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Tianjiao Dai

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Some vertex colouring problems and a generalisation of Hamilton-connectivity in graphs

*Certains problèmes de coloration des sommets et une généralisation
de la Hamilton-connectivité dans des graphes*

Thèse de doctorat de l'université Paris-Saclay

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de l'information et de la communication (STIC)

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Thèse préparée dans l'unité de recherche **Laboratoire interdisciplinaire des sciences du numérique (Université Paris-Saclay, CNRS)**, sous la direction de **Hao LI**, directeur de recherche, CNRS, et le co-encadrement de **François PIROT**, maître de conférences

Thèse soutenue à Paris-Saclay, le 17 octobre 2023, par

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Titre : Certains problèmes de coloration des sommets et une généralisation de la Hamilton-connectivité dans des graphes

Mots clés : Graphes planaires extérieurs 2-connexes; Graphes sans triangle; DP-colorations; Colorations fractionnaire; Hamilton-connectivité; Degré maximum; Somme des degrés.

Résumé : La décomposition des graphes fait référence au processus de décomposer un graphe complexe en composantes plus simples et plus petites, souvent dans le but d'analyser ou de résoudre des problèmes liés au graphe. Il s'agit d'un outil important pour représenter la structure globale et les propriétés d'une manière plus détaillée. Il est aussi également utile pour résoudre des problèmes impliquant la recherche de structures spécifiques dans un graphe. Il existe plusieurs types courants de techniques de décomposition de graphe largement utilisées en théorie des graphes et dans des domaines connexes, notamment la décomposition en arbres, la décomposition en blocs, la décomposition modulaire, la décomposition hiérarchique, etc. Cette thèse étudie deux types de décomposition de sommets d'un graphe : les colorations propres (décomposition en ensembles indépendants) et la Hamilton-connectivité (décomposition en chemins intérieurement disjoints entre deux ensembles où les chemins couvrent tous les sommets du graphe).

Title: Some vertex colouring problems and a generalisation of Hamilton-connectivity in graphs

Keywords: 2-connected outerplanar graphs; Triangle-free graphs; DP-colourings; Fractional colourings; Hamilton-connectivity; Maximum degree; Degree sum.

Abstract: The decomposition of graphs refers to the process of breaking down a complex graph into simpler, smaller components, often with the goal of analysing or solving problems related to the graph. It is an important tool to display the global structure and properties in a more fine-grained manner, and also useful in solving problems that involve finding specific structures in a graph. There are several common types of graph decomposition techniques that are widely used in graph theory and related fields, including tree decomposition, block decomposition, modular decomposition, hierarchical decomposition, etc. This thesis studies two kinds of vertex decomposition of a graph: proper colourings (decomposition into independent sets) and Hamilton-connectivity (decomposition into internally-disjoint paths between two sets where the paths cover all the vertices of graphs).

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Résumé

La décomposition des graphes fait référence au processus de décomposer un graphe complexe en composantes plus simples et plus petites, souvent dans le but d'analyser ou de résoudre des problèmes liés au graphe. Il s'agit d'un outil important pour représenter la structure globale et les propriétés d'une manière plus détaillée. Il est aussi également utile pour résoudre des problèmes impliquant la recherche de structures spécifiques dans un graphe. Parmi les nombreuses applications possibles, nous pourrions mentionner les problèmes d'enracinement, un réseau où plusieurs paires source-destination doivent communiquer. Pour garantir la résilience de la communication en cas de défaillance d'un nœud le long d'un lien, nous devons maintenir des liens de communication entre plusieurs paires source-destination. L'objectif dans un graphe est de trouver des chemins intérieurement disjoints entre ces paires, où chaque chemin devrait être unique et ne partager aucun nœud commun avec un autre chemin. Ainsi, si un chemin devient déconnecté dans une paire source-destination, nous pouvons trouver un nouveau chemin pour les relier. Il existe plusieurs types courants de techniques de décomposition de graphe largement utilisées en théorie des graphes et dans des domaines connexes, notamment la décomposition en arbres, la décomposition en blocs, la décomposition modulaire, la décomposition hiérarchique, etc. Cette thèse étudie deux types de décomposition de sommets d'un graphe : les colorations propres (décomposition en ensembles indépendants) et la Hamilton-connectivité (décomposition en chemins intérieurement disjoints entre deux ensembles où les chemins couvrent tous les sommets du graphe).

Dans le deuxième chapitre, nous contribuons aux extensions des colorations de sommets dans les graphes planaires. Les colorations par liste sont une généralisation des colorations propres, où chaque sommet choisit sa couleur dans une liste privée de couleurs autorisées [60]. En 2008, Hutchinson [59] a étudié le problème d'extension d'un problème de coloration de degré-liste sur des graphes planaires extérieurs. Elle a montré que si un graphe planaire extérieur biparti 2-connexe G a une liste de couleurs $L(v)$ pour chaque sommet v avec

$|L(v)| \geq \min\{\deg_G(v), 4\}$, alors G est L -colorable ; et si un graphe extérieur biparti maximal G avec au moins quatre sommets a une liste de couleurs $L(v)$ pour chaque sommet v avec $|L(v)| \geq \min\{\deg_G(v), 5\}$, alors G est L -colorable. Elle a également montré que les bornes inférieures sont optimales au sens où la borne inférieure 4 pour les graphes planaires extérieurs bipartis 2-connexes (resp. 5 pour les graphes planaires extérieurs maximaux) ne peuvent pas être remplacées par 3 (resp. 4). Nous visons à déterminer si ces bornes sont valables pour les DP-colorations, une généralisation des colorations par liste. Les DP-colorations ont été introduites par Dvořák et Postle [38] pour résoudre une conjecture sur les colorations par liste. Une fois qu'elles ont été proposées, elles ont attiré l'attention de nombreux chercheurs. Il y a une quantité significative de recherches visant à étendre les résultats des colorations par liste aux DP-colorations. Nos résultats indiquent que la première borne des résultats de Hutchinson n'est pas suffisante pour les DP-colorations. Cependant, la deuxième borne est suffisante pour tous les graphes planaires extérieurs 2-connexes.

Dans le troisième chapitre, nous nous intéressons aux colorations fractionnaires des graphes sans triangles avec un degré maximal au plus d , en mettant l'accent sur le cas $d = 4$. Nous désignons par $\chi_f(d, K_3)$ le supremum du nombre chromatique fractionnaire sur tous les graphes sans K_3 avec un degré maximal au plus d . Il a été établi par Dvořák, Sereni et Volec [39] que $\chi_f(3, K_3) = \frac{14}{5}$. Pour $d \in [16] \setminus \{3\}$, il existe une borne supérieure $\chi_f(d, K_3) \leq \frac{d+3}{2}$, qui découle de la borne fractionnaire de Reed, établie par Molloy et Reed [82]. On pense que cette borne n'est pas optimale pour tout $d \geq 3$. Pour $d \geq 17$, Pirot et Sereni [88] ont utilisé des distributions hardcore sur les ensembles indépendants des graphes sans triangles pour obtenir les meilleures bornes supérieures connues pour $\chi_f(d, K_3)$. Dans le cas $d = 4$, Jones [62] a prouvé que pour tout graphe sans triangle G de degré maximal 4 avec n sommets, la taille des ensembles indépendants maximum est au moins $\frac{4n}{13}$. Cela implique que $\chi_f(4, K_3) \leq \frac{13}{4}$ si nous restreignons sa définition à la classe des graphes vertex-transitifs. En général, $\chi_f(4, K_3)$ se situe entre 3.25 et 3.5 (selon la borne fractionnaire de Reed). Nous utilisons la méthodologie introduite par Pirot et Sereni [88] en utilisant des distributions de probabilités mixtes afin de prouver que $\chi_f(4, K_3) < 3.4663$.

Dans le quatrième chapitre, nous étudions une généralisation des problèmes

de connexion Hamiltonienne. Un graphe G est k -fan-connexe si pour chaque sommet $v \in V(G)$ et chaque sous-ensemble de sommets $U = \{u_1, u_2, \dots, u_k\}$ de $V(G) \setminus \{v\}$, il existe un ensemble de chemins intérieurement disjoints P_1, P_2, \dots, P_k tels que P_i est un chemin reliant v et u_i pour $1 \leq i \leq k$ et ces chemins couvrent tous les sommets de G . Cette notion a été introduite pour la première fois par Lin, Tan, Hsu et Hsu [76], qui est un concept de type Menger similaire à la connectivité englobante d'un graphe. Récemment, il y a eu un intérêt pour l'étude des conditions suffisantes pour les graphes k -fan-connexes. Inspirés par cette définition, nous proposons les graphes (k_1, k_2) -Hamilton-connexes. Un graphe G est appelé (k_1, k_2) -Hamilton-connexe, si pour tout deux sous-ensembles disjoints de sommets $X = \{x_1, x_2, \dots, x_{k_1}\}$ et $U = \{u_1, u_2, \dots, u_{k_2}\}$, il existe $k_1 k_2$ chemins intérieurement disjoints reliant x_i à u_j pour $1 \leq i \leq k_1$ et $1 \leq j \leq k_2$, qui couvrent tous les sommets de G . Il est évident que la $(1, k_2)$ -Hamilton-connexité est équivalente à la k_2 -fan-connexité. Soit $\sigma_2(G)$ la valeur minimale de $\deg(u) + \deg(v)$ sur toutes les paires $\{u, v\}$ de sommets non adjacents dans G . Nous prouvons qu'un graphe G à n sommets est $(2, k)$ -Hamilton-connexe si G est $(5k-4)$ -connexe avec $\sigma_2(G) \geq n + k - 2$ où $k \geq 2$. Nous prouvons également que si $\sigma_2(G) \geq n + k_1 k_2 - 2$ avec $k_1, k_2 \geq 2$, alors G est (k_1, k_2) -Hamilton-connexe. De plus, nous construisons deux graphes pour montrer que nos résultats sont optimaux.

Nous concluons cette thèse dans le Chapitre 5 et présentons quelques perspectives.

Abstract

The decomposition of graphs refers to the process of breaking down a complex graph into simpler, smaller components, often with the goal of analysing or solving problems related to the graph. It is an important tool to display the global structure and properties in a more fine-grained manner, and also useful in solving problems that involve finding specific structures in a graph. Among the many possible applications, we could mention routing problems, a network where multiple source-destination pairs need to communicate. To ensure the resilience of the communication in the case that a node fails along one link, we must maintain communication links within several source-destination pairs. The objective in a graph is to find internally-disjoint paths between these pairs where each path should be unique and share no common nodes with any other path. Thus, if some path becomes disconnected in a source-destination pair, we can find a new path to connect them. There are several common types of graph decomposition techniques that are widely used in graph theory and related fields, including tree decomposition, block decomposition, modular decomposition, hierarchical decomposition, etc. This thesis studies two kinds of vertex decomposition of a graph: proper colourings (decomposition into independent sets) and Hamilton-connectivity (decomposition into internally-disjoint paths between two sets where the paths cover all the vertices of graphs).

In the second chapter, we contribute to extensions of vertex colourings in planar graphs. List colourings are a generalisation of proper colourings, where every vertex picks its colour from a private list of allowed ones [60]. In 2008, Hutchinson [59] studied a degree-choosability problem on outerplanar graphs. She showed that if a 2-connected bipartite outerplanar graph G has a list of colours $L(v)$ for each vertex v with $|L(v)| \geq \min\{\deg_G(v), 4\}$, then G is L -colourable; and if a maximal outerplanar graph G with at least four vertices has a list of colours $L(v)$ for each vertex v with $|L(v)| \geq \min\{\deg_G(v), 5\}$, then G is L -colourable. She also showed the lower bounds are sharp in the sense that the lower bound 4 for 2-connected bipartite outerplanar graphs (resp. 5 for maximal outerplanar graph)

cannot be replaced by 3 (resp. 4). We aim to determine whether these bounds hold for DP-colourings, a generalisation of list colourings. DP-colourings were introduced by Dvořák and Postle [38] to solve a conjecture about list colourings. Once they were put forward, they caught the attention of many researchers. There is a significant amount of research to extend the results of list colourings to DP-colourings. Our findings indicate that the first bound of Hutchinson's results is not sufficient for DP-colourings. However, the second bound is sufficient for all 2-connected outerplanar graphs.

In the third chapter, we are interested in fractional colourings of triangle-free graphs of maximum degree at most d with a specific focus on the case $d = 4$. We denote $\chi_f(d, K_3)$ the supremum of the fractional chromatic number over all K_3 -free graphs of maximum degree at most d . It has been settled by Dvořák, Sereni, and Volec [39] that $\chi_f(3, K_3) = 14/5$. For $d \in [16] \setminus \{3\}$, there is an upper bound $\chi_f(d, K_3) \leq \frac{d+3}{2}$, which follows from the fractional Reed bound, established by Molloy and Reed [82]. It is believed that this bound is not tight for any $d \geq 3$. For $d \geq 17$, Pirot and Sereni [88] have used hard-core distributions on the independent sets of triangle-free graphs in order to derive the best-known upper bounds for $\chi_f(d, K_3)$. For the case $d = 4$, Jones [62] proved that for every triangle-free graph G of maximum degree 4 on n vertices, the size of maximum independent sets is at least $\frac{4n}{13}$. This implies that $\chi_f(4, K_3) \leq \frac{13}{4}$ if we restrict its definition to the class of vertex-transitive graphs. In general, $\chi_f(4, K_3)$ lies between 3.25 and 3.5 (by the fractional Reed bound). We use the methodology introduced by Pirot and Sereni [88] together with mixed probability distributions in order to prove that $\chi_f(4, K_3) < 3.4663$.

In the fourth chapter, we investigate a generalisation of Hamilton-connected problems. A graph G is k -fan-connected if for every vertex $v \in V(G)$ and every vertex subset $U = \{u_1, u_2, \dots, u_k\}$ of $V(G) \setminus \{v\}$, there is a set of internally-disjoint paths P_1, P_2, \dots, P_k such that P_i is a path connecting v and u_i for $1 \leq i \leq k$ and these paths cover all the vertices of G . This notion was first introduced by Lin, Tan, Hsu, and Hsu [76], which is a Menger-type concept similar to the spanning connectivity of a graph. Recently, there has been an interest in studying the sufficient conditions for k -fan-connected graphs. Inspired by this definition, we propose (k_1, k_2) -Hamilton-connected graphs. A graph G

is called (k_1, k_2) -Hamilton-connected, if for any two disjoint vertex subsets $X = \{x_1, x_2, \dots, x_{k_1}\}$ and $U = \{u_1, u_2, \dots, u_{k_2}\}$, there are $k_1 k_2$ internally-disjoint paths connecting x_i to u_j for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, which cover all the vertices of G . It is obvious that $(1, k_2)$ -Hamilton-connectedness is equivalent to k_2 -fan connectedness. Let $\sigma_2(G)$ be the minimum value of $\deg(u) + \deg(v)$ over all pairs $\{u, v\}$ of non-adjacent vertices in G . We prove that an n -vertex graph G is $(2, k)$ -Hamilton-connected if G is $(5k - 4)$ -connected with $\sigma_2(G) \geq n + k - 2$ where $k \geq 2$. We also prove that if $\sigma_2(G) \geq n + k_1 k_2 - 2$ with $k_1, k_2 \geq 2$, then G is (k_1, k_2) -Hamilton-connected. Moreover, we construct two graphs to show that our results are sharp.

We conclude this thesis in Chapter 5 and give some prospects.

1 - Introduction

Graphs are mathematical abstractions that represent any network that might possibly exist in the real world. The concept of graph theory started with the Königsberg Bridge's problem in 1735 for the first time in history [12]. It is related to various domains of mathematics, including algebra, topology, probabilities, and extremal combinatorics. For instance, Extremal Graph Theory focuses on finding graphs with extreme properties or determining bounds on certain graph parameters, and probabilistic graph theory investigates random processes on graphs, such as random walks. The probabilistic method is a technique for proving the existence of objects with specific properties by studying the typical behaviour of a random one. Graph Theory is rich in many problems of practical and/or theoretical interest; among the most classical ones we can mention: Shortest Path Problem, Minimum Spanning Tree Problem, Eulerian Path (Cycle) Problem, Graph Isomorphism and Homomorphism, Graph Colouring Problem and Hamilton-connectivity Problem that we are interested in, and so on. On the other hand, graphs are extensively used in many disciplines. For example, we can non-exhaustively mention: anatomy (neural circuit), biology (protein interaction network), chemistry (crystal structures), computer sciences (web, peer-to-peer networks), artificial intelligence (artificial neural network), statistics (Bayesian network), electricity (electrical grid), telecom (telecommunication network), transportation (road network, rail network), and urbanism (gas network, water distribution network).

In this thesis, we mainly study some vertex colouring problems and a generalisation of Hamilton-connectivity in graphs. In addition to mathematics, the colouring problem has a remarkable range of applications in other fields of science. All those seemingly unrelated topics — the efficiency of a computer processor, wireless communication networks, the number of tracks required in a railway station, and the number of frequencies that our smartphones should be able to capture — depend on a colouring problem. Hamilton-connectivity also has many applications in different fields, including network design and fault tolerance, DNA sequencing, robotics and automated path planning, and circuit design.

We introduce the fundamentals of graph theory in the first section of this chapter to make it easier to describe the context of our research topic. In the second section, we present the background and main classic results on proper vertex colourings, including variants of the chromatic number. We introduce the background and some insightful findings that will be related to our studies on Hamilton-connectivity in the third section. The final section will describe the contributions of the thesis and the outline of the manuscript.

1.1 . The Basics

Graphs in this thesis are simple unless otherwise stated. We follow Bondy and Murty [15] for undefined terms and notations in graph theory.

1.1.1 . Properties of planar graphs

A graph G is *planar* if G can be drawn in the plane with no pair of edges crossing. Such a drawing is called an *embedding* of G on the plane. Graphs already embedded on the plane are called *plane* graphs. *Faces* of a plane graph are regions bounded by a set of edges and which contain no other vertex or edge. The *degree* of a face f , denoted $\deg(f)$, is the number of edges along its boundary. Alternatively, it is the number of vertices along its boundary. Every edge in a planar graph is shared by exactly two faces.

In a plane graph, there is precisely one face that is unbounded; we call this the *infinite*, or *outer* face (sometimes one calls it external face), and all others are *finite* faces. An edge is called *external* (respectively, *internal*) when it lies (resp., does not lie) on the boundary of the external face.

A planar graph is called *outerplanar* if the graph can be embedded on the plane in such a way that all vertices of the graph lie on the outer face.

Planar graphs play an essential role in a variety of applications, making them an intriguing class of graphs to investigate. These graphs have specific properties that follow mainly from Euler's Formula, which establishes a relationship between the number of vertices, edges, and faces in a plane graph.

Theorem 1 (Euler's Formula) *Let G be a connected plane graph on n*

vertices, m edges, and f faces. Then

$$n + f - m = 2.$$

There are many applications to this formula, the most notorious one being certainly the Discharging Method. Although we do not rely on it in this thesis, it is one of the most efficient tools to derive results on planar graphs. We will now deduce the following simple result from Euler's Formula.

Lemma 1 *Every planar graph on $n \geq 3$ vertices has at most $3n - 6$ edges.*

Proof. Let G be a connected planar graph (otherwise we could add edges to make the graph connected). Let m be the number of edges and f be the number of faces of G . By Euler's Formula, we have $f = 2 - n + m$. A simple counting argument shows that $\sum_i \deg(f_i) = 2m$ where $\{f_i\}_i$ are the faces of G . Note that we count a bridge twice at every face with which they are incident. Since the number of edges in the boundary of each face is at least 3, we have $f \leq \frac{2m}{3}$. Thus $m \leq 3n - 6$, as desired. \square

As we know, the sum of the degrees of the vertices of a graph is precisely twice the number of edges of that graph. Thus, it follows from Lemma 1 that any planar graph has average degree less than 6. In particular:

Corollary 1 *Every planar graph has a vertex of degree at most 5.*

The *girth* of G , denoted $\text{girth}(G)$, is the size of a smallest cycle in G . More generally, the average degree of a planar graph is bounded by a decreasing function of its girth as follows.

Theorem 2 [89] *The average degree of a planar graph of girth at least g is less than $\frac{2g}{g-2}$.*

Since being planar is a monotone property, it means that any subgraph of a planar graph is also planar. Hence the degeneracy of a planar graph is at most 5.

Theorem 3 *For every planar graph G ,*

$$\delta^*(G) \leq 5,$$

and if moreover G is triangle-free (and so of girth $g \geq 4$),

$$\delta^*(G) \leq 3.$$

Recall that a planar graph is said to be outerplanar if it has a planar drawing in which all vertices lie on the outer face. For each outerplanar graph, we can find a vertex of degree at most 2.

Lemma 2 *Every edge-maximal outerplanar graph G on $n \geq 3$ contains a vertex of degree 2. In particular, every outerplanar graph has a vertex of degree at most 2.*

Proof. Let G be an edge-maximal outerplanar graph. We first claim that G is 2-connected. Toward a contradiction, suppose that G is not 2-connected. Let us take an outerplanar embedding of G . If G is not connected, then we could draw a non-crossing curve between any two vertices u and v in different components of G and still every vertex of G would be on the boundary of the outer face. So the graph $G + uv$ would be outerplanar, which contradicts that G is edge-maximal. If the connectivity of G is 1, then let v be a cut-vertex and let C_1 and C_2 be two components in $G - v$. Let $u_1 \in C_1$ and $u_2 \in C_2$ be neighbours of v , such that the drawing of the edges u_1v and vu_2 are leaving v right after each other when we go around v in a very small cycle. Two such neighbours certainly exist as v has neighbours both in C_1 and C_2 . Now it is possible to draw a non-crossing curve between u_1 and u_2 by following closely first the drawing of the edge u_1v and then the drawing of the edge vu_2 . This curve will close a triangle with the curves u_1v and vu_2 with no vertex in its interior, so the obtained drawing of $G + u_1u_2$ is outerplanar, contradicting our assumption of G . Hence G is 2-connected.

If we consider any planar drawing of G in which all vertices lie on the outer face, the outer face is a cycle (without repeated vertices), and all internal faces are triangles. Indeed, we assume there exists a face f_i and $\deg(f_i) > 3$. We claim that there exist two non-adjacent vertices v_j and v_k of f_i . Otherwise, the edge v_jv_k must be outside of f_i ; it contradicts the outerplanarity of G . So we can add the edge v_jv_k

to G without losing outerplanarity. It contradicts the maximality of G . A *chord* is an edge between two non-consecutive vertices of a given cycle. We consider a shortest chord xy of the outer face, the distance of x and y on the outer face is at least 2 because G is simple. So any vertex between x and y on the shortest path from x to y on the outer face must have degree 2.

□

Since outerplanar graphs form a hereditary family, this implies that outerplanar graphs are 2-degenerate.

1.1.2 . Connectivity in graphs

The basic result in the theory of connectivity was proved by Menger in 1927. Two s - t paths are *independent* if they have only the vertices s and t in common. We also say that the two s - t paths are *internally-disjoint paths*.

Theorem 4 (Menger, 1927 [79])

1. Let s and t be distinct nonadjacent vertices of a graph G . Then the minimal number of vertices separating s from t is equal to the maximal number of independent s - t paths.
2. Let s and t be distinct vertices of G . Then the minimal number of edges separating s from t is equal to the maximal number of edge-disjoint s - t paths.

It is often the case that Menger's Theorem needs to be applied as another version, the following corollary.

Corollary 2 For $k \geq 2$, a graph is k -connected if and only if it has at least two vertices and any two vertices can be joined by k independent paths. Also, for $k \geq 2$, a graph is k -edge-connected if and only if it has at least two vertices and any two vertices can be joined by k edge-disjoint paths.

A path is called a *Hamiltonian path* if it visits each vertex of G exactly once. If for every pair of vertices in G there is a Hamiltonian path between the two vertices, we say that G is *Hamilton-connected*.

A cycle is called a *Hamiltonian cycle* if it visits each vertex of G exactly once. If G has a Hamiltonian cycle, then G is a *Hamiltonian*.

1.2 . Colourings

The Four Colour Theorem is one of the most captivating and challenging episodes in the history of mathematics. It spans over a century and involves the contributions of numerous mathematicians and computer scientists. The origins of the Four Colour Theorem can be traced back to 1852 when Francis Guthrie, a mathematics student, posed the conjecture to his brother Frederick Guthrie. He noticed that he could colour the counties of England on a map using only four colours in such a way that no two adjacent counties shared the same colour. This observation led to the formulation of the Four Colour Conjecture. Many tried to prove or disprove this conjecture, but it remained unsolved for decades. In 1879, Alfred Kempe presented a proof, but it turned out to be flawed, as Percy Heawood pointed out in 1890. It took more than a century before a valid proof was established by Kenneth Appel and Wolfgang Haken from the University of Illinois, using the discharge method in 1976. Their groundbreaking proof relied heavily on computer analysis and verification, dealing with an exhaustive enumeration of numerous cases. The complexity of their proof sparked controversy, as some mathematicians were sceptical about the role of computers in mathematical proofs. In addition to the computer-checked parts, it contains a part that needs to be checked by a lengthy manual case analysis, and the researchers trying to do so actually found flaws (though likely fixable ones). In 1997, the first universally accepted proof of Four Colour Theorem is the one by Robertson, Sanders, Seymour, and Thomas [93]. They announced another proof, still using a computer, but simpler than Appel and Haken's in several respects. However, to this date, mathematicians still continue to work on the Four Colour Theorem, aiming to find more intuitive and concise proofs that require less reliance on computer verification.

The Four Colour Theorem is just one of the many challenges in the field of graph colouring, which deals with finding the optimal way to colour the vertices of a graph so that no two adjacent vertices have the same colour — it is known as a proper colouring. The minimum number of colours required for a proper

colouring of a graph is called its chromatic number, and the colouring problem is the process of determining the chromatic number of a given graph. Graph colouring has numerous applications across various fields. For example, graph colouring is used to schedule and assign time slots to events or activities in various domains such as schools, universities, conferences, and sports tournaments. Each event is represented as a vertex, and conflicts between events are represented as edges. By assigning distinct colours (time slots) to conflicting events, a conflict-free schedule can be generated.

However, the colouring problem is notoriously challenging, as it is NP-hard to solve or even approach with a non-optimal solution. This means that it is highly unlikely that we will be able to devise an effective method for determining the optimal colouring of any given graph. For this reason, a significant portion of graph colouring research focuses on finding large classes of graphs for which either the colouring problem is less difficult, or the chromatic number is substantially lower.

1.2.1 . Presentation and definition

We now introduce the definitions of some vertex colourings and their related basic properties.

Proper k -colourings

Definition 1. Let G be a graph.

1. A k -colouring of G is a function $c: V(G) \rightarrow [k]$ where $[k] = \{1, 2, \dots, k\}$. This can be seen as a function which associates to every vertex $v \in V(G)$ a colour among a palette of k possible ones. A *partial colouring* of G is a colouring of an induced subgraph of G .
2. A colouring c of G is *proper* if no two vertices of the same colour are adjacent;

$$\forall uv \in E(G), \quad c(u) \neq c(v).$$

When there exists a proper k -colouring of G , we say that G is k -colourable.

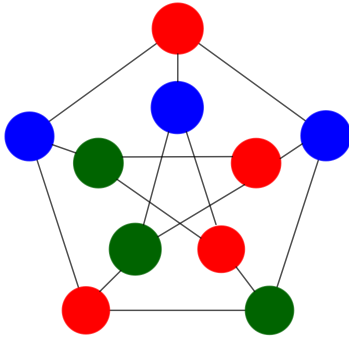


Figure 1.1: A proper 3-colouring of the Petersen graph

3. The *chromatic number* of G , denoted by $\chi(G)$, is the minimum k such that G is k -colourable.
4. For every colour $i \in [k]$ used in some k -colouring c of G , the subset of vertices $c^{-1}(\{i\})$ coloured with colour i is called a *colour class*, or *monochromatic class* of c . The colour classes of c yield a partition of $V(G)$, and if c is proper, every colour class is an independent set of G .

A graph G is k -colourable if and only if it is a subgraph of the complete k -partite graph $K_{k \times n}$. In particular, the class of bipartite graphs is exactly the class of 2-colourable graphs. Being k -colourable is a hereditary property;

$$\forall H \subseteq G, \quad \chi(H) \leq \chi(G).$$

There are many equivalent definitions of a proper k -colouring of a graph G . Here, we present a definition based on a linear program. The chromatic number $\chi(G)$ of a graph G is naturally defined as the solution of an integer linear program with one variable w_I for every of its independent sets $I \in \mathcal{I}(G)$.

$$\chi(G) = \min \sum_{I \in \mathcal{I}(G)} w_I,$$

$$\text{such that } \begin{cases} w_I \in \{0, 1\} & \text{for each } I \in \mathcal{I}(G) \\ \sum_{\substack{I \in \mathcal{I}(G) \\ v \in I}} w_I \geq 1 & \text{for each } v \in V(G). \end{cases} \quad (1.1)$$

List colourings

To tackle some graph colouring problems, Vizing [105], and independently Erdős, Rubin, and Taylor [40] introduced list colourings, a generalisation of proper colourings. In a k -colouring, each vertex picks a colour from a common set of colours $[k]$. Now we consider a generalisation of k -colourings to make each vertex choose its colour from a private list of allowed colours. These are list colourings.

Definition 2. Let G be a graph.

1. A *list assignment* L with a graph G is a mapping that maps each vertex v of G to a list of integers $L(v)$. A list assignment L is called a *t -list assignment* if $|L(v)| = t$ for every $v \in V(G)$. A list assignment L is called a *degree-list assignment* if $|L(v)| = \deg_G(v)$ for every $v \in V(G)$.
2. A graph G is *L -coloured* if there exists a proper colouring c of G such that

$$\forall v \in V(G), \quad c(v) \in L(v).$$
3. A graph G is *k -choosable* if G is L -coloured for every k -list assignment L .
4. The *choice number*, or *list chromatic number*, is the minimum k such that G is k -choosable. It is denoted $\chi_\ell(G)$.
5. A graph G is said to be *degree-choosable* if G is L -coloured for every degree-list assignment L .

Remark 1. The graph G has a proper k -colouring if and only if G has an L -colouring with $L(v) = [k]$. Hence we have

$$\forall G, \quad \chi_\ell(G) \geq \chi(G).$$

DP-colourings

List colourings helped establish a number of proper colouring results; however, it is usually much more difficult to establish upper bounds on the choice number than on the chromatic number. The methods used in proper colourings may not directly apply to list colourings. For instance, the operation of identifying vertices, which is common in proper colourings, may not be feasible in list colourings. Let

us illustrate this with an example of a proof of the Five Colour Theorem on planar graphs. We first observe that G contains a vertex v of degree at most 5. If $\deg(v) \leq 4$, then every 5-colouring of $G - v$ extends to a 5-colouring of G . If $\deg(v) = 5$, then we find two non-adjacent neighbours x and y of v , and let G' be the graph obtained from $G - v$ by identifying x and y to a new vertex w . Given a 5-colouring ϕ of G' , we then obtain a 5-colouring of G by giving both x and y the colour $\phi(w)$ and choosing a colour for v distinct from the colours of its neighbours. But in list colourings, vertex identification as a strategy is impractical, since different vertices can have various lists of available colours in list colourings. To solve this issue, and allow themselves to use vertex identification in the context of list colourings, Dvořák and Postle [38] have introduced the notion of correspondence colourings (also referred to as DP-colourings).

As we all know, a proper k -colouring of G is isomorphic to a (maximum) independent set of size $|V(G)|$ of the Cartesian product $G \square K_k$. List colourings are to find such an independent set in the setting of monochromatic matchings between two adjacent list assignments. While the idea behind correspondence colourings is to forbid more configurations, it is to find an independent set of size $n(G)$ for any matched colours between two adjacent list assignments. The lists do not matter for correspondence colourings (as long as all vertices use the same number of colours). Note that when performing vertex identification for DP-colourings, it is in general necessary to preserve the parallel edges if they arise.

Here we present the definition of DP-colourings in simple graphs from BernshTEYN, Kostochka, Pron [8]. DP-colourings in multigraphs will be described in Chapter 2.

Definition 3. Let G be a graph. A *cover* of G is a pair (L, H) , where L is a list assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\cup_{v \in V(G)} L(v)$, satisfying the following conditions.

1. For each $v \in V(G)$, $H[L(v)]$ is a complete graph.
2. For each $uv \in E(G)$, the edges between $L(u)$ and $L(v)$ form a matching (possibly empty).
3. For each distinct $u, v \in V(G)$ with $uv \notin E(G)$, there are no edges

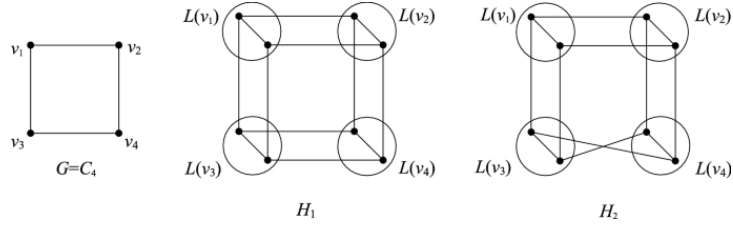


Figure 1.2: C_4 and two covers of it such that C_4 is (L, H_1) -colourable but not (L, H_2) -colourable.

between $L(u)$ and $L(v)$ in H .

Definition 4. Suppose G is a graph and (L, H) is a cover of G . An (L, H) -colouring of G is an independent set $I \subseteq V(H)$ of size $|V(G)|$. In this context, we refer to the vertices of H as the *colours*. G is said to be (L, H) -colourable if it admits an (L, H) -colouring.

Definition 5. Let G be a graph and $f: V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be an assignment of nonnegative integers to the vertices of G . G is DP - f -colourable if it is (L, H) -colourable whenever (L, H) is a cover of G and $|L(v)| \geq f(v)$ for all $v \in V(G)$. If G is DP - \deg_G -colourable, then G is said to be DP -degree-colourable.

Definition 6. The DP -chromatic number, $\chi_{DP}(G)$, is the minimum k such that G is (L, H) -colourable for each choice of (L, H) with $|L(v)| \geq k$ for all $v \in V(G)$.

In DP -colourings, we are mainly concerned with the correspondences between a colour of a vertex and the colours of its adjacent vertices. Hence we can arbitrarily rename the colours while updating the cover as follows. Let (L, H) be a cover for a graph G , and let v be a vertex in G . If $c_1 \in L(v)$ and $c_2 \notin L(v)$, then we can define a cover (L', H') where

$$L'(u) = \begin{cases} (L(v) \setminus \{c_1\}) \cup \{c_2\}, & \text{if } u = v, \\ L(u), & \text{otherwise.} \end{cases}$$

and for each $e \in E(G)$ incident with v , and $H'[L(v)]$ is obtained from $H[L(v)]$ by replacing the vertex c_1 by c_2 . We say (L', H') is obtained from (L, H) by *renaming*

(at a vertex v). We say that two covers (L, H) and (L', H') are *equivalent* if one can be obtained from the other by a sequence of renaming. The following are some useful facts about equivalent covers.

