha)} - \alpha\right)_{n\geqslant 0} \subset (\lambda_1, +\infty) \subset (0, +\infty)$, where λ_1 is the smallest eigenvalue, hence we deduce that $\left\|\mathcal{A}^{-1}\right\|_{\mathcal{L}(L^2(\rho_P))} \leqslant \lambda_1^{-1}$. □

Chapter 5

CLT for β ensembles at high-temperature, and for integrable systems: a transfer operator approach

Abstract In this paper, we prove a polynomial Central Limit Theorem for several integrable models, and for the β ensembles at high-temperatures with polynomial potential. Furthermore, we are able to relate the mean values, the variances and the correlations of the moments of these integrable systems with the one of the β ensembles. Moreover, we show that for several integrable models, the local functions' space-correlations decay exponentially fast.

5.1 Introduction

In this paper, we study eigenvalue fluctuations for several random matrix models related to some integrable dynamical systems and to the classical β ensembles in the high-temperature regime. Specifically, we consider random band matrices with fixed bandwidth and, under some mild assumptions, we prove a central limit theorem (CLT) for polynomial test functions for the empirical measure of the eigenvalues. In particular, we consider the following kind of matrices

- Type 1-*i*) **Periodic Jacobi matrices**, which are periodic tridiagonal matrix of the form

$$\begin{pmatrix} a_1 & b_1 & 0 & \dots & b_N \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ b_N & \dots & 0 & b_{N-1} & a_N \end{pmatrix}, \tag{5.1.1}$$

  for $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$, $\mathbf{b} = (b_1, \dots, b_N) \in \mathbb{R}_+^N$.

- Type 1-*ii*) **Antisymmetric Bidiagonal Periodic matrices**:

$$\begin{pmatrix} 0 & a_1 & 0 & \dots & -a_N \\ -a_1 & 0 & a_2 & \ddots & \vdots \\ 0 & -a_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ a_N & \dots & 0 & -a_{N-1} & 0 \end{pmatrix}, \tag{5.1.2}$$

for $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}_+^N$.

- Type 1-*iii*) **Periodic CMV (after Cantero, Moral and Velazquez) matrices**, which are $2N \times 2N$ unitary matrices given by

$$\mathcal{E} = \mathcal{L}\mathcal{M}\,, \tag{5.1.3}$$

where we define $\mathcal{L}$ and $\mathcal{M}$ in the following way. Let $\mathbf{a} = (a_1, \dots a_{2N})$ be complex numbers of the unit disk $\mathbb{D}$. Define the 2×2 unitary matrix Ξ_j

$$\Xi_j = \begin{pmatrix} \overline{a}_j & \rho_j \\ \rho_j & -a_j \end{pmatrix}, \quad j = 1, \dots, 2N\,, \quad \rho_j = \sqrt{1 - |a_j|^2}\,. \tag{5.1.4}$$

Then, $\mathcal{L}$ and $\mathcal{M}$ are the $2N \times 2N$ matrices

$$\mathcal{L} = \begin{pmatrix} \Xi_1 & & & \\ & \Xi_3 & & \\ & & \ddots & \\ & & & \Xi_{2N-1} \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} -a_{2N} & & & & \rho_{2N} \\ & \Xi_2 & & & \\ & & \ddots & & \\ & & & \Xi_{2N-2} & \\ \rho_{2N} & & & & \overline{a}_{2N} \end{pmatrix}. \tag{5.1.5}$$

The matrix $\mathcal{E}$ is a pentadiagonal periodic matrix and is unitary.

- Type 1-*iv*) **Two diagonals periodic matrices** given by

$$\begin{pmatrix} 0 & a_1 & 0 & \cdots & b_{N-r+1} & 0 & 0 & 0 \\ 0 & 0 & a_2 & \cdots & 0 & b_{N-r+2} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & a_{r-1} & 0 & 0 & b_N \\ b_1 & 0 & \cdots & \cdots & 0 & a_r & 0 & 0 \\ 0 & b_2 & 0 \cdots & & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & a_{N-1} \\ a_N & 0 & \cdots & b_{N-r} & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \\ r+1 \text{ row} \\ \longleftarrow \\ \\ \\ \end{matrix} \tag{5.1.6}$$

$\downarrow N - r$ column

Where $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^N$. In applications, we consider either $a_1 = a_2 = \ldots = a_N = 1$ or $b_1 = b_2 = \ldots = b_N = 1$.

We also consider the non-periodic counterparts of the previous matrices. More specifically:

- Type 2-*i*) **Jacobi matrices**, which are symmetric tridiagonal matrices

$$\begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ & & & b_{N-1} & a_N \end{pmatrix}, \tag{5.1.7}$$

where $\mathbf{a} \in \mathbb{R}^N$ and $\mathbf{b} \in \mathbb{R}_+^{N-1}$.

- Type 2-*ii*) **Bidiagonal Antisymmetric matrices**:

$$\begin{pmatrix} 0 & a_1 & 0 & & \\ -a_1 & 0 & a_2 & \ddots & \\ 0 & -a_2 & 0 & \ddots & 0 \\ & \ddots & \ddots & \ddots & a_{N-1} \\ & & 0 & -a_{N-1} & 0 \end{pmatrix}, \tag{5.1.8}$$

for $\mathbf{a} \in \mathbb{R}_+^{N-1}$.

- Type 2-*iii*) **CMV matrices**, $2N \times 2N$ unitary matrices of the form

$$\mathfrak{E} = \mathfrak{LM}, \tag{5.1.9}$$

where

$$\mathfrak{L} = \operatorname{diag}(\Xi_0, \Xi_2, \Xi_4, \dots, \Xi_{2N}) \qquad \text{and} \qquad \mathfrak{M} = \operatorname{diag}(\Xi_1, \Xi_3, \Xi_5 \dots, \Xi_{2N-1}), \tag{5.1.10}$$

and the blocks Ξ_j, $j = 1, \dots, 2N-1$ are defined in (5.1.4), while $\Xi_0 = (1)$ and $\Xi_{2N} = (\overline{a}_{2N})$ are 1×1 matrices.

The periodic matrices that we consider are the *Lax matrices* of some integrable models. These are particular dynamical systems that are Liouville integrable, and their integrability is proved obtaining a Lax pair (L, A) [Lax68] representation of the model, meaning that the equations of motions for each of these systems are equivalent to the following linear system for some matrices L, A

$$\dot{L} \equiv \frac{\mathrm{d}L}{\mathrm{d}t} = [L; A] = LA - AL\,. \tag{5.1.11}$$

This formulation is useful since it implies that $\{\mathrm{Tr}L^k\}_{k=1}^N$ are a system of independent constants of motion $\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}L^k = 0\right)$ for the system at hand, so the system is integrable in classical sense. We call these quantities *conserved fields*.

Specifically, the Toda lattice [Tod89] and the Exponential Toda lattice [GGGM23] have as Lax matrix a periodic Jacobi matrix, the Volterra lattice [GGGM23] has an antisymmetric periodic one, the Ablowitz-Ladik lattice [AL75] and the Schur flow [Gol06] have a periodic CMV one, and the family of Itoh–Narita–Bogoyavleskii [Bog88] lattices have a bidiagonal periodic one.

We endow the periodic matrices L of type 1 with the so-called *Generalized Gibbs Ensemble* of the corresponding dynamical system. The main property of these measures is that they are invariant with respect to the dynamics of the corresponding integrable system. These Generalized Gibbs Ensembles have the form

$$\mu_N^{(1)} = \frac{1}{Z_N^{(1)}(\alpha, G)} \left(\prod_{j=1}^N F(x_j, \alpha) \right) e^{-\mathrm{Tr}G(L)} \mathrm{d}\mathbf{x}\,, \tag{5.1.12}$$

where $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ for Jacobi matrices, $\mathbf{x} = \mathbf{a}$ for antisymmetric matrix, $\mathbf{x} = \mathbf{a}$ for periodic CMV matrices, $\mathbf{x} = \mathbf{a}$ (resp. $\mathbf{x} = \mathbf{b}$) for bidiagonal periodic matrices if $b_1 = \ldots = b_N = 1$ (resp. if $a_1 = \ldots = a_N = 1$) and G is a real-valued continuous function.

The non-periodic matrices of type 2 are related to the classical β ensembles, indeed both the real β ensemble and the Laguerre β ensemble [DE06] can be represented through a Jacobi matrix, the circular and the Jacobi ensemble have a representation in terms of CMV matrices [KN04], and the Antisymmetric β ensemble has a representation in terms of an antisymmetric matrix [DF10]. Specifically, we consider these ensembles in the high-temperature regime, meaning that the parameter β is not fixed, but scales

with the matrix size N as $\beta = \frac{2\alpha}{N}$, $\alpha \in \mathbb{R}_+$. The joint density for the entries of the matrix representation reads

$$\mu_N^{(2)} = \frac{1}{Z_N^{(2)}(\alpha, G)} \left(\prod_{j=1}^{N-1} F\left(x_j, \alpha \left(1 - \frac{j}{N} \right) \right) \right) R(x_N) e^{-\mathrm{Tr}G(L)} \mathrm{d}\mathbf{x}\,, \tag{5.1.13}$$

where $\mathbf{x} = (\mathbf{a}, \mathbf{b})$ for tridiagonal matrices, $\mathbf{x} = \mathbf{a}$ for the antisymmetric one and $\mathbf{x} = \mathbf{a}$ for CMV one.

As we already mentioned, we focus on the fluctuations (or *linear statistics*) around the equilibrium measure of these general models, where we choose the functions F and G so that the partition functions

$$\begin{aligned} Z_N^{(1)}(\alpha, G) &= \int_{X^N} \left(\prod_{j=1}^{N} F(x_j, \alpha) \right) e^{-\mathrm{Tr}G(L)} \mathrm{d}\mathbf{x} \\ Z_N^{(2)}(\alpha, G) &= \int_{X^N} \left(\prod_{j=1}^{N-1} F\left(x_j, \alpha \left(1 - \frac{j}{N} \right) \right) \right) R(x_N) e^{-\mathrm{Tr}G(L)} \mathrm{d}\mathbf{x} \end{aligned} \tag{5.1.14}$$

are finite for all N. Here X is a subset of $\mathbb{R}^d$ ($\mathbb{C}$ being identified with $\mathbb{R}^2$).

Specifically, we study the fluctuations of polynomial test functions $Q(z)$, i.e.

$$\int_{\mathbb{C}} Q \mathrm{d}\nu_N - \int_{\gamma} Q \mathrm{d}\nu\,, \tag{5.1.15}$$

where ν_N is the empirical measure of eigenvalues of L given by

$$\nu_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(L)}\,. \tag{5.1.16}$$

Here the $\lambda_j(L)$ are the eigenvalues of L and δ_x is the Dirac delta function centred at x, ν is the equilibrium measure (or *density of states*) of the system and γ is the support of the measure ν. In this paper, we are able to analyse the random variable (5.1.15) for polynomial potentials G, using a transfer operator technique.

The study of spectral properties of Lax matrices of integrable models was initiated by Spohn in [Spo20c], see also [Maz22]. In this paper, the author investigated time correlation functions for the Toda lattice. Applying the theory of *Generalized Hydrodynamic* [Doy20], Spohn argued that they have a ballistic behaviour, meaning that they have symmetrically located peaks, which travel in opposite directions at constant speed and decay as t^{-1} when $t \to \infty$. To obtain this result, Spohn had to compute the density of states of the Toda Lax matrix; he did it by connecting the Generalized Gibbs Ensemble of the Toda lattice to the real β ensemble in the high-temperature regime [ABG12]. After that, the Generalized Gibbs ensemble for the Ablowitz-Ladik lattice and the Schur flow were connected to the Circular β ensemble and the Jacobi β ensemble in the high-temperature regime [HL21, FM21] respectively by one of the present authors and T. Grava [GM23] and, independently, by H. Spohn [Spo20b]. For all these models, a large deviation principle for their mean density of states were developed in [MM23b, GM22]. Furthermore, in [GGGM23] the authors were able to connect the classical Gibbs ensemble for the Exponential Toda lattice and the Volterra one to the Laguerre [DE06] and to the Antisymmetric β ensemble [DF10] respectively.

As we mentioned, our study does not only involve integrable systems, but also the classical β ensembles. Specifically, we study the random variable (5.1.15) for these ensembles in the so-called high-temperature regime. The study of these quantities was initiated by Johansson in [Joh98], where the author obtained a CLT for the Gaussian unitary ensemble with polynomial potential, then generalized for other models and other values of β in [BG13b, Shc14a, DP12], and, more recently, in [BMP22], where the authors obtained also a rigidity result for the eigenvalues of the β ensembles. We mention also the work [NT18], where the author obtained a CLT for the Gaussian β ensemble in the high-temperature regime for a quadratic potential, the work [HL21] where the authors obtained a CLT for the Circular

β ensemble at high-temperature using a normal approximation method, and the recent paper [DGM23] where the authors obtained a CLT for the real β ensemble in the high-temperature regime for general confining potentials. Another relevant paper that it is worth mentioning is [AS21], where the authors studied Coulomb gases in dimension $\mathrm{d} \geqslant 2$ and studied the local laws for any temperature regime. Finally, we recall also the recent work [CJ21, CJL22] where the authors obtained a super-exponential bound for the convergence for the moments of the CUE, COE and CSE to a Gaussian vector.

Statement of the results. We come to precise statements of the main results of the present paper. We consider the previously mentioned family of random matrices and make the following assumptions.

Assumption 5.1.1. *The following hypotheses are valid throughout the paper:*

HP 1. $X \subseteq \mathbb{R}^d$;

HP 2. $F(x,\eta)$ *is such that for any* $\eta > 0$ $F(\cdot,\eta) \in C^1(X)$, *and for any* $x \in X, F(x,\cdot) \in C^\infty((0,+\infty))$;

HP 3. $F(x,\eta) > 0$ *almost surely for* $x \in X$, $\eta \in (0,\infty)$;

HP 4. $F(\cdot,\eta) \in L^1(X,\mathcal{B}) \cap L^2(X,\mathcal{B})$ *for all* $\eta \in (0,\infty)$, *and* $\partial_\eta F(\cdot,\eta) \in L^1(X,\mathcal{B}) \cap L^2(X,\mathcal{B})$; *moreover there exist a* $\mathrm{c} \in \mathbb{N}$ *and a compact set* $\mathcal{O} \subseteq X$ *such that*

- $||F(x,\eta)||_2 = O(\eta^{-\mathrm{c}})$
- *There exists* $d > 0$ *such that for all* $\eta > 0$, $\int_{\mathcal{O}} F(x,\eta)\mathrm{d}x \geqslant d\eta^{-\mathrm{c}}$

HP 5. The function $\mathrm{Tr}G(L)$, *where* L *is one of the matrices of type 1 or 2, is* ***circular****, meaning that there exists some* $k \in \mathbb{N}_{\geqslant 1}$ *and two functions* $W : X^k \times X^k \to \mathbb{C}$, $W_1 : X^\ell \times X^k \to \mathbb{C}$, $\widetilde{W}_1 : X^k \to \mathbb{C}$, $\widetilde{W}_2 : X^\ell \to \mathbb{C}$, *such that writing* $N = kM + \ell$ *with* $M \geqslant 0$ *and* $0 \leqslant \ell \leqslant k-1$, *we have*

$$\mathrm{Tr}G(L) = \begin{cases} \sum_{j=1}^{M-1} W(\mathbf{x}_j, \mathbf{x}_{j+1}) + W_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1) & \text{for type 1} \\ \sum_{j=1}^{M-1} W(\mathbf{x}_j, \mathbf{x}_{j+1}) + \widetilde{W}_1(\mathbf{x}_1) + W(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0) & \\ + \widetilde{W}_2(x_{kM+1}, \dots, x_{kM+\ell})) & \text{for type 2} \end{cases}, \tag{5.1.17}$$

here $\mathbf{x}_j = (x_{(j-1)k+1}, \dots, x_{jk})$. *In this case, we say that* W *is the* ***seed*** *of* G, *and* $W_1, \widetilde{W}_1, \widetilde{W}_2$ *are the* ***weeds****.*

HP 6. The real parts of $W, W_1, \widetilde{W}_1, \widetilde{W}_2$ *are lower bounded. Furthermore,* $\exp(-W) \in L^2(X^k \times X^k)$, $\exp(-W_1) \in L^2(X^k \times X^\ell \times X^k)$, $\exp\left(-\widetilde{W}_1\right) \in L^2(X^k)$, *and* $\exp\left(-\widetilde{W}_2\right) \in L^2(X^\ell)$

HP 7. Both integrals

$$\int_{X^k} \left(\partial_\eta \prod_{q=1}^k F(x_q,\eta)\right)^2 e^{-2W(\mathbf{x},\mathbf{y})}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\,,$$

$$\int_{X^k} \left(\partial_\eta \prod_{q=1}^k F\left(x_q, \eta\left(1 - \frac{(j-1)k+q}{N}\right)\right)\right)^2 e^{-2W(\mathbf{x},\mathbf{y})}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$

are finite.

HP 8. $R(x) \in L^2(X) \cap C^\infty(X)$, *and* $R(x) > 0$ *for all* $x \in X$, *or* $R(x) = \delta_y(x)$ *for some* $y \in X$.

Here $L^p(X,\mathcal{B})$ *is the usual* L^p *space.*

Remark 5.1.2. *The definition of circular function and seeds was introduced in this context in [GPP14].*

Under these assumptions, we are able to prove our main theorem:

Theorem 5.1.3. *Under hypotheses 5.1.1. Consider $\mu_N^{(1)}$, $\mu_N^{(2)}$ (5.1.12)-(5.1.13), and let $(S_N)_{N\geqslant 1}$ be a sequence of real random variables such that there exists a function $H : \mathbb{C} \to \mathbb{R}$ such that $\mathrm{Tr}H(L)$ is circular, it satisfies HP 6., and*

$$\mathbb{E}_1\left[e^{-itS_N}\right] = \frac{Z_N^{(1)}(\alpha, G+itH)}{Z_N^{(1)}(\alpha,G)}, \quad \mathbb{E}_2\left[e^{-itS_N}\right] = \frac{Z_N^{(2)}(\alpha, G+itH)}{Z_N^{(2)}(\alpha,G)}, \tag{5.1.18}$$

are finite, here $\mathbb{E}_1[\cdot]$, $\mathbb{E}_2[\cdot]$ are the mean values taken with respect to $\mu_N^{(1)}$, $\mu_N^{(2)}$ respectively. Furthermore, let W, h be the seeds of $\mathrm{Tr}G(L)$ and $\mathrm{Tr}H(L)$ respectively, and assume that $\int_{X^{2k}} |h(\mathbf{x},\mathbf{y})|^n e^{-2W(\mathbf{x},\mathbf{y})}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$ for $n = 2, 4, 6$ are finite. Then, there exist two continuous functions

$$A(x) : \mathbb{R}_+ \longrightarrow \mathbb{R}, \tag{5.1.19}$$

$$\sigma^2(x) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \tag{5.1.20}$$

such that under $\mu_N^{(1)}$ (5.1.12)

$$\left(S_N - NA(\alpha)\right)/\sqrt{N}$$

converges to a Gaussian distribution $\mathcal{N}(0, \sigma^2(\alpha))$ as N tends to infinity, and under $\mu_N^{(2)}$ (5.1.13),

$$\left(S_N - N\int_0^1 A(\alpha x)\mathrm{d}x\right)/\sqrt{N}$$

converges to a Gaussian distribution $\mathcal{N}(0, \int_0^1 \sigma^2(\alpha x)\mathrm{d}x)$ as N tends to infinity. Furthermore, defining the free energies $\mathcal{F}_N^{(1)}(\alpha, G), \mathcal{F}_N^{(2)}(\alpha, G)$ as

$$\mathcal{F}_N^{(1)}(\alpha,G) = -\lim_{N\to\infty} \frac{1}{N}\ln\left(Z_N^{(1)}(\alpha,G)\right), \tag{5.1.21}$$

$$\mathcal{F}_N^{(2)}(\alpha,G) = -\lim_{N\to\infty} \frac{1}{N}\ln\left(Z_N^{(2)}(\alpha,G)\right), \tag{5.1.22}$$

then

i. $\mathcal{F}^{(1)}(\alpha,G) = \partial_\alpha\left(\alpha\mathcal{F}^{(2)}(\alpha,G)\right)$

ii. $A(\alpha) = i\partial_t\mathcal{F}^{(1)}(\alpha, G+itH)_{|_{t=0}}$

iii. $\int_0^1 A(\alpha x) = i\partial_t\mathcal{F}^{(2)}(\alpha, G+itH)_{|_{t=0}}$

iv. $\sigma^2(\alpha) = \partial_t^2\mathcal{F}^{(1)}(\alpha, G+itH)_{|_{t=0}}$

v. $\int_0^1 \sigma^2(\alpha x)\mathrm{d}x = \partial_t^2\mathcal{F}^{(2)}(\alpha, G+itH)_{|_{t=0}}$

Remark 5.1.4. *In the central part of the proof we introduce a family of operators acting on $L^2(X^k)$ by*

$$\mathcal{L}_{t,\alpha}f(\mathbf{y}) = \int_{X^k} f(\mathbf{x})\prod_{j=1}^k F(x_j)e^{-(W+ith)(\mathbf{x},\mathbf{y})}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.$$

Because of our assumptions, each $\mathcal{L}_{t,\alpha}$ is Hilbert-Schmidt, meaning that the kernel

$$(\mathbf{x},\mathbf{y}) \mapsto \prod_{j=1}^k F(x_j)e^{-(W+ith)(\mathbf{x},\mathbf{y})}$$

is in $L^2(X^k \times X^k)$, thus $\mathcal{L}_{t,\alpha}$ is compact, and, as we show in the proof of Theorem 5.2.1, $\mathcal{L}_{0,\alpha}$ has a simple dominating eigenvalue. Hypothesis HP 7. and the assumption that the integrals

$$\int_{X^{2k}} |h(\mathbf{x},\mathbf{y})|^n e^{-2W(\mathbf{x},\mathbf{y})} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \quad n = 2,4,6$$

are finite is merely to ensure that $(\alpha, t) \mapsto \mathcal{L}_{t,\alpha}$ is regular. In particular, it is differentiable with respect to α, and three times differentiable with respect to t as an operator valued function, which in turn ensures that for t small enough, the operator $\mathcal{L}_{t,\alpha}$ has a simple, dominating eigenvalue $\lambda(t,\alpha)$ and that $t \mapsto \lambda(t,\alpha)$ is three times differentiable. Note that we only use the existence of a second derivative with respect to t in the proof of the main theorem, but use the existence of a third derivative in the proof of the Berry-Esseen bound, Theorem 5.4.5. We use the differentiability with respect to α in the proof of Theorem 5.2.2.

In the central part of our paper, we show how to use the previous result to obtain a polynomial CLT for the integrable models that we mentioned, and for the classical β ensemble in the high-temperature regime. Specifically, we use the previous theorem with G and H polynomials and

$$S_N = \frac{1}{N}\mathrm{Tr}H(L) \overset{(5.1.16)}{=} \int_{\mathbb{C}} H(x)\mathrm{d}\nu_N(x)\,.$$

The expectations we want to compute then reduce to

$$\mathbb{E}_j\left[e^{-itS_N}\right] = \frac{Z_N^{(j)}(\alpha, G + itH)}{Z_N^{(j)}(\alpha, G)}, \qquad j = 1,2\,.$$

Furthermore, as a by-product, we are also able to compute the so-called *susceptibility matrix* $\mathcal{C}$ for integrable models. This is the matrix of the space-correlation functions of the conserved fields, i.e.

$$\mathcal{C}_{m,n} = \lim_{N\to\infty} \frac{1}{N}\left(\mathbb{E}_1\left[\mathrm{Tr}L^m\mathrm{Tr}L^n\right] - \mathbb{E}_1\left[\mathrm{Tr}L^m\right]\mathbb{E}_1\left[\mathrm{Tr}L^n\right]\right), \tag{5.1.23}$$

where L is the Lax matrix of the integrable system at hand and the mean values are taken with respect to the corresponding Generalized Gibbs ensemble. The computation of such quantities is relevant to obtain the decay of the correlation functions for these integrable systems, as it is shown by Spohn in [Spo20a]. In particular, we can prove the following:

Theorem 5.1.5. *Under the same hypotheses as Theorem 5.1.3. Consider $\mu_N^{(1)}$ (5.1.12) and define the free energy $\mathcal{F}^{(1)}(\alpha, G)$ as in (5.1.21), then*

$$\mathcal{C}_{m,n} = \partial_{t_1}\partial_{t_2}\left(\mathcal{F}^{(1)}(\alpha, G + it_1x^m + it_2x^n)\right)_{|_{t_1=t_2=0}}. \tag{5.1.24}$$

Remark 5.1.6. *In view of Theorem 5.1.3, we can rewrite (5.1.24) as*

$$\mathcal{C}_{m,n} = \partial_\alpha\partial_{t_1}\partial_{t_2}\left(\alpha\left(\mathcal{F}^{(2)}(\alpha, G + it_1x^m + it_2x^n)\right)\right)_{|_{t_1=t_2=0}}. \tag{5.1.25}$$

In our context, the previous equality implies that we can compute the susceptibility matrix of the integrable systems that we are considering in terms of just the free energy of the corresponding classical β ensemble in the high-temperature regime.

Finally, considering the type 1 measures (5.1.12), we investigate the space-correlations for *local* functions, meaning that they depend only on a finite number (independent of N) of consecutive variables, proving the following

Theorem 5.1.7 (Decay of correlations)**.** *Let W be the seed of $\mathrm{Tr}G(L)$ and $I, J : X^k \to \mathbb{R}$ two local functions such that $\int_{X^k\times X^k}\left|I(\mathbf{x})\prod_{i=1}^k F(x_i,\alpha)e^{-W(\mathbf{x},\mathbf{y})}\right|^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} < \infty$, and analogously for $J(\mathbf{x})$. Write $N = kM + \ell$, and let $j \in \{1,\dots,M\}$. Then there exists some $0 < \mu < 1$ such that*

$$\mathbb{E}_1\left[I(\mathbf{x}_1)J(\mathbf{x}_j)\right] - \mathbb{E}_1\left[I(\mathbf{x}_1)\right]\mathbb{E}_1\left[J(\mathbf{x}_j)\right] = O(\mu^{M-j} + \mu^j)\,.$$

In particular, this result implies that the space-correlations between two local functions acting on two different parts of the chain decay exponentially fast according to the distance between the set of particles they are acting on. In section 5.3.1, we use the previous result to rigorously justify the assumption of H. Spohn on the decay of space-correlations between the *local conserved fields* and their currents [Spo21]; we mention also that one can follow exactly the same reasoning for all the other integrable systems that we consider.

The paper is organized as follows, in section 5.2 we state the theoretical results that lead to the proofs of Theorems 5.1.3 and 5.1.5. In section 5.3 we show how to apply our results to obtain a central limit theorem for several integrable systems and for the corresponding β ensembles in the high-temperature regime. A summary of these models can be found in Table 5.1. Specifically, we obtain a CLT for the Toda lattice and the real β ensemble, for the Exponential Toda lattice and the Laguerre β ensemble, the defocusing Ablowitz–Laddik lattice and the Circular β ensemble, the defocusing Schur flow and the Jacobi β ensemble, the Volterra lattice and the antisymmetric β ensemble, and for the families of Itoh–Narita–Bogoyavleskii (INB) multiplicative, and additive lattices. Furthermore, we apply Theorem 5.1.7 to the Toda lattice to derive the limiting currents of the conserved fields. In section 5.4, we prove the technical results we used in section 5.2, we prove Theorem 5.1.7 and deduce a Berry-Esseen type bound for all the previously considered integrable models. Finally, in section 5.5 we give some conclusions and outlooks for future developments on this topic.

