



**HAL**  
open science

# Statistical Properties of Quantized Toral Automorphisms

Nir Schwartz

► **To cite this version:**

Nir Schwartz. Statistical Properties of Quantized Toral Automorphisms. Spectral Theory [math.SP].  
Université Paris-Saclay, 2022. English. NNT : 2022UPASM026 . tel-04266949

**HAL Id: tel-04266949**

**<https://theses.hal.science/tel-04266949v1>**

Submitted on 1 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Statistical Properties of Quantized Toral Automorphisms

*Propriétés statistiques d'automorphismes quantiques du tore*

## Thèse de doctorat de l'université Paris-Saclay

École doctorale n° 574, Mathématiques Hadamard (EDMH)  
Spécialité de doctorat : Mathématiques fondamentales  
Graduate School : Mathématiques, Référent : Faculté des sciences d'Orsay

Thèse préparée au **Laboratoire de mathématiques d'Orsay** (Université Paris-Saclay, CNRS), sous la direction de **Stéphane NONNENMACHER**, Professeur

Thèse soutenue à Paris-Saclay, le 20 octobre 2022, par

**Nir SCHWARTZ**

## Composition du Jury

<b>Matthieu LÉAUTAUD</b> Professeur des universités, Université Paris-Saclay	Président
<b>Xiaolong HAN</b> Professeur des universités, Université d'État de Californie à Northridge (CSUN), Etats-Unis	Rapporteur et Examineur
<b>Gabriel RIVIÈRE</b> Professeur des universités, Université de Nantes	Rapporteur et Examineur
<b>Pär KURLBERG</b> Professeur des universités, École royale polytechnique (KTH), Suède	Examineur
<b>Stéphane NONNENMACHER</b> Professeur des universités, Université Paris-Saclay	Directeur de thèse

**Titre :** Propriétés statistiques d'automorphismes quantiques du tore

**Mots clés :** Chaos quantique, Transformation chaotique quantique, Ergodicité quantique, Délocalisation, Analyse semi-classique.

**Résumé :** Dans Chaos quantique, nous étudions la connexion entre les systèmes chaotiques classiques et leurs homologues quantiques. Plus précisément, comment les propriétés ergodiques des premiers se reflètent dans le spectre des seconds (dans la limite semi-classique). Dans le présent travail, on considère un célèbre modèle de jouet : on examine sur  $\mathbb{T}^2$  des automorphismes hyperboliques. On les quantifie en associant à chacun une famille de  $N \times N$  matrices unitaires, appelée "*un application de chat quantique*". On étudie les propriétés de délocalisation des états propres de ces applications chat quantiques dans la limite semi-classique  $N \rightarrow \infty$ . L'application de chat quantique est connue pour être *quantique ergodique*, c'est-à-dire que la plupart des états propres sont asymptotiquement équidistribués. Notre application n'est pas quantique *unique* ergodique,

c'est-à-dire qu'il existe une suite  $\{N_k\}_k$  pour laquelle les matrices présentent de grandes dégénérescences spectrales et admettent des états propres partiellement localisés. Néanmoins, on montre que les états propres satisfont une propriété plus faible, *délocalisation complète* sur  $\mathbb{T}^2$  dans la limite semi-classique. La preuve de ce résultat repose sur le principe d'incertitude fractale de Bourgain et Dyatlov. On considère alors le long de la suite  $\{N_k\}_k$  des bases propres aléatoires. Nous montrons qu'ils satisfont presque sûrement l'ergodicité quantique unique jusqu'à une petite échelle. Enfin, on prend en considération les propriétés statistiques de ces vecteurs propres aléatoires. Montrons qu'ils se comportent comme des vecteurs gaussiens standards dans  $\mathbb{C}^N$ .

**Title:** Statistical Properties of Quantized Toral Automorphisms

**Keywords:** Quantum Chaos, Quantum Chaotic Map, Quantum Ergodicity, Delocalization, Semiclassical Analysis.

**Abstract:** In Quantum chaos we study the connection between classical chaotic systems and their quantum counterparts. More specifically, how ergodic properties of the former are reflected in the spectrum of the latter (in the semiclassical limit). In the present work we consider a celebrated toy model: We consider on  $\mathbb{T}^2$  hyperbolic automorphisms. One quantizes these automorphisms by associating to each a family of  $N \times N$  unitary matrices, called "*a quantum cat map*". We study the delocalization properties of the eigenstates of these quantum cat maps in the semiclassical limit  $N \rightarrow \infty$ . The quantum cat map is known to be *quantum ergodic*, i.e., most eigenstates asymptotically equidistribute.

Our map is not quantum *unique* ergodic, i.e., there is a sequence  $\{N_k\}_k$  for which the matrices exhibit large spectral degeneracies and admit partially localized eigenstates. Nevertheless, we show that eigenstates satisfy a weaker property, *full delocalization* on  $\mathbb{T}^2$  in the semiclassical limit. The proof of this result relies on the fractal uncertainty principle of Bourgain and Dyatlov. We then consider along the sequence  $\{N_k\}_k$  random eigenbases. We show that almost surely they satisfy quantum unique ergodicity down to a small scale. Finally we consider the statistical properties of these random eigenvectors. We show they behave like standard Gaussian vectors in  $\mathbb{C}^N$ .

université  
PARIS-SACLAY

FACULTÉ  
DES SCIENCES  
D'ORSAY

FM  
JH

FONDATION MATHÉMATIQUE  
JACQUES HADAMARD



ÉCOLE DOCTORALE  
de mathématiques  
Hadamard (EDMH)



---

In memory of my grandfather, Yehuda Schwarc, an educator from the founding generation of the state of Israel.

“ Cada ciència ha mester las vocables per  
los quals mills sia manifestada; e car a  
aquesta ciència demostrativa sien mester  
vocables escurs e que los hòmens lecs no han  
en ús, e car nós façam aquest llibre als  
hòmens lecs per açò breument e ab plans  
vocables parlarem d’esta ciència ”

---

*Libre de Gentil*

Ramon Llull

# Acknowledgments

The composition of this thesis could never be possible, or that invigorating, without the consistent support I got from people to whom I owe a great debt of gratitude.

First, I wish to thank my advisor, Stéphane, whose contagious passion and enthusiasm made this work possible. Stéphane has immense mathematical knowledge and culture. Not only he came up with fascinating central problems in our domain, he dedicated to me uncountable hours discussing our projects, correcting my challenging drafts, forming and honing my approaches to them. Our discussions had a deep influence on my identity as a scholar and I am all hope I could employ the skills I acquired during our common working period in future projects. In addition I wish to thank him for the valuable advice he gave me concerning French bureaucracy and various ways to handle it.

In that context I wish to express my gratitude to the administrative personnel of Paris-Saclay and École nationale supérieure d'informatique pour l'industrie et l'entreprise (École Nationale Supérieure d'Informatique pour l'Industrie et l'Entreprise, especially to Valérie Linget, Amelie Coince and Mathilde Khetache, for their assistance with French formal duties.

Second, I wish to thank those who showed great interest in my job: Steven Zelditch who also invited me to talk in his seminar and expressed his thoughts on Fractal Uncertainty Principle, Gabriel Rivière who brought to my attention the works of Ledrappier, Xiaolong Han who discussed with me his splendid questions on the first part of this thesis and Suresh Eswarathasan who asked some good questions on the quantum map. All these discussions deeply contributed to my comprehension of the presented

work and allowed me to strengthen my grasp of the related mathematical infrastructure.

More importantly, during the work on the second half of this thesis I consulted with several other academics concerning probabilistic and number theoretic questions. I would like to use this opportunity to thank them all: to Paul Bourgade for our discussion on random matrix theory, Dubi Kelmer and Pär Kurlberg for our algebraic and number theoretic discussions concerning quantum periods and Nicolas Curien for his view on central limit theorems. Thanks for broadening my horizons and assisting me coping with assembling the pieces required for the deterministic hyperbolic example of Berry's model. An additional thanks should go to those who agreed to serve in the jury of this thesis. I am all hope you found the reading delightful.

I would like in this context to thank the scientific computing group of Institut de Mathématique d'Orsay for giving me access to Paris-Saclay's scientific cluster and replying my questions with great patience and great extent.

I am grateful for my friends (some of which are my colleagues) as well: to Hezi Halawi, Noy Aranova-Soffer and the Beaupères, my true friends, who accompany me through thick and thin. I am a lucky person to have you all in my life and words cannot express how much I appreciate you. In addition, I treasure my moments with Nhi Ngoc-Nguyen and Danielle Hilhorst as well, for our discussions, infinitely many cups of fancy tea I stole from their office and for their induction of positivity and light around them in Orsay and its surroundings. Moreover I cherish the talks I had with Tomer who believed in me and helped me resurrecting from the ashes. I wish also to thank my office mates in the past 4 years with whom I had several non-negative moments.



Last but not least I wish to share the gratitude from the bottom of the heart to my parents who roundly supported me when needed, gave me a decent level of education and taught me to strive to success and to never give up even when severely challenging situations and people hit one's wing. I am all hope this work will warm your heart.

# Contents

P	Preliminaries: dynamique hyperbolique classique et quantique	<b>6</b>
P.1	Rappel sur la mécanique classique . . . . .	6
P.2	Carte classique du chat d'Arnold et carte quantique du chat . . . . .	20
P.3	Relevé des résultats . . . . .	23
P	Preliminaries: classical and quantum hyperbolic dynamics	<b>34</b>
P.1	Reminder on classical mechanics . . . . .	34
P.2	Classical Arnold's cat map and the quantum cat map . . . . .	47
P.3	Statement of results . . . . .	50
1	Quantization of classical observables and anisotropic calculi	<b>61</b>
1.1	Hyperbolic automorphisms on $\mathbb{T}^2$ . . . . .	61
1.2	The quantum spaces . . . . .	63
1.3	Quantization of bi-periodic symbols . . . . .	65
1.4	Quantizing hyperbolic toral automorphisms . . . . .	68
1.5	Anisotropic calculi associated to Lagrangian foliations . . . . .	70
1.6	The properties of the quantization of anisotropic symbols . . . . .	74
1.7	The Quantum space of the torus phase space . . . . .	77
2	The complete delocalization of the semiclassical measures	<b>80</b>
2.1	Delocalization statements for Wigner distributions . . . . .	80
2.2	$\nu$ -porous sets used for proving Proposition 2.1.6 . . . . .	91
2.3	Introducing a partition of unity by smooth cut-offs on $\mathbb{R}^2$ . . . . .	92
2.4	A Proof relying on the minimality of the horocyclic flow . . . . .	94
2.5	A Proof relying on lattice-points counting . . . . .	97
2.6	Estimates for the uncontrolled region . . . . .	113
3	Small scale Probabilistic Quantum Unique Ergodicity	<b>123</b>
3.1	Quantum periods . . . . .	124
3.2	Construction of random eigenbases . . . . .	126
3.3	Small scale symbols . . . . .	127
3.4	A local Weyl law on the eigenspaces . . . . .	130
3.5	End of proof of Proposition 3.4.1 . . . . .	139
3.6	From Hanson-Wright inequality to probabilistic QUE . . . . .	141
4	The statistical distribution of random eigenstates of the quantum cat map	<b>148</b>
4.1	Random eigenstates for quantum cat maps . . . . .	149
4.2	The value distribution of random vectors . . . . .	160
A	Appendices and additions	<b>171</b>
A.1	Deriving an explicit formula for $\mathcal{M}_N(A)$ . . . . .	171
A.2	The quantum cat maps are quantum ergodic . . . . .	176

A.3 Expressing  $t_0, C_{\nu, \mathcal{R}, \mathcal{U}}^+$  . . . . . 181  
 A.4 Numerical porosity for DE cat map . . . . . 182  
 A.5 Averaging periodizations over lattice points . . . . . 183  
 A.6 Concentration of the norm of projections to  $V_{N, \nu}$  . . . . . 184

References . . . . . 191

## Listing of figures

P.1 La carte chat classique est fortement mélangeante . . . . . 10  
 P.2 Flux d’Anosov – illustration schématique . . . . . 12  
 P.3 Géodésiques dans  $\mathbb{H}$  . . . . . 14  
 P.4 Section de Poincaré . . . . . 16  
 P.5 Action de  $\gamma_{DE}$  sur le carré fondamental . . . . . 21

P.1 The classical cat map is a strongly mixing . . . . . 38  
 P.2 Anosov flow – schematic illustration . . . . . 40  
 P.3 Geodesics in  $\mathbb{H}$  . . . . . 42  
 P.4 Poincaré section . . . . . 44  
 P.5 Action of  $\gamma_{DE}$  on the fundamental square . . . . . 48

2.1 A partition of unity on  $\mathbb{T}^2$  . . . . . 83  
 2.2 Tiling  $\mathbb{R}^2$  by translations of  $\mathcal{S}^0$  (the orange tile). The green rectangles correspond to  $\mathcal{R}$  defined in section 2.5. . . . . 93  
 2.3 The parallelogram  $\mathcal{S}^0(\kappa)$  . . . . . 94  
 2.4 illustrating the proof of Proposition 2.4.1 . . . . . 97  
 2.5 A second partition of unity on  $\mathbb{T}^2$  . . . . . 98  
 2.6 The distances between rectangles and their images under  $\iota D^t$  in the  $y - \eta$  plane . . . 100  
 2.7 Positions of endpoints of  $I'$  in the proof of Lemma 2.5.3 . . . . . 101  
 2.8 The domain used in the proof of Lemma 2.5.4 . . . . . 103  
 2.9 Projecting the supports of symbols gives rise to  $\nu$ -porous sets . . . . . 107  
 2.10 the parallelogram  $\mathcal{D}^\sharp$  contains a large enough square . . . . . 110

3.1 The quantum periods of  $\mathcal{M}_N(\gamma_{DE})$  . . . . . 125  
 3.2 Obtaining  $\diamond_1$  from  ${}^t \square_1$  . . . . . 136  
 3.3 Change of summation lattice used for proving Lemma 3.4.4 . . . . . 137

4.1 HSV maps of  $\Delta_N$  . . . . . 151  
 4.2 Heatmaps for matrices . . . . . 153  
 4.3  $\ell^\infty$  norm of  $\left| \Pi_{\nu, N} - \frac{1}{P(N)} \right|$  and growth of  $\gcd(N, p(t))$  . . . . . 155  
 4.4  $\gcd(N, p(t))$  for an odd and for an even  $N$  . . . . . 156  
 4.5 Matrices contributing to the sum making  $\Pi_{\nu, N}$  differ depending on the parity of  $N$  . . 157  
 4.6 Empirical measures in the COE case . . . . . 166  
 4.7 Empirical measures in the CUE case . . . . . 167

# P

## Preliminaries: dynamique hyperbolique classique et quantique

Le présent travail étudie les propriétés spectrales des vecteurs propres du "application du chat quantique", un célèbre exemple de jouet d'un système quantique présentant une dynamique chaotique. Ces propriétés incluent mais sont non limitées à la (dé-)localisation et à l'équidistribution de séquences d'états propres du propagateur quantique. Commençons par rappeler les bases de la mécanique hamiltonienne et son pendant quantique. Nous présentons ensuite deux modèles centraux de systèmes chaotiques admettant des versions quantiques : Le flux géodésique, étant hyperbolique sur une surface de courbure négative et les automorphismes toraux hyperboliques également appelés "cat maps" d'après [AA67]. Nous énonçons ensuite nos résultats.