Fact 1 *Let (L, H) and (L', H') be equivalent covers for a graph G . Then G is (L, H) -colourable if and only if it is (L', H') -colourable.*

Fact 2 *Let (L, H) be a cover for a graph G such that for each vertex v , $|L(v)| = k$. Then there exists a cover (L', H') , equivalent to (L, H) , such that L' assigns the list $[k]$ to each vertex.*

So we may assume that each vertex is assigned the list $[k]$ when we study DP-colourings where every list has size k for some integer k . In fact, list colourings are a specific case of DP-colourings. Given a list L for a graph G , the vertex set of its cover $H = H(G, L)$ is $\{(v, c) | v \in V(G) \text{ and } c \in L(v)\}$, and two different vertices (v, c) and (v', c') are adjacent in H if and only if either $c = c'$ and $vv' \in E(G)$, or $v = v'$. The independence number of H is at most $|V(G)|$, since $V(H)$ is covered by $|V(G)|$ cliques. If H has an independent set I with $|I| = |V(G)|$, then for each $v \in V(G)$, there exists a unique $c \in L(v)$ such that $(v, c) \in I$. In addition, the same colour c is not chosen for any two adjacent vertices. So we have $\chi_{DP}(G) \geq \chi_\ell(G)$.

Fractional colourings

The notion of fractional colourings is an important concept in graph theory that is commonly used to extend the notion of graph colouring beyond integer values. It is a relaxation of the traditional chromatic number, allowing for real-valued weights or probabilities associated with each colouring of a graph. Fractional colourings have been extensively studied in combinatorial optimization, operations research, and computer science, among other fields, as they have various applications in scheduling, resource allocation, and network design. Many researchers and mathematicians have contributed to the theory and applications of fractional colourings.

Definition 7. Let G be a graph.

1. A proper $(a : b)$ -colouring of G is a function $c: V(G) \rightarrow \binom{[a]}{b}$ such that

$$\forall uv \in E(G), \quad c(u) \neq c(v).$$

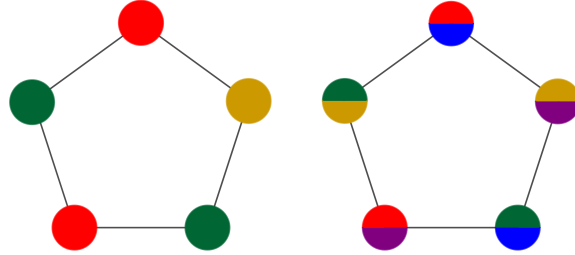


Figure 1.3: The chromatic number and the fractional chromatic number of C_5 , the right figure is also a proper $(5 : 2)$ -colouring of C_5 .

We say that c is a *fractional colouring* of G , of weight a/b .

2. The *fractional chromatic number* of G , denoted $\chi_f(G)$, is defined by

$$\chi_f(G) = \inf \left\{ \frac{a}{b} \mid \text{there exists a proper } (a : b)\text{-colouring of } G \right\}.$$

Remark 2. A proper k -colouring is the same as a proper $(k : 1)$ -colouring. Therefore,

$$\forall G, \quad \chi_f(G) \leq \chi(G).$$

The fractional chromatic number $\chi_f(G)$ can equivalently be defined as the fractional relaxation of the integer linear program (1.1) computing the chromatic number. Let G be a graph, we define $\mathcal{I}(G)$ to be the set of all independent sets of G , and the *fractional chromatic number* $\chi_f(G)$ of G is the solution of the following linear program.

$$\begin{aligned} \chi_f(G) &= \min \sum_{I \in \mathcal{I}(G)} w_I \\ \text{such that } &\begin{cases} w_I \in [0, 1] & \text{for each } I \in \mathcal{I}(G) \\ \sum_{\substack{I \in \mathcal{I}(G) \\ v \in I}} w_I \geq 1 & \text{for each } v \in V(G). \end{cases} \end{aligned} \quad (1.2)$$

A *fractional colouring of weight w* of G is any instance within the domain of the above linear program such that $\sum w_I = w$. This equivalent formulation

implies that the value of the fractional chromatic number is always attained as the weight of some fractional colouring. Therefore, the infimum in Definition 7 can be replaced with a minimum. Specifically, fractional chromatic numbers are always rational.

Fractional colourings can be defined as probability distributions on the independent sets of a graph G . A graph G has a fractional colouring of weight w if and only if there exists a probability distribution on independent sets of G such that

$$\forall v \in V(G), \quad \mathbb{P}[v \in \mathbf{I}] \geq \frac{1}{w}$$

for the independent set \mathbf{I} sampled from this distribution.

1.2.2 . Main classical results on proper vertex colourings

There is almost no hope of finding any algorithm of polynomial complexity computing the chromatic number of any graph, since it is strongly believed by the scientific community that $P \neq NP$, while the k -colouring problem is NP-complete. As the exact computation of the chromatic number of a given graph G appears infeasible in the general case, the initial step would be to bound it in terms of other parameters of G .

Naive bounds for the chromatic number

We first present two lower bounds of $\chi(G)$ that are sufficient in many applications to estimate the chromatic number of classical graphs, even though the chromatic number can be arbitrarily larger than both. We know that $\chi(K_n) = n$. As a result, any graph G that contains K_n as a subgraph must require at least n colours in a proper colouring;

$$\forall G, \quad \chi(G) \geq \omega(G).$$

A proper k -colouring of G is also a partition of $V(G)$ into k independent sets. Since the size of each independent set is at most $\alpha(G)$, we have $k \cdot \alpha(G) \geq |V(G)|$;

$$\forall G, \quad \chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

For a given graph G , it is possible to capture both lower bounds $\omega(G)$ and $\frac{|V(G)|}{\alpha(G)}$ of $\chi(G)$ in a stronger lower bound. The *Hall ratio* of G , denoted $\rho(G)$, and

defined by

$$\rho(G) := \max \left\{ \frac{|V(H)|}{\alpha(H)} \mid H \subseteq G \right\} \geq \max \left\{ \frac{|V(G)|}{\alpha(G)}, \omega(G) \right\}.$$

Since being k -colourable is a hereditary property, it holds that

$$\forall G, \quad \chi(G) \geq \rho(G).$$

For upper bounds, it is easy to bound from above the chromatic number of a graph G in terms of its maximum degree by considering a greedy colouring algorithm. Let G be a graph, and we order the vertices in $V(G)$ arbitrarily. We colour the vertices of G sequentially, assigning the smaller colour which does not appear in the neighbourhood of the considered vertex at each step when there exists one, or adding a new colour to the colouring otherwise. Since $|N_G(v)| \leq \Delta(G)$ where $\Delta(G)$ is the maximum degree of G , we have $\chi(G) \leq \Delta(G) + 1$ for every graph G .

Main classical results for the chromatic number

A well-known objective in graph colouring is to establish adequate conditions that can yield much-improved upper bounds on the chromatic number compared to the straightforward ones. Over the past few decades, considerable research has been conducted in this field, and presented below are some of the key findings in this domain.

Theorem 5 (Four Colour Theorem [92]) *For every planar graph G ,*

$$\chi(G) \leq 4.$$

Interestingly, the problem of deciding whether a planar graph is 3-colourable is NP-complete, as proved by Garey, Johnson and Stockmeyer [47]. Therefore, it seems wise to discuss some sufficient conditions for planar graphs to be 3-colourable. In 1959, Grötzsch's [51] proved the following theorem.

Theorem 6 (Grötzsch's theorem, 1959 [51]) *For every planar triangle-free graph G ,*

$$\chi(G) \leq 3.$$

In 1976, Steinberg first presented the following conjecture.

Conjecture 1 (Steinberg's conjecture [99]) *Every planar graph without 4-cycles and 5-cycles is 3-colourable.*

This problem has been attracting a substantial amount of attention among graph theorists. It is also one of the six graph theory problems ranked with the 4-star (highest) importance in the Open Problem Garden [1]. As a possible approach towards proving Steinberg's conjecture, Erdős in 1991 (see [99]) suggested determining the smallest k such that every planar graph with no cycles of length $4, \dots, k$ is 3-colourable. The best known upper bound for such a k is 7, which was proved by Borodin, Glebov, Raspaud and Salavatipour [19].

Theorem 7 (Borodin, Glebov, Raspaud and Salavatipour, 2005 [19]) *Every planar graph with no cycles of length from 4 to 7 is 3-colourable.*

In 2017, Cohen-Addad, Hebdige, Král, Li, and Salgado[27] constructed a planar graph to disprove Steinberg's conjecture. It implies that $k \geq 6$ for Erdős problem.

Theorem 8 (Cohen-Addad, Hebdige, Král, Li, and Salgado, 2017 [27]) *There is a planar graph with no cycles of length 4 or 5 that is not 3-colourable.*

There are also some insightful results about the upper bounds of the chromatic number in general graphs.

Theorem 9 (Bipartite graphs characterisation) *For every graph G ,*

$$\chi(G) \leq 2 \iff G \text{ contains no odd cycle.}$$

Theorem 10 (Brook's theorem [21]) *For every connected graph G , either G is a complete graph or an odd cycle, or*

$$\chi(G) \leq \Delta(G).$$

Theorem 11 (Johansson-Molloy theorem [61, 80]) *For every triangle-free graph G ,*

$$\chi(G) \leq (1 + o(1)) \frac{\Delta(G)}{\ln \Delta(G)}.$$

All these theorems are sharp, since infinite families of graphs satisfying their constraints and asymptotically reaching the upper bound exist for each of them. With the exception of the Johansson-Molloy theorem, the sharpness is precise for all of them, as the upper bound and the value achieved by the known infinite families of graphs coincide. However, in the case of the Johansson-Molloy theorem, an asymptotic multiplicative gap of 2 remains between the upper bound and the value attained by Bollobás [13]. He showed that for any integers g and $\Delta \geq 3$, there exists a graph with a maximum degree of Δ , girth at least g , and chromatic number at least $\frac{\Delta}{2 \ln \Delta}$.

There are still a lot of open problems related to the chromatic number of graphs. Extending Theorem 11 to any H -free graph is one of these issues.

Conjecture 2 (Alon, Krivelevich, Sudakov, 1999 [4]) *Let H be some fixed graph. There exists a constant $C_H > 0$ such that, for every H -free graph G ,*

$$\chi(G) \leq C_H \frac{\Delta(G)}{\ln \Delta(G)}.$$

To prove Conjecture 2, it is enough to demonstrate its validity when H is a complete graph, since any fixed H is a subgraph of $K_{n(H)}$, and thus any H -free graph is also $K_{n(H)}$ -free. This conjecture deals with the chromatic number's asymptotic value for graphs with a fixed clique number as $\Delta \rightarrow \infty$. The following conjecture, Reed conjecture, is relevant to the case where the maximum degree is arbitrarily near to the clique number.

Conjecture 3 (Reed, 1998 [91]) *For every graph G ,*

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.$$

The following represents one of the most notable partial outcomes in support of Reed's conjecture thus far.

Theorem 12 (Hurley, Verclos, and Kang, 2021 [58]) *There is some Δ_0 such that, for every graph G of maximum degree $\Delta(G) \geq \Delta_0$,*

$$\chi(G) \leq \lceil 0.881(\Delta(G) + 1) + 0.119\omega(G) \rceil.$$

1.2.3 . Extensions of proper vertex colourings

One can imagine numerous variants of the colouring problem, depending on the application context. Some are augmented variants of the problem, while others are simplified. In this section, we will discuss the most notorious.

Main results on list colourings

Since G has a proper k -colouring if and only if G has an L -colouring with $L(v) = [k]$, we have $\chi_\ell(G) \geq \chi(G)$ for every graph G . Clearly, every k -choosable graph is k -colourable, but the converse is not true in general. While the chromatic number of a bipartite graph G is at most 2, Erdős, Rubin, and Taylor [95] showed that the choice number of a bipartite graph G may be arbitrarily large by proving that $\chi_\ell(K_{n,n}) = (1 + o(1)) \log_2 n$. On the other hand, some well-known upper bounds on $\chi(G)$ in terms of vertex degrees hold for $\chi_\ell(G)$ as well. For example, Brook's theorem and the degeneracy upper bound hold for $\chi_\ell(G)$. By using a greedy algorithm, we have that:

Theorem 13 *If G is k -degenerate, then $\chi_\ell(G) \leq k + 1 \leq \Delta(G) + 1$.*

Alon proved in 2000 [2] that the choice number can also be bounded from below as a function of the minimum degree $\delta(G)$. Note that it also holds for the average degree.

Theorem 14 (Alon, 2000 [2]) *For every graph G ,*

$$\chi_\ell(G) \geq \left(\frac{1}{2} - o(1)\right) \cdot \log_2(\delta(G)).$$

This result was improved by Saxton and Thomason later.

Theorem 15 (Saxton, Thomason, 2015 [97]) *For every graph G ,*

$$\chi_\ell(G) \geq (1 + o(1)) \cdot \log_2(\delta(G)).$$

Back in 1998, Alon and Krivelevich [3] established that the lower bound stated in Theorem 15 is almost surely asymptotic to the choice number of a random bipartite graph. In the same paper, they proposed the following conjecture.

Conjecture 4 (Alon, Krivelevich, 1998 [3]) *There exists an absolute constant $C > 0$ such that for every bipartite graph G ,*

$$\chi_\ell(G) \leq C \ln \Delta(G).$$

While this conjecture asks a very natural question about a fundamental property of a highly studied graph class, surprisingly, researchers have made very little progress toward an answer.

In 2019, Molloy [81] further improved the celebrated result of Johansson [61] which extended Theorem 11 to list colourings.

Theorem 16 (Molloy, 2019 [81]) *For every triangle-free graph G ,*

$$\chi_\ell(G) \leq (1 + o(1)) \frac{\Delta(G)}{\ln \Delta(G)}.$$

It implies that the upper bound of the choice number of a bipartite graph is $(1 + o(1)) \frac{\Delta}{\ln \Delta}$. Recently, Bradshaw improved this upper bound of bipartite graphs in his thesis.

Theorem 17 (Bradshaw, 2022 [20]) *If G is a bipartite graph of sufficiently large maximum degree $\Delta(G)$, then,*

$$\chi_\ell(G) < 0.797 \frac{\Delta}{\ln \Delta}.$$

Theorem 17 makes only a modest improvement to the coefficient of the previously known upper bound and is still far away from the conjectured bound of $O(\ln \Delta)$. Whereas for planar graphs, the issue has been conclusively resolved. In 1994, Thomassen proved a version of the Four Colour Theorem for list colourings.

Theorem 18 (Thomassen, 1994 [101]) *For every planar graph G ,*

$$\chi_\ell(G) \leq 5.$$

Voigt [106] constructed a planar graph that needs more than four colours to be properly coloured, proving that not all planar graphs are 4-choosable.

Theorem 19 (Voigt, 1993 [106]) *There exist planar graphs which are not 4-choosable.*

Moreover, Gutner [52] established that the problem of determining whether a planar graph is 4-choosable is NP-hard. Thomassen [102] later gave a nice sufficient condition for a planar graph to be 3-choosable.

Theorem 20 (Thomassen, 1995 [102]) *For every planar graph G of girth at least 5,*

$$\chi_\ell(G) \leq 3.$$

The intermediate result between Theorem 18 and Theorem 20 pertains to the choice number of planar graphs without triangles. The conclusion is at most 4, since the degeneracy of triangle-free planar graphs is at most 3.

Main results on DP-colourings

Inspired by Steinberg's conjecture, Borodin [18] noted that it has remained open since 1996 whether every planar graph without cycles of lengths 4 to 8 is 3-choosable.

Conjecture 5 (Borodin, 2013 [18]) *Every planar graph without cycles of lengths 4 to 8 is 3-choosable.*

In order to prove such an upper bound for a class of planar graphs, Dvořák and Postle [38] proposed and heavily used a new tool — DP-colourings. They used vertex identification to deal with a main reduction of their proof. The lists do not matter for DP-colourings (as long as all vertices use the same number of colours). So they can assign every vertex the same list in order to perform vertex identification which is usually not possible in a list colouring settings.

Theorem 21 (Dvořák, Postle, 2018 [38]) *Every planar graph G without cycles of lengths 4 to 8 is 3-choosable.*

As mentioned above, we have $\chi_{DP}(G) \geq \chi_\ell(G)$ for every graph G . Some upper bounds on the choice number hold for the DP-chromatic number as well. For example, Dvořák and Postle observed that $\chi_{DP}(G) \leq k+1$ for every k -degenerate graph G . Indeed, we may choose a colour for each vertex v greedily from $L(v)$ avoiding the colours adjacent in its cover H to the colours already chosen for neighbours of v . Dvořák and Postle [38] noted that Theorem 18 and Theorem 20 immediately extend to DP-colourings.

Theorem 22 (Dvořák, Postle, 2018 [38]) *For every planar graph G ,*

$$\chi_{DP}(G) \leq 5.$$

Theorem 23 (Dvořák, Postle, 2018 [38]) *For every planar graph G of girth at least 5,*

$$\chi_{DP}(G) \leq 3.$$

In 2019, Bernshteyn verified Theorem 16 in the context of DP-colourings.

Theorem 24 (Bernshteyn, 2019 [10]) *For every triangle-free graph G ,*

$$\chi_{DP}(G) \leq (1 + o(1)) \frac{\Delta(G)}{\ln \Delta(G)}.$$

On the other hand, DP-colourings and list colourings are strikingly different. Recall that due to a celebrated result of Alon [2], the choice number of the graphs with average degree d is $\Omega(\log d)$, and this bound is sharp for “small” bipartite graphs. However, Bernshteyn proved that the DP-chromatic number of such graphs is close to linear in d .

Theorem 25 (Bernshteyn, 2016 [9]) *For every graph with average degree d ,*

$$\chi_{DP}(G) = \Omega\left(\frac{d}{\ln d}\right).$$

Important tools in the study of list colourings that do not generalise to the framework of DP-colourings are the orientation theorems of Alon and Tarsi [5] and the closely related Bondy–Boppana–Siegel lemma (see [5]). Indeed, they can be used to prove that even cycles are 2-choosable, while the DP-chromatic number of any cycle is 3, regardless of its length.

Theorem 26 (Alon, Tarsi, 1992 [5]) *Every planar bipartite graph is 3-choosable.*

However, Bernshteyn and Kostochka [11] showed that Theorem 26 does not hold for DP-colourings.

Theorem 27 (Bernshteyn, Kostochka, 2019 [11]) *There exists a bipartite planar graph G with*

$$\chi_{DP}(G) = 4.$$

Main results on fractional colourings

Firstly, let us analyse some observations about the bounds of the fractional chromatic number. As we mentioned previously, a k -colouring of graph G can be viewed as a specific type of fractional colouring of weight k of G , where the weight w_I is one if I is a monochromatic class in the k -colouring, w_I is 0 otherwise. So we have $\chi_f(G) \leq \chi(G)$. Furthermore, if W is a clique of G , then any fractional colouring of G must have a weight of at least $|W|$, and hence we have $\chi_f(G) \geq \omega(G)$. More generally, $\chi_f(G) \geq \chi_f(H)$ for any subgraph H of G .

Another lower bound on the fractional chromatic number of a graph G is related to its independence number $\alpha(G)$. Indeed, the total weight induced by an independent set I of G on its vertex set is at most $\omega_I \alpha(G)$, so we have

$$\begin{aligned} |V(G)| &\leq \sum_{v \in V(G)} w(v) \\ &\leq \sum_{I \in \mathcal{F}(G)} \alpha(G) w_I \\ &= \alpha(G) \cdot \sum_{I \in \mathcal{F}(G)} w_I. \end{aligned}$$

It means that the weight of any fractional colouring of G must be at least $\frac{|V(G)|}{\alpha(G)}$.

Recall that the *Hall ratio* $\rho(G)$ of G as $\rho(G) := \max\{\frac{|V(H)|}{\alpha(H)} \mid H \subseteq G\}$. By combining the two previously mentioned lower bounds, we have the following observations:

$$\max\left\{\frac{|V(G)|}{\alpha(G)}, \omega(G)\right\} \leq \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \Delta(G) + 1.$$

We may wonder how close the fractional chromatic number can deviate from the Hall ratio. In 2020, Dvořák, Ossona de Mendez, and Wu [37] conclude that the fractional chromatic number can not be bounded by any function of the Hall ratio.

If G is a perfect graph, then the clique number $\omega(G)$ equals to the chromatic number $\chi(G)$, and therefore also equal to the fractional chromatic number $\chi_f(G)$. A perfect graph G is a graph with the property that, for every induced subgraph H of G , $\omega(H) = \chi(H)$. Perfect graphs are a class of graphs that do not contain any odd holes or odd antiholes, as conjectured by Berge [7] in 1961 and proven by Chudnovsky, Robertson, Seymour, and Thomas, [26] in 2006. On the other hand, Brooks [21] established in 1941 that equality holds between $\chi(G)$ and $\Delta(G) + 1$

only for cliques and odd cycles. Since the fractional chromatic number of an odd cycle C_{2k+1} is $\frac{2k+1}{k}$, the only graphs G for which $\chi_f(G) = \Delta(G) + 1$ are cliques. Moreover, equality holds between the Hall ratio of G and its fractional chromatic number for example when G is vertex transitive.

Results on fractional colourings

Given a graph H , we let $\chi_f(d, H)$ be the supremum of the fractional chromatic numbers over all H -free graphs of maximum degree at most d . As we already mentioned in Theorem 11, it implies that $\chi_f(d, K_3) = (1+o(1))d/\ln d$ as $d \rightarrow \infty$, and one can infer from a study of random d -regular graphs by Bollobás [13] that $\chi_f(d, K_3) \geq \frac{d}{2 \ln d}$. However, there is still a significant range of degrees that are not covered by the bound for triangle-free graphs, i.e. when $\Delta < \Delta_\epsilon$, which is larger than $20^{2/\epsilon}$. The determination of the maximum value of $\chi_f(G)$ among triangle-free graphs with a maximum degree of 3 has been a long-standing open problem.

We define $\rho(d, g)$ to be the supremum of the Hall ratios over all graphs of maximum degree at most d and girth at least g . We also let $\rho(d, \infty)$ be the limit as $g \rightarrow \infty$ of $\rho(d, g)$. Note that if we fix d , then $\rho(d, g)$ is a non-increasing function of g . In symbols, $\rho(d, g) := \sup\{|V(G)|/\alpha(G) : G \text{ is a graph with } \Delta(G) \leq d \text{ and } \text{girth}(G) \geq g\}$, and $\rho(d, \infty) = \lim_{g \rightarrow \infty} \rho(d, g)$.

In 1979, Staton [98] established that $\rho(d, 4) \leq \frac{5d-1}{5}$, in particular implying that $\rho(3, 4) \leq \frac{14}{5}$. The two graphs depicted in Figure 1.4, called the graphs of Fajtlowicz and of Locke, have 14 vertices each, girth 5, and no independent set of order 6. It follows that $\rho(3, 4) = \frac{14}{5} = \rho(3, 5)$. It is known that the graphs of Fajtlowicz and of Locke are the only two cubic triangle-free and connected graphs with Hall ratio $\frac{14}{5}$. It follows from a result of Fraughnaugh and Locke [45] for graphs with more than 14 vertices completed by an exhaustive computer check on graphs with at most 14 vertices performed by Bajnok and Brinkmann [6].

The determination of $\chi_f(3, K_3)$ was finally settled in [39] by Dvořák, Sereni, and Volec.

Theorem 28 (Dvořák, Sereni, and Volec, 2014 [39]) $\chi_f(3, K_3) = \frac{14}{5}$.

In 1983, Jones [62] established that $\rho(4, 4) = \frac{13}{4}$. Only one connected graph is

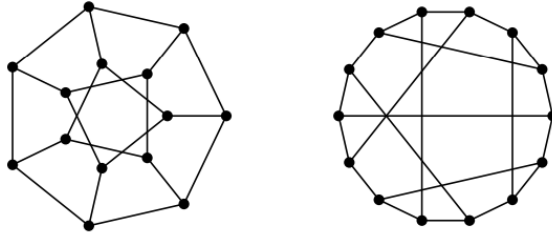


Figure 1.4: The two cubic triangle-free connected graphs with Hall ratio $\frac{14}{5}$.

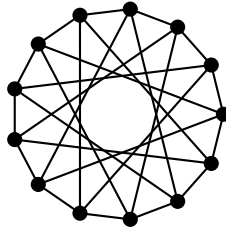


Figure 1.5: A graph certifying that $\chi_f(4, K_3) \geq 3.25$.

known to attain this value: it has 13 vertices and is presented in Figure 3.1. The value of $\rho(d, 4)$ when $d \geq 5$ is still unknown.

The best known upper bound in terms of clique number and maximum degree (when those two parameters are not too far apart) for the fractional chromatic number is due to Molloy and Reed [82], which solved Conjecture 3 in the fractional setting.

Theorem 29 (Molloy, Reed, 2002 [82]) *For every graph G ,*

$$\chi_f(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}.$$

It implies that $\chi_f(4, K_3)$ lies between 3.25 and 3.5. In Chapter 3, I will prove the following.

Theorem 30 (Dai, Ouyang, Pirot, and Sereni, 2023 [32]) $\chi_f(4, K_3) < 3.4663$.

For triangle-free d -degenerate graphs, the chromatic number of such a graph can be as large as $d + 1$. Notably, the only known examples of graphs reaching this bound have a much smaller fractional chromatic number. As a result, Harris has put forth the following conjecture.

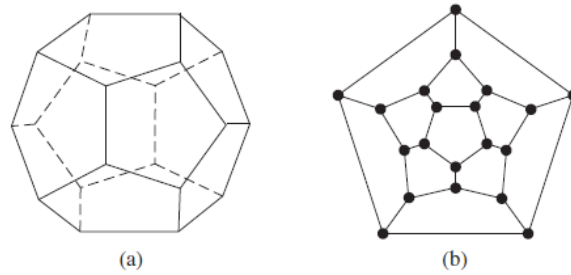


Figure 1.6: Hamilton's puzzle: the graph of the dodecahedron

Conjecture 6 (Harris, 2019 [53]) *There exists an absolute constant $C \geq \frac{1}{2}$ such that, for every triangle-free d -degenerate graph G ,*

$$\chi_f(G) \leq C \frac{d}{\ln d}.$$

1.3 . Hamilton-connectivity

In 1857, the Irish mathematician Sir William Hamilton (1805-1865) invented a game (Icosian Game, now also known as Hamilton's puzzle) of travelling around the edges of a graph from vertex to vertex. In a letter to his friend Graves, Hamilton described the game as a dodecahedron-based mathematical game. Every vertex of the dodecahedron is labelled with the name of a city, and the goal of the game is to find a cycle along the edges of the dodecahedron such that every vertex is visited exactly once, and the ending point is the same as the starting point; one is therefore looking for a Hamiltonian cycle of the dodecahedron (see Figure 1.6). Since then hamiltonicity, which aims at determining whether or not a graph G contains a Hamiltonian cycle, has been at the origin of a whole branch of graph theory.

Hamiltonicity has many applications in different fields, including network design and fault tolerance, DNA sequencing, robotics and automated path planning, circuit design, etc. Here we give a Traveling Salesman Problem as an example. Consider a graph with a set of cities and the distances between them. The task is to find a route that visits each city exactly once and returns to the starting city, minimising the total distance travelled. The goal of the graph is

to find a Hamiltonian cycle such that its distance is minimum. However, the Hamiltonian problem is NP-complete [48]. Finding a Hamiltonian cycle or path in a general graph is computationally tricky and may require significant computational resources.

1.3.1 . Main classic results of hamiltonicity and Hamilton-connectivity

Finding a good necessary and sufficient condition for a graph to have a Hamilton cycle (path) turns out to be impossible. Naturally, there has been much literature on graph theory studying sufficient conditions for hamiltonicity. Research on sufficient conditions for the existence of Hamilton cycles often involves conditions on the edge density, or equivalently, conditions related to the number of neighbouring vertices. If a graph G has k independent vertices with $k \geq 1$, we define:

$$\sigma_k(G) = \min_{S \subseteq V(G)} \left\{ \sum_{v \in S} \deg_G(v) \mid |S| = k, S \text{ is an independent set in } G \right\},$$

$$\bar{\sigma}_k(G) = \min_{S \subseteq V(G)} \left\{ \sum_{v \in S} \deg_G(v) - \left| \bigcap_{v \in S} N_G(v) \right| \mid |S| = k, S \text{ is an independent set in } G \right\}.$$

The earliest known result based on a degree condition was given by Dirac [35] in 1952.

Theorem 31 (Dirac, 1952 [35]) *Let G be a graph on n vertices. If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*

The lower bound of Dirac's theorem is known to be sharp. To illustrate this, let's consider a complete bipartite graph $K_{n,n+1}$ with $2n + 1$ distinct vertices. Clearly, the minimum degree of $K_{n,n+1}$ is n , and it is not Hamiltonian. Unless a few more conditions are added, Dirac's theorem cannot be relaxed without cancelling the conclusion.

Over time, numerous scientists extended Dirac's condition to encompass broader scenarios. In 1960, Ore [83] gave the following generalised theorem.

Theorem 32 (Ore, 1960 [83]) *Let G be a graph on n vertices. If $\sigma_2(G) \geq n$, then G is Hamiltonian.*

As well as Dirac's theorem, Ore's theorem also has a strict requirement on the bound of n in order to maintain its conclusion. An example for the sharp of Ore's theorem is the complete bipartite graph $K_{\frac{n-1}{2}, \frac{n+1}{2}}$. It's worth noting that Dirac's theorem is a consequence of Ore's theorem, which implies that Ore's condition implies Dirac's condition.

Two initial well-known theorems, according to Dirac and Ore, provide sufficient conditions based on degrees and degree sums. These are starting points for a common approach to theorems of hamiltonicity which uses degree and degree-like conditions together with other parameters in graphs. Studying the behaviour of 2-connected graphs, Flandrin, Jung, and Li proved the following result.

Theorem 33 (Flandrin, Jung, Li, 1991 [44]) *If G is a 2-connected graph with n vertices such that $\overline{\sigma}_3(G) \geq n$, then G is Hamiltonian.*

In the same paper [44], they also give a condition on $\overline{\sigma}_3(G)$ to show the existence of a Hamiltonian path.

Theorem 34 (Flandrin, Jung, Li, 1991 [44]) *If G is a connected graph with n vertices such that $\overline{\sigma}_3(G) \geq n - 1$, then G has a Hamiltonian path.*

The constant stream of results in hamiltonicity continues to supply us with new and interesting theorems and still further questions. One of them is about Hamilton-connectivity. In fact, determining whether a graph is Hamilton-connected is NP-complete [46]. Researchers try to find sufficient degree and neighbourhood union conditions for Hamilton-connected graphs. It was first introduced by Ore in 1963.

Theorem 35 (Ore, 1963 [84]) *If G is a graph with $\sigma_2(G) \geq n + 1$ where n is the order of G , then G is Hamilton-connected.*

In 1989, Flandrin, Gould, Jacobson, Schelp gave a sufficient degree sums condition of 3-connected graphs.

Theorem 36 (Flandrin, Gould, Jacobson, Schelp, 1989 [43]) *If G is 3-connected with n vertices such that $\sigma_2(G) \geq \frac{2n+1}{3}$, then G is Hamilton connected.*

This result was generalised by Wei in 1993.

Theorem 37 (Wei, 1993 [107]) *If G is a 3-connected graph with n vertices such that $\overline{\sigma}_3(G) \geq n + 1$, then G is Hamilton-connected.*

There are plenty of results about hamiltonicity and Hamilton-connectivity, here we have only provided partial elegant results. For more results, we would mention the excellent survey by Gould [49, 50].

1.3.2 . Extensions of Hamilton-connectivity

Hamilton-connectivity plays an important role in computer networks. The parallel computer's processors are connected through a network. Its architecture can be visualized as a graph, with the communication lines serving as the edges and the vertices as the processors. To ensure the resilience of the communication in the case that a node fails along one link, we must maintain communication links within several pairs of processors. The objective in a graph is to find internally-disjoint paths between these pairs where each path should be unique and share no common nodes with any other path. However, there are numerous needs that compete with one another when building the architecture of computer networks. It is almost impossible to design a network that is optimal from all aspects. One has to design a suitable network depending on the requirements and their properties. It inspired researchers to extend Hamilton-connectivity by the context needed for its application.

Spanning connectivity

In some specific application contexts [72, 73], Hsu in 1984 introduced the concept of a container and indicated that the container plays an essential role in evaluating the reliability and performance of interconnectivity networks [55].

Definition 8. Let G be a graph.

1. A k -container of G between u and v , $C_k(u, v)$, is a set of k -internally-disjoint paths between u and v .
2. A *spanning k -container*, (abbreviated as k^* -container), is a k -container that contains all vertices of $V(G)$.
3. A graph G is *k -container-connected* or *k^* -connected* if there exists a k^* -container between any two different vertices.

4. The *spanning connectivity* of G , $\kappa^*(G)$, is the maximum integer k such that G is w^* -connected for $1 \leq w \leq k$ if G is 1-container-connected.
5. A graph G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$.

Obviously, the complete graph K_n is super spanning connected if $n \geq 2$. Spanning connectivity is a hybrid concept of hamiltonicity and connectivity. We can obtain the following properties about the container.

Remark 3.

1. It follows from the Menger Theorem [79] that there is a k -container between any two distinct vertices of G if and only if G is k -connected.
2. A graph G is 1^* -connected if and only if it is Hamilton-connected, and a graph G is 2^* -connected if and only if it is Hamiltonian.
3. All 1^* -connected graphs except for K_1 and K_2 are 2^* -connected.

Dirac's theorem becomes that any graph G with at least three vertices and $\delta(G) \geq n(G)/2$ is 2^* -connected. Lin, Huang, and Hsu proved a k^* -connectivity analogue of Dirac's theorem as follows.

Theorem 38 (Lin, Huang, Hsu, 2007 [75]) *If G is a graph with $\delta(G) \geq \frac{n(G)+k-2}{2}$, then G is k^* -connected.*

In 2008, Hsu and Lin [57] gave a Ore-type sufficient condition of k^* -connected graphs.

Theorem 39 (Lin, Huang, Tan, Hsu, 2008 [75]) *Let k be a positive integer. If $\sigma_2(G) \geq n(G) + k - 2$, then G is k^* -connected.*

More results about the spanning connectivity of graphs have been studied extensively [56, 57, 72, 73, 74, 57, 96, 103].

k -fan connected graphs

In 2009, Lin, Tan, Hsu, and Hsu [76] gave a definition of k -fan connected graphs, which is a Menger-type concept similar to the spanning connectivity of a graph.

Definition 9. For any integer $k \geq 2$, let v be a vertex of a graph G and $U = \{u_1, u_2, \dots, u_k\}$ be a subset of $V(G) \setminus \{v\}$.

1. A (v, U) -fan is a set of internally-disjoint paths P_1, P_2, \dots, P_k such that P_i is a path connecting v and u_i for $1 \leq i \leq k$.
2. If a (v, U) -fan spans G , then it is called a *spanning (v, U) -fan* of G .
3. If G has a spanning (v, U) -fan for every vertex v of G and every subset U of $V(G) \setminus \{v\}$ with $|U| = k$, then we call G is *k -fan-connected* or *k_f^* -connected*.
4. The *spanning fan-connectivity* of a graph G , $\kappa_f^*(G)$, is defined as the largest integer k such that G is w_f^* -connected for $1 \leq w \leq k$ if G is a 1_f^* -connected graph.

Remark 4.

1. It is proved by Dirac [36] that a graph G is k -connected if and only if it has at least $k + 1$ vertices and there exists a (v, U) -fan for every choice of v and U with $|U| \leq k$ and $v \notin U$.
2. Every 1^* -connected graph is 1_f^* -connected. Moreover, every 1_f^* -connected graph that is not K_2 is 2_f^* -connected.

