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Adrien Séguret

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THÈSE DE DOCTORAT

DE L'UNIVERSITÉ PSL

Préparée à l'Université Paris-Dauphine

**Contrôle optimal et incitations pour des systèmes
décentralisés de type champ moyen**

Soutenue par

Adrien SEURET

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CONTRÔLE OPTIMAL ET INCITATIONS POUR DES SYSTÈMES DÉCENTRALISÉS DE TYPE CHAMP MOYEN**Résumé**

Cette thèse traite la résolution théorique et numérique de problèmes d'optimisation de type champ moyen. Un premier problème de contrôle champ moyen est formulé, avec contrainte de congestion, modélisant le chargement optimal d'une grande flotte de véhicules électriques. Les conditions d'optimalités sont identifiées comme un système d'équations aux dérivées partielles couplées, similairement à ceux obtenus dans les jeux à champ moyen. Le problème est résolu numériquement par un algorithme primal-dual. Un premier résultat sur la régularité des solutions est exploité pour caractériser le problème comme la limite d'un problème de contrôle avec grand nombre de véhicules. Les multiplicateurs de Lagrange associés à la contrainte de congestion sont bornés, permettant d'obtenir une meilleure régularité, de type Lipschitz, des solutions et d'approximer numériquement le problème dual.

Dans un second temps, une méthode d'approximation et de résolution numérique d'un problème de contrôle stochastique convexe de grande dimension est développée. Un algorithme d'optimisation distribué est proposé et la convergence vers une solution du problème est démontrée. La méthode et l'algorithme sont étudiés et appliqués à un problème de contrôle de processus markoviens déterministes par morceaux, une formulation équivalente au problème de contrôle champ moyen étudié en première partie.

Mots clés : contrôle champ moyen ; chargement intelligent ; approximation numérique ; contrôle stochastique ; optimisation distribuée ; flexibilité diffuse.

Abstract

This thesis studies the theoretical resolution and numerical approximation of mean field optimization problems. First, a mean field optimal control with congestion constraints is formulated, modeling the optimal charging of a large fleet of electrical vehicles. Optimality conditions are identified as a system of coupled partial differential equations, similarly to those obtained in mean field games. The problem is numerically solved by a primal-dual algorithm. A first result on the regularity of the solutions is exploited to characterize the problem as the mean field limit of an optimal control problem with a large number of vehicles. The Lagrange multipliers of the congestion constraint are proved to be bounded. This result enables to obtain a better regularity of Lipschitz type of the solutions and to numerically approximate the dual problem.

Secondly, an approximation method and a numerical method for a high dimensional convex stochastic control problem are developed. A distributed algorithm is proposed, whose convergence to a solution is proved. Finally, the method and the algorithm are analyzed and applied to a control problem of piecewise deterministic Markov processes, a formulation that is equivalent to the mean field control problem introduced in the first part.

Keywords: mean field control; smart charging; numerical approximation; stochastic control; distributed optimization; diffuse flexibility

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Chapter 1

Introduction

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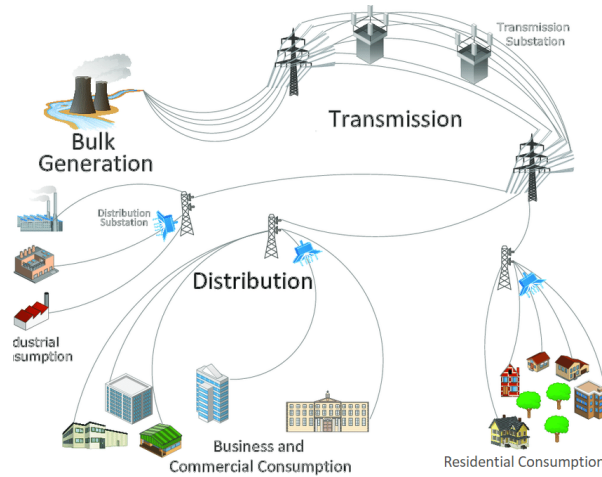


Figure 1.1 – A high-level structure of the current power grid. Source [257]

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The aim of this chapter is to show the motivations and the context of this dissertation, and to present the main contributions. Section 1.1 provides an overview of the on-going changes and new challenges arising in the electrical network. Next, after a presentation of the electric mobility, we sketch some reasons why it is crucial to coordinate the charge of electrical vehicle fleets in Section 1.2. Section 1.3 is devoted to the mathematical optimization problems and their resolution in energy management. Then, we provide a brief account on the theory of Mean Field Control (MFC for short). More precisely, we present some classical results about the MFC of Itô processes and some applications in flexibility management for the electrical grid. Finally, we summarize the contributions of each chapter in Section 1.4.

1.1 Context of the electrical network

1.1.1 Architecture of the electricity grid

The electrical network has a grid structure, connecting electricity production facilities with consumers. It allows the distribution of electricity, mainly in a three phase AC current. Among others, one can identify five main components of the grid: generation, transmission, substations, distribution and consumption. The generation refers to the means of electricity production. It is divided into two categories: centralized production by, for example, nuclear, gas, coal, hydro power plants, wind farms and solar panels fields, and decentralized production such as by rooftop solar panels. The centralized means of production are characterized by a large scale production and a great distance between the production site and the consumption site, while decentralized ones refer to local electricity production. The transmission part is made up of power lines and aims at transporting high voltage electricity over long distance, while minimizing losses. The substation part is composed of transformers and is meant for stepping up (for transmission) or reducing (for local distribution) the voltage. The distribution constitutes the part of the electrical network that serves the consumers. The consumption refers to industrial, commercial and residential use. An architecture of the grid is represented, in a simplified manner, in Figure 1.1. The work presented in this dissertation is focused on both the decentralized production and the consumption embedded in the distribution part of the electrical grid.

1.1.2 Production-consumption equilibrium

The proper functioning of the electricity grid depends on the balance between the power production and consumption. Since electricity can be stored to a very limited extent, demand and supply must be kept constantly in balance. Fluctuations and imbalance can occur due to unpredictable demand or to the randomness

of the electricity production of renewable source of energy (RES for short). In case of insufficient or excessive electricity supply compared to the demand, if the imbalance is uncontrolled, a blackout (or power outage) can occur. Such a situation can have dramatic effects, several hours or days can be necessarily to restore a functional network. In 2003, 56 million of Italians were plunged into darkness for several hours due to a faulty transmission of electricity from Switzerland to Italy [239]. To avoid such a situation, a system operator is responsible for electricity balancing. In Europe, an Electricity Balancing Guideline entered into force on December 2017, and provides operational and commercial rules to obtain efficient balancing mechanisms on the national and international level [224].

A common signal to measure the equilibrium between power consumption and production is the utility frequency. The generators connected to the same electrical synchronous grid must rotate at the same frequency. This frequency is set up at 60 Hz in the US and at 50 Hz in general in the rest of the world. A small imbalance between the generation and the consumption directly causes a deviation of the frequency. For this reason, fluctuations from the standard frequency are closely monitored: a decrease in the frequency indicates that the consumption is larger than the production, and an increase means the opposite. Frequency response refers to the mechanisms developed to stabilize the electric grid frequency.

Demand response (DR for short) denotes the regulation of power consumption from the consumer side in response to an external signal. As an example, peak and off peak hour pricing has been used for decades to coordinate the consumption and to avoid congestion effects. So far, industrial sites and “major” tertiary sites have played a significant role in DR for the frequency regulatory. Although adjusting the residential consumption in real time is a more difficult task, the development of smart grids makes it increasingly plausible for consumers to take part in the demand and frequency response, via domestic controllable loads.

1.1.3 Impact of the renewable energy source penetration

Next decades will be marked by deep changes on the electrical network. To deal with climate change, industrialized countries plan to close coal and gas power plant, and to compensate with an increase of RES in the electrical mix. While the share of RES was about 19.7% in 2019 in the European Union, it is planned to be at least 40% in 2030 [104].

The anticipation of the RES penetration in the energy generation mix is crucial to mitigate imbalance between the electricity production and consumption. The main concerns are about variable energy sources, such as online and offline wind generation power plants and solar panels. One can expect challenges in terms of balance (e.g. insufficient short- and long-term generation adequacy), stability (e.g. decreasing level of inertia and frequency control reserves), flow (e.g. missing visibility and controllability of RES generation) and quality (e.g. increasing harmonic distortions and local voltage excursions). A review of the impacts of RES on the distribution network can be found in [243] and the references therein. The economical impacts of RES penetration on the economy [98, 144, 153] and on the electricity price [143, 157, 197] are also important fields of study. Among the solutions proposed to mitigate these aftermaths, an efficient and coordinated control of local storage and flexible devices, such as water heaters and fridges, shows good results [132]. In this context, the control of the charge of electric vehicle has drawn attention over the last decade.

1.2 EV flexibility

The effort to diminish the gas emission also impacts transportation modes. While car transportation is responsible for 19% of the total greenhouse gas emission in the European Union in 2017 [102], electrical vehicles (EVs for short) are seen as a solution to decarbonize the transport. The actual proportion of EVs in the European Union among newly registered private vehicles is equal to 11% [103]. Since the European Union has approved the ban on new fossil-fuel cars from 2035, this share is expected to drastically increase over the next few years. Projection of the worldwide sales of EVs is displayed in Figure 1.2.

1.2.1 Challenges arised by the electrification of the vehicle fleets

While numerous benefits in terms of greenhouse gas emission are expected, the replacement of gas vehicles by electric ones induces a broad range of challenges to the electricity grid. The main concern is about the effects of uncoordinated load of a large population of EVs (i.e., EVs starting to charge their battery as soon as they are parked) on the electricity grid. It is now well understood how EVs can drastically increase the peak of consumption and affect the electrical network [89, 181, 205, 213, 267]. For instance, it has been assessed that

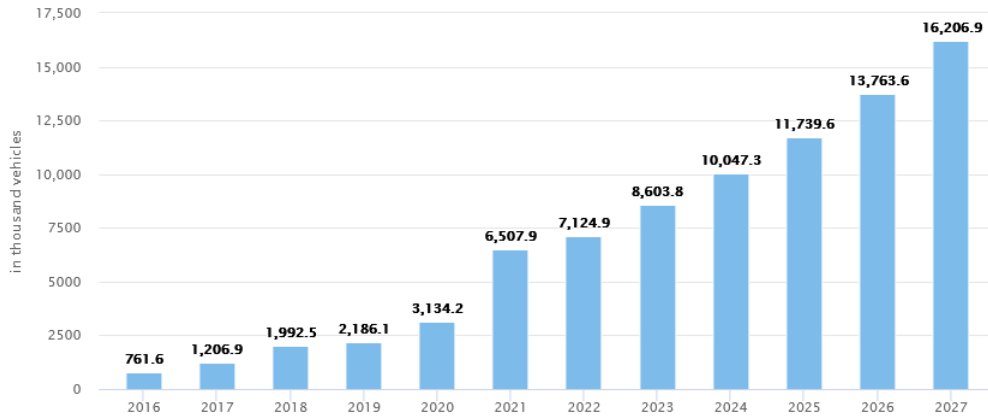


Figure 1.2 – Worldwide EV sales projection. Source: Statista, May 2022 [244]

peak demand can be increased in UK by around 17.9% if only 10% of the vehicles are EVs with uncoordinated load [214], and around 56% in Belgium for a share of 30% of uncoordinated EVs [70]. The installed production capacity may not match the peak demand [262], which may significantly increase the electricity price. According to [136], a penetration of 10% of the EVs can cause an increase of up to 22% of the monthly electric bill. Power loss and voltage deviation [186, 240] are also expected in the uncoordinated load scenarios. While voltage stability is crucial for the reliability of the power grids, a voltage deviation around 10% for a penetration of 30% of EVs in Belgium has been estimated in [70]. The overload of transformers can also result from uncoordinated EVs charge [6], leading to an increase of the internal temperature [219] and the aging [215] of the transformer. Harmonic distortions of the distribution network frequency [199], due to haphazard energy consumption of EVs, is also foreseeable. Such distortions can induce equipment ageing [6, 130, 215]. More details and references about the drawbacks of uncoordinated charge of EVs can be found in [134, 152, 240, 271]. To anticipate these consequences, the notion of smart charging has emerged over the last few years to designate the new methods dedicated to the dispatch of the available energy in the most effective way for the charging of EVs.

1.2.2 Potential of flexibility

While the previous paragraph is devoted to the issues caused by uncoordinated loads of EVs, we present in this paragraph why it is possible to coordinate EV charging and give details on the potential benefits. We expect that the coordination of the charge of EVs fleet would deal with the challenges cited above, and also have a positive impact on the electricity grid by providing services to balance the electrical network. Recent surveys about the statistical characteristics of the load charge of EVs during the days tend to show that plugged electrical vehicles (PEVs for short) are most of the time not charging. According to [270], in average, less than 20% of the time plugged is for actual charging. In [91], the authors correlate the flexibility of a PEV to the sojourn time of this PEV that is longer than the time required to (fully) charge the battery. Their definition of flexibility is close to the one in [154]. The potential of a DR resulting from the flexibility is computed. Based on real data, this potential is assimilated to the maximal load that can be reached or subtracted by a coordinated large fleet of PEVs. In [273], the authors first estimate the nominal consumption of a fleet of EVs, obtained by solving an optimization problem taken into account the price of electricity. Then, the (potential) flexibility for an EV at a time t is characterized by two variables: a level up (non negative variable) and a level down (negative variable) consumption. The impact of different electricity tariffs on the flexibility potential of electric vehicles is studied. More generally, the flexibility can be understood as a potential shift of energy consumption over the time.

Although the existence of an energy consumption flexibility of EVs has now been well established, such a flexibility has an impact on the electricity grid only when it is exploited within a collective behavior. It turns out that the total flexibility of a fleet in a coordinated scenario can be different from the sum of all the individual flexibility. Thus, it is crucial to dissociate the sum of individual flexibilities from the practically exploitable aggregate flexibility. Next paragraph describes the benefits for the electrical network of a coordination of the charge of EVs.

1.2.3 Quantification and exploitation of the flexibility

Studies on the flexibility coordination aims at understanding how the flexibility of an aggregate population of EVs exploits the potential of flexibility of each EV, in order to achieve a common goal (load flatten, RES consumption etc). Usually, flexibility coordination is analyzed through an optimization problem and is determined for a very specific situation. In [228], the authors first estimate the nominal consumption of a fleet of EVs based on real world data. Then, they exploit the flexibility of the EVs by solving an optimization problem, where the overall population of EVs is controlled in a centralized way to follow a signal of consumption (based on RES surplus) or to flatten the curve of consumption. Then, three flexibility indicators are analyzed, to indicate how much the flexibility potential of each EV has been exploited to optimize the criteria. The defined indicators are: the amount of transferable energy without threatening customer convenience, the fraction of time used for flexibility and the maximum allowable delay for the energy consumption. This in-depth analysis enables to understand which aspect of the EV flexibility is more useful, depending on the DR objectives.

The benefits of the coordination of EV charge has been studied extensively over the last decade. A coordination drastically reduces the above-described threats, but also facilitates the penetration of the RES in the electricity grid. The Vehicle-to-Grid (V2G for short) technology is a key element to understand the potential of EVs in ancillary services. It allows vehicles to discharge power out of the battery to the grid. It is assessed in [201] that a 64 kWh Nissan LEAF can furnish enough electricity for an average Japanese home's energy needs for four days. When combined with a rooftop solar panel, the authors in [258] estimate that a 32 kWh EV can supply up to 300 hours backup duration household. The increasing capacity of batteries leads to consider fleets of EVs as non negligible virtual battery, which can be exploited in case of power outage for instance. PEVs are expected to take part in ancillary services such as frequency and voltage regulation [180, 208, 250, 269, 278], peak shaving [259], valley filling [117], spinning reserve [182, 207] and demand side management [191]. The management of EV fleets to integrate RES is currently an important field of research. Some strategies and scenarios are explored such as peak shaving combined with renewable generation [183], frequency regulation in a context of RES [1, 150], regulation of voltage fluctuations due to RES [255], synergy with wind [100, 175, 217], synergy with photovoltaic [29, 248, 260], and generation cost reduction [227]. A recent survey from ENEDIS [101] shows that, in return for a discount, 63% of people in France are willing to postpone their recharging in order to relieve the power grid during peak consumption periods.

We underline that the benefits cited above highly depends on the quality of the adopted mathematical models and on the optimization methods. In the next paragraphs, we introduce the mathematical aspects of the coordination of large fleets of EVs, and more generally of the flexible load management.

1.3 Different mathematical tools for flexibility management in the literature

In a context of energy management, different models and mathematical approaches have been proposed to design efficient coordination mechanisms [263]. One can categorize coordination schemes into different classes of problems, such as: centralized control, distributed optimization or game theoretic models. Each category answers to different challenges and involves different mathematical tools. In most of models, each vehicle or flexible device is described by a state variable (level of the battery, internal temperature etc) and a control variable (electricity consumption or production level, charge or discharge of the battery etc). Accordingly on the scheme, this control can be transmitted by an external agent (such as an aggregator), or computed locally by each device, based an external signal (designed by an aggregator or the market). Different forms of control can depend on the characteristics of the electricity grid. The choice of the model and of the mathematical tools depends on several elements, such as the number of interacting devices on the electrical network, the non convexity of the problem, the computation time, the need of infrastructures in order to support communication with the devices, etc. Next paragraphs are devoted to a short review of the literature about schemes, optimization methods and algorithms used to coordinate flexible devices.

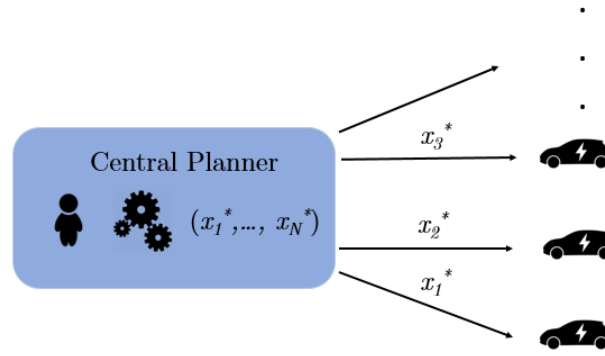


Figure 1.3 – Centralized optimization scheme

1.3.1 Centralized control

Centralized optimization control problems, for the energy management of N units of flexible devices, are often of the form:

$$\begin{aligned} \inf_{(x_1, \dots, x_N) \in \mathcal{X}^N} \quad & J(x_1, \dots, x_N), \\ \text{s.t.} \quad & G(x_1, \dots, x_N) \leq 0, \end{aligned} \quad (1.3.1)$$

where x_i is the control of the i^{th} device, \mathcal{X} is a vector space, J is the objective function and G is a constraint function. Problem (1.3.1) can model a smart charging problem with N EVs to charge in a parking over a period. The variable x_i denotes the charging power profile of the i^{th} EV. The constraint $G(x_1, \dots, x_N)$ is a coupling constraint at the level of the site such as the maximum power, and $J(x_1, \dots, x_N)$ is the sum of local costs of EVs (e.g. electricity consumption cost, penalty for insufficient charging) and of global level cost (e.g. cost of carbon dioxide emission). Historically, energy management problems close to Problem (1.3.1) were solved centrally, by a central planner coordinating the population of flexible devices. The central planner collects information from the devices, solves an optimization problem and sends the optimal control to each device (see Figure 1.3). See e.g. [216] for a centralized control problem in smart charging.

Convexity assumptions on the model (e.g. J and G being convex) enable to rely on convex optimization theory [99]. A large variety of existence and uniqueness results can be used to deal with Problem (1.3.1), as well as algorithms converging to a minimizer. If J is smooth enough with a Lipschitz continuous gradient, one can implement the gradient descent or projected gradient descent that are classically applied in such a framework. If J is twice-differentiable, Newton's method or quasi-Newton can also be applied. By the Dynamic Programming Principle [28], one can obtain a time decomposition of the problem, assuming the Markov property of the system. This method can be applied both in deterministic and in stochastic settings. Under continuous linear-convex assumptions, the Stochastic Dual Dynamic Programming algorithm (SDDP) [209] allows to get upper and lower bounds of the value function, using polyhedral approximations. This technique is efficient for central optimization problem having a medium sized population of devices $N \leq 30$ [59].

Note that in the non convex settings, heuristic algorithms can be directly applied to find a solution to Problem (1.3.1), such as genetic algorithm [125], simulated annealing [261] or machine learning algorithms [138].

The main issues encountered by centralized methods are scalability (e.g. w.r.t. communication), computation and privacy. When the number of devices N is large, a majority of the optimization methods may fail to solve Problem (1.3.1) (e.g. Dynamic Programming suffers from the curse of dimensionality [20]). A classical approach to tackle these issues is to decompose the centralized problem into a distributed optimization problem.

1.3.2 Distributed Optimization of a Centralized Problem

The last decades are characterized by a large penetration of flexible devices and local sources of energy generation, increasing the complexity and the dimensionality of Problem (1.3.1). In addition, some local information cannot be communicated to the central planner for privacy reasons. By assuming the convexity of the data, Problem (1.3.1) can be decomposed into problems of smaller sizes, leading to distributed schemes. In a distributed scheme, flexible devices can communicate with each other and with the central planner. Each device

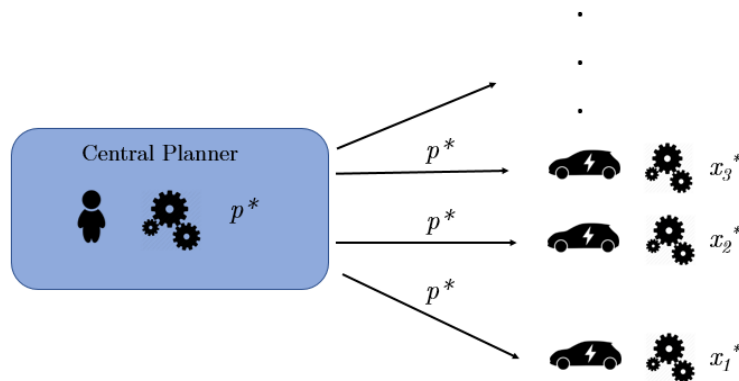


Figure 1.4 – Distributed optimization scheme

computes its own control, with a view to cooperate in minimizing a global objective function.

Most of decomposition techniques adopted to obtain a distributed problem from a centralized one, rely on dual decomposition. In the Lagrangian decomposition method, the constraint is relaxed and the Lagrangian function of the problem is introduced. The saddle-point of the Lagrangian can be numerically estimated by applying the Uzawa algorithm which is an iterative method. If the coupling form of the variables is additive, each iteration consists in solving decoupled sub-problems (w.r.t. the state and control variables of the agents). From a practical point of view, the central planner computes the optimal Lagrange multiplier (based on the knowledge of the agents parameter) and sends it to each agent that can compute locally its optimal control (see Figure 1.4). In a stochastic framework, the Dual Approximation Dynamic Programming (DADP) [123, 172] can be implemented to deal with stochastic constraint. This method is characterized by a price decomposition of the problem, where the stochastic constraints are projected on subspaces such that the associated Lagrangian multiplier is adapted for dynamic programming. Then the optimal multiplier is estimated by implementing Uzawa algorithm. DADP has been applied in different cases, such as storage management problem for electrical production in [14, 123] and hydro valley management [59]. In Chapter 6 of this dissertation, we provide a decomposition method for a stochastic optimization problem that is in the same vein as DADP. The Alternating Direction Method of Multipliers (ADMM) [115] is a decomposition algorithm based on the augmented Lagrangian that can be efficiently applied when the cost function is separable in the state and control variables. This method has been tested to solve smart charging problems in [220]. The Lagrangian decomposition may fail to solve Problem (1.3.1) when the objective function is not additive and separable w.r.t. the state and control variables. In such a case, it is possible to solve the problem with the Auxiliary Problem Principle method introduced in [73]. This approach consists in approximating the non additive cost function by an additive one and to solve successive auxiliary problems in a decentralized way. An algorithm is defined from this method and is proved to converge to the solution of the optimization problem [73]. This technique has been applied in [92] for the control of decentralized units of renewable energy sources.

It is possible to define distributed protocols to solve Problem (1.3.1) without relying on dual decomposition. Problem (1.3.1) can be decentralized by means of Benders decomposition [26]. This method, that consists in performing a resource decomposition, has been applied to smart charging in [272]. In the context of high dimensional multistage stochastic optimization problem, the author in [60] have provided theoretical bounds for the price and resource decomposition methods, as well as numerical simulations for microgrids optimal management.

Finally, if Problem (1.3.1) is unconstrained, gradient descent based techniques such as block coordinate descent [254] or random coordinate descent [198] may tackle this problem in a decentralized way, by computing the gradient only on some directions.

1.3.3 Bi-level optimization, game and pricing incentive approaches

In the centralized schemes and distributed approaches described in the previous paragraphs, the devices cooperate with the central planner in order to achieve a common goal. Although these models cover a large set of situations, they are not adapted to model competition among the devices. When each device has its own objective function to minimize and competes with the other devices to have an access to energy, a game

theoretical approach can be suitable to model this situation. Then, a common method for the central planner to coordinate the population of competing devices is to adopt price incentive approaches. The central planner aims at determining the best price to achieve a particular goal. This price can depend on the control decision of each device, introducing a game among the devices [68, 189, 279]. In a context of smart grids and demand response, the authors in [149] investigate the theoretical properties and computational aspects of the hourly billing mechanism. In a mean field framework, the author in [126] show the existence and uniqueness of the price that coordinates the agents to satisfy a global energy consumption constraint over the time in a electricity market framework.

Chapters 2 to 5 study a centralized control problem to coordinate the charging of a large fleet of EVs. In these chapters, the central planner computes the optimal control and communicates to each EV the instants where it has to switch to a certain mode of charging (idle, fast charging, V2G, etc...). In Chapters 6 and 7, centralized control problems are also formulated, but we propose a decomposition method to solve these problems. The method employed heavily rely on the Lagrangian decomposition and the Uzawa algorithm. Since Chapters 2 to 5 deal with a MFC problem, a brief presentation of MFC is given next.

1.3.4 Focus on McKean-Vlasov Optimal Control

A large part of this dissertation (Chapters 2-5 and 7) is focused on the optimal control of a large population of hybrid processes (hybrid in the sense that the state of the process is composed of a discrete variable and a continuous variable). Due to the large number of processes, classical optimization methods and algorithms may fail to solve this problem. To deal with the high dimensionality of the problem, a mean field control problem is introduced. Although our work in this thesis concerns hybrid processes, a short introduction about McKean-Vlasov optimal control problems is provided in this paragraph in order to expose the main questions, techniques and results arising in MFC theory. The finite population problem is formulated as well as the MFC problem. Some classical results on the characterization of the minimizers and on the convergence of the finite population problem are given. Finally, we mention some algorithms that have been successfully applied to solve MFC problems. These techniques have motivated and inspired the work presented in this thesis for the MFC of hybrid processes.

The finite population problem We consider the most basic settings for this formal introduction. Let $N \in \mathbb{N}$ Itô processes be controlled by a central planner and taking values in \mathbb{R}^d . The state of the process indexed by $i \in \{1, \dots, N\}$ is controlled by the feedback function $\alpha^i : [0, T] \times (\mathbb{R}^d)^N \mapsto \mathbb{R}^d$ and denoted by X^{i, α^i} . The dynamics is given, for any $t \in [0, T]$, by

$$dX_t^{i, \alpha^i} = \alpha_t^i dt + \sqrt{2} dW_t^i, \quad (1.3.2)$$

where W^1, \dots, W^N are independent Brownian motions in \mathbb{R}^d . The law of the initial state X_0^i is the given distribution $\mu^0 \in \mathcal{P}(\mathbb{R}^d)$. Let m^N be the empirical distribution of the population of processes, defined by $m^N(t) := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i, \alpha^i}}$. The cost function J^N is given by:

$$J^N(\alpha) := \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[\int_0^T \frac{\alpha_t^{i2}}{2} + F(t, m^N(t)) dt + G(X_T^{i, \alpha^i}, m^N(T)) \right], \quad (1.3.3)$$

where $\alpha := (\alpha^1, \dots, \alpha^N)$. The central planner aims at solving

$$\inf_{\alpha} J^N(\alpha). \quad (P_N)$$

This problem can be seen as an instance of Problem (1.3.1). The central planner can represent an aggregator controlling a collection of consumers or flexibility in a centralized way. The state variable X^i can typically denote the temperature of a heating or a cooling device, or the level of battery of a storing device. The control α usually refers to a power consumption or production. Functions F and G are coupling costs, encoding aggregate costs.

The mean field limit problem Though agents in Problem (P_N) are exchangeable (i.e. agents constitute

an homogeneous population, sharing the same dynamics parameters, initial distribution and local cost if any), it becomes almost impossible to solve the system for a very large number of agents. In the limit case $N \rightarrow \infty$, a classical way to understand the asymptotic behavior of the system is to consider its mean behavior. To this end, we focus on a representative agent, controlled by $\alpha : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, which evolves according to

$$dX_t^\alpha = \alpha_t dt + \sqrt{2}dW_t, \quad (1.3.4)$$

where W is a Brownian motions in \mathbb{R}^d and X_0 a random variable whose distribution is μ^0 . The objective function J is defined by

$$J(\alpha) := \mathbb{E} \left[\int_0^T \frac{\alpha_t^2}{2} + F(t, m(t)) dt + G(X_T^\alpha, m(T)) \right],$$

where $m(t)$ is the law of X_t . The mean field control problem is

$$\inf_{\alpha} J(\alpha). \quad (P_{\text{MFC}})$$

Problem (P_{MFC}) goes beyond the frame of the standard theory of stochastic control due to the non linearity w.r.t. the law $m(t)$ in the objective function.

The mean field limit approximation raises several questions:

- (i) How can the minimizers of Problem (P_{MFC}) be characterized?
- (ii) Which algorithms can be implemented to numerically solve Problem (P_{MFC}) ?
- (iii) Does the finite population control problem (P_n) converge to the mean field problem (P_{MFC}) and in which sens?
- (iv) Which regularity of the solutions can be expected when one considers constraints on the distribution m in Problem (P_{MFC}) ?

These questions emerged in the specific case of MFC of Itô processes and several answers have been provided.

Characterization of the minimizers of the mean field problem. We derive formally the optimality conditions of Problem (P_{MFC}) in some specific settings. We assume that there exists f and g such that, for any $t \in [0, T]$, $\frac{\delta F(t)}{\delta \mu} = f(t)$ and $\frac{\delta G}{\delta \mu} = g$, where the notion of derivative on the Wasserstein space of a map $H : \mathcal{P}(\mathbb{R}^d) \mapsto \mathbb{R}$ is the one introduced in [52]. Inspired by the seminal work of Benamou-Brenier in [21] in optimal transport, the variable $w := \alpha m$ is introduced to obtain a convex problem, equivalent to Problem (P_{MFC}) , and to apply Fenchel-Rockaffelar duality theorem [99]. The law m of the solution of the SDE (1.3.4) satisfies the following Fokker-Planck equation on $[0, T] \times \mathbb{R}^d$:

$$\partial_t m - \Delta m + \text{div}(w) = 0, \quad (1.3.5)$$

with the initial condition $m(0) = \mu^0$. Let \tilde{J} be defined by

$$\tilde{J}(m, w) := \int_0^T \int_{\mathbb{R}^d} \ell(w(t, x), m(t, x)) m(t, x) dx dt + \int_0^T F(t, m(t)) dt + \int_{\mathbb{R}^d} G(x, m(T, x)) dx, \quad (1.3.6)$$

where

$$\ell(a, b) := \begin{cases} \frac{a^2}{2b} & \text{if } b > 0 \text{ and } a \geq 0, \\ 0 & \text{if } a = b = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Problem (P_{MFC}) is equivalent to the following optimal control problem with PDE constraints:

$$\inf_{(m, w)} \tilde{J}(m, w) \quad (\tilde{P}_{\text{MFC}})$$

where (m, w) is a weak solution of (1.3.5).

One can apply a linearization procedure as in [39, Proposition 3] to Problem (\tilde{P}_{MFC}) . Then, relying on the convexity of the linearized problem, it is possible to introduce its Lagrangian and to apply Fenchel-Rockaffelar

duality theorem. In the same vein as in [21], one can show that if a couple (m, α) is a solution of (\tilde{P}_{MFC}) , there exists a function φ , which can be interpreted as the Lagrangian multiplier of (1.3.5), such that $\alpha = \nabla\varphi$ and (m, φ) is a solution to the following PDE system:

$$\begin{cases} -\partial_t\varphi - \Delta\varphi + \frac{1}{2}|\nabla\varphi|^2 = f(t, x, m(t)) & \text{in } (0, T) \times \mathbb{R}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m\nabla\varphi) = 0 & \text{in } (0, T) \times \mathbb{R}^d, \\ m(0) = \mu^0 \text{ and } \varphi(T) = g(x, m(T)) & \text{in } \mathbb{R}^d. \end{cases} \quad (1.3.7)$$

The existence of a solution of such a system is established in general frameworks, while the uniqueness depends on the monotonicity of F and G [167]. System (1.3.7) allows to deduce regularity results on the solution (m, α) of Problem (P_{MFC}) . In these specific settings, the solutions (φ, m) are bounded and belong to $C^{1,2}((0, T) \times \mathbb{R}^d)$, with one bounded continuous derivative in time and two bounded continuous derivatives in space [39, 82].

System (1.3.7) typically arises in the Mean Field Game (MFG for short) Theory. This class of problems, introduced by Lasry and Lions [166, 167, 168] and Huang, Malhamé and Caines [145, 146], describes the interaction among a large population of identical and rational agents in competition. The duality approach is close to the so-called variational approach used in MFG theory in [53], where the weak solution of the MFG system is characterized as the minimizer of some optimal control of Hamilton-Jacobi and Fokker-Planck equations. It has been widely applied to solve potential games [39]. This approach allows to use optimization techniques to prove the existence and uniqueness of the solution of MFG and MFC problems. We refer to [5, 25, 40, 54, 202] and the references therein. Note that under specific regularity, convexity and growth assumptions on the data, Problem (P_{MFC}) can be solved using an extension of the Pontryagin maximum principle [56]. This method introduces a forward-backward system of stochastic differential equations to express the necessary condition for optimality.

Chapter 2 of the dissertation defines a MFC problem of hybrid processes and aims at providing a system of optimality conditions (question (i)) that is similar to the system (1.3.7).

Connection with the finite population problem. The convergence of the finite population problem (P_N) to the MFC problem (P_{MFC}) is proved in the breakthrough [163], where the author relies on a relaxed formulation of both problems to show the convergence of the value of the finite population problem to the value of the MFC problem. This result is extended to the case with common noise in [94] and with interactions of the agents with the joint distribution of the state and control in [93]. In the specific case with constraints in law, the convergence of the value of the finite population problem with almost sure constraint to the value of the MFC problem has been proved in Chapter 5 of [80]. In a setting with both idiosyncratic and common noise in the dynamics of the agents, a convergence rate of the value function of the finite population problem to the value function of the MFC problem is obtained in [51]. The connection between the finite population problem and the mean field problem introduced in Chapter 2, is established in Chapter 5 of the dissertation (question (iii)).

Constraint in law. Due to the potential applications in economy, finance or energy management, Problem (P_{MFC}) has also been studied with additional constraints on the law of the distribution. In [81], the author provides the optimality conditions of a MFC problem with a terminal condition on the distribution. Optimality conditions and Lipschitz regularity results are given in [82] for MFC problems with constraints of the form $\Psi(m(t)) \leq 0$, for any $t \in [0, T]$, where Ψ is a smooth convex function over the Wasserstein space of probability measures. A similar problem is tackled with a level-set approach in [121]. This method enables to consider common noise in the dynamics and to perform numerical approximation. In the deterministic case, where there is no diffusion in the dynamics (1.3.4), MFG problems with hard congestion on the density is first introduced in [230], and solved [55] using a variational approach. In [35], the author formulates a Pontryagin principle stated in the Wasserstein space to solve the optimal control of a continuity equation with terminal and running constraints. Chapter 4 of this dissertation follows the path initiated in [46] and adapted in [82] for MFC problems of Itô processes, to obtain regularity results on the Lagrange multiplier and on the solutions (question (iv)) of the MFC problem defined in Chapter 2.

Numerical simulations. Different techniques can be used to numerically solve (P_{MFC}) . One can discretize the PDE system (1.3.7) using finite difference scheme [2] and then exploit the convex structure of the problem to apply a primal-dual algorithm. In the specific case of second order MFG with strongly convex running cost, a rate of convergence is obtained for the theta-scheme [33]. In [40, 41] the Chambolle-Pock algorithm [67] is implemented to solve variational MFG problems. Following works in optimal transport theory, the

Augmented Lagrangian algorithm can be performed to solve MFC problems [5, 23, 25]. MFG problems can also be numerically approximated by using an entropy minimizing approach and applying a variant of the Sinkhorn algorithm [24]. Based on the forward-backward stochastic differential equations characterization, the McKean-Vlasov control problem can be solved numerically using neural network techniques [120]. Relying on the application of the Chambolle-Pock algorithm to MFC problems in the articles cited above, numerical approximations of the MFC problem of hybrid processes introduced in Chapter 2 are performed and analyzed in Chapter 3 of the thesis (question (ii)). In Chapter 4, a numerical scheme, that is shown to converge, is proposed to numerically approximate the dual problem of the optimization introduced in Chapter 2.

Applications The mean field approximation of a population of controlled processes has been applied in a context of energy management and smart charging. In [124] the author develop a decentralized control scheme, based on a mean field approximation, to coordinate the consumption of Thermostatically Controlled Loads (TCLs). A MFG problem is formulated in [85] to model a large population of TCLs in competition. In [121], the author solves numerically an optimal storage of wind-generated electricity, modelled by a MFC problem of Itô processes with common noise and running constraint on the law of the processes. A similar problem of optimal storage is formulated in a MFG framework [7]. Mean field control and game models also arise in economics [3, 4]. Chapters 3,4 and 7 of the dissertation numerically solve MFC problems in a context of smart charging.

1.4 Contents of the manuscript and summary of the main results

In this PhD thesis, we first focus on the mathematical modelling for the optimal charging of a large fleet of electric vehicles. In Chapter 2, we formulate the optimal charging problem as a MFC problem of hybrid processes. We provide the characterization of the solutions as well as regularity results. Chapter 3 is dedicated to the numerical approximation of the mean field control problem investigated in Chapter 2 and its application to smart charging. Chapter 4 studies the regularity of the Lagrange multipliers introduced in Chapter 2. A L^∞ estimate is obtained for the Lagrange multipliers associated with the congestion constraint on the state distribution. This result is used to establish the convergence of a numerical scheme to solve the MFC problem introduced in Chapter 2. Chapter 5 aims at justifying the mean field approach adopted in the previous chapters of the dissertation, by showing the convergence, when the number of agent goes to infinity, of the value of the finite population control problem to the value of the MFC problem introduced in Chapter 2. The convergence of the solution of the finite population to a solution of the MFC problem is also provided. Another model is studied in Chapter 6, which considers a high dimensional convex stochastic control problem, with heterogenous agents. A modified problem is introduced, whose optimal control is under some reasonable assumptions, an ε -optimal solution of the original problem. We also present a decentralized algorithm whose convergence to the solution of the modified problem is established. Finally, Chapter 7 provides a decentralized approach for the mean field control problem of a population of Piecewise Deterministic Markov processes. The algorithm introduced in Chapter 6 is shown to converge in these particular settings and simulations to coordinate the charging of a large fleet of electric vehicles are presented.

Each of the chapters 2,3,5,6 and 7 is based on a journal article that has been accepted or submitted for publication, or for a conference proceeding reviewed and accepted. Chapter 4 is based on a working paper that will be soon submitted to a journal for publication.

- Chapter 2 is based on the preprint [233], submitted for publication.
- Chapter 3 is based on the conference paper [238] presented at the 2021 IEEE Conference on Innovative Smart Grid Technologies (ISGT).
- Chapter 4 is based on the working paper [235].
- Chapter 5 is based on the preprint [234], submitted for publication.
- Chapter 6 is based on the journal paper [236], accepted for publication in *Applied Mathematics & Optimization*.
- Chapter 7 is based on the preprint [237], submitted for publication.

1.4.1 Chapter 2: Optimality conditions of an optimal control problem for the continuity equation arising in smart charging

This chapter is focused on the mathematical modeling and the characterization of the solutions of the optimal charging of a large population of identical plug-in electric vehicles (PEVs) with hybrid state variables (continuous and discrete). A mean field optimal control problem of hybrid processes with congestion constraints is introduced and studied. Since the seminal work by Benamou and Brenier [21] in optimal transport, the analysis of the optimality conditions of mean field control problems heavily relies on the duality theory [110, 265]. This duality approach is revisited in the so-called variational approach used in MFG theory in [53], where the weak solution of the MFG system is characterized as the minimizer of some optimal control of Hamilton-Jacobi and Fokker-Planck equations. This approach allows to use optimization techniques to prove the existence and uniqueness of the solution of MFG and MFC problems. We refer to [5, 25, 40, 54, 202] and the references therein. The application of the variational approach is also motivated by the numerical resolution of MFG problems [23, 40, 41]. In this chapter we deal with a congestion constraint on the measure. MFG with density constraints is first investigated in [230] where the density of the population of a first order MFG problem does not exceed a given threshold, then in [190] where stationary second order MFGs are considered and in [55] for first order MFGs. A MFG problem in a finite state space and discrete time settings with congestion constraint is studied in [32], also by variational methods. The articles cited above looked only at problems with either continuous or discrete state variables, while we consider hybrid state variables.

More precisely, the problem takes the following form. We consider a finite time interval $[0, T]$ and a hybrid state space equal to the product $[0, 1] \times I$, where I is a finite space. The continuity equation is given by:

$$\begin{aligned} \partial_t m_i + \partial_s(m_i b_i) &= - \sum_{j \neq i} (\alpha_{i,j} m_i - \alpha_{j,i} m_j) & \text{on } I \times (0, T) \times (0, 1), \\ m_i(0) &= m_i^0 & \text{on } I \times [0, 1], \end{aligned} \quad (1.4.1)$$

where $m \in C^0([0, T], \mathcal{P}([0, 1] \times I))$, $m^0 \in \mathcal{P}([0, 1] \times I)$ is the given initial distribution, $b : I \times [0, 1] \rightarrow \mathbb{R}$ is a velocity field and is given, and the control $\alpha : I \times I \times [0, T] \times [0, 1] \rightarrow \mathbb{R}_+$ is a jump intensity. The distribution m is subject to the following congestion constraints:

$$m_i(t, [0, 1]) \leq D_i(t) \quad \forall (i, t) \in I \times [0, T], \quad (1.4.2)$$

where $D : [0, T] \rightarrow \mathbb{R}_+^*$ is given. The objective function J is defined as follows:

$$J(m, \alpha) := \sum_{i \in I} \int_0^T \int_0^1 (c_i(t, s) + \sum_{j \in I, j \neq i} L(\alpha_{i,j}(t, s))) m_i(t, ds) dt + \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds). \quad (1.4.3)$$

The purpose of this chapter is to study the optimization problem:

$$\begin{aligned} \inf_{m, \alpha} J(m, \alpha) \\ \text{where } (m, \alpha) \text{ is a weak solution of (1.4.1) and satisfies (1.4.2).} \end{aligned} \quad (P)$$

This work is motivated by the optimal charging of a population of PEVs controlled by a central planner. The continuous variable $s \in [0, 1]$ in (1.4.1) represents the level of battery of a PEV. The discrete variable $i \in I$ represents the mode of charging (e.g. idling, charging, discharging, etc...). For any $(t, s, i) \in [0, T] \times [0, 1] \times I$, the value $m_i(t, s)$ represents the proportion of PEVs at time t at state (s, i) . The given velocity field $b_i(s)$ denotes the power of charge or discharge of a PEV in mode i and with battery level s . For any $(t, s, i, j) \in [0, T] \times [0, 1] \times I \times I$, the value $\alpha_{i,j}(t, s)$ denotes the jump intensity of PEVs from the state (s, i) to the state (s, j) at time t . The congestion constraint (1.4.2) avoids high demand of energy at each moment over the period. The cost function L in (1.4.3) penalizes high values of α . It aims at avoiding the synchronization of jumps of the PEVs and the consequent instability of the electrical network. The value $c_i(t, s)$ corresponds to the instantaneous cost per PEV to pay at time $t \in [0, T]$ at state (s, i) ; $g_i(s)$ is the final cost per PEV to pay at state (s, i) .

The main result of this chapter is the following system of optimality conditions

Theorem 1.4.1. *Assume there exists $\varepsilon^0 > 0$ such that:*

$$\varepsilon^0 < D_i(t) - m_i^0([0, 1]) \quad \forall (t, i) \in [0, T] \times I,$$

then (m, α) is a solution to (P), if and only if there exists a pair $(\varphi, \lambda) \in (\text{Lip}([0, T] \times I \times [0, 1]) + BV([0, T] \times I)) \times \mathcal{M}^+([0, T] \times I)$ such that $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ and (φ, λ, m) is a weak solution of the following system on $[0, T] \times I \times [0, 1]$:

$$\left\{ \begin{array}{l} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \lambda_i + \sum_{j \in I, j \neq i} H(\varphi_j - \varphi_i) = 0 \\ \partial_t m_i + \partial_s(m_i b_i) + \sum_{j \neq i} (H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j) = 0 \\ m_i(0, s) = m_i^0(s), \varphi_i(T, s) = g_i(s) \\ \int_0^1 m_i(t, ds) - D_i(t) \leq 0, \lambda_i \geq 0 \\ \sum_{i \in I} \int_0^T \left(\int_0^1 m_i(t, ds) - D_i(t) \right) \lambda_i(dt) = 0 \end{array} \right. \quad (S)$$

where H is the Fenchel conjugate of L and H' its derivative.

The function φ can be understood as the Lagrange multiplier associated with the dynamic constraint (1.4.1), and the measure λ as the Lagrange multiplier associated with the congestion constraint (1.4.2). A duality approach is adopted to obtain (S). More explicitly, we relax the dynamics (1.4.1) and the congestion constraint (1.4.2). Given the Hamilton-Jacobi equation on $(0, T) \times (0, 1) \times I$:

$$\begin{aligned} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i(t, s) - \lambda_i + \sum_{j \in I, j \neq i} H(\varphi_j - \varphi_i) &\leq 0 && \text{on } (0, T) \times (0, 1) \times I, \\ \varphi_i(T) &\leq g_i && \text{on } [0, 1] \times I, \end{aligned} \quad (1.4.4)$$

the resulting relaxed problem is then expressed as the dual of the following convex problem

$$\begin{aligned} \inf_{(\varphi, \lambda)} \sum_{i \in I} \int_0^1 -\varphi_i(0, s) m_i^0(ds) + \int_0^T D_i(t) \lambda_i(dt) \\ \lambda \in \mathcal{M}^+([0, T] \times I) \end{aligned} \quad (D)$$

s.t. (φ, λ) sub solution of (1.4.4).

The regularity properties of $\varphi \in \text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)$, for any weak solution (λ, φ, m) of (S), is obtained under suitable assumptions on the data b, g and c . As a result, the optimal control α is bounded and Lipschitz continuous in space uniformly w.r.t. the time variable, and the measure m is in $\text{Lip}([0, 1], \mathcal{P}([0, 1] \times I))$. We show that, if the initial distribution m^0 is absolutely continuous w.r.t. the Lebesgue measure and has a smooth density, then the measure m is absolutely continuous w.r.t. the Lebesgue measure and its density is Lipschitz continuous. These regularity results will be exploited in Chapter 5 to establish the connection between the finite population problem and the mean field one. Finally, regularity results on the Lagrange multipliers (φ, λ) are enhanced in Chapter 4, where we show that the absolutely continuous part of λ is a L^∞ function and its singular part a Dirac mass at final time T .

1.4.2 Chapter 3: Computation and implementation of an optimal mean field control for smart charging

In this chapter, we propose an algorithm to solve numerically the mean field optimal control problem introduced in Chapter 2. An implementation procedure is presented and tested in a use case with a finite number of electrical vehicles. A mean field control problem, that is equivalent to problem (P), up to a change of variable, is discretized, with time and space steps respectively Δt and Δs (satisfying $T = N_T \Delta t$ and $1 = N_h \Delta s$). The variable $m_i^{k,\ell}$ denotes the proportion of vehicles in mode $i \in I$ with a state of charge (SoC for short) lying in $[\ell \Delta s, (\ell + 1) \Delta s)$ at time $t_k := k \Delta t$, $E_{i,j}^{k,\ell}$ is the frequency of switches from mode i to mode j for PEVs with a SoC lying in $[\ell \Delta s, (\ell + 1) \Delta s)$ at time t_k . A splitting method is applied to discretize the continuity equation. The variable $m_i^{k+\frac{1}{2},\ell}$ is introduced to first take into account the transfers of PEVs from one mode to another:

$$m_i^{k+\frac{1}{2},\ell} := m_i^{k,\ell} + \Delta t \sum_{j \in I \setminus \{i\}} E_{j,i}^{k,\ell} - E_{i,j}^{k,\ell}. \quad (1.4.5)$$

The convection (induced by the charging rate b) is then considered by applying an upwind scheme

$$\frac{m_i^{k+1,\ell} - m_i^{k+\frac{1}{2},\ell}}{\Delta t} + \frac{F_i^{k,\ell} - F_i^{k,\ell-1}}{h} = 0, \quad (1.4.6)$$

where $F_i^{k,\ell}$ depends on m_i^k and on the sign of b_i . The congestion constraint (1.4.2) is discretized by

$$\sum_{j=0}^{N_h} m_i^{k,j} \Delta s \leq D_i(t_k) \quad \forall i \in I, k \in \{0, \dots, N_T\}, \quad (1.4.7)$$

and a final constraint is considered instead of a final cost

$$\sum_{i \in I} \sum_{\ell=0}^{\bar{s}} m_i^{N_T,\ell} = 0 \quad \forall i \in I. \quad (1.4.8)$$

The discretized cost is of the form

$$\bar{J}(m, E) := \sum_{k=0}^{N_T-1} \sum_{i \in I} p_i^k \sum_{\ell=0}^{N_h-1} m_i^{k,\ell} \Delta s \Delta t + \sum_{k=0}^{N_T-1} \sum_{\substack{i,j \in I \\ i \neq j}} \Theta_{i,j}(m_i^k, E_{i,j}^k) \Delta t.$$

where p_i can be considered as an electricity cost or a reward, and $\Theta_{i,j} : \mathbb{R}^2 \mapsto \mathbb{R}$ is a convex function w.r.t. both variable. The discretized optimization problem is

$$\begin{aligned} & \inf_{m \in \mathbb{R}^{N_m}, E \in \mathbb{R}^{N_E}} \bar{J}(m, E) \\ & (m, E) \text{ are subject to } (1.4.5) - (1.4.8). \end{aligned} \quad (1.4.9)$$

This finite dimensional problem is solved using the Chambolle-Pock algorithm. This algorithm is introduced in [67] and belongs to the class of primal-dual algorithms. It has been successfully applied to finite dimensional MFG in [32] and to discretized MFG problems [40]. Two optimization problems having the structure of (1.4.9) but with different cost functions are numerically solved. The objective of these examples is to understand and analyze how the central planner responds to different peak and off-peak pricing, and to compare different ways to penalize transfers. We show that certain functions Θ allow to avoid both multiple switches per PEV and synchronization effects among the population of PEVs.

1.4.3 Chapter 4: Regularity of Lagrange multipliers for a mean field control problem arising in smart charging

This chapter is devoted to the regularity analysis of the Lagrange multiplier λ associated with the constraint (1.4.2) of Problem (P) . The regularity of the Lagrange multipliers is motivated in the literature for several reasons: to improve the regularity of the solution of some optimal control problems, to obtain second order optimality conditions or to establish the convergence of some numerical methods. Due to the continuity of the state variables, the Lagrange multipliers are expected to be Borel measures for pure state pointwise constraints, see for e.g. [61, 218]. For elliptic and parabolic control problems, the Lavrentiev regularization, consisting in transforming the pure state constraints into mixed control-state constraints, enables to prove the existence of Lagrange multipliers in L^2 or even in L^∞ , see [251] and the references therein. In this chapter we follow a penalization approach that has been first developed in the context of optimal control with state constraints in finite dimension in [47], where the authors apply the Maximum Principle on the penalized problem to obtain optimality conditions on the original problem with state constraints. Since then, this method has been exploited to MFG problems with constraints on agents state [12, 46]. Recently, the penalizing method has been extended to the optimal control of the Fokker-Planck equation with smooth density constraints in [82]. The author gives the system of optimality conditions for non-linear cost and constraint functionals of the measure variable, and shows the existence of a Lagrange multiplier in L^∞ and the Lipschitz continuity of the optimal controls. Such regularity results on the solutions of a constrained problem are consistent with the results obtained originally for control problem of finite dimensional processes [135].

Considering the optimization problem (P) introduced in Chapter 2 and the associated optimality conditions

stated in Theorem 1.4.1, the main result of this chapter is the following theorem.

Theorem 1.4.2. *If the parameter D of the congestion constraint (1.4.2) is time independent, then for any solution (m, α) of Problem (P), there exists $(\varphi, \lambda) \in \text{Lip}([0, T] \times [0, 1] \times I) \times \mathcal{M}^+([0, T] \times I)$ and $(\lambda^{ac}, \beta) \in L^\infty((0, T) \times I, \mathbb{R}_+) \times \mathbb{R}_+^{|I|}$ such that (φ, λ, m) is a weak solution of the system (S) and λ satisfies, for any $i \in I$,*

$$\lambda_i = \lambda_i^{ac} \mathcal{L} + \beta_i \delta_T,$$

where \mathcal{L} denotes the Lebesgue measure on $[0, T]$ and δ_T the Dirac measure at time $T \in \mathbb{R}_+$.

This result is obtained by introducing a penalized problem. First, for any $i \in I$ the function Ψ_i is defined, for any $\mu \in \mathcal{P}([0, 1] \times I)$, by:

$$\Psi_i(\mu) := \mu_i([0, 1]) - D_i. \quad (1.4.10)$$

Then, considering the penalization parameters $\varepsilon, \delta > 0$, the penalized objective function is given by

$$J_{\varepsilon, \delta}(m, \alpha) := J(m, \alpha) + \int_0^T \frac{1}{\varepsilon} \sum_{i \in I} \Psi_i^+(m(t)) dt + \frac{1}{\delta} \sum_{i \in I} \Psi_i^+(m(T)), \quad (1.4.11)$$

where, for any $i \in I$ and $t \in [0, T]$, $\Psi_i^+(m(t)) := \max(0, \Psi_i(m(t)))$. The penalized problem is:

$$\inf_{(m, \alpha)} J_{\varepsilon, \delta}(m, \alpha) \quad (P^{\varepsilon, \delta})$$

where (m, α) is a weak solution of (1.4.1).

The first step of the proof is to show that if a couple (m, α) is a solution of Problem $(P^{\varepsilon, \delta})$, then there exists $(\varphi, \lambda, \beta) \in \text{Lip}([0, T] \times I \times [0, 1]) \times L^\infty([0, T] \times I, \mathbb{R}_+) \times (\mathbb{R}_+)^{|I|}$ such that $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ on $\{m_i > 0\}$ and $(\varphi, \lambda, \beta, m)$ is a weak solution of the following system on $[0, T] \times [0, 1] \times I$:

$$\begin{cases} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \frac{\lambda_i}{\varepsilon} + \sum_{j \in I, j \neq i} H(\varphi_i - \varphi_j) = 0, \\ \partial_t m_i + \partial_s(m_i b_i) + \sum_{j \in I} H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j = 0, \\ m_i(0) = m_i^0, \varphi_i(T) = g_i + \frac{\beta_i}{\delta}, \end{cases} \quad (S^{\varepsilon, \delta})$$

where (λ, β) satisfies

$$\lambda_i(t) = \begin{cases} 0 & \text{if } \Psi_i(m(t)) < 0, \\ \in [0, 1] & \text{if } \Psi_i(m(t)) = 0, \\ 1 & \text{if } \Psi_i(m(t)) > 0, \end{cases} \quad \beta_i := \begin{cases} 0 & \text{if } \Psi_i(m(T)) < 0, \\ \in [0, 1] & \text{if } \Psi_i(m(T)) = 0, \\ 1 & \text{if } \Psi_i(m(T)) > 0. \end{cases}$$

The system of optimality conditions $(S^{\varepsilon, \delta})$ is obtained by following the linearization method developed in [39] and applied for penalized control problems [82].

The second step of the proof consists in showing that for $\varepsilon > 0$ and $\delta > 0$ small enough, any solution of Problem $(P^{\varepsilon, \delta})$ satisfies the congestion constraint (1.4.2), i.e. one has for any solution (m, α) of Problem $(P^{\varepsilon, \delta})$ that $\Psi_i(m(t)) \leq 0$, for any $(i, t) \in I \times [0, T]$. This result follows the proof by contradiction developed in [47]. Consequently, for $\varepsilon > 0$ and $\delta > 0$ small enough, Problem $(P^{\varepsilon, \delta})$ and Problem (P) have the same solutions.

A first consequence of Theorem 1.4.2 is that any optimal control α of Problem (P) is in $\text{Lip}([0, T] \times [0, 1] \times I^2)$. Secondly, it enables to propose a numerical scheme to solve Problem (D). An explicit finite difference scheme is used to discretize the Hamilton-Jacobi equation (1.4.4) and the convergence of the solutions of the discretized problems to a solution of Problem (D) is proved. Finally, we are able to build a sequence of weak solutions of the continuity equation (1.4.1) $\{(m^n, \alpha^n)\}_n$, based on the solutions of the discretized problems, that converges to a solution of Problem (P), when the time and space steps converge to 0.

1.4.4 Chapter 5: Mean field approximation of an optimal control problem for the continuity equation arising in smart charging

This chapter deals with the optimal control of a finite population of hybrid processes, modeling the optimal charging of a large population of identical PEVs. We prove the convergence, when the number of PEVs goes to infinity, of the solutions and that of the value of a sequence of the finite population problems respectively to a solution and the value of Problem (P), introduced in Chapter 2. The connection between the optimal control of a large population of agents and a mean field control problem is first established in [112], in the specific case where the dynamics of each agent is described by an ODE. The convergence of the value and that of the solutions of the optimal control of a finite population of interacting McKean-Vlasov dynamics, to the value and that of a solution of a mean field optimal control problem, is proved in [163] in fairly general settings (the results hold for degenerate diffusion). Convergence results in stochastic settings can be found in [51, 93, 94]. In deterministic settings, the Eulerian, Lagrangian and Kantorovich formulations of a finite population and a mean field control problems are studied in [62], and the convergence of the value functions of the finite population Lagrangian and Eulerian problems are established. In a finite state space setting, the rate of convergence of the value function of the finite population problem to the value function of the mean field problem is proved to be of order $1/\sqrt{n}$ in [64]. While problems in the literature deal with either continuous or discrete state variables, this work addresses the analysis of the mean field limit of hybrid processes. A particularity of the model developed in this chapter is that the nature of the dynamics in the finite population problem is different from that in the mean field problem. While the switches are controlled and deterministic in the finite population setting, the jumps of the discrete variable of each process are stochastic and the control is on the transition rate in the mean field problem. Also, a congestion constraint is considered in the optimization problem, which is unusual among the existing literature studying mean field limit of control problems (see however [66] for results in deterministic settings and [80, Chapter 4] for results in stochastic settings).

In the finite population model, we consider a population of n PEVs ($n \in \mathbb{N}^*$) controlled by a central planner, with the same state space $[0, 1] \times I$ where I is a finite set, as in Chapter 2, and with a time interval $[0, T]$. A time and space discretization depending on n is introduced, with time step Δt^n and space step Δs^n (both indexed by the superscript n), such that $N_T^n := T/\Delta t^n$ and $N_s^n := 1/\Delta s^n$ are integers. The time mesh is $\{0, t_1^n, \dots, t_{k+1}^n, \dots, T\}$ with $t_k^n := k\Delta t^n$, for any $k \in \{0, \dots, N_T^n\}$. The space mesh is $\{0, y_1^n, \dots, y_p^n, \dots, 1\}$ with $y_p^n := p\Delta s^n$, for any $p \in \{0, \dots, N_s^n\}$. Each vehicle ℓ is described by its state variable $x_t^\ell := (i_t^\ell, s_t^\ell) \in I \times [0, 1]$, with a given initial datum $x_0^\ell = (i_0^\ell, s_0^\ell)$. The discrete variable i^ℓ , denoting the mode of charging, can switch deterministically and only at fixed times in $\{t_1^n, \dots, t_{N_T^n-1}^n\}$, while the continuous variable s^ℓ , representing the SoC, is governed by an ODE depending on the mode of charging. Between two jumps of the variable i^ℓ , i.e. within each interval $[t_k^n, t_{k+1}^n)$, the dynamics of x^ℓ is deterministic and is given by:

$$i_t^\ell = i_{t_k^n}^\ell, \quad \frac{ds_t^\ell}{dt} = b(i_t^\ell, s_t^\ell) \quad \text{for any } t \in [t_k^n, t_{k+1}^n) \text{ and with } s_{t_k^n}^\ell = \lim_{\tau \uparrow t_k^n} s_\tau^\ell. \quad (1.4.12)$$

The population of n processes is subject to the same congestion constraints as (1.4.2) in Chapter 2,

$$\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_i(i_t^\ell) \leq D_i(t) \quad \forall (i, t) \in I \times [0, T]. \quad (1.4.13)$$

We denote by (\mathbf{i}, \mathbf{s}) the set of n trajectories $\{(i^\ell, s^\ell)\}_{1 \leq \ell \leq n}$ satisfying (1.4.12) and (1.4.13). The objective function \mathcal{J}^n for the finite population problem is defined by:

$$\mathcal{J}^n(\mathbf{i}, \mathbf{s}) := \frac{1}{n} \sum_{\ell=1}^n \int_0^T c(t, i_t^\ell, s_t^\ell) dt + g(i_T^\ell, s_T^\ell) + \sum_{i, j \in I, j \neq i} \sum_{k=1}^{N_T^n-1} \sum_{p=0}^{N_s^n-1} L \left(\frac{Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s})}{\Delta t} \right) \Delta t Q_i^{k,p}(\mathbf{i}, \mathbf{s}),$$

where $Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s})$ is the proportion of PEVs among the processes with a state in $\{j\} \times [y_p^n, y_{p+1}^n)$ that switch their discrete state i to j at time t_k^n , and $Q_i^{k,p}(\mathbf{i}, \mathbf{s})$ is the proportion of PEVs among the overall population of processes that has a state in $\{i\} \times [y_p^n, y_{p+1}^n)$ at time t_k^{n-} (just before the jumps). The purpose of this chapter

is to study the mean field limit of the following finite population optimal control problem:

$$\inf_{(\mathbf{i}, \mathbf{s})} \mathcal{J}^n(\mathbf{i}, \mathbf{s}). \quad (P_n)$$

The main result of this chapter is the following theorem, where Problem (P) is introduced in Chapter 2.

Theorem 1.4.3. *There exists $K > 0$ such that, for any $n \in \mathbb{N}$,*

$$\text{val}(P_n) - \frac{K}{n^{\frac{1}{3}}} \leq \text{val}(P) \leq \liminf_{N \rightarrow \infty} \text{val}(P_N).$$

To prove the second inequality, i.e. $\text{val}(P) \leq \liminf_{N \rightarrow \infty} \text{val}(P_N)$, we introduce for any set of n trajectories (\mathbf{i}, \mathbf{s}) the empirical distribution m^n of the processes and the empirical measure E^n of the switches:

$$m^n(t) := \frac{1}{n} \sum_{\ell=1}^n \delta_{(i_t^\ell, s_t^\ell)} \quad \text{and} \quad E_{i,j}^n := \frac{1}{n} \sum_{\ell=1}^n \sum_{k=1}^{N_T^n - 1} \mathbb{1}_i(i_{t_k}^\ell) \mathbb{1}_j(i_{t_k}^\ell) \delta_{t_k} \otimes \delta_{s_{t_k}^\ell}. \quad (1.4.14)$$

We show that, for any (\mathbf{i}, \mathbf{s}) , (m^n, E^n) is a weak solution of the continuity equation

$$\partial_t m_i^n + \partial_s(m_i^n b_i) = - \sum_{i,j \in I, j \neq i} (E_{i,j}^n - E_{j,i}^n) \quad \text{on } I \times (0, T) \times (0, 1). \quad (1.4.15)$$

We introduce the cost function

$$\begin{aligned} J^n(m, E) := & \sum_{i \in I} \int_0^T \int_0^1 c_i(t, s) m_i(t, ds) dt + \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds) \\ & + \sum_{i,j \in I, i \neq j} \sum_{k=1}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} L \left(\frac{E_{i,j}(\{t_k\}, [y_p^n, y_{p+1}^n])}{\Delta t^n m_i(t_k^-, [y_p^n, y_{p+1}^n])} \right) \Delta t^n m_i(t_k^-, [y_p^n, y_{p+1}^n]). \end{aligned}$$

By applying a superposition principle, which is a reminiscent of the one in [62], we show that the problem

$$\begin{aligned} & \inf_{(m, E)} J^n(m, E) \\ & \text{s.t. } (m, E) \text{ is a weak solution of (1.4.15) and } m \text{ satisfies (1.4.2),} \end{aligned} \quad (1.4.16)$$

is equivalent to Problem (P_n) . By compactness and lower semi-continuity arguments, we obtain that

$$\text{val}(P) \leq \liminf_{n \rightarrow \infty} \inf_{(m, E)} J^n(m, E) = \liminf_{n \rightarrow \infty} \inf_{(\mathbf{i}, \mathbf{s})} \mathcal{J}^n(\mathbf{i}, \mathbf{s}). \quad (1.4.17)$$

Then, the first inequality in Theorem 1.4.3, i.e. $\text{val}(P_n) - \frac{K}{n^{\frac{1}{3}}} \leq \text{val}(P)$, is obtained by using the regularity results derived in Chapter 2. We show that by implementing a mean field optimal control α to a finite population of n hybrid processes, one can obtain the following estimate between the empirical distribution of the finite population m^n and the optimal mean field distribution m ,

$$\mathcal{W}(m(t), m^n(t)) = \mathcal{O}\left(\frac{1}{n^{\frac{1}{3}}}\right) \quad \forall t \in [0, T],$$

where \mathcal{W} is the 1-Wasserstein distance. The previous estimate and the equivalence between problems (1.4.16) and (P_n) enables to obtain the inequality

$$\inf_{(\mathbf{i}, \mathbf{s})} \mathcal{J}^n(\mathbf{i}, \mathbf{s}) - \frac{C}{n^{\frac{1}{3}}} \leq \inf_{(m, \alpha)} J(m, \alpha). \quad (1.4.18)$$

The convergence of the value of Problem (P_n) to the value of Problem (P) is a consequence of the previous inequality and of (1.4.17). In addition, by compactness arguments, we show that a subsequence of solutions $\{(m^n, E^n)\}_n$ of the Problem (1.4.16) weakly converges to $(m, \alpha m)$, where (m, α) is a solution of (P) .

1.4.5 Chapter 6: Decomposition of convex high dimensional aggregative stochastic control problems

We consider the framework of convex high dimensional stochastic control problems, in which the controls are aggregated in the cost function. As a first contribution, we introduce a modified problem, whose optimal control is under some reasonable assumptions an ε -optimal solution of the original problem. As a second contribution, we present a decentralized algorithm whose convergence to the solution of the modified problem is established. Finally, we study the application of the developed tools in an engineering context, studying a coordination problem for large populations of domestic thermostatically controlled loads (TCLs). We highlight that, while we propose in Chapter 3 a deterministic and centralized algorithm to solve an optimal control problem in infinite dimension with aggregated state constraints introduced in Chapter 2, this chapter develops a stochastic and decentralized algorithm. Finally, contrary to the model presented in Chapter 2, the homogeneity of the agents is not assumed.

This work is motivated by its potential applications to the distributed coordination of flexible electrical appliances, to support power system operation in a context of increasing penetration of renewables. This chapter is related to the work of De Paola *et al.* [84], where a distributed solution is presented for the operation of a population of $n = 2 \times 10^7$ refrigerators providing frequency support and load shifting. The potential practical application of our work also considers a large population of TCLs which, as extension to [84], have stochastic dynamics. The proposed approach is able to minimize the overall system costs in a distributed way, with each TCL determining its optimal power consumption profile in response to price signals.

The considered problem belongs to the class of stochastic control: looking for strategies minimizing the expectation of an objective function under specific constraints. The Dynamic Programming Principle [28] and the SDDP are not suitable in our settings due to the number of agents. To tackle this type of high dimensional problems, it is natural to investigate decomposition techniques in the spirit of the Dual Approximation Dynamic Programming (DADP) [123, 172]. This approach is characterized by a price decomposition of the problem, where the stochastic constraints are projected on subspaces such that the associated Lagrangian multiplier is adapted for dynamic programming. Then the optimal multiplier is estimated by implementing Uzawa's algorithm. In this chapter, in the same vein as DADP, we propose a price decomposition approach restricted to deterministic prices. This new approach takes advantage of the large population number in order to introduce an auxiliary problem where the coupling term is purely deterministic. The present chapter aims at solving problem (P_1) involving a large number n of agents indexed by $i \in \{1, \dots, n\}$, of the form:

$$(P_1) \quad \begin{cases} \min_{u \in \mathcal{U}} J(u) \\ J(u) := \mathbb{E} \left[F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i(\omega^i, \omega^{-i}) \right) + \frac{1}{n} \sum_{i=1}^n G_i(u^i(\cdot, \omega^{-i}), \omega^i) \right]. \end{cases}$$

Here the noise $\omega := (\omega^1, \dots, \omega^n)$ belongs to $\Omega := \prod_{i=1}^n \Omega^i$, where $(\Omega^i, \mathcal{F}^i, \mu^i)$ is a probability space, and $(\Omega, \mathcal{F}, \mu)$ is the corresponding product probability space. Each decision variable u^i is \mathcal{F} -measurable (and not only \mathcal{F}^i -measurable), square summable with value in a Hilbert space \mathbb{U} . Also, $\mathcal{U} := \prod_{i=1}^n \mathcal{U}_i$ where \mathcal{U}_i is, for $i = 1$ to n , a closed convex subset of $L^2(\Omega, \mathbb{U})$. The cost function is the sum of a coupling term $F_0 : \mathbb{U} \rightarrow \mathbb{R}$, function of the *aggregate strategies* $\frac{1}{n} \sum_{i=1}^n u^i$, and *local terms* functions of the local decision u^i and local noise ω^i with

$G_i : L^2(\Omega^i, \mathbb{U}) \times \Omega^i \rightarrow \mathbb{R}$. The numerical difficulty of Problem (P_1) is related to the randomness of the aggregate term $\frac{1}{n} \sum_{i=1}^n u^i$ involved in the coupling function F_0 . We introduce the set of decentralized controls:

$$\hat{\mathcal{U}} := \prod_{i=1}^n \hat{\mathcal{U}}_i, \text{ where } \hat{\mathcal{U}}_i := \{u^i \in \mathcal{U}_i \mid u^i \text{ is } \mathcal{T}^1 \otimes \dots \otimes \mathcal{F}^i \otimes \mathcal{T}^{i+1} \otimes \dots \otimes \mathcal{T}^n \text{-measurable}\}, \quad (1.4.19)$$

where \mathcal{T}^j is the trivial σ -field $\{\emptyset, \Omega^j\}$. If the control u is decentralized, since the variance of the sum of independent random variables is equal to the sum of the variances of these random variables, the aggregate term $\frac{1}{n} \sum_{i=1}^n u^i$ can be approximated by $\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i$ when n is large enough. The following approximation of

Problem (P_1) is introduced:

$$(P_2) \quad \begin{cases} \min_{u \in \mathcal{U}} \tilde{J}(u) \\ \tilde{J}(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i \right) + \frac{1}{n} \mathbb{E} \left[\sum_{i=1}^n G_i(u^i(\cdot, \omega^{-i}), \omega^i) \right]. \end{cases}$$

We first show that, without loss of optimality in Problem (P_2) , one can restrict the control set \mathcal{U} to $\hat{\mathcal{U}}$. We denote by (\hat{P}_2) , the minimization problem associated with the objective function \tilde{J} and with the feasible set $\hat{\mathcal{U}}$. Problem (\hat{P}_2) can be written as:

$$(\hat{P}'_2) \quad \begin{cases} \min_{u \in \hat{\mathcal{U}}, v \in \mathbb{U}} \bar{J}(u, v), \\ \bar{J}(u, v) := F_0(v) + \frac{1}{n} \mathbb{E} \left[\sum_{i=1}^n G_i(u^i, \omega^i) \right], \\ \text{s.t. } \frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i - v = 0. \end{cases}$$

As a first contribution, this chapter shows that under some convexity and regularity assumptions on F_0 and $(G_i)_{i \in \{1, \dots, n\}}$, any solution of Problem (P_2) is an ε_n -solution of (P_1) , with $\varepsilon_n \rightarrow 0$ when $n \rightarrow \infty$. In addition, we will see that an approach of price decomposition for (P_2) , based on the formulation (\hat{P}'_2) , is tractable for dynamical problems, since the minimization of the Lagrangian obeys to dynamic programming principle. Since computing the dual cost of (\hat{P}'_2) is expensive, we propose *Stochastic Uzawa* and *Sampled Stochastic Uzawa* algorithms relying on the Robbins-Monroe algorithm, in the spirit of the stochastic gradient. Their convergence is established, based on the proof provided by [119] for the convergence of the stochastic gradient in a Hilbert space. We check the effectiveness of the *Stochastic Uzawa* algorithm on a linear quadratic Gaussian framework, and we apply the *Sampled Stochastic Uzawa* algorithm to a model of power system, inspired by the work of A. De Paola *et al.* [84].

1.4.6 Chapter 7: A decentralized algorithm for a MFC problem of Piecewise Deterministic Markov Processes

This chapter aims at extending the decentralized approach developed in Chapter 6 to a non-convex framework, with unbounded controls, allowing for more realistic dynamics ruling the evolution of each agent state. In particular, we are interested in the aggregative control of Piecewise Deterministic Markov Processes (PDMPs for short) in a mean field control framework. PDMPs are introduced in [83] as a class of non-diffusion stochastic models, mixing random jumps and deterministic dynamics between jumps. Our first contribution lies in the originality of the proof of a saddle point existence which follows a completely different path from the one developed in Chapter 6. In particular, we make use of regularity results on the solution of the Hamilton-Jacobi equation arising in optimal control of PDMPs developed in Chapters 2 and 4, to show the existence of a saddle point of our Lagrangian problem. Our second contribution consists in proving that the *Stochastic Uzawa Algorithm* proposed in Chapter 6 is still providing a converging sequence of controls in this new setting involving the PDMP dynamics which violates the convexity conditions originally exploited. Finally, an application to the smart charging of a fleet of PEVs by an aggregator illustrates the performance and the interest of the approach. Similarly to Chapters 2-5, the state of each EV is composed of a discrete and a continuous variable and a mean field assumption is considered. However, while the MFC problem of hybrid state variables in Chapters 2 and 5 is formulated as an optimal control of partial differential equation problem, and numerically approximated in a deterministic and centralized manner, this chapter focuses on stochastic techniques and relies on the decentralized and stochastic algorithm proposed in Chapter 6 to solve this MFC problem.

More precisely, we consider the time interval $[0, T]$ with $T > 0$ and a population of n independent and identically distributed processes, controlled by a central planner via a common feedback control α that belongs to the space of feedbacks \mathbb{A} . The state of an agent, controlled by the jump intensity α , is given at time t by $X_t^\alpha = (Y_t^\alpha, Z_t^\alpha)$ with $Y_t^\alpha \in I$ and $Z_t^\alpha \in [0, 1]$. As in Chapter 2, the discrete variable Y^α represents the charging mode of a vehicle and the continuous variable Z^α represents the SoC. The process X^α is said to be a PDMP(b, α), if Y^α is a jump process with values in I switching spontaneously at jump times $\{T_k^\alpha\}_{k \in \mathbb{N}}$, given by a Poisson process with intensity α , while Z^α follows a deterministic dynamics between two consecutive jumps

$Z_t^\alpha = \phi_{T_k^\alpha, Z_{T_k^\alpha}^\alpha}(Y_{T_k^\alpha}^\alpha, t)$ for $t \in [T_k^\alpha, T_{k+1}^\alpha)$, where the flow ϕ is the unique solution of the ordinary differential equation:

$$\partial_t \phi_{\tau, z}(j, t) = b(j, \phi_{\tau, z}(j, t)) \text{ with } \phi_{\tau, z}(j, \tau) = z,$$

and b is a function given. The objective function J is defined by:

$$J(\alpha) := \int_0^T f(t, \mathbb{E}[p(t, X_t^\alpha)]) dt + \int_0^T \mathbb{E} \left[c(t, X_t^\alpha) + \sum_{j \in I} L(\alpha_j(t, X_t^\alpha)) \right] dt + \mathbb{E}[g(X_T^\alpha)],$$

where X^α is a PDMP(b, α) and the function f represents a coupling cost depending on the expectation of $p(t, X_t^\alpha)$ (p being a smooth function). This chapter is dedicated to the following problem:

$$\min_{\alpha \in \mathbb{A}} J(\alpha). \tag{P_{pdmp}}$$

Considering a non linear function f , the cost function J is nonlinear w.r.t $\mathbb{E}[p(t, X_t^\alpha)]$, consequently Problem (P_{pdmp}) goes beyond the scope of optimal control of PDMP. The existence of a solution to Problem (P_{pdmp}) is established. We propose to numerically solve Problem (P_{pdmp}) by applying the *Stochastic Uzawa Algorithm* introduced in Chapter 6. This algorithm generates the sequence of control $\{\alpha^k\}_k$ whose convergence to a solution of Problem (P_{pdmp}) is proved. This result can not be obtained by a direct application of the convergence results in Chapter 6, which relies on additional assumptions that are not verified in this framework that involves unbounded controls and PDMP dynamics.

1.5 Contenu du manuscrit et présentation des principaux résultats (en français)

Cette thèse porte principalement sur la modélisation mathématique d'un problème de chargement optimal d'une flotte de véhicules électriques. Nous formulons le problème de chargement optimal comme un problème de contrôle champ moyen au Chapitre 2. Nous obtenons une caractérisation des solutions ainsi que des résultats de régularité. Le Chapitre 3 est dédié à l'approximation numérique du problème de contrôle champ moyen examiné au Chapitre 2, et à son application à un problème de charge intelligente. Le Chapitre 4 étudie la régularité des multiplicateurs de Lagrange introduits au Chapitre 2. Une estimation L^∞ est obtenue pour le multiplicateur de Lagrange associé à la contrainte de congestion sur la distribution des états. Ce résultat est exploité pour démontrer la convergence d'un schéma numérique introduit au Chapitre 2. Le Chapitre 5 a pour objectif de justifier l'approche champ moyen adoptée aux deux précédents chapitres du manuscrit, en démontrant la convergence, quand le nombre d'agents tend vers l'infini, de la valeur du problème de contrôle avec un nombre fini d'agents vers la valeur du problème de contrôle optimal introduit au Chapitre 2. La convergence de la suite de solutions des problèmes avec population finie vers une solution du problème de contrôle champ moyen est également obtenue. Un autre modèle est étudié au Chapitre 6, dans lequel un problème convexe stochastique de grande dimension, avec une population d'agents hétérogènes, est étudié. Une version modifiée du problème est introduite, dont le contrôle optimal est, sous certaines conditions raisonnables, un contrôle ε -optimal pour le problème initial. Nous présentons également un algorithme décentralisé dont la convergence vers la solution du problème modifié est établie. Enfin, le Chapitre 7 propose une approche décentralisée pour le problème de contrôle champ moyen de Processus de Markov Déterministes par Morceaux (PDMP). La convergence de l'algorithme introduit au Chapitre 6 est démontrée dans ce cadre particulier et des simulations illustrant la coordination du chargement d'une flotte importante de véhicules électriques sont présentées. Chacun des chapitres 2,3,5,6 et 7 est basé sur un article de recherche qui a été accepté ou soumis pour publication dans une revue scientifique ou à une conférence. Le Chapitre 4 est basé sur un document de travail qui sera bientôt soumis à une revue scientifique.

- Le Chapitre 2 est basé sur la prépublication [233], soumise à une revue pour publication.
- Le Chapitre 3 est basé sur l'article de conférence [238] présenté lors de la conférence 2021 IEEE Conference on Innovative Smart Grid Technologies (ISGT).
- Le Chapitre 4 est basé sur le document de travail [235].
- Le Chapitre 5 est basé sur la prépublication [234], soumise à une revue pour publication.
- Le Chapitre 6 est basé sur l'article [236], publié dans la revue *Applied Mathematics & Optimization*.
- Le Chapitre 7 est basé sur la prépublication [237], soumise à une revue pour publication.

1.5.1 Chapitre 2 : Conditions d'optimalité d'un problème de contrôle optimal de l'équation de continuité survenant dans un problème de charge intelligente

Ce chapitre se focalise sur la modélisation mathématique et la caractérisation des solutions d'un problème de chargement optimal d'une flotte importante de véhicules électriques rechargeables (PEVs), avec variables d'état hybrides (continue et discrète). Un problème de contrôle champ moyen de processus hybrides avec contraintes de congestion est introduit puis étudié. Depuis le travail fécond de Benamou et Brenier [21] en transport optimal, l'analyse des conditions d'optimalité d'un problème de contrôle champ moyen s'appuie largement sur le principe de dualité [110, 265]. L'approche par dualité est revisitée par l'approche variationnelle utilisée en Jeux à Champ Moyen (MFG) dans [53], où la solution faible d'un système MFG est caractérisée comme la solution d'un problème de contrôle optimal d'une équation d'Hamilton-Jacobi et d'une équation de continuité. Cette approche permet d'exploiter des techniques d'optimisation pour obtenir des résultats d'existence et d'unicité à des problèmes de contrôle et de jeux à champ moyen. Nous nous référons à [5, 25, 40, 54, 202] et aux références qui y figurent. L'utilisation de l'approche variationnelle est également motivée par l'approximation numérique des problèmes de MFG [23, 40, 41]. Dans ce chapitre nous considérons une contrainte de congestion sur la mesure. Les problèmes MFGs avec contraintes de congestion ont été pour la première fois étudiés dans [230], où la densité de la population d'agents ne peut dépasser un certain seuil, puis dans [190] où un système MFG du second ordre stationnaire est examiné et dans [55] pour les MFGs du premier ordre. La méthode variationnelle est également appliquée dans [32] pour résoudre un problème de MFG avec espace d'état fini, temps discret

et contraintes de congestion. Contrairement aux articles cités ci-dessus qui considèrent des problèmes avec variables d'état continues ou discrètes, nous étudions dans ce chapitre un problème avec variables hybrides.

Plus précisément, le problème se formule de la façon suivante. Nous considérons un intervalle de temps fini $[0, T]$ et un espace d'état hybride égal au produit $[0, 1] \times I$, où I est un espace d'état fini. L'équation de continuité est donnée par :

$$\begin{aligned} \partial_t m_i + \partial_s(m_i b_i) &= - \sum_{j \neq i} (\alpha_{i,j} m_i - \alpha_{j,i} m_j) \quad \text{sur } I \times (0, T) \times (0, 1), \\ m_i(0) &= m_i^0 \quad \text{sur } I \times [0, 1], \end{aligned} \quad (1.5.1)$$

où $m \in C^0([0, T], \mathcal{P}([0, 1] \times I))$, $m^0 \in \mathcal{P}([0, 1] \times I)$ est donnée et est la distribution initiale, $b : I \times [0, 1] \rightarrow \mathbb{R}$ est un champ de vitesse, et le contrôle $\alpha : I \times I \times [0, T] \times [0, 1] \rightarrow \mathbb{R}_+$ est une intensité de sauts. La distribution m est soumise à la contrainte de congestion suivante :

$$m_i(t, [0, 1]) \leq D_i(t) \quad \forall (i, t) \in I \times [0, T], \quad (1.5.2)$$

où $D : [0, T] \rightarrow \mathbb{R}_+^*$ est donnée. La fonction objectif J est définie par :

$$J(m, \alpha) := \sum_{i \in I} \int_0^T \int_0^1 (c_i(t, s) + \sum_{j \in I, j \neq i} L(\alpha_{ij}(t, s))) m_i(t, ds) dt + \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds). \quad (1.5.3)$$

L'objectif de ce chapitre est d'étudier le problème d'optimisation :

$$\begin{aligned} \inf_{m, \alpha} J(m, \alpha) \\ \text{où } (m, \alpha) \text{ est une solution faible de (1.5.1) et satisfait (1.5.2).} \end{aligned} \quad (P)$$

Ce travail est motivé par le chargement optimal d'une population de PEVs contrôlée par un planificateur central. La variable continue $s \in [0, 1]$ dans (1.5.1) représente le niveau de batterie du véhicule. La variable discrète $i \in I$ représente le mode chargement (par exemple inactif, chargement, déchargement, etc...). Pour tout $(t, s, i) \in [0, T] \times [0, 1] \times I$, la valeur $m_i(t, s)$ représente la proportion de PEVs à l'instant t et à l'état (s, i) . Le champ de vitesse $b_i(s)$ désigne la puissance de charge du PEV en mode i et avec un niveau de batterie s . Pour tout $(t, s, i, j) \in [0, T] \times [0, 1] \times I \times I$, la quantité α correspond à l'intensité de sauts des PEVs depuis l'état (s, i) vers l'état (s, j) à l'instant t . La contrainte de congestion (1.5.2) permet d'éviter de fortes demandes en puissance à chaque instant au cours de la période. La fonction objectif (1.5.3) pénalise les grandes valeurs de α . Cette pénalisation a pour objectif d'éviter la synchronisation des sauts des PEVs et l'instabilité sur le réseau électrique qui en résulte. La quantité $c_i(t, s)$ correspond au coût instantané pour un PEV, à l'instant t et à l'état (s, i) et $g_i(s)$ est le coût final par PEV à l'état (s, i) . Le principal résultat de ce chapitre est le système de conditions d'optimalité suivant.

Theorem 1.5.1. *S'il existe $\varepsilon^0 > 0$ tel que :*

$$\varepsilon^0 < D_i(t) - m_i^0([0, 1]) \quad \forall (t, i) \in [0, T] \times I,$$

alors (m, α) est une solution de (P), si et seulement si, il existe une paire $(\varphi, \lambda) \in (\text{Lip}([0, T] \times I \times [0, 1]) + BV([0, T] \times I)) \times \mathcal{M}^+([0, T] \times I)$ tel que $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ et (φ, λ, m) est une solution faible du système suivant sur $[0, T] \times I \times [0, 1]$:

$$\left\{ \begin{array}{l} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \lambda_i + \sum_{j \in I, j \neq i} H(\varphi_j - \varphi_i) = 0 \\ \partial_t m_i + \partial_s(m_i b_i) + \sum_{j \neq i} (H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j) = 0 \\ m_i(0, s) = m_i^0(s), \varphi_i(T, s) = g_i(s) \\ \int_0^1 m_i(t, ds) - D_i(t) \leq 0, \lambda_i \geq 0 \\ \sum_{i \in I} \int_0^T \left(\int_0^1 m_i(t, ds) - D_i(t) \right) \lambda_i(dt) = 0 \end{array} \right. \quad (S)$$

où H est la conjuguée de Fenchel de L et H' sa dérivée.

La fonction φ peut être comprise comme le multiplicateur de Lagrange associé à la contrainte sur la dynamique du système (1.5.1), et la mesure λ comme le multiplicateur de Lagrange associé à la contrainte de congestion (1.5.2). Le système (S) est obtenu via une approche par dualité. Plus précisément, la dynamique (1.5.1) et la contrainte de congestion (1.5.2) sont relaxées. Étant donné l'équation d'Hamilton-Jacobi sur $(0, T) \times (0, 1) \times I$:

$$\begin{aligned} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i(t, s) - \lambda_i + \sum_{j \in I, j \neq i} H(\varphi_j - \varphi_i) &\leq 0 && \text{sur } (0, T) \times (0, 1) \times I, \\ \varphi_i(T) &\leq g_i && \text{sur } [0, 1] \times I, \end{aligned} \quad (1.5.4)$$

le problème relaxé obtenu est ensuite exprimé comme le problème dual du problème convexe suivant :

$$\begin{aligned} \inf_{(\varphi, \lambda)} \sum_{i \in I} \int_0^1 -\varphi_i(0, s) m_i^0(ds) + \int_0^T D_i(t) \lambda_i(dt) \\ \text{s.t. } (\varphi, \lambda) \text{ sous solution de (1.5.4).} \end{aligned} \quad (D)$$

Pour toute solution faible (λ, φ, m) de (D), la régularité de $\varphi \in \text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)$ est obtenue sous certaines hypothèses sur les données du problème b, g et c . Il en résulte que le contrôle optimal α est borné et Lipschitz continu par rapport à la variable d'espace, de façon uniforme par rapport à la variable temporelle, et la distribution m appartient à $\text{Lip}([0, 1], \mathcal{P}([0, 1] \times I))$. De plus, nous montrons que si la distribution initial m^0 est absolument continue par rapport à la mesure de Lebesgue et a une densité continûment dérivable, alors la mesure m est absolument continue par rapport à la mesure de Lebesgue et a une densité Lipschitz continue. Ces résultats de régularité sont exploités au Chapitre 5 afin d'établir un lien entre le problème avec population finie et le problème de limite champ moyen. Enfin, les résultats de régularité sur les multiplicateurs de Lagrange (φ, λ) sont améliorés au Chapitre 4, où nous montrons que la partie absolument continue de λ est une fonction L^∞ et que sa partie singulière est une masse de Dirac à l'instant final T .

1.5.2 Chapitre 3 : Calcul et implémentation d'un contrôle optimal champ moyen pour un problème de charge intelligente

Dans ce chapitre, nous proposons un algorithme afin de résoudre numériquement le problème de contrôle champ moyen introduit au Chapitre 2. Une procédure d'implémentation est présentée et testée dans une étude de cas avec un nombre fini de véhicules. Un problème de contrôle champ moyen, qui est équivalent au problème (P), à un changement de variables près, est discrétisé, avec comme pas en temps et espace respectifs Δt et Δs (satisfaisant $T = N_T \Delta t$ et $1 = N_h \Delta s$). La variable $m_i^{k, \ell}$ désigne la proportion de véhicules en mode $i \in I$ avec un niveau de batterie (SoC) compris dans $[\ell \Delta s, (\ell + 1) \Delta s)$ à l'instant $t_k := k \Delta t$, $E_{i,j}^{k, \ell}$ est la fréquence de saut du mode i vers le mode j pour les PEVs avec un SoC compris dans $[\ell \Delta s, (\ell + 1) \Delta s)$ à l'instant t_k . L'équation de continuité est discrétisée en considérant deux étapes successives. La variable $m_i^{k+\frac{1}{2}, \ell}$ est tout d'abord introduite pour prendre en compte les transferts des PEVs vers et depuis le mode i :

$$m_i^{k+\frac{1}{2}, \ell} := m_i^{k, \ell} + \Delta t \sum_{j \in I \setminus \{i\}} E_{j,i}^{k, \ell} - E_{i,j}^{k, \ell}. \quad (1.5.5)$$

La convection (induite par la vitesse de chargement b) est ensuite prise en compte en appliquant un schéma de type *upwind*

$$\frac{m_i^{k+1, \ell} - m_i^{k+\frac{1}{2}, \ell}}{\Delta t} + \frac{F_i^{k, \ell} - F_i^{k, \ell-1}}{h} = 0, \quad (1.5.6)$$

où $F_i^{k, \ell}$ dépend de m_i^k et du signe de b_i . La contrainte de congestion (1.5.2) est discrétisée par :

$$\sum_{j=0}^{N_h} m_i^{k, j} \Delta s \leq D_i(t_k) \quad \forall i \in I, k \in \{0, \dots, N_T\}, \quad (1.5.7)$$

et une contrainte finale, et non un coût final, est considérée

$$\sum_{i \in I} \sum_{\ell=0}^s m_i^{N_T, \ell} = 0 \quad \forall i \in I. \quad (1.5.8)$$

Le coût discrétisé est de la forme :

$$\bar{J}(m, E) := \sum_{k=0}^{N_T-1} \sum_{i \in I} p_i^k \sum_{\ell=0}^{N_h-1} m_i^{k, \ell} \Delta s \Delta t + \sum_{k=0}^{N_T-1} \sum_{\substack{i, j \in I \\ i \neq j}} \Theta_{i, j}(m_i^k, E_{i, j}^k) \Delta t.$$

où p peut être considéré comme un coût de consommation de l'électricité ou comme une récompense, et $\Theta_{i, j} : \mathbb{R}^2 \mapsto \mathbb{R}$ est une fonction convexe par rapport aux deux variables. Le problème d'optimisation discrétisé est de la forme :

$$\inf_{\substack{m \in \mathbb{R}^{N_m}, E \in \mathbb{R}^{N_E} \\ (m, E) \text{ satisfait (1.4.5) - (1.4.8)}}} \bar{J}(m, E) \quad (1.5.9)$$

Ce problème d'optimisation de dimension finie est résolu numériquement en appliquant l'algorithme de Chambolle-Pock. Cet algorithme est introduit dans [67] et appartient à la classe des algorithmes primal-dual. Il a été implémenté avec succès à un problème de MFG de dimension finie [32] et à un problème de MFG discrétisé [40]. Deux problèmes d'optimisation ayant la même structure que (1.5.9), mais avec des fonctions de coût différentes, sont résolus numériquement. L'objectif de ces exemples est de comprendre et d'analyser comment le planificateur central répond à une tarification heures pleines - heures creuses, et de comparer les résultats selon la façon dont les transferts ont été pénalisés. Nous montrons que certains types de fonction Θ permettent d'éviter à la fois de nombreux changements de mode de chargement par PEV et des effets de synchronisation parmi la population de PEVs.