### P.1 Rappel sur la mécanique classique

Le point de départ de notre discussion est l'étude des systèmes dynamiques. Pour simplifier nous limitons notre traitement dans la page suivante au mouvement d'une particule dans *l'espace des positions*  $\mathbb{R}^d$  muni de coordonnées  $x = (x_1, \dots, x_d)$ . On note la position de la particule dans le temps  $t$  par  $x(t)$ . Nous supposons en outre que la force  $F(x(t))$  agissant sur la particule est *conservative*, c'est-à-dire  $F(x(t)) = -\nabla V(x(t))$  avec  $V \in C^\infty(\mathbb{R}^d)$  signifiant l'énergie potentielle de la particule. L'étude des systèmes dynamiques a commencé à partir de la théorie des équations différentielles. Afin de décrire la dynamique, on prend le *formalisme hamiltonien*, dans lequel on considère un système de 2 équations différentielles ordinaires du premier ordre des équations appelées *équations du mouvement de Hamilton* données par,

$$\begin{cases} \frac{dx_j}{dt} = \frac{\partial H}{\partial \xi_j}(x(t), \xi(t)) \\ \frac{d\xi_j}{dt} = -\frac{\partial H}{\partial x_j}(x(t), \xi(t)) \end{cases} \quad (\text{P.1})$$

La fonction  $H$  résolvant ce système modélise l'énergie totale conservée de la particule et est appelée hamiltonien du système. Ici l'espace de tous les états ("configurations") du système est donné par

$$T^*\mathbb{R}^d = \{(x_1, \dots, x_d, \xi_1, \dots, \xi_d) : x_j, \xi_j \in \mathbb{R}\}$$

avec  $x = (x_1, \dots, x_d)$  étant les coordonnées de position tandis que  $\xi = (\xi_1, \dots, \xi_d)$  représente la quantité de mouvement de la particule.

**Exemple P.1.1.** Soit une particule non relativiste de masse  $m$  sur laquelle agit une force conservatrice  $F$ ,

son énergie totale est donnée par l'hamiltonien,

$$H(x, \xi) = \frac{\|\xi\|_2^2}{2m} + V(x).$$

Le premier terme est l'énergie cinétique de la particule et le second est l'énergie potentielle.

Supposons que pour chaque  $(x(0), \xi(0)) \in T^*\mathbb{R}^d$  le système (P.1) admette une solution pour chaque  $t \in \mathbb{R}$ . Alors pour toute valeur de  $(x(0), \xi(0)) \in T^*\mathbb{R}^d$  on obtient une *trajectoire de la dynamique*, soit  $((x(t), \xi(t)))_{t \in \mathbb{R}}$ .

Nous nous intéresserons à l'étude des propriétés des trajectoires correspondant aux temps grands (ie, "l'évolution en temps long du système"). Pour cela on représente les trajectoires par un *flux*,  $\{\Phi^t\}_{t \in \mathbb{R}}$ , une famille à un paramètre de difféomorphismes  $\Phi^t : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ , prenant pour valeur  $(x(0), \xi(0))$  à son état initial et  $(x(t), \xi(t))$  au temps  $t$ . Nous pouvons vérifier que pour chaque  $t_1, t_2 \in \mathbb{R}$  le flux satisfait

$$\Phi^0 = \text{Id}_{\mathbb{R}^{2d}} \quad \Phi^{t_1+t_2} = \Phi^{t_1} \circ \Phi^{t_2}. \quad (\text{P.2})$$

Relions géométriquement le flux et le système. Munir  $\mathbb{R}^{2d}$  de la forme bilinéaire  $\omega$  appelée *la forme symplectique*, donnée par

$$\omega(u, v) = u^t \left( \begin{array}{c|c} 0_d & \text{Id}_d \\ \hline -\text{Id}_d & 0_d \end{array} \right) v, \quad u, v \in \mathbb{R}^{2d}$$

ou considéré comme une forme différentielle,

$$\omega = \sum_{j=1}^d dx_j \wedge d\xi_j.$$

Le système ci-dessus correspond à  $X_H$ , le champ vectoriel unique sur  $\mathbb{R}^{2d}$  satisfaisant pour chaque  $v \in \mathbb{R}^{2d}$

$$\langle \nabla H, v \rangle_{\mathbb{R}^{2d}} = \omega(X_H, v)$$

et générant le flux  $\Phi^t$ . En fait,

$$X_H(x, \xi) = \left( \frac{\partial H}{\partial \xi_1}, \dots, \frac{\partial H}{\partial \xi_d}, -\frac{\partial H}{\partial x_1}, \dots, -\frac{\partial H}{\partial x_d} \right) (x, \xi)$$

et est appelé un *Champ vectoriel hamiltonien* et  $\Phi^t$  est appelé un *flot hamiltonien*. Les ensembles de niveaux de  $H$ , sur lesquels la particule a une énergie fixe  $E > 0$ , sont appelés *coquilles d'énergie*,

$$\mathcal{S}(E) := \{(x, \xi) \in \mathbb{R}^{2d} : H(x, \xi) = E\}.$$

On sait que les couches d'énergie sont préservées par le flux, donc on peut restreindre la dynamique à un certain fixe  $\mathcal{S}(E)$ .

La théorie des systèmes dynamiques s'intéresse au comportement en temps long des trajectoires

$(x(t), \xi(t))_{t \in \mathbb{R}}$  issues de (P.1). Nous ne restreindrons pas notre attention uniquement à  $(\mathbb{R}^{2d}, \omega)$  mais plutôt à une variété symplectique lisse, c'est-à-dire la paire  $(M, \omega)$ , où  $\omega$  est une forme 2 différentielle fermée non dégénérée, ayant des patches de coordonnées locales isomorphes à  $(\mathbb{R}^{2d}, \omega)$ . Le flot hamiltonien  $\{\Phi^t\}_{t \in \mathbb{R}}$  pourrait être remplacé par un difféomorphisme symplectique  $\Phi : M \rightarrow M$  non nécessairement associé à une équation différentielle ordinaire. Cette carte va générer la dynamique à travers des trajectoires  $(x(n), \xi(n))_{n \in \mathbb{Z}}$  décrivant l'évolution temporelle du système. Autrement dit, on distingue la dynamique continue, correspondant à un flux  $\{\Phi^t\}$ , et la dynamique discrète, induite par l'itération de l'application  $\Phi$  (et donc correspondant à des temps discrets  $n \in \mathbb{Z}$ ).

Comme nous l'expliquons dans la subsection P.1.4 ci-dessous, dans certaines circonstances, il est possible d'obtenir à partir de la dynamique continue, une "contrepartie discrète" correspondante par restriction à un sous-espace de faible dimension.

### P.1.1 Ergodicité et mélange

Nous définissons maintenant une structure supplémentaire sur l'espace des phases  $M$ . Considérons le triplet  $(M, \omega, \Phi)$  pour un inversible  $\Phi : M \rightarrow M$ . En définissant sur  $M$  une  $\sigma$ -algèbre  $\mathcal{B}$  et une  $\Phi$ -mesure de probabilité invariante  $\mu$  (ie, pour tout  $A \in \mathcal{B}$   $\mu(\Phi^{-1}(A)) = \mu(A)$ ), on obtient un *espace de mesure* sur lequel agit l'application  $\Phi$ .

**Exemple P.1.2.** *Considérons la variété symplectique  $(\mathbb{R}^{2d}, \omega)$  et munissons-la de la mesure de Liouville  $\mu_{\text{Liouville}} = \frac{\omega}{d!}$ . Tout difféomorphisme symplectique  $\Phi$  préserve  $\mu_{\text{Liouville}}$  (ce qui signifie que  $\mu_{\text{Liouville}}$  est  $\Phi$ -invariant).*

On se réfère au quadruplet  $(M, \mathcal{B}, \mu, \Phi)$  une *transformation préservant la mesure*. Étant donné une telle transformation, nous aimerions comprendre le comportement à long terme de  $\Phi$ -orbits.

Le premier résultat est dû à Poincaré qui a prouvé que si  $\mu$  est une mesure de probabilité, pour chaque  $E \in \mathcal{B}$  pour presque chaque  $y \in A$  il existe une suite  $\{n_k\}_k$  pour lequel  $\Phi^{n_k}(y) \in A$ .

**Définition P.1.3.** *Soit  $(\mathbb{R}^{2d}, \mathcal{B}, \mu, \Phi)$  une transformation préservant la mesure. On dit que  $\Phi$  est ( $\mu$ -)ergodique si pour tout ensemble mesurable  $A \in \mathcal{B}$ , si  $\Phi^{-1}(A) = A$  alors  $\mu(A) \in \{0, 1\}$ .*

**Exemple P.1.4.** *Soit  $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$  muni de la mesure euclidienne  $dx$ . Pour certains  $\alpha \in \mathbb{R}$ , considérons la rotation du cercle  $\Phi_\alpha(x) = x + \alpha \pmod{1}$  en préservant  $dx$ . Il est naturel de se demander s'il est ergodique. Si  $\alpha = \frac{j}{k}$  pour un certain  $j \in \mathbb{Z}, k \in \mathbb{N}^*$  l'ensemble  $A = \bigcup_{m=0}^{k-1} \left[ \frac{m}{k}, \frac{m+\frac{1}{2}}{k} \right]$  est  $dx$ -invariant de mesure  $\frac{1}{2}$  et donc pour ces valeurs de  $\alpha$  l'application  $\Phi_\alpha$  n'est pas ergodique. Au contraire,  $\Phi_\alpha$  est ergodique si  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ .*

Notre prochain objectif est de formuler des conditions pour "l'équidistribution asymptotique" : en commençant par  $A \in \mathcal{B}$ , nous aimerions que  $\Phi^t(A)$  soit "équidistribué" par rapport à  $\mu$  comme  $t \rightarrow \infty$ . À cette fin, nous introduisons une autre propriété théorique de la mesure (chaotique) appelée *mélange*,

**Définition P.1.5.** On dit que  $(\mathbb{R}^{2d}, \mathcal{B}, \mu, \Phi)$  est un mélange (fort) si pour chaque mesurable définit  $A, B \subset \mathcal{B}$

$$\lim_{n \rightarrow \infty} \mu(B \cap \Phi^n(A)) = \mu(A) \mu(B),$$

où  $\Phi^n = \underbrace{\Phi \circ \dots \circ \Phi}_{n \text{ fois}}$ . Nous testons combien de  $\Phi^n(A)$  intersectent avec  $n$ 'importe quel sous-ensemble  $B \in \mathcal{B}$ .

**Remarque P.1.6.** Un mélange fort implique une ergodicité : si  $A \in \mathcal{B}$  est un ensemble  $\Phi$ -invariant, alors pour tout  $n \in \mathbb{Z}$ ,  $\mu(A \cap \Phi^n(A)) = \mu(A)$ . Si  $(M, \mathcal{B}, \mu, \Phi)$  est un mélange fort,  $\lim_{n \rightarrow \infty} \mu(A \cap \Phi^n(A)) = \mu^2(A)$ , donc  $\mu(A) \in \{0, 1\}$ .

Ci-dessous nous utiliserons un critère équivalent de mélange fort et d'ergodicité dû à Von Neumann :

**Théorème P.1.1** (Von Neumann, Theorem 2.21 in [EW10]). *Considérons la transformation préservant la mesure  $(\mathbb{R}^{2d}, \mathcal{B}, \mu, \Phi)$ .*

1. *Le système est ergodique si et seulement si pour tout  $f \in L^2(M, \nu)$*

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{k=0}^{N-1} f \circ \Phi^k - \int_{\mathbb{R}^{2d}} f d\mu \right\|_{L^2(M, \mu)} = 0.$$

2. *Il mélange fortement si et seulement si pour tout  $f, g \in L^2(M, \mu)$*

$$\lim_{N \rightarrow \infty} \int_M f \circ \Phi^N \cdot g d\mu = \int_M f d\mu \cdot \int_M g d\mu.$$

**Exemple P.1.7.** Nous utiliserons ci-dessous le fait que la dynamique sur  $M = \mathbb{T}^{2d}$  défini par un automorphisme  $\gamma \in SL_d(\mathbb{Z})$  avec  $\text{Spec}(\gamma) \cap \mathbb{S}_{\mathbb{C}}^1 = \emptyset$  se mélange fortement (pour preuve cf. [KH97, Proposition 4.2.12] et Figure P.1).

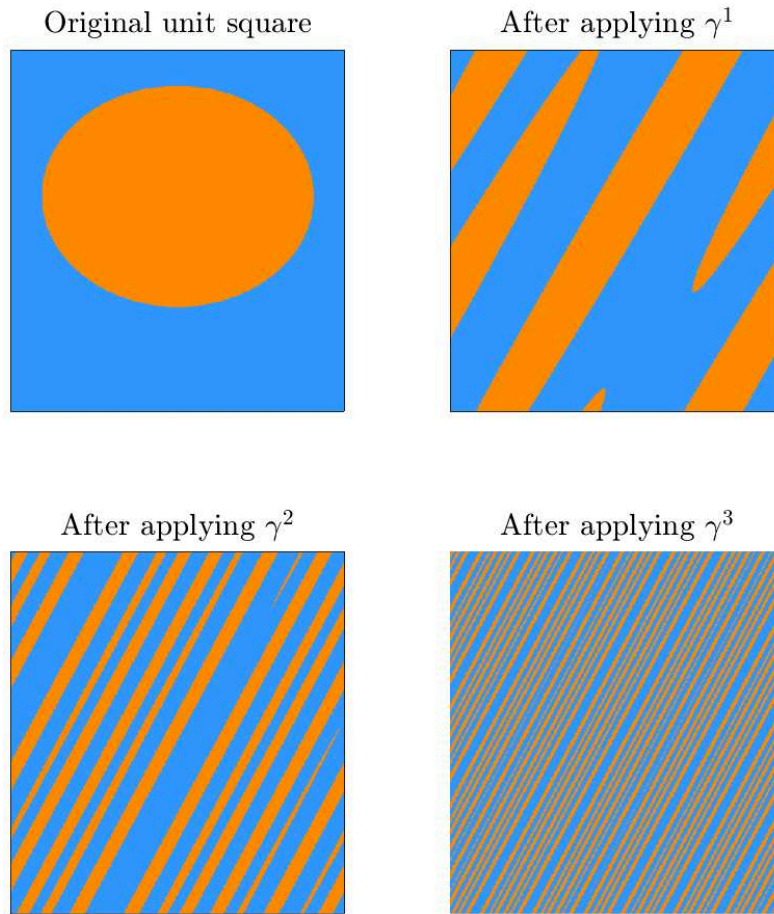
Un critère équivalent d'ergodicité a été donné par Birkhoff,

**Théorème P.1.2** (Théorème ergodique ponctuel de Birkhoff, section 4.1 dans [KH97]). *Soit  $(M, \mathcal{B}, \mu, \Phi)$  une transformation préservant les probabilités. Si  $f \in L^1(M, \mu)$  la limite*

$$\bar{f}(y) := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(\Phi^k(y))$$

*existe  $\mu$ -presque partout et  $\frac{1}{N} \sum_{k=0}^{N-1} f(\Phi^k(y)) \xrightarrow{L^1} \bar{f}(y)$ . La fonction  $\bar{f}$  est  $\Phi$ -invariante, absolument intégrable et  $\int \bar{f} d\mu = \int f d\mu$ .*

*Si  $\Phi$  est ergodique, alors  $\bar{f}(y) = \int f d\mu$  pour  $\mu$ -presque chaque  $y \in M$ . En d'autres termes, presque chaque fois que la moyenne le long d'une trajectoire converge vers la moyenne  $\mu$ -.*



**Figure P.1:** Les itérations d'un cluster de points  $\Omega$  se répartissent le long des branches instables, « remplissant »  $\mathbb{T}^2$  pendant un temps assez long. Conséquence :  $\gamma_{DE}$  (voir Exemple P.2.1 ci-dessous) est mélangeante et donc ergodique par rapport à la mesure de Liouville.