Lin, Tan, Hsu and Hsu [76] discussed the relationship among $\kappa_f^*(G)$, $\kappa^*(G)$, and $\kappa(G)$.

Theorem 40 (Lin, Tan, Hsu, Hsu, 2009 [76]) *For any 1_f^* -connected graph, $\kappa_f^*(G) \leq \kappa^*(G) \leq \kappa(G)$. Moreover, $\kappa_f^*(G) \leq \kappa^*(G) \leq \kappa(G) = n(G) - 1$ if and only if G is a complete graph.*

Furthermore, they presented an Ore-type sufficient condition for a graph to be κ_f^* -connected.

Theorem 41 (Lin, Tan, Hsu, Hsu, 2009 [76]) *Let G be a graph and $k \geq 2$ be an integer. If $\sigma_2(G) \geq n(G) + k - 1$, then G is κ_f^* -connected.*

Recently, Li, Maezawa, and Tian proved a stronger result as follows.

Theorem 42 (Li, Maezawa, Tian [70]) *Let G be an n -vertex graph. If G is $(k + 1)$ -connected and $\bar{\sigma}_3(G) \geq n + k - 1$ with $k \geq 2$, then G is k -fan-connected.*

k -linked connected graphs

We introduce a significant definition, k -linked graphs, which is also related to the connectivity of graphs. Linkage structure plays an important role in the study of graph minors.

Definition 10. For an integer $k \geq 2$, a graph is k -linked if it has at least $2k$ vertices and for every sequence $s_1, s_2, \dots, s_k, t_1, t_2, \dots, t_k$ of distinct vertices there exist disjoint paths P_1, \dots, P_k such that the ends of P_i are s_i and t_i for $i = 1, 2, \dots, k$.

Clearly, every k -linked graph is k -connected. The converse is not true, however, which brings up the natural question of how much connectivity, as a function $f(k)$, is necessary to ensure that a graph is k -linked. After a series of papers by Jung [63], Larman and Mani [69], Mader [78], and Robertson and Seymour [94], the first linear upper bound for f , namely $f(k) \leq 22k$, was proved by Bollobás and Thomason [14].

In 2006, Kawarabayashi, Kostochka, and Yu [64] improved the bound of $f(k)$ to $12k$.

Theorem 43 ([64]) *Every $2k$ -connected graph G with $e(G) \geq 6kn(G)$ is k -linked. In particular, every $12k$ -connected graph is k -linked.*

They also determined sharp minimum degree and degree sum conditions for a graph G of order at least $2k$ to be k -linked.

Theorem 44 ([64]) *Let G be a graph on $n \geq 2k$ vertices.*

$$\delta(G) \geq \begin{cases} \frac{n+2k-3}{2}, & \text{if } n \geq 4k - 1, \\ \frac{n+5k-5}{3}, & \text{if } 3k \leq n \leq 4k - 2, \\ n - 1, & \text{if } 2k \leq n \leq 3k - 1. \end{cases}$$

or

$$\sigma_2(G) \geq \begin{cases} n + 2k - 3, & \text{if } n \geq 4k - 1, \\ \frac{2(n+5k)}{3} - 3, & \text{if } 3k \leq n \leq 4k - 2, \\ 2n - 3, & \text{if } 2k \leq n \leq 3k - 1. \end{cases}$$

then G is k -linked. These bounds are the best possible.

Thomas and Wollan [100] used the bound $\delta(G) \geq \frac{n+2k-3}{2}$ to give sufficient conditions for a graph to be k -linked in terms of connectivity.

Theorem 45 ([100]) *If G is $10k$ -connected, then G is k -linked.*

(k_1, k_2) -Hamilton-connected graphs

Motivated by spanning connectivity and k -fan connected graphs, we define (k_1, k_2) -Hamilton-connected graphs.

Definition 11. A graph G is (k_1, k_2) -Hamilton-connected, if for any two disjoint vertex subsets $X = \{x_1, x_2, \dots, x_{k_1}\}$ and $U = \{u_1, u_2, \dots, u_{k_2}\}$, there are $k_1 k_2$ internally-disjoint paths connecting x_i to u_j for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, which span the whole graph.

Remark 5.

1. $(1, k)$ -Hamilton-connectivity is equivalent to k -fan-connectivity.
2. Both $(1, 1)$ -Hamilton-connectivity and $(1, 2)$ -Hamilton-connectivity are equivalent to Hamilton-connectivity.
3. (k, k) -Hamilton-connectivity implies k -link.

In 2022, Dai, Li, Ouyang, and Tian studied $(2, k)$ -Hamilton-connected graphs.

Theorem 46 (Dai, Li, Ouyang, Tian, [30]) *Let G be an n -vertex graph. If G is $(5k - 2)$ -connected and $\sigma_2(G) \geq n + k - 2$ with $k \geq 2$, then G is $(2, k)$ -Hamilton-connected. Moreover, the bound of $\sigma_2(G)$ is sharp.*

Another result obtained is about (k_1, k_2) -Hamilton-connectivity which is also a tight sufficient condition.

Theorem 47 (Dai, Li, Ouyang, Tian, [30]) *Let G be an n -vertex graph. If $\sigma_2(G) \geq n + k_1k_2 - 2$ with $k_1, k_2 \geq 2$, then G is (k_1, k_2) -Hamilton-connected. Moreover, the bound of $\sigma_2(G)$ is sharp.*

The two theorems above will be stated Theorem 59 and Theorem 58 in Chapter 4.

1.4 . Contribution and outline of the thesis

In this section, we summarise the main works and the organization of this thesis.

(1) We extend the bound of list colourings on 2-connected bipartite outerplanar graphs to DP-colourings. In 2008, Hutchinson [59] showed that

- if a 2-connected bipartite outerplanar graph G has a list of colors $L(v)$ for each vertex v with $|L(v)| \geq \min\{\deg_G(v), 4\}$, then G is L -colorable; and
- if a maximal outerplanar graph G with at least four vertices has a list of colors $L(v)$ for each vertex v with $|L(v)| \geq \min\{\deg_G(v), 5\}$, then G is L -colorable.

She also showed the lower bounds are sharp in the sense that the lower bound 4 for 2-connected bipartite outerplanar graphs (resp. 5 for maximal outerplanar graph) cannot be replaced by 3 (resp. 4). Our findings indicate that the first bound of Hutchinson's results is not sufficient for DP-colourings. Moreover, we give a sharp bound on DP-colourings for all the 2-connected outerplanar graphs. This work is illustrated precisely in Chapter 2.

[Corresponding paper: *On DP-colourings of outerplanar graphs*, Submitted to Discrete Math. in Oct. 2022, with Jie Hu, Hao Li, and Shun-ichi Maezawa.]

- (2) We are interested in fractional colourings of triangle-free graphs of maximum degree at most d with a specific focus on the case $d = 4$.

Given a graph H , we let $\chi_f(d, H)$ be the supremum of the fractional chromatic numbers over all H -free graphs of maximum degree at most d . It has been settled by Dvořák, Sereni, and Volec [39] that $\chi_f(3, K_3) = 14/5$. For $d \in [16] \setminus \{3\}$, there is an upper bound $\chi_f(d, K_3) \leq \frac{d+3}{2}$, which follows from the fractional Reed bound, established by Molloy and Reed [82]. It is believed that this bound is not tight for any $d \geq 3$. For $d \geq 17$, Pirot and Sereni [88] have used hard-core distributions on the independent sets of triangle-free graphs in order to derive the best-known upper bounds for $\chi_f(d, K_3)$. For the case $d = 4$, Jones [62] proved that for every triangle-free graph G of maximum degree 4 on n vertices, the size of maximum independent sets is at least $\frac{4n}{13}$. This implies that $\chi_f(4, K_3) \leq \frac{13}{4}$ if we restrict its definition to the class of vertex-transitive graphs. In general, $\chi_f(4, K_3)$ lies between 3.25 and 3.5 (by the fractional Reed bound). We use the methodology introduced by Pirot and Sereni [88] together with mixed probability distributions in order to prove that $\chi_f(4, K_3) < 3.4663$. This work is shown in Chapter 3. [Corresponding paper: *Beyond the fractional Reed bound for triangle-free graphs*, in preprint, with Qiancheng Ouyang, François Pirot, and Jean-Sébastien Sereni.]

- (3) Inspired by the definition of spanning connectivity and k -fan connected graphs, we study (k_1, k_2) -Hamilton-connected graphs. We prove that an n -vertex graph G is $(2, k)$ -Hamilton-connected if G is $(5k - 4)$ -connected with $\sigma_2(G) \geq n + k - 2$ where $k \geq 2$. We also prove that if $\sigma_2(G) \geq n + k_1k_2 - 2$ with $k_1, k_2 \geq 2$, then G is (k_1, k_2) -Hamilton-connected. Moreover, we construct two graphs to show that our results are sharp. This work is discussed in Chapter 4.

[Corresponding paper: *On $(2, k)$ -Hamilton-connected graphs*, Submitted to Discrete Applied Math. in Aug. 2022, with Hao Li, Qiancheng Ouyang, and Zengxian Tian.]

Remark 6. Some of my works are not included in this thesis (see [33, 29, 31, 85]), which have been (or will be) contained in theses of my collaborators. We will introduce these works briefly in the last chapter Appendix.

2 - On DP-colourings of outerplanar graphs

In Chapter 1, we have introduced the definition of DP-colourings on simple graphs and related results. This chapter focuses mainly on our work with DP-colourings.

2.1 . Introduction

For list colourings, we have known that researchers try to make some upper bounds on $\chi(G)$ in terms of vertex degrees hold for $\chi_\ell(G)$ as well. For example, Brook's theorem and the degeneracy upper bound hold for $\chi_\ell(G)$. Furthermore, Borodin [16, 17] and independently Erdős, Rubin, and Taylor [40] generalised Brook's theorem to degree-list assignments. Recall that a list assignment L is called a *degree-list assignment* if $|L(v)| \geq \deg_G(v)$ for every $v \in V(G)$. A graph G is said to be *degree-choosable* if G admits an L -colouring for every degree-list assignment L . A *block* of G , or 2-connected component, is a maximal (therefore induced) 2-connected subgraph of G .

Theorem 48 ([16, 17, 40], a simple proof in [68]) *A connected graph G is not degree-choosable if and only if G is a Gallai tree, that is, each block of G is isomorphic to K_n for some integer n or C_n for some odd integer n .*

Hutchinson [59] studied the extension problem of a degree-list colouring problem on outerplanar graphs. By induction, an outerplanar graph is easily seen to be 3-choosable since it contains vertices of degree 2. The question she asked involves a mixture of these two hypotheses: if a 2-connected outerplanar graph satisfies $|L(v)| \geq \min\{\deg(v), 4\}$ for every vertex v , is it L -list colourable provided it does not form a Gallai tree of maximum degree at most 4? She showed that the answer is yes when the graph is bipartite, but the answer is no in general due to an example of Kostochka [67]; Hutchinson [59] showed the lower bounds are sharp in the sense that the lower bound 4 in Theorem 49 (resp. 5 in Theorem 50) cannot be replaced by 3 (resp. 4). Note that all conditions of Hutchison's results are needed.

Theorem 49 ([59]) *Let G be a 2-connected, outerplanar bipartite graph. For any list assignments L with $|L(v)| \geq \min\{\deg_G(v), 4\}$ for every vertex $v \in V(G)$, G has an L -colouring.*

Theorem 50 ([59]) *Let G be a 2-connected edge-maximal outerplanar graph with at least three vertices that is not K_3 with identical 2-lists at each vertex. For any list assignments L with $|L(v)| \geq \min\{\deg_G(v), 5\}$ for every vertex $v \in V(G)$, G has an L -colouring.*

We mainly study whether Theorems 49 and 50 hold in the setting of DP-colourings. To better illustrate our results, we first propose a definition of DP-colourings that applies to multigraphs and graphs.

Definition 12. Let G be a multigraph (possibly having multiple edges but no loops) and L be a list assignment of G .

1. For each edge uv in $E(G)$, let $M_{L,uv}$ be the union of $\mu_G(uv)$ matchings between $\{u\} \times L(u)$ and $\{v\} \times L(v)$ where $\mu_G(uv)$ is the multiplicity of uv in G . Note that if u and v are not connected by an edge in G , then $\mu_G(uv) = 0$ and $M_{L,uv}$ is an empty set. With abuse of notation, we sometimes regard $M_{L,uv}$ as a bipartite graph between $\{u\} \times L(u)$ and $\{v\} \times L(v)$ of maximum degree at most $\mu_G(uv)$. Let $\mathcal{M}_L = \{M_{L,uv} : uv \in E(G)\}$, which is called a *matching assignment over L* . Then a graph H is said to be the \mathcal{M}_L -cover of G if it satisfies the following conditions:

- (i) The vertex set of H is $\bigcup_{u \in V(G)} (\{u\} \times L(u)) = \{(u, c) : u \in V(G), c \in L(u)\}$.
- (ii) For every $u \in V(G)$, the set $\{u\} \times L(u)$ induces a clique in H .
- (iii) For every two vertices u and v in G , $\{u\} \times L(u)$ and $\{v\} \times L(v)$ induce in H the graph obtained from $M_{L,uv}$ by adding those edges defined in (ii).

(See Figure 2.1 for an example)

2. An \mathcal{M}_L -colouring of G is an independent set I in the \mathcal{M}_L -cover with $|I| = |G|$. G is said to be \mathcal{M}_L -colourable if it admits an \mathcal{M}_L -colouring.

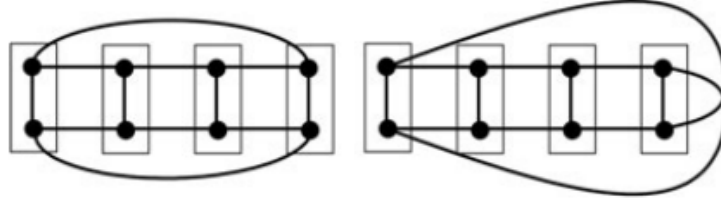


Figure 2.1: Two examples of the \mathcal{M}_L -cover of C_4 such that $|L(u)| = 2$ for any vertex u . Each thin rectangle represents $\{u\} \times L(u)$ for some vertex u . In fact, the cycle C_4 admits an \mathcal{M}_L -colouring for the left, but it does not for the right

3. The *DP-chromatic number*, denoted by $\chi_{DP}(G)$, is the minimum integer t such that G admits an \mathcal{M}_L -colouring for each t -list assignment L and each matching assignment \mathcal{M}_L over L .
4. A graph G is said to be *DP-degree-colourable* if G admits an \mathcal{M}_L -colouring for each degree-list assignment L and each matching assignment \mathcal{M}_L over L .

Remark 7.

- (1) When G is a simple graph and $M_{L,uv} = \{(u, c)(v, c) : c \in L(u) \cap L(v)\}$ for every edge uv in G , then G has an L -colouring if and only if G admits an \mathcal{M}_L -colouring. Thus

$$\forall G, \quad \chi_{DP}(G) \geq \chi_\ell(G).$$

- (2) When $|L(u)| = [k]$ for each $u \in V(G)$ (that is, when we consider a proper k -colouring), then the \mathcal{M}_L -cover of G is isomorphic to the graph $G \square K_k$, which is the Cartesian product of G and the complete graph K_k . The *Cartesian product* of graphs G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$.

We want to generalise Theorems 49 and 50 to hold in the DP-colouring setting. The following problem corresponds to Theorem 49.

Problem 1 Let G be a 2-connected, outerplanar bipartite graph and L be a list assignment with $|L(v)| \geq \min\{\deg_G(v), 4\}$ for every vertex $v \in V(G)$. Does G admit an \mathcal{M}_L -colouring for any matching assignment over L unless G is an even cycle?

The answer to this problem is no, as we show by finding counterexamples. Note that even cycles satisfying the condition of Problem 1 do not admit an \mathcal{M}_L -colouring for some matching assignments over L by Theorem 52 (ii) for $t = 1$.

Theorem 51 *There exist infinitely many counterexamples to Problem 1.*

For the setting on DP-colourings of Theorem 49, our result is stronger than it. In fact, we solve all the 2-connected outerplanar graphs, not only bipartite graphs. Before giving the statement of our result, we need to introduce some special graphs and a related result. For two graphs G and H and a vertex u of G , *blowing up u to H* is the operation of replacing u by H and joining each vertex of H to every neighbour in G of u . Let n and t be positive integers.

- The *ladder of length n* is $C_n \square K_2$. The left graph in Figure 2.1 is the ladder of length 4.
- A graph G is the *Möbius ladder of length n* if it is obtained from the $P_n \square K_2$ with $P_n = v_1 \dots v_n$ and $V(K_2) = \{a, b\}$ by adding the edges $(v_1, a)(v_n, b)$ and $(v_1, b)(v_n, a)$. The right graph in Figure 2.1 is the Möbius ladder of length 4.
- The *t -fat ladder of length n* is obtained from the ladder of length n by blowing up every vertex to a complete graph K_t .
- The *t -fat Möbius ladder of length n* is obtained from the Möbius ladder of length n by blowing up every vertex to a complete graph K_t .

For a graph G and an integer t , we denote by G^t the multigraph obtained from G by replacing every edge of G with t multiple edges. A multigraph G is called a *complete multigraph*, if the graph obtained from G by replacing every set of multiple edges with a single edge is a complete graph. Kim and Ozeki obtained the following result (in fact they showed a result stronger than Theorem 52).

Theorem 52 ([65]) *Let G be a 2-connected multigraph not isomorphic to a complete multigraph with order at least four and let L be a list assignment of G with $|L(v)| \geq \deg_G(v)$ for each $v \in V(G)$. Then G does not admit an \mathcal{M}_L -colouring if and only if G is isomorphic to C_n^t for some integers n and t such that the following hold.*

- (i) *If n is an odd integer, then \mathcal{M}_L -cover is isomorphic to t -fat ladder of length n .*
- (ii) *If n is an even integer, then \mathcal{M}_L -cover is isomorphic to t -fat Möbius ladder of length n .*

Given a graph G and a list assignment L of G , if an \mathcal{M}_L -cover of G satisfies one of the two conditions in Theorem 52, then we say (G, L, \mathcal{M}_L) is *bad*. We prove the following main result in this chapter, which is a generalisation of Theorem 50.

Theorem 53 *Let G be a 2-connected outerplanar graph. Let L be a list assignment of G with $|L(v)| \geq \min\{\deg_G(v), 5\}$ for every $v \in V(G)$. Then G admits an \mathcal{M}_L -colouring for each matching assignment \mathcal{M}_L over L unless (G, L, \mathcal{M}_L) is *bad*.*

The possible bad triples (G, L, \mathcal{M}_L) in Theorem 53 are as follows: either

- (i) G is isomorphic to C_n for an odd integer n and its \mathcal{M}_L -cover is isomorphic to the ladder of length n ; or
- (ii) G is isomorphic to C_n for an even integer n and \mathcal{M}_L -cover is isomorphic to the Möbius ladder of length n .

Let G be a graph, L be a list assignment of G , and \mathcal{M}_L be a matching assignment over L . To simplify notation, we can regard the vertex set of the \mathcal{M}_L -cover of G as $\cup_{u \in V(G)} \tilde{L}(u)$ instead of $\cup_{u \in V(G)} \{u\} \times L(u)$. Similarly, $\tilde{L}'(u)$ denote $\{u\} \times L'(u)$ for $L'(u) \subseteq L(u)$.

2.2 . The proof of Theorem 51

In this section, we prove Theorem 51: There exist infinitely many counterexamples to Problem 1.

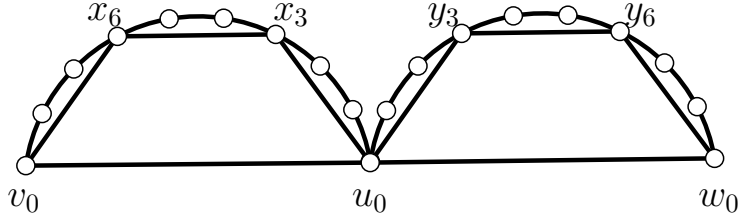


Figure 2.2: The graph H_0 .

We begin this section with an easy but useful fact about independent sets in Möbius ladders. It is clear to obtain the following fact by the definition of the Möbius ladder of length k .

Fact 3 *Let M be a Möbius ladder of length k . Then the size of any independent set of M is at most $k - 1$ when k is even.*

We are now ready to prove Theorem 51 by constructing a graph G in this section. Let L be a list assignment with $|L(v)| \geq \min\{\deg_G(v), 4\}$ for every vertex $v \in V(G)$. Our goal is to prove that G does not admit an \mathcal{M}_L -colouring for a given matching assignment over L .

In fact, G consists of a series of subgraphs. Firstly, we construct a base subgraph H_0 (see Figure 2.2) of G . Let H_0 be a connected outerplanar graph satisfying the following conditions:

- (i) The vertex set of H_0 is $\{u_0, v_0, w_0, x_1, x_2, \dots, x_8, y_1, y_2, \dots, y_8\}$.
- (ii) There exists the walk $v_0x_8x_7 \dots x_1u_0y_1y_2 \dots y_8w_0u_0v_0$ and it is an outer walk of H_0 .
- (iii) The edge set of H_0 is the edges of the outer walk of (ii) and $\{u_0x_3, x_3x_6, v_0x_6, u_0y_3, y_3y_6, w_0y_6\}$.

Note that H_0 is symmetric with respect to u_0 . Let L be a list assignment with $|L(v)| \geq \min\{\deg_{H_0}(v), 4\}$ for every vertex $v \in V(H_0)$. For $1 \leq i \leq 4$, let u_0^i , v_0^i , and w_0^i be elements of $\tilde{L}(u_0)$, $\tilde{L}(v_0)$, and $\tilde{L}(w_0)$, respectively. Similarly, for $1 \leq i \leq 4$, let x_3^i , x_6^i , y_3^i , and y_6^i be elements of $\tilde{L}(x_3)$, $\tilde{L}(x_6)$, $\tilde{L}(y_3)$, and $\tilde{L}(y_6)$, respectively. We construct an \mathcal{M}_L -cover J_0 of H_0 such that

- (i) Each of $J_0[\{u_0^1, u_0^2, x_3^1, x_3^2\} \cup \tilde{L}(x_1) \cup \tilde{L}(x_2)]$, $J_0[\{x_3^3, x_3^4, x_6^3, x_6^4\} \cup \tilde{L}(x_4) \cup \tilde{L}(x_5)]$, and $J_0[\{v_0^1, v_0^2, x_6^1, x_6^2\} \cup \tilde{L}(x_7) \cup \tilde{L}(x_8)]$ induces a Möbius ladder of length four.
- (ii) Each of $J_0[\{u_0^3, u_0^4, y_3^1, y_3^2\} \cup \tilde{L}(y_1) \cup \tilde{L}(y_2)]$, $J_0[\{y_3^3, y_3^4, y_6^3, y_6^4\} \cup \tilde{L}(y_4) \cup \tilde{L}(y_5)]$, and $J_0[\{w_0^1, w_0^2, y_6^1, y_6^2\} \cup \tilde{L}(y_7) \cup \tilde{L}(y_8)]$ induces a Möbius ladder of length four.
- (iii) $u_0^1 v_0^4, u_0^2 v_0^3, u_0^3 w_0^4, u_0^4 w_0^3 \in E(J_0)$.

Note that J_0 is symmetric with respect to $\tilde{L}(u_0)$.

Claim 2.2.1 *Let I_0 be an independent set of J_0 with $|I_0| = |H_0|$. If u_0^i is in I_0 for some $i \in \{1, 2\}$, then v_0^{i+2} is in I_0 . If u_0^i is in I_0 for some $i \in \{3, 4\}$, then w_0^i is in I_0 .*

Proof. By symmetry, we may assume that $u_0^1 \in I_0$. Since $J_0[\{u_0^1, u_0^2, x_3^1, x_3^2\} \cup \tilde{L}(x_1) \cup \tilde{L}(x_2)]$ is a Möbius ladder of length four, it follows from Lemma 3 that $x_3^1, x_3^2 \notin I_0$ and so either x_3^3 or x_3^4 is in I_0 . Since $J_0[\{x_3^3, x_3^4, x_6^3, x_6^4\} \cup \tilde{L}(x_4) \cup \tilde{L}(x_5)]$ is a Möbius ladder of length four, it follows from Lemma 3 that $x_6^3, x_6^4 \notin I_0$ and so either x_6^1 or x_6^2 is in I_0 . Since $J_0[\{v_0^1, v_0^2, x_6^1, x_6^2\} \cup \tilde{L}(x_7) \cup \tilde{L}(x_8)]$ is a Möbius ladder of length four, it follows from Lemma 3 that $v_0^1, v_0^2 \notin I_0$ and so either v_0^3 or v_0^4 is in I_0 . Since $u_0^1 \in I_0$ and $u_0^1 v_0^4 \in E(G)$, v_0^3 is in I_0 . □

For $i \geq 1$, let H_i be a graph isomorphic to H_0 and let u_i, v_i , and w_i be vertices of H_i corresponding to u_0, v_0 , and w_0 , respectively. For $i \geq 1$ and $1 \leq j \leq 4$, let J_i be an \mathcal{M}_L -cover of H_i isomorphic to J_0 and let u_i^j, v_i^j , and w_i^j be vertices of J_i corresponding to u_0^j, v_0^j , and w_0^j , respectively. Let G_9 be the graph obtained from H_0, \dots, H_9 by adding the edges $v_i w_{i+1}$ and $u_i u_{i+1}$ for each $0 \leq i \leq 8$ and $u_0 u_3, u_3 u_6, u_6 u_9, u_9 u_0$. We construct an \mathcal{M}_L -cover J of G_9 such that

- (i) $v_i^3 w_{i+1}^4, v_i^4 w_{i+1}^3 \in E(J)$ for each $0 \leq i \leq 8$,
- (ii) $u_i^1 u_{i+1}^3, u_i^2 u_{i+1}^4 \in E(J)$ for each $0 \leq i \leq 8$,
- (iii) $J[\{u_0^1, u_0^2, u_3^1, u_3^2, u_6^1, u_6^2, u_9^1, u_9^2\}]$ induce a Möbius ladder of length four.

Let G'_9 be the graph isomorphic to G_9 . G'_9 also consists of ten base subgraphs that are isomorphic to H_0 . For convenience, let H_{i+10} be the subgraph in G'_9 corresponding to H_i for $0 \leq i \leq 9$ and w_{10} corresponds to v_0 . Let J' be an \mathcal{M}_L -cover of G'_9 isomorphic to J replacing u_0^1 and u_0^2 with u_{10}^3 and u_{10}^4 , respectively.

Let G be a graph obtained from G_9 and G'_9 by identifying H_0 and H_{10} and connecting v_9 and v_{19} by a path P with odd order in the sense the path in G connecting v_9 and v_{19} not containing u_0 is a path with odd order (see Figure 2.3). Then the outer walk of G is a cycle and contains every vertex of G . Moreover, every finite face of G is of even length. It means that all cycles of G have even length as well. Hence G is a 2-connected outerplanar bipartite graph. Let K be an \mathcal{M}_L -cover of G obtained from J and J' by identifying $J[\cup_{u \in V(H_0)} \tilde{L}(u)]$ and $J'[\cup_{u \in V(H_{10})} \tilde{L}(u)]$. We show that G does not admit any \mathcal{M}_L -colouring. Suppose that G admits an \mathcal{M}_L -colouring. Let I be an independent set of K with $|I| = |G|$. By symmetry, we may assume that either u_0^1 or u_0^2 is in I .

Claim 2.2.2 For each $0 \leq i \leq 9$, either u_i^1 or u_i^2 is in I .

Proof. We prove this claim by induction on i . When $i = 0$, this claim holds. Suppose that $i \geq 1$. By Claim 2.2.1 and the induction hypothesis, either v_{i-1}^3 or v_{i-1}^4 is in I . Suppose that either u_i^3 or u_i^4 is in I . Then either w_i^3 or w_i^4 is in I by Claim 2.2.1. However this contradicts to Lemma 3 since $K[u_{i-1}^1, u_{i-1}^2, v_{i-1}^3, v_{i-1}^4, w_i^3, w_i^4, u_i^3, u_i^4]$ is isomorphic to a Möbius ladder of length four. Hence this claim holds. \square

By Claim 2.2.2, we can deduce a contradiction Lemma 3 since $K[\{u_0^1, u_0^2, u_3^1, u_3^2, u_6^1, u_6^2, u_9^1, u_9^2\}]$ is isomorphic to a Möbius ladder of length four. \square

2.3 . The proof of Theorem 53

In this section, we prove our main result Theorem 53 by a technical but stronger result Theorem 54.

We need to introduce more notations so as to state Theorem 54. Let L be a list assignment of G , and let \mathcal{M}_L and \mathcal{M}'_L be matching assignments over L . We

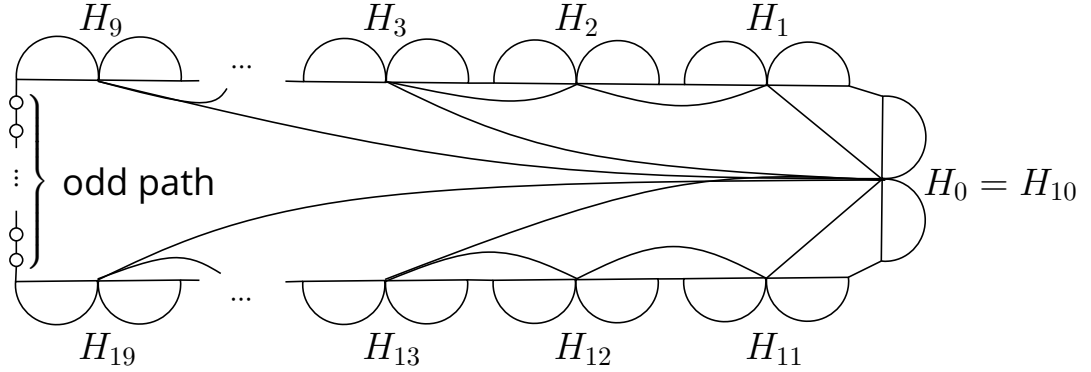


Figure 2.3: The graph G

call the union of an \mathcal{M}_L -cover and an \mathcal{M}'_L -cover an $(\mathcal{M}_L \cup \mathcal{M}'_L)$ -cover. We say G has an $(\mathcal{M}_L \cup \mathcal{M}'_L)$ -colouring if there is an independent set I in $(\mathcal{M}_L \cup \mathcal{M}'_L)$ -cover with $|I| = |G|$. If an $(\mathcal{M}_L \cup \mathcal{M}'_L)$ -cover of G satisfies one of the two conditions in Theorem 52 the result of Kim and Ozeki, then we say $(G, L, \mathcal{M}_L \cup \mathcal{M}'_L)$ is *bad*.

Theorem 54 *Let G be a 2-connected outerplanar graph and let C be the outer face of G . Let L be a list assignment of G with $|L(u)| \geq \min\{\deg_G(u), 5\}$ for every $u \in V(G)$ and let \mathcal{M}_L be an arbitrary matching assignment over L . Let \mathcal{M}_L^1 and \mathcal{M}_L^2 be two arbitrary matching assignments over L in C such that for each $u_1u_2 \in E(C)$,*

- (i) *the number of vertices in $\tilde{L}(u_i)$ incident with an edge in $\mathcal{M}_{L,u_1u_2}^1 \cup \mathcal{M}_{L,u_1u_2}^2$ is at most two for each $i \in \{1, 2\}$,*
- (ii) *if $\mathcal{M}_{L,u_1u_2}^1 \cup \mathcal{M}_{L,u_1u_2}^2 \neq \emptyset$, then $|L(u_i)| \geq \min\{\deg_G(u_i) + 1, 5\}$ for each $i \in \{1, 2\}$,*
- (iii) *if there exists a vertex $w \in \tilde{L}(u_i)$ incident with three edges in $\mathcal{M}_{L,u_1u_2} \cup \mathcal{M}_{L,u_1u_2}^1 \cup \mathcal{M}_{L,u_1u_2}^2$, then $|L(u_{3-i})| \geq 5$, and*
- (iv) *for consecutive three vertices x, y, z on the outer face, if $\mathcal{M}_{L,xy}^1 \cup \mathcal{M}_{L,xy}^2 \neq \emptyset$ and $\mathcal{M}_{L,yz}^1 \cup \mathcal{M}_{L,yz}^2 \neq \emptyset$, then $|L(y)| \geq \min\{\deg_G(y) + 2, 5\}$.*

Then G admits an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring unless $(G, L, \mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ is bad.

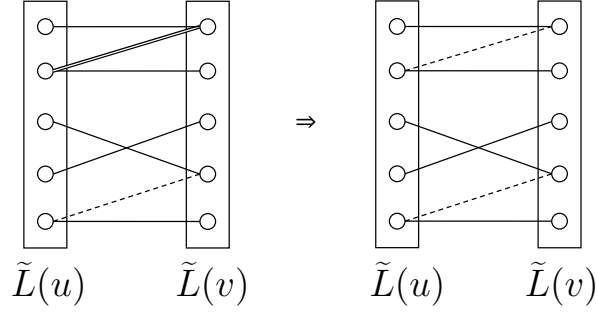


Figure 2.4: u and v are vertices lying on the outer cycle of G and the black lines, dotted lines, and doublet lines represent the edges in \mathcal{M}_L , \mathcal{M}_L^1 , \mathcal{M}_L^2 , respectively.

Let us make a few remarks on Theorem 54 before providing the proof.

- (i) In Theorem 54, \mathcal{M}_L^1 and \mathcal{M}_L^2 are two arbitrary matching assignments over L . In fact, it is possible that $\mathcal{M}_L^1 = \mathcal{M}_L^2$, and we may assume that $\mathcal{M}_L^2 = \emptyset$ in this case.
- (ii) We may assume that

$$(|\mathcal{M}_L|, |\mathcal{M}_L^1|, |\mathcal{M}_L^2|) \text{ is as large as possible in lexicographic order.}$$

Figure 2.4 shows an example. The left matching assignments in Figure 2.4 do not satisfy the assumption above since the edge in \mathcal{M}_L^2 in the left matching assignments can be regarded as an edge in \mathcal{M}_L^1 , and it increases $(|\mathcal{M}_L|, |\mathcal{M}_L^1|, |\mathcal{M}_L^2|)$.

- (iii) We construct a multigraph G_m and a matching assignment \mathcal{M}'_L over L such that G_m admits an \mathcal{M}'_L -colouring if and only if G admits an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring. For every edge $e = uv \in E(C)$, we replace e with a multiple edge e' such that the multiplicity of e' is the number of matching assignments between $\tilde{L}(u)$ and $\tilde{L}(v)$. Note that the number of matching assignments between $\tilde{L}(u)$ and $\tilde{L}(v)$ is unique by remark (ii). Figure 2.5 shows an example. We can regard $\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2$ as a matching assignment \mathcal{M}'_L of G_m over L . Then G_m admits an \mathcal{M}'_L -colouring if and only if G admits an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring.