Integrable System (Type 1)	β**-ensemble at high-temperature** (Type 2)
Toda lattice	Real
Defocusing Ablowitz-Ladik lattice	Circular
Exponential Toda lattice	Laguerre
Defocusing Schur flow	Jacobi
Volterra lattice	Antisymmetric

Table 5.1: Integrable systems and random matrix ensembles

5.2 Nagaev–Guivarc'h theory: a transfer operator approach

In this section, we prove Theorem 5.1.3-5.1.5, to do that we need to develop the fluctuations' theory of Nagaev–Guivarc'h [Nag57, GH88, HH01, Gou15] through transfer operator methods, see for example [KRB01, KS75, PB89].

The proof of these theorems is divided into 3 main parts. In the first one, we compute $\mathbb{E}_1\left[e^{-it\mathrm{Tr}H}\right], \mathbb{E}_2\left[e^{-it\mathrm{Tr}H}\right]$ through transfer operator techniques. Since the proof of these results is technical, we postpone it to section 5.4. Our proof follows the same line as the corresponding one in [GM23]. In the second part, we prove a slight generalization of Nagaev–Guivarc'h theorems [Gou15]. In the last part, we combine the previous two results to complete the proof of Theorem 5.1.3-5.1.5.

In view of the hypotheses 5.1.1, we consider the following decomposition of $N = kM + \ell$, in this

notation, we can rewrite our measures as

$$\mu^{(1)}_{kM+\ell} = \frac{1}{Z^{(1)}_{kM+\ell}(\alpha, W)} \prod_{j=1}^{kM+\ell} F(x_j, \alpha) \prod_{j=1}^{M-1} \exp\left(-W(\mathbf{x}_j, \mathbf{x}_{j+1})\right) \tag{5.2.1}$$

$$\times \exp\left(-W_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1)\right) \prod_{j=1}^{kM+\ell} \mathrm{d}x_j \tag{5.2.2}$$

$$\mu^{(2)}_{kM+\ell} = \frac{1}{Z^{(2)}_{kM+\ell}(\alpha, W)} \prod_{j=1}^{kM+\ell-1} F\left(x_j, \alpha\left(1 - \frac{j}{kM+\ell}\right)\right) \prod_{j=1}^{M-1} \exp\left(-W(\mathbf{x}_j, \mathbf{x}_{j+1})\right) \tag{5.2.3}$$

$$\times \exp\left(-\widetilde{W}_1(\mathbf{x}_1) - W(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0)\right) \tag{5.2.4}$$

$$\times \exp\left(-\widetilde{W}_2(x_{kM+1}, \dots, x_{kM+\ell})\right) R(x_N) \prod_{j=1}^{kM+\ell} \mathrm{d}x_j\,, \tag{5.2.5}$$

Where $\mathbf{x}_j = (x_{k(j-1)+1}, \dots, x_{kj})$ and the partition functions become:

$$Z^{(1)}_{kM+\ell}(\alpha, W) = \int_{X^{kM+\ell}} \prod_{j=1}^{kM+\ell} F(x_j, \alpha) \prod_{j=1}^{M-1} \exp\left(-W(\mathbf{x}_j, \mathbf{x}_{j+1})\right) \tag{5.2.6}$$

$$\times \exp\left(-W_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1)\right) \prod_{j=1}^{kM+\ell} \mathrm{d}x_j \tag{5.2.7}$$

$$Z^{(2)}_{kM+\ell}(\alpha, W) = \int_{X^{kM+\ell}} \prod_{j=1}^{kM+\ell-1} F\left(x_j, \alpha\left(1 - \frac{j}{kM+\ell}\right)\right) \prod_{j=1}^{M-1} \exp\left(-W(\mathbf{x}_j, \mathbf{x}_{j+1})\right) \tag{5.2.8}$$

$$\times \exp\left(-W(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0) - \widetilde{W}_2(x_{kM+1}, \dots, x_{kM+\ell})\right) \tag{5.2.9}$$

$$\times \exp\left(-\widetilde{W}_1(\mathbf{x}_1)\right) R(x_N) \prod_{j=1}^{kM+\ell} \mathrm{d}x_j\,, \tag{5.2.10}$$

we set $Z^{(s)}_N(\alpha, W) \equiv Z^{(s)}_N(\alpha, G)$, $s = 1, 2$.
On the space $L^2(X^k, \mathcal{B}^k)$ we introduce the standard scalar product for $f, g \in L^2(X^k, \mathcal{B}^k)$ as

$$\langle f; g\rangle = \int_{X^k} f(x)\overline{g(x)}\mathrm{d}\mathbf{x}\,. \tag{5.2.11}$$

Furthermore, for $I, J \subset \mathbb{R}$, and E a normed space, denote by $C^{s,d}(I \times J, E)$ the functions $f : I \times J \to X$ that are C^s (respectively C^d) with respect to the first (respectively the second) variable. If $s = d$, then we set $C^d(I \times J, E) \equiv C^{d,d}(I \times J, E)$, and if the normed space $E = \mathbb{C}$ we just omit it.

Transfer operator for partitions functions. As we already stated, in the first part of the section, we apply the transfer operator method in order to compute $\mathbb{E}_1\left[e^{-itH}\right], \mathbb{E}_2\left[e^{-itH}\right]$. In particular, we prove the following theorems

Theorem 5.2.1. *Under Assumptions 5.1.1. Consider a real function $H : \mathbb{C} \to \mathbb{R}$ such that $\mathrm{Tr}H(L)$ is circular, and let W be the seed of $\mathrm{Tr}G(L) + itH(L)$, thus $W(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \mathbf{y}) + itU(\mathbf{x}, \mathbf{y})$ for $V, U : X^k \times X^k \to \mathbb{R}$ seeds of $\mathrm{Tr}G(L), \mathrm{Tr}H(L)$. Furthermore, assume that $U \in L^d(X^{2k}, \exp(-2V))$, with $\mathbb{N} \ni d \geqslant 6$. Then, there exists an $\varepsilon > 0$, and two complex valued functions $\lambda(y, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$ and $c_{k,\ell}(y, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon])$ such that for all $q \in \mathbb{N}$:*

$$\mathbb{E}_1\left[e^{-it\mathrm{Tr}H}\right] = \frac{Z^{(1)}_{kM+\ell}(\alpha, t)}{Z^{(1)}_{kM+\ell}(\alpha, 0)} = c_{k,\ell}(\alpha, t)\lambda(\alpha, t)^{M-2}\left(1 + o(M^{-q})\right)\,, \quad \text{as } M \to \infty\,, \tag{5.2.12}$$

for $|t| < \varepsilon$, *here* $Z^{(1)}_{kM+\ell}(\alpha, t) \equiv Z^{(1)}_{kM+\ell}(\alpha, V + itU)$. *Furthermore,*

$$\lambda(x, 0) = 1\,, \tag{5.2.13}$$
$$c_{k,\ell}(x, 0) = 1\,. \tag{5.2.14}$$

Moreover, there exist two functions $\widetilde{c}_{k,\ell}(\alpha, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon])$ *and* $\widetilde{\lambda}(\alpha, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$ *such that there exist two constants* $C_1, C_2 > 0$ *such that for all* $q \in \mathbb{N}$*:*

$$\begin{aligned} &C_1 < \widetilde{c}_{k,\ell}(\alpha, t) < C_2\,, \\ &\lambda(\alpha, t) = \frac{\widetilde{\lambda}(\alpha, t)}{\widetilde{\lambda}(\alpha, 0)}\,, \\ &Z^{(1)}_{kM+\ell}(\alpha, t) = \widetilde{c}_{k,\ell}(\alpha, t)\widetilde{\lambda}(\alpha, t)^{M-2}\left(1 + o(M^{-q})\right)\,. \end{aligned} \tag{5.2.15}$$

In the next theorem, we prove an analogue decomposition of the partition function for the second type of measure. This decomposition involves the same function $\lambda(y, t)$ as in Theorem 5.2.1.

Theorem 5.2.2. *Under Assumptions 5.1.1. Consider a real function* $H : \mathbb{C} \to \mathbb{R}$ *such that* $\mathrm{Tr}H(L)$ *is circular (HP 5.), and let* W *be the seed of* $\mathrm{Tr}G(L) + itH(L)$*, thus* $W(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \mathbf{y}) + itU(\mathbf{x}, \mathbf{y})$ *for* $V, U : X^k \times X^k \to \mathbb{R}$ *seeds of* $\mathrm{Tr}G(L), \mathrm{Tr}H(L)$*. Furthermore, assume that* $U \in L^d(X^{2k}, \exp(-2V))$*, with* $\mathbb{N} \ni d \geqslant 6$.

Then, there exists an $\varepsilon > 0$ *and* $c_{k,\ell,M}(y, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon])$*, such that, with* λ *given by Theorem 5.2.1,*

$$\mathbb{E}_2\left[e^{-it\mathrm{Tr}H}\right] = \frac{Z^{(2)}_{kM+\ell}(\alpha, t)}{Z^{(2)}_{kM+\ell}(\alpha, 0)} = c_{k,\ell,M}(\alpha, t)\prod_{j=1}^{M-2}\lambda\left(\alpha\frac{j}{M}, t\right)(1 + o_M(1)) \tag{5.2.16}$$

for $|t| < \varepsilon$. *Furthermore,*

$$\lambda(x, 0) = 1 \tag{5.2.17}$$
$$\lim_{t\to 0} c_{k,\ell,M}(\alpha, t) = 1 \quad \textit{uniformly in } M \tag{5.2.18}$$
$$\textit{the remainder } o_M(1) \textit{ is independent of } t \in [-\varepsilon, \varepsilon]. \tag{5.2.19}$$

Moreover, there exist two functions $\widetilde{c}_{k,\ell,M}(\alpha, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$*,* $\widetilde{\lambda}(\alpha, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$*, and three constants* $C_1, C_2 > 0$ *and* $p \in \mathbb{N}$ *such that*

$$\begin{aligned} &C_1 N^p < \widetilde{c}_{k,\ell,M}(\alpha, t) < C_2 N^p\,, \\ &\lambda(\alpha, t) = \frac{\widetilde{\lambda}(\alpha, t)}{\widetilde{\lambda}(\alpha, 0)}\,, \\ &Z^{(2)}_{kM+\ell}(\alpha, t) = \widetilde{c}_{k,\ell,M}(\alpha, t)\prod_{j=1}^{M-2}\widetilde{\lambda}\left(\alpha\frac{j}{M}, t\right)(1 + o_M(1))\,. \end{aligned} \tag{5.2.20}$$

Since the proof of these results is technical, we postpone it to Section 5.4.

Generalization of Nagaev–Guivarc'h method. In this second part, we need to generalize some standard results from the fluctuation theory of Nagaev–Guivarc'h [Gou15] to our situation. Specifically, we prove the following:

Theorem 5.2.3. *Let $(X_n)_{n\geqslant 1}$ be a sequence of real random variables with partial sums $(S_n)_{n\geqslant 1} \in \mathbb{R}$. Assume that there exists $\varepsilon > 0$, two functions $\lambda(t) \in C^1([0,\varepsilon))$, $c(t) \in C^0([0,\varepsilon))$ and $h_n(t) \in C^0([0,\varepsilon))$, such that for all $t \in [-\varepsilon,\varepsilon]$, and all $n \geqslant 1$ we have*

$$\mathbb{E}\left[e^{-itS_n}\right] = c(t)\lambda(t)^m\left(1+h_n(t)\right), \tag{5.2.21}$$

Where $\lim_{n\to\infty} n/m = k \in \mathbb{N}$.
Furthermore, assume that:

a. *there exists A, $\sigma^2 \in \mathbb{C}$ such that $\lambda(t) = \exp\left(-iAt - \sigma^2 t^2/2 + o(t^3)\right)$ as $t \to 0$;*

b. *$h_n \xrightarrow{n\to\infty} 0$ uniformly in $[-\varepsilon,\varepsilon]$, and $h_n(0) = 0$;*

c. *$c(0) = 1$.*

Then $A \in \mathbb{R}$, $\sigma^2 \geqslant 0$, and $(S_n - nA/k)/\sqrt{n}$ converges to a Gaussian distribution $\mathcal{N}(0,\sigma^2/k)$ as n tends to infinity.

Proof. First, evaluating (5.2.21) at $t = 0$, we deduce that $\lambda(0) = 1$. Then, we use the asymptotic expansion of $\lambda(t)$, and properties b.-c. to prove that

$$\mathbb{E}\left[\exp\left(-it\frac{S_n - mA}{n}\right)\right] \xrightarrow[n\to\infty]{} 1\,.$$

Thus, by Lévy theorem [Wil91], we deduce that $S_n/n - A/k$ converges in distribution to 0. So, since S_n is real, then $A \in \mathbb{R}$. Exploiting again the asymptotic expansion of $\lambda(t)$ and properties b.-c., we show that $\mathbb{E}\left[\exp\left(-it\frac{S_n - mA}{\sqrt{n}}\right)\right]$ converges to the function $\exp\left(-\frac{\sigma^2 t^2}{2k}\right)$. By Lévy theorem [Wil91], this must be the characteristic function of a real random variable, proving that $\sigma^2 \geqslant 0$, and that $(S_n - nA/k)/\sqrt{n}$ converges to a Gaussian distribution $\mathcal{N}(0,\sigma^2/k)$. □

Further, we prove the following:

Theorem 5.2.4. *Let $(X_n)_{n\geqslant 1}$ be a sequence of random variables with partial sums $(S_n)_{n\geqslant 1} \in \mathbb{R}$. Assume that there exists $\varepsilon > 0$ and functions $\lambda(x,t) \in C^{1,0}([0,1)\times\mathbb{R})$, $c_n(t) \in C^0(\mathbb{R})$, and $h_n(t)$ continuous in 0, such that for all $t \in [-\varepsilon,\varepsilon]$, and all $n \geqslant 1$ we have*

$$\mathbb{E}\left[e^{-itS_n}\right] = c_n(t)\left(\prod_{j=1}^{m}\lambda\left(j/m,t\right)\right)\left(1+h_n(t)\right), \tag{5.2.22}$$

where $\lim_{n\to\infty} n/m = k$.
Furthermore, assume that:

a. *there exists two continuous functions $A(x)$, $\sigma^2(x) : [0,1] \to \mathbb{C}$ such that $\lambda(x,t) = \exp\left(-iA(x)t - \sigma^2(x)t^2/2 + o(t^2)\right)$ as $t \to 0$;*

b. *$||h_n||_\infty \xrightarrow{n\to\infty} 0$ uniformly in $[-\varepsilon,\varepsilon]$, and $h_n(0) = 0$;*

c. *$c_n(0) = 1$ and $\lim_{n\to\infty} c_n(t/\sqrt{n}) = \lim_{n\to\infty} c_n(t/n) = 1$.*

Then $\int_0^1 A(x)\mathrm{d}x \in \mathbb{R}$, $\int_0^1 \sigma^2(x)\mathrm{d}x \in \mathbb{R}_+$, and $\sqrt{n}\left(\frac{S_n}{n} - \frac{\int_0^1 A(x)\mathrm{d}x}{k}\right)$ converges to a Gaussian distribution $\mathcal{N}\left(0, \frac{\int_0^1 \sigma^2(x)\mathrm{d}x}{k}\right)$ as n tends to infinity.

Proof. First, let $t = \widetilde{t}/n$, then by hypothesis c.

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-i\widetilde{t}\left(S_n/n - \frac{1}{n}\sum_{\ell=1}^{m} A(\ell/m)\right)}\right] = \lim_{n\to\infty} \mathbb{E}\left[e^{-i\widetilde{t}\left(S_n/n - \frac{\int_0^1 A(x)\mathrm{d}x}{k}\right)}\right] = 1\,. \tag{5.2.23}$$

Thus, by Levy theorem, $S_n/n \to \frac{\int_0^1 A(x)\mathrm{d}x}{k}$ almost surely, thus, since $S_n \in \mathbb{R}$, this implies that $\int_0^1 A(x)\mathrm{d}x \in \mathbb{R}$. Consider now $t = \widetilde{t}/\sqrt{n}$, following the same reasoning one conclude that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[e^{-i\widetilde{t}\sqrt{n}\left(S_n/n - \frac{1}{n}\sum_{\ell=1}^{n} A(\ell/m)\right)}\right] &= \lim_{n\to\infty} \mathbb{E}\left[e^{-i\widetilde{t}\sqrt{n}\left(S_n/n - \frac{\int_0^1 A(x)\mathrm{d}x}{k}\right)}\right] \\ &= \lim_{n\to\infty} e^{-\frac{\widetilde{t}^2}{2n}\sum_{\ell=1}^{m}\sigma^2(\ell/m)} = e^{-\frac{\widetilde{t}^2}{2k}\int_0^1 \sigma^2(x)\mathrm{d}x}\,,\end{aligned} \tag{5.2.24}$$

thus, by Lévy theorem [Wil91], $e^{-\frac{\widetilde{t}^2}{2k}\int_0^1 \sigma^2(x)\mathrm{d}x}$ must be the characteristic function of a real random variable, proving that $\int_0^1 \sigma^2(x) \in \mathbb{R}_+$. □

Proof of Theorem 5.1.3-5.1.5. We are now ready to prove Theorem 5.1.3-5.1.5, for convenience, we split the proof into two Lemmas, which combined give the full proof of our results.

Lemma 5.2.5. *Under hypotheses 5.1.1. Consider $\mu^{(1)}_{kM+\ell}$, $\mu^{(2)}_{kM+\ell}$ (5.2.1)-(5.2.3), and let $H : \mathbb{C} \to \mathbb{R}$ such that $\mathrm{Tr}H(L)$ is circular (HP.5) with seed U, so that $W + itU$ is the seed of $\mathrm{Tr}G(L) + itH(L)$. Let $S_{kM+\ell} = \mathrm{Tr}H(L)$. Then*

$$\mathbb{E}_1\left[e^{-itS_{kM+\ell}}\right] = \frac{Z^{(1)}_{kM+\ell}(\alpha,t)}{Z^{(1)}_{kM+\ell}(\alpha,0)}, \quad \mathbb{E}_2\left[e^{-itS_{kM+\ell}}\right] = \frac{Z^{(2)}_{kM+\ell}(\alpha,t)}{Z^{(2)}_{kM+\ell}(\alpha,0)}\,, \tag{5.2.25}$$

where $\mathbb{E}_1[\cdot]$, $\mathbb{E}_2[\cdot]$ are the mean values taken with respect to $\mu^{(1)}_{kM+\ell}$, $\mu^{(2)}_{kM+\ell}$ respectively. Furthermore, assume that $U \in L^d(X^{2k}, \exp(-2W))$, with $\mathbb{N} \ni d \geqslant 3$. Then, there exist four continuous functions

$$A(x) : \mathbb{R}_+ \longrightarrow \mathbb{R}\,, \qquad \widetilde{A}(x) : \mathbb{R}_+ \longrightarrow \mathbb{R}\,, \tag{5.2.26}$$

$$\sigma(x) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+\,, \qquad \widetilde{\sigma}(x) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+\,, \tag{5.2.27}$$

such that under $\mu^{(1)}_{kM+\ell}$,

$$\left(S_{kM+\ell} - (kM+\ell)A(\alpha)/k\right)/\sqrt{kM+\ell}$$

converges to a Gaussian distribution $\mathcal{N}(0, \sigma^2(\alpha)/k)$ as M tends to infinity, and under $\mu^{(2)}_{kM+\ell}$,

$$\left(S_{kM+\ell} - (kM+\ell)\widetilde{A}(\alpha)/k\right)/\sqrt{kM+\ell}$$

converges to a Gaussian distribution $\mathcal{N}(0, \widetilde{\sigma}^2(\alpha)/k)$ as M tends to infinity.

The proof of the previous result is a trivial application of Theorem 5.2.1-5.2.2-5.2.3-5.2.4. Furthermore, we can interpret the previous relations through the *free energies* of $\mu^{(1)}_{kM+\ell}$, $\mu^{(2)}_{kM+\ell}$ (5.2.1) -(5.2.3):

Lemma 5.2.6. *Under the same hypotheses and notation of Lemma 5.2.5. Consider the two measures $\mu^{(1)}_{kM+\ell}$, $\mu^{(2)}_{kM+\ell}$ (5.2.1)-(5.2.3), and define the free energies as*

$$\mathcal{F}^{(1)}(\alpha, W) = -\lim_{M\to\infty} \frac{\ln(Z^{(1)}_{kM+\ell}(\alpha, W))}{kM+\ell}\,, \quad \mathcal{F}^{(2)}(\alpha, W) = -\lim_{M\to\infty} \frac{\ln(Z^{(2)}_{kM+\ell}(\alpha, W))}{kM+\ell}\,, \tag{5.2.28}$$

then, using the same notation as in Lemma 5.2.5, the following holds:

i. $\mathcal{F}^{(1)}(\alpha, W) = \partial_\alpha \left(\alpha \mathcal{F}^{(2)}(\alpha, W)\right)$

ii. $A(\alpha) = ik\partial_t \mathcal{F}^{(1)}(\alpha, W + itU)_{|t=0}$

iii. $\tilde{A}(\alpha) = ik\partial_t \mathcal{F}^{(2)}(\alpha, W + itU)_{|t=0}$

iv. $\sigma(\alpha) = k\partial_t^2 \mathcal{F}^{(1)}(\alpha, W + itU)_{|t=0}$

v. $\tilde{\sigma}(\alpha) = k\partial_t^2 \mathcal{F}^{(2)}(\alpha, W + itU)_{|t=0}$

Remark 5.2.7. *The previous theorem implies that*

$$A(\alpha) = \partial_\alpha(\alpha \tilde{A}(\alpha))\,, \qquad \sigma^2(\alpha) = \partial_\alpha(\alpha \tilde{\sigma}^2(\alpha))\,. \tag{5.2.29}$$

Proof. To prove *i.*, we can just compute the free energy of $\mu^{(1)}_{kM+\ell}$, $\mu^{(2)}_{kM+\ell}$ using Theorem 5.2.1-5.2.2. For $\mathcal{F}^{(1)}(\alpha, W)$ we deduce immediately that

$$\begin{aligned} \mathcal{F}^{(1)}(\alpha, W) &= -\lim_{M\to\infty} \frac{1}{kM+\ell} \ln(Z^{(1)}_{kM+\ell}) \overset{(5.2.15)}{=} -\lim_{M\to\infty} \frac{M}{kM+\ell} \ln(\tilde{\lambda}(\alpha, 0)) \\ &= -\frac{1}{k} \ln(\tilde{\lambda}(\alpha, 0))\,. \end{aligned} \tag{5.2.30}$$

The proof for $\mathcal{F}^{(2)}(\alpha, W)$ follows in the same way. We now prove $ii. - iv.$ First, we notice that following the notation of Theorem 5.2.3 - 5.2.1 :

$$c(t) = c_{k,\ell}(\alpha, t)\,, \quad h_n = 1 + o(M^{-q})\,, \quad \lambda(t) = \lambda(\alpha, t)\,, \tag{5.2.31}$$

thus to compute explicitly $A(\alpha, W)$, $\sigma(\alpha, W)$ we have just to expand $\lambda(\alpha, t)$ around $t = 0$

$$\lambda(\alpha, t) = 1 + t\partial_t \lambda(\alpha, 0) + \frac{t^2}{2}\partial_t^2 \lambda(\alpha, 0) + o(t^3)\,, \tag{5.2.32}$$

which implies that

$$\lambda(\alpha, t) = \exp\left(t\partial_t \ln(\lambda(\alpha, t))_{|t=0} + \frac{t^2}{2}\partial_t^2 \ln(\lambda(\alpha, t))_{|t=0} + o(t^3)\right)\,. \tag{5.2.33}$$

This implies that $A(\alpha) = -i\partial_t \ln(\lambda(\alpha, t))_{|t=0}, \sigma^2(\alpha) = -\partial_t^2 \ln(\lambda(\alpha, t))_{|t=0}$. From (5.2.15), we deduce that $\partial_t \ln(\lambda(\alpha, t))_{|t=0} = \partial_t \ln(\tilde{\lambda}(\alpha, t))_{|t=0}$, thus from the previous expressions and the explicit form of the free energy (5.2.30) we conclude.

To prove $iii. - v.$ we proceed in the same way, thus following the notation of Theorem 5.2.4 - 5.2.2:

$$c_n(t) = c_{k,\ell,M}(\alpha, t)\,, \quad h_n = 1 + o(1)\,, \quad \lambda(j/M, t) = \lambda\left(\alpha\frac{j}{M}, t\right)\,. \tag{5.2.34}$$

Thus, as in (5.2.33) except that $\alpha \to \alpha\frac{j}{M}$, we expand $\lambda\left(\alpha\frac{j}{M}, t\right)$ around $t = 0$, leading to

$$\begin{aligned} \tilde{A}(\alpha) &= \partial_t \left(\int_0^1 \ln(\lambda(\alpha x, t))\mathrm{d}x\right)_{|t=0}\,, \\ \tilde{\sigma}^2(\alpha) &= \partial_t^2 \left(\int_0^1 \ln(\lambda(\alpha x, t))\mathrm{d}x\right)_{|t=0} \end{aligned} \tag{5.2.35}$$

which concludes the proof. □

Remark 5.2.8. *We notice that the Lemma 5.2.6, and Lemma 5.2.5 imply that we can compute the expected values, and the variances of $S_{kM+\ell}$ according to $\mu^{(1)}_{kM+\ell}$, $\mu^{(2)}_{kM+\ell}$ just computing derivatives of the corresponding free energy. This property is broadly used in the physics literature, but we lacked of a precise statement, and of a proof for the general result. Furthermore, we can compute the expected value, and the variance of $S_{kM+\ell}$ according to $\mu^{(1)}_{kM+\ell}$ starting from the corresponding values for $\mu^{(2)}_{kM+\ell}$. Thus, we have reduced all this problem to the computation of the free energy of $\mu^{(2)}_{kM+\ell}$.*

The proof of both Theorem 5.1.3 and Theorem 5.1.5 follows from the four previous lemmas. Thus, we have completed the proof of our main theorems, and now we show how to apply them to some integrable models, and the β ensembles in the high-temperature regime.

5.3 Application

In this section, we show how to apply Theorem 5.1.3 to obtain a CLT for some integrable systems and for the classical β ensembles in the high-temperature regime. Namely, we prove a CLT for the systems of table 5.1.

Specifically, we are able to prove that all the integrable systems in table 5.1 in the periodic case have a Generalized Gibbs ensemble of the form $\mu^{(1)}_{kM+\ell}$ (5.2.1), that is the reason of the label "type 1". Meanwhile, the β ensembles at high-temperatures are characterized by a probability distribution of the form $\mu^{(2)}_{kM+\ell}$ (5.2.3), that is the reason for the label "type 2". In this way, we proved a further connection between the theory of integrable systems and Random Matrix Theory. Indeed, in view of Theorem 5.1.3 and Theorem 5.1.5, for any integrable system in the previous table, we can relate its free energy, moments, variances and covariances with the corresponding quantities of the random matrix model on the same line. Moreover, in the final part of this section, we consider the family of INB lattices that do not have a known β ensemble counterpart. Despite that, we are still able to derive the existence of a polynomial central limit theorem. Finally, applying Theorem 5.1.7, we are able to show that for the Toda lattice the space-correlations between the local conserved fields and the currents decay exponentially.