Nous remarquons que même si les théorèmes de Birkhoff et de Von Neumann ont été formulés pour des systèmes à temps discret, nous pourrions remplacer la sommation par l'intégration pour obtenir des caractérisations des propriétés ergodiques et de mélange pour des systèmes à temps continu.

### P.1.2 Hyperbolicité, propriété d'Anosov et flux géodésique

Soit  $N$  une variété riemannienne lisse et compacte sans bord (c'est-à-dire "une variété fermée"). Dorénavant la dynamique que nous considérons sera hyperbolique, notion rappelée ci-dessous, et  $C^1$  (continuellement différentiable). Pour simplifier la présentation on se réfère aux coordonnées locales de la



co-sphère<sup>1</sup> figent  $M = S^*N$  comme  $y$  et aux coordonnées locales sur  $N$  comme  $x$ .

Considérons d'abord la notion d'hyperbolicité uniforme ("Anosov") pour les cartes,

**Définition P.1.8** (difféomorphisme d'Anosov). *Supposons que  $M = SN$  est une variété riemannienne de norme  $\|\cdot\|$ . L'application  $\Phi$  est appelée un difféomorphisme d'Anosov s'il existe  $\Phi$ -sous-fibrés invariants  $E^s, E^u \subset TM$  tel que pour chaque  $y \in M$*

$$T_y M = E^s(y) \oplus E^u(y)$$

tel qu'il existe  $\lambda > 1$  et  $C > 0$  tel que pour tout  $t \geq 0$  si  $v \in E^u(y)$  alors  $\|(d\Phi^t(y))v\| \leq C\lambda^t \|v\|$  et si  $v \in E^s(y)$  alors  $\|(d\Phi^t(y))v\| \leq C\lambda^{-t} \|v\|$ . En d'autres termes, il a une direction d'expansion exponentielle décrite par le sous-paquet instable  $E^u$  et se contractant de manière exponentielle direction donnée par le sous-fibré stable  $E^s$ .

On peut souvent réaliser  $E^u$  et  $E^s$  en utilisant variétés instables et stables,

**Définition P.1.9.** *La variété de  $y \in M$  est l'ensemble*

$$V^s(y) := \left\{ y' \in M : \lim_{t \rightarrow \infty} d(\Phi^t(y), \Phi^t(y')) = 0 \right\}.$$

et de même sa variété instable est l'ensemble

$$V^u(y) := \left\{ y' \in M : \lim_{t \rightarrow -\infty} d(\Phi^t(y), \Phi^t(y')) = 0 \right\}.$$

Pour les cartes classiques définies ci-dessous (cf. section P.2)  $E^{u/s}(y) = T_y V^{u/s}(y)$ .

La notion d'hyperbolicité et de propriété d'Anosov peut également être traitée lors de l'étude des systèmes à temps continu.

**Définition P.1.10** (Flux hyperbolique et Anosov). *Supposons que  $M = S^*N$  est une variété riemannienne et notons la distance induite de la métrique par  $d$ . Le flux  $\{\Phi^t\}$  est appelé un flux Anosov s'il existe des  $\{\Phi^t\}$ -sous-paquets invariants  $E^s, E^u \subset TM$  tel que pour chaque  $y \in M$*

$$T_y M = E^s(y) \oplus E^u(y) \oplus E^0(y)$$

où dénotant  $X := \frac{d\Phi^t}{dt} \Big|_{t=0}$ ,  $E^0(x) = \text{Span}(X(y)) = \mathbb{R}X$  et il existe  $\lambda > 1$  et  $C > 0$  tels que pour tout  $t \geq 0$  si  $v \in E^u(y)$  alors  $\|(d\Phi^t(y))v\| \leq C\lambda^t \|v\|$  et si  $v \in E^s(y)$  alors  $\|(d\Phi^t(y))v\| \leq C\lambda^{-t} \|v\|$ .

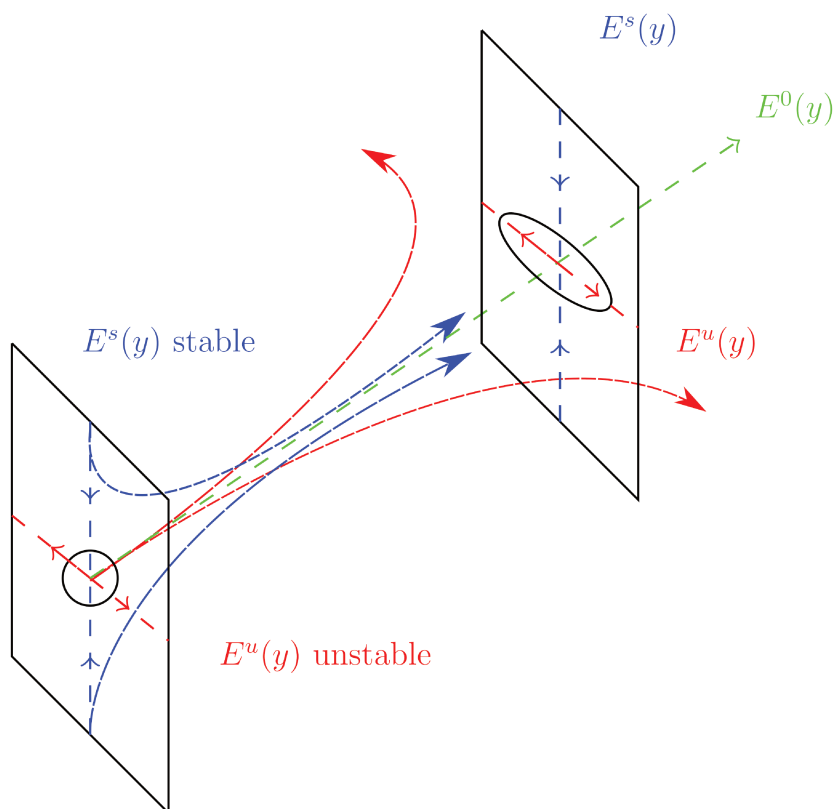
Faisons maintenant une digression et rappelons la définition d'un flot géodésique : Écrivez les coordonnées locales sur  $N$  sous la forme  $(x_1, \dots, x_d)$ . Dénotons la métrique riemannienne sur  $N$  par

---

<sup>1</sup>Rappelons que étant donnée une variété riemannienne  $(N, g)$ ,  $g$  se relève en une métrique  $\hat{g}$  sur  $TN$  appelée la métrique Sasaki. Dans ces notations,

$$S^*N := \{(x, \xi) : x \in N, \xi \in S_x^*N\}$$

avec  $S_x^*N$  étant le double espace de les vecteurs tangents de longueur unitaire en  $x$ .



**Figure P.2:** Une illustration schématique d'un flux d'Anosov, décomposant le faisceau (co-)tangent à chaque point en une distribution exponentiellement en expansion, en contraction exponentielle et tangentielle.

$g(x) = (g_{jk}(x))$  et (en utilisant la convention de sommation d'Einstein) sa matrice inverse par  $g^{-1}(x) = (g^{kn}(x))$ . Les *symboles de Christoffel* sont alors définis par

$$\Gamma_{jk}^i(x) = \frac{1}{2}g^{im}(x) \cdot \left( \frac{\partial g_{jm}}{\partial x_k} + \frac{\partial g_{mk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_m} \right)(x).$$

Une géodésique est alors la  $C^1$ -courbe  $c = (c_1, \dots, c_d)$  partant de  $c(0) \in N$  à une vitesse non nulle  $c'(0) \in N$  et résolvant l'équation différentielle ordinaire

$$c''_i + \Gamma_{j,k}^i c'_j c'_k = 0.$$

Notons que chaque  $(c(0), c'(0)) \in TN$  définit une géodésique unique. Le flux géodésique  $\Phi^t$  mappe chacun de ces points sur  $(c(t), c'(t))$  avec  $c(t)$  étant le point sur  $c$  satisfaisant  $d(c(0), c(t)) = t \|c'(0)\|_{c(0)} := t \sqrt{g_{c(0)}(c'(0), c'(0))}$ . En fait, un calcul direct montre que  $\|c'(t)\|$ , la vitesse de toute  $c$  géodésique, est constante et nous pouvons la normaliser en reformulant  $\{\Phi^t\}$  en flux sur le fibré unitaire tangent  $SN$  (ou plutôt sur  $S^*N$ ). Il existe une correspondance naturelle entre les géodésiques

proches (orientées) et les orbites proches de  $\Phi^t$ . Plus de détails sur le flux géodésique peuvent être trouvés dans des manuels comme [Kli11] ou [Pat99].

Nous nous intéressons à une classe de variétés généralisant les surfaces de selle<sup>2</sup> dans  $\mathbb{R}^d$ . Grosso modo, lorsque  $(N, g)$  est une surface, en tout point  $x$  on peut définir une matrice appelée *la seconde forme fondamentale* (cf. [Lee06]) dont les valeurs propres  $\kappa_1(x), \kappa_2(x)$  décrivent l'écart minimal et maximal au point  $x$  par rapport au plan. Le produit  $\kappa_1 \kappa_2(x)$  est appelé la *courbure gaussienne* à laquelle nous nous référons désormais simplement comme "la" courbure d'une surface. Cette définition peut être généralisée aux variétés. Comme cette thèse ne traite pas d'exemples de grande dimension, nous ne mentionnons que la généralisation en quelques mots. Lorsque nous prenons  $v \in \mathbb{R}^{2d}$  et le "transportons en parallèle" autour d'une boucle (courbe  $C^1$  avec des extrémités identiques), il pointera à nouveau dans la direction d'origine en revenant à sa position initiale. Riemann a suggéré de capturer l'étendue de l'échec de cette propriété sur  $M$  en considérant *Tensor de courbure de Riemann*,

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w,$$

où  $u, v \in T_y M, w$  sont des vecteurs tangents et  $[u, v]$  est le crochet de Lie de  $u$  et  $v$ . Alors on définit la *courbure en coupe de  $M$*  par

$$K(u, v) := \frac{\langle R(u, v)v, u \rangle}{\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2}.$$

Revenant à notre discussion principale en dimensions inférieures, un exemple bien étudié de dynamique hyperbolique se produit sur des variétés de courbure négative constante -1 :

**Théorème P.1.3** (par exemple [KH97, Théorème 17.6.2]). *Supposons que  $(N, g)$  est une variété riemannienne fermée de courbure sectionnelle négative, le flot géodésique sur<sup>3</sup>  $M = S^*N$  est un flux Anosov et fortement mélangeant par rapport à la mesure invariante naturelle sur  $M$  dite Liouville mesure  $\mu_{\text{Liouville}}$  et définie par  $\mu_{\text{Liouville}} := \det(g_{ij}) dy$ .*

Comme  $\mu_{\text{Liouville}}|_M$  est fini, on peut supposer en normalisant que  $\mu_{\text{Liouville}}(M) = 1$ , c'est-à-dire qu'il s'agit d'une probabilité invariante mesure.

Nous consacrons la sous-section suivante à rappeler brièvement quelques détails élémentaires concernant les surfaces hyperboliques de courbure négative constante.

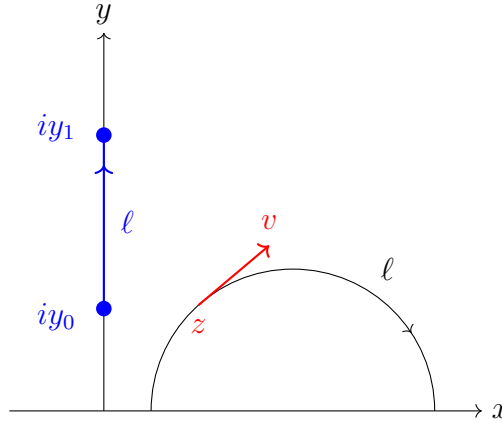
### P.1.3 Surfaces de courbure négative constante

Les dynamiques que nous présentons ci-dessous dans chapitre 1 sont un jouet modèle de dynamique bien étudié que nous rappelons brièvement dans cette sous-section (pour un traitement plus rigoureux cf. par exemple [Lan12] ou [EW10]) : On commence par rappeler des détails sur le demi-plan supérieur de Poincaré,

$$\mathbb{H} := \{z = x + iy : y > 0\}.$$

<sup>2</sup>par exemple  $f(x_1, x_2) = x_1^2 - x_2^2$

<sup>3</sup>Dénotant  $H(x, \xi) = \frac{\|\xi\|^2}{2}$  cet espace peut être réalisé comme  $H^{-1}(\frac{1}{2})$



**Figure P.3:** Les deux "types" possibles de géodésiques dans  $\mathbb{H}$  : soit sur l'axe  $y$  – soit sur des demi-cercles. Ici  $z$  indique un point complexe sur la géodésique tandis que  $v$  est la direction de la vitesse le long de celui-ci. La variété stable de  $(z, v)$  est obtenue à partir des géodésiques verticales voisines et sa variété instable à partir des géodésiques semi-circulaires.