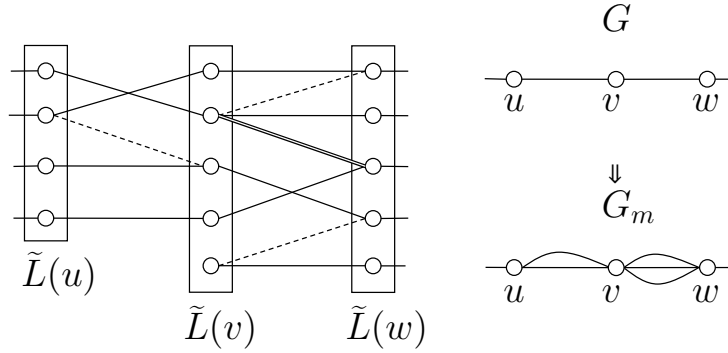


Figure 2.5: $u, v,$ and w are vertices lying on the outer cycle of G and the black lines, dotted lines, and doublet lines represent the edges in $\mathcal{M}_L, \mathcal{M}_L^1, \mathcal{M}_L^2,$ respectively. The number of matching assignments between $\tilde{L}(u)$ and $\tilde{L}(v)$ (resp. $\tilde{L}(v)$ and $\tilde{L}(w)$) is two (resp. three) and so the multiplicity of uv (resp. vw) is two (resp. three) in $G_m.$

- (iv) If $\mathcal{M}_L^1 \cup \mathcal{M}_L^2 = \emptyset$ in Theorem 54, then it corresponds to Theorem 53. Since if $\mathcal{M}_L^1 \cup \mathcal{M}_L^2 = \emptyset,$ then we do not need to consider conditions (I)–(IV) in Theorem 54. Hence we have only to prove Theorem 54.

Proof. We prove the theorem by induction on the number of faces of $G.$ Suppose that G does not admit any $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring. Let G_m be a multigraph defined as in remark (iii).

Case 1. The number of faces of G is two.

In this case, G is a cycle. If $|L(v)| = 2$ for each $v \in V(G),$ we have $\mathcal{M}_L^1 \cup \mathcal{M}_L^2 = \emptyset$ by assumption (II), then G admits an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring unless $(G, L, \mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ is bad by Theorem 52 (the result of Kim and Ozeki). Hence we may assume that there exists a vertex $v \in V(G)$ with $|L(v)| \geq 3.$ Then $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover is isomorphic to neither a t -fat ladder nor a t -fat Möbius ladder by assumption (I) of Theorem 54. Suppose that there is no vertex $u \in V(G)$ with $|L(u)| \geq 5.$ The number of matching assignments in the $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover is at most two by assumption (III) of Theorem 54. Recall that for an edge $uv \in E(C),$ the number of matching assignments between $\tilde{L}(u)$ and $\tilde{L}(v)$ in $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover corresponds to the multiplicity of uv in $G_m.$ For a vertex

$v \in V(G)$ with $|L(v)| = 4$ and a vertex $w \in \tilde{L}(v)$, the number of edges in the $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover incident with w is at most four since the number of matching assignments in the $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover is at most two. Moreover, for a vertex $v \in V(G)$ with $|L(v)| = 3$ and a vertex $w \in \tilde{L}(v)$, the number of edges in $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover incident with w is at most three by assumption (IV). Hence for each vertex $v \in V(G)$ and each vertex $w \in \tilde{L}(v)$, the number of edges in $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover incident with w is at most $|L(v)|$. By Theorem 52, G_m admits an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring and so G admits an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring. Hence there is a vertex $u \in V(G)$ with $|L(u)| \geq 5$.

We prove the case by induction on the number of vertices of G . Suppose that $|G| = 3$. Write $V(G) = \{v_1, v_2, v_3\}$ and we may assume that $|L(v_1)| \geq 5$. By assumptions (I) and (II) of Theorem 54, there is a vertex $w_2 \in \tilde{L}(v_2)$ incident with at most one edge in the $M_{L, v_1 v_2} \cup M_{L, v_1 v_2}^1 \cup M_{L, v_1 v_2}^2$. We show that there is a vertex $w_3 \in \tilde{L}(v_3)$ not adjacent to w_2 in the $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover. Suppose that w_2 is adjacent to every vertex in $\tilde{L}(v_3)$. By assumption (II) of Theorem 54, $|L(v_3)| \geq 3$. Then w_2 is adjacent to three vertices in $L(v_3)$. However, by assumption (III) of Theorem 54, $|L(v_3)| \geq 5$ and so there is a vertex $w_3 \in \tilde{L}(v_3)$ not adjacent to w_2 , a contradiction. Hence there is a vertex $w_3 \in \tilde{L}(v_3)$ not adjacent to w_2 in the $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover. Since $|L(v_1)| \geq 5$ and w_3 is adjacent to at most three vertices in $\tilde{L}(v_1)$, there is a vertex $w_1 \in \tilde{L}(v_1)$ adjacent to neither w_2 nor w_3 . Then $\{w_1, w_2, w_3\}$ is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring, a contradiction.

Suppose that $|G| \geq 4$. Recall that we choose $u \in V(G)$ with $|L(u)| \geq 5$. Let u_1 and u_2 be the neighbours of u in G and let U_i be the subset of $L(u_i)$ such that the degree of each vertex in U_i is at least two in the $\mathcal{M}_{L, u u_i} \cup \mathcal{M}_{L, u u_i}^1 \cup \mathcal{M}_{L, u u_i}^2$ for each $i = 1, 2$ (see Figure 2.6). Let $G' := G - u + u_1 u_2$ and let L' be a list assignment of G' such that $L'(v) = L(v)$ for every $v \in V(G')$. Let $\mathcal{M}_{L'}, \mathcal{M}_{L'}^1, \mathcal{M}_{L'}^2$ be matching assignments over L' obtained from $\mathcal{M}_L, \mathcal{M}_L^1, \mathcal{M}_L^2$, respectively, by deleting the edges incident with the vertices in $\tilde{L}(u)$ and adding all

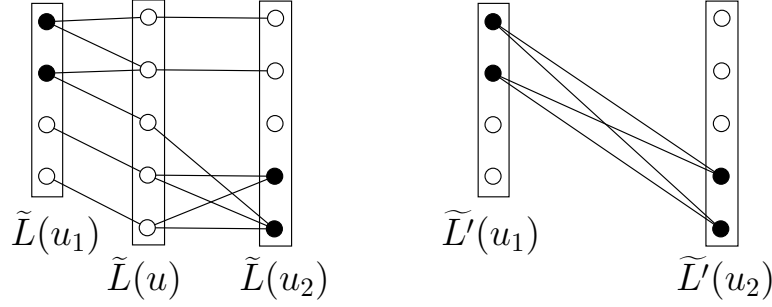


Figure 2.6: The left graph is a part of the $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover. The right graph is a part of the $(\mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ -cover. The black vertices are in U_1 and U_2 .

edges between U_1 and U_2 to $\mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1$. Note that we do not need to add any edges in $\mathcal{M}_{L'}^2$, since $|U_i| \leq 2$ for each $i \in \{1, 2\}$. We show that $(G', L', \mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ is not bad. If $U_1 = \emptyset$ or $U_2 = \emptyset$, then there is no edge between $\tilde{L}'(u_1)$ and $\tilde{L}'(u_2)$. So $(G', L', \mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ is not bad. We may assume that $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$, so for each $i = 1, 2$, we have $|L(u_i)| \geq 3$ by assumption (II) of Theorem 54 and the definition of U_i . For each $i = 1, 2$, there is a vertex $w \in \tilde{L}(u_i)$ such that the degree of w in $\mathcal{M}_{L, u_1 u_2}^1 \cup \mathcal{M}_{L, u_1 u_2}^2$ is zero. Hence $(G', L', \mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ is not bad.

By the induction hypothesis, G' has an $(\mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ -colouring I' . Then $I' \cap U_1 = \emptyset$ or $I' \cap U_2 = \emptyset$ by the construction above. Since the number of vertices in $\tilde{L}(u)$ adjacent to one of $I' \cap (\tilde{L}(u_1) \cup \tilde{L}(u_2))$ is at most four and $|L(u)| \geq 5$, there is a vertex $v \in \tilde{L}(u)$ not adjacent to $I' \cap (\tilde{L}(u_1) \cup \tilde{L}(u_2))$. Then $I' \cup \{v\}$ is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring.

Case 2. The number of faces of G is at least three.

Let v_0, v_1, \dots, v_{m+1} be consecutive vertices on the outer face such that $\deg_G(v_i) = 2$ for $1 \leq i \leq m$, $\deg_G(v_0) \geq 3$, and $\deg_G(v_{m+1}) \geq 3$; note that $v_0 v_1 \dots v_{m+1}$ is an ear of G . Let H be the subgraph of the $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover such that $V(H) = \tilde{L}(v_0) \cup \tilde{L}(v_1) \cup \dots \cup \tilde{L}(v_{m+1})$ and $E(H) = \bigcup_{0 \leq i \leq m} (\mathcal{M}_{L, v_i v_{i+1}} \cup \mathcal{M}_{L, v_i v_{i+1}}^1 \cup \mathcal{M}_{L, v_i v_{i+1}}^2)$. For an integer i with $1 \leq i \leq m$, we call a pair $\{a_i, b_i\}$ of two vertices in $\tilde{L}(v_i)$ a *good pair* if one of a_i and b_i is degree at most one in $H[\tilde{L}(v_i) \cup \tilde{L}(v_{i+1})]$. Note that

there is a good pair in $\tilde{L}(v_i)$ for every $1 \leq i \leq m$ by assumptions (I) and (II) of Theorem 54. Write $P = v_1v_2 \dots v_m$. Let $G' := G - P$ and L' be a list assignment of G' such that $L'(u) = L(u)$ for every $u \in V(G')$. Let

$$\begin{aligned}\mathcal{M}_{L'} &:= \mathcal{M}_L - \bigcup_{0 \leq i \leq m} M_{L, v_i v_{i+1}} \\ \mathcal{M}_{L'}^1 &:= \mathcal{M}_L^1 - \bigcup_{0 \leq i \leq m} M_{L, v_i v_{i+1}}^1 \\ \mathcal{M}_{L'}^2 &:= \mathcal{M}_L^2 - \bigcup_{0 \leq i \leq m} M_{L, v_i v_{i+1}}^2.\end{aligned}$$

Note that $|L'(v_i)| \geq \min\{\deg_{G'}(v_i) + 1, 5\}$ for each $i \in \{0, m+1\}$ and the edge v_0v_{m+1} is on the outer face of G' .

We now provide a sketch of our proof of Case 2. We obtain an $(\mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ -colouring of G' by induction first. We find two independent sets A and B in H consisting of good pairs (Claims 2.3.1). More precisely, independent sets A and B have the property that every pair $\{a, b\}$ with $a \in A \cap \tilde{L}(v_j)$ and $b \in B \cap \tilde{L}(v_j)$ is a good pair. One of A and B will be a part of an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring of G . In order to show that adding one of A and B to an $(\mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ -colouring of G' preserves independence, we analyse $H[\tilde{L}(v_0) \cup \tilde{L}(v_1)]$ and $H[\tilde{L}(v_m) \cup \tilde{L}(v_{m+1})]$ (Claims 2.3.2, 2.3.3, 2.3.5, and 2.3.6).

Claim 2.3.1 *Let i be a fixed integer with $1 \leq i \leq m$. For every good pair $\{a_i, b_i\}$ of two vertices in $\tilde{L}(v_i)$, there exist vertex sets $A = \{a_i, a_{i+1}, \dots, a_m\}$ and $B = \{b_i, b_{i+1}, \dots, b_m\}$ such that*

- (i) $a_j, b_j \in \tilde{L}(v_j)$ for all j such that $i \leq j \leq m$,
- (ii) $\{a_j, b_j\}$ is a good pair for all j such that $i \leq j \leq m$, and
- (iii) A and B are independent sets in H .

Proof. We show by induction on j that we can take $a_j, b_j \in \tilde{L}(v_j)$ for each $i \leq j \leq m$ such that $\{a_i, a_{i+1}, \dots, a_j\}$ and $\{b_i, b_{i+1}, \dots, b_j\}$ are independent sets in H and a_j and b_j satisfy conditions (i), (ii), (iii) of the claim. When $j = i$, the claim holds trivially. So we may assume that $j > i$. Let A_j and B_j be subsets of maximum size of $L(v_j)$ such

that $N_H(a_{j-1}) \cap A_j = \emptyset$ and $N_H(b_{j-1}) \cap B_j = \emptyset$. We show that we can take a good pair $\{a_j, b_j\}$ in $\tilde{L}(v_j)$, which means that we can find $a_j, b_j \in \tilde{L}(v_j)$ such that one of a_j and b_j has degree at most one in $H[\tilde{L}(v_j) \cup \tilde{L}(v_{j+1})]$. By induction hypothesis, $\{a_{j-1}, b_{j-1}\}$ is a good pair and so we may assume that a_{j-1} has degree at most one in $H[\tilde{L}(v_{j-1}) \cup \tilde{L}(v_j)]$.

Suppose that the degree of b_{j-1} in $H[\tilde{L}(v_{j-1}) \cup \tilde{L}(v_j)]$ is also at most one. If $H[\tilde{L}(v_j) \cup \tilde{L}(v_{j+1})]$ is a matching, then we can take any two vertices in $\tilde{L}(v_j)$ as a good pair. There must exist two vertices $a_j, b_j \in \tilde{L}(v_j)$ with $a_{j-1}a_j, b_{j-1}b_j \notin E(H)$ since $|L(v_j)| \geq 2$. Hence $H[\tilde{L}(v_j) \cup \tilde{L}(v_{j+1})]$ has some edges in $\mathcal{M}_L^1 \cup \mathcal{M}_L^2$ and $|L(v_j)| \geq 3$ by assumption (II) of the theorem. If there is no vertex in $\tilde{L}(v_j)$ that is adjacent to both a_{j-1} and b_{j-1} , then $|A_j \cup B_j| = |L(v_j)| \geq 3$. If there is a vertex in $\tilde{L}(v_j)$ that is adjacent to both a_{j-1} and b_{j-1} , then $|L(v_j)| \geq 4$ by assumption (IV) of Theorem 54, thus $|A_j \cup B_j| \geq |L(v_j)| - 1 \geq 3$. Hence $|A_j| \geq 2$, $|B_j| \geq 2$, and $|A_j \cup B_j| \geq 3$. By assumption (I) of the theorem, we can take the desired two vertices to be $a_j \in A_j$ and $b_j \in B_j$ (see the left side of Figure 2.7). Hence we may assume that the degree of b_{j-1} in $H[\tilde{L}(v_{j-1}) \cup \tilde{L}(v_j)]$ is at least two.

By assumption (II) of the theorem, $|L(v_j)| \geq 3$. Note that $|A_j| \geq 2$ and $|B_j| \geq 1$. If $H[\tilde{L}(v_j) \cup \tilde{L}(v_{j+1})]$ is a matching, then we can take any two vertices in $\tilde{L}(v_j)$ as a good pair, and there exist two vertices $a_j, b_j \in \tilde{L}(v_j)$ with $a_{j-1}a_j, b_{j-1}b_j \notin E(H)$. Hence $H[\tilde{L}(v_j) \cup \tilde{L}(v_{j+1})]$ has some edges in $\mathcal{M}_L^1 \cup \mathcal{M}_L^2$ and so $|L(v_j)| \geq 4$ by assumption (IV) of the theorem. Then $|A_j| \geq 3$ and $|B_j| \geq 2$. By assumption (I) of the theorem, we can take the desired two vertices $a_j \in A_j$ and $b_j \in B_j$ (see the right side of Figure 2.7). Now we complete the proof of Claim 2.3.1. \square

Recall that $G' = G - \cup_{i=1}^m v_i$, L' is the list assignment of G' , $|L'(v_i)| \geq \min\{\deg_{G'}(v_i) + 1, 5\}$ for each $i \in \{0, m+1\}$, and the edge v_0v_{m+1} is on the outer face of G' .

Claim 2.3.2 *One of $H[\tilde{L}(v_0) \cup \tilde{L}(v_1)]$ and $H[\tilde{L}(v_m) \cup \tilde{L}(v_{m+1})]$ has some edges in $\mathcal{M}_L^1 \cup \mathcal{M}_L^2$.*

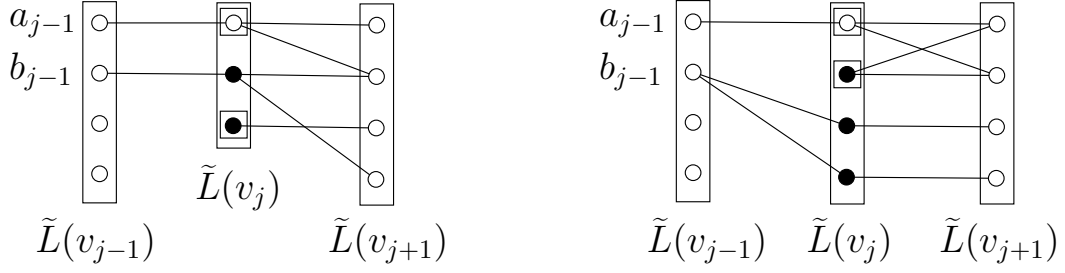


Figure 2.7: The left graph is a part of the $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover when the case that the degree of b_{j-1} in $H[\tilde{L}(v_{j-1}) \cup \tilde{L}(v_j)]$ is one and the right one is the other case. The black vertices are in A_j and the vertices surrounded by squares are in B_j .

Proof. Suppose that both $H[\tilde{L}(v_0) \cup \tilde{L}(v_1)]$ and $H[\tilde{L}(v_m) \cup \tilde{L}(v_{m+1})]$ are matchings. Let $\{a_1, b_1\}$ be a good pair of two vertices in $\tilde{L}(v_1)$. By Claim 2.3.1, there exist two vertex sets A and B with $a_1 \in A$ and $b_1 \in B$ satisfying conditions (i)–(iii) of Claim 2.3.1. Write $\{a_m\} = A \cap \tilde{L}(v_m)$ and $\{b_m\} = B \cap \tilde{L}(v_m)$. Let $c_0 \in \tilde{L}(v_0) \cap N_H(a_1)$, $d_0 \in \tilde{L}(v_0) \cap N_H(b_1)$, $c_{m+1} \in \tilde{L}(v_{m+1}) \cap N_H(a_m)$, and $d_{m+1} \in \tilde{L}(v_{m+1}) \cap N_H(b_m)$ (possibly some of c_0, d_0, c_{m+1} , and d_{m+1} do not exist). We add the edges $c_0 d_{m+1}$ and $d_0 c_{m+1}$ to $\mathcal{M}_{L'}^1$ if they are not in $\mathcal{M}_{L'}^1$ (see Figure 2.8). Since the graph induced by the edges connecting $\tilde{L}(v_0)$ and $\tilde{L}(v_{m+1})$ in the $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -cover is a matching and we add at most two edges between $\tilde{L}(v_0)$ and $\tilde{L}(v_{m+1})$. Hence we have $(G', L', \mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ is not bad. It is clear that G' satisfies the assumptions in Theorem 54. By the induction hypothesis, G' has an $(\mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ -colouring I' . If either $d_0 \in I'$ or $d_{m+1} \in I'$, then $I := I' \cup A$ otherwise $I := I' \cup B$. Then I is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring of G . This is a contradiction. \square

We may assume that $H[\tilde{L}(v_0) \cup \tilde{L}(v_1)]$ has some edges in $\mathcal{M}_L^1 \cup \mathcal{M}_L^2$. Let U_0 and U_{m+1} be vertex sets such that $U_i \subseteq \tilde{L}(v_i)$ and the degree of each vertex in U_i is at least two in H for each $i \in \{0, m+1\}$.

Claim 2.3.3 For a vertex $u_0 \in \tilde{L}(v_0)$ with degree at most one in $H[\tilde{L}(v_0) \cup$

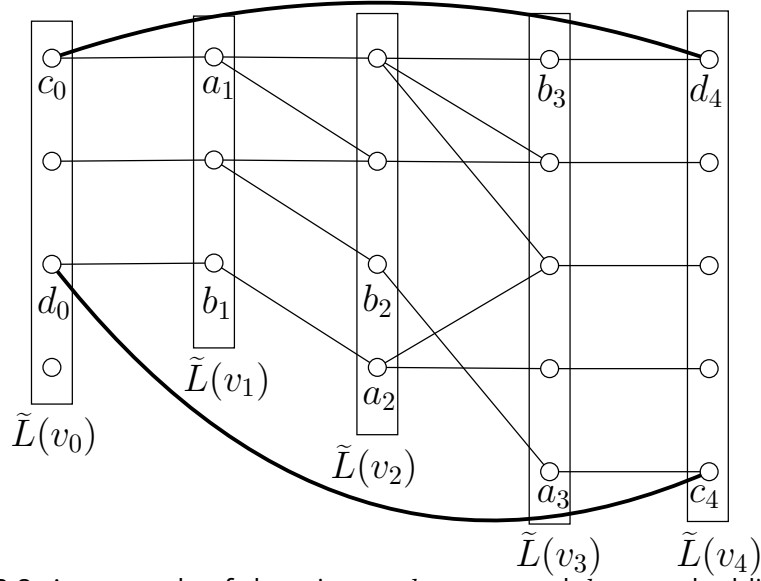


Figure 2.8: An example of choosing c_0, d_0, c_{m+1} , and d_{m+1} and adding the edges c_0d_{m+1} and d_0c_{m+1} for $m = 3$.

$\tilde{L}(v_1)]$, there exists a good pair $\{a_1, b_1\}$ of two vertices in $\tilde{L}(v_1)$ such that a_1 and b_1 are both not adjacent to u_0 .

Proof. If $|L(v_1)| \geq 4$, then there are at least three vertices in $\tilde{L}(v_1)$ not adjacent to u_0 and so we can take a good pair by assumption (I). Hence we may assume that $|L(v_1)| = 3$. Since $H[\tilde{L}(v_0) \cup \tilde{L}(v_1)]$ has some edges in $\mathcal{M}_L^1 \cup \mathcal{M}_L^2$, it follows from assumption (IV) that $H[\tilde{L}(v_1) \cup \tilde{L}(v_2)]$ is a matching. Then we can take a good pair. \square

Recall that U_0 is the vertex set such that $U_0 \subseteq \tilde{L}(v_0)$ and the degree of each vertex in U_0 is at least two in H . For $1 \leq i \leq m$, if all the vertices in U_{i-1} are adjacent to the same vertices in $\tilde{L}(v_i)$ and $|\tilde{L}(v_i) \setminus N_H(U_{i-1})| \leq 2$, then we define $U_i = \tilde{L}(v_i) \setminus N_H(U_{i-1})$ (if U_{i-1} is not defined, then we do not define U_i). Note that $U_i \neq \emptyset$ by the assumptions (II)-(IV). Let j_{max} be the largest integer j in $0 \leq j \leq m$ such that $U_{j_{max}}$ is defined. If $j_{max} \geq 1$, we take an arbitrary vertex u_i from U_i for each $0 \leq i \leq j_{max} - 1$ and let U be the set of such vertices (see Figure 2.9). Since we deal with the case that there is an $(\mathcal{M}_{L''} \cup \mathcal{M}_{L''}^1 \cup \mathcal{M}_{L''}^2)$ -colouring I' of G' such that the degree of the vertex in $I' \cap \tilde{L}(v_{m+1})$ is at most one

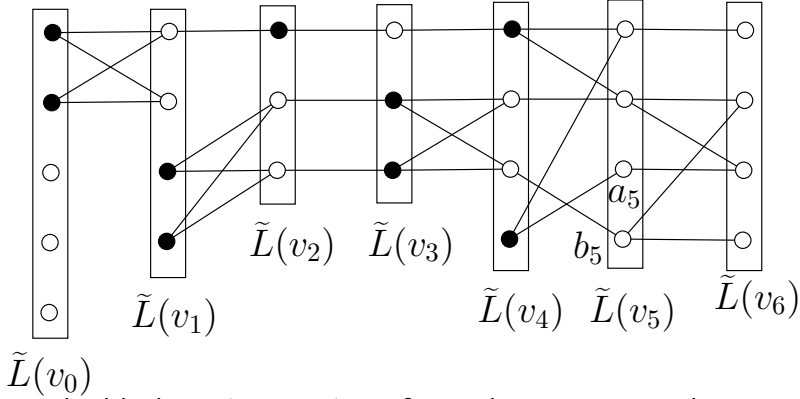


Figure 2.9: The black vertices are in U_i for each $1 \leq i \leq m$ and $j_{max} = 4 \neq m$.

in H several times after this, we will show a technical claim to handle those situations.

Claim 2.3.4 *Let L'' be a list assignment of G' and $L''(u) = L(u)$ for every $u \in V(G')$. Let $\mathcal{M}_{L''}, \mathcal{M}_{L''}^1, \mathcal{M}_{L''}^2$ be matching assignments over L'' satisfying assumptions (I)–(IV). Suppose that there is an $(\mathcal{M}_{L''} \cup \mathcal{M}_{L''}^1 \cup \mathcal{M}_{L''}^2)$ -colouring I' of G' such that the degree of the vertex in $I' \cap \tilde{L}(v_{m+1})$ is at most one in H . Then the following properties hold.*

- (i) *The degree of the vertex in $I' \cap \tilde{L}(v_0)$ is at least two in H .*
- (ii) *$j_{max} = m$.*

Proof. (i) Suppose that the vertex in $I' \cap \tilde{L}(v_0)$ is degree at most one in H . Note that $I' \cap \tilde{L}(v_{m+1})$ is a set that contains a single vertex. By Claim 2.3.3, there exists a good pair $\{a_1, b_1\}$ of two vertices in $\tilde{L}(v_1)$ such that a_1 and b_1 are not adjacent to the vertex in $I' \cap \tilde{L}(v_0)$. By Claim 2.3.1, there exist two vertex sets $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_m\}$ satisfying the conditions (i)–(iii) of Claim 2.3.1. Since the degree of the vertex in $I' \cap \tilde{L}(v_{m+1})$ is at most one, one of a_m and b_m , say a_m , is not adjacent to the vertex in $I' \cap \tilde{L}(v_{m+1})$. Then $I' \cup A$ is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring of G , a contradiction.

(ii) Suppose that $j_{max} \leq m - 1$. We prove the following subclaim.

Subclaim 2.3.4.1 *For any good pair $\{a_{j_{max}+1}, b_{j_{max}+1}\}$ of two vertices in $\tilde{L}(v_{j_{max}+1})$, either $a_{j_{max}+1}$ or $b_{j_{max}+1}$ is adjacent to every vertex in $U_{j_{max}}$.*

Proof. Suppose that there is a good pair $\{a_{j_{max}+1}, b_{j_{max}+1}\}$ of two vertices in $\tilde{L}(v_{j_{max}+1})$ such that both $a_{j_{max}+1}$ and $b_{j_{max}+1}$ are not adjacent to one of vertices in $U_{j_{max}}$. By Claim 2.3.1, there exist vertex sets $A = \{a_{j_{max}+1}, \dots, a_m\}$ and $B = \{b_{j_{max}+1}, \dots, b_m\}$ satisfying conditions (i)–(iii) of Claim 2.3.1. Since the vertex in $I' \cap \tilde{L}(v_{m+1})$ is degree one in H , one of a_m and b_m , say a_m , is not adjacent to the vertex in $I' \cap \tilde{L}(v_{m+1})$ in H . Let $u_{j_{max}}$ be a vertex in $U_{j_{max}}$ not adjacent to $a_{j_{max}+1}$. Since $U_0 \cap I' \neq \emptyset$ by Claim 2.3.4 (i), $I' \cup U \cup \{u_{j_{max}}\} \cup A$ is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring of G . This is a contradiction. \square

Suppose that $\tilde{L}[L(v_{j_{max}}) \cup \tilde{L}(v_{j_{max}+1})]$ is a matching. If $|U_{j_{max}}| \geq 2$, then for any good pair $\{a_{j_{max}+1}, b_{j_{max}+1}\}$ of two vertices in $\tilde{L}(v_{j_{max}+1})$, both $a_{j_{max}+1}$ and $b_{j_{max}+1}$ are not adjacent to one of the vertices in $U_{j_{max}}$. This contradicts Subclaim 2.3.4.1. Hence we may assume that $|U_{j_{max}}| = 1$. Since $H[\tilde{L}(v_{j_{max}}) \cup \tilde{L}(v_{j_{max}+1})]$ is a matching, we have $|\tilde{L}(v_{j_{max}+1}) \setminus N_H(U_{j_{max}})| \geq 3$ by the definition of j_{max} . We have a good pair $\{a_{j_{max}+1}, b_{j_{max}+1}\}$ in $\tilde{L}(v_{j_{max}+1})$ such that both $a_{j_{max}+1}$ and $b_{j_{max}+1}$ are not adjacent to the vertices in $U_{j_{max}}$. This contradicts Subclaim 2.3.4.1. Hence $H[\tilde{L}(v_{j_{max}}) \cup \tilde{L}(v_{j_{max}+1})]$ is not a matching and so $|L(v_{j_{max}+1})| \geq 3$ by assumption (II) of Theorem 54.

Suppose $|L(v_{j_{max}+1})| \geq 4$. We show that $|\cup_{u_j \in U_{j_{max}}} (\tilde{L}(v_{j_{max}+1}) \setminus N_H(u_j))| \geq 3$. If $|L(v_{j_{max}+1})| = 4$, then by assumption (IV) of Theorem 54 and the definition of $U_{j_{max}}$, the number of vertices that are both adjacent to the vertices of $U_{j_{max}}$ in $\tilde{L}(v_{j_{max}+1})$ is at most one. If $|L(v_{j_{max}+1})| = 5$, then the number of vertices that are both adjacent to the vertices of $U_{j_{max}}$ in $\tilde{L}(v_{j_{max}+1})$ is at most two. If $|L(v_{j_{max}+1})| \geq 6$, then the number of vertices that are both adjacent to the vertices of $U_{j_{max}}$ in $\tilde{L}(v_{j_{max}+1})$ is at most three. Hence in both cases, $|\cup_{u_j \in U_{j_{max}}} (\tilde{L}(v_{j_{max}+1}) \setminus N_H(u_j))| \geq 3$ holds. This together with assumption (I) implies that there is a good pair $\{a_{j_{max}+1}, b_{j_{max}+1}\} \subseteq \cup_{u_j \in U_{j_{max}}} (\tilde{L}(v_{j_{max}+1}) \setminus N_H(u_j))$ of two vertices in $\tilde{L}(v_{j_{max}+1})$ such that both $a_{j_{max}+1}$ and $b_{j_{max}+1}$ are not adjacent to one of vertices in $U_{j_{max}}$. This contradicts Subclaim 2.3.4.1. Hence we may assume $|L(v_{j_{max}+1})| = 3$.

By assumption (IV), $H[\tilde{L}(v_{j_{max}+1}) \cup \tilde{L}(v_{j_{max}+2})]$ is a matching and so every pair of two vertices in $\tilde{L}(v_{j_{max}+1})$ is a good pair. Hence we can take a good pair $\{a_{j_{max}+1}, b_{j_{max}+1}\}$ of two vertices in $\tilde{L}(v_{j_{max}+1})$ such that both $a_{j_{max}+1}$ and $b_{j_{max}+1}$ are not adjacent to one of the vertices in $U_{j_{max}}$. This contradicts Subclaim 2.3.4.1. \square

Claim 2.3.5 *Both $H[\tilde{L}(v_0) \cup \tilde{L}(v_1)]$ and $H[\tilde{L}(v_m) \cup \tilde{L}(v_{m+1})]$ have some edges in $\mathcal{M}_L^1 \cup \mathcal{M}_L^2$.*

Proof. Suppose that $H[\tilde{L}(v_m) \cup \tilde{L}(v_{m+1})]$ has no edges in $\mathcal{M}_L^1 \cup \mathcal{M}_L^2$. By Claim 2.3.2, $H[\tilde{L}(v_0) \cup \tilde{L}(v_1)]$ has some edges in $\mathcal{M}_L^1 \cup \mathcal{M}_L^2$.

Subclaim 2.3.5.1 $|L(v_0)| \geq 5$.

Proof. Suppose that $|L(v_0)| \leq 4$. By the assumption (II) of Theorem 54, $\deg_G(v_0) \geq 3$ and we obtain $\deg_G(v_0) = 3$ and $\deg_{G'}(v_0) = 2$. Let L'' be a list assignment of G' such that

$$L''(u) = \begin{cases} L'(u) - \{c \in L(u) : (u, c) \in U_0\}, & \text{if } u = v_0, \\ L'(u), & \text{otherwise.} \end{cases}$$

Let $\mathcal{M}_{L''}, \mathcal{M}_{L''}^1, \mathcal{M}_{L''}^2$ be matching assignments over L'' obtained from $\mathcal{M}_{L'}, \mathcal{M}_{L'}^1, \mathcal{M}_{L'}^2$, respectively, by deleting the edges incident with the vertices in U_0 . Note that $1 \leq |U_0| \leq 2$ by assumption (I) of Theorem 54, so $2 \leq |L''(v_0)| \leq 3$, $|L''(v_{m+1})| \geq \deg_G(v_{m+1}) \geq 3$, and the graph induced by the edges connecting $\tilde{L}''(v_0)$ and $\tilde{L}''(v_{m+1})$ is a matching. Suppose that $(G', L'', \mathcal{M}_{L''} \cup \mathcal{M}_{L''}^1 \cup \mathcal{M}_{L''}^2)$ is bad, then the $(\mathcal{M}_{L''} \cup \mathcal{M}_{L''}^1 \cup \mathcal{M}_{L''}^2)$ -cover of G' must be isomorphic to a ladder of odd length or a Möbius ladder of even length by assumption (I) of Theorem 54. Since $|L''(v_{m+1})| \geq 3$, the $(\mathcal{M}_{L''} \cup \mathcal{M}_{L''}^1 \cup \mathcal{M}_{L''}^2)$ -cover of G' is not isomorphic to a ladder of odd length or a Möbius ladder of even length, a contradiction. Hence $(G', L'', \mathcal{M}_{L''} \cup \mathcal{M}_{L''}^1 \cup \mathcal{M}_{L''}^2)$ is not bad. By the induction hypothesis, G' has an $(\mathcal{M}_{L''} \cup \mathcal{M}_{L''}^1 \cup \mathcal{M}_{L''}^2)$ -colouring I' . Since the vertices in $I' \cap (\tilde{L}(v_0) \cup \tilde{L}(v_{m+1}))$ are degree at most one in H , this contradicts Claim 2.3.4 (i). Hence the subclaim holds. \square

Since $H[\tilde{L}(v_m) \cup \tilde{L}(v_{m+1})]$ is a matching, it follows from Claim 2.3.4 (ii) that we obtain $j_{max} = m$. Suppose that $|U_m| \geq 2$. Since $H[\tilde{L}(v_m) \cup \tilde{L}(v_{m+1})]$ is a matching, one of the vertices in U_m , say u_m , is not adjacent to the vertex in $I' \cap \tilde{L}(v_{m+1})$. Then $I' \cup U \cup \{u_m\}$ is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring of G . This is a contradiction. Hence $|U_m| = 1$ and write $U_m = \{u_m\}$. Let w_{m+1} be the vertex in $\tilde{L}(v_{m+1})$ adjacent to u_m . We add the edges between U_0 and w_{m+1} to $\mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2$. Since $|L(v_0)| \geq 5$, such new $\mathcal{M}_{L'}, \mathcal{M}_{L'}^1, \mathcal{M}_{L'}^2$ satisfy the assumptions of Theorem 54 and $(G', L', \mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ is not bad. By the induction hypothesis, G' has an $(\mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ -colouring I'' . By Claim 2.3.4 (i), $I'' \cap U_0 \neq \emptyset$ and so $w_{m+1} \notin I''$. However $I'' \cup U \cup \{u_m\}$ is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring of G , a contradiction. □

Claim 2.3.6 $|L(v_0)| \geq 5$ and $|L(v_{m+1})| \geq 5$.