1.5.3 Chapitre 4 : Régularité des multiplicateurs de Lagrange d'un problème de contrôle champ moyen survenant dans un problème de charge intelligente

Ce chapitre est dédié à l'analyse de la régularité du multiplicateur de Lagrange λ associé à la contrainte de congestion (1.5.2) du Problème (P). L'analyse de la régularité des multiplicateurs de Lagrange est présente dans la littérature pour différentes raisons : pour améliorer la régularité des solutions de problèmes de contrôle optimal, pour obtenir des conditions d'optimalité du second ordre ou pour établir des résultats de convergence pour certaines méthodes numériques. En raison de la continuité de la variable d'état par rapport à la variable temporelle, les multiplicateurs de Lagrange pour les contraintes sur état sont a priori des mesures de Borel, voir par exemple [61, 218]. Pour les problèmes de contrôle elliptique et parabolique, la régularisation de Lavrentiev, consistant à modifier des contraintes sur état en des contraintes mixtes contrôle-état, permet de prouver l'existence de multiplicateurs de Lagrange dans L^2 ou même dans L^∞ , voir [251] et les références y figurants. Dans ce chapitre, nous suivons l'approche par pénalisation qui fut d'abord développée pour un problème de contrôle optimal de dimension finie avec contraintes sur l'état dans [47], où les auteurs appliquent le Principe de Maximum au problème pénalisé pour obtenir des conditions d'optimalité au problème initial avec contraintes sur l'état. Depuis, cette méthode a été exploitée pour des problèmes MFG avec contraintes de densité sur l'état des agents [12, 46]. Récemment, la méthode par pénalisation a été étendue au problème de contrôle optimal d'une équation de Fokker-Planck avec contraintes sur la densité dans [82]. L'auteur obtient un système de conditions d'optimalité pour des contraintes et des fonctions de coûts non linéaires par rapport à la distribution des états, et montre l'existence d'un multiplicateur de Lagrange dans L^∞ ainsi que la continuité Lipschitz des contrôles optimaux. De tels résultats de régularité sur des problèmes avec contraintes sont consistants avec des résultats obtenus précédemment dans la littérature pour des problèmes de contrôle sur des processus de dimension finie [135].

Le principal résultat de ce chapitre porte sur le problème d'optimisation (P) introduit au Chapitre 2 et sur le système de conditions d'optimalité énoncé dans le Théorème 1.4.1.

Theorem 1.5.2. *Si la fonction D , intervenant dans la contrainte de congestion (1.4.2), est indépendante du temps, alors pour toute solution (m, α) du Problème (P) il existe $(\varphi, \lambda) \in \text{Lip}([0, T] \times [0, 1] \times I) \times \mathcal{M}^+([0, T] \times I)$ et $(\lambda^{ac}, \beta) \in L^\infty((0, T) \times I, \mathbb{R}_+) \times \mathbb{R}_+^{[I]}$, tel que (φ, λ, m) est une solution faible du système (S) et λ satisfait,*

pour tout $i \in I$,

$$\lambda_i = \lambda_i^{ac} \mathcal{L} + \beta_i \delta_T,$$

où \mathcal{L} désigne la mesure de Lebesgue sur $[0, T]$ et δ_T la mesure de Dirac à l'instant $T \in \mathbb{R}_+$.

Ce résultat est obtenu en introduisant un problème pénalisé. Pour cela, la fonction Ψ_i est définie pour tout $i \in I$ et tout $\mu \in \mathcal{P}([0, 1] \times I)$ par :

$$\Psi_i(\mu) := \mu_i([0, 1]) - D_i. \quad (1.5.10)$$

Étant donné les paramètres de pénalisation $\varepsilon, \delta > 0$, la fonction objectif du problème pénalisé est donnée par :

$$J_{\varepsilon, \delta}(m, \alpha) := J(m, \alpha) + \int_0^T \frac{1}{\varepsilon} \sum_{i \in I} \Psi_i^+(m(t)) dt + \frac{1}{\delta} \sum_{i \in I} \Psi_i^+(m(T)), \quad (1.5.11)$$

où pour tout $i \in I$ et $t \in [0, T]$, $\Psi_i^+(m(t)) := \max(0, \Psi_i(m(t)))$. Le problème pénalisé est :

$$\inf_{(m, \alpha)} J_{\varepsilon, \delta}(m, \alpha) \quad (P^{\varepsilon, \delta})$$

où (m, α) est une solution faible de (1.4.1).

La première étape de la preuve est de montrer que si le couple (m, α) est une solution du Problème $(P^{\varepsilon, \delta})$, alors il existe $(\varphi, \lambda, \beta) \in \text{Lip}([0, T] \times I \times [0, 1]) \times L^\infty([0, T] \times I, \mathbb{R}_+) \times (\mathbb{R}_+)^{|I|}$ tel que $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ sur $\{m_i > 0\}$ et $(\varphi, \lambda, \beta, m)$ est une solution faible, sur $[0, T] \times [0, 1] \times I$, du système

$$\begin{cases} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \frac{\lambda_i}{\varepsilon} + \sum_{j \in I, j \neq i} H(\varphi_i - \varphi_j) = 0, \\ \partial_t m_i + \partial_s(m_i b_i) + \sum_{j \in I} H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j = 0, \\ m_i(0) = m_i^0, \varphi_i(T) = g_i + \frac{\beta_i}{\delta}, \end{cases} \quad (S^{\varepsilon, \delta})$$

où (λ, β) satisfait

$$\lambda_i(t) = \begin{cases} 0 & \text{si } \Psi_i(m(t)) < 0, \\ \in [0, 1] & \text{si } \Psi_i(m(t)) = 0, \\ 1 & \text{si } \Psi_i(m(t)) > 0, \end{cases} \quad \beta_i := \begin{cases} 0 & \text{si } \Psi_i(m(T)) < 0, \\ \in [0, 1] & \text{si } \Psi_i(m(T)) = 0, \\ 1 & \text{si } \Psi_i(m(T)) > 0. \end{cases}$$

Le système de conditions d'optimalité $(S^{\varepsilon, \delta})$ est obtenu en suivant la méthode de linéarisation développée dans [39] et appliquée à un problème de contrôle pénalisé dans [82]. La seconde étape de la preuve consiste à montrer que pour $\varepsilon > 0$ et $\delta > 0$ suffisamment petits, toute solution du Problème $(P^{\varepsilon, \delta})$ satisfait la contrainte de congestion $(P^{\varepsilon, \delta})$, c'est à dire, pour toute solution (m, α) du Problème $(P^{\varepsilon, \delta})$, on a $\Psi_i(m(t)) \leq 0$, pour tout $(i, t) \in I \times [0, T]$. Ce résultat suit les étapes de la preuve par contradiction développée dans [47]. Par conséquent, pour $\varepsilon > 0$ et $\delta > 0$ suffisamment petits, le Problème $(P^{\varepsilon, \delta})$ et le Problème (P) ont les mêmes solutions. Une première conséquence du Théorème 1.5.2 est que tout contrôle optimal α pour le Problème (P) est dans $\text{Lip}([0, T] \times [0, 1] \times I^2)$. Deuxièmement, ce théorème permet de proposer un schéma numérique pour résoudre le Problème (D) . Un schéma explicite de différences finies est utilisé pour discrétiser l'équation d'Hamilton-Jacobi (1.5.4) et la convergence des solutions du problème discrétisé vers une solution du Problème (D) est démontrée. Enfin, à partir des solutions du problème discrétisé, nous proposons une suite $\{(m^n, \alpha^n)\}_n$ de solutions faibles de l'équation de continuité (1.5.1), qui convergent vers une solution du Problème (P) , quand les pas en temps et espace du problème discrétisé tendent vers 0.

1.5.4 Chapitre 5 : Approximation champ moyen d'un problème de contrôle optimal d'une équation de continuité intervenant dans un problème de charge intelligente

Ce chapitre traite d'un problème de contrôle optimal d'une population finie de processus hybrides, modélisant le chargement optimal d'une flotte importante et homogène de PEVs. Nous démontrons la convergence, quand le nombre de PEVs tend vers l'infini, de la suite de solutions et de la suite des valeurs des problèmes d'optimisation

avec population finie, vers respectivement une solution et la valeur du Problème (P), introduit au Chapitre 2. Le lien entre une suite de problèmes de contrôle optimal d'une grande population d'agents et un problème de contrôle champ moyen fut établi pour la première fois dans [112], dans le cas spécifique où la dynamique de chaque agent est décrite par une équation aux dérivées ordinaires (EDO). La convergence de la valeur et des solutions du problème de contrôle optimal d'une population finie en interaction via une dynamique de type McKean-Vlasov, vers la valeur et une solution d'un problème de contrôle champ moyen, est démontré dans [163], dans un cadre général (les résultats sont également valables pour des diffusions dégénérées). Des résultats de convergence dans un cadre stochastique peuvent être trouvés dans [51, 93, 94]. Dans un cadre déterministe, les formulations Eulérienne, Lagrangienne et de Kantorovich d'un problème de contrôle avec population finie et la comparaison avec la limite de type champ moyen du problème sont étudiées dans [62], et la convergence de la fonction valeur, des problèmes de type Lagrangien et Eulérien avec population finie, est démontrée quand le nombre d'agents tend vers l'infini. Le taux de convergence de la fonction valeur du problème avec un nombre fini d'agents vers la fonction valeur du problème de contrôle champ moyen est de l'ordre de $1/\sqrt{n}$, dans le cas d'un problème avec espace d'état fini [64]. Alors qu'une majeure partie des problèmes traités dans la littérature ont des variables d'état uniquement discrètes ou uniquement continues, notre travail porte sur l'analyse de la limite champ moyen de processus hybrides. Une spécificité du modèle étudiée dans ce chapitre porte sur la nature de la dynamique des agents du problème de contrôle avec population finie, qui se révèle être différente de la dynamique des agents du problème de contrôle champ moyen. Alors que les instants de sauts sont contrôlés et déterministes dans le problème avec population finie, les sauts de la variable discrète sont stochastiques et le contrôle est sur le taux de transition dans le problème champ moyen. De plus, une contrainte de congestion est considérée dans le problème d'optimisation étudié, ce qui est inhabituel parmi la littérature existante sur les limites de champ moyen de problèmes de contrôle (voir cependant [66] pour des résultats dans un cadre déterministe et [80, Chapter 4] pour des résultats dans un cadre stochastique).

Dans le modèle avec population finie, nous considérons une population de n PEVs ($n \in \mathbb{N}^*$), contrôlée par un planificateur central, ayant le même espace d'état $[0, 1] \times I$, où I est un ensemble fini, comme au Chapitre 2, et avec comme horizon en temps $[0, T]$. Une discrétisation en temps et en espace dépendant de n est introduite, avec comme pas en temps Δt^n et en espace Δs^n (tous deux indexés par l'exposant n), tel que $N_T^n := T/\Delta t^n$ et $N_s^n := 1/\Delta s^n$ sont des entiers. Le maillage en temps est $\{0, t_1^n, \dots, t_{k+1}^n, \dots, T\}$ avec $t_k^n := k\Delta t^n$, pour tout $k \in \{0, \dots, N_T^n\}$. Le maillage en espace est $\{0, y_1^n, \dots, y_p^n, \dots, 1\}$ avec $y_p^n := p\Delta s^n$, pour tout $p \in \{0, \dots, N_s^n\}$. Chaque véhicule ℓ est décrit par la variable d'état $x_t^\ell := (i_t^\ell, s_t^\ell) \in I \times [0, 1]$, avec comme état initial $x_0^\ell = (i_0^\ell, s_0^\ell)$. La variable discrète i^ℓ , désignant le mode de chargement, peut sauter de façon déterministe et uniquement à certains instants $\{t_1^n, \dots, t_{N_T^n-1}^n\}$, alors que la variable continue s^ℓ , représentant le niveau de batterie (SoC), est régie par une EDO dans laquelle intervient le mode de chargement. Entre deux instants de sauts de la variable discrète i^ℓ , c'est-à-dire au sein de chaque intervalle $[t_k^n, t_{k+1}^n)$, la dynamique de x^ℓ est déterministe et donnée par :

$$i_t^\ell = i_{t_k^n}^\ell, \quad \frac{ds_t^\ell}{dt} = b(i_t^\ell, s_t^\ell) \text{ pour tout } t \in [t_k^n, t_{k+1}^n) \text{ et avec } s_{t_k^n}^\ell = \lim_{\tau \uparrow t_k^n} s_\tau^\ell. \quad (1.5.12)$$

La population de n processus est soumise à la même contrainte de congestion que (1.5.2) du Chapitre 2,

$$\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_i(i_t^\ell) \leq D_i(t) \quad \forall (i, t) \in I \times [0, T]. \quad (1.5.13)$$

Nous désignons par (\mathbf{i}, \mathbf{s}) , l'ensemble des n trajectoires $\{(i^\ell, s^\ell)\}_{1 \leq \ell \leq n}$ satisfaisant (1.5.12) et (1.5.13). La fonction objectif \mathcal{J}^n du problème avec une population finie est donnée par :

$$\mathcal{J}^n(\mathbf{i}, \mathbf{s}) := \frac{1}{n} \sum_{\ell=1}^n \int_0^T c(t, i_t^\ell, s_t^\ell) dt + g(i_T^\ell, s_T^\ell) + \sum_{i,j \in I, i \neq j} \sum_{k=1}^{N_T^n-1} \sum_{p=0}^{N_s^n-1} L \left(\frac{Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s})}{\Delta t} \right) \Delta t Q_i^{k,p}(\mathbf{i}, \mathbf{s}),$$

où $Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s})$ est la proportion de PEVs dont la variable d'état discrète saute de i vers j , parmi les processus avec un état dans $\{i\} \times [y_p^n, y_{p+1}^n)$, à l'instant t_k^n , et $Q_i^{k,p}(\mathbf{i}, \mathbf{s})$ est la proportion de PEVs dont la variable d'état est dans $\{i\} \times [y_p^n, y_{p+1}^n)$ parmi l'ensemble de la population à l'instant t_k^{n-} (avant de potentiels sauts). L'objectif de ce chapitre est d'étudier la limite de champ moyen du problème de contrôle optimal avec population finie

suivant :

$$\inf_{(\mathbf{i}, \mathbf{s})} \mathcal{J}^n(\mathbf{i}, \mathbf{s}). \quad (P_n)$$

Le principal résultat de ce chapitre est le théorème suivant, où le Problème (P) est introduit au Chapitre 2.

Theorem 1.5.3. *Il existe $K > 0$ tel que, pour tout $n \in \mathbb{N}$,*

$$\text{val}(P_n) - \frac{K}{n^{\frac{1}{3}}} \leq \text{val}(P) \leq \liminf_{N \rightarrow \infty} \text{val}(P_N).$$

Pour obtenir la seconde inégalité, c'est à dire $\text{val}(P) \leq \liminf_{N \rightarrow \infty} \text{val}(P_N)$, nous introduisons pour tout ensemble de n trajectoires (\mathbf{i}, \mathbf{s}) , la distribution empirique m^n de la population de processus et la mesure empirique E^n des sauts :

$$m^n(t) := \frac{1}{n} \sum_{\ell=1}^n \delta_{(i_\ell^\ell, s_\ell^\ell)} \text{ et } E_{i,j}^n := \frac{1}{n} \sum_{\ell=1}^n \sum_{k=1}^{N_T^n-1} \mathbb{1}_i(i_{t_k}^\ell) \mathbb{1}_j(i_{t_k}^\ell) \delta_{t_k}^\ell \otimes \delta_{s_{t_k}^\ell}. \quad (1.5.14)$$

Nous montrons que, pour tout (\mathbf{i}, \mathbf{s}) , (m^n, E^n) est une solution faible de l'équation de continuité

$$\partial_t m_i^n + \partial_s(m_i^n b_i) = - \sum_{i,j \in I, j \neq i} (E_{i,j}^n - E_{j,i}^n) \text{ on } I \times (0, T) \times (0, 1). \quad (1.5.15)$$

Nous introduisons la fonction de coût

$$\begin{aligned} J^n(m, E) := & \sum_{i \in I} \int_0^T \int_0^1 c_i(t, s) m_i(t, ds) dt + \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds) \\ & + \sum_{i,j \in I, i \neq j} \sum_{k=1}^{N_T^n-1} \sum_{p=0}^{N_s^n-1} L \left(\frac{E_{i,j}(\{t_k\}, [y_p^n, y_{p+1}^n])}{\Delta t^n m_i(t_k^{n-}, [y_p^n, y_{p+1}^n])} \right) \Delta t^n m_i(t_k^{n-}, [y_p^n, y_{p+1}^n]). \end{aligned}$$

En appliquant le principe de superposition, qui est une variante de celui établie dans [62], nous montrons que le problème

$$\begin{aligned} & \inf_{(m, E)} J^n(m, E) \\ & \text{tel que } (m, E) \text{ est une solution faible de (1.5.15) et } m \text{ satisfait (1.5.2)}, \end{aligned} \quad (1.5.16)$$

est équivalent au Problème (P_n) . L'utilisation d'arguments de compacité et de continuité semi-inférieure permet d'obtenir :

$$\text{val}(P) \leq \liminf_{n \rightarrow \infty} \inf_{(m, E)} J^n(m, E) = \liminf_{n \rightarrow \infty} \inf_{(\mathbf{i}, \mathbf{s})} \mathcal{J}^n(\mathbf{i}, \mathbf{s}). \quad (1.5.17)$$

De plus, la première inégalité dans le Théorème 1.5.3, c'est à dire $\text{val}(P_n) - \frac{K}{n^{\frac{1}{3}}} \leq \text{val}(P)$, est obtenue en utilisant les résultats de régularité établis au Chapitre 2. Nous montrons qu'en implémentant un contrôle de type champ moyen α à une population de n processus hybrides, nous pouvons obtenir l'estimation suivante entre la distribution empirique de la population finie m^n et la distribution optimale du problème champ moyen m ,

$$\mathcal{W}(m(t), m^n(t)) = \mathcal{O}\left(\frac{1}{n^{\frac{1}{3}}}\right) \quad \forall t \in [0, T],$$

où \mathcal{W} est la distance de Wasserstein-1. L'inégalité précédente et l'équivalence entre les problèmes (1.5.16) et (P_n) permettent d'obtenir l'inégalité

$$\inf_{(\mathbf{i}, \mathbf{s})} \mathcal{J}^n(\mathbf{i}, \mathbf{s}) - \frac{C}{n^{\frac{1}{3}}} \leq \inf_{(m, \alpha)} J(m, \alpha).$$

La convergence de la valeur du Problème (P_n) vers la valeur du Problème (P) est une conséquence de l'inégalité précédente et de (1.5.17). De plus, par des arguments de compacité, nous montrons qu'une sous-suite des solutions $\{(m^n, E^n)\}_n$ du Problème (1.5.16) converge faiblement vers $(m, \alpha m)$, où (m, α) est une solution du Problème (P) .

1.5.5 Chapitre 6 : Décomposition d'un problème stochastique agrégatif convexe de grande dimension

Nous considérons le cadre des problèmes de contrôle stochastique convexe et de grande dimension, dans lesquels les contrôles sont agrégés dans la fonction de coût. En guise de première contribution, nous introduisons un problème modifié, dont le contrôle optimal est, sous certaines hypothèses standards, une solution ε -optimal du problème originel. La seconde contribution est la présentation d'un algorithme décentralisé dont la convergence vers une solution du problème modifié est démontrée. Enfin, nous étudions l'application des outils développés dans un contexte d'ingénierie, en étudiant un problème de coordination de grandes populations d'appareils de consommation domestiques contrôlés par thermostat (TCL). Alors qu'au Chapitre 3 un algorithme déterministe et centralisé est proposé pour résoudre un problème de contrôle optimal en dimension infinie avec des contraintes d'état agrégées (introduit au chapitre 2), ce chapitre développe un algorithme stochastique et décentralisé. Enfin, contrairement au modèle présenté au chapitre 2, l'homogénéité des agents n'est pas supposée. Ce travail est motivé par ses applications potentielles au contrôle distribué de flexibilités diffuses, afin de soutenir l'équilibre du système électrique dans un contexte d'augmentation des énergies renouvelables. Ce chapitre est lié aux travaux de De Paola *et al.* [84], dans lequel une solution distribuée est présentée pour le contrôle d'une population de $n = 2 \times 10^7$ réfrigérateurs, assurant de la réponse en fréquence et du déplacement de charge. L'application de notre travail à ce problème de contrôle prend également en compte une grande population de TCL qui, comme extension au modèle de [84], ont une dynamique stochastique. L'approche proposée est capable de minimiser les coûts globaux du système de manière distribuée, chaque TCL déterminant son profil optimal de consommation d'énergie en réponse à un signal de prix.

Le problème étudié dans ce chapitre appartient à la classe du contrôle stochastique : on cherche des stratégies minimisant l'espérance d'une fonction objective en prenant en compte certaines contraintes. Le principe de la programmation dynamique [28] et la SDDP ne sont pas adaptés à notre situation en raison du grand nombre d'agents. Afin de traiter ce type de problèmes de grande dimension, il est naturel d'étudier les techniques de décomposition dans l'esprit de la DADP [123, 172]. Cette approche se caractérise par une décomposition par les prix du problème, où les contraintes stochastiques sont projetées sur des sous-espaces choisis tels que le multiplicateur de Lagrange associé est adapté à la programmation dynamique. Le multiplicateur optimal est ensuite estimé numériquement en implémentant l'algorithme d'Uzawa. Dans ce chapitre, dans la même veine que la DADP, nous proposons une approche de décomposition par les prix mais limitée aux prix déterministes. Cette nouvelle approche tire parti du grand nombre d'agents, permettant d'introduire un problème auxiliaire dans lequel le terme de couplage est purement déterministe.

Ce chapitre a pour but de résoudre le Problème (P_1) , impliquant un grand nombre d'agents n indexés par $i \in \{1, \dots, n\}$, et de la forme :

$$(P_1) \quad \begin{cases} \min_{u \in \mathcal{U}} J(u) \\ J(u) := \mathbb{E} \left[F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i(\omega^i, \omega^{-i}) \right) + \frac{1}{n} \sum_{i=1}^n G_i(u^i(\cdot, \omega^{-i}), \omega^i) \right]. \end{cases}$$

Le bruit $\omega := (\omega^1, \dots, \omega^n)$ appartient à $\Omega := \prod_{i=1}^n \Omega^i$, où $(\Omega^i, \mathcal{F}^i, \mu^i)$ est un espace probabilitisé, et $(\Omega, \mathcal{F}, \mu)$ est l'espace probabilitisé produit correspondant. Chaque variable de décision u^i est \mathcal{F} -mesurable (et non uniquement \mathcal{F}^i -mesurable), de carré sommable à valeur dans l'espace de Hilbert \mathbb{U} . On définit $\mathcal{U} := \prod_{i=1}^n \mathcal{U}_i$, où \mathcal{U}_i est, pour $i \in \{1, \dots, n\}$, un sous ensemble de $L^2(\Omega, \mathbb{U})$ convexe et fermé. La fonction de coût est la somme d'un terme de couplage $F_0 : \mathbb{U} \rightarrow \mathbb{R}$, qui est fonction de l'agrégat des stratégies $\frac{1}{n} \sum_{i=1}^n u^i$, et de termes locaux, qui sont fonctions des décisions locales u^i et du bruit local ω^i , avec $G_i : L^2(\Omega^i, \mathbb{U}) \times \Omega^i \rightarrow \mathbb{R}$. La difficulté dans la résolution numérique du Problème (P_1) est liée à l'aléa du terme agrégatif $\frac{1}{n} \sum_{i=1}^n u^i$, qui intervient dans la fonction couplante F_0 . Nous introduisons l'ensemble des contrôles décentralisés :

$$\hat{\mathcal{U}} := \prod_{i=1}^n \hat{\mathcal{U}}_i, \text{ où } \hat{\mathcal{U}}_i := \{u^i \in \mathcal{U}_i \mid u^i \text{ is } \mathcal{T}^1 \otimes \dots \otimes \mathcal{F}^i \otimes \mathcal{T}^{i+1} \otimes \dots \otimes \mathcal{T}^n - \text{mesurable}\}, \quad (1.5.18)$$

où \mathcal{T}^j est la tribu triviale $\{\emptyset, \Omega^j\}$. Si le contrôle u est décentralisé, puisque la variance d'une somme de variables aléatoires indépendantes est égale à la somme des variances de ces variables, le terme agrégé $\frac{1}{n} \sum_{i=1}^n u^i$ peut être approximé par $\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i$ pour n suffisamment grand. L'approximation suivante du Problème (P_1) est

introduite :

$$(P_2) \quad \begin{cases} \min_{u \in \mathcal{U}} \bar{J}(u) \\ \bar{J}(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i \right) + \frac{1}{n} \mathbb{E} \left[\sum_{i=1}^n G_i(u^i(\cdot, \omega^{-i}), \omega^i) \right]. \end{cases}$$

Nous montrons tout d'abord, sans perte d'optimalité du Problème (P_2) , que l'on peut restreindre l'ensemble des contrôles admissibles \mathcal{U} à $\hat{\mathcal{U}}$. Nous désignons par (\hat{P}_2) , le problème d'optimisation lié à la minimisation de la fonction \bar{J} sur l'ensemble $\hat{\mathcal{U}}$. Le Problème (\hat{P}_2) peut être écrit comme :

$$(\hat{P}_2) \quad \begin{cases} \min_{u \in \hat{\mathcal{U}}, v \in \mathbb{U}} \bar{J}(u, v), \\ \bar{J}(u, v) := F_0(v) + \frac{1}{n} \mathbb{E} \left[\sum_{i=1}^n G_i(u^i, \omega^i) \right], \\ \text{sous contrainte } \frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i - v = 0. \end{cases}$$

La première contribution de ce chapitre est d'établir que sous certaines hypothèses de convexité et de régularité sur F_0 et $(G_i)_{i \in \{1, \dots, n\}}$, toute solution du Problème (P_2) est une ε_n -solution de (P_1) , avec $\varepsilon_n \rightarrow 0$ quand $n \rightarrow \infty$.

De plus, nous montrons qu'une approche de décomposition par les prix du Problème (P_2) , basée sur la formulation du Problème (\hat{P}_2) , peut être résolue par programmation dynamique, puisque la minimisation du Lagrangien satisfait le principe de la programmation dynamique. Le calcul du coût dual de (\hat{P}_2) étant coûteux, nous proposons les algorithmes *Stochastic Uzawa* et *Sampled Stochastic Uzawa* qui s'appuient sur l'algorithme de Robbins-Monroe, dans l'esprit du gradient stochastique. La convergence de ces deux algorithmes est établie, en s'appuyant sur la preuve fournie par [119] pour la convergence du gradient stochastique dans un espace de Hilbert. Nous vérifions l'efficacité de l'algorithme *Stochastic Uzawa* dans un cadre Linéaire Quadratique Gaussien, et nous appliquons l'algorithme *Sampled Stochastic Uzawa* à un problème de coordination d'une population importante de TCLs, en s'inspirant des travaux d'A. De Paola *et al.* [84].

1.5.6 Chapitre 7 : Un algorithme décentralisé pour un problème de contrôle champ moyen de processus de Markov déterministes par morceaux

Ce chapitre vise à étendre l'approche décentralisée développée au Chapitre 6 à un cadre non convexe, avec des contrôles non bornés, permettant une dynamique plus réaliste modélisant l'évolution de l'état de chaque agent. En particulier, nous nous intéressons au contrôle agrégé des processus de Markov déterministes par morceaux (PDMP) dans un cadre de contrôle champ moyen. Les PDMP sont introduits dans [83] comme une classe de modèles stochastiques non diffusifs, mélangeant des sauts aléatoires et une dynamique déterministe entre les sauts. Notre première contribution réside dans l'originalité de la preuve de l'existence d'un point selle qui suit une méthode complètement différente que celle développée dans le Chapitre 6. En particulier, nous utilisons les résultats de régularité sur la solution de l'équation de Hamilton-Jacobi qui apparaît dans le contrôle optimal des PDMP obtenus dans les Chapitres 2 et 4, afin de montrer l'existence d'un point selle de notre problème Lagrangien. Notre deuxième contribution consiste à prouver que l'algorithme d'Uzawa stochastique proposé dans le Chapitre 6 fournit toujours une suite convergente de contrôles, dans ce cadre spécifique impliquant la dynamique des PDMPs qui ne respecte pas les conditions de convexité exploitées au Chapitre 6. Enfin, nous présentons une application de cette méthode et de l'algorithme à la recharge intelligente d'une flotte de PEVs par un agrégateur afin d'illustrer la performance et l'intérêt de l'approche. Similairement aux Chapitres 2-5, l'état de chaque véhicule électrique est composé d'une variable discrète et d'une variable continue et une hypothèse de type champ moyen sur la population est prise en compte. Cependant, alors que le problème de contrôle champ moyen des variables d'état hybrides dans les Chapitres 2 et 5 est formulé comme un problème de contrôle optimal d'une équation différentielle aux dérivées partielles, et est approché numériquement par un algorithme déterministe et centralisé, ce chapitre se concentre sur les techniques d'optimisation stochastiques et s'appuie sur l'algorithme décentralisé et stochastique proposé au Chapitre 6 pour résoudre ce problème de contrôle champ moyen.

Plus précisément, nous considérons l'intervalle de temps $[0, T]$ avec $T > 0$ et une population de n processus indépendants identiquement distribués, contrôlés par un planificateur central via un contrôle commun α qui

appartient à l'espace de contrôle feedback \mathbb{A} . L'état d'un agent, contrôlé par l'intensité de sauts α , est donné à l'instant t par $X_t^\alpha = (Y_t^\alpha, Z_t^\alpha)$ avec $Y_t^\alpha \in I$ et $Z_t^\alpha \in [0, 1]$. Comme au Chapitre 2, la variable discrète Y^α représente le mode de chargement du véhicule et la variable continue Z^α représente le niveau de batterie. Un processus X^α est un PDMP(b, α) si Y^α est un processus à sauts, à valeurs dans I , sautant spontanément à des instants aléatoires $\{T_k^\alpha\}_{k \in \mathbb{N}}$, donnés par un processus de Poisson avec intensité α , alors que Z^α a une dynamique déterministe entre deux sauts consécutifs $Z_t^\alpha = \phi_{T_k^\alpha, Z_{T_k^\alpha}^\alpha}(Y_{T_k^\alpha}^\alpha, t)$ pour $t \in [T_k^\alpha, T_{k+1}^\alpha)$, où le flot ϕ est l'unique solution de l'EDO :

$$\partial_t \phi_{\tau, z}(j, t) = b(j, \phi_{\tau, z}(j, t)) \text{ avec } \phi_{\tau, z}(j, \tau) = z,$$

et b est une fonction donnée. La fonction objectif J est définie par :

$$J(\alpha) := \int_0^T f(t, \mathbb{E}[p(t, X_t^\alpha)]) dt + \int_0^T \mathbb{E} \left[c(t, X_t^\alpha) + \sum_{j \in I} L(\alpha_j(t, X_t^\alpha)) \right] dt + \mathbb{E}[g(X_T^\alpha)],$$

où X^α est un PDMP(b, α) et la fonction f représente un coût de couplage, dépendant de l'espérance de $p(t, X_t^\alpha)$ (p étant une fonction continûment dérivable). Ce chapitre est dédié au problème suivant

$$\min_{\alpha \in \mathbb{A}} J(\alpha). \tag{P_{pdm}}_p$$

La fonction f est supposée non linéaire, et ainsi la fonction de coût J est également non linéaire par rapport à $\mathbb{E}[p(t, X_t^\alpha)]$. Ainsi, le Problème (P_{pdm}) va au-delà du cadre du contrôle optimal de PDMP. L'existence d'une solution au Problème (P_{pdm}) est établie. Nous proposons une approche numérique pour résoudre le Problème (P_{pdm}), en appliquant l'algorithme *Stochastic Uzawa* introduit au Chapitre 6. Cet algorithme permet de générer une suite de contrôles $\{\alpha^k\}_k$ dont la convergence vers une solution au Problème (P_{pdm}) est démontrée. Ce résultat ne peut être obtenu par une application directe des résultats de convergence établis au Chapitre 6, qui s'appuient sur des hypothèses supplémentaires de convexité qui ne sont pas vérifiées dans le cadre présent, qui implique des contrôles non bornés et des dynamiques de PDMP.

List of publications, proceedings and preprints

- A.Seguret, C. Wan and C. Alasseur. "A mean field control approach for smart charging with aggregate power demand constraints". In *2021 IEEE PES Innovative Smart Grid Technologies Europe (ISGT Europe)* (pp. 01-05). IEEE. 2021
- A. Seguret, T. Le Corre, N. Oudjane. "A decentralized algorithm for a Mean Field Control problem of Piecewise Deterministic Markov Processes". In <https://hal.science/hal-03910622/>, 2022.
- A.Seguret. "An optimal control problem for the continuity equation arising in smart charging". In <https://hal.science/hal-03353917v3>, 2022.
- A.Seguret. "Mean field approximation of an optimal control problem for the continuity equation arising in smart charging". In <https://hal.science/hal-03910634/>, 2022.
- A. Seguret, C. Alasseur, J.F Bonnans, A. De Paola, N. Oudjane and V. Trovato. "Decomposition of convex high dimensional aggregative stochastic control problems". In *Applied Mathematics & Optimization*, forthcoming, 2023.
- A.Seguret. "Regularity of Lagrange multipliers for a mean field control problem arising in smart charging", Unpublished Manuscript, 2023.

Chapter 2

Optimality conditions of an optimal control problem for the continuity equation arising in smart charging

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2.1 Introduction

This article studies the optimal control of a first order continuity equation with a reaction term under state constraints. Let us consider a finite time interval $[0, T]$ and a mixed state space equal to the product $[0, 1] \times I$, where I is a finite space, the cardinality of which is denoted by $|I|$. The continuity equation is given by:

$$\begin{aligned} \partial_t m_i(t, s) + \partial_s(m_i(t, s)b_i(s)) &= - \sum_{j \neq i} (\alpha_{i,j}(t, s)m_i(t, s) - \alpha_{j,i}(t, s)m_j(t, s)) \quad (i, t, s) \in I \times (0, T) \times (0, 1), \\ m_i(0, s) &= m_i^0(s) \quad (i, s) \in I \times [0, 1], \end{aligned} \tag{2.1.1}$$

where $m : [0, T] \rightarrow \mathcal{P}([0, 1] \times I)$ is a curve of probability distribution, $m^0 \in \mathcal{P}([0, 1] \times I)$ is the given initial distribution, $b : I \times [0, 1] \rightarrow \mathbb{R}$ is a velocity field, and the control $\alpha : I \times I \times [0, T] \times [0, 1] \rightarrow \mathbb{R}_+$ is a jump intensity. The notion of weak solution of (2.1.1) is specified in Definition 2.3.1. We further assume that b vanishes at the boundary of $[0, 1]$ so that the mass conservation in $I \times [0, 1]$ is guaranteed without forcing a boundary condition.

The distribution m is subject to the following congestion constraints:

$$m_i(t, [0, 1]) \leq D_i(t) \quad \forall (i, t) \in I \times [0, T], \quad (2.1.2)$$

where $D : [0, T] \rightarrow \mathbb{R}_+^*$ is given. The objective function J is defined as follows:

$$J(m, \alpha) := \sum_{i \in I} \int_0^T \int_0^1 (c_i(t, s) + \sum_{j \in I, j \neq i} L(\alpha_{ij}(t, s))) m_i(t, ds) dt + \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds). \quad (2.1.3)$$

Our purpose is to study the optimization problem:

$$\inf_{m, \alpha} J(m, \alpha), \text{ where } (m, \alpha) \text{ is a weak solution of (2.1.1) and satisfies (2.1.2)}. \quad (2.1.4)$$

Our work is initially motivated by the optimal charging of a population of PEVs controlled by a central planner. The continuous variable $s \in [0, 1]$ in (2.1.1) represents the level of battery of a PEV. The discrete variable $i \in I$ represents the mode of charging (e.g. idling, charging, discharging, etc...). For any $(t, s, i) \in [0, T] \times [0, 1] \times I$, the value $m_i(t, s)$ represents the proportion of PEVs at time t at state (s, i) . The given velocity field $b_i(s)$ denotes the power of charge or discharge of a PEV in mode i and with battery level s . For any $(t, s, i, j) \in [0, T] \times [0, 1] \times I \times I$, the value $\alpha_{i,j}(t, s)$ denotes the jump intensity of PEVs from the state (s, i) to the state (s, j) at time t . The control α is required to be a non-negative measurable function. The congestion constraint (2.1.2) avoids high demand of energy at each moment over the period. The cost function L penalizes high values of α . It aims at avoiding the synchronization of jumps of the PEVs and the consequent instability of the electrical network. The value $c_i(t, s)$ corresponds to the cost per PEV to pay at time $t \in [0, T]$ at state (s, i) ; $g_i(s)$ is the final cost per PEV to pay at state (s, i) . Numerical results of Problem (2.1.4) applied to smart charging can be found in [238]. Problem (2.1.4) can be interpreted heuristically as an approximation of the limit case $N \rightarrow \infty$ of an optimal switching problem of N PEVs. Combinatorial techniques as well as optimal control tools may fail to solve problems with large population of PEVs, due to the curse of dimensionality [20]. To overcome these difficulties, a continuum of PEVs can be considered, leading to the techniques of the optimal control of PDE. The connection between the finite population problem and the mean field problem is addressed in a companion paper [234] by the author.

2.1.1 Contributions, methodology and literature

Contributions This paper makes three main contributions.

First we prove the existence of solutions of Problem (2.1.4).

Second, we derive optimality conditions of Problem (2.1.4). More precisely, let H denote the Fenchel conjugate of L and H' denote the derivative of H , we show that, if (m, α) is a solution of (2.1.4), then there exists a pair (φ, λ) such that, for any $i, j \in I$, $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$, and (φ, λ, m) is a weak solution of the following system:

$$\left\{ \begin{array}{ll} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \lambda_i + \sum_{j \in I, j \neq i} H(\varphi_i - \varphi_j) = 0 & \text{on } (0, T) \times (0, 1) \times I \\ \partial_t m_i + \partial_s (m_i b_i) + \sum_{j \neq i} H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j = 0 & \text{on } (0, T) \times (0, 1) \times I \\ m_i(0, s) = m_i^0(s), \varphi_i(T, s) = g_i(s) & \text{on } (0, 1) \times I \\ \int_0^1 m_i(t, ds) - D_i(t) \leq 0, \lambda \geq 0 & \text{on } [0, T] \times I \\ \sum_{i \in I} \int_0^T \left(\int_0^1 m_i(t, ds) - D_i(t) \right) \lambda_i(dt) = 0. & \end{array} \right. \quad (2.1.5)$$

The function φ is the Lagrange multiplier associated to the dynamic constraint (2.1.1), and the measure λ is associated to the congestion constraint (2.1.2). The first equation in (2.1.5) is a backward Hamilton-Jacobi equation. The existence, uniqueness and characterization of weak solutions of the backward Hamilton-Jacobi equation are investigated in the paper. The second equation in (2.1.5) is a forward continuity equation, where the control α , defined by $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$, is optimal. The measure λ is non-negative and finite. The last

equality in (2.1.5) ensures that the congestion constraint (2.1.2) is satisfied.

Third, we obtain regularity property for any weak solution (λ, φ, m) of (2.1.5). We prove that, under suitable assumptions on the data b, g and c , the multiplier φ is in $\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)$. As a result, the optimal control α is bounded and Lipschitz continuous in space uniformly w.r.t. the time variable and the measure m is in $\text{Lip}([0, 1], \mathcal{P}([0, 1] \times I))$. We show that if the initial distribution m^0 is absolutely continuous w.r.t. the Lebesgue measure and has a smooth density, then the measure m is absolutely continuous w.r.t. the Lebesgue measure and has Lipschitz continuous density.

Methodology The existence of an optimal solution is established by compactness arguments (see for e.g. [21]). We adopt a duality approach to obtain (2.1.5). More explicitly, we relax the dynamics (2.1.1) and the congestion constraint (2.1.2). The resulting relaxed problem is then expressed as the dual of another convex problem. We show that the system (2.1.5) is the optimality condition of Problem (2.1.4).

Literature Solving the optimal control of a Fokker-Planck equation by means of the duality theory has been well known since decades [110, 265]. Our work follows the method developed in the seminal work by Benamou and Brenier [21] for optimal transport problems. In [21], a continuity equation is controlled with initial and final constraints; optimality conditions are obtained as a system of PDEs close to (2.1.5). Similar method and results also in optimal transport are derived in [50]. More recently, this approach was applied to solve on optimal control problem of a Fokker-Planck equation under state constraints in the Wasserstein space [81, 82], where Lipschitz regularity results of the optimal control are proved.

The optimality conditions (2.1.5) typically arises in the Mean Field Game (MFG for short) Theory. This class of problems, introduced by Lasry and Lions [166, 167, 168] and Huang, Malhamé and Caines [145, 146], describes the interaction among a large population of identical and rational agents in competition. Mean Field Control (MFC for short) and MFG theories have been extensively used over the last few years as a mathematical tool in electrical engineering. More specifically, the optimal control of PDEs applied to smart charging can be found in [170, 241], and to the management of a population of thermostatically controlled loads in [122, 195].

Conversely, the duality approach is close to the so-called variational approach used in MFG theory in [53], where the weak solution of the MFG system is characterized as the minimizer of some optimal control of Hamilton-Jacobi and Fokker-Planck equations. This approach allows to use optimization techniques to prove the existence and uniqueness of the solution of MFG and MFC problems. We refer to [5, 25, 40, 54, 202] and the references therein. Besides, the variational approach allows to apply optimization algorithms to numerically solve MFG problems [23, 41, 40].

Note that different optimality conditions for control problems in the space of probability measures can be derived by using a kind of Pontryagin Maximum Principle [36, 35].

A particularity of this paper is to deal with a congestion constraint (2.1.2) on the measure. Two kinds of congestion effects are explored in the MFG and MFC frameworks. On the one hand, "soft congestion" increases the cost of velocity of the agents in areas with high density. On the other hand, "hard congestion" constraints impose density constraints, e.g. $m \leq \bar{m}$ at any point (t, s) . The variational approach yields good results when applied to MFC [5] and MFG with "soft congestion" in a stationary framework [105], as well as to MFG problems dealing with "hard congestion" constraints. This was first investigated in [230] where the density of the population did not exceed a given threshold, then in [190] where stationary second order MFGs were considered. In [55], a price which is imposed on the saturated zone to make the density satisfy the constraints is obtained. In the same vein as the work of Benamou and Brenier [21], "hard congestion" constraints were examined in optimal transport [45]. We highlight that our paper deals with aggregate "hard congestion" constraints on the measure m (2.1.2), i.e., our constraint is less restrictive than a constraint of the type $m \leq \bar{m}$ a.e..

Besides, we consider a mixed state space with continuous and discrete state variable. This setting has seldomly been investigated in the MFG literature. Indeed, the articles cited in the paragraphs above looked only at continuous state variables. The resulting continuity equation (2.1.1) contains a term of reaction, indicating mass transfers between states in I . Such PDEs also arised in [10] to model the mean field limit of Piecewise Deterministic Markov Processes (PDMP for short). The velocity was controlled in [10]. Here, we control the intensity of the jump α , but the velocity b is given. A MFG problem with discrete time and state space was explored in [129] and applied to socio-economic sciences in [127]. The uniqueness of the solution of a finite state MFG was discussed in [17] and the convergence of the N -player game to the mean field model as $N \rightarrow \infty$ was obtained in [128]. Mixed state spaces in a MFG framework were studied in [108, 107], where a major player can switch his state on a finite state space and minor players decide their stopping time. A MFG problem in a finite state space and discrete time settings with "hard congestion" was studied in [32], also by variational methods.

Some of our regularity results, namely that α is Lipschitz continuous w.r.t. s , and the density of m w.r.t. the Lebesgue measure is Lipschitz continuous when m^0 is absolutely continuous w.r.t. the Lebesgue measure, are unusual. We believe that it is mainly due to the linearity of the Hamilton-Jacobi equation w.r.t. $\partial_s \varphi$. These results will be used in a companion work [234] to quantify the mean field limit of the model. The time regularity of φ may not be improved as far as we have no more regularity results on λ . The function φ is discontinuous at each atom of the measure λ . Regularity results about the multiplier of the density constraint can be found in the literature: in [55], the authors showed some BV estimates on the pressure, whereas L^∞ estimates for the price were proved in [82] and [169] in the special case of a quadratic Hamiltonian in a MFG problem. The Sobolev regularity of the solution of a first order MFG was established in [177] and improved in [232]; also see [133].

Organization of the paper The paper is organized as follows. In Section 2.2, we present our assumptions and main results. The existence of a solution of Problem (2.1.4) is established in Section 2.3. In Section 2.4, which is independent of the other sections, we show the existence, uniqueness and regularity of weak solution of the Hamilton-Jacobi equation in (2.1.5). In Section 2.5, we return to Problem (2.1.4) and develop the duality approach. We formulate its Lagrangian relaxation and show it is the dual problem of another convex problem. We obtain the optimality conditions (2.1.5) of Problem (2.1.4) in Section 2.6. The Lipschitz continuity of the value of Problem (2.1.4) w.r.t. the data (m^0, D) is proved in Section 2.7. We recall basic statements about weak solutions of (2.1.1) in Appendix 2.8.

2.2 Assumptions and main results

2.2.1 Notations and Assumptions

Notations The space of Borel, positive and bounded measures on a space A is denoted by $\mathcal{M}^+(A)$ and the space of Borel probability measures on a space A is denoted by $\mathcal{P}(A)$. For any measure $\mu \in \mathcal{M}([0, T])$ and $0 \leq t_1 < t_2 \leq T$, we set $\int_{t_1}^{t_2} \mu(dt) := \mu([t_1, t_2])$. Given a set \mathcal{S} , for any function f defined on $I \times \mathcal{S}$ and any measure $\mu \in \mathcal{M}(\mathcal{S} \times I)$, we use the notations $f_i(x) := f(i, x)$ for any $(i, x) \in I \times \mathcal{S}$ and $\mu_i(\mathcal{S}) := \mu(\{i\} \times \mathcal{S})$ for any $(i, \mathcal{S}) \in I \times \mathcal{B}(\mathcal{S})$, where $\mathcal{B}(\mathcal{S})$ denotes the Borel algebra. Similarly, for any function g defined on $I \times I \times \mathcal{S}$ and any measure $\nu \in \mathcal{M}(\mathcal{S} \times I \times I)$, we use the notations $g_{i,j}(x) := g(i, j, x)$ for any $(i, j, x) \in I \times I \times \mathcal{S}$ and $\nu_{i,j}(\mathcal{S}) := \nu(\{i\} \times \{j\} \times \mathcal{S})$ for any $(i, j, \mathcal{S}) \in I^2 \times \mathcal{B}(\mathcal{S})$. If \mathcal{S} is a metric space, let $\text{Lip}(\mathcal{S})$ denote the vector space of bounded and Lipschitz continuous maps $f : \mathcal{S} \rightarrow \mathbb{R}$. For any $\mu \in C^0([0, T], \mathcal{P}([0, 1]))$, let $L_\mu^2([0, T] \times \mathbb{R}) := \{f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, \int_0^T \int_0^1 f(t, s)^2 \mu(t, ds) dt < +\infty\}$. We denote by \mathcal{W} the Wasserstein distance on $\mathcal{P}([0, 1] \times I)$, defined by $\mathcal{W}(\mu, \rho) := \sup \{\sum_{i \in I} \int_0^1 \varphi(\mu - \rho) \mid \varphi \text{ is } 1\text{-Lipschitz from } [0, 1] \times I \text{ to } \mathbb{R}\}$. We recall that if a function φ is 1-Lipschitz continuous from $[0, 1] \times I$ to \mathbb{R} , then $|\varphi(x, i) - \varphi(x, j)| \leq 1$ for any $i, j \in I$. The dual of a normed space X is denoted by X^* . We consider the space $\Omega := \mathcal{P}([0, 1] \times I) \times C^0(I \times [0, T], \mathbb{R}_+^*)$. For any $\varepsilon > 0$, we define the subspace Ω_ε of Ω , by

$$\Omega_\varepsilon := \left\{ (\mu, f) \in \Omega \mid \varepsilon < \inf_{t \in [0, T], i \in I} f_i(t) - \mu_i([0, 1]) \right\}. \quad (2.2.1)$$

We consider the following spaces for any $\delta > 0$:

$$\begin{aligned} \mathcal{M}_\delta^+([0, T] \times I) &:= \{ \mu \in \mathcal{M}^+([0, T] \times I) \mid \mu([0, T] \times I) \leq \delta \}, \\ C_\delta^0([0, T] \times I, \mathbb{R}_+) &:= \left\{ f \in C^0([0, T] \times I, \mathbb{R}_+) \mid \int_0^T \sum_{i \in I} f_i(t) dt \leq \delta \right\}. \end{aligned}$$

For any $\delta > 0$, the space $C_\delta^0([0, T] \times I, \mathbb{R}_+)$ can be considered as a subspace of $\mathcal{M}_\delta^+([0, T] \times I)$ in the sense that, for any $f \in C_\delta^0([0, T] \times I, \mathbb{R}_+)$, we have $f\mathcal{L} \in \mathcal{M}_\delta^+([0, T] \times I)$, where \mathcal{L} is the Lebesgue measure on $[0, T]$.

Assumptions The following assumptions are in force throughout the paper.

1. For any $i \in I$, $b_i \in C^1(\mathbb{R})$ with $b_i(s) = 0$ for any $s \notin (0, 1)$.
2. The initial distribution of (2.1.1) $m^0 \in \mathcal{P}(\mathbb{R} \times I)$ is such that $\text{supp}(m_i^0) \subset [0, 1]$ for any $i \in I$.
3. There exists $\varepsilon^0 > 0$ such that the parameter D of (2.1.2) and m^0 satisfy $(m^0, D) \in \Omega_{\varepsilon^0}$.
4. For any $i \in I$, it is assumed that $c_i \in C^1([0, T] \times [0, 1])$ and $g_i \in C^1([0, 1])$.

5. $L : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ is a convex function, defined by:

$$L(x) := \begin{cases} l(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $l \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing strongly convex function bounded from above by a quadratic function. More explicitly, there exists $C > 0$ such that for any $x \in \mathbb{R}_+$:

$$\frac{x^2}{C} - C \leq l(x) \leq C(x^2 + 1),$$

where the first inequality is due to the strong convexity of l .

The function H being the Fenchel conjugate of L , by Assumption 5, H is non-decreasing, non-negative, and H' is Lipschitz continuous on \mathbb{R} . For any $\lambda \in \mathcal{M}^+([0, T] \times I)$, $\psi \in L^2([0, T] \times [0, 1] \times I)$, $(t, \tau, s, i, j) \in [0, T]^2 \times [0, 1] \times I^2$, we define H^λ by:

$$H^\lambda(i, j, t, \tau, s, \psi) := H\left((\psi_i - \psi_j)(\tau, S_i^{t,s}(\tau)) + \int_\tau^T (\lambda_i - \lambda_j)(dr)\right). \quad (2.2.2)$$

By Assumption 1, for any $s \in [0, 1]$, $i \in I$ and $t \in [0, T]$, there exists a unique $S_i^{t,s}$ satisfying the ODE below

$$\begin{cases} \frac{dS_i^{t,s}(\tau)}{d\tau} = b_i(S_i^{t,s}(\tau)), & \tau \in [0, T], \\ S_i^{t,s}(t) = s. \end{cases} \quad (2.2.3)$$

Since $b \in C^1([0, 1] \times I)$, we recall that the flow S_i satisfies the following equation for any $(i, \tau, t, s) \in I \times (0, T) \times (0, T) \times (0, 1)$:

$$\partial_x S_i^{\tau,x}(t) = \exp\left(\int_\tau^t b'_i(S_i^{\tau,x}(r))dr\right). \quad (2.2.4)$$

Remark 2.2.1. *The main role of Assumptions 1 and 2 is to ensure that the support of the weak solution of (2.1.1) is contained in $[0, 1]$ over the period $[0, T]$ (cf. Lemma 2.8.1 in Appendix 2.8). Assumption 3 provides an estimate on $\lambda([0, T] \times I)$ for any weak solution (φ, λ, m) of (2.1.5) and ensures that the control $\alpha = 0$ is an admissible control. Correspondingly, the feasible set of Problem (2.1.4) is not empty. Regularity results of the weak solutions of the system (2.1.5) are derived thanks to the assumptions formulated on c and g in Assumption 4.*

2.2.2 Main results

We introduce, for a given $\lambda \in \mathcal{M}^+([0, T] \times I)$, the Hamilton-Jacobi equation on $(0, T) \times (0, 1) \times I$:

$$\begin{aligned} -\partial_t \varphi_i(t, s) - b_i(s) \partial_s \varphi_i(t, s) - c_i(t, s) - \lambda_i(t) + \sum_{j \in I, j \neq i} H((\varphi_j - \varphi_i)(t, s)) &= 0 & (t, s, i) \in (0, T) \times (0, 1) \times I, \\ \varphi_i(T, s) = g_i(s) & & (s, i) \in [0, 1] \times I. \end{aligned} \quad (2.2.5)$$

We define the function \tilde{A} for any $(\varphi, \lambda) \in (\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)) \times \mathcal{M}^+([0, T] \times I)$ by:

$$\tilde{A}(\varphi, \lambda) := \sum_{i \in I} \int_0^1 -\varphi_i(0^+, s) m_i^0(ds) + \int_0^T D_i(t) \lambda_i(dt). \quad (2.2.6)$$

The following Theorem summarizes the main results of the paper.

Theorem 2.2.1. *Problem (2.1.4) has a solution. Furthermore, the minimizers have the following properties:*

1. *If (m, α) is a minimizer of Problem (2.1.4) and $(\varphi, \lambda) \in (\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)) \times \mathcal{M}^+([0, T] \times I)$ is such that $\tilde{A}(\varphi, \lambda) = -J(m, \alpha)$, then (φ, λ, m) is a weak solution of (2.1.5) in the sense of Definition 2.6.1, and $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ on $\{m_i > 0\}$ for any $i, j \in I$.*

2. Conversely, if (φ, λ, m) is a weak solution of (2.1.5) in the sense of Definition 2.6.1, then there exists α , defined for any $i, j \in I$ by: $\alpha_{i,j} := H'(\varphi_i - \varphi_j)$ on $\{m_i > 0\}$, such that (m, α) is a minimizer of (2.1.4) and $\tilde{A}(\varphi, \lambda) = -J(m, \alpha)$.
3. If (m, α) is a minimizer of Problem (2.1.4), then for any $i, j \in I$ $\alpha_{i,j}$ and $\partial_s \alpha_{i,j}$ are both in $L^\infty([0, T] \times [0, 1])$, and $m \in \text{Lip}([0, T], \mathcal{P}([0, 1] \times I))$.

The existence of a solution of Problem (2.1.4) is stated in Lemma 2.3.3. The characterization of a solution, i.e., Theorem 2.2.1.1-2, is given by Theorem 2.6.1. For this, we adopt a duality approach which is developed in Section 2.5. In particular, we introduce a convex problem, the dual of which is Problem (2.1.4) up to a change of variable (cf. Theorem 2.5.1). The Lipschitz continuity of m stated in Theorem 2.2.1.3 is deduced from the regularity of φ , which is derived in Section 2.4, and Proposition 2.8.1. The proof of Theorem 2.2.1 is finally given in Section 2.6.2.

Further regularity results on the solution of Problem (2.1.4) are obtained with additional conditions on the initial distribution m^0 .

Proposition 2.2.1. *If the initial distribution m^0 is absolutely continuous w.r.t. the Lebesgue measure with a density in $C^1([0, 1] \times I)$, then any solution (m, α) of Problem (2.1.4) is such that m is absolutely continuous w.r.t. the Lebesgue measure on $[0, T] \times [0, 1]$, and has a density in $\text{Lip}([0, T] \times [0, 1] \times I)$.*

The proof of Proposition 2.2.1 is given in Section 2.6.2.

2.3 Existence of an optimal solution

We introduce in this section a convex problem that is equivalent to (2.1.4). Standard compactness arguments are used to show the existence of an optimal solution (see for e.g. [21]). The definition of a weak solution of (2.1.1) is specified below.

Definition 2.3.1. *A pair (m, α) satisfies (2.1.1) in the weak sense if $t \in [0, T] \mapsto m(t, \cdot) \in \mathcal{P}(\mathbb{R} \times I)$ is continuous, for any $i, j \in I$ with $i \neq j$, it holds that $\alpha_{i,j} \in L^2_{m_i}([0, T] \times \mathbb{R})$ and for any test function $\phi \in C_c^\infty([0, T] \times \mathbb{R} \times I)$, we have:*

$$\begin{aligned} & \sum_{i \in I} \int_{\mathbb{R}} \phi_i(T, s) m_i(T, ds) - \phi_i(0, s) m_i^0(ds) \\ &= \int_0^T \int_{\mathbb{R}} \sum_{i \in I} (\partial_t \phi_i(t, s) + b_i(s) \partial_s \phi_i(t, s)) m_i(t, ds) + \sum_{j \in I, j \neq i} (\phi_j(t, s) - \phi_i(t, s)) \alpha_{i,j}(t, s) m_i(t, ds) dt. \end{aligned}$$

Remark 2.3.1. *Using Assumptions 1 and 2, Lemma 2.8.1 in Appendix 2.8 states that, for any weak solution (m, α) of (2.1.1) in the sense of Definition 2.3.1, the measure $m_i(t, \cdot)$ has its support contained in $[0, 1]$ for any $(t, i) \in [0, T] \times I$. Thus, we will consider throughout the paper only weak solutions (m, α) of (2.1.1) satisfying $m(t, \cdot) \in \mathcal{P}([0, 1] \times I)$ for any $t \in [0, T]$.*

Problem (2.1.4) being not convex w.r.t. the variables (m, α) , for any $i, j \in I$ with $i \neq j$, we make a change of variables $E_{i,j} := \alpha_{i,j} m_i$. We now rewrite the continuity equation (2.1.1):

$$\begin{aligned} \partial_t m_i(t, s) + \partial_s (m_i(t, s) b_i(s)) &= - \sum_{j \in I, j \neq i} (E_{i,j}(t, s) - E_{j,i}(t, s)) \quad (i, t, s) \in I \times (0, T) \times (0, 1) \\ m_i(0, s) &= m_i^0(s) \quad (i, s) \in I \times [0, 1], \end{aligned} \tag{2.3.1}$$

where $E_{i,j} \in \mathcal{M}^+([0, T] \times [0, 1])$ is such that $E_{i,j} \ll m_i$, with $\frac{dE_{i,j}}{dm_i} = \alpha_{i,j}$ and $\frac{dE_{i,j}}{dm_i} \in L^2_{m_i}([0, T] \times [0, 1])$. For any $(m^0, D) \in \Omega$ satisfying Assumption 3, we introduce the set:

$$\begin{aligned} \mathcal{S}(m^0, D) &:= \\ & \left\{ (m, E) \text{ such that } E_{i,j} \ll m_i \forall i, j \in I, (m, \alpha) \text{ satisfies (2.3.1) in the weak sense, where} \right. \\ & \left. \alpha_{i,j} := \frac{dE_{i,j}}{dm_i}, \text{ with additional constraints: } \int_0^1 m_i(t, ds) \leq D_i(t) \quad \forall (i, t) \in I \times [0, T], \text{ and } \frac{dE_{i,j}}{dm_i} \geq 0 \right\}. \end{aligned} \tag{2.3.2}$$

From Assumption 3, the set $\mathcal{S}(m^0, D)$ is not empty. Indeed, denoting by $(m, 0)$ a weak solution of (2.1.1) with control $\alpha \equiv 0$, the distribution m satisfies that, for any $(t, i) \in [0, T] \times I$, $m_i(t, [0, 1]) = m_i^0([0, 1]) < D_i(t)$. Thus, $(m, 0) \in \mathcal{S}(m^0, D)$. We define the function \tilde{J} on $\mathcal{S}(m^0, D)$ by:

$$\tilde{J}(m, E) := \sum_{i \in I} \int_0^T \int_0^1 c_i(t, s) m_i(t, ds) dt + \int_0^T \int_0^1 \sum_{i, j \in I, j \neq i} L \left(\frac{dE_{i,j}}{dm_i}(t, s) \right) m_i(t, ds) dt + \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds), \quad (2.3.3)$$

where the function L is defined in Assumption 5. Since $m(t)$ is a probability measure for any $t \in [0, T]$, by Assumption 4, the first and last integrals in (2.3.3) are well defined. Since $\frac{dE_{i,j}}{dm_i} \in L^2_{m_i}([0, T] \times [0, 1])$ and l is bounded by a quadratic function according to Assumption 5, the second integral in (2.3.3) is also well defined. Thus, for any $(m, E) \in \mathcal{S}(m^0, D)$, the quantity $\tilde{J}(m, E)$ is finite. The following optimization problem is considered:

$$\inf_{(m, E) \in \mathcal{S}(m^0, D)} \tilde{J}(E, m). \quad (2.3.4)$$

For any $\gamma > 0$, we define the subset $\mathcal{S}_\gamma(m^0, D)$ of $\mathcal{S}(m^0, D)$ by:

$$\mathcal{S}_\gamma(m^0, D) := \left\{ (m, E) \in \mathcal{S}(m^0, D) \mid \sum_{(i,j) \in I, i \neq j} \int_0^T \int_0^1 L \left(\frac{dE_{i,j}}{dm_i}(t, s) \right) m_i(t, ds) dt \leq \gamma \right\}. \quad (2.3.5)$$

For any $(m, E) \in \mathcal{S}_\gamma(m^0, D)$, the next lemma provides a Hölder regularity property on m .

Lemma 2.3.1. *For any $\gamma > 0$, there exists a positive constant C_γ such that, for any $(m, E) \in \mathcal{S}_\gamma(m^0, D)$, m is $\frac{1}{2}$ -Hölder continuous with constant C_γ from $[0, T]$ to $\mathcal{P}([0, 1] \times I)$ (endowed with the Wasserstein distance \mathcal{W}).*

Proof. The proof relies on classical arguments for the time regularity of weak solutions of the continuity equation (see [9]) and on the estimate, for any $t, \tau \in [0, T]$ with $t \leq \tau$,

$$\begin{aligned} \left(\int_t^\tau \int_0^1 E_{i,j}(t, ds) dt \right)^2 &\leq (\tau - t) \int_t^\tau \int_0^1 \sum_{i, j \in I, j \neq i} \left(\frac{dE_{i,j}}{dm_i}(t, s) \right)^2 m_i(\tau, ds) d\tau \\ &\leq C \int_0^T \int_0^1 L \left(\frac{dE_{i,j}}{dm_i}(t, s) \right) m_i(t, ds) dt + C \\ &\leq C_\gamma, \end{aligned}$$

where the first inequality is obtained by Cauchy-Schwarz inequality, $C_\gamma := C(\gamma + 1)$ and $C > 0$ is the constant defined in Assumption 5. \square

The next lemma is useful to show that any minimizing sequence of (2.3.4) is relatively compact.

Lemma 2.3.2. *For any $\gamma > 0$, the subset $\mathcal{S}_\gamma(m^0, D)$ is relatively compact in $C([0, T], \mathcal{P}([0, 1] \times I)) \times \mathcal{M}^+([0, T] \times [0, 1] \times I^2)$, where $\mathcal{M}^+([0, T] \times [0, 1] \times I^2)$ is endowed with the weak*-convergence.*

Proof. The compactness is derived from Lemma 2.3.1 and standard arguments in Optimal Transport [231]. \square

Lemma 2.3.3. *Problem (2.3.4) admits a solution. Consequently, Problem (2.1.4) has a solution.*

Proof. Existence of a solution of (2.3.4) is a consequence of Lemma 2.3.2 and of the l.s.c. of \tilde{J} w.r.t. the topology induced by $C([0, T], \mathcal{P}([0, 1] \times I)) \times \mathcal{M}([0, T] \times [0, 1] \times I^2)$ (where $\mathcal{M}([0, T] \times [0, 1] \times I^2)$ is endowed with the weak*-convergence). Since Problem (2.1.4) is equivalent to Problem (2.3.4) up to a change of variable, the existence of a solution is straightforward. \square

2.4 Analysis of the Hamilton-Jacobi equation

The purpose of this section is to study weak solutions of the Hamilton-Jacobi equation (2.2.5) when λ is in $\mathcal{M}^+([0, T] \times I)$. It is divided into three subsections. Section 2.4.1 is devoted to the analysis of the following

equation on $(0, T) \times (0, 1) \times I$

$$-\partial_t \psi_i(t, s) - b_i(s) \partial_s \psi_i(t, s) - c_i(t, s) + \sum_{j \in I, j \neq i} H \left((\psi_i - \psi_j)(t, s) + \int_t^T (\lambda_i - \lambda_j)(dr) \right) = 0, \quad (2.4.1)$$

$$\psi_i(T, \cdot) = g_i.$$

for a given $\lambda \in C_\delta^0([0, T] \times I, \mathbb{R}_+)$. Then, equation (2.4.1) is studied for a given $\lambda \in \mathcal{M}_\delta^+([0, T] \times I)$ in Section 2.4.2. Finally, we obtain in Section 2.4.3 the existence and the uniqueness of the solution to (2.2.5) by using the solution of (2.4.1).

2.4.1 The Hamilton-Jacobi equation for continuous valued data

The main result of this subsection is the following.

Proposition 2.4.1. *For any $\lambda \in C_\delta^0([0, T] \times I, \mathbb{R}_+)$, there exists a unique $\psi^\lambda \in C^1([0, T] \times [0, 1] \times I)$ satisfying, for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,*

$$\psi_i^\lambda(t, s) = \int_t^T \sum_{j \in I, j \neq i} -H^\lambda(i, j, t, \tau, s, \psi^\lambda) + c_i(\tau, S_i^{t,s}(\tau)) d\tau + g_i(S_i^{t,s}(T)). \quad (2.4.2)$$

In addition, ψ^λ is the unique classical solution of (2.4.1) on $[0, T] \times [0, 1] \times I$.

We recall that the definition of H^λ is given in (2.2.2). The next lemma states that a function ψ satisfies (2.4.2) if and only if it is a classical solution of (2.4.1).

Lemma 2.4.1. *For any $\lambda \in C^0([0, T] \times I, \mathbb{R}_+)$ and $t_0 \in [0, T]$, a function $\psi \in C^1((t_0, T] \times [0, 1] \times I)$ satisfies equation (2.4.2) on $[t_0, T] \times [0, 1] \times I$ if and only if it is a solution of (2.4.1) on $(t_0, T] \times [0, 1] \times I$.*

Proof. If $\psi \in C^1((t_0, T] \times [0, 1] \times I)$ satisfies (2.4.2), then $\psi(T, \cdot) = g$ and by computing the partial derivatives in space and time of ψ , one obtains that ψ is a solution of (2.4.1) on $(t_0, T] \times (0, 1) \times I$.

Conversely, if $\psi \in C^1((t_0, T] \times [0, 1] \times I)$ is a solution of (2.4.1) then, by the method of characteristics one deduces that ψ is a solution of (2.4.2) on $[t_0, T] \times [0, 1] \times I$. \square

Before proving Proposition (2.4.1), we need the following estimate on solutions of the Hamilton-Jacobi equation.

Lemma 2.4.2 (A priori estimate). *For any $\delta > 0$, there exists $M > 0$ such that, for any $\lambda \in C_\delta^0([0, T] \times I, \mathbb{R}_+)$ and any $t_0 \in [0, T]$, if $\psi \in C^1((t_0, T] \times [0, 1] \times I)$ satisfies (2.4.1) on $(t_0, T] \times (0, 1) \times I$, then $\|\psi\|_\infty < M$.*

Proof. Let us define $M := \|g\|_\infty + T(\|c\|_\infty + |I|H(\delta)) + 1$ and, for any $(t, s, i) \in [t_0, T] \times [0, 1] \times I$,

$$\underline{u}_i(t, s) := -\|g\|_\infty - (T - t)(\|c\|_\infty + |I|H(\delta)),$$

$$\bar{u}_i(t, s) := \|g\|_\infty + (T - t)(\|c\|_\infty + |I|H(\delta)).$$

We note that $\|\underline{u}\|_\infty < M$ and $\|\bar{u}\|_\infty < M$. One has, for any $(t, s, i) \in (t_0, T] \times [0, 1] \times I$,

$$-\partial_t \underline{u}_i(t, s) \leq - \sum_{j \in I, j \neq i} H \left(\int_\tau^T (\lambda_i - \lambda_j)(r) dr \right) + c_i(t, s)$$

$$-\partial_t \bar{u}_i(t, s) \geq - \sum_{j \in I, j \neq i} H \left(\int_\tau^T (\lambda_i - \lambda_j)(r) dr \right) + c_i(t, s). \quad (2.4.3)$$

We will show that ψ is bounded by \underline{u} and \bar{u} . For any $t \in (t_0, T]$, we define:

$$\gamma(t) = \max_{(x,i) \in [0,1] \times I} (\underline{u}_i - \psi_i)(t, x) \quad \text{and} \quad (x_t, i_t) \in \arg \max_{(x,i) \in [0,1] \times I} (\underline{u}_i - \psi_i)(t, x).$$

Since \underline{u} is independent of x and i , we have $(x_t, i_t) \in \arg \min_{(x,i) \in [0,1] \times I} \psi_i(t, x)$. On the one hand, if $x_t \in \{0, 1\}$, then $b_{i_t}(x_t) = 0$. On the other hand, if $x_t \in (0, 1)$, then $\partial_s \psi_{i_t}(t, x_t) = 0$. Therefore, one has $b_{i_t}(x_t) \partial_s (\underline{u}_{i_t}(t, x_t) -$

$\psi_{i_t}(t, x_t) = 0$. Since \underline{u} and ψ are Lipschitz continuous in time uniformly in (s, i) , γ is also Lipschitz continuous and thus differentiable a.e. on $(t_0, T]$. Using the Envelop Theorem [193, Theorem 1], γ is absolutely continuous on $(t_0, T]$ and, for a.e. $t \in (t_0, T]$, inequality (2.4.3) and equality (2.4.1) give:

$$\begin{aligned} \gamma'(t) &= \partial_t(\underline{u}_{i_t}(t, x_t) - \psi_{i_t}(t, x_t)) \\ &\geq \sum_{j \in I, j \neq i_t} H \left(\int_{\tau}^T (\lambda_i - \lambda_j)(r) dr \right) - H \left((\psi_i - \psi_j)(\tau, S_i^{t,s}(\tau)) + \int_{\tau}^T (\lambda_i - \lambda_j)(r) dr \right). \end{aligned}$$

Since for any $j \in I$ we have $\psi_{i_t}(t, x_t) \leq \psi_j(t, x_t)$ and $\underline{u}_{i_t}(t, x_t) = \underline{u}_j(t, x_t)$, the fact that H is non decreasing implies:

$$H \left(\psi_{i_t}(t, x_t) - \psi_j(t, x_t) + \int_{\tau}^T (\lambda_{i_t} - \lambda_j)(r) dr \right) \leq H \left(\int_{\tau}^T (\lambda_{i_t} - \lambda_j)(r) dr \right),$$

and thus, $\gamma'(t) \geq 0$. Since $\gamma(T) < 0$, we deduce that $\psi > \underline{u}$ on $(t_0, T] \times [0, 1] \times I$. With similar arguments, one obtains $\bar{u} > \psi$ on $(t_0, T] \times [0, 1] \times I$. Therefore, according to the definition of \underline{u} and \bar{u} , we have, for any $(t, s, i) \in (t_0, T] \times [0, 1] \times I$:

$$-M < \psi_i(t, s) < M. \quad (2.4.4)$$

□

Proposition 2.4.1 will be proved by a fixed point argument. For this, we need several lemmas. We fix the constant $M > 0$ associated to $\delta > 0$ from Lemma 2.4.2. The constant $\kappa > 0$ is defined below and depends only on $M, \|b'\|_{\infty}, T$ and $|I|$. We consider the space

$$C^{0,1}([0, T] \times [0, 1] \times I) := \{f \in C^0([0, T] \times [0, 1] \times I) \mid \partial_s f_i \in C^0([0, T] \times [0, 1] \times I)\},$$

endowed with the norm $\|\cdot\|_{0,1}^{\kappa}$ defined by:

$$\|f\|_{0,1}^{\kappa} := \|f\|_{\infty}^{\kappa} + \|\partial_s f\|_{\infty}^{\kappa},$$

where, for any $h \in C^0([0, T] \times [0, 1] \times I)$,

$$\|h\|_{\infty}^{\kappa} := \sup_{(t,s,i) \in [0,T] \times [0,1] \times I} |h_i(t, s) e^{-\kappa(T-t)}|.$$

The space $(C^{0,1}([0, T] \times [0, 1] \times I), \|\cdot\|_{0,1}^{\kappa})$ is a Banach space. We fix $C_0 > 0$ and $C_1 > 0$ to be chosen below, where C_0 depends on $\|\partial_s c\|_{\infty}, \|g'\|_{\infty}, \|b'\|_{\infty}$ and C_1 depends on M, δ, T and $|I|$. We look for a solution in the space Σ defined by:

$$\Sigma := \left\{ f \in C^{0,1}([0, T] \times [0, 1] \times I) \mid \|f\|_{\infty} \leq M + 1 \text{ and } \|\partial_s f(t, \cdot)\|_{\infty} \leq C_0 e^{C_1(T-t)} \forall t \in [0, T] \right\}.$$

Remark 2.4.1. *The set Σ is bounded and closed and therefore complete w.r.t. the topology induced by the norm $\|\cdot\|_{0,1}^{\kappa}$.*

In this subsection, we are looking for a solution of (2.4.1) as a fixed point of the map $\Gamma^{\lambda} := (\Gamma_i^{\lambda})_{i \in I}$, which is defined on $C^{0,1}([0, T] \times [0, 1] \times I)$ by

$$\Gamma_i^{\lambda}(\phi)(t, s) := \int_t^T \sum_{j \in I, j \neq i} -H^{\lambda}(i, j, t, \tau, s, \phi) + c_i(\tau, S_i^{t,s}(\tau)) d\tau + g_i(S_i^{t,s}(T)). \quad (2.4.5)$$

Since the function H is not necessarily bounded on \mathbb{R} , we need to introduce a smooth truncation $F \in C^1(\mathbb{R}, [-M - 1, M + 1])$ to obtain a fixed point in Σ . The function F is defined on \mathbb{R} such that $F' \geq 0$, $\|F'\|_{\infty} \leq 1$ and:

$$F(x) := \begin{cases} -M - 1 & \text{if } x \leq -(M + 2), \\ x & \text{if } -M - 1/2 \leq x \leq M + 1/2, \\ M + 1 & \text{if } M + 2 \leq x. \end{cases} \quad (2.4.6)$$

Finally, we define the function Π^λ by:

$$\forall \phi \in \Sigma, \Pi^\lambda(\phi) := (\Pi_1^\lambda(\phi), \dots, \Pi_{|I|}^\lambda(\phi)) \text{ where } \Pi_i^\lambda(\phi) := (F \circ \Gamma_i^\lambda)(\phi) \quad \forall i \in I.$$

The following lemma states that Π^λ maps Σ into itself.

Lemma 2.4.3. *For a suitable choice of the constants C_0 and C_1 , one has $\Pi^\lambda(\phi) \in \Sigma$ for any $\phi \in \Sigma$.*

Proof. Let $\phi \in \Sigma$ and, for any $i \in I$, $\sigma_i := \Gamma_i^\lambda(\phi)$. We have $\sigma \in C^{0,1}([0, T] \times [0, 1] \times I)$. We need to show that, for any $t \in [0, T]$, $\|\partial_s \sigma(t, \cdot)\|_\infty \leq C_0 e^{C_1(T-t)}$. According to the definition of Γ_i^λ , we have, for any $(i, t) \in I \times [0, T]$,

$$\begin{aligned} \|\partial_s \sigma_i(t, \cdot)\|_\infty &\leq \|\partial_s S\|_\infty K \int_t^T \sum_{i, j \in I} \|\partial_s(\phi_j - \phi_i)(t, \cdot)\|_\infty dt + T \|\partial_s S\|_\infty \|\partial_s c\|_\infty + \|\partial_s S\|_\infty \|g'\|_\infty \\ &\leq \frac{C_0 C}{C_1} e^{C_1(T-t)} + C, \end{aligned}$$

where $K := \sup_{x \in [-2M-\delta, 2M+\delta]} |H'(x)|$ and $C > 0$ is a positive constant which depends on $\|\partial_s S\|_\infty, \|\partial_s c\|_\infty, \|g'\|_\infty, |I|, T$ and K . Choosing carefully C_0 and C_1 depending on C , one obtains that, for any $(i, t) \in I \times [0, T]$,

$$\|\partial_s \sigma_i(t, \cdot)\|_\infty \leq C_0 e^{C_1(T-t)}.$$

Finally, from the definition of F in (2.4.6), the function $\Pi(\phi) = (F(\sigma_1), \dots, F(\sigma_{|I|}))$ is in Σ . \square

The existence of a fixed point of Π^λ in Σ is established in the following lemma.

Lemma 2.4.4. *For a suitable choice of the constant κ , the function Π^λ admits a unique fixed point $\psi^\lambda \in \Sigma$. In addition, we have $\psi^\lambda \in C^1([0, T] \times [0, 1] \times I)$.*

Proof. Let $\phi^1, \phi^2 \in \Sigma$. Since ϕ^1 and ϕ^2 are bounded by $M + 1$, one has, for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,

$$\begin{aligned} |\Gamma_i^\lambda(\phi^1)(t, s) - \Gamma_i^\lambda(\phi^2)(t, s)| &\leq \sum_{j \in I, j \neq i} \int_t^T |H^\lambda(i, j, t, \tau, s, \phi^1) - H^\lambda(i, j, t, \tau, s, \phi^2)| d\tau \\ &\leq C \sum_{j \in I} \int_t^T |\phi_j^1(\tau, S_i^{t,s}(\tau)) - \phi_j^2(\tau, S_i^{t,s}(\tau))| d\tau \\ &\leq \|\phi^1 - \phi^2\|_\infty^\kappa \frac{C e^{\kappa(T-t)}}{\kappa}, \end{aligned} \tag{2.4.7}$$

where $C > 0$ is a constant depending on $\delta, |I|$ and $\sup_{x \in [-2(M+1)-\delta, 2(M+1)+\delta]} |H'(x)|$. Since H' is Lipschitz continuous on $[-2(M+1), 2(M+1)]$, one has, for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,

$$\begin{aligned} |\partial_s(\Gamma_i^\lambda(\phi^1)(t, s) - \Gamma_i^\lambda(\phi^2)(t, s))| &\leq \sum_{j \in I, j \neq i} \int_t^T |\partial_s H^\lambda(i, j, t, \tau, s, \phi^1) - \partial_s H^\lambda(i, j, t, \tau, s, \phi^2)| d\tau \\ &\leq \|\phi^1 - \phi^2\|_{0,1}^\kappa \frac{K e^{\kappa(T-t)}}{\kappa}, \end{aligned} \tag{2.4.8}$$

where $K > 0$ is a constant that depends on $|I|, T, \|b'\|_\infty, M, C_0, C_1$ and on the bound and the Lipschitz constant of H' on $[-2(M+1)-\delta, 2(M+1)+\delta]$. From (2.4.7) and (2.4.8), one obtains that:

$$\|\Gamma^\lambda(\phi^1) - \Gamma^\lambda(\phi^2)\|_{0,1}^\kappa \leq \max(C_0/\kappa, K/\kappa) \|\phi^1 - \phi^2\|_{0,1}^\kappa.$$

Choosing $\kappa > \max(C_0, K)$ and the function F being non expensive, one deduces that the function Π^λ is a contraction on Σ and that it admits a unique fixed point $\psi^\lambda \in \Sigma$. By the definitions of Π^λ and Σ , it is straightforward that $\psi^\lambda \in C^1([0, T] \times [0, 1] \times I)$. \square

We now turn to the proof of Proposition 2.4.1.

Proof of Proposition 2.4.1. Let $t_0 \in [0, T]$ be the minimum time such that $\|\psi^\lambda(t, \cdot)\|_\infty \leq M + 1/2$ for any $t \in [t_0, T]$. Since $\|\psi_i^\lambda(T, \cdot)\|_\infty = \|g_i\|_\infty < M$ for any $i \in I$ and ψ^λ is continuous, the time t_0 is smaller than T . The function ψ^λ is a fixed point of Γ^λ on $[t_0, T]$ and thus, ψ^λ is a solution of (2.4.2) on $[t_0, T] \times [0, 1] \times I$. By Lemma 2.4.1, we deduce that ψ^λ satisfies (2.4.1) on $(t_0, T] \times [0, 1] \times I$. If $t_0 = 0$, then the conclusion follows. If $t_0 > 0$, then $\|\psi^\lambda(t_0, \cdot)\|_\infty = M + 1/2$. By Lemma 2.4.2, one also has $\|\psi^\lambda(t_0, \cdot)\|_\infty \leq M$. Hence, there is a contradiction. Therefore, ψ^λ is a solution of (2.4.2) and a classical solution of (2.4.1) on $[0, T] \times [0, 1] \times I$. \square

Remark 2.4.2. *One can show that $\|\partial_t \psi^\lambda\|_\infty + \|\partial_s \psi^\lambda\|_\infty$ is bounded by a function that is non-decreasing w.r.t. the variable δ . Indeed, by Lemma 2.4.3 and its proof, $\|\partial_s \psi^\lambda\|_\infty$ is bounded by $C_0 e^{C_1 T}$ where C_0 and C_1 depend on M and on δ by the definition of M in Lemma 2.4.2. By Lemma 2.4.1, ψ^λ is a classical solution of (2.4.1). Therefore,*

$$\|\partial_t \psi^\lambda\|_\infty \leq \|b\|_\infty \|\partial_s \psi^\lambda\|_\infty + \|c\|_\infty + \sup_{x \in [-2M - \delta, 2M + \delta]} |H(x)|.$$

Thus, $\|\partial_t \psi^\lambda\|_\infty$ is bounded by a non-decreasing function of δ .

2.4.2 The Hamilton-Jacobi equation for measure valued data

In this subsection, we prove

Proposition 2.4.2. *For any $\lambda \in \mathcal{M}^+([0, T] \times I)$, there exists a unique $\psi^\lambda \in \text{Lip}([0, T] \times [0, 1] \times I)$ satisfying that, for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,*

$$\psi_i^\lambda(t, s) = \int_t^T \sum_{j \in I, j \neq i} -H^\lambda(i, j, t, \tau, s, \psi^\lambda) + c_i(\tau, S_i^{t,s}(\tau)) d\tau + g_i(S_i^{t,s}(T)). \quad (2.4.9)$$

In addition, the map $\lambda \mapsto \psi^\lambda$ is continuous from $\mathcal{M}^+([0, T] \times I)$, endowed with the weak topology, to $C^0([0, T] \times [0, 1] \times I)$, endowed with the norm $\|\cdot\|_\infty$.

We recall that the definition of H^λ , with $\lambda \in \mathcal{M}^+([0, T] \times I)$, is given in (2.2.2). The proof of Proposition 2.4.2 relies on the results of the previous subsection. We define the map $\Theta : C_\delta^0([0, T] \times I, \mathbb{R}_+) \rightarrow C^1([0, T] \times [0, 1] \times I)$ by:

$$\Theta : \lambda \mapsto \psi^\lambda, \quad (2.4.10)$$

where ψ^λ is given by Proposition 2.4.1. We know from Proposition 2.4.1 that Θ is well defined on $C^0([0, T] \times I, \mathbb{R}_+)$. We want to show that Θ can be continuously extended to a function defined on $\mathcal{M}_\delta^+([0, T] \times I)$ with values in $\text{Lip}([0, T] \times [0, 1] \times I)$. We define the distance \mathcal{D} on $\mathcal{M}^+([0, T] \times I)$ by

$$\mathcal{D}(\lambda, \mu) := \int_0^T \sum_{i \in I} \left| \int_t^T (\lambda_i - \mu_i)(d\tau) \right| dt + \sum_{i \in I} \left| \int_0^T (\lambda_i - \mu_i)(dt) \right|. \quad (2.4.11)$$

Remark 2.4.3. *If a sequence $\{\lambda^n\}_n$ in $\mathcal{M}_\delta^+([0, T] \times I)$ converges w.r.t. the weak topology in $\mathcal{M}^+([0, T] \times I)$ to $\lambda \in \mathcal{M}_\delta^+([0, T] \times I)$, then $\lim_{n \rightarrow \infty} \mathcal{D}(\lambda^n, \lambda) = 0$. Indeed, since $\{\lambda^n\}_n$ weakly converges to λ , then $\lim_{n \rightarrow \infty} (\lambda_i^n - \lambda_i)([t, T]) = 0$ for any $i \in I$, a.e. $t \in [0, T]$ and $t = 0$. Applying the dominated convergence theorem, the conclusion follows.*

The next remark will be useful to extend the domain of Θ .

Remark 2.4.4. *The space $C_\delta^0([0, T] \times I, \mathbb{R}_+)$ is dense in $\mathcal{M}_\delta^+([0, T] \times I)$ w.r.t. the topology induced by \mathcal{D} . More precisely, for any $\lambda \in \mathcal{M}_\delta^+([0, T] \times I)$, there exists a sequence $\{\lambda^n\}_n$ in $C_\delta^0([0, T] \times I, \mathbb{R}_+)$, such that $\lim_{n \rightarrow \infty} \mathcal{D}(\lambda, \lambda^n \mathcal{L}) = 0$ (where \mathcal{L} is the Lebesgue measure on $[0, T]$).*

Lemma 2.4.5. *The map Θ can be extended to a Lipschitz continuous map from $\mathcal{M}_\delta^+([0, T] \times I)$, endowed with distance \mathcal{D} , to $C^0([0, T] \times [0, 1] \times I)$, endowed with the norm $\|\cdot\|_\infty$. In addition we have $\Theta(\lambda) \in \text{Lip}([0, T] \times [0, 1] \times I)$ for any $\lambda \in \mathcal{M}_\delta^+([0, T] \times I)$.*

Proof. We need to show that there exists a constant $C > 0$ such that, for any $\lambda^1, \lambda^2 \in C_\delta^0([0, T] \times I, \mathbb{R}_+)$, we have: $\|\Theta(\lambda^1) - \Theta(\lambda^2)\|_\infty \leq C\mathcal{D}(\lambda^1, \lambda^2)$. Since H is locally Lipschitz, there exists a constant $K > 0$ such that,

for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,

$$\begin{aligned}
 & |\Theta(\lambda^1)_i(t, s) - \Theta(\lambda^2)_i(t, s)| \\
 &= |\psi_i^{\lambda^1}(t, s) - \psi_i^{\lambda^2}(t, s)| \\
 &= \left| \sum_{j \in I, j \neq i} \int_t^T H^{\lambda^1}(i, j, t, \tau, s, \psi^{\lambda^1}) - H^{\lambda^2}(i, j, t, \tau, s, \psi^{\lambda^2}) d\tau \right| \\
 &\leq |I| \sum_{j \in I} K \int_t^T \left(|\psi_j^{\lambda^1}(\tau, S_i^{t,s}(\tau)) - \psi_j^{\lambda^2}(\tau, S_i^{t,s}(\tau))| + \left| \int_\tau^T \lambda_j^1(r) dr - \int_\tau^T \lambda_j^2(r) dr \right| \right) d\tau.
 \end{aligned}$$

Taking the supremum over $I \times [0, 1]$ yields:

$$\begin{aligned}
 & \|\psi^{\lambda^1}(t, \cdot) - \psi^{\lambda^2}(t, \cdot)\|_\infty \\
 &\leq |I|^2 \int_t^T K \|\psi^{\lambda^1}(\tau, \cdot) - \psi^{\lambda^2}(\tau, \cdot)\|_\infty d\tau + |I|^2 K \sup_{i \in I} \left| \int_t^T \lambda_i^1(r) dr - \int_t^T \lambda_i^2(r) dr \right| d\tau \\
 &\leq |I|^2 \int_t^T K \|\psi^{\lambda^1}(\tau, \cdot) - \psi^{\lambda^2}(\tau, \cdot)\|_\infty d\tau + |I|^2 K \mathcal{D}(\lambda^1, \lambda^2).
 \end{aligned}$$

Then, by applying Gronwall Lemma to $t \mapsto \|\psi^{\lambda^1}(t, \cdot) - \psi^{\lambda^2}(t, \cdot)\|_\infty$, one has, for any $t \in [0, T]$,

$$\|\psi^{\lambda^1}(t, \cdot) - \psi^{\lambda^2}(t, \cdot)\|_\infty \leq C \mathcal{D}(\lambda^1, \lambda^2), \quad (2.4.12)$$

where the constant $C > 0$ depends on δ, K, T and $|I|$. Therefore,

$$\|\Theta(\lambda^1) - \Theta(\lambda^2)\|_\infty = \|\psi^{\lambda^1} - \psi^{\lambda^2}\|_\infty \leq C \mathcal{D}(\lambda^1, \lambda^2).$$

From the previous inequality and Remark 2.4.4, the map Θ can be continuously extended to a Lipschitz continuous map from $\mathcal{M}_\delta^+([0, T] \times I)$ to $C^0([0, T] \times [0, 1] \times I)$.

Finally, we approximate $\lambda \in \mathcal{M}_\delta^+([0, T] \times I)$ by a sequence in $\{\lambda^n\}_n$ in $C_\delta^0([0, T] \times I, \mathbb{R}_+)$ w.r.t. the distance \mathcal{D} . One has that $\{\Theta(\lambda^n)\}_n$ is uniformly Lipschitz continuous on $[0, T] \times [0, 1] \times I$ according to Remark 2.4.2. Thus, $\Theta(\lambda)$ is in $\text{Lip}([0, T] \times [0, 1] \times I)$. \square

Proof of Proposition 2.4.2. The proposition is a direct consequence of Lemma 2.4.5. \square

Remark 2.4.5. *By Remark 2.4.2 and Lemma 2.4.5, one can show that for any $\lambda \in \mathcal{M}^+([0, T] \times I)$, the solution ψ^λ of (2.4.9) is such that, $\max(\|\psi^\lambda\|_\infty, \|\partial_s \psi^\lambda\|_\infty)$ depends only on H, b, c, g and $\lambda([0, T] \times I)$.*

2.4.3 Analysis of weak solution of the Hamilton-Jacobi equation (2.5)

For any $\lambda \in \mathcal{M}^+([0, T] \times I)$, we consider the function $L^\lambda \in BV([0, T] \times I)$ defined for any $i \in I$ and any $t \in [0, T]$ by:

$$L_i^\lambda(t) := \lambda_i([t, T]). \quad (2.4.13)$$

We introduce the notion of weak solution for equation (2.2.5).

Definition 2.4.1. *For a given $\lambda \in \mathcal{M}^+([0, T] \times I)$, a function φ defined from $[0, T] \times [0, 1] \times I$ to \mathbb{R} is a weak solution of equation (2.2.5) if $\varphi - L^\lambda$ is in $\text{Lip}([0, T] \times [0, 1] \times I)$ and if, for any test function $f \in C^1([0, T] \times [0, 1] \times I)$,*

$$\begin{aligned}
 & \int_0^1 \varphi_i(0, s) f_i(0, s) ds - \int_0^1 g_i(s) f_i(T, s) ds + \int_0^T \int_0^1 (\partial_t f_i(t, s) + \partial_s(f_i(t, s) b_i(s))) \varphi_i(t, s) ds dt \\
 &+ \int_0^T \int_0^1 \left(\sum_{j \in I, j \neq i} H(\varphi_i(t, s) - \varphi_j(t, s)) - c_i(t, s) \right) f_i(t, s) dt ds - \sum_{i \in I} \int_0^T \int_0^1 f_i(t, s) ds \lambda_i(dt) \\
 &= 0,
 \end{aligned} \quad (2.4.14)$$

where $\varphi_i(0, \cdot)$ is understood in the sense of trace.