5.3.1 The Toda lattice, and the real β ensemble at high-temperature

In this subsection, we focus on the Toda lattice, which is an integrable model, and its relation with the real β ensemble in the high-temperature regime. The connection between these two systems was first realized by Spohn in [Spo20c], see also [Maz22, Spo20a]. In this seminal paper, the author was able to compute the density of states for the Toda lattice when the initial data is sampled according to a Generalized Gibbs ensemble in terms of one of the Gaussian β ensemble in the high-temperature regime. This was further developed in [GM22] where the authors obtained a Large Deviations Principle for the Toda lattice, and they connect it to the one for the real β ensemble in the high-temperature regime. In this paper, we further develop this analysis, obtaining a CLT theorem for these two systems, and connecting them. This result is particularly relevant in the context of the so-called Generalized Hydrodynamics, a recent physical theory that allows computing the correlation functions for classical integrable models, for an introduction to the subject see [Doy20, Spo21]. According to this theory, one of the main ingredients to compute the correlation functions for the integrable model at hand is to be able to calculate the correlation functions for the conserved fields at time 0. Thanks to our result, we are able to access these quantities. We show how to do it at the end of this subsection. We mention also the recent work [GKM$^+$23], where the authors made molecular dynamics simulations of the correlation functions of the Toda lattice, and they compared them with the predictions of linear Generalized Hydrodynamics, showing an astonishing agreement.

The Toda lattice. The classical Toda chain [Tod89] is the dynamical system described by the following Hamiltonian:

$$H_T(\mathbf{p},\mathbf{q}) := \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N} V_T(q_{j+1}-q_j)\,, \quad V_T(x) = e^{-x}+x-1\,, \tag{5.3.1}$$

with periodic boundary conditions $q_{j+N} = q_j + \Omega \quad \forall\, j \in \mathbb{Z}$, $\Omega > 0$. Its equations of motion take the form

$$\dot{q}_j = \frac{\partial H_T}{\partial p_j} = p_j, \quad \dot{p}_j = -\frac{\partial H_T}{\partial q_j} = V_T'(q_{j+1}-q_j) - V_T'(q_j - q_{j-1}), \;\; j = 1,\dots,N\,. \tag{5.3.2}$$

It is well known that the Toda chain is an integrable system [Tod89, Hen74], one way to prove it is to put the Toda equations in Lax pair form. This was obtained by Flaschka [Fla74b], and Manakov [Man74] through the following *non-canonical* change of coordinates:

$$a_j := -p_j\,, \qquad b_j := e^{\frac{1}{2}(q_j - q_{j+1})} \equiv e^{-\frac{1}{2}r_j}, \qquad 1 \leqslant j \leqslant N\,, \tag{5.3.3}$$

where $r_j = q_{j+1} - q_j$ is the relative distance.

Defining the Lax operator L as the periodic Jacobi matrix [vM76]

$$L := \begin{pmatrix} a_1 & b_1 & 0 & \dots & b_N \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ b_N & \dots & 0 & b_{N-1} & a_N \end{pmatrix}, \tag{5.3.4}$$

and the antisymmetric matrix B

$$B := \begin{pmatrix} 0 & b_1 & 0 & \dots & -b_N \\ -b_1 & 0 & b_2 & \ddots & \vdots \\ 0 & -b_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ b_N & \dots & 0 & -b_{N-1} & 0 \end{pmatrix}, \tag{5.3.5}$$

a straightforward calculation shows that the equations of motions (5.3.2) are equivalent to

$$\dot{L} = [B; L]\,, \tag{5.3.6}$$

where $[B; L] = BL - LB$ is the commutator of two matrices. This form implies that $\mathrm{Tr}L^k$, $k = 1, \dots, N$ are constants of motions for the Toda lattice, so the system is integrable. We call these quantities *conserved fields*.

On the phase space $\mathbb{R}^N \times \mathbb{R}^N_+$, we introduce the *Generalized Gibbs Ensemble*

$$\mathrm{d}\mu_T := \frac{1}{Z_N^T(\alpha, P)} \prod_{j=1}^{N} b_j^{2\alpha-1} \mathbb{1}_{b_j>0} e^{-\mathrm{Tr}P(L)} \mathrm{d}\mathbf{a}\,\mathrm{d}\mathbf{b}\,, \tag{5.3.7}$$

where $P(x)$ is a polynomial of even degree with positive leading coefficients, and $\alpha > 0$ is a pressure parameter.

Our aim is to obtain a central limit theorem for the conserved fields when the initial data is sampled according to (5.3.7). So, we want to apply Theorem 5.1.3 to this model. To do that, we need some preparation. First, we recall the following result about the structure of the trace of periodic Jacobi matrices which was proved in [GMMP20]:

Theorem 5.3.1 (cf. Theorem 3.1 [GMMP20])**.** *For any $1 \leqslant m \leqslant N-1$, consider the matrix L given by (5.3.4). One has*

$$\mathrm{Tr}L^m = \sum_{j=1}^{N} h_j^{(m)}\,, \tag{5.3.8}$$

where $h_j^{(m)} := [L^m]_{jj}$ is given explicitly by

$$h_j^{(m)}(\mathbf{p}, \mathbf{r}) = \sum_{(\mathbf{n},\mathbf{q}) \in \mathcal{A}^{(m)}} \rho^{(m)}(\mathbf{n}, \mathbf{q}) \prod_{i=-\widetilde{m}}^{\widetilde{m}-1} a_{j+1}^{q_i} \prod_{i=-\widetilde{m}+1}^{\widetilde{m}-1} b_{j+i}^{2n_i}\,, \tag{5.3.9}$$

where it is understood $a_j \equiv a_{j \bmod N+1}$, $b_j \equiv b_{j \bmod N+1}$ *and* $\mathcal{A}^{(m)}$ *is the set*

$$\begin{aligned}\mathcal{A}^{(m)} := \Big\{ (\mathbf{n},\mathbf{q}) \in \mathbb{N}_0^{\mathbb{Z}} \times \mathbb{N}_0^{\mathbb{Z}} \,:\quad & \sum_{i=-\widetilde{m}}^{\widetilde{m}-1} (2n_i + q_i) = m, \\ & \forall i \geqslant 0, \quad n_i = 0 \Rightarrow n_{i+1} = q_{i+1} = 0, \\ & \forall i < 0, \quad n_{i+1} = 0 \Rightarrow n_i = q_i = 0 \Big\}.\end{aligned} \tag{5.3.10}$$

The quantity $\widetilde{m} := \lfloor 1 \rfloor$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\rho^{(m)}(\mathbf{n},\mathbf{q}) \in \mathbb{N}$ *is given by*

$$\rho^{(m)}(\mathbf{n},\mathbf{q}) := \binom{n_{-1}+n_0+q_0}{q_0} \binom{n_{-1}+n_0}{n_0} \prod_{\substack{i=-\widetilde{m} \\ i\neq -1}}^{\widetilde{m}-1} \binom{n_i+n_{i+1}+q_{i+1}-1}{q_{i+1}} \binom{n_i+n_{i+1}-1}{n_{i+1}}. \tag{5.3.11}$$

This Theorem immediately leads to the following Corollary

Corollary 5.3.2. *Fix* $m \in \mathbb{N}$, *and consider the matrix* L (5.3.4). *Then for* N *big enough, there exists some* $k = k(m) \in \mathbb{N}$, *and two polynomial functions* $V : \mathbb{R}^{2k} \times \mathbb{R}_+^{2k} \to \mathbb{R}$, $V_1 : \mathbb{R}^{2k+\ell} \times \mathbb{R}_+^{2k+\ell} \to \mathbb{R}$ *such that*

$$\mathrm{Tr} L^m = \sum_{j=1}^{M-1} V(\mathbf{a}_j, \mathbf{b}_j, \mathbf{a}_{j+1}, \mathbf{b}_{j+1}) + V_1(\mathbf{a}_M, a_{kM+1}, \dots, a_{kM+\ell}, \mathbf{a}_1, \mathbf{b}_M, b_{kM+1}, \dots, b_{kM+\ell}, \mathbf{b}_1), \tag{5.3.12}$$

where $N = kM + \ell$, $\mathbf{a}_j = (a_{(j-1)k+1}, a_{(j-1)k+2}, \dots a_{jk})$, *and similarly for* $\mathbf{b}_j$.

Remark 5.3.3. *In other words, the function* $\mathrm{Tr} L^m$ *is circular in the sense of Hypotheses 5.1.1, HP. 5. Furthermore, we notice that the local potential* $V(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2)$ *is bounded from below, this can be proved using the explicit formula in Theorem 5.3.1 or applying the properties of super-Motzkin paths used for the proof of the theorem in [GMMP20].*

We apply the previous Corollary to the Gibbs measure of the Toda lattice (5.3.7), so it can be written as

$$\begin{aligned}\mathrm{d}\mu_T = \frac{1}{Z_N^T(\beta,P)} \prod_{j=1}^N b_j^{2\alpha-1} \mathbb{1}_{b_j>0} \exp\Big(- \sum_{j=1}^M V(\mathbf{a}_j, \mathbf{b}_j, \mathbf{a}_{j+1}, \mathbf{b}_{j+1}) \\ - V_1(\mathbf{a}_M, a_{kM+1}, \dots, a_{kM+\ell}, \mathbf{a}_1, \mathbf{b}_M, b_{kM+1}, \dots, b_{kM+\ell}, \mathbf{b}_1)\Big) \mathrm{d}\mathbf{a}\, \mathrm{d}\mathbf{b}\,.\end{aligned} \tag{5.3.13}$$

We would like to apply Theorem 5.1.3 to the previous density with $F(b) = b^{2\alpha-1}$, and $W = V$, but in this case $F \notin L^2(\mathbb{R}_+)$, so we have to take care of this issue. To do it, we fix $\varepsilon > 0$, and consider the following measure

$$\begin{aligned}\mathrm{d}\mu_T = \frac{1}{Z_N^T(\beta,P)} \prod_{j=1}^N b_j^{2\alpha-1} e^{-\varepsilon(a_j^2+b_j^2)} \mathbb{1}_{a_j>0} \exp\Big(- \sum_{j=1}^M V(\mathbf{a}_j, \mathbf{b}_j, \mathbf{a}_{j+1}, \mathbf{b}_{j+1}) \\ - V_1(\mathbf{a}_M, a_{kM+1}, \dots, a_{kM+\ell}, \mathbf{a}_1, \mathbf{b}_M, b_{kM+1}, \dots, b_{kM+\ell}, \mathbf{b}_1) \\ + \varepsilon \sum_{j=1}^N a_j^2 + b_j^2\Big) \mathrm{d}\mathbf{a}\, \mathrm{d}\mathbf{b}\,,\end{aligned} \tag{5.3.14}$$

this is exactly the same measure as before, but, following the notation of Theorem 5.1.3, we can now set

$$F(a,b,\alpha) = b^{2\alpha-1}e^{-\varepsilon(a^2+b^2)}\,,$$
$$W(\mathbf{a}_j,\mathbf{b}_j,\mathbf{a}_{j+1},\mathbf{b}_{j+1}) = V(\mathbf{a}_j,\mathbf{b}_j,\mathbf{a}_{j+1},\mathbf{b}_{j+1}) - \frac{\varepsilon}{2}\sum_{n=1}^{2k} a^2_{(j-1)k+n} + b^2_{(j-1)k+n}\,. \tag{5.3.15}$$

These functions satisfy the hypotheses of Theorem 5.1.3, so we can apply it and deduce the following

Corollary 5.3.4 (CLT for the Toda lattice)**.** *Consider the Lax matrix L* (5.3.4) *of the Toda lattice distributed according to the Generalized Gibbs Ensemble* (5.3.7)*, and assume that $P(x)$ is a polynomial of even degree with positive leading order coefficient. Then, defining the Free energy $\mathcal{F}_T(\alpha,P)$ as*

$$\mathcal{F}_T(\alpha,P) = -\lim_{N\to\infty}\frac{1}{N}\ln(Z_N^T(\alpha,P))\,, \tag{5.3.16}$$

for all $j\in\mathbb{N}$ fixed, we have the following weak limit

$$\lim_{N\to\infty}\frac{\mathrm{Tr}L^j - \mathbb{E}\left[\mathrm{Tr}L^j\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0,\sigma^2)\,. \tag{5.3.17}$$

Where

$$\mathbb{E}\left[\mathrm{Tr}L^j\right] = iN\partial_t\mathcal{F}_T(\alpha,P+itx^j)_{|_{t=0}}\,,\quad \sigma^2 = \left|\partial_t^2\mathcal{F}_T(\alpha,P+itx^j)_{|_{t=0}}\right|. \tag{5.3.18}$$

Moreover, we can also apply Theorem 5.1.5 to compute the correlation between the conserved fields at time zero, indeed the theorem immediately implies that

$$\lim_{N\to\infty}\frac{\mathbb{E}\left[\mathrm{Tr}L^j\mathrm{Tr}L^n\right] - \mathbb{E}\left[\mathrm{Tr}L^j\right]\mathbb{E}\left[\mathrm{Tr}L^n\right]}{N} = \partial_{t_1}\partial_{t_2}\mathcal{F}_T(\alpha,P+it_1x^j+it_2x^n)_{|_{t_1,t_2=0}}\,, \tag{5.3.19}$$

where the mean value is taken with respect to the Gibbs measure of the Toda lattice (5.3.7). We notice that this implies that we can compute the susceptibility matrix of the Toda lattice (5.1.24) in terms of the derivative of the Free energy.

The Toda chain's currents

Since the conserved fields are local quantities, they must satisfy a local conservation law. Following the notation of [Spo22b], we define

$$Q_j^{[n],N} = L^n_{j,j}\,, \tag{5.3.20}$$

where $L\in\mathrm{Mat}(N,\mathbb{R})$ is (5.3.4). We can easily compute the evolution equation for such quantities as

$$\frac{d}{dt}Q_j^{[n],N} = (BL^n - L^nB) = b_{j-1}L^n_{j,j-1} - b_jL^n_{j+1,j}\,. \tag{5.3.21}$$

Defining $J_j^{[n],N} = b_{j-1}L^n_{j,j-1}$, we have

$$\frac{d}{dt}Q_j^{[n],N} = J_j^{[n],N} - J_{j+1}^{[n],N}$$

and we say that $J_j^{[n],N}$ is the current of the local conserved field $Q_j^{[n],N}$. In particular, defining the matrix $L^{\downarrow}$ as

$$L^{\downarrow}_{i,j} = \begin{cases} L_{i,j} & \text{if } j<i \text{ or } i=1, j=N \\ 0 & \text{otherwise} \end{cases} \tag{5.3.22}$$

we can recast the previous definition as

$$J_j^{[n],N} = (L^n L^{\downarrow})_{j,j} \,. \tag{5.3.23}$$

We notice that both $Q_j^{[n],N}$ and $J_j^{[n],N}$ depend on time, and we adopt the convention that if not explicitly written the evaluation is at time 0. Furthermore, we define

$$Q^{[n],N} = \sum_{j=1}^{N} Q_j^{[n],N} \,, \qquad J^{[n],N} = \sum_{j=1}^{N} J_j^{[n],N} \,, \tag{5.3.24}$$

and we refer to $Q^{[n],N}$ as the n^{th}-conserved field, and to $J^{[n],N}$ as the n^{th}-total current.

The evaluation of the expected values of both the currents $J_j^{[n],N}$ and the total current $J^{[n],N}$ according to the Generalized Gibbs ensemble (5.3.7) is one of the crucial steps to apply the theory of Generalized Hydrodynamics to the Toda lattice, as it is explained in [Spo22b]. In this manuscript, the author used some heuristic arguments to explicitly derive the expression for these quantities, here we rigorously justify his argument applying Theorem 5.1.7.

First, we extend the definition of $Q_j^{[n],N}$ and $J_j^{[n],N}$ for $n = 0$, setting $Q_j^{[0],N} = r_j$, and $J_j^{[0],N} = -p_j = -Q_j^{[1],N}$. We notice that $\sum_{j=1}^{N} J_j^{[0],N} = -\sum_{j=1}^{N} Q_j^{[1],N}$ is still a conserved field. We are now in position to show how to compute the limiting Toda average current

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[J^{[n],N} \right] , \tag{5.3.25}$$

in terms of the susceptibility matrix (5.1.24) of the Toda chain, so in particular of the derivative of the Free energy (5.3.19). Indeed, we prove the following:

Lemma 5.3.5. *Consider the Lax matrix L* (5.3.4) *of the Toda lattice distributed according to the Generalized Gibbs Ensemble* (5.3.7)*, and assume that $P(x)$ is a polynomial of even degree with positive leading order coefficient. Then, for any fixed $n \in \mathbb{N}$, and $\alpha \in \mathbb{R}_+$ defining the total currents $J^{[n],N}$ as in* (5.3.24) *we have the following equality*

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[J^{[n],N} \right] = \int_0^{\alpha} \partial_{t_1}\partial_{t_2} \mathcal{F}_T(s, P + it_1 x + it_2 x^n)_{|_{t_1,t_2=0}} \mathrm{d}s. \tag{5.3.26}$$

Proof. In view of the cyclic structure of the measure μ_T and of the total current, we deduce that

$$\frac{1}{N} \mathbb{E}\left[J^{[n],N} \right] = \mathbb{E}\left[J_1^{[n],N} \right] . \tag{5.3.27}$$

Furthermore, for any fixed N, we deduce, by differentiating with respect to the parameter α, the following equality

$$\partial_\alpha \mathbb{E}\left[J_1^{[n],N} \right] = -\mathrm{Cov}\left(J_1^{[n],N} \; ; \; \sum_{j=1}^{N} r_j \right) = -\sum_{j=1}^{N} \mathrm{Cov}\left(J_1^{[n],N} \; ; \; Q_j^{[0],N} \right) , \tag{5.3.28}$$

where we defined for any functions $f, g \in L^2(X^N, \mu_T)$

$$\mathrm{Cov}(f\,;\,g) = \mathbb{E}\left[fg\right] - \mathbb{E}\left[f\right]\mathbb{E}\left[g\right] . \tag{5.3.29}$$

We show now that the following limits coincide

$$\lim_{N\to\infty} \sum_{j=1}^{N} \mathrm{Cov}\left(J_1^{[n],N} \,;\, Q_j^{[0],N} \right) = \lim_{N\to\infty} \sum_{j=1}^{N} \mathrm{Cov}\left(J_1^{[0],N} \,;\, Q_{N-j+2}^{[n],N} \right) . \tag{5.3.30}$$

Indeed, for any $n, m \geqslant 0$ and $t \in \mathbb{R}$

$$\begin{aligned}\mathrm{Cov}\left(J_{j+1}^{[n],N}(t) - J_j^{[n],N}(t)\ ;\ Q_1^{[m],N}(0)\right) &= -\frac{d}{dt}\mathrm{Cov}\left(Q_j^{[n],N}(t)\ ;\ Q_1^{[m],N}(0)\right)\\ &= -\frac{d}{dt}\mathrm{Cov}\left(Q_1^{[n],N}(0)\ ;\ Q_{N-j+2}^{[m],N}(-t)\right)\\ &= \mathrm{Cov}\left(Q_1^{[n],N}(0)\ ;\ J_{N-j+3}^{[m],N}(-t) - J_{N-j+2}^{[m],N}(-t)\right),\end{aligned} \tag{5.3.31}$$

where we used that $s \mapsto Q_j^{[n],N}(t+s)Q_1^{[m],N}(s)$ is constant in law under the Toda dynamic, and the periodicity of the matrix L (5.3.4). Denoting the difference operator $\partial_j f(j) = f(j+1) - f(j)$, equation (5.3.31) shows that

$$\partial_j\left(\mathrm{Cov}\left(J_j^{[n],N}(t)\ ;\ Q_1^{[m],N}(0)\right) - \mathrm{Cov}\left(Q_1^{[n],N}(0);\ J_{N-j+2}^{[m],N}(-t)\right)\right) = 0 \tag{5.3.32}$$

Evaluating the previous expression at $t = 0$, we deduce that there is some constant c_N, independent of j, such that

$$\mathrm{Cov}\left(J_j^{[n],N}\ ;\ Q_1^{[m],N}\right) - \mathrm{Cov}\left(Q_1^{[n],N};\ J_{N-j+2}^{[m],N}\right) = c_N\,. \tag{5.3.33}$$

Furthermore, since both $Q_j^{[n],N}$, and $J_j^{[m],N}$ are local quantities, in view of Theorem 5.1.7, we deduce that $\lim_{N\to\infty} Nc_N = 0$. So, evaluating the previous expression for $m = 0$, we deduce (5.3.30). Thus, in the large N limit, we can recast (5.3.28) as

$$\lim_{N\to\infty}\partial_\alpha\mathbb{E}\left[J_1^{[n],N}\right] = -\lim_{N\to\infty}\sum_{j=1}^{N}\mathrm{Cov}\left(J_1^{[0],N}\ ;\ Q_j^{[n],N}\right) = \lim_{N\to\infty}\sum_{j=1}^{N}\mathrm{Cov}\left(Q_1^{[1],N}\ ;\ Q_j^{[n],N}\right). \tag{5.3.34}$$

Moreover, in view of the periodicity properties ot the conserved fields and (5.3.19)

$$\lim_{N\to\infty}\partial_\alpha\mathbb{E}\left[J_1^{[n],N}\right] = \lim_{N\to\infty}\frac{1}{N}\mathrm{Cov}\left(Q^{[1],N}\ ;\ Q^{[n],N}\right) = \partial_{t_1}\partial_{t_2}\mathcal{F}_T(\alpha, P + it_1x + it_2x^n)_{|_{t_1,t_2=0}}\,. \tag{5.3.35}$$

Noticing that $\lim_{\alpha\to 0}\mathbb{E}\left[J_1^{[n],N}\right] = 0$, and that we can always uniformly bound $\mathbb{E}\left[J_1^{[n],N}\right]$ by a constant independent of N, the previous equation implies that

$$\lim_{N\to\infty}\mathbb{E}\left[J_1^{[n],N}\right] = \int_0^\alpha \partial_{t_1}\partial_{t_2}\mathcal{F}_T(s, P + it_1x + it_2x^n)_{|_{t_1,t_2=0}}\mathrm{d}s. \tag{5.3.36}$$

So, we conclude. □

The real β-ensemble in the high-temperature regime. The real β-ensemble is the probability measure on $\mathbb{R}^N$ given by

$$\mathrm{d}\mathbb{P}_H(\lambda_1, \ldots, \lambda_N) = \frac{1}{\mathfrak{Z}_N^H(\beta, P)}\prod_{i<j}|\lambda_j - \lambda_i|^\beta e^{-\sum_{j=1}^N P(\lambda_j)}\mathrm{d}\boldsymbol{\lambda}\,, \tag{5.3.37}$$

where $\beta > 0$ and P is a continuous function such that the partition function

$$\mathfrak{Z}_N^H(\beta, P) = \int_{\mathbb{R}^N}\prod_{i<j}|\lambda_j - \lambda_i|^\beta e^{-\sum_{j=1}^N P(\lambda_j)}\mathrm{d}\boldsymbol{\lambda}$$

is finite. This is the case if P grows to infinity fast enough, namely if for some $\beta' > \max(1, \beta)$,

$$\liminf_{|x|\to\infty}\frac{P(x)}{N\beta'\ln|x|} > 1\,, \tag{5.3.38}$$

see [AGZ10, equation (2.6.2)].

Dumitriu and Edelman showed in [DE02] that the β-ensemble admits a tridiagonal representation

$$H = \begin{pmatrix} a_1 & b_1 & & & 0 \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & b_{N-1} & a_N \end{pmatrix}, \tag{5.3.39}$$

where the entries of the matrix are distributed according to the following probability measure

$$\mathrm{d}\mu_H = \frac{1}{Z_N^H(\beta, P)} \prod_{j=1}^{N-1} b_j^{\beta(N-j)-1} \mathbb{1}_{b_j \geqslant 0} \exp\left(-\mathrm{Tr}P(H)\right) \mathrm{d}\mathbf{a}\mathrm{d}\mathbf{b}\,. \tag{5.3.40}$$

Then, the eigenvalues of H are distributed according to $\mathrm{d}\mathbb{P}_H$ (5.3.37). An important example is the case $P(x) = x^2/2$ for which we recover the classical *Gaussian β Ensemble*, see [AGZ10, Section 2.5], and the distribution μ_H factorizes in the following way: the entries of H can be seen to be independent (modulo the symmetry of the matrix), Gaussian $\mathcal{N}(0,1)$ on the diagonal, and the law of the off-diagonal elements is given by renormalized chi variables

$$b_j \sim \frac{1}{\sqrt{2}} \chi_{(N-j)\beta}\,,$$

where the variable X is χ_κ-distributed if its law is given by the density function

$$f(x) = \frac{x^{\kappa-1} e^{-x^2/2}}{2^{\kappa/2-1}\Gamma(\kappa/2)}\,.$$

We are interested in the so-called *high-temperature regime* for this model, specifically, we are interested in the infinite size N limit, in such a way that $\beta = \frac{2\alpha}{N}$ for some $\alpha > 0$. In this regime the probability distribution (5.3.40) becomes

$$\mathrm{d}\mu_H = \frac{1}{Z_N^H(\beta)} \prod_{j=1}^{N-1} b_j^{2\alpha\left(1-\frac{j}{N}\right)-1} \mathbb{1}_{b_j \geqslant 0} \exp\left(-\mathrm{Tr}P(H)\right) \mathrm{d}\mathbf{a}\mathrm{d}\mathbf{b}\,. \tag{5.3.41}$$

This regime has drawn a lot of attention from the random matrix and statistical physics communities lately. Introducing the empirical measure by

$$\mathrm{d}\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}\,,$$

this model was first considered in [ABG12], where the authors were able to compute the limiting empirical measure for this model when $P(x) = x^2/2$. Recently, Garcia-Zelada showed in [GZ19] that under a general choice of P, the sequence of empirical measures satisfies a large deviation principle with strictly convex rate function, ensuring the convergence of $\hat{\mu}_N$. Although the limiting measure is not explicit, its density ρ_α^P satisfies for almost every x the nonlinear equation

$$P(x) - 2\alpha \int_{\mathbb{R}} \log|x-y| \rho_\alpha^P(y)\mathrm{d}y + \log \rho_\alpha^P(x) = \lambda_\alpha^P$$

for some constant λ_α^P, see [GM22, Lemma 3.2] for example.

The fluctuations of the eigenvalues in the bulk and at the edge of a configuration were studied for example in [BGP15, NT18, NT20, Pak18, Lam21]. These fluctuations were shown to be described by Poisson statistics in this regime. With the choice $P(x) = x^2/2$, Nakano and Trinh proved in [NT18] a

Central Limit theorem for this ensemble, namely they proved that for smooth enough $f : \mathbb{R} \to \mathbb{R}$, the random variables

$$\sqrt{N}\left(\int_{\mathbb{R}} f \mathrm{d}\hat{\mu}_N - \int_{\mathbb{R}} f \rho_\alpha^P \mathrm{d}x\right)$$

converge towards a centred Gaussian variable with variance depending both on α and P. In [DGM23], the authors showed this central limit theorem for general confining potentials and smooth enough, decaying at infinity test functions. In this paper, we consider the case where P is a polynomial of even degree $\geqslant 2$. We deduce here from Section 5.2 a central limit theorem for polynomial test functions.

Indeed, in view of Corollary 5.3.2, following the same reasoning as in the case of Toda lattice, we can apply Theorem 5.1.3 to the real β ensemble in the high-temperature regime, thus we deduce that

Corollary 5.3.6 (CLT for Gaussian β ensemble)**.** *Consider the matrix representation* (5.3.39) *of the real β ensemble in the high-temperature regime, and let $P(x)$ be a polynomial of even degree with positive leading order coefficient. Then, defining the Free energy $\mathcal{F}_H(\alpha, P)$ as*

$$\mathcal{F}_H(\alpha, P) = -\lim_{N\to\infty} \frac{1}{N} \ln(Z_N^T(\alpha, P)), \tag{5.3.42}$$

for all $j \in \mathbb{N}$ fixed, we have the following weak limit

$$\lim_{N\to\infty} \frac{\mathrm{Tr}H^j - \mathbb{E}\left[\mathrm{Tr}H^j\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0, \sigma^2), \tag{5.3.43}$$

where

$$\mathbb{E}\left[\mathrm{Tr}H^j\right] = iN\partial_t \mathcal{F}_H(\alpha, P + itx^j)_{|_{t=0}}, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_H(\alpha, P + itx^j)_{|_{t=0}}|. \tag{5.3.44}$$

Thus, we obtained a central limit theorem for the real β ensemble in the high-temperature regime with polynomial potential.