Proof. Suppose that one of $|L(v_0)|$ and $|L(v_{m+1})|$ is at most four. We may assume that $|L(v_{m+1})| \leq 4$. Let L'' be a list assignment of G' such that

$$L''(u) = \begin{cases} L'(u) - \{c \in L'(u) : (u, c) \in U_{m+1}\} & \text{if } u = v_{m+1} \\ L'(u) & \text{otherwise.} \end{cases}$$

Let $\mathcal{M}_{L''}, \mathcal{M}_{L''}^1, \mathcal{M}_{L''}^2$ be matching assignments over L'' obtained from $\mathcal{M}_{L'}, \mathcal{M}_{L'}^1, \mathcal{M}_{L'}^2$, respectively, by deleting the edges incident with the vertices in U_{m+1} . We have $2 \leq |L''(v_{m+1})| \leq 3$, $|L''(v_0)| \geq 4$, and the graph induced by the edges connecting $\tilde{L}''(v_0)$ and $\tilde{L}''(v_{m+1})$ is a matching. Hence, $(G', L'', \mathcal{M}_{L''} \cup \mathcal{M}_{L''}^1 \cup \mathcal{M}_{L''}^2)$ is not bad and satisfies the assumptions of the theorem. By the induction hypothesis, G' has an $(\mathcal{M}_{L''} \cup \mathcal{M}_{L''}^1 \cup \mathcal{M}_{L''}^2)$ -colouring I' . Since the vertex in $I' \cap \tilde{L}(v_{m+1})$ is degree one in H , it follows from Claim 2.3.4 (ii) that we obtain $j_{max} = m$.

We show that $|U_m| \geq 2$. By Claim 2.3.5 and assumption (II) of the theorem, $|L(v_m)| \geq 3$. If $|L(v_m)| = 4$, then $|N_H(U_{m-1}) \cap \tilde{L}(v_m)| \leq 2$ by assumption (III) of the theorem and so $|U_m| \geq 2$. If $|L(v_m)| \geq 5$, then $|N_H(U_{m-1}) \cap \tilde{L}(v_m)| \leq 3$ and so $|U_m| \geq 2$. Suppose that $|L(v_m)| = 3$. By

Claim 2.3.5 and assumption (IV) of the theorem, $|N_H(U_{m-1}) \cap \tilde{L}(v_m)| = 1$ and so $|U_m| \geq 2$. Hence $|U_m| \geq 2$. This together with $I' \cap U_{m+1} = \emptyset$ implies that there is a vertex u_m in U_m not adjacent to the vertex in $I' \cap \tilde{L}(v_{m+1})$. Then $I' \cup U \cup \{u_m\}$ is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring of G . This is a contradiction. \square

We add the edges between U_0 and U_{m+1} to $\mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2$. By Claim 2.3.6 and assumption (I) of Theorem 54, $(G', L', \mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ is not bad and satisfies the assumptions of Theorem 54. By the induction hypothesis, G' has an $(\mathcal{M}_{L'} \cup \mathcal{M}_{L'}^1 \cup \mathcal{M}_{L'}^2)$ -colouring I' . Note that either $U_0 \cap I' = \emptyset$ or $U_{m+1} \cap I' = \emptyset$. We may assume that $U_0 \cap I' = \emptyset$.

Suppose that $m = 1$. By Claim 2.3.5 and assumption (IV) of Theorem 54, $|L(v_1)| \geq 4$. Since $U_0 \cap I' = \emptyset$, there is at most one vertex in $\tilde{L}(v_1)$ adjacent to the vertex in $I' \cap \tilde{L}(v_0)$. If $|L(v_1)| = 4$, then there are at most two vertices in $\tilde{L}(v_1)$ adjacent to the vertex in $I' \cap \tilde{L}(v_{m+1})$. If $|L(v_1)| \geq 5$, then there are at most three vertices in $\tilde{L}(v_1)$ adjacent to the vertex in $I' \cap L(v_{m+1})$. In the both cases, there is at least one vertex in $\tilde{L}(v_1)$ not adjacent to $I' \cap (\tilde{L}(v_0) \cup \tilde{L}(v_{m+1}))$. Then we add such a vertex to I' and it is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring of G . This is a contradiction. Hence $m \geq 2$.

We show that there exists a good pair $\{a_1, b_1\}$ of two vertices in $\tilde{L}(v_1)$ such that a_1 and b_1 are not adjacent to $I' \cap \tilde{L}(v_0)$. If $|L(v_1)| \geq 4$, then since $U_0 \cap I' = \emptyset$, there are at least three vertices in $\tilde{L}(v_1)$ not adjacent to the vertex in $I' \cap \tilde{L}(v_0)$ and so we can take a desired good pair by assumption (I) of Theorem 54. Hence we may assume that $|L(v_1)| = 3$. By Claim 2.3.5 and assumption (IV) of Theorem 54, $H[\tilde{L}(v_1) \cup \tilde{L}(v_2)]$ is a matching. Hence the pair of two vertices not adjacent to the vertex in $I' \cap \tilde{L}(v_0)$ is a desired good pair.

By Claim 2.3.1, there exist vertex sets $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_m\}$ satisfying conditions (i)–(iii) of Claim 2.3.1. Suppose that $|L(v_m)| = 3$. Then by assumption (III) of Theorem 54, there is a vertex u_m in $\tilde{L}(v_m)$ not adjacent to the vertex in $I' \cap \tilde{L}(v_{m+1})$. By Claim 2.3.5 and assumption (IV) of Theorem 54, $H[\tilde{L}(v_{m-1}) \cup \tilde{L}(v_m)]$ is a matching

and so one of a_{m-1} and b_{m-1} , say a_{m-1} , is not adjacent to u_m . Then $I' \cup \{u_m\} \cup (A \setminus \{a_m\})$ is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring of G , a contradiction. Hence $|L(v_m)| \geq 4$. By condition (ii) of Claim 2.3.1, one of a_{m-1} and b_{m-1} , say a_{m-1} , is degree one in $H[\tilde{L}(v_{m-1}) \cup \tilde{L}(v_m)]$. If $|L(v_m)| = 4$, then there are at most two vertices in $\tilde{L}(v_m)$ adjacent to the vertex in $I' \cap \tilde{L}(v_{m+1})$ by assumption (III) of Theorem 54. If $|L(v_m)| \geq 5$, then there are at most three vertices in $\tilde{L}(v_m)$ adjacent to the vertex in $I' \cap \tilde{L}(v_{m+1})$. In both cases, there are at least two vertices in $\tilde{L}(v_m)$ not adjacent to the vertex in $I' \cap \tilde{L}(v_{m+1})$ and so there is a vertex w_m in $\tilde{L}(v_m)$ not adjacent to the vertices in $I' \cap \tilde{L}(v_{m+1}) \cup \{a_{m-1}\}$. Then $I' \cup \{w_m\} \cup (A \setminus \{a_m\})$ is an $(\mathcal{M}_L \cup \mathcal{M}_L^1 \cup \mathcal{M}_L^2)$ -colouring of G , a contradiction. So we complete the proof of Theorem 54. \square

3 - Beyond the fractional Reed bound for triangle-free graphs

Given a graph H , we let $\chi_f(d, H)$ be the supremum of the fractional chromatic numbers over all H -free graphs of maximum degree at most d . In this chapter, we study $\chi_f(4, K_3)$.

3.1 . Introduction

When H is a complete graph, the study of $\chi_f(d, H)$ falls in the domain of Ramsey theory, a domain which emerged in the 1930s following seminal results by van der Waerden [104] and by Ramsey [90], and has attracted a lot of attention ever since. An important result in this case is due to Molloy and Reed [82, Theorem 21.7, p. 244]: known as “the fractional Reed bound”, it states that $\chi_f(d, K_n) \leq \frac{d+n}{2}$ for all integers $d, n \geq 2$.

In this chapter, we focus on the case $H = K_3$, which is closely related to *off-diagonal Ramsey numbers*. It has been established [39] that $\chi_f(3, K_3) = 14/5$. The same question for larger values of the maximum degree is still open. At one end of the spectrum, we know that $\chi_f(4, K_3)$ lies between 3.25 (see Figure 3.1) and 3.5 (by the fractional Reed bound). At the other end of the spectrum, one has $\chi_f(d, K_3) \leq (1 + o(1)) d / \ln d$ as $d \rightarrow \infty$, which is a consequence of a result by Molloy [80], and one can infer from a study of random d -regular graphs by Bollobás [13] that $\chi_f(d, K_3) \geq \frac{d}{2 \ln d}$.

A first study of $\chi_f(d, K_3)$ has been made by Pirot and Sereni [88], with the

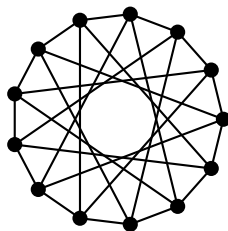


Figure 3.1: A graph certifying that $\chi_f(4, K_3) \geq 3.25$.

help of a so-called *greedy fractional colouring algorithm* (GFCA). This algorithm takes as input a graph G and a probability distribution π on the independent sets of any induced subgraph of G , and returns a fractional colouring of G whose weight is bounded by a function of π and G . They proved the following.

Theorem 55 (Piroot & Sereni, 2021) *For every integer d ,*

$$\chi_f(d, K_3) \leq 1 + \min_{k \in \mathbb{N}} \inf_{\lambda > 0} \frac{(1 + \lambda)^k + \lambda(1 + \lambda)d}{\lambda(1 + k\lambda)}.$$

The upper bound in Theorem 55 which can be effectively computed from this formula improves on the fractional Reed bound as soon as $d \geq 17$. When $d \in [1, 16]$, the bound still follows $\frac{d+3}{2}$.

We use the GFCA to obtain the following result.

Theorem 56 $\chi_f(4, K_3) < 3.4663$.

3.1.1 . Notations

Let G be a given graph. If J is a subset of vertices of G , then we write $N_G(J)$ for the set of vertices that are not in J and have a neighbour in J , while $N_G[J]$ is $N_G(J) \cup J$. We omit the graph subscript when there is no ambiguity, write $N(v)$ for $N(\{v\})$, and we sometimes write $N_X(v)$ instead of $N(v) \cap X$, for any subset of vertices $X \subseteq V(G)$. As mentioned earlier, the set of all independent sets of G is $\mathcal{I}(G)$. If w is a mapping from $\mathcal{I}(G)$ to \mathbb{R} , then for every vertex $v \in V(G)$ we set

$$w[v] := \sum_{\substack{I \in \mathcal{I}(G) \\ v \in I}} w(I).$$

Further, if \mathcal{I} is a collection of independent sets of G , then $w(\mathcal{I}) := \sum_{I \in \mathcal{I}} w(I)$. If I is an independent set of a graph G , a vertex v is *covered* by I if v belongs to I or has a neighbour in I . A vertex that is not covered by I is *uncovered* (by I).

3.1.2 . Intuition of our method

In order to apply the GFCA to a graph G , one needs to have some probability distribution over the independent sets of any given induced subgraph H of G . When this probability distribution has a relatively uniform coverage of the closed neighbourhoods in H — this is referred to as (α, β) -*local occupancy* in works using

the GFCA — one can deduce an upper bound on the fractional chromatic number of G .

In previous works using the GFCA, the input probability distribution has always been the hard-core distribution on some family of independent sets of H . In this paper, we construct our random independent sets in several steps, where at each step we lie within the subgraph of H induced by the vertices uncovered by the random independent set constructed so far. The first step is the hard-core distribution with fugacity λ , and the last step is given by a specific fractional colouring with local demand (the demand is a decreasing function of the degree).

3.2 . Prerequisites

In this section we introduce the notions that will be needed to derive our results.

3.2.1 . Greedy fractional colouring algorithm

Our results on fractional colouring are obtained using a greedy algorithm analysed in a recent work [34]. This algorithm is a generalisation of an algorithm first described in the book of Molloy and Reed [82, p. 245] for the uniform distribution over maximum independent sets. The setting here is, for each induced subgraph H of the graph we wish to fractionally colour, a probability distribution over the independent sets of H .

Lemma 3 (Davies *et al.*, 2020) *Let G be a graph given with couples (α_v, β_v) for every vertex $v \in V(G)$. For every induced subgraph H of G , let \mathbf{I}_H be a random independent set of H drawn according to a given probability distribution, and assume that*

$$\alpha_v \mathbb{P}[v \in \mathbf{I}_H] + \beta_v \mathbb{E}[|\mathbf{I}_H \cap N(v)|] \geq 1,$$

for every $v \in V(H)$. Then the GFCA produces a fractional colouring w of G which certifies that

$$\chi_f(G) \leq \max_{v \in V(G)} \alpha_v + \beta_v \deg(v).$$

Algorithm 1: The Greedy Fractional Colouring Algorithm
(GFCA)

```

for  $I \in \mathcal{I}(G)$  do
   $w(I) \leftarrow 0$ 
 $H \leftarrow G$ 
while  $|V(H)| > 0$  do
   $\iota \leftarrow \min_{v \in V(H)} \frac{1 - w[v]}{\mathbb{P}[v \in \mathbf{I}_H]}$ 
  for  $I \in \mathcal{I}(H)$  do
     $w(I) \leftarrow w(I) + \mathbb{P}[\mathbf{I}_H = I] \iota$ 
   $H \leftarrow H - \{v \in V(H) : w[v] = 1\}$ 

```

We note that in Lemma 3, although there is one probability distribution on each induced subgraph, the couple (α_v, β_v) associated with each vertex is fixed once and for all, which somewhat ties together the different probability distributions involved.

In our setting (the class of triangle-free graphs of bounded degree), the only way to distinguish vertices when we look only at their first neighbourhood is by considering their degree (and possibly those of their neighbours). In particular, if G is a regular graph, then there is no way of distinguishing its vertices in our setting. For that reason, we will only apply Lemma 3 with all couples $(\alpha_v, \beta_v)_{v \in V(G)}$ equal. To ensure optimality, our task is to solve the following linear program.

Definition 13. Let G be a graph, and let φ map each (induced) subgraph $H \subseteq G$ to a random independent set $\varphi(H) \in \mathcal{I}(H)$. We let $\text{LP}_\varphi(G)$, which we call the *linear program associated with* (G, φ) , be defined as follows.

$$\text{LP}_\varphi(G): \begin{cases} \text{Minimise} & \alpha + \beta \Delta \\ \text{such that} & \alpha \mathbb{P}[v \in \varphi(H)] + \beta \mathbb{E}[|\varphi(H) \cap N(v)|] \geq 1, \quad \text{for all } H \subseteq G \\ & v \in V(H); \\ & \alpha, \beta \geq 0. \end{cases}$$

First, we need to compute the constraints of the linear program described above. To that end, we define the following.

Definition 14. Let H be a graph and \mathbf{I} a random independent set of H . For every vertex $v \in V(H)$, we let

$$e_{\mathbf{I}}(v) := \begin{pmatrix} \mathbb{P}[v \in \mathbf{I}] \\ \mathbb{E}[|N(v) \cap \mathbf{I}|] \end{pmatrix}$$

be the *constraint* of \mathbf{I} on v .

With the above definition in hand, the constraints of the linear program are all of the form $(\alpha, \beta) \cdot e_{\varphi(H)}(v) \geq 1$, for every $H \subseteq G$ and $v \in V(H)$.

We note that if we generate the whole set of constraints $\{e_{\varphi(H)}(v) : H \subseteq G, v \in V(H)\}$, we may end up with a huge number of constraints. It is however possible to considerably reduce the number of constraints by restricting to a specific subset. This requires the introduction of a partial ordering on the set of constraints. In the end, we will keep only minimal constraints with respect to that order, and argue that this does not affect the result of the linear program.

Definition 15. Given two constraints $e_0 = (p_0, q_0)^T$ and $e_1 = (p_1, q_1)^T$, we say that e_0 is *tighter* than e_1 , or equivalently that e_1 is *looser* than e_0 , if they satisfy $p_0 \leq p_1$ and $p_0 + q_0 \leq p_1 + q_1$. We denote it $e_0 \preceq e_1$.

We can now introduce a new linear program obtained by keeping only minimal constraints with respect to (\preceq) from the complete linear program.

Definition 16. Let G be a graph, and let φ map each (induced) subgraph $H \subseteq G$ to a random independent set $\varphi(H) \in \mathcal{I}(H)$. We denote \mathcal{E}_φ the set of minimal constraints with respect to (\preceq) among $\{e_{\varphi(H)}(v) : H \subseteq G, v \in V(H)\}$. We let $\text{LP}_\varphi^*(G, \varphi)$, which we call the *reduced linear program associated with (G, φ)* , be defined as follows.

$$\text{LP}_\varphi^*(G) : \begin{cases} \text{Minimise} & \alpha + \beta \Delta \\ \text{such that} & \alpha p + \beta q \geq 1, \quad \text{for each } \begin{pmatrix} p \\ q \end{pmatrix} \in \mathcal{E}_\varphi; \\ & \alpha, \beta \geq 0. \end{cases}$$

We now argue that solving $\text{LP}_\varphi^*(G)$ instead of $\text{LP}_\varphi(G)$ is enough for our purpose. This is justified by the following claim.

Claim 3.2.1 *If the solution of the linear program $\text{LP}_\varphi^*(G)$ is at most $\Delta(G) + 1$, then it matches the solution of $\text{LP}_\varphi(G)$.*

Proof. Let (α, β) realise the solution of the linear program $\text{LP}_\varphi^*(G)$, which we assume to be at most $\Delta + 1$. So, in particular, the constraint generated by a subgraph $H = \{v\}$ of G must be satisfied, i.e. $\alpha \geq 1$. If we now assume for the sake of contradiction that $\beta > \alpha$, then the solution of the linear program will be more than $\Delta(G) + 1$, which contradicts the assumption. So we have $\alpha \geq \beta$. We now argue that every constraint $e_{\mathbf{I}_H}(v) = (x, y)^T$ for $H \subseteq G$ and $v \in V(H)$ is satisfied. Indeed, there is a constraint $e' = (x', y')^T$ that is minimal with respect to (\preceq) and such that $e' \preceq e_{\mathbf{I}_H}(v)$. The constraint e' appears in the linear program, so it is satisfied by assumption; we have $\alpha x' + \beta y' \geq 1$. So we have

$$\begin{aligned} \alpha x + \beta y &= (\alpha - \beta)x + \beta(x + y) \\ &\geq (\alpha - \beta)x' + \beta(x' + y') \\ &= \alpha x' + \beta y' \geq 1. \end{aligned}$$

This ends the proof. □

3.2.2 . Hard-core model

The probability distribution that we are going to use as a setting of Lemma 3 uses the hard-core distribution over the independent sets of a graph, which has the Spatial Markov Property. Given a family \mathcal{I} of independent sets of a graph H , and a positive real λ , a random independent set \mathbf{I} drawn according to the hard-core distribution at fugacity λ over \mathcal{I} is such that

$$\mathbb{P}[\mathbf{I} = I] = \frac{\lambda^{|I|}}{Z_{\mathcal{I}}(\lambda)},$$

for every $I \in \mathcal{I}$, where $Z_{\mathcal{I}}(\lambda) = \sum_{J \in \mathcal{I}} \lambda^{|J|}$ is the *partition function* associated with \mathbf{I} .

Note that when $\mathcal{I} = \mathcal{I}(H)$, and $\lambda \rightarrow \infty$, the hard-core distribution converges towards the uniform distribution over the maximum independent sets of H .

Lemma 4 (Spatial Markov Property) *Given a graph H , and a real $\lambda > 0$, let \mathbf{I} be drawn according to the hard-core distribution at fugacity λ over the independent sets $\mathcal{I}(H)$ of H . Let $X \subseteq V(H)$ be any given subset of vertices, and J be any possible outcome of $\mathbf{I} \setminus X$. Then, conditioned on the fact that $\mathbf{I} \setminus X = J$, the random independent set $\mathbf{I} \cap X$ follows the hard-core distribution at fugacity λ over the independent sets of $H[X \setminus N(J)]$.*

Proof. Conditioned on the fact that $\mathbf{I} \setminus X = J$, let \mathcal{I}_J be the set of possible realisations of $\mathbf{I} \cap X$. Now, for every $I \in \mathcal{I}_J$,

$$\begin{aligned} \mathbb{P}[\mathbf{I} \cap X = I \mid \mathbf{I} \setminus X = J] &= \mathbb{P}[\mathbf{I} = I \cup J] = \frac{\lambda^{|I \cup J|}}{\sum_{I' \in \mathcal{I}_J} \lambda^{|I' \cup J|}} = \frac{\lambda^{|I|}}{\sum_{I' \in \mathcal{I}_J} \lambda^{|I'|}} \cdot \frac{\lambda^{|J|}}{\lambda^{|J|}} \\ &= \frac{\lambda^{|I|}}{\sum_{I' \in \mathcal{I}_J} \lambda^{|I'|}} = \frac{\lambda^{|I|}}{Z_{\mathcal{I}_J}(\lambda)}. \end{aligned}$$

□

The proof of this result is standard and follows from a simple consideration of the marginal probabilities. It remains valid when we fix $\lambda = \infty$, i.e. the uniform distribution over the maximum independent sets of any graph H has the Spatial Markov Property.

Among the many consequences of the Spatial Markov Property, one can observe that for every vertex $v \in V(H)$,

$$\mathbb{P}[v \in \mathbf{I}] = \lambda \mathbb{P}[v \text{ is uncovered by } \mathbf{I}]. \quad (3.1)$$

3.3 . Maximum degree 4

In this section, we focus on optimising the upper bound of $\chi_f(4, K_3)$. To that end, we use a 2-step procedure to construct the random independent sets that feed the GFCA. The first step follows the hard-core distribution, and the second step consists of a fractional colouring with local demand of the uncovered vertices of degree at most 4. Before describing the procedure in more detail, we introduce the necessary terminology.

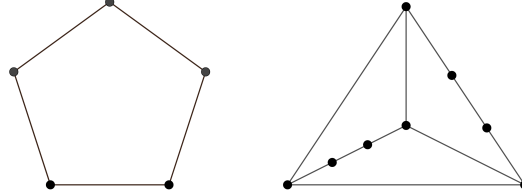


Figure 3.2: Dangerous graphs

3.3.1 . Setting up the finishing step

Let G be a graph. Given a function $f: V(G) \rightarrow \mathbb{Q}^+$, an f -fractional colouring of G is a random independent set \mathbf{I} of G such that $\mathbb{P}[v \in \mathbf{I}] \geq f(v)$ for every vertex $v \in V(G)$. An f -fractional colouring of G is also called a *fractional colouring of G with local demand f* . Given a vertex v , the value of $f(v)$ is called the *demand of v* .

We say that a subgraph H of a given subcubic graph G is *dangerous* if it is isomorphic to C_5 or to K_4^+ (the complete graph on 4 vertices where two non-adjacent edges have been subdivided twice each, see Figure 3.2). A vertex is *dangerous* if it has degree 2 in a dangerous graph. A vertex $v \in V(G)$ is *special* if it belongs to a dangerous subgraph of G , and has degree 2 in G . Given a subset B of the dangerous vertices of G , we say that a vertex v of degree 2 in a dangerous subgraph H of G is *nailed by B* if either v belongs to B , or v is not special (i.e has degree 3 in G). A *nail B* of G is a subset of the dangerous vertices of G such that for every dangerous subgraph H of G , at least 2 vertices of degree 2 in H are nailed by B .

In order to analyse the final step of our procedure, we use the following result [39].

Theorem 57 (Dvořák, Sereni, Volec; 2014) *Let G be a triangle-free subcubic graph, and let B be a nail of G . For every vertex $v \in B$, we let $f_B(v) = \frac{7-\deg(v)}{14}$, and for every vertex $v \notin B$, we let $f_B(v) = \frac{8-\deg(v)}{14}$.*

Then G has an f_B -fractional colouring.

We begin by showing that Theorem 57 has the following result as a corollary.

Corollary 3 *Let G be a triangle-free graph, and let*

$$f_G(v) = \begin{cases} 1 & \text{if } \deg(v) = 0; \\ 1/2 & \text{if } v \text{ belongs to an isolated edge,} \\ 1 - f_G(u) \geq 4/7 & \text{otherwise if } N(v) = \{u\}; \\ 11/28 & \text{if } v \text{ is special and has a special neighbour,} \\ 3/7 & \text{otherwise if } \deg(v) = 2; \\ 5/14 & \text{if } \deg(v) = 3; \\ 0 & \text{if } \deg(v) \geq 4 \end{cases}$$

for every vertex $v \in V(G)$. Then G has an f_G -fractional colouring.

Proof. Let G' be obtained from G by deleting all vertices of degree at least 4. Since the demand of the vertices of degree at least 4 in G is 0, and since the demand of a vertex increases if its degree decreases (as can be seen by the definition of f_G), any $f_{G'}$ -fractional colouring of G' is in particular an f_G -fractional colouring of G . Therefore, in the rest of the proof, we may assume that G is subcubic (its maximum degree is at most 3).

If G contains a connected component C isomorphic to C_5 , then any f_G -fractional colouring of $G \setminus V(C)$, extends to an f_G -fractional colouring of G with an independent $2/5$ -fractional colouring of $G[C]$. Indeed, the vertices $v \in C$ are all special, hence we have demand $f_G(v) = 11/28 < 2/5$. We may therefore assume that G contains no isolated C_5 .

Let S be the set of special vertices in G . Since G contains no isolated C_5 , it means that each copy of C_5 in G contains one non-special vertex. It follows that $G[S]$ consists of a union of paths of length at most 3. We disregard from $G[S]$ the isolated vertices, which yields a linear forest F of minimum degree 1, that we can properly 2-colour. Let \mathbf{B} be a

uniform random colour class of F ; this is a dominating set of $V(F)$, and moreover $\mathbb{P}[v \in \mathbf{B}] = 1/2$ for every vertex $v \in V(F)$.

We now claim that \mathbf{B} is a nail of G . Assume for the sake of contradiction that there exists a dangerous subgraph H of G with at most 1 nailed vertex. Let $M = \{u_1v_1, u_2v_2\}$ be a matching of the degree-2 vertices in H . For $i \in \{1, 2\}$, if both u_i and v_i are special, then they are adjacent in F and therefore have a different colour; hence \mathbf{B} contains one of them, i.e. one of them is nailed. On the other hand, if one of u_i and v_i is non-special, then it is nailed. So H contains one nailed vertex among $\{u_1, v_1\}$, and one nailed vertex among $\{u_2, v_2\}$, a contradiction.

Let \tilde{G} be the subgraph of G induced by the vertices of degree 2 or 3. We may apply Theorem 57 in order to obtain an $f_{\mathbf{B}}$ -fractional colouring of \tilde{G} ; this is a random independent set \mathbf{I}_0 such that $\mathbb{P}[v \in \mathbf{I}_0] \geq \mathbb{E}[f_{\mathbf{B}}(v)]$ if $\deg(v) \in \{2, 3\}$. We observe that the vertices in $V(F)$ are precisely the special vertices which have (at least) one special neighbour. For each such vertex v , we recall that $\mathbb{P}[v \in \mathbf{B}] = 1/2$, hence $\mathbb{E}[f_{\mathbf{B}}(v)] = \frac{1}{2} \cdot \frac{5}{14} + \frac{1}{2} \cdot \frac{3}{7} = \frac{11}{28}$. Moreover, we have $f_{\mathbf{B}}(v) = \frac{5}{14}$ if $\deg(v) = 3$ and $f_{\mathbf{B}}(v) = \frac{3}{7}$ if $\deg(v) = 2$ and $v \notin V(F)$, regardless of the random outcome of \mathbf{B} .

There remains to treat the case of vertices of degree 0 or 1. For every vertex v such that $\mathbb{P}[v \in \mathbf{I}_0] > f_G(v)$, we remove it from \mathbf{I}_0 with probability $1 - \frac{f_G(v)}{\mathbb{P}[v \in \mathbf{I}_0]}$; this yields a new random independent set \mathbf{I}_1 where $\mathbb{P}[v \in \mathbf{I}_1] = f_G(v)$ for every vertex v of degree at least 2. We construct \mathbf{I}_2 by adding to \mathbf{I}_1 all isolated vertices, as well as one extremity chosen uniformly at random from each isolated edge, and finally every other vertex v of degree 1 whenever it is uncovered by \mathbf{I}_1 . It is now straightforward to check that \mathbf{I}_2 is indeed a fractional f_G -colouring of G . \square

3.3.2 . A description of the probability distribution

We are now ready to describe our fractional colouring procedure for a given triangle-free graph G of maximum degree 4. We will use the GFCA with a probability distribution over the independent sets of any induced subgraph H of G

obtained as follows.

- (i) We fix some real value $\lambda > 0$ (which will be optimised later on). We let \mathbf{I}_0 be drawn according to the hard-core distribution at fugacity λ over the independent sets of H . Let us write this $\mathbf{I}_0 \leftarrow \text{hc}_\lambda(\mathcal{I}(H))$.
- (ii) We let $\mathbf{H} := H \setminus N[\mathbf{I}_0]$ be the subgraph of H induced by the vertices uncovered by \mathbf{I}_0 . We apply Corollary 3 to \mathbf{H} in order to obtain an $f_{\mathbf{H}}$ -fractional colouring \mathbf{I}_1 of \mathbf{H} .
- (iii) We return the random independent set $\mathbf{I}_0 \cup \mathbf{I}_1$.

It turns out that this procedure is not enough to guarantee that $\chi_f(G) < 3.5$. In order to fall below that threshold, we need to alter our last step and give a non-zero demand $f(v)$ for some vertices v of degree 4, namely the ones which have at least one degree-4 neighbour. Before describing how to do that, we describe the analysis of the current procedure. We will then explain how to alter it gradually in order to obtain the promised upper bounds on $\chi_f(G)$.

3.3.3 . The analysis of the probability distribution

Let G be a graph given with a function φ which maps each subgraph H from a given family of subgraphs of G to a random independent set of H . We introduce some notation to bound from below the marginal probabilities associated with φ on a vertex of given degree.

Definition 17. For every integer $d \leq \Delta(G)$, we let

$$\mu_\varphi(d) := \min\{\mathbb{P}[v \in \varphi(H)] : H \subseteq G, v \in V(H), \deg_H(v) = d\}.$$

To avoid a non-uniform behaviour of the probability distribution we construct over the independent sets, we will always make it so that μ_φ is a non-increasing function over the integers.

Next, we introduce the notion of *refined constraint*, which will be helpful to compute constraints generated by a random independent set $\mathbf{I} = \mathbf{I}_0 \cup \mathbf{I}_1$ constructed in two steps, where \mathbf{I}_0 follows the hard-core distribution.

Definition 18. For every $H \in \text{dom}(\varphi)$ and $v \in V(H)$, we let

$$e_{\varphi, H}^+(v) := \begin{pmatrix} \mathbb{P}[v \in \varphi(H)] \\ \mathbb{E}[|N(v) \cap \varphi(H)|] \\ \sum_{u \in N_H(v)} \mu_{\varphi}(\deg_H(u) - 1) \end{pmatrix}$$

be the *refined constraint of φ on v* .

We extend the order (\succsim) to refined constraints as follows.

Definition 19. Given two refined constraints $e_0 = (p_0, q_0, r_0)^T$ and $e_1 = (p_1, q_1, r_1)^T$, we say that e_0 is *tighter* than e_1 , or equivalently that e_1 is *looser* than e_0 , if they satisfy $p_0 \leq p_1$, $p_0 + q_0 \leq p_1 + q_1$, and $r_0 \leq r_1$. We denote it $e_0 \succsim e_1$.

For every integer $d \leq \Delta(G)$, we denote $\mathcal{E}_{\varphi}^+(d)$ the set of minimal refined constraints in

$$\left\{ e_{\varphi, H}^+(v) : H \in \text{dom}(\varphi), v \in V(H), \deg_H(v) = d \right\},$$

with respect to (\succsim). We now show how to rely on this set of refined constraints in order to compute a random independent set constructed in two steps as follows.