- Remark 2.4.6.** 1. There is no boundary condition in (2.4.14). This is due to the fact that $b(0) = b(1) = 0$, involving a null incoming flow in the domain $[0, 1]$.
2. Since φ is in $BV([0, T] \times I) + \text{Lip}([0, T] \times [0, 1] \times I)$, $\varphi(0, \cdot)$ exists in the sense of trace. In addition, φ is bounded and thus, it belongs to $L^2([0, T] \times [0, 1] \times I)$ so that the integrals in (2.4.14) exist.
3. The final condition in (2.2.5) is misleading. Indeed, any weak solution φ of (2.2.5) in the sense of Definition 2.4.1 satisfies $\varphi_i(T, \cdot) = g_i(\cdot) + L_i^\lambda(T)$, where $\varphi_i(T, \cdot)$ and $L_i^\lambda(T)$ are in the sense of trace. Thus, $\varphi_i(T, \cdot) \geq g_i(\cdot)$.

The main result of this subsection is the following.

Theorem 2.4.1. Let $\lambda \in \mathcal{M}^+([0, T] \times I)$. A function $\varphi \in \text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)$ is the unique weak solution of (2.2.5) in the sense of Definition 2.4.1, if and only if it satisfies that, for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,

$$\varphi_i(t, s) = \int_t^T \sum_{j \in I, j \neq i} -H((\varphi_i - \varphi_j)(\tau, S_i^{t,s}(\tau))) + c_i(\tau, S_i^{t,s}(\tau)) d\tau + L_i^\lambda(t) + g_i(S_i^{t,s}(T)). \quad (2.4.15)$$

Theorem 2.4.1 is a consequence of Lemmas 2.4.7 and 2.4.8. Lemma 2.4.7 shows that a function satisfying (2.4.15) is a weak solution of (2.2.5). Lemma 2.4.8 shows the converse. In what follows, for any $\lambda \in \mathcal{M}^+([0, T] \times I)$, we define ψ^λ as in Proposition 2.4.2 and define $\varphi^\lambda \in \text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)$ by:

$$\varphi^\lambda := \psi^\lambda + L^\lambda. \quad (2.4.16)$$

Remark 2.4.7. For any $\lambda \in \mathcal{M}^+([0, T] \times I)$, φ^λ is a solution of (2.4.15) on $[0, T] \times [0, 1] \times I$ if and only if ψ^λ is a solution of (2.4.9) on $[0, T] \times [0, 1] \times I$. According to Remark 2.4.5, the quantity $\max(\|\varphi^\lambda\|_\infty, \|\partial_s \varphi^\lambda\|_\infty)$ depends on $\lambda([0, T] \times I)$.

Lemma 2.4.6. For any $\lambda \in \mathcal{M}^+([0, T] \times I)$, there exists a sequence $\{(\lambda^n, \varphi^n)\}_n$ such that

- (i) $\{\lambda^n\}_n$ is in $C^0([0, T] \times I, \mathbb{R}_+)$ and converges to λ w.r.t. the weak topology in $\mathcal{M}^+([0, T] \times I)$,
- (ii) for any $n \in \mathbb{N}$, φ^n is a classical solution of (2.2.5) associated to λ^n on $(0, T) \times (0, 1) \times I$
- (iii) we have $\lim_{n \rightarrow \infty} \|\varphi^\lambda(t, \cdot) - \varphi^n(t, \cdot)\|_\infty = 0$ for a.e. $t \in [0, T]$ and $\lim_{n \rightarrow \infty} \|\varphi^\lambda(0, \cdot) - \varphi^n(0, \cdot)\|_\infty = 0$.

Proof. We consider a sequence $\{\lambda^n\}_n$ in $C^0([0, T] \times I, \mathbb{R}_+)$ that weakly converges to λ . We set $\psi^n := \psi^{\lambda^n}$, where ψ^{λ^n} is defined by Proposition 2.4.2, and $\varphi^n := \psi^n + L^{\lambda^n}$. By Lemma 2.4.1, for any $n \in \mathbb{N}$, ψ^n is a classical solution of (2.4.1). Then, for any $n \in \mathbb{N}$, φ^n is a classical solution of (2.2.5) associated to λ^n on $(0, T) \times (0, 1) \times I$. The weak convergence of $\{\lambda^n\}_n$ implies:

$$\lim_{n \rightarrow \infty} L^{\lambda^n}(t) = L^\lambda(t) \text{ for a.e. } t \in [0, T] \quad \text{and} \quad \lim_{n \rightarrow \infty} L^{\lambda^n}(0) = L^\lambda(0).$$

By Proposition 2.4.2, $\{\psi^n\}_n$ converges to ψ^λ w.r.t. the norm $\|\cdot\|_\infty$. Then, the two previous equalities imply that:

$$\lim_{n \rightarrow \infty} \|\varphi^\lambda(t, \cdot) - \varphi^n(t, \cdot)\|_\infty = 0 \text{ for a.e. } t \in [0, T] \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\varphi^\lambda(0, \cdot) - \varphi^n(0, \cdot)\|_\infty = 0.$$

□

Lemma 2.4.7. For any $\lambda \in \mathcal{M}^+([0, T] \times I)$, φ^λ is a weak solution of (2.2.5) in the sense of Definition 2.4.1.

Proof. We consider a sequence $\{\lambda^n, \varphi^n\}_n$ defined as in Lemma 2.4.6. For any $n \in \mathbb{N}$, φ^n is a classical solution of (2.2.5). Thus, for any test function $f \in C^1([0, T] \times [0, T] \times I)$:

$$\begin{aligned} & \int_0^1 \varphi_i^n(0, s) f_i(0, s) ds - \int_0^1 g_i(s) f_i(T, s) ds + \int_0^T \int_0^1 (\partial_t f_i(t, s) + \partial_s(f_i(t, s) b_i(s))) \varphi_i^n(t, s) ds dt \\ & + \int_0^T \int_0^1 \left(\sum_{j \in I, j \neq i} H(\varphi_i^n(t, s) - \varphi_j^n(t, s)) - c_i(t, s) \right) f_i(t, s) dt ds - \sum_{i \in I} \int_0^T \int_0^1 f_i(t, s) ds \lambda_i^n(t) dt \\ & = 0. \end{aligned} \quad (2.4.17)$$

The conclusion follows by using Lemma 2.4.6, the continuity of H , the limit of (2.4.17) when n tends to infinity and by applying the dominated convergence theorem. \square

The next lemma states the converse of Lemma 2.4.7.

Lemma 2.4.8. *For any $(\lambda, \varphi) \in \mathcal{M}^+([0, T] \times I) \times (\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I))$, if φ is a weak solution of (2.2.5) associated to λ in the sense of Definition 2.4.1, then φ satisfies (2.4.15) for any $(t, s, i) \in [0, T] \times [0, 1] \times I$.*

Proof. Let φ be a weak solution of (2.2.5) associated to λ . Let $\beta \in C^1([0, T] \times [0, 1] \times I, \mathbb{R})$ be a test function. Then,

$$\begin{aligned} & \int_0^1 \varphi_i(0, s) \beta_i(0, s) ds - \int_0^1 g_i(s) \beta_i(T, s) ds + \int_0^T \int_0^1 (\partial_t \beta_i(t, s) + \partial_s(\beta_i(t, s) b_i(s))) \varphi_i(t, s) ds dt \\ & + \int_0^T \int_0^1 \left(\sum_{j \in I, j \neq i} H(\varphi_i(t, s) - \varphi_j(t, s)) - c_i(t, s) \right) \beta_i(t, s) dt ds - \sum_{i \in I} \int_0^T \int_0^1 \beta_i(t, s) ds \lambda_i(dt) \\ & = 0. \end{aligned} \quad (2.4.18)$$

We choose the function β such that there exist $\theta \in C^\infty([0, T] \times [0, 1] \times I)$ and $\xi \in C^\infty([0, 1] \times I)$ satisfying:

$$\begin{aligned} \partial_t \beta_i(t, s) + \partial_s(\beta_i(t, s) b_i(s)) &= \theta_i(t, s) & \text{for any } (t, s, i) \in (0, T) \times (0, 1) \times I, \\ \beta_i(0, \cdot) &= \xi_i(\cdot) & \text{for any } (s, i) \in [0, 1] \times I. \end{aligned} \quad (2.4.19)$$

The function β is given by for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,

$$\beta_i(t, s) = \int_0^t \theta_i(\tau, S_i^{t,s}(\tau)) \exp\left(-\int_\tau^t b'_i(S_i^{t,s}(r)) dr\right) d\tau + \xi_i(S_i^{t,s}(0)) \exp\left(-\int_0^t b'_i(S_i^{t,s}(\tau)) d\tau\right).$$

To simplify (2.4.18), we introduce, for any $i \in I$ the following functions ν_i and π_i , satisfying $\beta_i = \nu_i + \pi_i$: for any $(t, s) \in [0, T] \times [0, 1]$,

$$\nu_i(t, s) := \int_0^t \theta_i(\tau, S_i^{t,s}(\tau)) \exp\left(-\int_\tau^t b'_i(S_i^{t,s}(r)) dr\right) d\tau \quad \text{and} \quad \pi_i(t, s) := \xi_i(S_i^{t,s}(0)) \exp\left(-\int_0^t b'_i(S_i^{t,s}(\tau)) d\tau\right).$$

Setting $h(t, s) := \sum_{j \in I, j \neq i} H(\varphi_i(t, s) - \varphi_j(t, s)) - c_i(t, s)$ for any $(t, s, i) \in [0, T] \times [0, 1] \times I$, we have $h \in L^1((0, T) \times (0, 1))$. By switching the order of integration, and applying the change of variable $x = S_i^{t,s}(\tau)$ and equality (2.2.4), one has:

$$\begin{aligned} \int_0^T \int_0^1 h(t, s) \nu_i(t, s) dt ds &= \int_0^T \int_0^1 \int_0^t h(t, s) \theta_i(\tau, S_i^{t,s}(\tau)) \exp\left(-\int_\tau^t b'_i(S_i^{t,s}(r)) dr\right) d\tau ds dt \\ &= \int_0^T \int_0^1 \theta_i(\tau, x) \int_\tau^T h(t, S_i^{\tau,x}(t)) \exp\left(-\int_\tau^t b'_i(S_i^{\tau,x}(r)) dr\right) \partial_x S_i^{\tau,x}(t) dt dx d\tau, \\ &= \int_0^T \int_0^1 \theta_i(\tau, x) \int_\tau^T h(t, S_i^{\tau,x}(t)) dt dx d\tau. \end{aligned} \quad (2.4.20)$$

By the same computations, for any $i \in I$, one has:

$$\int_0^1 g_i(s) \nu_i(T, s) ds = \int_0^T \int_0^1 g_i(S_i^{\tau,x}(T)) \theta_i(\tau, x) ds d\tau, \quad (2.4.21)$$

$$\int_0^T \int_0^1 h(t, s) \pi_i(t, s) dt ds = \int_0^1 \xi_i(x) \int_0^T h(t, S_i^{0,x}(t)) dt dx, \quad (2.4.22)$$

$$\int_0^1 g_i(s) \pi_i(T, s) ds = \int_0^1 \xi_i(x) g_i(S_i^{0,x}(T)) dx, \quad (2.4.23)$$

and:

$$\int_0^T \int_0^1 \beta_i(t, s) ds \lambda_i(dt) = \int_0^T \int_0^1 \xi_i(\tau, x) L_i^\lambda(\tau) dx d\tau + \int_0^1 \xi_i(x) L_i^\lambda(0) dx. \quad (2.4.24)$$

Using (2.4.20), (2.4.22), (2.4.23), (2.4.24) and (2.4.19), equality (2.4.18) becomes:

$$\begin{aligned} & \int_0^1 \xi_i(s) \left(\varphi_i(0, s) + \int_0^T \sum_{j \in I, j \neq i} H((\varphi_i - \varphi_j)(\tau, S_i^{0,s}(\tau))) - c_i(\tau, S_i^{0,s}(\tau)) d\tau - L_i^\lambda(0) - g_i(S_i^{0,s}(T)) \right) ds \\ & + \int_0^T \int_0^1 \theta_i(t, s) \left(\varphi_i(t, s) + \int_t^T \sum_{j \in I, j \neq i} H((\varphi_i - \varphi_j)(\tau, S_i^{t,s}(\tau))) - c_i(\tau, S_i^{t,s}(\tau)) d\tau - L_i^\lambda(t) - g_i(S_i^{t,s}(T)) \right) ds dt \\ & = 0. \end{aligned}$$

This equality holds for any test functions θ_i and ξ_i . Then, one has for any $s \in [0, 1]$:

$$\varphi_i(0, s) = \int_0^T \sum_{j \in I, j \neq i} -H((\varphi_i - \varphi_j)(\tau, S_i^{0,s}(\tau))) + c_i(\tau, S_i^{0,s}(\tau)) d\tau + L_i^\lambda(0) + g_i(S_i^{0,s}(T)),$$

and for any $(t, s) \in [0, T] \times [0, 1]$:

$$\varphi_i(t, s) = \int_t^T \sum_{j \in I, j \neq i} -H((\varphi_i - \varphi_j)(\tau, S_i^{t,s}(\tau))) + c_i(\tau, S_i^{t,s}(\tau)) d\tau + L_i^\lambda(t) + g_i(S_i^{t,s}(T)).$$

□

With the above lemmas, the proof of Theorem 2.4.1 is straightforward.

Proof of Theorem 2.4.1. The proof is a direct consequence of Lemmas 2.4.7 and 2.4.8. The uniqueness of a weak solution is deduced by Remark 2.4.7. Indeed, since ψ^λ is the unique solution of (2.4.9), φ^λ is the unique solution of (2.4.15) and thus, the unique weak solution of (2.2.5) in the sense of Definition 2.4.1. □

2.5 The dual problem

In this section, an optimization problem (2.5.3) is introduced. Using tools from convex analysis [99], we show that this problem is in duality with (2.3.4). We consider the following spaces:

$$E_0 = C^1([0, T] \times [0, 1] \times I) \times C^0([0, T] \times I) \text{ and } E_1 := C^0([0, T] \times [0, 1] \times I) \times C^0([0, T] \times [0, 1] \times I^2).$$

Given $(\varphi, \lambda) \in E_0$, we consider the following inequality:

$$\begin{aligned} -\partial_t \varphi_i(t, s) - b_i(s) \partial_s \varphi_i(t, s) - c_i(t, s) - \lambda(t) + \sum_{j \in I, j \neq i} H((\varphi_i - \varphi_j)(t, s)) &\leq 0 \quad \text{on } (0, T) \times (0, 1) \times I, \\ \varphi_i(T, \cdot) &\leq g_i \quad \text{on } (0, 1) \times I. \end{aligned} \quad (2.5.1)$$

The set \mathcal{K}_0 is defined by: $\mathcal{K}_0 := \{(\varphi, \lambda) \in E_0, \lambda \geq 0 \text{ and } \varphi \text{ satisfies (2.5.1) associated to } \lambda\}$. We introduce the function A , defined on \mathcal{K}_0 by :

$$A(\varphi, \lambda) := \sum_{i \in I} \int_0^1 -\varphi_i(0, s) m_i^0(ds) + \int_0^T \lambda_i(t) D_i(t) dt, \quad (2.5.2)$$

and the following problem is considered:

$$\inf_{(\varphi, \lambda) \in \mathcal{K}_0} A(\varphi, \lambda). \quad (2.5.3)$$

Lemma 2.5.1. $\inf_{(\varphi, \lambda) \in \mathcal{K}_0} A(\varphi, \lambda)$ is finite.

Proof. Let $(\varphi, \lambda) \in \mathcal{K}_0$. Since φ is in $C^1([0, T] \times [0, 1] \times I)$ and satisfies (2.5.1), using that H is non negative, we have, for any $(i, s) \in I \times [0, 1]$, $\varphi_i(0, s) \leq T\|c\|_\infty + \int_0^T \lambda_i(\tau) d\tau + \|g\|_\infty$. Setting $Q := -|I|(T\|c\|_\infty + \|g\|_\infty)$, one has:

$$Q + \sum_{i \in I} \int_0^T \lambda_i(t) \left(D_i(t) - \int_0^1 m_i^0(ds) \right) dt \leq A(\varphi, \lambda).$$

Since $\lambda \geq 0$, we deduce from the Assumption 3 and previous inequality that $Q \leq \inf_{(\varphi, \lambda) \in \mathcal{K}_0} A(\varphi, \lambda)$. \square

We consider the linear and bounded function $\Lambda : E_0 \rightarrow E_1$ defined by: $\Lambda(\varphi, \lambda) := (\partial_t \varphi + b \partial_s \varphi + \tilde{\lambda}, \Delta \varphi)$, where $\partial_t \varphi + b \partial_s \varphi := (\partial_t \varphi_i + b_i \partial_s \varphi_i)_{i \in I}$, $\Delta \varphi := (\Delta \varphi_{i,j})_{(i,j) \in \bar{I}}$ with $\Delta \varphi_{i,j} = \varphi_j - \varphi_i$ and, for any $(s, i) \in [0, 1] \times I$, $\tilde{\lambda}_i(\cdot, s) := \lambda_i(\cdot)$. The linear function $\Lambda^* : E_1^* \rightarrow E_0^*$ is the adjoint operator of Λ . The functional \mathcal{F} is defined, for any $(\varphi, \lambda) \in E_0$, by

$$\mathcal{F}(\varphi, \lambda) := \begin{cases} \sum_{i \in I} \int_0^1 -\varphi_i(0, s) m_i^0(ds) + \int_0^T D_i(t) \lambda_i(t) dt & \text{if } \varphi_i(T, \cdot) \leq g_i \text{ and } \lambda_i \geq 0 \quad \forall i \in I, \\ +\infty & \text{otherwise.} \end{cases}$$

Using that:

$$\begin{aligned} & \langle (m, E), \Lambda(\varphi, \lambda) \rangle_{E_1^*, E_1} \\ &= \sum_{i \in I} \int_0^1 \int_0^T (\partial_t \varphi_i(t, s) + b_i(s) \partial_s \varphi_i(t, s)) m_i(ds, t) + \sum_{j \in I, j \neq i} (\varphi_j(t, s) - \varphi_i(t, s)) E_{i,j}(t, ds) dt \\ &+ \sum_{i \in I} \int_0^T \int_0^1 m_i(t, ds) \tilde{\lambda}_i(t, s) dt, \end{aligned}$$

defining \mathcal{F}^* as the Fenchel conjugate of \mathcal{F} , we have:

$$\mathcal{F}^*(\Lambda^*(m, E)) := \begin{cases} \int_0^1 \sum_{i \in I} g_i(s) m_i(T, ds) & \text{if } (m, E) \text{ is a weak solution of (2.3.1)} \\ & \text{and } \int_0^1 m_i(t, ds) \leq D_i(t) \quad \forall (t, i) \in [0, T] \times I, \\ +\infty & \text{otherwise.} \end{cases}$$

For any $(x, y) \in E_1$, the functional \mathcal{G} is defined by:

$$\mathcal{G}(x, y) := \begin{cases} 0 & \text{if } -c_i(t, s) - x_i(t, s) + \sum_{j \in I, j \neq i} H(-y_{i,j}(t, s)) \leq 0 \quad \forall (t, s, i) \in (0, T) \times (0, 1) \times I, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, for any $(\varphi, \lambda) \in E_0$, it holds:

$$\mathcal{G}(\Lambda(\varphi, \lambda)) := \begin{cases} 0 & \text{if } -c_i(t, s) - \partial_t \varphi_i(t, s) - b_i(t, s) \partial_s \varphi_i(t, s) - \tilde{\lambda}_i(t, s) + \sum_{j \in I, j \neq i} H(-\Delta \varphi_{i,j}(t, s)) \leq 0 \\ & \forall (t, s, i) \in (0, T) \times (0, 1) \times I, \\ +\infty & \text{otherwise.} \end{cases}$$

Using that $L(x) = H^*(x)$, one can show, as in [21] for the quadratic case, that, for any $(v, w) \in \mathbb{R}^2$,

$$\sup_{a, b \in \mathbb{R}} \{av + bw; a + H(b) \leq 0\} = \begin{cases} L\left(\frac{w}{v}\right)v & \text{if } v > 0 \text{ and } w \geq 0, \\ 0 & \text{if } v = 0 \text{ and } w = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, with similar computations as in [53, Lemma 4.3], for any $(m, E) \in E'_1$, we have:

$$\begin{aligned}
& \mathcal{G}^*(-(m, E)) \\
&= \sup_{(x,y) \in E_1} \sum_{i \in I} \int_0^T \int_0^1 -x_i(t, s) m_i(t, ds) dt - \sum_{j \neq i} y_{i,j}(t, s) E_{i,j}(t, ds) dt - \mathcal{G}(x, y) \\
&= \sup_{(x,y) \in E_1} \sum_{i \in I} \int_0^T \int_0^1 (-x_i(t, s) - c_i(t, s) + c_i(t, s)) m_i(t, ds) - \sum_{j \neq i} y_{i,j}(t, s) E_{i,j}(t, ds) dt - \mathcal{G}(x, y) \\
&= \sum_{i \in I} \int_0^T \int_0^1 c_i(t, s) m_i(t, ds) dt + \sup_{(x,y) \in E_1} \sum_{i \in I} \int_0^T \int_0^1 x_i(t, s) m_i(t, ds) + \sum_{j \neq i} y_{i,j}(t, s) E_{i,j}(t, ds) dt \\
&\quad - \mathcal{G}(-x - c, -y) \\
&= \begin{cases} \int_0^T \int_0^1 \sum_{i \in I} c_i(t, s) m_i(t, ds) + \sum_{j \neq i} L \left(\frac{dE_{i,j}}{dm_i}(t, s) \right) m_i(t, ds) dt & \text{if } m > 0, E \geq 0 \text{ and } E \ll m, \\ 0 & \text{if } m = 0 \text{ and } E = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5.4}
\end{aligned}$$

The following lemma shows the constraint qualification for Problem (2.5.3).

Lemma 2.5.2. *There exists $(\varphi, \lambda) \in E_0$ such that $\mathcal{F}(\varphi, \lambda) < \infty$ and \mathcal{G} is continuous at $\Lambda(\varphi, \lambda)$.*

Proof. Let φ and λ be such that, for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,

$$\varphi_i(t, s) = -\max_{i \in I} (\|g_i\|_\infty) - 1,$$

and

$$\lambda_i(t) := \|c_i\|_\infty + 1,$$

Functions φ and λ being constant, it holds that $(\varphi, \lambda) \in E_0$ and $\mathcal{F}(\varphi, \lambda) < \infty$. Also, from the choice of φ and λ , it follows that, for any $i \in I$, $s \in [0, 1]$ and $t \in [0, T]$,

$$-c_i(t, s) - \partial_t \varphi_i(t, s) - b_i(t, s) \partial_s \varphi_i(t, s) - \lambda_i(t, s) + \sum_{j \in I, j \neq i} H(\varphi_i(t, s) - \varphi_j(t, s)) < 0.$$

Thus, \mathcal{G} is continuous at $\Lambda(\varphi, \lambda)$. □

The main result on the duality of this section is the following.

Theorem 2.5.1. *We have:*

$$\inf_{(\varphi, \lambda) \in \mathcal{K}_0} A(\varphi, \lambda) = - \inf_{(m, E) \in \mathcal{S}(m^0, D)} \tilde{J}(m, E)$$

Proof. One can observe that:

$$\inf_{(\varphi, \lambda) \in \mathcal{K}_0} A(\varphi, \lambda) = \inf_{(\varphi, \lambda) \in E_0} \mathcal{F}(\varphi, \lambda) + \mathcal{G}(\Lambda(\varphi, \lambda)),$$

and

$$\inf_{(m, E) \in \mathcal{S}(m^0, D)} \tilde{J}(m, E) = \inf_{(m, E) \in E'_1} \mathcal{F}(\Lambda^*(m, E)) + \mathcal{G}^*(-(m, E)),$$

Using Lemmas 2.5.2 and 2.5.1, the conclusion follows by applying the Fenchel-Rockafellar duality theorem [99]. □

For any $(\varphi, \lambda) \in (\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)) \times \mathcal{M}^+([0, T] \times I)$, we define the function

$$\tilde{A}(\varphi, \lambda) := \sum_{i \in I} \int_0^1 -\varphi_i(0, s) m_i^0(ds) + \int_0^T D_i(t) \lambda_i(dt). \tag{2.5.5}$$

Proposition 2.5.1. *There exists $\lambda \in \mathcal{M}^+([0, T] \times I)$ such that:*

$$\tilde{A}(\lambda, \varphi^\lambda) = \inf_{(\phi, \mu) \in \mathcal{K}_0} A(\phi, \mu),$$

where φ^λ is defined in (2.4.16).

Before proving this result, we need the following lemma.

Lemma 2.5.3. *For any $K > 0$, there exists a constant $C > 0$ such that, for any $(\varphi, \lambda) \in \mathcal{K}_0$ satisfying $A(\varphi, \lambda) \leq K$, we have:*

$$\sum_{i \in I} \int_0^T \lambda_i(t) dt \leq C.$$

Proof. Let $K \in \mathbb{R}_+^*$ and $(\varphi, \lambda) \in \mathcal{K}_0$ be such that $A(\varphi, \lambda) \leq K$. Since $(\varphi, \lambda) \in \mathcal{K}_0$, one can show, as in the proof of Lemma 2.5.1, that, for any $(i, s) \in I \times [0, 1]$,

$$\varphi_i(0, s) \leq T\|c\|_\infty + \|g\|_\infty + \sum_{i \in I} \int_0^T \lambda_i(t) dt.$$

Therefore, recalling that $A(\varphi, \lambda) \leq K$:

$$\sum_{i \in I} \int_0^T \lambda_i(t) (D_i(t) - m_i([0, 1])) dt \leq K + T\|c\|_\infty + \|g\|_\infty.$$

From Assumption 3, there exists $\varepsilon^0 > 0$ such that $D_i(t) - m_i([0, 1]) > \varepsilon^0$ for any $(t, i) \in [0, T] \times I$. Setting $C := (K + T\|c\|_\infty + \|g\|_\infty)/\varepsilon^0$, we have:

$$\sum_{i \in I} \int_0^T \lambda_i(t) dt \leq C.$$

□

We are now ready to prove Proposition 2.5.1.

Proof of Proposition 2.5.1. Let $\{(\phi^n, \lambda^n)\}_n$ be a minimizing sequence of (2.5.3). For any $n \in \mathbb{N}$, we consider φ^n the classical solution of (2.2.5) associated to λ^n . Since φ^n is a classical solution and ϕ^n satisfies (2.5.1) on $[0, T] \times [0, 1] \times I$, we can easily check that the following comparison holds: $\varphi^n \leq \phi^n$. Thus, $A(\varphi^n, \lambda^n) \leq A(\phi^n, \lambda^n)$ for any $n \in \mathbb{N}$. Therefore, $\{(\varphi^n, \lambda^n)\}_n$ is also a minimizing sequence and, there exist $K > 0$ and $n^0 > 0$ such that for any $n \geq n^0$ one has $A(\varphi^n, \lambda^n) \leq K$. From Lemma 2.5.3, the sequence $\{\sum_{i \in I} \int_0^T \lambda_i^n(t) dt\}_n$ is uniformly bounded. Thus, a subsequence of $\{\lambda^n\}_n$ weakly converges to a measure $\lambda \in \mathcal{M}^+([0, T] \times I)$ w.r.t. the weak topology in $\mathcal{M}^+([0, T] \times I)$. We set $\varphi^\lambda := \Theta(\lambda) + L^\lambda$ and $\varphi^n := \Theta(\lambda^n) + L^{\lambda^n}$ for any $n \in \mathbb{N}$, where Θ is defined in (2.4.10). By the same arguments as in the proof of Lemma 2.4.6, one has $\lim_{n \rightarrow \infty} \|\varphi^\lambda(0, \cdot) - \varphi^n(0, \cdot)\|_\infty = 0$ up to a subsequence of $\{\varphi^n\}_n$ and, therefore, we have:

$$\lim_{n \rightarrow \infty} A(\varphi^n, \lambda^n) = \tilde{A}(\varphi^\lambda, \lambda),$$

and the conclusion follows. □

2.6 Characterization of the minimizers

The purpose of this section is to define and characterize the solutions of Problem (2.3.4). We show that the following system gives optimality conditions for (2.3.4):

$$\left\{ \begin{array}{ll} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \lambda_i + \sum_{j \in I, j \neq i} H(\varphi_i - \varphi_j) = 0 & \text{on } (0, T) \times (0, 1) \times I, \\ \partial_t m_i + \partial_s(m_i b_i) + \sum_{j \neq i} H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j = 0 & \text{on } (0, T) \times (0, 1) \times I, \\ m_i(0, s) = m_i^0(s), \varphi_i(T, s) = g_i(s) & \text{on } (0, 1) \times I, \\ \int_0^1 m_i(t, ds) - D_i(t) \leq 0, \lambda \geq 0 & \text{on } [0, T] \times I, \\ \sum_{i \in I} \int_0^T \left(\int_0^1 m_i(t, ds) - D_i(t) \right) \lambda_i(dt) = 0. & \end{array} \right. \quad (2.6.1)$$

The notion of weak solutions of system (2.6.1) is given in the following definition.

Definition 2.6.1. A triplet $(\varphi, \lambda, m) \in (\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)) \times \mathcal{M}^+([0, T] \times I) \times C^0([0, T], \mathcal{P}([0, 1] \times I))$ is called a weak solution of (2.6.1) if it satisfies the following conditions:

1. The function φ is a weak solution of (2.2.5), associated to λ in the sense of Definition 2.4.1;
2. m satisfies the continuity equation:

$$\partial_t m_i + \partial_s(m_i b_i) + \sum_{j \neq i} H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j = 0, \quad m_i(0, \cdot) = m_i^0,$$

in the sense of Definition 2.3.1, with $\alpha_{i,j} := H'(\varphi_i - \varphi_j)$;

3. it holds that, for any $t \in [0, T]$,

$$\int_0^1 m_i(t, ds) - D_i(t) \leq 0 \quad \text{and} \quad \sum_{i \in I} \int_0^T \left(\int_0^1 m_i(t, ds) - D_i(t) \right) \lambda_i(dt) = 0.$$

Remark 2.6.1. Since $\varphi \in \text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)$ and H' is Lipschitz continuous, the control $\alpha_{i,j} := H'(\varphi_i - \varphi_j)$ is bounded on $[0, T] \times [0, 1] \times I$ and $\partial_s \alpha_{i,j} \in L^\infty([0, T] \times [0, 1])$. Thus, $\alpha_{i,j}$ is in $L^2_{m_i}([0, T] \times [0, 1])$ and the forward equation in (2.6.1) makes sense.

The following theorem states the optimality conditions of Problem (2.3.4).

Theorem 2.6.1. 1. If $(m, E) \in \mathcal{S}(m^0, D)$ is a minimizer of Problem (2.3.4), and φ a weak solution of (2.2.5) in the sense of Definition 2.4.1 associated to λ satisfying $\tilde{A}(\varphi, \lambda) = \inf_{(\phi, \mu) \in \mathcal{K}_0} A(\phi, \mu)$, then (φ, λ, m)

is a weak solution of (2.6.1) and $\frac{dE_{i,j}}{dm_i} = H'(\varphi_i - \varphi_j)$ on $\{m_i > 0\}$ for any $i, j \in I$.

2. Conversely, if (φ, λ, m) is a weak solution of (2.6.1), then $\tilde{A}(\varphi, \lambda) = \inf_{(\phi, \mu) \in \mathcal{K}_0} A(\phi, \mu)$ and there exists E ,

defined for any $i, j \in I$ by $\frac{dE_{i,j}}{dm_i} := H'(\varphi_i - \varphi_j)$, such that $(m, E) \in \mathcal{S}(m^0, D)$ is a minimizer of (2.3.4).

Remark 2.6.2. If (φ, λ, m) is a weak solution of (2.6.1), then (φ, λ) is a minimizer of a relaxed version of Problem (2.5.3), i.e. (φ, λ) is the minimum of \tilde{A} over the space $(\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)) \times \mathcal{M}^+([0, T] \times I)$.

2.6.1 Proof of Theorem 6.1

Before the proof of Theorem 2.6.1, we make the following remark.

Remark 2.6.3. For any $\lambda \in \mathcal{M}^+([0, T] \times I)$ and any $(m, E) \in \mathcal{S}(m^0, D)$ one has $-\tilde{J}(m, E) \leq \tilde{A}(\varphi^\lambda, \lambda)$. Indeed, considering a sequence $\{(\lambda^n, \varphi^n)\}_n$ defined as in Lemma 2.4.6 and using the proof of Theorem 2.5.1, we get: $-\tilde{J}(m, E) \leq A(\varphi^n, \lambda^n)$ and, therefore, $-\tilde{J}(m, E) \leq \tilde{A}(\varphi^\lambda, \lambda)$.

Proof of Theorem 2.6.1. 1. By Theorem 2.5.1, one has:

$$\inf_{(\hat{\varphi}, \hat{\lambda}) \in \mathcal{K}_0} A(\hat{\varphi}, \hat{\lambda}) = - \inf_{(\hat{m}, \hat{E}) \in \mathcal{S}(m^0, D)} \tilde{J}(\hat{m}, \hat{E}),$$

and thus

$$\sum_{i \in I} \int_0^1 g_i m_i(T) - \varphi_i(0) m_i^0 + \int_0^T D_i \lambda_i + \int_0^T \int_0^1 \left(c_i + \sum_{j \neq i} L \left(\frac{dE_{i,j}}{dm_i} \right) \right) m_i = 0. \quad (2.6.2)$$

We want to show that $E_{i,j} = H'(\varphi_i - \varphi_j) m_i$. We consider a sequence $\{(\lambda^n, \varphi^n)\}_n$ defined as in Lemma 2.4.6. For any $n \in \mathbb{N}$, φ^n is smooth enough to be a test function for the weak formulation of (2.1.1) satisfied by m . According to Lemma 2.4.6 and the fact that $D \in C^0([0, T] \times I)$, it holds that, for any $n \in \mathbb{N}$ and $i \in I$,

$$\begin{aligned} & \sum_{i \in I} \int_0^1 g_i m_i(T) - \varphi_i(0) m_i^0 + \int_0^T \lambda_i D_i \\ &= \lim_{n \rightarrow \infty} \sum_{i \in I} \int_0^1 g_i m_i(T) - \varphi_i^n(0) m_i^0 + \int_0^T D_i \lambda_i^n \\ &= \lim_{n \rightarrow \infty} \sum_{i \in I} \int_0^T \int_0^1 \left(\partial_t \varphi_i^n + b_i \partial_s(\varphi_i^n) + \sum_{j \in I, j \neq i} (\varphi_j^n - \varphi_i^n) \frac{dE_{i,j}}{dm_i} \right) m_i + \int_0^T \lambda_i^n D_i \\ &= \lim_{n \rightarrow \infty} \sum_{i \in I} \int_0^T \int_0^1 \left(-c_i + \sum_{j \in I, j \neq i} H(\varphi_i^n - \varphi_j^n) + \sum_{j \in I, j \neq i} (\varphi_j^n - \varphi_i^n) \frac{dE_{i,j}}{dm_i} \right) m_i + \int_0^T \lambda_i^n \left(D_i - \int_0^1 m_i \right). \end{aligned}$$

By the previous equality and (2.6.2),

$$\lim_{n \rightarrow \infty} \sum_{i \in I} \int_0^T \int_0^1 \left(\sum_{j \in I, j \neq i} H(\varphi_i^n - \varphi_j^n) + L \left(\frac{dE_{i,j}}{dm_i} \right) + (\varphi_j^n - \varphi_i^n) \frac{dE_{i,j}}{dm_i} \right) m_i + \int_0^T \lambda_i^n \left(D_i - \int_0^1 m_i \right) = 0.$$

According to Lemma 2.4.6, for a.e. $t \in [0, T]$, the sequence $\{\varphi(t, \cdot)\}_n$ converges uniformly to φ . By the continuity of H and the dominated convergence theorem, we get $\int_0^T \int_0^1 m_i H(\varphi_i^n - \varphi_j^n)$ converges to $\int_0^T \int_0^1 m_i H(\varphi_i - \varphi_j)$ for any $i, j \in I$. Since (m, E) is a solution of (2.3.4), $\tilde{J}(m, E)$ is finite. Then one can show that, for any $i, j \in I$, $\int_0^T \int_0^1 E_{i,j} < \infty$. Applying dominated convergence theorem, we can show that: $\int_0^T \int_0^1 (\varphi_j^n - \varphi_i^n) E_{i,j}$ converges to $\int_0^T \int_0^1 (\varphi_j - \varphi_i) E_{i,j}$. Since, for any $i \in I$, the map $t \mapsto D_i(t) - \int_0^1 m_i(t, ds)$ is continuous, the weak convergence of λ^n to λ in $\mathcal{M}^+([0, T] \times I)$ gives:

$$\lim_{n \rightarrow \infty} \sum_{i \in I} \int_0^T \lambda_i^n \left(D_i - \int_0^1 m_i \right) = \sum_{i \in I} \int_0^T \lambda_i \left(D_i - \int_0^1 m_i \right).$$

Thus,

$$\int_0^T \int_0^1 \left(\sum_{j \in I, j \neq i} H(\varphi_i - \varphi_j) + L \left(\frac{dE_{i,j}}{dm_i} \right) + (\varphi_j - \varphi_i) \frac{dE_{i,j}}{dm_i} \right) m_i + \int_0^T \lambda_i \left(D_i - \int_0^1 m_i \right) = 0. \quad (2.6.3)$$

Since $\lambda \geq 0$ and $\int_0^1 m_i(t, ds) \leq D_i(t)$ for any $t \in [0, T]$, one has, for any $i \in I$ and $t \in [0, T]$,

$$0 \leq \int_0^T \left(D_i(t) - \int_0^1 m_i(t, ds) \right) \lambda_i(dt). \quad (2.6.4)$$

Recalling that $L^*(p) = H(p)$, we have $L(p) + H(q) - pq \geq 0$ for any $p, q \in \mathbb{R}$. Thus, by inequality (2.6.4) and

equality (2.6.3), one deduces

$$\sum_{j \in I, j \neq i} H(\varphi_i - \varphi_j) + L\left(\frac{dE_{i,j}}{dm_i}\right) + (\varphi_j - \varphi_i) \frac{dE_{i,j}}{dm_i} = 0 \quad m - \text{a.e.}$$

Therefore,

$$\frac{dE_{i,j}}{dm_i}(t, s) = H'(\varphi_i(t, s) - \varphi_j(t, s)) \quad m - \text{a.e.}, \quad (2.6.5)$$

and inequality (2.6.4) becomes an equality. Thus, $\int_0^T \left(D_i - \int_0^1 m_i(t, ds)\right) \lambda_i(dt) = 0$. By equality (2.6.5), the properties of H and the fact that $\varphi \in \text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)$, one has $\frac{dE_{i,j}}{dm_i} \in L^\infty([0, T] \times [0, 1])$ and $\partial_s \left(\frac{dE_{i,j}}{dm_i}\right) \in L^\infty((0, T) \times (0, 1))$. Thus, by Proposition 2.8.1 in Appendix, we deduce that $m \in \text{Lip}([0, T], \mathcal{P}([0, 1] \times I))$.

2. We assume now that (λ, φ, m) is a weak solution of (2.6.1). Since φ is in $\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)$ and λ is a finite measure, the quantity $\tilde{A}(\varphi, \lambda)$ is well defined. We want to show that $\tilde{A}(\varphi, \lambda) + \tilde{J}(m, E) = 0$. We approximate (λ, φ) by the sequence $\{(\lambda^n, \varphi^n)\}_n$ defined in Lemma 2.4.6.(1). For any n , φ^n is smooth enough to be considered as a test function for the equation (2.1.1) satisfied in the weak sense by m . We have, for any $i \in I$,

$$\sum_{i \in I} \int_0^1 g_i m_i(T) - \varphi_i^n(0) m_i^0 + \sum_{i \in I} \int_0^T \int_0^1 -m_i b_i \partial_s \varphi_i^n - m_i \partial_t \varphi_i^n + \sum_{j \in I, j \neq i} (\varphi_i^n - \varphi_j^n) H'(\varphi_i - \varphi_j) m_i = 0. \quad (2.6.6)$$

For any $i \in I$, φ_i^n is a classical solution of (2.2.5) associated to λ^n . Multiplying (2.2.5) by m_i , summing over I and integrating over $[0, T] \times [0, 1]$, we have:

$$\sum_{i \in I} \int_0^T \int_0^1 -m_i \partial_t \varphi_i^n - m_i b_i \partial_s \varphi_i^n - m_i c_i - m_i \lambda^n + \sum_{j \in I, j \neq i} H(\varphi_i^n - \varphi_j^n) m_i = 0. \quad (2.6.7)$$

Combining (2.6.6) and (2.6.7) yields

$$\sum_{i \in I} \int_0^1 g_i m_i(T) - \varphi_i^n(0) m_i^0 + \sum_{i \in I} \int_0^T \int_0^1 c_i m_i + \lambda_i^n m_i + m_i \left(\sum_{\substack{j \in I, \\ j \neq i}} H'(\varphi_i - \varphi_j) (\varphi_i^n - \varphi_j^n) - H(\varphi_i^n - \varphi_j^n) \right) = 0.$$

Since (φ, λ, m) is a weak solution of (2.6.1), by Lemma 2.4.6, and letting n tend to infinity, one deduces:

$$\sum_{i \in I} \int_0^1 g_i m_i(T) - \varphi_i(0) m_i^0 + \sum_{i \in I} \int_0^1 D_i \lambda_i + \sum_{i \in I} \int_0^T \int_0^1 c_i m_i + m_i \left(\sum_{\substack{j \in I, \\ j \neq i}} H'(\varphi_i - \varphi_j) (\varphi_i - \varphi_j) - H(\varphi_i - \varphi_j) \right) = 0.$$

By the definition of L and H , we have:

$$\sum_{i \in I} \int_0^1 g_i m_i(T) - \varphi_i(0) m_i^0 + \sum_{i \in I} \int_0^1 D_i \lambda_i + \sum_{i \in I} \int_0^T \int_0^1 c_i m_i + m_i \left(\sum_{j \in I, j \neq i} L(H'(\varphi_i - \varphi_j)) \right) = 0.$$

By the definition of \tilde{A} in (2.5.5) and \tilde{J} in (2.3.3), we have $\tilde{A}(\varphi, \lambda) + \tilde{J}(m, E) = 0$. Finally, by Remark 2.6.3, one deduces that (m, E) is a minimizer of (2.3.4). \square

2.6.2 Proof of Theorem 2.1

We are now ready to prove our main theorem by using Theorem 1 and applying the change of variable $\alpha_{i,j} := \frac{dE_{i,j}}{m_i}$.

Proof of Theorem 2.2.1. The existence of a solution to Problem (2.1.4) is given by Lemma 2.3.3.

1 This statement is proved by Theorem 2.6.1.1

2 This point is given by Theorem 2.6.1.2.

3 The uniform bound on α and $\partial_s \alpha$ are deduced by Theorem 2.2.1.1, using the fact that H has a globally Lipschitz continuous gradient and that φ is in $\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)$. The time regularity of m is obtained by Proposition 2.8.2 in Appendix 2.8. \square

We now prove Proposition 2.2.1.

Proof of Proposition 2.2.1. Let (m, α) be a solution of Problem (2.1.4) and μ^0 be the density of m^0 w.r.t. the Lebesgue measure. By a fixed point argument, it is easy to check that there exists a unique solution $\mu \in \text{Lip}([0, T] \times [0, 1] \times I)$ of the following equation on $[0, T] \times [0, 1] \times I$:

$$\begin{aligned} \mu_i(t, s) = & \mu_i^0(S_i^{t,s}(0)) + \int_0^t \mu_i(\tau, S_i^{t,s}(\tau)) \partial_s b_i(S_i^{t,s}(\tau)) d\tau \\ & + \int_0^t \sum_{j \in I, j \neq i} -\alpha_{i,j}(\tau, S_i^{t,s}(\tau)) \mu_j(\tau, S_i^{t,s}(\tau)) + \alpha_{j,i}(\tau, S_i^{t,s}(\tau)) \mu_j(\tau, S_i^{t,s}(\tau)) d\tau. \end{aligned} \quad (2.6.8)$$

Denote by \mathcal{L} the Lebesgue measure on $[0, T] \times [0, 1]$. The conclusion follows by proving that $\mu \mathcal{L}$ is the unique weak solution of (2.1.1). \square

2.7 Sensitivity analysis of the value of the optimization problem w.r.t. the data

In this section we study how the value of Problem (2.1.4) depends on the initial distribution $m^0 \in \mathcal{P}([0, 1] \times I)$ and on the parameter $D \in C^0([0, T] \times I, \mathbb{R}_+^*)$ of the constraint (2.1.2). We endow the space $\Omega := \mathcal{P}([0, 1] \times I) \times C^0(I \times [0, T], \mathbb{R}_+^*)$ with the distance \mathcal{D}_Ω defined by:

$$\mathcal{D}_\Omega((m^0, D), (\bar{m}^0, \bar{D})) := \mathcal{W}(m^0, \bar{m}^0) + \|D - \bar{D}\|_\infty,$$

where \mathcal{W} is the Wasserstein distance on $\mathcal{P}([0, 1] \times I)$. We recall that the definition of Ω_ε is given in (2.2.1). For any $\varepsilon > 0$ we consider the function $\mathcal{V} : \Omega_\varepsilon \rightarrow \mathbb{R}$ defined by:

$$\mathcal{V}(m^0, D) := \inf_{(m, E) \in \mathcal{S}(m^0, D)} \tilde{J}(m, E), \quad (2.7.1)$$

where the set $\mathcal{S}(m^0, D)$ is defined in (2.3.2).

The main result of this section is the following proposition, which shows the Lipschitz continuity of the value of the problem (2.1.4) w.r.t. the initial distribution and the congestion constraint (2.1.2).

Proposition 2.7.1. *For any $\varepsilon > 0$, \mathcal{V} is Lipschitz continuous on Ω_ε w.r.t. the distance \mathcal{D}_Ω .*

To prove the proposition, we need to introduce some lemmas. For any $(m^0, D) \in \Omega_\varepsilon$, we consider the function $A[m^0, D] : C^0([0, T] \times [0, 1] \times I) \times C^0([0, T] \times I, \mathbb{R}_+) \rightarrow \mathbb{R}$ defined by:

$$A[m^0, D](\varphi, \lambda) := \sum_{i \in I} \int_0^1 -\varphi_i(0, s) m_i^0(ds) + \int_0^T \lambda_i(t) D_i(t) dt.$$

The following result gives some properties of the function \mathcal{V} .

Lemma 2.7.1. *For any $\varepsilon > 0$, the function \mathcal{V} is bounded independently of ε , convex and l.s.c. on Ω_ε .*

Proof. Let $\varepsilon > 0$ and $C := (T\|c\|_\infty + \|g\|_\infty)$. By the definition of \mathcal{V} in (2.7.1), one can show, for any $(m^0, D) \in \Omega_\varepsilon$, that $|\mathcal{V}(m^0, D)| \leq C$. For any $(m^0, D) \in \Omega_\varepsilon$, the duality result in Theorem 2.5.1 gives

$$\mathcal{V}(m^0, D) = \sup_{(\varphi, \lambda) \in \mathcal{K}_0} -A[m^0, D](\varphi, \lambda). \quad (2.7.2)$$

Since \mathcal{V} is the supremum of continuous and linear functions, we deduce that \mathcal{V} is convex and l.s.c. on Ω_ε . \square

For any $\varepsilon > 0$ and $(m^0, D) \in \Omega_\varepsilon$, we know by Proposition 2.5.1 that there exists $(\varphi^{m^0, D}, \lambda^{m^0, D}) \in (\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)) \times \mathcal{M}^+([0, T] \times I)$ such that $\varphi^{m^0, D}$ is a weak solution of (2.2.5), in the sense of Definition 2.4.1, associated to $\lambda^{m^0, D}$ and such that $(\varphi^{m^0, D}, \lambda^{m^0, D})$ satisfies:

$$\sum_{i \in I} \int_0^1 \varphi_i^{m^0, D}(0, s) m_i^0(ds) - \int_0^T D_i(t) \lambda_i^{m^0, D}(dt) = - \inf_{(\varphi, \lambda) \in \mathcal{K}_0} A[m^0, D](\varphi, \lambda) = \mathcal{V}(m^0, D).$$

The next lemma provides an estimate on $\varphi^{m^0, D}$ and $\lambda^{m^0, D}$ for any $\varepsilon > 0$ and $(m^0, \lambda) \in \Omega_\varepsilon$.

Lemma 2.7.2. *For any $\varepsilon > 0$, there exists $C > 0$ such that, for any $(m^0, D) \in \Omega_\varepsilon$,*

$$\max(\|\varphi^{m^0, D}\|_\infty, \|\partial_s \varphi^{m^0, D}\|_\infty, \lambda^{m^0, D}(I \times [0, T])) \leq C.$$

Proof. Let $\varepsilon > 0$ and $(m^0, D) \in \Omega_\varepsilon$. According to Lemma 2.7.1, there exists a constant $K > 0$, independent of ε , such that $\sum_{i \in I} \int_0^1 \varphi_i^{m^0, D}(0, s) m_i^0(ds) - \int_0^T D_i(t) \lambda_i^{m^0, D}(dt) < K$. Thus, by using same arguments as in the proof of Lemma 2.5.3 and setting $\tilde{K} := (K + T\|c\|_\infty + \|g\|_\infty)/\varepsilon$, one obtains:

$$\lambda^{m^0, D}(I \times [0, T]) \leq \tilde{K}. \quad (2.7.3)$$

By Remark 2.4.7 and the previous inequality, there exists a constant \tilde{C} , which depends on \tilde{K} , such that $\|\varphi^{m^0, D}\|_\infty$ and $\|\partial_s \varphi^{m^0, D}\|_\infty$ are bounded by \tilde{C} . The conclusion follows by setting $C := \max(\tilde{C}, \tilde{K})$. \square

We are now ready to prove the Lipschitz regularity of \mathcal{V} .

Proof of Proposition 2.7.1. Let $\varepsilon > 0$ and $(m^0, D), (\bar{m}^0, \bar{D}) \in \Omega_\varepsilon$. By the definition of \mathcal{V} in (2.7.1) and Lemma 2.7.2, one has:

$$\begin{aligned} & \mathcal{V}(m^0, D) \\ & \leq \sum_{i \in I} \int_0^1 \varphi_i^{m^0, D} \bar{m}_i^0(ds) - \sum_{i \in I} \int_0^T \bar{D}_i(t) \lambda_i^{m^0, D}(dt) + \|\partial_s \varphi^{m^0, D}\|_\infty \mathcal{W}(m^0, \bar{m}^0) + \lambda^{m^0, D}(I \times [0, T]) \|D - \bar{D}\|_\infty \\ & \leq \mathcal{V}(\bar{m}^0, \bar{D}) + C \mathcal{D}_\Omega((m^0, D), (\bar{m}^0, \bar{D})), \end{aligned}$$

where $C > 0$ is a constant defined in Lemma 2.7.2. Similarly, we have:

$$\mathcal{V}(\bar{m}^0, \bar{D}) \leq \mathcal{V}(m^0, D) + C \mathcal{D}_\Omega((m^0, D), (\bar{m}^0, \bar{D})).$$

The conclusion follows. \square

2.8 Appendix

Some properties of the weak solution of the continuity equation (2.1.1) are derived in this subsection. Assumptions in Section 2.1 are in force in the Appendix. A first result on the support of the solution is established in Lemma 2.8.1.

Lemma 2.8.1. *For any weak solution (α, m) of (2.1.1) in the sense of Definition 2.3.1, $m(t)$ has a support contained in $[0, 1] \times I$ for any $t \in [0, T]$.*

Proof. Let $\varepsilon > 0$ and $\varphi^\varepsilon \in C_c^\infty([0, T] \times \mathbb{R} \times I)$ such that, for any $t \in [0, T]$ and $i \in I$,

$$\varphi_i^\varepsilon(t, s) \in [0, 1], \quad \forall s \in \mathbb{R}; \quad \varphi_i^\varepsilon(t, s) = 0, \quad \forall s \in \mathbb{R} \setminus (-1 - \varepsilon, 2 + \varepsilon); \quad \text{and} \quad \varphi_i^\varepsilon(t, s) = 1 \quad \forall s \in [-1, 2].$$

Since m is a weak solution of (2.1.1), b satisfies Assumption 1, and $\varphi_i = \varphi_j$ for any $i, j \in I$, we deduce that, for any $t \in (0, T)$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \sum_{i \in I} \varphi_i^\varepsilon(t, s) m_i(t, ds) &= \int_{\mathbb{R}} \sum_{i \in I} \partial_s \varphi_i^\varepsilon(t, s) b_i(s) m_i(t, ds) \\ &= \int_{-1-\varepsilon}^{-1} \sum_{i \in I} \partial_s \varphi_i^\varepsilon(t, s) b_i(s) m_i(t, ds) + \int_2^{2+\varepsilon} \sum_{i \in I} \partial_s \varphi_i^\varepsilon(t, s) b_i(s) m_i(t, ds) \\ &= 0. \end{aligned} \quad (2.8.1)$$

By (2.8.1) and the continuity of m , we deduce that $t \mapsto \int_{\mathbb{R}} \sum_{i \in I} \varphi_i^\varepsilon(t, s) m_i(t, ds)$ is constant on $[0, T]$. Let ε

tend to $+\infty$, it holds that $t \mapsto \int_{\mathbb{R}} \sum_{i \in I} m_i(t, ds)$ is constant over $[0, T]$. Then, we have for any $t \in (0, T)$,

$\int_{\mathbb{R}} \sum_{i \in I} m_i(t, ds) = \int_{\mathbb{R}} \sum_{i \in I} m_i^0(ds) = 1$. Now let us show that $\int_0^1 \sum_{i \in I} m_i(t, ds) = 1$. Let $\varepsilon > 0$ and ψ^ε be another test function in $C_c^\infty([0, T] \times \mathbb{R} \times I)$ such that, for any $t \in [0, T]$ and $i \in I$,

$$\begin{aligned} \psi_i^\varepsilon(t, s) &= 0, \quad \forall s \in \mathbb{R} \setminus (-\varepsilon, 1 + \varepsilon); \quad \partial_s \psi_i^\varepsilon(t, s) \geq 0, \quad \forall s \in (-\varepsilon, 0); \quad \partial_s \psi_i^\varepsilon(t, s) \leq 0, \quad \forall s \in (1, \varepsilon); \\ \text{and } \psi_i^\varepsilon(t, s) &= 1, \quad \forall s \in [0, 1]. \end{aligned}$$

By the same computation as in (2.8.1) and Assumption 1, one has, for any $t \in (0, T)$,

$$\frac{d}{dt} \int_{\mathbb{R}} \sum_{i \in I} \psi_i^\varepsilon(t, s) m_i(t, ds) = \int_{-\varepsilon}^0 \sum_{i \in I} \partial_s \psi_i^\varepsilon(t, s) b_i(s) m_i(t, ds) + \int_1^{1+\varepsilon} \sum_{i \in I} \partial_s \psi_i^\varepsilon(t, s) b_i(s) m_i(t, ds) \geq 0.$$

Thus, $t \mapsto \int_{\mathbb{R}} \sum_{i \in I} \psi_i^\varepsilon(t, s) m_i(t, ds)$ is non-decreasing on $[0, T]$. Taking the limit $\varepsilon \rightarrow 0$, the map $t \mapsto \int_0^1 \sum_{i \in I} m_i(t, ds)$

is also non-decreasing on $[0, T]$. Finally, for any $t \in [0, T]$: $1 = \int_0^1 \sum_{i \in I} m_i(0, ds) \leq \int_0^1 \sum_{i \in I} m_i(t, ds) \leq$

$$\int_{\mathbb{R}} \sum_{i \in I} m_i(t, ds) = 1. \quad \square$$

For any pair of weak solution (m, α) of (2.1.1), the next lemma provides some regularity on m if α and $\partial_s \alpha$ are bounded.

Lemma 2.8.2. *Let $m^0 \in \mathcal{P}([0, 1] \times I)$ and (m, α) be a weak solution of (2.1.1) in the sense of Definition 2.3.1, with $\alpha \in L^\infty([0, T] \times I \times I \times [0, 1], \mathbb{R}_+)$. Then, m belongs to $\text{Lip}([0, T], \mathcal{P}([0, 1] \times I))$ with a Lipschitz constant independent of m^0 .*

Proof. As in Lemma 2.3.1, following standard arguments and computations from Optimal Transport [9], one can show that, for any $t, \tilde{t} \in [0, T]$,

$$\mathcal{W}(m(t, \cdot), m(\tilde{t}, \cdot)) \leq |t - \tilde{t}| (\|I\| \|b\|_\infty + \|\alpha\|_\infty).$$

The conclusion follows. \square

Finally, the next Proposition states that, for any α , the existence and uniqueness of an m such that (m, α) is a weak solution of (2.1.1).

Proposition 2.8.1. *Let $m^0 \in \mathcal{P}([0, 1] \times I)$ and $\alpha \in L^\infty([0, T] \times I \times I \times [0, 1])$ satisfy $\partial_s \alpha \in L^\infty([0, T] \times I \times I \times [0, 1])$. Then, there exists a unique $m \in \text{Lip}([0, T], \mathcal{P}([0, 1] \times I))$ such that (m, α) is a weak solution of (2.1.1) in the sense of Definition 2.3.1.*

Proof. The existence and uniqueness of a weak solution are proved in [71] for controls α that are continuous in space and time independent. The extension of this result to bounded controls that are measurable in time is straightforward. \square

Chapter 3

Computation and implementation of an optimal mean field control for smart charging

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3.1 Introduction

EVs penetration in the global automotive market is a phenomenon observed since few years. Integration of electric mobility needs to be prepared, in the context of energy transition and penetration of renewable energy. If correctly managed, PEV fleets can be considered as a non negligible virtual battery, and thus, be a resource to strengthen the stability and the performance of the electrical network. PEV fleets can take part in ancillary services such as frequency and voltage regulation [250], peak shaving [259], valley filling [117] and [182], spinning reserve [207] and demand side management [191]. However, a non coordinated power demand of PEVs, can disrupt the energy balance on the electrical network during peak hours [181].

The present paper analyses the optimal strategy, for a central planner aiming at controlling a large population of PEVs and minimizing the overall operational costs, in a context of limited power supply and mitigation of synchronization effects of the power demand. Combinatorial techniques as well as optimal control tools fail to solve problems with large population of PEVs, due to the curse of dimension [20]. To overcome these difficulties, a continuum of PEVs is considered in this paper, leading to optimal control of PDE techniques. The SoC is a relative quantity that indicates the ratio of the remaining capacity to the present maximum available capacity of the battery. The population is assumed to be homogeneous, in the sense that the batteries of the PEVs share the same characteristic (battery capacity, charging rate, etc.). The problem is solved numerically, in discrete time and space settings, by applying the Chambolle-Pock algorithm [67]. The state of each vehicle is composed

of two variables: the SoC of its battery $s \in [0, 1]$ and its charging mode $i \in \mathcal{K}$. The charging mode of a vehicle indicates its charging rate (not charging, charging in fast or slow mode, etc.). Besides, all the charging terminals are supposed to be identical with discrete charging rates.

3.1.1 Relative literature

Both convex optimization methods [117], [276] and non convex optimization methods [159] have been applied to determine the optimal charging scheduling for a fleet of PEVs. Heuristic approaches have also been proposed to get minimum charging cost [49]. To deal with large fleets of PEVs in a market based framework, the mean field game theory has been applied in the context of smart charging [77], [204]. Also, the control-oriented aggregate load PDE modeling has been first investigated to model a large number of homogeneous PEVs, in a framework different from optimization. It has been successfully applied in [15] where the aggregate charging power of PEVs is adapted to highly intermittent renewable power, and in [97] to perform renewable power tracking. Optimal control of PDE for smart charging has been studied in a linear optimization framework in [170], where PEV fleet provide regulation and vehicle-to-grid (V2G) services, and in [241] where the optimal dispatch of an autonomous fleet of taxis is analyzed, to serve passengers and electric power demand during outages. Optimal control of PDE is adopted in a linear quadratic framework in [171], combined with Model Predictive Control (MPC) techniques, to solve the optimization problem of smart charging in real time.

3.1.2 Our contributions

In this paper, a strategy to charge a large population of PEVs, over a finite period of time, is given by optimal control of PDE techniques. Among the existing literature, [170] and [171] are most close to our paper. However, contributions are different in several regards. Our paper innovates in the mitigation of charging cycles per PEVs and synchronization effect in a PDE-based model framework. Indeed, the synchronization of PEVs can disrupt energy balance on the electrical network [256], while high charging frequency, is responsible for stronger battery aging and degradation [88]. Two different cost functions, used to mitigate the effects described above, are analyzed in the two case studies, at the end of the paper. This penalization leads us to consider convex cost functions in contrast to linear and quadratic ones in [170] and [171]. A constraint on the aggregate of the load is considered as well as constraints on the SoC of the PEVs at the final moment of the period, while [170] considers an aggregate constraint on the V2G service in the linear case only. Furthermore, to the best of our knowledge, our paper is the first one to apply the Chambolle-Pock algorithm, to solve a constraint problem of optimal control of PDE in the context of EV smart charging.

The remainder of this paper is organized as follows. Section 3.2 presents the evolution of the system in a continuous time and space and then in discrete time and space settings. The optimization problem is formulated in Section 3.3 and the Chambolle-Pock algorithm is introduced in Section 3.4. Section 3.5 is dedicated to two case studies, in a peak and off-peak hours pricing framework, and the results are analyzed.

3.2 EV population dynamics

3.2.1 Evolution of the system in continuous settings

The density of the PEVs at time $t \in [0, T]$, in mode $i \in \mathcal{K}$ and SoC $s \in [0, 1]$, is denoted by $m_i(t, s)$, and its corresponding flow switching to mode j $E_{i,j}(t, s)$. The instantaneous charging rate for an PEV with a SoC s and in mode i is $b_i(s)$. The set \mathcal{K} is finite, and $I := \text{card}(\mathcal{K})$ denotes the cardinality of \mathcal{K} . By choosing the flow E , the central planner controls how PEVs switch from one charging mode to another. The total population of EV is supposed constant over the period $[0, T]$, with no departures or arrivals. From the conservation law, and using similar arguments as in [171], the evolution of m_i over the period $[0, T]$ is given for all $i \in \mathcal{K}$ by:

$$\partial_t m_i(t, s) + \partial_s (m_i(t, s) b_i(s)) = - \sum_{j \in \mathcal{K} \setminus \{i\}} (E_{i,j}(t, s) - E_{j,i}(t, s)). \quad (3.2.1)$$

with the initial condition :

$$m_i(0, \cdot) = \bar{m}_i^0, \quad (3.2.2)$$

where \bar{m}_i^0 is the given initial distribution of the SoC for EVs in mode i . Two constraints are examined in this paper. A congestion constraint, for which for any mode i , the total proportion of EV is bounded above by

$\bar{D}_i \geq 0$:

$$\int_0^1 m_i(t, s) ds \leq \bar{D}_i \quad \forall t \in [0, T]. \quad (3.2.3)$$

Inequality (3.2.3) stands for the limited power constraint. A minimal SoC $\underline{s} \geq 0$ for all EV at the final moment T is imposed:

$$\sum_{i \in \mathcal{K}} \int_0^{\underline{s}} m_i(T, s) ds = 0. \quad (3.2.4)$$

This constraint ensures that all EV are sufficiently charged at the end of the period.

3.2.2 Discrete settings

The continuous case leads to optimal control of partial differential equations, which is beyond of the scope of this paper, and will be discussed in another coming article. The dynamics of the PEVs population are now considered in a discrete settings to get a finite dimensional optimization problem that can be solved numerically. Let $h > 0$ and $\Delta t > 0$ be such that N_h and N_T are integers, with $N_h := 1/h$ and $N_T := T/\Delta t$. For any $k \in \{0, \dots, N_T\}$ and $\ell \in \{0, \dots, N_h - 1\}$, $\tilde{m}_i^{k, \ell}$ denotes the proportion of vehicles in mode $i \in \mathcal{K}$ with a SoC lying in $[\ell h, (\ell + 1)h)$ at time t_k , $\tilde{E}_{i,j}^{k, \ell}$ the flow from mode i to mode j for PEVs with a SoC lying in $[\ell h, (\ell + 1)h)$ at time t_k . The space and time steps h and Δt are taken to satisfy the Courant–Friedrichs–Lewy condition (see [142]), knowing that b_i has a constant sign:

$$\max_{i \in \mathcal{K}} (\|b_i\|_\infty) \Delta t \leq h.$$

This inequality makes sure that $\tilde{m}_i^{k+1, \ell}$ is positive in (3.2.6) and (3.2.7). The evolution of \tilde{m} is described by the discretization of (3.2.1). A splitting method is applied here. The variable $\tilde{m}_i^{k+\frac{1}{2}, \ell}$ is introduced to first take into account the transfers from one mode to another:

$$\tilde{m}_i^{k+\frac{1}{2}, \ell} := \tilde{m}_i^{k, \ell} + \Delta t \sum_{j \in \mathcal{K} \setminus \{i\}} \tilde{E}_{j,i}^{k, \ell} - \tilde{E}_{i,j}^{k, \ell}. \quad (3.2.5)$$

The evolution of the batteries is then considered. An upwind scheme, commonly used in finite volume methods [142], is applied. The scheme depends on the sign of b_i , supposed constant. If $b_i \geq 0$, then:

$$\frac{\tilde{m}_i^{k+1, \ell} - \tilde{m}_i^{k+\frac{1}{2}, \ell}}{\Delta t} + \frac{F_i^{k, \ell} - F_i^{k, \ell-1}}{h} = 0, \quad (3.2.6)$$

where $F_i^{k, \ell} = b_i(x_{\ell+\frac{1}{2}}) \tilde{m}_i^{k+\frac{1}{2}, \ell}$ and $x_\ell := (\ell + 1/2)h$. If $b_i \leq 0$:

$$\frac{\tilde{m}_i^{k+1, \ell} - \tilde{m}_i^{k+\frac{1}{2}, \ell}}{\Delta t} + \frac{B_i^{k, \ell} - B_i^{k, \ell-1}}{h} = 0, \quad (3.2.7)$$

where $B_i^{k, \ell} = b_i(x_{\ell+\frac{1}{2}}) \tilde{m}_i^{k+\frac{1}{2}, \ell+1}$. The initial condition is given by:

$$\tilde{m}_i^{0, \ell} = \frac{1}{h} \int_{x_\ell - h/2}^{x_\ell + h/2} \tilde{m}_i^0(s) ds \quad (3.2.8)$$

To guarantee the positivity of $\tilde{m}_i^{k+\frac{1}{2}, \ell}$ in (3.2.5), values of $\{\tilde{E}_{i,j}^{k, \ell}\}_{i,j}$ are constrained by:

$$\sum_{j \in \mathcal{K}, j \neq i}^{N_h-1} \tilde{E}_{i,j}^{k, \ell} \Delta t - \tilde{m}_i^{k, \ell} \leq 0. \quad (3.2.9)$$

For any $i \in \mathcal{K}$ and $k \in \{0, \dots, N_T\}$, the constraint (3.2.3) is discretized by:

$$\sum_{j=0}^{N_h} \tilde{m}_i^{k,j} h \leq \bar{D}_i, \quad (3.2.10)$$

Finally, the constraint of the minimal final SoC is expressed by:

$$\sum_{i \in \mathcal{K}} \sum_{\ell=0}^{\underline{s}^h} \tilde{m}_i^{N_T, \ell} = 0, \quad (3.2.11)$$

where $\underline{s}^h = \lceil \underline{s}/h \rceil$, and $\lfloor a \rfloor$ is the largest integer smaller than or equal to a .

3.3 The optimization problem

The optimization problem has the following form:

$$\begin{aligned} & \inf_{\tilde{m} \in \mathbb{R}^{N_m}, \tilde{E} \in \mathbb{R}^{N_E}} J(\tilde{m}, \tilde{E}) \\ & \text{subject to (3.2.5) - (3.2.11),} \end{aligned} \quad (3.3.1)$$

where:

$$J(\tilde{m}, \tilde{E}) := \sum_{k=0}^{N_T-1} \sum_{i \in \mathcal{K}} p_i^k \sum_{\ell=0}^{N_h-1} \tilde{m}_i^{k, \ell} h \Delta t + \sum_{k=0}^{N_T-1} \sum_{\substack{i, j \in \mathcal{K} \\ i \neq j}} \Theta_{i,j}(\tilde{m}_i^k, \tilde{E}_{i,j}^k) \Delta t.$$

The objective function J is the sum of two terms: the first one is the cost of electricity and the second one penalizes transfers. The quantity p_i^k is the average cost of electricity (in €/kWh) over the period $[t_k, t_{k+1}]$ for charging in mode i . The convex functions $\Theta_{i,j}$ are considered to deal with synchronization of PEVs and high charging frequency. Two examples are exposed in Section 3.5, where different choices of $\Theta_{i,j}$ are investigated to penalize the flow of transfers. The cost function J is convex but not necessarily linear or quadratic. The next section presents the algorithm applied to solve (3.3.1).

3.4 The Chambolle-Pock algorithm

The Chambolle-Pock algorithm is a first-order primal-dual algorithm for non-smooth, convex optimization problems with known saddle-point structure. This algorithm shows good results on similar optimization problems to (3.3.1) in [40].

The Chambolle-Pock algorithm is introduced in [67] to solve an optimization problem of the form:

$$\min_{y \in \mathbb{R}^N} \varphi(y) + \psi(y), \quad (3.4.1)$$

and its associated dual:

$$\min_{\sigma \in \mathbb{R}^N} \varphi^*(-\sigma) + \psi^*(\sigma), \quad (3.4.2)$$

where $\varphi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\psi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper, convex and lower semi-continuous (l.s.c.) functions. Let \hat{y} and $\hat{\sigma}$ be respective solutions of (3.4.1) and (3.4.2). The proximal operator prox_f of a function f at a point $x \in \mathbb{R}^N$ is defined by:

$$\text{prox}_f(x) := \arg \min_{z \in \mathbb{R}^N} \frac{1}{2} \|z - x\|_2^2 + f(z).$$

Problem (3.4.1) can be solved using the following primal-dual algorithm:

1. Fix $\theta \in [0, 1]$, $\gamma > 0$ and $\tau > 0$ such that $\gamma\tau < 1$.
2. Initialization: $(y^0, \hat{y}^0, \sigma^0) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$.

3. For any iteration k , knowing $(y^k, \tilde{y}^k, \sigma^k)$, the following steps are executed:

- $\sigma^{k+1} = \text{prox}_{\gamma\psi^*}(\sigma^k + \gamma\tilde{y}^k)$
- $y^{k+1} = \text{prox}_{\tau\varphi}(y^k - \tau\sigma^{k+1})$
- $\tilde{y}^{k+1} = y^{k+1} + \theta(y^{k+1} - y^k)$

Using [67, Theorem 1], (y^k, σ^k) converges to $(\hat{y}, \hat{\sigma})$. The following variables are introduced: $N_m := (N_T + 1)N_h I$ and $N_E := N_T N_h I (I - 1)$. The functions $\varphi : \mathbb{R}^{N_m} \times \mathbb{R}^{N_E} \rightarrow \mathbb{R}$ and $\psi : \mathbb{R}^{N_m} \times \mathbb{R}^{N_E} \rightarrow \mathbb{R}$ are defined by:

$$\varphi(\tilde{m}, \tilde{E}) := \begin{cases} J(\tilde{m}, \tilde{E}) & \text{if } (\tilde{m}, \tilde{E}) \text{ satisfy (3.2.9) – (3.2.11)} \\ +\infty & \end{cases}$$

and

$$\psi(\tilde{m}, \tilde{E}) := \begin{cases} 0 & \text{if } (\tilde{m}, \tilde{E}) \text{ satisfy (3.2.5) – (3.2.8)} \\ +\infty & \end{cases}$$

The function ψ constraints the variables (\tilde{m}, \tilde{E}) to satisfy the discretized dynamics. The function φ ensures charging requests and system constraints are satisfied.

3.5 Case study: Response to peak and off peak hours pricing problem

This section presents two optimization problems having the structure of (3.3.1) but with different cost functions. The Chambolle-Pock algorithm is applied in each example, with parameters $\theta = \gamma = 0.5$ and $\tau = 1.8$. In both cases, a central planner controls a large fleet of PEVs located on the same node of the distribution grid. The site has a non flexible and deterministic electrical demand due to other (non-EV) electric usages: it may include some EV charging and other electricity consumption that are not controlled by the central planner. Only two modes of charging are considered: $\mathcal{K} := \{0, 1\}$. At each time step, the central planner can decide the proportion of PEVs being charged in mode 0 i.e., no charging, or in mode 1. A power $P_{ON} = 20\text{kW}$ is delivered to each EV in mode 1 while no power in mode 0. The charging rate in mode 1 is defined by: $b_1(s) = 1/9000$ for all $s \in [0, 0.75)$ and $b_1(s) = (1 - s)/2700$ for all $s \in [0.75, 1]$. An EV needs approximately 3h to perform a full charging cycle in mode 1. To take into account the self-discharge for non charging vehicles, it is set: $b_0(0) = 0$ and $b_0(s) = 2.6 \times 10^{-6}$ for any $s \in (0, 1]$. A fixed population of PEVs is considered over a period of $[0, T]$, with $T = 5\text{h}$. The discretization parameters are $\Delta t = 450$ seconds and $h = 0.05$. All PEVs must have a SoC greater or equal to $\underline{s} = 0.7$ at the end of the period. It is assumed that price of electricity depends on peak and off-peak hours. Both problems share the same structure but have distinct transfers costs, price signals and congestion constraints. The objective of these examples is to understand and analyze how the central planner responses to different prices and to different ways to penalize transfers. In the first example, the cost penalizes at each time step a quadratic form of the aggregate transfers of PEVs in the same mode, while in the second one it penalizes at each time step the aggregate of transfers of PEVs in the same mode with the same SoC. It is assumed that the central planner knows in advance: the SoC of each vehicle at time 0, the average price of electricity and the maximal proportion of EV in mode 1 \bar{D}_1^k .

3.5.1 Example 1: Aggregative switching costs

In this example, the functions $\Theta_{1,0}$ and $\Theta_{0,1}$, are defined by:

$$\Theta_{i,j}(\tilde{m}_i^k, \tilde{E}_{i,j}^k) := \theta \left(\sum_{\ell=0}^{N_h-1} \tilde{E}_{i,j}^{k,\ell} h \right)^2, \quad (3.5.1)$$

where $\theta = 10$. This cost aggregates the control \tilde{E} in the SoC space. At each time step, this function penalizes large values of $\sum_{\ell=0}^{N_h-1} \tilde{E}_{i,j}^{k,\ell}$. The price of electricity p used for the simulation is displayed in Figure 3.2. The entire time period is composed of three different periods: two periods of peak hours and one period of off-peak hours. The minimum SoC at the end of the period is $\underline{s} = 0.7$ and the maximum proportion of vehicles that can be in mode 1 is constant over the time: $\bar{D}_1 = 1/2$. The proportion of vehicle per charging mode over the period $[0, T]$ is displayed in Figure 3.1. The comparison of Figures 3.1 and 3.2 shows that PEVs charge mainly during the

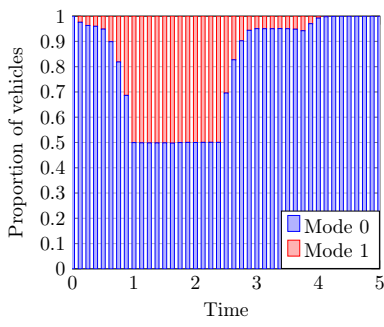


Figure 3.1 – Proportion of vehicles per mode of charging over the period

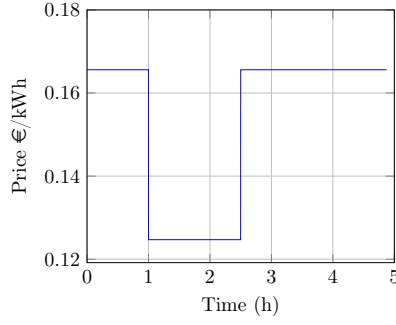


Figure 3.2 – Price of electricity over the period

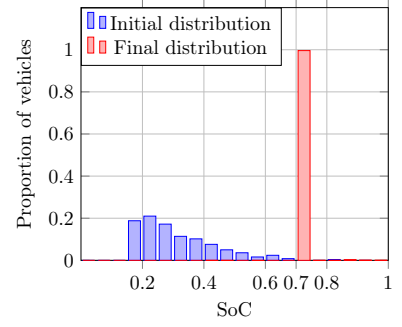


Figure 3.3 – Initial and final distribution of the SoC

off-peak hours, when the price is lower. The maximum proportion \bar{D}_1 of PEVs in mode 1 is attained during all the off-peak period. The constraint is binding during all the off-peak hours. The transfers from one mode to another are progressive over the time, avoiding large penalization. The average number of changes per EV (obtained by summing all the flows E over the time and space state) equals 2.75. It indicates that around 2/3 of PEVs switch only twice, from mode 0 to 1 and then from mode 1 to 0 once charged, and that 1/3 of PEVs switch twice to mode 1. The initial and final empirical distribution of the SoC of the EV batteries are displayed in Figure 3.3. At time $t = 0$ h all SoC are between 0.2 and 0.6 while at time $t = 5$ h almost all the SoC are at level 0.7.

3.5.2 Example 2: Local switching costs

In this example, the functions $\Theta_{1,0}$ and $\Theta_{0,1}$, are defined by:

$$\Theta_{i,j}(\tilde{m}_i^k, \tilde{E}_{i,j}^k) := \sum_{\ell=0}^{N_h-1} f(\tilde{m}_i^{k,\ell}, \tilde{E}_{i,j}^{k,\ell})h, \quad (3.5.2)$$

where

$$f(a, b) := \begin{cases} \theta_{i,j}b + \tilde{\theta}_{i,j} \frac{b^2}{a} & \text{if } a > 0 \text{ and } b \geq 0, \\ 0 & \text{if } a = 0 \text{ and } b = 0, \\ +\infty & \text{otherwise,} \end{cases} \quad (3.5.3)$$

with $\theta_{i,j} = 0.04$ and $\tilde{\theta}_{i,j} = 20$. Such functions are known to be convex and lower semi-continuous [22]. These costs are considered as *local* because not aggregated w.r.t. the SoC space. The functions $\Theta_{i,j}$ penalize large control values of \tilde{E} (through the term $\tilde{\theta}_{i,j}b^2/a$) as well as small ones (through the term $\theta_{i,j}b$). Thus, large flows and numerous small flows between modes are penalized, at each time and space step. The considered price of electricity p over the period is displayed in Figure 3.5. The price signal is composed of two periods of peak hours (between 0h and 1h and between 2h30 and 3h30) and two periods of off-peak hours. No more than 1/3 of PEVs can be in mode 1 at the same time ($\bar{D}_1 = 1/3$). The initial and final distributions of the SoC of PEVs are displayed in Figure 3.6. While all PEVs start in mode 0 with a SoC between 0.2 and 0.6, the control computed by the algorithm enables to charge all EV to a SoC of 0.7. Thus, the minimum SoC constraint at final time is satisfied. Figure 3.4 shows the proportion of PEVs per mode over the period. The constraint of at most 1/3 of PEVs in mode 1 is satisfied. As in the previous case, one can observe, on Figures 3.4 and 3.5, that the PEVs charge only during the off-peak hours. No synchronization effects among the EV population are observed. Transfers are progressive over the time. For instance, the first off-peak period starts at time $t = 1$ h: at that moment the proportion of vehicles at mode 1 starts to increase progressively. It reaches the upper limit \bar{D}_1 , before the mid-term of the first off-peak period. The average number of changes per EV equals 1.89. It shows that majority of PEVs switch only two times. These observations highlight that the penalization on the transfers manage to avoid synchronization effects and excessive changes between different charging modes. The Chambolle-Pock Algorithm has required 5 times fewer iterations in the second example than in the first one. This is due to the quadratic term in (3.5.1) penalizing the transfers from the same mode. However, this makes the cost function (3.5.1) more adapted to situations, where it is primordial to avoid synchronization effects.

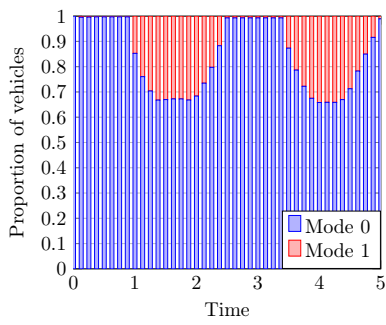


Figure 3.4 – Proportion of vehicles per mode of charging over the period

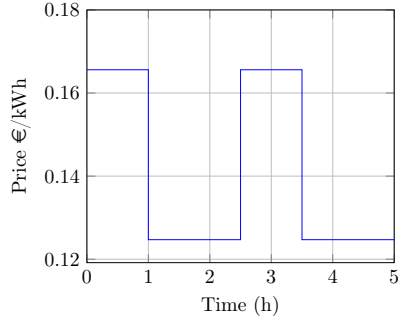


Figure 3.5 – Price of electricity over the period

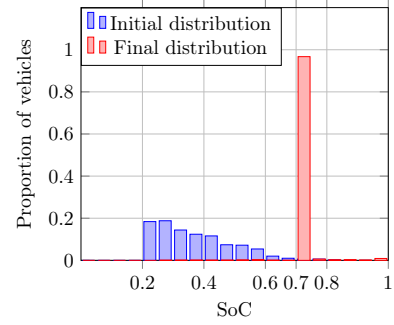


Figure 3.6 – Initial and final distribution of the SoC

3.6 Conclusion

This paper proposes a PDE-based model to control a large fleet of PEVs. The problem solved minimizes the cost of electricity consumption as well as the number of transfers between modes of charging. Aggregate constraints on the distribution of PEVs at any time and minimal SoC constraint at the final moment are considered. The Chambolle-Pock algorithm is applied to obtain numerical solution in the two case studies. Thanks to this algorithm, we are able to deal with non quadratic and non differentiable objective functions. Two different transfer penalization functions are studied. In both cases, the constraints are satisfied, the aggregate consumption of PEVs is important only during the off-peak hours and the average number of transfers per PEV is low.

Chapter 4

Regularity of Lagrange multipliers for a mean field control problem arising in smart charging

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4.1 Introduction

This article studies the regularity of solutions for an optimal control problem of a first order continuity equation with a reaction term, under congestion constraints. A numerical approximation of the problem is also proposed and analyzed. This work is motivated by its potential application to optimally charge a very large population of plug-in electric vehicles (PEVs) controlled by a central planner. A finite time interval $[0, T]$ and a mixed state space equal to the product $[0, 1] \times I$, where I is a finite space, are considered. The objective function J is defined as follows:

$$J(m, \alpha) := \sum_{i \in I} \int_0^T \int_0^1 (c_i(t, s) + \sum_{j \in I, j \neq i} L(\alpha_{ij}(t, s))) m_i(t, ds) dt + \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds), \quad (4.1.1)$$

where $m : [0, T] \rightarrow \mathcal{P}([0, 1] \times I)$ is a continuous curve of probability distribution and the control $\alpha : I \times I \times [0, T] \times [0, 1] \rightarrow \mathbb{R}_+$ is a jump intensity, that is square-integrable w.r.t. m . The continuity equation is given by:

$$\begin{aligned} \partial_t m_i(t, s) + \partial_s(m_i(t, s)b_i(s)) &= - \sum_{j \in I, j \neq i} (\alpha_{i,j}(t, s)m_i(t, s) - \alpha_{j,i}(t, s)m_j(t, s)) \quad (i, t, s) \in I \times (0, T) \times (0, 1), \\ m_i(0, s) &= m_i^0(s) \quad (i, s) \in I \times [0, 1], \end{aligned} \quad (4.1.2)$$

where $m^0 \in \mathcal{P}([0, 1] \times I)$ is the given initial distribution and $b : I \times [0, 1] \rightarrow \mathbb{R}$ is a velocity field vanishing at the boundary of $[0, 1]$ so that the mass conservation in $I \times [0, 1]$ is guaranteed without forcing a boundary condition. The notion of weak solution of (4.1.2) is specified in Definition 4.2.1. The distribution m is subject to the following congestion constraints:

$$m_i(t, [0, 1]) \leq D_i \quad \forall (i, t) \in I \times [0, T], \quad (4.1.3)$$

where $D_i > 0$ is given. We introduce the admissible set, which depends on the initial distribution m^0 and on the congestion parameter D ,

$$\mathcal{E}(m^0, D) := \{(m, \alpha) \text{ such that } \alpha \geq 0, (m, \alpha) \text{ satisfies (4.1.3) and is a weak solution of (4.1.2)}\}. \quad (4.1.4)$$

In [233], we have studied the optimization problem:

$$\inf_{(m, \alpha) \in \mathcal{E}(m^0, D)} J(m, \alpha). \quad (P)$$

As explained in [233], a pair (m, α) is a solution of (P), if and only if there exists $(\varphi, \lambda) \in (\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)) \times \mathcal{M}^+([0, T] \times I)$ such that $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ on $\{m_i > 0\}$ and (φ, λ, m) is a weak solution of the following system

$$\left\{ \begin{array}{ll} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \lambda_i + \sum_{j \in I, j \neq i} H(\varphi_i - \varphi_j) = 0 & \text{on } (0, T) \times (0, 1) \times I \\ \partial_t m_i + \partial_s(m_i b_i) + \sum_{j \neq i} H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j = 0 & \text{on } (0, T) \times (0, 1) \times I \\ m_i(0, s) = m_i^0(s), \varphi_i(T, s) = g_i(s) & \text{on } (0, 1) \times I \\ \int_0^1 m_i(t, ds) - D_i \leq 0, \lambda \geq 0 & \text{on } [0, T] \times I \\ \sum_{i \in I} \int_0^T \left(\int_0^1 m_i(t, ds) - D_i \right) \lambda_i(dt) = 0. & \end{array} \right. \quad (4.1.5)$$

The function φ is the Lagrange multiplier associated to the dynamic constraint (4.1.2), and the measure λ is associated to the congestion constraint (4.1.3). The first equation in (4.1.5) is a backward Hamilton-Jacobi equation and the second a forward continuity equation, where the control α , defined by $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$, is optimal. The measure λ is non-negative and finite. The last equality in (4.1.5) ensures that the congestion constraint (4.1.3) is satisfied. This characterization further enables to deduce regularity properties on the solutions of Problem (P). We recall these results in Section 4.2.1. Our work aims at obtaining better regularity results by studying the regularity of the Lagrange multiplier λ in (4.1.5) associated with the congestion constraint (4.1.3). Our results are then exploited to numerically approximate the solutions of Problem (P).

Problem (P) models the optimal charge a very large population of PEVs controlled by a central planner (see [234, 233, 238]). It is proved in [234] that Problem (P) is the mean field limit of a problem arising in smart charging. The convergence of the solution and that of the value of a sequence of finite population problems respectively to a solution and the value of Problem (P) are established. In [238], Problem (P) is numerically solved and implemented to a finite population of PEVs in a centralized manner. More explicitly, the central planner solves the centralized optimal charging problem (P) and then sends the control to each PEV (time of switches and the mode of charging to select for each PEV). However, there is no proof of convergence of solutions of the discretized problem to a solution of Problem (P). Based on the main results of this article (stated in section 4.2.2), we propose in Section 4.6.2 a numerical approximation of Problem (P) for a smart charging problem, that differs from the numerical approach developed in [238]. While the numerical approximation of Problem (P) in [238] relies on a primal-dual algorithm, a dual approach is adopted in Section 4.6.2, which

enables to estimate the Lagrange multiplier associated with the constraint (4.1.3).

4.1.1 Contributions and Literature

Contributions

The first and main contribution of this work is to show that there exists a Lagrange multiplier λ , associated with the congestion constraints (4.1.3), belonging to L^∞ . As a second contribution, we show that any optimal control α of Problem (P) is Lipschitz continuous. Finally, as a third contribution, this paper proposes a numerical method to approximate solutions of Problem (P), and establishes the convergence. This method enables to estimate, with a convergence rate, the value of Problem (P).

Methodology

A penalized problem is first introduced, where the constraint on the density (4.1.3) is relaxed and replaced by an additional penalizing term in the objective function. We then characterize the solution of this penalized problem as the weak solution of a system of two coupled PDEs similar to (4.1.5). Then, we show that, for sufficiently large penalizing terms, the penalized problem and Problem (P) have the same solutions. The regularity of the Lagrange multipliers are derived from the system of optimality conditions of the penalized problem. The Lipschitz continuity of any optimal control α of Problem (P) is a direct consequence of the regularity of the Lagrange multipliers.

To numerically approximate solutions of Problem (P), we first propose to numerically solve a convex optimization problem (Problem (D) that is introduced in Section 4.2.1) and whose dual problem is Problem (P). We introduce an explicit finite difference numerical scheme to discretize the Hamilton-Jacobi equation of Problem (D). The convergence of the sequence of solutions of the discretized problems to a solution of Problem (D), when the time and space steps converge to 0, is obtained by compactness arguments and by exploiting the L^∞ bound of the Lagrange multiplier of the congestion constraint (4.1.3). Then, the solutions of the discretized problems are used to build a sequence that converges to a solution of Problem (P), when the time and space steps converge to 0.

Literature review

The system (4.1.5) of optimality conditions of Problem (P) are derived in [233] using a Lagrangian relaxation and duality arguments. This system of PDEs is close to the systems arising in Mean Field Game (MFG) Theory (see [145, 146, 166, 167, 168]). Relying on the so-called variational approach, MFG problems with constraints on the density are studied for first order MFG in [55] and second order MFG in [190]. Other techniques, inspired from the optimal control theory in finite dimension, have been extended to infinite dimensional control problems with state constraints, to characterize the solutions. Firstly, in [151] the authors study the notion of viscosity solutions in the Wasserstein space to obtain an Hamilton-Jacobi-Bellman equation associated with the control problem. The reader can refer to [63] and the references therein. Secondly, the Pontryagin principle approach has been adapted to the framework of control problems in the space of probability measures [35, 36, 38].

The optimal control of a Fokker-Planck equation is strongly related to stochastic optimal control problems with constraints on the probability distribution. The optimality conditions of a stochastic control problem with terminal constraints in law are given in [81]. In the specific case of constraints on the expectation, necessary and sufficient optimality conditions are proved in [211], and first and second order optimality conditions are given in [114, 113]. Problem (P) can be formulated as a mean field control problem of Piecewise Deterministic Markov Processes (PDMPs), where the continuity equation (4.1.2) describes the evolution of the probability measure [71]. In [74, 75], the authors consider the optimal control of PDMP where the constraints are written as infinite horizon expected discounted costs. In [187, 188], an optimal control of hybrid processes is studied, where the constraints concern the jumping instants. Finally, a mean field control problem of PDMP with constraints on the expectation is studied and numerically solved in [237].

This work is mainly concerned with the regularity of the Lagrange multipliers arising from the system of optimality conditions obtained in [233]. An extensive literature exists about the regularity of the Lagrange multipliers for constrained control problems. These studies are motivated by the analysis of second order optimality conditions or by the investigation of the convergence of some numerical methods. Due to the continuity of the state variables, the Lagrange multipliers are expected to be Borel measures for pure state pointwise constraints (see for e.g. [61, 218]). For elliptic and parabolic control problems, the Lavrentiev regularization, consisting in transforming the pure state constraints into mixed control-state constraints, enables to prove the existence of Lagrange multipliers in L^2 or even in L^∞ (see [11, 27, 192, 223] and the references

therein). For first order MFG problems with hard congestion (e.g. $\rho \leq 1$), prior works on the incompressible Euler's equations enable to show that, the absolutely continuous part of the Lagrange multiplier associated with the congestion constraint is in $L_{\text{loc}}^{\frac{d}{d-1}}((0, T) \times \mathbb{T}^d)$, while its singular part is supported on $\{T\} \times \mathbb{T}^d$ [55]. When the Hamiltonian is quadratic, a L^∞ estimate on the absolutely continuous part of the Lagrange multiplier is found in [169], using techniques from optimal transport. In [13], the authors show the Lipschitz continuity of a Lagrange multiplier associated with a market clearing condition in a MFG problem of first order. This result is obtained by introducing a variational formulation on the space of Radon positive measures.