Furthermore, we are in place to apply the second part of our result; indeed, we deduce the following identities

$$\begin{aligned} &\partial_\alpha(\alpha\partial_t \mathcal{F}_H(\alpha, P + itx^j)_{|_{t=0}}) = \partial_t \mathcal{F}_T(\alpha, P + itx^j)_{|_{t=0}}, \\ &\partial_\alpha(\alpha\partial_t^2 \mathcal{F}_H(\alpha, P + itx^j)_{|_{t=0}}) = \partial_t^2 \mathcal{F}_T(\alpha, P + itx^j)_{|_{t=0}} \end{aligned} \tag{5.3.45}$$

so we are able to compute both the moments and their variances of the Toda lattice starting from the one of the real β ensemble at high-temperature.

Remark 5.3.7. *Applying the second part of Theorem 5.1.3, we deduce the following equality valid for the currents of the Toda lattice:*

$$\lim_{N\to\infty} \mathbb{E}\left[J_1^{[n],N}\right] = \int_0^\alpha \partial_{t_1}\partial_{t_2} \mathcal{F}_T(s, P + it_1x + it_2x^n)_{|_{t_1,t_2=0}} \mathrm{d}s = \alpha\partial_{t_1}\partial_{t_2} \mathcal{F}_H(\alpha, P + it_1x + it_2x^n)_{|_{t_1,t_2=0}}. \tag{5.3.46}$$

5.3.2 The exponential Toda lattice, and the Laguerre β ensemble at high-temperature

In this subsection, we focus on the Exponential Toda lattice and its relation with the Laguerre β ensemble in the high-temperature regime [FM21]. These two systems were considered in [GGGM23]. In this paper, the authors considered the classical Gibbs ensemble for the Exponential Toda lattice and were able to compute the density of states for this model connecting it to the Laguerre α ensemble [Maz22], which is related to the classical β one in the high-temperature regime. Here we consider both the Generalized Gibbs ensemble for the integrable lattice and the Laguerre β ensemble at high-temperature with polynomial potential, and we obtain a CLT for both systems, furthermore, we connect the two in the same way as we did for the Toda lattice and the real β ensemble.

The exponential Toda lattice. The exponential Toda lattice is the Hamiltonian system on $\mathbb{R}^{2N}$ described by the Hamiltonian

$$H_E(\mathbf{p},\mathbf{q}) = \sum_{j=1}^{N} e^{-p_j} + \sum_{j=1}^{N} e^{q_j - q_{j+1}}\,, \quad p_j, q_j \in \mathbb{R}\,, \tag{5.3.47}$$

with canonical Poisson bracket. Here, we consider periodic boundary conditions

$$q_{j+N} = q_j + \Omega, \quad p_{j+N} = p_j, \qquad \forall\, j \in \mathbb{Z}, \tag{5.3.48}$$

and $\Omega \geqslant 0$ is an arbitrary constant. The equations of motion are given in Hamiltonian form as

$$\begin{aligned} \dot{q_j} &= \frac{\partial H_E}{\partial p_j} = -e^{-p_j}\,, \\ \dot{p_j} &= -\frac{\partial H_E}{\partial q_j} = e^{q_{j-1}-q_j} - e^{q_j - q_{j+1}}\,. \end{aligned} \tag{5.3.49}$$

Following [GGGM23], we perform the non-canonical change of coordinates

$$x_j = e^{-\frac{p_j}{2}}, \quad y_j = e^{\frac{q_j - q_{j+1}}{2}} = e^{-\frac{r_j}{2}}, \quad r_j = q_{j+1} - q_j, \quad j = 1, \dots, N, \tag{5.3.50}$$

to obtain a Lax Pair for this system. Indeed, in these variables, the Hamiltonian (5.3.47) transform into

$$H_E(\mathbf{x},\mathbf{y}) = \sum_{j=1}^{N} (x_j^2 + y_j^2)\,, \tag{5.3.51}$$

and the Hamilton's equations (5.3.49) become

$$\dot{x_j} = \frac{x_j}{2}\left(y_j^2 - y_{j-1}^2\right), \quad \dot{y_j} = \frac{y_j}{2}\left(x_{j+1}^2 - x_j^2\right), \quad j = 1, \dots, N, \tag{5.3.52}$$

where $x_{N+1} = x_1$, $y_0 = y_N$.

Let us introduce the matrices $L, A \in \mathrm{Mat}(N)$ as

$$L = \begin{pmatrix} x_1^2 + y_N^2 & x_1 y_1 & & & x_N y_N \\ x_1 y_1 & x_2^2 + y_1^2 & x_2 y_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & x_{N-1} y_{N-1} \\ x_N y_N & & & x_{N-1} y_{N-1} & x_N^2 + y_{N-1}^2 \end{pmatrix}, \tag{5.3.53}$$

$$A = \frac{1}{2}\begin{pmatrix} 0 & x_1 y_1 & & & -x_N y_N \\ -x_1 y_1 & 0 & x_2 y_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & x_{N-1} y_{N-1} \\ x_N y_N & & & -x_{N-1} y_{N-1} & 0 \end{pmatrix}, \tag{5.3.54}$$

The system of equations (5.3.52) admits the Lax representation

$$\dot{L} = [A, L]. \tag{5.3.55}$$

Hence, the quantities $H_m = \mathrm{Tr} L^m$, $m = 1, \dots, N$ are constants of motion as well as the eigenvalues of L. For this integrable model, we define the generalized Gibbs ensemble as

$$\mathrm{d}\mu_{ET} = \frac{1}{Z_N^{H_E}(\alpha,\gamma,P)} \prod_{j=1}^{N} x_j^{2\frac{\alpha}{\gamma}-1} y_j^{2\alpha-1} \mathbb{1}_{x_j \geqslant 0} \mathbb{1}_{y_j \geqslant 0} e^{-\mathrm{Tr} P(L)} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\,, \tag{5.3.56}$$

where $\alpha, \gamma > 0$, and P is a real valued polynomial with positive leading coefficient. $Z_N^{H_E}(\alpha, \gamma, P)$ is the normalization constant.

Remark 5.3.8. *The definition of our Gibbs ensemble is slightly different from the one given in [GGGM23], indeed there the authors were considering just the classical Gibbs ensemble for this model, so the case $P(x) = x/2$.*

We notice that the structure of (5.3.56) resembles the one of $\mu^{(1)}_{kM+\ell}$ (5.2.1), thus we want to apply Theorem 5.1.3. To do this, we have to identify the functions F, W. First, as an application of Theorem 5.3.1, we obtain the following corollary

Corollary 5.3.9. *Fix $m \in \mathbb{N}$, and consider the matrix L (5.3.53). Then for N big enough, there exists some $k = k(m) \in \mathbb{N}$, and two polynomial functions $V : \mathbb{R}_+^{2k} \times \mathbb{R}_+^{2k} \to \mathbb{R}$ and $V_1 : \mathbb{R}_+^{2k+\ell} \times \mathbb{R}_+^{2k+\ell} \to \mathbb{R}$ such that*

$$\begin{aligned} \mathrm{Tr}L^m = &\sum_{j=1}^{M} V(\mathbf{x}_j, \mathbf{y}_j, \mathbf{x}_{j+1}, \mathbf{y}_{j+1}) \\ &+ V_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1, \mathbf{y}_M, y_{kM+1}, \dots, y_{kM+\ell}, \mathbf{y}_1)\,, \end{aligned} \tag{5.3.57}$$

where $N = kM + \ell$.

As in the Toda lattice case, if we naively set $F(x, y) = x^{2\frac{\alpha}{\gamma}-1} y^{2\alpha-1}$, this would not fit in the hypotheses of our theorem, since this is not an $L^2(\mathbb{R}_+^2)$ function. As in the previous case, we have just to consider a slight modification of the measure:

$$\mathrm{d}\mu_{ET} = \frac{1}{Z_N^{H_E}(\alpha, \gamma, P)} \prod_{j=1}^{N} x_j^{2\frac{\alpha}{\gamma}-1} y_j^{2\alpha-1} \exp\left(-\varepsilon \frac{x_j^2 + y_j^2}{2}\right) \mathbb{1}_{x_j \geqslant 0} \mathbb{1}_{y_j \geqslant 0} e^{-\mathrm{Tr}P(L) + \varepsilon \frac{x_j^2+y_j^2}{2}} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\,, \tag{5.3.58}$$

for fixed $\varepsilon > 0$, but small. In this way, defining $F(x, y, \alpha) = x^{2\frac{\alpha}{\gamma}-1} y^{2\alpha-1} \exp(-\varepsilon \frac{x^2+y^2}{2})$, and $W(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) = V(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) - \frac{\varepsilon}{2} \sum_{j=1}^{2k} x_j^2 + y_j^2$ we are in the same hypotheses as Theorem 5.1.3, thus we deduce the following corollary

Corollary 5.3.10 (CLT for the Exponential Toda lattice)**.** *Consider the Lax matrix L (5.3.53) of the Exponential Toda lattice distributed according to the Generalized Gibbs Ensemble (5.3.56). Then, defining the Free energy $\mathcal{F}_{HE}(\alpha, \gamma, P)$ as*

$$\mathcal{F}_{ET}(\alpha, \gamma, P) = -\lim_{N \to \infty} \frac{1}{N} \ln(Z_N^{H_E}(\alpha, \gamma, P))\,, \tag{5.3.59}$$

for all $j \in \mathbb{N}$ fixed, we have the following weak limit

$$\lim_{N \to \infty} \frac{\mathrm{Tr}L^j - \mathbb{E}\left[\mathrm{Tr}L^j\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0, \sigma^2)\,, \tag{5.3.60}$$

where

$$\mathbb{E}\left[\mathrm{Tr}L^j\right] = iN\partial_t \mathcal{F}_{ET}(\alpha, \gamma, P + itx^j)_{|_{t=0}}\,, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_{ET}(\alpha, \gamma, P + itx^j)_{|_{t=0}}| \tag{5.3.61}$$

The Laguerre β ensemble in the high-temperature regime. The Laguerre β ensemble is a random matrix ensemble introduced by Dumitriu and Edelman in [DE02]. It has the following matrix representation

$$Q = \begin{pmatrix} x_1^2 & x_1y_1 & & & \\ x_1y_1 & x_2^2+y_1^2 & x_2y_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & x_{N-1}y_{N-1} \\ & & & x_{N-1}y_{N-1} & x_N^2+y_{N-1}^2 \end{pmatrix}, \tag{5.3.62}$$

where the entries of Q are distributed according to

$$\mathrm{d}\mu_L = \frac{1}{Z_N^L(\beta)} \prod_{j=1}^{N} x_j^{\beta(M-j+1)-1} \mathbb{1}_{x_j \geqslant 0} \prod_{j=1}^{N-1} y^{\beta(N-j)-1} \mathbb{1}_{y_j \geqslant 0} \exp\left(-\mathrm{Tr}P(Q)\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\,, \tag{5.3.63}$$

where M is such that $\lim_{N\to\infty} N/M = \gamma \in (0,1]$, and P can be any continuous function such that the partition function is well-defined, for our purpose we consider $P(x)$ to be a polynomial.

The remarkable property of this ensemble is that it is possible to explicitly compute the joint eigenvalue density as

$$\mathrm{d}\mathbb{P}_L = \frac{1}{\mathfrak{Z}_N^L(\beta,P)} \prod_{j=1}^{N} \lambda_j^{\frac{\beta}{2}(M-N+1)-1} \mathbb{1}_{\lambda_j \geqslant 0} \prod_{j<i} |\lambda_j - \lambda_i|^\beta e^{-\sum_{j=1}^N P(\lambda_j)} \mathrm{d}\boldsymbol{\lambda}\,. \tag{5.3.64}$$

We are interested in the so-called high-temperature limit, i.e. when $\beta = \frac{2\alpha}{N}$, $\alpha \in \mathbb{R}_+$, which was considered in [ABMV13], where the authors were able to compute the density of states for the particular case $P(x) = x/2$.

In this regime, the density (5.3.63) takes the form

$$\mathrm{d}\mu_L = \frac{1}{Z_N^L(\alpha,\gamma,P)} \prod_{j=1}^{N} x_j^{2\frac{\alpha}{\gamma}\left(1-\frac{j+1}{N}\right)-1} \mathbb{1}_{x_j \geqslant 0} \prod_{j=1}^{N-1} y^{2\alpha\left(1-\frac{j}{N}\right)-1} \mathbb{1}_{y_j \geqslant 0} \exp\left(-\mathrm{Tr}P(Q)\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\,. \tag{5.3.65}$$

The structure of this density resembles the one of $\mathrm{d}\mu^{(2)}_{kM+\ell}$(5.2.3), indeed proceeding as in the case of the Exponential Toda lattice, we deduce the following corollary

Corollary 5.3.11 (CLT for Laguerre β ensemble)**.** *Consider the matrix representation* (5.3.62) *of the Laguerre β ensemble in the high-temperature regime, and let $P(x)$ be a real polynomial of degree at least* 1*. Then, defining the Free energy $\mathcal{F}_L(\alpha,\gamma,P)$ as*

$$\mathcal{F}_L(\alpha,\gamma,P) = -\lim_{N\to\infty} \frac{1}{N} \ln(Z_N^L(\alpha,\gamma,P))\,, \tag{5.3.66}$$

for all $j \in \mathbb{N}$ fixed, we have the following weak limit

$$\lim_{N\to\infty} \frac{\mathrm{Tr}Q^j - \mathbb{E}\left[\mathrm{Tr}Q^j\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0,\sigma^2)\,, \tag{5.3.67}$$

where

$$\mathbb{E}\left[\mathrm{Tr}Q^j\right] = iN\partial_t \mathcal{F}_L(\alpha,\gamma,P+itx^j)_{|_{t=0}}\,, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_L(\alpha,\gamma,P+itx^j)_{|_{t=0}}| \tag{5.3.68}$$

Which is the perfect analogue of the result for the Exponential Toda lattice. Furthermore, we are in position to apply the second part of our result, indeed we can deduce the following identities

$$\begin{aligned} \partial_\alpha(\alpha\partial_t \mathcal{F}_L(\alpha,\gamma,P+itx^j)_{|_{t=0}}) &= \partial_t \mathcal{F}_{ET}(\alpha,\gamma,P+itx^j)_{|_{t=0}}\,, \\ \partial_\alpha(\alpha\partial_t^2 \mathcal{F}_L(\alpha,\gamma,P+itx^j)_{|_{t=0}}) &= \partial_t \mathcal{F}_{ET}(\alpha,\gamma,P+itx^j)_{|_{t=0}}\,, \end{aligned} \tag{5.3.69}$$

thus, we can compute all the quantities involved in the previous theorems from the Free Energy of the Laguerre ensemble.

5.3.3 The Volterra lattice, and the antisymmetric β ensemble at high-temperature

In this subsection, we focus on the Volterra lattice and its relation with the Antisymmetric β ensemble [DF10] in the high-temperature regime [FM21]. These two systems were considered in [GGGM23]. In this paper, the authors considered the classical Gibbs ensemble for the Volterra lattice and were able to compute the density of states for this model connecting it to the Antisymmetric α ensemble [FM21], which is related to the classical β one introduced by Dumitriu and Forrester [DF10].

The Volterra Lattice. The Volterra lattice (or *discrete KdV equation*) is the following systems of N coupled ODEs

$$\dot{a_j} = a_j \left(a_{j+1} - a_{j-1}\right), \qquad j = 1, \dots, N, \tag{5.3.70}$$

here $a_j \in \mathbb{R}_+$ for $j = 1, \dots, N$, and we consider periodic boundary conditions $a_j = a_{j+N}$ for all $j \in \mathbb{Z}$. Volterra introduced it to study evolution of populations in a hierarchical system of competing species. This system was considered by Kac and van Moerbeke in [KvM75], who solved it explicitly using a discrete version of the inverse scattering transform introduced by Flaschka [Fla74a].

Introducing on the phase space $\mathbb{R}_+^N$ the following Poisson bracket

$$\{a_j, a_i\}_{\text{Volt}} = a_j a_i (\delta_{i,j+1} - \delta_{i,j-1}), \tag{5.3.71}$$

and defining the Hamiltonian $H_1 = \sum_{j=1}^N a_j$, we can rewrite the equations of motion (5.3.70) in Hamiltonian form as

$$\dot{a}_j = \{a_j, H_1\}_{\text{Volt}}. \tag{5.3.72}$$

An elementary constant of motion for the system is $H_0 = \prod_{j=1}^N a_j$ which is independent of H_1.

The Volterra lattice is a completely integrable system, and it admits several equivalents *Lax representations*, see e.g. [KvM75, Mos75, GGGM23]. We use the one presented in [GGGM23]. Specifically, we introduce the matrices $L, A \in \text{Mat}(\mathbb{R}, N)$ as

$$L = \begin{pmatrix} 0 & \sqrt{a_1} & & & -\sqrt{a_N} \\ -\sqrt{a_1} & 0 & \sqrt{a_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{a_{N-1}} \\ \sqrt{a_N} & & & -\sqrt{a_{N-1}} & 0 \end{pmatrix}, \tag{5.3.73}$$

$$A = \frac{1}{2} \sum_{j=1}^{N} \sqrt{a_j a_{j+1}} (E_{j,j+2} - E_{j+2,j}), \tag{5.3.74}$$

where $E_{r,s}$ is defined as $(E_{r,s})_{ij} = \delta_r^i \delta_s^j$ and $E_{j+N,i} = E_{j,i+N} = E_{j,i}$. Then, it follows that the equations of motion (5.3.72) are equivalent to

$$\dot{L} = [L; A]. \tag{5.3.75}$$

In view of this Lax pair, we know that $\text{Tr}L^k$ are constant of motion for the model.

Following [GGGM23], we introduce the Generalized Gibbs Ensemble of the Volterra lattice (5.3.70) as

$$\mathrm{d}\mu_{\text{Volt}}(\mathbf{a}) = \frac{e^{\text{Tr}P(L)} \prod_{j=1}^N a_j^{\alpha-1} \mathbb{1}_{a_j>0} \mathrm{d}\mathbf{a}}{Z_N^{\text{Volt}}(\alpha, P)}, \tag{5.3.76}$$

where $\alpha > 0$, $P(x)$ is a polynomial of the form $P(x) = (-1)^j x^{2j} + \text{l.o.t}$, otherwise the previous measure is not normalizable, moreover, we notice that, in view of the antisymmetric nature of L, $\text{Tr}L^{2j+1} = 0$.

For this reason, we perform the change of coordinates $\sqrt{a_j} = x_j$, where we take just the positive root, so the previous measure read

$$\mathrm{d}\mu_{\mathrm{Volt}}(\mathbf{x}) = \frac{e^{\mathrm{Tr}P(L)} \prod_{j=1}^{N} x_j^{2\alpha-1} \mathbb{1}_{x_j>0} \mathrm{d}\mathbf{x}}{Z_N^{\mathrm{Volt}}(\alpha, P)} \,. \tag{5.3.77}$$

This Generalized Gibbs ensemble resembles the structure of $\mu^{(1)}_{kM+\ell}$ (5.2.1), we have just to identify F, W. We notice that it is possible to generalize Theorem 5.3.1 also for the antisymmetric situation, so we deduce the following Corollary:

Corollary 5.3.12. *Fix $m \in \mathbb{N}$, and consider the matrix L* (5.3.73)*. Then for N big enough, there exists a $k = k(m) \in \mathbb{N}$, and two polynomial functions $V : \mathbb{R}_+^k \times \mathbb{R}_+^k \to \mathbb{R}$, $V_1 : \mathbb{R}_+^k \times \mathbb{R}_+^\ell \times \mathbb{R}_+^k \to \mathbb{R}$ such that*

$$\mathrm{Tr}L^m = \sum_{j=1}^{M} V(\mathbf{x}_j, \mathbf{x}_{j+1}) + V_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1) \,, \tag{5.3.78}$$

where $N = kM + \ell$.

Thus, following the same kind of reasoning as in the Toda lattice, section 5.3.1, and the Exponential Toda lattice, section 5.3.2, we deduce the following:

Corollary 5.3.13 (CLT for Volterra lattice)**.** *Consider the Lax matrix L* (5.3.73) *of the Volterra lattice distributed according to the Generalized Gibbs Ensemble* (5.3.76)*. Then, defining the Free energy $\mathcal{F}_{Volt}(\alpha, P)$ as*

$$\mathcal{F}_{Volt}(\alpha, P) = -\lim_{N\to\infty} \frac{1}{N} \ln(Z_N^{Volt}(\alpha, P)) \,, \tag{5.3.79}$$

for all $j \in \mathbb{N}$ fixed, we have the following weak limit

$$\lim_{N\to\infty} \frac{\mathrm{Tr}L^j - \mathbb{E}\left[\mathrm{Tr}L^j\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0, \sigma^2) \,, \tag{5.3.80}$$

where

$$\mathbb{E}\left[\mathrm{Tr}L^j\right] = iN\partial_t \mathcal{F}_{Volt}(\alpha, P - itx^j)_{|_{t=0}} \,, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_{Volt}(\alpha, P - itx^j)_{|_{t=0}}| \,. \tag{5.3.81}$$

The Antisymmetric β ensemble in the high-temperature regime The Antisymmetric β ensemble is a random matrix ensemble introduced by Dumitriu and Forrester in [DF10]; it has the following matrix representation

$$Q = \begin{pmatrix} 0 & x_1 & & & & \\ -x_1 & 0 & x_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & x_{N-1} & \\ & & & -x_{N-1} & 0 & \end{pmatrix} , \tag{5.3.82}$$

and the entries of the matrix Q are distributed according to

$$\mathrm{d}\mu_{AG} = \frac{1}{Z_N^{AG}(\beta, P)} \prod_{j=1}^{N-1} x_j^{\beta(N-j)-1} \mathbb{1}_{x_j \geqslant 0} \exp(\mathrm{Tr}P(Q)) \mathrm{d}\mathbf{x} \,, \tag{5.3.83}$$

here $P(x)$ can be any function that makes (5.3.83) normalizable, but for our purpose we will consider $P(x)$ polynomial of the form $P(x) = (-1)^j x^{2j} + \text{l.o.t.}$

As in the previous cases, we are interested in the high-temperature regime for this model, so we set $\beta = \frac{2\alpha}{N}$, and we rewrite the previous density as

$$\mathrm{d}\mu_{AG} = \frac{1}{Z_N^{AG}(\alpha, P)} \prod_{j=1}^{N-1} x_j^{\alpha\left(1-\frac{j}{N}\right)-1} \mathbb{1}_{x_j \geqslant 0} \exp(\mathrm{Tr}P(Q))\mathrm{d}\mathbf{x}\,. \tag{5.3.84}$$

This regime was introduced in [GGGM23], where the author computed the density of states for this model in the case $P(x) = x^2/2$. The structure of this last density (5.3.84) resembles the one of $\mu^{(2)}_{kM+\ell}$(5.2.3), indeed proceeding as in the case of the Volterra lattice, we deduce the following corollary

Corollary 5.3.14 (CLT for Antisymmetric Gaussian β ensemble)**.** *Consider the matrix representation* (5.3.82) *of the Antisymmetric β ensemble in the high-temperature regime, endowed with the probability distribution* $\mathrm{d}\mu_{AG}$ (5.3.84)*, and let $P(x)$ be a polynomial of the form $P(x) = (-1)^j x^{2j}$ + l.o.t.. Then, defining the Free energy $\mathcal{F}_{AG}(\alpha, P)$ as*

$$\mathcal{F}_{AG}(\alpha, P) = -\lim_{N\to\infty} \frac{1}{N} \ln(Z_N^{AG}(\alpha, P))\,, \tag{5.3.85}$$

for all $j \in \mathbb{N}$ fixed, we have the following weak limit

$$\lim_{N\to\infty} \frac{\mathrm{Tr}Q^j - \mathbb{E}\left[\mathrm{Tr}Q^j\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0, \sigma^2)\,. \tag{5.3.86}$$

where

$$\mathbb{E}\left[\mathrm{Tr}Q^j\right] = iN\partial_t \mathcal{F}_{AG}(\alpha, P - itx^j)_{|_{t=0}}\,, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_{AG}(\alpha, P - itx^j)_{|_{t=0}}|\,. \tag{5.3.87}$$

Which is the perfect analogue of the result for the Volterra lattice.

Remark 5.3.15. *In view of Theorem 5.1.3, we deduce the following identities*

$$\begin{aligned} \partial_\alpha(\alpha\partial_t \mathcal{F}_{AG}(\alpha, P - itx^j)_{|_{t=0}}) &= \partial_t \mathcal{F}_{AG}(\alpha, P - itx^j)_{|_{t=0}}\,, \\ \partial_\alpha(\alpha\partial_t^2 \mathcal{F}_{AG}(\alpha, P - itx^j)_{|_{t=0}}) &= \partial_t \mathcal{F}_{AG}(\alpha, P - itx^j)_{|_{t=0}} \end{aligned} \tag{5.3.88}$$

5.3.4 The defocusing Ablowitz-Ladik lattice, and the Circular β ensemble at high-temperature

In this subsection, we focus on the defocusing Ablowitz-Ladik lattice, and its relation to the Circular β ensemble at high-temperature [HL21, GM23, Spo22b]. This relation was highlighted by one of the present authors and T. Grava [GM23], and independently by H. Spohn [Spo22b]. In these papers, the authors were able to characterize the density of states of the Ablowitz-Ladik lattice in terms of the one of the Circular β ensemble in the high-temperature regime. Moreover, in [GM23] the authors were able to compute explicitly the density of states in the case of linear potential in terms of the solution of the Double Confluent Heun Equation [DLMF] highlighting a connection with the Painlevé equations [LN21, FIKN06]. In [MM23b], the two present authors obtained a large deviations principles for the empirical spectral measure for any continuous and bounded potential.