En chaque point  $z \in \mathbb{H}$  le fibré tangent  $T_z\mathbb{H} = \{z\} \times \mathbb{C} \cong \mathbb{C}$  et on y définit la métrique riemannienne

$$d_z(v, w) = \frac{1}{y^2} (\operatorname{Re} v \cdot \operatorname{Re} w + \operatorname{Im} v \operatorname{Im} w)$$

avec  $z = x + iy$ . Cette métrique induit une distance sur  $\mathbb{H}$  donnée par

$$d(z_0, z_1) := \cosh^{-1} \left( 1 + \frac{|z_0 - z_1|^2}{2 \operatorname{Im}(z_0) \operatorname{Im}(z_1)} \right).$$

En d'autres termes, c'est la longueur d'une géodésique reliant  $z_0$  à  $z_1$ . L'action de groupe sur  $\mathbb{H}$  est donnée par une action transitive du groupe  $\operatorname{PSL}_2(\mathbb{R})$ , à savoir donné une matrice  $\mathfrak{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{PSL}_2(\mathbb{R})$  son action sur  $z$  est exprimé en termes de transformation Möbius,

$$\mathfrak{g}z = \frac{az + b}{cz + d}.$$

On rappelle que  $\mathfrak{g}z \in \mathbb{H}$  puisque d'un calcul direct  $\operatorname{Im}(\mathfrak{g}z) = \frac{\operatorname{Im} z}{|cz+d|^2}$ . Cela nous permet par un abus de notation d'associer à chaque  $\mathfrak{g}$  une fonction  $\mathfrak{g} : \mathbb{H} \rightarrow \mathbb{H}$  donnée par  $\mathfrak{g}(z) = \mathfrak{g}z$ . On note sa dérivée par rapport à  $z$  par  $\mathfrak{g}'(z) = \frac{1}{(cz+d)^2}$ . On peut alors étendre l'action à  $S\mathbb{H}$  en

$$\mathfrak{g} \cdot (z, v) := D\mathfrak{g}(z, v) = (\mathfrak{g}(z), \mathfrak{g}'(z) \cdot v) = \left( \frac{az + b}{cz + d}, \frac{v}{(cz + d)^2} \right).$$

Pour un  $z \in \mathbb{H}$ , fixe l'application  $(D\mathfrak{g})_z := D\mathfrak{g}(z, \cdot)$  est une application linéaire  $(D\mathfrak{g})_z : T_z\mathbb{H} \rightarrow$

$T_{g(z)}\mathbb{H}$  soulevant l'action de  $\mathbb{H}$  à  $T\mathbb{H}$ . Une propriété utile de cette action est qu'il s'agit d'une isométrie à la fois sur  $\mathbb{H}$ , c'est-à-dire,  $d(gz_0, gz_1) = d(z_0, z_1)$ , et sur  $S\mathbb{H}$  (pour une levée appropriée de  $d$ ). Cela signifie qu'on peut associer à chaque géodésique de  $S\mathbb{H}$  une géodésique de  $\mathbb{H}$ , image d'une courbe fixe "confortable" sous un élément  $g \in \mathrm{PSL}_2(\mathbb{Z})$ . Les géodésiques dans  $\mathbb{H}$  sont des demi-cercles perpendiculaires à l'axe réel. Nous permettons à l'une des extrémités du demi-cercle d'être  $i\infty$ , auquel cas la géodésique est une droite verticale parallèle à l'axe imaginaire. La "géodésique de référence" est la géodésique verticale passant par  $i$  et paramétrée par  $\gamma : \mathbb{R}_+ \rightarrow i\mathbb{R}$  donnée par  $\gamma(t) = ie^t$ . En notant que

$$\mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{R})}(i) = \mathrm{PSO}_2(\mathbb{R})$$

donne une identification  $\mathbb{H} = \mathrm{PSL}_2(\mathbb{R}) / \mathrm{PSO}_2(\mathbb{R})$ . Combiné avec la transitivité de l'action, cela donne  $S\mathbb{H} = \mathrm{PSL}_2(\mathbb{R})$ . On obtient une correspondance entre les éléments de  $S\mathbb{H}$  et les géodésiques à vitesse unitaire données par

$$\begin{array}{ccc} g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) & \longrightarrow & \mathbf{D}g(i, i) := (g(i), g'(i) \cdot i) = \left( \frac{ai+b}{ci+d}, \frac{i}{(ci+d)^2} \right) = (z, v) \in S\mathbb{H} \\ & \searrow & \downarrow \\ & & \gamma(t) := g(ie^{it}) = \frac{aie^{it}+b}{cie^{it}+d} \end{array}$$

Le flot géodésique  $\Phi^t : S\mathbb{H} \rightarrow S\mathbb{H}$  est alors défini comme le mappage prenant  $(z_{t_0}, v_{t_0}) = g(ie^{it_0})$  vers  $(z_{t+t_0}, v_{t+t_0}) := g(ie^{i(t+t_0)})$  au temps  $t$ .

Les variétés instables et stables sont données en termes de *flux horocycliques* correspondants, c'est-à-dire en écrivant  $(z, v) = \mathbf{D}g(i, i)$  pour un certain  $g \in \mathrm{PSL}_2(\mathbb{Z})$

$$V^s((z, v)) = \left\{ D \left( g \cdot \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \right) (i, i) : t \in \mathbb{R} \right\}, \quad V^u((z, v)) = \left\{ D \left( g \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) (i, i) : t \in \mathbb{R} \right\}$$

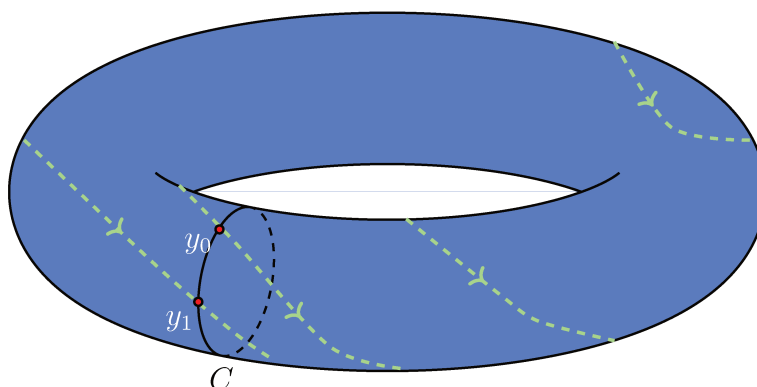
A partir d'un calcul direct,  $\mathbb{H}$  a une courbure négative constante  $-1$  et il sert comme modèle d'hyperbolicité : une surface est dite *hyperbolique* si chaque voisinage ouvert de celui-ci est isométrique à  $\mathbb{H}$ .

**Exemple P.1.11.** *En compactant  $\mathbb{H}$  en considérant son quotient sous un sous-groupe co-compact  $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$  on obtient une surface hyperbolique  $X = \mathbb{H} / \Gamma$ . Chaque géodésique de  $\mathbb{H}$  est enroulée autour de  $\Gamma$ , donnant lieu à des variétés instables/stables et des sous-paquets de  $SX$  ainsi le flux géodésique est Anosov également lorsque l'on considère le quotient.*

#### P.1.4 Passage de la dynamique continue à la dynamique discrète

Il existe un lien entre la dynamique continue et la dynamique discrète. Au début du XXe siècle, Poincaré propose la *première carte de récurrence* également connue sous le nom de *carte de Poincaré* afin d'analyser les solutions d'équations différentielles ordinaires comme (P.1) : Soit  $X$  un vecteur champ sur  $M$  sans point fixe, c'est-à-dire  $X(y) \neq 0$  pour chaque  $y \in M$ . Étant donné le flot  $\{\Phi^t\}$  sur  $M$ , généré par celui-ci, on peut construire une famille de sous-variétés de bord  $\Sigma_j \subset M = SN$  telle que

1.  $\mathrm{co-dim}\Sigma = 1$ .



**Figure P.4:** La restriction de la dynamique continue sur un tore à une section de Poincaré  $C$  génère une dynamique discrète de basse dimension

2. Chacun des  $\Sigma_j$  est transversal à  $X$ .
3. pour chaque  $y \in M$ , le flux  $\{\Phi^t\}_{t>0}$  croise  $\Sigma$  pour un certain  $t > 0$ .

Supposons qu'il existe  $T > 0$  tel que pour tout  $y \in M$  on puisse trouver  $\tau(y) < T$  pour lequel  $\Phi^{\tau(y)}(y) \in \Sigma := \bigcup_{j=1}^J \Sigma_j$ . En fixant  $y_0 \in M$ , le flux coupe  $\Sigma$  au premier temps de traversée  $y_1$  soit  $\Phi^t(y_0) = y_1 \in \Sigma \cap M$ . En itérant le processus, nous obtenons une séquence de points  $\{y_j\}_{j \in \mathbb{N}}$ , bien que  $t_{j+1} - t_j$  puisse varier, ce qui peut être considéré comme une trajectoire de dynamique discrète commençant de  $y_0$  et qui est donnée par *application de retour de Poincaré*  $R : \Sigma \rightarrow \Sigma$  donnée par  $R(y_0) = y_1, \dots, R^n(y_0) = R(y_{n-1}) = y_n$ .

**Exemple P.1.12.** Soit  $N$  une variété riemannienne proche de courbure négative avec  $\dim N = d$ . Considérons le flot géodésique  $\Phi^t$  sur son fibré sphérique  $SN$ . Alors en prenant  $\Sigma \subset SN$  comme une réunion d'hyper-surfaces  $\Sigma_j$  de dimension  $\dim \Sigma_j = 2d - 2$  on obtient par  $R : \Sigma \rightarrow \Sigma$  la dynamique discrète.

La dynamique discrète dans cet exemple est hyperbolique.

### P.1.5 Dynamique quantique

Au cours des 19e et 20e siècles, les physiciens ont mené des expériences qui conduisent à formuler la mécanique dans un nouveau langage « quantique » : Au début du XIXe siècle, James Clerk Maxwell, Ludwig Boltzmann et d'autres se sont appuyés sur les idées de John Dalton et Amedeo Avogadro pour établir la théorie cinétique des gaz. En 1900, Max Planck a introduit l'idée que l'énergie est *quantifiée*. Il a conjecturé que pour certains types de systèmes, l'énergie peut (théoriquement) être divisé en "énergie" discrète éléments"  $\varepsilon$ , chacun d'eux proportionnel à une fréquence  $\nu$  via la formule  $\varepsilon = 2\pi\nu\hbar$  avec  $\hbar \approx 10^{-34} Js$  appelée *constante de Planck*. Suivant les idées présentées lors du premier congrès Solvay de 1911, les scientifiques ont remplacé les fonctions sur le espace des phases (*observables classiques*), par des opérateurs et des fonctions d'onde caractérisant le comportement du système. En effet, en mécanique quantique, l'état du système est décrite en termes de norme unitaire *fonction d'onde*  $u(t)$ . La particule est décrite comme une onde dont la position précise à tout le temps ne peut pas être déterminé de

manière déterministe, mais seulement en fonction d'un densité de probabilité  $|u(x)|^2$  appelée *la densité de probabilité de position*. Cette l'incertitude sur la position a été formulée par Heisenberg en 1918 dans l'un des piliers de la mécanique quantique moderne :

**Théorème P.1.4** (Principe d'incertitude de Heisenberg, 1918). *Une fonction d'onde ne peut pas être localisée à la fois dans position et en élan. Plus précisément, soit  $u \in \mathcal{S}(\mathbb{R}^d)$  avec  $\|u\|_{L^2(\mathbb{R}^d)} = 1$ . En désignant la transformée de Fourier ( $\hbar$ -rescaled) par*

$$(\mathcal{F}_\hbar u)(\xi) := \int_{\mathbb{R}^d} u(x) e^{-\frac{i}{\hbar}\langle x, \xi \rangle} dx,$$

On a

$$\frac{\hbar}{2} \|u\|_{L^2} \cdot \|\mathcal{F}_\hbar u\|_2 \leq \|x_j u\|_2 \cdot \xi_j \mathcal{F}_\hbar u, \quad \begin{array}{l} x = (x_1, \dots, x_d) \\ \xi = (\xi_1, \dots, \xi_d) \end{array}.$$

Une question immédiate serait alors "quel lien peut-on trouver entre la formulation classique et celle quantique ? Sont classiques propriétés reflétées dans le niveau quantique ?". Plus formellement, nous aimerions comprendre comment les propriétés ergodiques d'un système classique dictent les propriétés spectrales de sa quantification. Nous allons en fait construire un *calcul de symboles*, c'est-à-dire des règles systématiques pour manipuler les opérateurs pseudo-différentiels. Un processus dans lequel on associe à chaque observable classique (parfois appelé *un symbole*)  $a \in C^\infty(\mathbb{R}_{x,\xi}^{2d}, \mathbb{R})$  un opérateur pseudo-différentiel  $A : L^2(\mathbb{R}_x^d) \rightarrow L^2(\mathbb{R}_x^d)$  s'appelle *quantification*. Comme  $A$  peut être un opérateur illimité, nous demandons en plus que  $\overline{\text{Dom}(A)} = L^2(\mathbb{R}^d)$ . Même si plusieurs quantifications similaires existent, nous considérerons ci-dessous une procédure de quantification fixe utile pour nos besoins nous notons  $a \mapsto A = \text{Op}_\hbar^W(a) = \text{Op}_\hbar(a)$ .

**Exemple P.1.13.** *Rappel Example P.1.1 d'une observable classique décrivant l'énergie d'une particule se déplaçant dans  $\mathbb{R}^d$ ,*

$$p(x, \xi) = \frac{\|\xi\|^2}{2} + V(x)$$

avec  $V \in C^\infty(\mathbb{R}^d)$ . Il induit au flot hamiltonien  $\Phi^t = \exp(tH_p)$  avec  $H_p$  étant le champ vectoriel hamiltonien généré par  $p$ ; son action sur  $g$  est donnée par  $H_p g := \{p, g\} = \frac{\partial p}{\partial x} \frac{\partial g}{\partial \xi} - \frac{\partial p}{\partial \xi} \frac{\partial g}{\partial x}$ , jouant le rôle du champ vectoriel hamiltonien du système. La quantification de l'hamiltonien classique, est appelé l'hamiltonien quantique du système est donné par

$$P_\hbar = -\frac{\hbar^2 \Delta}{2} + V(x) \tag{P.3}$$

avec son domaine  $\text{Dom}(P_\hbar)$  étant un sous-espace dense de  $L^2(\mathbb{R}^d)$ . En fait, la dynamique d'une particule quantique massive (non relativiste) est régi par l'équation de Schrödinger dépendante du temps sur  $L^2(\mathbb{R}^d)$ , c'est à dire,

$$i\hbar \partial_t u(x, t) = P_\hbar u(x, t).$$

On pose souvent des hypothèses sur  $V(x)$  pour que  $P$  soit un opérateur auto-adjoint. Si  $V$  croît polynomialement et  $\lim_{|x| \rightarrow \infty} V(x) = \infty$  l'opérateur est auto-adjoint et l'équation de Schrödinger a un spectre

réel discret non borné  $\{E_{\hbar,j}\}$  permettant d'écrire

$$P_{\hbar}u_{\hbar,j} = E_{\hbar,j}u_{\hbar,j}$$

avec  $u_{\hbar,j}$  les états stationnaires du système et  $E_{\hbar,j}$  les fréquences propres correspondantes. Dans ce scénario, on s'intéresse aux données spectrales :

1. La distribution des états propres dans un intervalle fixe  $I$ .
2. La (micro-)localisation spatiale des états propres  $u_{\hbar,j}$  et leurs domaines nodaux.
3. La distribution de valeur de  $u_{\hbar,j}$ .

Afin de quantifier le flot  $\Phi^t$ , on introduit le propagateur quantique,  $U(t) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ , un opérateur unitaire défini formellement par  $U(t) := \exp\left(-\frac{itP_{\hbar}}{\hbar}\right)$  ou de manière équivalente dans la représentation en base propre,

$$U(t)u_0 := \sum_{j \in \mathbb{N}} e^{-\frac{itE_{\hbar,j}}{\hbar}} \langle u_{\hbar,j}, u_0 \rangle u_{\hbar,j}$$

Remplacer  $\hbar$  par un  $h \rightarrow 0$  positif (appelé le *semi-classique paramètre*), nous étudions le comportement à long terme (à haute énergie) de l'opérateur pseudo-différentiel (dans le cas précédent de  $-h^2\Delta_{\mathbb{R}^d} + V$ ).

Au cours des 30 dernières années, il y a eu un intérêt croissant pour l'étude des propriétés des systèmes quantiques (en fait ceux mentionnés ci-dessus) dans la *limite de haute fréquence*, ce qui correspond mathématiquement à prendre  $h \rightarrow 0$ . Le domaine étudiant les propriétés spectrales d'un système chaotique pour ces faibles valeurs de  $h$  est appelé *chaos quantique*. En fait, il y a eu beaucoup d'intérêt pour les systèmes hyperboliques et pour montrer que les propriétés spectrales et dynamiques classiques se reflètent dans la contrepartie correspondante. Le domaine qui étudie les phénomènes spectraux quantiques dans ces contextes hyperboliques est appelé *Quantum Chaos*.