In this article, we follow a penalization approach that has been first developed in the context of optimal control with state constraints in finite dimension in [47], where the authors apply the Maximum Principle on the penalized problem to obtain optimality conditions on the original problem with state constraints. This method is exploited in [46] to derive the smooth regularity of the solutions of the optimal control problem with state constraints, extending some well known results about the Lipschitz regularity of the solutions [31, 116, 135]. The authors use this result to extend regularity results of MFG problems with constraints on agents' state. The penalization technique is used in [12] to obtain optimality conditions and regularity results of a MFG problem inspired by crowd motion (agents seeking to exit a region in minimal time) with state constraints. Finally, the penalizing method has been extended to the optimal control of the Fokker-Planck equation with smooth density constraints in [82]. The author gives the system of optimality conditions for non-linear cost and constraint functionals of the measure variable, and shows the existence of a Lagrange multiplier in L^∞ and the Lipschitz continuity of optimal controls. The same author establishes in his PhD thesis [80] the convergence of the sequence of the value of a finite population problem with almost sure constraints to the value of the mean field control problem with constraints in the Wasserstein space, when the number of agents goes to infinity.

The numerical method adopted in this work relies on a convex problem that is in duality with Problem (P). This approach is close to primal-dual algorithms. Such algorithms have been successfully implemented to numerically approximate mean field control problems with congestion constraints. In [41, 40], proximal algorithms enable to handle second order MFG with local constraints on the measure. The ADMM algorithm is used to solve MFG problems of first or second order with constraints on the distribution in [25, 40]. Note that ADMM and Chambolle-Pock algorithms show good results for finite dimensional MFG with congestion constraints [32]. Such constrained MFG problems can also be numerically approximated by using an entropy minimizing approach and applying a variant of the Sinkhorn algorithm [24]. A level set reformulation [30] of constrained mean field control problems is studied in [121], and is shown to be particularly suitable for numerical approximations using machine learning algorithms.

The rest of this paper is organized as follows. In Section 4.2 we provide some preliminary results and the main results of the paper. The penalized problem is introduced in Section 4.3 and its optimality conditions are given. The link between Problem (P) and the penalized problem is established in Section 4.4 for smooth initial distribution. Section 4.5 is devoted to the proof of the main results. In Section 4.6, the regularity results on the Lagrange multipliers are used to find a numerical approximation of the solutions of Problem (P). Finally, some results on the solutions of the continuity equation (4.1.2) are given in Appendix 4.7.

4.2 Main results and assumptions

Notations The space of Borel, positive and bounded measures on a space A is denoted by $\mathcal{M}^+(A)$ and the space of Borel probability measures on a space A is denoted by $\mathcal{P}(A)$. For any measure $\mu \in \mathcal{M}^+([0, T])$ and $0 \leq t_1 < t_2 \leq T$, we set $\int_{t_1}^{t_2} \mu(dt) := \mu([t_1, t_2])$. We denote by $\delta_T \in \mathcal{M}^+([0, T])$ the Dirac mass at T . Given a set \mathcal{S} and the finite space I , for any function f defined on $I \times \mathcal{S}$ and any measure $\mu \in \mathcal{M}(\mathcal{S} \times I)$, we use the notations $f_i(x) := f(i, x)$ for any $(i, x) \in I \times \mathcal{S}$ and $\mu_i(\mathcal{S}) := \mu(\{i\} \times \mathcal{S})$ for any $(i, \mathcal{S}) \in I \times \mathcal{B}(\mathcal{S})$, where $\mathcal{B}(\mathcal{S})$ denotes the Borel algebra. Similarly, for any function g defined on $I \times I \times \mathcal{S}$ and any measure $\nu \in \mathcal{M}(\mathcal{S} \times I \times I)$, we use the notations $g_{i,j}(x) := g(i, j, x)$ for any $(i, j, x) \in I \times I \times \mathcal{S}$ and $\nu_{i,j}(\mathcal{S}) := \nu(\{i\} \times \{j\} \times \mathcal{S})$ for any $(i, j, \mathcal{S}) \in I^2 \times \mathcal{B}(\mathcal{S})$. If \mathcal{S} is a metric space, let $\text{Lip}(\mathcal{S})$ denote the vector space of bounded and Lipschitz continuous maps $f : \mathcal{S} \rightarrow \mathbb{R}$. For any $\mu \in C^0([0, T], \mathcal{P}([0, 1]))$, let $L_\mu^2([0, T] \times \mathbb{R}) := \{f : [0, T] \times \mathbb{R} \mapsto \mathbb{R}, \int_0^T \int_0^1 f(t, s)^2 \mu(t, ds) dt < +\infty\}$. The Lebesgue measure on \mathbb{R} is denoted by \mathcal{L} . We denote by \mathcal{W} the Wasserstein distance on $\mathcal{P}([0, 1] \times I)$, defined by $\mathcal{W}(\mu, \rho) := \sup \{\sum_{i \in I} \int_0^1 \varphi(\mu - \rho) \mid \varphi \text{ is } 1 - \text{Lipschitz from } [0, 1] \times I \text{ to } \mathbb{R}\}$. We recall that if a function φ is 1-Lipschitz continuous from $[0, 1] \times I$ to \mathbb{R} , then $|\varphi(x, i) - \varphi(x, j)| \leq 1$ for any $i, j \in I$. For any optimization problem (P), we denote by $\text{val}(\mathbf{P})$ the value of this problem.

Assumptions The following assumptions are in force throughout the paper.

1. For any $i \in I$, $b_i \in C^2(\mathbb{R})$ with $b_i(s) = 0$ for any $s \notin (0, 1)$.
2. We assume that $\text{supp}(m_i^0) \subset [0, 1]$ for any $i \in I$.
3. The parameter D is in $(\mathbb{R}_+^*)^{|I|}$ and there exists $\varepsilon^0 > 0$ such D and m^0 satisfy for any $i \in I$, $\varepsilon^0 < D_i - m_i^0([0, 1])$.
4. For any $i \in I$, it is assumed that $c_i \in C^1([0, T] \times [0, 1])$ and $g_i \in C^1([0, 1])$.
5. $L : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ is a convex function, defined by:

$$L(x) := \begin{cases} l(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $l \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing strongly convex function bounded from above by a quadratic function. More explicitly, there exists $C > 0$ such that for any $x \in \mathbb{R}_+$:

$$\frac{x^2}{C} - C \leq l(x) \leq C(x^2 + 1),$$

where the first inequality is due to the strong convexity of l .

6. Let H be the Fenchel conjugate of L . We assume that there exists $C_H > 0$ such that for any $x \in \mathbb{R}_+^*$:

$$\frac{1}{C_H} \leq H''(x) \leq C_H. \quad (4.2.1)$$

The function H being the Fenchel conjugate of L , by Assumption 5, H is non-decreasing, non-negative, and H' is Lipschitz continuous on \mathbb{R} .

Remark 4.2.1. As explained in [233], the main role of Assumptions 1 and 2 is to ensure that the support of the weak solution of (4.1.2) is contained in $[0, 1]$ over the period $[0, T]$. Assumption 3 provides an estimate on $\lambda([0, T] \times I)$ for any weak solution (φ, λ, m) of (4.1.5) and ensures that the control $\alpha = 0$ is an admissible control. Correspondingly, the feasible set of Problem (P) is not empty. It is also used to obtain uniform bounds, independently of the penalization parameters of the penalized problem, on $\|\alpha\|_\infty$ and $\|\partial_s \alpha\|_\infty$ for any optimal control α of the penalized problem. It is possible to extend the results of the paper to a congestion parameter D that is continuous and piecewise linear. The Assumptions 4 and 5 on c , g and L enable to obtain regularity results of the weak solutions of the Hamilton-Jacobi equation in the system (4.3.5). Finally, the bounds on H'' in Assumption 6 are used to obtain estimates on $\partial_t \alpha$, for any optimal control α of the penalized problem.

4.2.1 Preliminary results

The main results obtained in [233] on Problem (P) are recalled in this section. We start by stating the notion of weak solution of the continuity equation (4.1.2).

Definition 4.2.1. A pair (m, α) satisfies (4.1.2) in the weak sense if $m \in C^0([0, T], \mathcal{P}([0, 1] \times I))$, for any $i, j \in I$ with $i \neq j$, it holds that $\alpha_{i,j} \in L^2_{m_i}([0, T] \times [0, 1], \mathbb{R}_+)$ and for any test function $\phi \in C_c^\infty([0, T] \times [0, 1] \times I)$, we have:

$$\begin{aligned} & \sum_{i \in I} \int_0^1 \phi_i(T, s) m_i(T, ds) - \phi_i(0, s) m_i^0(ds) \\ &= \int_0^T \int_0^1 \sum_{i \in I} (\partial_t \phi_i(t, s) + b_i(s) \partial_s \phi_i(t, s)) m_i(t, ds) + \sum_{j \in I, j \neq i} (\phi_j(t, s) - \phi_i(t, s)) \alpha_{i,j}(t, s) m_i(t, ds) dt. \end{aligned}$$

For a given $\lambda \in \mathcal{M}^+([0, T] \times I)$, the following Hamilton-Jacobi equation is considered on $(0, T) \times (0, 1) \times I$:

$$\begin{aligned} -\partial_t \varphi_i(t, s) - b_i(s) \partial_s \varphi_i(t, s) - c_i(t, s) - \lambda_i(t) + \sum_{j \in I, j \neq i} H((\varphi_j - \varphi_i)(t, s)) &= 0 \quad (t, s, i) \in (0, T) \times (0, 1) \times I, \\ \varphi_i(T, s) = g_i(s) & \quad (s, i) \in [0, 1] \times I. \end{aligned} \quad (4.2.2)$$

For any $\lambda \in \mathcal{M}^+([0, T] \times I)$, the function $L^\lambda \in BV([0, T] \times I)$ is defined, for any $i \in I$ and any $t \in [0, T]$, by:

$$L_i^\lambda(t) := \lambda_i([t, T]). \quad (4.2.3)$$

We introduce the notion of weak sub-solution and solution for equation (4.2.2).

Definition 4.2.2. For a given $\lambda \in \mathcal{M}^+([0, T] \times I)$, a function φ defined from $[0, T] \times [0, 1] \times I$ to \mathbb{R} is a weak solution (sub-solution) of equation (4.2.2) if $\varphi - L^\lambda$ is in $\text{Lip}([0, T] \times [0, 1] \times I)$ and if, for any test function $f \in C^1([0, T] \times [0, 1] \times I)$ ($f \in C^1([0, T] \times [0, 1] \times I, \mathbb{R}_+)$),

$$\begin{aligned} & \int_0^1 \varphi_i(0, s) f_i(0, s) ds - \int_0^1 g_i(s) f_i(T, s) ds + \int_0^T \int_0^1 (\partial_t f_i(t, s) + \partial_s(f_i(t, s) b_i(s))) \varphi_i(t, s) ds dt \\ & + \int_0^T \int_0^1 \left(\sum_{j \in I, j \neq i} H(\varphi_i(t, s) - \varphi_j(t, s)) - c_i(t, s) \right) f_i(t, s) dt ds - \sum_{i \in I} \int_0^T \int_0^1 f_i(t, s) ds \lambda_i(dt) \\ & = (\leq) 0, \end{aligned} \quad (4.2.4)$$

where $\varphi_i(0, \cdot)$ is understood in the sense of trace.

Remark 4.2.2. The final condition in (4.2.2) is misleading. Indeed, any weak solution (sub-solution) φ of (4.2.2), associated with a measure $\lambda \in \mathcal{M}^+([0, T] \times I)$, in the sense of Definition 4.2.2 satisfies $\varphi_i(T, \cdot) = (\leq) g_i(\cdot) + L_i^\lambda(T)$, where $\varphi_i(T, \cdot)$ and $L_i^\lambda(T)$ are in the sense of trace.

By [233, Theorem 4.1], if φ is a weak solution of (4.2.2) associated with the measure $\lambda \in \mathcal{M}^+([0, T] \times I)$, then it satisfies, for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,

$$\varphi_i(t, s) = \int_t^T \sum_{j \in I, j \neq i} -H((\varphi_i - \varphi_j)(\tau, S_i^{t,s}(\tau))) + c_i(\tau, S_i^{t,s}(\tau)) d\tau + L_i^\lambda(t) + g_i(S_i^{t,s}(T)), \quad (4.2.5)$$

where $S_i^{t,s}$ is the unique solution of the ODE below

$$\begin{cases} \frac{dS_i^{t,s}(\tau)}{d\tau} = b_i(S_i^{t,s}(\tau)), & \tau \in [0, T], \\ S_i^{t,s}(t) = s. \end{cases} \quad (4.2.6)$$

We introduce the problem

$$\inf_{(\varphi, \lambda) \in \mathcal{M}^+([0, T] \times I)} \sum_{i \in I} \int_0^1 -\varphi_i(0^+, s) m_i^0(ds) + \int_0^T D_i \lambda_i(dt), \quad (D)$$

where φ is a weak sub-solution of (4.2.2) associated with λ

where $\varphi_i(0^+, \cdot)$ is understood in the sense of trace.

The notion of weak solution of the PDE system (4.1.5) is given in the next definition.

Definition 4.2.3. A triplet $(\varphi, \lambda, m) \in (\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I)) \times \mathcal{M}^+([0, T] \times I) \times C^0([0, T], \mathcal{P}([0, 1] \times I))$ is called a weak solution of (4.1.5) if it satisfies the following conditions:

1. The function φ is a weak solution of (4.2.2), associated with λ in the sense of Definition 4.2.2;
2. m satisfies the continuity equation:

$$\partial_t m_i + \partial_s(m_i b_i) + \sum_{j \neq i} H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j = 0, \quad m_i(0, \cdot) = m_i^0,$$

in the sense of Definition 4.2.1, with $\alpha_{i,j} := H'(\varphi_i - \varphi_j)$ on $\{m_i > 0\}$;

3. it holds that, for any $t \in [0, T]$,

$$\int_0^1 m_i(t, ds) - D_i(t) \leq 0 \quad \text{and} \quad \sum_{i \in I} \int_0^T \left(\int_0^1 m_i(t, ds) - D_i(t) \right) \lambda_i(dt) = 0.$$

We have the following results on the optimality conditions of Problem (P), from [233, Theorem 2.1, Remark 6.2].

Theorem 4.2.1. *Problem (P) has a solution and satisfies*

$$\text{val}(P) = -\text{val}(D). \quad (4.2.7)$$

Furthermore, the minimizers have the following properties:

1. If (m, α) is a minimizer of Problem (P) and (φ, λ) a minimizer of Problem (D), then (φ, λ, m) is a weak solution of (4.1.5) in the sense of Definition 4.2.3, and $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ on $\{m_i > 0\}$ for any $i, j \in I$.
2. Conversely, if (φ, λ, m) is a weak solution of (4.1.5) in the sense of Definition 4.2.3, then there exists α , defined for any $i, j \in I$ by: $\alpha_{i,j} := H'(\varphi_i - \varphi_j)$ on $\{m_i > 0\}$, such that (m, α) is a minimizer of (P) and (φ, λ) a minimizer of Problem (D).
3. If (m, α) is a minimizer of Problem (P), then for any $i, j \in I$ $\alpha_{i,j}$ and $\partial_s \alpha_{i,j}$ are both in $L^\infty([0, T] \times [0, 1])$, and $m \in \text{Lip}([0, T], \mathcal{P}([0, 1] \times I))$.
4. If the initial distribution m^0 is absolutely continuous w.r.t. the Lebesgue measure with a density in $C^1([0, 1] \times I)$, then any solution (m, α) of Problem (P) is such that m is absolutely continuous w.r.t. the Lebesgue measure on $[0, T] \times [0, 1]$, and has a density in $\text{Lip}([0, T] \times [0, 1] \times I)$.

In this article, we show that there exists (φ, λ, m) that is a weak solution of (4.1.5), where the absolutely continuous part of λ w.r.t. the Lebesgue measure is in $L^\infty([0, T] \times I)$ and its singular part a Dirac mass at time $t = T$, and φ is in $\text{Lip}([0, T] \times [0, 1] \times I)$. Consequently, any optimal control α of Problem (P) is in $\text{Lip}([0, T] \times [0, 1] \times I^2)$. Based on these results, we provide in Section 4.6 a numerical scheme to solve Problem (D) and approximate solutions of Problem (P).

4.2.2 Main results

The main result of this paper is the following theorem, providing regularity results on a solution of Problem (D).

Theorem 4.2.2. *There exists a solution $(\varphi^*, \lambda^*) \in \text{Lip}([0, T] \times [0, 1] \times I) \times \mathcal{M}^+([0, T] \times I)$ of Problem (D) such that there exists $(\lambda, \beta) \in L^\infty([0, T] \times I, \mathbb{R}_+) \times (\mathbb{R}_+)^{|I|}$ satisfying, for any $i \in I$:*

$$\lambda_i^* = \lambda_i \mathcal{L} + \beta_i \delta_T.$$

In addition, $\|\lambda\|_\infty$ and $\|\beta\|_\infty$ only depend on $|I|, b, c, g, L, D$ and ε^0 .

Remark 4.2.3. *The fact that the singular part of the Lagrange multiplier is supported on $\{T\}$ and the L^∞ regularity of the absolutely continuous part of the Lagrange multiplier are expected. Similar results are established for first order MFG problems with constraints of the type $\rho \leq 1$ a.e. in [55, 169] and for mean field control problems with smooth constraints in law [82].*

Remark 4.2.4. *We show in Corollary 4.5.1 that if the final cost function g is independent w.r.t. the discrete variable, i.e. there exists $f \in C^1([0, 1])$ such that $g_i = f$ for any $i \in I$, then $\beta = 0$. From a practical perspective, this assumption makes sense. In the optimal charging problem of a fleet of PEVs, PEVs are penalized only on the basis of their state of charge, independently on their mode of charging, at the end of the period.*

Remark 4.2.5. *The time continuity of the Lagrange multiplier λ^* can not be a priori expected. Indeed, one can consider the following counter example in dimension 1 (there is no continuous variable in the state of each agent):*

- $I := \{0, 1\}$
- $c_0 > 0, c_1 < 0$ and $g_0 = g_1 = 0$
- $D_0 > 1$ and $D_1 < 1$.
- $m_1^0 = D_1 - \varepsilon$ and $m_0^0 = 1 - m_1^0$, with ε such that $0 < \varepsilon \leq D_1$.

Then, one can show that $\lambda_1(t) = \mathbb{1}_{m_1(t)=D_1}(t)(c_0 - c_1)$.

A direct consequence of Theorem (4.2.2) is to improve the regularity results on the solution of Problem (P) given in Theorem 4.2.1.

Corollary 4.2.1. *If (m, α) is a minimizer of Problem (P), then $\alpha \in \text{Lip}([0, T] \times [0, 1] \times I^2)$.*

Proof. This result is a consequence of Theorem 4.2.2 and point 1 of Theorem 4.2.1. \square

4.3 The penalized problem

The penalized problem is introduced in this section and the optimality conditions of this problem are derived. We first introduce, for any $i \in I$ the function Ψ_i , defined for any $\mu \in \mathcal{P}([0, 1] \times I)$, by:

$$\Psi_i(\mu) := \mu_i([0, 1]) - D_i. \quad (4.3.1)$$

A distribution $m \in C^0([0, T], \mathcal{P}([0, 1] \times I))$ satisfies constraint (4.1.3) if and only if $\Psi_i(m(t)) \leq 0$ for any (i, t) in $I \times [0, T]$. By the definition of Ψ , one has for any $\mu \in \mathcal{P}([0, 1] \times I)$ and $i \in I$,

$$\frac{\delta \Psi_i}{\delta m}(\mu, j, s) = \mathbb{1}_i(j) - \mu_i([0, 1]). \quad (4.3.2)$$

For any $\varepsilon, \delta > 0$, we consider the function

$$J_{\varepsilon, \delta}(m, \alpha) := J(m, \alpha) + \int_0^T \frac{1}{\varepsilon} \sum_{i \in I} \Psi_i^+(m_i(t)) dt + \frac{1}{\delta} \sum_{i \in I} \Psi_i^+(m_i(T)), \quad (4.3.3)$$

where $\Psi_i^+(\mu) = \max(0, \Psi_i(\mu))$, for any $\mu \in \mathcal{P}([0, 1] \times I)$ and $i \in I$. The penalized problem is:

$$\begin{aligned} & \inf_{(m, \alpha)} J_{\varepsilon, \delta}(m, \alpha) \\ & (m, \alpha) \text{ is a weak solution of (4.1.2)}. \end{aligned} \quad (P^{\varepsilon, \delta})$$

Lemma 4.3.1. *For any $\varepsilon, \delta > 0$, Problem $(P^{\varepsilon, \delta})$ has a solution.*

Proof. The proof is essentially the same as [233, Lemma 3.3]. \square

By the previous lemma, for any $\varepsilon, \delta > 0$, there exists a solution $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ of Problem $(P^{\varepsilon, \delta})$. Let $(\lambda, \beta) \in L^\infty((0, T) \times I, \mathbb{R}_+) \times (\mathbb{R}_+)^{|I|}$, we consider the Hamilton-Jacobi equation on $[0, T] \times [0, 1] \times I$

$$\begin{aligned} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \sum_{j \in I} \frac{\lambda_j^{\varepsilon, \delta}(t)}{\varepsilon} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(t), i, s) + \sum_{j \in I, j \neq i} H(\varphi_i - \varphi_j) &= 0, \\ \varphi_i(T, \cdot) &= g_i + \sum_{j \in I} \frac{\beta_j^{\varepsilon, \delta}}{\delta} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(T), i, \cdot). \end{aligned} \quad (4.3.4)$$

A weak solution of (4.3.4) is a function in $\text{Lip}([0, T] \times [0, 1] \times I)$ that satisfies (4.3.4) in the sense of distributions.

The main result of this section is the following proposition, which provides optimality conditions of Problem $(P^{\varepsilon, \delta})$.

Proposition 4.3.1. *For any $\varepsilon, \delta > 0$, Problem $(P^{\varepsilon, \delta})$ has a solution, and for any solution $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$, there exists $(\varphi^{\varepsilon, \delta}, \lambda^{\varepsilon, \delta}, \beta^{\varepsilon, \delta}) \in \text{Lip}([0, T] \times [0, 1] \times I) \times L^\infty([0, T] \times I, \mathbb{R}_+) \times (\mathbb{R}_+)^{|I|}$, such that $(\varphi^{\varepsilon, \delta}, \lambda^{\varepsilon, \delta}, m^{\varepsilon, \delta})$ is a weak solution of the following system on $[0, T] \times [0, 1] \times I$,*

$$\left\{ \begin{aligned} -\partial_t \varphi_i^{\varepsilon, \delta} - b_i \partial_s \varphi_i^{\varepsilon, \delta} - c_i - \sum_{j \in I} \frac{\lambda_j^{\varepsilon, \delta}}{\varepsilon} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(t), i, s) + \sum_{j \in I, j \neq i} H(\varphi_i^{\varepsilon, \delta} - \varphi_j^{\varepsilon, \delta}) &= 0 & \text{on } (0, T) \times (0, 1) \times I \\ \partial_t m_i^{\varepsilon, \delta} + \partial_s(m_i^{\varepsilon, \delta} b_i) + \sum_{j \neq i} H'(\varphi_i^{\varepsilon, \delta} - \varphi_j^{\varepsilon, \delta}) m_i^{\varepsilon, \delta} - H'(\varphi_j^{\varepsilon, \delta} - \varphi_i^{\varepsilon, \delta}) m_j^{\varepsilon, \delta} &= 0 & \text{on } (0, T) \times (0, 1) \times I \\ m_i^{\varepsilon, \delta}(0, s) = m_i^0(s), \varphi_i^{\varepsilon, \delta}(T, s) = g_i(s) + \sum_{j \in I} \frac{\beta_j^{\varepsilon, \delta}}{\delta} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(T), i, s) & & \text{on } (0, 1) \times I, \end{aligned} \right. \quad (4.3.5)$$

where φ solves the Hamilton-Jacobi in the weak sense and the continuity equation is understood in the sense of Definition 4.2.1. The optimal control feedback $\alpha^{\varepsilon,\delta}$ and $\varphi^{\varepsilon,\delta}$ satisfy $dt \otimes m^{\varepsilon,\delta}(t, ds)$ -almost everywhere,

$$\alpha_{i,j}^{\varepsilon,\delta} = H'(\varphi_i^{\varepsilon,\delta} - \varphi_j^{\varepsilon,\delta}), \quad (4.3.6)$$

and $(\lambda^{\varepsilon,\delta}, \beta^{\varepsilon,\delta})$ is such that,

$$\lambda_i^{\varepsilon,\delta}(t) := \begin{cases} 0 & \text{if } m_i^{\varepsilon,\delta}(t, [0, 1]) < D_i \\ \in [0, 1] & \text{if } m_i^{\varepsilon,\delta}(t, [0, 1]) = D_i \\ 1 & \text{if } m_i^{\varepsilon,\delta}(t, [0, 1]) > D_i, \end{cases} \quad \text{and} \quad \beta_i^{\varepsilon,\delta} := \begin{cases} 0 & \text{if } m_i^{\varepsilon,\delta}(T, [0, 1]) < D_i \\ \in [0, 1] & \text{if } m_i^{\varepsilon,\delta}(T, [0, 1]) = D_i \\ 1 & \text{if } m_i^{\varepsilon,\delta}(T, [0, 1]) > D_i. \end{cases} \quad (4.3.7)$$

Remark 4.3.1. The system of optimality conditions (4.3.5) is close to the system (4.1.5). However, any weak solution $(\varphi^{\varepsilon,\delta}, \lambda^{\varepsilon,\delta}, m^{\varepsilon,\delta})$ of the system (4.3.5) does not necessarily satisfy

$$\int_0^1 m_i^{\varepsilon,\delta}(t, ds) - D_i \leq 0 \quad \text{and} \quad \sum_{i \in I} \int_0^T \left(\int_0^1 m_i^{\varepsilon,\delta}(t, ds) - D_i \right) \bar{\lambda}_i(t) dt = 0,$$

where $\bar{\lambda}_i(t) := \sum_{j \in I} \frac{\lambda_j^{\varepsilon,\delta}}{\varepsilon} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon,\delta}(t), i, s)$.

The next Lemma, which states that any solution of the penalized problem is a solution of a linearized problem, is a key step to prove Proposition 4.3.1. The following remark will be useful for the proof of this lemma.

Remark 4.3.2. As stated in [233, Section 3], by applying the change of variable $E_{i,j} := \alpha_{i,j} m_i$, it is possible to introduce a convex problem, where the minimization is over the couples (m, E) and the objective function is $\tilde{J}(m, E) := J(m, \frac{dE}{dm})$, that is equivalent to Problem (P). One can show that the function \tilde{J} is weakly lower semi continuous w.r.t. the couple of variables (m, E) .

Lemma 4.3.2. Let $\varepsilon, \delta > 0$ and $(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta})$ be a solution of $(P^{\varepsilon,\delta})$, then there exist $\lambda^{\varepsilon,\delta} \in L^\infty([0, T] \times I, \mathbb{R}_+)$ and $\beta^{\varepsilon,\delta} \in (\mathbb{R}_+)^{|I|}$ such that

$$\lambda_i^{\varepsilon,\delta}(t) := \begin{cases} 0 & \text{if } m_i^{\varepsilon,\delta}(t, [0, 1]) < D_i \\ \in [0, 1] & \text{if } m_i^{\varepsilon,\delta}(t, [0, 1]) = D_i \\ 1 & \text{if } m_i^{\varepsilon,\delta}(t, [0, 1]) > D_i, \end{cases} \quad \text{and} \quad \beta_i^{\varepsilon,\delta} := \begin{cases} 0 & \text{if } m_i^{\varepsilon,\delta}(T, [0, 1]) < D_i \\ \in [0, 1] & \text{if } m_i^{\varepsilon,\delta}(T, [0, 1]) = D_i \\ 1 & \text{if } m_i^{\varepsilon,\delta}(T, [0, 1]) > D_i, \end{cases} \quad (4.3.8)$$

and $(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta})$ minimizes the linearized cost function $J_{\varepsilon,\delta}^l$, defined by:

$$J_{\varepsilon,\delta}^l(m, \alpha) := J(m, \alpha) + \sum_{i \in I} \int_0^T \int_0^1 \left(\sum_{j \in I} \frac{\lambda_j^{\varepsilon,\delta}(t)}{\varepsilon} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon,\delta}(t), i, s) \right) m_i(t, ds) dt + \sum_{i \in I} \int_0^1 \sum_{j \in I} \left(\frac{\beta_j^{\varepsilon,\delta}}{\delta} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon,\delta}(T), i, s) \right) m_i(T, ds). \quad (4.3.9)$$

over the pair (m, α) satisfying the continuity equation (4.1.2) in the sense of Definition 4.2.1.

Proof. The proof follows the same steps as the proof of [82, Lemma 3.2]. For the sake of completeness, we reproduce it, by adapting this proof to our particular framework.

We consider a family $\{u^n\}_{n \in \mathbb{N}}$ of functions in $C^2([0, 1] \times I)$ such that for any $\mu, \bar{\mu} \in \mathcal{P}([0, 1] \times I)$, we have:

$$\mu = \bar{\mu} \Leftrightarrow \forall n \in \mathbb{N} \sum_{i \in I} \int_0^1 u_i^n(s) (\mu_i - \bar{\mu}_i)(ds) = 0.$$

We define the function $q : \mathcal{P}([0, 1] \times I) \times \mathcal{P}([0, 1] \times I) \mapsto \mathbb{R}$ by

$$q(\mu, \bar{\mu}) := \sum_{n \in \mathbb{N}} \frac{\left| \sum_{i \in I} \int_0^1 u_i^n(s) (\mu_i - \bar{\mu}_i)(ds) \right|^2}{2^n (1 + \|u^n\|_\infty^2 + \|Du^n\|_\infty^2)}.$$

It is straightforward that the function q satisfies, for any $\mu, \bar{\mu} \in \mathcal{P}([0, 1] \times I)$,

$$q(\mu, \bar{\mu}) \geq 0,$$

and

$$q(\mu, \bar{\mu}) = 0 \text{ if and only if } \mu = \bar{\mu}.$$

One can observe that, for any $\mu, \bar{\mu} \in \mathcal{P}([0, 1] \times I)$,

$$\frac{\delta q}{\delta \mu}(\mu, \bar{\mu}, i, x) = \sum_{n \in \mathbb{N}} \frac{\sum_{j \in I} \int_0^1 u_j^n(s) (\mu_j - \bar{\mu}_j)(ds)}{2^n (1 + \|u^n\|_\infty^2 + \|Du^n\|_\infty^2)} \left(u_i^n(x) - \sum_{j \in I} \int_0^1 u_j^n(s) \mu_j(ds) \right),$$

and

$$\sum_{i \in I} \int_0^1 \frac{\delta q}{\delta \mu}(\mu, \bar{\mu}, i, x) \mu_i(dx) = 0 \text{ and } \frac{\delta q}{\delta \mu}(\mu, \mu, \cdot, \cdot) = 0.$$

We consider the optimization problem

$$\begin{aligned} & \inf_{(\alpha, m)} J_{\varepsilon, \delta}(m, \alpha) + \int_0^T q(m(t), m^{\varepsilon, \delta}(t)) dt \\ & \text{where } (m, \alpha) \text{ is a weak solution of (4.1.2).} \end{aligned} \quad (4.3.10)$$

By the definition and properties of q , is straightforward that $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ is a solution of the above problem and that for any other solution (m, α) , we have $m^{\varepsilon, \delta} = m$.

For any $h > 0$, let $\gamma_h : \mathbb{R} \rightarrow \mathbb{R}_+$ be such that:

$$\begin{cases} \gamma_h \in C^2(\mathbb{R}), \\ \gamma_h(x) = \max(0, x) \text{ in } \mathbb{R} \setminus [-h, h], \\ \sup_{x \in \mathbb{R}} |\gamma_h'(x)| \leq 1, \\ \lim_{h \rightarrow 0} \sup_{x \in \mathbb{R}} |\gamma_h(x) - \max(0, x)| \rightarrow 0 \leq 1. \end{cases}$$

For any $i \in I$ and $h > 0$, we define the function $\Psi_{i, h}(\mu) := \gamma_h(\Psi_i(\mu))$, for any $\mu \in \mathcal{P}([0, 1] \times I)$. The following regularized function is introduced

$$J_{\varepsilon, \delta, h}(m, \alpha) := J(m, \alpha) + \frac{1}{\varepsilon} \sum_{i \in I} \int_0^1 \Psi_{i, h}(m(t)) dt + \frac{1}{\delta} \sum_{i \in I} \Psi_{i, h}(m(T)).$$

Applying similar arguments as in the proof of [233, Lemma 3.3], one can show that, for any $h > 0$, the problem

$$\begin{aligned} & \inf_{(\alpha, m)} J_{\varepsilon, \delta, h}(m, \alpha) + \int_0^T q(m(t), m^{\varepsilon, \delta}(t)) dt \\ & \text{where } (m, \alpha) \text{ is a weak solution of (4.1.2),} \end{aligned} \quad (4.3.11)$$

has a solution $(m^{\varepsilon, \delta, h}, \alpha^{\varepsilon, \delta, h})$. Following [233, Section 3] and Remark 4.3.2, by compactness arguments, there exists (m', α') such that $m' \in C^0([0, T], \mathcal{P}([0, 1] \times I))$ and $\alpha' \in L^2_{m'}([0, T] \times [0, 1] \times I^2, \mathbb{R}_+)$, $\{(m^{\varepsilon, \delta, h}, \alpha^{\varepsilon, \delta, h})\}_h$, and $\{m^{\varepsilon, \delta, h}\}_h$ converges, up to a subsequence, w.r.t. the Wasserstein distance uniformly in time to m' and $\{\alpha^{\varepsilon, \delta, h}, m^{\varepsilon, \delta, h}\}_h$ weakly converges to $\alpha' m'$ in $\mathcal{M}^+([0, T] \times [0, 1] \times I^2)$, when h converges to 0. Now we want to show that m' is equal to $m^{\varepsilon, \delta}$. To this end, it is enough to obtain:

$$\int_0^T q(m'(t), m^{\varepsilon, \delta}(t)) dt = 0.$$

Since $(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h})$ is a solution of (4.3.11), one has:

$$\begin{aligned}
& J_{\varepsilon,\delta}(m', \alpha') + \int_0^T q(m'(t), m^{\varepsilon,\delta}(t))dt - J_{\varepsilon,\delta}(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta}) \\
&= J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h}) + \int_0^T q(m^{\varepsilon,\delta,h}(t), m^{\varepsilon,\delta}(t))dt - J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta}) \\
&\quad + J_{\varepsilon,\delta}(m', \alpha') - J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h}) + \int_0^T q(m'(t), m^{\varepsilon,\delta}(t))dt - \int_0^T q(m^{\varepsilon,\delta,h}(t), m^{\varepsilon,\delta}(t))dt \\
&\quad + J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta}) - J_{\varepsilon,\delta}(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta}) \\
&\leq J_{\varepsilon,\delta}(m', \alpha') - J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h}) + \int_0^T q(m'(t), m^{\varepsilon,\delta}(t))dt - \int_0^T q(m^{\varepsilon,\delta,h}(t), m^{\varepsilon,\delta}(t))dt \\
&\quad + J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta}) - J_{\varepsilon,\delta}(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta}).
\end{aligned} \tag{4.3.12}$$

The weak convergence, uniformly in time, of a subsequence of $\{m^{\varepsilon,\delta,h}\}_h$ to m' , implies that

$$\lim_{h \rightarrow 0} \int_0^T q(m'(t), m^{\varepsilon,\delta}(t))dt - \int_0^T q(m^{\varepsilon,\delta,h}(t), m^{\varepsilon,\delta}(t))dt = 0.$$

By the definition of γ_h , one has

$$\lim_{h \rightarrow 0} J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta}) - J_{\varepsilon,\delta}(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta}) = 0.$$

By the two previous equalities, one gets when h tends to 0 in (4.3.12),

$$J_{\varepsilon,\delta}(m', \alpha') + \int_0^T q(m'(t), m^{\varepsilon,\delta}(t))dt - J_{\varepsilon,\delta}(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta}) \leq J_{\varepsilon,\delta}(m', \alpha') - \liminf_{h \rightarrow 0} J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h}). \tag{4.3.13}$$

We seek to show that $J_{\varepsilon,\delta}(m', \alpha') - \liminf_{h \rightarrow 0} J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h}) \leq 0$. Since

$$\lim_{h \rightarrow 0} \frac{1}{\varepsilon} \sum_{i \in I} \int_0^T (\Psi_i^h(m^{\varepsilon,\delta,h}(t)) - \Psi_i^+(m^{\varepsilon,\delta,h}(t)))dt + \frac{1}{\delta} \sum_{i \in I} (\Psi_i^h(m^{\varepsilon,\delta,h}(T)) - \Psi_i^+(m^{\varepsilon,\delta,h}(T))) = 0,$$

one deduces that $\liminf_{h \rightarrow 0} J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h}) = \liminf_{h \rightarrow 0} J_{\varepsilon,\delta}(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h})$. One can prove that $J_{\varepsilon,\delta}$ is convex and l.s.c. w.r.t. to the couple of variables $(m, \alpha m)$. Then, by Remark 4.3.2, the weak convergence, up to a subsequence, of $\{(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h} m^{\varepsilon,\delta,h})\}_h$ to $(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta} m^{\varepsilon,\delta})$, yields $\liminf_{h \rightarrow 0} J_{\varepsilon,\delta,h}(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h}) \geq J_{\varepsilon,\delta}(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta})$.

Therefore, by taking the limit in inequality (4.3.13), one gets

$$\int_0^T q(m'(t), m^{\varepsilon,\delta}(t))dt = 0,$$

and thus, $m' = m^{\varepsilon,\delta}$.

Now we need to prove that (m', α') minimizes the function (4.3.9) over the pair (m, α) satisfying the continuity equation (4.1.2) in the sense of Definition 4.2.1. We introduce the linearized cost $J_{\varepsilon,\delta,h}^l$, defined by:

$$\begin{aligned}
& J_{\varepsilon,\delta,h}^l(m, \alpha) := \\
& J(m, \alpha) + \sum_{i \in I} \int_0^T \int_0^1 \left(\sum_{j \in I} \frac{\lambda_j^{\varepsilon,\delta}(t)}{\varepsilon} \frac{\delta \Psi_j^h}{\delta m}(m^{\varepsilon,\delta,h}(t), i, s) \right) m_i(t, ds)dt + \sum_{i \in I} \int_0^1 \sum_{j \in I} \left(\frac{\beta_j^{\varepsilon,\delta}}{\delta} \frac{\delta \Psi_j^h}{\delta m}(m^{\varepsilon,\delta,h}(T), i, s) \right) m_i(T, ds).
\end{aligned}$$

Since the function $J_{\varepsilon,\delta,h}$ is convex w.r.t. the couple $(m, \alpha m)$ (see Remark 4.3.2), and using the convexity and differentiability of q , one can show that $(m^{\varepsilon,\delta,h}, \alpha^{\varepsilon,\delta,h})$ minimizes the problem

$$\inf_{(m, \alpha)} J_{\varepsilon,\delta,h}^l(m, \alpha) + \int_0^T \frac{\delta q}{\delta \mu}(m^{\varepsilon,\delta,h}(t), m^{\varepsilon,\delta}(t))dt \tag{4.3.14}$$

where (m, α) is a weak solution of (4.1.2),

where $\frac{\delta q}{\delta \mu}$ is the derivative of q w.r.t. the first probability measure.

By the definition of γ_h , for any $h > 0$ and $i \in I$, the map $t \mapsto \gamma'_h(\Psi_i(m^{\varepsilon, \delta, h}(t)))$ is in $L^\infty([0, T])$. Then by convergence of $\{m^{\varepsilon, \delta, h}\}_h$ to m' , the sequence of maps $t \mapsto \gamma'_h(\Psi_i(m^{\varepsilon, \delta, h}(t)))$ weakly* converges in $L^\infty([0, T])$, up to a sub-sequence, to a map $\lambda_i \in L^\infty([0, T])$ that satisfies (4.3.8), when h tends to 0. By 4.3.2, the convergence of $\{m^{\varepsilon, \delta, h}\}_h$ to m' w.r.t. the Wasserstein distance uniformly in time and the fact that $m' = m^{\varepsilon, \delta}$, one has, for any $i \in I$ and any $m \in C^0([0, T], \mathcal{P}([0, 1] \times I))$,

$$\lim_{h \rightarrow 0} \left\| \sum_{j \in I} \int_0^1 \left(\frac{\delta \Psi_i}{\delta m}(m^{\varepsilon, \delta, h}(\cdot), j, s) - \frac{\delta \Psi_i}{\delta m}(m^{\varepsilon, \delta}(\cdot), j, s) \right) m_j(\cdot, ds) \right\|_\infty = 0.$$

Thus, one deduces, that up to a sub-sequence, for any $i \in I$ and $m \in C^0([0, T], \mathcal{P}([0, 1] \times I))$,

$$\begin{aligned} \lim_{h \rightarrow 0} \int_0^T \sum_{j \in I} \int_0^1 \frac{\delta \Psi_i^h}{\delta m}(m^{\varepsilon, \delta, h}(t), j, s) m_j(t, ds) dt &= \lim_{h \rightarrow 0} \int_0^T \gamma'_h(\Psi_i(m^{\varepsilon, \delta, h}(t))) \sum_{j \in I} \int_0^1 \frac{\delta \Psi_i}{\delta m}(m^{\varepsilon, \delta, h}(t), j, s) m_j(t, ds) dt \\ &= \int_0^T \lambda_i(t) \sum_{j \in I} \int_0^1 \frac{\delta \Psi_i}{\delta m}(m^{\varepsilon, \delta}(t), j, s) m_j(t, ds) dt. \end{aligned}$$

Applying similar arguments, one can show that, for any $i \in I$, there exists $\beta_i \in (\mathbb{R}_+)^{|I|}$ satisfying (4.3.8) and that $\frac{1}{\delta} \sum_{j \in I} \int_0^1 \frac{\delta \Psi_i^h}{\delta m}(m^{\varepsilon, \delta, h}(T), j, s) m_j(T, ds)$ converges, up to a sub-sequence, to $\frac{\beta_i}{\delta} \sum_{j \in I} \int_0^1 \frac{\delta \Psi_i}{\delta m}(m^{\varepsilon, \delta}(T), j, s) m_j(T, ds)$, for any $m \in C^0([0, T], \mathcal{P}([0, 1] \times I))$, when h tends to 0. Therefore, for any pair (m, α) weak solution of the continuity equation (4.1.2), the sequence $\{J_{\varepsilon, \delta, h}^l(m, \alpha)\}_h$ converges, up to a sub-sequence, to $J_{\varepsilon, \delta}^l(m, \alpha)$ (where the function $J_{\varepsilon, \delta}^l$ is defined in (4.3.9)) when h tends to 0. The quantity $\frac{\delta q}{\delta \mu}$ in (4.3.14) is equal to 0 when h tends to 0, by the convergence of $\{m^{\varepsilon, \delta, h}\}_h$ to m' and the fact that $\frac{\delta q}{\delta \mu}(m, m, i, x) = 0$ for any $(i, x) \in I \times [0, 1]$. Then, one deduces that (m', α') is an infimum of

$$\inf_{(m, \alpha)} J_{\varepsilon, \delta}^l(m, \alpha) \tag{4.3.15}$$

where (m, α) is a weak solution of (4.1.2),

Since $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ is a solution to the penalized problem $(P^{\varepsilon, \delta})$ and that $m' = m^{\varepsilon, \delta}$, one has

$$\sum_{i, j \in I, j \neq i} \int_0^T \int_0^1 L(\alpha_{i, j}^{\varepsilon, \delta}(t, s)) m_i^{\varepsilon, \delta}(t, ds) \leq \sum_{i, j \in I, j \neq i} \int_0^T \int_0^1 L(\alpha'_{i, j}(t, s)) m_i^{\varepsilon, \delta}(t, ds),$$

and thus, $J_{\varepsilon, \delta}^l(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta}) \leq J_{\varepsilon, \delta}^l(m^{\varepsilon, \delta}, \alpha')$. Since $(m^{\varepsilon, \delta}, \alpha')$ is a solution of (4.3.15), $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ is also a solution of (4.3.15). \square

Remark 4.3.3. By the proof of Proposition 4.3.2, one has $\sum_{i \in I} \int_0^1 \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) = J_{\varepsilon, \delta}^l(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$. In addition, by the definition of $\frac{\delta \Psi_j}{\delta m}$ and of $J_{\varepsilon, \delta}^l$, we have $J_{\varepsilon, \delta}^l(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta}) = J(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ and thus, $\sum_{i \in I} \int_0^1 \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) = J(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$

We can now prove the main result of the section.

Proof of Proposition 4.3.1. The existence of a solution is given by Lemma 4.3.1. Let $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ be a solution of Problem $(P^{\varepsilon, \delta})$ and $(\lambda^{\varepsilon, \delta}, \beta^{\varepsilon, \delta})$ be defined from Lemma 4.3.2. By [233, Theorem 4.1], there exists $\varphi^{\varepsilon, \delta} \in \text{Lip}([0, T] \times [0, 1] \times I)$ that is a weak solution of (4.3.4) associated with $(\lambda^{\varepsilon, \delta}, \beta^{\varepsilon, \delta})$.

One can approximate $\lambda^{\varepsilon, \delta}$ by a sequence $\{\lambda^{\varepsilon, n}\}_n$ in $C^0([0, T] \times I, \mathbb{R}_+)$, such that $\{\lambda^{\varepsilon, n}\}_n$ converges to $\lambda^{\varepsilon, \delta}$ w.r.t. the norm $\|\cdot\|_2$. Then by [233, Lemma 4.6], there exists a sequence $\{\varphi^{\varepsilon, n}\}_n$ in $C^1([0, T] \times [0, 1] \times I)$ such

that, for any $n \in \mathbb{N}$, $\varphi^{\varepsilon, n}$ is a weak solution of (4.3.4) associated with $(\lambda^{\varepsilon, n}, \beta^{\varepsilon, \delta})$. By [238, Lemma 4.6], one can show that $\{\varphi^{\varepsilon, n}\}_n$ converges to $\varphi^{\varepsilon, \delta}$ w.r.t. the norm $\|\cdot\|_\infty$. Since $m^{\varepsilon, \delta}$ is a weak solution of (4.1.2), taking, for any $n \in \mathbb{N}$, $\varphi^{\varepsilon, n}$ as a test function, one obtains by considering the limit n to infinity, that

$$\begin{aligned} \sum_{i \in I} \int_0^T \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) &= \sum_{\substack{i, j \in I \\ j \neq i}} \int_0^T \int_0^1 \left((\varphi_i^{\varepsilon, \delta}(t, s) - \varphi_j^{\varepsilon, \delta}(t, s)) \alpha_{i, j}^{\varepsilon, \delta}(t, s) - H(\varphi_i^{\varepsilon, \delta}(t, s) - \varphi_j^{\varepsilon, \delta}(t, s)) \right) m_i^{\varepsilon, \delta}(t, s) \\ &\quad + \sum_{i \in I} \left(\int_0^T \int_0^1 c_i(t, s) m_i^{\varepsilon, \delta}(t, ds) dt + \int_0^1 g_i(s) m_i^{\varepsilon, \delta}(T, ds) \right). \end{aligned} \quad (4.3.16)$$

By Assumption 5, L is convex and H is the Fenchel conjugate of L , then one deduces that, for any $(i, j, t, s) \in I^2 \times [0, T] \times [0, 1]$,

$$(\varphi_i^{\varepsilon, \delta}(t, s) - \varphi_j^{\varepsilon, \delta}(t, s)) \alpha_{i, j}^{\varepsilon, \delta}(t, s) - H(\varphi_i^{\varepsilon, \delta}(t, s) - \varphi_j^{\varepsilon, \delta}(t, s)) \leq L(\alpha_{i, j}^{\varepsilon, \delta}(t, s)).$$

By the previous inequality and using that, for any $i \in I$ and $t \in [0, T]$, $\sum_{j \in I} \int_0^1 \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}, i, s) m_i^{\varepsilon, \delta}(t, ds) = 0$, equality (4.3.16) becomes

$$\sum_{i \in I} \int_0^T \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) \leq J_{\varepsilon, \delta}^l(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta}), \quad (4.3.17)$$

where the equality holds if and only if one has, $dt \otimes m^{\varepsilon, \delta}(t, ds)$ -almost everywhere,

$$\alpha_{i, j}^{\varepsilon, \delta}(t, s) = H'(\varphi_i^{\varepsilon, \delta}(t, s) - \varphi_j^{\varepsilon, \delta}(t, s)). \quad (4.3.18)$$

To prove that (4.3.18) hold $dt \otimes m^{\varepsilon, \delta}(t, ds)$ -almost everywhere, we need show the converse inequality of (4.3.17). We consider the control $\bar{\alpha}_{i, j} := H'(\varphi_i^{\varepsilon, \delta} - \varphi_j^{\varepsilon, \delta})$ and the distribution \bar{m} such that $(\bar{m}, \bar{\alpha})$ is a weak solution of (4.1.2). The existence of such \bar{m} is guaranteed by the fact that $\bar{\alpha} \in \text{Lip}([0, T] \times [0, 1] \times I^2)$ and by Lemma (4.7.2) in Appendix. Since $(\bar{m}, \bar{\alpha})$ is a weak solution of (4.1.2), by applying similar arguments as the ones to obtain (4.3.16), one obtains, by approximating $\varphi^{\varepsilon, \delta}$ with a sequence of test functions $\{\varphi^{\varepsilon, n}\}_n$, that can be considered as a sequence of test functions for the continuity equation (4.1.2),

$$\begin{aligned} \sum_{i \in I} \int_0^T \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) &= \sum_{i \in I} \int_0^T \int_0^1 \left(c_i(t, s) + \sum_{j \in I} \frac{\lambda_j^{\varepsilon, \delta}(t)}{\varepsilon} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(t), i, s) \right) \bar{m}_i(t, s) \\ &\quad + \sum_{i, j \in I, j \neq i} \int_0^T \int_0^1 \left((\varphi_i^{\varepsilon, \delta}(t, s) - \varphi_j^{\varepsilon, \delta}(t, s)) \bar{\alpha}_{i, j}(t, s) - H(\varphi_i^{\varepsilon, \delta}(t, s) - \varphi_j^{\varepsilon, \delta}(t, s)) \right) \bar{m}_i(t, s) \\ &\quad + \sum_{i \in I} \int_0^T \left(g_i(s) + \sum_{j \in I} \frac{\beta_j^{\varepsilon, \delta}}{\delta} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(T), i, s) \right) \bar{m}_i(T, ds). \end{aligned} \quad (4.3.19)$$

By the definition of H and $\bar{\alpha}$, one has $dt \otimes m^{\varepsilon, \delta}(t, ds)$ -almost everywhere,

$$(\varphi_i^{\varepsilon, \delta}(t, s) - \varphi_j^{\varepsilon, \delta}(t, s)) \bar{\alpha}_{i, j}(t, s) - H(\varphi_i^{\varepsilon, \delta}(t, s) - \varphi_j^{\varepsilon, \delta}(t, s)) = L(\bar{\alpha}_{i, j}(t, s)).$$

By previous equality, equality (4.3.19) becomes

$$\sum_{i \in I} \int_0^T \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) = J_{\varepsilon, \delta}^l(\bar{\alpha}, \bar{m}).$$

By inequality (4.3.17) and the previous inequality, one deduces that

$$\sum_{i \in I} \int_0^T \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) = J_{\varepsilon, \delta}^l(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta}),$$

and thus, one has $dt \otimes m^{\varepsilon, \delta}(t, ds)$ -almost everywhere

$$\alpha_{i,j}^{\varepsilon, \delta} = H'(\varphi_i^{\varepsilon, \delta} - \varphi_j^{\varepsilon, \delta}).$$

The system (4.3.5) is obtained by using previous equality in the continuity equation (4.1.2) satisfied by $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$. \square

4.4 Link between the constrained and the penalized problems

We assume in this section that the initial distribution m^0 of the continuity equation (4.1.2) has a density in $C^1([0, 1] \times I)$ w.r.t. the Lebesgue measure. The main result of this section is the following theorem. It states that for $\delta > 0$ and $\varepsilon > 0$ small enough, Problem $(P^{\varepsilon, \delta})$ and Problem (P) have the same solutions. The constants in these sections are denoted by $C > 0$ and might change from line to line. We highlight that all the constants in this section depend on the data of the problem (i.e. the functions b, L, c and g and the constant $\varepsilon^0 > 0$ defined in Assumption 3). The only dependence w.r.t. to the initial distribution m^0 is through the term ε^0 in Assumption 3. Finally, we use the same notations as in Section 4.3, for any $\delta, \varepsilon > 0$, we denote by $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ a solution of Problem $(P^{\varepsilon, \delta})$.

Theorem 4.4.1. *If the initial distribution m^0 has a density in $C^1([0, 1] \times I)$ w.r.t. the Lebesgue measure, then there exists $\varepsilon^*, \delta^* > 0$ such that for any $(\varepsilon, \delta) \in (0, \varepsilon^*) \times (0, \delta^*)$, Problem $(P^{\varepsilon, \delta})$ and Problem (P) have the same solutions.*

This theorem is proved later in this section. The key point of the proof consists in showing that for $\delta > 0$ and $\varepsilon > 0$ small enough, any solution $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ of Problem $(P^{\varepsilon, \delta})$ satisfies the congestion constraint (4.1.3). By Proposition 4.3.1 in Section 4.3 and Theorem 4.4.1, one obtains the following optimality conditions of Problem (P) .

Corollary 4.4.1. *If the initial distribution m^0 has a density in $C^1([0, 1] \times I)$ w.r.t. the Lebesgue measure, then for any solution (m, α) of Problem (P) , there exists $(\varphi, \lambda, \beta) \in \text{Lip}([0, T] \times I \times [0, 1]) \times L^\infty([0, T] \times I, \mathbb{R}_+) \times (\mathbb{R}_+)^{|I|}$ such that $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ on $\{m_i > 0\}$ and (φ, λ, m) is a weak solution of the following system on $[0, T] \times [0, 1] \times I$:*

$$\left\{ \begin{array}{l} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \sum_{j \in I} \lambda_j \frac{\delta \Psi_j}{\delta m}(m(t), i, s) + \sum_{j \in I, j \neq i} H(\varphi_i - \varphi_j) = 0 \quad \text{on } (0, T) \times (0, 1) \times I, \\ \partial_t m_i + \partial_s(m_i b_i) + \sum_{j \in I, j \neq i} H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j = 0 \quad \text{on } (0, T) \times (0, 1) \times I, \\ m_i(0) = m_i^0, \varphi_i(T) = g_i + \sum_{j \in I} \beta_j \frac{\delta \Psi_j}{\delta m}(m(T), i, \cdot) \quad \text{on } (0, 1) \times I, \end{array} \right. \quad (4.4.1)$$

where φ is a weak solution of the Hamilton-Jacobi equation associated with (λ, β) and m satisfies the continuity equation in the sense of Definition 4.2.1. In addition, (λ, β) satisfies

$$\lambda_i(t) = \begin{cases} 0 & \text{if } m_i(t, [0, 1]) < D_i, \\ \lambda_i(t) \in \mathbb{R}_+ & \text{if } m_i(t, [0, 1]) = D_i, \end{cases} \quad \text{and} \quad \beta_i := \begin{cases} 0 & \text{if } m_i(T, [0, 1]) < D_i, \\ \beta_i \in \mathbb{R}_+ & \text{if } m_i(T, [0, 1]) = D_i. \end{cases}$$

In addition, $\|\lambda\|_\infty$ and $\|\beta\|_\infty$ only depend on $|I|, b, c, g, L, D$ and ε^0 .

The proof of the above corollary is at the end of this section.

The assumption, that m^0 has a density in $C^1([0, 1] \times I)$ w.r.t. the Lebesgue measure, enables to apply Lemma 4.7.2 in Appendix 4.7 and to simplify the computation of the time derivatives of the function $t \mapsto \Psi_i(m^{\varepsilon, \delta}(t))$. However, this regularity condition on the initial distribution is not crucial, Theorem 4.4.1 and Corollary 4.4.1 can be extended to any initial distribution in $\mathcal{P}([0, 1] \times I)$. For the sake of concision, we omit this extension.

The next Lemma provides uniform estimates for the solutions of Problem $(P^{\varepsilon, \delta})$ and for the weak solutions of the system (4.3.5) in Proposition (4.3.1).

Lemma 4.4.1. *There exists $C > 0$, such that for any $\varepsilon, \delta > 0$ and any solution $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ of Problem $(P^{\varepsilon, \delta})$, the triplet $(\varphi^{\varepsilon, \delta}, \lambda^{\varepsilon, \delta}, \beta^{\varepsilon, \delta})$ defined in Proposition 4.3.1 satisfies:*

$$\sum_{i \in I} \int_0^T \frac{\lambda_i^{\varepsilon, \delta}(t)}{\varepsilon} dt + \sum_{i \in I} \frac{\beta_i^{\varepsilon, \delta}}{\delta} \leq \frac{C}{\varepsilon^0}, \quad (4.4.2)$$

$$\|\partial_s \varphi^{\varepsilon, \delta}\|_\infty, \|\varphi^{\varepsilon, \delta}\|_\infty \leq C, \quad (4.4.3)$$

$$\|\partial_s \alpha^{\varepsilon, \delta}\|_\infty, \|\alpha^{\varepsilon, \delta}\|_\infty \leq C, \quad (4.4.4)$$

where $\varepsilon^0 > 0$ is defined in Assumption 3.

Proof. Let $\rho \in C^1([0, T] \times [0, 1] \times I)$ be a solution of the continuity equation $\partial_t \rho_i + \partial_s(b_i \rho_i) = 0$ on $[0, T] \times [0, 1] \times I$, with initial distribution m^0 . Since $\varphi^{\varepsilon, \delta}$ is a weak solution of the Hamilton-Jacobi equation, one has using ρ as a test function:

$$\begin{aligned} & \sum_{i \in I} \int_0^1 \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) - \left(g_i(s) + \sum_{j \in I} \frac{\beta_j^{\varepsilon, \delta}}{\delta} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(T), i, s) \right) \rho_i(T, s) ds \\ &= \sum_{i \in I} \int_0^T \int_0^1 \left(c_i(t, s) + \sum_{j \in I} \frac{\lambda_j^{\varepsilon, \delta}(t)}{\varepsilon} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(t), i, s) - \sum_{j \in I, j \neq i} H((\varphi_i^{\varepsilon, \delta} - \varphi_j^{\varepsilon, \delta})(t, s)) \right) \rho_i(t, s) dt ds. \end{aligned}$$

By rearranging the terms in the previous equality, one has:

$$\begin{aligned} & - \sum_{i \in I} \int_0^1 \sum_{j \in I} \frac{\beta_j^{\varepsilon, \delta}}{\delta} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(T), i, s) \rho_i(T, s) ds - \sum_{i \in I} \int_0^T \int_0^1 \sum_{j \in I} \frac{\lambda_j^{\varepsilon, \delta}(t)}{\varepsilon} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(t), i, s) \rho_i(t, s) ds dt \\ &= - \sum_{i \in I} \int_0^1 \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) + \sum_{i \in I} \int_0^T \int_0^1 \left(c_i(t, s) - \sum_{j \in I, j \neq i} H((\varphi_i^{\varepsilon, \delta} - \varphi_j^{\varepsilon, \delta})(t, s)) \right) \rho_i(t, s) dt ds + \sum_{i \in I} \int_0^1 g_i(s) \rho_i(T, s) ds. \end{aligned}$$

Since, for any $y, x \in \mathbb{R}$ we have $-H(x) \leq xy + L(y)$,

$$\begin{aligned} & - \sum_{i \in I} \int_0^1 \sum_{j \in I} \frac{\beta_j^{\varepsilon, \delta}}{\delta} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(T), i, s) \rho_i(T, s) ds - \sum_{i \in I} \int_0^T \int_0^1 \sum_{j \in I} \frac{\lambda_j^{\varepsilon, \delta}(t)}{\varepsilon} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(t), i, s) \rho_i(t, s) ds dt \\ & \leq - \sum_{i \in I} \int_0^1 \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) + \sum_{i \in I} \int_0^T \int_0^1 \left(c_i(t, s) + \sum_{j \in I, j \neq i} L(0) \right) \rho_i(t, s) dt ds + \sum_{i \in I} \int_0^1 g_i(s) \rho_i(T, s) ds \\ & \leq - \sum_{i \in I} \int_0^1 \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) + J(\rho, 0). \end{aligned}$$

By Remark 4.3.3, one has $\sum_{i \in I} \int_0^1 \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) = J(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$. Since $L(0) = 0$ and g and c are uniformly bounded, there exists a constant $C > 0$, independent of ε and δ , such that:

$$-C \leq \inf_{(m, \alpha)} J(m, \alpha) \leq J(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta}),$$

where (m, α) weak solution of (4.1.2)

consequently, one has $-\sum_{i \in I} \int_0^1 \varphi_i^{\varepsilon, \delta}(0, s) m_i^0(ds) \leq C$. Thus, there exists $C > 0$ such that

$$-\sum_{i \in I} \int_0^1 \sum_{j \in I} \frac{\beta_j^{\varepsilon, \delta}}{\delta} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(T), i) \rho_i(T) ds - \sum_{i \in I} \int_0^T \int_0^1 \sum_{j \in I} \frac{\lambda_j^{\varepsilon, \delta}}{\varepsilon} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(t), i, s) \rho_i(t, s) ds dt \leq C. \quad (4.4.5)$$

Since, for any $j \in I$, Ψ_j is convex, one has, for any $t \in [0, T]$:

$$\Psi_j(m^{\varepsilon, \delta}(t)) - \Psi_j(\rho(t)) \leq -\sum_{i \in I} \int_0^1 \sum_{j \in I} \frac{\delta \Psi_j}{\delta m}(m^{\varepsilon, \delta}(t), i, s) \rho_i(t, s) ds.$$

By Assumption 3, one has for any $t \in [0, T]$ and $j \in I$, that $\Psi_j(\rho(t)) < -\varepsilon^0 < 0$. Since $\beta_j^{\varepsilon, \delta} \geq 0$ and $\lambda_j^{\varepsilon, \delta}(t) \geq 0$, for any $j \in I$ and $t \in [0, T]$, inequality (4.4.5) becomes

$$\sum_{i \in I} \int_0^1 \sum_{j \in I} (\varepsilon^0 + \Psi_j(m^{\varepsilon, \delta}(T))) \frac{\beta_j^{\varepsilon, \delta}}{\delta} \rho_i(T) ds + \sum_{i \in I} \int_0^T \int_0^1 \sum_{j \in I} \frac{\lambda_j^{\varepsilon, \delta}}{\varepsilon} (\varepsilon^0 + \Psi_j(m^{\varepsilon, \delta}(t))) \rho_i(t) ds dt \leq C.$$

By definition of λ and β in (4.3.8), we have $\beta_j^{\varepsilon, \delta} \Psi_j(m^{\varepsilon, \delta}(T)) \geq 0$ and $\lambda_j^{\varepsilon, \delta}(t) \Psi_j(m^{\varepsilon, \delta}(t)) \geq 0$, for any $j \in I$ and $t \in [0, T]$. Then, the previous inequality gives:

$$\sum_{i \in I} \int_0^1 \sum_{j \in I} \varepsilon^0 \frac{\beta_j^{\varepsilon, \delta}}{\delta} \rho_i(T) ds + \sum_{i \in I} \int_0^T \int_0^1 \sum_{j \in I} \frac{\lambda_j^{\varepsilon, \delta}}{\varepsilon} \varepsilon^0 \rho_i(t) ds dt \leq C.$$

Estimate (4.4.2) follows by dividing the previous inequality by ε^0 .

By [233, Remark 4.7]), the quantities $\|\partial_s \varphi\|_\infty$ and $\|\varphi\|_\infty$ depend on the data of the problem and on $\sum_{i \in I} \int_0^T \lambda_i^{\varepsilon, \delta}(t)$. Thus, inequality (4.4.3) is derived from estimate (4.4.2). Since $\alpha_{i,j}^{\varepsilon, \delta} = H'(\varphi_i^{\varepsilon, \delta} - \varphi_j^{\varepsilon, \delta})$, inequality (4.4.4) is deduced from the property of Lipschitz continuity of H' and from (4.4.3). \square

We consider the function $\bar{\Psi} : \mathcal{P}([0, 1] \times I) \rightarrow \mathbb{R}_+$, defined by:

$$\bar{\Psi}(\mu) = \sum_{i \in I} \Psi_i^+(\mu). \quad (4.4.6)$$

The constant $\varepsilon^* > 0$ is defined in Lemma 4.4.5 and $\delta^* > 0$ in Lemma 4.4.2. Theorem 4.4.1 is proved using a contradiction argument. We assume that there exist $(\varepsilon, \delta) \in (0, \varepsilon^*) \times (0, \delta^*)$, a solution $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ of Problem $(P^{\varepsilon, \delta})$ and an instant $t \in (0, T)$, such that one has:

$$\bar{\Psi}(m_i^{\varepsilon, \delta}(t)) > 0. \quad (4.4.7)$$

Then, we can define $0 < \underline{t}^{\varepsilon, \delta} < t^{\varepsilon, \delta} < \bar{t}^{\varepsilon, \delta} \leq T$ such that:

$$\begin{aligned} \bar{\Psi}(m^{\varepsilon, \delta}(\underline{t}^{\varepsilon, \delta})) &= 0, \\ \bar{\Psi}(m^{\varepsilon, \delta}(t)) &\geq 0 \quad \forall t \in [\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta}], \\ \bar{\Psi}(m^{\varepsilon, \delta}(t^{\varepsilon, \delta})) &= \sup_{t \in [\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta}]} \bar{\Psi}(m^{\varepsilon, \delta}(t)), \\ \bar{\Psi}(m^{\varepsilon, \delta}(\bar{t}^{\varepsilon, \delta})) &= 0 \text{ or } \bar{t}^{\varepsilon, \delta} = T. \end{aligned} \quad (4.4.8)$$

The contradiction is established in Proposition 4.4.1, where it is proven that for any $\varepsilon < \varepsilon^*$ and $\delta < \delta^*$, there is no $t \in (0, T)$ satisfying (4.4.7). We first show, in the next lemma, that for any $\delta < \delta^*$, the constraint (4.1.3) is satisfied at the final time. Then, by the estimates obtained in Lemma 4.4.1, other estimates are derived in Lemma 4.4.3 on any $\partial_t \alpha^{\varepsilon, \delta}$. These results are exploited in Lemma 4.4.5 to show that for any $\varepsilon < \varepsilon^*$, the second order time derivative of $\bar{\Psi}$ is non negative a.e. on $(\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$. Finally, the proof of Proposition 4.4.1 is a direct

consequence of this lemma.

Lemma 4.4.2. *There exists $\delta^* > 0$ such that for any $\varepsilon > 0$ and $\delta \in (0, \delta^*)$, we have $\bar{\Psi}(m^{\varepsilon, \delta}(T)) \leq 0$.*

Proof. This is a direct consequence of Lemma 4.4.1. If $\bar{\Psi}(m^{\varepsilon, \delta}(T)) > 0$, then by (4.3.7) $\beta_i^{\varepsilon, \delta} = 1$, and for any $\delta < \delta^* := \frac{D_i \varepsilon^0}{C}$, where C is defined in Lemma 4.4.1, inequality (4.4.2) is not satisfied. \square

By the definition of Ψ_i in (4.3.1) and using the continuity equation (4.1.2) satisfied by $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$, one has for any $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$:

$$\frac{d}{dt} \Psi_i(m^{\varepsilon, \delta}(t)) = \sum_{j \in I, j \neq i} \int_0^1 -\alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds) + \alpha_{j,i}^{\varepsilon, \delta}(t, s) m_j^{\varepsilon, \delta}(t, ds). \quad (4.4.9)$$

By (4.3.6), the fact that $\varphi^{\varepsilon, \delta} \in \text{Lip}([0, T] \times [0, 1] \times I)$ and Assumption 5 on L , one has $\alpha^{\varepsilon, \delta} \in \text{Lip}([0, T] \times [0, 1] \times I^2)$. Then, by Lemma 4.7.2 in Appendix, for any $j \in I$, the function $t \mapsto \int_0^1 \alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, s) ds$ is Lipschitz continuous on $[0, T]$ and therefore, $t \mapsto \Psi_i(m^{\varepsilon, \delta}(t))$ is in $C^{1,1}(\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$, for any $i \in I$. Consequently, for a.e. $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$, one has:

$$\frac{d^2}{dt^2} \Psi_i(m^{\varepsilon, \delta}(t)) = \sum_{j \in I, j \neq i} \frac{d}{dt} \int_0^1 -\alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds) + \alpha_{j,i}^{\varepsilon, \delta}(t, s) m_j^{\varepsilon, \delta}(t, ds). \quad (4.4.10)$$

Based on (4.4.10), we want to obtain bounds on the second order time derivative of each function Ψ_i . We recall that by Proposition 4.3.1 and (4.3.2), $\varphi^{\varepsilon, \delta}$ is a weak solution, on $[0, T] \times [0, 1] \times I$, of

$$-\partial_t \varphi_i^{\varepsilon, \delta} - b_i \partial_s \varphi_i^{\varepsilon, \delta} - c_i - \frac{\lambda_i^{\varepsilon, \delta}(t)}{\varepsilon} (1 - m_i^{\varepsilon, \delta}(t, [0, 1])) + \sum_{j \in I, j \neq i} \frac{\lambda_j^{\varepsilon, \delta}(t)}{\varepsilon} m_j^{\varepsilon, \delta}(t, [0, 1]) + \sum_{j \in I, j \neq i} H(\varphi_i^{\varepsilon, \delta} - \varphi_j^{\varepsilon, \delta}) = 0. \quad (4.4.11)$$

By (4.4.11) and the uniform bounds of $\{\|\varphi^{\varepsilon, \delta}\|_\infty\}_\varepsilon$ and $\{\|\partial_s \varphi^{\varepsilon, \delta}\|_\infty\}_\varepsilon$ in Lemma 4.4.1, there exists $C > 0$ such that for a.e. $t \in [0, T]$ and any $(i, j, s) \in I \times I \times [0, 1]$:

$$\partial_t (\varphi_i^{\varepsilon, \delta} - \varphi_j^{\varepsilon, \delta})(t, s) \leq C + \frac{\lambda_j^{\varepsilon, \delta}(t) - \lambda_i^{\varepsilon, \delta}(t)}{\varepsilon}. \quad (4.4.12)$$

We set

$$\bar{\varepsilon} := \min\left(\frac{1}{C^2}, \frac{1}{(1 + |I|)^2}\right), \quad (4.4.13)$$

where the constant $C > 0$ comes from inequality (4.4.12). For any $(i, j, t) \in I^2 \times [\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta}]$, we define:

$$\mathbb{A}_{i,j}^{\varepsilon, \delta}(t) := \{s \in [0, 1] \mid \varphi_i^{\varepsilon, \delta}(t, s) > \varphi_j^{\varepsilon, \delta}(t, s)\}. \quad (4.4.14)$$

Further, for any $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$, we consider:

$$\begin{aligned} I^+(t) &:= \{i \in I \mid \Psi_i(m^{\varepsilon, \delta}(t)) > 0\}, \\ I^0(t) &:= \{i \in I \mid \Psi_i(m^{\varepsilon, \delta}(t)) = 0\}, \\ I^-(t) &:= \{i \in I \mid \Psi_i(m^{\varepsilon, \delta}(t)) < 0\}. \end{aligned} \quad (4.4.15)$$

At any time $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$, we consider the set $\{i_1, \dots, i_k, \dots, i_{|I^0(t)|}\}$ satisfying:

$$\{i_1, \dots, i_k, \dots, i_{|I^0(t)|}\} = I^0(t) \quad \text{and} \quad 0 \leq \lambda_{i_1}^{\varepsilon, \delta}(t) \leq \lambda_{i_2}^{\varepsilon, \delta}(t) \leq \dots \leq \lambda_{i_{|I^0(t)|}}^{\varepsilon, \delta}(t) \leq 1.$$

At any time $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$, if $I^0(t) = \emptyset$, then we define $I^{0,-}(t) := I^{0,+}(t) := \emptyset$, otherwise the sets $I^{0,-}(t)$ and $I^{0,+}(t)$ are defined as follows:

- if $\lambda_{i_1}(t) > \sqrt{\bar{\varepsilon}}$, then $I^{0,-}(t) = \emptyset$ and $I^{0,+}(t) = I^0(t)$.

- else, if $\lambda_{i_{|I^0(t)|}}(t) < 1 - \sqrt{\varepsilon}$, then $I^{0,-}(t) = I^0(t)$ and $I^{0,+}(t) = \emptyset$.
 - else, let $k^* \in \arg \max_{k \in \{1, \dots, |I^0(t)|-1\}} \lambda_{i_{k+1}}(t) - \lambda_{i_k}(t)$, we set: $I^{0,-}(t) = \{i_1, \dots, i_{k^*}\}$ and $I^{0,+}(t) = \{i_{k^*+1}, \dots, i_{|I^0(t)|}\}$.
- One can observe that, for any $\varepsilon \in (0, \frac{1}{(1+|I|)^2})$, it holds $\sqrt{\varepsilon} \leq \lambda_{i_{k^*+1}}(t) - \lambda_{i_{k^*}}(t)$.

By the definition of $I^{0,+}$ and $I^{0,-}$, we have $I^0 = I^{0,+} \cup I^{0,-}$ and $I^{0,+} \cap I^{0,-} = \emptyset$. In addition, for any $\varepsilon \in (0, \frac{1}{(1+|I|)^2})$, we have:

$$\lambda_j^{\varepsilon,\delta}(t) - \lambda_i^{\varepsilon,\delta}(t) \leq -\sqrt{\varepsilon}, \quad \forall (i, j) \in (I^+(t) \cup I^{0,+}(t)) \times (I^-(t) \cup I^{0,-}(t)). \quad (4.4.16)$$

The definition of the sets $I^{0,+}(t)$ and $I^{0,-}(t)$ is mainly motivated by the following lemma.

Lemma 4.4.3. *There exists $C > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, where $\bar{\varepsilon}$ is defined in (4.4.13), for a.e. $t \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta})$, any $i \in I^+(t) \cup I^{0,+}(t)$, and any $j \in I^-(t) \cup I^{0,-}(t)$, one has:*

1. for a.e. $s \in \mathbb{A}_{i,j}^{\varepsilon,\delta}(t)$,

$$\partial_t \alpha_{i,j}^{\varepsilon,\delta}(t, s) \leq C - \frac{1}{C\sqrt{\varepsilon}},$$

2. for a.e. $s \in \mathbb{A}_{j,i}^{\varepsilon,\delta}(t)$

$$-C + \frac{1}{C\sqrt{\varepsilon}} \leq \partial_t \alpha_{j,i}^{\varepsilon,\delta}(t, s).$$

Proof. Let $i \in I^+(t) \cup I^{0,+}(t)$ and $j \in I^-(t) \cup I^{0,-}(t)$. By inequality (4.4.16), inequality (4.4.12) becomes, for any $\varepsilon \in (0, 1/C)$,

$$\partial_t (\varphi_i^{\varepsilon,\delta} - \varphi_j^{\varepsilon,\delta})(t, s) \leq C - \frac{1}{\sqrt{\varepsilon}} < 0. \quad (4.4.17)$$

By the previous inequality, for any $\varepsilon \in (0, 1/C)$, the set $\{(t, s) \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta}) \times [0, 1] \mid \varphi_i^{\varepsilon,\delta}(t, s) = \varphi_j(t, s)\}$ is negligible. Then by the definition of $\alpha^{\varepsilon,\delta}$ in (4.3.6), one has, for any $i, j \in I$ and for a.e. $(t, s) \in [0, T] \times [0, 1]$,

$$\partial_t \alpha_{i,j}^{\varepsilon,\delta} = \begin{cases} H''(\varphi_i^{\varepsilon,\delta} - \varphi_j^{\varepsilon,\delta}) \partial_t (\varphi_i^{\varepsilon,\delta} - \varphi_j^{\varepsilon,\delta}) & \text{if } \varphi_i^{\varepsilon,\delta} > \varphi_j^{\varepsilon,\delta} \\ 0 & \text{else.} \end{cases}$$

By the bounds on H'' in (4.2.1) and (4.4.17), there exists $C > 0$ such that one has, for a.e. $t \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta})$, any $i \in I^+(t) \cup I^{0,+}(t)$, any $j \in I^-(t) \cup I^{0,-}(t)$ and a.e. $s \in \mathbb{A}_{i,j}^{\varepsilon,\delta}(t)$,

$$\partial_t \alpha_{i,j}^{\varepsilon,\delta}(t, s) \leq C - \frac{1}{C\sqrt{\varepsilon}}.$$

Similar arguments give, for a.e. $t \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta})$, any $i \in I^+(t) \cup I^{0,+}(t)$, any $j \in I^-(t) \cup I^{0,-}(t)$ and a.e. $s \in \mathbb{A}_{i,j}^{\varepsilon,\delta}(\cdot)$:

$$-C + \frac{1}{C\sqrt{\varepsilon}} \leq \partial_t \alpha_{j,i}^{\varepsilon,\delta}(t, s).$$

□

We need the following lemma on the time derivative of $\frac{d}{dt} \Psi_i(m^{\varepsilon,\delta}(t))$.

Lemma 4.4.4. *For any non empty $P \subset I$ and a.e. $t \in [0, T]$, one has:*

$$\sum_{i \in P} \frac{d^2}{dt^2} \Psi_i(m^{\varepsilon,\delta}(t)) = \sum_{\substack{i \in P, \\ j \in I \setminus P}} \frac{d}{dt} \int_0^1 -\alpha_{i,j}^{\varepsilon,\delta}(t, s) m_i^{\varepsilon,\delta}(t, ds) + \alpha_{j,i}^{\varepsilon,\delta}(t, s) m_j^{\varepsilon,\delta}(t, ds).$$

Proof. Let $P \subset I$ with $P \neq \emptyset$. By equality (4.4.10), one has, for a.e. $t \in [0, T]$,

$$\begin{aligned}
& \sum_{i \in P} \frac{d^2}{dt^2} \Psi_i(m^{\varepsilon, \delta}(t)) \\
&= \sum_{i \in P} \sum_{j \in I, j \neq i} \frac{d}{dt} \int_0^1 -\alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds) + \alpha_{j,i}^{\varepsilon, \delta}(t, s) m_j^{\varepsilon, \delta}(t, ds) \\
&= \sum_{i \in P} \sum_{j \in P, j \neq i} \frac{d}{dt} \int_0^1 -\alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds) + \alpha_{j,i}^{\varepsilon, \delta}(t, s) m_j^{\varepsilon, \delta}(t, ds) \\
&\quad + \sum_{i \in P} \sum_{j \in I \setminus P} \frac{d}{dt} \int_0^1 -\alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds) + \alpha_{j,i}^{\varepsilon, \delta}(t, s) m_j^{\varepsilon, \delta}(t, ds).
\end{aligned} \tag{4.4.18}$$

Observing that

$$\begin{aligned}
& \sum_{i \in P} \sum_{j \in P, j \neq i} \frac{d}{dt} \int_0^1 -\alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds) + \alpha_{j,i}^{\varepsilon, \delta}(t, s) m_j^{\varepsilon, \delta}(t, ds) \\
&= \sum_{i \in P} \sum_{j \in P, j \neq i} \frac{d}{dt} \int_0^1 -\alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds) + \sum_{i \in P} \sum_{j \in P, j \neq i} \frac{d}{dt} \int_0^1 \alpha_{j,i}^{\varepsilon, \delta}(t, s) m_j^{\varepsilon, \delta}(t, ds) \\
&= \sum_{i \in P} \sum_{j \in P, j \neq i} \frac{d}{dt} \int_0^1 -\alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds) + \sum_{j \in P} \sum_{i \in P, j \neq i} \frac{d}{dt} \int_0^1 \alpha_{j,i}^{\varepsilon, \delta}(t, s) m_j^{\varepsilon, \delta}(t, ds) \\
&= 0,
\end{aligned}$$

equality (4.4.18) becomes

$$\sum_{i \in P} \frac{d^2}{dt^2} \Psi_i(m^{\varepsilon, \delta}(t)) = \sum_{\substack{i \in P, \\ j \in I \setminus P}} \frac{d}{dt} \int_0^1 -\alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds) + \alpha_{j,i}^{\varepsilon, \delta}(t, s) m_j^{\varepsilon, \delta}(t, ds).$$

□

Since $t \mapsto \Psi_i(m^{\varepsilon, \delta}(t))$ is in $C^1([0, T])$ with a Lipschitz continuous derivative, by the definition of $\bar{\Psi}$ in (4.4.6), the function $t \mapsto \bar{\Psi}(m^{\varepsilon, \delta}(t))$ is semiconvex on $[0, T]$ with a linear modulus. By Alexandroff's Theorem [48], the map $t \mapsto \bar{\Psi}(m^{\varepsilon, \delta}(t))$ is twice differentiable a.e. on $[0, T]$

Lemma 4.4.5. *There exists ε^* such that, for any $\varepsilon \in (0, \varepsilon^*)$ and a.e. $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$, one has:*

$$0 \leq \frac{d^2}{dt^2} \bar{\Psi}(m^{\varepsilon, \delta}(t)).$$

Proof. For any $i \in I$, since $t \mapsto \Psi_i(m^{\varepsilon, \delta}(t))$ is in $C^1([0, T])$ and $t \mapsto \frac{d}{dt} \Psi_i(m^{\varepsilon, \delta}(t))$ is in $\text{Lip}([0, T])$, one deduces that:

$$\frac{d^2}{dt^2} \Psi_i(m^{\varepsilon, \delta}(t)) = 0 \text{ for a.e. } t \in \{\tau \in [0, T] \mid \Psi_i(m^{\varepsilon, \delta}(t)) = 0\}.$$

Thus, for a.e. $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$ and any $i \in I^{0,+}(t)$, one has

$$\frac{d^2}{dt^2} \Psi_i(m^{\varepsilon, \delta}(t)) = 0,$$

and thus,

$$\frac{d^2}{dt^2} \bar{\Psi}(m^{\varepsilon, \delta}(t)) = \sum_{i \in I^+(t)} \frac{d^2}{dt^2} \Psi_i(m^{\varepsilon, \delta}(t)) = \sum_{i \in I^+(t) \cup I^{0,+}(t)} \frac{d^2}{dt^2} \Psi_i(m^{\varepsilon, \delta}(t)).$$

By Lemma 4.4.4, one deduces, for a.e. $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$,

$$\frac{d^2}{dt^2} \bar{\Psi}(m^{\varepsilon, \delta}(t)) = \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} \frac{d}{dt} \int_0^1 -\alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds) + \alpha_{j,i}^{\varepsilon, \delta}(t, s) m_j^{\varepsilon, \delta}(t, ds). \quad (4.4.19)$$

For any $i, j \in I, i \neq j$, we define the functions $A_{i,j} : [\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta}] \mapsto \mathbb{R}$ by:

$$A_{i,j}(t) := \int_0^1 \alpha_{i,j}^{\varepsilon, \delta}(t, s) m_i^{\varepsilon, \delta}(t, ds). \quad (4.4.20)$$

The quantity $A_{i,j}(t)$ can be understood as the instantaneous mass transferred from i to j . By (4.4.19) and the definition of $A_{i,j}$, it is straightforward that for a.e. $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$, one has:

$$\frac{d^2}{dt^2} \bar{\Psi}(m^{\varepsilon, \delta}(t)) = \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} -\frac{d}{dt} A_{i,j}(t) + \frac{d}{dt} A_{j,i}(t). \quad (4.4.21)$$

We want to control the sign of $A_{j,i}$ and $A_{i,j}$ in the right hand sign of the previous inequality, using Lemma 4.4.3. Since $(m^{\varepsilon, \delta}, \alpha^{\varepsilon, \delta})$ is a weak solution of the continuity equation (4.1.2), taking $[0, T] \times [0, 1] \times I \ni (t, s, p) \mapsto \mathbb{1}_i(p) \alpha_{i,j}^{\varepsilon, \delta}(t, s)$ as a test function, one deduces that for a.e. $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$:

$$\begin{aligned} \frac{d}{dt} A_{i,j}(t) &= \int_0^1 \left(\partial_t \alpha_{i,j}^{\varepsilon, \delta}(t, s) + b_i(s) \partial_s (\alpha_{i,j}^{\varepsilon, \delta}(t, s)) - \alpha_{i,j}^{\varepsilon, \delta}(t, s) \sum_{k \in I, k \neq i} \alpha_{i,k}^{\varepsilon, \delta}(t, s) \right) m_i^{\varepsilon, \delta}(t, ds) \\ &\quad + \int_0^1 \alpha_{i,j}^{\varepsilon, \delta}(t, s) \sum_{k \in I, k \neq i} \alpha_{k,i}^{\varepsilon, \delta}(t, s) m_k^{\varepsilon, \delta}(t, ds) \end{aligned}$$

By Lemma 4.7.2 in Appendix, $m_i^{\varepsilon, \delta}(t)$ is absolutely continuous w.r.t. the Lebesgue measure on $[0, 1]$. We consider the constant $\varepsilon^* > 0$ to be smaller than $\bar{\varepsilon}$, that is defined in (4.4.13). Using that for a.e. $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$, the set $\{s \in [0, 1] \mid \varphi_i^{\varepsilon, \delta}(t, s) = \varphi_j^{\varepsilon, \delta}(t, s)\}$ is negligible, by the bounds on $\|\alpha_{i,j}^{\varepsilon, \delta}\|_\infty$ and $\|\partial_s \alpha_{i,j}^{\varepsilon, \delta}\|_\infty$ from Lemma 4.4.1 and on $\partial_t \alpha_{i,j}$ from Lemma 4.4.3, and using that $I = (I^-(t) \cup I^{0,-}(t)) \cup (I^+(t) \cup I^{0,+}(t))$, previous equality gives:

$$\begin{aligned} \frac{d}{dt} A_{i,j}(t) &\leq \int_0^1 \mathbb{1}_{\mathbb{A}_{i,j}^{\varepsilon, \delta}(t)}(s) \left(C - \frac{1}{C\sqrt{\varepsilon}} \right) m_i^{\varepsilon, \delta}(t, ds) + \int_0^1 \alpha_{i,j}^{\varepsilon, \delta}(t, s) \sum_{\substack{k \in I^+(t) \cup I^{0,+}(t) \\ k \neq i}} \alpha_{k,i}^{\varepsilon, \delta}(t, s) m_k^{\varepsilon, \delta}(t, ds) \\ &\quad + \int_0^1 \alpha_{i,j}^{\varepsilon, \delta}(t, s) \sum_{k \in I^-(t) \cup I^{0,-}(t)} \alpha_{k,i}^{\varepsilon, \delta}(t, s) m_k^{\varepsilon, \delta}(t, ds). \end{aligned}$$

Summing previous inequality over $i \in I^+(t) \cup I^{0,+}(t)$ and $j \in I^-(t) \cup I^{0,-}(t)$ and rearranging the terms, one obtains:

$$\begin{aligned} \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} \frac{d}{dt} A_{i,j}(t) &\leq \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} \int_0^1 \left(\mathbb{1}_{\mathbb{A}_{i,j}^{\varepsilon, \delta}(t)}(s) \left(C - \frac{1}{C\sqrt{\varepsilon}} \right) + \sum_{\substack{k \in I^+(t) \cup I^{0,+}(t) \\ k \neq i}} \alpha_{k,j}^{\varepsilon, \delta}(t, s) \alpha_{i,k}^{\varepsilon, \delta}(t, s) \right) m_i^{\varepsilon, \delta}(t, ds) \\ &\quad + \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} \int_0^1 \alpha_{i,j}^{\varepsilon, \delta}(t, s) \sum_{k \in I^-(t) \cup I^{0,-}(t)} \alpha_{k,i}^{\varepsilon, \delta}(t, s) m_k^{\varepsilon, \delta}(t, ds). \end{aligned} \quad (4.4.22)$$

For any $i, j, k \in I$ and any $t \in [0, T]$, the quantity $\alpha_{k,j}^{\varepsilon, \delta}(t, s) \alpha_{i,k}^{\varepsilon, \delta}(t, s)$ vanishes over $[0, 1] \setminus \mathbb{A}_{i,j}^{\varepsilon, \delta}(t)$. Indeed, if $\alpha_{k,j}^{\varepsilon, \delta}(t, s) \alpha_{i,k}^{\varepsilon, \delta}(t, s) > 0$, then $s \in \mathbb{A}_{k,j}^{\varepsilon, \delta}(t) \cap \mathbb{A}_{i,k}^{\varepsilon, \delta}(t)$ and $\varphi_j^{\varepsilon, \delta}(t, s) < \varphi_k^{\varepsilon, \delta}(t, s) < \varphi_i^{\varepsilon, \delta}(t, s)$ and thus, $s \in \mathbb{A}_{i,j}^{\varepsilon, \delta}(t)$. In addition, by Lemma 4.4.1, we have $\|\alpha_{i,j}^{\varepsilon, \delta}\|_\infty \leq C$. Therefore, there exists $C > 0$ such that the previous

inequality becomes:

$$\begin{aligned} \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} \frac{d}{dt} A_{i,j}(t) &\leq \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} \int_0^1 \mathbb{1}_{\mathbb{A}_{i,j}^{\varepsilon,\delta}}(s) \left(C - \frac{1}{C\sqrt{\varepsilon}}\right) m_i^{\varepsilon,\delta}(t, ds) \\ &+ \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} \int_0^1 \alpha_{i,j}^{\varepsilon,\delta}(t, s) \sum_{k \in I^-(t) \cup I^{0,-}(t)} \alpha_{k,i}^{\varepsilon,\delta}(t, s) m_k^{\varepsilon,\delta}(t, ds). \end{aligned} \quad (4.4.23)$$

The constant $\varepsilon^* > 0$ is taken as small enough to have $(C - \frac{1}{C\sqrt{\varepsilon}}) < 0$ for any $\varepsilon \in (0, \varepsilon^*)$. In addition, by the uniform bound on $\alpha^{\varepsilon,\delta}$ in Lemma 4.4.1, one deduces that, for a.e. $t \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta})$, any $i \in I^+(t) \cup I^{0,+}(t)$ and any $j \in I^-(t) \cup I^{0,-}(t)$,

$$\left(C - \frac{1}{C\sqrt{\varepsilon}}\right) \int_0^1 \mathbb{1}_{\mathbb{A}_{i,j}^{\varepsilon,\delta}}(s) m_i^{\varepsilon,\delta}(t, ds) \leq \left(C - \frac{1}{C\sqrt{\varepsilon}}\right) \int_0^1 \frac{\alpha_{i,j}^{\varepsilon,\delta}(t, s)}{C} m_i^{\varepsilon,\delta}(t, ds) \leq \left(C - \frac{1}{C\sqrt{\varepsilon}}\right) A_{i,j}(t). \quad (4.4.24)$$

By the definition of $A_{j,i}$ in (4.4.20), the non negativity of $\alpha^{\varepsilon,\delta}$ and its uniform bound in Lemma 4.4.1, there exists $C > 0$ such that one has, for a.e. $t \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta})$, any $i \in I^+(t) \cup I^{0,+}(t)$ and any $j \in I^-(t) \cup I^{0,-}(t)$,

$$\int_0^1 \alpha_{i,j}^{\varepsilon,\delta}(t, s) \sum_{k \in I^-(t) \cup I^{0,-}(t)} \alpha_{k,i}^{\varepsilon,\delta}(t, s) m_k^{\varepsilon,\delta}(t, ds) \leq C \sum_{k \in I^-(t) \cup I^{0,-}(t)} A_{k,i}(t). \quad (4.4.25)$$

By inequalities (4.4.24) and (4.4.25), inequality (4.4.22) becomes, for a.e. $t \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta})$,

$$\sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} \frac{d}{dt} A_{i,j}(t) \leq \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} \left(C + \frac{1}{C\sqrt{\varepsilon}}\right) A_{i,j}(t) + C \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} A_{j,i}(t). \quad (4.4.26)$$

Now we want to obtain a lower bound of $\frac{d}{dt} A_{j,i}(t)$ for a.e. $t \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta})$, any $i \in I^+(t) \cup I^{0,+}(t)$ and any $j \in I^-(t) \cup I^{0,-}(t)$. Since $(m^{\varepsilon,\delta}, \alpha^{\varepsilon,\delta})$ is a weak solution of the continuity equation (4.1.2), taking $[0, T] \times [0, 1] \times I \ni (t, s, p) \mapsto \mathbb{1}_i(p) \alpha_{i,j}^{\varepsilon,\delta}(t, s)$ as a test function, one deduces that for a.e. $t \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta})$,

$$\begin{aligned} \frac{d}{dt} A_{j,i}(t) &= \int_0^1 \left(\partial_t \alpha_{j,i}^{\varepsilon,\delta}(t, s) + b_j(s) \partial_s (\alpha_{j,i}^{\varepsilon,\delta}(t, s)) - \alpha_{j,i}^{\varepsilon,\delta}(t, s) \sum_{k \in I, k \neq j} \alpha_{j,k}^{\varepsilon,\delta}(t, s) \right) m_j^{\varepsilon,\delta}(t, ds) \\ &+ \int_0^1 \alpha_{j,i}^{\varepsilon,\delta}(t, s) \sum_{k \in I, k \neq j} \alpha_{k,j}^{\varepsilon,\delta}(t, s) m_k^{\varepsilon,\delta}(t, ds) \end{aligned}$$

Since, for any $t \in [0, T]$, one has $\int_0^1 \alpha_{j,i}^{\varepsilon,\delta}(t, s) \sum_{k \in I, k \neq j} \alpha_{k,j}^{\varepsilon,\delta}(t, s) m_k^{\varepsilon,\delta}(t, ds) \geq 0$, by Lemmas 4.4.3 and 4.4.1, there exists $C > 0$ such that previous inequality yields, for a.e. $t \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta})$,

$$\left(-C + \frac{1}{C\sqrt{\varepsilon}}\right) \int_0^1 \mathbb{1}_{\mathbb{A}_{j,i}^{\varepsilon,\delta}}(s) m_j^{\varepsilon,\delta}(t, ds) \leq \frac{d}{dt} A_{j,i}(t).$$

By previous inequality and applying similar arguments as the ones to obtain inequality (4.4.24), there exists $C > 0$ such that, for $\varepsilon^* > 0$ small enough, one has, for a.e. $t \in (\underline{t}^{\varepsilon,\delta}, \bar{t}^{\varepsilon,\delta})$, any $i \in I^+(t) \cup I^{0,+}(t)$ and any $j \in I^-(t) \cup I^{0,-}(t)$,

$$\left(-C + \frac{1}{C\sqrt{\varepsilon}}\right) A_{j,i}(t) \leq \frac{d}{dt} A_{j,i}(t). \quad (4.4.27)$$

By inequalities 4.4.26 and 4.4.27, there exists $C > 0$ such that inequality 4.4.21 becomes, for a.e. $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$,

$$\left(-C + \frac{1}{C\sqrt{\varepsilon}} \right) \left(\sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} A_{i,j}(t) + \sum_{\substack{i \in I^+(t) \cup I^{0,+}(t) \\ j \in I^-(t) \cup I^{0,-}(t)}} A_{j,i}(t) \right) \leq \frac{d^2}{dt^2} \bar{\Psi}(m^{\varepsilon, \delta}(t)).$$

The constant $\varepsilon^* > 0$ is defined as small enough to have for any $\varepsilon \in (0, \varepsilon^*)$,

$$0 \leq -C + \frac{1}{C\sqrt{\varepsilon}}.$$

Since $A_{i,j} \geq 0$ and $A_{j,i} \geq 0$, for any $t \in (\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$, if $\bar{\Psi}(m^{\varepsilon, \delta}(\cdot))$ is twice differentiable at t , one has

$$0 \leq \frac{d^2}{dt^2} \bar{\Psi}(m^{\varepsilon, \delta}(t)).$$

□

Remark 4.4.1. Since the map $t \mapsto \bar{\Psi}(m^{\varepsilon, \delta}(t))$ is semiconvex on $[0, T]$, previous lemma implies that $t \mapsto \bar{\Psi}(m^{\varepsilon, \delta}(t))$ is convex on $(\underline{t}^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$.