The defocusing Ablowitz-Ladik lattice. The defocusing Ablowitz-Ladik (dAL) lattice is defined by the following system of nonlinear equations

$$i\dot{a}_j = -(a_{j+1} + a_{j-1} - 2a_j) + |a_j|^2(a_{j-1} + a_{j+1})\,, \tag{5.3.89}$$

where $a_j(t) \in \mathbb{C}$. We assume N-periodic boundary conditions $a_{j+N} = a_j$, for all $j \in \mathbb{Z}$. The dAL lattice was introduced by Ablowitz and Ladik [AL75, AL76] as the spatial integrable discretization of the defocusing cubic nonlinear Schrödinger Equation for the complex function $\psi(x,t)$, $x \in S^1$ and $t \in \mathbb{R}$:

$$i\partial_t \psi(x,t) = -\partial_x^2 \psi(x,t) + 2|\psi(x,t)|^2 \psi(x,t). \tag{5.3.90}$$

As for the others dynamical systems that we considered, the dAL is an integrable system. Its integrability was proved by Ablowitz and Ladik by discretizing the 2×2 Zakharov-Shabat Lax pair [AKN74, AL75] of the cubic nonlinear Schrödinger equation. Furthermore, Nenciu and Simon [Nen05, Sim05] constructed a new Lax pair for this lattice. Following their construction we double the size of the chain according to the periodic boundary condition, thus we consider a chain of $2N$ particles $a_1,\dots,a_{2N}$ such that $a_j = a_{j+N}$ for $j=1,\dots,N$. Define the 2×2 unitary matrix Ξ_j

$$\Xi_j = \begin{pmatrix} \overline{a}_j & \rho_j \\ \rho_j & -a_j \end{pmatrix}, \quad j=1,\dots,2N, \quad \rho_j = \sqrt{1-|a_j|^2} \tag{5.3.91}$$

and the $2N\times 2N$ matrices

$$\mathcal{M} = \begin{pmatrix} -a_{2N} & & & & & \rho_{2N} \\ & \Xi_2 & & & & \\ & & \Xi_4 & & & \\ & & & \ddots & & \\ & & & & \Xi_{2N-2} & \\ \rho_{2N} & & & & & \overline{a}_{2N} \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} \Xi_1 & & & & \\ & \Xi_3 & & & \\ & & \ddots & & \\ & & & \Xi_{2N-1} & \end{pmatrix}. \tag{5.3.92}$$

Now let us define the unitary Lax matrix

$$\mathcal{E} = \mathcal{L}\mathcal{M}, \tag{5.3.93}$$

that has the structure of a 5-band periodic diagonal matrix. The matrix $\mathcal{E}$ is a periodic CMV matrix [CMV05]. The equations of motion (5.3.89) are equivalent to the following Lax equation for the matrix $\mathcal{E}$:

$$\dot{\mathcal{E}} = i\left[\mathcal{E}, \mathcal{E}^+ + (\mathcal{E}^+)^\dagger\right], \tag{5.3.94}$$

where † stands for hermitian conjugate and

$$\mathcal{E}^+_{j,k} = \begin{cases} \frac{1}{2}\mathcal{E}_{j,j} & j=k \\ \mathcal{E}_{j,k} & k=j+1 \bmod 2N \text{ or } k=j+2 \bmod 2N \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.95}$$

Since the matrix $\mathcal{E}$ is a periodic band matrix with fixed bandwidth, we can follow the same reasoning as in the previous cases and conclude the following

Lemma 5.3.16. *Fix $m\in\mathbb{N}$, and consider the matrix $\mathcal{E}$ (5.3.93). Then for N big enough, there exists a $k=k(m)\in\mathbb{N}$, and two polynomials $V:\mathbb{C}^k_+\times\mathbb{C}^k_+\to\mathbb{R}$, and $V:\mathbb{C}^k_+\times\mathbb{C}^\ell_+\times\mathbb{C}^k_+\to\mathbb{R}$ such that*

$$\mathrm{Tr}\mathcal{E}^m = \sum_{j=1}^{M} V(\mathbf{a}_j,\mathbf{a}_{j+1}) + V_1(\mathbf{a}_M,\mathbf{a}_{kM+1},\dots,a_{kM+\ell},\mathbf{a}_1), \tag{5.3.96}$$

where $N=kM+\ell$.

Following [Spo22b, GM23, MM23b], we notice that the quantity $K_0=\prod_{j=1}^N(1-|a_j|^2)$ is conserved, so this means that if $|a_j(0)|<1$ for all $j=1,\dots,N$ then $|a_j(t)|<1$ for all $j=1,\dots,N$ for all $t\in\mathbb{R}$, so we can consider $\mathbb{D}^N$ as our phase space, here $\mathbb{D}=\{z\in\mathbb{C}\,|\,|z|<1\}$. On this phase space, we introduce the Generalized Gibbs ensemble for the defocusing AL lattice as

$$\mathrm{d}\mu_{\mathrm{dAL}} = \frac{\prod_{j=1}^N\left(1-|a_j|^2\right)^{\alpha-1}\mathbb{1}_{a_j\in\mathbb{D}}\exp\left(-\mathrm{Tr}\left(P\left(\mathcal{E}\right)\right)\right)\mathrm{d}^2\mathbf{a}}{Z_N^{\mathrm{dAL}}(\alpha,P)}, \tag{5.3.97}$$

where P is a real-valued Polynomial, meaning that there exists a polynomial $\widetilde{P}$ such that $P=\Re(\widetilde{P})$. In view of Lemma 5.3.16, we are in the hypotheses of Theorem 5.1.3, thus we deduce the following:

Corollary 5.3.17 (CLT for defocusing Ablowitz–Ladik lattice)**.** *Consider the Lax matrix* $\mathcal{E}$ (5.3.93) *of the defocusing Ablowitz–Ladik lattice distributed according to the Generalized Gibbs Ensemble* (5.3.97)*. Then, defining the Free energy* $\mathcal{F}_{dAL}(\alpha, P)$ *as*

$$\mathcal{F}_{dAL}(\alpha, P) = -\lim_{N\to\infty} \frac{\ln(Z_N^{dAL}(\alpha, P))}{2N}\,, \tag{5.3.98}$$

for all $j \in \mathbb{N}$ *fixed, we have the following weak limit*

$$\lim_{N\to\infty} \frac{\mathrm{Tr}\mathcal{E}^j - \mathbb{E}\left[\mathrm{Tr}\mathcal{E}^j\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0, \sigma^2)\,, \tag{5.3.99}$$

where

$$\mathbb{E}\left[\mathrm{Tr}\mathcal{E}^j\right] = iN\partial_t \mathcal{F}_{dAL}(\alpha, P + itx^j)_{|_{t=0}}\,, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_{dAL}(\alpha, P + itx^j)_{|_{t=0}}|\,. \tag{5.3.100}$$

The circular β ensemble at high-temperature. The circular β ensemble was introduced by Killip and Nenciu in [KN04]; as the other β ensembles that we considered, it possesses a matrix representation. Consider the two block diagonal matrices

$$\mathfrak{M} = \mathrm{diag}\,(\Xi_1, \Xi_3, \Xi_5 \ldots,) \quad \text{and} \quad \mathfrak{l} = \mathrm{diag}\,(\Xi_0, \Xi_2, \Xi_4, \ldots)\,, \tag{5.3.101}$$

where the block Ξ_j, $j = 1, \ldots, N-1$ are defined in (5.3.91), while $\Xi_0 = (1)$ and $\Xi_N = (\overline{\alpha}_N)$ are 1×1 matrices. Then, we define $\mathfrak{E}$ as follows

$$\mathfrak{E} = \mathfrak{l}\mathfrak{M}. \tag{5.3.102}$$

The entries of this matrix are distributed according to

$$\mathrm{d}\mu_C = \frac{\prod_{j=1}^{N-1}\left(1-|a_j|^2\right)^{\beta(N-j)-1} \mathbb{1}_{a_j\in\mathbb{D}} \exp\left(-\mathrm{Tr}\left(P\left(\mathfrak{E}\right)\right)\right) \prod_{j=1}^{N-1} \mathrm{d}^2 a_j \frac{\mathrm{d}a_N}{ia_N}}{Z_N^{\mathrm{dAL}}(\beta, P)}\,. \tag{5.3.103}$$

As for the other β ensembles, one can explicitly compute the joint eigenvalue density for this ensemble as

$$\mathrm{d}\mathbb{P}_C = \frac{1}{\mathfrak{Z}_N^C(\beta, P)} \prod_{j<\ell} |e^{i\theta_j} - e^{i\theta_\ell}|^\beta \mathbb{1}_{\theta_j\in\mathbb{T}} e^{-\sum_{j=1}^N P(e^{i\theta_j})} \mathrm{d}\boldsymbol{\theta}\,, \tag{5.3.104}$$

here $\mathbb{T} = [-\pi, \pi)$, $e^{i\theta_j}$ are the eigenvalues of $\mathfrak{E}$, and P can be any continuous function that makes the measure normalizable. We restrict our attention to the class of real polynomial $P(z)$.

We are interested in the high-temperature limit for this ensemble [GM23, Spo22b], so we set $\beta = \frac{2\alpha}{N}$, obtaining

$$\mathrm{d}\mu_C = \frac{\prod_{j=1}^{N-1}\left(1-|a_j|^2\right)^{2\alpha\left(1-\frac{j}{N}\right)-1} \mathbb{1}_{a_j\in\mathbb{D}} \exp\left(-\mathrm{Tr}\left(P\left(\mathfrak{E}\right)\right)\right) \prod_{j=1}^{N-1} \mathrm{d}^2 a_j \frac{\mathrm{d}a_N}{ia_N}}{Z_N^{\mathrm{dAL}}(\alpha, P)}\,. \tag{5.3.105}$$

So, in view of Lemma 5.3.16, we are in the hypotheses of Theorem 5.1.3, so we deduce the following

Corollary 5.3.18 (CLT for Circular β ensemble)**.** *Consider the matrix representation* (5.3.102) *of the Circular* β *ensemble in the high-temperature regime, endowed with the probability distribution* $\mathrm{d}\mu_C$ (5.3.105)*, and let* $P(x)$ *be a real-valued polynomial. Then, defining the Free energy* $\mathcal{F}_C(\alpha, P)$ *as*

$$\mathcal{F}_C(\alpha, P) = -\lim_{N\to\infty} \frac{\ln(Z_N^C(\alpha, P))}{2N}\,, \tag{5.3.106}$$

for all $j \in \mathbb{N}$ *fixed, we have the following weak limit*

$$\lim_{N\to\infty} \frac{\mathrm{Tr}\mathfrak{E}^j - \mathbb{E}\left[\mathrm{Tr}\mathfrak{E}^j\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0, \sigma^2)\,, \tag{5.3.107}$$

where

$$\mathbb{E}\left[\mathrm{Tr}\mathfrak{C}^j\right] = iN\partial_t\mathcal{F}_C(\alpha,\gamma,P+itx^j)_{|_{t=0}}\,,\quad \sigma^2 = |\partial_t^2\mathcal{F}_C(\alpha,P+itx^j)_{|_{t=0}}|\,. \tag{5.3.108}$$

Remark 5.3.19. *We notice that*

- *Hardy and Lambert in [HL21] already proved a CLT theorem for the Circular β ensemble in the high-temperature regime for a wider class of functions and potentials than we can consider with our result. Nevertheless, we highlight the fact that in our case we can explicitly compute the means, and the variances in terms of the Free energy.*
- *The following identities hold in view of the last part of Theorem 5.1.3*

$$\begin{aligned}\partial_\alpha(\alpha\partial_t\mathcal{F}_C(\alpha,P+itx^j)_{|_{t=0}}) &= \partial_t\mathcal{F}_{dAL}(\alpha,P+itx^j)_{|_{t=0}}\,,\\ \partial_\alpha(\alpha\partial_t^2\mathcal{F}_C(\alpha,P+itx^j)_{|_{t=0}}) &= \partial_t\mathcal{F}_{dAL}(\alpha,P+itx^j)_{|_{t=0}}\,.\end{aligned} \tag{5.3.109}$$

This relation was already proven in [GM23] with the same kind of argument that we followed.

5.3.5 The defocusing Schur flow, and the Jacobi β ensemble at high-temperature

In this subsection, we focus on the defocusing Schur flow [Gol06], and its relation to the Jacobi β ensemble at high-temperature [FM21]. This relation was first noticed in [Spo22b], and then the two present authors obtained a large deviations principles for the empirical spectral measure for the defocusing Schur flow, and they were able to link it to the one of the Jacobi β ensemble in the high-temperature regime [MM23b].

The defocusing Schur flow. The defocusing Schur flow is the system of ODEs [Gol06]

$$\dot{a_j} = \rho_j^2(a_{j+1}-a_{j-1})\,,\quad \rho_j = \sqrt{1-|a_j|^2}\,, \tag{5.3.110}$$

and, as before, we consider periodic boundary conditions, namely $a_j = a_{j+N}$ for all $j\in\mathbb{Z}$.

We notice that, if one chooses an initial data such that $a_j(0)\in\mathbb{R}$ for all $j=1,\dots,N$, then $a_j(t)\in\mathbb{R}$ for all times. Moreover, it is straightforward to verify that $K_0 = \prod_{j=1}^N\left(1-|a_j|^2\right)$ is conserved along the Schur flow. This implies that we can choose as phase space for the Schur flow the N-cube $\mathbb{I}^N$, where $\mathbb{I} := (-1,1)$. Furthermore, it was shown in [Gol06], that the Schur flow has the same Lax matrix as the focusing Ablowitz–Laddik lattice.

Following [Spo22b, GGGM23], on $\mathbb{I}^N$ we define the finite volume limit GGE as

$$\mathrm{d}\mu_{\mathrm{dS}}(\mathbf{a}) = \frac{\prod_{j=1}^N\left(1-a_j^2\right)^{\alpha-1}\mathbb{1}_{a_j\in\mathbb{I}}\exp\left(-\mathrm{Tr}\left(P\left(\mathcal{E}\right)\right)\right)\mathrm{d}\mathbf{a}}{Z_N^{\mathrm{dS}}(\alpha,P)}\,, \tag{5.3.111}$$

where $P(x):\mathbb{R}\to\mathbb{R}$ is a polynomial. Thanks to Lemma 5.3.16, we can apply Theorem 5.1.3 obtaining a CLT theorem for the defocusing Schur flow

Corollary 5.3.20 (CLT for defocusing Schur flow)**.** *Consider the Lax matrix $\mathcal{E}$* (5.3.93) *of the defocusing Schur flow distributed according to the Generalized Gibbs Ensemble* (5.3.111)*. Then, defining the Free energy $\mathcal{F}_{dS}(\alpha,P)$ as*

$$\mathcal{F}_{dS}(\alpha,P) = -\lim_{N\to\infty}\frac{\ln(Z_N^{dS}(\alpha,P))}{2N}\,, \tag{5.3.112}$$

for all $j\in\mathbb{N}$ fixed, we have the following weak limit

$$\lim_{N\to\infty}\frac{\mathrm{Tr}\mathcal{E}^j-\mathbb{E}\left[\mathrm{Tr}\mathcal{E}^j\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0,\sigma^2)\,, \tag{5.3.113}$$

where

$$\mathbb{E}\left[\mathrm{Tr}\mathcal{E}^j\right] = iN\partial_t\mathcal{F}_{dS}(\alpha,P+itx^j)_{|_{t=0}}\,,\quad \sigma^2 = |\partial_t^2\mathcal{F}_{dS}(\alpha,P+itx^j)_{|_{t=0}}|\,. \tag{5.3.114}$$

The Jacobi β ensemble in the high-temperature regime. The Jacobi β ensemble is a random matrix ensemble introduced by Killip and Nenciu in [KN04]. It has two slightly different matrix representations. The first one is the same as the Circular β ensemble (5.3.102), but the distribution of the entries of the matrix is

$$\mathrm{d}\mu_J(\mathbf{a}) = \frac{\prod_{j=1}^{2N-1}\left(1-a_j^2\right)^{\beta(2N-j)/4-1}\prod_{j=1}^{2N-1}(1-a_j)^{\mathrm{a}+1-\beta/4}(1+(-1)^j a_j)^{\mathrm{b}+1-\beta/4}\mathbb{1}_{a_j\in\mathbb{I}}\exp\left(-\mathrm{Tr}\left(P\left(\mathfrak{E}\right)\right)\right)\mathrm{d}\mathbf{a}}{Z_N^J(\beta,P)}\,, \tag{5.3.115}$$

where $\mathrm{a},\mathrm{b}>-1$, $P(x)$ is a real value polynomial. We notice that we are considering an *even* number of random variables, and $a_j\in\mathbb{R}$; for these reasons, all the eigenvalues of $\mathfrak{E}$ come in pairs, meaning that if $e^{i\theta}$ is an eigenvalue, then $e^{-i\theta}$ is another one. Exploiting this symmetry, Killip and Nenciu found another matrix representation for this ensemble

$$J=\begin{pmatrix} c_1 & b_1 & & & \\ b_1 & c_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ & & & b_{N-1} & c_N \end{pmatrix}, \quad \begin{cases} b_j = \left((1-a_{2j-2})(1-a_{2j-1}^2)(1+a_{2j})\right)^{1/2} \\ c_j = (1-a_{2j-2})a_{2j-1} - (1+a_{2j-2})a_{2j-3} \end{cases}, \tag{5.3.116}$$

where $a_0=a_{2N}=-1$, and the eigenvalues $\{\lambda_j\}_{j=1}^N$ of J are related to the one of $\mathfrak{E}$ as $\lambda_j=\cos(\theta_j)$.

Also, in this case, it is possible to compute explicitly the joint eigenvalue density for this model as

$$\mathrm{d}\mathbb{P}_J = \frac{1}{\mathfrak{Z}_N^J(\beta,P)}\prod_{j<\ell}|\cos(\theta_j)-\cos(\theta_\ell)|^\beta\,\mathbb{1}_{\theta_j\in\mathbb{T}}e^{-2\sum_{j=1}^N P(\cos(\theta_j))}\mathrm{d}\boldsymbol{\theta}\,. \tag{5.3.117}$$

As in the previous cases, we are interested in the high-temperature regime for this ensemble, so we wet $\beta=\frac{2\alpha}{N}$, thus the measure (5.3.115) read

$$\mathrm{d}\mu_J(\mathbf{a}) = \frac{\prod_{j=1}^{2N-1}\left(1-a_j^2\right)^{\alpha\left(1-\frac{j}{2N}\right)}\prod_{j=1}^{2N-1}(1-a_j)^{\mathrm{a}+1-\frac{\alpha}{2N}}(1+(-1)^j a_j)^{\mathrm{b}+1-\frac{\alpha}{2N}}\mathbb{1}_{a_j\in\mathbb{I}}\exp\left(-\mathrm{Tr}\left(P\left(\mathfrak{E}\right)\right)\right)\mathrm{d}\mathbf{a}}{Z_N^J(\beta,P)}\,. \tag{5.3.118}$$

This regime was considered in [TT21] and in the recent paper [NTT23], where the authors established a CLT for polynomial test functions in the absence of external potential ($P=0$ in (5.3.118)) by considering orthogonal polynomials, obtaining an explicit recurrence relation for the limiting variance.

Again, thanks to Lemma 5.3.16, we can apply Theorem 5.1.3 deducing the following

Corollary 5.3.21 (CLT for Jacobi β ensemble in the high-temperature)**.** *Consider the matrix representation* $\mathfrak{E}$ (5.3.102) *of the Jacobi* β *ensemble in the high-temperature regime* (5.3.118) *. Then, defining the Free energy* $\mathcal{F}_J(\alpha,P)$ *as*

$$\mathcal{F}_J(\alpha,P) = -\lim_{N\to\infty}\frac{\ln(Z_N^J(\alpha,P))}{N}\,, \tag{5.3.119}$$

for all $j\in\mathbb{N}$ *fixed, we have the following weak limit*

$$\lim_{N\to\infty}\frac{\mathrm{Tr}\mathfrak{E}^j-\mathbb{E}\left[\mathrm{Tr}\mathfrak{E}^j\right]}{\sqrt{N}}\rightharpoonup\mathcal{N}(0,\sigma^2)\,, \tag{5.3.120}$$

where

$$\mathbb{E}\left[\mathrm{Tr}\mathfrak{E}^j\right] = iN\partial_t\mathcal{F}_J(\alpha,P+itx^j)_{|_{t=0}}\,,\quad \sigma^2=|\partial_t^2\mathcal{F}_J(\alpha,P+itx^j)_{|_{t=0}}|\,. \tag{5.3.121}$$

Remark 5.3.22. *We notice that for N even, for $\mathrm{a}+\mathrm{b}=-1+\frac{\beta}{4}$ we can apply the final part of Theorem 5.1.3, thus we deduce that*

$$\begin{aligned}\partial_\alpha(\alpha\partial_t\mathcal{F}_J(\alpha,P+itx^j)_{|_{t=0}}) &= \partial_t\mathcal{F}_{dS}(\alpha,P+itx^j)_{|_{t=0}}\,,\\ \partial_\alpha(\alpha\partial_t^2\mathcal{F}_J(\alpha,P+itx^j)_{|_{t=0}}) &= \partial_t^2\mathcal{F}_{dS}(\alpha,P+itx^j)_{|_{t=0}}\end{aligned} \tag{5.3.122}$$

5.3.6 The Itoh–Narita–Bogoyavleskii lattices

In this section, we apply our results to two families of integrable lattices with short-range interaction that generalize the Volterra one (5.3.70). These families are described in [Bog91] (see also [Bog88, Ito75, Nar82]).

One is called *additive Itoh–Narita–Bogoyavleskii (INB) r-lattice* and is defined by the following equations

$$\dot{a}_i = a_i\left(\sum_{j=1}^{r} a_{i+j} - \sum_{j=1}^{r} a_{i-j}\right), \quad i=1,\dots,N,\ N\geqslant r\in\mathbb{N}. \tag{5.3.123}$$

The second family is called the *multiplicative Itoh–Narita–Bogoyavleskii (INB) r-lattice* and is defined by the equations

$$\dot{a}_i = a_i\left(\prod_{j=1}^{r} a_{i+j} - \prod_{j=1}^{r} a_{i-j}\right), \quad i=1,\dots,N,\ N\geqslant r\in\mathbb{N}. \tag{5.3.124}$$

In both cases we consider the periodicity condition $a_{j+N}=a_j$. We notice that setting $r=1$, we recover in both cases the Volterra lattice. Moreover, both families admit the KdV equation as continuum limits, see [Bog91].

In both cases the interaction is short-range, but in the additive case (5.3.123) the nonlinearity is quadratic as in the Volterra lattice, instead in the multiplicative one (5.3.124) it is of polynomial order.

As we already mentioned, both families are integrable for all $r\in\mathbb{N}$, indeed both families admits a Lax pair formulation. For the additive INB lattice (5.3.123), it reads

$$L^{(+,r)} = \sum_{i=1}^{N}\left(a_{i+r}E_{i+r,i}+E_{i,i+1}\right) \tag{5.3.125}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & a_{N-r} & 0 & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & a_{N-r+1} & 0 & 0\\ 0 & 0 & 0 & 1 & \cdots & 0 & a_{N-r+2} & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ a_{r+1} & 0 & \cdots & \cdots & 0 & 1 & 0 & 0\\ 0 & a_{r+2} & 0\cdots & & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 1\\ 1 & 0 & \cdots & a_{N-r-1} & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{5.3.126}$$

($\downarrow N-r$ column; $r+1$ row $\longleftarrow$)

$$A^{(+,r)} = \sum_{i=1}^{N}\left(\sum_{j=0}^{r} a_{i+j}\right)E_{i,i} + E_{i,i+r+1}\,, \tag{5.3.127}$$

we recall that we are always considering periodic boundary conditions, so for all $j\in\mathbb{Z}$, $a_{j+N}=a_j$ and $E_{i,j+N}=E_{i+N,j}=E_{i,j}$. In this notation, the equations of motion (5.3.123) are equivalent to

$$\dot{L}^{(+,r)} = \left[L^{(+,r)};A^{(+,r)}\right]. \tag{5.3.128}$$

Analogously, the multiplicative INB r-lattices have a Lax Pair formulation, which reads

$$L^{(\times,r)} = \sum_{i=1}^{N} \left(a_i E_{i,i+1} + E_{i+r,i}\right), \tag{5.3.129}$$

$$= \begin{pmatrix} 0 & a_1 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_3 & \cdots & 0 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 1 & 0 & \cdots & \cdots & 0 & a_r & 0 & 0 \\ 0 & 1 & 0\cdots & & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & a_{N-1} \\ a_N & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix} \tag{5.3.130}$$

with $\downarrow N-r$ column above the matrix and $r+1$ row $\longleftarrow$ next to the last row.

$$A^{(\times,r)} = \sum_{i=1}^{N} \left(\prod_{j=0}^{r} a_{i+j}\right) E_{i,i+r+1}\,. \tag{5.3.131}$$

Following the construction made in [GGGM23], where the authors numerically computed the density of states for these two families of lattices, we introduce the generalized Gibbs ensemble for these models as

$$\mathrm{d}\mu_{+,r} = \frac{\exp(-\mathrm{Tr}P(L^{(+,r)}))\prod_{j=1}^{N} a_j^{\alpha-1}\mathbb{1}_{a_j\geqslant 0}\mathrm{d}\mathbf{a}}{Z_N^{(+,r)}(a,P)}\,, \tag{5.3.132}$$

$$\mathrm{d}\mu_{\times,r} = \frac{\exp(-\mathrm{Tr}P(L^{(\times,r)}))\prod_{j=1}^{N} a_j^{\alpha-1}\mathbb{1}_{a_j\geqslant 0}\mathrm{d}\mathbf{a}}{Z_N^{(\times,r)}(\alpha,P)}\,, \tag{5.3.133}$$

where $P(x)$ is a polynomial. Moreover, enforcing the result of [GGGM23]

Lemma 5.3.23. *Fix $\ell \in \mathbb{N}$. Then for N large enough*

$$\mathrm{Tr}(L^{(+,r)})^{\ell} = \mathrm{Tr}(L^{(\times,r)})^{\ell} = 0\,, \tag{5.3.134}$$

if ℓ is not an integer multiple of $r+1$.

we can consider just the polynomials $P(x)$ such that $P(x) = x^{j(r+1)} + \text{l.o.t.}$ for some $j \in \mathbb{N}$.

Due to the local structure of $L^{(+,r)}, L^{(\times,r)}$, one can deduce the following:

Lemma 5.3.24. *Fix $m \in \mathbb{N}$, and consider the matrices $L^{(+,r)}, L^{(\times,r)}$ (5.3.125)-(5.3.129). Then for N big enough, there exist $k^{(+,r)} = k^{(+,r)}(m), k^{(\times,r)} = k^{(\times,r)}(m) \in \mathbb{N}$, and four polynomial functions $V^{(+,r)} : \mathbb{R}_+^{k^{(+,r)}} \times \mathbb{R}_+^{k^{(+,r)}} \to \mathbb{R}$, $V^{(\times,r)} : \mathbb{R}_+^{k^{(\times,r)}} \times \mathbb{R}_+^{k^{(\times,r)}} \to \mathbb{R}$, $V_1^{(+,r)} : \mathbb{R}_+^{k^{(+,r)}} \times \mathbb{R}_+^{\ell^{(+,r)}} \times \mathbb{R}_+^{k^{(+,r)}} \to \mathbb{R}$, $V_1^{(\times,r)} : \mathbb{R}_+^{k^{(\times,r)}} \times \mathbb{R}_+^{\ell^{(\times,r)}} \times \mathbb{R}_+^{k^{(\times,r)}} \to \mathbb{R}$ such that*

$$\begin{aligned} \mathrm{Tr}(L^{(+,r)})^m = &\sum_{j=1}^{M^{(+,r)}} V^{(+,r)}(\mathbf{x}_j, \mathbf{x}_{j+1}) \\ &+ V_1^{(+,r)}(\mathbf{x}_{M^{(+,r)}}, x_{k^{(+,r)}M^{(+,r)}+1}, \dots, x_{k^{(+,r)}M^{(+,r)}+\ell^{(+,r)}}, \mathbf{x}_1)\,, \end{aligned} \tag{5.3.135}$$

$$\begin{aligned} \mathrm{Tr}(L^{(\times,r)})^m = &\sum_{j=1}^{M^{(\times,r)}} V^{(\times,r)}(\mathbf{x}_j, \mathbf{x}_{j+1}) \\ &+ V_1^{(\times,r)}(\mathbf{x}_{M^{(\times,r)}}, x_{k^{(\times,r)}M^{(\times,r)}+1}, \dots, x_{k^{(\times,r)}M^{(\times,r)}+\ell^{(\times,r)}}, \mathbf{x}_1)\,, \end{aligned} \tag{5.3.136}$$

where $N = k^{(+,r)}M^{(+,r)} + \ell^{(+,r)} = k^{(\times,r)}M^{(\times,r)} + \ell^{(\times,r)}$.