**Exemple P.1.14.** En prenant comme espace des phases  $S^*N$  où  $N = \mathbb{H}/\Gamma$  et  $\Gamma \leq PSL_2(\mathbb{R})$  un sous-groupe co-compact, le flux géodésique sur  $S^*N$  est quantifié par la opérateur de Laplace-Beltrami semi-classique  $-\Delta_g$  pour  $g$  étant la métrique sur  $N$ . C'est bien sait que son spectre est discret, que ses valeurs propres satisfont  $\text{Spec}(-\Delta_g) = \{\lambda_j^2\}$  pour la séquence  $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$  et que le correspondant les états propres  $\{\varphi_j\}_j$  forment une base orthonormée de  $L^2(N)$ . On étudie souvent le problème aux valeurs propres

$$-h^2\Delta_g\varphi_j = \lambda_j^2\varphi_j$$

dans la limite haute fréquence  $j \rightarrow \infty$ .

#### P.1.6 Ergodicité quantique et ergodicité quantique unique

Let  $(N, g)$  une surface compacte de courbure négative constante. Nous développons  $L^2(N)$  comme une somme directe des espaces propres résultant du choix d'une base propre orthonormée  $\{\varphi_j\}_j$ . La



correspondance quantique-classique établit un lien entre le flux classique et l'opérateur quantique. Cette connexion est donnée en termes du théorème d'Egorov. Les premiers résultats sur la distribution spatiale des modes propres haute fréquence sont dus à Shnirelman, Zelditch et Colin de Verdière ([Shn74],[Zel87],[CdV85]) déclarant que presque chaque  $\varphi_j$  est asymptotiquement équiréparti sur  $M$  dans la limite des hautes fréquences, une propriété appelée *ergodicité quantique*. Une formulation équivalente en est qu'il existe un ensemble  $J \subset \mathbb{N}$  de densité 1 tel que pour tout  $f \in C(M)$ ,

$$\int_N f(x) |\varphi_j(x)|^2 dx \xrightarrow[j \in J]{j \rightarrow \infty} \int_N f(x) dx. \quad (\text{P.4})$$

Rudnick et Sarnak ont supposé que les modes propres de  $-\Delta_g$  satisfont la propriété *quantum unique ergodicité* (QUE), ce qui signifie que  $J = \mathbb{N}$ , c'est-à-dire que (P.4) est valable pour la séquence complète  $\{\varphi_j\}$ . L'ergodicité quantique et QUE peuvent être tous deux reformulés en élevant les modes propres aux distributions  $dW_j \in \mathcal{D}'(T^*M)$  analogues aux distributions de Wigner sur  $T^*M$  qui sont définies par

$$a \mapsto \langle \text{Op}_h(a) \varphi_j, \varphi_j \rangle, \quad a \in C_c^\infty(T^*M), h = h_j = \lambda_j^{-1}. \quad (\text{P.5})$$

On peut toujours extraire une sous-suite convergente  $\{dW_{j_k}\}_{k \geq 1}$ , convergeant au sens distributionnel vers une mesure de probabilité sur  $S^*N$ . Les limites faibles-\* de  $\{dW_j\}_j$  sont appelées mesures semi-classiques, notées  $\mu_{sc}$ , et sont invariantes par rapport au flot géodésique. Ils représentent la micro-localisation de  $\{\varphi_{j_k}\}_{k \geq 1}$  sur l'espace des phases. Dans ces notations, l'ergodicité quantique revient à l'existence d'une sous-suite  $J \subset \mathbb{N}$  de densité 1 telle que  $dW_j \xrightarrow[j \in J]{j \rightarrow \infty} \mu_{\text{Liouv}}$  de  $S^*N$  et QUE signifie que la séquence complète converge faiblement vers la mesure de Liouville. Nous notons que puisque le flux géodésique est Anosov, il existe de nombreuses mesures de probabilité invariantes par rapport à lui en dehors de Lebesgue, par exemple des mesures supportées sur un ensemble fractal invariant ou sur des géodésiques fermées.

Tant que QUE reste ouvert, on peut se demander laquelle de ces myriades de mesures de probabilité, étant invariante sous le flot géodésique, peut être obtenue comme bornes faibles-\* de  $\{dW_j\}_j$ . Devant une telle mesure, on peut aussi se demander où elle est localisée. Il existe plusieurs résultats concernant les contraintes que les mesures semi-classiques doivent satisfaire.

Certaines de ces contraintes sont exprimées en termes d'entropie de Kolmogorov-Sinai quantifiant la complexité théorique de l'information d'une mesure invariante donnée. Cette entropie renseigne sur la localisation de la mesure : plus l'entropie est élevée, plus la mesure est délocalisée. Par exemple la mesure de Liouville est la mesure d'entropie maximale 1 alors qu'une mesure delta sur une géodésique fermée est d'entropie minimale 0. Anantharaman a prouvé dans [Ana08] que l'entropie de Kolmogorov-Sinai de chaque  $\mu_{sc}$  est positive. Conjointement avec Nonnenmacher, ils ont montré dans [ANo7b] que l'entropie est en fait bornée par le bas par la moitié de l'entropie maximale. Ces limites signifient que  $\mu_{sc}$  ne peut pas être "trop" localisé. Néanmoins, il existe des mesures invariantes de flux avec une entropie élevée qui sont supportées sur des sous-ensembles propres invariants. Un résultat récent dû à Dyatlov, Jin et Nonnenmacher, [DJ18, DJN22], indique que pour les surfaces hyperboliques, toute mesure semi-classique est entièrement supportée sur  $S^*N$ , et de plus que pour tout  $\emptyset \text{ouvert} \neq \Omega \subset S^*N$  il existe une constante  $C_\Omega$  indépendante du choix de  $\mu_{sc}$  telle que  $\mu_{sc}(\Omega) > C_\Omega$ . Leur méthode de preuve repose sur le principe d'incertitude fractale introduit dans [BD18] (voir Proposition 2.6.2). De plus, combiné avec le principe de continuation unique ils en déduisent que pour tout  $\Omega \neq \emptyset$  ouvert il

existe une constante  $c_\Omega$  telle que pour chaque mode propre  $\varphi_j$

$$\int_\Omega |\varphi_j|^2 dx \geq c_\Omega \int_N |\varphi_j|^2 dx = c_\Omega. \quad (\text{P.6})$$

## P.2 Carte classique du chat d'Arnold et carte quantique du chat

Dans la présente section, nous ne faisons qu'effleurer les détails essentiels nécessaires à la présentation des résultats. Dans chapitre 1, nous donnons un traitement plus complet du système classique et de sa quantification. Cette thèse est consacrée à l'étude de la carte du chat, un modèle de jouet de dynamique en temps discret sur le tore à 2 dimensions  $\mathbb{T}^2$ , ayant des propriétés dynamiques analogues à celles de la géodésique s'écoulent sur des collecteurs courbés négativement. Cette classe d'exemples a été popularisée par Arnold en 1967 (cf. [AA67]) et porte donc son nom, Arnold's cat maps. Ces applications chat sont des difféomorphismes d'Anosov agissant sur  $\mathbb{T}^2$  et préservant sa mesure de Lebesgue<sup>4</sup>. Plus précisément, chacun d'eux est une matrice  $\gamma \in \text{SL}_2(\mathbb{Z})$  de valeurs propres  $\lambda = \lambda_u, \lambda_s = \lambda^{-1}$  telle que  $|\lambda| > 1$  ayant une direction propre instable (en expansion)  $V_u$  et une stable (en contraction)  $V_s$ . Pour des raisons techniques expliquées ci-dessous, nous supposons en outre que la dynamique classique est donnée par des matrices en damier mod 2.

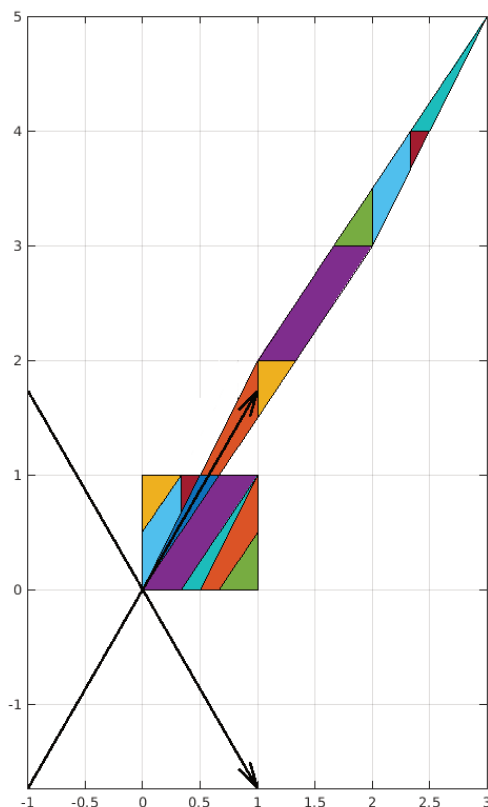
**Exemple P.2.1.** *Un exemple numérique canonique d'une telle matrice a été donné par Hannay-Berry dans [HB80] et plus tard dans les travaux de Keating et Degli-Esposti,  $\gamma_{DE} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ . Notez que nous ne pouvons pas utiliser l'exemple original dans le travail d'Arnold (cf. [AA67]),  $\gamma_{Arnold} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$  pour les raisons expliquées dans 1.4.2 ci-dessous. Nous illustrons l'action de  $\gamma_{DE}$  sur le carré de l'unité fondamentale dans Figure P.5. Ses valeurs propres sont  $\lambda_u = 2 + \sqrt{3}$ ,*

*$\lambda_s = 2 - \sqrt{3}$ . Pour la carte du chat de Degli-Esposti, les espaces propres correspondants sont donnés explicitement par le transport parallèle des vecteurs*

$$v_u(\gamma) = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad v_s(\gamma) = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}.$$

Passant au niveau quantique, l'espace quantique est un sous-espace de Hilbert  $\mathcal{H}_N \subset \mathcal{S}'(\mathbb{R})$  isomorphe à  $\mathbb{C}^N$  avec  $N = \frac{1}{2\pi h}$ . Nous considérons  $N$  comme notre paramètre semi-classique, en remplaçant la limite  $h \rightarrow 0$  par  $N \rightarrow \infty$ . Notez que (P.5) donne un opérateur  $\text{Op}_h(a) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ . Bouzouina et De Bièvre montrent dans la [BDB96, Proposition 2.3] que  $L^2(\mathbb{R})$  se décompose en une intégrale directe d'espaces de Hilbert de dimension finie et en déduisent qu'il en va de même pour la  $\text{Op}_h$ -quantification. Comme  $\mathcal{H}_N$  apparaît comme un intégrand direct dans la décomposition de  $L^2(\mathbb{R})$ , cela signifie qu'on peut construire à partir de tout opérateur  $\text{Op}_h(a)$  une quantification sur  $\mathcal{H}_N$ ,  $\text{Op}_N(a) : \mathcal{H}_N \rightarrow \mathcal{H}_N$  : Soit une fonction  $f \in L^2(\mathbb{R})$  dont la projection sur  $\mathcal{H}_N$  est notée  $f_N \in \mathcal{H}_N$ ,  $\text{Op}_N(a) f_N$  est la projection de  $\text{Op}_h(a) f$  sur  $\mathcal{H}_N$ ,

<sup>4</sup>En fait, en équipant  $\mathbb{T}^2$  de la forme symplectique  $\omega$ , les applications chat sont des symplectomorphismes.



**Figure P.5:** L'action de  $\gamma_{DE}$  sur  $\mathbb{T}^2$  peut être visualisée comme prenant mod 1 son action sur  $[0, 1]^2$ . Les flèches correspondent à ses directions instable/stable.

$$\begin{array}{ccc}
 f \in L^2(\mathbb{R}) & \hookrightarrow & f_N \in \mathcal{H}_N \\
 \downarrow & & \downarrow \text{dotted} \\
 \text{Op}_h(a) f \in L^2(\mathbb{R}) & \hookrightarrow & \text{Op}_N(a) f_N \in (\mathcal{H}_N)
 \end{array}$$

La dynamique classique est quantifiée par une famille de matrices unitaires  $\{\mathcal{M}_N(\gamma)\}$  appelée *la carte du chat quantique* obtenue à partir de la projection de l'évaluation d'une représentation métaplectique<sup>5</sup> (cf. [HB80]),  $\mathcal{M}_h(\gamma)$ , à  $\mathcal{H}_N$ . Pour le choix standard de base pour  $\mathcal{H}_N$  donné après Claim 1.2.2 ci-dessous,

**Exemple P.2.2.** Dans le cas de la carte de Degli-Esposti ci-dessus, on peut exprimer  $\mathcal{M}_N(\gamma) = \left( \mathcal{M}_N(\gamma)_{jk} \right)_{j,k} \in \mathbb{C}^{N \times N}$  explicitement par des sommes théoriques numériques. Par exemple pour  $\gamma_{DE}$ , pour chaque  $N \in 2\mathbb{N} + 1$ , jusqu'à un certain facteur de phase scalaire (son expression en tant que quotient des sommes de

<sup>5</sup>rappelons que tandis que  $\mathcal{M}_h(\gamma)$  est initialement un opérateur sur  $L^2(\mathbb{R})$ , on peut l'étendre à un opérateur sur  $\mathcal{S}'(\mathbb{R})$  puis le projeter sur  $\mathcal{H}_N$ .

Gauss est développée dans [KR00] et [DEG03]),

$$\mathcal{M}_N (\gamma_{DE})_{jk} = \frac{1}{\sqrt{N}} \exp \left( \frac{2\pi i}{N} (k^2 - kj + j^2) \right).$$

La carte chat quantique satisfait une relation d'entrelacement exacte : pour chaque  $a \in C^\infty (\mathbb{T}^2)$  et  $N \in \mathbb{N}^*$

$$\text{Op}_N (a \circ \gamma) = \mathcal{M}_N^{-1} (\gamma) \text{Op}_N (a) \mathcal{M}_N (\gamma).$$

Cette relation permet de faire le lien entre le système classique et son homologue quantique.

### P.2.1 Périodes quantiques

Une propriété particulière des matrices  $\mathcal{M}_N$  est leur quasi-idempotence : pour tout  $N$ , il existe une période quantique  $P(N)$  telle que

$$\mathcal{M}_N^{P(N)} = e^{i\kappa_N} \text{Id}_N, \quad \text{pour certains } \kappa_N \in [-\pi, \pi].$$

(ici  $P(N)$  sera le plus petit entier de ce type). Par conséquent, les valeurs propres de  $\mathcal{M}_N$  sont données par  $P(N)$ -ièmes racines de  $e^{i\kappa_N}$ . Si  $P(N) < N$ , cela implique une certaine dégénérescence spectrale de  $\mathcal{M}_N$ . La période quantique se comporte de manière assez erratique avec  $N$  [Kea91a], et  $\frac{1}{N} \sum_{n=1}^N P(n) \asymp N$ . Plus précisément, il existe  $C > 0$  tel que la période par laquelle il est borné

$$2 \frac{\log N}{\log |\lambda|} - C \leq P(N) \leq CN \log \log N, \quad (\text{P.7})$$

et il existe une sous-suite dense  $\mathcal{N} \subset \mathbb{N}$  telle que les périodes  $P(N) \geq N^{1/2} e^{(\log(N)^\delta)}$  [KR01a] (en supposant l'hypothèse de Riemann généralisée, on peut montrer que  $P(N) \geq N^{1-o(1)}$  le long d'une densité une sous-séquence [Kuro3]).