Proposition 4.4.1. For any $(\varepsilon, \delta) \in (0, \varepsilon^*) \times (0, \delta^*)$, $i \in I$ and $t \in [0, T]$, one has $\Psi_i(m^{\varepsilon, \delta}(t)) \leq 0$.

Proof. Proof by contradiction. Assume there exists $(\underline{t}^{\varepsilon, \delta}, t^{\varepsilon, \delta}, \bar{t}^{\varepsilon, \delta})$ as defined in (4.4.8) for a certain $i \in I$. Then, there exists $\tau \in (\underline{t}^{\varepsilon, \delta}, t^{\varepsilon, \delta})$ such that

$$\frac{d}{dt} \bar{\Psi}(m^{\varepsilon, \delta}(t)) \Big|_{t=\tau} > 0.$$

By Lemma 4.4.5 and the semiconvexity of $\bar{\Psi}$, one has $\frac{d^2}{dt^2} \bar{\Psi}(m^{\varepsilon, \delta}(\cdot)) \geq 0$ in the sense of distribution on $(\underline{t}^{\varepsilon, \delta}, t^{\varepsilon, \delta})$. Thus, by previous inequality one deduces that, for a.e. $t \in [\tau, \bar{t}^{\varepsilon, \delta}]$,

$$\frac{d}{dt} \bar{\Psi}_i(m^{\varepsilon, \delta}(t)) \Big|_{t=\tau} > 0.$$

Thus, $0 < \bar{\Psi}(m^{\varepsilon, \delta}(t^{\varepsilon, \delta})) < \bar{\Psi}(m^{\varepsilon, \delta}(\bar{t}^{\varepsilon, \delta}))$. By Lemma 4.4.2 and the definition of $\bar{t}^{\varepsilon, \delta}$ and $t^{\varepsilon, \delta}$ in (4.4.8), one obtains a contradiction. □

The main results of this section is a direct consequence of the proposition above.

Proof of Theorem 4.4.1. This theorem is obtained by Proposition 4.4.1. □

Proof of Corollary 4.4.1. This result is a direct consequence of Theorem 4.4.1 and of Proposition 4.3.1. The upper bounds of $\|\lambda\|_\infty$ and $\|\beta\|_\infty$ w.r.t. the data of the problem, is derived from the definition of ε^* in Lemma 4.4.5 and δ^* in Lemma 4.4.2. □

4.5 Proof of the main theorem

The main theorem of the paper, stated in Section 4.2.2, can now be proved.

Proof of Theorem 4.2.2. We proceed in two steps. First, by Corollary 4.4.1 we prove the result for any initial distribution m^0 having a density in $C^1([0, 1] \times I)$ w.r.t. the Lebesgue measure. Then, we extend this result for any initial distribution m^0 in the space $\mathcal{P}([0, 1] \times I)$.

Let the initial distribution m^0 have a density in $C^1([0, 1] \times I)$ w.r.t. the Lebesgue measure and let (m, α) be a solution of Problem (P). Let $(\varphi, \lambda, \beta) \in \text{Lip}([0, T] \times I \times [0, 1]) \times L^\infty([0, T] \times I, \mathbb{R}_+) \times (\mathbb{R}_+)^{|I|}$ be defined from Corollary 4.4.1. We consider $\lambda^* \in \mathcal{M}^+([0, T] \times I)$, defined, for any $i \in I$, by

$$\lambda_i^* := \lambda_i \mathcal{L} + \beta_i \delta_T. \tag{4.5.1}$$

By the definitions of λ and of λ^* , and the fact that $\frac{\delta\Psi_i}{\delta m}(m(t), i, s) = 1 - m_i(t, [0, 1])$, it is straightforward that, for any $t \in [0, T]$,

$$0 \leq \lambda_i^*(t) \quad \text{and} \quad \lambda_i^*(t)(D_i - m_i(t, [0, 1])) = 0.$$

Let φ^* be the weak solution of the Hamilton-Jacobi equation (4.2.2) associated with λ^* , in the sense of Definition (4.2.2). To show that $(\lambda^*, \varphi^*, m)$ is a weak solution of (4.1.5), it is enough to prove that $\alpha_{i,j} = H'(\varphi_i^* - \varphi_j^*)$ on $\{m_i > 0\}$.

By Theorem 4.4.1, we have $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ on $\{m_i > 0\}$ thus, one needs to verify, for any $(i, j, t, s) \in I^2 \times [0, T] \times [0, 1]$, that $\varphi_i(t, s) - \varphi_j(t, s) = \varphi_i^*(t, s) - \varphi_j^*(t, s)$. Since φ is weak solution of the Hamilton-Jacobi equation in the PDE system (4.4.1), by [233, Theorem 4.1], one has, for any $(i, j, t, s) \in I^2 \times [0, T] \times [0, 1]$,

$$\begin{aligned} & \varphi_i(t, s) - \varphi_j(t, s) \\ &= \sum_{p \in I} \int_t^T -H((\varphi_i - \varphi_p)(\tau, S_i^{t,s}(\tau))) + H((\varphi_j - \varphi_p)(\tau, S_j^{t,s}(\tau))) + \lambda_p(t) \left(\frac{\delta\Psi_p}{\delta m}(m(t), i, s) - \frac{\delta\Psi_p}{\delta m}(m(t), j, s) \right) d\tau \\ & \quad + \int_t^T c_i(\tau, S_i^{t,s}(\tau)) - c_j(\tau, S_j^{t,s}(\tau)) d\tau + g_i(S_i^{t,s}(T)) - g_j(S_j^{t,s}(T)) + \sum_{p \in I} \beta_p \left(\frac{\delta\Psi_p}{\delta m}(m(T), i, \cdot) - \frac{\delta\Psi_p}{\delta m}(m(T), j, \cdot) \right), \end{aligned} \quad (4.5.2)$$

where the flow $S_i^{t,s}$ is defined in (4.2.6). By equality (4.3.2) satisfied by $\frac{\delta\Psi_p}{\delta m}$, one has, for any $t \in [0, T]$,

$$\begin{aligned} & \sum_{p \in I} \lambda_p(t) \left(\frac{\delta\Psi_p}{\delta m}(m(t), i, s) - \frac{\delta\Psi_p}{\delta m}(m(t), j, s) \right) = \lambda_i(t) - \lambda_j(t), \\ & \sum_{p \in I} \beta_p \left(\frac{\delta\Psi_p}{\delta m}(m(T), i, \cdot) - \frac{\delta\Psi_p}{\delta m}(m(T), j, \cdot) \right) = \beta_i - \beta_j, \end{aligned}$$

and equality (4.5.2) becomes

$$\begin{aligned} & \varphi_i(t, s) - \varphi_j(t, s) \\ &= \int_t^T \sum_{p \in I} -H((\varphi_i - \varphi_p)(\tau, S_i^{t,s}(\tau))) + H((\varphi_j - \varphi_p)(\tau, S_j^{t,s}(\tau))) d\tau + \int_t^T \lambda_i(t) - \lambda_j(t) d\tau \\ & \quad + \int_t^T c_i(\tau, S_i^{t,s}(\tau)) - c_j(\tau, S_j^{t,s}(\tau)) d\tau + g_i(S_i^{t,s}(T)) - g_j(S_j^{t,s}(T)) + \beta_i - \beta_j, \end{aligned} \quad (4.5.3)$$

Since φ^* is a weak solution of (4.2.2) associated with λ^* , by (4.2.5) and the fact that $L_i^{\lambda^*}(T) = \beta_i$ (where L^{λ^*} is defined in (4.2.3) and $L^{\lambda^*}(T)$ is understood in the sense of trace), one deduces, for any $(i, j, t, s) \in I^2 \times [0, T] \times [0, 1]$,

$$\begin{aligned} & \varphi_i^*(t, s) - \varphi_j^*(t, s) \\ &= \int_t^T \sum_{p \in I} -H((\varphi_i^* - \varphi_p^*)(\tau, S_i^{t,s}(\tau))) + H((\varphi_j^* - \varphi_p^*)(\tau, S_j^{t,s}(\tau))) d\tau + \int_t^T \lambda_i(t) - \lambda_j(t) d\tau \\ & \quad + \int_t^T c_i(\tau, S_i^{t,s}(\tau)) - c_j(\tau, S_j^{t,s}(\tau)) d\tau + g_i(S_i^{t,s}(T)) - g_j(S_j^{t,s}(T)) + \beta_i - \beta_j. \end{aligned} \quad (4.5.4)$$

By Assumption 5, H is locally Lipschitz continuous, then equalities (4.5.3) and (4.5.4) give, for any $t \in [0, T]$,

$$\sum_{i,j \in I} \|(\varphi_j - \varphi_i - \varphi_i^* + \varphi_j^*)(t, \cdot)\|_\infty \leq |I|C \sum_{i,j \in I} \int_t^T \|(\varphi_j - \varphi_i - \varphi_i^* + \varphi_j^*)(\tau, \cdot)\|_\infty d\tau,$$

where C is a Lipschitz constant of H on $[-2 \max(\|\varphi\|_\infty, \|\varphi^*\|_\infty), 2 \max(\|\varphi\|_\infty, \|\varphi^*\|_\infty)]$. Applying Gronwall Lemma to the previous inequality, one deduces that $\|(\varphi_j - \varphi_i) - (\varphi_i^* - \varphi_j^*)\|_\infty = 0$ for any $i, j \in I$. Thus, (φ^*, λ^*) is a solution of Problem (D). The uniform bounds on $\|\lambda\|_\infty$ and $\|\beta\|_\infty$ w.r.t. the data of the problem

is given by Corollary 4.4.1.

We now extend the result to any initial distribution $m^0 \in \mathcal{P}([0, 1] \times I)$. One can approximate m^0 by a sequence of measure $\{m^{0,n}\}_n$, converging to m^0 w.r.t. the weak topology in $\mathcal{P}([0, 1] \times I)$, where for any $n \in \mathbb{N}$, $m^{0,n}$ has a density in $C^1([0, 1] \times I)$ w.r.t. the Lebesgue measure. By Corollary (4.4.1) and using the first step of the proof, there exists the sequences $\{(\lambda^n, \varphi^{*,n}, \beta^n)\}_n$ and $\{\lambda^{*,n}, \varphi^{*,n}\}_n$ such that for any n , λ^n is in $L^\infty((0, T), \times I, \mathbb{R}_+)$, $\lambda^{*,n}$ is defined by (4.5.1) from λ^n , $\beta^n \in (\mathbb{R}_+)^{|I|}$ and $\varphi^{*,n}$ is a solution of the Hamilton-Jacobi equation (4.2.2), associated with $\lambda^{*,n}$.

For $N \in \mathbb{N}$ large enough, $m^{0,n}$ satisfies Assumption 3 for any $n \geq N$. Thus, By Corollary 4.4.1, there exists $C > 0$, that depends on the data of the problem, such that, for any $n \geq N$,

$$\|\lambda^n\|_\infty, \|\beta^n\|_n \leq C.$$

Then, there exist $(\lambda, \beta) \in L^\infty((0, T) \times I, \mathbb{R}_+) \times (\mathbb{R}_+)^{|I|}$ such that $\{\lambda^n\}_n$ weakly* converges, up to a sub sequence, to λ in $L^\infty((0, T) \times I, \mathbb{R}_+)$, and $\{\beta^n\}_n$ converges, up to a sub sequence, to β in $(\mathbb{R}_+)^{|I|}$, where both λ and β satisfy

$$\|\lambda\|_\infty, \|\beta\|_n \leq C.$$

We denote by λ^* the measure defined from λ and β as in (4.5.1). We denote by φ^* the weak solution of (4.2.2) associated with λ^* . One can easily show that

$$\lim_{n \rightarrow \infty} \int_0^T \sum_{i \in I} \left| \int_t^T (\lambda_i^{*,n} - \lambda_i^*)(d\tau) \right| dt + \sum_{i \in I} \left| \int_0^T (\lambda_i^{*,n} - \lambda_i^*)(dt) \right| = 0.$$

Then by using same arguments as in the proof of [233, Lemma 4.6], one can show that $\{\varphi^{*,n}\}_n$ converges, up to a sub sequence, to φ^* , in the sense that

$$\lim_{n \rightarrow \infty} \|\varphi^{*,n}(0) - \varphi^*(0)\|_\infty = \lim_{n \rightarrow \infty} \|\varphi^{*,n}(t) - \varphi^*(t)\|_\infty = 0 \quad \text{for a.e. } t \in (0, T].$$

Then, one has:

$$\lim_{n \rightarrow \infty} \sum_{i \in I} \int_0^1 -\varphi_i^{*,n}(0^+, s) m_i^{0,n}(ds) + \int_0^T D_i \lambda_i^{*,n}(dt) = \sum_{i \in I} \int_0^1 -\varphi_i^*(0^+, s) m_i^0(ds) + \int_0^T D_i \lambda_i^*(dt).$$

Recalling that, for any $n \in \mathbb{N}$, $(\varphi^{*,n}, \lambda^{*,n})$ is a solution of Problem (D), and that, by [233, Proposition 7.1] the value of Problem (D) is continuous w.r.t. the initial distribution m^0 in the weak topology on $\mathcal{P}([0, 1] \times I)$, one deduces by previous equality that (φ^*, λ^*) is a solution of Problem (D) with initial distribution m^0 . \square

The next corollary provides a sufficient condition to have $\beta = 0$ (where β is defined in Theorem 4.4.1).

Corollary 4.5.1. *Let (m, α) be a solution of (P) and let $(\varphi^*, \lambda^*, \lambda, \beta)$ be defined from Theorem 4.2.2. If there exists $f \in C^1([0, 1])$ such that, $g_i = f$ for any $i \in I$, then $\beta = 0$.*

Proof. We consider the set $I^0(T)$ defined in (4.4.15) and the functions $(\Psi_i)_{i \in I}$ defined in (4.3.1). By the definition of β in Theorem 4.2.2 and the system (4.1.5) satisfied by $(\varphi^*, \lambda^*, m)$, it is straightforward that $\beta_i = 0$ for any $i \in I \setminus I^0(T)$. We want to show by a contradiction argument, that $\beta_i = 0$, for any $i \in I$. By Lemma 4.4.4, one has for a.e. $t \in [0, T]$

$$\begin{aligned} \sum_{i \in I^0(T)} \frac{d}{dt} \Psi_i(m(t)) &= \sum_{\substack{i \in I^0(t) \\ j \in I \setminus I^0(T)}} \int_0^1 -\alpha_{i,j}(t, s) m_i(t, ds) + \alpha_{j,i}(t, s) m_j(t, ds) \\ &= \sum_{\substack{i \in I^0(t) \\ j \in I \setminus I^0(T)}} \int_0^1 -H'(\varphi_i(t, s) - \varphi_j(t, s)) m_i(t, ds) + H'(\varphi_j(t, s) - \varphi_i(t, s)) m_j(t, ds). \end{aligned} \tag{4.5.5}$$

Since φ^* is a weak solution of (4.2.2) associated with λ^* , by (4.2.5) and the assumption that $f = g_i$, for any $i \in I$, we have,

$$\varphi_i^*(T, s) = f(s) + L_i^{\lambda^*}(T),$$

where $L_i^{\lambda^*}(T)$ is defined in (4.2.3) and is understood in the sense of trace. Since $\lambda_i^* = \lambda_i \mathcal{L} + \delta_T \beta_i$, previous equality becomes for any $i \in I$

$$\varphi_i^*(T, s) = f(s) + \beta_i \quad \forall s \in [0, 1].$$

Thus, one has for any $i \in I^0(T)$ and $j \in I \setminus I^0(T)$, that,

$$\varphi_i^*(T, s) - \varphi_j^*(T, s) = \beta_i \geq 0 \quad \forall s \in [0, 1]. \quad (4.5.6)$$

We prove the result by a proof by contradiction. Assume that there exists $i^* \in I^0(T)$ such that $\beta_{i^*} > 0$. Then, by (4.5.6), for any $s \in [0, 1]$ and $j \in I \setminus I^0(T)$, one has $\varphi_{i^*}^*(T, s) - \varphi_j^*(T, s) > 0$. Since i^* is in $I^0(t)$, $m_{i^*}(T, [0, 1]) = D_{i^*}$. For any $i \in I$ and $P \subset I$, the map

$$t \mapsto \sum_{j \in P} \int_0^1 H'(\varphi_i^*(t, s) - \varphi_j^*(t, s)) m_i(t, ds),$$

is Lipschitz continuous on $[0, T]$. Consequently, there exists $\nu > 0$ and $\bar{t}_1 \in [0, T)$ such that, for any $t \in (\bar{t}_1, T]$,

$$\sum_{j \in I \setminus I^0(t)} \int_0^1 -H'(\varphi_{i^*}^*(t, s) - \varphi_j^*(t, s)) m_{i^*}(t, ds) < -\nu,$$

and

$$\sum_{j \in I \setminus I^0(t)} \int_0^1 H'(\varphi_j^*(t, s) - \varphi_{i^*}^*(t, s)) m_j(t, ds) = 0.$$

Similarly, by (4.5.6) and applying similar arguments of Lipschitz continuity, one can show that there exists $\bar{t}_2 \in (\bar{t}_1, T)$, such that for any $i \in I^0(T) \setminus \{i^*\}$ and $t \in (\bar{t}_2, T]$, one has

$$\sum_{j \in I \setminus I^0(t)} \int_0^1 -H'(\varphi_i^*(t, s) - \varphi_j^*(t, s)) m_i(t, ds) < \frac{\nu}{2|I|},$$

and

$$\sum_{j \in I \setminus I^0(t)} \int_0^1 H'(\varphi_j^*(t, s) - \varphi_{i^*}^*(t, s)) m_j(t, ds) < \frac{\nu}{2|I|}.$$

Using the four last inequalities, equality (4.5.5) becomes, for a.e. $t \in (\bar{t}_2, T]$

$$\sum_{i \in I^0(T)} \frac{d}{dt} \Psi_i(m(t)) < 0.$$

Since $\sum_{i \in I^0(T)} \Psi_i(m(T)) = 0$, previous inequality gives the contradiction. \square

4.6 Numerical approximation

In this section we exploit Theorem 4.2.2 to numerically approximate Problem (D) . We first introduce Problem (\bar{D}) that is equivalent to Problem (D) . Then, this problem is discretized and we show the convergence of the solutions of the discretized problems to a solution of Problem (\bar{D}) , when the time and space steps converge to 0. This convergence results is used to build a sequence converging to a solution of Problem (P) and to provide an estimate on the value of Problem (P) . Finally, some numerical simulations are displayed in Section 4.6.2.

4.6.1 Convergence results

Given $(\lambda, \beta) \in L^\infty((0, T) \times I) \times (\mathbb{R}_+)^{|I|}$, we introduce the following Hamilton-Jacobi equation on $(0, T) \times (0, 1) \times I$:

$$\begin{aligned} -\partial_t \varphi_i(t, s) - b_i(s) \partial_s \varphi_i(t, s) - c_i(t, s) - \lambda_i(t) + \sum_{j \in I, j \neq i} H((\varphi_j - \varphi_i)(t, s)) &= 0 & (t, s, i) \in (0, T) \times (0, 1) \times I, \\ \varphi_i(T, s) &= g_i(s) + \beta_i & (s, i) \in [0, 1] \times I. \end{aligned} \quad (4.6.1)$$

A weak solution φ of equation (4.6.1) is a function in $\text{Lip}([0, T] \times [0, 1] \times I)$ that is a solution of (4.6.1) in the sense of distributions. We have the following key result.

Lemma 4.6.1. *Problem (D) is equivalent to the following problem:*

$$\begin{aligned} \inf_{(\varphi, \lambda, \beta)}, & \sum_{i \in I} \int_0^1 -\varphi_i(0, s) m_i^0(ds) + \int_0^T D_i \lambda_i(t) dt + \beta_i D_i, \\ (\lambda, \beta) \in L^\infty((0, T) \times I, \mathbb{R}_+) \times (\mathbb{R}_+)^{|I|} & \\ \|\lambda\|_\infty \leq 1/\varepsilon^*, \|\beta\|_1 \leq C/\varepsilon^0 & \end{aligned} \quad (\bar{D})$$

where φ is a weak sub-solution of (4.6.1) associated with (λ, β)

where the constants $C > 0$ and $\varepsilon^* > 0$ depend on the data of the problem and are respectively defined in Lemma 4.4.1 and in Lemma 4.4.5, and the constant $\varepsilon^0 > 0$ is defined by Assumption 3.

Proof. This is a direct consequence of Theorem 4.2.2. \square

We propose a time and space discretization of Problem (\bar{D}) , based on an explicit finite difference scheme of the Hamilton-Jacobi equation (4.6.1). Let $\Delta t > 0$ and $\Delta s > 0$ be such that $N_T := T/\Delta t$ and $N_s := 1/\Delta s$ are integers. The discretization parameters are defined such that there exists $C > 0$ and $a > 1$, independent of Δt and Δs , such that Δt and Δs satisfy

$$\frac{\Delta t}{\Delta s} \leq C(\Delta t)^a. \quad (4.6.2)$$

Let $(\lambda, \beta) \in \mathbb{R}_+^{N_T \times |I|} \times \mathbb{R}_+^{|I|}$. Let $i \in I$, $k \in \{0, \dots, N_T - 1\}$ and $\ell \in \{0, \dots, N_s\}$, then we consider the system of inequalities on $\varphi \in \mathbb{R}^{(N_T+1) \times (N_s+1) \times |I|}$

$$\left\{ \begin{array}{ll} \begin{aligned} & -\frac{\varphi_{i,k+1,\ell} - \varphi_{i,k,\ell}}{\Delta t} - b_i(\ell \Delta s) \frac{\varphi_{i,k+1,\ell+1} - \varphi_{i,k+1,\ell}}{\Delta s} - c_i((k+1)\Delta t, \ell \Delta s) \\ & - \lambda_{i,k} + \sum_{j \in I, j \neq i} H(\varphi_{i,k+1,\ell} - \varphi_{j,k+1,\ell}) \leq 0 \end{aligned} & \text{if } b_i(\ell \Delta s) \geq 0, \\ \\ \begin{aligned} & -\frac{\varphi_{i,k+1,\ell} - \varphi_{i,k,\ell}}{\Delta t} - b_i(\ell \Delta s) \frac{\varphi_{i,k+1,\ell} - \varphi_{i,k+1,\ell-1}}{\Delta s} - c_i((k+1)\Delta t, \ell \Delta s) \\ & - \lambda_{i,k} + \sum_{j \in I, j \neq i} H(\varphi_{i,k+1,\ell} - \varphi_{j,k+1,\ell}) \leq 0 \end{aligned} & \text{if } b_i(\ell \Delta s) < 0, \\ \\ \varphi_{i,N_T,\ell} \leq g_i(\ell \Delta s) + \beta_i. \end{array} \right. \quad (4.6.3)$$

For any $\Delta t > 0$ and $\Delta s > 0$, we consider the optimization problem

$$\begin{aligned} \inf_{(\varphi, \lambda, \beta)}, & \sum_{i \in I} \left(\sum_{\ell=0}^{N_s-1} -\varphi_{i,0,\ell} m_i^0([\ell \Delta s, (\ell+1)\Delta s]) + \sum_{k=0}^{N_T-1} D_i \lambda_{i,k} + \beta_i D_i \right), \\ (\lambda, \beta) \in (\mathbb{R}_+)^{N_T \times |I|} \times (\mathbb{R}_+)^{|I|} & \\ \|\lambda\|_\infty \leq 1/\varepsilon^*, \|\beta\|_1 \leq C/\varepsilon^0 & \end{aligned} \quad (\bar{D}^{\Delta t, \Delta s})$$

where φ satisfies (4.6.3) associated with (λ, β)

For a given $(\lambda, \beta) \in \mathbb{R}_+^{N_T \times |I|} \times \mathbb{R}_+^{|I|}$, one can prove a comparison principle for the system (4.6.3). More precisely, let φ satisfy (4.6.3) and $\bar{\varphi}$ also satisfy (4.6.3) but with equalities, then one can show that $\varphi \leq \bar{\varphi}$. Note that

the distinction between $b_i(\ell\Delta s) \geq 0$ and $b_i(\ell\Delta s) < 0$ in (4.6.3) is important to get the comparison principle. Since the objective function of Problem $(\bar{D}^{\Delta t, \Delta s})$ is non increasing w.r.t. the variable φ , one can restrict the admissible set in Problem $(\bar{D}^{\Delta t, \Delta s})$ to variables φ satisfying (4.6.3) with equalities. In addition, since the admissible set for the pair (λ, β) is compact, one we can restrict the admissible set for the variables $(\varphi, \lambda, \beta)$ in Problem $(\bar{D}^{\Delta t, \Delta s})$ to a compact set. By the continuity of the objective function, Problem $(\bar{D}^{\Delta t, \Delta s})$ admits at least a solution. For any $\Delta t > 0$ and $\Delta s > 0$, we denote by $(\varphi^{\Delta t, \Delta s}, \lambda^{\Delta t, \Delta s}, \beta^{\Delta t, \Delta s})$ a solution of Problem $(\bar{D}^{\Delta t, \Delta s})$. By the comparison principle for the system (4.6.3), $\varphi^{\Delta t, \Delta s}$ satisfies

$$\left\{ \begin{array}{ll} -\frac{\varphi_{i,k+1,\ell}^{\Delta t, \Delta s} - \varphi_{i,k,\ell}^{\Delta t, \Delta s}}{\Delta t} - b_i(\ell\Delta s) \frac{\varphi_{i,k+1,\ell+1}^{\Delta t, \Delta s} - \varphi_{i,k+1,\ell}^{\Delta t, \Delta s}}{\Delta s} - c_i((k+1)\Delta t, \ell\Delta s) & \text{if } b_i(\ell\Delta s) \geq 0, \\ -\lambda_{i,k}^{\Delta t, \Delta s} + \sum_{j \in I, j \neq i} H(\varphi_{i,k+1,\ell}^{\Delta t, \Delta s} - \varphi_{j,k+1,\ell}^{\Delta t, \Delta s}) = 0 & \\ \\ -\frac{\varphi_{i,k+1,\ell}^{\Delta t, \Delta s} - \varphi_{i,k,\ell}^{\Delta t, \Delta s}}{\Delta t} - b_i(\ell\Delta s) \frac{\varphi_{i,k+1,\ell}^{\Delta t, \Delta s} - \varphi_{i,k+1,\ell-1}^{\Delta t, \Delta s}}{\Delta s} - c_i((k+1)\Delta t, \ell\Delta s) & \text{if } b_i(\ell\Delta s) < 0, \\ -\lambda_{i,k}^{\Delta t, \Delta s} + \sum_{j \in I, j \neq i} H(\varphi_{i,k+1,\ell}^{\Delta t, \Delta s} - \varphi_{j,k+1,\ell}^{\Delta t, \Delta s}) = 0 & \\ \\ \varphi_{i,N_T,\ell}^{\Delta t, \Delta s} = g_i(\ell\Delta s) + \beta_i^{\Delta t, \Delta s}. & \end{array} \right. \quad (4.6.4)$$

We define $\bar{\beta}^{\Delta t, \Delta s} := \beta^{\Delta t, \Delta s}$ and the bounded function $\bar{\lambda}^{\Delta t, \Delta s} \in L^\infty([0, T] \times I, \mathbb{R}_+)$ by

$$\bar{\lambda}_i^{\Delta t, \Delta s}(t) := \sum_{k=0}^{N_T-1} \lambda_{i,k}^{\Delta t, \Delta s} \mathbb{1}_{[k\Delta t, (k+1)\Delta t)}(t) \text{ for any } (i, t) \in I \times [0, T]. \quad (4.6.5)$$

Let $\bar{\varphi}^{\Delta t, \Delta s}$ be the weak solution of (4.6.1) associated with $(\bar{\lambda}^{\Delta t, \Delta s}, \bar{\beta}^{\Delta t, \Delta s})$. The next theorem states the convergence of $\{(\bar{\varphi}^{\Delta t, \Delta s}, \bar{\lambda}^{\Delta t, \Delta s}, \bar{\beta}^{\Delta t, \Delta s})\}_{(\Delta t, \Delta s)}$, up to a subsequence, to a solution of Problem (D) .

Theorem 4.6.1. *The sequence $\{(\bar{\varphi}^{\Delta t, \Delta s}, \bar{\lambda}^{\Delta t, \Delta s}, \bar{\beta}^{\Delta t, \Delta s})\}_{(\Delta t, \Delta s)}$ converges, up to a sub sequence, to a solution $(\bar{\varphi}, \bar{\lambda}, \bar{\beta})$ of Problem (D) , in the sense that,*

$$\begin{aligned} \lim_{\Delta t, \Delta s \rightarrow 0} \|\bar{\varphi}^{\Delta t, \Delta s} - \bar{\varphi}\|_\infty &= 0 \\ \bar{\lambda}^{\Delta t, \Delta s} &\xrightarrow[\Delta t, \Delta s \rightarrow 0]{*} \bar{\lambda} \quad \text{in } L^\infty([0, T] \times I), \\ \lim_{\Delta t, \Delta s \rightarrow 0} \|\bar{\beta}^{\Delta t, \Delta s} - \bar{\beta}\|_\infty &= 0. \end{aligned}$$

Proof. By the definition of Problem $(\bar{D}^{\Delta t, \Delta s})$, $\{\|\lambda^{\Delta t, \Delta s}\|_\infty\}_{(\Delta t, \Delta s)}$ is uniformly bounded and thus, $\{\|\bar{\lambda}^{\Delta t, \Delta s}\|_\infty\}_{(\Delta t, \Delta s)}$ is also uniformly bounded. Then, one can extract a sub-sequence of $\{\bar{\lambda}^{\Delta t, \Delta s}\}_{(\Delta t, \Delta s)}$ weakly* converging w.r.t. the L^∞ topology, up to a sub sequence, to an element of $L^\infty((0, T) \times I, \mathbb{R}_+)$ denoted by $\bar{\lambda}$, when $(\Delta t, \Delta s)$ converges to 0. By the uniform bound of the sequence $\{\bar{\lambda}^{\Delta t, \Delta s}\}_{(\Delta t, \Delta s)}$, one has $\|\bar{\lambda}\|_\infty \leq 1/\varepsilon^*$. Similarly, there exists $\bar{\beta}$ such that a sub-sequence of $\{\bar{\beta}^{\Delta t, \Delta s}\}_{(\Delta t, \Delta s)}$ converges to $\bar{\beta}$ in $\mathbb{R}_+^{|I|}$ when $(\Delta t, \Delta s)$ converges to 0. We denote by $\bar{\varphi}$ the weak solution of (4.6.1) associated with $(\bar{\lambda}, \bar{\beta})$. Using the weak* convergence of $\{\bar{\lambda}^{\Delta t, \Delta s}\}_{(\Delta t, \Delta s)}$ to $\bar{\lambda}$ and the convergence of $\{\bar{\beta}^{\Delta t, \Delta s}\}_{(\Delta t, \Delta s)}$ to $\bar{\beta}$, by using the same arguments as in the proof of [233, Lemma 4.6] and the continuity of $\bar{\varphi}$ and $\varphi^{\Delta t, \Delta s}$, one can show that $\{\bar{\varphi}^{\Delta t, \Delta s}\}_{(\Delta t, \Delta s)}$ converges uniformly to $\bar{\varphi}$. Now we want to show that $(\bar{\varphi}, \bar{\lambda}, \bar{\beta})$ is a solution of Problem (D) . To do so, we first show that the value of Problem $(\bar{D}^{\Delta t, \Delta s})$ converges to the value of the objective function of Problem (D) evaluated in $(\bar{\varphi}, \bar{\lambda}, \bar{\beta})$, and then, that it converges also to the value of Problem (D) , when $(\Delta t, \Delta s)$ converges to 0.

Let $\Delta t > 0$ and $\Delta s > 0$. By [233, Theorem 4.1], $\bar{\varphi}^{\Delta t, \Delta s}$ satisfies for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,

$$\bar{\varphi}_i^{\Delta t, \Delta s}(t, s) = \int_t^T \sum_{j \in I, j \neq i} -H((\bar{\varphi}_i^{\Delta t, \Delta s} - \bar{\varphi}_j^{\Delta t, \Delta s})(\tau, S_i^{t, s}(\tau))) + c_i(\tau, S_i^{t, s}(\tau)) + \bar{\lambda}^{\Delta t, \Delta s}(\tau) d\tau + g_i(S_i^{t, s}(T)) + \beta_i^{\Delta t, \Delta s}, \quad (4.6.6)$$

where the flow $S_i^{t, s}$ is defined in (4.2.6). There exists a constant $C > 0$, independent of Δt and Δs , such that, since $\bar{\varphi}^{\Delta t, \Delta s} \in C^1([0, T] \times [0, 1] \times I)$, $\bar{\varphi}^{\Delta t, \Delta s}$ satisfies the following inequality for any $k \in \{0, \dots, N_T - 1\}$, $(\ell, i) \in \{0, \dots, N_s - 1\} \times I$:

$$\begin{aligned} & \left| \bar{\varphi}_i^{\Delta t, \Delta s}(k\Delta t, \ell\Delta s) - \bar{\varphi}_i^{\Delta t, \Delta s}((k+1)\Delta t, \ell\Delta s) + \Delta t \sum_{i \neq j} H\left(\bar{\varphi}_i^{\Delta t, \Delta s}((k+1)\Delta t, \ell\Delta s) - \bar{\varphi}_j^{\Delta t, \Delta s}((k+1)\Delta t, \ell\Delta s)\right) \right. \\ & \quad \left. - \Delta t b_i(\ell\Delta s) \frac{\bar{\varphi}_i^{\Delta t, \Delta s}((k+1)\Delta t, (\ell+1)\Delta s) - \bar{\varphi}_i^{\Delta t, \Delta s}((k+1)\Delta t, \ell\Delta s)}{\Delta s} - \Delta t (c_i((k+1)\Delta t) + \bar{\lambda}_i^{\Delta t, \Delta s}(k\Delta t)) \right| \\ & \leq C \left(\Delta t + \Delta s + \frac{(\Delta t)^2}{\Delta s} \right). \end{aligned} \quad (4.6.7)$$

Note that previous equality is still valid if the term $\frac{\bar{\varphi}_i^{\Delta t, \Delta s}((k+1)\Delta t, (\ell+1)\Delta s) - \bar{\varphi}_i^{\Delta t, \Delta s}((k+1)\Delta t, \ell\Delta s)}{\Delta s}$ is replaced by $\frac{\bar{\varphi}_i^{\Delta t, \Delta s}((k+1)\Delta t, \ell\Delta s) - \bar{\varphi}_i^{\Delta t, \Delta s}((k+1)\Delta t, (\ell-1)\Delta s)}{\Delta s}$.

By the Assumptions 4 and 5 on c and L , the definition of $\bar{\lambda}^{\Delta t, \Delta s}$ in (4.6.5), and equalities (4.6.3) and (4.6.7), there exists a constant $C > 0$, independent of Δt and Δs , such that one has for any $k \in \{0, \dots, N_T - 1\}$,

$$\begin{aligned} & \sup_{\substack{i \in I, \\ \ell \in \{0, \dots, N_s\}}} |\bar{\varphi}_i^{\Delta t, \Delta s}(k\Delta t, \ell\Delta s) - \varphi_{i, k, \ell}^{\Delta t, \Delta s}| \\ & \leq \left(1 + C\Delta t + C\frac{\Delta t}{\Delta s}\right) \sup_{\substack{i \in I, \\ \ell \in \{0, \dots, N_s\}}} |\bar{\varphi}_i^{\Delta t, \Delta s}((k+1)\Delta t, \ell\Delta s) - \varphi_{i, k+1, \ell}^{\Delta t, \Delta s}| \\ & \quad + C \left(\Delta t + \Delta s + \frac{(\Delta t)^2}{\Delta s} \right). \end{aligned}$$

Using that for any $(i, \ell) \in I \times \{0, \dots, N_s\}$ one has $\bar{\varphi}_i^{\Delta t, \Delta s}(T, \ell\Delta s) = \varphi_{i, N_T, \ell}^{\Delta t, \Delta s}$, one deduces by comparison, for any $\Delta t > 0$ and $\Delta s > 0$ small enough and satisfying (4.6.2), that there exists a constant $C > 0$ such that

$$\sup_{\substack{i \in I, \\ \ell \in \{0, \dots, N_s\}, \\ k \in \{0, \dots, N_k\}}} |\bar{\varphi}_i^{\Delta t, \Delta s}(k\Delta t, \ell\Delta s) - \varphi_{i, k, \ell}^{\Delta t, \Delta s}| \leq C(\Delta t + \Delta s). \quad (4.6.8)$$

By [233, Remark 4.7], the quantity $\|\partial_s \bar{\varphi}^{\Delta t, \Delta s}\|_\infty$ depends on the data of the problem and on $\sum_{i \in I} \int_0^T \bar{\lambda}_i^{\Delta t, \Delta s}(t) dt$.

Thus, by the uniform bound on $\{\|\lambda^{\Delta t, \Delta s}\|_\infty\}_{\Delta t, \Delta s}$, there exists a constant C , independent of Δt and Δs , such that $\{\|\partial_s \bar{\varphi}^{\Delta t, \Delta s}\|_\infty\}_{(\Delta t, \Delta s)}$ is bounded by C . Thus, by (4.6.8), there exists a constant $C > 0$ such that

$$\begin{aligned} & \left| \sum_{i \in I} \sum_{\ell=0}^{N_s-1} -\varphi_{i, 0, \ell}^{\Delta t, \Delta s} m_i^0([\ell\Delta s, (\ell+1)\Delta s]) + \sum_{k=0}^{N_T-1} D_i \lambda_{i, k}^{\Delta t, \Delta s} + \beta_i^{\Delta t, \Delta s} D_i \right. \\ & \quad \left. - \sum_{i \in I} \int_0^1 -\bar{\varphi}_i^{\Delta t, \Delta s}(0, s) m_i^0(ds) + \int_0^T D_i \bar{\lambda}_i^{\Delta t, \Delta s}(t) dt + \bar{\beta}_i^{\Delta t, \Delta s} D_i \right| \leq C(\Delta t + \Delta s). \end{aligned} \quad (4.6.9)$$

Using the convergence of $\{(\bar{\varphi}^{\Delta t, \Delta s}, \bar{\lambda}^{\Delta t, \Delta s}, \bar{\beta}^{\Delta t, \Delta s})\}_{(\Delta t, \Delta s)}$ to $(\bar{\varphi}, \bar{\lambda}, \bar{\beta})$, previous inequality gives

$$\begin{aligned} & \lim_{(\Delta t, \Delta s) \rightarrow 0} \sum_{i \in I} \sum_{\ell=0}^{N_s-1} -\varphi_{i,0,\ell}^{\Delta t, \Delta s} m_i^0([\ell \Delta s, (\ell+1)\Delta s]) + \sum_{k=0}^{N_T-1} D_i \lambda_{i,k}^{\Delta t, \Delta s} + \beta_i^{\Delta t, \Delta s} D_i \\ &= \lim_{(\Delta t, \Delta s) \rightarrow 0} \sum_{i \in I} \int_0^1 -\bar{\varphi}_i^{\Delta t, \Delta s}(0, s) m_i^0(ds) + \int_0^T D_i \bar{\lambda}_i^{\Delta t, \Delta s}(t) dt + \bar{\beta}_i^{\Delta t, \Delta s} D_i \\ &= \sum_{i \in I} \int_0^1 -\bar{\varphi}_i(0, s) m_i^0(ds) + \int_0^T D_i \bar{\lambda}_i(t) dt + \bar{\beta}_i D_i. \end{aligned}$$

Since $(\varphi^{\Delta t, \Delta s}, \lambda^{\Delta t, \Delta s}, \beta^{\Delta t, \Delta s})$ is a solution of Problem $(\bar{D}^{\Delta t, \Delta s})$, by previous equality, one has

$$\lim_{(\Delta t, \Delta s) \rightarrow 0} \text{val}(\bar{D}^{\Delta t, \Delta s}) = \sum_{i \in I} \int_0^1 -\bar{\varphi}_i(0, s) m_i^0(ds) + \int_0^T D_i \bar{\lambda}_i(t) dt + \bar{\beta}_i D_i. \quad (4.6.10)$$

To conclude the proof, we need to show that the value of Problem $(\bar{D}^{\Delta t, \Delta s})$ converges to the value of Problem (\bar{D}) when $(\Delta t, \Delta s)$ goes to 0. Let $(\hat{\varphi}, \hat{\lambda}, \hat{\beta})$ be a solution of Problem (\bar{D}) . For any $\Delta t, \Delta s > 0$, satisfying (4.6.2), we consider $(\hat{\varphi}^{\Delta t, \Delta s}, \hat{\lambda}^{\Delta t, \Delta s}, \hat{\beta}^{\Delta t, \Delta s}) \in (\mathbb{R}_+)^{(N_T+1) \times (N_s+1) \times |I|} \times (\mathbb{R}_+)^{N_T \times |I|} \times (\mathbb{R}_+)^{|I|}$, such that $\hat{\lambda}_{i,k}^{\Delta t, \Delta s} := \int_{k\Delta t}^{(k+1)\Delta t} \hat{\lambda}_i(t) dt$, $\hat{\beta}_i^{\Delta t, \Delta s} := \hat{\beta}_i$, and $\hat{\varphi}$ satisfies (4.6.3) associated with $(\hat{\lambda}^{\Delta t, \Delta s}, \hat{\beta}^{\Delta t, \Delta s})$. Applying similar computations as the ones to obtain (4.6.8), there exist $C > 0$, independent of Δt and Δs , such that one has

$$\sup_{\substack{i \in I, \\ \ell \in \{0, \dots, N_s\}, \\ k \in \{0, \dots, N_k\}}} |\hat{\varphi}_i(k\Delta t, (\ell+1)\Delta s) - \hat{\varphi}_{i,k,\ell}^{\Delta t, \Delta s}| \leq C(\Delta t + \Delta s),$$

and thus, there exists $C > 0$, independent of Δt and Δs , such that

$$\begin{aligned} & \left| \sum_{i \in I} \sum_{\ell=0}^{N_s-1} -\hat{\varphi}_{i,0,\ell}^{\Delta t, \Delta s} m_i^0([\ell \Delta s, (\ell+1)\Delta s]) + \sum_{k=0}^{N_T-1} D_i \hat{\lambda}_{i,k}^{\Delta t, \Delta s} + \hat{\beta}_i^{\Delta t, \Delta s} D_i \right. \\ & \left. - \int_0^1 -\hat{\varphi}_i(0, s) m_i^0(ds) + \int_0^T D_i \hat{\lambda}_i(t) dt + \hat{\beta}_i D_i \right| \leq C(\Delta t + \Delta s). \end{aligned} \quad (4.6.11)$$

Since $(\hat{\varphi}, \hat{\lambda}, \hat{\beta})$ is a solution of Problem $(\bar{D}^{\Delta t, \Delta s})$, previous inequality yields

$$\text{val}(\bar{D}^{\Delta t, \Delta s}) \leq \text{val}(\bar{D}) + C(\Delta t + \Delta s).$$

Equality (4.6.10) and previous equality give, when $(\Delta t, \Delta s)$ converges to 0,

$$\sum_{i \in I} \int_0^1 -\bar{\varphi}_i(0, s) m_i^0(ds) + \int_0^T D_i \bar{\lambda}_i(t) dt + \bar{\beta}_i D_i \leq \text{val}(\bar{D}),$$

Thus, $(\bar{\varphi}, \bar{\lambda}, \bar{\beta})$ is a solution of Problem (\bar{D}) . □

For any $\Delta t, \Delta s > 0$, given a solution $(\varphi^{\Delta t, \Delta s}, \lambda^{\Delta t, \Delta s}, \beta^{\Delta t, \Delta s})$ of Problem $(\bar{D}^{\Delta t, \Delta s})$, we define the function $\bar{\alpha}^{\Delta t, \Delta s} \in \text{Lip}([0, T] \times [0, 1] \times I^2)$ by

$$\bar{\alpha}_{i,j}^{\Delta t, \Delta s} := H'(\bar{\varphi}_i^{\Delta t, \Delta s} - \bar{\varphi}_j^{\Delta t, \Delta s}). \quad (4.6.12)$$

By Lemma 4.7 in Appendix 4.7, we denote by $\bar{m}^{\Delta t, \Delta s}$ the unique distribution in $\text{Lip}([0, T], \mathcal{P}([0, 1] \times I))$ such that $(\bar{m}^{\Delta t, \Delta s}, \bar{\alpha}^{\Delta t, \Delta s})$ is a weak solution of equation (4.1.2) associated with the initial distribution m^0 . The next corollary shows the convergence of $\{(\bar{m}^{\Delta t, \Delta s}, \bar{\alpha}^{\Delta t, \Delta s})\}_{(\Delta t, \Delta s)}$ to a solution of Problem (P) .

Corollary 4.6.1. *Let $\Delta t > 0$ and $\Delta s > 0$ satisfying (4.6.2). Then there exists a solution $(\bar{m}, \bar{\alpha})$ of Problem (P), such that, up to a sub-sequence of $\{(\bar{m}^{\Delta t, \Delta s}, \bar{\alpha}^{\Delta t, \Delta s})\}_{(\Delta t, \Delta s)}$, one has*

$$\lim_{\Delta t, \Delta s \rightarrow 0} \|\bar{\alpha}^{\Delta t, \Delta s} - \bar{\alpha}\|_{\infty} = 0 \quad (4.6.13)$$

$$\lim_{\Delta t, \Delta s \rightarrow 0} \|\mathcal{W}(\bar{m}^{\Delta t, \Delta s}(\cdot), \bar{m}(\cdot))\|_{\infty} = 0, \quad (4.6.14)$$

$$\lim_{\Delta t, \Delta s \rightarrow 0} J(\bar{m}^{\Delta t, \Delta s}, \bar{\alpha}^{\Delta t, \Delta s}) = J(\bar{m}, \bar{\alpha}). \quad (4.6.15)$$

Proof. Let $\bar{\alpha} \in \text{Lip}([0, T] \times [0, 1] \times I^2)$ be defined for any $i, j \in I$, by

$$\bar{\alpha}_{i,j} := H'(\bar{\varphi}_i - \bar{\varphi}_j).$$

The uniform convergence of $\{\alpha^{\Delta t, \Delta s}\}_{\Delta t, \Delta s}$ to $\bar{\alpha}$ is a consequence of the definition of $\alpha^{\Delta t, \Delta s}$ in (4.6.12) and of Theorem 4.6.1.

By the convergence of $\{\bar{\alpha}^{\Delta t, \Delta s}\}_{(\Delta t, \Delta s)}$ and [233, Lemma 8.2], the set $\{\bar{m}^{\Delta t, \Delta s}\}_{(\Delta t, \Delta s)}$ is uniformly Lipschitz from $[0, T]$ to $\mathcal{P}([0, 1] \times I)$, where $\mathcal{P}([0, 1] \times I)$ is endowed with the Wasserstein distance. Then, by the Arzela-Ascoli Theorem, there exists $\bar{m} \in \text{Lip}([0, T], \mathcal{P}([0, 1] \times I))$ such that the set $\{\bar{m}^{\Delta t, \Delta s}\}_{(\Delta t, \Delta s)}$ satisfies, up to a sub-sequence, (4.6.14). Further, \bar{m} is the only distribution such that $(\bar{m}, \bar{\alpha})$ is a weak solution of (4.1.2) with initial distribution m^0 . Equality (4.6.15) is a direct consequence of equalities (4.6.13) and (4.6.14) and of the definition of J in (4.1.1).

Since $(\bar{\varphi}, \bar{\lambda})$ is a solution of (D), by the definition of $\bar{\alpha}$ and the continuity equation (4.1.2) satisfied by $(\bar{m}, \bar{\alpha})$, (φ, λ, m) is a weak solution of the system (4.1.5). Thus, by Theorem 4.2.1, $(\bar{m}, \bar{\alpha})$ is a solution of Problem (P). \square

Although the sequence $\{(\bar{m}^{\Delta t, \Delta s}, \bar{\alpha}^{\Delta t, \Delta s})\}_{(\Delta t, \Delta s)}$ converges to a solution of Problem (P), there is no guaranty, for any $\Delta t, \Delta s > 0$, that $(\bar{m}^{\Delta t, \Delta s}, \bar{\alpha}^{\Delta t, \Delta s})$ satisfies the constraint (4.1.3).

Finally, we have the following estimate, that is of practical interest to numerically approximate the value of Problem (P).

Corollary 4.6.2. *There exists a constant $C > 0$ such that, for any $\Delta t > 0$ and $\Delta s > 0$ satisfying (4.6.2), one has*

$$-val(\bar{D}^{\Delta t, \Delta s}) - C(\Delta t + \Delta s) \leq val(P) \leq -val(\bar{D}^{\Delta t, \Delta s}) + C(\Delta t + \Delta s).$$

Proof. By Lemma 4.6.1 and the equality on the value of Problem (P) in Theorem 4.2.1, this result is a direct consequence of inequalities (4.6.9) and (4.6.11) in the proof of Theorem 4.6.1. \square

Although the value of the constant C in Corollary 4.6.2 is not given, it can be traced back through the computation.

Remark 4.6.1. *Since there is no convergence rate in the convergence of $\{J(\bar{m}^{\Delta t, \Delta s}, \bar{\alpha}^{\Delta t, \Delta s})\}_{\Delta t, \Delta s}$ to $J(\bar{m}, \bar{\alpha})$ in (4.6.15) and that it is numerically possible to compute $val(\bar{D}^{\Delta t, \Delta s})$ and $J(\bar{m}^{\Delta t, \Delta s}, \bar{\alpha}^{\Delta t, \Delta s})$, the main interest of Corollary 4.6.2 is to make it possible to verify if $J(\bar{m}^{\Delta t, \Delta s}, \bar{\alpha}^{\Delta t, \Delta s})$ is close to $val(P)$.*

4.6.2 Numerical results

This section illustrates the numerical method developed to approximate Problem (P) with an example of smart charging. A central planner aims at optimally charging a large fleet of plug-in electric vehicles (PEVs) over a finite period horizon $[0, T]$, with $T = 5h$. It is assumed that no PEV arrives after $t = 0$ and leaves before $t = T$. In this use case, the discrete variable taking values in I represents the charging mode of PEVs and the continuous variable taking value in $[0, 1]$ represents the state of charge (SoC) of PEVs relatively to the maximum energy capacity of the battery. It is assumed that the set of charging modes is $I = \{0, 1\}$, where 0 stands for idle mode and 1 for charging. The transport term b in the continuity equation represents the charging rate. In each situation, each mode $i \in I$ is characterized by its charging rate $b(i, \cdot)$ such that

- $i = 0$ corresponds to a non-charging mode with $b_0(s) = 0$,
- $i = 1$ corresponds to a charging mode with $b_1(s) = 0.7 \times (1 - s)$.

The initial distribution of the states m^0 is concentrated on the mode 0 (i.e. $m_0^0([0, 1]) = 1$) and is displayed in Figure 4.3. The running cost c represents a cost of electricity consumption and is such that $c_0 = 0$ and c_1 follows a peak and off-peak periods pricing, see Figure 4.2. The function L penalizing the transfers is defined by $L(\alpha) := \frac{\alpha^2}{2}$. The final cost g penalizes low value of SoC: $g_i(s) := 40 \times \max(0, 0.75 - s)^2$. Finally, the congestion parameter D is set as follows: $D_0 = 1$ and $D_1 = 0.2$, meaning that all PEVs can be at the same time in mode 0 (and thus $\lambda_0^{\Delta t, \Delta s} = 0$) and at most 20% of the PEVs in mode 1. The discretization parameters are $\Delta s = 0.05$ and $\Delta t = 0.05$. The discretized problem $(\bar{D}^{\Delta t, \Delta s})$ is solved using the solver *minimize* from the library *scipy.optimize* [266]. The solution $\lambda_1^{\Delta t, \Delta s}$ is displayed in Figure 4.1 (blue curve). From $\lambda_1^{\Delta t, \Delta s}$, we can build $\bar{\lambda}_1^{\Delta t, \Delta s}$ from (4.6.5), and then $\bar{\varphi}^{\Delta t, \Delta s}$ and $(\bar{m}^{\Delta t, \Delta s}, \bar{\alpha}^{\Delta t, \Delta s})$ from (4.6.12). The proportion of PEVs in mode 1 over the time ($t \mapsto \bar{m}_1^{\Delta t, \Delta s}(t, [0, 1])$) is also displayed in Figure 4.1 (red curve). One can observe on this figure that the congestion constraint (4.1.3) is saturated for the mode 1 during the off-peak periods (between $t = 1h$ and $t = 2h$ and between $t = 3h15$ and $t = 4h15$ in Figure 4.2). As expected, $\lambda_1^{\Delta t, \Delta s}$ is non-zero when the constraint is saturated. The distributions of the level of battery at the initial and final time are plotted in Figure 4.3, showing the charging of the battery. The average SoC is equal to 0.4 at initial time and to 0.58 at the end of the period, with less than 3% of PEVs having less than 0.4 level of battery at the final time.

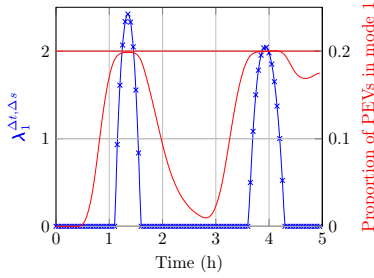


Figure 4.1 – Optimal Lagrangian multiplier $\lambda_1^{\Delta t, \Delta s}$ and proportion of PEVs in mode 1 over the time

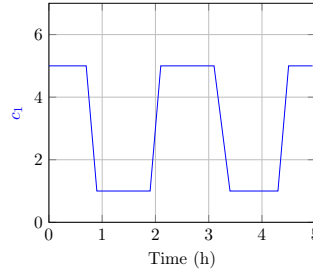


Figure 4.2 – Price of electricity c_1 over the time

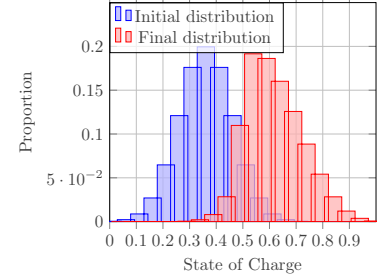


Figure 4.3 – Marginal distribution of the State of Charge (s) at initial and final time

4.7 Appendix

4.7.1 Weak solution of the continuity equation

The following lemma ensures the existence and uniqueness of a weak solution (m, α) of (4.1.2) whenever α is in $\text{Lip}([0, T] \times [0, 1] \times I^2)$.

Lemma 4.7.1. *Let $m^0 \in \mathcal{P}([0, 1] \times I)$ and $\alpha \in \text{Lip}([0, T] \times [0, 1] \times I \times I)$. Then, there exists a unique $m \in \text{Lip}([0, T], \mathcal{P}([0, 1] \times I))$ such that (m, α) is a weak solution of (4.1.2) in the sense of Definition 4.2.1.*

Proof. The existence and uniqueness of a weak solution are proved in [71] for controls α that are continuous in space and time independent. The extension of this result to bounded controls that are measurable in time is straightforward. \square

The next lemma provides regularity result of a weak solution (m, α) in the specific case of an absolute continuous initial distribution m^0 .

Lemma 4.7.2. *Let $m^0 \in \mathcal{P}([0, 1] \times I)$ be absolutely continuous w.r.t. the Lebesgue measure on $[0, 1]$ and with a density in $C^1([0, 1])$. Then, for any $\alpha \in \text{Lip}([0, T] \times [0, 1] \times I^2)$, there exists a unique distribution m such that (m, α) is a solution of the continuity equation (4.1.2) in the sense of Definition 4.2.1. The distribution m is absolutely continuous w.r.t. the Lebesgue measure on $[0, T] \times [0, 1] \times I$, with a density (also denoted m) belonging to $\text{Lip}([0, T] \times [0, 1] \times I)$.*

Proof. The existence and uniqueness of a weak solution (α, m) are given by Lemma 4.7.1.

We want to show that m is absolutely continuous w.r.t. the Lebesgue measure, with a density belonging to $\text{Lip}([0, T] \times [0, 1] \times I)$. Considering the Banach space $\text{Lip}([0, T] \times [0, 1] \times I)$, endowed with the norm $\|f\|_L :=$

$L_f + \|f'\|_\infty$, where L_f is the least Lipschitz constant of f , by applying a fixed point argument, one obtains that there exists a unique function $\bar{m} \in \text{Lip}([0, T] \times [0, 1] \times I)$ satisfying, for any $(t, s, i) \in [0, T] \times [0, 1] \times I$,

$$\begin{aligned} \bar{m}_i(t, s) = & \bar{m}_i(0, S_i^{t,s}(0)) + \int_0^t \left(-b'(S_i^{t,s}(\tau)) - \sum_{j \in I, j \neq i} \alpha_{i,j}(\tau, S_i^{t,s}(\tau)) \right) \bar{m}_i(\tau, S_i^{t,s}(\tau)) d\tau, \\ & + \int_0^t \sum_{j \in I, i \neq j} \alpha_{j,i}(\tau, S_i^{t,s}(\tau)) \bar{m}_j(\tau, S_j^{t,s}(\tau)), \end{aligned}$$

where $S_i^{s,t}[0, T] \mapsto [0, 1]$ is the solution of the ODE:

$$\frac{d}{dt} S_i^{s,t}(\tau) = b_i(S_i^{s,t}) \text{ and } S_i^{s,t}(t) = s.$$

Then, one can prove that (\bar{m}, α) is a weak solution of (4.1.2). The regularity of the density of m follows by the uniqueness of the weak solution of (4.1.2). \square

Chapter 5

Mean field approximation of an optimal control problem for the continuity equation arising in smart charging

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5.1 Introduction

This work is motivated by the optimal charging of a very large population of plug-in electric vehicles (PEVs) controlled by a central planner. Each PEV is characterised by two variables: a continuous one representing the state of charge (SoC) of the battery, and a discrete one denoting the mode of charging of the PEV (e.g. idling, charging, discharging, etc...). The central planner determines when and to which mode of charging of each PEV switches. In addition to the charging cost, the objective function also contains a term penalizing the switches, in order to avoid both excessive jumps per PEV and synchronization effects, i.e., simultaneous switches of a large proportion of PEVs. An optimization problem is considered, where the distribution of the population is subject to a congestion constraint to avoid large proportion of PEVs having the same regime. Such a control problem typically arises in parking lots powered by solar energy that can be found in malls, airports, stadiums, hospitals and other facilities with large parking areas [90]. Since the number n of PEVs is very large, both combinatorial techniques and optimal control tools may fail to solve the problems, due to the curse of dimensionality [20]. To overcome these difficulties, one can approximate the problem of n PEVs by considering a continuum of PEVs, leading to the techniques of optimal control of PDEs and those of convex optimization. The resulting limit mean field control problem was studied in [233] and numerically solved in [238]. Note that several articles have already dealt with smart charging problems within a mean field limit framework [76, 196, 206]. Our paper

aims to justify the mean field approximation by proving the convergence of the finite population optimization problem to the mean field problem, when n tends to infinity.

We point out three important features in our modelling of the PEV charging. First, only a finite number of charging rates are allowed, because charging is mostly done at discrete rates [200]. This feature was also adopted for example in [78, 118, 245]. However, these papers did not systematically take into account the switching cost and congestion constraints, which are the second and third features of our modelling. Indeed, penalizing switches is crucial because, on the one hand, multiple changes in charging regime causes more intensive battery aging and degradation [88, 184] whereas, on the other hand, the synchronization of switches of PEVs can disrupt energy balance on the electrical network [256] and increase instability of distribution transformers [268]. Finally, congestion constraints enable to avoid voltage drops and overloading of transformers [155] caused by uncoordinated large fleets of PEVs.

The main contribution of this work is the convergence, as n tends to infinity, of the value of the finite population problem to the value of the mean field control problem (Theorem 5.2.1). We also prove the convergence (up to a subsequence) of optimal solutions of the finite population problem to a solution of the mean field control problem (Theorem 5.2.2).

Let us make some remarks on the method of proof. The finite population problem is first defined in a Lagrangian point of view, namely that the evolution of the population is described by the trajectory of each process (PEV). Then, an Eulerian formulation of the problem, which characterizes the evolution of the population by its state distribution, velocity field and the distribution of the switches of its discrete state, is introduced and proved to be equivalent to the Lagrangian formulation (Corollary 5.3.1). This result is obtained thanks to a superposition principle (Theorem 5.3.1), that is a direct adaptation of the one in [62]. The superposition principle states that any curve of probability measure, solution of the continuity equation, can be represented as the transport of the initial distribution along an ODE flow. This result was first introduced in [8] in the Euclidean setting and extended to general metric spaces in [178].

The convergence of the solution and that of the value of the finite population problem in the Eulerian formulation are achieved in two steps. First, we prove that the lower limit, as n tends to infinity, of the value of the finite population problem is larger than the value of the mean field control problem, due to the compactness and lower semi-continuity arguments (Theorem 5.4.1). Second, we show that the value of the finite population problem is bounded above by the value of the mean field control problem up to a term of order $\mathcal{O}(n^{-1/3})$ (Theorem 5.4.2). To obtain this bound, a mean field optimal control is implemented to a finite population of processes and an estimate of the Wasserstein distance between the empirical distribution of the finite population and the optimal mean field distribution is derived (Theorem 5.4.3). This estimate strongly relies on the regularity of the optimal control of the mean field control problem. The Lipschitz continuity w.r.t. the space variable, uniformly in time, of the optimal control has been established in a companion paper [233]. Similar results of Lipschitz regularity of the optimal controls have been studied in the context of mean field control problem [37, 82].

The *mean field limit* of a system of n interacting agents is defined as the asymptotic behaviour of the system when n tends to infinity. The empirical distribution of the system of n agents can be approximated by a distribution that is a solution of a Vlasov type equation. In the stochastic setting, one often refers this asymptotic behavior to the notion of propagation of chaos [246]. In the context of optimal control with deterministic dynamics, the convergence of the solutions of the finite population problem to a solution of the mean field control problem, was first proved in [112]. The authors applied Γ -convergence techniques [79] and restricted the result to the particular case of feedback control functions that are locally Lipschitz continuous in space. The convergence of the value of the optimal control of a finite population of interacting McKean-Vlasov dynamics to the value of a mean field optimal control problem was proved in [163] in fairly general settings (the results hold for degenerate diffusion). The convergence results in [112] were obtained without the restrictions on the control in [111]. In [62], the Eulerian, Lagrangian and Kantorovich formulation of the finite population and the mean field problems are introduced, and the convergence of the value functions of the finite population Lagrangian and Eulerian problems were established. More convergence properties in various deterministic settings can be found in [43, 139] and in the references therein. In a stochastic setting, the results of [163] were extended to the case with common noise in [94] and with interaction of the agents with joint distribution of the state and control in [93]. More recently, in a setting with idiosyncratic and common noise in the dynamic of the agents, a convergence rate of the value function of the finite population problem to the value function of the mean field limit problem was derived in [51]. In a finite state space setting, this rate was proved to be of order $1/\sqrt{n}$ in [64]. Similar results on the value function were obtained in [18] in the case

of mean field control problem with regime switching in the state dynamics. The Γ -convergence of a control problem of hybrid processes was proved in [131] in the very specific framework of multi-line traffic. Several convergence properties of the finite population model to the mean field one are given in [194] for the discrete time setting with common noise.

Mean field control problems are strongly connected with the mean field game (MFG) problems. This class of games, introduced by Lasry and Lions [166, 167, 168] and Huang, Malhamé and Caines [145, 146], describes the interaction among a large population of identical and rational agents in competition. It was first proved that one can construct ε -Nash equilibrium in the n -player game from mean field models [57, 56, 65, 145]. The convergence of the Nash equilibrium system to the MFG system is closely related to the well-posedness of the so-called "master equation". Such a property was proved in general settings, with common noise, in the breakthrough of [52]. The convergence was studied in [109, 162] in the open-loop control framework and extended in [164] for closed-loop Nash equilibrium, expanding results obtained in [52]. Finally, convergence results in the finite state settings were obtained in [19, 65, 128, 158]

While problems in the literature deal with either continuous or discrete state variables, this work addresses the analysis of the mean field limit of hybrid processes. Also, a congestion constraint is considered in the optimization problem, which is unusual among the existing literature studying mean field limit of control problems (see however [66] for a Γ -convergence result of an n -agent system to a mean field control problem with L^∞ upper bound on the density of the population). Finally, a particularity of the model in this paper is that the nature of the dynamics in the finite population problem is different from that in the mean field problem. While the switches are controlled and deterministic in the finite population setting, the jumps of the discrete variable of each process are stochastic and the control is on the transition rate in the mean field problem .

The paper is organized as follows. In Section 5.2, we present our assumptions, the n -agent optimal control problem, the mean field control problem and the main results. The equivalence between the finite population problem and its Eulerian formulation is established in Section 5.3, as well as the superposition principle. Finally, the convergence of the solution and that of the value of the finite population problem to the solution and the value of the mean field problem are proved in Section 5.4.

5.2 Main results

Notations The space of Borel, positive and bounded measures on a space A is denoted by $\mathcal{M}^+(A)$ and the space of Borel probability measures on a space A is denoted by $\mathcal{P}(A)$. For any measure $\mu \in \mathcal{M}([0, T])$ and $0 \leq t_1 < t_2 \leq T$, we set $\int_{t_1}^{t_2} \mu(dt) := \mu([t_1, t_2])$. Given a set \mathcal{S} , for any function f defined on $\mathcal{S} \times I$ and any measure $\mu \in \mathcal{M}(\mathcal{S} \times \mathcal{S} \times I)$, we use the notations $f_i(x) := f(x, i)$ for any $(x, i) \in \mathcal{S} \times I$ and $\mu_i(S) := \mu(S \times i)$ for any $(S, i) \in \mathcal{B}(\mathcal{S}) \times I$, where $\mathcal{B}(\mathcal{S})$ denotes the Borel algebra. Similarly, for any function g defined on $\mathcal{S} \times I \times I$ and any measure $\nu \in \mathcal{M}(\mathcal{S} \times \mathcal{S}' \times I \times I)$, where \mathcal{S}' is a set, we use the notations $g_{i,j}(x) := g(x, i, j)$ for any $(x, i, j) \in \mathcal{S} \times I^2$ and $\nu_{i,j}(S, S') := \nu(S \times S' \times \{i\} \times \{j\})$ for any $(S, S', i, j) \in \mathcal{B}(\mathcal{S}) \times \mathcal{B}(\mathcal{S}') \times I^2$. If \mathcal{S} is a metric space, let $\text{Lip}(\mathcal{S})$ denote the vector space of bounded and Lipschitz continuous maps $f : \mathcal{S} \rightarrow \mathbb{R}$. For any $\mu \in C^0([0, T], \mathcal{P}([0, 1]))$, let $L_\mu^2([0, T] \times [0, 1]) := \{f : [0, T] \times [0, 1] \mapsto \mathbb{R}, \int_0^T \int_0^1 f(t, s)^2 \mu(t, ds) dt < +\infty\}$. We denote by \mathcal{W} the Wasserstein distance on $\mathcal{P}([0, 1] \times I)$, defined by $\mathcal{W}(\mu, \rho) := \sup \{ \sum_{i \in I} \int_0^1 \varphi(\mu - \rho) \mid \varphi \text{ is } 1\text{-Lipschitz from } [0, 1] \times I \text{ to } \mathbb{R} \}$. We recall that if a function φ is 1-Lipschitz continuous from $[0, 1] \times I$ to \mathbb{R} , then $|\varphi(x, i) - \varphi(x, j)| \leq 1$ for any $i, j \in I$. For any metric space (X, d) , we denote by $D([0, T], X)$ the set of cadlag functions from $[0, T]$ to X and by $AC([0, T], X)$ the set of absolutely continuous functions. For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer truncation of x . For any $n \in \mathbb{N}$, we introduce $\mathcal{P}^n([0, 1] \times I)$, the set of empirical probability measures on the space $[0, 1] \times I$ defined by

$$\mathcal{P}^n([0, 1] \times I) := \left\{ \mu = \frac{1}{n} \sum_{\ell=1}^n \delta_{(x^\ell, i^\ell)} \text{ for some } (x^\ell, i^\ell) \in [0, 1] \times I \right\},$$

and similarly we introduce $\mathcal{M}^n([0, 1])$ defined by:

$$\mathcal{M}^n([0, 1]) := \left\{ \mu = \frac{1}{n} \sum_{\ell=1}^{\bar{n}} \delta_{x^\ell} \text{ for some } x^\ell \in [0, 1] \text{ and } \bar{n} \leq n \right\}. \quad (5.2.1)$$

5.2.1 The n PEVs control problem

We consider a population of n PEVs ($n \in \mathbb{N}^*$), a state space $[0, 1] \times I$, where I is a finite set, and a time interval $[0, T]$. We consider a time and space discretization depending on n , with time step Δt^n and space step Δs^n (indexed by the superscript n), such that $N_T^n := T/\Delta t^n$ and $N_s^n := 1/\Delta s^n$ are integers. The time mesh is $\{0, t_1^n, \dots, t_{k+1}^n, \dots, T\}$ with $t_k^n := k\Delta t^n$, for any $k \in \{0, \dots, N_T^n\}$. The space mesh is $\{0, y_1^n, \dots, y_p^n, \dots, 1\}$ with $y_p^n := p\Delta s^n$, for any $p \in \{0, \dots, N_s^n\}$. For the sake of simplicity, we write t_k and y_p instead of t_k^n and y_p^n , and Δs and Δt instead of Δs^n and Δt^n .

For any $\ell \in \{1, \dots, n\}$ and $t \in [0, T]$, the vehicle ℓ is described by its state variable $x_t^\ell := (i_t^\ell, s_t^\ell) \in I \times [0, 1]$, with a given initial datum $x_0^\ell = (i_0^\ell, s_0^\ell)$. The discrete variable i^ℓ , denoting the mode of charging, can switch deterministically and only at fixed times in $\{t_1, \dots, t_{N_T^n-1}\}$, while the continuous variable s^ℓ , representing the SoC, is governed by an ODE depending on the mode of charging. Between two jumps of the variable i^ℓ , i.e. within each interval $[t_k, t_{k+1})$, the dynamics of x^ℓ is deterministic and is given by:

$$i_t^\ell = i_{t_k}^\ell, \quad \frac{ds_t^\ell}{dt} = b(i_t^\ell, s_t^\ell) \text{ for any } t \in [t_k, t_{k+1}) \text{ and with } s_{t_k}^\ell = \lim_{\tau \uparrow t_k} s_\tau^\ell, \quad (5.2.2)$$

where the initial state (i_0^ℓ, s_0^ℓ) is given. Note that there are no switches of i^ℓ at times $t = 0$ and $t = T$. We further assume that b belongs to $C^1(I \times \mathbb{R})$ and vanishes at the boundary of $[0, 1]$, so that the ODE satisfied by s^ℓ has a unique solution and s^ℓ lies in $[0, 1]$. By (5.2.2), the map $t \mapsto (i_t^\ell, s_t^\ell)$ is cadlag. Given $\bar{m} \in \mathcal{P}^n([0, 1] \times I)$, we introduce the set of processes:

$$\begin{aligned} \mathcal{X}^n(\bar{m}) := & \left\{ \{(i^\ell, s^\ell)\}_{\ell \in \{1, \dots, n\}} \in (D([0, T], I) \times C^0([0, T], [0, 1]))^n \mid \text{the empirical distribution} \right. \\ & \left. \text{of } \{(i_0^\ell, s_0^\ell)\}_\ell \text{ is equal to } \bar{m} \text{ and } (i^\ell, s^\ell) \text{ satisfies (5.2.2)} \right\}. \end{aligned} \quad (5.2.3)$$

We denote by (\mathbf{i}, \mathbf{s}) a generic element of $\mathcal{X}^n(\bar{m})$. The population of n processes is subject to the following congestion constraints:

$$\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_i(i_t^\ell) \leq D_i(t) \quad \forall (i, t) \in I \times [0, T], \quad (5.2.4)$$

where $D : [0, T] \rightarrow \mathbb{R}_+^*$ is given. The admissible set $\mathcal{T}^n(\bar{m}, D)$ is defined by:

$$\mathcal{T}^n(\bar{m}, D) := \{(\mathbf{i}, \mathbf{s}) \in \mathcal{X}^n(\bar{m}) \mid (\mathbf{i}, \mathbf{s}) \text{ satisfies (5.2.4)}\}. \quad (5.2.5)$$

Let \mathcal{J}^n be the objective function, defined by:

$$\mathcal{J}^n(\mathbf{i}, \mathbf{s}) := \frac{1}{n} \sum_{\ell=1}^n \int_0^T c(t, i_t^\ell, s_t^\ell) dt + g(i_T^\ell, s_T^\ell) + \sum_{i, j \in I, j \neq i} \sum_{k=1}^{N_T^n-1} \sum_{p=0}^{N_s^n-1} L \left(\frac{Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s})}{\Delta t} \right) \Delta t Q_i^{k,p}(\mathbf{i}, \mathbf{s}), \quad (5.2.6)$$

where $Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s})$ is the proportion of PEVs among the processes with a state in $[y_p, y_{p+1}) \times \{i\}$ that switch their discrete state i to j at time t_k , and $Q_i^{k,p}(\mathbf{i}, \mathbf{s})$ is the proportion of PEVs among the overall population of processes that has a state in $[y_p, y_{p+1}) \times \{i\}$ at time t_k^- (just before the jumps). More precisely, $Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s})$ and $Q_i^{k,p}(\mathbf{i}, \mathbf{s})$ are defined by:

$$\begin{aligned} Q_i^{k,p}(\mathbf{i}, \mathbf{s}) &:= \frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_i(i_{t_k}^\ell) \mathbb{1}_{[y_p, y_{p+1})}(s_{t_k}^\ell), \\ Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s}) &:= \begin{cases} 0 & \text{if } Q_i^{k,p}(\mathbf{i}, \mathbf{s}) = 0, \\ \frac{1}{n Q_i^{k,p}(\mathbf{i}, \mathbf{s})} \sum_{\ell=1}^n \mathbb{1}_i(i_{t_k}^\ell) \mathbb{1}_j(i_{t_k}^\ell) \mathbb{1}_{[y_p, y_{p+1})}(s_{t_k}^\ell) & \text{otherwise.} \end{cases} \end{aligned}$$

Our purpose is to study the mean field limit of the following finite population optimal control problem:

$$\inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}). \quad (5.2.7)$$

We recall that this work is initially motivated by the optimal charging of a population of plug-in electrical vehicles (PEVs) controlled by a central planner. The given velocity field $b(i, s)$ denotes the power of charge or discharge of a PEV in mode i and with battery level s . The congestion constraint (5.2.4) aims at avoiding high demand of energy at each moment over the period. The value $c(t, i, s)$ in (5.2.6) corresponds to the running cost of a PEV at state (s, i) and at time $t \in [0, T]$; $g(i, s)$ is the final cost per PEV at state (s, i) . The switching cost $L\left(\frac{Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s})}{\Delta t}\right) \Delta t Q_i^{k,p}(\mathbf{i}, \mathbf{s})$ penalizes large values of $Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s})$, i.e., large proportion of PEVs switching from a state in $[y_p, y_{p+1}) \times \{i\}$ to a state in $[y_p, y_{p+1}) \times \{j\}$ at time t_k . The normalization by Δt avoids frequent jumps. Finally, the multiplication of $L(\cdot)$ by $\Delta t Q_i^{k,p}(\mathbf{i}, \mathbf{s})$ normalizes the transition cost and avoids its explosion when n tends to infinity. The switching costs in (5.2.7) showed good numerical results in [238].

Remark 5.2.1. *The definition of \mathcal{J}^n in (5.2.6) does not take into account the jumps of PEVs with SoC equal to 1. However, by Assumptions 1 and 2 (specified in Section 5.2.3) it is not possible for a PEV to have a SoC equal to 1, which justifies our choice.*

Remark 5.2.2. *Since the number of admissible trajectories is finite and by Assumption 3 (given in Section 5.2.3) non empty, the infimum is always attained in (5.2.7). Actually, Assumption 3 ensures that the n processes with no switches for the discrete variable is admissible. We deduce that, for any $n \in \mathbb{N}$,*

$$\inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}) \leq T \|c\|_\infty + \|g\|_\infty. \quad (5.2.8)$$

5.2.2 The mean field control problem

This section defines the limit model when n tends to infinity. Let $(\bar{m}, D) \in \mathcal{P}([0, 1] \times I) \times C^0([0, T] \times I, \mathbb{R}_+^*)$ satisfy $\text{supp}(\bar{m}_i) \subset (0, 1)$, for any $i \in I$, and

$$\sup_{t \in [0, T], i \in I} D_i(t) - \bar{m}_i([0, 1]) \leq \varepsilon^0,$$

where the constant $\varepsilon^0 > 0$ is defined further in Assumption 3. We consider the continuity equation on the pair (m, E) , defined on the domain $(0, T) \times (0, 1) \times I$:

$$\begin{aligned} \partial_t m_i(t, s) + \partial_s(m_i(t, s)b_i(s)) &= - \sum_{i, j \in I, j \neq i} (E_{i,j}(t, s) - E_{j,i}(t, s)) \quad (i, t, s) \in I \times (0, T) \times (0, 1) \\ m_i(0, s) &= \bar{m}_i(s) \quad (i, s) \in I \times (0, 1). \end{aligned} \quad (5.2.9)$$

We introduce the density constraint:

$$m_i(t, [0, 1]) \leq D_i(t) \quad \forall (i, t) \in I \times [0, T], \quad (5.2.10)$$

and the admissible set

$$\mathcal{S}(\bar{m}, D) := \left\{ (m, E) \mid m \in C^0([0, T], \mathcal{P}([0, 1] \times I)), E \in \mathcal{M}^+([0, T] \times [0, 1] \times I^2), E_{i,j} \ll m_i \text{ and } \frac{dE_{i,j}}{dm_i} \in L^2_{m_i}(0, T) \forall i, j \in I, (m, E) \text{ is a weak solution of (5.2.9) and satisfies (5.2.10)} \right\}.$$

The definition of weak solution of (5.2.9) is given by Definition 5.3.1 in Section 5.3. The objective function J is defined for any $(m, E) \in \mathcal{S}(\bar{m}, D)$ by:

$$J(m, E) := \sum_{i \in I} \int_0^T \int_0^1 \left(c_i(t, s) + \sum_{j \in I, j \neq i} L\left(\frac{dE_{i,j}}{dm_i}(t, s)\right) \right) m_i(t, ds) dt + \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds).$$

The mean field control problem is:

$$\inf_{(m, E) \in \mathcal{S}(\bar{m}, D)} J(m, E). \quad (5.2.11)$$

This problem has been studied in [233], where optimality conditions and regularity results on the solutions are established. The main result of the paper, giving the convergence of Problem (5.2.7) to Problem (5.2.11), is

described in the next section.

5.2.3 Convergence result

Throughout the paper, we assume the following:

General assumptions

1. For any $i \in I$, $b_i \in C^1(\mathbb{R})$ with $b_i(s) = 0$ for any $s \notin (0, 1)$.
2. For any $n \in \mathbb{N}$ and $i \in I$, we assume that $\bar{m}^n \in \mathcal{P}^n([0, 1] \times I)$ and $\text{supp}(\bar{m}_i^n) \subset (0, 1)$.
3. There exists $\varepsilon^0 > 0$ such that, for any $n \in \mathbb{N}$, $\varepsilon^0 \leq \inf_{(i,t) \in I \times [0,T]} D_i(t) - \bar{m}_i^n([0, 1])$.
4. For any $i \in I$, $c_i \in C^1([0, T] \times [0, 1])$ and $g_i \in C^1([0, 1])$.
5. $L : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ is a convex function, defined by:

$$L(x) := \begin{cases} l(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

where $l \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing strongly convex function, bounded from above by a quadratic function, i.e. there exist $C > 0$ such that for any $x \in \mathbb{R}_+$:

$$\frac{x^2}{C} - C \leq l(x) \leq C(x^2 + 1),$$

where the first inequality is due to the strong convexity of l . By convention: $L(\frac{0}{0}) \cdot 0 := 0$.

Remark 5.2.3. *The main role of Assumptions 1 and 2 is to ensure that the solution of the ODE in (5.2.2) takes values in $[0, 1]$. In addition, the superposition principle formulated in Section 5.3 relies on the regularity of b stated in Assumption 1. Assumption 3 ensures that the n trajectories with no switch of discrete variable i are admissible trajectories. Correspondingly, the feasible set $\mathcal{T}^n(\bar{m}, D)$ of Problem (5.2.7) is not empty. This assumption also enables to build in Section 5.4.2 admissible trajectories based on a solution of the mean field control problem. It is possible to replace Assumption 3 by less restrictive conditions. However, for the sake of clarity, we restrict the analysis to the case with Assumption 3. Regularity results of the solution of the mean field control problem (5.2.11) are derived from the properties of c , g and L given in Assumptions 4 and 5. Assumption 5 enables to obtain the compactness of the solutions of the finite population problem and to apply Γ -convergence techniques in Section 5.4.1.*

In the following theorem, we state the convergence of the value of Problem (5.2.7) to the value of Problem (5.2.11) as n tends to infinity.

Theorem 5.2.1. *There exist constants $\tilde{C}_1, \tilde{C}_2 > 1$ such that, if*

$$\frac{1}{\tilde{C}_1 n^{\frac{1}{3}}} \leq \Delta t, \Delta s \leq \frac{\tilde{C}_1}{n^{\frac{1}{3}}}, \quad (5.2.12)$$

and

$$\Delta t < \frac{1}{\tilde{C}_2 |I|}, \quad (5.2.13)$$

hold for any $n \in \mathbb{N}^*$, then,

1. If $\{\bar{m}^n\}_n$ weakly converges to $\bar{m} \in \mathcal{P}([0, 1] \times I)$, then

$$\inf_{(m,E) \in \mathcal{S}(\bar{m}, D)} J(m, E) \leq \liminf_{n \rightarrow \infty} \inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}^n, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}). \quad (5.2.14)$$

2. There exists $C > 0$ such that for any $n \in \mathbb{N}$:

$$\inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}^n, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}) - \frac{C}{n^{\frac{1}{3}}} \leq \inf_{(m,E) \in \mathcal{S}(\bar{m}^n, D)} J(m, E), \quad (5.2.15)$$

where C depends on the data.

The proof of Theorem 5.2.1 is given in Section 5.4.3. To obtain this result, an Eulerian formulation of Problem 5.2.7 is introduced and proved to be equivalent to Problem 5.2.7 (Corollary 5.3.1) thanks to a superposition principle (Theorem 5.3.1). Then, the first part of Theorem 5.2.1 is obtained in Section 5.4 by applying compactness arguments on the sequences of solutions of the Eulerian problem, while the second part directly derives from Theorem 5.4.2. Inequality (5.2.15) relies on regularity results of the solution of Problem (5.2.11), that are used to build an admissible control for the Eulerian version of Problem 5.2.7.

Remark 5.2.4. *Improving the qualitative inequality (5.2.14) into a quantitative one, as in (5.2.15), seems a difficult task. The main difficulties come from the constraint (5.2.4) and the lack of regularity of the objective function \mathcal{J}^n w.r.t. to the set of trajectories (\mathbf{s}, \mathbf{i}) .*

Remark 5.2.5. *To obtain the inequality in Theorem 5.2.1.1, one needs that $\lim_{n \rightarrow \infty} \Delta t = \lim_{n \rightarrow \infty} \Delta s = 0$. This condition is ensured by inequality (5.2.12). Inequality (5.2.15), on the other hand, requires (5.2.12) and (5.2.13).*

For any $n \in \mathbb{N}$, let $(\mathbf{i}^n, \mathbf{s}^n) \in \mathcal{T}^n(\bar{m}^n, D)$ be a solution of Problem (5.2.7). We define the empirical distribution of the processes m^n and the empirical measure of the switches E^n by:

$$m^n(t) := \frac{1}{n} \sum_{\ell=1}^n \delta_{(i_t^{n,\ell}, s_t^{n,\ell})}, \quad (5.2.16)$$

and

$$E_{i,j}^n := \frac{1}{n} \sum_{\ell=1}^n \sum_{k=1}^{N_T^n - 1} \mathbb{1}_i(i_{t_k}^{n,\ell}) \mathbb{1}_j(i_{t_k}^{n,\ell}) \delta_{t_k} \otimes \delta_{s_{t_k}^{n,\ell}}. \quad (5.2.17)$$

The next result states the weak convergence of $\{(m^n, E^n)\}$ to a solution of Problem (5.2.11).

Theorem 5.2.2. *Under the assumptions of Theorem 5.2.1, if $\{\bar{m}^n\}_n$ weakly converges to $\bar{m} \in \mathcal{P}([0, 1] \times I)$, there exists a subsequence of $\{(m^n, E^n)\}_n$ and a solution (m^*, E^*) of Problem (5.2.11) such that, $\{E^n\}_n$ weakly converges to E^* and $\{m^n(t)\}_n$ converges in Wasserstein distance uniformly in $t \in [0, T]$ to $m^*(t)$.*

The proof of Theorem 5.2.2 is given in Section 5.4.3.

5.3 Equivalence between Eulerian and Lagrangian finite population control problems

In this section, an Eulerian formulation of Problem (5.2.7) is introduced. The equivalence between the Lagrangian and the Eulerian formulation relies on a superposition principle, adapted to the problem and stated in Theorem 5.3.1.

Definition 5.3.1. *We say that a pair $(m, E) \in D([0, T], \mathcal{P}([0, 1] \times I)) \times \mathcal{M}^+([0, T] \times [0, 1] \times I \times I)$ satisfies (5.2.9) in the weak sense if, for any test function $\phi \in C^1([0, T] \times [0, 1] \times I)$, we have:*

$$\begin{aligned} & \sum_{i \in I} \int_0^1 \phi_i(T, s) m_i(T, ds) - \phi_i(0, s) \bar{m}_i(ds) \\ &= \int_0^T \int_0^1 \sum_{i \in I} (\partial_t \phi_i(t, s) + b_i(s) \partial_s \phi_i(t, s)) m_i(t, ds) + \sum_{j \in I, j \neq i} \phi_i(t, s) (E_{j,i} - E_{i,j})(dt, ds). \end{aligned}$$

Let $n \in \mathbb{N}$, $(m, E) \in D([0, T], \mathcal{P}^n([0, 1] \times I)) \times \mathcal{M}^+([0, T] \times [0, 1] \times I^2)$, we consider the following:

$$\text{supp}(E_{i,j}) \subset \{t_q\}_{1 \leq q \leq N_T^n - 1} \times [0, 1], \quad \forall i, j \in I, \quad (5.3.1)$$

and

$$\sum_{j \in I, j \neq i} E_{i,j}(\{t_k\}, B) \leq m_i(t_k, B), \quad \forall t_k \in \{t_q\}_{0 \leq q \leq N_T^n} \text{ and } \forall B \in \mathcal{B}([0, 1]). \quad (5.3.2)$$

The two constraints above are used to characterize the empirical distribution m of the set of processes $\{(i^\ell, s^\ell)\}_{1 \leq \ell \leq n}$ and to characterize the distribution of the jumps E associated to $\{i^\ell\}_{1 \leq \ell \leq n}$. Constraint (5.3.1) implies that the support of E is concentrated on the nodes of the time mesh. Thus, the jumps only occur at times in the set $\{t_1, \dots, t_{N_T-1}\}$. Constraint (5.3.2) ensures that the number of switches does not exceed the number of vehicles at each time and position.

Finally, for any $n \in \mathbb{N}$ and $\bar{m} \in \mathcal{P}^n([0, 1] \times I)$, we define the set $\mathcal{Q}^n(\bar{m})$ by:

$$\begin{aligned} \mathcal{Q}^n(\bar{m}) := & \{(m, E) \in D([0, T], \mathcal{P}^n([0, 1] \times I)) \times \mathcal{M}^+([0, T] \times [0, 1] \times I^2) \text{ such that } (m, E) \\ & \text{is a weak solution of (5.2.9) with initial distribution } \bar{m}, (m, E) \text{ satisfies} \\ & \text{(5.3.1) and (5.3.2), and } E_{i,j}(\{t_k\}, \cdot) \in \mathcal{M}^n([0, 1]) \forall k \in \{0, \dots, N_T^n\}\}. \end{aligned} \quad (5.3.3)$$

The theorem below is the main result of this section. It highlights the equivalence between the Lagrangian and the Eulerian points of view, when describing the evolution of the system of n PEVs. It is reminiscent of the superposition principle [9].

Theorem 5.3.1 (Superposition principle).