Thus, proceeding as we have done for the others systems previously considered, we obtain the following:

Corollary 5.3.25 (CLT for INB lattices)**.** *Consider the Lax matrices $L^{(+,r)}, L^{(\times,r)}$ (5.3.125)-(5.3.129) of the additive and multiplicative INB lattices respectively distributed according to their Generalized Gibbs Ensemble* (5.3.132)-(5.3.133)*. Then, defining the Free energies $\mathcal{F}_{+,r}(\alpha,P), \mathcal{F}_{\times,r}(\alpha,P)$ as*

$$\mathcal{F}_{+,r}(\alpha,P) = -\lim_{N\to\infty} \frac{1}{N} \ln(Z_N^{(+,r)}(\alpha,P)), \tag{5.3.137}$$

$$\mathcal{F}_{\times,r}(\alpha,P) = -\lim_{N\to\infty} \frac{1}{N} \ln(Z_N^{(\times,r)}(\alpha,P)), \tag{5.3.138}$$

$$\tag{5.3.139}$$

for all $j \in \mathbb{N}$ fixed, we have the following weak limit

$$\lim_{N\to\infty} \frac{\mathrm{Tr}(L^{(+,r)})^{(r+1)j} - \mathbb{E}\left[\mathrm{Tr}(L^{(+,r)})^{(r+1)j}\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0,\sigma_{+,r}^2), \tag{5.3.140}$$

$$\lim_{N\to\infty} \frac{\mathrm{Tr}(L^{(\times,r)})^{(r+1)j} - \mathbb{E}\left[\mathrm{Tr}(L^{(\times,r)})^{(r+1)j}\right]}{\sqrt{N}} \rightharpoonup \mathcal{N}(0,\sigma_{\times,r}^2), \tag{5.3.141}$$

where

$$\mathbb{E}\left[\mathrm{Tr}(L^{(+,r)})^{(r+1)j}\right] = iN\partial_t \mathcal{F}_{+,r}(\alpha, P+itx^{(r+1)j})_{|_{t=0}}, \quad \sigma_{+,r}^2 = |\partial_t^2 \mathcal{F}_{+,r}(\alpha, P+itx^{(r+1)j})_{|_{t=0}}|, \tag{5.3.142}$$

$$\mathbb{E}\left[\mathrm{Tr}(L^{(\times,r)})^{(r+1)j}\right] = iN\partial_t \mathcal{F}_{\times,r}(\alpha, P+itx^{(r+1)j})_{|_{t=0}}, \quad \sigma_{\times,r}^2 = |\partial_t^2 \mathcal{F}_{\times,r}(\alpha, P+itx^{(r+1)j})_{|_{t=0}}|. \tag{5.3.143}$$

Remark 5.3.26. *We recall that in [GGGM23], it was shown that the density of states for this model has support on the complex plane, but despite that all the moments of the Generalized Gibbs ensemble are reals. Furthermore, in this case, we lack a β ensemble to compare with.*

5.4 Technical Results

In this section, we prove the technical results that we used to prove our main Theorems 5.1.3-5.1.5, the proof follows the same line as the proof of [GM23, Proposition 4.2], and we prove Theorem 5.1.7, whose proof uses the same machinery as the latter proofs. In the last part, we prove a Berry-Esseen type bound for the type 1 measure $\mu_N^{(1)}$. We start by proving Theorem 5.2.1 and Theorem 5.2.2.

To prove these results, we follow the same ideas as in [Gou15, Theorem 2.4]. In particular, we enforce the following proposition, which can be easily deduced from [Gou15, Proposition 2.3]:

Proposition 5.4.1. *Let $\lambda(0)$ be an isolated eigenvalue of the operator $\mathcal{L}_0$ with multiplicity one, and assume that the family of operators $t \to \mathcal{L}_t$ depends on t in a C^d way, with $d \geqslant 3$. Then, $\lambda(t)$, the corresponding eigenprojection π_t and its eigenfunction φ_t are C^d with respect to t.*

Moreover, assume that the rest of the spectrum of $\mathcal{L}_0$ it is contained in a disk of radius $|\lambda(0)| - \delta$. Writing $\mathcal{Q}_t = (I - \pi_t)\mathcal{L}_t$, so that $\mathcal{L}_t = \lambda(t)\pi_t + \mathcal{Q}_t$. For any $r > |\lambda(0)| - \delta$, there exists a constant $C > 0$ independent of t, n such that $|\mathcal{Q}_t^n| \leqslant Cr^n$ for all $n \in \mathbb{N}$.

Applying the previous proposition, we can prove both Theorem 5.2.1 and Theorem 5.2.2. For the reader's convenience, we report the two statements here.

Theorem 5.4.2. *Under Assumptions 5.1.1. Consider a real function $H : \mathbb{C} \to \mathbb{R}$ such that $\mathrm{Tr}H(L)$ is circular (HP 5.), and let W be the seed of $\mathrm{Tr}G(L) + itH(L)$, thus $W(\mathbf{x},\mathbf{y}) = V(\mathbf{x},\mathbf{y}) + itU(\mathbf{x},\mathbf{y})$ for $V, U : X^k \times X^k \to \mathbb{R}$ the seeds of $\mathrm{Tr}G(L), \mathrm{Tr}H(L)$. Furthermore, assume that $U \in L^{2d}(X^{2k}, \exp(-2V))$, with $\mathbb{N} \ni d \geqslant 3$. Then, there exists an $\varepsilon > 0$, and two complex valued functions $\lambda(y,t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon,\varepsilon])$, and $c_{k,\ell}(y,t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon,\varepsilon])$ such that for all $q \in \mathbb{N}$:*

$$\mathbb{E}_1\left[e^{-it\mathrm{Tr}H(L)}\right] = \frac{Z^{(1)}_{kM+\ell}(\alpha,t)}{Z^{(1)}_{kM+\ell}(\alpha,0)} = c_{k,\ell}(\alpha,t)\lambda(\alpha,t)^{M-2}\left(1+o(M^{-q})\right), \quad \text{as } M \to \infty, \tag{5.4.1}$$

for $|t| < \varepsilon$, here $Z^{(1)}_{kM+\ell}(\alpha,t) \equiv Z^{(1)}_{kM+\ell}(\alpha, V+itU)$. Furthermore,

$$\lambda(x,0) = 1 \tag{5.4.2}$$
$$c_{k,\ell}(x,0) = 1\,. \tag{5.4.3}$$

Moreover, there exist two functions $\widetilde{c}_{k,\ell}(\alpha,t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon,\varepsilon])$ and $\widetilde{\lambda}(\alpha,t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon,\varepsilon])$ such that there exist two constants $C_1, C_2 > 0$ such that for all $q \in \mathbb{N}$:

$$\begin{aligned} &C_1 < \widetilde{c}_{k,\ell}(\alpha,t) < C_2\,,\\ &\lambda(\alpha,t) = \frac{\widetilde{\lambda}(\alpha,t)}{\widetilde{\lambda}(\alpha,0)}\,,\\ &Z^{(1)}_{kM+\ell}(\alpha,t) = \widetilde{c}_{k,\ell}(\alpha,t)\widetilde{\lambda}(\alpha,t)^{M-2}\left(1+o(M^{-q})\right). \end{aligned} \tag{5.4.4}$$

Proof. Define the kernel operator (depending on $k \in \mathbb{N}$, $\alpha > 0$ and $t \in \mathbb{R}$) $\mathcal{L}_{t,\alpha} : L^2(X^k) \to L^2(X^k)$ as

$$\mathcal{L}_{t,\alpha} f(\mathbf{y}) = \int_{X^k} f(\mathbf{x}) \prod_{q=1}^{k} F(x_q,\alpha) e^{-W(\mathbf{y},\mathbf{x})} \mathrm{d}\mathbf{x}\,. \tag{5.4.5}$$

Then, for all $k \in \mathbb{N}$, $\alpha > 0$ and $t \in \mathbb{R}$, $\mathcal{L}_{t,\alpha}$ is a Hilbert-Schmidt operator [Kat95], meaning that the function $(\mathbf{x},\mathbf{y}) \mapsto \prod_{q=1}^{k} F(x_q,\alpha)e^{-V(\mathbf{y},\mathbf{x})}$ is $L^2(X^k \times X^k)$, and so it is compact. Moreover, since the kernel is positive, we can apply a generalization of Jentzsch's theorem [Zaa83, Theorem 137.4] in combinations with Proposition 5.4.1 deducing that there exist two functions $(y,t) \mapsto \widetilde{\lambda}(y,t) \in C^{1,d}(\mathbb{R}^+ \times [-\varepsilon,\varepsilon])$, $(y,t) \mapsto \varphi(\cdot,y,t) \in C^{1,d}(\mathbb{R}^+ \times [-\varepsilon,\varepsilon], L^2(X^k))$, and an operator $\mathcal{Q}_t : L^2(X^k) \to L^2(X^k)$ such that

$$\mathcal{L}_{t,\alpha}\phi(\mathbf{y}) = \widetilde{\lambda}(\alpha,t)\langle\phi, \varphi(\cdot,\alpha,t)\rangle\varphi(\mathbf{y},\alpha,t) + \mathcal{Q}_t\phi(\mathbf{y}), \quad \forall\,\phi \in L^2(X^k) \tag{5.4.6}$$

where $\widetilde{\lambda}(y,0) > 0$, $\varphi(\mathbf{x},y,0) > 0$ is the associated eigenfunction of $\mathcal{L}_{t,\alpha}$ with $|\varphi|_2 = 1$ and there exists a $\delta > 0$ such that $||\mathcal{Q}_t|| \leqslant |\widetilde{\lambda}(\alpha,t)| - \delta$, denoting by $\langle\cdot,\cdot\rangle$ the standard scalar product in $L^2(X^k)$.

For $\mathbf{x} \in X^k$ define $G_{\mathbf{x}}(\mathbf{y})$ as

$$G_{\mathbf{x}}(\mathbf{y}) = \begin{cases} \int_{X^\ell} \prod_{j=kM+1}^{kM+\ell} F(x_j,\alpha) \exp\left(-W_1(\mathbf{y}, x_{kM+1}, \ldots, x_{kM+\ell}, \mathbf{x})\right) \prod_{j=kM+1}^{kM+\ell} \mathrm{d}x_j\,, & \ell > 0\,,\\ \exp\left(-W(\mathbf{y},\mathbf{x})\right), & \ell = 0\,, \end{cases} \tag{5.4.7}$$

and the linear operator $\mathcal{S}_t : L^2(X^k \times X^k) \to \mathbb{C}$ as

$$\mathcal{S}_t\varphi = \int_{X^k \times X^k} \prod_{j=1}^{2k} F(x_j,\alpha) \exp\left(-W(\mathbf{x}_1,\mathbf{x}_2)\right) \varphi(\mathbf{x}_1,\mathbf{x}_2) \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2\,, \tag{5.4.8}$$

we notice that $|\mathcal{S}_t| \leqslant c\,|F|_2^{2k}$, and so it is continuous.

In this notation, we can recast (5.2.6), applying $\mathcal{S}_t$ to $(\mathbf{x},\mathbf{y}) \mapsto \left(\mathcal{L}_{t,\alpha}^{M-2} G_{\mathbf{x}}\right)(\mathbf{y})$, as

$$Z^{(1)}_{kM+\ell}(\alpha,t) = \mathcal{S}_t\left(\left(\mathcal{L}_{t,\alpha}^{M-2} G_{\mathbf{x}_1}\right)(\mathbf{x}_2)\right) = \widetilde{\lambda}^{M-2}(\alpha,t)\mathcal{S}_t\big(\langle \varphi(\cdot,\alpha,t)\,; G_{\mathbf{x}_1}\rangle \varphi(\mathbf{x}_2,\alpha,t)\big) + \mathcal{S}_t\big(\mathcal{Q}_t^{M-2} G_{\mathbf{x}_1}(\mathbf{x}_2)\big)\,, \tag{5.4.9}$$

where here and in the sequel, if $h \in L^2(X^k \times X^k)$, we write abusively $\mathcal{S}_t(h(\mathbf{x},\mathbf{y}))$ for $\mathcal{S}_t(h)$. Defining

$$\begin{aligned} c_{k,\ell}(\alpha,t) &= \frac{\mathcal{S}_t\big(\langle \varphi(\cdot,\alpha,t)\,; G_{\mathbf{x}_1}\rangle \varphi(\mathbf{x}_M,\alpha,t)\big)}{\mathcal{S}_0\big(\langle \varphi(\cdot,\alpha,0)\,; G_{\mathbf{x}_1}\rangle \varphi(\mathbf{x}_M,\alpha,0)\big)}\,, \\ \lambda(\alpha,t) &= \frac{\widetilde{\lambda}^{M-2}(\alpha,t)}{\widetilde{\lambda}^{M-2}(\alpha,0)}\,, \end{aligned} \tag{5.4.10}$$

and since in view of Proposition 5.4.1 $|\mathcal{Q}_t^n| \leqslant (|\widetilde{\lambda}(t)| - \delta)^n$ we conclude. □

Theorem 5.4.3. *Under Assumptions 5.1.1. Consider a real function $H : \mathbb{C} \to \mathbb{R}$ such that $\mathrm{Tr}H(L)$ is circular, and let W be the seed of $\mathrm{Tr}G(L) + itH(L)$, thus $W(\mathbf{x},\mathbf{y}) = V(\mathbf{x},\mathbf{y}) + itU(\mathbf{x},\mathbf{y})$ for $V, U : X^k \times X^k \to \mathbb{R}$ seeds of $\mathrm{Tr}G(L), \mathrm{Tr}H(L)$. Furthermore, assume that $U \in L^{2d}(X^{2k}, \exp(-2V))$, with $\mathbb{N} \ni d \geqslant 3$. Then there exists an $\varepsilon > 0$ and two scalar functions $\lambda(y,t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon,\varepsilon], \mathbb{C})$, $c_{k,\ell,M}(y,t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon,\varepsilon])$, such that*

$$\mathbb{E}_2\left[e^{-it\mathrm{Tr}H(L)}\right] = \frac{Z^{(2)}_{kM+\ell}(\alpha,t)}{Z^{(2)}_{kM+\ell}(\alpha,0)} = c_{k,\ell,M}(\alpha,t) \prod_{j=1}^{M-2} \lambda\left(\alpha\frac{j}{M},t\right)(1+o_M(1)) \tag{5.4.11}$$

for $|t| < \varepsilon$. Furthermore,

$$\lambda(x,0) = 1 \tag{5.4.12}$$

$$\lim_{t\to 0} c_{k,\ell,M}(\alpha,t) = 1 \quad \textit{uniformly in } M \tag{5.4.13}$$

$$\textit{the remainder } o_M(1) \textit{ is independent of } t \in [-\varepsilon,\varepsilon]. \tag{5.4.14}$$

Moreover, there exist two functions $\widetilde{c}_{k,\ell,M}(\alpha,t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon,\varepsilon])$, $\widetilde{\lambda}(\alpha,t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon,\varepsilon])$, and three constants $C_1, C_2 > 0$ and $p \in \mathbb{N}$ such that

$$\begin{aligned} &C_1 N^p < \widetilde{c}_{k,\ell,M}(\alpha,t) < C_2 N^p\,, \\ &\lambda(\alpha,t) = \frac{\widetilde{\lambda}(\alpha,t)}{\widetilde{\lambda}(\alpha,0)}\,, \\ &Z^{(2)}_{kM+\ell}(\alpha,t) = \widetilde{c}_{k,\ell,M}(\alpha,t) \prod_{j=1}^{M-2} \widetilde{\lambda}\left(\alpha\frac{j}{M},t\right)(1+o_M(1))\,. \end{aligned} \tag{5.4.15}$$

Proof. Define the family of kernel operators (depending on $k \in \mathbb{N}$, $\alpha > 0$ and $t \in \mathbb{R}$) $\mathcal{L}^{(j)}_{t,\alpha} : L^2(X^k) \to L^2(X^k)$ as

$$\mathcal{L}^{(j)}_{t,\alpha} f(\mathbf{y}) = \int_{X^k} f(\mathbf{x}) \prod_{q=1}^{k} F\left(x_q, \alpha\left(1 - \frac{(j-1)k+q}{N}\right)\right) e^{-W(\mathbf{y},\mathbf{x})} \mathrm{d}\mathbf{x}\,. \tag{5.4.16}$$

Then, for all $k \in \mathbb{N}$, $\alpha > 0$ and $t \in \mathbb{R}$, the kernel of $\mathcal{L}^{(j)}_{t,\alpha}$ is in $L^2(X^k \times X^k)$, thus it is a Hilbert-Schmidt operator, and so it is compact. Moreover, since for $t = 0$ the kernel is positive, we can apply a generalization of Jentzsch's theorem [Zaa83, Theorem 137.4] in combinations with Proposition 5.4.1

deducing that there exist two functions $\widehat{\lambda} = \widehat{\lambda}(y,t) \in C^{1,d}(\mathbb{R}^+ \times [-\varepsilon,\varepsilon])$, $(y,t) \mapsto \varphi(\cdot,y,t) \in C^{1,d}(\mathbb{R}^+ \times [-\varepsilon,\varepsilon], L^2(X^k))$ and an operator $\mathcal{Q}_t^{(j)} : L^2(X^k) \to L^2(X^k)$ such that $\forall\, \phi \in L^2(X^k)$, $\forall |t| < \varepsilon$,

$$\mathcal{L}_{t,\alpha}^{(j)}\phi(\mathbf{y}) = \widehat{\lambda}\left(\alpha\left(1-\frac{j}{M}\right),t\right)\pi_t^{(j)}\phi(\mathbf{y}) + \mathcal{Q}_t^{(j)}\phi(\mathbf{y}) \tag{5.4.17}$$

with

$$\pi_t^{(j)}\phi(\mathbf{y}) = \left\langle \phi; \varphi\left(\cdot,\alpha\left(1-\frac{j}{M}\right),t\right)\right\rangle \varphi\left(\mathbf{y},\alpha\left(1-\frac{j}{M}\right),t\right),$$

where $\widehat{\lambda}\left(\alpha\left(1-\frac{j}{M}\right),t\right)$ is the biggest eigenvalue (in modulus) of $\mathcal{L}_{t,\alpha}^{(j)}$, $\widehat{\lambda}(y,0) > 0$, $\varphi(\cdot,y,0) > 0$, $|\varphi(\cdot,y,t)|_2 = 1$ and there exists a $\delta_j > 0$ such that $\left|\mathcal{Q}_t^{(j)}\right| \leqslant |\widehat{\lambda}\left(\alpha\left(1-\frac{j}{M}\right),t\right)| - \delta_j$, and we recall that we denote by $\langle\cdot,\cdot\rangle$ the standard scalar product in $L^2(X^k)$. Furthermore, with $\widetilde{\lambda}$ the function of Theorem 5.4.2, we have

$$\widehat{\lambda}\left(\alpha\left(1-\frac{j}{M}\right),t\right) = \widetilde{\lambda}\left(\alpha\left(1-\frac{j}{M}\right),t\right) + O\left(\frac{1}{M}\right), \tag{5.4.18}$$

Where the $O\left(\frac{1}{M}\right)$ term is uniform in $t \in (-\varepsilon,\varepsilon)$. Indeed, recalling that $\mathcal{L}_{t,\alpha}$ is defined in (5.4.5), by the integrability assumptions on U and on $\partial_\alpha F$ (HP 7. of Assumptions 5.1.1), we have

$$\left|\mathcal{L}_{t,\alpha}^{(j)} - \mathcal{L}_{t,\alpha(1-j/M)}\right| \leqslant C_t\frac{\alpha}{M} \tag{5.4.19}$$

where $C_t \geqslant 0$ is bounded on $[-\varepsilon,\varepsilon]$. We then deduce (5.4.18) by applying Proposition 5.4.1.

Define the function h_t on X^k by

$$h_t(\mathbf{x}) = \int_{X^{k+\ell}} \prod_{j=1}^{k+\ell-1} F\left(x_{k(M-1)+j}, \alpha\left(1-\frac{j+k(M-1)}{kM+\ell}\right)\right) \exp\big(-W(\mathbf{x},\mathbf{x}_M)\big)\times$$

$$\times \exp\big(-W(\mathbf{x}_M, x_{kM+1},\dots,x_{kM+\ell},0,\dots,0) - W(x_{kM+1},\dots,x_{kM+\ell},0,\dots,0)\big) R(x_N) \prod_{j=1}^{k+\ell} \mathrm{d}x_{k(M-1)+j}\,. \tag{5.4.20}$$

Note that in view of assumptions 5.1.1, $\|h_t\|_\infty = O(N^{\mathbf{c}(\ell+k)})$.

We recall that $W = V + itU$ so h_t depends on t , and the linear operator $\mathcal{S}_t : L^2(X^k) \to \mathbb{C}$ as

$$\mathcal{S}_t(\psi) = \int_{X^k} \prod_{j=1}^{k} F\left(x_j, \alpha\left(1-\frac{j}{kM+\ell}\right)\right) e^{-W(0,\dots,0,\mathbf{x})}\psi(\mathbf{x}) \prod_{j=1}^{k} \mathrm{d}x_j\,. \tag{5.4.21}$$

We notice that, again in view of the assumptions 5.1.1 the operator $\mathcal{S}_t$ is uniformly bounded in k,ℓ for all $t \in \mathbb{R}$, and it is continuous in t in the operator norm sense.

In this notation, we can rewrite $Z_{kM+\ell}^{(2)}(\alpha,t)$ as

$$Z_{kM+\ell}^{(2)}(\alpha,t) = \mathcal{S}_t\left(\mathcal{L}_{t,\alpha}^{(2)}\dots\mathcal{L}_{t,\alpha}^{(M-1)}h_t\right) = \mathcal{S}_t\left(\prod_{j=2}^{M-1}\mathcal{L}_{t,\alpha}^{(j)}h_t\right) \tag{5.4.22}$$

Applying the decomposition (5.4.17), it follows that we can decompose the previous expression as

$$
\begin{aligned}
Z^{(2)}_{kM+\ell}(\alpha,t) = &\prod_{j=2}^{M-1} \widehat{\lambda}_t\left(\alpha\left(1-\frac{j}{M}\right)\right) \mathcal{S}_t\left(\pi_t^{(2)}\dots\pi_t^{(M-1)}h_t\right) \\
&+ \mathcal{S}_t\left(\mathcal{L}^{(2)}_{t,\alpha}\dots\mathcal{L}^{(M-2)}_{t,\alpha}\mathcal{Q}_t^{(M-1)}h_t\right) \\
&+ \sum_{n=2}^{M-2}\prod_{j=n+1}^{M-1} \widehat{\lambda}_t\left(\alpha\left(1-\frac{j}{M}\right)\right) \mathcal{S}_t\left(\mathcal{L}^{(2)}_{t,\alpha}\dots\mathcal{L}^{(n-1)}_{t,\alpha}\mathcal{Q}_t^{(n)}\pi_t^{(n+1)}\dots\pi_t^{(M-1)}h_t\right),
\end{aligned}
\tag{5.4.23}
$$

where we arranged the terms of the product of the $\mathcal{L}^{(j)}_{t,\alpha}$'s by order of the first appearance from the right of a factor $\mathcal{Q}_t^{(j)}$ (the first term being the product where no $\mathcal{Q}_t^{(j)}$ appears). We notice that

$$
\mathcal{S}_t\left(\pi_t^{(2)}\dots\pi_t^{(M-1)}h_t\right) = \langle\varphi_t^{1/M};h_t\rangle\prod_{i=2}^{M-2}\langle\varphi_t^{\alpha(1-i/M)};\varphi_t^{\alpha(1-(i+1)/M)}\rangle\mathcal{S}_t\left(\varphi_t^{\alpha\left(1-\frac{2}{M}\right)}\right),
$$

where we set $\varphi_t^{\alpha(1-i/M)} \equiv \varphi\left(\cdot,\alpha(1-i/M),t\right)$ to shorten the notation. Furthermore, the ratio

$$
\prod_{i=2}^{M-1}\frac{\langle\varphi_t^{\alpha\left(1-\frac{i}{M}\right)};\varphi_t^{\alpha\left(1-\frac{i+1}{M}\right)}\rangle}{\langle\varphi_0^{\alpha\left(1-\frac{i}{M}\right)};\varphi_0^{\alpha\left(1-\frac{i+1}{M}\right)}\rangle}
$$

converges uniformly to 1 in $M \geqslant 1$, $t \in (-\varepsilon,\varepsilon)$. This is due to the fact that

$$
\langle\varphi_t^{\alpha\left(1-\frac{i}{M}\right)};\varphi_t^{\alpha\left(1-\frac{i+1}{M}\right)}\rangle = 1 + O\left(\frac{\alpha}{M}\right),
$$

because of (5.4.19) and Proposition 5.4.1, thus the product

$$
\prod_{i=2}^{M-1}\langle\varphi_t^{\alpha\left(1-\frac{i}{M}\right)};\varphi_t^{\alpha\left(1-\frac{i+1}{M}\right)}\rangle
$$

stays bounded below and above uniformly on $M \geqslant 1$, $t \in (-\varepsilon,\varepsilon)$.

Denoting the first term of (5.4.23) by $f(\alpha,t)$, and the second and third terms by $g_1(\alpha,t)$ and $g_2(\alpha,t)$, we can rewrite (5.4.11) as

$$
\frac{Z^{(2)}_{kM+\ell}(\alpha,t)}{Z^{(2)}_{kM+\ell}(\alpha,0)} = \frac{f(\alpha,t)}{f(\alpha,0)}\left(\frac{1+\frac{g_1(\alpha,t)}{f(\alpha,t)}+\frac{g_2(\alpha,t)}{f(\alpha,t)}}{1+\frac{g_1(\alpha,0)}{f(\alpha,0)}+\frac{g_2(\alpha,0)}{f(\alpha,0)}}\right).
\tag{5.4.24}
$$

Thus, to prove our result we need to show that there exist 3 constants c_1, c_2, c_3 independent of M such that for all $t \in (-\varepsilon,\varepsilon)$,

$$
\left|\frac{g_1(\alpha,t)}{f(\alpha,t)}\right| \leqslant c_1\,, \tag{5.4.25}
$$

$$
\left|\frac{g_2(\alpha,t)}{f(\alpha,t)}\right| \leqslant c_2\,, \tag{5.4.26}
$$

$$
\left|\frac{\left\langle\varphi_t^{(1/M)};h_t\right\rangle\mathcal{S}_t\varphi_t^{\alpha\left(1-\frac{2}{M}\right)}}{\left\langle\varphi_0^{(1/M)};h_0\right\rangle\mathcal{S}_0\varphi_0^{\alpha\left(1-\frac{2}{M}\right)}}\right| \leqslant c_3\,. \tag{5.4.27}
$$

If we are able to show this, then defining

$$c_{k,\ell,M}(\alpha,t) = \frac{\left\langle \varphi_t^{(1/M)}; h_t \right\rangle \mathcal{S}_t \varphi_t^{\alpha\left(1-\frac{2}{M}\right)}}{\left\langle \varphi_0^{(1/M)}; h_0 \right\rangle \mathcal{S}_0 \varphi_0^{\alpha\left(1-\frac{2}{M}\right)}} \prod_{i=2}^{M-1} \frac{\langle \varphi_t^{\alpha\left(1-\frac{i}{M}\right)}; \varphi_t^{\alpha\left(1-\frac{i+1}{M}\right)} \rangle}{\langle \varphi_0^{\alpha\left(1-\frac{i}{M}\right)}; \varphi_0^{\alpha\left(1-\frac{i+1}{M}\right)} \rangle}, \qquad (5.4.28)$$
$$\lambda(y,t) = \frac{\widetilde{\lambda}(y,t)}{\widetilde{\lambda}(y,0)},$$

we obtain (5.4.11) with the wanted properties. Notice that in the definition of λ we took $\dfrac{\widetilde{\lambda}(y,t)}{\widetilde{\lambda}(y,0)}$ instead of $\dfrac{\widehat{\lambda}(y,t)}{\widehat{\lambda}(y,0)}$. This is indeed possible because of equation (5.4.18).