Bouzouina et De Bièvre ont prouvé que [BDB96] a montré que, le long de la suite complète  $N \in \mathbb{N}$ , quel que soit le choix des bases propres de  $\mathcal{M}_N$ , l'application quantique  $(\mathcal{M}_N)_{N \in \mathbb{N}^*}$  satisfait l'ergodicité quantique. En restreignant les  $N$  à une séquence spécifique de nombres premiers, Degli Esposti, Graffi, Isola [DEGI95] ont montré QUE pour les  $\mathcal{M}_N$ . Kurlberg-Rudnick [KR01a] et Bourgain [Bou07], ont montré que QUE tient dès qu'on se restreint à la densité une sous-suite  $\mathcal{N}$  de périodes  $P(N) \geq N^\epsilon$ . Kurlberg-Rudnick a également montré dans [KR00] que, si l'on impose, pour tout  $N \in \mathbb{N}^*$ , les états propres de  $\mathcal{M}_N$  à être des états propres conjoints d'une famille d'opérateurs "Hecke", alors les bases propres correspondantes satisfont QUE pour la suite complète  $N \in \mathbb{N}$  (résultat similaire à celui de Lindenstrauss sur les surfaces arithmétiques [Lino6]).

À l'autre extrême du spectre des périodes, Bonechi-De Bièvre a présenté dans [BDB00] une sous-séquence infinie  $\mathcal{N}_2 \subset \mathbb{N}$  (de densité asymptotique nulle) telle que les périodes soient *minimales*, à savoir saturer la borne inférieure dans (P.7) :

$$P(N) = 2 \frac{\log N}{\log |\lambda|} + O(1), \quad \mathcal{N}_2 \ni N \rightarrow \infty.$$

Cette petite période induit d'énormes multiplicités spectrales  $\asymp \frac{N}{\log N}$ , ce qui donne une grande liberté pour choisir nos états propres à l'intérieur de chaque espace propre. Profitant de cette liberté, Faure, Nonnenmacher et De Bièvre [FNDB03] ont exposé une suite d'états propres de  $(\mathcal{M}_N)_{N \in \mathcal{N}_2}$  qui ne sont pas asymptotiquement équidistribués sur  $\mathbb{T}^2$ , mais présentent plutôt une localisation partielle (appelée "cicatrice") sur une orbite périodique donnée de  $\gamma$ . Ces états propres ont été les premiers contre-exemples à QUE pour un système Anosov. Quelques années plus tard, Kelmer [Kel10] étudia les automorphismes hyperboliques quantifiés de tores de dimension supérieure  $\mathbb{T}^{2d}$ , et exhiba des séquences d'états propres parfaitement localisés sur certains sous-tors co-isotropes de  $\mathbb{T}^{2d}$ , en utilisant un mécanisme différent de celui de [FNDB03].

L'application de Baker, un autre exemple bien connu d'application chaotique sur le tore bidimensionnel, a été quantifiée par Balasz-Voros [BV89] et beaucoup étudiée numériquement et analytiquement, principalement dans la communauté de la physique théorique. Malgré des complications techniques dues à la discontinuité de la carte, l'ergodicité quantique a pu être prouvée pour ce système [DENW06]. Une quantification alternative de cette carte a été proposée dans [AN07a], dans laquelle des contre-exemples à QUE ont été donnés.

Citons également les travaux de Chang, Krüger, Schubert et Troubetzkoy [CKST08], qui ont construit des cartes quantiques associées à des cartes ergodiques sur l'intervalle, qui présentaient également des états propres non équirépartis.

### P.3 Relevé des résultats

La thèse est consacrée à la démonstration et à la présentation de trois types de résultats, tous concernant les (séquences d')états propres et leur délocalisation et leurs propriétés statistiques. Le premier résultat concerne la délocalisation complète de telles séquences (sans restriction sur les périodes quantiques), tandis que les deux autres concernent des cartes quantiques de grandes dégénérescences spectrales et étudient la délocalisation et les propriétés statistiques d'états propres aléatoires.

#### P.3.1 La délocalisation complète des mesures semi-classiques

Bouzouina et De Bièvre ont prouvé dans [BDB96] un analogue de l'ergodicité quantique pour l'application chat quantique, ie, il existe un ensemble  $J \subset \mathbb{N}$  de densité 1 tel que dans la limite  $N \in \infty$  les mesures de Wigner de presque tous les états propres (dont l'adaptation rigoureuse à  $\mathcal{H}_N$  est donnée dans ??) convergent faiblement vers  $\mu_{\text{Liouv}}$ . On rappelle une preuve de ce résultat dans section A.2. Cependant, l'ergodicité quantique n'empêche pas les sous-séquences exceptionnelles de converger vers une mesure  $\gamma$ -invariante différente de  $\mu_{\text{Liouv}}$  sur  $\mathbb{T}^2$ . Un travail de Kurlberg et Rudnick ([KR00]) a construit des bases propres "jointes" composées de vecteurs propres mutuels d'opérateurs  $\mathcal{M}_N(\gamma)$  et "Hecke" sur  $\mathbb{T}^2$  (analogue à celles définies pour les surfaces arithmétiques compactes). Ils ont prouvé que ces bases propres satisfont l'ergodicité quantique unique. En 2003, Faure, Nonnenmacher et De Bièvre ([FNDB03]) ont démontré que pour la carte quantifiée du chat, l'ergodicité quantique unique ne tient pas. Pour chaque orbite périodique  $\mathcal{P}$  de  $\gamma$ , ils ont trouvé une limite faible-\* de la forme  $\mu_{\text{sc}} = \frac{1}{2}\mu_{\text{Liouv}} + \frac{1}{2}\delta_{\mathcal{P}}$ . Faure et Nonnenmacher ont montré plus tard dans [FN04] que pour chaque  $\mu_{\text{sc}}$  de la carte du chat le poids de sa composante atomique est inférieur ou égal au poids de sa composante de Liouville, ce qui implique que la mesure semi-classique ne peut pas être portée entièrement sur une orbite périodique.

Anantharaman et Nonnenmacher ont obtenu un résultat sur l'entropie des mesures semi-classiques pour un autre modèle de jouet sur  $\mathbb{T}^2$ , l'application de Baker quantifiée de Walsh (cf. [AN07a]), à savoir que toute mesure semi-classique a pour au moins la moitié de l'entropie maximale (ie, Lebesgue).

Brooks [Bro10] a donné une caractérisation connexe des mesures semi-classiques pour la carte du chat. Il a interpolé entre la contrainte de [FN04] et l'analogue de la borne inférieure de [AN07a] pour la carte chat. Il a prouvé que les composantes ergodiques de haute entropie des mesures semi-classiques ont au moins autant de poids que celles de faible entropie.

Une partie centrale de la thèse est consacrée à prouver un résultat analogue à [DJ18] pour la carte quantique du chat, que chaque mesure semi-classique  $\mu_{sc}$  associée à  $\gamma$  est entièrement pris en charge sur  $\mathbb{T}^2$  ce qui signifie que toutes les séquences d'états propres  $\{\varphi_N\}_N$  sont entièrement délocalisées sur  $\mathbb{T}^2$  pour  $N$  suffisamment grand,

**Théorème P.3.1.** *Soit  $\gamma \in \tilde{\Gamma}(2)$  (où  $\tilde{\Gamma}(2)$  est défini dans (1.2)) une matrice hyperbolique quantifiée dans la famille  $\{\mathcal{M}_N(\gamma)\}_N$ . Soit  $\mu_{sc}$  une mesure semi-classique associée. Alors pour tout ouvert  $\emptyset \neq \Omega \subset \mathbb{T}^2$  il existe une constante  $c_\Omega > 0$  indépendante de  $\mu_{sc}$  tel que  $\mu_{sc}(\Omega) > c_\Omega$ .*

On peut l'énoncer en termes de  $\varphi_N \in \mathbb{C}^N$  par le choix de la base canonique ci-dessous 1.4.2 et l'identification (1.23) ci-dessous :

**Théorème P.3.2** (La "délocalisation" des fonctions propres). *Soit  $\gamma \in \tilde{\Gamma}(2)$ . Fixez  $0 \leq \alpha_1 < \alpha_2 \leq 1$ . Alors il existe une constante  $c_{\alpha_1, \alpha_2} > 0$  et un indice  $N_0 \in \mathbb{N}^*$  tels que pour tout  $N > N_0$  et vecteur propre normalisé  $\varphi_N = (\varphi_{N,k})_{k=1}^N$  de  $\mathcal{M}_N(\gamma)$*

$$\frac{1}{N} \sum_{k \in [\lfloor \alpha_1 N \rfloor, \lceil \alpha_2 N \rceil]} |\varphi_{N,k}|^2 > c_{\alpha_1, \alpha_2}. \quad (\text{P.8})$$

Les deux théorèmes sont déduits de Theorem 2.1.2 (que nous prouvons au chapitre 2).

Nous mentionnons que l'on peut poser des questions similaires pour des analogues de dimension supérieure de la dynamique hyperbolique des cartes de chat, c'est-à-dire des cartes hyperboliques  $\gamma \in \text{Sp}(2d, \mathbb{Z})$ . Rivière (cf. [Riv11]) a généralisé le résultat de [AN07b] pour ces paramètres de dimension supérieure en délimitant explicitement l'entropie de  $\mu_{sc}$  par le bas en termes de spectre de  $\gamma$ . De plus Kelmer a obtenu dans [Kel10] que si  $\gamma$  appartient à un certain sous-groupe de  $\text{Sp}(2d, \mathbb{Z})$ , on peut construire des vecteurs propres des opérateurs quantiques correspondants  $\mathcal{M}_N(\gamma)$  dont la mesure semi-classique associée est la mesure de Liouville sur un sous-espace lagrangien invariant de  $\mathbb{T}^{2d}$ , réfutant QUE pour  $\gamma$  ainsi qu'un analogue de Theorem P.3.1 en dimension haute.

Récemment Dyatlov et Jézéquel ont étendu (cf. [DJ21]) notre résultat à des applications chat de dimension arbitraire  $\gamma \in \text{Sp}(2d, \mathbb{Z})$  ayant une unique valeur propre simple et dont le polynôme caractéristique est irréductible sur  $\mathbb{Q}$  (ce qui rend le résultat du travail de Kelmer inapplicable). Leur méthode de preuve repose également sur le principe d'incertitude fractale à une dimension (Proposition 2.6.2 ci-dessous).

P.3.2 Ergodicité quantique (unique) probabiliste et à petite échelle

Outre le théorème d'ergodicité quantique, qui relie le comportement ergodique d'une dynamique classique (déterministe) à l'équidistribution des états propres, une autre façon de construire des états quantiques équidistribués consiste à considérer des systèmes avec de grands espaces propres, et en sélectionnant des états propres au hasard dans ces espaces propres. Cela a été fait principalement sur la sphère unitaire à 2 dimensions, sur laquelle l'espace  $V_l \subset L^2(\mathbb{S}^2)$  des harmoniques sphériques de niveau  $l \in \mathbb{N}$  (et de valeur propre  $l(l+1)$ ) est de dimension  $2l+1$  (la dimension  $\dim V_l \sim \sqrt{l(l+1)}$  est essentiellement la dimension maximale autorisée par la formule asymptotique spectrale de Weyl). Si l'on prend, pour chaque  $l \geq 0$ , une base orthonormée aléatoire de  $V_l$  (en utilisant la mesure naturelle de Haar), alors Zelditch [Zel92] a montré que presque sûrement, la base propre aléatoire globale de  $L^2(\mathbb{S}^2)$  satisfait l'ergodicité quantique. Vanderkam a amélioré le résultat dans [Van97], montrant que cette base propre aléatoire globale satisfait en fait QUE. Notons qu'on utilise ici les notions de QE ou QUE, bien qu'il n'y ait pas d'ergodicité dans le flot géodésique classique ; on dirait plutôt que les vecteurs propres le long de la base propre aléatoire deviennent équirépartis sur  $S^*\mathbb{S}^2$  dans la limite des hautes fréquences. Il a été démontré que les harmoniques sphériques aléatoires présentent d'autres propriétés statistiques intéressantes, par exemple, elles convergent statistiquement, dans la limite des hautes fréquences, vers le modèle d'onde aléatoire monochromatique proposé par Berry pour décrire les états propres chaotiques [Ber77]. Nous reviendrons sur son modèle dans subsection P.3.3 ci-dessous.

Burq et Lebeau ont généralisé la construction au compact arbitraire Variétés riemanniennes : pour une suite croissante de fréquences  $\lambda_{j_N} \rightarrow \infty$  ils ont considéré les intervalles spectraux  $I_N$  dans le spectre de  $\sqrt{\Delta_N}$ , centré sur  $\lambda_{j_N}$ , tel que le sous-espace spectral correspondant  $V_{I_N}$  a une dimension croissante  $d_N$  comme  $N \rightarrow \infty$ . Ils ont ensuite construit des combinaisons linéaires aléatoires de ces états propres, de manière à former un état aléatoire (normalisé)  $u_N \in V_{I_N}$ . À condition que les intervalles  $I_N$  soient suffisamment larges (ils supposent que  $|I_{N_n}| \gg 1$  quand  $N \rightarrow \infty$ ), ils montrent qu'avec une probabilité totale, la séquence  $(u_N)_N$  satisfait QUE [BL13, Thm 3]. Notez que dans leur construction, les  $u_N$  sont pas des états propres du Laplacien, mais plutôt des quasimodes. Il n'y a pas hypothèse dynamique sur le flot géodésique sur  $M$ . L'ingrédient principal de leur preuve est la loi de Weyl locale

$$\frac{1}{d_{N_n}} \text{Tr} (\Pi_{N_n} A \Pi_{N_n}) \xrightarrow{n \rightarrow \infty} \frac{1}{\text{Vol}(S^*N)} \int_{S^*N} \sigma(A) d\mu_{\text{Liouv}}, \quad (\text{P.9})$$

où  $\Pi_N$  est le projecteur orthogonal sur  $V_{I_N}$  et  $\sigma(A)$  est le symbole principal de  $A$ .

Zelditch [Zel14] a développé cette idée de manière plus générale paramètre. Il considère une suite d'espaces vectoriels  $(\mathcal{H}_N \subset L^2(N))_{N \in \mathbb{N}}$  orthogonaux entre eux, de dimensions  $d_N \rightarrow \infty$ , en supposant que pour tout pseudodifférentiel opérateur  $A$  d'ordre zéro sur  $M$ , les opérateurs projetés  $\Pi_N A \Pi_N$  satisfont une loi de Weyl généralisée du type (P.9), comme dans les réglages précédents de Burq-Lebeau. Il prouve ensuite qu'en choisissant des bases orthonormées au hasard sur les  $\mathcal{H}_N$ , la base aléatoire globale pour  $\bigoplus_N \mathcal{H}_N$  satisfait presque sûrement QE. Maples [Map13] a amélioré le résultat de Zelditch en une instruction QUE presque sûre similaire à celle de [BL13].

L'amélioration du QE probabiliste au QUE probabiliste provient des méthodes probabilistes util-

isées. Zelditch calcule les variances quantiques associées aux bases propres  $(u_{N,k})_{k=1,\dots,d_N}$  :

$$\text{Var}_N := \frac{1}{d_N} \sum_{k=1}^N |\langle u_{N,k}, Au_{N,k} \rangle - \omega(A)|^2, \quad \text{où} \quad \omega(A) = \frac{1}{\text{Vol}(S^*M)} \int_{S^*M} \sigma(A) d\mu_{\text{Liouville}},$$

et prouve QE par le fait que les variances tombent à zéro lorsque  $N \rightarrow \infty$ . Les méthodes de [Van97, BL13, Mapi13] montrent des estimations plus précises pour  $\mathbb{P}(|\langle u_{N,k}, Au_{N,k} \rangle - \omega(A)| \geq \epsilon)$ , permettant de contrôler tous les états ensemble.