1. For any $n \in \mathbb{N}$, $\bar{m} \in \mathcal{P}^n([0, 1] \times I)$ and $(\mathbf{i}, \mathbf{s}) \in \mathcal{X}^n(\bar{m})$, there exists a pair $(m^n, E^n) \in \mathcal{Q}^n(\bar{m})$ such that m^n is the empirical distribution of the processes (\mathbf{i}, \mathbf{s}) and E^n is that of the jumps over the period $[0, T]$, i.e. m^n satisfies (5.2.16) and E^n satisfies (5.2.17).
2. For any $n \in \mathbb{N}$, $\bar{m} \in \mathcal{P}^n([0, 1] \times I)$, and $(m, E) \in \mathcal{Q}^n(\bar{m})$, there exists $(\mathbf{i}, \mathbf{s}) \in \mathcal{X}^n(\bar{m})$, such that m satisfies (5.2.16) and E satisfies (5.2.17).

We stress that the processes (\mathbf{i}, \mathbf{s}) in 2 is not necessary unique. The proof of the theorem is given in section 5.3.2.

5.3.1 Eulerian problem formulation

In this subsection, we describe the Eulerian formulation of the optimal control problem and show that it is equivalent to the Lagrangian Problem (5.2.7).

We define, for any $\bar{m} \in \mathcal{P}^n([0, 1] \times I)$ and $D \in C^0([0, T] \times I, \mathbb{R}_+^*)$, the set $\mathcal{S}^n(\bar{m}, D)$ by:

$$\mathcal{S}^n(\bar{m}, D) := \{(m, E) \in \mathcal{Q}^n(\bar{m}) \mid (m, E) \text{ satisfies (5.2.10)}\}. \quad (5.3.4)$$

Remark 5.3.1. For all $\bar{m} \in \mathcal{P}^n([0, 1] \times I)$ and $D \in C^0([0, T] \times I, \mathbb{R}_+^*)$ satisfying Assumption 3, the sets $\mathcal{S}^n(\bar{m}, D)$ and $\mathcal{S}(\bar{m}, D)$ are different. On the one hand, if (m, E) belongs to $\mathcal{S}(\bar{m}, D)$, then $E_{i,j} \ll m_i$ for any $i, j \in I$ and we show below that m is continuous in time. On the other hand, if (m, E) belongs to $\mathcal{S}^n(\bar{m}, D)$ and $E \neq 0$, then by (5.3.1) $E_{i,j}$ is supported by a set of discrete times (and therefore $E_{i,j} \ll m_i$ cannot hold); moreover m is discontinuous w.r.t. the time variable.

For any $n \in \mathbb{N}$, we define the cost function J^n , for any $(m, E) \in \mathcal{S}^n(\bar{m}, D)$ by:

$$\begin{aligned} J^n(m, E) := & \sum_{i \in I} \int_0^T \int_0^1 c_i(t, s) m_i(t, ds) dt + \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds) \\ & + \sum_{i,j \in I, i \neq j} \sum_{k=1}^{N_T^n-1} \sum_{p=0}^{N_s^n-1} L \left(\frac{E_{i,j}(\{t_k\}, [y_p, y_{p+1}])}{\Delta t m_i(t_k^-, [y_p, y_{p+1}])} \right) \Delta t m_i(t_k^-, [y_p, y_{p+1}]). \end{aligned} \quad (5.3.5)$$

For any $n \in \mathbb{N}$, we consider the optimization problem:

$$\inf_{(m, E) \in \mathcal{S}^n(\bar{m}, D)} J^n(m, E). \quad (5.3.6)$$

The following corollary is a direct consequence of Theorem 5.3.1.

Corollary 5.3.1. One has:

$$\inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}) = \inf_{(m, E) \in \mathcal{S}^n(\bar{m}, D)} J^n(m, E).$$

Proof. Let $(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}, D)$, then from Theorem 5.3.1.1, there exists $(m, E) \in \mathcal{Q}^n(\bar{m})$ such that m is the empirical distribution of (\mathbf{i}, \mathbf{s}) satisfying (5.2.16) and E is the empirical distribution of the jumps satisfying (5.2.17). Since $(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}, D)$, we have for any $i \in I$ and $t \in [0, T]$

$$m_i(t, [0, 1]) = \frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_i(i_t^\ell) \leq D_i(t) \quad \forall (i, t) \in I \times [0, T],$$

and thus, $(m, E) \in \mathcal{S}^n(\bar{m}, D)$. For any $j \in I$ with $i \neq j$, $k \in \{1, \dots, N_T^n - 1\}$ and $p \in \{0, \dots, N_s^n - 1\}$, one has:

$$m_i(t_k^-, [y_p, y_{p+1}]) = \frac{1}{n} \sum_{\ell=1}^n \delta_{(i_{t_k^-}^\ell, s_{t_k^-}^\ell)}(\{i\} \times [y_p, y_{p+1}]) = Q_i^{k,p}(\mathbf{i}, \mathbf{s}),$$

and

$$E_{i,j}(\{t_k\}, [y_p, y_{p+1}]) = \frac{1}{n} \sum_{\ell=1}^n \sum_{k=1}^{N_T^n} \mathbb{1}_i(i_{t_k^-}^\ell) \mathbb{1}_j(i_{t_k}^\ell) \delta_{s_{t_k}^\ell}([y_p, y_{p+1}]) = Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s}) Q_i^{k,p}(\mathbf{i}, \mathbf{s}).$$

From the two previous equalities, one has for any $k \in \{1, \dots, N_T^n - 1\}$ and $p \in \{0, \dots, N_s^n - 1\}$:

$$L \left(\frac{E_{i,j}(\{t_k\}, [y_p, y_{p+1}])}{\Delta t m_i(t_k^-, [y_p, y_{p+1}])} \right) \Delta t m_i(t_k^-, [y_p, y_{p+1}]) = L \left(\frac{Q_{i,j}^{k,p}(\mathbf{i}, \mathbf{s})}{\Delta t} \right) \Delta t Q_i^{k,p}(\mathbf{i}, \mathbf{s}).$$

By (5.2.16) and (5.2.17), we have:

$$\frac{1}{n} \sum_{\ell=1}^n \int_0^T c(t, i_t^\ell, s_t^\ell) dt = \sum_{i \in I} \int_0^T \int_0^1 c_i(t, s) m_i(t, ds) dt \quad \text{and} \quad \frac{1}{n} \sum_{\ell=1}^n g(i_T^\ell, s_T^\ell) = \sum_{i \in I} \int_0^1 g_i(s) m_i(T, ds).$$

Therefore, it holds:

$$\mathcal{J}^n(\mathbf{i}, \mathbf{s}) = J^n(m, E), \tag{5.3.7}$$

and

$$\inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}) \geq \inf_{(m, E) \in \mathcal{S}^n(\bar{m}, D)} J^n(m, E).$$

Using Theorem 5.3.1.2 and similar computations as in the first part of this proof, one can obtain the reverse inequality:

$$\inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}) \leq \inf_{(m, E) \in \mathcal{S}^n(\bar{m}, D)} J^n(m, E),$$

and the conclusion follows. \square

Remark 5.3.2. *By Remark 5.2.2, there exists a solution to Problem (5.2.7). Thus, by Theorem 5.3.1 and Corollary 5.3.1, there exists a solution to Problem (5.3.6).*

5.3.2 Proof of the superposition principale

Construction of the empirical distribution and of the jump measure from n PEVs trajectories

In this section, we fix $n \in \mathbb{N}^*$, $\bar{m} \in \mathcal{P}^n([0, 1] \times I)$ and $(\mathbf{i}, \mathbf{s}) \in \mathcal{X}^n(\bar{m})$, where $\mathcal{X}^n(\bar{m})$ is defined in (5.2.3).

Lemma 5.3.1. *The pair (m^n, E^n) defined in (5.2.16) and (5.2.17) from (\mathbf{i}, \mathbf{s}) is a weak solution of (5.2.9) and satisfies (5.3.1) and (5.3.2).*

Proof. Using the definition of E^n in (5.2.17) and m^n in (5.2.16), (5.3.1) and (5.3.2) hold. We now show that (m^n, E^n) is a weak solution of (5.2.9). Since, for any $\ell \in \{1, \dots, n\}$, the function $t \mapsto (i_t^\ell, s_t^\ell)$ is cadlag, one has

$m^n \in D([0, T], \mathcal{P}^n([0, 1] \times I))$. Let $\varphi \in C^1([0, T] \times [0, 1] \times I)$. One deduces:

$$\begin{aligned}
& \sum_{i \in I} \int_0^T \int_0^1 (\partial_t \varphi_i(t, s) + b_j(s) \partial_s \varphi_i(t, s)) m_i^n(t, ds) dt \\
&= \frac{1}{n} \sum_{i \in I} \sum_{\ell=1}^n \sum_{k=0}^{N_T^n - 1} \int_{t_k}^{t_{k+1}} (\partial_t \varphi_i(t, s_{t_k}^\ell) + b_i(s_{t_k}^\ell) \partial_s \varphi_i(t, s_{t_k}^\ell)) \mathbb{1}_i(i_{t_k}^\ell) dt \\
&= \frac{1}{n} \sum_{i \in I} \sum_{\ell=1}^n \sum_{k=0}^{N_T^n - 1} (\varphi_i(t_{k+1}, s_{t_{k+1}}^\ell) - \varphi_i(t_k, s_{t_k}^\ell)) \mathbb{1}_i(i_{t_k}^\ell) \\
&= \frac{1}{n} \sum_{i \in I} \sum_{\ell=1}^n \varphi_i(T, s_T^\ell) \mathbb{1}_i(i_0^\ell) - \varphi_i(0, s_0^\ell) \mathbb{1}_i(i_0^\ell) + \frac{1}{n} \sum_{i, j \in I, j \neq i} \sum_{\ell=1}^n \sum_{k=1}^{N_T^n - 1} (\varphi_i(t_k, s_{t_k}^\ell) - \varphi_j(t_k, s_{t_k}^\ell)) \mathbb{1}_i(i_{t_k}^\ell) \mathbb{1}_j(i_{t_k}^\ell) \\
&= \sum_{i \in I} \int_0^1 (\varphi_i(T, s) m_i^n(T, ds) - \varphi_i(0, s) \bar{m}_i^n(ds)) + \int_0^T \int_0^1 \sum_{i, j \in I} (\varphi_i(t, s) - \varphi_j(t, s)) E_{i, j}^n(ds, dt),
\end{aligned}$$

and the conclusion follows. \square

Construction of n PEVs trajectories from a couple of measure (m, E)

In this subsection, we prove the converse result of Section 5.3.2. Given $n \in \mathbb{N}$, $\bar{m} \in \mathcal{P}^n(I \times [0, 1])$ and $(m, E) \in \mathcal{Q}^n(\bar{m})$, where $\mathcal{Q}^n(\bar{m})$ is defined in (5.3.3), we show that there exists $(\mathbf{i}, \mathbf{s}) \in \mathcal{X}^n(\bar{m})$ such that m , the empirical distribution of (\mathbf{i}, \mathbf{s}) , satisfies (5.2.16) and E , the empirical distribution of the jumps, satisfies (5.2.17).

First, a relation between $E_{i, j}$ and m_i at any time t_k is stated in the following lemma.

Lemma 5.3.2. *The measure E satisfies for any $i \in I$ and any $k \in \{1, \dots, N_T^n - 1\}$:*

$$\sum_{j \in I, j \neq i} (E_{i, j}(\{t_k\}, ds) - E_{j, i}(\{t_k\}, ds)) = m_i(t_k^-, ds) - m_i(t_k, ds). \quad (5.3.8)$$

Proof. Let $\varphi \in C^1([0, 1])$, $\varepsilon > 0$, $k \in \{1, \dots, N_T^n - 1\}$ and $\xi^{k, \varepsilon} \in \text{Lip}([0, T])$ be such that $\xi^{k, \varepsilon}(t) = 0$ outside $[t_k - \varepsilon, t_k + \varepsilon]$ and:

$$\xi^{k, \varepsilon}(t) = 1 + \frac{t - t_k}{\varepsilon} \quad \forall t \in [t_k - \varepsilon, t_k] \quad \text{and} \quad \xi^{k, \varepsilon}(t) = 1 - \frac{t - t_k}{\varepsilon} \quad \forall t \in [t_k, t_k + \varepsilon].$$

For any $i \in I$, even though $(t, s, j) \mapsto \xi^{k, \varepsilon}(t) \varphi(s) \mathbb{1}_i(j)$ is not in $C^1([0, T] \times [0, 1] \times I)$, it can be considered as a test function for the equation (5.2.9) because $(t, s) \mapsto \xi^{k, \varepsilon}(t) \partial_s \varphi(s)$ is in $C^0([0, T] \times [0, 1])$ and $(t, s) \mapsto \partial_t \xi^{k, \varepsilon}(t) \varphi(s)$ is in $L^\infty([0, T] \times [0, 1])$. Thus,

$$\int_{t_k - \varepsilon}^{t_k + \varepsilon} \int_0^1 (\varphi(s) \partial_t \xi^{k, \varepsilon}(t) + b_i(s) \xi^{k, \varepsilon}(t) \partial_s \varphi(s)) m_i(t, ds) dt = \sum_{j \in I, j \neq i} \int_0^1 \varphi(s) (E_{i, j}(\{t_k\}, ds) - E_{j, i}(\{t_k\}, ds)). \quad (5.3.9)$$

On the one hand, using the dominated convergence theorem one has:

$$\lim_{\varepsilon \rightarrow 0} \int_{t_k - \varepsilon}^{t_k + \varepsilon} \int_0^1 b_i(s) \xi^{k, \varepsilon}(t) \partial_s \varphi(s) m_i(t, ds) dt = 0. \quad (5.3.10)$$

On the other hand, one gets:

$$\int_{t_k - \varepsilon}^{t_k + \varepsilon} \int_0^1 \varphi(s) \partial_t \xi^{k, \varepsilon}(t) m_i(t, ds) dt = \frac{1}{\varepsilon} \left(\int_{t_k - \varepsilon}^{t_k} \int_0^1 \varphi(s) m_i(t, ds) dt - \int_{t_k}^{t_k + \varepsilon} \int_0^1 \varphi(s) m_i(t, ds) \right) dt.$$

Since m is cadlag, one obtains by considering the limit $\varepsilon \rightarrow 0$ in the previous equality:

$$\lim_{\varepsilon \rightarrow 0} \int_{t_k - \varepsilon}^{t_k + \varepsilon} \int_0^1 \varphi(s) \partial_t \xi^{k, \varepsilon}(t) m_i(t, ds) = \int_0^1 \varphi(s) m_i(t_k^-, ds) - \int_0^1 \varphi(s) m_i(t_k, ds).$$

Therefore

$$\sum_{j \in I, j \neq i} \int_0^1 \varphi(s) (E_{i,j}(\{t_k\}, ds) - E_{j,i}(\{t_k\}, ds)) = \int_0^1 \varphi(s) (m_i(t_k^-, ds) - m_i(t_k, ds)).$$

Since φ is arbitrary, equality (5.3.8) is satisfied. \square

We define $\hat{m} : [0, T] \mapsto \mathcal{P}([0, 1])$ by: $\hat{m}(t) := \sum_i m_i(t)$. The next lemma states a superposition principle for \hat{m} .

Lemma 5.3.3. *There exists $\{z^\ell\}_{1 \leq \ell \leq n}$ such that, for any $\ell \in \{1, \dots, n\}$, $z^\ell \in AC([0, T], [0, 1])$ is solution of:*

$$\frac{dz_t^\ell}{dt} = v(t, z_t^\ell) \quad \text{for a.e. } t \in [0, T], \quad (5.3.11)$$

where $v : [0, T] \times [0, 1] \rightarrow [0, 1]$ is a Borel vector field and \hat{m} satisfies:

$$\hat{m}(t) = \frac{1}{n} \sum_{\ell=1}^n \delta_{z^\ell(t)} \quad \forall t \in [0, T].$$

Proof. We first show that \hat{m} is in $\text{Lip}([0, T], \mathcal{P}^n([0, 1]))$. Since $m(t) \in \mathcal{P}^n([0, 1] \times I)$, for any $t \in [0, T]$, on has $\hat{m}(t) \in \mathcal{P}^n([0, 1])$. Since (m, E) is a weak solution (5.2.9), taking a test function $\varphi \in C^1(0, 1)$ that is 1-Lipschitz continuous, one has:

$$\frac{d}{dt} \int_0^1 \varphi(s) \hat{m}(t, ds) = \int_0^1 (\partial_s \varphi(s)) \sum_{i \in I} b_i(s) m_i(t, ds).$$

Since φ is 1-Lipschitz continuous, one deduces:

$$\left| \frac{d}{dt} \int_0^1 \varphi(s) \hat{m}(t, ds) \right| \leq \|b\|_\infty,$$

and therefore, for any $t, \tau \in [0, T]$:

$$\mathcal{W}(\hat{m}(t), \hat{m}(\tau)) \leq |t - \tau| \|b\|_\infty.$$

Thus, $\hat{m} \in \text{Lip}([0, T], \mathcal{P}^n([0, 1]))$ and the conclusion follows from [62, Theorem C.1]. \square

The following lemma gives for each time interval (t_k, t_{k+1}) a superposition principle for m .

Lemma 5.3.4. *For any $k \in \{0, \dots, N_T^n - 1\}$, there exists $\{(i^{k,\ell}, s^{k,\ell})\}_{\ell \in \{1, \dots, n\}}$ such that for any $\ell \in \{1, \dots, n\}$ $(i^{k,\ell}, s^{k,\ell}) \in C^0([t_k, t_{k+1}], I) \times C^1([t_k, t_{k+1}], [0, 1])$ and any $t \in (t_k, t_{k+1})$:*

$$m(t) = \frac{1}{n} \sum_{\ell=1}^n \delta_{(i_t^{k,\ell}, s_t^{k,\ell})}. \quad (5.3.12)$$

In addition, for any $\ell \in \{1, \dots, n\}$ $(i^{k,\ell}, s^{k,\ell})$ satisfies for any $t \in [t_k, t_{k+1}]$:

$$i_t^{k,\ell} = i_{t_k}^{k,\ell} \quad \text{and} \quad \frac{ds_t^{k,\ell}}{dt} = b(i_t^{k,\ell}, s_t^{k,\ell}), \quad (5.3.13)$$

where the distribution of $\{(i_{t_k}^{k,\ell}, s_{t_k}^{k,\ell})\}_{\ell \in \{1, \dots, n\}}$ is $m(t_k)$.

Proof. For any $k \in \{1, \dots, N_T^n - 1\}$ and test function $\varphi \in C_c^1((t_k, t_{k+1}) \times [0, 1] \times I)$, by (5.2.9) and using that $\text{supp}(E)$ satisfies (5.3.1), one has:

$$\sum_{i \in I} \int_{t_k}^{t_{k+1}} \int_0^1 (\partial_t \varphi_i(t, s) + b_i(s) \partial_s \varphi_i(t, s)) m_i(ds, t) = 0.$$

The existence of $\{(i^{k,\ell}, s^{k,\ell})\}_{\ell \in \{1, \dots, n\}}$, which satisfies the dynamics (5.3.13) and equality (5.3.12), follows from [62, Theorem C.1]. \square

Since $m \in D([0, T], \mathcal{P}([0, 1] \times I))$ and, for any $\ell \in \{1, \dots, n\}$, $(i^{k, \ell}, s^{k, \ell}) \in C^0([t_k, t_{k+1}], I) \times C^1([t_k, t_{k+1}], [0, 1])$, one deduces from the previous lemma that, for any $k \in \{0, \dots, N_T^n - 1\}$,

$$m(t_k) = \frac{1}{n} \sum_{\ell=1}^n \delta_{(i_{t_k}^{k, \ell}, s_{t_k}^{k, \ell})}. \quad (5.3.14)$$

The next lemma shows that, for any $k \in \{1, \dots, N_T^n - 1\}$, the number of indices $\ell \in \{1, \dots, n\}$ satisfying $s_{t_k}^{k, \ell} = s$ is equal to the number of indices $\ell \in \{1, \dots, n\}$ satisfying $s_{t_k}^{k+1, \ell} = s$.

Lemma 5.3.5. *For any $k \in \{0, \dots, N_T^n - 1\}$, one has:*

$$\frac{1}{n} \sum_{\ell=1}^n \delta_{s_{t_{k+1}}^{k, \ell}} = \frac{1}{n} \sum_{\ell=1}^n \delta_{s_{t_{k+1}}^{k+1, \ell}},$$

where $s^{k, \ell}$ and $s^{k+1, \ell}$ are defined in Lemma 5.3.4.

Proof. By Lemmas 5.3.3 and 5.3.4, summing (5.3.12) over I one deduces that, for any $t \in (t_k, t_{k+1})$,

$$\frac{1}{n} \sum_{\ell=1}^n \delta_{s_t^{k, \ell}} = \sum_{i \in I} m_i(t) = \hat{m}(t) = \frac{1}{n} \sum_{\ell=1}^n \delta_{z_t^\ell},$$

where z^ℓ is defined in Lemma 5.3.3. Since z^ℓ is continuous at t_{k+1} and, $s^{k, \ell}$ and $s^{k+1, \ell}$ are respectively cadlag on $[t_k, t_{k+1}]$ and $[t_{k+1}, t_{k+2}]$, the previous equality holds for $t = t_{k+1}$ and gives the result. \square

Lemma 5.3.6. *For any $k \in \{1, \dots, N_T^n - 1\}$, there exist two collections $\{T_{i,j}^{k,+}(s)\}_{i,j \in I, s \in [0,1]}$ and $\{T_{i,j}^{k,-}(s)\}_{i,j \in I, s \in [0,1]}$ of subsets of $\{1, \dots, n\}$ satisfying:*

1. $\text{card}(T_{i,j}^{k,+}(s)) = \text{card}(T_{i,j}^{k,-}(s)) = nE_{i,j}(t_k, \{s\})$, for any $i, j \in I$ with $i \neq j$ and for any $s \in [0, 1]$.
2. $\{T_{i,j}^{k,+}(s)\}_{i,j \in I, s \in [0,1]}$ and $\{T_{i,j}^{k,-}(s)\}_{i,j \in I, s \in [0,1]}$ are partitions of $\{1, \dots, n\}$.

Remark 5.3.3. *For some $s \in [0, 1]$ and $i, j \in I$, the sets $T_{i,j}^{k,+}(s)$ or $T_{i,j}^{k,-}(s)$ can be empty. We maintain the terminology of "partition" in this case for the sake of simplicity.*

Proof. Let $k \in \{1, \dots, N_T^n - 1\}$, $i \in I$ and $s \in [0, 1]$. Let us set:

$$R_i^{k,-}(s) := \left\{ \ell \in \{1, \dots, n\} \mid (i_{t_k}^{k-1, \ell}, s_{t_k}^{k-1, \ell}) = (i, s) \right\},$$

and

$$R_i^{k,+}(s) := \left\{ \ell \in \{1, \dots, n\} \mid (i_{t_k}^{k, \ell}, s_{t_k}^{k, \ell}) = (i, s) \right\}.$$

The set $R_i^{k,-}(s)$ represents the set of indices of processes defined in Lemma 5.3.4 that have a state equal to (i, s) at time t_k^- , just before a possible jump, while $R_i^{k,+}(s)$ represents the set of indices of processes with a state equal to (i, s) at time t_k . By the definition of $R_i^{k,-}(s)$ and $R_i^{k,+}(s)$, and by Lemma 5.3.5, we have, for any $k \in \{1, \dots, N_T^n - 1\}$,

$$R_i^{k,-}(s) \cap R_i^{k,-}(s) = \emptyset \quad \text{and} \quad R_i^{k,+}(s) \cap R_i^{k,+}(s) = \emptyset \quad \text{for any } i, j \in I \text{ with } i \neq j, \quad (5.3.15)$$

$$\text{card}(R_i^{k,-}(s)) = nm_i(t_k^-, \{s\}) \quad \text{and} \quad \text{card}(R_i^{k,+}(s)) = nm_i(t_k, \{s\}), \quad (5.3.16)$$

$$\bigcup_{s \in [0,1]} \bigcup_{i \in I} R_i^{k,-}(s) = \bigcup_{s \in [0,1]} \bigcup_{i \in I} R_i^{k,+}(s) = \{1, \dots, n\}. \quad (5.3.17)$$

Hence, $\{R_i^{k,-}(s)\}_{i \in I, s \in [0,1]}$ and $\{R_i^{k,+}(s)\}_{i \in I, s \in [0,1]}$ are partitions of $\{1, \dots, n\}$. The rest of the proof consists in constructing the sets $\{T_{i,j}^{k,-}(s)\}_{j \in I}$ and $\{T_{i,j}^{k,+}(s)\}_{j \in I}$ that are respectively a partition of $R_i^{k,-}(s)$ and $R_i^{k,+}(s)$. We define, for any $s \in [0, 1]$, $i, j \in I$ with $j \neq i$,

$$c_{i,j}^k(s) := nE_{i,j}(t_k, \{s\}). \quad (5.3.18)$$

We define, for any $i \in I$, $c_{i,i}^k(s) := nm_i(t_k, \{s\}) - \sum_{j \in I, j \neq i} c_{j,i}^k(s)$. By Lemma 5.3.2, one has:

$$c_{i,i}^k(s) = nm_i(t_k^-, \{s\}) - \sum_{j \in I, j \neq i} c_{i,j}^k(s) \geq 0, \quad (5.3.19)$$

where the inequality is obtained by (5.3.2). Since $m(t_k) \in \mathcal{P}^n([0, 1] \times I)$ and $E_{i,j}(t_k, \cdot) \in \mathcal{M}^n([0, 1])$, one has $c_{i,j}^k(s) \in \mathbb{N}$ for any $i, j \in I$. By (5.3.16) and (5.3.19), one has, for any $i \in I$ and $s \in [0, 1]$ such that $m_i(t_k^-, \{s\}) > 0$,

$$\sum_{j \in I} c_{i,j}^k(s) = \text{card}(R_i^{k,-}(s)) \text{ and } \sum_{j \in I} c_{j,i}^k(s) = \text{card}(R_i^{k,+}(s)). \quad (5.3.20)$$

Sorting the elements of $R_i^{k,-}(s)$ and $R_i^{k,+}(s)$ in ascending order, we can now define the collections of subsets $\{T_{i,j}^{k,+}(s)\}_{i,j \in I, s \in [0,1]}$ and $\{T_{i,j}^{k,-}(s)\}_{i,j \in I, s \in [0,1]}$. For any $i \in I$ and $s \in [0, 1]$, $T_{i,j}^{k,-}(s)$ and $T_{i,j}^{k,+}(s)$ are defined iteratively for $j = 1, \dots, |I|$:

- if $c_{i,j}^k(s) = 0$, then $T_{i,j}^{k,-}(s) = T_{i,j}^{k,+}(s) = \emptyset$;
- otherwise, $T_{i,j}^{k,-}(s)$ is the set of the $c_{i,j}^k(s)$ with smallest indices of $R_i^{k,-}(s) \setminus \bigcup_{1 \leq q < j} T_{i,q}^{k,-}(s)$ and $T_{i,j}^{k,+}(s)$ is equal to the set of the $c_{i,j}^k(s)$ with smallest indices of $R_i^{k,+}(s) \setminus \bigcup_{1 \leq q < j} T_{i,q}^{k,+}(s)$,

we used the convention $\bigcup_{1 \leq q < 1} T_{i,q}^{k,-}(s) = \bigcup_{1 \leq q < 1} T_{i,q}^{k,+}(s) = \emptyset$. By (5.3.20) and their construction, the sets $\{T_{i,j}^{k,+}(s)\}_{j \in I}$ and $\{T_{i,j}^{k,-}(s)\}_{j \in I}$ are well defined and are respectively a partition of $R_i^{k,-}(s)$ and $R_i^{k,+}(s)$. Since $\{R_i^{k,-}(s)\}_{i \in I, s \in [0,1]}$ and $\{R_i^{k,+}(s)\}_{s \in [0,1]}$ are both partitions of $\{1, \dots, n\}$, the conclusion follows. \square

We denote by $\mathfrak{S}(n)$ the set of permutation in $\{1, \dots, n\}$. The next lemma enables to express $E_{i,j}(t^k, \cdot)$ in terms of the processes $\{(i^{k,\ell}, s^{k,\ell})\}_{\ell \in \{1, \dots, n\}}$.

Lemma 5.3.7. *For any $k \in \{1, \dots, N_T^n - 1\}$, there exists $\sigma^k \in \mathfrak{S}(n)$ such that the measure E satisfies, for any $i, j \in I$ and $s \in [0, 1]$:*

$$E_{i,j}(t^k, \{s\}) = \frac{1}{n} \text{card} \left(\left\{ \ell \in \{1, \dots, n\} \mid (i_{t_k^-}^{k-1,\ell}, s_{t_k^-}^{k-1,\ell}) = (i, s) \text{ and } (i_{t_k}^{k,\sigma^k(\ell)}, s_{t_k}^{k,\sigma^k(\ell)}) = (j, s) \right\} \right),$$

where the processes $\{(i^{k,\ell}, s^{k,\ell})\}_{\ell \in \{1, \dots, n\}}$ are defined in Lemma 5.3.4,

Proof. Let $\sigma^k \in \mathfrak{S}(n)$ be such that, for any $s \in [0, 1]$ and $i, j \in I$, if $c_{i,j}^k(s) > 0$, the restriction of σ^k to $T_{i,j}^{k,-}(s)$ is a bijective map from $T_{i,j}^{k,-}(s)$ to $T_{i,j}^{k,+}(s)$, where $T_{i,j}^{k,-}(s)$ and $T_{i,j}^{k,+}(s)$ are defined in Lemma 5.3.6. The existence of such a permutation is guaranteed by the properties of $\{T_{i,j}^{k,+}(s)\}_{i,j \in I, s \in [0,1]}$ and $\{T_{i,j}^{k,-}(s)\}_{i,j \in I, s \in [0,1]}$ established in Lemma 5.3.6. By the construction of σ^k , one deduces that, for any non empty set $T_{i,j}^{k,-}(s)$,

$$T_{i,j}^{k,-}(s) = \{\ell \in R_i^{k,-}(s) \mid \sigma^k(\ell) \in R_j^{k,+}(s)\},$$

where $R_j^{k,+}(s)$ and $R_j^{k,-}(s)$ are defined in the proof of Lemma 5.3.6. Finally, one has, for any $s \in [0, 1]$ and $i, j \in I$,

$$\begin{aligned} E_{i,j}(t_k, \{s\}) &= \frac{c_{i,j}^k(s)}{n} = \frac{1}{n} \text{card}(T_{i,j}^k(s)) \\ &= \frac{1}{n} \text{card} \left(\left\{ \ell \in R_i^{k,-}(s) \mid \sigma^k(\ell) \in R_j^{k,+}(s) \right\} \right) \\ &= \frac{1}{n} \text{card} \left(\left\{ \ell \in \{1, \dots, n\} \mid (i_{t_k^-}^{k-1,\ell}, s_{t_k^-}^{k-1,\ell}) = (i, s) \text{ and } (i_{t_k}^{k,\sigma^k(\ell)}, s_{t_k}^{k,\sigma^k(\ell)}) = (j, s) \right\} \right). \end{aligned}$$

\square

The next lemma shows the existence of n processes, such that m is the empirical measure of these processes and E is the empirical measure of the jumps of the processes.

Lemma 5.3.8. *There exists $(\mathbf{i}, \mathbf{s}) \in \mathcal{X}^n(\bar{m})$, such that m satisfies (5.2.16) and E satisfies (5.2.17).*

Proof. For any $k \in \{1, \dots, N_T^n - 1\}$, we consider the permutation $\hat{\sigma}^k \in \mathfrak{S}(n)$, defined by:

$$\hat{\sigma}^k := \sigma^k \circ \sigma^{k-1} \circ \dots \circ \sigma^1, \quad (5.3.21)$$

where σ^k is defined for any $k \in \{1, \dots, N_T^n - 1\}$ in Lemma 5.3.7. For $k = 0$, we define $\hat{\sigma}^0$ by $\hat{\sigma}^0(\ell) = \ell$, for any $\ell \in \{1, \dots, n\}$. For any $\ell \in \{1, \dots, n\}$, we consider the processes (i^ℓ, s^ℓ) defined on $[0, T]$ by:

$$i_t^\ell := i_t^{k, \hat{\sigma}^k(\ell)} \text{ and } s_t^\ell := s_t^{k, \hat{\sigma}^k(\ell)} \text{ for any } t \in [t_k, t_{k+1}), \text{ and } i_T^\ell := \lim_{t \rightarrow T} i_t^\ell, \quad (5.3.22)$$

where $\{(i^{k, \ell}, s^{k, \ell})\}_{\ell \in \{1, \dots, n\}}$ is defined in Lemma 5.3.4. Thus, one has for any $k \in \{0, \dots, N_T^n - 1\}$ and $t \in [t_k, t_{k+1})$:

$$\frac{ds_t^\ell}{dt} = \frac{ds_t^{k, \hat{\sigma}^k(\ell)}}{dt} = b(i_t^{k, \hat{\sigma}^k(\ell)}, s_t^{k, \hat{\sigma}^k(\ell)}) = b(i_t^\ell, s_t^\ell),$$

and

$$i_t^\ell = i_t^{k, \hat{\sigma}^k(\ell)} = i_{t_k}^{k, \hat{\sigma}^k(\ell)} = i_{t_k}^\ell.$$

By the definitions $\hat{\sigma}^k$ and σ^k in Lemma 5.3.7, one gets for any $\ell \in \{1, \dots, n\}$:

$$s_{t_{k+1}}^{k, \ell} = s_{t_{k+1}}^{k+1, \sigma^k(\ell)},$$

and therefore:

$$s_{t_{k+1}}^\ell = \lim_{\tau \uparrow t_{k+1}} s_\tau^\ell.$$

Thus, one has for any $t \in [0, T)$:

$$m^n(t) := \frac{1}{n} \sum_{\ell=1}^n \delta_{(i_t^\ell, s_t^\ell)}.$$

Since there is no jump at time $t = T$ and since i^ℓ and s^ℓ are continuous at T , equality also holds for $t = T$. Finally, by Lemma 5.3.7, one has:

$$E_{i,j} = \frac{1}{n} \sum_{\ell=1}^n \sum_{k=1}^{N_T^n - 1} \mathbb{1}_i(i_{t_k}^{k-1, \ell}) \mathbb{1}_j(i_{t_k}^{k, \sigma^k(\ell)}) \delta_{t_k} \otimes \delta_{s_{t_k}^{k-1, \ell}} = \frac{1}{n} \sum_{\ell=1}^n \sum_{k=1}^{N_T^n - 1} \mathbb{1}_i(i_{t_k}^\ell) \mathbb{1}_j(i_{t_k}^\ell) \delta_{t_k} \otimes \delta_{s_{t_k}^\ell}.$$

Taking $(\mathbf{i}, \mathbf{s}) := \{(i^\ell, s^\ell)\}_{\ell \in \{1, \dots, n\}}$, one has $(\mathbf{i}, \mathbf{s}) \in \mathcal{X}^n(\bar{m})$ and the conclusion follows. \square

We are now able to prove Theorem 5.3.1.

Proof of Theorem 5.3.1. This is a direct consequence of Lemmas 5.3.1 and 5.3.8. \square

5.4 The convergence result

This section is devoted to the proofs of Theorem 5.2.1.1, reformulated in Theorem 5.4.1 and proved in Section 5.4.1, Theorem 5.2.1.2 reformulated in Theorem 5.4.2 and proved in Section 5.4.2, and Theorem 5.2.2 proved in Section 5.4.3. We assume in this section that the time and space parameters Δt and Δs are such that (5.2.12) and (5.2.13) are satisfied.

5.4.1 Γ -lower limit result

We reformulate the first result of the paper, Theorem 5.2.1.1 in Theorem 5.4.1.

Theorem 5.4.1. *For any sequence $\{\bar{m}^n\}_n$ weakly converging to a measure $\bar{m} \in \mathcal{P}([0, 1] \times I)$ and satisfying Assumptions 2 and 3, we have:*

$$\inf_{(m, E) \in \mathcal{S}(\bar{m}, D)} J(m, E) \leq \liminf_{n \rightarrow \infty} \inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}^n, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}). \quad (5.4.1)$$

The rest of this subsection is dedicated to the proof of Theorem 5.4.1. The proof is based on Lemmas 5.4.1 and 5.4.2, for which we need the following preliminary results.

By Corollary 5.3.1 and Remark 5.3.2 we know that there exists a sequence $\{(m^n, E^n)\}_n$ such that, for any $n \in \mathbb{N}$, $(m^n, E^n) \in \mathcal{S}^n(\bar{m}^n, D)$ and

$$J^n(m^n, E^n) = \inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}^n, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}).$$

The proof of Theorem 5.4.1 relies on the existence of a limit point (m^*, E^*) of the sequence $\{(m^n, E^n)\}_n$ that belongs to the set $\mathcal{S}(\bar{m}, D)$. By inequality (5.2.8), we have

$$\sum_{i, j \in I} \sum_{k=1}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} L \left(\frac{E_{i,j}^n(\{t_k\}, [y_p, y_{p+1}])}{\Delta t m_i(t_k^-, [y_p, y_{p+1}])} \right) \Delta t m_i(t_k^-, [y_p, y_{p+1}]) \leq C_0, \quad (5.4.2)$$

where the constant $C_0 > 0$ only depends on T , $\|c\|_\infty$ and $\|g\|_\infty$. Inequality (5.4.2) implies that $\{E_{i,j}^n; ([0, T] \times [0, 1])\}_n$ is uniformly bounded. Indeed, let $t, \bar{t} \in [0, T]$, with $t \leq \bar{t}$. If there is no $t_q \in \{0, \dots, k\Delta t, \dots, T\}$ satisfying $t \leq t_q \leq \bar{t}$, then by (5.3.1) $\sum_{i,j \in I} E_{i,j}^n([t, \bar{t}], [0, 1]) = 0$. Otherwise, let $k_1, k_2 \in \{0, \dots, N_T^n\}$ be such that $t_{k_1} - \Delta t < t \leq t_{k_1} \leq t_{k_2} \leq \bar{t} < t_{k_2} + \Delta t$. By (5.3.1) and Cauchy-Schwartz inequality, one has:

$$\begin{aligned} & \left(\sum_{i,j \in I} E_{i,j}^n([t, \bar{t}], [0, 1]) \right)^2 \\ &= \left(\sum_{i,j \in I} \sum_{k=k_1}^{k_2} \sum_{p=0}^{N_s^n - 1} E_{i,j}^n(\{t_k\}, [y_p, y_{p+1}]) \right)^2 \\ &\leq \left(\sum_{i,j \in I} \sum_{k=k_1}^{k_2} \sum_{p=0}^{N_s^n - 1} \Delta t m_i^n(t_k^-, [y_p, y_{p+1}]) \right) \left(\sum_{i,j \in I} \sum_{k=k_1}^{k_2} \sum_{p=0}^{N_s^n - 1} \frac{(E_{i,j}^n(\{t_k\}, [y_p, y_{p+1}]))^2}{\Delta t m_i^n(t_k^-, [y_p, y_{p+1}])} \right) \\ &\leq |I| \Delta t (k_2 - k_1 + 1) \sum_{i,j \in I} \sum_{k=k_1}^{k_2} \sum_{p=0}^{N_s^n - 1} C \left(L \left(\frac{E_{i,j}^n(\{t_k\}, [y_p, y_{p+1}])}{\Delta t m_i^n(t_k^-, [y_p, y_{p+1}])} \right) + C \right) \Delta t m_i^n(t_k^-, [y_p, y_{p+1}]) \\ &\leq C |I| (\bar{t} - t + \Delta t) (C_0 + C), \end{aligned} \quad (5.4.3)$$

where the two last inequalities are obtained by Assumption 5 and inequality (5.4.2). Taking $t = 0$ and $\bar{t} = T$, one obtains:

$$\sum_{i,j \in I} E_{i,j}^n([0, T], [0, 1]) \leq \sqrt{CT|I|}, \quad (5.4.4)$$

$C >$ is a constant that depends on Assumption 5 and on C_0 . By (5.4.4), there exists a limit point $E^* \in \mathcal{M}^+([0, T] \times [0, 1] \times I^2)$ of the sequence $\{E^n\}_n$, w.r.t. the weak topology induced by $\mathcal{M}^+([0, T] \times [0, 1] \times I^2)$.

Now we are ready to prove the two lemmas needed for the proof of Theorem 5.4.1. First Lemma states that $\{m^n\}_n$ admits a limit point $m^* \in C^0([0, T], \mathcal{P}([0, 1] \times I))$.

Lemma 5.4.1. *There exists $m^* \in C^0([0, T], \mathcal{P}([0, 1] \times I))$ such that, up to a subsequence, $\{m^n\}_n$ uniformly converges in time to m^* w.r.t. the Wasserstein distance \mathcal{W} . In addition, (m^*, E^*) is a weak solution of (5.2.9) with initial data \bar{m} , and it satisfies (5.2.10).*

Proof. By (5.4.3) and the continuity equation (5.2.9) satisfied by (m^n, E^n) , one has, for any $t, \bar{t} \in [0, T]$ and $n \in \mathbb{N}$,

$$\mathcal{W}(m^n(t), m^n(\bar{t})) \leq \sqrt{|t - \bar{t}| + \Delta t} (\sqrt{T} \|b\|_\infty + \sqrt{C|I|}).$$

By adapting the Ascoli-Arzelà Theorem, one can deduce the existence of $m^* : [0, T] \rightarrow \mathcal{P}([0, 1] \times I)$, that is

1/2-Hölder continuous in time w.r.t. the distance \mathcal{W} and, such that, up to a subsequence, $\{m^n\}_n$ uniformly converges in time to m^* w.r.t. to \mathcal{W} . Finally, since (5.2.9) and (5.2.10) are linear w.r.t. the couple (m, E) and that, for any $n \in \mathbb{N}$, (m^n, E^n) is a weak solution of (5.2.9) and satisfies (5.2.10), one can deduce, by the respective weak convergence of (m^n, E^n) and $\{\bar{m}^n\}_n$ to (m^*, E^*) and \bar{m} , that (m^*, E^*) is also a weak solution of (5.2.9) with initial distribution \bar{m} , and satisfies (5.2.10). \square

The next lemma states the absolute continuity of E^* w.r.t. the measure m^* .

Lemma 5.4.2. *For any $i, j \in I$, $E_{i,j}^*$ is absolutely continuous w.r.t. the measure m_i^* . Denoting by $\alpha_{i,j}^*$ the Radon-Nikodym derivative of $E_{i,j}^*$ w.r.t. the measure m_i^* , we have $\alpha_{i,j}^* \in L^2_{m_i^*}([0, T] \times [0, 1])$.*

Proof. We define for any $n \in \mathbb{N}$ and $i, j \in I$, the curve of measure $\tilde{m}^n : [0, T] \rightarrow \mathcal{P}([0, 1] \times I)$ by:

$$\tilde{m}_i^n(t, ds) := \sum_{k=0}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} \mathbb{1}_{[t_k, t_{k+1})}(t) \mathbb{1}_{[y_p, y_{p+1})}(s) \frac{m_i^n(t_k^-, [y_p, y_{p+1}))}{\Delta s} ds. \quad (5.4.5)$$

By the uniform weak convergence of $\{m^n\}_n$ to m^* in Lemma 5.4.1 and the definition of \tilde{m}^n , one can show that for any $t \in [0, T]$, $\{\tilde{m}^n(t)\}_n$ weakly converges to $m^*(t)$ in $\mathcal{P}([0, 1] \times I)$. For any $n \in \mathbb{N}$ and $i, j \in I$, we introduce $\alpha_{i,j}^n : [0, T] \times [0, 1] \rightarrow \mathbb{R}_+$:

$$\alpha_{i,j}^n(t, s) := \frac{E_{i,j}^n(\{t_k\}, [y_p, y_{p+1}))}{\Delta t m_i^n(t_k^-, [y_p, y_{p+1}))} \quad \text{where } t \in [t_k, t_{k+1}) \quad \text{and } s \in [y_p, y_{p+1}), \quad (5.4.6)$$

and we define the measure $\tilde{E}_{i,j}^n$ on $[0, T] \times [0, 1]$ by: $\tilde{E}_{i,j}^n(dt, ds) := \alpha_{i,j}^n \tilde{m}_i^n(t, ds) dt$. One can observe that:

$$\begin{aligned} \sum_{i,j \in I, i \neq j} \tilde{E}_{i,j}^n([0, T] \times [0, 1]) &= \sum_{i,j \in I, i \neq j} \sum_{k=0}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} \int_{t_k}^{t_{k+1}} \int_{y_p}^{y_{p+1}} \frac{E_{i,j}^n(\{t_k\}, [y_p, y_{p+1}))}{\Delta s \Delta t} dt ds \\ &= \sum_{i,j \in I, i \neq j} E_{i,j}^n([0, T] \times [0, 1]). \end{aligned} \quad (5.4.7)$$

Using the previous equality and (5.4.4), $\{\tilde{E}_{i,j}^n\}_n$ is tight. Thus, there exists a measure $\tilde{E}_{i,j}$ such that a subsequence of $\{\tilde{E}_{i,j}^n\}_n$ weakly converges to $\tilde{E}_{i,j}$. We define the function $\Theta : \mathcal{M}^+([0, T] \times [0, 1] \times I) \times \mathcal{M}^+([0, T] \times [0, 1] \times I \times I) \rightarrow \mathbb{R} \cup \{+\infty\}$ by:

$$\Theta(m, E) : \begin{cases} \sum_{i,j \in I, i \neq j} \int_0^T \int_0^1 L(\alpha_{i,j}(t, s)) m_i(dt, ds) & \text{if } E_{i,j} \ll m_i \text{ and } \alpha_{i,j} := \frac{dE_{i,j}}{dm_i} \text{ with } \alpha_{i,j} \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The function Θ is convex and l.s.c. w.r.t. the weak topology in $\mathcal{M}^+([0, T] \times [0, 1] \times I) \times \mathcal{M}^+([0, T] \times [0, 1] \times I^2)$ (see e.g. [231, Proposition 5.18]). From the definition of \tilde{E}^n , it turns out that $\tilde{E}_{i,j}^n \ll \tilde{m}_i^n$ with $\frac{d\tilde{E}_{i,j}^n}{d\tilde{m}_i^n} = \alpha_{i,j}^n$.

We have:

$$\begin{aligned} \Theta(\tilde{m}^n, \tilde{E}^n) &= \sum_{i,j \in I} \int_0^T \int_0^1 L(\alpha_{i,j}^n(t, s)) \tilde{m}_i^n(t, s) dt ds \\ &= \sum_{i,j \in I} \sum_{k=0}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} L\left(\frac{E_{i,j}^n(\{t_k\}, [y_p, y_{p+1}))}{\Delta t m_i^n(t_k^-, [y_p, y_{p+1}))}\right) m_i^n(t_k^-, [y_p, y_{p+1})) \Delta t \\ &\leq C_0. \end{aligned}$$

where the constant $C_0 > 0$ is defined in (5.4.2). Using that $(m_i^n, \tilde{E}_{i,j}^n)$ weakly converges to $(m_i^*, \tilde{E}_{i,j})$ and by the previous inequality, one deduces that $\Theta(m^*, \tilde{E}) \leq C_0$ and thus, for any $i, j \in I$: $\tilde{E}_{i,j} \ll m_i^*$. We now prove that $\tilde{E}_{i,j} = E_{i,j}^*$ by showing that $\tilde{E}_{i,j}^n - E_{i,j}^n$ weakly converges to 0 in $\mathcal{M}^+([0, T] \times [0, 1] \times I^2)$. By the definition of \tilde{E}^n and (5.4.4) one has $\tilde{E}_{i,j}^n([0, T] \times [0, 1] \times I^2) = E_{i,j}^n([0, T] \times [0, 1] \times I^2) \leq \sqrt{CT|I|}$ for any $n \in \mathbb{N}$. Let

$\varphi : [0, T] \times [0, 1] \rightarrow \mathbb{R}$ be a 1-Lipschitz continuous function. Using (5.4.4), we have:

$$\begin{aligned}
& \int_0^T \int_0^1 \varphi(t, s) (\alpha_{i,j}^n(t, s) \tilde{m}_i^n(t, ds) dt - E_{i,j}^n(dt, ds)) \\
&= \sum_{k=0}^{N_T^n-1} \sum_{p=0}^{N_s^n-1} \int_{t_k}^{t_{k+1}} \int_{y_p}^{y_{p+1}} \varphi(t, s) \left(\frac{E_{i,j}^n(\{t_k\}, [y_p, y_{p+1}])}{\Delta s \Delta t} ds dt - E_{i,j}^n(dt, ds) \right) \\
&\leq \sum_{k=0}^{N_T^n-1} \sum_{p=0}^{N_s^n-1} \int_{t_k}^{t_{k+1}} \int_{y_p}^{y_{p+1}} \frac{\Delta s + \Delta t}{\Delta s \Delta t} E_{i,j}^n(\{t_k\}, [y_p, y_{p+1}]) ds dt \\
&\leq (\Delta s + \Delta t) E_{i,j}([0, T] \times [0, T]) \\
&\leq \sqrt{CT|I|}(\Delta s + \Delta t),
\end{aligned} \tag{5.4.8}$$

and therefore $\mathcal{W}(\tilde{E}_{i,j}^n, E_{i,j}^n) \rightarrow 0$. We deduce that $E_{i,j}^* = \tilde{E}_{i,j}$ and we can define the Radon-Nikodym derivative $\alpha_{i,j}^* := \frac{dE_{i,j}^*}{dm_i^*}$. Since $\Theta(m_i^*, E_{i,j}^*) < \infty$, we deduce for any $i, j \in I$ that $\alpha_{i,j}^* \in L_{m_i^*}^2([0, T] \times [0, 1])$ from Assumption 5. \square

We can now prove Theorem 5.4.1.

Proof of Theorem 5.4.1. By Lemmas 5.4.1 and 5.4.2, one has that $(m^*, E^*) \in \mathcal{S}(\bar{m}, D)$. Since $c \in C^1([0, T] \times [0, 1] \times I)$ and $g \in C^1([0, 1] \times I)$, by the w.l.s.c. of Θ defined in Lemma 5.4.2, we have:

$$\inf_{(m,E) \in \mathcal{S}(\bar{m}, D)} J(m, E) \leq J(m^*, E^*) \leq \liminf_{n \rightarrow \infty} J^n(m^n, E^n) = \liminf_{n \rightarrow \infty} \inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}^n, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}),$$

and the conclusion follows. \square

5.4.2 Upper bound of the value of the finite population problem

We reformulate the second main result of this paper, Theorem 5.2.1.2, in Theorem 5.4.2 and gives a proof.

Theorem 5.4.2. *There exist $C > 0$ such that, for any sequence $\{\bar{m}^n\}_n$ in $\mathcal{P}^n([0, 1] \times I)$ satisfying Assumptions 2 and 3, one has, for any $n \in \mathbb{N}^*$,*

$$\inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}^n, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}) - \frac{C}{n^{\frac{1}{3}}} \leq \inf_{(m,E) \in \mathcal{S}(\bar{m}^n, D)} J(m, E). \tag{5.4.9}$$

Theorem 5.4.2 provides an upper bound on the value of Problem (5.2.7). Note that the constants in this theorem depend on ε^0 (defined in Assumption 3) and not on the choice of the sequence $\{\bar{m}^n\}_n$. To prove this theorem, we first show in Section 5.4.2 how to implement an optimal control of the mean field control problem to a finite population of PEVs, such that the constraint (5.2.4) is satisfied (Corollary 5.4.1). Then, in Section 5.4.2 we give an estimate of the Wasserstein distance between the resulting empirical distribution of the finite population of processes and the mean field distribution. Finally, we finish the proof of Theorem 5.4.2 in Section 5.4.2.

Transfer procedure for a finite number of PEVs using a mean field control

The goal of this subsection is to present how a bounded mean field control α can be implemented for a finite population of PEVs. We provide a convergence rate of the empirical distribution to the mean field distribution. Let us fix a sequence $\{\bar{m}^n\}_n$ of initial distribution in (5.2.9) satisfying Assumptions 2 and 3. We highlight that, all the constants in this section depend on ε^0 and not on the choice of the sequence $\{\bar{m}^n\}_n$. Let $N \in \mathbb{N}$ be such that,

$$N > \left(\frac{2C^*}{\varepsilon^0} \right)^3, \tag{5.4.10}$$

where $C^* > 0$ is introduced later in (5.4.17). For any $n \in \mathbb{N}$, we consider $(m^{*,n}, E^{*,n})$, a minimizer of J over the set $\mathcal{S}(\bar{m}^n, D - C^*/n^{\frac{1}{3}})$. We define α^n by $\alpha_{i,j}^n := \frac{dm_i^{*,n}}{dE_{i,j}^{*,n}}$, for any $i, j \in I$. By Assumption 3 and the definition

of N in (5.4.10), one has for any $n \geq N$, and $(i, t) \in I \times [0, T]$,

$$(D_i(t) - \frac{C^*}{n^{\frac{1}{3}}}) - m_i^n(0, [0, 1]) \leq \frac{\varepsilon^0}{2}. \quad (5.4.11)$$

By the previous inequality and [233, Theorem 2.1], we have $\alpha^n \in L^\infty([0, T] \times [0, 1] \times I^2)$ and, for any $(i, j, t, s) \in I^2 \times [0, T] \times [0, 1]$,

$$\alpha_{i,j}^n(t, s) = H' \left(\beta_{i,j}^n(t, s) + \int_t^T (\lambda_i^n - \lambda_j^n)(d\tau) \right), \quad (5.4.12)$$

where $\beta^n \in \text{Lip}([0, T] \times [0, 1] \times I^2)$ and $\lambda^n \in \mathcal{M}^+([0, T] \times I)$. By Assumption 5, H' is Lipschitz continuous on \mathbb{R} and thus, α^n is Lipschitz continuous in space uniformly in time. By [233, Lemma 5.3], the upper bound of $\lambda^n([0, T] \times I)$ only depends on ε^0 in (5.4.11) and on the data of the problem. In addition, by [233, Theorem 2.1], for any $n \in \mathbb{N}^*$, there exists $\varphi^n(\text{Lip}([0, T] \times [0, 1] \times I) + BV([0, T] \times I))$ such that $\alpha_{i,j}^n = H'(\varphi_i^n - \varphi_j^n)$, where by [233, Remark 4.5], upper bounds on $\|\varphi^n\|_\infty$ and $\|\partial_s \varphi^n\|_\infty$ depend on the data of the problem and on $\lambda^n([0, T] \times I)$. Since (5.4.11) is satisfied for any $n > N$, there exists $\bar{C} > 0$ that depends on ε^0 , such that, for any $n > N$,

$$\max(\|\alpha^n\|_\infty, \|\partial_s \alpha^n\|_\infty, \lambda^n([0, T] \times I)) \leq \bar{C}. \quad (5.4.13)$$

Transfer procedure We consider $n > N$ PEVs with an initial state distribution \bar{m}^n . At each time step $t_k \in \{\Delta t, \dots, \Delta t(N_T^n - 1)\}$ and for any $p \in \{0, \dots, N_s^n - 1\}$, we apply the following steps:

- We define $V_i^{k,p}$, the set of indices of PEVs in the mode of charging $i \in I$ with a SoC in the range $[y_p, y_{p+1})$ at time t_k , and set $N_i^{k,p} := \text{card}(V_i^{k,p})$.
- The number of PEVs in $V_i^{k,p}$ transferred from the mode of charging $i \in I$ to the mode $j \in I$ at time t_k is denoted by $a_{i,j}^n(k, p)$ and is defined by: $a_{i,j}^n(k, p) := \lfloor \Delta t \alpha_{i,j}^n(t_k, y_p) N_i^{k,p} \rfloor$. The transferred vehicles are the ones with the lowest indices. We denote by $T_{i,j}^{k,p}$ the set of indices of the transferred vehicles. We have: $T_{i,j}^{k,p} \subset V_i^{k,p}$.

Remark 5.4.1. By inequalities (5.2.13) and (5.4.13), one has, for any $n \in \mathbb{N}$, $\Delta t \leq \frac{1}{|I| \|\alpha^n\|_\infty}$. It implies that,

for any i, k, ℓ , $\sum_{j=1, j \neq i}^n a_{i,j}^n(k, p) \leq N_i^{k,p}$. Thus, the maximal amount of PEVs with a state in $\{i\} \times [y_p, y_{p+1})$ that can be transferred is bounded by the total number of PEVs with a state in $i \times [y_p, y_{p+1})$.

Remark 5.4.2. One can observe that PEVs with a SoC equal to 1 are not taken into account in the transfer procedure given above. This is without consequence because, following Remark 5.2.1 and Assumption 2, there is never a PEV with a SoC equal to 1.

Recall that $\bar{m}^n \in \mathcal{P}^n([0, 1] \times I)$ is the initial distribution of the states of the population of PEVs. The procedure defined above enables to construct a unique set of n processes $\{(i^\ell, s^\ell)\}_{\ell \in \{1, \dots, n\}} \in \mathcal{X}^n(\bar{m}^n)$. For any $\ell \in \{1, \dots, n\}$ and $t \in [0, T]$, i_t^ℓ and s_t^ℓ denote respectively the mode of charging and the SoC of the ℓ^{th} PEV at time t . By Theorem 5.3.1, the pair (m^n, E^n) defined by (5.2.16) and (5.2.17) belongs to $\mathcal{Q}^n(\bar{m}^n)$.

There exists a pair $(m^n, E^n) \in \mathcal{Q}^n(\bar{m}^n)$ that is the empirical distribution of the states of the population of PEVs and of the jumps and that satisfies (5.2.16) and (5.2.17). In addition, the following equalities hold, for any $k \in \{0, \dots, N_T^n - 1\}$, $p \in \{0, \dots, N_s^n - 1\}$ and $i, j \in I$,

$$m_i^n(t_k^-, [y_p, y_{p+1})) = \frac{N_i^{k,p}}{n}, \quad (5.4.14)$$

$$E_i^n([t_k, t_{k+1}), [y_p, y_{p+1})) = \frac{a_{i,j}^n(k, p)}{n}. \quad (5.4.15)$$

The theorem below gives an estimate of the Wasserstein distance between $m^{*,n}$ and m^n .

Theorem 5.4.3. *There exists $C > 0$, such that for any $n > N$ and $t \in [0, T]$,*

$$\mathcal{W}(m^{*,n}(t), m^n(t)) \leq \frac{C}{n^{\frac{1}{3}}}. \quad (5.4.16)$$

We postpone the proof to Section 5.4.2. As we show below, the rate $n^{-\frac{1}{3}}$ in the right hand-side of (5.4.16) comes from inequality (5.2.12).

Next result shows that the previous theorem enables to find strategies, based on a mean field optimal control, satisfying the constraint (5.2.4). We now set:

$$C^* = C + 1, \quad (5.4.17)$$

where $C > 0$ is the constant defined in Theorem 5.4.3.

Corollary 5.4.1. *One has, for any $n \geq N$,*

$$\frac{1}{n^{\frac{1}{3}}} \leq D_i(t) - m_i^n(t, [0, 1]) \quad \forall (i, t) \in I \times [0, T]. \quad (5.4.18)$$

Proof. Observing that

$$D_i(t) - m_i^{*,n}(t, [0, 1]) - \mathcal{W}(m^{*,n}(t), m^n(t)) \leq D_i(t) - m_i^n(t, [0, 1]),$$

by Theorem 5.4.3 and since $(m^{*,n}, E^{*,n}) \in \mathcal{S}(\bar{m}^n, D - \frac{C^*}{n^{\frac{1}{3}}})$, one deduces, for any $(i, t) \in I \times [0, T]$,

$$\frac{C^* - C}{n^{\frac{1}{3}}} \leq D_i(t) - m_i^n(t, [0, 1]).$$

The conclusion follows from the previous inequality and inequality (5.4.17). \square

Proof of Theorem 5.4.3

We start by stating some preliminary results.

Lemma 5.4.3. *For any function $\varphi \in C^1([0, T] \times [0, 1] \times I)$, we have, for any $i \in I$,*

$$\begin{aligned} & \int_0^T \int_0^1 (\partial_t \varphi_i(t, s) + b_i(t) \partial_s \varphi_i(t, s)) m_i^n(t, ds) dt \\ &= \int_0^1 \varphi_i(T, s) m_i^n(T, ds) - \varphi_i(0, s) \bar{m}_i^n(ds) + \frac{1}{n} \sum_{k=1}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} \sum_{j \in I, j \neq i} \left(\sum_{\ell \in T_{i,j}^{k,p}} \varphi_i(t_k, s_{t_k}^\ell) - \sum_{\ell \in T_{j,i}^{k,p}} \varphi_i(t_k, s_{t_k}^\ell) \right). \end{aligned}$$

Proof. This lemma is a direct consequence of the proof of Lemma 5.3.1. \square

We introduce $\check{m}^n \in L^1([0, T] \times [0, 1] \times I)$ and $\check{\alpha}^n \in L^\infty([0, T] \times [0, 1] \times I)$ defined, for any $i, j \in I$, $t \in [t_k, t_{k+1})$ and $s \in [y_p, y_{p+1})$, by:

$$\check{m}_i^n(t, s) := \frac{1}{\Delta s} m_i^n(t_k, [y_p, y_{p+1})) \text{ and } \check{\alpha}_{i,j}^n(t, s) := \alpha_{i,j}^n(t_k, y_p). \quad (5.4.19)$$

We observe that: $\check{m}_i^n(t, s) = N_i^{k,p} / (\Delta s n)$ for any $t \in [t_k, t_{k+1})$ and $s \in [y_p, y_{p+1})$.

Lemma 5.4.4. *There exists $C > 0$ such that for any $n \geq N$ and any function $\varphi \in C^0([0, T] \times [0, 1] \times I)$ and $i \in I$, we have:*

$$\left| \int_0^T \int_0^1 \varphi_i(t, s) (\alpha_{i,j}^n(t, s) m_i^n(t, ds) - \check{\alpha}_{i,j}^n(t, s) \check{m}_i^n(t, s) ds) dt \right| \leq C \|\varphi\|_\infty (\Delta s + \Delta t). \quad (5.4.20)$$

Proof. We observe for any $i, j \in I$:

$$\begin{aligned}
& \left| \int_0^T \int_0^1 \varphi_i(t, s) (\alpha_{i,j}^n(t, s) m_i^n(t, ds) - \check{\alpha}_{i,j}^n(t, s) \check{m}_i^n(t, s) ds) dt \right| \\
&= \left| \sum_{p=0}^{N_s^n-1} \sum_{k=0}^{N_T^n-1} \int_{t_k}^{t_{k+1}} \left(\int_{y_p}^{y_{p+1}} \varphi_i(t, s) \alpha_{i,j}^n(t, s) m_i^n(t, ds) - \int_{y_p}^{y_{p+1}} \varphi_i(t, s) \alpha_{i,j}^n(t_k, y_p) \frac{N_i^{k,p}}{n \Delta s} ds \right) dt \right| \\
&\leq \|\varphi\|_\infty \sum_{p=0}^{N_s^n-1} \sum_{k=0}^{N_T^n-1} \int_{t_k}^{t_{k+1}} \left| \left(\sum_{\ell \in V_i^{k,p}} \frac{\alpha_{i,j}^n(t, s_t^\ell)}{n} \right) - \alpha_{i,j}^n(t_k, y_p) \frac{N_i^{k,p}}{n} \right| dt \\
&= \|\varphi\|_\infty \sum_{p=0}^{N_s^n-1} \sum_{k=0}^{N_T^n-1} \int_{t_k}^{t_{k+1}} \left| \sum_{\ell \in V_i^{k,p}} \left(\frac{\alpha_{i,j}^n(t, s_t^\ell)}{n} - \frac{\alpha_{i,j}^n(t_k, y_p)}{n} \right) \right| dt.
\end{aligned} \tag{5.4.21}$$

By equality (5.4.12), and the uniform Lipschitz continuity property of H' and β , one has, for any $\ell \in V_i^{k,p}$ and $t \in [t_k, t_{k+1})$:

$$\begin{aligned}
|\alpha_{i,j}^n(t, s_t^\ell) - \alpha_{i,j}^n(t_k, y_p)| &\leq C \left(|\beta_{i,j}^n(t, s_t^\ell) - \beta_{i,j}^n(t_k, y_p)| + \int_{t_k}^t (\lambda_i^n + \lambda_j^n)(d\tau) \right) \\
&\leq C \left(\Delta s + \Delta t + \int_{t_k}^t (\lambda_i^n + \lambda_j^n)(d\tau) \right),
\end{aligned} \tag{5.4.22}$$

where the second inequality is deduced by the fact that, if $s_{t_k}^\ell \in [y_p, y_{p+1})$, then $|s_t^\ell - y_p| \leq (\Delta s + \Delta t \|b\|_\infty)$ for any $t \in [t_k, t_{k+1})$. Summing the previous inequality over $\ell \in V_i^{k,p}$, one has, for any $t \in [t_k, t_{k+1})$,

$$\left| \sum_{\ell \in V_i^{k,p}} \left(\frac{\alpha_{i,j}^n(t, s_t^\ell)}{n} - \frac{\alpha_{i,j}^n(t_k, y_p)}{n} \right) \right| \leq \frac{N_i^{k,p}}{n} C \left(\Delta s + \Delta t + \int_{t_k}^t (\lambda_i^n + \lambda_j^n)(d\tau) \right). \tag{5.4.23}$$

By (5.4.23) and (5.4.13), inequality (5.4.21) becomes:

$$\begin{aligned}
& \left| \int_0^T \int_0^1 \varphi_i(t, s) (\alpha_{i,j}^n(t, s) m_i^n(t, ds) - \check{\alpha}_{i,j}^n(t, s) \check{m}_i^n(t, s) ds) dt \right| \\
&\leq \|\varphi\|_\infty \sum_{p=0}^{N_s^n-1} \sum_{k=0}^{N_T^n-1} \int_{t_k}^{t_{k+1}} \frac{N_i^{k,p}}{n} C \left(\Delta s + \Delta t + \int_{t_k}^t (\lambda_i^n + \lambda_j^n)(d\tau) \right) dt \\
&\leq \|\varphi\|_\infty \sum_{k=0}^{N_T^n-1} \sum_{p=0}^{N_s^n-1} \frac{N_i^{k,p}}{n} C \left(\Delta s + \Delta t + \int_{t_k}^{t_{k+1}} (\lambda_i^n + \lambda_j^n)(d\tau) \right) \Delta t \\
&\leq \|\varphi\|_\infty C (\Delta s + \Delta t).
\end{aligned}$$

□

Lemma 5.4.5. *For any $i \in I$, $n \geq N$ and any function $\varphi \in C^1([0, T] \times [0, 1] \times I)$, with Lipschitz constant denoted by $\gamma_\varphi > 0$, we have:*

$$\begin{aligned}
& \left| \int_0^T \int_0^1 (\partial_t \varphi_i(t, s) + b_i(s) \partial_s \varphi_i(t, s)) m_i^n(t, ds) dt - \int_0^1 \varphi_i(T, s) m_i^n(T, ds) - \varphi_i(0, s) \bar{m}_i^n(ds) \right. \\
& \quad \left. - \int_0^T \int_0^1 \varphi_i(t, s) (\check{\alpha}_{i,j}^n(t, s) \check{m}_i^n(t, s) - \check{\alpha}_{j,i}^n(t, s) \check{m}_j^n(t, s)) dt ds \right| \\
&\leq C (\gamma_\varphi + \|\varphi\|_\infty) \left(\Delta s + \Delta t + \frac{1}{n \Delta s \Delta t} \right).
\end{aligned} \tag{5.4.24}$$

Proof. Let $\varphi \in C^1([0, T] \times [0, 1] \times I)$. We observe, for any $i, j \in I$,

$$\begin{aligned}
& \int_0^T \int_0^1 \varphi_i(t, s) (\check{\alpha}_{i,j}^n(t, s) \check{m}_i^n(t, s) - \check{\alpha}_{j,i}^n(t, s) \check{m}_j^n(t, s)) dt ds \\
&= \sum_{k=0}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} \int_{t_k}^{t_{k+1}} \int_{y_p}^{y_{p+1}} \varphi_i(t, s) \left(\alpha_{i,j}^n(t_k, y_p) \frac{N_i^{k,p}}{n \Delta s} - \alpha_{j,i}^n(t_k, y_p) \frac{N_j^{k,p}}{n \Delta s} \right) dt ds \\
&= \sum_{p=0}^{N_s^n - 1} \sum_{k=0}^{N_T^n - 1} \left(\Delta t \alpha_{i,j}^n(t_k, y_p) \frac{N_i^{k,p}}{n \Delta s} - \Delta t \alpha_{j,i}^n(t_k, y_p) \frac{N_j^{k,p}}{n \Delta s} \right) \frac{1}{\Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} \varphi_i(t, s) dt ds.
\end{aligned} \tag{5.4.25}$$

Combining Lemma 5.4.3 and the previous equality, we get:

$$\begin{aligned}
& \int_0^T \int_0^1 (\partial_t \varphi_i(t, s) + b_i(s) \partial_s \varphi_i(t, s)) m_i^n(t, ds) dt - \int_0^1 \varphi_i(T, s) m_i^n(T, ds) - \varphi_i(0, s) \bar{m}_i^n(ds) \\
& - \int_0^T \int_0^1 \varphi_i(t, s) (\check{\alpha}_{i,j}^n(t, s) \check{m}_i^n(t, s) - \check{\alpha}_{j,i}^n(t, s) \check{m}_j^n(t, s)) dt ds \\
&= \frac{1}{n} \sum_{k=0}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} \sum_{j \in I, j \neq i} \left(\sum_{\ell \in T_{i,j}^{k,p}} \varphi_i(t_k, s_{t_k}^\ell) - \sum_{\ell \in T_{j,i}^{k,p}} \varphi_i(t_k, s_{t_k}^\ell) \right) \\
& - \sum_{p=0}^{N_s^n - 1} \sum_{k=0}^{N_T^n - 1} \sum_{j \in I, j \neq i} \left(\Delta t \alpha_{i,j}^n(t_k, y_p) \frac{N_i^{k,p}}{n \Delta s} - \Delta t \alpha_{j,i}^n(t_k, y_p) \frac{N_j^{k,p}}{n \Delta s} \right) \frac{1}{\Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} \varphi_i(t, s) dt ds.
\end{aligned} \tag{5.4.26}$$

We recall that there is no transfer at time $t_0 = 0$ in the transfer procedure described in Section 5.4.2. Thus, to find an upper bound of (5.4.26), two cases are considered below: $k = 0$ and $k \in \{1, \dots, N_T^n - 1\}$.

— If $k = 0$, we have, for any $i, j \in I$ and $p \in \{0, \dots, N_s^n - 1\}$, that $T_{i,j}^{k,p} = \emptyset$. Thus, $\sum_{\ell \in T_{i,j}^{k,p}} \frac{1}{n} \varphi_i(t_k, s_{t_k}^\ell) = 0$

and

$$\begin{aligned}
& \sum_{p=0}^{N_s^n - 1} \sum_{j \in I, j \neq i} \left| \sum_{\ell \in T_{i,j}^{k,p}} \frac{1}{n} \varphi_i(t_k, s_{t_k}^\ell) - \Delta t \alpha_{i,j}^n(t_k, y_p) \frac{N_i^{k,p}}{n} \frac{1}{\Delta s \Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} \varphi_i(t, s) dt ds \right| \\
&= \sum_{p=0}^{N_s^n - 1} \sum_{j \in I, j \neq i} \left| \Delta t \alpha_{i,j}^n(t_k, y_p) \frac{N_i^{k,p}}{n} \frac{1}{\Delta s \Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} \varphi_i(t, s) dt ds \right| \\
&\leq \bar{C} \|\varphi\|_\infty \Delta t,
\end{aligned} \tag{5.4.27}$$

where the constant $\bar{C} > 0$ is defined in (5.4.13).

— For $k \in \{1, \dots, N_T^n - 1\}$, recalling the definition a^n in Section 5.4.2, we have, for any i, j, k, p ,

$$|a_{i,j}^n(k, p) - \Delta t \alpha_{i,j}^n(t_k, y_p) N_i^{k,p}| \leq 1,$$

and thus, one obtains, for any $k \in \{0, \dots, N_T^n - 1\}$ and $p \in \{0, \dots, N_s^n - 1\}$,

$$\left| \left(a_{i,j}^n(k, p) - \Delta t \alpha_{i,j}^n(t_k, y_p) N_i^{k,p} \right) \frac{1}{n \Delta s \Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} \varphi_i(t, s) dt ds \right| \leq \frac{\|\varphi\|_\infty}{n}.$$

Since, for any i, j, p , $\text{card}(T_{i,j}^{k,p}) = a_{i,j}^n(k, p)$, one has from the previous inequality, for any $p \in \{0, \dots, N_s^n -$

1\},

$$\begin{aligned}
& \left| \sum_{\ell \in T_{i,j}^{k,p}} \frac{1}{n} \varphi_i(t_k, s_{t_k}^\ell) - \Delta t \alpha_{i,j}^n(t_k, y_p) \frac{N_i^{k,p}}{n} \frac{1}{\Delta s \Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} \varphi_i(t, s) dt ds \right| \\
& \leq \left| \sum_{\ell \in T_{i,j}^{k,p}} \frac{1}{n \Delta s \Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} \varphi_i(t_k, s_{t_k}^\ell) dt ds - \frac{\alpha_{i,j}^n(k, p)}{n \Delta s \Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} \varphi_i(t, s) dt ds \right| + \frac{\|\varphi\|_\infty}{n} \\
& \leq \left| \sum_{\ell \in T_{i,j}^{k,p}} \left(\frac{1}{n \Delta s \Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} (\varphi_i(t_k, s_{t_k}^\ell) - \varphi_i(t, s)) dt ds \right) \right| + \frac{\|\varphi\|_\infty}{n}.
\end{aligned} \tag{5.4.28}$$

By the Lipschitz continuity of φ , the definition of $\alpha_{i,j}^n(k, p)$ and inequality (5.4.13), one gets:

$$\begin{aligned}
& \left| \sum_{\ell \in T_{i,j}^{k,p}} \left(\frac{1}{n \Delta s \Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} (\varphi_i(t_k, s_{t_k}^\ell) - \varphi_i(t, s)) dt ds \right) \right| \leq \frac{\alpha_{i,j}^n(k, p) \gamma_\varphi (\Delta t + \Delta s)}{n} \\
& \leq \frac{(\bar{C} N_i^{k,p} \Delta t + 1) \gamma_\varphi (\Delta t + \Delta s)}{n}.
\end{aligned}$$

Inequality (5.4.28) becomes:

$$\begin{aligned}
& \left| \sum_{\ell \in T_{i,j}^{k,p}} \frac{1}{n} \varphi_i(t_k, s_{t_k}^\ell) - \Delta t \alpha_{i,j}^n(t_k, y_p) \frac{N_i^{k,p}}{n} \frac{1}{\Delta s \Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} \varphi_i(t, s) dt ds \right| \\
& \leq \frac{(\bar{C} N_i^{k,p} \Delta t + 1) \gamma_\varphi (\Delta t + \Delta s) + \|\varphi\|_\infty}{n}.
\end{aligned} \tag{5.4.29}$$

By a similar computation, one can show that:

$$\begin{aligned}
& \left| \sum_{\ell \in T_{j,i}^{k,p}} \frac{1}{n} \varphi_i(t_k, s_{t_k}^\ell) - \Delta t \alpha_{j,i}^n(t_k, y_p) \frac{N_j^{k,p}}{n} \frac{1}{\Delta s \Delta t} \int_{y_p}^{y_{p+1}} \int_{t_k}^{t_{k+1}} \varphi_i(t, s) dt ds \right| \\
& \leq \frac{(\bar{C} N_j^{k,p} \Delta t + 1) \gamma_\varphi (\Delta t + \Delta s) + \|\varphi\|_\infty}{n}.
\end{aligned} \tag{5.4.30}$$

Finally, by inequality (5.4.27) and summing (5.4.29) and (5.4.30) over k, p and j , equality (5.4.26) becomes:

$$\begin{aligned}
& \left| \int_0^T \int_0^1 (\partial_t \varphi_i(t, s) + b_i(s) \partial_s \varphi_i(t, s)) m_i^n(t, ds) dt - \int_0^1 \varphi_i(T, s) m_i^n(T, ds) - \varphi_i(0, s) \bar{m}_i^n(ds) \right. \\
& \quad \left. - \sum_{j \in I, j \neq i} \int_0^T \int_0^1 \varphi_i(t, s) (\check{\alpha}_{i,j}^n(t, s) \check{m}_i^n(t, s) - \check{\alpha}_{j,i}^n(t, s) \check{m}_j^n(t, s)) ds dt \right| \\
& \leq \bar{C} \|\varphi\|_\infty \Delta t + \sum_{j \in I, j \neq i} \sum_{k=1}^{N_T^n - 1} \sum_{p=1}^{N_s^n - 1} \frac{(\bar{C} (N_i^{k,p} + N_j^{k,p}) \Delta t + 2) \gamma_\varphi (\Delta t + \Delta s) + 2 \|\varphi\|_\infty}{n} \\
& \leq C (\gamma_\varphi + \|\varphi\|_\infty) (\Delta s + \Delta t + \frac{1}{n \Delta s \Delta t}),
\end{aligned} \tag{5.4.31}$$

where the constant C depends on \bar{C}, T and $|I|$. □

Lemma 5.4.6. *There exists $C > 0$ such that, for any $i \in I$, $n \geq N$ and any function $\varphi \in C^1([0, T] \times [0, 1] \times I)$,*

with Lipschitz constant denoted by $\gamma_\varphi > 0$, we have:

$$\begin{aligned} & \left| - \int_0^1 \varphi_i(T, s) m_i^n(T, ds) - \varphi_i(0, s) m_i^n(0, ds) + \int_0^T \int_0^1 (\partial_t \varphi_i(t, s) + b_i(s) \partial_s \varphi_i(t, s)) m_i^n(t, ds) dt \right. \\ & \left. - \sum_{j \in I, j \neq i} \int_0^T \int_0^1 \varphi_i(t, s) (\alpha_{i,j}^n(t, s) m_i^n(t, s) - \alpha_{j,i}^n(t, s) m_j^n(t, ds)) dt \right| \\ & \leq C(\|\varphi\|_\infty + \gamma_\varphi) (\Delta t + \Delta s + \frac{1}{n \Delta s \Delta t}). \end{aligned} \quad (5.4.32)$$

Proof. This results is a direct consequence of Lemmas 5.4.4 and 5.4.5. \square

Remark 5.4.3. Lemma 5.4.6 also holds when T is replaced by any $t \in (0, T]$.

Remark 5.4.4. The term $\frac{T}{n \Delta s \Delta t}$ on the r.h.s. of the inequality in Lemma 5.4.6, implies to choose carefully the time and space steps, depending on the number of agents n . To this end, inequality (5.2.12) is crucial.

We can now prove Theorem 5.4.3.

Proof of Theorem 5.4.3. Let $\psi \in C^1([0, 1] \times I)$ and $\varphi \in C^1([0, T] \times [0, 1] \times I)$ be the classical solution of the PDE:

$$\begin{aligned} \partial_\tau \varphi_i(\tau, s) + b_i(s) \partial_s \varphi_i(\tau, s) &= 0 & (\tau, s, i) \in [0, t] \times [0, 1] \times I \\ \varphi_i(t, s) &= \psi_i(s) & (s, i) \in \times [0, 1] \times I. \end{aligned} \quad (5.4.33)$$

One has $\|\varphi\|_\infty = \|\psi\|_\infty$, and denoting by γ_ψ the Lipschitz constant of ψ , for any $t \in [0, T]$ $\varphi_i(t, \cdot)$ is Lipschitz continuous with Lipschitz constant $\gamma_\psi e^{T \|b'\|_\infty}$. Since $(m^{*,n}, E^{*,n})$ is a weak solution of the continuity equation (5.2.9) with initial distribution \bar{m}^n , and φ a classical solution of (5.4.33), using that $\alpha_{i,j}^n = \frac{dE_{i,j}^{*,n}}{dm_i^{*,n}}$, we have:

$$\begin{aligned} & \int_0^1 \psi_i(s) m_i^{*,n}(t, ds) \\ &= \int_0^1 \varphi_i(0, s) m_i^n(0, s) ds - \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi_i(\tau, s) (\alpha_{i,j}^n(\tau, s) m_i^{*,n}(\tau, ds) - \alpha_{j,i}^n(\tau, s) m_j^{*,n}(\tau, ds)) d\tau. \end{aligned} \quad (5.4.34)$$

From Lemma 5.4.6 and Remark 5.4.3, we have, for any $t \in (0, T]$:

$$\begin{aligned} \int_0^1 \psi_i(s) m_i^n(t, ds) &\leq \int_0^1 \varphi_i(0, s) m_i^n(0, ds) - \sum_{j \in I, j \neq i} \int_0^t \int_0^1 \varphi_i(\tau, s) (\alpha_{i,j}^n(\tau, s) m_i^n(\tau, s) - \alpha_{j,i}^n(\tau, s) m_j^n(\tau, ds)) d\tau \\ &\quad + C(\|\psi\|_\infty + \gamma_\psi e^{T \|b'\|_\infty}) (\Delta t + \Delta s + \frac{1}{n \Delta s \Delta t}). \end{aligned} \quad (5.4.35)$$

From the previous equality and inequality, one deduces:

$$\begin{aligned} \sum_{i \in I} \int_0^1 \psi_i(s) (m_i^{*,n}(t, ds) - m_i^n(t, ds)) &\leq - \sum_{i \in I} \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi_i(\tau, s) \alpha_{i,j}^n(\tau, s) (m_i^{*,n}(\tau, ds) - m_i^n(\tau, ds)) d\tau \\ &\quad + \sum_{i \in I} \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi_i(\tau, s) \alpha_{j,i}^n(\tau, s) (m_j^{*,n}(\tau, ds) - m_j^n(\tau, ds)) d\tau \\ &\quad + C(\|\psi\|_\infty + \gamma_\psi) (\Delta t + \Delta s + \frac{1}{n \Delta s \Delta t}). \end{aligned} \quad (5.4.36)$$

By inequality (5.4.13), $\{\alpha^n(t, \cdot)\}_{n > N, t \in [0, T]}$ is uniformly Lipschitz continuous with constant \bar{C} . Thus, for any $t \in [0, T]$, $(s, i, j) \mapsto \varphi_i(t, s) \alpha_{i,j}^n(t, s)$ is Lipschitz on $[0, 1] \times I \times I$ with Lipschitz constant $\bar{C}(\|\psi\|_\infty + \gamma_\psi e^{T \|b'\|_\infty})$. We deduce that, for any $\tau \in [0, T]$:

$$\sum_{i \in I} \int_0^1 \sum_{j \in I, j \neq i} \varphi_i(\tau, s) \alpha_{i,j}^n(\tau, s) (m_i^{*,n}(\tau, ds) - m_i^n(\tau, ds)) d\tau \leq \bar{C} |I| (\|\psi\|_\infty + \gamma_\psi e^{T \|b'\|_\infty}) \mathcal{W}(m^{*,n}(\tau), m^n(\tau)).$$

Using the previous inequality, equality (5.4.36) becomes:

$$\begin{aligned} \sum_{i \in I} \int_0^1 \psi_i(s) (m_i^{*,n}(t, ds) - m_i^n(t, ds)) &\leq 2\bar{C}|I|(\|\psi\|_\infty + \gamma_\psi e^{T\|b'\|_\infty}) \int_0^t \mathcal{W}(m^{*,n}(\tau), m^n(\tau)) d\tau \\ &\quad + C(\|\psi\|_\infty + \gamma_\psi e^{T\|b'\|_\infty})(\Delta t + \Delta s + \frac{1}{n\Delta s\Delta t}). \end{aligned}$$

Consider $\eta \in \text{Lip}([0, 1] \times I)$ with a Lipschitz constant equal to 1 and satisfying $\eta_i(0) = 0$ for a certain $i \in I$ (to bound $\|\eta\|_\infty$). Let $\{\psi^m\}_m$ be a sequence in $C^1([0, 1] \times I)$ that is uniformly bounded w.r.t. the Lipschitz norm and that approximates η w.r.t. the norm $\|\cdot\|_\infty$. Since the previous inequality holds for any function ψ in $C^1([0, 1] \times I)$, one deduces that:

$$\sum_{i \in I} \int_0^1 \eta_i(s) (m_i^{*,n}(t, ds) - m_i^n(t, ds)) \leq C \left(\int_0^t \mathcal{W}(m^{*,n}(\tau), m^n(\tau)) d\tau + \Delta t + \Delta s + \frac{1}{n\Delta s\Delta t} \right),$$

where $C > 0$ depends on \bar{C} and $|I|$. Since η is arbitrary, one has:

$$\mathcal{W}(m^{*,n}(t), m^n(t)) \leq C \left(\int_0^t \mathcal{W}(m^{*,n}(\tau), m^n(\tau)) d\tau + \Delta t + \Delta s + \frac{1}{n\Delta s\Delta t} \right).$$

Applying the Gronwall Lemma and usig inequality (5.2.12), there exists $C > 0$ such that one gets, for any $t \in (0, T]$,

$$\mathcal{W}(m^{*,n}(t), m^n(t)) \leq \frac{C}{n^{\frac{1}{3}}}.$$

□

Proof of Theorem 5.4.2

We need the following preliminary results.

Lemma 5.4.7. *There exists $C > 0$ such that, for any $n \in \mathbb{N}$, the function defined on $[0, T]$, by*

$$t \mapsto \sum_{i, j \in I, j \neq i} \int_0^1 L(\alpha_{i,j}^n(t, s)) m_i^{*,n}(t, ds) \quad (5.4.37)$$

has a total variation on $[0, T]$ lower than C .

Proof. For any $i, j \in I$ and $t_1, t_2 \in [0, T]$ with $t_2 > t_1$, we have:

$$\begin{aligned} &\left| \int_0^1 L(\alpha_{i,j}^n(t_2, s)) m_i^{*,n}(t_2, ds) - \int_0^1 L(\alpha_{i,j}^n(t_1, s)) m_i^{*,n}(t_1, ds) \right| \\ &\leq \left| \int_0^1 L(\alpha_{i,j}^n(t_2, s)) (m_i^{*,n}(t_2, ds) - m_i^{*,n}(t_1, ds)) \right| + \left| \int_0^1 (L(\alpha_{i,j}^n(t_1, s)) - L(\alpha_{i,j}^n(t_2, s))) m_i^{*,n}(t_1, ds) \right|. \end{aligned}$$

By (5.4.13) one has that $\{\alpha_{i,j}^n\}_i$ and $\{\partial_s \alpha_{i,j}^n\}_n$ are uniformly bounded. By Assumption 5, one gets that L is locally Lipschitz continuous and by [233, Theorem 2.1] that $m^{*,n} \in \text{Lip}([0, T], \mathcal{P}([0, 1] \times I))$, with a Lipschitz constant that only depends on $\|b\|_\infty, T$ and $\|\alpha^n\|_\infty$. Thus, there exists a constant $C > 0$ independent of n such that:

$$\sum_{i \in I} \left| \int_0^1 L(\alpha_{i,j}^n(t_2, s)) (m_i^{*,n}(t_2, ds) - m_i^{*,n}(t_1, ds)) \right| \leq C|t_2 - t_1|. \quad (5.4.38)$$

By (5.4.12), inequality 5.4.13 and the locally Lipschitz continuity of L , one gets

$$\begin{aligned} \left| \int_0^1 (L(\alpha_{i,j}^n(t_1, s)) - L(\alpha_{i,j}^n(t_2, s))) m_i^{*,n}(t_1, ds) \right| &\leq C \int_0^1 |\alpha_{i,j}^n(t_1, s) - \alpha_{i,j}^n(t_2, s)| m_i^{*,n}(t_1, ds) \\ &\leq C \left(|t_1 - t_2| + \int_{t_1}^{t_2} (\lambda_i^n + \lambda_j^n)(d\tau) \right). \end{aligned} \quad (5.4.39)$$

By inequalities 5.4.13, (5.4.38) and (5.4.39), we deduce that the total variation of the function defined in (5.4.37) on $[0, T]$ is bounded by a constant. \square

Lemma 5.4.8. *There exists $C > 0$ such that, for any $n \geq N$,*

$$J^n(m^n, E^n) \leq J(m^{*,n}, E^{*,n}) + \frac{C}{n^{\frac{1}{3}}}. \quad (5.4.40)$$

Proof of Lemma 5.4.8. By equality (5.4.15), one deduces, for any $k \in \{1, \dots, N_T^n - 1\}$, $i \in I$ and $p \in \{1, \dots, N_s^n - 1\}$ satisfying $m_i^n(t_k^-, [y_p, y_{p+1})) > 0$,

$$\left| \frac{E^n([t_k, t_{k+1}), [y_p, y_{p+1}))}{\Delta t m_i^n(t_k^-, [y_p, y_{p+1}))} - \alpha_{i,j}^n(t_k, y_p) \right| \leq \frac{1}{n \Delta t m_i^n(t_k^-, [y_p, y_{p+1}))}. \quad (5.4.41)$$

The function L being locally Lipschitz continuous, one has:

$$\begin{aligned} & \sum_{i,j} \sum_{k=1}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} L\left(\frac{E_{i,j}^n([t_k, t_{k+1}), [y_p, y_{p+1}))}{\Delta t m_i^n(t_k^-, [y_p, y_{p+1}))}\right) m_i^n(t_k^-, [y_p, y_{p+1})) \Delta t \\ & \leq \sum_{i,j} \sum_{k=0}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} L(\alpha_{i,j}^n(t_k, x_\ell)) m_i^n(t_k^-, [y_p, y_{p+1})) \Delta t + \frac{C}{n \Delta s \Delta t}, \end{aligned} \quad (5.4.42)$$

Now we compare $\sum_{i,j} \sum_{k=0}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} L(\alpha_{i,j}^n(t_k, y_p)) m_i^n(t_k^-, [y_p, y_{p+1})) \Delta t$ and $\sum_{i,j} \int_0^T \int_0^1 L(\alpha_{i,j}^n(t, s)) m_i^{*,n}(t, ds) dt$. Recall that, for any $i, j \in I$, $s \mapsto L(\alpha_{i,j}^n(t, s))$ is Lipschitz continuous on $[0, 1]$ uniformly in time. We have:

$$\begin{aligned} \sum_{i,j} \sum_{k=0}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} L(\alpha_{i,j}^n(t_k, y_p)) m_i^n(t_k^-, [y_p, y_{p+1})) \Delta t & \leq \sum_{i,j} \sum_{k=0}^{N_T^n - 1} \int_0^1 L(\alpha_{i,j}^n(t_k, s)) m_i^n(t_k^-, ds) \Delta t + C \Delta s \\ & \leq \sum_{i,j} \sum_{k=0}^{N_T^n - 1} \int_0^1 L(\alpha_{i,j}^n(t_k, s)) m_i^{*,n}(t_k^-, ds) \Delta t + \frac{C}{n^{\frac{1}{3}}}, \end{aligned} \quad (5.4.43)$$

where the last inequality is obtained by Theorem 5.4.3 and the fact that $\Delta s = \mathcal{O}(n^{-\frac{1}{3}})$. From Lemma 5.4.7, there exists $C > 0$ such that $t \mapsto \int_0^1 L(\alpha_{i,j}^n(t, s)) m_i^{*,n}(t, ds)$ has a total variation on $[0, T]$ lower than C . Applying classical results on approximation of Riemann sum of bounded total variation functions, we have, for any $i, j \in I$:

$$\sum_{k=0}^{N_T^n - 1} \int_0^1 L(\alpha_{i,j}^n(t_k, s)) m_i^{*,n}(t_k, ds) \Delta t \leq \int_0^T \int_0^1 L(\alpha_{i,j}^n(t, s)) m_i^{*,n}(t, ds) dt + \frac{C}{n^{\frac{1}{3}}}, \quad (5.4.44)$$

where we used that $\Delta t = \mathcal{O}(\frac{1}{n^{\frac{1}{3}}})$. By inequalities (5.4.42), (5.4.43) and (5.4.44), and by the definition of α ,

$$\begin{aligned} & \sum_{i,j} \sum_{k=1}^{N_T^n - 1} \sum_{p=0}^{N_s^n - 1} L\left(\frac{E_{i,j}^n([t_k, t_{k+1}), [y_p, y_{p+1}))}{\Delta t m_i^n(t_k^-, [y_p, y_{p+1}))}\right) m_i^n(t_k^-, [y_p, y_{p+1})) \Delta t \\ & \leq \int_0^T \int_0^1 L(\alpha_{i,j}^n(t, s)) m_i^{*,n}(t, ds) dt + \frac{C}{n^{\frac{1}{3}}} \\ & = \int_0^T \int_0^1 L\left(\frac{dE_{i,j}^{*,n}}{dm_i^{*,n}}(t, s)\right) m_i^{*,n}(t, ds) dt + \frac{C}{n^{\frac{1}{3}}}. \end{aligned}$$

From the previous inequality and Theorem 5.4.3, using that $g \in C^1([0, 1] \times I)$, $c \in C^1([0, T] \times [0, 1] \times I)$ and

the definition of J^n and J , one has:

$$J^n(m^n, E^n) \leq J(m^{*,n}, E^{*,n}) + \frac{C}{n^{\frac{1}{3}}}.$$

□

We now turn to the proof of Theorem 5.4.2.