First, we focus on (5.4.26). The term $g_2(\alpha,t)$ is given by

$$\langle \varphi^{(1/M)}; h_t \rangle \sum_{n=2}^{M-2} \prod_{j=n+1}^{M-1} \widehat{\lambda}\left(\alpha\left(1-\frac{j}{M}\right), t\right) \prod_{i=1}^{M-n-2} \langle \varphi_t^{\frac{\alpha i}{M}}; \varphi_t^{\frac{\alpha(i+1)}{M}} \rangle \mathcal{S}_t \left(\mathcal{L}_{t,\alpha}^{(2)} \dots \mathcal{L}_{t,\alpha}^{(n-1)} \mathcal{Q}_t^{(n)} \varphi_t^{\alpha\left(1-\frac{n+1}{M}\right)} \right).$$

Because $\varphi(\mathbf{x},y,t)$ is regular with respect to y, we deduce that there exists a function $(y,t) \mapsto \psi(\cdot,y,t) \in C^\infty(\mathbb{R}^+ \times [-\varepsilon,\varepsilon], L^2(X^k))$ with $\left|\psi_t^{\alpha(1-n/M)}\right|_2$ uniformly bounded in n, M and t such that

$$\mathcal{Q}_t^n \left(\varphi_t^{\alpha\left(1-\frac{n+1}{M}\right)}\right) = Q_t^{(n)} \left(\varphi_t^{\alpha\left(1-\frac{n}{M}\right)}\right) + \frac{1}{M} \mathcal{Q}_t^{(n)} \psi_t^{\alpha(1-n/M)} = \frac{1}{M} \mathcal{Q}_t^{(n)} \psi_t^{\alpha(1-n/M)}. \qquad (5.4.29)$$

given this equality, it is trivial to prove (5.4.26), recalling that for any t, j, $\dfrac{1}{\widehat{\lambda}_t^{(j)}} \mathcal{L}_{t,\alpha}^{(j)}$ has operator norm smaller than one.

For (5.4.25), it suffices to show that there exists a constant c_2 independent of M such that

$$\left\| \frac{\mathcal{Q}_t^{(M-1)} h_t}{\left\langle \varphi_t^{(1/M)}, h_t \right\rangle} \right\| \leqslant c_2. \qquad (5.4.30)$$

From the assumptions, (5.4.17) and the definition of h_t (5.4.20), we deduce that there exists a constant d_1 such that

$$\left\| \mathcal{Q}_j^{(M-1)} h_t \right\| \leqslant d_1 \left(\lambda\left(\frac{\alpha}{M}\right) - \delta_1\right) M^{\mathrm{c}(k+\ell)}, \qquad (5.4.31)$$

on the other hand, in view of the previous proof and the assumptions, we conclude that, for t small enough, there exists a constant d_2 such that

$$\left| \left\langle \varphi_t^{(1/M)}, h_t \right\rangle \right| \geqslant d_2 M^{\mathrm{c}(k+\ell)}. \qquad (5.4.32)$$

Indeed, for $t=0$, $\left\langle \varphi_0^{(1/M)}, h_0 \right\rangle$ is given by

$$\int_{X^{2k+\ell}} \varphi_0^{(1/M)}(\mathbf{x}) \prod_{j=1}^{k+\ell-1} F\left(x_{k(M-1)+j}, \alpha\left(1-\frac{k(M-1)+j}{kM+\ell}\right)\right) r(\mathbf{x},\mathbf{x}_M,\widetilde{\mathbf{x}}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}_M\mathrm{d}\widetilde{\mathbf{x}},$$

where we denoted $\widetilde{\mathbf{x}} = (x_{kM+1}, \dots, x_{kM+\ell})$ and

$$r(\mathbf{x},\mathbf{x}_M,\widetilde{\mathbf{x}}) = e^{-W(\mathbf{x},\mathbf{x}_M) - W(\mathbf{x}_M, x_{kM+1},\dots,x_{kM+\ell},0,\dots,0) - W(x_{kM+1},\dots,x_{kM+\ell},0,\dots,0)}.$$

By Assumptions 5.1.1 HP.4, and positivity of $\varphi_0^{(1/M)}$,

$$\left\langle \varphi_0^{(1/M)}, h_0 \right\rangle \geqslant (d(N/\alpha)^{\mathrm{c}})^{k+\ell-1} \int_{\mathcal{O}} \varphi_0^{(1/M)}(\mathbf{x})\mathrm{d}\mathbf{x} \inf_{\mathcal{O}^{2k+\ell}} r \geqslant c_{k,\ell} M^{k+\ell} \int_{\mathcal{O}} \varphi^{(1/M)}(\mathbf{x})\mathrm{d}\mathbf{x}.$$

By continuity of $\eta \mapsto \varphi_0^{\eta}$, this last integral converges to $\int_{\mathcal{O}} \varphi_0^{(1/M)}(\mathbf{x})\mathrm{d}\mathbf{x} > 0$, thus we conclude for the case $t = 0$. Finally, we conclude on (5.4.32) for t small enough by continuity.

Combining the two previous estimates, and setting $p = \mathrm{c}(k + \ell)$ we deduce (5.4.30), which leads to (5.4.26). The proof of (5.4.27) is analogous, thus we conclude. □

We now turn on the proof of Theorem 5.1.7, which we rewrite here for convenience.

Theorem 5.4.4 (Decay of correlations)**.** *Let W be the seed of* $\mathrm{Tr}G(L)$ *and $I, J : X^k \to \mathbb{R}$ two local functions such that $\int_{X^k \times X^k} \left| I(\mathbf{x}) \prod_{i=1}^k F(x_i, \alpha) e^{-W(\mathbf{x},\mathbf{y})} \right|^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} < \infty$, and analogously for $J(\mathbf{x})$. Write $N = kM + \ell$, and let $j \in \{1, \dots, M\}$. Then there exists some $0 < \mu < 1$ such that*

$$\mathbb{E}_1\left[I(\mathbf{x}_1)J(\mathbf{x}_j)\right] - \mathbb{E}_1\left[I(\mathbf{x}_1)\right]\mathbb{E}_1\left[J(\mathbf{x}_j)\right] = O(\mu^{M-j} + \mu^j)\,.$$

Proof. Let $\mathcal{L} = \mathcal{L}_{0,\alpha}$ with $\mathcal{L}_{t,\alpha}$ given by (5.4.5). Furthermore, define $\mathcal{L}^{(J)}$

$$\mathcal{L}^{(J)}\phi(\mathbf{y}) = \int_{X^k} \phi(\mathbf{x}) \prod_{i=1}^k F(x_i, \alpha) J(\mathbf{x}) e^{-V(\mathbf{y},\mathbf{x})} = \mathcal{L}(J\phi)(\mathbf{y})\,,$$

and $\mathcal{L}^{(I)}$ analogously. With $G_{\mathbf{x}}^{(I)}(\mathbf{y}) = I(\mathbf{x})G_{\mathbf{x}}(\mathbf{y})$, G given in (5.4.7), we have for $j \geqslant 3$

$$\begin{aligned}\mathbb{E}_1\left[I(\mathbf{x}_1)J(\mathbf{x}_j)\right] &= \frac{\mathcal{S}_0\left(\left(\mathcal{L}^{M-j}\mathcal{L}^{(J)}\mathcal{L}^{j-3}G_{\mathbf{x}_1}^{(I)}\right)(\mathbf{x}_2)\right)}{\mathcal{S}_0(\mathcal{L}^{M-2}G_{\mathbf{x}_1}(\mathbf{x}_2))} \\ &= \frac{\widetilde{\lambda}^{M-j}(\alpha,0)\mathcal{S}_0(\pi_0\mathcal{L}^{(J)}\mathcal{L}^{j-3}G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2)) + O(\widetilde{\lambda}^{j-3}r^{M-j}))}{\widetilde{\lambda}^{M-2}(\alpha,0)\mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)) + O(r^{M-2})}\,,\end{aligned}$$

where $\mathcal{S}_t$ is defined in (5.4.8), and we used the decomposition

$$\mathcal{L}_0^k = \widetilde{\lambda}^k(\alpha,0)\pi_0 + \mathcal{Q}_0^k\,,$$

where π_0 is the orthogonal projection on the (one dimensional) eigenspace associated with $\widetilde{\lambda}(\alpha,0)$, and $\mathcal{Q}_0$ is an operator such that $\|\mathcal{Q}_0^k\| \leqslant Cr^k$ for some $0 < r < \widetilde{\lambda}$. Similarly,

$$\mathcal{S}_0(\pi_0\mathcal{L}^{(J)}\mathcal{L}^{j-3}G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2)) = \widetilde{\lambda}^{j-3}(\alpha,0)\mathcal{S}_0(\pi_0\mathcal{L}^{(J)}\pi_0 G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2)) + O(r^{j-3}).$$

We deduce

$$\mathbb{E}_1\left[I(\mathbf{x}_1)J(\mathbf{x}_j)\right] = \frac{\mathcal{S}_0((\pi_0\mathcal{L}^{(J)}\pi_0 G_{\mathbf{x}_1}^{(I)})(\mathbf{x}_2)) + O((r/\widetilde{\lambda})^{M-j} + (r/\widetilde{\lambda})^{j-3})}{\widetilde{\lambda}\left(\mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)) + O((r/\widetilde{\lambda})^{M-2})\right)}\,.$$

Similarly, we deduce

$$\mathbb{E}_1[I(\mathbf{x}_1)]\mathbb{E}_1[J(\mathbf{x}_j)] = \frac{\mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2))\mathcal{S}_0\left(\pi_0\mathcal{L}^{(J)}\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)\right) + O((r/\widetilde{\lambda})^{M-j} + (r/\widetilde{\lambda})^{j-3})}{\widetilde{\lambda}\left(\mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2))^2 + O((r/\widetilde{\lambda})^{M-2})\right)}\,. \tag{5.4.33}$$

By a direct computation, recalling that $\pi_0\phi = \langle\varphi_1, \phi\rangle\varphi_1$ where φ_1 is the eigenfunction associated with $\widetilde{\lambda}$, we deduce the following

$$\mathcal{S}_0\left(\pi_0\mathcal{L}^{(J)}\pi_0 G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2)\right) = \langle\mathcal{L}^{(J)}\varphi_1, \varphi_1\rangle \int \langle G_{\mathbf{x}}, \varphi_1\rangle I(\mathbf{x})F(\mathbf{x})\varphi_1(\mathbf{x})\mathrm{d}\mathbf{x}\,,$$

$$\mathcal{S}\left(\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)\right) = \int \langle G_{\mathbf{x}}, \varphi_1\rangle \varphi_1(\mathbf{x}) F(\mathbf{x}) \mathrm{d}\mathbf{x}\,, \quad \mathcal{S}_0\left(\pi_0 G^{(I)}_{\mathbf{x}_1}(\mathbf{x}_2)\right) = \int \langle G_{\mathbf{x}}, \varphi_1\rangle I(\mathbf{x}) F(\mathbf{x}) \varphi_1(\mathbf{x}) \mathrm{d}\mathbf{x}\,,$$

and

$$\mathcal{S}_0\left(\pi_0 \mathcal{L}^{(J)} \pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)\right) = \langle \mathcal{L}^{(J)}\varphi_1, \varphi_1\rangle \int \langle G_{\mathbf{x}}, \varphi_1\rangle F(\mathbf{x}) \varphi_1(\mathbf{x}) \mathrm{d}\mathbf{x}\,.$$

These formulas imply that

$$\mathcal{S}_0((\pi_0 \mathcal{L}^{(J)} \pi_0 G^{(I)}_{\mathbf{x}_1})(\mathbf{x}_2)) = \frac{\mathcal{S}_0(\pi_0 G^{(I)}_{\mathbf{x}_1}(\mathbf{x}_2)) \mathcal{S}_0\left(\pi_0 \mathcal{L}^{(J)} \pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)\right)}{\mathcal{S}_0\left(\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)\right)}\,,$$

and so

$$\mathbb{E}_1\left[I(\mathbf{x}_1) J(\mathbf{x}_j)\right] - \mathbb{E}_1\left[I(\mathbf{x}_1)\right] \mathbb{E}_1\left[J(\mathbf{x}_j)\right] = O((r/\widetilde{\lambda})^{M-j} + (r/\widetilde{\lambda})^{j-3})\,.$$

□

Finally, we prove a Berry-Esseen bound type theorem for the measure $\mu^{(1)}_{kM+\ell}$:

Theorem 5.4.5. *Under Hypotheses 5.1.1. Consider the measure $\mu^{(1)}_{kM+\ell}$, G satisfying assumptions 5.1.1 and $H : \mathbb{C} \to \mathbb{R}$ such that $\mathrm{Tr}H(L)$ is cyclic (HP. 5) with seed h and weed h_1 such that $h, h_1 \in L^d(X^{2k}, \exp(-2W))$, with $\mathbb{N} \ni d \geqslant 6$, so that*

$$\mathbb{E}_1\left[e^{-it\mathrm{Tr}H(L)}\right] = \frac{Z^{(1)}_{kM+\ell}(\alpha, G + itH)}{Z^{(1)}_{kM+\ell}(\alpha, G)}\,. \tag{5.4.34}$$

Then, there exists $A \in \mathbb{R}$, $\sigma, C > 0$ such that if $Y \sim \mathcal{N}(0, \sigma^2)$ we have for any interval J of the real line

$$\left|\mathbb{P}\left(\left[\mathrm{Tr}H(L) - (kM+\ell)A\right]/\sqrt{(kM+\ell)} \in J\right) - \mathbb{P}\left(Y \in J\right)\right| \leqslant \frac{C}{\sqrt{(kM+\ell)}}\,. \tag{5.4.35}$$

Proof. We adapt the arguments of [Gou15, Theorem 3.7]. By [Fel71, Lemma XVI.3.2], there exists a constant C such that for any X real random variable, and Y Gaussian random variable, for any interval $J \subset \mathbb{R}$ and for any $T > 0$, we have

$$|\mathbb{P}\left(X \in J\right) - \mathbb{P}\left(Y \in J\right)| \leqslant C \int_0^T \frac{|\mathbb{E}[e^{-itX}] - e^{-\sigma^2 t^2/2}|}{t} \mathrm{d}t + \frac{C}{T}.$$

We take $X = \left(\mathrm{Tr}H(L) - (kM+\ell)A\right)/\sqrt{kM+\ell}$. We are going to show that, taking $T = \varepsilon\sqrt{(kM+\ell)}$ for some small enough ε, the last integral remains bounded by $\frac{C_{k,\ell}}{\sqrt{(kM+\ell)}}$, where $C_{k,\ell}$ is a constant depending on k, ℓ. Recall $N = kM + \ell$. By Theorem 5.1.3, there exists an $A \in \mathbb{R}, \sigma > 0$ such that as N goes to infinity X converges to $\mathcal{N}(0, \sigma^2)$. Since t^{-1} is not integrable at 0, we consider the special interval $[0, N^{-1}]$. In this interval, we have the following estimate, denoting by W the seed of $\mathrm{Tr}G(L)$:

$$\begin{aligned}
\left|\mathbb{E}\left[e^{-itX}\right] - e^{-it\sqrt{N}A}\right| &\overset{(5.2.8)}{=} \frac{|Z^{(1)}_{kM+\ell}\left(\alpha, G + i\frac{t}{\sqrt{N}}H\right) - Z^{(1)}_{kM+\ell}\left(\alpha, G\right)|}{Z^{(1)}_{kM+\ell}\left(\alpha, G\right)} \\
&= \frac{1}{Z^{(1)}_{kM+\ell}\left(\alpha, G\right)} \Big| \sum_{p=1}^{M-1} \int_{X^{kM+\ell}} \mathfrak{F}(\mathbf{x}) \prod_{j=1}^{p-1} e^{i\frac{t}{\sqrt{N}} h(\mathbf{x}_j, \mathbf{x}_{j+1})} \left(e^{i\frac{t}{\sqrt{N}} h(\mathbf{x}_p, \mathbf{x}_{p+1})} - 1\right) \prod_{j=1}^{N} \mathrm{d}x_j \\
&\quad + \int_{X^{kM+\ell}} \mathfrak{F}(\mathbf{x}) \left(e^{i\frac{t}{\sqrt{N}} h_1(\mathbf{x}_M, x_{kM+1}, \ldots, x_{kM+\ell}, \mathbf{x}_1)} - 1\right) \Big|\,,
\end{aligned} \tag{5.4.36}$$

with the convention that the empty product is equal to one. Here we defined

$$\begin{aligned}\mathfrak{F}(\mathbf{x}) = \prod_{j=1}^{kM+\ell} F(x_j,\alpha) \exp\left(-\sum_{j=1}^{M-1} W(\mathbf{x}_j,\mathbf{x}_{j+1}) - W_1(\mathbf{x}_M, x_{kM+1},\dots,x_{kM+\ell},\mathbf{x}_1)\right)\\ \times \exp\left(\frac{it}{\sqrt{N}} h_1(\mathbf{x}_M, x_{kM+1},\dots,x_{kM+\ell},\mathbf{x}_1)\right),\end{aligned} \tag{5.4.37}$$

where h_1 is the weed of H in the sense of HP 5 of Hyphotheses 5.1.1. Thus, since $|e^{i\frac{t}{\sqrt{N}}h(\mathbf{x}_p,\mathbf{x}_{p+1})}-1| \leqslant |h(\mathbf{x}_p,\mathbf{x}_{p+1})|N^{-1/2}t$, we deduce the following inequality

$$\left|\mathbb{E}_1\big[e^{-itX}\big] - e^{-it\sqrt{N}A}\right| \leqslant \mathbb{E}_1\left[|h(\mathbf{x}_1,\mathbf{x}_2)|\right] t\sqrt{N} + \frac{t}{\sqrt{N}}\mathbb{E}_1\left[|h_1(\mathbf{x}_M, x_{kM+1},\dots,x_{kM+\ell},\mathbf{x}_1)|\right], \tag{5.4.38}$$

and this last term is by assumption bounded by $Ct\sqrt{N}$ for some C independent of N and t. Thus integrating for $t \in [0, N^{-1}]$ we deduce the following

$$\begin{aligned}&\int_0^{\frac{1}{N}} \frac{|\mathbb{E}_1\big[e^{-itX}\big] - e^{-\sigma^2t^2/2}|}{t}\mathrm{d}t\\ &\quad\leqslant \int_0^{\frac{1}{N}} \frac{\left|\mathbb{E}_1\left[e^{-itX}\right] - e^{-it\sqrt{N}A}\right| + \left|e^{-it\sqrt{N}A}-1\right| + \left|1-e^{-\sigma^2t^2/2}\right|}{t}\mathrm{d}t\\ &\quad\overset{(5.4.38)}{\leqslant} \int_0^{\frac{1}{N}} \frac{C\sqrt{N}t + t\sqrt{N}A + \sigma^2t^2/2}{t}\mathrm{d}t \leqslant \frac{C_1}{\sqrt{N}},\end{aligned} \tag{5.4.39}$$

for some constant C_1.

We now consider the integral on $[1/N, \varepsilon\sqrt{N}]$. Here we use the spectral decomposition of $\mathbb{E}_1[e^{itX}]$. Since $h \in L^d(X^{2k}, \exp(-2W))$ for some $d \geqslant 6$, we deduce (following Remark 5.1.4) applying Proposition 5.4.1, and from Theorem 5.2.1, that there exist two continuous functions $p(t) \in C^0([-\varepsilon,\varepsilon])$ and $c_{k,\ell}(y,t) \in C^{1,d}(\mathbb{R}\times[-\varepsilon,\varepsilon])$ for some $\varepsilon > 0$, such that $c_{k,\ell}(y,0) = 1$ and $\|p\|_\infty < +\infty$, such that for $q \geqslant 1$

$$\int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{|\mathbb{E}_1\big[e^{-itX}\big] - e^{-\sigma^2t^2/2}|}{t}\mathrm{d}t = \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{\left|c_{k,\ell}(\alpha,t/\sqrt{N})e^{-\sigma^2t^2/2+t^3p(t/\sqrt{N})/\sqrt{N}}(1+o(N^{-q})) - e^{-\sigma^2t^2/2}\right|}{t}\mathrm{d}t, \tag{5.4.40}$$

thus we have the following estimate

$$\begin{aligned}\int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{\left|\mathbb{E}_1\big[e^{-itX}\big] - e^{-\sigma^2t^2/2}\right|}{t}\mathrm{d}t &\leqslant \left|\left|c_{k,\ell}(\alpha,\cdot)(1+o(M^{-q}))\right|\right|_{\infty,[0,\varepsilon]} \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{\left|\left(1-e^{t^3p(t/\sqrt{N})/\sqrt{N}}\right)e^{-\sigma^2t^2/2}\right|}{t}\mathrm{d}t\\ &\quad+\int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{\left|1-c_{k,\ell}(\alpha,t/\sqrt{N})(1+o(M^{-q}))\right|e^{-\sigma^2t/2}}{t}\mathrm{d}t,\end{aligned} \tag{5.4.41}$$

where $||\cdot||_{\infty,[0,\varepsilon]}$ in the L^∞ norm on $[0,\varepsilon]$.

We notice that $||c_{k,\ell}(\alpha,\cdot)(1+o(M^{-q}))||_{\infty,[0,\varepsilon]}$ is uniformly bounded in N. Moreover,

$$\begin{aligned}\int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{e^{-\sigma^2t^2/2}}{t}\left|e^{t^3p(t/\sqrt{N})/\sqrt{N}}-1\right|\mathrm{d}t &\leqslant \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{e^{-\sigma^2t^2/2}}{t\sqrt{N}} e^{t^3\|p\|_{\infty,[0,\varepsilon]}/\sqrt{N}} t^3\|p\|_{\infty,[0,\varepsilon]}\mathrm{d}t\\ &\leqslant \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{e^{-\sigma^2t^2/2}}{\sqrt{N}} e^{t^2\varepsilon\|p\|_{\infty,[0,\varepsilon]}} t^2\|p\|_{\infty,[0,\varepsilon]}\mathrm{d}t,\end{aligned} \tag{5.4.42}$$

where in the first inequality we used the bound $|e^x - 1| \leqslant |x|e^{|x|}$. Since for ε small enough $||p||_\infty \varepsilon < \sigma^2/4$, thus integrating, we deduce that

$$\int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{\left|e^{-\sigma^2 t^2/2 + t^3 p(t/\sqrt{N})/\sqrt{N}} - e^{-\sigma^2 t^2/2}\right|}{t} \mathrm{d}t = O\left(\frac{1}{\sqrt{N}}\right). \tag{5.4.43}$$

To conclude, we have to show that the last integral is of order $N^{-1/2}$. Since $c_{k,\ell}(\alpha, t)$ is C^1 in t, and $c_{k,\ell}(\alpha, 0) = 1$, it is easy to deduce that there exists a constant C such that

$$\int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{\left|1 - c_{k,\ell}(\alpha, t/\sqrt{N})(1 + o(M^{-q}))\right| e^{-\sigma^2 t/2}}{t} \mathrm{d}t \leqslant \frac{C}{\sqrt{N}} \tag{5.4.44}$$

so we conclude. □

5.5 Conclusion and Outlooks

In this paper, we proved a general Central Limit Theorem type result and we apply it to several models in random matrix theory and integrable systems. By doing this, we strengthen the connection between these two subjects. Specifically, we could connect the expected values and the variances of the moments of each classical β ensemble in the high-temperature regime with one specific integrable model, see Table 5.1.

The results that we have obtained are relevant for two main reasons. Under the random matrix theory perspective, we were able to develop a general framework to prove polynomial central limit theorems for the classical β ensemble in the high-temperature regime, based on their band matrix representation and on the transfer operator technique. Under the integrable systems' theory point of view, our result enables the explicit computation of the so-called *susceptibility* matrix, which is a fundamental object in the theory of Generalized Hydrodynamics in order to compute the correlation functions for integrable models. Furthermore, we are able to prove rigorously the exponential decay of correlation for short-range interacting systems with polynomial potential.

It would be fascinating to generalize our result to a wider class of potential and functions and to obtain a Berry-Esseen bound for the classical β ensemble in the high-temperature regime. Furthermore, defining a new β ensemble related to the INB lattice would be interesting. Finally, we point out that it would be interesting to obtain large deviation principles for the Exponential Toda lattice and the Volterra one in the spirit of [GM22, MM23b].

Acknowledgements
The authors warmly thank Alice Guionnet for giving the initial idea of the proof of the polynomial CLT, and Herbert Spohn for the useful discussions about the applications of our result.
R.M. is supported by ERC Project LDRAM : ERC-2019-ADG Project 884584.
G.M. is financed by the KAM grant number 2018.0344

Appendix A

Large deviations

We present here definitions and results about large deviation theory that we use throughout this manuscript. For a detailed and general introduction to this topic, we refer to [DZ10].

A.1 Definition and first results

Definition A.1.1. *Let $(\mathcal{X}, \mathcal{T})$ be a topological space. Let $I : \mathcal{X} \to [0, \infty]$. We say that I is a **rate function** if it is lower semicontinuous (that is, for all $\alpha \geqslant 0$, $I^{-1}([0, \alpha])$ is closed). We say that I is a **good rate function** is for all $\alpha \geqslant 0$, $I^{-1}([0, \alpha])$ is compact.*

*Let $(\mu_N)_{n\in\mathbb{N}}$ be a sequence of probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{T}))$, where $\mathcal{B}(\mathcal{T})$ is the Borel sigma algebra. We say that $(\mu_N)_{N\in\mathbb{N}}$ satisfies a **large deviation principle (LDP)** at scale $v(N)$, where $\lim_N v(N) = +\infty$, and with rate function I if, for all $A \in \mathcal{B}(\mathcal{T})$, we have*

$$-\inf_{x\in\mathring{A}} I(x) \leqslant \liminf_N \frac{\log(\mu_N(A))}{v(N)} \leqslant \limsup_N \frac{\log(\mu_N(A))}{v(N)} \leqslant -\inf_{x\in\overline{A}} I(x)\,,$$

with $\mathring{A}$ the interior of A and $\overline{A}$ its closure.

Remark A.1.2. *The inequality*

$$-\inf_{x\in\mathring{A}} I(x) \leqslant \liminf_N \frac{\log(\mu_N(A))}{v(N)}$$

*is called the **large deviations lower bound,** and the inequality*

$$\limsup_N \frac{\log(\mu_N(A))}{v(N)} \leqslant -\inf_{x\in\overline{A}} I(x)$$

*is the **large deviations upper bound**.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and (X_N) be a sequence of $\mathcal{X}$-valued random variables, defined on $(\Omega, \mathcal{F})$. We say that $(X_N)_{N\in\mathbb{N}}$ satisfies a large deviation principle at scale $v(N)$ and rate function I if the sequence of the laws, defined by $\mu_N(A) = \mathbb{P}(X_N \in A)$, does.

In the previous definition, we assume I to be lower semi continuous to ensure uniqueness of the rate function:

Recall that the topological space $\mathcal{X}$ is *Hausdorff* if for all distinct x, y there exist disjoint neighborhoods of x and y, and it is *regular* if for all C closed and all $x \in C^c$, there exist disjoint open sets O_1 and O_2 such that $x \in O_1$ and $F \subset O_2$. We then have:

Proposition A.1.3. *Assume that $(\mu_N)_N$ satifies a large deviation principle at scale $v(N)$ for two rate functions I_1 and I_2, and that $(\mathcal{X}, \mathcal{T})$ is Hausdorff and regular, then $I_1 = I_2$.*

We now introduce a weaker form of large deviation principle.