Lemma 5 *Let G be a triangle-free graph. Let \mathbf{I} be a random independent set of G constructed in two steps, i.e. $\mathbf{I} = \mathbf{I}_0 \cup \varphi(G \setminus N[\mathbf{I}_0])$, where \mathbf{I}_0 is drawn from the hard-code distribution at fugacity $\lambda > 0$ from $\mathcal{I}(G)$, and φ maps any subgraph $H \subseteq G$ to a random independent set of H . Then each constraint $e_{\mathbf{I}}(v)$ is looser than some convex combination of the vectors*

$$\left\{ \frac{1}{\lambda + (1 + \lambda)^d} \begin{pmatrix} \lambda + p \\ (1 + \lambda)^{d-1}(d\lambda + r) + q - r \end{pmatrix} : d \in \{0, 1, \dots, \Delta(G)\} \text{ and } \begin{pmatrix} p \\ q \\ r \end{pmatrix} \in \mathcal{E}_{\varphi}^+(d) \right\}.$$

Proof. Given a possible realisation J of $\mathbf{I}_0 \setminus N[v]$, let E_J be the random event that $\mathbf{I}_0 \setminus N[v] = J$. By the Spatial Markov Property, given a vertex $v \in V(G)$, if we condition on the event E_J , then $\mathbf{I}_0 \cap N[v]$ follows the hard-core distribution at fugacity λ on the independent sets of a star $K_{1,d}$ centred in v , where $d = |N(v) \setminus N(J)|$. A classical analysis of the

hard-core model yields that

$$\begin{aligned}\mathbb{P}[v \in \mathbf{I}_0 \mid E_J] &= \frac{\lambda}{\lambda + (1 + \lambda)^d}, \quad \text{and} \\ \mathbb{P}[u \in \mathbf{I}_0 \mid E_J] &= \frac{\lambda(1 + \lambda)^{d-1}}{\lambda + (1 + \lambda)^d} \quad \text{for every } u \in N(v) \setminus N(J).\end{aligned}$$

If we further condition on the event that v is uncovered by \mathbf{I}_0 , then we have $\mathbf{I}_0 = J$. Let us write this event E_J^0 ; we have $\mathbb{P}[E_J^0 \mid E_J] = \frac{1}{\lambda + (1 + \lambda)^d}$. If we write $H_1 = G \setminus N[J]$, then the fact that E_J^0 holds means that $G \setminus N[\mathbf{I}_0] = H_1$. Note that we have $v \in V(H_1)$, and $\deg_{H_1}(v) = d$. So

there exists $\begin{pmatrix} p \\ q \\ r \end{pmatrix} \in \mathcal{E}_\varphi^+(d)$ such that $\begin{pmatrix} p \\ q \\ r \end{pmatrix} \preceq e_{\varphi, H_1}^+(v) = \begin{pmatrix} p' \\ q' \\ r' \end{pmatrix}$. It follows

that

$$\begin{aligned}\mathbb{P}[v \in \mathbf{I}_1 \mid E_J^0] &= \mathbb{P}[v \in \varphi(H_1)] = p', \quad \text{and so} \\ \mathbb{P}[v \in \mathbf{I} \mid E_J] &= \mathbb{P}[v \in \mathbf{I}_0 \mid E_J] + \mathbb{P}[v \in \mathbf{I}_1 \mid E_J^0] \mathbb{P}[E_J^0 \mid E_J] \\ &= \frac{\lambda}{\lambda + (1 + \lambda)^d} + \frac{p'}{\lambda + (1 + \lambda)^d} \geq \frac{\lambda + p}{\lambda + (1 + \lambda)^d}. \quad (3.2)\end{aligned}$$

We now let E_J^* be the random event that $N(v) \cap \mathbf{I}_0 \neq \emptyset$, given the event E_J . Observe that $E_J \equiv E_J^0 \vee E_J^* \vee "v \in \mathbf{I}_0"$, and that $\mathbb{P}[u \in V(G \setminus N[\mathbf{I}_0])] = 0$ under the event " $v \in \mathbf{I}_0$ " and 1 under the event E_J^0 . We have

$$\mathbb{P}[u \in V(G \setminus N[\mathbf{I}_0]) \mid E_J] = \frac{1}{1 + \lambda} (1 - \mathbb{P}[v \in \mathbf{I}_0 \mid E_J]) = \frac{(1 + \lambda)^{d-1}}{\lambda + (1 + \lambda)^d},$$

and so

$$\begin{aligned}\mathbb{P}[u \in V(G \setminus N[\mathbf{I}_0]) \wedge E_J^* \mid E_J] &= \mathbb{P}[u \in V(G \setminus N[\mathbf{I}_0]) \mid E_J] - \mathbb{P}[E_J^0 \mid E_J] \\ &= \frac{(1 + \lambda)^{d-1} - 1}{\lambda + (1 + \lambda)^d}.\end{aligned}$$

If H'_1 is any realisation of $G \setminus N[\mathbf{I}_0]$ under the condition E_J^* , then $\deg_{H'_1}(u) \leq \deg_{H_1}(u) - 1$ for every $u \in N(v) \cap V(H'_1)$, since $v \notin V(H'_1)$. In particular, $\mathbb{P}[u \in \varphi(H'_1)] \geq \mu_\varphi(\deg_{H_1}(u) - 1)$. It follows that

$$\begin{aligned}\mathbb{P}[u \in \mathbf{I}_1 \wedge E_J^* \mid E_J] &\geq \mathbb{P}[u \in V(G \setminus N[\mathbf{I}_0]) \wedge E_J^* \mid E_J] \mu_\varphi(\deg_{H_1}(u) - 1) \\ &\geq \frac{(1 + \lambda)^{d-1} - 1}{\lambda + (1 + \lambda)^d} \mu_\varphi(\deg_{H_1}(u) - 1),\end{aligned}$$

and so

$$\begin{aligned} \mathbb{P}[u \in \mathbf{I}_1 \mid E_J] &= \mathbb{P}[u \in \mathbf{I}_1 \wedge E_J^* \mid E_J] + \mathbb{P}[u \in \mathbf{I}_1 \mid E_J^0] \mathbb{P}[E_J^0 \mid E_J] \\ &\geq \frac{1}{\lambda + (1 + \lambda)^d} \left(((1 + \lambda)^{d-1} - 1) \mu_\varphi(\deg_{H_1}(u) - 1) + \mathbb{P}[u \in \varphi(H_1)] \right). \end{aligned}$$

We now infer that

$$\begin{aligned} \mathbb{E}[|N(v) \cap \mathbf{I}| \mid E_J] &= \sum_{u \in N_{H_1}(v)} (\mathbb{P}[u \in \mathbf{I}_0 \mid E_J] + \mathbb{P}[u \in \mathbf{I}_1 \mid E_J]) \\ &\geq \frac{d\lambda(1 + \lambda)^{d-1}}{\lambda + (1 + \lambda)^d} + \sum_{u \in N_{H_1}(v)} \frac{1}{\lambda + (1 + \lambda)^d} \left(((1 + \lambda)^{d-1} - 1) \mu_\varphi(\deg_{H_1}(u) - 1) + \mathbb{P}[u \in \varphi(H_1)] \right). \end{aligned}$$

So we have

$$\begin{aligned} \mathbb{P}[v \in \mathbf{I} \mid E_J] + \mathbb{E}[|N(v) \cap \mathbf{I}| \mid E_J] &\geq \frac{\lambda + p'}{\lambda + (1 + \lambda)^d} + \frac{d\lambda(1 + \lambda)^{d-1}}{\lambda + (1 + \lambda)^d} \\ &\quad + \sum_{u \in N_{H_1}(v)} \frac{1}{\lambda + (1 + \lambda)^d} \left(((1 + \lambda)^{d-1} - 1) \mu_\varphi(\deg_{H_1}(u) - 1) + \mathbb{P}[u \in \varphi(H_1)] \right) \\ &\geq \frac{\lambda + p + (1 + \lambda)^{d-1}(d\lambda + r) + q - r}{\lambda + (1 + \lambda)^d}. \end{aligned} \tag{3.3}$$

We finish the proof with the laws of total probability and expectation, from which we infer that the constraint $e_{\mathbf{I}}(v)$ is looser than some convex combination of the vector formed with the right-hand sides of (3.2) and (3.3).

□

Now, we need to compute the sets $\mathcal{E}_\varphi^+(d)$ for each integer $d \leq 4$, where φ is the finishing step described in Section 3.3.1.

Lemma 6 *If G is a triangle-free graph of maximum degree at most 4, and φ maps each subgraph H of G to the f_H -fractional colouring given by Corollary 3, then we have*

$$\mu_\varphi(0) = 1, \quad \mu_\varphi(1) = 1/2, \quad \mu_\varphi(2) = 11/28, \quad \mu_\varphi(3) = 5/14, \quad \mu_\varphi(4) = 0;$$

$$\begin{aligned}
\mathcal{E}_\varphi^+(0) &= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}; \\
\mathcal{E}_\varphi^+(1) &= \left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4/7 \\ 3/7 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 9/14 \\ 5/14 \\ 11/28 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 5/14 \end{pmatrix} \right\}; \\
\mathcal{E}_\varphi^+(2) &= \left\{ \begin{pmatrix} 11/28 \\ 3/4 \\ 25/28 \end{pmatrix}, \begin{pmatrix} 3/7 \\ 5/7 \\ 11/14 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 19/14 \end{pmatrix}, \begin{pmatrix} 4/7 \\ 3/7 \\ 6/7 \end{pmatrix}, \begin{pmatrix} 9/14 \\ 5/14 \\ 3/4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 5/7 \end{pmatrix} \right\}; \\
\mathcal{E}_\varphi^+(3) &= \left\{ \begin{pmatrix} 5/14 \\ 15/14 \\ 33/28 \end{pmatrix}, \begin{pmatrix} 11/28 \\ 3/4 \\ 35/28 \end{pmatrix}, \begin{pmatrix} 3/7 \\ 5/7 \\ 8/7 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 12/7 \end{pmatrix}, \begin{pmatrix} 4/7 \\ 3/7 \\ 17/14 \end{pmatrix}, \begin{pmatrix} 9/14 \\ 5/14 \\ 31/28 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 15/14 \end{pmatrix} \right\}; \\
\mathcal{E}_\varphi^+(4) &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 10/7 \end{pmatrix} \right\}.
\end{aligned}$$

Proof. The values of $\mu_\varphi(d)$ for each $d \leq 4$ follow readily from the definition of f_H .

Computing $\mathcal{E}_\varphi^+(1)$ is straightforward: given a vertex $v \in V(H)$ of degree 1, its unique neighbour u has degree $\deg_H(u) \in \{1, \dots, 4\}$. By definition of f_H , the constraint in v is of the form

$$\begin{pmatrix} 1 - f_H(u) \\ f_H(u) \\ \mu_\varphi(\deg_H(u) - 1) \end{pmatrix}.$$

When the degree of u is fixed, the worst case is when $f_H(u)$ is maximised (since this decreases the demand for v); hence for the case $\deg_H(u) = 2$ we keep the value $f_H(u) = 3/7$.

To compute $\mathcal{E}_\varphi^+(2)$, we can first reuse $\mathcal{E}_\varphi^+(1)$ to treat the case where the root vertex v has (at least) one neighbour u of degree 4, by adding $5/14$ to the last coordinate of the constraint associated to the pattern obtained by removing u . Note that this can be done more generally for every degree. There remains to consider the patterns where both neighbours u_1, u_2 of v have degree at most 3. The minimum total demand of the neighbourhood (i.e. the second coordinate of the

constraint) is $5/7$ and is reached when both u_1 and u_2 have degree 3, but in that case v cannot be special and so its demand is $3/7$; this yields the constraint $(3/7, 5/7, 11/14)^T$. The minimum demand for v is $\mu_\varphi(2) = 11/28$ and is reached when v is special; in that case it must have a special neighbour of demand $11/28$, and in the worst case its other neighbour has degree 3 and demand $5/14$. This yields the constraint $(11/28, 3/4, 25/28)^T$.

To compute $\mathcal{E}_\varphi^+(3)$, we reuse $\mathcal{E}_\varphi^+(2)$ as explained above, and then consider the patterns where all neighbours of v have degree at most 3. The minimum demand for v is $\mu_\varphi(3) = 5/14$, and the minimum total demand for its neighbourhood is $3\mu_\varphi(3) = 15/14$; this is reached when all its neighbours have degree 3. This yields the constraint $(\frac{5}{14}, \frac{15}{14}, \frac{33}{28})^T$.

To compute $\mathcal{E}_\varphi^+(4)$, the worst case is when v has four degree-4 neighbours, which yields the constraint $e_0 = (0, 0, \frac{10}{7})^T$. Note that e_0 is tighter than all constraints that could be inherited from $\mathcal{E}_\varphi^+(3)$, so these are discarded from $\mathcal{E}_\varphi^+(4)$ (since we keep only minimal constraints with respect to (\succsim)).

□

In Figure 3.3, we present the patterns rooted in a degree-3 vertex yielding the refined constraints in $\mathcal{E}_\varphi^+(3)$

3.3.4 . Giving non-zero demand to (some) degree-4 vertices

Let H be any subgraph of G (which we recall is a triangle-free graph of maximum degree 4). Let $S_{\leq 3} \subseteq V(H)$ be the set of vertices of degree at most 3, and let $S_4 = V(H) \setminus S_{\leq 3}$ be the set of vertices of degree 4. We further partition S_4 into $S_4^0 \cup S_4^+$, where S_4^0 is the set of isolated vertices in $G[S_4]$ (and hence each vertex in S_4^+ has a neighbour in S_4).

Claim 3.3.1 *There exists a random set $\mathbf{X} \subseteq S_4$ such that $H \setminus \mathbf{X}$ is deterministically subcubic, and $\mathbb{P}[v \in \mathbf{X}] = 1/2$ for every $v \in S_4^+$.*

Proof. Let us construct a random dominating set \mathbf{X} of $H[S_4]$, so that $H \setminus \mathbf{X}$ is indeed subcubic. Let F be a spanning forest of $H[S_4^+]$, and c be a proper 2-colouring of F . We let \mathbf{X}^+ be a random colour class of

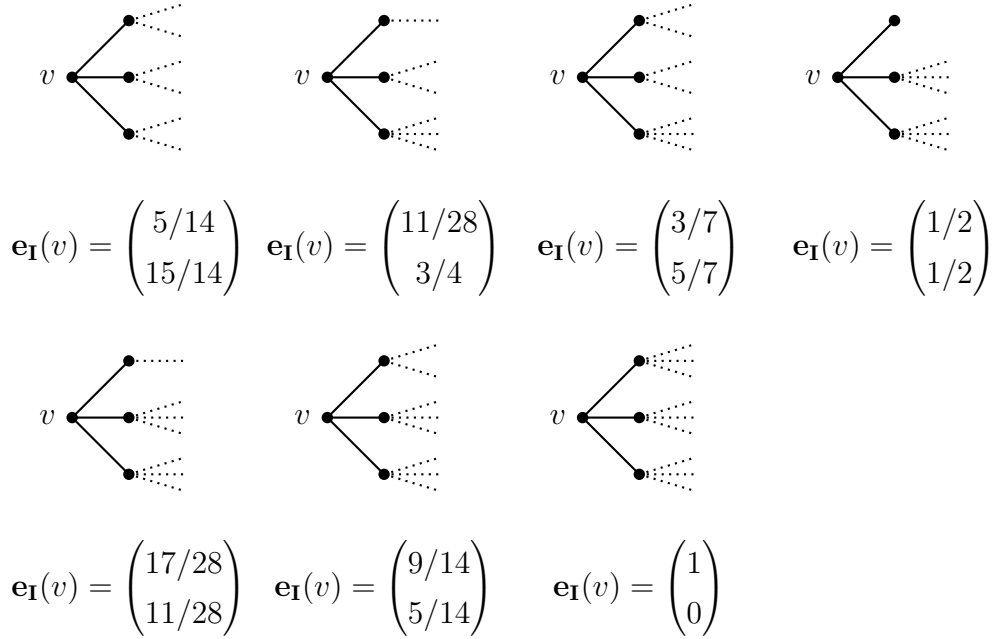


Figure 3.3: The patterns rooted in a degree-3 vertex that yield a minimal refined constraint

c , and $\mathbf{X} := \mathbf{X}^+ \cup S_4^0$. This is a random dominating set of $H[S_4]$, and indeed we have $\mathbb{P}[v \in \mathbf{X}] = 1/2$ for every vertex $v \in S_4^+$. \square

Let \mathbf{X} be the random set promised by Claim 3.3.1, so that $\mathbf{H}' := H \setminus \mathbf{X}$ is a (random) subcubic triangle-free graph. We are now ready to define the new finishing step.

Let $\varphi': H \mapsto \varphi(\mathbf{H}')$ for every $H \subseteq G$, where φ is the finishing step described in Section 3.3.3. Let us analyse the constraints that it generates.

For every vertex $v \in S_4$, we have

$$\mathbb{P}[v \in \varphi'(H)] \geq \mathbb{E}[\mu_\varphi(\deg_{\mathbf{H}'}(v)) \mid v \notin \mathbf{X}] \mathbb{P}[v \notin \mathbf{X}], \quad \text{and}$$

$$\mathbb{E}[|N(v) \cap \varphi'(H)|] \geq \sum_{u \in N(v)} \mathbb{E}[\mu_\varphi(\deg_{\mathbf{H}'}(u)) \mid u \notin \mathbf{X}] \mathbb{P}[u \notin \mathbf{X}].$$

For every $v \in S_4^0$, every neighbour of v has degree at most 3 in H and belongs to $V(\mathbf{H}')$. Since we always have $v \notin V(\mathbf{H}')$, we infer that $\deg_{\mathbf{H}'}(u) \leq 2$ for every $u \in N(v)$. Since $\mu_\varphi(2) \geq 11/28$, we conclude that $(0, 11/7, 11/7)^T \preceq e_{\varphi', H}^+(v)$.

For every $v \in S_4^+$, we have

$$\mathbb{E}[\mu_\varphi(\deg_{\mathbf{H}'}(v)) \mid v \notin \mathbf{X}] \geq \mu_\varphi(3) \geq \frac{5}{14},$$

and so

$$\mathbb{P}[v \in \varphi'(H)] \geq \frac{5}{28}. \quad (3.4)$$

Since moreover all degree-4 neighbours of v are also in S_4^+ , (3.4) holds for them as well; hence

$$\mathbb{E}[|N(v) \cap \varphi'(H)|] \geq 4 \times \frac{5}{28} = \frac{5}{7}.$$

We recall that $\mu_\varphi(3) \geq 5/14$, so we conclude that $(5/28, 5/7, 10/7)^T \preceq e_{\varphi', H}^+(v)$.

The two constraints that we have derived are reached by the patterns drawn in Figure 3.4. We conclude that

$$\mathcal{E}_{\varphi'}^+(4) = \left\{ \begin{pmatrix} 0 \\ 11/7 \\ 11/7 \end{pmatrix}, \begin{pmatrix} 5/28 \\ 5/7 \\ 10/7 \end{pmatrix} \right\}.$$

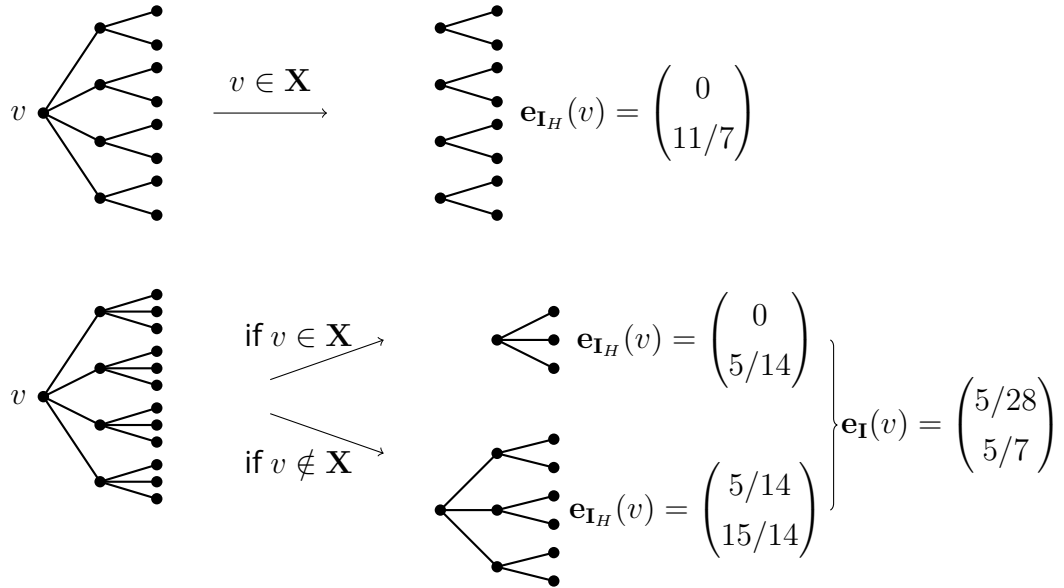


Figure 3.4: Two possible patterns rooted at v with $N_G[v] \cap S_4 \neq \emptyset$, and their related constraints

Finally, we argue that we may set $\mathcal{E}_{\varphi'}^+(d) := \mathcal{E}_{\varphi}^+(d)$ for every integer $d \leq 3$.

Claim 3.3.2 For every vertex $v \in V(H)$ of degree at most 3, the refined constraint $e_{\varphi', H}^+(v)$ is looser than a convex combination of constraints from $\cup_{d \leq 3} \mathcal{E}_{\varphi}^+(d)$.

Proof. This is straightforward once one notices that, by definition of φ' , $e_{\varphi', H}^+(v) = \mathbb{E} [e_{\varphi, H'}^+(v)]$. \square

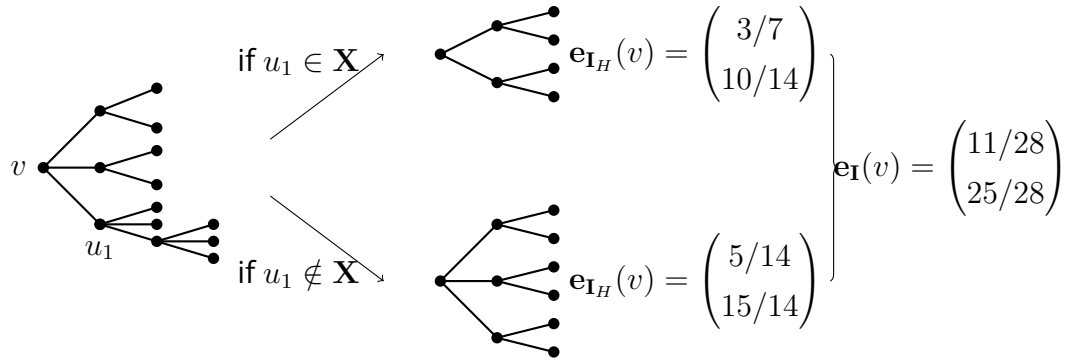


Figure 3.5: A possible pattern rooted at v with $N_G[v] \cap S_4 \neq \emptyset$, and its related constraints

We are now ready to prove the main result of this section. Let us fix $\lambda = 0.51$. Given an induced subgraph $H \subseteq G$, we let $\varphi_0(H) := \mathbf{I}_0 \cup \varphi'(H \setminus N[\mathbf{I}_0])$, where $\mathbf{I}_0 \leftarrow \text{hc}_\lambda(H)$. We construct the linear program $\text{LP}_{\varphi_0}^*(G)$ with the set of constraints depicted in Table 3.1, which we obtain by combining Lemma 5 with the values of $\mathcal{E}_{\varphi}^+(d)$ computed above. The solution to that program is reached when $\alpha = 1.8980861$ and $\beta = 0.39205135$; this implies that $\chi_f(4, K_3) < 3.4663$.

$$\begin{aligned}
\mathcal{E}_{\varphi_0}(0) &= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}; \\
\mathcal{E}_{\varphi_0}(1) &= \left\{ \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\}; \\
\mathcal{E}_{\varphi_0}(2) &= \left\{ \begin{pmatrix} 0.4131956355890983 \\ 0.8171186900623122 \end{pmatrix}, \begin{pmatrix} 0.3875949146727972 \\ 0.8623039624795835 \end{pmatrix}, \begin{pmatrix} 0.3619941937564962 \\ 0.9792992570670792 \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} 0.3235931123820447 \\ 0.9840353904365947 \end{pmatrix}, \begin{pmatrix} 0.3363934728401952 \\ 0.9516504784774741 \end{pmatrix} \right\}; \\
\mathcal{E}_{\varphi_0}(3) &= \left\{ \begin{pmatrix} 0.29164468339150756 \\ 1.3313975884273366 \end{pmatrix}, \begin{pmatrix} 0.2735750004056644 \\ 1.3841637731984464 \end{pmatrix}, \begin{pmatrix} 0.2555053174198213 \\ 1.5641504645155335 \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} 0.22840079294105667 \\ 1.4770428472298294 \end{pmatrix}, \begin{pmatrix} 0.23743563443397822 \\ 1.4333115039516406 \end{pmatrix}, \begin{pmatrix} 0.21936595144813514 \\ 1.5352254194759454 \end{pmatrix} \right\}; \\
\mathcal{E}_{\varphi_0}(4) &= \left\{ \begin{pmatrix} 0.12061460778924579 \\ 1.966740049953671 \end{pmatrix}, \begin{pmatrix} 0.08933488585220072 \\ 2.178014577640148 \end{pmatrix} \right\}.
\end{aligned}$$

Table 3.1: The constraints of the final linear program, with $\lambda := 0.51$.

4 - On $(2,k)$ -Hamilton-connected graphs

In this chapter, we introduce our work on (k_1, k_2) -Hamilton connected graphs.

4.1 . Introduction

Motivated by spanning connectivity and k -fan connected graphs, we define (k_1, k_2) -Hamilton-connected graphs. Recall that a graph G is (k_1, k_2) -Hamilton-connected, if for any two disjoint vertex subsets $X = \{x_1, x_2, \dots, x_{k_1}\}$ and $U = \{u_1, u_2, \dots, u_{k_2}\}$, there exist $k_1 k_2$ internally-disjoint paths connecting x_i to u_j for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, which span the whole graph. We note that $(1, k)$ -Hamilton-connectivity is equivalent to k -fan-connectivity. It is easy to see that both $(1, 1)$ -Hamilton-connectivity and $(1, 2)$ -Hamilton-connectivity are equivalent to Hamilton-connectivity. Clearly (k, k) -Hamilton-connectivity implies k -linkedness.

Firstly, we give a sufficient condition for a graph to be $(2, k)$ -Hamilton-connected.

Theorem 58 *Let G be an n -vertex graph. If G is $(5k - 2)$ -connected and $\sigma_2(G) \geq n + k - 2$ with $k \geq 2$, then G is $(2, k)$ -Hamilton-connected. Moreover, the bound of $\sigma_2(G)$ is sharp.*

We construct a graph to state that $\sigma_2(G) \geq n + k - 2$ of Theorem 58 is sharp as follows. Let k and m be integers with $m \geq 3k - 2$. Let G be a graph of order $n = 3k + 2m + 3$ composed of G_1 and $G_2 = K_{2k+m}$ where G_1 is a set of $k + m + 3$ independent vertices such that each vertex of G_1 is adjacent to each vertex of G_2 . The graph G is $(5k - 2)$ -connected and $\sigma_2(G) \geq n + k - 3$. But for two disjoint subsets $X = \{x_1, x_2\}$ and $U = \{u_1, u_2, \dots, u_k\}$ from the same part of G_1 , we can not find $2k$ internally-disjoint paths connecting x_i and u_j for $1 \leq i \leq 2$ and $1 \leq j \leq k$ which span G .

Another result obtained is about (k_1, k_2) -Hamilton-connectivity which is also a tight sufficient condition.

Theorem 59 *Let G be an n -vertex graph. If $\sigma_2(G) \geq n + k_1k_2 - 2$ with $k_1, k_2 \geq 2$, then G is (k_1, k_2) -Hamilton-connected. Moreover, the bound of $\sigma_2(G)$ is sharp.*

We construct a graph G of order n certifying the sharpness of Theorem 59. For any $k_1, k_2 \geq 2$ and $n = k_1k_2 + k_1 + k_2 - 1$, G consists of $G_1 = K_{k_1}$, $G_2 = K_{k_1k_2-1}$, and $G_3 = K_{k_2}$ such that each vertex of G_1 is adjacent to each vertex of G_2 and each vertex of G_3 is adjacent to each vertex of G_2 . We have $\sigma_2(G) = n + k_1k_2 - 3$. But for $X = V(G_1)$ and $U = V(G_3)$, we can not find k_1k_2 internally-disjoint paths connecting each pair $\{x, u\}$ where $x \in X$ and $u \in U$.

The remainder of this chapter is organized as follows. In Section 4.2, we prove Theorem 58. The first part of this section is to prove G is $(2, k)$ -connected. In the second part of it, we prove that G is $(2, k)$ -Hamilton-connected. In section 4.3, we show Theorem 59. In Section 4.4, we present a polynomial-time algorithm to find a spanning (X, U) -connection given two disjoint vertex subsets $X = \{x, y\}$ and $U = \{u_1, u_2, \dots, u_k\}$ with $k \geq 2$ for any graph satisfying the hypotheses of Theorem 58.

4.2 . Proof of Theorem 58

For the purpose of proof, we need the following preliminaries. In fact, the following notations and terminology are also suitable for the proof of Theorem 59. Let G be a graph. For two disjoint vertex subsets $X = \{x_1, x_2, \dots, x_{k_1}\}$ and $U = \{u_1, u_2, \dots, u_{k_2}\}$ with $k_1, k_2 \geq 2$, an (X, U) -connection of G is a set of k_1k_2 internally-disjoint paths $S_{i,j}$ connecting x_i to u_j for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. Specially, we call G is (k_1, k_2) -connected if for every k_1 -subset X of $V(G)$ and k_2 -subset U of $V(G) \setminus \{X\}$, G contains an (X, U) -connection as a subgraph. If an (X, U) -connection spans G , then it is a *spanning* (X, U) -connection of G . Let the orientation of $S_{i,j}$ be from x_i to u_j . For any subset W of $V(S_{i,j})$, we use W^- (or W^+) to denote the set of predecessors (or successors) of W in $S_{i,j}$, respectively. For any vertex $v \in V(S_{i,j})$, we use v^- (or v^+) to denote the predecessor (or successor) of v in $S_{i,j}$, respectively. For the endpoints of $S_{i,j}$, it has either no predecessor or no successor.

The following lemma is also necessary for the proof.

Lemma 7 ([71]) *Let $P = u_1u_2u_3 \cdots u_p$ be a path in a graph G . Let w_1 and w_2 be two vertices in $V(G) \setminus V(P)$ such that $(N_G(w_1) \cap (V(P) \setminus \{u_1\})) \cap N_G(w_2) = \emptyset$. Then $|N_G(w_1) \cap V(P)| + |N_G(w_2) \cap V(P)| \leq p + 1$. Moreover, if $|N_G(w_1) \cap V(P)| + |N_G(w_2) \cap V(P)| = p + 1$, then $w_1u_1, w_2u_p \in E(G)$.*

4.2.1 . The graph is $(2, k)$ -connected

Lemma 8 *Let G be an n -vertex graph. If G is $(5k - 2)$ -connected and $\sigma_2(G) \geq n + k - 2$ with $k \geq 2$, then G is $(2, k)$ -connected.*

Proof. Suppose to the contrary that G is not $(2, k)$ -connected, but for any $e \in E(\overline{G})$, the graph $G + e$ is $(2, k)$ -connected where \overline{G} is the complement of G .

Let $\{x, y\}$ be any pair of vertices in G and $U = \{u_1, u_2, \dots, u_k\}$ be a subset of $V(G) \setminus \{x, y\}$. There exist $2k - 1$ internally-disjoint paths $P_1, P_2, \dots, P_k, Q_2, Q_3, \dots, Q_k$ connecting $\{x, y\}$ and U . Without loss of generality, let P_i be the path connecting x to u_i for $1 \leq i \leq k$ and Q_j be the path connecting y to u_j for $2 \leq j \leq k$. Let F be a subgraph of G consisting of all these $2k - 1$ paths. Assume that the order of F is minimal, denoted by assumption (I).

For the purpose of proof, let the orientation of P_i be from x to u_i for $1 \leq i \leq k$ and the orientation of Q_j be from y to u_j for $2 \leq j \leq k$. We denote $P_i = P_i[x, u_i]$ and $P'_i = P_i \setminus \{x, u_i\} = P'_i(x, u_i)$ for $1 \leq i \leq k$. Let $Q_j = Q_j[y, u_j]$, $Q'_j = Q_j \setminus \{y, u_j\} = Q'_j(y, u_j)$ for $2 \leq j \leq k$, and $R = G - F$ (see Figure 4.1). Note that P'_i and Q'_j may be empty for $1 \leq i \leq k$ and $2 \leq j \leq k$.

Claim 4.2.1 *For $2 \leq i \leq k$, $|N_G(u_1) \cap V(Q'_i)| \leq 1$ and $|N_G(y) \cap (V(Q'_i) \cup \{u_i\})| = 1$.*

Proof. It is clear that $|N_G(y) \cap (V(Q'_i) \cup \{u_i\})| = 1$ for $2 \leq i \leq k$ by assumption (I).

Toward a contradiction, suppose that there exists an integer i for $2 \leq i \leq k$ such that $|N_G(u_1) \cap Q'_i| \geq 2$. Without loss of generality, $i = 2$. Let v_1 and v_2 be the neighbours of u_1 in Q'_2 . The orientation of $Q'_2[v_1, v_2]$

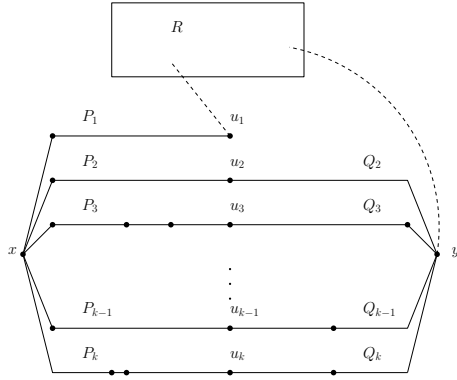


Figure 4.1: The maximal counterexample of a $(2, k)$ -Hamilton-connected graph.

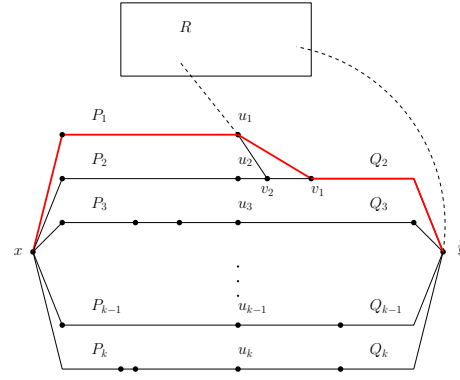


Figure 4.2: F' with fewer vertices than F

is from v_1 to v_2 . We have a new subgraph F' , which is the union of $2k - 1$ internally-disjoint paths connecting $\{x, y\}$ and U with fewer vertices than F after we add u_1v_1 to F and delete v_2 from F (see Figure 4.2). This contradicts assumption (I). So $|N_{Q'_i}(u_1)| \leq 1$ for $2 \leq i \leq k$. \square

Claim 4.2.2 *There exists an integer i for $2 \leq i \leq k$ such that $|Q'_i| \geq 2$.*

Proof. Suppose that $|Q'_i| \leq 1$ for each $2 \leq i \leq k$. Let G' be a graph with order n' obtained by removing x and all vertices of Q_i except y for $2 \leq i \leq k$ from G . Hence we have $n' = n - k - \sum_{i=2}^k |Q'_i|$ and $\sigma_2(G') \geq \sigma_2(G) - 2(k + \sum_{i=2}^k |Q'_i|) \geq n' - k - 1$. It is easy to see that there exist at most $k - 1$ vertices not dominated by u_1 and y . G' is still connected after deleting these vertices. So we obtain a path P in G' from u_1 to y such that $|P| \leq 4$.

Let F be obtained from G by removing y , all internal vertices of P , and all vertices of Q'_i for $2 \leq i \leq k$. F is $(4k - 4)$ -connected. Hence we have an (x, U) -fan in F . Then we obtain an (X, U) -connection consisting of this (x, U) -fan, P , and $\cup_{i=2}^k V(Q'_i)$ where $X = \{x, y\}$. It is a contradiction. \square

Claim 4.2.3 *There exists an integer i for $1 \leq i \leq k$ such that $|N_{P'_i}(u_1) \cap N_{P'_i}(y)| \geq 1$. Furthermore, we have $|P'_i| = 1$ and $|N_{P'_i}(u_1) \cap N_{P'_i}(y)| = 1$.*

Proof. Suppose that $N_{P'_i}(u_1) \cap N_{P'_i}(y) = \emptyset$ for $1 \leq i \leq k$. Then $d_{P'_i}(u_1) + d_{P'_i}(y) \leq |P'_i|$ for $1 \leq i \leq k$ and $d_{\{x\}}(u_1) + d_{\{x\}}(y) \leq 2$. Since $N_R(u_1) \cap N_R(y) = \emptyset$, we have $d_R(u_1) + d_R(y) \leq |R|$. By Claim 4.2.1, we have $d_{Q'_i \cup \{u_i\}}(u_1) + d_{Q'_i \cup \{u_i\}}(y) \leq |Q'_i| + 2$ for $2 \leq i \leq k$. To be specific, $d_{Q'_i \cup \{u_i\}}(u_1) + d_{Q'_i \cup \{u_i\}}(y) \leq 3 \leq |Q'_i| + 1$ when $|Q'_i| \geq 2$ for $2 \leq i \leq k$. If $0 \leq |Q'_i| \leq 1$, then $d_{Q'_i \cup \{u_i\}}(u_1) + d_{Q'_i \cup \{u_i\}}(y) \leq |Q'_i| + 2$ for $2 \leq i \leq k$. By Claim 4.2.2, we assume that there exists an integer j such that $d_{Q'_j \cup \{u_j\}}(u_1) + d_{Q'_j \cup \{u_j\}}(y) \leq |Q'_j| + 1$. Since u_1 and y are not adjacent and $\sigma_2(G) \geq n + k - 2$, we have

$$n+k-2 \leq d(u_1)+d(y) \leq \sum_{i=1}^k |P'_i| + \sum_{\substack{2 \leq i \leq k \\ i \neq j}} (|Q'_i|+2) + (|Q'_j|+1) + |R| + 2 = n+k-3.$$

We deduce a contradiction. Now there exists an integer i for $1 \leq i \leq k$ such that $|N_{P'_i}(u_1) \cap N_{P'_i}(y)| \geq 1$. Let v be any vertex in $N_{P'_i}(u_1) \cap N_{P'_i}(y)$. If $P'_i \setminus \{v\} \neq \emptyset$, then it contradicts assumption (I). So $|P'_i| = 1$ and $|N_{P'_i}(u_1) \cap N_{P'_i}(y)| = 1$. \square

Claim 4.2.4 For $1 \leq i \leq k$, $d_{P'_i}(u_1) + d_{P'_i}(y) \leq |P'_i| + 1$.