Proof of Theorem 5.4.2. By [233, Proposition 7.1], the value of Problem (5.2.11) is Lipschitz continuous, with Lipschitz constant C_L , w.r.t. the initial distribution \bar{m} and the congestion constraint D . Then, since $(m^{*,n}, E^{*,n})$ is a minimizer of J over the set $\mathcal{S}(\bar{m}^n, D - C^*/n^{\frac{1}{3}})$, one has:

$$J(m^{*,n}, E^{*,n}) \leq \inf_{(m,E) \in \mathcal{S}(\bar{m}^n, D)} J(m, E) + \frac{C_L C^*}{n^{\frac{1}{3}}}.$$

By the previous inequality and Lemma 5.4.8:

$$J^n(m^n, E^n) \leq \inf_{(m,E) \in \mathcal{S}(\bar{m}^n, D)} J(m, E) + \frac{C + C_L C^*}{n^{\frac{1}{3}}}. \quad (5.4.45)$$

By Corollary 5.4.1, one has for any $i \in I$, $t \in [0, T]$ and $n \geq N$, that:

$$0 \leq D_i(t) - m_i^n(t, [0, 1]),$$

yielding that $(m^n, E^n) \in \mathcal{S}^n(\bar{m}^n, D)$. It follows from Corollary 5.3.1 and inequality (5.4.45) that there exist $N, C > 0$ such that, for any sequence $\{\bar{m}^n\}_n$ in $\mathcal{P}^n([0, 1] \times I)$ satisfying Assumptions 2 and 3, one has, for any $n > N$,

$$\inf_{(\mathbf{i}, \mathbf{s}) \in \mathcal{T}^n(\bar{m}^n, D)} \mathcal{J}^n(\mathbf{i}, \mathbf{s}) - \frac{C}{n^{\frac{1}{3}}} \leq \inf_{(m,E) \in \mathcal{S}(\bar{m}^n, D)} J(m, E). \quad (5.4.46)$$

One can consider a constant $C > 0$ large enough such that previous inequality holds for any $n \in \mathbb{N}^*$. □

5.4.3 Proof of Theorem 5.2.2

In this section we provide a proof of Theorem 5.2.2, based on the results obtained above in Section 5.4.

Proof of Theorem 5.2.2. The convergence of a subsequence of $\{(m^n, E^n)\}$ to (m^*, E^*) is given by Lemma 5.4.1. One obtains that (m^*, E^*) is optimal for Problem (5.2.11) by Theorem 5.2.1 and the proof of Theorem 5.4.1

□

Chapter 6

Decomposition of convex high dimensional aggregative stochastic control problems

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6.1 Introduction

The present article aims at solving a high dimensional stochastic control problem (P_1) involving a large number n of agents indexed by $i \in \{1, \dots, n\}$, of the form:

$$(P_1) \quad \begin{cases} \min_{u \in \mathcal{U}} J(u) \\ J(u) := \mathbb{E} \left[F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i(\omega^i, \omega^{-i}) \right) + \frac{1}{n} \sum_{i=1}^n G_i(u^i(\cdot, \omega^{-i}), \omega^i) \right]. \end{cases} \quad (6.1.1)$$

Here the noise $\omega := (\omega^1, \dots, \omega^n)$ belongs to $\Omega := \prod_{i=1}^n \Omega^i$, where $(\Omega^i, \mathcal{F}^i, \mu^i)$ is a probability space, and $(\Omega, \mathcal{F}, \mu)$ is the corresponding product probability space. Let $\omega^{-i} := (\omega^1, \dots, \omega^{i-1}, \omega^{i+1}, \dots, \omega^n)$ denote an

element of the space $\Omega^{-i} := \prod_{j=1, j \neq i}^n \Omega^j$. The associated product probability space is $(\Omega^{-i}, \mathcal{F}^{-i}, \mu^{-i})$, where $\mathcal{F}^{-i} := \otimes_{j=1, j \neq i}^n \mathcal{F}^j$ and $\mu^{-i} := \prod_{j=1, j \neq i}^n \mu^j$. Each decision variable u^i is a random variable (i.e. is \mathcal{F} -measurable), square summable with value in a Hilbert space \mathbb{U} so that $u := (u^1, \dots, u^n)$ belongs to $L^2(\Omega, (\mathbb{U})^n)$. The function $\omega^i \mapsto u^i(\omega^i, \omega^{-i})$ is denoted by $u^i(\cdot, \omega^{-i})$ and is a.s. (in ω^{-i}) \mathcal{F}^i -measurable and belongs to $L^2(\Omega^i, \mathbb{U})$. Also, $\mathcal{U} := \prod_{i=1}^n \mathcal{U}_i$ where \mathcal{U}_i is, for $i = 1$ to n , a closed convex subset of $L^2(\Omega, \mathbb{U})$. In the application to dynamical problems, the constraint $u^i \in \mathcal{U}_i$ includes the constraint of adaptation of u^i to some filtration. If each u^i is a random variable of ω^i , for $i = 1$ to n , we say that u is a decentralized decision variable.

The cost function is the sum of a *coupling term* $F_0 : \mathbb{U} \rightarrow \mathbb{R}$, function of the *aggregate strategies* $\frac{1}{n} \sum_{i=1}^n u^i$, and *local terms* functions of the local decision u^i and local noise ω^i with $G_i : L^2(\Omega^i, \mathbb{U}) \times \Omega^i \rightarrow \mathbb{R}$. This framework aims at containing stochastic optimal control problems, where the states of the agents are driven by independent noises (see equations (6.5.5) and (6.5.2) developed in Section 6.5).

6.1.1 Motivations

This work is motivated by its potential applications to distributed coordination of large populations of small agents, with relevant real-world implications in different sectors, from communication networks to power systems. The application developed in this paper deals with the coordination of flexible electrical appliances, to support power system operation in a context of increasing penetration of renewables. Among other appliances, thermostatically controlled loads (e.g. refrigerators, air conditioners etc.) have been investigated in the last few years, for their intrinsic flexibility and potential for network support. Several papers have already assessed the potential of demand-side response actions for frequency response services of TCLs [242] and how the population recovers from significant perturbations [69]. The coordination of TCLs can be performed in a centralized way, like in [137]. However, this approach raises concerns with respect to the communication requirements and customer privacy. A common objective can be reached in a fully distributed approach, like in [253], where each TCL is able to calculate its own actions (ON/OFF switching) to pursue a common objective. This paper is related to the work of De Paola *et al.* [84], where each agent represents a flexible TCL device. In [84] a distributed solution is presented for the operation of a population of $n = 2 \times 10^7$ refrigerators providing frequency support and load shifting. They adopt a game-theory framework, modelling the TCLs as price-responsive rational agents that schedule their energy consumption and allocate their frequency response provision in order to minimize their operational costs. The potential practical application of our work also considers a large population of TCLS which, as extension to [84], have stochastic dynamics. The proposed approach is able to minimize the overall system costs in a distributed way, with each TCL determining its optimal power consumption profile in response to price signals.

6.1.2 Related literature

The considered problem belongs to the class of stochastic control: looking for strategies minimizing the expectation of an objective function under specific constraints. One of the main approaches proposed in the literature to tackle this problem is to use random trees: this consists in replacing the almost sure constraints, induced by non-anticipativity, by a finite number of constraints, in order to get a finite set of scenarios (see [140] and [226]). Once the tree structure is built, the problem is solved by different decomposition methods such as scenario decomposition [221] or dynamic splitting [229]. The main objective of the scenario method is to reduce the problem to an approximated deterministic one. The present paper focuses on high dimensional noise problems with a large number of time steps, for which this approach is not feasible.

The idea of reducing a single high-dimensional problem to a large number of smaller problems with lower dimension has been widely studied in the deterministic case. In deterministic and stochastic problems there is the possibility of using time decomposition thanks to the Dynamic Programming Principle [28], taking advantage of the Markov property of the system. However, this method requires a specific time structure of the cost function and is not suitable for problems with high-dimensional state spaces. Under continuous linear-convex assumptions, one can deal with the curse of dimensionality by using the Stochastic Dual Dynamic Programming algorithm (SDDP) [209] to get upper and lower bounds of the value function, using polyhedral approximations. The almost-sure convergence of a broad class of SDDP algorithms has been proven [212], and complexity of the algorithm can be estimated, in the specific case of Lipschitz continuous value function [165] or by using a regularization of the value functions [277]. In [173, 203], a stopping criteria based on a dual version of SDDP,

which gives a deterministic upper-bound for the primal problem, is proposed. SDDP is well-adapted for medium sized population problems ($n \leq 30$), whereas it fails for problems with large populations ($n > 1000$) such as the ones considered in this paper. To tackle this type of high dimensional problems, it is natural to investigate decomposition techniques in the spirit of the Dual Approximation Dynamic Programming (DADP) [123, 172]. This approach is characterized by a price decomposition of the problem, where the stochastic constraints are projected on subspaces such that the associated Lagrangian multiplier is adapted for dynamic programming. Then the optimal multiplier is estimated by implementing Uzawa's algorithm. To this end in [172], the Uzawa's algorithm, formulated in a Hilbert setting, is extended to a Banach space. DADP has been applied in different cases, such as storage management problem for electrical production in [14, 123] and hydro valley management [59]. The idea of approaching the primal and dual problems by restricting or relaxing the set of decision variables has also been proposed in the context of stochastic programming [42, 160] to provide upper and lower bounds for the considered problem. In the proposed paper, in the same vein as DADP, we propose a price decomposition approach restricted to deterministic prices. This new approach takes advantage of the large population number in order to introduce an auxiliary problem where the coupling term is purely deterministic.

6.1.3 Contributions

The numerical difficulty of Problem (P_1) is related to the randomness of the aggregate term $\frac{1}{n} \sum_{i=1}^n u^i$ involved in the coupling function F_0 . Let us introduce the set of decentralized controls:

$$\hat{\mathcal{U}} := \prod_{i=1}^n \hat{\mathcal{U}}_i, \quad (6.1.2)$$

where $\hat{\mathcal{U}}_i := \{u^i \in \mathcal{U}_i \mid u^i \text{ is } \mathcal{T}^1 \otimes \dots \otimes \mathcal{F}^i \otimes \mathcal{T}^{i+1} \otimes \dots \otimes \mathcal{T}^n \text{ - measurable}\},$

where \mathcal{T}^j is the trivial σ -field $\{\emptyset, \Omega^j\}$. Note that by construction, we can identify $\hat{\mathcal{U}}_i$ with \mathcal{F}^i -measurable functions defined on Ω^i . In addition, two decentralized controls $u^i \in \hat{\mathcal{U}}_i$ and $u^j \in \hat{\mathcal{U}}_j$, $i \neq j$, are independent random variables. If the control u is decentralized, since the variance of the sum of independent random variables is equal to the sum of the variances of these random variables, the aggregate term $\frac{1}{n} \sum_{i=1}^n u^i$ can be approximated by $\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i$ when n is large enough. Let us consider then the following approximation of Problem (P_1):

$$(P_2) \quad \begin{cases} \min_{u \in \hat{\mathcal{U}}} \tilde{J}(u) \\ \tilde{J}(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i \right) + \frac{1}{n} \mathbb{E} \left[\sum_{i=1}^n G_i(u^i(\cdot, \omega^{-i}), \omega^i) \right]. \end{cases} \quad (6.1.3)$$

A first step consists in showing that, without loss of optimality in Problem (P_2), one can restrict the control set \mathcal{U} to $\hat{\mathcal{U}}$.

Theorem 6.2.1 states the equivalence between Problem (P_2) and its decentralized version (\hat{P}_2) defined by:

$$(\hat{P}_2) \quad \begin{cases} \min_{u \in \hat{\mathcal{U}}} \tilde{J}(u) \\ \tilde{J}(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i \right) + \frac{1}{n} \mathbb{E} \left[\sum_{i=1}^n G_i(u^i, \omega^i) \right]. \end{cases} \quad (6.1.4)$$

Through the article, the circumflex symbol $\hat{\cdot}$ will be used to denote minimization problems w.r.t. decentralized controls. Problem (\hat{P}_2) can be written as:

$$(\hat{P}_2') \quad \begin{cases} \min_{u \in \hat{\mathcal{U}}, v \in \mathbb{U}} \tilde{J}(u, v), \\ \tilde{J}(u, v) := F_0(v) + \frac{1}{n} \mathbb{E} \left[\sum_{i=1}^n G_i(u^i, \omega^i) \right], \\ \text{s.t. } g(u, v) = 0, \end{cases} \quad (6.1.5)$$

where $g: \mathcal{U} \times \mathbb{U} \rightarrow \mathbb{U}$ is defined by

$$g(u, v) := \frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i - v. \quad (6.1.6)$$

Observe that, for any $u^i \in \hat{\mathcal{U}}_i$, $G_i(u^i, \cdot)$ is independent of \mathcal{F}^{-i} . As a first contribution, this paper shows that under some convexity and regularity assumptions on F_0 and $(G_i)_{i \in \{1, \dots, n\}}$, any solution of Problem (P_2) is an ε_n -solution of (P_1) , with $\varepsilon_n \rightarrow 0$ when $n \rightarrow \infty$. In addition, we will see that an approach of price decomposition for (\hat{P}_2) , based on the formulation (\hat{P}'_2) , is tractable for dynamical problems, since the problem of minimizing the Lagrangian with deterministic dual variables can be decomposed in subproblems which are solvable by Dynamic Programming.

Since computing the dual cost of (\hat{P}'_2) is expensive, we propose *Stochastic Uzawa* and *Sampled Stochastic Uzawa* algorithms relying on the Robbins-Monroe algorithm, in the spirit of the stochastic gradient. Their convergence is established, relying on the proof provided by [119] for the convergence of the stochastic gradient in a Hilbert space. We check the effectiveness of the *Stochastic Uzawa* algorithm on a linear quadratic Gaussian framework, and we apply the *Sampled Stochastic Uzawa* algorithm to a model of power system, inspired by the work of A. De Paola *et al.* [84].

6.1.4 Assumptions

Various assumptions needed in the article are listed in this subsection.

- Assumption 1.** (i) Each set \mathcal{U}_i is bounded, i.e. there exists $M > 0$ such that $\mathbb{E}\|u^i\|_{\mathbb{U}}^2 \leq M^2$, for $i \in \{1, \dots, n\}$.
- (ii) The function $u^i \mapsto G_i(u^i(\cdot, \omega^{-i}), \omega^i)$ is a.s. non negative, convex and lower semi continuous (l.s.c. for short). And, for any $u^i \in \mathcal{U}_i$, the function $\omega \mapsto G_i(u^i(\cdot, \omega^{-i}), \omega^i)$ is measurable.
- (iii) The function F_0 is l.s.c. and proper.
- (iv) Problem (P_1) is feasible.

Assumption 2. The function F_0 is convex.

Assumption 3. The function F_0 is Gâteaux differentiable with c -Lipschitz derivative.

- Assumption 4.** (i) The function $u^i \mapsto G_i(u^i, \omega^i)$ is for a.a. $\omega^i \in \Omega^i$ strictly convex on $\hat{\mathcal{U}}_i$.
- (ii) The function F_0 has at least quadratic growth, i.e. there exist $C_1, C_2 > 0$ such that for any $v \in \mathbb{U}$:

$$C_1\|v\|_{\mathbb{U}}^2 - C_2 \leq F_0(v).$$

Remark 6.1.1. By Lemma 6.9.1 in Appendix 6.9, if F_0 satisfies Assumption 3 then, F_0 has at most quadratic growth, i.e. there exists $C > 0$ such that for any $v \in \mathbb{U}$ one has:

$$F_0(v) \leq C(\|v\|_{\mathbb{U}}^2 + 1).$$

We denote by $\{\rho_k\}_{k \in \mathbb{N}^*}$ the sequence of step sizes used in the *Stochastic Uzawa* and *Sampled Stochastic Uzawa* algorithms in Section 6.4.

- Assumption 5.** The sequence $\{\rho_k\}_{k \in \mathbb{N}^*}$ is such that: $\rho_k > 0$, $\sum_{k=1}^{\infty} \rho_k = \infty$ and $\sum_{k=1}^{\infty} (\rho_k)^2 < \infty$.

Note that a sequence of the form $\rho_k := \frac{a}{b+k}$, with $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+$, satisfies Assumption 5.

- Assumption 6.** (i) F_0 is strongly convex.
- (ii) There exists $\delta > 0$ such that for any $i \in \{1, \dots, n\}$ and for a.a. $\omega^i \in \Omega^i$, the function $\hat{\mathcal{U}}_i \ni u^i \mapsto G_i(u^i, \omega^i)$ is strongly convex with modulus of convexity greater or equal to δ .

Obviously, Assumption 6 is stronger than Assumption 4.

6.2 Approximating the optimization problem

In this section, the link between the values of problems (P_1) and (P_2) is analyzed.

Lemma 6.2.1. Let Assumptions 1 and 2 hold. Then Problem (P_1) has a solution, i.e. J reaches its minimum over \mathcal{U} .

Proof. The existence of a minimum is proved by considering a minimizing sequence (which exists since (P_1) is feasible) $\{u_k\}$ of J over \mathcal{U} . The set \mathcal{U} being bounded and weakly close, there exists a subsequence $\{u_{k_\ell}\}$ which weakly converges to a certain $u^* \in \mathcal{U}$. Using Assumptions 1.(ii) and convexity of F_0 , it follows that $\liminf J(u_{k_\ell}) \geq J(u^*)$ and thus u^* is a solution of (P_1) . \square

We obtain the following Corollary about (P_2) .

Corollary 6.2.1. *If Assumptions 1 and 2 are satisfied, then Problem (P_2) has a solution and its value is lower or equal to the value of Problem (P_1) i.e.:*

$$\inf_{u \in \mathcal{U}} \tilde{J}(u) \leq \inf_{u \in \mathcal{U}} J(u).$$

Proof. Assumption 1.(iv) and convexity of F_0 imply that (P_2) is feasible. By using the same techniques as in the proof of Lemma 6.2.1, one can prove that (P_2) admits a solution. Using the convexity of F_0 and Jensen's inequality, one has for any centralized control $u \in \mathcal{U}$:

$$F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i\right) \leq \mathbb{E}\left[F_0\left(\frac{1}{n} \sum_{i=1}^n u^i\right)\right],$$

and the conclusion follows from the definition of (P_1) in (6.1.1) and (P_2) in (6.1.3) \square

We have the following key result.

Theorem 6.2.1. *If Assumption 1 is satisfied, then the decentralized Problem (\hat{P}_2) has the same value as the centralized Problem (P_2) i.e.:*

$$\inf_{u \in \hat{\mathcal{U}}} \tilde{J}(u) = \inf_{u \in \mathcal{U}} \tilde{J}(u). \quad (6.2.1)$$

Proof. Since $\hat{\mathcal{U}} \subset \mathcal{U}$, it is immediate that $\inf_{u \in \hat{\mathcal{U}}} \tilde{J}(u) \leq \inf_{u \in \mathcal{U}} \tilde{J}(u)$.

Fix $i \in \{1, \dots, n\}$, using the definition of conditional expectation, we define $\tilde{u}^i \in L^2(\Omega^i, \mathbb{U})$ for any $u^i \in \mathcal{U}_i$ by:

$$\tilde{u}^i(\omega^i) := \mathbb{E}[u^i(\omega^i, \omega^{-i}) | \omega^i] = \int_{\Omega^{-i}} u^i(\omega^i, \omega^{-i}) d\mu^{-i}(\omega^{-i}) \quad \text{for any } \omega^i \in \Omega^i.$$

Since G_i is a.s. convex w.r.t. the first variable, Jensen's inequality gives:

$$G_i(\tilde{u}^i, \omega^i) \leq \int_{\Omega^{-i}} G_i(u^i(\cdot, \omega^{-i}), \omega^i) d\mu^{-i}(\omega^{-i}) = \mathbb{E}[G_i(u^i(\cdot, \omega^{-i}), \omega^i) | \omega^i] \quad \text{a.s.} \quad (6.2.2)$$

On the other hand $(u^1, \dots, u^n) \mapsto F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i\right)$ is invariant when taking the conditional expectation, thus:

$$F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i\right) = F_0\left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} \tilde{u}^i\right).$$

Taking the expectation of (6.2.2), we have $\inf_{u \in \hat{\mathcal{U}}} \tilde{J}(u) \leq \inf_{u \in \mathcal{U}} \tilde{J}(u)$, and the conclusion follows. \square

Remark 6.2.1. *In the applications to stochastic control problems (in discrete and continuous time) we have the constraint of having progressively measurable control policies. Since the set of progressively measurable policies is closed and convex, this enters in the above framework. In particular, the decentralized policy \tilde{u}^i constructed in the above proof is progressively measurable.*

Remark 6.2.2. *By Theorem 6.2.1, for any $\varepsilon > 0$ there exists an ε -optimal solution of Problem (P_2) that is a decentralized control.*

Before stating the next result, we need to introduce Problem (\hat{P}_1) , that corresponds to the minimization of J , defined in (6.1.1), over the set of decentralized controls $\hat{\mathcal{U}}$:

$$(\hat{P}_1) \left\{ \min_{u \in \hat{\mathcal{U}}} J(u). \right.$$

Proposition 6.2.1. *If Assumption 1 is satisfied and F_0 is Lipschitz continuous with constant γ , then any solution of (\hat{P}_2) is an ε -optimal solution of (\hat{P}_1) and, conversely, any solution of (\hat{P}_1) is an ε -optimal solution of (\hat{P}_2) , with $\varepsilon = 2\gamma M/\sqrt{n}$.*

Proof. Since F_0 is Lipschitz continuous with Lipschitz constant γ , it holds for any $x, y \in \mathbb{U}$: $|F_0(x) - F_0(y)| \leq \gamma \|x - y\|_{\mathbb{U}}$. We set for any $u \in \mathcal{U}$:

$$\mathbf{u}^i := u^i - \mathbb{E} u^i. \quad (6.2.3)$$

Using the Lipschitz continuity of F_0 , one has for any $u \in \hat{\mathcal{U}}$:

$$\begin{aligned} \left| \mathbb{E} \left[F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) - F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i \right) \right] \right| &\leq \mathbb{E} \left| F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) - F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i \right) \right| \\ &\leq \frac{\gamma}{n} \mathbb{E} \left\| \sum_{i=1}^n \mathbf{u}^i \right\|_{\mathbb{U}}. \end{aligned}$$

Using the Jensen's inequality, for any $u \in \hat{\mathcal{U}}$, the mutual independence of the centered variables \mathbf{u}_i and \mathbf{u}_j for any $j \neq i$ and $\mathbb{E} \|\mathbf{u}_i\|_{\mathbb{U}}^2 \leq M^2$, we get:

$$\frac{\gamma}{n} \mathbb{E} \left\| \sum_{i=1}^n \mathbf{u}^i \right\|_{\mathbb{U}} \leq \frac{\gamma}{n} \mathbb{E} \left[\left\| \sum_{i=1}^n \mathbf{u}^i \right\|_{\mathbb{U}}^2 \right]^{\frac{1}{2}} \leq \frac{\gamma}{n^{\frac{1}{2}}} M. \quad (6.2.4)$$

Let \hat{u} denote a minimizer of (\hat{P}_2) , then using (6.2.4) for the first and last inequality, for any $u \in \hat{\mathcal{U}}$ it holds:

$$J(\hat{u}) \leq \tilde{J}(\hat{u}) + \frac{\gamma}{n^{\frac{1}{2}}} M \leq \tilde{J}(u) + \frac{\gamma}{n^{\frac{1}{2}}} M \leq J(u) + \frac{2\gamma}{n^{\frac{1}{2}}} M. \quad (6.2.5)$$

Similarly, if u^* is a solution of (\hat{P}_1) , then for any $u \in \hat{\mathcal{U}}$ one has:

$$\tilde{J}(u^*) \leq J(u^*) + \frac{\gamma}{n^{\frac{1}{2}}} M \leq J(u) + \frac{\gamma}{n^{\frac{1}{2}}} M \leq \tilde{J}(u) + \frac{2\gamma}{n^{\frac{1}{2}}} M. \quad (6.2.6)$$

□

Theorem 6.2.2. *Let Assumptions 1, 2 and 3 be satisfied. Then any solution of Problem (\hat{P}_2) is an ε -optimal solution (where $\varepsilon = cM^2/n$) of Problem (P_1) .*

Proof. From Corollary 6.2.1 and Theorem 6.2.1, one has for any $\hat{u} \in \hat{\mathcal{U}}$ solution of (\hat{P}_2) that:

$$\tilde{J}(\hat{u}) \leq \inf_{u \in \hat{\mathcal{U}}} J(u). \quad (6.2.7)$$

Since F_0 is convex, differentiable, with a c -Lipschitz derivative, one can derive a.s.:

$$\begin{aligned} &F_0 \left(\frac{1}{n} \sum_{i=1}^n \hat{u}^i \right) - F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} \hat{u}^i \right) \\ &\leq \frac{1}{n} \langle \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n \hat{u}^i \right), \sum_{i=1}^n \mathbf{u}^i \rangle_{\mathbb{U}} \\ &= \frac{1}{n} \langle \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n \hat{u}^i \right) - \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} \hat{u}^i \right), \sum_{i=1}^n \mathbf{u}^i \rangle_{\mathbb{U}} + \frac{1}{n} \langle \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} \hat{u}^i \right), \sum_{i=1}^n \mathbf{u}^i \rangle_{\mathbb{U}} \\ &\leq \frac{c}{n^2} \left\| \sum_{i=1}^n \mathbf{u}^i \right\|_{\mathbb{U}}^2 + \frac{1}{n} \langle \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} \hat{u}^i \right), \sum_{i=1}^n \mathbf{u}^i \rangle_{\mathbb{U}}, \end{aligned} \quad (6.2.8)$$

where \mathbf{u}^i is defined from \hat{u} as in (6.2.3). From the definition of \mathbf{u}^i , one obtains:

$$\mathbb{E} \left[\left\langle \nabla F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} \hat{u}^i \right), \sum_{i=1}^n \mathbf{u}^i \right\rangle_{\mathbb{U}} \right] = 0.$$

Since $\hat{u} \in \hat{\mathcal{U}}$, controls are mutually independent and bounded a.s. by M , one gets as in (6.2.4):

$$\frac{c}{n^2} \mathbb{E} \left\| \sum_{i=1}^n \mathbf{u}^i \right\|_{\mathbb{U}}^2 \leq \frac{c}{n} M^2. \quad (6.2.9)$$

Taking the expectation of the first and last terms of (6.2.8) and then incorporating (6.2.9) and the equality above, one obtains:

$$J(\hat{u}) - \tilde{J}(\hat{u}) \leq \frac{c}{n} M^2.$$

From previous inequality and (6.2.7), we get:

$$J(\hat{u}) \leq \inf_{u \in \mathcal{U}} J(u) + \frac{c}{n} M^2. \quad (6.2.10)$$

□

Remark 6.2.3. Observe that the value of the centralized Problem (P_1) on the l.h.s. of the inequality (6.2.11) below is upper bounded by the following decentralized problem on the r.h.s of this inequality i.e.

$$\inf_{u \in \mathcal{U}} J(u) \leq \inf_{u \in \hat{\mathcal{U}}} J(u). \quad (6.2.11)$$

Ref. [60] obtains an upper bound for the decentralized problem and a lower bound for the centralized problem. The upper bound is provided by a resource decomposition approach (with deterministic quantities) while the lower bound is obtained by a price decomposition approach with deterministic prices (see Equation (28) of [60]). Theorem 6.2.2 provides an upper bound for Problem (P_1) with an a priori quantification of the deviation from the optimal value which vanishes when the number of agents grows to infinity. Moreover, in Section 6.4 we provide an original algorithm that allows to approach the solution of the decentralized problem.

Remark 6.2.4. Let \hat{u} and u^* be respectively the optimal solutions of problems (\hat{P}_2) and (P_1). From Jensen's inequality and by definition of \hat{u} we have:

$$-J(u^*) \leq -\tilde{J}(u^*) \leq -\tilde{J}(\hat{u}).$$

Adding $J(\hat{u})$, one has:

$$0 \leq J(\hat{u}) - J(u^*) \leq J(\hat{u}) - \tilde{J}(u^*) \leq J(\hat{u}) - \tilde{J}(\hat{u}). \quad (6.2.12)$$

Inequality (6.2.12) allows to compute an upper bound of the "optimality" error $J(\hat{u}) - J(u^*)$, by evaluating $J(\hat{u}) - \tilde{J}(\hat{u})$.

6.3 Dualization and Decentralization of Problem (P_2)

The *Lagrangian* function associated to the constrained optimization Problem (\hat{P}'_2), defined in (6.1.5), is: $L : \hat{\mathcal{U}} \times \mathbb{U} \times \mathbb{U} \rightarrow \bar{\mathbb{R}}$ defined by:

$$L(u, v, \lambda) := \tilde{J}(u, v) + \langle \lambda, \frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i - v \rangle_{\mathbb{U}}. \quad (6.3.1)$$

The dual Problem (D) associated with (\hat{P}'_2) is:

$$(D) \quad \max_{\lambda \in \mathbb{U}} \mathcal{W}(\lambda), \quad \text{where } \mathcal{W}(\lambda) := \min_{u \in \hat{\mathcal{U}}, v \in \mathbb{U}} L(u, v, \lambda). \quad (6.3.2)$$

For any $\lambda \in \mathbb{U}$, it holds:

$$\mathcal{W}(\lambda) = -F_0^*(\lambda) + \frac{1}{n} \sum_{i=1}^n \min_{u^i \in \hat{\mathcal{U}}_i} \mathbb{E}[G_i(u^i, \omega^i)] + \langle \lambda, \mathbb{E} u^i \rangle_{\mathbb{U}}, \quad (6.3.3)$$

where, for any real valued function F defined on \mathbb{U} , F^* stands for its Fenchel conjugate, defined for $x \in \mathbb{U}$ by $F^*(x) := \sup_{y \in \mathbb{U}} \langle x, y \rangle_{\mathbb{U}} - F(y)$. The problem is said to be qualified if it is still feasible after a small perturbation of the constraint, in the following sense:

$$\text{There exists } \varepsilon > 0 \text{ such that } \mathcal{B}_{\mathbb{U}}(0, \varepsilon) \subset g(\hat{\mathcal{U}}, \mathbb{U}), \quad (6.3.4)$$

where $\mathcal{B}_{\mathbb{U}}(0, \varepsilon)$ is the open ball of radius ε in \mathbb{U} , g has been defined in (6.1.6) and $g(\hat{\mathcal{U}}, \mathbb{U})$ is the image by g of $\hat{\mathcal{U}} \times \mathbb{U}$.

Lemma 6.3.1. *If Assumption 1 holds, then Problem (\hat{P}'_2) is qualified. If Assumption 2 is also satisfied, then problems (\hat{P}'_2) and (D) have the same value, the set of dual solutions S is nonempty and bounded and any primal solution \hat{u} satisfies both $W(\hat{\lambda}) = \tilde{J}(\hat{u})$ and $(\hat{u}, \hat{v}) \in \arg \min_{u \in \hat{\mathcal{U}}, v \in \mathbb{U}} L(\hat{\lambda}, u, v)$, with $\hat{\lambda} \in S$.*

Proof. By Assumption 1.(iv), there exists \tilde{u} feasible for Problem (P_1) . Then using the definition of g in (6.1.6)

$$\mathcal{B}_{\mathbb{U}}(0, \varepsilon) \subset \mathbb{U} = g(\tilde{u}, \mathbb{U}) \subset g(\hat{\mathcal{U}}, \mathbb{U}). \quad (6.3.5)$$

The qualification of (\hat{P}'_2) follows. The conclusion follows by [34, Theorem 2.165]. \square

Since the set of admissible controls $\hat{\mathcal{U}} = \hat{\mathcal{U}}_1 \times \dots \times \hat{\mathcal{U}}_n$ is a Cartesian product, if G_i is strictly convex with respect to its first variable, then each component \hat{u}^i of the solution \hat{u} of Problem (\hat{P}'_2) , can be uniquely determined by solving the following subproblem:

$$\hat{u}^i = \arg \min_{u^i \in \hat{\mathcal{U}}_i} \left\{ \mathbb{E} \left[G_i(u^i, \omega^i) + \langle \hat{\lambda}, u^i \rangle_{\mathbb{U}} \right] \right\},$$

where $\hat{\lambda} \in S$.

Remark 6.3.1. *By using the same argument as in Theorem 6.2.1, one can prove, for any $\lambda \in \mathbb{U}$:*

$$\begin{aligned} & \min_{u^i \in \hat{\mathcal{U}}_i} \left\{ \mathbb{E} \left[G_i(u^i, \omega^i) + \langle \lambda, u^i \rangle_{\mathbb{U}} \right] \right\} \\ &= \min_{u^i \in \hat{\mathcal{U}}_i} \left\{ \mathbb{E} \left[G_i(u^i(\cdot, \omega^{-i}), \omega^i) + \langle \lambda, u^i \rangle_{\mathbb{U}} \right] \right\}. \end{aligned} \quad (6.3.6)$$

6.4 Stochastic Uzawa and Sampled Stochastic Uzawa algorithms

This section aims at proposing an algorithm to find a solution of the dual problem (6.3.2).

6.4.1 Preliminary results

Though the below result is well-known and can be found in [141] for functions defined on finite vector spaces, we adapt the proof to the Hilbert space setting for the sake of completeness.

Lemma 6.4.1. *If Assumption 1 and 2 are satisfied, then Assumption 3 holds iff F_0^* is strongly convex.*

Proof. (i) Let Assumption 3 holds. Since F_0 is proper, convex and l.s.c., F_0^* is l.s.c. proper. From the Lipschitz property of the gradient of F_0 , it holds that $\text{dom}(F_0) = \mathbb{U}$.

Let $s, \tilde{s} \in \text{dom}(F_0^*)$ such that there exist $\lambda_s \in \partial F_0^*(s)$ and $\lambda_{\tilde{s}} \in \partial F_0^*(\tilde{s})$. From the differentiability, l.s.c. and convexity of F_0 , it follows that: $s = \nabla F_0(\lambda_s)$ and $\tilde{s} = \nabla F_0(\lambda_{\tilde{s}})$. By Assumption 3 and the extended Baillon-Haddad theorem [210, Theorem 3.1], ∇F_0 is cocoercive. In other words:

$$\begin{aligned} \langle s - \tilde{s}, \lambda_s - \lambda_{\tilde{s}} \rangle_{\mathbb{U}} &= \langle \nabla F_0(\lambda_s) - \nabla F_0(\lambda_{\tilde{s}}), \lambda_s - \lambda_{\tilde{s}} \rangle_{\mathbb{U}} \\ &\geq \frac{1}{c} \|\nabla F_0(\lambda_s) - \nabla F_0(\lambda_{\tilde{s}})\|_{\mathbb{U}}^2 \\ &= \frac{1}{c} \|s - \tilde{s}\|_{\mathbb{U}}^2, \end{aligned} \quad (6.4.1)$$

where c is the Lipschitz constant of ∇F_0 defined in Assumption 3. Therefore ∂F_0^* is strongly monotone, which implies the strong convexity of F_0^* .

(ii) Conversely, assume that F_0^* is proper and strongly convex. Then there exist $\alpha, \beta > 0$ and $\gamma \in \mathbb{U}$ such that for any $s \in \text{dom}(F_0^*)$: $F_0^*(s) \geq \alpha \|s\|_{\mathbb{U}}^2 + \langle \gamma, \alpha \rangle_{\mathbb{U}} - \beta$, and F_0 being convex, l.s.c. and proper, for any $\lambda \in \mathbb{U}$ it holds:

$$F_0(\lambda) \leq \sup_{s \in \mathbb{U}} \langle s, \lambda - \gamma \rangle_{\mathbb{U}} - \alpha \|s\|_{\mathbb{U}}^2 + \beta = \|\lambda - \gamma\|^2 / (4\alpha) + \beta. \quad (6.4.2)$$

Thus, F_0 is proper and uniformly upper bounded over bounded sets and therefore is locally Lipschitz. In addition, from the strong convexity of F_0^* and the convexity of F_0 , for any $\lambda \in \mathbb{U}$, $\partial F_0(\lambda)$ is a singleton. Thus F_0 is everywhere Gâteaux differentiable.

Let $\lambda, \mu \in \mathbb{U}$. Since F_0^* is strongly convex, the functions $F_0^*(s) - \langle \lambda, s \rangle_{\mathbb{U}}$ (resp. $F_0^*(s) - \langle \mu, s \rangle_{\mathbb{U}}$) has a unique minimum point s_λ (resp. s_μ), characterized by: $\lambda \in \partial F_0^*(s_\lambda)$ and $\mu \in \partial F_0^*(s_\mu)$. From the strong convexity of F_0^* , the strong monotonicity of ∂F_0^* holds: $\langle \mu - \lambda, s_\mu - s_\lambda \rangle_{\mathbb{U}} \geq \frac{1}{c} \|s_\mu - s_\lambda\|_{\mathbb{U}}^2$, where $c > 0$ is a constant related to the strong convexity of F_0^* . Using that $s_\lambda = \nabla F_0(\lambda)$ and $s_\mu = \nabla F_0(\mu)$, it holds:

$$\langle \mu - \lambda, \nabla F_0(\mu) - \nabla F_0(\lambda) \rangle_{L^2(0,T)} \geq \frac{1}{c} \|\nabla F_0(\mu) - \nabla F_0(\lambda)\|_{L^2(0,T)}^2, \quad (6.4.3)$$

meaning that ∇F_0 is cocoercive. Applying the Cauchy–Schwarz inequality to the left hand side of the previous inequality, the Lipschitz property of ∇F_0 follows. \square

Lemma 6.4.2. *If Assumptions 1, 2 and 3 hold, then \mathcal{W} is strongly concave.*

Proof. For any $\lambda \in \mathbb{U}$, the expression of $\mathcal{W}(\lambda)$ is given by (6.3.3), where for any $i \in \{1, \dots, n\}$, $\lambda \mapsto \inf_{u^i \in \mathcal{U}_i} \mathbb{E} G_i(u^i, \omega^i) + \langle \lambda, \mathbb{E} u^i \rangle_{\mathbb{U}}$ is concave and, from Lemma 6.4.1, $-F_0^*$ is strongly concave. Since the sum of a concave function and of a strongly concave function is strongly concave, the result follows. \square

6.4.2 Analysis of the algorithms

Assumptions 1, 2 and 4 are supposed to hold throughout Section 6.4.2. For all $i \in \{1, \dots, n\}$, and $\lambda \in \mathbb{U}$, we define the optimal control $u^i(\lambda)$:

$$u^i(\lambda) := \arg \min_{u^i \in \mathcal{U}_i} \left\{ \mathbb{E} [G_i(u^i, \omega^i) + \langle \lambda, u^i \rangle_{\mathbb{U}}] \right\}, \quad (6.4.4)$$

which is well defined since $u^i \rightarrow \mathbb{E} G_i(u^i, \omega^i)$ is strictly convex.

For any $\lambda \in \mathbb{U}$, the subset $V(\lambda)$ is defined by:

$$V(\lambda) := \arg \min_{v \in \mathbb{U}} \{F_0(v) - \langle \lambda, v \rangle_{\mathbb{U}}\}. \quad (6.4.5)$$

Since F_0 is convex and has at least quadratic growth, $V(\lambda)$ is a non empty subset of \mathbb{U} and is reduced to a singleton if F_0 is strictly convex. For any $\lambda \in \mathbb{U}$, we denote by $v(\lambda)$ an element of $V(\lambda)$, and for any $v(\lambda) \in V(\lambda)$, one has $v(\lambda) \in \partial F_0^*(\lambda)$.

Uzawa's algorithm seems particularly fitting for this problem. However, at each dual iteration k and any $i \in \{1, \dots, n\}$, one would have to compute the quantities $\mathbb{E}[u^i(\lambda^k)]$ for the update of λ^{k+1} , which is hard in practice. Therefore two algorithms are proposed where, at each iteration k , λ^{k+1} is updated thanks to a realization of $u^i(\lambda^k)$.

We introduce the function $f : \mathbb{U} \rightarrow \mathbb{U}$ defined by:

$$f(\lambda) := g(u(\lambda), v(\lambda)) = \frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i(\lambda) - v(\lambda). \quad (6.4.6)$$

By Assumption 4.(ii), F_0 has at least quadratic growth, whence F_0^* has at most quadratic growth. Indeed, using the definition of the Fenchel's conjugate, the fact that F_0 has at least quadratic growth, and the Cauchy-Schwarz inequality, there exist $\bar{C}_1, \bar{C}_2 > 0$ such that for any $\lambda \in \mathbb{U}$:

$$F_0^*(\lambda) \leq \sup_{\mu \in \mathbb{U}} \|\mu\|_{\mathbb{U}} \|\lambda\|_{\mathbb{U}} - \bar{C}_1 \|\mu\|_{\mathbb{U}}^2 + \bar{C}_2 = \frac{\|\lambda\|_{\mathbb{U}}^2}{2\bar{C}_1} + \bar{C}_2.$$

Then using Lemma 6.9.1 in Appendix 6.9 and that $v(\lambda) \in \partial F_0^*(\lambda)$, there exists $C > 0$ such that for any $\lambda \in \mathbb{U}$:

$$\|v(\lambda)\|_{\mathbb{U}} \leq C(\|\lambda\|_{\mathbb{U}} + 1). \quad (6.4.7)$$

Using the definition of \mathcal{U} , one has $\frac{1}{n} \sum_{i=1}^n \|\mathbb{E} u^i(\lambda)\|_{\mathbb{U}} \leq M$ for any $\lambda \in \mathbb{U}$. Therefore, from the definition of f in (6.4.6), there exist $M_1, M_2 > 0$ such that for any $\lambda \in \mathbb{U}$ one has:

$$\|f(\lambda)\|_{\mathbb{U}}^2 \leq M_1 + M_2 \|\lambda\|_{\mathbb{U}}^2. \quad (6.4.8)$$

For any $\lambda \in \mathbb{U}$, we denote by $\partial(-\mathcal{W}(\lambda))$ the subgradient of $-\mathcal{W}$ at λ . Therefore, for any $\lambda \in \mathbb{U}$:

$$\partial(-\mathcal{W}(\lambda)) \ni -f(\lambda). \quad (6.4.9)$$

The iterative algorithm, proposed as an approximation scheme for $\lambda^* \in \arg \max_{\lambda} \mathcal{W}(\lambda)$, is summarized in the *Stochastic Uzawa* Algorithm 1.

Algorithm 1 Stochastic Uzawa

- 1: Initialization $\lambda^0 \in \mathbb{U}$, set $\{\rho_k\}_{k \in \mathbb{N}^*}$ satisfying Assumption 5.
 - 2: $k \leftarrow 0$.
 - 3: **for** $k = 0, 1, \dots$ **do**
 - 4: $v^k \leftarrow v(\lambda^k)$ where $v(\lambda^k) \in V(\lambda^k)$, this set being defined in (6.4.5).
 - 5: $u^{i,k} \leftarrow u^i(\lambda^k)$ where $u^i(\lambda^k)$ is defined in (6.4.4) for any $i \in \{1, \dots, n\}$.
 - 6: Generate n independent noises $(\omega^{1,k+1}, \dots, \omega^{n,k+1})$, independent also of $\{\omega^{i,p} : 1 \leq i \leq n, p \leq k\}$.
 - 7: Compute the associated control realization $(u^{1,k}(\omega^{1,k+1}), \dots, u^{n,k}(\omega^{n,k+1}))$.
 - 8: $Y^{k+1} \leftarrow \frac{1}{n} \sum_{i=1}^n u^{i,k}(\omega^{i,k+1}) - v^k$.
 - 9: $\lambda^{k+1} \leftarrow \lambda^k + \rho_k Y^{k+1}$.
-

At any dual iteration k of Algorithm 1, Y^{k+1} is an estimator of $\mathbb{E}[\frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(\omega^{i,k+1}) - v(\lambda^k)]$. An alternative approach, proposed in the *Sampled Stochastic Uzawa* Algorithm 2, consists in performing less simulations at each iteration, by taking $m < n$, at the risk of performing more dual iterations, to estimate the quantity $\mathbb{E}[\frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(\omega^{i,k+1}) - v(\lambda^k)]$.

Algorithm 2 Sampled Stochastic Uzawa

- 1: Initialization of m a positive integer and $\check{\lambda}^0 \in \mathbb{U}$, set $\{\rho_k\}_{k \in \mathbb{N}^*}$ satisfying Assumption 5.
 - 2: $k \leftarrow 0$.
 - 3: **for** $k = 0, 1, \dots$ **do**
 - 4: $v^k \leftarrow v(\check{\lambda}^k)$ where $v(\check{\lambda}^k) \in V(\check{\lambda}^k)$, this set being defined in (6.4.5).
 - 5: Generate m i.i.d. discrete random variables I_1^k, \dots, I_m^k uniformly in $\{1, \dots, n\}$.
 - 6: $u^{I_j^k, k} \leftarrow u^{I_j^k}(\check{\lambda}^k)$ where $u^{I_j^k}(\check{\lambda}^k)$ is defined in (6.4.4) for any $j \in \{1, \dots, m\}$.
 - 7: Generate m independent noises $(\omega^{1,k+1}, \dots, \omega^{m,k+1})$, independent also of $\{\omega^{i,p} : 1 \leq i \leq m, p \leq k\}$.
 - 8: Compute the associated control realization $(u^{I_1^k, k}(\omega^{1,k+1}), \dots, u^{I_m^k, k}(\omega^{m,k+1}))$.
 - 9: $\check{Y}^{k+1} \leftarrow \frac{1}{m} \sum_{j=1}^m u^{I_j^k, k}(\omega^{I_j^k, k+1}) - v^k$
 - 10: $\check{\lambda}^{k+1} \leftarrow \check{\lambda}^k + \rho_k \check{Y}^{k+1}$.
-

The complexity of the *Sampled Stochastic Uzawa* Algorithm 2 is proportional to $m \times K$, where K is the total number of dual iterations and m the number of simulations performed at each iteration. The error $\mathbb{E} \|\lambda^{k+1} - \lambda^*\|_{\mathbb{U}}^2$ for $\lambda^* \in S$ (we recall that S is defined by $S := \arg \max_{\lambda \in \mathbb{U}} \mathcal{W}(\lambda)$ and from Lemma 6.3.1 S is non empty) is the sum of the square of the bias (which only depends on K and not on m) and the variance (which

both depends on K and m). Therefore, this algorithm enables a bias variance trade-off for a given complexity. Similarly, for a given error, it enables to optimize the complexity of the algorithm.

The following result establishes the convergence of the *Stochastic Uzawa Algorithm 3*:

Lemma 6.4.3. *Let Assumptions 1, 2, 4 and 5 hold and let $\{\lambda_k\}_k$ be a sequence of multipliers generated by Algorithm 3. Then:*

- (i) $\{\|\lambda^k - \lambda\|_{\mathbb{U}}^2\}$ converges a.s., for all $\lambda \in S$.
- (ii) $\mathcal{W}(\lambda^k) \xrightarrow[k \rightarrow \infty]{} \max_{\lambda \in \mathbb{U}} \mathcal{W}(\lambda)$ a.s.
- (iii) $\{\lambda^k\}$ weakly converges to some $\bar{\lambda} \in S$ in \mathbb{U} a.s.
- (iv) If Assumption 3 holds, then a.s. $\{\lambda^k\}$ converges to $\bar{\lambda}$ in \mathbb{U} , with $S := \{\bar{\lambda}\}$.

The proof follows from [119, Theorem 3.6]. The cited reference (changing minimization in maximization) is interested in the maximization of a function \mathcal{W} of the specific form $\mathcal{W}(\lambda) = \mathbb{E} \mathbf{W}(\lambda, \omega)$, where $\mathbf{W}(\cdot, \omega)$ is concave a.s. in ω . However, in our setting we cannot in general exhibit such a representation for the dual function \mathcal{W} , defined in (6.3.2). Using the definition of $u(\lambda)$ in (6.4.4) and $v(\lambda)$ in (6.4.5), we have $\mathcal{W}(\lambda) = \mathbb{E} \mathbf{W}(\lambda, \omega)$, where

$$\lambda \mapsto \mathbf{W}(\lambda, \omega) := F_0(v(\lambda)) + \frac{1}{n} \sum_{i=1}^n G_i(u^i(\lambda), \omega^i) + \langle \lambda, \frac{1}{n} \sum_{i=1}^n u^i(\lambda) - v(\lambda) \rangle_{\mathbb{U}}$$

Note that $\mathbf{W}(\cdot, \omega)$ is not a concave function of λ for a.a. $\omega \in \Omega$. Although our setting does not enter in the framework considered in [119], the proof of Lemma 6.4.3 follows from an obvious adaptation of the one in [119, Theorem 3.6]. It is enough to provide the first steps of the proof.

Proof of Lemma 6.4.3. First consider point (i). Let $\lambda \in S$. For any k , \mathcal{G}_{k+1} is the filtration defined by:

$$\mathcal{G}_{k+1} := \sigma(\{\omega^{i,p}\} : 1 \leq i \leq n, p \leq k+1). \quad (6.4.10)$$

Using the definition of $Y^{k+1} \in \mathbb{U}$ line 8 in the *Stochastic Uzawa Algorithm 1*, we have:

$$\begin{aligned} \|\lambda^{k+1} - \lambda\|_{\mathbb{U}}^2 &= \|\lambda^k + \rho_k Y^{k+1} - \lambda\|_{\mathbb{U}}^2 \\ &= \|\lambda^k - \lambda\|_{\mathbb{U}}^2 + 2\rho_k \langle \lambda^k - \lambda, Y^{k+1} \rangle_{\mathbb{U}} \\ &\quad + (\rho_k)^2 \|Y^{k+1}\|_{\mathbb{U}}^2. \end{aligned} \quad (6.4.11)$$

Since Y^{k+1} is independent from \mathcal{G}_k , it follows that:

$$\mathbb{E}[\|Y^{k+1}\|_{\mathbb{U}}^2 | \mathcal{G}_k] = \mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^n u^i(\lambda^k)(\omega^{i,k+1}) - v(\lambda^k) \right\|_{\mathbb{U}}^2. \quad (6.4.12)$$

Using previous equality and the inequality (6.4.8), one can easily show that there exists $M_3, M_4 > 0$ such that, for any $k \in \mathbb{N}$, one has:

$$\mathbb{E}[\|Y^{k+1}\|_{\mathbb{U}}^2 | \mathcal{G}_k] \leq M_1 + M_2 \|\lambda^k\|_{\mathbb{U}}^2 \leq M_3 + M_4 \|\lambda^k - \lambda\|_{\mathbb{U}}^2 \quad (6.4.13)$$

Since λ^k is \mathcal{G}_k -measurable and that $\mathbb{E}[Y^{k+1} | \mathcal{G}_k] = f(\lambda^k)$, we have that:

$$\begin{aligned} &\mathbb{E}[\|\lambda^{k+1} - \lambda\|_{\mathbb{U}}^2 | \mathcal{G}_k] \\ &= \|\lambda^k - \lambda\|_{\mathbb{U}}^2 + 2\rho_k \mathbb{E}[\langle \lambda^k - \lambda, Y^{k+1} \rangle_{\mathbb{U}} | \mathcal{G}_k] + (\rho_k)^2 \mathbb{E}[\|Y^{k+1}\|_{\mathbb{U}}^2 | \mathcal{G}_k] \\ &\leq \|\lambda^k - \lambda\|_{\mathbb{U}}^2 + 2\rho_k \langle \lambda^k - \lambda, f(\lambda^k) \rangle_{\mathbb{U}} + (\rho_k)^2 (M_3 + M_4 \|\lambda^k - \lambda\|_{\mathbb{U}}^2) \\ &\leq \|\lambda^k - \lambda\|_{\mathbb{U}}^2 (1 + M_4 \rho_k^2) + (\rho_k)^2 M_3 - 2\rho_k (\mathcal{W}(\lambda) - \mathcal{W}(\lambda^k)). \end{aligned} \quad (6.4.14)$$

In the last inequality, we used the concavity of \mathcal{W} and (6.4.9). The rest of the proof follows [119, Theorem 3.6]. \square

Recalling the definition of $\bar{J}(u, v)$ in (6.1.5) and of $\bar{\lambda}$ in Lemma 6.4.3.(iii), we define \bar{u} :

$$\bar{u} := \arg \min_{u \in \hat{\mathcal{U}}} \left\{ \mathbb{E} \left[\sum_{i=1}^n G_i(u^i, \omega^i) + \langle \bar{\lambda}, u^i \rangle_{\mathbb{U}} \right] \right\}. \quad (6.4.15)$$

Under Assumption 4, G_i is strictly convex w.r.t. the first variable, and then \bar{u} is well defined. If F_0 is strictly convex, then $V(\bar{\lambda})$ is a singleton and we can write:

$$\bar{v} := \arg \min_{v \in \mathbb{U}} \{ F_0(v) + \langle \bar{\lambda}, v \rangle_{\mathbb{U}} \}. \quad (6.4.16)$$

Remark 6.4.1. *If F_0 is convex, by Lemma 6.3.1, there is no duality gap associated to the Lagrangian L defined in (6.3.1). Further, if F_0 is strictly convex, then $(\bar{u}, \bar{v}, \bar{\lambda})$ is the unique saddle point associated to the Lagrangian L . Indeed, by Assumption 3, $\bar{\lambda}$ is the unique solution of the dual problem (D), by Assumption 4.(i), \bar{u} is unique and by strict convexity of F_0 , \bar{v} is also the unique minimizer in the right hand side of (6.4.16).*

Theorem 6.4.1. *Let the Assumptions 1, 2, 3, 4 and 5 hold, then we have:*

(i) $\{u(\lambda^k)\}$ weakly converges a.s. to \bar{u} .

Furthermore, if F_0 is strictly convex, then from Remark 6.4.1, \bar{u} is the unique minimizer of \tilde{J} in $\hat{\mathcal{U}}$ and:

(ii) $\tilde{J}(u(\lambda^k)) \xrightarrow[k \rightarrow \infty]{} \tilde{J}(\bar{u})$ a.s.

(iii) $\limsup_{k \rightarrow \infty} J(u(\lambda^k)) \leq \inf_{u \in \hat{\mathcal{U}}} J(u) + 2\varepsilon$ a.s. where $\varepsilon = cM^2/n$.

Proof. Proof of point (i). By Lemma 6.4.3.(iv), the sequence $\{\lambda^k\}$ is bounded in \mathbb{U} . Thus, using inequality (6.4.7) one deduces that $\{v(\lambda^k)\}$ is also bounded in \mathbb{U} . Since the sequence $\{(u(\lambda^k), v(\lambda^k))\}$ is bounded in $\hat{\mathcal{U}} \times \mathbb{U}$, there exists a weakly convergent subsequence $\{(u(\lambda^{\theta_k}), v(\lambda^{\theta_k}))\}$ such that:

$$(u(\lambda^{\theta_k}), v(\lambda^{\theta_k})) \xrightarrow[k \rightarrow \infty]{} (u^\theta, v^\theta) \in \hat{\mathcal{U}} \times \mathbb{U}. \quad (6.4.17)$$

Using the definition of $\lambda \mapsto u(\lambda)$ in (6.4.4), it holds for any $k > 0$:

$$\begin{aligned} & \mathbb{E} [G_i(\bar{u}^i, \omega^i) + \langle \lambda^{\theta_k}, \bar{u}^i \rangle_{\mathbb{U}}] \\ & \geq \mathbb{E} [G_i(u^i(\lambda^{\theta_k}), \omega^i) + \langle \lambda^{\theta_k}, u^i(\lambda^{\theta_k}) \rangle_{\mathbb{U}}]. \end{aligned} \quad (6.4.18)$$

Using that $u^i \mapsto G_i(u^i, \omega^i)$ is a.s. w.l.s.c. on $\hat{\mathcal{U}}_i$ and the a.s. convergence of $\{\lambda^k\}$, resulting from Lemma 6.4.3.(iv), we have from (6.4.18) when $k \rightarrow \infty$:

$$\mathbb{E} [G_i(\bar{u}^i, \omega^i) + \langle \bar{\lambda}, \bar{u}^i \rangle_{\mathbb{U}}] \geq \mathbb{E} [G_i(u^{i,\theta}, \omega^i) + \langle \bar{\lambda}, u^{i,\theta} \rangle_{\mathbb{U}}]. \quad (6.4.19)$$

Since \bar{u} is unique, it follows $u^\theta = \bar{u}$ and (6.4.19) is an equality. Using that every weakly convergent subsequence of $\{u(\lambda^k)\}$ has the same weak limit \bar{u} , (i) is deduced.

Proof of point (ii).

From point (i) and (6.4.19), it follows for any $i \in \{1, \dots, n\}$:

$$\lim_{k \rightarrow \infty} \mathbb{E} G_i(u^i(\lambda^k), \omega^i) = \mathbb{E} G_i(\bar{u}^i, \omega^i). \quad (6.4.20)$$

Using (6.4.17), the w.l.s.c. of F_0 , equation (6.4.16), and applying the same previous argument to $\{v(\lambda^k)\}$, it holds that:

$$\lim_{k \rightarrow \infty} F_0(v(\lambda^k)) - \langle \lambda^k, v(\lambda^k) \rangle_{\mathbb{U}} = F_0(\bar{v}) - \langle \bar{\lambda}, \bar{v} \rangle_{\mathbb{U}}, \quad (6.4.21)$$

and $v(\lambda^k) \xrightarrow[k \rightarrow \infty]{} \bar{v}$.

From the two previous equalities and the a.s. convergence of $\{\lambda^k\}$, it follows:

$$\lim_{k \rightarrow \infty} F_0(v(\lambda^k)) = F_0(\bar{v}). \quad (6.4.22)$$

Using that $(\bar{u}, \bar{v}, \bar{\lambda})$ is a saddle point, it follows:

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E} \bar{u}^i = \bar{v}. \quad (6.4.23)$$

From (6.4.22) and (6.4.23), it holds:

$$\lim_{k \rightarrow \infty} F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i(\lambda^k) \right) = F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} \bar{u}^i \right). \quad (6.4.24)$$

Then adding (6.4.20) and (6.4.24): $\lim_{k \rightarrow \infty} \tilde{J}(u(\lambda^k)) = \tilde{J}(\bar{u})$.

Proof of point (iii). From point (ii), inequality (6.2.10) and Theorem 6.2.2, it holds:

$$\limsup_{k \rightarrow \infty} J(u(\lambda^k)) \leq \limsup_{k \rightarrow \infty} \tilde{J}(u(\lambda^k)) + \varepsilon = \inf_{u \in \mathcal{U}} \tilde{J}(u) + \varepsilon \leq \inf_{u \in \mathcal{U}} J(u) + 2\varepsilon. \quad (6.4.25)$$

□

Lemma 6.4.4. *Let Assumptions 1 and 6.(i) hold, then the function $\lambda \mapsto v(\lambda)$ is Lipschitz on \mathbb{U} .*

Proof. From the definition of v in (6.4.5), we have for any $\lambda \in \mathbb{U}$ that $\lambda \in \partial F_0(v(\lambda))$. Thus, for any $\lambda, \mu \in \mathbb{U}$, we have from the strong convexity of F_0 :

$$\begin{cases} F_0(v(\mu)) & \geq F_0(v(\lambda)) + \langle \lambda, v(\mu) - v(\lambda) \rangle_{\mathbb{U}} + \alpha \|v(\mu) - v(\lambda)\|_{\mathbb{U}}^2 \\ F_0(v(\lambda)) & \geq F_0(v(\mu)) + \langle \mu, v(\lambda) - v(\mu) \rangle_{\mathbb{U}} + \alpha \|v(\lambda) - v(\mu)\|_{\mathbb{U}}^2. \end{cases} \quad (6.4.26)$$

Adding the two previous inequalities, after simplifications, we get:

$$\langle \lambda - \mu, v(\lambda) - v(\mu) \rangle_{\mathbb{U}} \geq 2\alpha \|v(\lambda) - v(\mu)\|_{\mathbb{U}}^2. \quad (6.4.27)$$

Applying the Cauchy-Schwarz inequality and simplifying by $\|v(\lambda) - v(\mu)\|_{\mathbb{U}}$, we get the desired Lipschitz inequality. □

Lemma 6.4.5. *If Assumptions 1 and 6.(ii) hold, the function $\lambda \mapsto u(\lambda)$ is Lipschitz on \mathbb{U} .*

Proof. The proof is similar to the proof of Lemma 6.4.4. □

Theorem 6.4.2. *Let Assumptions 1, 3, 5, and 6 hold, then: $u(\lambda^k) \xrightarrow[k \rightarrow \infty]{} u(\bar{\lambda})$ a.s.*

Proof. The convergence follows from the Lipschitz property of $\lambda \mapsto u(\lambda)$ (as a result of Assumption 6) associated with the a.s. convergence of $\{\lambda^k\}$. □

Remark 6.4.2. *Note that Lemma 6.4.3 and Theorems 6.4.1 and 6.4.2 still hold when replacing λ^k by $\check{\lambda}^k$ and Y^k by \check{Y}^k (as defined respectively in line 9 and 10 of the Sampled Stochastic Uzawa Algorithm 2). This can be proved by the same argument, using that \check{Y}^k is bounded a.s. and $\mathbb{E}[\check{Y}^k | \check{\mathcal{G}}_k] = f(\check{\lambda}^k)$ for any k , where:*

$$\check{\mathcal{G}}_k = \sigma \left(\{W^{I_\ell^p, p}\} : 1 \leq \ell \leq m, p \leq k \right) \vee \sigma \left(\{I_\ell^p\} : 1 \leq \ell \leq m, p \leq k \right), \quad (6.4.28)$$

with $W^{I_\ell^k, p}$ and I_ℓ^p defined respectively in line 7 and 5 of the Sampled Stochastic Uzawa Algorithm 2.

Remark 6.4.3. *We wish to emphasize that the proposed Algorithm 2 is particularly suitable for practical distributed implementations in agent-based scenarios. In these cases, the quantity λ can be interpreted as a common price signal that is broadcast to the independent agents, which in turn compute independently their optimal solution $u(\lambda)$ on the basis of their local parameters (step 5).*

To illustrate the results, we consider in the next section stochastic control problems in both continuous and discrete time settings.

6.5 Application to stochastic control

6.5.1 Continuous time setting

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which $W = (W^i)_{i=1, \dots, n}$ is a $n \times d$ -dimensional Brownian motion such that, for any $t \in [0, T]$ and $i \in \{1, \dots, n\}$, W_t^i takes value in \mathbb{R}^d and generates the filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. In the considered notation, \mathbb{P} stands for the Wiener measure associated with this filtration and \mathbb{F} for the augmented filtration by all \mathbb{P} -null sets. The following notations are used:

$$\mathbb{X} := \{\varphi : \Omega \rightarrow \mathcal{C}([0, T], \mathbb{R}^d) \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted, } \|\varphi\|_{\infty, 2} := \mathbb{E} \sup_{\substack{1 \leq k \leq d \\ s \in [0, T]}} |\varphi_k(s)|^2 < \infty\},$$

$$\mathbb{U} := L^2([0, T], \mathbb{R}^p) := \{\varphi : [0, T] \rightarrow \mathbb{R}^p \mid \int_0^T \sum_{k=1}^p |\varphi_k(t)|^2 dt < \infty\},$$

For any $i \in \{1, \dots, n\}$, the feasible set of controls is defined by:

$$\mathcal{U}_i := \{v : \Omega \times [0, T] \rightarrow \mathbb{R}, v(\cdot) \text{ is } \mathbb{F}\text{-prog. measurable, } v(\omega) \in \mathbb{U} \text{ and } v_t(\omega) \in [-M_i, M_i]^p, \text{ for a.a. } (t, \omega) \in [0, T] \times \Omega\}, \quad (6.5.1)$$

and we set $M := \max_{i \in \{1, \dots, n\}} M_i$, where $M_i > 0$.

Each local agent $i = 1, \dots, n$ is supposed to control its state variable through the control process $u^i \in \mathcal{U}_i$ and is subject to independent uncertainties. More specifically, the state process of each agent, $X^{i, u^i} = (X_t^{i, u^i})_{t \in [0, T]}$, for $i = 1, \dots, n$ takes values in \mathbb{R}^d and follows the dynamics for $i \in \{1, \dots, n\}$:

$$\begin{cases} dX_t^{i, u^i} &= \mu_i(t, u_t^i(\cdot, W^i), X_t^{i, u^i}) dt + \sigma_i(t, X_t^{i, u^i}) dW_t^i, \text{ for } t \in [0, T], \\ X_0^{i, u^i} &= x_0^i \in \mathbb{R}^d; \end{cases} \quad (6.5.2)$$

We assume that, for any i , there exist five functions $\alpha_i \in L^\infty([0, T], \mathbb{R}^{d \times p})$, $\beta_i, \theta_i \in L^\infty([0, T], \mathbb{R}^{d \times d})$, $\gamma_i \in L^\infty([0, T], \mathbb{R}^d)$ and $\xi_i \in L^\infty([0, T], \mathbb{R}^{d \times d \times d})$ such that, for any $(t, \nu, x) \in [0, T] \times [-M, M]^p \times \mathbb{R}^d$:

$$\mu_i(t, \nu, x) = \alpha_i(t)\nu + \beta_i(t)x + \gamma_i(t) \text{ and } \sigma_i(x, t) = \xi_i(t)x + \theta_i(t). \quad (6.5.3)$$

Without loss of generality, the initial states x_0^i are supposed to be deterministic. The process X^{i, u^i} is \mathbb{F} -progressively measurable. For all i , \mathcal{F}^i stands for the natural filtration of the Brownian motion W^i .

On the well-posedness of (P_1)

In this section, we discuss some conditions under which Problem (P_1) is well-posed.

Lemma 6.5.1. *Let $i \in \{1, \dots, n\}$ and $v \in \mathcal{U}_i$ be a control process. The map $v \mapsto X^{i, v}$ is linear continuous from \mathcal{U}_i to \mathbb{X} and there exists a unique process $X^{i, v} \in \mathbb{X}$ satisfying (6.5.2) (in the strong sense) such that, for any $p \in [1, \infty)$:*

$$\mathbb{E} \left[\sup_{\substack{0 \leq t \leq T \\ 1 \leq k \leq d}} |X_{k, t}^{i, v}|^r \right] < C(r, T, x_0, K) < \infty. \quad (6.5.4)$$

Proof. The proof for the existence and uniqueness of a solution of (6.5.2) relies on [185, Theorem 3.6, Chapter 2]. The inequality is a result of [185, Theorem 4.4, Chapter 2]. \square

Let $F_0 : \mathbb{U} \rightarrow \mathbb{R}$ be a proper, convex and lower semi continuous function, satisfying Assumptions 3 and 4.(ii). For any $i \in \{1, \dots, n\}$, we assume that there exists F_i such that the local cost G_i is of the form:

$$u^i \mapsto G_i(u^i(\cdot, \omega^{-i}), \omega^i) = F_i(u^i(\omega^i, \omega^{-i}), X^{i, u^i}(\omega^i)), \quad (6.5.5)$$

where $F_i : \mathbb{U} \times \mathcal{C}([0, T] \times \mathbb{R}^d) \rightarrow \mathbb{R}_+$ is a proper and lower semi continuous function. Additional assumptions are formulated below.

Assumption 7. For any $i \in \{1, \dots, n\}$:

- (i) F_i is jointly convex w.r.t. both variables and strictly convex w.r.t. the first variable.
- (ii) there exists a positive integer r such that F_i has r -polynomial growth, i.e there exists $K > 0$ such that for any $x^i \in \mathcal{C}([0, T], \mathbb{R}^d)$ and $u^i \in \mathbb{U}$: $|F_i(u^i, x^i)| \leq K(1 + \sup_{\substack{0 \leq t \leq T \\ 0 \leq k \leq n}} |x_{k,t}^i|^r)$.

Note that if Assumption 7.(i) is satisfied, using the definition of F_0 and F_i above in this section, Assumptions 1 to 4 hold.

Remark 6.5.1. It is worth highlighting the following aspects regarding Assumption 7:

1. Assumption 7.(i) is satisfied if there exist
2. Observe that Assumption 7 satisfies Assumptions 1.(ii) and 4.(i)

From now on, Assumption 7 is in force. Now the optimization problems (P_1^c) and (\hat{P}_2^c) can be clearly defined:

$$(P_1^c) \begin{cases} \inf_{u \in \mathcal{U}} J^c(u) \\ J^c(u) := \mathbb{E} \left[F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i(\omega) \right) + \frac{1}{n} \sum_{i=1}^n F_i(u^i(\omega), X^{i,u^i}(\omega^i)) \right], \end{cases} \quad (6.5.6)$$

and

$$(\hat{P}_2^c) \begin{cases} \inf_{u \in \tilde{\mathcal{U}}} \tilde{J}^c(u) \\ \tilde{J}^c(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i \right) + \frac{1}{n} \mathbb{E} \left[\sum_{i=1}^n F_i(u^i(\omega), X^{i,u^i}(\omega^i)) \right], \end{cases} \quad (6.5.7)$$

Using the results of Section 6.2, we can state the following corollary:

Corollary 6.5.1. (i) Problems (P_1^c) and (\hat{P}_2^c) admit both a unique solution.

(ii) Any optimal solution of Problem (\hat{P}_2^c) is an ε -optimal solution, where $\varepsilon = cM^2/n$, of Problem (P_1^c) .

Proof. The proof of point (i) is a specific case of Lemma 6.2.1. Similarly, point (ii) is a particular case of Theorem 6.2.2. \square

Remark 6.5.2. A practical example of this type of stochastic optimization problem is illustrated in Section 6.7, which considers the interactions between a large population of price-responsive self-interested domestic appliances and a central system operator which has to meet the prescribed levels of demand at minimum generation costs.

6.5.2 Discrete time setting

The main results of the paper are instantiated to the discrete time setting in this subsection. The following notations are used.

- Let $n \in \mathbb{N}^*$ be the number of agents, $d, p \in \mathbb{N}^*$ the dimension respectively of their state and control variables at any time step, and $T \in \mathbb{N}^*$ the finite time horizon.
- For any matrix M , its transpose is denoted by M^\top .
- We consider a global noise process as a sequence of independent random variables (W_1, \dots, W_T) , where for any $t \in \{1, \dots, T\}$, W_t is a vector of d -dimensional independent random variables, with finite variance, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$: $W_t := (W_t^1, \dots, W_t^d)$, with $W_t^i \in \mathbb{R}^d$. For any $i \in \{1, \dots, n\}$ and $t \in \{1, \dots, T\}$ we define $\mathcal{F}_t^i := \sigma(W_1^i, \dots, W_t^i)$ and $\mathcal{F}_t := \otimes_{i=1}^n \mathcal{F}_t^i$.
- The space \mathbb{X} is defined by:

$$\mathbb{X} := \{x = (x_0, \dots, x_T) \mid \forall t \in \{0, \dots, T\}, \mathbb{R}^d \ni x_t \text{ is } \mathcal{F}_t\text{-measurable and } \mathbb{E} \|x_t\|_2^2 < \infty\}. \quad (6.5.8)$$

— For any $i \in \{1, \dots, n\}$, we define the space of control \mathcal{U}^i of agent i by:

$$\mathcal{U}^i := \{u^i = (u_0^i, \dots, u_{T-1}^i) \mid \forall t \in \{0, \dots, T-1\}, \mathbb{R}^p \ni u_t^i \text{ is } \mathcal{F}_t\text{-measurable and } u_k^i(\omega) \in [-M, M]^p \text{ } \mathbb{P}\text{-a.s.}\}, \quad (6.5.9)$$

where $M > 0$. We finally set $\mathcal{U} := \prod_{i=1}^n \mathcal{U}^i$.

— For any $i \in \{1, \dots, n\}$, $X^{i,u^i} := (X_0^{i,u^i}, \dots, X_T^{i,u^i}) \in \mathbb{X}$ is the state trajectory of agent i controlled by $u^i \in \mathcal{U}^i$. We have the following dynamics:

$$\begin{cases} X_{t+1}^{i,u^i} &= A^i X_t^{i,u^i} + B^i u_t^i + W_{t+1}^i, \quad \text{for } t \in \{0, \dots, T-1\}, \\ X_0^{i,u^i} &= x_0 \in \mathbb{R}^d, \end{cases} \quad (6.5.10)$$

where $A^i \in \mathbb{R}^{d \times d}$ and $B^i \in \mathbb{R}^{d \times p}$.

Let $F_0 : \mathbb{R}^{p \times T} \rightarrow \bar{\mathbb{R}}$ be proper, lower semi continuous, convex and satisfy Assumptions 3 and 4.(ii). Similarly to the previous subsection, we assume that, for any i , there exists a function $F_i : \mathbb{R}^{p \times T} \times \mathbb{R}^{d \times T} \rightarrow \mathbb{R}$ such that G_i and F_i satisfy (6.5.5), and F_i satisfies Assumption 7 for an integer r such that $\mathbb{E} \|W_t\|^r$ is finite for any $t \in \{1, \dots, T\}$.

Now, for any $n \in \mathbb{T}^*$, the optimization problems (P_1^d) and (\hat{P}_2^d) can be clearly defined:

$$(P_1^d) \begin{cases} \inf_{u \in \mathcal{U}} J^d(u) \\ J^d(u) := \mathbb{E} \left[F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i \right) + \frac{1}{n} \sum_{i=1}^n F_i(u^i, X^{i,u^i}) \right], \end{cases} \quad (6.5.11)$$

and

$$(\hat{P}_2^d) \begin{cases} \inf_{u \in \mathcal{U}} \tilde{J}^d(u) \\ \tilde{J}^d(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i \right) + \frac{1}{n} \mathbb{E} \left[\sum_{i=1}^n F_i(u^i, X^{i,u^i}) \right]. \end{cases} \quad (6.5.12)$$

In the same spirit as in the previous subsection, we have the following results, which will be useful for the next section.

Corollary 6.5.2. (i) Problems (P_1^d) and (\hat{P}_2^d) admit both a unique solution.

(ii) Any optimal solution of Problem (\hat{P}_2^d) is an ε -optimal solution, where $\varepsilon = cM^2/n$, of Problem (P_1^d) .

Proof. The proof of point (i) is analogous to the one of Lemma 6.2.1. Similarly, proof of point (ii) is analogous to the one of Theorem 6.2.2. \square

One can implement the *Stochastic Uzawa* (Algo 1) and the *Sampled Stochastic Uzawa* (Algo 2) in this discrete time setting with Lemma 6.4.3 and Theorems 6.4.1 and 6.4.2 still ensuring the algorithm convergence.

6.6 A numerical example: the LQG (Linear Quadratic Gaussian) problem

This section aims at illustrating numerically the convergence of the *Stochastic Uzawa* (Algo 3) on a simple example. The speed of convergence of the algorithm is evaluated according to the number of dual iterations and of agents. A linear quadratic formulation is considered, with n agents in a discrete setting Problem (\hat{P}_2^{LQG}) . We use the notations of Section 6.5.2.

This framework constitutes a simple test case, since the (deterministic) Uzawa's algorithm can be performed, and one can compare the resulting multiplier estimate with the one provided by the *Stochastic Uzawa* algorithm. All the assumptions required for the convergence of the *Stochastic Uzawa* (Algo 3) are satisfied for Problem (\hat{P}_2^{LQG}) . Moreover, the optimal solutions to the local problems (line 5 of Algorithm 1) can be resolved analytically.

Problem (\hat{P}_2^{LQG}) is similar to (\hat{P}_2^d) defined in (6.5.12) but, in this specific case, the function F_0 is a quadratic function of the aggregate strategies of the agents

$$F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i \right) := \frac{\nu}{2} \sum_{t=0}^T \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u_t^i - r_t \right)^2, \quad (6.6.1)$$

where $\nu > 0$ and $\{r_t\}$ is a deterministic target sequence. Similarly, the cost term F_i of the individual agents is expressed as a quadratic function of their state X^{i,u^i} and control u^i

$$F_i(u^i, X^{i,u^i}) := \frac{1}{2} \left(\sum_{t=0}^T d_i (X_t^{i,u^i})^2 + q_i (u_t^i)^2 \right) + \frac{d_i^f}{2} (X_T^{i,u^i})^2, \quad (6.6.2)$$

where $q_i > 0$ and $d_i > 0$ for any $i \in \{1, \dots, n\}$. Defining the matrices $D = \text{diag}(d_1, \dots, d_n)$, $Q = \text{diag}(q_1, \dots, q_n)$ and $D^f = \text{diag}(d_1^f, \dots, d_n^f)$, we get:

$$\sum_{i=1}^n F_i(u^i, X^{i,u^i}) = \frac{1}{2} \left(\sum_{t=0}^T X_t^{u^\top} D X_t^u + u_t^\top Q u_t \right) + \frac{1}{2} X_T^{u^\top} D^f X_T^u, \quad (6.6.3)$$

where, for any $t \in \{0, \dots, T\}$, $X_t^u := (X_t^{1,u^1}, \dots, X_t^{n,u^n}) \in \mathbb{R}^n$ is the controlled state vector of all the agents. Now the optimization Problem (\hat{P}_2^{LQG}) is clearly defined.

To find the optimal multiplier and control of (\hat{P}_2^{LQG}) , the *Stochastic Uzawa* Algorithm 3 is applied. In this specific case, the lines 4 and 6 take respectively the following form at any dual iteration k :

$$u^i(\lambda^k) := \arg \min_{u^i \in \hat{U}^i} \left\{ \mathbb{E} \left[\frac{1}{2} \left(\sum_{t=0}^T d_i (X_t^{i,u^i})^2 + q_i (u_t^i)^2 + \lambda_t^k u_t^i \right) + \frac{d_i^f}{2} (X_T^{i,u^i})^2 \right] \right\}, \quad (6.6.4)$$

$$v(\lambda^k) := \arg \min_{v \in \mathbb{R}^T} \left\{ \left(\sum_{t=0}^T \nu (v_t - r_t)^2 - \lambda_t^k v_t \right) \right\}. \quad (6.6.5)$$

The optimization problem (6.6.4) solved by each local agent also falls within the LQG framework. One can solve these problems using the results of [249]. The resolution via Riccati equations of (6.6.4) shows that $u^i(\lambda^k)$ is a linear function of the state X^{i,u^i} and of the price λ^k . Therefore, in this specific example, one can explicitly compute $\mathbb{E}[u_t^i(\lambda^k) | \mathcal{G}_k]$ for any t , with \mathcal{G}_k as defined in (6.4.10).

Within this described framework, it is possible to implement the (deterministic) Uzawa's algorithm and use it as a reference to evaluate the performances of the *Stochastic Uzawa* algorithm.

Different population sizes n are considered, with n ranging between 1 and 10^4 . Similarly, the algorithm is stopped after different numbers of dual iteration k , ranging between 10 and 10^4 . In order to evaluate the bias and variance of the *Stochastic Uzawa* algorithm, this has been performed over $J = 1000$ runs.

It is possible to define a Problem (\hat{P}_2^{LQG}) and a dual Problem (D^{LQG}) from Problem (\hat{P}_2^{LQG}) following the same approach presented in (6.1.5) and (6.3.2) for the definition of (\hat{P}_2') and (D) , respectively, from (\hat{P}_2) . It can be shown that, for any n , there exists a unique optimal multiplier $\bar{\lambda}^n$, solution of (D^{LQG}) . For any n , the quantity $\lambda^{k,n,j}$ denotes the dual price computed during the j^{th} simulations ($j = 1, \dots, J$) of the *Stochastic Uzawa* algorithm, after k dual iterations.

For any n , the deterministic multiplier $\bar{\lambda}^n$ is obtained by applying Uzawa's algorithm, after 10^4 dual iterations. To this end, we applied the *Stochastic Uzawa* Algorithm 3, where we ignored the line 8 and we replaced the update of λ^k line 9 by: $\bar{\lambda}^{k+1} \leftarrow \bar{\lambda}^k + \rho_k \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i(\bar{\lambda}^k) - v(\bar{\lambda}^k) \right)$.

At each dual iteration k , the computation of $\mathbb{E} u^i(\lambda^k)$ is straightforward in this specific case, since $u^i(\lambda^k)$ is a linear function of X^{i,u^i} and λ^k , as explained in the previous subsection.

The multipliers $\lambda^{k,n,j}$ and $\bar{\lambda}^n$, obtained by applying the *Stochastic Uzawa* and *Uzawa algorithms*, respectively, are now compared. For any k and n , let $b_{k,n}$, $v_{k,n}$ and $\ell_{k,n}$ denote an estimation of the bias, the variance and the L^2 norm of the error, respectively, as computed via Monte Carlo method with J simulations. For any k and

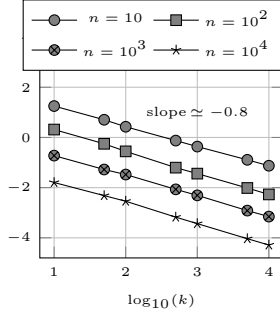


Figure 6.1 – Variance term $\log_{10}(v_{k,n})$, expressed as a function of k , for different number of agents n .

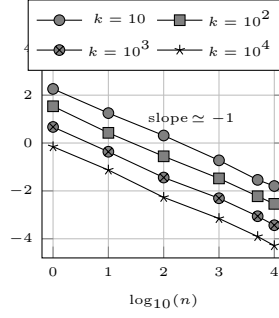


Figure 6.2 – Variance term $\log_{10}(v_{k,n})$, expressed as a function of n , for different number of iterations k .

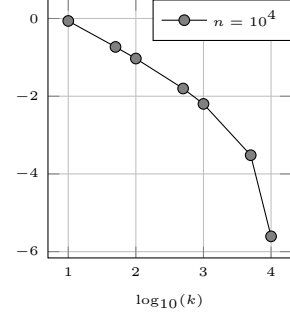


Figure 6.3 – Bias term $\log_{10}(\|b_{k,n}\|_2^2)$, expressed as a function of k , given the number of agents $n = 10^4$.

n , these quantities are defined as follows:

$$b_{k,n} := \frac{1}{J} \sum_{j=1}^J \lambda^{k,n,j} - \bar{\lambda}^n,$$

$$v_{k,n} := \frac{1}{J} \sum_{j=1}^J \|\lambda^{k,n,j} - \bar{\lambda}^n - b_{k,n}\|_2^2,$$

$$\ell_{k,n} := v_{k,n} + \|b_{k,n}\|_2^2.$$

Since numerical simulations are based on finite dimensional approximations, it is relevant to compare the empirical convergence rates, shown in Figures 6.1-6.3, with the associated theoretical asymptotic rates presented in the literature for a finite dimensional setting.

In Figure 6.1, we observe a behavior in $1/k^\alpha$ (with $\alpha \simeq 0.8$) of the variance $v_{k,n}$ w.r.t. the number of iterations k . This rate of convergence is consistent with [95, Theorem 2.2.12, Chapter 2], where the best asymptotic convergence rate for the Robbins-Monro algorithm is proved to be of the order of $1/k$ (for the quadratic error).

In Figure 6.2 we observe a behavior in $1/n^\beta$ (with $\beta \simeq 1$) of the variance $v_{k,n}$ w.r.t. the number of agents n . This is expected, following [95, Theorem 2.2.12, Chapter 2] and the observation that the variance of Y^{k+1} is of the order of $1/n$ for any iteration k .

Finally, in Figure 6.3, we note that the bias $\|b_{k,n}\|_2^2$ decreases faster than $1/k$ w.r.t. the number of iterations k . Thus, for a large number of iterations ($k > 0$), the dominant term impacting the error $\ell_{k,n}$ is the variance $v_{k,n}$.

6.7 Price-based coordination of a large population of thermostatically controlled loads

The goal of this section is to demonstrate the applicability of the presented approach for the coordination of TCLs in the context of flexible power systems. In particular, the problem analyses the daily operation of a power system with a large penetration of price-responsive TCLs, adopting a modelling framework similar to [84]. Two distinct elements are considered: i) a system operator, which must schedule a portfolio of generation assets in order to satisfy the energy demand at a minimum cost, and ii) a population of price-responsive TCLs that individually determine their ON/OFF power profile in response to price, with the objective of minimizing their operating cost while fulfilling users' requirements. Note that the operations of the two elements are interconnected, since the aggregate power consumption of the TCLs will modify the system-level demand profile that needs to be accommodated by the system operator.

6.7.1 Formulation of the problem

In the considered problem, the function F_0 represents the minimized power production cost and corresponds to the resolution of an Unit Commitment (UC) problem. The UC determines generation scheduling decisions (in terms of energy production and frequency response (FR) provision) in order to minimize the short term operating cost of the system while matching generation and demand. The demand quantity is the sum of an inflexible deterministic term (denoted for any time instant $t \in [0, T]$ by $\bar{D}(t)$) and of a stochastic component $n \times U_{TCL}(t)$, i.e. the product of the population size n and the average demand profile $U_{TCL}(t)$ of the TCL population.

For simplicity, a Quadratic Programming (QP) formulation in a discrete time setting is adopted for the UC problem. The central planner disposes of Z generation technologies (gas, nuclear, wind) and schedules their production and allocates response by slot of 30 min every day. For any $j \in \{1, \dots, Z\}$ and $\ell \in \{1, \dots, 48\}$, the quantities $H_j(t_\ell)$, $G_j(t_\ell)$ and $R_j(t_\ell)$ denote respectively the commitment, the power production and the frequency-response from unit j during the time interval $[t_\ell, t_{\ell+1}]$ (all expressed in MWh). The associated vectors are denoted by $H(t_\ell) = [H_1(t_\ell), \dots, H_Z(t_\ell)]$, $G(t_\ell) = [G_1(t_\ell), \dots, G_Z(t_\ell)]$ and $R(t_\ell) = [R_1(t_\ell), \dots, R_Z(t_\ell)]$.

The cost sustained at time t_ℓ by unit j is linear with respect to the commitment $H_j(t_\ell)$ and quadratic with respect to generation $G_j(t_\ell)$ and can be expressed as $c_{1,j}H_j(t_\ell)G_j^{Max}(t_\ell) + c_{2,j}G_j(t_\ell) + c_{3,j}G_j(t_\ell)^2$. In this cost expression, G_j^{Max} denotes the production limit allocated by each generation technology, $c_{1,j}$ [£/MWh] is the no-load cost term, whereas $c_{2,j}$ [£/MWh] and $c_{3,j}$ [$\text{£/MW}^2\text{h}$] are the production cost coefficients of the generation technology j . The optimization of F_0 must satisfy the following constraints for all $\ell \in \{1, \dots, 48\}$ and $j \in \{1, \dots, Z\}$:

$$\sum_{j=1}^Z G_j(t_\ell) - \int_{t_\ell}^{t_{\ell+1}} (\bar{D}(t) + nU_{TCL}(t))dt = 0, \quad (6.7.1)$$

$$0 \leq H_j(t_\ell) \leq 1, \quad (6.7.2)$$

$$R_j(t_\ell) - r_j H_j(t_\ell) G_j^{max}(t_\ell) \leq 0, \quad (6.7.3)$$

$$R_j(t_\ell) - s_j (H_j(t_\ell) G_j^{max}(t_\ell) - G_j(t_\ell)) \leq 0, \quad (6.7.4)$$

$$\Delta G_L - \Lambda (\bar{D}(t_\ell) + n(\bar{U}_{TCL}(t_\ell) - \bar{R}_{TCL}(t_\ell))) \Delta f_{qss}^{max} - \hat{R}(t_\ell) \leq 0, \quad (6.7.5)$$

$$2\Delta G_L t_{ref} t_d - t_{ref}^2 \hat{R}(t_\ell) - 4\Delta f_{ref} t_d \hat{H}(t_\ell) \leq 0, \quad (6.7.6)$$

$$\bar{q}(t) - \hat{H}(t) \hat{R}(t) \leq 0 \quad (6.7.7)$$

$$\mu r_j H_j(t_\ell) G_j^{max}(t_\ell) - G_j(t_\ell) \leq 0, \quad (6.7.8)$$

where (6.7.1) equals production and aggregated demand (i.e. the system inelastic demand \bar{D} and the TCL flexible demand nU_{TCL}). The quantities \hat{R} and \hat{H} denote the total reserve and inertia of the system, respectively, and are defined for any $\ell \in \{1, \dots, 48\}$ as:

$$\hat{R}(t_\ell) = \sum_{j=1}^Z R_j(t_\ell) + nR_{TCL}(t_\ell),$$

$$\hat{H}(t_\ell) = \sum_{j=1}^Z \frac{h_j H_j(t_\ell) G_j^{max} - h_L \Delta G_L}{f_0}.$$

In (6.7.2) it is supposed that, for any generation technology j , the capacity of the single power plant is significantly smaller than the total installed capacity. As a result, it is reasonable to consider the continuous relaxation of the UC problem by assuming $H_j(t_\ell) \in [0, 1]$.

The amount of response allocated by each generation technology is limited by the headroom $r_j H_j(t_\ell) G_j^{max}(t_\ell)$ in (6.7.3) and by the slope s_j linking the FR with the dispatch level (6.7.4). Constraints (6.7.5) to (6.7.8) deal with frequency response provision and R_{TCL} (the mean of FR allocated by TCLs). They guarantee secure frequency deviations following sudden generation loss ΔG_L . Inequality (6.7.5) allocates enough FR (with delivery time t_d) such that the quasi-steady-state frequency remains above Δf_{qss}^{max} , with Λ accounting for the damping effect introduced by the loads [161]. The constraint (6.7.7) imposes the maximum tolerable frequency deviation Δf_{nad} , following the formulation and methodology presented in [247] and [252]. The rate of change of

frequency is taken into account in (6.7.6) where at t_{rcf} the frequency deviation remains above Δf_{ref} . Constraint (6.7.8) prevents trivial unrealistic solutions that may arise in the proposed formulation, such as high values of committed generation $H_j(t_\ell)$ in correspondence with low (even zero) generation dispatch $G_j(t_\ell)$. The reader can refer to [84] for more details on the UC problem.

The solution F_0 of the UC problem can be defined by the following optimization problem:

$$F_0(U_{TCL}, R_{TCL}) := \min_{H, G, R} \sum_{\ell=1}^{48} \sum_{j=1}^Z c_{1,j} H_j(t_\ell) G_j^{max}(t_\ell) + c_{2,j} G_j(t_\ell) + c_{3,j} G_j(t_\ell)^2, \quad (6.7.9)$$

subject to equations (6.7.1)-(6.7.8).

Note that the formulation of the present problem does not fulfill all the assumptions presented in Section 6.4. In particular, the function F_0 is not strictly convex, as instead supposed in Theorem 6.4.1.(ii).(iii). Nevertheless, the numerical simulations of Section 6.7.2 shows that the proposed approach is still able to achieve convergence.

Regarding the modelling of the individual price-responsive TCLs, each TCL $i \in \{1, \dots, n\}$ is characterized at any time $t \in [0, T]$ by its temperature state X_t^{i, u^i} [$^\circ C$] and by its power consumption control u_t^i [W]. The thermal dynamic X_t^{i, u^i} of a single TCL i is given by:

$$\begin{cases} dX_t^{i, u^i} &= -\frac{1}{\gamma_i} (X_t^{i, u^i} - X_{OFF}^i + \zeta_i u_t^i) dt + \sigma_i dW_t^i, \quad \text{for } t \in [0, T], \\ X_{0, u^i}^i &= x_0^i \in \mathbb{R}, \end{cases} \quad (6.7.10)$$

where:

- γ_i is its thermal time constant [s].
- X_{OFF}^i is the ambient temperature [$^\circ C$].
- ζ_i is the heat exchange parameter [$^\circ C/W$].
- σ_i is a positive constant [$(^\circ C)s^{\frac{1}{2}}$],
- W^i is a Brownian Motion [$s^{\frac{1}{2}}$], independent from W^j for any $j \neq i$.

For any $i \in \{1, \dots, n\}$, the set of control \mathcal{U}_i is defined by:

$$\mathcal{U}_i := \left\{ v : \Omega \times [0, T] \rightarrow \mathbb{R}, v(\cdot) \text{ is } \mathbb{F} - \text{prog. measurable}, \right. \\ \left. v(\omega) \in \mathbb{U} \text{ and } v_t(\omega) \in \{0, P_{ON, i}\}, \text{ for a.a. } (t, \omega) \in [0, T] \times \Omega \right\}, \quad (6.7.11)$$

The TCLs dynamics in (6.7.10) have been derived according to [156], with the addition of the stochastic term $\sigma_i dW_t^i$ to account for the influence of the environment (opening/closing of the fridge, environment temperature, etc.) on the evolution of the TCL temperature.