### P.3.2.1 Équidistribution à petite échelle

Les déclarations d'ergodicité quantique utilisent des observables fixes pour tester l'équidistribution des fonctions propres dans la limite semi-classique. Dans le cas du Laplacien sur une variété riemannienne  $(M, g)$ , un renforcement récent de QE porte sur les poids des fonctions propres dans certains ensembles semiclassiquement petits sur  $M$ , par exemple dans les boules  $B(x, r(\lambda_j))$  de rayons  $r(\lambda_j)$  tels que  $r(\lambda_j) \searrow 0$  dans la limite semi-classique  $\lambda_j \rightarrow \infty$ . L'ergodicité quantique à petite échelle (SSQE), par rapport aux échelles  $(r(\lambda_j))_{\lambda_j}$ , fait alors référence au fait que, pour une sous-séquence de densité un,

$$\int_{B(x, r(\lambda_j))} |\varphi_j(x)|^2 d\text{Vol}_M(x) = \frac{\text{Vol}(B(x, r(\lambda_j)))}{\text{Vol}(M)} + o(r(\lambda_j)^d) \quad \text{lorsque } N \rightarrow \infty.$$

uniformément par rapport aux choix  $x_N$  des centres des boules.

Cette forme de délocalisation à petite échelle a été introduite presque simultanément par Han [Han15] et Hezari-Rivière [HR16] dans le contexte de variétés à courbure négative, avec de petites échelles de décroissance logarithmique  $r(N) \sim |\log \lambda_N|^{-\alpha}$ . Han considère également la microlocalisation de  $u_N$  sur de petites régions d'espace de phase, en utilisant des opérateurs pseudodifférentiels  $A_N$  avec des symboles  $\sigma(A_N)$  dépendant explicitement du paramètre semi-classique, de manière à être pris en charge sur des régions d'espace de phase plus petites comme  $N \in \infty$ .

Hezari-Rivière [HR17] et Lester-Rudnick [LR17] ont étudié la concentration à petite échelle des états propres du Laplacien sur tore  $\mathbb{T}^d$ , atteignant les échelles algébriques  $r(N) \sim \lambda_N^{-\alpha}$  pour certains  $\alpha > 0$ ; dans cette géométrie, puisque le flux géodésique conserve l'angle, la délocalisation ne se produit que dans la variable de position, tandis que les états propres peuvent être très concentrés dans l'espace de Fourier.

Dans [Han15], Han considérait le Laplacien sur une variété riemannienne  $d$ -dimensionnelle sur lequel un groupe d'isométries agit transitivement (ceci généralise la sphères rondes, ou les tores plats), et considérée comme une sous-séquence  $\mathcal{N} \subset \mathbb{N}$  tel que, le long de cette sous-suite, les fréquences propres  $(\lambda_j)_{j \in \mathcal{N}}$  du Laplacien admettent des multiplicités "logarithmiquement grandes"  $m_{\lambda_j}$  :

$$\liminf_{\mathcal{N} \ni j \rightarrow \infty} \frac{m_{\lambda_j}}{\log \lambda_j} > 0.$$

Il a considéré une base propre aléatoire sur le sous-espace spectral  $\bigoplus_{j \in \mathcal{N}} V_{\lambda_j}$ , et a montré que, pour tout



$\alpha \in (0, \frac{1}{2a})$ , alors presque sûrement la base propre est équidistribuée sur  $M$ , à l'échelle  $r(\lambda_j) = m_{\lambda_j}^{-\alpha}$  dans la limite  $j \in \infty$ .

Revenons maintenant à notre configuration de l'application chat quantique sur  $\mathbb{T}^2$  brièvement décrite dans section P.2. Han a récemment [Han18] prouvé plusieurs résultats à petite échelle délocalisation des états propres de  $\mathcal{M}_N$ . Donner une définition de SSQE dans ce contexte, un doit approximer la fonction caractéristique sur les balles  $B(\rho, r(N)) \subset \mathbb{T}^2$  centré en  $\rho \in \mathbb{T}^2$  et de petite taille rayon  $r(N)$ , par des fonctions lisses  $\chi_{\rho, r(N)}$  vivant dans classe de symboles légèrement exotiques sur  $\mathbb{T}^2$ . Han a approximé  $\mathbb{1}_{B(\rho), r}$  par deux familles de polynômes trigonométriques  $b_{\rho, r}^{\pm}$  dont les coefficients de Fourier satisfont certaines conditions de contrôle ; on utilisera plutôt une famille de fonctions  $\chi_{\rho, r}$  de petits supports, définis dans Equation 3.9 dessous. Cette fonction satisfait des propriétés importantes : premièrement, elles se rapprochent de la fonction caractéristique sur  $B(\rho, r)$ , et satisfont en fait  $\int_{\mathbb{T}^2} \chi_{\rho, r} d\rho = \pi r(N)^2 + \text{reste}$ . Deuxièmement, de manière analogue aux fonctions de Han, nous avons un contrôle sur leurs coefficients de Fourier (par exemple dans (3.10) et (3.11)) ce qui nous permet de prouver le SSQE.

Une fois que l'on a fixé la classe des fonctions caractéristiques lissées, la définition de Q(U)E à petite échelle se lit comme suit.

**Définition P.3.1** (Ergodicité quantique (unique) à petite échelle pour  $\mathcal{M}_N$ ). *Soit  $\nu := (r(N))_{N \in \mathbb{N}^*}$  soit une séquence de microscopiques échelles, c'est-à-dire  $\lim_{N \rightarrow \infty} r(N) = 0$  avec  $r(N) > N^{-\frac{1}{2}}$ . Fixer une sous-séquence infinie  $\mathcal{N} \subset \mathbb{N}$ , et choisissez les bases propres correspondantes  $\mathcal{B}_N$  pour  $(\mathcal{M}_N)_{N \in \mathcal{N}}$ .*

1. *On dit que ergodicité quantique à l'échelle  $\nu$  tient le long de la séquence de bases propres  $(\mathcal{B}_N)_{N \in \mathcal{N}}$ , si pour chaque  $N \in \mathcal{N}$ , il existe un sous-ensemble  $S(N) \subset \{1, \dots, N\}$  tel que  $\frac{\#S(N)}{N} \xrightarrow{\mathcal{N} \ni N \rightarrow \infty} 1$ , et l'estimation suivante est valable, uniformément par rapport aux choix de  $\rho \in \mathbb{T}^2$  :*

$$\langle \text{Op}_N(\chi_{\rho, r(N)}) \varphi_{j, N}, \varphi_{j, N} \rangle = \pi r^2(N) + o(r^2(N)) \text{ as } \mathcal{N} \ni N \rightarrow \infty, \quad j \in S(N).$$

2. *On dit que QUE tient à l'échelle  $r(N)$  le long de la suite des bases propres  $(\mathcal{B}_N)_{N \in \mathcal{N}}$  si l'estimation ci-dessus tient le long de la pleine bases propres  $(\mathcal{B}_N)_{N \in \mathcal{N}}$ .*
3. *On dit que QE (resp. QUE) est vérifiée à l'échelle  $\nu$  le long de la suite  $N \in \mathcal{N}$  si les estimations ci-dessus sont valables pour toute séquence de bases propres  $(\mathcal{B}_N)_{N \in \mathcal{N}}$  de  $(\mathcal{M}_N)_{N \in \mathcal{N}}$ .*

Ici  $\text{Op}_N(\chi)$  est la quantification de l'observable  $\chi \in C^\infty(\mathbb{T}^2)$  sur l'espace quantique de niveau  $N$ , voir chapitre 1. Lorsque nous discutons de la petite échelle Q(U)E, nous prenons toujours  $r(N) \gg N^{-\frac{1}{2}}$  pour nous assurer que  $\chi$  est dans une classe de symboles "bien comportés".

[Han18] montre plusieurs formes de SSQE pour le chat quantique map <sup>6</sup> :

1. pour la séquence complète  $N \in \mathbb{N}^*$  des paramètres de Planck et toutes les bases propres, la carte du chat quantique satisfait QE à l'échelle  $r(N) = |\log N|^{-\alpha}$ . pour tout  $\alpha \in [0, 1/4)$ .

---

<sup>6</sup>Bien que Han utilise une famille de fonctions de test différente de notre  $\chi_{\rho, r(N)}$ , ses preuves peuvent être facilement adaptées à nos fonctions de test.

2. prend la séquence  $\mathcal{N} \subset \mathbb{N}$  de densité un correspondant aux "longues" périodes quantiques  $P(N) \geq N^{1/2} e^{(\log N)^\delta}$ ; alors pour tout choix de bases propres de  $(\mathcal{M}_N)_{N \in \mathcal{N}}$ , QE vaut à l'échelle  $r(N) = N^{-\alpha}$  pour tout  $0 \leq \alpha < 1/16$ .
3. considère la séquence complète  $N \in \mathbb{N}$  mais ne considère que les bases propres conjointes de  $\mathcal{M}_N$  et les opérateurs de Hecke; alors ces bases propres de Hecke satisfont QE à l'échelle  $r(N) = N^{-\alpha}$  pour tout  $0 \leq \alpha < 1/12$ .

Alors que la preuve de la première partie est similaire à celle de [Han15, HR16] pour les variétés de courbure négative, les deux derniers points sont spécifiques à la carte chat quantique, et utilisent directement les estimations obtenues par [KR00, KR01a].

### P.3.2.2 Nos résultats SSQUE probabilistes

Dans le présent travail, nous considérons la carte de chat quantique  $\mathcal{M}_N$ , et restreignons les valeurs de  $N$  aux sous-séquences présentant des périodes très courtes. plus précisément on fixe un  $\alpha \in [2, 4)$ , et une suite  $\mathcal{N}_\alpha$  telle que pour tout  $N$  dans cette suite

$$P(N) \leq \alpha \frac{\log N}{\log |\lambda|} + O(1) \quad \mathcal{N}_\alpha \ni N \rightarrow \infty. \quad (\text{P.10})$$

Au facteur  $\alpha$  près, ces périodes sont minimales, et toute sous-suite  $\mathcal{N}_\alpha$  est nécessairement de densité nulle à l'intérieur de  $\mathbb{N}$ . Pourtant, comme nous l'avons mentionné ci-dessus, il existe des sous-séquences infinies satisfaisant ces bornes.

Comme expliqué dans Lemma 3.4.2 des périodes aussi courtes impliquent d'énormes dégénérescences spectrales : nous allons en fait montrer que le long de cette séquence  $\mathcal{N}_\alpha$ , tous les espaces propres  $V_{\nu, N} = \ker(\mathcal{M}_N - \nu)$  de  $\mathcal{M}_N$  ont des dimensions comparables

$$d_{\nu, N} := \dim(V_{\nu, N}) \sim \frac{N}{P(N)} \asymp \frac{N}{\log N}.$$

Au lieu de chercher les états propres les plus localisés, comme cela a été fait dans [FNDB03], nous allons plutôt considérer les états propres typiques dans  $V_{\nu, N}$ , en construisant l'ensemble des bases propres orthonormées aléatoires sur  $V_{N, \nu}$ , comme expliqué dans section 3.2 dans divers paramètres. En rassemblant les bases propres aléatoires de tous les  $\{\nu \in \text{Spec}(\mathcal{M}_N)\}$  nous obtenons une loi de probabilité  $\mathbb{P}_N$  sur l'ensemble des bases propres de  $\mathcal{M}_N$ . En joignant ces bases propres pour tout  $N \in \mathcal{N}_\alpha$ , on obtient une loi de probabilité globale  $\mathbb{P} = \prod_{N \in \mathcal{N}_\alpha} \mathbb{P}_N$  sur l'ensemble global des bases propres de  $(\mathcal{M}_N)_{N \in \mathcal{N}_\alpha}$ .

Notre résultat principal énonce une forme probabiliste de QUE pour ces bases propres aléatoires :

**Théorème P.3.3** (QUE probabiliste pour les cartes de chats quantiques à courte période). *Pour  $2 \leq \alpha < 4$ , considérons une séquence infinie  $\mathcal{N}_\alpha \subset \mathbb{N}$  avec de courtes périodes quantiques satisfaisant la borne (P.10), et construisons l'ensemble correspondant de bases propres aléatoires. Alors,*

1. presque sûrement les bases propres aléatoires de  $(\mathcal{M}_N)_{N \in \mathcal{N}_\alpha}$  satisfont QUE.

2. Plus précisément, si on prend  $0 \leq \delta < 1/2 - \alpha/8$ , alors presque sûrement les bases propres aléatoires de  $(\mathcal{M}_N)_{N \in \mathcal{N}_\alpha}$  satisfait SSQUE à l'échelle  $r(N) = N^{-\delta}$ .

Notez que, dans le cas  $\alpha = 2$ , notre séquence  $\mathcal{N}_2$  était construit par [BDB00], et utilisé dans [FNDB03], pour montrer la existence d'états propres partiellement localisés (« marqués »). Ce qui précède théorème montre que les états propres typiques de  $(\mathcal{M}_N)_{N \in \mathcal{N}_2}$  sont plutôt équirépartis jusqu'à l'échelle  $N^{-1/4}$ . La cicatrisation phénomène décrit dans [FNDB03] est donc très atypique, aussi à l'intérieur des espaces propres.

Au vu des résultats “abstrait” de [Map13, Zel14], les principales étape pour montrer notre QUE probabiliste est de prouver un Weyl local loi sur les espaces propres  $V_{\nu,N}$  de  $(\mathcal{M}_N)_{N \in \mathcal{N}_\alpha}$ . Nous le prouvons dans section 3.4 en nous appuyant sur une méthode de Keating, utilisant l'heuristique de la formule de la trace de Gutzwiller sur des orbites périodiques classiques. Cette La loi de Weyl locale montre que la “moyenne quantique” de  $\text{Op}_N(a)$  sur le espaces propres  $V_{\nu,N}$ , converge vers la moyenne classique de  $a$  sur  $\mathbb{T}^2$ ; dans un sens, cela dit que les espaces propres  $V_{\nu,N}$  sont, globalement, quantique ergodique dans la limite  $N \in \infty$ . En utilisant une inégalité mesure-concentration due à Hanson et Wright (apparaissant sous une forme moderne dans Equation 3.32), nous obtenons que la probabilité qu'un élément de la matrice s'écarte de la moyenne classique correspondante est exponentiellement faible. Comme nous l'expliquons dans subsection 3.6.3.2 ci-dessous, pour terminer la preuve dans le cas à petite échelle, il faut exiger des conditions de contrôle supplémentaires sur la série de Fourier de  $\chi_{\rho,r(N)}$ .