Proof. Obviously, $d_{P'_i}(u_1) + d_{P'_i}(y) \leq |P'_i|$ when $N_{P'_i}(u_1) \cap N_{P'_i}(y) = \emptyset$ for $1 \leq i \leq k$. If there exists an integer i such that $N_{P'_i}(u_1) \cap N_{P'_i}(y) \neq \emptyset$ for $1 \leq i \leq k$, then $|N_{P'_i}(u_1) \cap N_{P'_i}(y)| = 1$ by Claim 4.2.3. So $d_{P'_i}(u_1) + d_{P'_i}(y) \leq |P'_i| + 1$ for $1 \leq i \leq k$. \square

Without loss of generality, we assume that $v \in N_{P'_2}(u_1) \cap N_{P'_2}(y)$ and $P'_2 = \{v\}$ by Claim 4.2.3.

For $2 \leq i \leq k$, if $|Q'_i| = 0$, then $d_{Q'_i \cup \{u_i\}}(u_1) + d_{Q'_i \cup \{u_i\}}(y) \leq 2$. By the same argument as above, $d_{Q'_i \cup \{u_i\}}(u_1) + d_{Q'_i \cup \{u_i\}}(y) \leq 3$ for $2 \leq i \leq k$ when $|Q'_i| \geq 1$. Since $N_R(u_1) \cap N_R(y) = \emptyset$, we have $d_R(u_1) + d_R(y) \leq |R|$.

By Claim 4.2.4, we have $d_{P'_1 \cup \{x\}}(u_1) + d_{P'_1 \cup \{x\}}(y) \leq |P'_1 \cup \{x\}| + 1$ and $d_{P'_i}(u_1) + d_{P'_i}(y) \leq |P'_i| + 1$ for $2 \leq i \leq k$. So

$$n+k-2 \leq d(u_1) + d(y) \leq \sum_{i=2}^k (|P'_i| + 1) + |P'_1 \cup \{x\}| + 1 + 3(k-1) + |R|.$$

Hence we have

$$\sum_{i=2}^k |Q'_i| \leq 2k - 2. \quad (4.1)$$

Let G^* be the subgraph of G with $P_i^* = P'_i$ for $1 \leq i \leq k$ and $i \neq 2$, $Q_j^* = Q'_j$ for $2 \leq j \leq k$, and $Q_1^* = yv + vu_1$. We have $\sum_{i=1}^k |P'_i| - |P'_2| \leq 2k - 2$ by the same argument as mentioned above. By Claim 4.2.3, we have

$$\sum_{i=1}^k |P'_i| \leq 2k - 1. \quad (4.2)$$

By (4.1) and (4.2), we have $\sum_{i=1}^k |P'_i| + \sum_{i=2}^k |Q'_i| + |U \setminus \{u_1\}| + |\{x\}| \leq 5k - 3$. Since G is $(5k - 2)$ -connected, u_1 and y are connected through a path in R . It is a contradiction. \square

4.2.2 . The graph is $(2, k)$ -Hamilton-connected

Let G be a graph satisfying the conditions of Theorem 58. For any pair of vertices x and y of G , let $U = \{u_1, u_2, \dots, u_k\}$ be a subset of $V(G) \setminus \{x, y\}$. By Lemma 8, it follows that there exist $2k$ internally-disjoint paths $P_1, P_2, \dots, P_k, Q_1, Q_2, \dots, Q_k$ connecting $\{x, y\}$ and U where P_i is the path connecting x to u_i and Q_i is the path connecting y to u_i for $1 \leq i \leq k$. Let $P_i = P_i[x, u_i]$, $P'_i = P'_i(x, u_i)$, $Q_i = Q_i[u_i, y]$, and $Q'_i = Q'_i(u_i, y)$ for $1 \leq i \leq k$. We assume $S_i = P_i \cup Q_i$ and the orientation of S_i is from x to y for $1 \leq i \leq k$. Let $S = \cup_{1 \leq i \leq k} S_i$ be the union of k internally-disjoint paths connecting x to y for $u_i \in S_i \setminus \{x, y\}$ of G such that $|V(S)|$ is maximum and $R = G - S$.

If $|V(S)| = |V(G)|$, then Theorem 58 is proved. So let H be a component of R . For $1 \leq i \leq k$, let a and b be any two vertices of P_i (or Q_i), if P_{ab} is a path in P_i (or Q_i) connecting a to b such that $N_G(V(H)) \cap P_{ab} = \{a, b\}$, then we call the path P_{ab} a *segment* of S . For any segment P_{ab} , we denote aHb a path connecting a and b through H . For any segment, the order of a segment is at least 3 by the maximality of S .

Let P_{ab} be a segment of S such that $N_G(V(H)) \cap P_{ab} = \{a, b\}$ and w be an internal vertex of P_{ab} , if there exist two vertices $c, d \in N_G(w)$ such that $cd \in E(S) \setminus E(P_{ab})$, then w is called an *insertable vertex* of P_{ab} , and we call w

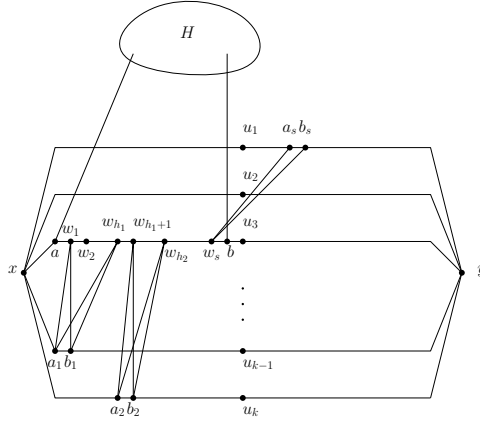


Figure 4.3: P_{ab} is a segment of S . $P_{ab}[w_1, w_{h_1}]$ is inserted by an edge $a_1 b_1$.
 $P_{ab}[w_{h_1+1}, w_{h_2}]$ is inserted by an edge $a_2 b_2$.

is inserted by cd . If c and d do not exist, then we call the internal vertex w is a *non-insertable vertex*. Suppose that w_1, w_2, \dots, w_s are insertable vertices of the segment P_{ab} in order along P_{ab} . Let

$$h_1 := \max \{i : w_i \text{ can be inserted by an edge which } w_1 \text{ can be inserted by}\}$$

and we suppose that w_1 and w_{h_1} can be inserted by an edge $a_1 b_1$. Let

$$h_2 := \max \{i : w_i \text{ can be inserted by an edge which } w_{h_1+1} \text{ can be inserted by}\}$$

and we suppose that w_{h_1+1} and w_{h_2} can be inserted by an edge $a_2 b_2$. Continuing in the same procedure, we have $h_t = s$ for some $t \geq 1$. Then we insert $P_{ab}[w_1, w_{h_1}]$ between a_1 and b_1 , $P_{ab}[w_{h_1+1}, w_{h_2}]$ between a_2 and $b_2, \dots, P_{ab}[w_{h_{t-1}+1}, w_{h_t}]$ between a_t and b_t (see Figure 4.3). We call such an operation a *segment insertion* of $P_{ab}[w_1, w_s]$ and denote it by $SI[P_{ab}[w_1, w_s]]$.

Claim 4.2.5 *Every segment of S contains a non-insertable vertex.*

Proof. Suppose to the contrary that there exists a segment $P_{w_1 w_s} = w_1 w_2 \dots w_s$ not containing any non-insertable vertex. Let T be the resulting graph of S after using a segment insertion $SI[P_{w_1 w_s}[w_2, w_{s-1}]$. Then $T \cup w_1 H w_s$ is the set of k internally-disjoint paths connecting x to y such that $u_i \in S_i \setminus \{x, y\}$ for $1 \leq i \leq k$ with the order at least $|V(S)| + 1$, which contradicts the maximality of S . \square

Claim 4.2.6 *There exists a segment in P_i or Q_i for $1 \leq i \leq k$.*

Proof. Since G is $(5k - 2)$ -connected, G is also $(2k + 1)$ -connected. If $|V(S)| \geq 2k + 1$, then for any vertex $v \in V(H)$ and any $(2k + 1)$ -subset $W = \{w_1, w_2, \dots, w_{2k+1}\}$ of $V(S)$, there exists a (v, W) -fan of G . For each path from v to w_j with $1 \leq j \leq 2k + 1$, let w'_j be the first vertex such that $w'_j \in S$. By the pigeonhole principle, at least two of them belong to the same P_i or Q_i . It implies that there exists a segment in P_i or Q_i for $1 \leq i \leq k$. If $|V(S)| \leq 2k$, then we obtain a segment in P_i or Q_i for $1 \leq i \leq k$ by the connectivity of G . \square

By Claim 4.2.5 and Claim 4.2.6, without loss of generality, we assume that $r_1, r_2 \in N_G(V(H)) \cap V(P_1)$ such that there is no vertex of $P_1[r_1^+, r_2^-]$ which has a neighbour in H , and w is a non-insertable vertex of $P_1[r_1, r_2]$ such that there is no non-insertable vertex of $P_1[r_1, w^-]$. Clearly, $P_1[r_1, r_2]$ is a segment of S . Let $P_1[r_1, w] = y_0 y_1 \dots y_m$ where $y_0 = r_1$ and $y_m = w$. For any vertex $z \in N_G(V(H)) \cap V(S_1) \setminus \{r_1\}$, let z^+ be a successor of z in S_1 . We obtain the following claim.

Claim 4.2.7 *y_i is not adjacent to z^+ for $1 \leq i \leq m$.*

Proof. By induction on i for $1 \leq i \leq m$. Suppose that $z^+ y_1 \in E(G)$. Then $S + z^+ y_1 - r_1 y_1 - z z^+ \cup r_1 H z$ is the union of k internally-disjoint paths connecting x to y . The order of these k internally-disjoint paths at least $|V(S)| + 1$, a contradiction. We suppose that this claim holds for $1 \leq j \leq i - 1$ with $i \geq 2$. Suppose $z^+ y_i \in E(G)$. Let T be a resulting graph of S after using a segment insertion $SI[P_1[y_1, y_{i-1}]]$. It follows from the induction hypothesis of this claim that y_j is not inserted by $z z^+$ for $1 \leq j \leq i - 1$. Then $T + z^+ y_i - z z^+ \cup r_1 H z$ contradicts the choice of S . \square

By Claim 4.2.7, we use the segment insertion $SI[P_1[y_1, y_{m-1}]]$ to S . Then we get a resulting graph T such that $T = \cup_{1 \leq i \leq k} T_i$ where T_2, T_3, \dots, T_k are internally-disjoint paths connecting x and y , respectively such that $u_i \in T_i \setminus \{x, y\}$

for $2 \leq i \leq k$, and T_1 is obtained by S_1 removing all y_i for $1 \leq i \leq m-1$. Note that $V(S) = V(T)$ and $\{r_1, r_2, u_1\} \subseteq V(T_1)$. Let v be any vertex of $V(H)$. It is easy to see that

$$d_R(w) + d_R(v) \leq |R| - 1. \quad (4.3)$$

By the choices of r_1 and r_2 , we have $wv \notin E(G)$. So

$$d_{T_1[w, r_2]}(w) + d_{T_1[w, r_2]}(v) \leq |T_1[w, r_2]|. \quad (4.4)$$

Since w is a non-insertable vertex of $P_1[r_1, r_2]$, for $2 \leq i \leq k$ we have

$$d_{T_i \setminus \{x, y\}}(w) \leq \begin{cases} \frac{|T_i \setminus \{x, y\}| + 1}{2}, & \text{if } T_i \setminus \{x, y\} \text{ is odd,} \\ \frac{|T_i \setminus \{x, y\}|}{2}, & \text{otherwise.} \end{cases} \quad (4.5a)$$

$$d_{T_i \setminus \{x, y\}}(v) \leq \frac{|T_i \setminus \{x, y\}| + 1}{2}. \quad (4.5b)$$

By the maximality of S , the neighbours of v in S are not adjacent. For $2 \leq i \leq k$, we know that $d_{T_i \setminus \{x, y\}}(v) \leq \frac{|T_i \setminus \{x, y\}| + 1}{2}$. So we have

$$d_{T_i \setminus \{x, y\}}(w) + d_{T_i \setminus \{x, y\}}(v) \leq |T_i \setminus \{x, y\}| + 1. \quad (4.6)$$

We now prove the following claim in order to obtain the degree sum of w and v in $T_1[x, r_1]$ and $T_1[r_2^+, y]$.

Claim 4.2.8 $xv \notin E(G)$ and $vy \notin E(G)$.

Proof. Case 1: $k = 2$. We assume that $xv \in E(G)$ by symmetry. If $vy \in E(G)$, then we have $\{x_2^+, y_2^-\} \notin N_G(v)$ where x_2^+ is the successor of x in T_2 and y_2^- is the predecessor of y in T_2 . Hence we have

$$d_{T_2 \setminus \{x, y\}}(v) \leq \begin{cases} \frac{|T_2 \setminus \{x, y\}| - 1}{2}, & \text{if } T_2 \setminus \{x, y\} \text{ is odd,} \\ \frac{|T_2 \setminus \{x, y\}| - 2}{2}, & \text{otherwise,} \end{cases}$$

and so

$$d_{T_2 \setminus \{x, y\}}(w) + d_{T_2 \setminus \{x, y\}}(v) \leq \begin{cases} |T_2 \setminus \{x, y\}|, & \text{if } T_2 \setminus \{x, y\} \text{ is odd,} \\ |T_2 \setminus \{x, y\}| - 1, & \text{otherwise.} \end{cases} \quad (4.7)$$

Note that $(N_G(w) \cap V(T_1[x, r_1])) \cap (N_G(v) \cap V(T_1[x, r_1]))^+ = \emptyset$. Otherwise, let $a = N_G(v) \cap V(T_1[x, r_1])$ and $T'_1 = T_1 - aa^+ + wa^+ + aHr_1$. So $T' = T_2 \cup T'_1$ is the union of k internally-disjoint paths connecting x to y such that $u_2 \in T_2 - \{x, y\}$ and $u_1 \in T'_1$ with the order at least $|V(S)| + 1$, this contradicts the maximality of S . By Lemma 7, we have

$$d_{T_1[x, r_1]}(w) + d_{T_1[x, r_1]}(v) \leq |T_1[x, r_1]| + 1, \quad (4.8)$$

and the equality of (4.8) holds only if $wx \in E(G)$. The equality of (4.5a) holds only if $wx_2^+ \in E(G)$. Since w is a non-insertable vertex, the two equalities can not hold in the meanwhile.

Let r_2^+ be the successor of r_2 in T_1 . By Claim 4.2.7, $wr_2^+ \notin E(G)$. We also know that $vr_2^+ \notin E(G)$. By the same argument as above, $(N_G(w) \cap V(T_1[r_2^+, y])) \cap (N_G(v) \cap V(T_1[r_2^+, y]))^+ = \emptyset$. Hence we have

$$d_{T_1[r_2^+, y]}(w) + d_{T_1[r_2^+, y]}(v) \leq |T_1[r_2^+, y]|. \quad (4.9)$$

Since now $\sigma_2(G) \geq n$, by (4.3), (4.4), (4.7), (4.8), and (4.9), we have

$$\begin{aligned} n &\leq d(w) + d(v) \\ &= d_{T_1[x, r_1]}(w) + d_{T_1[x, r_1]}(v) \\ &\quad + d_{T_1[w, r_2]}(w) + d_{T_1[w, r_2]}(v) + d_{T_1[r_2^+, y]}(w) + d_{T_1[r_2^+, y]}(v) \\ &\quad + d_{T_2 \setminus \{x, y\}}(w) + d_{T_2 \setminus \{x, y\}}(v) + d_R(w) + d_R(v) \\ &\leq n - 1. \end{aligned}$$

This is a contradiction.

If $vy \notin E(G)$, then we have

$$d_{T_2 \setminus \{x, y\}}(v) \leq \begin{cases} \frac{|T_2 \setminus \{x, y\}| - 1}{2}, & \text{if } T_2 \setminus \{x, y\} \text{ is odd,} \\ \frac{|T_2 \setminus \{x, y\}|}{2}, & \text{otherwise.} \end{cases}$$

and so

$$d_{T_2 \setminus \{x, y\}}(w) + d_{T_2 \setminus \{x, y\}}(v) \leq |T_2 \setminus \{x, y\}|. \quad (4.10)$$

Let r_2^+ be the successor of r_2 in T_1 . By Lemma 7, we have $d_{T_1[r_2^+, y]}(w) + d_{T_1[r_2^+, y]}(v) \leq |T_1[r_2^+, y]|$, and the equality holds only if $vy \in E(G)$. We have

$$d_{T_1[r_2^+, y]}(w) + d_{T_1[r_2^+, y]}(v) \leq |T_1[r_2^+, y]| - 1. \quad (4.11)$$

So $n \leq d(w) + d(v) \leq n - 1$ by (4.10) and (4.11), it is a contradiction.

Case 2: $k \geq 3$. We assume that $xv \in E(G)$ by symmetry. For $3 \leq i \leq k$, we have $d_{T_i \setminus \{x, y\}}(w) + d_{T_i \setminus \{x, y\}}(v) \leq |T_i \setminus \{x, y\}|$. By the same argument as above, we have $d_{T_1[x, r_1]}(w) + d_{T_1[x, r_1]}(v) \leq |T_1[x, r_1]| + 1$, and $d_{T_1[r_2^+, y]}(w) + d_{T_1[r_2^+, y]}(v) \leq |T_1[r_2^+, y]|$. So we obtain $d(w) + d(v) \leq n$. This contradicts $\sigma_2(G) \geq n + k - 2$ for $k \geq 3$. \square

By Lemma 7 and Claim 4.2.8, we have

$$d_{T_1[x, r_1]}(w) + d_{T_1[x, r_1]}(v) \leq |T_1[x, r_1]|. \quad (4.12)$$

Let r_2^+ be the successor of r_2 in T_1 . By Claim 4.2.7, $wr_2^+ \notin E(G)$. Note that $vr_2^+ \notin E(G)$. By the same argument as above, we have $(N_G(w) \cap V(T_1[r_2^+, y])) \cap (N_G(v) \cap V(T_1[r_2^+, y]))^+ = \emptyset$. By Lemma 7 and Claim 4.2.8, we have

$$d_{T_1[r_2^+, y]}(w) + d_{T_1[r_2^+, y]}(v) \leq |T_1[r_2^+, y]| - 1. \quad (4.13)$$

Since $\sigma_2(G) \geq n + k - 2$, by (4.3), (4.4), (4.6), (4.12) and (4.13), we have

$$\begin{aligned} n + k - 2 &\leq d(w) + d(v) \\ &= d_{T_1[w, r_2]}(w) + d_{T_1[w, r_2]}(v) \\ &\quad + d_{T_1[r_2^+, y]}(w) + d_{T_1[r_2^+, y]}(v) + d_{T_1[x, r_1]}(w) + d_{T_1[x, r_1]}(v) \\ &\quad + \sum_{i=2}^k (d_{T_i \setminus \{x, y\}}(w) + d_{T_i \setminus \{x, y\}}(v)) + d_R(w) + d_R(v) \\ &\leq n + k - 3. \end{aligned}$$

This is a contradiction. Now we finish the proof of Theorem 58. \square

4.3 . The proof of Theorem 59

The proof of Theorem 59 is same as Theorem 58 mainly except the constraint of connectivity and degree sum. So we prove some claims without connectivity in this part.

Lemma 9 *Let G be a graph with order $n \geq 4$. For any two integers $k_1 \geq 2$ and $k_2 \geq 2$, if $\sigma_2(G) \geq n + k_1k_2 - 2$, then G is (k_1, k_2) -connected.*

Proof. Suppose to the contrary that G is not (k_1, k_2) -connected, but for any $e \in E(\overline{G})$, the graph $G + e$ is (k_1, k_2) -connected where \overline{G} is the complement of G .

Let $X = \{x_1, x_2, \dots, x_{k_1}\}$ be a subset of $V(G)$ and $U = \{u_1, u_2, \dots, u_{k_2}\}$ be a subset of $V(G) \setminus X$. There exist $k_1k_2 - 1$ internally-disjoint paths connecting X to U . Without loss of generality, we denote these $k_1k_2 - 1$ internally-disjoint paths $(\cup_{1 \leq i \leq k_1 - 1, 1 \leq j \leq k_2} S_{i,j}) \cup (\cup_{j=2}^{k_2} S_{k_1,j})$. Let $S_{i,j}$ be the path connecting x_i to u_j for $1 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$, and $S_{k_1,j}$ be the path connecting x_{k_1} to u_j for $2 \leq j \leq k_2$. Let F be a subgraph of G consists of all these $k_1k_2 - 1$ paths. Assume that the order of F is minimum, denoted by assumption (II).

For the purpose of proof, let the orientation of $S_{i,j}$ be from x_i to u_j for $1 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$, and $S_{k_1,j}$ be from x_{k_1} to u_j for $2 \leq j \leq k_2$. We denote $S_{i,j} = S_{i,j}[x_i, u_j]$ and $S'_{i,j} = S_{i,j} \setminus \{x_i, u_j\} = S'_{i,j}(x_i, u_j)$ for $1 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$. Let $S_{k_1,j} = S_{k_1,j}[x_{k_1}, u_j]$ and $S'_{k_1,j} = S_{k_1,j} \setminus \{x_{k_1}, u_j\} = S'_{k_1,j}(x_{k_1}, u_j)$ for $2 \leq j \leq k_2$, and $R = G - F$ (see Figure 4.4). Note that $S'_{i,j}$ and $S'_{k_1,2}, S'_{k_1,3}, \dots, S'_{k_1,k_2}$ may be empty for $1 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$.

We obtain Claim 4.3 and Claim 4.3.2 by the same arguments as with Claim 4.2.1 and Claim 4.2.4.

For $2 \leq j \leq k_2$, $|N_G(u_1) \cap V(S'_{k_1,j})| \leq 1$ and $|N_G(x_{k_1}) \cap (V(S'_{k_1,j}) \cup \{u_j\})| = 1$.

Claim 4.3.1 *There are two integers i and j for $1 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$ such that $|N_{S'_{i,j}}(x_{k_1}) \cap N_{S'_{i,j}}(u_1)| \geq 1$. Furthermore, $|N_{S'_{i,j}}(x_{k_1}) \cap N_{S'_{i,j}}(u_1)| = 1$.*

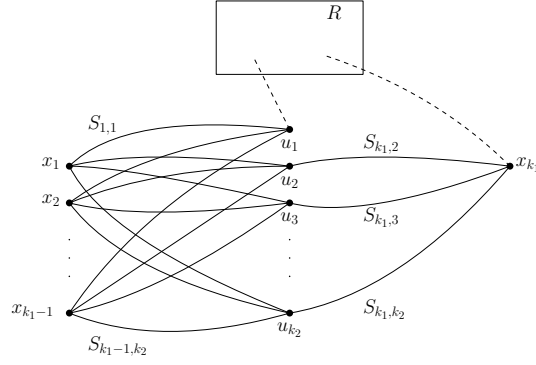


Figure 4.4: The maximal counterexample of a (k_1, k_2) -Hamilton-connected graph.

Proof. Suppose that $N_{S'_{i,j}}(x_{k_1}) \cap N_{S'_{i,j}}(u_1) = \emptyset$ for $1 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$. Then $d_{S'_{i,j}}(u_1) + d_{S'_{i,j}}(x_{k_1}) \leq |S'_{i,j}|$ for $1 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$, and $d_{\{x_i\}}(u_1) + d_{\{x_i\}}(x_{k_1}) \leq 2$. Since $N_R(u_1) \cap N_R(x_{k_1}) = \emptyset$, we have $d_R(u_1) + d_R(x_{k_1}) \leq |R|$. By Claim 4.3, we have $d_{S'_{k_1,j} \cup \{u_j\}}(u_1) + d_{S'_{k_1,j} \cup \{u_j\}}(x_{k_1}) \leq |S'_{k_1,j}| + 2$ for $2 \leq j \leq k_2$. Since u_1 and x_{k_1} are not adjacent and $\sigma_2(G) \geq n + k_1 k_2 - 2$, we have

$$\begin{aligned} n + k_1 k_2 - 2 &\leq d(u_1) + d(x_{k_1}) \\ &\leq \sum_{i=1}^{k_1-1} \sum_{j=1}^{k_2} |S'_{i,j}| + \sum_{j=2}^{k_2} (|S'_{k_1,j}| + 2) + |R| + 2(k_1 - 1) \\ &= n + k_1 + k_2 - 4. \end{aligned}$$

We deduce a contradiction, since $k_1(k_2 - 1) > k_2 - 2$ when $k_1 \geq 2$ and $k_2 \geq 2$. Now there are two integers i and j for $1 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$ such that $|N_{S'_{i,j}}(x_{k_1}) \cap N_{S'_{i,j}}(u_1)| \geq 1$. Let v be any vertex in $N_{S'_{i,j}}(x_{k_1}) \cap N_{S'_{i,j}}(u_1)$. If $S'_{i,j} \setminus \{v\} \neq \emptyset$, then it contradicts Assumption (II). So $|S'_{i,j}| = 1$ and $|N_{S'_{i,j}}(x_{k_1}) \cap N_{S'_{i,j}}(u_1)| = 1$. \square

Claim 4.3.2 For $1 \leq i \leq k_1 - 1$, $1 \leq j \leq k_2$, $d_{S'_{i,j}}(u_1) + d_{S'_{i,j}}(x_{k_1}) \leq |S'_{i,j}| + 1$.

It is easy to see that $d_R(u_1) + d_R(x_{k_1}) \leq |R|$. By Claim 4.3, we have $d_{S'_{k_1,j} \cup \{u_j\}}(u_1) + d_{S'_{k_1,j} \cup \{u_j\}}(x_{k_1}) \leq |S'_{k_1,j}| + 2$ for $2 \leq j \leq k_2$. We also have $d_{S'_{i,1} \cup \{x_i\}}(u_1) + d_{S'_{i,1} \cup \{x_i\}}(x_{k_1}) \leq |S'_{i,1}| + 2$ for $1 \leq i \leq k_1 - 1$.

We now infer that

$$\begin{aligned}
n + k_1 k_2 - 2 &\leq d(u_1) + d(x_{k_2}) \\
&= \sum_{i=1}^{k_1-1} \sum_{j=2}^{k_2} (d_{S'_{i,j}}(u_1) + d_{S'_{i,j}}(x_{k_2})) + \sum_{j=2}^{k_2} (d_{S'_{k_2,j} \cup \{u_j\}}(u_1) + d_{S'_{k_2,j} \cup \{u_j\}}(x_{k_2})) \\
&\quad + \sum_{i=1}^{k_1-1} (d_{S'_{i,1} \cup \{x_i\}}(u_1) + d_{S'_{i,1} \cup \{x_i\}}(x_{k_2})) + d_R(u_1) + d_R(x_{k_2}) \\
&\leq \sum_{i=1}^{k_1-1} \sum_{j=2}^{k_2} (|S'_{i,j}| + 1) + \sum_{j=2}^{k_2} (|S'_{k_1,j}| + 2) + \sum_{i=1}^{k_1-1} (|S'_{i,1}| + 2) + |R| \\
&= \sum_{i=1}^{k_1-1} \sum_{j=2}^{k_2} |S'_{i,j}| + (k_1 - 1)(k_2 - 1) + \sum_{j=2}^{k_2} |S'_{k_1,j}| + 2(k_2 - 1) + \sum_{i=1}^{k_1-1} |S'_{i,1}| \\
&\quad + 2(k_1 - 1) + |R| \\
&= n + k_1 k_2 - 3,
\end{aligned}$$

this is a contradiction. So we complete the proof of Lemma 9. \square

Let G be a graph satisfying the conditions of Theorem 59. For any k_1 -subset $X = \{x_1, x_2, \dots, x_{k_1}\}$ of $V(G)$, let $U = \{u_1, u_2, \dots, u_{k_2}\}$ be a k_2 -subset of $V(G) \setminus X$. By Lemma 9, it follows that there exist $k_1 k_2$ internally-disjoint paths $S_{i,j}$ connecting x_i to u_j for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. Let the orientation of $S_{i,j}$ be from x_i to u_j for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. We denote $S_{i,j} = S_{i,j}[x_i, u_j]$, $S'_{i,j} = S_{i,j} \setminus \{x_i, u_j\} = S'_{i,j}(x_i, u_j)$ for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. Let $S = \cup_{1 \leq i \leq k_1, 1 \leq j \leq k_2} S_{i,j}$ be the union of these $k_1 k_2$ internally-disjoint paths such that $|V(S)|$ is maximum, and $R = G - S$.

If $|V(S)| = |V(G)|$, then Theorem 59 is proved. So let H be a component of R . We obtain the following claim by the same argument with Claim 4.2.5.

Claim 4.3.3 *Every segment of S contains a non-insertable vertex.* \square

Claim 4.3.4 *There exists a segment in $S_{i,j}$ for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$.*

Proof. Suppose that there is no segment in $S_{i,j}$ for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. Let z be any vertex of $V(H)$. We consider the following cases.

Case 1: $N_G(V(H)) \cap X \neq \emptyset$. Without loss of generality, we suppose that $x_1 \in N_G(V(H))$. Let $x_{1,1}^+$ be a successor of x_1 in $S_{1,1}$. By the maximality of S , $x_{1,1}^+$ and z are not adjacent.

It is clear that there exist two integers i and j with $2 \leq i \leq k_1$ and $1 \leq j \leq k_2$ such that $N_G(V(H)) \cap (V(S'_{i,j}) \cup \{x_i\}) \neq \emptyset$. Otherwise, we can obtain the contradiction obviously by calculating the sum degree of $x_{1,1}^+$ and z . Without loss of generality, we assume that $N_G(V(H)) \cap (V(S'_{k_1,1}) \cup \{x_{k_1}\}) \neq \emptyset$.

For any vertex $w \in N_G(V(H)) \cap (V(S'_{k_1,1}) \cup \{x_{k_1}\})$, let w^- be a predecessor of w in $S_{k_1,1}$. We have $x_{1,1}^+ w^- \notin E(G)$. If $x_{1,1}^+ w^- \in E(G)$, then it contradicts the maximality of S . So we obtain

$$d_{S_{1,1} \cup S_{k_1,1}}(z) + d_{S_{1,1} \cup S_{k_1,1}}(x_{1,1}^+) \leq \begin{cases} |S_{1,1} \cup S_{k_1,1}| + 1, & \text{if } y \in N_G(V(H)), \\ |S_{1,1} \cup S_{k_1,1}|, & \text{otherwise.} \end{cases}$$

For $2 \leq j \leq k_2$, we have

$$d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(z) + d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(x_{1,1}^+) \leq \begin{cases} |S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}|, & \text{if } y \in N_G(V(H)), \\ |S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}| + 1, & \text{otherwise.} \end{cases}$$

For $2 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$, we obtain

$$d_{S'_{i,j} \cup \{x_i\}}(z) + d_{S'_{i,j} \cup \{x_i\}}(x_{1,1}^+) \leq \sum_{i=2}^{k_1-1} \sum_{j=1}^{k_2} |S'_{i,j} \cup \{x_i\}| + 1.$$

By the maximality of S , we have $d_R(x_{1,1}^+) + d_R(z) \leq |R| - 1$. So

$$\begin{aligned}
n + k_1 k_2 - 2 &\leq d(x_{1,1}^+) + d(z) \\
&= d_{S_{1,1} \cup S_{k_1,1}}(z) + d_{S_{1,1} \cup S_{k_1,1}}(x_{1,1}^+) + \sum_{j=2}^{k_2} (d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(z) \\
&\quad + d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(x_{1,1}^+)) + \sum_{i=2}^{k_1-1} \sum_{j=1}^{k_2} (d_{S'_{i,j} \cup \{x_i\}}(z) + d_{S'_{i,j} \cup \{x_i\}}(x_{1,1}^+)) \\
&\quad + d_R(x_{1,1}^+) + d_R(z) \\
&\leq |S_{1,1} \cup S_{k_1,1}| + 1 + \sum_{j=2}^{k_2} (|S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}| + 1) + \sum_{i=2}^{k_1-1} \sum_{j=1}^{k_2} |S'_{i,j} \cup \{x_i\}| \\
&\quad + k_2(k_1 - 2) + |R| - 1 \\
&= n + 1 + (k_2 - 1) + k_1 k_2 - 2k_2 - 1 \\
&= n + k_1 k_2 - k_2 - 1,
\end{aligned}$$

this is a contradiction.

Case 2: $N_G(V(H)) \cap X = \emptyset$.

For $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, we have $|N_G(V(H)) \cap (V(S'_{i,j}) \cup \{u_j\})| \leq 1$. Without loss of generality, we suppose $v \in (V(S'_{1,1}) \cup \{u_1\}) \cap N_G(V(H))$. Let v^- be a predecessor of v in $S_{1,1}$. Note that v^- and z are not adjacent. By the maximality of S , we have $N_G(v^-) \cap (N_G(z) \cap (V(S'_{k_1,1}) \cup \{x_{k_1}\}))^+ = \emptyset$. So $d_{S_{1,1} \cup S_{k_1,1}}(v^-) + d_{S_{1,1} \cup S_{k_1,1}}(z) \leq |S_{1,1} \cup S_{k_1,1}|$ by Lemma 7. When $N_G(V(H)) \cap V(S'_{1,j}) \neq \emptyset$ for $2 \leq j \leq k_2$. Let w be any vertex of $N_G(V(H)) \cap V(S'_{1,j})$ and w^+ be a successor of w in $S'_{1,j}$. By the maximality of S , $v^- w^+ \notin E(G)$. For $2 \leq j \leq k_2$, we obtain

$$d_{S'_{1,j}}(v^-) + d_{S'_{1,j}}(z) \leq \begin{cases} |S'_{1,j}| + 1, & \text{if } w^+ = u_i, \\ |S'_{1,j}|, & \text{otherwise.} \end{cases}$$

Since $|N_G(V(H)) \cap (V(S'_{k_1,j}) \cup \{u_j\})| \leq 1$, we have

$$d_{S'_{k_1,j} \cup \{u_j\}}(v^-) + d_{S'_{k_1,j} \cup \{u_j\}}(z) \leq \begin{cases} |S'_{k_1,j}| + 1, & \text{if } w^+ = u_i, \\ |S'_{k_1,j} \cup \{u_i\}| + 1, & \text{otherwise.} \end{cases}$$

So $d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(v^-) + d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(z) \leq |S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}| + 1$ for $2 \leq j \leq k_2$.

If $N_G(V(H)) \cap V(S'_{1,j}) = \emptyset$ for $2 \leq j \leq k_2$, then $d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(v^-) + d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(z) \leq |S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}| + 1$.