By combining the objective functions of the systems, the system operator has to solve the following optimization problem:

$$(P_1^{TCL}) \begin{cases} \inf_{u \in \mathcal{U}} J(u) \\ J(u) := \mathbb{E} \left[F_0 \left(\frac{1}{n} \sum_{i=1}^n u^i, \frac{1}{n} \sum_{i=1}^n r_i(u^i, X^{i, u^i}) \right) \right] \\ + \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \int_0^T f_i(u_s^i, X_s^{i, u^i}) ds + \gamma_i (X_T^{i, u^i} - \bar{X}^i)^2 \right], \end{cases} \quad (6.7.12)$$

The term $r_i(u^i, X^{i, u^i})$ denotes the maximum amount of FR allocated by the TCL i at time s and can be expressed as:

$$r_i(u^i, X^{i, u^i})(s) := u_s^i \frac{X_s^{i, u^i} - X_{min}^i}{X_{max}^i - X_{min}^i}. \quad (6.7.13)$$

The discomfort term of the single TCL i at time s is denoted by $f_i(u_s^i, X_s^{i, u^i})$, which takes the following expression:

$$f_i(u_s^i, X_s^{i, u^i}) := \alpha_i (X_s^{i, u^i} - \bar{X}^i)^2 + \beta_i ((X_{min}^i - X_s^{i, u^i})_+^2 + (X_s^{i, u^i} - X_{max}^i)_+^2), \quad (6.7.14)$$

where:

- $\alpha_i(X_s^{i,u^i} - \bar{X}^i)^2$ is a discomfort term penalizing temperature deviations from some comfort target \bar{X} [$^{\circ}C$], considering α_i [$\pounds/h(^{\circ}C)^2$] as a discomfort term parameter.
- $\beta_i((X_s^{i,u^i} - X_{\min}^i)_+^2 + (X_{\max}^i - X_s^{i,u^i})_+^2)$ is a penalization term meant to maintain the temperature within the interval $[X_{\min}^i, X_{\max}^i]$, considering the cost parameter β_i [$\pounds/s(^{\circ}C)^2$] and the maximum function $(a)_+ = \max(0, a)$.
- $\gamma_i(X_T^{i,u^i} - \bar{X}^i)^2$ is a terminal cost term meant to impose soft periodic constraints by quadratically penalizing the deviations of the final temperature state X_T^{i,u^i} with respect to the initial temperature value \bar{X}^i , considering the cost parameter γ [$\pounds/s(^{\circ}C)^2$].

Note that the control set \mathcal{U} is not convex. We can mention a possible relaxation of the problem by taking the control in the interval $[0, P_{ON,i}]$.

In order to solve (P_1^{TCL}) , the modified Problem (P_2^{TCL}) is studied:

$$(P_2^{TCL}) \quad \begin{cases} \inf_{u \in \mathcal{U}} \tilde{J}(u) \\ \tilde{J}(u) := F_0 \left(\frac{1}{n} \sum_{i=1}^n \mathbb{E} u^i, \frac{1}{n} \sum_{i=1}^n \mathbb{E} r_i(u^i, X^{i,u^i}) \right) \\ + \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \int_0^T f_i(u_s^i, X_s^{i,u^i}) ds + \gamma_i (X_T^{i,u^i} - \bar{X}^i)^2 \right]. \end{cases} \quad (6.7.15)$$

6.7.2 Decentralized implementation

The *Sampled Stochastic Uzawa* Algorithm 2 is applied to solve (P_2^{TCL}) , with $m = 317$ simulations per iteration. At each iteration k , the lines 4 and 6 of Algorithm 2 correspond to the solution of a deterministic UC problem and of an Hamilton Jacobi Bellman (HJB) equation, respectively. The time steps $\Delta t = 7.6$ s and temperature steps $\Delta T = 0.15^{\circ}C$ are chosen for the discretization of the HJB equation. Let us note that, at line 6, each TCL solves its own local problem on the basis of the received price signal $\lambda^k = (p^k, \rho^k)$:

$$\inf_{u^i \in \mathcal{U}_i} \int_0^T f_i(u_s^i, X_s^{i,u^i}) + u_s^i p_s^k - r_i(u^i, X^{i,u^i})(s) \rho_s^k ds, \quad (6.7.16)$$

where $f_i(u_s^i, X_s^{i,u^i})$ is a discomfort term defined in (6.7.14), $u_s^i p_s^k$ can be interpreted as consumption cost and $r_i(u^i, X^{i,u^i})(s) \rho_s^k$ as fee awarded for FR provision. This implementation has a practical sense: each TCL uses local information and the received price signals to schedule its power consumption on the time interval $[0, T]$, with the objective of minimizing its overall costs. It follows that, with the proposed approach, it is possible to optimize the total system costs in (P_1^{TCL}) in a distributed manner, with each TCL acting independently and pursuing its own cost minimization.

6.7.3 Results

In the proposed case study, the considered generation technologies available in the system are nuclear, combined cycle gas turbines (CCGT), open cycle gas turbines (OCGT) and wind. The characteristics and parameters of the UC in this simulation are the same as in [84].

It is assumed that the population of TCLs corresponds to $n = 2 \times 10^7$ fridges with built-in freeze compartment that operate in the system according to the proposed price-based control scheme. For any TCL i , we set the consumption parameter $P_{ON,i} = 180W$. The values of the TCL dynamic parameters γ_i and X_{OFF}^i of (6.7.10) are equal to the ones considered in [84]. The initial temperatures of the TCLs are selected randomly according to a uniform probability distribution, considering temperature values between $-21^{\circ}C$ and $-14^{\circ}C$. For any TCL i , the parameters of the individual cost function f_i , defined in (6.7.14), are: $\alpha_i = 0.2 \times 10^{-4}$ $\pounds/s(^{\circ}C)^2$, $\beta_i = 50 \pounds/s(^{\circ}C)^2$, $\bar{X}^i = -17.5^{\circ}C$ and $X_{\max}^i = -14^{\circ}C$, $X_{\min}^i = -21^{\circ}C$. The parameter β_i is intentionally taken very large to ensure that the TCL temperature remains within the interval $[X_{\max}^i, X_{\min}^i]$. Note that the individual problems solved by the TCLs are distinct than the ones in [84] (different terms and parameters).

Simulations are performed for different volatility values $\sigma_i := 0, 1, 2$ (all the TCLs have the same volatility in the simulations), with σ_i defined as in (6.7.10). The *Sampled Stochastic Uzawa* Algorithm is stopped after 75 iterations.

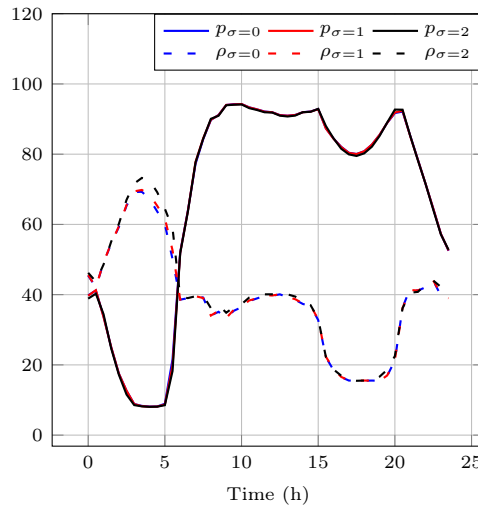
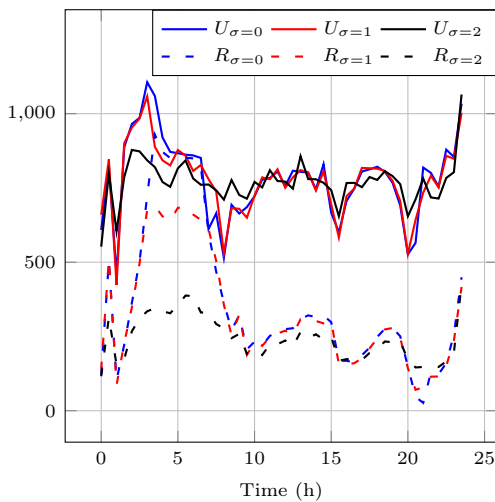


Figure 6.4 – Total power consumption U and allocated response R (MW) of the TCLs after 75 algorithm iterations. Figure 6.5 – Electricity price p and response availability price ρ (£/MWh) after 75 algorithm iterations.

The resulting profiles of total power consumption $U = nU_{TCL}$ and total allocated response $R = nR_{TCL}$ by the TCLs population are reported in Figure 6.4, while the resulting electricity prices p and response availability prices ρ are shown in Figure 6.5. As observed in [84], the total consumption U is higher when the electricity price p is lower. Conversely, the total allocated response R is higher when the FR remuneration price ρ is also higher. This can be observed in particular during the first hours of the day, between 0 and 6 h. The power consumption U exhibits smaller oscillations during the rest of the day, as the internal temperature of the TCLs is maintained within feasible levels. Although the prices do not seem to be particularly sensitive with respect to the volatility parameter σ , the power consumption U and frequency response R are highly correlated to the volatility of the TCLs temperature.

The TCLs impact on system commitment decisions and consequent energy/FR dispatch levels is also analyzed and displayed in Figures 6.6 and 6.7. In this analysis, the “flexibility scenario”, obtained with the proposed optimization strategy and considering flexible price-responsive TCLs, is compared to a “business-as-usual” scenario where the TCL do not respond to external price signals and do not perform any optimization of their costs. In the “business-as-usual”, we impose $R_{TCL}(t) = 0$ and we assume that the TCLs operate exclusively according to their internal temperature X^{i,u^i} . They switch ON ($u^i(t) = P_{ON,i}$) when they reach their maximum feasible temperature X_{max}^i and they switch back OFF again ($u^i(t) = 0$) when they reach the minimum temperature X_{min}^i . In Figure 6.6, we can clearly observe that TCL’s flexibility allows to increase the contribution of wind to the energy balance of the system while decreasing the contribution of CCGT both in energy and frequency response. In the “business-as-usual” scenario, without frequency support by the TCL, the optimal solution envisages a further curtailment of wind output in favor of an increase in CCGT generation, as wind does not provide any FR. As expected, the influence of the TCL on the system is larger when the temperature volatility is lower.

A comparison of the system costs (i.e. UC solution) between the “flexibility scenario” (FS) and the “Business-as-usual” (BAU) framework is provided in Table 6.1. As expected, costs are lower in the FS, as the flexibility of the TCLs positively supports system operation, allowing to replace gas generation from OCGT and CCGT plants with cheaper wind energy. The reduction is higher (about 1.9%) for $\sigma = 0$ with respect to the cases with $\sigma = 1$ or $\sigma = 2$ (about 1.6% and 1.2%, respectively). This confirms previous indications that TCLs tend to be more flexible when the volatility of their internal temperature is lower.

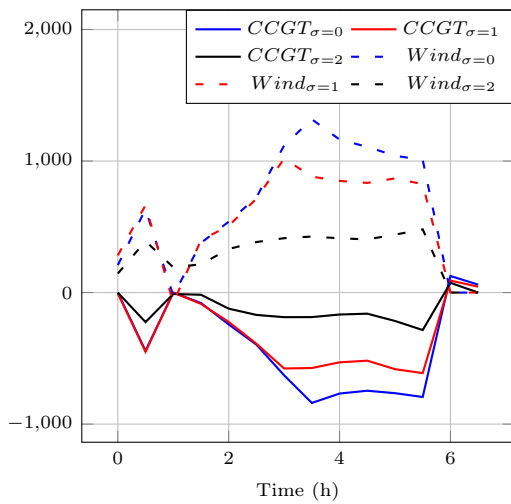


Figure 6.6 – Deviation of generation profiles (MW) from the “business-as-usual” scenario during the first hours of the day, considering three different values of temperature volatility σ .

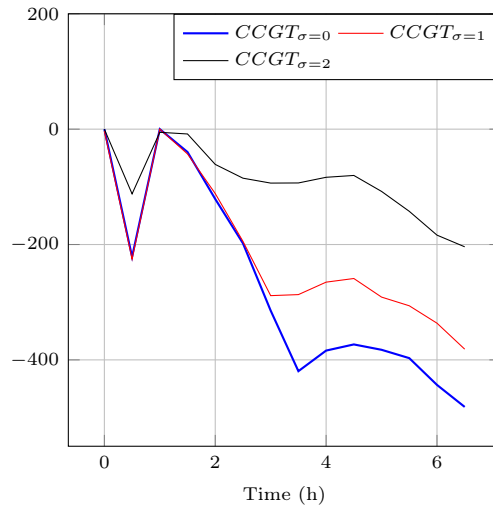


Figure 6.7 – Deviation of Frequency Response (MW) allocated by CCGT technology with respect to the “business-as-usual” scenario during the first hours of the day, considering three different values of temperature volatility σ .

	$\sigma = 0$	$\sigma = 1$	$\sigma = 2$
BAU	2.770×10^7	2.770×10^7	2.772×10^7
FS	2.719×10^7	2.725×10^7	2.740×10^7

Table 6.1 – Minimized system costs in (£)

6.8 Conclusions

Randomness and high dimensionality usually make the resolution of an optimization problem quite difficult. However, in the specific case of convex aggregative control problems, we have shown that, under independent noise assumptions, one can take advantage of the high dimension to approximate accurately the original Problem (P_1) by a decentralized Problem (\hat{P}_2), whose numerical resolution is more tractable. We highlight the fact that the approximation error is of order $\frac{1}{n}$, where n is the number of agents. The extension of this approach to stochastic control problems with common noise or to non convex problems may be challenging but interesting topics for further work.

6.9 Appendix

Lemma 6.9.1. *Let H be a Hilbert space and $f : H \rightarrow \mathbb{R}$ be l.s.c. and convex. The function f has at most quadratic growth if and only if its subgradient has linear growth.*

Proof. Let the subgradient have linear growth, that is, $\|q\|_H \leq c_1(1 + \|x\|_H)$ whenever $x \in H$ and $q \in \partial f(x)$. Then $f(x) \leq f(0) + \langle q, x \rangle_H \leq f(0) + \|q\|_H \|x\|_H \leq c_2(1 + \|x\|_H^2)$, so that f has at most quadratic growth.

Conversely, let f have at most quadratic growth. Since f is convex, one has for all $x \in H$ and $q_0 \in \partial f(0)$:

$$f(x) \geq f(0) + \langle q_0, x \rangle_H \geq -c_3(1 + \|x\|_H^2),$$

where $c_3 > 0$ depends only on $f(0)$ and q_0 . Then, using the growth assumption on f and the inequality above, one gets for all $x, y \in H$ and $q \in \partial f(x)$:

$$c_4(1 + \|y\|_H^2) \geq f(y) \geq f(x) + \langle q, y - x \rangle_H \geq -c_3(1 + \|x\|_H^2) + \langle q, y - x \rangle_H.$$

Take $y = x + \alpha q$, with $\alpha \in (0, 1)$, we get

$$2c_4(1 + \|x\|_H^2 + \alpha^2 \|q\|_H^2) \geq -c_3(1 + \|x\|_H^2) + \alpha \|q\|_H^2$$

so that $(\alpha - 2c_4\alpha^2)\|q\|_H^2 \leq (2c_4 + c_3)(1 + \|x\|_H^2)$. Take $\alpha = 1/(4c_4)$, then $\alpha - 2c_4\alpha^2 = 1/(8c_4) > 0$ and then

$$\|q\|_H^2 \leq 8c_4(2c_4 + c_3)(1 + \alpha\|x\|_H^2)$$

and the conclusion follows. □

Chapter 7

A decentralized algorithm for a MFC problem of Piecewise Deterministic Markov Processes

Outline of the current chapter

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7.1 Introduction

This paper focuses on a specific family of stochastic control problems called "large and aggregative". The problem consists in optimizing the strategies of a large number of agents, in a random environment, while the decisions of all the agents interact via the objective function because the latter depends on an "aggregative" term as the sum of the decisions of all the agents. This type of problem occurs naturally in several fields of application such as energy management [148], telecommunications [179], portfolio management [174], robot coordination [58] among others.

To deal with the curse of dimensionality in such high dimensional stochastic control problems, methods based on the value function, have been developed in the convex setting. For instance, Stochastic Dual Dynamic Programming [209] relies on polyhedral approximations of the value function providing good performances for medium sized problems with a number of agents $N \leq 30$. In our case, where the number of agents is very large $N \geq 1000$, we look more specifically for decentralized solutions, allowing both to respect the privacy of each agent and to reduce the complexity of the problem. In the particular setting, where controls are bounded and strong convexity conditions are satisfied, [236] proposed an iterative stochastic algorithm providing a decentralized solution to such problems. In the same spirit as the Dual Approximate Dynamic Programming [59], this approach relies on a Lagrangian decomposition technique to obtain a decentralized solution, but it takes advantage of a mean field approximation to ensure the validity of the approach when the number of agents is sufficiently large.

This paper aims at extending the decentralized approach developed in [236] to a non-convex framework, with

unbounded controls, allowing for more realistic dynamics ruling the evolution of each agent state. In particular, we are interested in the aggregative control of Piecewise Deterministic Markov Processes (PDMPs for short), where the state of each agent is modelled by a PDMP controlled by an unbounded jumps intensity. PDMPs were introduced in [83] as a class of non-diffusion stochastic models, mixing random jumps and deterministic dynamics between jumps. This kind of process is used to model a wide broad of phenomena or situations, such as system reliability and maintenance [87, 274], oil production [275], biology [176, 225] etc. . . The approach proposed in this paper results in a decentralized algorithm, where at each iteration, each agent is intended to solve its own small dimensional optimal control problem of PDMP. This type of problem is well known and has been studied through dynamic programming and related equations [16, 75, 86, 147, 264]. Due to the large number of agents, the problem is formulated as a Mean Field Control (MFC for short) problem. Relying on the law of large numbers, the MFC theory consists in controlling a representative agent of the population, where the interaction between the agents is approximated by the interaction of this representative agent with the probability distribution of its state or control [56].

This work is particularly motivated by demand-side management in power systems to help balancing between energy production and consumption. This problem is indeed critical today, due to the increasing share of uncontrollable energies (wind and solar) in the electricity generation mix, which requires to compensate the uncontrollable character of the production by controlling the demand. Controlling the sum of the consumption of each flexible consumer in order to balance the electrical system has already been investigated successfully in the specific framework of Quadratic Kullback-Leibler control problems [44]. In [238] and [171], a mean field assumption is also considered to control the charging of a large fleet of electrical vehicles (EVs), leading to the optimal control of partial differential equation problems.

The novelty of the present paper is to extend the Lagrangian approach proposed in [236], to a particular non-convex setting with unbounded controls by first establishing the existence of a saddle point and then ensuring the convergence of the decentralized algorithm. The first contribution lies in the originality of the proof of a saddle point existence which follows a completely different path than the one developed in [236]. In particular, we make use of regularity results on the solution of the Hamilton Jacobi equation arising in optimal control of PDMPs developed in [233] to show the existence of a saddle point of our Lagrangian problem. The second contribution consists in proving that the Stochastic Uzawa Algorithm proposed in [236] is still providing a converging sequence of controls, in this new setting involving the PDMP dynamics which violates the convexity conditions originally exploited. Finally, an application to the smart charging of an electric vehicles fleet by an aggregator illustrates the performance and the interest of the approach to coordinate the charge of a large number of electric vehicles in order to track a given target power consumption profile for the whole population of electrical vehicles.

The outline of this paper is as follows. In Section 7.2, we formulate the optimization problem, the assumptions and the main results. Section 7.3 presents a dual approach of this problem. The Stochastic Uzawa algorithm is presented in Section 7.4 and is proved to converge in this setting. Section 7.5 presents simulations of the coordination of power consumption of a large fleet of electrical vehicles.

In the rest of this section we will list some frequently used notation.

Notation Let \mathcal{X} be the state space defined by $\mathcal{X} := I \times [0, 1]$, where I is a finite set, of cardinality $d \in \mathbb{N}^*$. The space $D([0, T], I)$ is the space of càdlàg functions from $[0, T]$ to I . The space $C^1(\mathcal{X}, \mathbb{R})$ denotes the set of real-valued continuously differentiable functions defined on \mathcal{X} . The set of \mathbb{R}_+^d -valued Lipschitz continuous functions defined on $[0, T] \times \mathcal{X}$ is denoted by $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$. The space of Borel probability measures on the space \mathcal{X} is denoted by $\mathcal{P}(A)$. We recall the definition of the Wasserstein distance, denoted by \mathbb{W} , on the space $\mathcal{P}(\mathcal{X})$:

$$\mathbb{W}(\mu, \rho) := \sup \left\{ \sum_{i \in I} \int_0^1 \varphi(i, s) (\mu(i, ds) - \rho(i, ds)) \mid \varphi \text{ is } 1\text{-Lipschitz from } \mathcal{X} \text{ to } \mathbb{R} \right\}.$$

We recall that if a function φ is 1-Lipschitz continuous from \mathcal{X} to \mathbb{R} , then $|\varphi(i, x) - \varphi(j, x)| \leq 1$ for any $i, j \in I$ and $x \in [0, 1]$. The space $C([0, T], \mathcal{P}(\mathcal{X}))$ is endowed with the distance $\mathbb{W}_{\|\cdot\|_\infty}$ defined by:

$$\mathbb{W}_{\|\cdot\|_\infty}(m^1, m^2) := \sup_{t \in [0, T]} \mathbb{W}(m^1(t), m^2(t)). \quad (7.1.1)$$

Let \mathbb{H} be an Hilbert space and F be a real valued function defined on \mathbb{H} . The convex conjugate of F is denoted

by F^* and is defined for any $x \in \mathbb{H}$ by $F^*(x) := \sup_{y \in \mathbb{H}} \langle x, y \rangle_{\mathbb{H}} - F(y)$.

7.2 Problem formulation and main results

We consider the time interval $[0, T]$ with $T > 0$ and a population of N independent and identically distributed processes, controlled by a central planner via a common feedback control α . The space of feedbacks, denoted by \mathbb{A} , is defined by

$$\mathbb{A} := \{\alpha \in C^0([0, T] \times \mathcal{X}, \mathbb{R}_+^d) : \forall i \in I, \alpha_i(\cdot, i, \cdot) = 0\}, \quad (7.2.1)$$

and is endowed with the norm $\|\cdot\|_\infty$. The state of an agent, controlled by the jump intensity α , is given at time t by $X_t^\alpha = (Y_t^\alpha, Z_t^\alpha)$ with $Y_t^\alpha \in I$ and $Z_t^\alpha \in [0, 1]$. In the smart charging application in Section 7.5, Y^α represents the charging mode of a vehicle, specifying its discrete charging rate (e.g. $i = 0$ for not charging, $i = 1$ for charging and $i = -1$ for discharging). The continuous variable Z^α represents the state of charge (SoC for short) of an EV, that is the ratio of energy charged into the battery w.r.t. the total capacity of the battery. We say that the process X^α is a PDMP(b, α) controlled by a feedback control α , if $X^\alpha = (Y^\alpha, Z^\alpha)$ where Y^α is a jump process with values in I switching spontaneously, at jump times $\{T_k^\alpha\}_{k \in \mathbb{N}}$ given by a Poisson process with intensity α , while Z^α follows a deterministic dynamics between two consecutive jumps $Z_t^\alpha = \phi_{T_k^\alpha, Z_{T_k^\alpha}^\alpha}(Y_{T_k^\alpha}^\alpha, t)$ for $t \in [T_k^\alpha, T_{k+1}^\alpha)$. For any $(\tau, t, j, z) \in [0, T] \times [\tau, T] \times \mathcal{X}$, the flow ϕ is the unique solution of the ordinary differential equation:

$$\partial_t \phi_{\tau, z}(j, t) = b(j, \phi_{\tau, z}(j, t)) \text{ with } \phi_{\tau, z}(j, \tau) = z, \quad (7.2.2)$$

where b is a function given. To be more specific, following the definition of a PDMP given in [96], knowing T_k^α and $X_{T_k^\alpha}^\alpha = (Y_{T_k^\alpha}^\alpha, Z_{T_k^\alpha}^\alpha)$, one obtains $(T_{k+1}^\alpha, X_{T_{k+1}^\alpha}^\alpha)$ as follows:

$$\begin{cases} T_{k+1, j} := \inf \left\{ t \geq T_k : E_{k+1, j} < \int_{T_k}^t \alpha_j(r, X_{T_k}^\alpha) dr \right\} \\ T_{k+1}^\alpha := \min_{j \in \{1, \dots, d\}} T_{k+1, j} \\ Y_{T_{k+1}^\alpha}^\alpha = \min \{ j \in \{1, \dots, d\} : T_{k+1, j} = T_{k+1}^\alpha \} \\ Z_{T_{k+1}^\alpha}^\alpha := \phi_{Y_{T_k}^\alpha, Z_{T_k}^\alpha}(Z_{T_k}^\alpha, T_{k+1}^\alpha) \\ X_{T_{k+1}^\alpha}^\alpha = (Y_{T_{k+1}^\alpha}^\alpha, Z_{T_{k+1}^\alpha}^\alpha), \end{cases}$$

where $\{E_{k, j}\}_{k, j}$ are independent random variables following an exponential distribution of parameter 1, independent of X_0^α . Indeed, the control process $\alpha = (\alpha_j)_{j \in I}$ is such that $\alpha_j(t, x)$ represents the intensity rate, at time $t \in [0, T]$, of jumping into mode $j \in I$ when coming from state $x \in \mathcal{X}$. We initialize $T_0 = 0$ and at time $t = 0$, the law of the couple of random variables (Y_0^α, Z_0^α) is given by $m^0 \in \mathcal{P}(\mathcal{X})$. The cost function J^N for the N -agents problem is defined for any $\alpha \in \mathbb{A}$ by:

$$J^N(\alpha) := \mathbb{E} \left[\int_0^T f \left(t, \frac{1}{N} \sum_{n=1}^N p(t, X_t^{n, \alpha}) \right) dt + \frac{1}{N} \sum_{n=1}^N G(\alpha, X^{n, \alpha}) \right], \quad (7.2.3)$$

where $\{X^{n, \alpha}\}_{n \in \{1, \dots, N\}}$ are supposed to be independent PDMP(b, α) driven by d independent exponential variables. Indeed, for any $n \in \{1, \dots, N\}$, $X^{n, \alpha}$ is supposed to be controlled by the common feedback function $\alpha \in \mathbb{A}$ only depending on the agent state X^n . Besides, the function G is common to every agent and defined as the sum of a running and a terminal cost, such that for any $x := (y, z) \in D([0, T], I) \times C^0([0, T], [0, 1])$, by:

$$G(\alpha, x) := \int_0^T \left[c(t, x_t) + \sum_{j \in I} L(\alpha_j(t, x_t)) \right] dt + g(x_T).$$

From a practical point of view, the function f represents a coupling cost depending on the aggregate quantity $\frac{1}{N} \sum_{n=1}^N p(t, X_t^{n, \alpha})$, and G the individual cost. In this paper, we focus on the mean field limit control problem, corresponding to the problem with an infinite population. Thus, we introduce the function J defined, for any

$\alpha \in \mathbb{A}$, by:

$$J(\alpha) := \int_0^T f(t, \mathbb{E}[p(t, X_t^\alpha)]) dt + \mathbb{E}[G(\alpha, X^\alpha)], \quad (7.2.4)$$

where X^α is a PDMP(b, α). This paper is dedicated to the following problem:

$$\min_{\alpha \in \mathbb{A}} J(\alpha). \quad (7.2.5)$$

As the cost function (7.2.4) is nonlinear w.r.t. the expectation term, $\mathbb{E}[p(t, X_t^\alpha)]$, via the coupling cost f , Problem (7.2.5) goes beyond the scope of optimal control of PDMP. We propose to numerically solve Problem (7.2.5) by applying the Stochastic Uzawa Algorithm 3, that is detailed in Section 7.4. This algorithm is introduced in [236] and is a direct application of the stochastic gradient descent algorithm [119]. The main result of this paper, stated in Theorem 7.2.1, ensures the convergence of the Stochastic Uzawa Algorithm 3 to the solution of Problem (7.2.5).

Throughout the paper, we assume the following:

General assumptions

1. The vector field $b \in C^1(\mathcal{X}, \mathbb{R})$ is assumed to vanish at the boundary: $b(j, 0) = b(j, 1) = 0$ for any $j \in I$.
2. The function $p \in C^1([0, T] \times \mathcal{X}, \mathbb{R})$ and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function being lower semi-continuous (l.s.c. for short), strictly convex and differentiable w.r.t. the second variable. In addition, there exists a constant $C_f > 0$ such that, for any $(t, x) \in [0, T] \times \mathbb{R}$,

$$\frac{x^2}{2C_f} - C_f \leq f(t, x) \leq C_f \frac{x^2}{2} + C_f. \quad (7.2.6)$$

3. For any $i \in I$, it is assumed that $c_i \in C^1([0, T] \times [0, 1])$ and $g_i \in C^1([0, 1])$.
4. The function $L : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ is convex and defined by:

$$L(x) := \begin{cases} l(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $l \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing strongly convex function bounded from above by a quadratic function. More explicitly, there exists $C > 0$ such that for any $x \in \mathbb{R}_+$:

$$\frac{x^2}{C} - C \leq l(x) \leq C(x^2 + 1),$$

where the first inequality is due to the strong convexity of l . We denote by H be the convex conjugate of L . Note that by Assumption 4, H is non-decreasing, non-negative, and H' is Lipschitz continuous on \mathbb{R} .

Remark 7.2.1. *The main role of Assumption 1 is to ensure that the flow defined in (7.2.2) exists and takes values in $[0, 1]$. Assumption 2 ensures that $\int_0^T f(t, \mathbb{E}[p(t, X_t^\alpha)]) dt$ in (7.2.4) is well defined for any $\alpha \in \mathbb{A}$. The continuity of the map $\alpha \mapsto \mathbb{E}[G(\alpha, X^\alpha)]$ is an automatic consequence of the properties of c, g and L given in Assumptions 3 and 4.*

Remark 7.2.2. *The boundary condition on b may seem restrictive. However, if $\overline{\text{supp}(m^0)} \subset I \times (0, 1)$ and the vector field b is such that, for any $i \in I$, the sign of $b(i, \cdot)$ is constant, $b(i, 0) \geq 0$ and $b(i, 1) \leq 0$, then there exists $\varepsilon > 0$ such that $\mathbb{P}(Z_t^{n, \alpha} \in (\varepsilon, 1 - \varepsilon)) = 1$ for any $n \in \{1, \dots, N\}$, any $\alpha \in \mathbb{A}$ and any $t \in [0, T]$. Therefore, it is possible to approximate such a vector field b by a smooth function, vanishing at the boundary without modifying the trajectory of $Z_t^{n, \alpha}$.*

The main results of the paper are summarized in the following theorem.

Theorem 7.2.1. *Problem 7.2.5 has a solution. Let $\{\alpha^k\}_{k \in \mathbb{N}}$ be a sequence in \mathbb{A} generated by Algorithm 3 (line 5), then the following assertions hold*

- (i) *The sequence $\{\alpha^k\}_{k \in \mathbb{N}}$ converges a.s. to an element of $\arg \min_{\alpha \in \mathbb{A}} J(\alpha)$ w.r.t. the norm $\|\cdot\|_\infty$.*

(ii) The sequence $\{J(\alpha^k)\}_{k \in \mathbb{N}}$ converges a.s. to $\min_{\alpha \in \mathbb{A}} J(\alpha)$.

This theorem can not be obtained by a direct application of [236], which relies on additional assumptions, that are not verified in the present framework, which involves unbounded controls and PDMP dynamics. The proof of Theorem 7.2.1 is given in Section 7.4. Before, we introduce and analyze in the next section a dual problem proved to be equivalent, in some sense, to Problem 7.2.5.

7.3 Dual approach

In the same vein as [236], a Lagrangian decomposition approach is adopted to obtain a decentralized algorithm. Contrary to [236], the specific assumptions of convexity, that ensure the absence of duality gap and the convergence of the Stochastic Uzawa Algorithm, are not satisfied here. In Sections 7.3 and 7.4, we propose new theoretical arguments allowing to demonstrate the validity of the approach proposed in [236] in this specific framework of control of PDMP.

Let F be defined for any $v \in L^2(0, T)$ by:

$$F(v) := \int_0^T f(t, v(t)) dt.$$

Owing to the properties of f , one deduces that function F is l.s.c., strictly convex, differentiable with a Lipschitz continuous gradient, and it has at least quadratic growth.

Lemma 7.3.1. *The problem*

$$\begin{cases} \min_{\alpha \in \mathbb{A}, v \in L^2(0, T)} \bar{J}(\alpha, v), \\ \bar{J}(\alpha, v) := F(v) + \mathbb{E}[G(\alpha, X^\alpha)], \\ \text{s.t. } \mathbb{E}[p(t, X_t^\alpha)] - v(t) = 0 \text{ a.e. on } [0, T], \end{cases} \quad (7.3.1)$$

is equivalent to Problem (7.2.5).

Proof. The fact that $p \in C^1([0, T] \times \mathcal{X}, \mathbb{R})$ implies that the map $t \mapsto \mathbb{E}[p(t, X_t^\alpha)]$ is in $L^2(0, T)$. The equivalence between the two problems then follows from the definitions of J and \bar{J} . \square

Let us introduce the Lagrangian $\mathcal{L} : \mathbb{A} \times L^2(0, T) \times L^2(0, T) \rightarrow \mathbb{R}$:

$$\mathcal{L}(\alpha, v, \lambda) := \mathcal{L}_1(\alpha, \lambda) + \mathcal{L}_2(v, \lambda), \quad (7.3.2)$$

where

$$\begin{aligned} \mathcal{L}_1(\alpha, \lambda) &:= G(\alpha, X^\alpha) + \int_0^T \mathbb{E}[p(t, X_t^\alpha)] \lambda(t) dt, \\ \mathcal{L}_2(v, \lambda) &:= F(v) - \int_0^T v(t) \lambda(t) dt, \end{aligned}$$

and the associated dual function $\mathcal{W} : L^2(0, T) \rightarrow \mathbb{R}$:

$$\mathcal{W}(\lambda) := \inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda) + \inf_{v \in L^2(0, T)} \mathcal{L}_2(v, \lambda). \quad (7.3.3)$$

The dual problem associated with (7.3.1) consists of the following maximization problem:

$$\max_{\lambda \in L^2(0, T)} \mathcal{W}(\lambda). \quad (7.3.4)$$

The following lemma gives the existence of a unique solution of the dual problem (7.3.4). It is derived from the assumptions on F and G .

Lemma 7.3.2. *There exists a unique $\bar{\lambda} \in L^2(0, T)$ such that $\bar{\lambda} = \arg \max_{\lambda \in L^2(0, T)} \mathcal{W}(\lambda)$.*

Proof. Since F has a Lipschitz continuous gradient, the function $\lambda \mapsto F^*(\lambda) := \sup_{v \in L^2(0, T)} \langle v, \lambda \rangle - F(v)$ is strongly convex and l.s.c. [141], and so is the function $\lambda \mapsto - \inf_{v \in L^2(0, T)} \mathcal{L}_2(v, \lambda)$. Similarly, for any $\lambda \in L^2(0, T)$, the map $\lambda \mapsto - \inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$ is convex and l.s.c.. One deduces that $-\mathcal{W}$ is strongly convex and l.s.c. on $L^2(0, T)$. Thus, problem (7.3.4) has a unique solution. \square

The below proposition is a key result, that enables to show the convergence of a sequence $\{\alpha^k\}_k$ generated by Algorithm 7.4 defined in the next section, to a solution of Problem (7.2.5).

Theorem 7.3.1. *For any $\lambda \in L^2(0, T)$, $\arg \min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$ is not empty. In addition, there exists a selection*

$$\lambda \mapsto \alpha[\lambda] \in \arg \min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda), \quad (7.3.5)$$

such that the map $L^2(0, T) \ni \lambda \mapsto \alpha[\lambda] \in \mathbb{A}$ is Lipschitz continuous on any bounded subset of $L^2(0, T)$.

The selection $\alpha[\lambda] \in \arg \min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$ is first specified for any $\lambda \in L^\infty(0, T)$ in Lemma 7.3.4, and then extended to any $\lambda \in L^2(0, T)$ in the proof of Theorem 7.3.1. Theorem 7.3.1 allows to prove that the dual function \mathcal{W} is Gâteaux differentiable in $L^2(0, T)$, that is the main argument of the proof of Theorem 7.3.2. Theorem 7.3.2 gives the existence of a saddle point of the Lagrangian function \mathcal{L} . This result provides a solution of Problem (7.2.5).

Theorem 7.3.2. *There is no duality gap associated with Problem (7.3.4), i.e.,*

$$\max_{\lambda \in L^2(0, T)} \mathcal{W}(\lambda) = \min_{\alpha \in \mathbb{A}, v \in L^2(0, T)} \bar{J}(\alpha, v).$$

Besides, $\arg \min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \bar{\lambda})$ and $\arg \min_{v \in L^2(0, T)} \mathcal{L}_2(v, \bar{\lambda})$ are not empty and, for any $\bar{\alpha} \in \arg \min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \bar{\lambda})$ and $\bar{v} \in \arg \min_{v \in L^2(0, T)} \mathcal{L}_2(v, \bar{\lambda})$, $((\bar{\alpha}, \bar{v}), \bar{\lambda})$ is a saddle point of the Lagrangian \mathcal{L} and $\bar{\alpha}$ is a solution of Problem (7.2.5).

The main argument for the proof of Theorem 7.3.2 is that the map $\lambda \mapsto \mathcal{W}(\lambda)$ is Gâteaux differentiable in $L^2(0, T)$. We show that $\lambda \mapsto \inf_{v \in L^2(0, T)} \mathcal{L}_2(v, \lambda)$ and $\lambda \mapsto \inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$ are both differentiable. The second result is more difficult to prove. Different intermediary results are needed. First, properties of the sub problem w.r.t. the variable v in the dual problem (7.3.4) are given in Lemma 7.3.3. Then, properties of the sub problem w.r.t. the variable α are proved in Lemma 7.3.4 in the special case where $\lambda \in L^\infty(0, T)$. These properties are extended in Theorem 7.3.1 to the case where $\lambda \in L^2(0, T)$, with the help of Lemma 7.3.5 and of Lemma 7.6.1 in Appendix 7.6. Finally, Lemma 7.3.6 shows that the map $\lambda \mapsto \inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$ is Gâteaux differentiable in $L^2(0, T)$. These lemmas are stated and proved below.

7.3.1 Proof of Theorem 7.3.1

The next Lemma shows that, for any $\lambda \in L^2(0, T)$, there exists a unique solution $v[\lambda]$ of the sub problem and that the map $\lambda \mapsto \mathcal{L}_2(v[\lambda], \lambda)$ is Gâteaux differentiable.

Lemma 7.3.3.

- (i) *For any $\lambda \in L^2(0, T)$, there exists a unique $v[\lambda]$ satisfying $v[\lambda] = \arg \min_{v \in L^2(0, T)} \mathcal{L}_2(v, \lambda)$.*
- (ii) *The map $\lambda \mapsto \min_{v \in L^2(0, T)} \mathcal{L}_2(v, \lambda)$ admits a Gâteaux derivative $\lambda \mapsto D\mathcal{L}_2(v[\lambda], \lambda)(\cdot)$ in $L^2(0, T)$, which satisfies that, for any $\mu \in L^2(0, T)$:*

$$D\mathcal{L}_2(v[\lambda], \lambda)(\mu) = - \int_0^T v[\lambda](\tau) \mu(\tau) d\tau. \quad (7.3.6)$$

Proof. (i) Since the map $v \mapsto F(v)$ is l.s.c., strictly convex with at least quadratic growth, one can deduce the existence and uniqueness of $v[\lambda] := \arg \min_{v \in L^2(0,T)} \mathcal{L}_2(v, \lambda)$.

(ii) Since the function F is proper, strictly convex and differentiable, classical results from convex analysis give that its convex conjugate F^* is differentiable [141]. Therefore, the map $\lambda \mapsto \mathcal{L}_2(v[\lambda], \lambda)$ is differentiable on $L^2(0, T)$, with derivative $\lambda \mapsto v[\lambda]$ \square

Regularity results about $\alpha[\lambda]$, defined in (7.3.5), are stated in the next Lemma for any $\lambda \in L^\infty(0, T)$.

Lemma 7.3.4. *Let $\lambda \in L^\infty(0, T)$. Then, one has:*

- (i) *The value function u associated with the stochastic control problem $\inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$ is the unique solution of the associated Hamilton-Jacobi-Bellman equation (in the sense of [147, Theorem 3.4]) on $[0, T] \times I \times [0, 1]$:*

$$\begin{aligned} & -\partial_t u(t, i, z) - b(i, z) \partial_z u(t, i, z) - c(t, i, z) - \lambda(t) p(t, i, z) \\ & = \inf_{(a_{i,j})_{j \geq 0}} \sum_{j \in I, j \neq i} L(a_{i,j}) + (u(t, j, z) - u(t, i, z)) a_{i,j}, \end{aligned} \quad (7.3.7)$$

with the terminal condition: $u(T, i, z) = g(i, z)$ for any $(i, z) \in \mathcal{X}$.

- (ii) *We set the selection $\alpha[\lambda] \in \inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$ to be given, for any $i, j \in I$ and $(t, z) \in [0, T] \times [0, 1]$, by,*

$$\alpha[\lambda]_j(t, i, z) = H'(u(t, i, z) - u(t, j, z)).$$

Then, the function $\alpha[\lambda]$ is in $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$.

- (iii) *The function $L^\infty(0, T) \ni \lambda \mapsto \alpha[\lambda] \in \mathbb{A}$ is Lipschitz continuous, w.r.t. the norm $\|\cdot\|_2$, on any bounded subset of $L^\infty(0, T)$.*

Remark 7.3.1. *The proof of Lemma 7.3.4 essentially relies on [147, Theorem 3.4]. The assumptions needed for the application of this theorem require bounded running cost in the minimization problem $\min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$. Thus, we only consider $\lambda \in L^\infty(0, T)$ and not in $L^2(0, T)$ in the statement of Lemma 7.3.4. The extension of the results of Lemma 7.3.4 for $\lambda \in L^2(0, T)$ is stated in Theorem 7.3.1.*

Proof. (i) The problem $\inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$ is a stochastic control problem of PDMP. Let u be the value function of this problem, defined on $[0, T] \times I \times [0, 1]$ by:

$$\begin{aligned} u(t, i, z) := & \inf_{\alpha \in \mathbb{A}} \int_t^T \sum_{j \in I, j \neq i} \mathbb{E}[H(\alpha_j(\tau, X_\tau^\alpha)) \mid X_t^\alpha = (i, z)] d\tau \\ & + \int_t^T \mathbb{E}[c(\tau, X_\tau^\alpha) + \lambda(\tau) p(\tau, X_\tau^\alpha) \mid X_t^\alpha = (i, z)] d\tau \\ & + \mathbb{E}[g(X_T^\alpha) \mid X_t^\alpha = (i, z)] \quad \text{for a.e. } (t, z) \in (0, T) \times (0, 1). \end{aligned}$$

By [147, Theorem 3.4], the value function u is the unique function satisfying for a.e. $(t, z, i) \in (0, T) \times I \times (0, 1)$:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{u(t, i, z) - u(t+h, i, \phi_{t,z}(i, t+h))}{h} & = c(t, i, z) + p(t, i, z) \lambda(t) \\ & + \inf_{(a_{i,j})_{j \geq 0}} \sum_{j \in I, j \neq i} L(a_{i,j}) + (u(t, j, z) - u(t, i, z)) a_{i,j}, \end{aligned}$$

with $u(T, i, z) = g(i, z)$ on \mathcal{X} . By [147] and the definition of ϕ in (7.2.2), u satisfies for any $i \in I$ and a.e. $(t, z) \in [0, T] \times [0, 1]$:

$$\lim_{h \rightarrow 0} \frac{u(t, i, z) - u(t+h, i, \phi_{t,z}(i, t+h))}{h} = -(\partial_t u(t, i, z) + b(i, z) \partial_z u(t, i, z)).$$

The conclusion follows from the two previous equalities.

(ii) By taking the infimum on the r.h.s. of the equation (7.3.7) and recalling that H is the convex conjugate of L , one obtains that u satisfies for a.e. $(t, z, i) \in (0, T) \times I \times (0, 1)$:

$$-\partial_t u(t, i, z) - b(i, z) \partial_z u(t, i, z) - c_i(t, i, z) - \lambda(t) p(t, i, z) = - \sum_{j \in I, j \neq i} H(u(t, i, z) - u(t, j, z)), \quad (7.3.8)$$

with $u(T, i, z) = g(i, z)$ on \mathcal{X} and where $x^+ := \max(0, x)$. By [233], since $\lambda \in L^\infty$ and $p \in C^1([0, T] \times \mathcal{X}, \mathbb{R})$, there exists a unique function \bar{u} which satisfies for a.e. $t \in [0, T]$ and for any $(i, z) \in \mathcal{X}$:

$$\begin{aligned} \bar{u}(t, i, z) = & \int_t^T \sum_{j \in I} -H(\bar{u}(\tau, i, \phi_{t,z}(i, \tau)) - \bar{u}(\tau, j, \phi_{t,z}(i, \tau))) d\tau \\ & + \int_t^T c(\tau, i, \phi_{t,z}(i, \tau)) + \lambda(\tau) p(\tau, i, \phi_{t,z}(i, \tau)) d\tau + g(i, \phi_{t,z}(i, T)). \end{aligned}$$

Since \bar{u} is in $\text{Lip}([0, T] \times \mathcal{X})$, the set of real valued Lipschitz continuous functions defined on $[0, T] \times \mathcal{X}$, \bar{u} satisfies, for a.e. $(t, i, z) \in [0, T] \times \mathcal{X}$, the equation (7.3.8). Therefore, \bar{u} is the value function associated with the stochastic control problem $\inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$. It comes that $u = \bar{u}$. Setting, for any $(t, z, i, j) \in [0, T] \times [0, 1] \times I \times I$,

$$\alpha[\lambda]_j(t, i, z) := \arg \min_{a \geq 0} L(a) + (u(t, j, z) - u(t, i, z))a = H'(u(t, i, z) - u(t, j, z)), \quad (7.3.9)$$

by [147, Theorem 3.4], $\alpha[\lambda]$ is in $\arg \min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$. Since $u \in \text{Lip}([0, T] \times \mathcal{X})$, $\alpha[\lambda]$ is in $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$.

(iii) Let $k > 0$ and $\lambda, \mu \in L^\infty(0, T)$ be such that $\|\lambda\|_2 \leq k$ and $\|\mu\|_2 \leq k$. Then, by the same arguments as in the proof of [233, Lemma 4.5], there exists a constant $K(k, T) > 0$ such that for any $(t, i, j, z) \in [0, T] \times I \times I \times [0, 1]$:

$$|\alpha_j[\lambda](t, i, s) - \alpha_j[\mu](t, i, s)| \leq K(k, T) \|p\|_\infty \|\lambda - \mu\|_2.$$

The result follows from taking the supremum over $[0, T] \times \mathcal{X}$ of the l.h.s. of the previous inequality. \square

Lemma 7.3.5 enables to show the continuity of the map $(\alpha, \lambda) \mapsto \mathcal{L}_1(\alpha, \lambda)$.

Lemma 7.3.5. *The map $(\alpha, \lambda) \mapsto \mathcal{L}_1(\alpha, \lambda)$ is continuous on $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d) \times L^2(0, T)$, where $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d) \times L^2(0, T)$ is endowed with the norm $(\alpha, \lambda) \mapsto \|\alpha\|_\infty + \|\lambda\|_2$.*

Proof. Recalling the expression of H , one has for any $(\alpha, \lambda) \in \text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d) \times L^2(0, T)$:

$$\begin{aligned} \mathcal{L}_1(\alpha, \lambda) = & \int_0^T \int_0^1 \sum_{i \in I} c(t, i, z) m^\alpha(t, i, dz) dt + \int_0^T \int_0^1 \sum_{i \in I} \lambda(t) p(t, i, z) m^\alpha(t, i, dz) dt \\ & + \int_0^T \int_0^1 \sum_{i, j \in I} L(\alpha_j(t, i, z)) m^\alpha(t, i, dz) dt + \int_0^1 \sum_{i \in I} g(i, z) m^\alpha(T, i, dz). \end{aligned}$$

Let $\{(\alpha^n, \lambda^n)\}_n$ be a sequence in $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d) \times L^2(0, T)$, converging to (α, λ) . The convergence of $\{m^{\alpha^k}\}_k$ to m^α , w.r.t. the distance $\mathbb{W}_{\|\cdot\|_\infty}$, is derived from Lemma 7.6.1 in Appendix 7.6. Thus, by the regularity of c, g, p and L in Assumptions 2-4, the maps $t \mapsto \sum_{i \in I} \int_0^1 p(t, i, z) m^{\alpha^n}(t, i, dz)$, $t \mapsto \sum_{i \in I} \int_0^1 c(t, i, z) m^{\alpha^n}(t, i, dz)$, $t \mapsto \sum_{i \in I} \int_0^1 L(\alpha_j^n(t, i, z)) m^{\alpha^n}(t, i, dz)$ converge, w.r.t. the norm $\|\cdot\|_\infty$, when n tends to infinity, respectively to $t \mapsto \sum_{i \in I} \int_0^1 p(t, i, z) m^\alpha(t, i, dz)$, $t \mapsto \sum_{i \in I} \int_0^1 c(t, i, z) m^\alpha(t, i, dz)$, $t \mapsto \sum_{i \in I} \int_0^1 L(\alpha_j(t, i, z)) m^\alpha(t, i, dz)$. Similarly $\sum_{i \in I} \int_0^1 g(i, z) m^{\alpha^n}(T, i, dz)$ converges when n tends to infinity to $\sum_{i \in I} \int_0^1 g(i, z) m^\alpha(T, i, dz)$. The dominated convergence theorem and the convergence of $\{\lambda^k\}_k$ to λ w.r.t. the norm $\|\cdot\|_2$ achieves the proof. \square

We are now ready to prove Theorem 7.3.1. The results obtained in Lemma 7.3.4 for any $\lambda \in L^\infty(0, T)$ can now be extended to any $\lambda \in L^2(0, T)$.

Proof of Theorem 7.3.1. Let $\lambda \in L^2(0, T)$. Since $L^\infty(0, T)$ is dense in $L^2(0, T)$, one can extract a sequence $\{\lambda^k\}_k$ in $L^\infty(0, T)$ converging to $\lambda \in L^2(0, T)$.

By [233, Remark 4.5], for any $k \in \mathbb{N}^*$, the solution u^k of (7.3.7) associated with λ^k is such that upper bounds on $\|u^k\|_\infty$ and $\|\partial_s u^k\|_\infty$ depend on the data of the problem and on $\|\lambda^k\|_2$. Since $\{\lambda_k\}_k$ is converging w.r.t. the norm $L^2(0, T)$, the sequence $\{\|\lambda^k\|_2\}_k$ is uniformly bounded. Thus, by the Lipschitz continuity of the map H' , guaranteed by Assumption 4, and by equality (7.3.9), the sequence $\{\alpha[\lambda^k]\}_k$ is uniformly bounded and uniformly Lipschitz. By the Ascoli-Arzelà Theorem, there exists $\hat{\alpha} \in \mathbb{A}$ such that $\{\alpha[\lambda^k]\}_k$ uniformly converges, up to a subsequence, to $\hat{\alpha} \in \mathbb{A}$ w.r.t. the norm $\|\cdot\|_\infty$. In addition, since $\{\alpha[\lambda^k]\}_k$ is uniformly Lipschitz continuous, $\hat{\alpha}$ is also Lipschitz continuous on $[0, T] \times \mathcal{X}$. Let $\alpha \in \mathbb{A}$, by the definition of $\lambda \mapsto \alpha[\lambda]$, one has, for any $k \in \mathbb{N}$,

$$0 \leq \mathcal{L}_1(\alpha, \lambda^k) - \mathcal{L}_1(\alpha[\lambda^k], \lambda^k).$$

According to Lemma 7.3.5, $(\alpha, \lambda) \mapsto \mathcal{L}_1(\alpha, \lambda)$ is continuous on $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d) \times L^2(0, T)$. Therefore by taking the limit $k \rightarrow \infty$ in the previous inequality, one gets:

$$0 \leq \mathcal{L}_1(\alpha, \lambda) - \mathcal{L}_1(\hat{\alpha}, \lambda).$$

Thus, $\hat{\alpha} \in \arg \min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$. Defining $\alpha[\lambda] := \hat{\alpha}$ and using the uniform convergence of $\{\alpha[\lambda^k]\}_k$ to $\alpha[\lambda]$, we deduce that $\lambda \mapsto \alpha[\lambda]$ is continuous from $L^2(0, T)$ to \mathbb{A} . Since $L^\infty(0, T)$ is dense in $L^2(0, T)$ and $\lambda \mapsto \alpha[\lambda]$ is continuous in $L^2(0, T)$, the Lipschitz continuity of $\lambda \mapsto \alpha[\lambda]$ on any bounded subset of $L^2(0, T)$ is derived from Lemma 7.3.4.(iii). \square

7.3.2 Proof of Theorem 7.3.2

Note that rewriting Problem (7.2.5) w.r.t. to the distribution of the states, one can prove Theorem 7.3.2 by using a change of variable, as it has been done in [233] for a similar problem. To avoid additional notations, we have preferred to prove Theorem 7.3.2 by means of Theorem 7.3.1.

The next lemma states that the map $\lambda \mapsto \inf_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$ is Gâteaux differentiable in $L^2(0, T)$.

Lemma 7.3.6. *The map $\lambda \mapsto \min_{\alpha \in \mathbb{A}} \mathcal{L}_1(\alpha, \lambda)$ admits a Gâteaux derivative $\lambda \mapsto D\mathcal{L}_1(\alpha[\lambda], \lambda)(\cdot)$ in $L^2(0, T)$, such that, for any $\mu \in L^2(0, T)$:*

$$D\mathcal{L}_1(\alpha[\lambda], \lambda)(\mu) = \int_0^T \mathbb{E} \left[p(\tau, X_\tau^{\alpha[\lambda]}) \right] \mu(\tau) d\tau.$$

Proof. For any $\lambda \in L^2(0, T)$, we recall that the definition of the Gâteaux derivative $\lambda \mapsto D\mathcal{L}_1(\alpha[\lambda], \lambda)(\cdot)$ in the direction $\mu \in L^2(0, T)$, if it exists, is:

$$D\mathcal{L}_1(\alpha[\lambda], \lambda)(\mu) := \lim_{h \rightarrow 0} \frac{\mathcal{L}_1(\alpha[\lambda + h\mu], \lambda + h\mu) - \mathcal{L}_1(\alpha[\lambda], \lambda)}{h}.$$

For any $h \in [0, T]$, we have:

$$\begin{aligned} \int_0^T \mathbb{E} \left[p(\tau, X_\tau^{\alpha[\lambda + h\mu]}) \right] \mu(\tau) d\tau &= \frac{\mathcal{L}_1(\alpha[\lambda + h\mu], \lambda + h\mu) - \mathcal{L}_1(\alpha[\lambda + h\mu], \lambda)}{h} \\ &\leq \frac{\mathcal{L}_1(\alpha[\lambda + h\mu], \lambda + h\mu) - \mathcal{L}_1(\alpha[\lambda], \lambda)}{h} \\ &\leq \frac{\mathcal{L}_1(\alpha[\lambda], \lambda + h\mu) - \mathcal{L}_1(\alpha[\lambda], \lambda)}{h} \\ &= \int_0^T \mathbb{E} \left[p(\tau, X_\tau^{\alpha[\lambda]}) \right] \mu(\tau) d\tau. \end{aligned} \tag{7.3.10}$$

Since $\lim_{h \rightarrow 0} \lambda + h\mu = \lambda$ in $L^2(0, T)$, it turns out from Theorem 7.3.1 that $\lim_{h \rightarrow 0} \alpha[\lambda + h\mu] = \alpha[\lambda] \in \text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$

w.r.t. the norm $\|\cdot\|_\infty$. Since $p \in C^1([0, T] \times \mathcal{X}, \mathbb{R})$, one deduces by Lemma 7.6.1 in Appendix 7.6:

$$\begin{aligned} \lim_{h \rightarrow 0} \int_0^T \mathbb{E} \left[p(\tau, X_\tau^{\alpha[\lambda+h\mu]}) \right] \mu(\tau) d\tau &= \lim_{h \rightarrow 0} \int_0^T \sum_{i \in I} \int_0^1 p(\tau, i, z) m^{\alpha[\lambda+h\mu]}(\tau, i, dz) \mu(\tau) d\tau \\ &= \int_0^T \sum_{i \in I} \int_0^1 p(\tau, i, z) m^{\alpha[\lambda]}(\tau, i, dz) \mu(\tau) d\tau \\ &= \int_0^T \mathbb{E} \left[p(\tau, X_\tau^{\alpha[\lambda]}) \right] \mu(\tau) d\tau. \end{aligned}$$

The conclusion follows by taking the limit $h \rightarrow 0$ in (7.3.10). \square

We now turn to the proof of Theorem 7.3.2.

Proof of Theorem 7.3.2. According to Lemma 7.3.3.(ii) and Lemma 7.3.6, one has that \mathcal{W} is Gâteaux differentiable in $L^2(0, T)$. For any $\lambda \in L^2(0, T)$, the Gâteaux differential of \mathcal{W} at λ is denoted by $D\mathcal{W}(\lambda)(\cdot)$ and is given in the direction $\mu \in L^2(0, T)$ by:

$$D\mathcal{W}(\lambda)(\mu) = \int_0^T \left(\mathbb{E} \left[p(\tau, X_\tau^{\alpha[\lambda]}) \right] - v[\lambda](\tau) \right) \mu(\tau) d\tau. \quad (7.3.11)$$

By the definition of $\bar{\lambda}$ given in Lemma 7.3.2, it results that $0 \in \partial(-\mathcal{W}(\bar{\lambda}))$. Since $-\mathcal{W}$ is convex and Gâteaux differentiable, its sub-differential is reduced to a singleton and is equal to $-D\mathcal{W}$. One has, for any $\mu \in L^2(0, T)$, by the previous equality:

$$\int_0^T \left(\mathbb{E} \left[p(\tau, X_\tau^{\alpha[\bar{\lambda}]}) \right] - v[\bar{\lambda}](\tau) \right) \mu(\tau) d\tau = \langle \partial(-\mathcal{W}(\bar{\lambda})), \mu \rangle = 0.$$

Therefore, one has for a.e. $t \in [0, T]$:

$$\mathbb{E} \left[p(\tau, X_t^{\alpha[\bar{\lambda}]}) \right] - v[\bar{\lambda}](t) = 0.$$

By the previous equality, one has $\mathcal{L}(\alpha[\bar{\lambda}], v[\bar{\lambda}], \bar{\lambda}) = \mathcal{L}(\alpha[\bar{\lambda}], v[\bar{\lambda}], \lambda)$ for any $\lambda \in L^2(0, T)$ and, for any $\alpha \in \mathbb{A}$ and $v \in L^2(0, T)$, satisfying $\mathbb{E}[p(t, X_t^\alpha)] - v(t) = 0$ a.e on $[0, T]$, one has:

$$J(\alpha[\bar{\lambda}]) = \bar{J}(\alpha[\bar{\lambda}], v[\bar{\lambda}]) = \mathcal{L}(\alpha[\bar{\lambda}], v[\bar{\lambda}], \bar{\lambda}) \leq \mathcal{L}(\alpha, v, \bar{\lambda}) = \bar{J}(\alpha, v) = J(\alpha).$$

\square

7.4 Stochastic Uzawa Algorithm

While the existence of a unique solution $\bar{\lambda}$ of the dual problem (7.3.4) is established in Lemma 7.3.2, we propose in this section an iterative algorithm converging to $\bar{\lambda}$. This algorithm is directly derived from the gradient descent in Hilbert space [119], and has been introduced in [236] to solve dual problems. This algorithm is an adaptation of the Uzawa Algorithm in stochastic optimization settings. The sub gradient of the opposite of the dual function \mathcal{W} is estimated by Monte Carlo simulations. Convexity assumptions introduced in [236] are not satisfied here.

We consider a sequence $\{\rho_k\}_k$ satisfying $\sum_{k=0}^{\infty} \rho_k = \infty$ and $\sum_{k=0}^{\infty} \rho_k^2 < \infty$ to ensure the convergence. Typically, consider the sequence defined by $\rho_k = \frac{a}{b+k}$ with $a > 0$ and $b > 0$ chosen empirically to accelerate the convergence.

The next Lemma shows that it is possible at any iteration k to compute $\alpha[\lambda^k]$ defined in line 5 of the Algorithm 3.

Lemma 7.4.1. *For any $k \in \mathbb{N}$, $\lambda^k \in L^\infty(0, T)$ a.s..*

Algorithm 3 Stochastic Uzawa

-
- 1: Initialization $\lambda^0 \in L^\infty(0, T)$, set $\{\rho_k\}$ and $M \in \mathbb{N}^*$
 - 2: $k \leftarrow 0$.
 - 3: **for** $k = 0, 1, \dots$ **do**
 - 4: $v^k \leftarrow v[\lambda^k]$ where $v[\lambda^k]$ is defined in Lemma 7.3.3.
 - 5: $\alpha^k \leftarrow \alpha[\lambda^k]$ where $\alpha[\lambda^k]$ is defined in Lemma 7.3.4.
 - 6: Generate M independent states realizations $(X^{1, \alpha^k}, \dots, X^{M, \alpha^k})$ independent of previous iterations simulations.
 - 7: $U^{k+1} \leftarrow v^k - \frac{1}{M} \sum_{j=1}^M p(\cdot, X^{j, \alpha^k})$.
 - 8: $\lambda^{k+1} \leftarrow \lambda^k + \rho_k U^{k+1}$.
-

Proof. The proof is done by induction. For $k = 0$, λ^0 is initialized as an element of $L^\infty(0, T)$. Assume now that $\lambda^k \in L^\infty(0, T)$ for an integer k . Since $p \in C^1([0, T] \times \mathcal{X}, \mathbb{R})$, one has that a.s. $\frac{1}{M} \sum_{j=1}^M p(\cdot, X^{j, \alpha^k}) \in L^\infty(0, T)$.

We show now that $v[\lambda^k] \in L^\infty(0, T)$. Since f is l.s.c., it is a normal integrand. By the exchange property [222, Theorem 14.60], one has:

$$\inf_{v \in L^2(0, T)} \int_0^T f(t, v(t)) - v(t) \lambda^k(t) dt = \int_0^T \inf_{v \in \mathbb{R}} f(t, v) - v \lambda^k(t) dt.$$

and for a.e. $t \in [0, T]$:

$$v[\lambda^k](t) \in \arg \min_{v \in \mathbb{R}} f(t, v) - v \lambda^k(t). \quad (7.4.1)$$

For any $t \in [0, T]$, $f^*(t, \cdot)$ denotes the convex conjugate of $f(t, \cdot)$. Since for any $t \in [0, T]$, $f(t, \cdot)$ is strictly convex and by inequality (7.2.6) has at most quadratic growth, $f^*(t, \cdot)$ is diferentiable, with linear growth [236, Lemma A.1]. Then, there exists a constant $C > 0$, such that for any $t \in [0, T]$ one has:

$$|\nabla f^*(t, \lambda(t))| \leq C(|\lambda(t)| + 1),$$

where $\nabla f^*(t, \cdot)$ is the derivative of $z \mapsto f^*(t, z)$. By (7.4.1), one has, for a.e. $t \in [0, T]$, that $v[\lambda^k](t) = \nabla f^*(t, \lambda(t))$. By previous inequality, one deduces that

$$\|v[\lambda^k]\|_\infty \leq C(\|\lambda^k\|_\infty + 1). \quad (7.4.2)$$

The conclusion follows from the definition of λ^{k+1} at line 8 of Algorithm 3, previous inequality and the fact that a.s. $\frac{1}{M} \sum_{j=1}^M p(\cdot, X^{j, \alpha^k}) \in L^\infty(0, T)$. \square

The next Lemma provides an estimate on the norm of the gradient of \mathcal{W} , that is crucial to show the convergence of Algorithm 3.

Lemma 7.4.2. *There exists $C > 0$ such that the sequence $\{U^{k+1}\}_k$ generated by Algorithm 3 (line 7) satisfies, for any $k \in \mathbb{N}$,*

$$\mathbb{E}[\|U^{k+1}\|_2] \leq C(\|\lambda^k\|_2 + 1).$$

Proof. By Assumption 2, p is in $C^1([0, T] \times \mathcal{X}, \mathbb{R})$ thus, there exists $C > 0$, such that, for any $k \in \mathbb{N}$ and $j \in \{1, \dots, M\}$:

$$\mathbb{E}[\|p(\cdot, X^{j, \alpha^k})\|_2] \leq C.$$

The conclusion follows from previous inequality, inequality (7.4.2) in the proof of Lemma 7.4.1 and the definition of U^{k+1} at line 7 of Algorithm 3. \square

The convergence of Algorithm 3 is stated in the next lemma.

Lemma 7.4.3. *The sequence $\{\lambda^k\}_k$, generated by Algorithm 3, converges to $\bar{\lambda}$ a.s. in $L^2(0, T)$.*

Proof. Since \mathcal{W} is concave and Gateau differentiable (proof of Theorem 7.3.2), by Lemma 7.4.2, the proof is a simple adaptation of the one in [119, Theorem 3.6], as it has been done in [236]. \square

Before to give the proof of the main result of this paper, we need the following regularity result on the function J .

Lemma 7.4.4. *The map J defined in (7.2.4) is continuous on $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$.*

Proof. By the definition of m^α , the map J can be rewritten as:

$$\begin{aligned} J(\alpha) = & \int_0^T f\left(t, \sum_{i \in I} \int_0^1 p(t, i, z) m^\alpha(t, i, dz)\right) dt + \int_0^T \int_0^1 \sum_{i \in I} c(t, i, z) m^\alpha(t, i, dz) dt \\ & + \int_0^T \int_0^1 \sum_{i, j \in I} L(\alpha_j(t, i, z)) m^\alpha(t, i, dz) dt + \int_0^1 \sum_{i \in I} g(i, z) m^\alpha(T, i, dz). \end{aligned}$$

By the regularity of the functions p, c, g and L in Assumptions 2-4 and the continuity of the map $\alpha \mapsto m^\alpha$ on $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$ stated in Lemma 7.6.1 in Appendix 7.6, one obtains the continuity of J over $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$ w.r.t. the norm $\|\cdot\|_\infty$. \square

We can now prove Theorem 7.2.1.

Proof of Theorem 7.2.1. This is a direct consequence of Theorem 7.3.2.

(i) This is a direct consequence of Lemma 7.4.3 and Theorem 7.3.1.

(ii) This is a direct consequence of Theorem 7.2.1.(i) and Lemma 7.4.4. \square

7.5 Simulations

This section illustrates the results with an example of smart charging. In this use case, the discrete variable taking values in I represents the charging mode and the continuous variable taking value in $[0, 1]$ represents the state of charge (SoC) of electrical vehicles (EV) relatively to the maximum energy capacity of the battery.

7.5.1 Definition of the use Case

We consider a large fleet of EVs controlled by a central planner during their charging period $[0, T]$. The goal of the central planner is to provide ancillary services to the transmission grid by controlling the aggregate consumption profile of the fleet on the time horizon $[0, T]$. More specifically, the central planner aims at making the consumption profile of the fleet to be close to a given profile $r = (r_t)_{0 \leq t \leq T}$, supposed to be known on the whole period $[0, T]$. In our simulation we compare two situations. In the first case known as V1G, it is assumed that the vehicle batteries can only draw electricity from the grid, then the set of charging modes is $I = \{0, 1\}$, where 0 stands for idle mode and 1 for charging. In the second case known as V2G, it is assumed that the batteries can either draw or inject electricity into the grid, then the charging modes are $I = \{-1, 0, 1\}$, where -1 corresponds to injection mode. In each situation, each mode $i \in I$ is characterized by its charging rate $b(i, \cdot)$ such that

- $i = -1$ corresponds to a V2G mode with $b(-1, z) = -\frac{a}{4}$ for $z \geq 0.25$ and $b(-1, z) = -az$ for $z \leq 0.25$,
with $a = \frac{2}{5}$.
- $i = 0$ corresponds to a non-charging mode with $b(0, z) = 0$,
- $i = 1$ corresponds to a charging mode with $b(1, z) = \frac{a}{4}$ for $z \leq 0.75$ and $b(1, z) = a(1 - z)$ for $z \geq 0.75$,
with $a = \frac{2}{5}$,

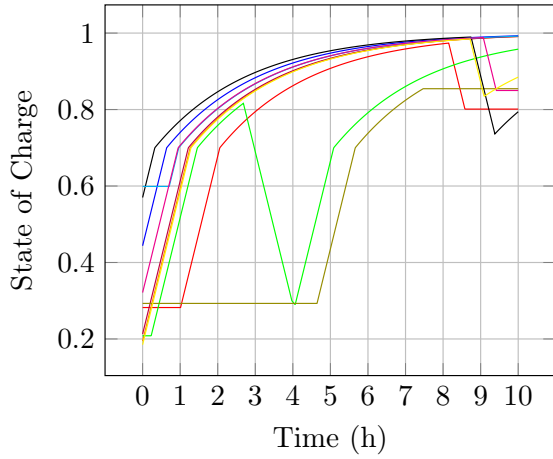


Figure 7.1 – Representation of the SoC of 10 PDMP

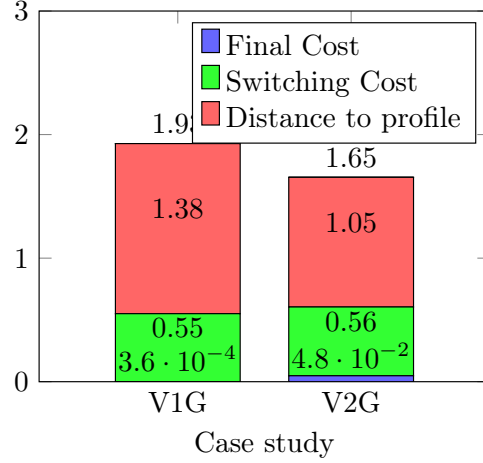


Figure 7.2 – Decomposition of the cost function

The charging rate b of the batteries has been designed based on industrial data [106]. The state of each EV, controlled by α , is represented by a PDMP $X^\alpha = (Y^\alpha, Z^\alpha)$ with Y_t^α , the charging mode and Z_t^α , the SoC of the vehicle at time t . We assume that the power consumption of a vehicle at state $(i, z) \in \mathcal{X}$ is equal to its instantaneous charging rate $b(i, z)$. At time $t = 0$, Y_0^α equals 0 or 1 with probability 1/2, while $Z_0^\alpha = \min(1, |z_0| + 0.15)$ where $z_0 \sim \mathcal{N}(0, 0.2)$.

To ensure customer satisfaction, the final cost $g(i, z) := 30 \times (1 - e^{5(z-0.75)})^+$ penalizes vehicles with a final SoC z lower than 75%. As previously stated, high values of α are penalized through the cost $\alpha^2/2$ in order to avoid high frequencies of jumps damaging the batteries. For any $t \in [0, T]$,

$$\begin{cases} c(t, i, z) & = 0 \\ p(t, i, z) & := b(i, z) \\ l(a) & := \frac{a^2}{2} \\ F(v) & := \kappa \int_0^T (v(t) - r(t))^2 dt \end{cases}$$

with $\kappa = 1000$ in order to ensure that the overall consumption is close to the profile r .

The target profile r is a slight modification of the nominal behaviour of the fleet of EVs (without control), with the same energy consumed over the fixed time horizon $[0, T]$. The nominal behaviour corresponds to the situation where the EVs are not required to fit the target profile and seek only to satisfy their own comfort (namely $F(v) = 0$). The idea is to define a realistic target profile so that it is possible for the fleet to follow this profile while satisfying their charging needs. We consider a population of $N = 10^5$ EVs. The optimal control of Problem (7.2.5) is computed using Algorithm 3.

7.5.2 Algorithm parameters

For the implementation of the algorithm, the line 5 is computed by discretization of the Hamilton Jacobi-Bellman-Equation III.5 associated with $\alpha[\lambda]$ defined in III.6 with $N_t = 1000$ regular points of time and $N_s = 400$ regular points of SoC. The Lagrangian multipliers λ^k are obtained after $k = 1000$ iterations and with $M = 1000$ realizations and are displayed in Figure 7.4. The initial multiplier is $\lambda^0 = 0$ and the stepsize sequence is such that $\rho_k = \min(30, \frac{500}{1+k})$. For each multiplier λ , the associated optimal strategy $\alpha[\lambda]$ is computed.

7.5.3 Results

Four periods on Figure 7.3 are distinguished, depending on whether the target profile is above or below the nominal consumption. The reference profile is displayed in green. EVs are encouraged to consume more than the nominal consumption during periods P1 and P3, and to consume less during periods P2 and P4. First,

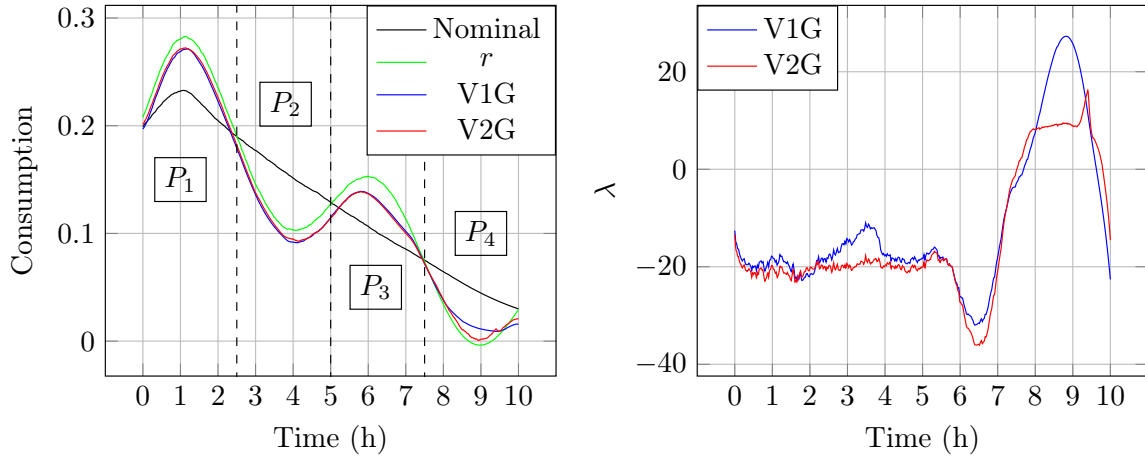


Figure 7.3 – Consumption for V1G and V2G cases compared to the profile and nominal Figure 7.4 – Optimal Lagrangian multiplier λ obtained

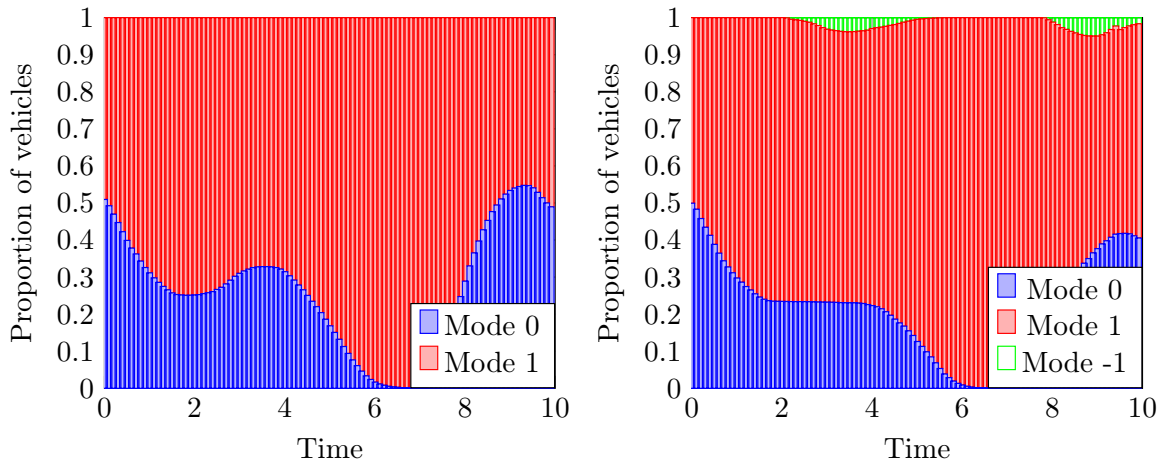


Figure 7.5 – Evolution of the proportion of vehicles for V1G Figure 7.6 – Evolution of the proportion of vehicles for V2G

we observe that both V1G (blue line) and V2G (red line) consumptions are close to the profile r (green line). The main difference lies in P_4 , where the possibility for the EVs to perform V2G allows the fleet consumption to stay closer to the target profile. As expected, the profile is better tracked with the V2G mode, due to the additional degree of liberty. On Figure 7.6 one can observe that V2G is used during periods P_2 and P_4 , when the profile r is lower than the nominal consumption. Ten trajectories of SoC, generated in the V2G use case, are represented in Figure 7.1. The green line is an example of a PDMP switching from mode 0 to mode 1 around $t = 0.2h$, from mode 1 to mode -1 around $t = 2.6h$ and then from mode -1 to mode 1 around $t = 4h$. The initial and final distribution of the SoC of the fleet are displayed on Figures 7.7 and 7.8. While the initial distribution is the same for both scenarios, one can observe that their final distribution are very closed, and that very few EVs have a SoC lower than 0.75% in both cases. Thus, the comfort of each agent is weakly impacted by the V2G mode.

To conclude, the interest of adding V2G is confirmed by the comparison of the optimal cost of each scenario (Figure 7.2). These optimal costs are decomposed into three parts: the switching cost $(\frac{1}{2} \int_0^T \sum_{j \neq I} \alpha_j(t, X_t^{i,\alpha})^2 dt)$, the final cost (g) and the distance to profile (F). The low value of the final cost (3.6×10^{-4} in the V1G use case and 4.8×10^{-2} in the V2G one) shows that, despite the tracking objective, EVs have enough battery at the end of the period, as previously stated. Moreover, this performance is achieved without significant loss in switching costs, thus preserving fleet batteries.

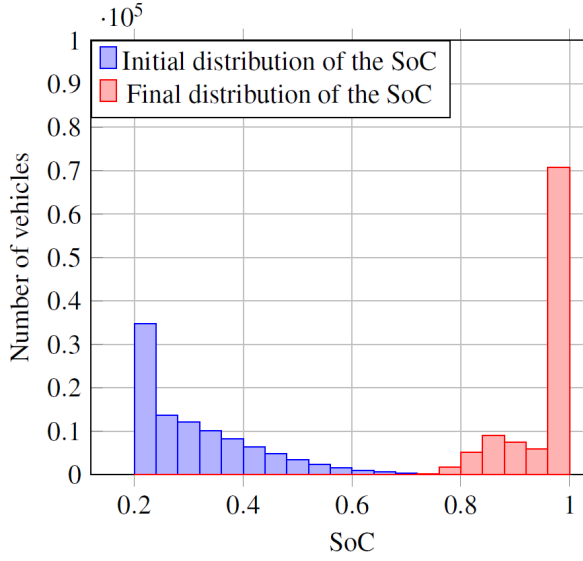


Figure 7.7 – Initial and Final SoC in the V1G case

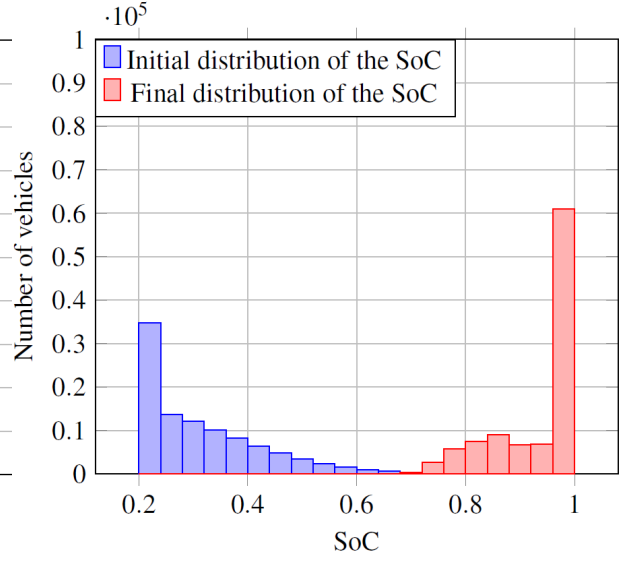


Figure 7.8 – Initial and Final SoC in the V2G case

7.6 Appendix

In this section, we provide some classical results about the distribution m^α of a process X^α that is a PDMP(b, α) (the definition of a PDMP(b, α) is given in Section 7.2), when α is in $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$. We introduce the continuity equation on $[0, T] \times \mathcal{X}$:

$$\begin{aligned} \partial_t m(t, i, z) + \partial_z (b(i, z)m(t, i, z)) &= - \sum_{j \neq i} (\alpha_j(t, i, z)m(t, i, z) - \alpha_i(t, j, z)m(t, j, z)) \quad (t, i, z) \in (0, T) \times \mathcal{X}, \\ m_i(0, z) &= m_i^0(z) \quad (i, z) \in \mathcal{X}, \end{aligned} \quad (7.6.1)$$

where $m^0 \in \mathcal{P}(\mathcal{X})$ is given. The next definition gives the characterization of weak solutions of (7.6.1).

Definition 7.6.1. A pair (m, α) satisfies (7.6.1) in the weak sense if $t \in [0, T] \mapsto m(t, \cdot) \in \mathcal{P}(\mathcal{X})$ is continuous, for any $i, j \in I$ with $i \neq j$, it holds that $\alpha \in L_{m_i}^2([0, T] \times \mathcal{X}, \mathbb{R}^d)$ and for any test function $\phi \in C_c^\infty([0, T] \times \mathcal{X})$, we have:

$$\begin{aligned} &\sum_{i \in I} \int_0^1 \phi_i(T, i, z)m(T, i, dz) - \phi(0, i, z)m_i^0(dz) \\ &= \int_0^T \int_0^1 \sum_{i \in I} (\partial_t \phi(t, i, z) + b(i, z)\partial_z \phi(t, i, z))m(t, i, dz) + \sum_{j \in I, j \neq i} (\phi(t, j, z) - \phi(t, i, z))\alpha_j(t, i, z)m(t, i, dz)dt. \end{aligned}$$

Proposition 7.6.1. Let $m^0 \in \mathcal{P}([0, 1] \times I)$, $\alpha \in \text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$ and X^α be a PDMP(b, α). Then, the distribution m^α of X^α is the unique distribution such that (m^α, α) is a weak solution of (7.6.1) in the sense of Definition 7.6.1.

Proof. This result is proved in [72] for controls α that are continuous in space and time independent. The extension of this result to bounded controls that are measurable in time is straightforward. \square

Continuity results of the map $\alpha \mapsto m^\alpha$ is presented in the next lemma.

Lemma 7.6.1. The map $\alpha \mapsto m^\alpha$ is continuous from $\text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$ endowed with the norm $\|\cdot\|_\infty$ to $C([0, T], \mathcal{P}(\mathcal{X}))$, endowed with the distance $\mathbb{W}_{\|\cdot\|_\infty}$.

Proof. Let $\psi \in C^1(\mathcal{X})$ and let $\varphi \in C^1([0, T] \times \mathcal{X})$ be the classical solution of the PDE:

$$\begin{aligned} \partial_\tau \varphi(\tau, i, z) + b(i, z)\partial_s \varphi(\tau, i, z) &= 0 \quad (\tau, i, z) \in [0, t] \times \mathcal{X} \\ \varphi(t, i, z) &= \psi(i, z) \quad (i, z) \in \mathcal{X}. \end{aligned} \quad (7.6.2)$$

One has $\|\varphi\|_\infty = \|\psi\|_\infty$, and denoting by γ_ψ the Lipschitz constant of ψ , for any $t \in [0, T]$ $\varphi(t, \cdot, \cdot)$ is Lipschitz continuous with Lipschitz constant $\gamma_\psi e^{T\|b'\|_\infty}$.

Let $\alpha, \bar{\alpha} \in \text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$. Since X^α is a PDMP(b, α) and $X^{\bar{\alpha}}$ is a PDMP($b, \bar{\alpha}$), by Proposition 7.6.1, (m^α, α) and $(m^{\bar{\alpha}}, \bar{\alpha})$ are both weak solutions of the continuity equation (7.6.1) on $[0, T] \times I \times [0, 1]$. Then, by equality (7.6.2), one has:

$$\begin{aligned}
& \int_0^1 \sum_{i \in I} \psi(i, z) (m^\alpha(t, i, dz) - m^{\bar{\alpha}}(t, i, dz)) \\
&= \sum_{i \in I} \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi(i, \tau, z) \left(-\alpha_j(\tau, i, z) m^\alpha(\tau, i, dz) + \alpha_i(\tau, j, z) m^\alpha(\tau, j, dz) \right) d\tau \\
&\quad + \sum_{i \in I, j \neq i} \int_0^t \int_0^1 \sum_{j \in I} \varphi(i, \tau, z) \left(\bar{\alpha}_j(\tau, i, z) m^{\bar{\alpha}}(\tau, i, dz) - \bar{\alpha}_i(\tau, j, z) m^{\bar{\alpha}}(\tau, j, dz) \right) d\tau \\
&= \sum_{i \in I, j \neq i} \int_0^t \int_0^1 \sum_{j \in I} \varphi(i, \tau, z) \alpha_j(\tau, i, z) \left(m^{\bar{\alpha}}(\tau, i, dz) - m^\alpha(\tau, i, dz) \right) d\tau \\
&\quad + \sum_{i \in I} \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi(i, \tau, z) \alpha_j(\tau, i, z) \left(m^\alpha(\tau, j, dz) - m^{\bar{\alpha}}(\tau, j, dz) \right) d\tau \\
&\quad + \sum_{i \in I} \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi(i, \tau, z) \left(\bar{\alpha}_j(\tau, i, z) - \alpha_j(\tau, i, z) \right) m^{\bar{\alpha}}(\tau, i, dz) d\tau \\
&\quad + \sum_{i \in I} \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi(i, \tau, z) \left(\alpha_i(\tau, j, z) - \bar{\alpha}_i(\tau, j, z) \right) m^{\bar{\alpha}}(\tau, j, dz) d\tau
\end{aligned} \tag{7.6.3}$$

Using that $\varphi(t, \cdot, \cdot)$ is Lipschitz continuous with Lipschitz constant $\gamma_\psi e^{T\|b'\|_\infty}$ and that $\alpha \in \text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$, there exists $L > 0$ such that:

$$\sum_{i \in I} \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi(i, \tau, z) \alpha_j(\tau, i, z) \left(m^{\bar{\alpha}}(\tau, i, dz) - m^\alpha(\tau, i, dz) \right) d\tau \leq L \gamma_\psi e^{T\|b'\|_\infty} \int_0^T \mathcal{W}(m^\alpha(\tau), m^{\bar{\alpha}}(\tau)) d\tau, \tag{7.6.4}$$

$$\sum_{i \in I} \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi(i, \tau, z) \alpha_j(\tau, i, z) \left(m^\alpha(\tau, j, dz) - m^{\bar{\alpha}}(\tau, j, dz) \right) d\tau \leq L \gamma_\psi e^{T\|b'\|_\infty} \int_0^T \mathcal{W}(m^\alpha(\tau), m^{\bar{\alpha}}(\tau)) d\tau. \tag{7.6.5}$$

Since $\alpha, \bar{\alpha} \in \text{Lip}([0, T] \times \mathcal{X}, \mathbb{R}_+^d)$ and that $\|\varphi\|_\infty \leq \|\psi\|_\infty$, one has:

$$\sum_{i \in I} \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi(i, \tau, z) \left(\bar{\alpha}_j(\tau, i, z) - \alpha_j(\tau, i, z) \right) m^{\bar{\alpha}}(\tau, i, dz) d\tau \leq |I| \|\psi\|_\infty \|\alpha - \bar{\alpha}\|_\infty, \tag{7.6.6}$$

$$\sum_{i \in I} \int_0^t \int_0^1 \sum_{j \in I, j \neq i} \varphi(i, \tau, z) \left(\alpha_i(\tau, j, z) - \bar{\alpha}_i(\tau, j, z) \right) m^{\bar{\alpha}}(\tau, j, dz) d\tau \leq |I| \|\psi\|_\infty \|\alpha - \bar{\alpha}\|_\infty. \tag{7.6.7}$$

By inequalities (7.6.4)-(7.6.7), inequality (7.6.3) becomes:

$$\int_0^1 \sum_{i \in I} \psi(i, z) (m^\alpha(t, i, dz) - m^{\bar{\alpha}}(t, i, dz)) \leq 2|I| \|\psi\|_\infty \|\alpha - \bar{\alpha}\|_\infty + 2L \gamma_\psi e^{T\|b'\|_\infty} \int_0^t \mathcal{W}(m^\alpha(\tau), m^{\bar{\alpha}}(\tau)) d\tau.$$

Since ψ is arbitrary and in $C^1(\mathcal{X})$, previous inequality becomes, for any $t \in [0, T]$,

$$\mathbb{W}(m^\alpha(t), m^{\bar{\alpha}}(t)) \leq 2|I|^2 \|\psi\|_\infty \|\alpha - \bar{\alpha}\|_\infty + 2L e^{T\|b'\|_\infty} \int_0^t \mathcal{W}(m^\alpha(\tau), m^{\bar{\alpha}}(\tau)) d\tau.$$

By Gronwall inequality, one deduces that there exists a constant $C > 0$ such that, for any $t \in [0, T]$,

$$\mathbb{W}(m^\alpha(t), m^{\bar{\alpha}}(t)) \leq L \|\alpha - \bar{\alpha}\|_\infty.$$

Thus, $\mathbb{W}_{\|\cdot\|_\infty}(m^\alpha, m^{\bar{\alpha}}) \leq L \|\alpha - \bar{\alpha}\|_\infty$ and the continuity of $\alpha \mapsto m^\alpha$ follows. \square

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RÉSUMÉ

Cette thèse traite la résolution théorique et numérique de problèmes d'optimisation de type champ moyen. Un premier problème de contrôle champ moyen est formulé, avec contrainte de congestion, modélisant le chargement optimal d'une grande flotte de véhicules électriques. Les conditions d'optimalités sont identifiées comme un système d'équations aux dérivées partielles couplées, similairement à ceux obtenus dans les jeux à champ moyen. Le problème est résolu numériquement par un algorithme primal-dual. Un premier résultat sur la régularité des solutions est exploité pour caractériser le problème comme la limite d'un problème de contrôle avec grand nombre de véhicules. Les multiplicateurs de Lagrange associés à la contrainte de congestion sont bornés, permettant d'obtenir une meilleure régularité, de type Lipschitz, des solutions et d'approximer numériquement le problème dual.

Dans un second temps, une méthode d'approximation et de résolution numérique d'un problème de contrôle stochastique convexe de grande dimension est développée. Un algorithme d'optimisation distribué est proposé et la convergence vers une solution du problème est démontrée. La méthode et l'algorithme sont étudiés et appliqués à un problème de contrôle de processus markoviens déterministes par morceaux, une formulation équivalente au problème de contrôle champ moyen étudié en première partie.

MOTS CLÉS

Contrôle champ moyen ; Chargement intelligent ; Approximation numérique ; Contrôle stochastique ; Optimisation distribuée ; Flexibilité diffuse.

ABSTRACT

This thesis studies the theoretical resolution and numerical approximation of mean field optimization problems. First, a mean field optimal control with congestion constraints is formulated, modeling the optimal charging of a large fleet of electrical vehicles. Optimality conditions are identified as a system of coupled partial differential equations, similarly to those obtained in mean field games. The problem is numerically solved by a primal-dual algorithm. A first result on the regularity of the solutions is exploited to characterize the problem as the mean field limit of an optimal control problem with a large number of vehicles. The Lagrange multipliers of the congestion constraint are proved to be bounded. This result enables to obtain a better regularity of Lipschitz type of the solutions and to numerically approximate the dual problem.

Secondly, an approximation method and a numerical method for a high dimensional convex stochastic control problem are developed. A distributed algorithm is proposed, whose convergence to a solution is proved. Finally, the method and the algorithm are analyzed and applied to a control problem of piecewise deterministic Markov processes, a formulation that is equivalent to the mean field control problem introduced in the first part.

KEYWORDS

Mean field control; Smart charging; Numerical approximation; Stochastic control; Distributed optimization; Diffuse flexibility.