Definition A.1.4. *Assume that the compact sets of $(\mathcal{X}, \mathcal{T})$ are in $\mathcal{B}(\mathcal{T})$. The sequence of probability measures $(\mu_N)_{N\in\mathbb{N}}$ is said to satisfy a* ***weak large deviation principle*** *at scale $v(N)$ and rate function I if the large deviation lower bound (Remark A.1.2) holds, and if the large deviation lower bound holds for all compacts K such that $\sup_{x\in K} I(x) < +\infty$.*

A weak large deviation principle can be strengthened into a full one through the following property.

Definition A.1.5. *Suppose that the compacts of $(\mathcal{X}, \mathcal{T})$ are in $\mathcal{B}(\mathcal{T})$. The sequence $(\mu_N)_N$ is said to be* ***exponentially tight*** *at scale $v(N)$ if for all $M > 0$, there exists a compact K_M such that*

$$\limsup_N \frac{1}{v(N)} \log(\mu_N(K_M^c)) < -M\,.$$

We then have the following result.

Proposition A.1.6. *Assume that $(\mu_N)_{N\in\mathbb{N}}$ satisfies a weak large deviation principle at scale $v(N)$ and with rate function I, and assume that it is exponentially tight. Then, I is a good rate function and $(\mu_N)_{N\in\mathbb{N}}$ satisfies a full large deviation principle at scale $v(N)$ and good rate function I.*

The previous result is convenient in practice because of the following result.

Proposition A.1.7. *Let $\mathcal{A}$ be a base of the topology $\mathcal{T}$. Suppose that for all $x \in \mathcal{X}$, we have*

$$\sup_{A\in\mathcal{A}|x\in A} \left[-\limsup_N \frac{1}{v(N)} \log \mu_N(A)\right] = \sup_{A\in\mathcal{A}|x\in A} \left[-\liminf_N \frac{1}{v(N)} \log \mu_N(A)\right]\,.$$

Then $(\mu_N)_{N\in\mathbb{N}}$ satisfies a weak LDP at scale $v(N)$ with rate function I given by

$$I(x) = \sup_{A\in\mathcal{A}|x\in A} \left[-\limsup_N \frac{1}{v(N)} \log \mu_N(A)\right]\,.$$

An important example is the one of a metric space: to meet this criterium in a metric space, it suffices to have, for all $x \in \mathcal{X}$, denoting by $B_x(\delta)$ the ball centered at x and with radius δ,

$$\lim_{\delta\to 0} \limsup_N \frac{1}{v(N)} \log \mu_N(B_x(\delta)) = \lim_{\delta\to 0} \liminf_N \frac{1}{v(N)} \log \mu_N(B_x(\delta)) =: -I(x)\,.$$

A.2 Varadhan's Lemma and Bryc's inverse Lemma

One of the principal applications of large deviations is the derivation of the asymptotic behavior of Laplace integrals $\int_{\mathcal{X}} e^{v(N)\varphi(x)} d\mu_N(x)$ for continuous φ.

Proposition A.2.1 (Varadhan's Lemma)**.** *Let $(\mu_N)_{N\in\mathbb{N}}$ be the law of the sequence of random variables $(X_N)_{N\in\mathbb{N}}$. Assume it satisfies a LDP at scale $v(N)$ with good rate function I. Let $\varphi : \mathcal{X} \to \mathbb{R}$ satisfying the following condition:*

$$\lim_{M\to\infty} \limsup_N \frac{1}{v(N)} \log \mathbb{E}\left[e^{v(N)\varphi(X_N)\mathbf{1}_{\{\varphi(X_N)\geqslant M\}}}\right] = -\infty$$

and the following moment condition for some $\gamma > 1$:

$$\limsup_N \frac{1}{v(N)} \log \mathbb{E}\left[e^{\gamma v(N)\varphi(X_N)}\right] < \infty\,.$$

Then,

$$\lim_N \frac{1}{v(N)} \log \mathbb{E}\left[e^{v(N)\varphi(X_N)}\right] = \sup_{x\in\mathcal{X}}\{\varphi(x) - I(x)\}\,.$$

This result can be thought of as an extension of the Laplace method for real integrals. It is of particular interest in applications since it allows to link the computation of the *partition function* and the *free energy* of a model to a large deviation principle for this model.

The next proposition gives a way to reconstruct the rate function of a sequence satisfying a large deviation principle by means of its Laplace integrals.

Proposition A.2.2 (Bryc's inverse Lemma)**.** *Let $(\mu_N)_{N\in\mathbb{N}}$ be an exponentially tight sequence of measures on $(\mathcal{X}, \mathcal{B}(\mathcal{T}))$, and such that for any $f \in \mathcal{C}_b(\mathcal{X}, \mathbb{R})$ the following limit exists:*

$$\Lambda_f := \lim_{N\to\infty} \frac{1}{v(N)} \log \int_{\mathcal{X}} e^{v(N)f(x)} d\mu_N(x)\,.$$

Then $(\mu_N)_{N\in\mathbb{N}}$ satisfies a large deviation principle at scale $v(N)$ with good rate function

$$I(x) = \sup_{f\in\mathcal{C}_b(\mathcal{X})} \{f(x) - \Lambda_f\}\,.$$

A.3 Cramér and Sanov theorems

We now state two important results of large deviations theory, which establish large deviation principles for respectively *i.i.d* sequences, and empirical measures of *i.i.d* sequences.

Let $(\mathcal{X}, \mathcal{T})$ be a topological vector space. For μ a probability measure on $\mathcal{X}$, and for $\lambda \in \mathcal{X}^*$ a continuous linear form on $\mathcal{X}$, define

$$\Lambda(\lambda) = \log \int_{\mathcal{X}} e^{\lambda(x)} d\mu(x)\,.$$

Also define its Fenchel-Legendre transform Λ^*, defined on $\mathcal{X}$ by

$$\Lambda^*(x) = \sup_{\lambda\in\mathcal{X}^*} \{\lambda(x) - \Lambda(\lambda)\} \in [0, \infty]\,.$$

We have:

Theorem A.3.1 (Camér Theorem)**.** *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with values in a locally convex vector space $(\mathcal{X}, \mathcal{T})$ and with law μ. Denote by μ_N the law of $(X_1 + \ldots + X_N)/N$. Assume:*

1. *There exists a convex, closed set $\mathcal{E} \subset \mathcal{X}$ with $\mu(\mathcal{E}) = 1$ and such that the induced topology on $\mathcal{E}$ is compatible with a metric d such that $(\mathcal{E}, d)$ is a Polish space (that is, separable and complete).*

2. *The closed convex hull of any compact $K \subset \mathcal{E}$ is compact.*

Then, the sequence $(\mu_N)_{N\in\mathbb{N}}$ satisfies a weak LDP at scale N, with rate function Λ^*.

The next result, due to Sanov, can be seen as a consequence of Cramér's Theorem in the set of finite, signed measures on a Polish space Σ. The set $\mathcal{E} \subset \mathcal{P}(\Sigma)$ is then the set of probability measures.

Let Σ be a polish space, and let μ be a probability measure on Σ.

Theorem A.3.2 (Sanov's Theorem)**.** *Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. with law μ. Let $\hat{\mu}_N$ be the empirical measure of the X_i's:*

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i}\,.$$

Then, the sequence of random measures $(\hat{\mu}_N)_{N\in\mathbb{N}}$ satisfies a large deviation principle at scale N and with good rate function $H(\cdot|\mu)$ given by

$$H(\nu|\mu) = \begin{cases} \int_\Sigma f \log f d\mu \text{ if } \nu \text{ is absolutely continuous with respect to } \mu \text{ with density } f = \frac{d\nu}{d\mu} \\ +\infty \text{ otherwise.} \end{cases}$$

The function $H(\cdot|\mu)$ is called the ***relative entropy*** *with respect to μ.*

Bibliography

[ABA07] P. Auscher and B. Ben Ali. Maximal inequalities and riesz transform estimates on L^p spaces for schrödinger operators with nonnegative potentials. In *Annales de l'institut Fourier*, volume 57, pages 1975–2013, 2007.

[ABG12] R. Allez, J. P. Bouchaud, and A. Guionnet. Invariant beta ensembles and the gauss-wigner crossover. *Physical Review Letters*, Aug. 2012.

[ABMV13] R. Allez, J. P. Bouchaud, S. N. Majumdar, and P. Vivo. Invariant β-Wishart ensembles, crossover densities and asymptotic corrections to the Marčenko-Pastur law. *J. Phys. A Math. Theor.*, 46(1):1–26, 2013.

[AG88] P. Arbenz and G. H. Golub. On the spectral decomposition of hermitian matrices modified by low rank perturbations with applications. *SIAM J. Matrix Anal. Appl.*, 9(1):40–58, 1988.

[AGZ10] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[AKN74] M.J. Ablowitz, D.J. Kaup, and A.C. Newell. Coherent pulse propagation, a dispersive, irreversible phenomenon. *Journal of Mathematical Physics*, 15(11):1852–1858, 1974.

[AL75] M. J. Ablowitz and J. F. Ladik. Nonlinear differential- difference equations. *Journal of Mathematical Physics*, 16(3):598–603, 1975.

[AL76] M. J. Ablowitz and J. F. Ladik. Nonlinear differential–difference equations and fourier analysis. *Journal of Mathematical Physics*, 17(6):1011–1018, 1976.

[AS21] S. Armstrong and S. Serfaty. Local laws and rigidity for Coulomb gases at any temperature. *Ann. Probab.*, 49(1):46–121, 2021.

[BAG97] G. Ben Arous and A. Guionnet. Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Related Fields*, 108(4):517–542, 1997.

[BBCG08] D. Bakry, F. Barthe, P. Cattiaux, and A. Guillin. A simple proof of the Poincaré inequality for a large class of probability measures including the log-concave case. *Electron. Commun. Probab.*, 13:60–66, 2008.

[BDWY18] Alvise Bastianello, Benjamin Doyon, Gerard Watts, and Takato Yoshimura. Generalized hydrodynamics of classical integrable field theory: the sinh-gordon model. *SciPost Physics*, 4(6):045, 2018.

[Ber18] R. J. Berman. On large deviations for gibbs measures, mean energy and gamma-convergence. *Constructive Approximation*, 48:3–30, 2018.

[BFG15] F. Bekerman, A. Figalli, and A. Guionnet. Transport maps for β-matrix models and universality. *Communications in mathematical physics*, 338(2):589–619, 2015.

[BG99] T. Bodineau and A. Guionnet. About the stationary states of vortex systems. In *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, volume 35, pages 205–237. Elsevier, 1999.

[BG13a] G. Borot and A. Guionnet. Asymptotic expansion of β matrix models in the one-cut regime. *Comm. Math. Phys.*, 317(2):447–483, 2013.

[BG13b] G. Borot and A. Guionnet. Asymptotic expansion of beta matrix models in the several-cut regime. *arxiv 1303.1045*, 2013.

[BGK16] G. Borot, A. Guionnet, and K. K. Kozlowski. *Asymptotic expansion of a partition function related to the sinh-model*. Mathematical Physics Studies. Springer, [Cham], 2016.

[BGP15] F. Benaych-Georges and S. Péché. Poisson statistics for matrix ensembles at large temperature. *Journal of Statistical Physics*, 161(3):633–656, 2015.

[BGS09] E Bogomolny, Olivier Giraud, and C Schmit. Random matrix ensembles associated with lax matrices. *Physical review letters*, 103(5):054103, 2009.

[BGS11] E Bogomolny, O Giraud, and C Schmit. Integrable random matrix ensembles. *Nonlinearity*, 24(11):3179, 2011.

[BLS18] F. Bekerman, T. Leblé, and S. Serfaty. Clt for fluctuations of β-ensembles with general potential. *Electronic Journal of Probability*, 23:1–31, 2018.

[BMP22] P. Bourgade, K. Mody, and M. Pain. Optimal local law and central limit theorem for β-ensembles. *Communications in Mathematical Physics*, 390(3):1017–1079, 2022.

[BNR09] P. Bourgade, A. Nikeghbali, and A. Rouault. Circular Jacobi ensembles and deformed Verblunsky coefficients. *Int. Math. Res. Not. IMRN*, (23):4357–4394, 2009.

[Bog88] O. I. Bogoyavlensky. Integrable discretizations of the KdV equation. *Phys. Lett. A*, 134:34–38, 1988.

[Bog91] O. I. Bogoyavlensky. Algebraic constructions of integrable dynamical systems-extensions of the Volterra system. *Russ. Math Surveys*, 46:1–64, 1991.

[Caf09] M. Cafasso. Matrix biorthogonal polynomials on the unit circle and non-abelian Ablowitz-Ladik hierarchy. *J. Phys. A*, 42(36):365211, 20, 2009.

[CHM18] Djalil Chafaï, Adrien Hardy, and Mylène Maïda. Concentration for coulomb gases and coulomb transport inequalities. *Journal of Functional Analysis*, 275(6):1447–1483, 2018.

[CJ21] K. Courteaut and K. Johansson. Multivariate normal approximation for traces of orthogonal and symplectic matrices, 2021.

[CJL22] K. Courteaut, K. Johansson, and G. Lambert. From Berry-Esseen to super-exponential, 2022.

[CL97] E. Cépa and D. Lépingle. Diffusing particles with electrostatic repulsion. *Probability theory and related fields*, 107(4):429–449, 1997.

[CMV05] M. J. Cantero, L. Moral, and L. Velázquez. Minimal representations of unitary operators and orthogonal polynomials on the unit circle. *Linear algebra and its applications*, 408:40–65, 2005.

[Dav96] E. B. Davies. *Spectral theory and differential operators*, volume 42. Cambridge University Press, 1996.

[DE97] P. Dupuis and R. S. Ellis. *A weak convergence approach to the theory of large deviations.* Chichester: John Wiley & Sons, 1997.

[DE02] I. Dumitriu and A. Edelman. Matrix models for beta ensembles. *J. Math. Phys.*, 43:5830–5847, 2002.

[DE06] I. Dumitriu and A. Edelman. Global spectrum fluctuations for the β-Hermite and β-Laguerre ensembles via matrix models. *J. Math. Phys.*, 47:063302, 36, 2006.

[Dei99] P. A. Deift. *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, volume 3 of *Courant Lecture Notes in Mathematics.* New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[DF10] I. Dumitriu and P. J. Forrester. Tridiagonal realization of the antisymmetric gaussian β-ensemble. *Journal of mathematical physics*, 51(9):093302, 2010.

[DGM23] C. Dworaczek Guera and R. Memin. Clt for real β-ensembles at high temperature. *arXiv preprint arXiv:2301.05516*, 2023.

[DHLM21] David Dereudre, Adrien Hardy, Thomas Leblé, and Mylène Maïda. Dlr equations and rigidity for the sine-beta process. *Communications on Pure and Applied Mathematics*, 74(1):172–222, 2021.

[DLMF] *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/, Release 1.1.3 of 2021-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[Doy20] B. Doyon. Lecture Notes On Generalised Hydrodynamics. *SciPost Phys. Lect. Notes*, page 18, 2020.

[DP12] I. Dumitriu and E. Paquette. Global fluctuations for linear statistics of β-Jacobi ensembles. *Random Matrices Theory Appl.*, 1(4):1250013, 60, 2012.

[DS17] Benjamin Doyon and Herbert Spohn. Dynamics of hard rods with initial domain wall state. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(7):073210, 2017.

[DZ10] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[EL06] N. M. Ercolani and G. Lozano. A bi-Hamiltonian structure for the integrable, discrete non-linear Schrödinger system. *Physica D*, 218:105–121, 2006.

[Fel71] W. Feller. *An introduction to probability theory and its applications, Part II.* Wiley, second edition, 1971.

[FIKN06] A. S. Fokas, A. R. Its, A. A. Kapaev, and V. Yu. Novokshenov. *Painlevé transcendents*, volume 128 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2006. The Riemann-Hilbert approach.

[Fla74a] H. Flaschka. On the Toda lattice. II. Inverse-scattering solution. *Progr. Theoret. Phys.*, 51:703–716, 1974.

[Fla74b] H. Flaschka. The toda lattice. II. existence of integrals. *Physical Review B*, 9(4):1924, 1974.

[FM21] P. J. Forrester and G. Mazzuca. The classical β-ensembles with β proportional to $1/N$: from loop equations to Dyson's disordered chain. *J. Math. Phys.*, 62(7):Paper No. 073505, 22, 2021.

[FST73] J. Ford, S. D. Stoddard, and J.S. Turner. On the integrability of the toda lattice. *Progress of Theoretical Physics*, 50(5):1547–1560, 1973.

[GGGM23] T. Grava, M. Gisonni, G. Gubbiotti, and G. Mazzuca. Discrete integrable systems and random Lax matrices. *J. Stat. Phys.*, 190(1):Paper No. 10, 35, 2023.

[GGK12] I. Gohberg, S. Goldberg, and N. Krupnik. *Traces and determinants of linear operators*, volume 116. Birkhäuser, 2012.

[GH88] Y. Guivarc'h and J. Hardy. Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov. *Ann. Inst. H. Poincaré Probab. Statist.*, 24(1):73–98, 1988.

[Gio22] A. Giorgilli. *Notes on Hamiltonian Dynamical Systems*, volume 102. Cambridge University Press, 2022.

[GKM$^+$23] T. Grava, T. Kriecherbauer, G. Mazzuca, K.T-R. McLaughlin, C.B. Mendl, and H. Spohn. Equilibrium Spacetime Correlations of the Toda Lattice on the Hydrodynamic Scale, 2023.

[GM22] A. Guionnet and R. Memin. Large deviations for gibbs ensembles of the classical toda chain. *Electron. J. Probab.*, 27:1–29, 2022.

[GM23] T. Grava and G. Mazzuca. Generalized gibbs ensemble of the ablowitz–ladik lattice, circular β-ensemble and double confluent heun equation. *Communications in Mathematical Physics*, pages 1–41, 2023.

[GMMP20] T. Grava, A. Maspero, G. Mazzuca, and A. Ponno. Adiabatic invariants for the FPUT and Toda chain in the thermodynamic limit. *Comm. Math. Phys.*, 380(2):811–851, 2020.

[GN09] M. Gekhtman and I. Nenciu. Multi-hamiltonian structurefor the finite defocusing ablowitz-ladik equation. *Comm. Pure Appl. Math.*, 62(1):147–182, 2009.

[Gol06] L. B. Golinskii. Schur flows and orthogonal polynomials on the unit circle. *Sb. Math.*, 197(7-7):1145 – 1165, 2006.

[Gou15] S. Gouëzel. Limit theorems in dynamical systems using the spectral method. In *Hyperbolic dynamics, fluctuations and large deviations*, volume 89 of *Proc. Sympos. Pure Math.*, pages 161–193. Amer. Math. Soc., Providence, RI, 2015.

[GPP14] A. Giorgilli, S. Paleari, and T. Penati. An Extensive Adiabatic Invariant for the Klein–Gordon Model in the Thermodynamic Limit. *Annales Henri Poincaré*, 16(4):897–959, April 2014.

[Gui09] A. Guionnet. *Large random matrices: lectures on macroscopic asymptotics*, volume 1957 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Lectures from the 36th Probability Summer School held in Saint-Flour, 2006.

[Gui19] A. Guionnet. *Asymptotics of random matrices and related models: the uses of Dyson-Schwinger equations*, volume 130. American Mathematical Soc., 2019.

[GZ19] D. García-Zelada. A large deviation principle for empirical measures on Polish spaces: application to singular Gibbs measures on manifolds. *Ann. Inst. Henri Poincaré Probab. Stat.*, 55(3):1377–1401, 2019.

[Hen74] M. Henón. Integrals of the toda lattice. *Physical Review B*, 9(4):1921, 1974.

[HH01] H. Hennion and L. Hervé. *Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness*, volume 1766 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[HL21] A. Hardy and G. Lambert. Clt for circular beta-ensembles at high temperature. *Journal of Functional Analysis*, 280(7):108869, 2021.

[Ito75] Y. Itoh. An H-theorem for a system of competing species. *Proc. Japan Acad.*, 51:374–379, 1975.

[Joh98] K. Johansson. On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.*, 91:151–204, 1998.

[Kat95] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[Kin09] F. W. King. *Hilbert Transforms: Volume 1 (Encyclopedia of Mathematics and its Applications)*. 1 edition, 2009.

[KN04] R. Killip and I. Nenciu. Matrix models for circular ensembles. *Int. Math. Res. Not.*, (50):2665–2701, 2004.

[KN07] R. Killip and I. Nenciu. CMV: the unitary analogue of Jacobi matrices. *Comm. Pure Appl. Math.*, 60:1148–1188, 2007.

[KRB01] P. G. Kevrekidis, K.Ø. Rasmussen, and A. R. Bishop. The Discrete Nonlinear Schrödinger Equation: A Survey of Recent Results. *Int. J. Mod. Phys. B*, 15(21):2833–2900, aug 2001.

[KS75] J. A. Krumhansl and J. R. Schrieffer. Dynamics and statistical mechanics of a one-dimensional model Hamiltonian for structural phase transitions. *Phys. Rev. B*, 11(9):3535–3545, may 1975.

[KS09] R. Killip and M. Stoiciu. Eigenvalue statistics for CMV matrices: from Poisson to clock via random matrix ensembles. *Duke Math. J.*, 146(3):361–399, 2009.

[KvM75] M. Kac and P. van Moerbeke. On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices. *Advances in Math.*, 16:160–169, 1975.

[Lam21] G. Lambert. Poisson statistics for gibbs measures at high temperature. 57(1):326–350, 2021.

[Lax68] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[LLW19] G. Lambert, M. Ledoux, and C. Webb. Quantitative normal approximation of linear statistics of β-ensembles. *Ann. of Probab.*, 47:2619–2685, 2019.

[LN21] O. Lisovyy and A. Naidiuk. Accessory parameters in confluent Heun equations and classical irregular conformal blocks. *Lett. Math. Phys.*, 111(6):Paper No. 137, 28, 2021.

[Man74] S. V. Manakov. Complete integrability and stochastization of discrete dynamical systems. *Ž. Èksper. Teoret. Fiz.*, 67(2):543–555, 1974.

[Maz22] G. Mazzuca. On the mean density of states of some matrices related to the beta ensembles and an application to the Toda lattice. *J. Math. Phys.*, 63(4):Paper No. 043501, 13, 2022.

[MM23a] G. Mazzuca and R. Memin. Clt for β-ensembles at high-temperature, and for integrable systems: a transfer operator approach. *arXiv preprint arXiv:2304.10323*, 2023.

[MM23b] G. Mazzuca and R. Memin. Large deviations for ablowitz-ladik lattice, and the schur flow. *Electronic Journal of Probability*, 28:1–29, 2023.

[MMS14] M. Maïda and É. Maurel-Segala. Free transport-entropy inequalities for non-convex potentials and application to concentration for random matrices. *Probability Theory and Related Fields*, 159(1):329–356, 2014.

[Mos75] J. Moser. Three integrable Hamiltonian systems connected with isospectral deformations. *Advances in Math.*, 16:197–220, 1975.

[Nag57] S. V. Nagaev. Some limit theorems for stationary Markov chains. *Teor. Veroyatnost. i Primenen.*, 2:389–416, 1957.

[Nar82] K. Narita. Soliton solution to extended Volterra equation. *J. Phys. Soc. Japan*, 51:1682–1685, 1982.

[Nen05] I. Nenciu. Lax pairs for the Ablowitz-Ladik system via orthogonal polynomials on the unit circle. *Int. Math. Res. Not.*, 2005(11):647–686, 2005.

[NT18] F. Nakano and K. D. Trinh. Gaussian beta ensembles at high temperature: eigenvalue fluctuations and bulk statistics. *J.Stat.Phys*, 173:295–321, 2018.

[NT20] F. Nakano and K. D. Trinh. Poisson statistics for beta ensembles on the real line at high temperature. *J. Stat. Phys.*, 179(2):632–649, 2020.

[NTT23] F. Nakano, H. D. Trinh, and K. D. Trinh. Beta Jacobi ensembles and associated Jacobi polynomials, II, 2023.

[OF22] Stefano Olla and Pablo Ferrari. Diffusive fluctuations in hard rods system. *arXiv preprint arXiv:2210.02079*, 2022.

[Pak18] C. Pakzad. Poisson statistics at the edge of gaussian beta-ensembles at high temperature. *arXiv preprint arXiv:1804.08214*, 2018.

[PB89] M. Peyrard and A. R. Bishop. Statistical mechanics of a nonlinear model for DNA denaturation. *Phys. Rev. Lett.*, 62(23):2755–2758, 1989.

[QNCVDL84] G. R. W. Quispel, F. W. Nijhoff, H. W. Capel, and J. Van Der Linden. Linear integral equations and nonlinear difference-difference equations. *Physica A: Statistical Mechanics and its Applications*, 125(2-3):344–380, 1984.

[RS78] M. Reed and B. Simon. Methods of modern mathematical physics, vol. 4, 1978.

[Shc13] M. Shcherbina. Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime. *J. Stat. Phys.*, 151(6):1004–1034, 2013.

[Shc14a] M. Shcherbina. Asymptotic expansions for *beta* matrix models and their applications to the universality conjecture. In *Random matrix theory, interacting particle systems, and integrable systems*, volume 65 of *Math. Sci. Res. Inst. Publ.*, pages 463–482. Cambridge Univ. Press, New York, 2014.

[Shc14b] M. Shcherbina. Change of variables as a method to study general β-models: bulk universality. *Journal of Mathematical Physics*, 55(4):043504, 2014.

[Sim05] B. Simon. *Orthogonal Polynomials on the Unit Circle*, volume 54.1 of *Colloquium Publications*. American Mathematical Society, Providence, Rhode Island, 2005.

[Spo20a] H. Spohn. Ballistic space-time correlators of the classical toda lattice. *J. Phys.*, A53:265004, 2020.

[Spo20b] H. Spohn. The collision rate ansatz for the classical toda lattice. *Phys Rev.*, E101:060103, 2020.

[Spo20c] H. Spohn. Generalized Gibbs Ensembles of the Classical Toda Chain. *J. Stat. Phys.*, 180(1-6):4–22, 2020.

[Spo21] H. Spohn. Hydrodynamic equations for the toda lattice. *ArXiv 2101.06528*, 2021.

[Spo22a] H. Spohn. Hydrodynamic equations for the ablowitz–ladik discretization of the nonlinear schrödinger equation. *Journal of Mathematical Physics*, 63(3):033305, 2022.

[Spo22b] H. Spohn. Hydrodynamic equations for the Ablowitz-Ladik discretization of the nonlinear Schrödinger equation. *J. Math. Phys.*, 63(3):Paper No. 033305, 21, 2022.

[Tod67] Morikazu Toda. Vibration of a chain with nonlinear interaction. *Journal of the Physical Society of Japan*, 22(2):431–436, 1967.

[Tod89] M. Toda. *Theory of nonlinear lattices*, volume 20 of *Springer Series in Solid-State Sciences*. Springer-Verlag, Berlin, second edition, 1989.

[TT21] H. D. Trinh and K. D. Trinh. Beta Jacobi Ensembles and Associated Jacobi Polynomials. *J. Stat. Phys.*, 185(1):Paper No. 4, 2021.

[vM76] P. van Moerbeke. The spectrum of Jacobi matrices. *Invent. Math.*, 37(1):45–81, 1976.

[Wig55] E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Annals Math.*, 62:548–564, 1955.

[Wil91] D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.

[Win14] A. Winkelbauer. Moments and absolute moments of the normal distribution. *arXiv:1209.4340v2*, 2014.

[Wis28] J. Wishart. The generalized product moment distribution in samples from a normal multivariate population. *Biometrika*, 20A:32–52, 1928.

[Zaa83] A. C. Zaanen. *Riesz spaces. II*, volume 30 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983.

[Zha17] D. Zhang. Tridiagonal random matrix: Gaussian fluctuations and deviations. *J. Theoret. Probab.*, 30(3):1076–1103, 2017.

[ZS72] V. E. Zkharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP*, 34(1):62–69, 1972.