Hence for $2 \leq j \leq k_2$, we have

$$d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(v^-) + d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(z) \leq |S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}| + 1.$$

For $2 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$, we have $d_{S'_{i,j} \cup \{x_i\}}(v^-) + d_{S'_{i,j} \cup \{x_i\}}(z) \leq |S'_{i,j} \cup \{x_i\}| + 1$. By the maximality of S , $d_R(v^-) + d_R(z) \leq |R| - 1$. So

$$\begin{aligned} n + k_1 k_2 - 2 &\leq d(v^-) + d(z) \\ &= d_{S_{1,1} \cup S_{k_1,1}}(v^-) + d_{S_{1,1} \cup S_{k_1,1}}(z) + \sum_{j=2}^{k_2} (d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(v^-) \\ &\quad + d_{S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}}(z)) + \sum_{i=2}^{k_1-1} \sum_{j=1}^{k_2} (d_{S'_{i,j} \cup \{x_i\}}(v^-) + d_{S'_{i,j} \cup \{x_i\}}(z)) \\ &\quad + d_R(v^-) + d_R(z) \\ &\leq |S_{1,1} \cup S_{k_1,1}| + \sum_{j=2}^{k_2} (|S'_{1,j} \cup S'_{k_1,j} \cup \{u_j\}| + 1) + \sum_{i=2}^{k_1-1} \sum_{j=1}^{k_2} (|S'_{i,j} \cup \{x_i\}| + 1) \\ &\quad + |R| - 1 \\ &= n + k_1 k_2 - k_2 - 2. \end{aligned}$$

It is a contradiction. \square

By Claim 4.3.3 and Claim 4.3.4, without loss of generality, we assume that $r_1, r_2 \in N_G(V(H)) \cap V(S_{1,1})$ such that there is no vertex of $S_{1,1}[r_1^+, r_2^-]$ which has a neighbour in H , and w is a non-insertable vertex of $S_{1,1}[r_1, r_2]$ such that there is no non-insertable vertex of $S_{1,1}[r_1, w^-]$. Clearly, $S_{1,1}[r_1, r_2]$ is a segment of S . Let $S_{1,1}[r_1, w] = y_0 y_1 \dots y_m$ where $y_0 = r_1$ and $y_m = w$. We denote $S_{1,1,k_1} = S_{1,1} \cup S_{k_1,u_1}$ and the orientation of $S_{1,1,k_1}$ is from x_1 to x_{k_1} . For any vertex $z \in N_G(V(H)) \cap V(S_{1,1,k_1}) \setminus \{r_1\}$, let z^+ be a successor of z in $S_{1,1,k_1}$. We obtain the following claim with the same argument of Claim 4.2.7.

Claim 4.3.5 y_i is not adjacent to z^+ for $1 \leq i \leq m$. \square

By Claim 4.3.5, we use the segment insertion $SI[P_1[y_1, y_{m-1}]]$ to S . Then we get a resulting graph T such that $T = \cup_{\substack{1 \leq i \leq k_1 \\ 1 \leq j \leq k_2}} T_{i,j}$ where $T_{1,1}$ is obtained by $S_{1,1}$ removing all y_i for $1 \leq i \leq m-1$. We denote $T_{1,1,k_1} = S_{1,1,k_1} \setminus \{y_1, y_2, \dots, y_{m-1}\}$ and the orientation of $T_{1,1,k_1}$ is from x_1 to x_{k_1} . It is clear that $V(S) = V(T)$ and $\{r_1, r_2, u_1\} \subseteq V(T_{1,1})$. Let v be any vertex of $V(H)$.

We note that $(N_G(w) \cap V(T_{1,1,k_1}[x_1, r_1])) \cap (N_G(v) \cap V(T_{1,1,k_1}[x_1, r_1]))^+ = \emptyset$. Hence we have

$$d_{T_{1,1,k_1}[x_1, r_1]}(w) + d_{T_{1,1,k_1}[x_1, r_1]}(v) \leq |T_{1,1,k_1}[x_1, r_1]| + 1.$$

Let r_2^+ be the successor of r_2 in $T_{1,1,k_1}$. By Claim 4.3.5, $wr_2^+ \notin E(G)$. Note that $vr_2^+ \notin E(G)$ and $(N_G(w) \cap V(T_{1,1,k_1}[r_2^+, x_{k_1}])) \cap (N_G(v) \cap V(T_{1,1,k_1}[r_2^+, x_{k_1}]))^+ = \emptyset$. So we have

$$d_{T_{1,1,k_1}[r_2^+, x_{k_1}]}(w) + d_{T_{1,1,k_1}[r_2^+, x_{k_1}]}(v) \leq |T_{1,1,k_1}[r_2^+, x_{k_1}]|.$$

We have the following facts likewise the analysis of (4.3), (4.4), and (4.6): $d_R(w) + d_R(v) \leq |R| - 1$ and $d_{T_{1,1,k_1}[w, r_2]}(w) + d_{T_{1,1,k_1}[w, r_2]}(v) \leq |T_{1,1,k_1}[w, r_2]|$. For $2 \leq j \leq k_2$, we have

$$d_{T_{1,j} \cup T_{x_{k_1}, j} \setminus \{x_1, x_{k_1}\}}(w) + d_{T_{1,j} \cup T_{x_{k_1}, j} \setminus \{x_1, x_{k_1}\}}(v) \leq |T_{1,j} \cup T_{x_{k_1}, j} \setminus \{x_1, x_{k_1}\}| + 1.$$

For $2 \leq i \leq k_1 - 1$ and $1 \leq j \leq k_2$, we have $d_{T_{i,j} \setminus \{u_j\}}(w) + d_{T_{i,j} \setminus \{u_j\}}(v) \leq |T_{i,j} \setminus \{u_j\}| + 1$.

We now infer that

$$\begin{aligned} n + k_1 k_2 - 2 &\leq d(w) + d(v) \\ &\leq \sum_{j=2}^{k_2} (|T_{1,j} \cup T_{x_{k_1}, j} \setminus \{x_1, x_{k_1}\}| + 1) + \sum_{i=2}^{k_1-1} \sum_{j=1}^{k_2} (|T_{i,j} \setminus \{u_j\}| + 1) \\ &\quad + |T_{1,1,k_1}[x_1, r_1]| + 1 + |T_{1,1,k_1}[r_2^+, x_{k_1}]| + |T_{1,1,k_1}[w, r_2]| + |R| - 1 \\ &\leq n + k_1 k_2 - k_2 - 1. \end{aligned}$$

This is a contradiction.

The proof of Theorem 59 is completed. \square

4.4 . Algorithm to find a spanning (X, U) -connection given two disjoint vertex subsets $X = \{x, y\}$ and $U = \{u_1, u_2, \dots, u_k\}$

Algorithm 2:

input : a $(5k - 4)$ -connected graph $G = (V, E)$ with $\sigma_2(G) \geq n + k$,

$\{x, y, u_1, u_2, \dots, u_k\} \in V$

output: a spanning (X, U) -connection where $X = \{x, y\}$ and

$U = \{u_1, u_2, \dots, u_k\}$

$P_1, P_2, \dots, P_{2k} \leftarrow \emptyset$

$S \leftarrow \cup_{i=1}^{2k} P_i$

while P_i or P_{i+k} is empty for some $1 \leq i \leq k$ **do**

if $|S|$ is minimal in terms of u_i (see the proof of Claim 4.2.3 and

 Claim 4.2.1) **then**

 Find the common neighbour of u_i and x (or y) outside S (by our proof it must exist).

else

 Do the operation of minimising $|S|$.

while there is a segment I in P_i for $1 \leq i \leq 2k$. **do**

if there is a non-insertable vertex w in I **then**

 Find the path from x to y through w and the vertices outside S .

else

 Do the segment insertion on I .

We present an algorithm to find a spanning (X, U) -connection given two disjoint vertex subsets $X = \{x, y\}$ and $U = \{u_1, u_2, \dots, u_k\}$ with $k \geq 2$ for a graph satisfying the condition of Theorem 58 as follows. The time complexity of the algorithm is $O(kn^2) + O(n^3)$.

5 - Conclusions and prospects

We first describe some open questions related to our results in the thesis. Then we would like to mention the prospects of my research.

5.1 . Open problem

5.1.1 . Generalisation of Hamilton-connectivity

In Chapter 4, we prove that every graph G with n vertices is $(2, k)$ -Hamilton-connected if G is $(5k - 4)$ -connected and $\sigma_2(G) \geq n + k - 2$ where $k \geq 2$. The connectivity of G is mainly used to prove the existence of the $(2, k)$ -connection. We want to improve this result if we remove the connectivity condition.

Another direction is to study sufficient conditions of (k_1, k_2) -Hamilton connected graphs. We have given a sufficient condition of $(2, k)$ -Hamilton connected graphs. We will continue to consider the case when $k_1 \geq 3$.

5.1.2 . Fractional colourings on triangle-free graphs

In Chapter 3, we know that $\chi_f(4, K_3)$ lies between 3.25 and 3.5. Although we have improved the bound of $\chi_f(4, K_3)$ to 3.466, there is still room for improvement. The same question for larger values of the maximum degree is still open.

In fact, we have done some work on further loosening of the constraints of degree-4 vertices. It is a well-known fact that, given any graph G , if H is the bipartite spanning subgraph of G induced by a maximum cut, then $\deg_H(v) \geq \deg_G(v)/2$ for every vertex $v \in V(G)$ (this actually holds even if H is induced by a maximal cut, up to moving any vertex from one part to the other). By using this method, we could improved 3.4663 to 3.456.

5.2 . Future research

5.2.1 . Proper conflict-free colourings

Motivated by a frequency assignment problem in cellular networks, Even, Lotker, Ron, and Smorodinsky [41] introduced the notion of conflict-free colourings of hypergraphs. A colouring σ of a hypergraph \mathcal{H} is *conflict-free* if for every edge $e \in E(\mathcal{H})$ there exists a colour appearing exactly once in e . Pach and Tardos [86] studied this notion and proved that every hypergraph with fewer than $\binom{s}{2}$ edges (for some integer s) has a conflict-free colouring with fewer than s colours. Kostochka, Kumbhat, and Luczak [66] further studied conflict-free colouring for uniform hypergraphs.

A *proper conflict-free colouring* of G (*pcf-colouring* for short) is a proper colouring of G such that for every non-isolated vertex v , there is a colour appearing exactly once among the neighbours of v . We let $\chi_{pcf}(G)$ be the smallest integer k such that there exists a pcf k -colouring of G . This notion is the combination of proper colouring and the pointed conflict-free chromatic parameter introduced by Cheilaris [24].

The notion of pcf colourings of graphs was formally introduced by Fabrici, Lužar, Rindošová, and Soták [42], where they investigated the pcf colourings of planar and outerplanar graphs, among many other related variants of a proper conflict-free colouring. They proved that $\chi_{pcf}(G) \leq 8$ for all planar graphs and $\chi_{pcf}(G) \leq 5$ for all outerplanar graphs. Plenty of further studies in pcf colourings of sparse graphs can be found in [23, 25, 42, 54, 77].

Caro, Petruševski, and Škrekovski [23] proposed the following conjecture about pcf colourings.

Conjecture 7 (Caro, Petruševski, Škrekovski [23, Conjecture 6.4]) *If G is a connected graph of maximum degree $\Delta \geq 3$, then $\chi_{pcf}(G) \leq \Delta + 1$.*

As a first step toward their conjecture, Caro, Petruševski, and Škrekovski [23] proved that for such a graph G , $\chi_{pcf}(G) \leq \lfloor 2.5\Delta \rfloor$. Recently, it has been observed by Cranston and Liu [28] that $\chi_{pcf}(G) \leq \Delta(G) + \delta^*(G) + 1$ where $\delta^*(G)$ denote the *degeneracy* of G , that is $\delta^*(G) = \max_{H \subseteq G} \delta(H)$ (they actually more generally proved that there always exists a pcf $(\Delta(\mathcal{H}) + \delta^*(G) + 1)$ -colouring of any given pair (G, \mathcal{H})). They further reduced the gap to Conjecture 7 by proving that $\chi_{pcf}(G) \leq \lceil 1.6550826\Delta + \sqrt{\Delta} \rceil$, given that Δ is large enough. So it is very interesting to study Conjecture 7 by using probabilistic method to have an asymptotical bound

as $\Delta \rightarrow \infty$.

5.2.2 . Odd colouring

A proper vertex-colouring of a graph G is said to be *odd* if for each non-isolated vertex there is a colour appearing an odd number of times on its neighbourhood. It is introduced by Petruševski and Škrekovski [87]. The *odd chromatic number* of a graph G , denoted by $\chi_o(G)$, is the minimum $k \in \mathbb{N}$ such that there exists an odd colouring $\sigma: V(G) \rightarrow [k]$. Since odd colourings are a weakening of pcf colourings, it always holds that $\chi_o(G) \leq \chi_{pcf}(G)$ for every graph G .

In the last couple of years, there has been some interest in determining the extremal value of χ_o in various classes of graphs. Petruševski and Škrekovski [87] showed that $\chi_o(G) \leq 9$ for every planar graph G with a proof that relies on the discharging method. Furthermore, they conjectured that this bound may be reduced to 5. If true, this would be tight, since $\chi_o(C_5) = 5$.

Caro, Petruševski, and Škrekovski [22] also studied various properties of the odd chromatic number of general graphs; in particular, they proved the following facts: every graph of maximum degree three has an odd 4-colouring; every graph, except for C_5 , of maximum degree Δ has an odd 2Δ -colouring. Moreover, they presented a conjecture for general graphs.

Conjecture 8 (Caro, Petruševski, Škrekovski [22, Conjecture 5.5]) *If G is a connected graph of maximum degree $\Delta \geq 3$, then $\chi_o(G) \leq \Delta + 1$.*

Recently, Dai, Ouyang, and Pirot [31] used the probabilistic method to prove that for every graph G with maximum degree Δ , $\chi_o(G) \leq \Delta + O(\ln \Delta)$ as $\Delta \rightarrow \infty$. We also prove that $\chi_o(G) \leq \lfloor 3\Delta/2 \rfloor + 2$ for every Δ . We will study this conjecture in the near future.

Bibliography

- [1] *Problem Garden*, <http://www.openproblemgarden.org/>.
- [2] N. Alon. Degrees and choice numbers. *Random Structures & Algorithms*, 16(4):364–368, 2000.
- [3] N. Alon and M. Krivelevich. The choice number of random bipartite graphs. *Annals of Combinatorics*, 2(4):291–297, 1998.
- [4] N. Alon, M. Krivelevich, and B. Sudakov. Coloring graphs with sparse neighborhoods. *Journal of Combinatorial Theory, Series B*, 77(1):73–82, 1999.
- [5] N. Alon and M. Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12:125–134, 1992.
- [6] B. Bajnok and G. Brinkmann. On the independence number of triangle free graphs with maximum degree three. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 26:237–254, 1998.
- [7] C. Berge. Farbung von graphen, deren samtliche bzw. deren ungerade kreise starr sind. *Wissenschaftliche Zeitschrift*, 1961.
- [8] A. Y. Bernshtein, A. V. Kostochka, and S. P. Pron'. On dp-coloring of graphs and multigraphs. *Sibirskii Matematicheskii Zhurnal*, 58(1):36–47, 2017.
- [9] A. Bernshteyn. The asymptotic behavior of the correspondence chromatic number. *Discrete Mathematics*, 339(11):2680–2692, 2016.
- [10] A. Bernshteyn. The johansson-molloy theorem for DP-coloring. *Random Structures & Algorithms*, 54(4):653–664, 2019.
- [11] A. Y. Bernshteyn and A. V. Kostochka. On differences between DP-coloring and list coloring. *Siberian Advances in Mathematics*, 29:183–189, 2019.
- [12] N. Biggs, E. K. Lloyd, and R. J. Wilson. *Graph Theory, 1736–1936*. Oxford University Press, 1986.

- [13] B. Bollobás. The independence ratio of regular graphs. *Proceedings of the American Mathematical Society*, pages 433–436, 1981.
- [14] B. Bollobás and A. Thomason. Highly linked graphs. *Combinatorica*, 16:313–320, 1996.
- [15] J. A. Bondy and U. S. R. Murty. *Graph theory*. Springer Publishing Company, Incorporated, 2008.
- [16] O. Borodin. *Problems of colouring and of covering the vertex set of a graph by induced subgraphs*. PhD thesis, Novosibirsk State University, Novosibirsk, Russian, 1979.
- [17] O. V. Borodin. Criterion of chromaticity of a degree prescription. In *IV All-Union Conference on Theoretical Cybernetics (Novosibirsk)*, pages 127–128, 1977.
- [18] O. V. Borodin. Colorings of plane graphs: a survey. *Discrete Mathematics*, 313(4):517–539, 2013.
- [19] O. V. Borodin, A. N. Glebov, A. Raspaud, and M. R. Salavatipour. Planar graphs without cycles of length from 4 to 7 are 3-colorable. *Journal of Combinatorial Theory, Series B*, 93(2):303–311, 2005.
- [20] P. Bradshaw. *Graph colorings with local restrictions*. PhD thesis, Simon Fraser University, 2022.
- [21] R. L. Brooks. On colouring the nodes of a network. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 37, pages 194–197. Cambridge University Press, 1941.
- [22] Y. Caro, M. Petruševski, and R. Škrekovski. Remarks on odd colorings of graphs. *arXiv:2201.03608*, 2022.
- [23] Y. Caro, M. Petruševski, and R. Škrekovski. Remarks on proper conflict-free colorings of graphs. *arXiv:2203.01088*, 2022.
- [24] P. Cheilaris. *Conflict-free coloring*. PhD thesis, City University of New York, 2009.

- [25] E.-K. Cho, I. Choi, H. Kwon, and B. Park. Proper conflict-free coloring of sparse graphs. *arXiv:2203.16390*, 2022.
- [26] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Annals of mathematics*, pages 51–229, 2006.
- [27] V. Cohen-Addad, M. Hebdige, Z. Li, E. Salgado, et al. Steinberg’s conjecture is false. *Journal of Combinatorial Theory, Series B*, 122:452–456, 2017.
- [28] D. W. Cranston and C.-H. Liu. Proper conflict-free coloring of graphs with large maximum degree. *arXiv:2211.02818*, 2022.
- [29] T. Dai, H. Li, Y. Manoussakis, and Q. Ouyang. Properly colored cycles in edge-colored complete graphs. submitted for publication.
- [30] T. Dai, H. Li, Q. Ouyang, and Z. Tian. On $(2,k)$ -hamilton-connected graphs. submitted for publication.
- [31] T. Dai, Q. Ouyang, and F. Pirot. New bounds for odd colourings of graphs. submitted for publication.
- [32] T. Dai, Q. Ouyang, F. Pirot, and J. Sereni. Beyond the fractional reed bound for triangle-free graphs. preprint.
- [33] T. Dai, G. Wang, D. Yang, and G. Yu. Strong list-chromatic index of subcubic graphs. *Discrete Mathematics*, 341(12):3434–3440, 2018.
- [34] E. Davies, R. de Joannis de Verclos, R. J. Kang, and F. Pirot. Coloring triangle-free graphs with local list sizes. *Random Structures & Algorithms*, 57(3):730–744, 2020.
- [35] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, 3(1):69–81, 1952.
- [36] G. A. Dirac. In abstrakten graphen vorhandene vollständige 4-graphen und ihre unterteilungen. *Mathematische Nachrichten*, 22(1-2):61–85, 1960.
- [37] Z. Dvořák, P. O. de Mendez, and H. Wu. 1-subdivisions, the fractional chromatic number and the hall ratio. *Combinatorica*, 40:759–774, 2020.

- [38] Z. Dvořák and L. Postle. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *Journal of Combinatorial Theory, Series B*, 129:38–54, 2018.
- [39] Z. Dvořák, J.-S. Sereni, and J. Volec. Subcubic triangle-free graphs have fractional chromatic number at most $14/5$. *Journal of the London Mathematical Society*, 89(3):641–662, 2014.
- [40] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proc. West Coast Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, volume 26, pages 125–157, 1979.
- [41] G. Even, Z. Lotker, D. Ron, and S. Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM Journal on Computing*, 33(1):94–136, 2003.
- [42] I. Fabrici, B. Lužar, S. Rindošová, and R. Soták. Proper conflict-free and unique-maximum colorings of planar graphs with respect to neighborhoods. *arXiv:2202.02570*, 2022.
- [43] R. J. Faudree, R. J. Gould, M. S. Jacobson, and R. H. Schelp. Neighborhood unions and hamiltonian properties in graphs. *Journal of Combinatorial Theory, Series B*, 47(1):1–9, 1989.
- [44] E. Flandrin, H. Jung, and H. Li. Hamiltonism, degree sum and neighborhood intersections. *Discrete mathematics*, 90(1):41–52, 1991.
- [45] K. Fraughnaugh and S. C. Locke. $11/30$ (finding large independent sets in connected triangle-free 3-regular graphs). *Journal of Combinatorial Theory, Series B*, 65(1):51–72, 1995.
- [46] M. Garey and D. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1983. p.199.
- [47] M. Garey, D. Johnson, Stockmeyer, and L.J. Some simplified np-complete graph problems. *theoretical computer science*. *Theoretical Computer Science*, 1:237–267, 1976.

- [48] M. R. Garey, D. S. Johnson, and R. E. Tarjan. The planar hamiltonian circuit problem is np-complete. *SIAM Journal on Computing*, 5(4):704–714, 1976.
- [49] R. J. Gould. Updating the hamiltonian problem—a survey. *Journal of Graph Theory*, 15(2):121–157, 1991.
- [50] R. J. Gould. Advances on the hamiltonian problem—a survey. *Graphs and Combinatorics*, 19(1):7–52, 2003.
- [51] H. Grotzsch. Ein dreifarbensatz fur dreikreisfreie netze auf der kugel. *Wiss. Z. Martin Luther Univ. Halle-Wittenberg, Math. Nat. Reihe*, 8:109–120, 1959.
- [52] S. Gutner. The complexity of planar graph choosability. *Discrete Mathematics*, 159(1-3):119–130, 1996.
- [53] D. G. Harris. Some results on chromatic number as a function of triangle count. *SIAM Journal on Discrete Mathematics*, 33(1):546–563, 2019.
- [54] R. Hickingbotham. Odd colourings, conflict-free colourings and strong colouring numbers. *arXiv:2203.10402*, 2022.
- [55] D. F. Hsu. On container width and length in graphs, groups, and networks—dedicated to professor paul erdős on the occasion of his 80th birthday—. *IEICE transactions on fundamentals of electronics, communications and computer sciences*, 77(4):668–680, 1994.
- [56] H.-C. Hsu, C.-K. Lin, H.-M. Hung, and L.-H. Hsu. The spanning connectivity of the (n, k) -star graphs. *International Journal of Foundations of Computer Science*, 17(02):415–434, 2006.
- [57] L.-H. Hsu and C.-K. Lin. *Graph theory and interconnection networks*. CRC press, 2008.
- [58] E. Hurley, R. de Joannis de Verclos, and R. J. Kang. An improved procedure for colouring graphs of bounded local density. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 135–148. SIAM, 2021.

- [59] J. P. Hutchinson. On list-coloring outerplanar graphs. *Journal of Graph Theory*, 59(1):59–74, 2008.
- [60] T. R. Jensen and B. Toft. Graph coloring problems, 1995.
- [61] A. Johansson. Asymptotic choice number for triangle free graphs. Technical report, 91-95 DIMACS, 1996.
- [62] K. F. Jones. Independence in graphs with maximum degree four. *Journal of Combinatorial Theory, Series B*, 37(3):254–269, 1984.
- [63] H. A. Jung. Eine verallgemeinerung des n-fachen zusammenhangs für graphen. *Mathematische Annalen*, 187:95–103, 1970.
- [64] K.-i. Kawarabayashi, A. Kostochka, and G. Yu. On sufficient degree conditions for a graph to be-linked. *Combinatorics, Probability and Computing*, 15(5):685–694, 2006.
- [65] S.-J. Kim and K. Ozeki. A note on a brooks’ type theorem for DP-coloring. *Journal of Graph Theory*, 91(2):148–161, 2019.
- [66] A. Kostochka, M. Kumbhat, and T. Łuczak. Conflict-free colourings of uniform hypergraphs with few edges. *Combinatorics, Probability and Computing*, 21(4):611–622, 2012.
- [67] A. V. Kostochka. personal communication.
- [68] A. V. Kostochka, M. Stiebitz, and B. Wirth. The colour theorems of brooks and gallai extended. *Discrete Mathematics*, 162(1-3):299–303, 1996.
- [69] D. G. Larman and P. Mani. On the existence of certain configurations within graphs and the 1-skeletons of polytopes. *Proceedings of the London Mathematical Society*, 3(1):144–160, 1970.
- [70] H. Li, S. Maezawab, and Z. Tian. New sufficient condition for graphs to be k -fan-connected. submitted for publication.
- [71] H. Li and Z. Tian. On pancyclic 2-connected graphs. submitted for publication.

- [72] C.-K. Lin, H.-M. Huang, D. F. Hsu, and L.-H. Hsu. On the spanning w -wide diameter of the star graph. *Networks: An International Journal*, 48(4):235–249, 2006.
- [73] C.-K. Lin, H.-M. Huang, and L.-H. Hsu. The super connectivity of the pancake graphs and the super laceability of the star graphs. *Theoretical Computer Science*, 339(2-3):257–271, 2005.
- [74] C.-K. Lin, H.-M. Huang, and L.-H. Hsu. On the spanning connectivity of graphs. *Discrete Mathematics*, 307(2):285–289, 2007.
- [75] C.-K. Lin, H.-M. Huang, J. J. Tan, and L.-H. Hsu. On spanning connected graphs. *Discrete Mathematics*, 308(7):1330–1333, 2008.
- [76] C.-K. Lin, J. J. Tan, D. F. Hsu, and L.-H. Hsu. On the spanning fan-connectivity of graphs. *Discrete applied mathematics*, 157(7):1342–1348, 2009.
- [77] C.-H. Liu. Proper conflict-free list-coloring, subdivisions, and layered treewidth. *arXiv:2203.12248*, 2022.
- [78] W. Mader. Homomorphieeigenschaften und mittlere kantendichte von graphen. *Mathematische Annalen*, 174(4):265–268, 1967.
- [79] K. Menger. Zur allgemeinen kurventheorie. *Fundamenta mathematicae*, 10(1):96–115, 1927.
- [80] M. Molloy. The list chromatic number of graphs with small clique number. *Journal of Combinatorial Theory, Series B*, 134:264–284, 2019.
- [81] M. Molloy. The list chromatic number of graphs with small clique number. *Journal of Combinatorial Theory, Series B*, 134:264–284, 2019.
- [82] M. Molloy and B. Reed. *Graph colouring and the probabilistic method*, volume 23. Springer Science & Business Media, 2002.
- [83] O. Ore. A note on hamiltonian circuits. *American Mathematical Monthly*, 67:55, 1960.

- [84] O. Ore. Hamilton connected graphs. *Journal de Mathématiques Pures et Appliquées*, 42(2127):70, 1963.
- [85] Q. Ouyang, T. Dai, and Y. Ma. A hypergraph approach for logic-based abduction. *DBKDA 2023*, page 31, 2023.
- [86] J. Pach and G. Tardos. Conflict-free colourings of graphs and hypergraphs. *Combinatorics, Probability and Computing*, 18(5):819–834, 2009.
- [87] M. Petruševski and R. Škrekovski. Colorings with neighborhood parity condition. *Discrete Applied Mathematics*, 321:385–391, 2022.
- [88] F. Pirot and J.-S. Sereni. Fractional chromatic number, maximum degree, and girth. *SIAM Journal on Discrete Mathematics*, 35(4):2815–2843, 2021.
- [89] F. F. Pirot. *Colouring sparse graphs*. PhD thesis, Radboud University and Université de Lorraine, 2019.
- [90] F. P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, 30, 1930.
- [91] B. Reed. ω , δ , and χ . *Journal of Graph Theory*, 27(4):177–212, 1998.
- [92] N. Robertson, D. Sanders, P. Seymour, and R. Thomas. A new proof of the four-colour theorem. *Electronic research announcements of the American Mathematical Society*, 2(1):17–25, 1996.
- [93] N. Robertson, D. Sanders, P. Seymour, and R. Thomas. The four-colour theorem. *journal of combinatorial theory, Series B*, 70(1):2–44, 1997.
- [94] N. Robertson and P. D. Seymour. Graph minors. xiii. the disjoint paths problem. *Journal of combinatorial theory, Series B*, 63(1):65–110, 1995.
- [95] A. Rubin, H. Taylor, and P. Erdős. Choosability in graphs. In *Proc. West Coast Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, volume 26, pages 125–157, 1979.
- [96] E. Sabir et al. Generalizations of the classics to spanning connectedness. *Applied Mathematics and Computation*, 403:126167, 2021.

- [97] D. Saxton and A. Thomason. Hypergraph containers. *Inventiones mathematicae*, 201(3):925–992, 2015.
- [98] W. Staton. Some ramsey-type numbers and the independence ratio. *Transactions of the American Mathematical Society*, 256:353–370, 1979.
- [99] R. Steinberg. The state of the three color problem. In *Annals of discrete mathematics*, volume 55, pages 211–248. Elsevier, 1993.
- [100] R. Thomas and P. Wollan. An improved linear edge bound for graph linkages. *European Journal of Combinatorics*, 26(3-4):309–324, 2005.
- [101] C. Thomassen. Every planar graph is 5-choosable. *Journal of Combinatorial Theory Series B*, 62(1):180–181, 1994.
- [102] C. Thomassen. 3-list-coloring planar graphs of girth 5. *Journal of Combinatorial Theory, Series B*, 64(1):101–107, 1995.
- [103] C.-H. Tsai, J. J. Tan, and L.-H. Hsu. The super-connected property of recursive circulant graphs. *Information Processing Letters*, 91(6):293–298, 2004.
- [104] B. L. van der Waerden. Beweis einer Baudetschen Vermutung. *Nieuw Arch. Wiskd., II. Ser.*, 15:212–216, 1927.
- [105] V. G. Vizing. Vertex colorings with given colors. *Diskret. Analiz*, 29:3–10, 1976.
- [106] M. Voigt. List colourings of planar graphs. *Discrete Mathematics*, 120(1-3):215–219, 1993.
- [107] B. Wei. Hamiltonian paths and hamiltonian connectivity in graphs. *Discrete mathematics*, 121(1-3):223–228, 1993.

Publications

International journals and preprints

- (1) Tianjiao Dai, Guanghui Wang, Donglei Yang, Gexin Yu, Strong list-chromatic index of subcubic graphs, *Discrete Mathematics*, 341 (2018) 3434–3440.
- (2)* Tianjiao Dai, Jie Hu, Hao Li, Shun-ichi Maezawa, On DP-coloring of outerplanar graphs, submitted to *Discrete Mathematics*.
- (3)* Tianjiao Dai, Qiancheng Ouyang, François Pirot, Jean-Sébastien Sereni, Beyond the fractional Reed bound for triangle-free graphs, preprint.
- (4)* Tianjiao Dai, Hao Li, Qiancheng Ouyang, Zengxian Tian, On $(2, k)$ -Hamilton-connected graphs, submitted to *Discrete Applied Mathematics*.
- (5) Tianjiao Dai, Qiancheng Ouyang, François Pirot, New bounds for odd colourings of graphs, submitted to *Electronic Journal of Combinatorics*.
- (6) Tianjiao Dai, Hao Li, Yannis Manoussakis, Qiancheng Ouyang, Properly colored cycles in edge-colored complete graphs, submitted to *Discrete Mathematics*.

International conferences

- (1) Qiancheng Ouyang, Tianjiao Dai, Yue Ma. A Hypergraph Approach for Logic-based Abduction, In *15th International Conference on Advances in Databases, Knowledge, and Data Applications (DBKDA 2023)*, Barcelona, Spain, 2023.

Note that the papers marked with * form the main content of this thesis. Other papers are introduced briefly in Appendix.

Appendix

Some of my works are not included in this thesis (see [33, 31, 29, 85]), which have been (or will be) contained in theses of my collaborators or my master's thesis. I introduce these works briefly here.

- (1) We study strong k -edge-colouring of subcubic graphs in [33]. A *strong k -edge-colouring* of a graph G is an edge-colouring with k colours in which every colour class is an induced matching. The strong chromatic index of G , denoted by $\chi'_s(G)$, is the minimum k for which G has a strong k -edge-colouring. In 1985, Erdős and Nešetřil conjectured that $\chi'_s(G) \leq \frac{5}{4}\Delta(G)^2$, where $\Delta(G)$ is the maximum degree of G . When G is a graph with maximum degree at most 3, the conjecture was verified independently by Andersen and Horák, Qing, and Trotter. In this paper, we consider the list version of strong edge-colouring. In particular, we show that every subcubic graph has strong list-chromatic index at most 11 and every planar subcubic graph has strong list-chromatic index at most 10.
- (2) We consider odd colouring of graphs [31]. A proper vertex-colouring of a graph G is said to be odd if for each non-isolated vertex there is a colour appearing an odd number of times on its neighbourhood. The odd chromatic number of a graph G , denoted by $\chi_o(G)$, is the minimum $k \in \mathbb{N}$ such that there exists an odd colouring $\sigma: V(G) \rightarrow [k]$. In this paper, we use the probabilistic method to prove that for every graph G with maximum degree Δ , $\chi_o(G) \leq \Delta + O(\ln \Delta)$ as $\Delta \rightarrow \infty$. We also prove that $\chi_o(G) \leq \lfloor 3\Delta/2 \rfloor + 2$ for every Δ . If moreover the minimum degree δ of G is sufficiently large, we have $\chi_o(G) \leq \chi(G) + O(\Delta \ln \Delta / \delta)$ and $\chi_o(G) = O(\chi(G) \ln \Delta)$, and the latter bound is tight up to some multiplicative constant. Finally, given an integer $h \geq 1$, we study the generalisation of these results to h -odd colourings, where each vertex v must have $\min\{\deg(v), h\}$ colours appearing an odd number of times on its neighbourhood.
- (3) As an analogy of the well-known anti-Ramsey problem, we study the existence

of properly coloured cycles of given length in an edge-colored complete graph in [29]. Let $\text{pr}(K_n, G)$ be the maximum number of colours in an edge-colouring of K_n with no properly coloured copy of G . In this paper, we determine the exact threshold for cycles $\text{pr}(K_n, C_l)$, which proves a conjecture proposed by Fang, Győri, and Xiao, that the maximum number of colours in an edge-colouring of K_n with no properly coloured copy of C_l is $\max \left\{ \binom{l-1}{2} + n - l + 1, \left\lfloor \frac{l-1}{3} \right\rfloor n - \binom{\lfloor \frac{l-1}{3} \rfloor + 1}{2} + 1 + r_{l-1} \right\}$, where C_l is a cycle on l vertices, $l - 1 \equiv r_{l-1} \pmod{3}$, and $0 \leq r_{l-1} \leq 2$. It is a slight modification of a previous conjecture posed by Manoussakis, Spyrtatos, Tuza and Voigt. Also, we consider the maximal colouring of K_n whether a properly coloured cycle can be extended by exact one more vertex.

- (4)** We present a hypergraph approach for logic-based abduction in [85]. Abduction reasoning, which finds possible hypotheses from existing observations, has been studied in many different areas. We consider an abduction problem that takes into account a user's interest. We propose a new approach to solving such an abduction problem based on a hypergraph representation of an ontology and obtain a linear algorithm for a description logic.