### P.3.3 Propriétés statistiques des états propres aléatoires individuels

Dans la dernière partie de la thèse, nous discutons d'un thème évoqué précédemment : la caractérisation statistique des états propres *individuels* d'un système chaotique quantique. Dans les sous-sections précédentes, nous avons eu à l'esprit les fluctuations des états propres soit à des échelles macroscopiques (en considérant QE et QUE) ou à des échelles microscopiques supérieures à  $\frac{1}{N}$  (en considérant SSQUE). Lorsqu'il s'agit de fonctions propres individuelles, on étudie les fluctuations sur une échelle de Planck, c'est-à-dire  $\approx \frac{1}{N}$ . En 1977, Berry ([Ber77]) a proposé un *Random Wave Model*. À savoir, il a conjecturé que pour les états propres de systèmes chaotiques  $\psi$  devrait ressembler à une combinaison aléatoire d'ondes planes ayant la même fréquence  $\lambda \asymp N$  mais diverses directions arbitraires  $\xi_k$ ,

$$\psi(x) \approx \text{Re} \psi_{RW,\lambda}(x) := \sum_{j=1}^{J(\lambda)} \alpha_j e^{i\lambda \xi_k x}$$

où  $\alpha_j$  sont des variables aléatoires complexes indépendantes identiquement distribuées. La distribution de valeur de  $\psi_{RW,\lambda}(x)$  devrait être dans ce cas gaussienne. Son travail reposait sur des champs aléatoires gaussiens monochromatiques sur  $\mathbb{R}^2$ , définis par la fonction de covariance à 2 points

$$\mathbb{E}(\psi_{RW,\lambda}(x) \psi_{RW,\lambda}(x + \lambda y)) = J_0(\|y\|),$$

pour proposer un lien entre les fluctuations “à l'échelle de Planck” et l'équidistribution globale de la fonction de Wigner (avec ergodicité quantique).

Suite à l'observation de Berry, plusieurs découvertes numériques ont soutenu son modèle et proposé un lien avec la théorie des ensembles matriciels aléatoires. Les premières tentatives reliaient les ensembles gaussiens et circulaires de matrices et le spectre des hamiltoniens quantiques ou opérateurs de Floquet (cf. [FM86, HKS87, BGS84]). Suivi plus tard par le numérique d'Izraïlev (cf. [Izr87]), Kuś, Mostowski et Haake (cf. [KMH88]) ont suggéré une relation entre les ensembles matriciels circulaires (c'est-à-dire l'ensemble orthogonal circulaire, COE, et l'ensemble Ensemble Unitaire Circulaire, CUE) et les états propres de l'opérateur de Floquet des sommets périodiquement frappés. Ils ont montré que les statistiques des vecteurs propres concordaient avec celles des ensembles matriciels circulaires de Dyson.

Le modèle de Berry a été confirmé numériquement pour plusieurs systèmes dynamiques. Eckhardt a construit dans [Eck86] des états propres explicites pour les automorphismes toraux quantifiés correspondant à des valeurs de  $N$  de dégénérescence spectrale *faible* et présentant des statistiques matricielles aléatoires. Bäcker dans [Bäc03] est arrivé à des résultats similaires en perturbant la carte classique du chat par la "carte standard" (préservant la zone) et en quantifiant la nouvelle carte.

Plus tard, plusieurs travaux ont établi des versions faibles du modèle de Berry dans des fenêtres spectrales de variation : pour énoncer ces résultats, nous posons  $(M, g)$  une variété riemannienne compacte et lisse de dimension  $\dim M \geq 2$  et considérons les fonctions propres de l'opérateur de Laplace-Beltrami, soit  $-\Delta_g \varphi_j = \lambda_j^2 \varphi_j$ . Canzani et Hanin ont montré dans [CH15, CH18] que sous d'autres conditions sur les géodésiques, en fixant des  $R > 0$ , en définissant

$$\mathcal{H}_N := \bigoplus_{\lambda_k \in [N, N+1]} \ker (\Delta_g - \lambda_k^2), \quad d(N) := \dim \mathcal{H}_N$$

et en prenant un vecteur normal réel standard  $(\alpha_1, \dots, \alpha_{d(N)}) \sim \mathcal{N}_{\mathbb{R}}(0, \text{Id}_{d(N)})$ , les "vagues aléatoires redimensionnées",

$$\phi_N := \frac{1}{\sqrt{\dim \mathcal{H}}} \sum_{\lambda_j \in [N, N+1]} \alpha_j \varphi_j$$

satisfaire le modèle de Berry dans une certaine mesure. Han et Tacy [HT20] considèrent également des combinaisons aléatoires d'états propres dans des fenêtres spectrales : sous une contrainte sur les géodésiques, en prenant une fenêtre  $[N, N+1]$  ayant  $d(N)$  valeurs propres à l'intérieur et  $(\tilde{\alpha}_1, \dots, \tilde{\alpha}_{d(N)})$  étant une variable aléatoire sur  $\mathbb{S}^{d(N)-1}$ , il existe  $C > 0$  tel que pour tout  $x \in M, r > 0$

$$\mathbb{E} \left[ \int_{B(x,r)} |\tilde{\phi}_N|^2 d\text{Vol}_M \right] = C \frac{\text{Vol}(B(x,r))}{\text{Vol}(M)} \quad \text{où } \tilde{\phi}_N = \sum_{\lambda_k \in [N, N+1]} \alpha_k \varphi_k$$

le reste étant indépendant de  $x$ . Si  $r^{-1} = o(N)$ , la variance est  $o(\text{Vol}(B(x,r))^2)$ .

### P.3.3.1 Nos résultats sur les propriétés statistiques des états propres de la carte du chat quantique

Les trois travaux que nous venons d'évoquer s'appuyaient sur une grande fenêtre spectrale de taille constante correspondant à la somme directe de plusieurs espaces propres, alors que nous nous concentrons sur des fenêtres "logarithmiquement petites" peuplées d'états propres générant un grand espace propre *unique*  $V_\lambda$  de  $\dim V_\lambda \asymp \frac{N}{\log N}$  (comme nous en discutons dans Proposition 3.4.1 ci-dessous).

Nous utilisons la carte du chat quantique pour produire le premier exemple à notre connaissance d'une famille de *vrais états propres* de système chaotique quantifié satisfaisant une version miniature des statistiques du modèle de Berry. Nous ne considérons qu'une séquence spécifique de  $N$ s ayant une période quantique minimale. On note l'ensemble de ces  $N$ s par  $\mathcal{N}$ . Pour étudier les états propres aléatoires, nous partons d'un vecteur gaussien aléatoire  $(\alpha_{1,N}, \dots, \alpha_{N,N}) \sim \mathcal{N}(0, \text{Id}_N)$ . Comme nous l'expliquons ci-dessous, nous prenons  $\alpha_N = \alpha_N^{\mathbb{C}} \sim \mathcal{N}_{\mathbb{C}}(0, \text{Id}_N)$  sauf si  $\gamma$  possède des symétries, auquel cas nous considérons  $\alpha_N = \alpha_N^{\mathbb{R}} \sim \mathcal{N}_{\mathbb{R}}(0, \text{Id}_N)$ . Projection de ce vecteur gaussien aléatoire sur  $V_{\nu,N}$ . On note l'état propre aléatoire obtenu par  $v_N$ . Afin de simplifier la présentation on considère son multiple  $\beta_N = \sqrt{P(N)}v_N := (\beta_{1,N}, \dots, \beta_{N,N})$ . Avant d'énoncer nos résultats, nous définissons la  $\ell^\infty$ -norm d'une matrice  $A = (a_{jk})_{j,k \in \llbracket 1, N \rrbracket}$  comme étant  $\|A\|_{\ell^\infty} = \sup_{j,k} |a_{jk}|$ . Comme première proposition, nous montrons que  $\beta_N$  ressemble statistiquement à un vecteur gaussien aléatoire standard,

**Théorème.** *Soit  $\gamma \in \tilde{\Gamma}(2)$ . Considérons le projecteur  $\Pi_{\nu,N}$  correspondant à l'espace propre  $V_{N,\nu}$  où  $\nu \in \text{Spec}(\mathcal{M}_N)$ ,  $N \in \mathcal{N}$ . Alors il existe une suite de matrices  $(\Delta_N)_N$  avec  $\sup_N \|\Delta_N\|_{\ell^\infty} < \infty$  telle que*

1.

$$\Pi_{\nu,N} = \frac{1}{P(N)}I_N + \frac{1}{\sqrt[4]{N}}\Delta_N.$$

2. *Construisant un état propre aléatoire  $\beta_N$  comme ci-dessus et notant la matrice de covariance  $\text{cov}(\beta_N)$*

$$\text{cov}(\beta_N) = I_N + \frac{P(N)}{\sqrt[4]{N}}\Delta_N.$$

La preuve repose sur l'observation que les éléments de la matrice de  $\Pi_{\nu,N}$  sont proportionnels à ceux de la matrice de covariance puis sur leur étude dans Theorem 4.1.1. La preuve numérique (Figure 4.2a, Figure 4.2b) montre que le théorème ne peut pas être étendu à des valeurs paires de  $N$ s car les projecteurs spectraux appropriés ne sont plus proches de l'identité.

Notre deuxième résultat concerne les états propres normalisés aléatoires de  $\mathcal{M}_N$  lorsque  $N \in \mathcal{N}$ . On prouve dans Theorem 4.2.3 que leur distribution de valeur converge en probabilité vers la variable aléatoire normale standard,

**Théorème.** *Soit  $\gamma \in \tilde{\Gamma}(2)$ ,  $N \in \mathcal{N}$  et soit  $\beta_N$  un état propre aléatoire de  $\mathcal{M}_{N_k}(\gamma)$ .*

1. *if  $\beta_N = \beta_N^{\mathbb{R}}$  (c'est-à-dire obtenu en projetant une gaussienne standard réelle), Considérons le vecteur aléatoire normalisé*

$$\tilde{\beta}_N = \left( \tilde{\beta}_{1,N}, \dots, \tilde{\beta}_{N,N} \right) := \frac{\sqrt{2d(\nu, N)P(N)}}{\|\beta_N\|} (\beta_{1,N}, \dots, \beta_{N,N}).$$

*Définir les mesures empiriques*

$$\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{\beta_{j,N}} \quad \tilde{\mu}_N = \frac{1}{N} \sum_{j=1}^N \delta_{\tilde{\beta}_{j,N}}.$$

(a) Il existe une constante  $C_\gamma^{\mathbb{C}} > 0$  telle que  $\mathcal{N} \ni N \rightarrow \infty$ , pour tout rectangle complexe  $R \subset \mathbb{C}$  et  $\varepsilon \in (0, \frac{1}{8})$

$$\mathbb{P} \left( \left| \tilde{\mu}_N(R) - \mu_{\mathcal{N}_{\mathbb{C}}(0,1)}(R) \right| \geq C_\gamma^{\mathbb{C}} N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}. \quad (\text{P.11})$$

(b) Il existe une constante  $c_\gamma^{\mathbb{C}} > 0$  telle que  $\mathcal{N} \ni N \rightarrow \infty$ , pour tout rectangle complexe  $R \subset \mathbb{C}$  et  $\varepsilon \in (0, \frac{1}{8})$

$$\mathbb{P} \left( \left| \mu_N(R) - \mu_{\mathcal{N}_{\mathbb{C}}(0,1)}(R) \right| \geq c_\gamma^{\mathbb{C}} N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}. \quad (\text{P.12})$$

2. if  $\beta_N = \beta_N^{\mathbb{R}}$  (c'est-à-dire obtenu en projÉtant une gaussienne standard réelle), Considérons le vecteur aléatoire normalisé

$$\tilde{\beta}_N = \left( \tilde{\beta}_{1,N}, \dots, \tilde{\beta}_{N,N} \right) := \frac{\sqrt{d(\nu, N) P(N)}}{\|\beta_N\|} (\beta_{1,N}, \dots, \beta_{N,N}).$$

Définir les mesures empiriques

$$\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{\beta_{j,N}} \quad \tilde{\mu}_N(I) = \frac{1}{N} \sum_{j=1}^N \delta_{\tilde{\beta}_{j,N}}.$$

(a) Il existe une constante  $C_\gamma^{\mathbb{R}} > 0$  telle que  $\mathcal{N} \ni N \rightarrow \infty$ , pour tout intervalle  $I \subset \mathbb{R}$  et  $\varepsilon \in (0, \frac{1}{8})$

$$\mathbb{P} \left( \left| \tilde{\mu}_N(I) - \mu_{\mathcal{N}_{\mathbb{R}}(0,1)}(I) \right| \geq C_\gamma^{\mathbb{R}} N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}. \quad (\text{P.13})$$

(b) Il existe une constante  $c_\gamma^{\mathbb{R}} > 0$  telle que  $\mathcal{N} \ni N \rightarrow \infty$ , pour tout intervalle  $I \subset \mathbb{R}$  et  $\varepsilon \in (0, \frac{1}{8})$

$$\mathbb{P} \left( \left| \mu_N(I) - \mu_{\mathcal{N}_{\mathbb{R}}(0,1)}(I) \right| \geq c_\gamma^{\mathbb{R}} N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}. \quad (\text{P.14})$$

On déduit les déclarations pour  $\tilde{\beta}_N$ s en montrant que la norme de  $\beta_N$  ne fluctue pas beaucoup et donc la probabilité que  $\beta_{j,N}$  soit dans un ensemble est asymptotiquement identique à la probabilité  $\tilde{\beta}_{j,N}$  est dedans.

Nous aimerions mentionner un résultat précédent dû à Kurlberg et Rudnick dans [KR01b] qui ont étudié la distribution des valeurs d'un état propre de Hecke normalisé  $\psi_N = (\psi_{1,N}, \dots, \psi_{N,N})$  de  $\mathcal{M}_N$  chaque fois que  $N$  est un nombre premier divisé<sup>7</sup>. A savoir, pour ces valeurs de  $N$  ils se sont demandé quelles étaient les limites des mesures empiriques de  $\psi_N$

$$\mu_N(I) := \frac{1}{N} \sum_{j=1}^N \mathbb{1}_I(|\psi_{j,N}|), \quad \text{où } I \subset [0, 1]$$

<sup>7</sup>ce qui signifie que  $\gamma$  est diagonalisable mod  $N$  pour chacun d'eux, permettant d'appliquer des méthodes à la théorie des sommes et caractères exponentielles.

peut être. Ils ont trouvé que,

**Théorème P.3.4** (Kurlberg-Rudnick, [KR01b]). *En prenant une séquence de  $L^2$ -états propres de Hecke normalisés de la carte de chat quantique  $\psi_{\chi, N}$ , comme  $N \rightarrow \infty$  le long d'une séquence de nombres premiers divisés, pour chaque  $I \subset [0, 1]$*

$$\frac{1}{2} \mu_N(\psi_{\chi, N}; I) \rightarrow \mu_{sc}(I) := \frac{4}{\pi} \int_I \sqrt{1-x^2} dx.$$

# P

## Preliminaries: classical and quantum hyperbolic dynamics

The present work investigates the spectral properties of eigenvectors of the "quantum cat map", a celebrated toy example of a quantum system exhibiting chaotic dynamics. Such properties include but are not limited to the (de-)localization and the equidistribution of sequences of eigenstates of the quantum propagator. Let us begin by recalling the basics of Hamiltonian mechanics and its quantum counterpart. We then present two central models of chaotic systems admitting quantum versions: The geodesic flow, being hyperbolic on a surface of constant negative curvature and hyperbolic toral automorphisms also called "cat maps" after [AA67]. We then state our results.

### P.1 Reminder on classical mechanics

The starting point for our discussion is the study of dynamical systems. For simplicity we restrict our treatment in the next page to the motion of a particle in *the position space*  $\mathbb{R}^d$  equipped with coordinates  $x = (x_1, \dots, x_d)$ . We denote the particle's position in time  $t$  by  $x(t)$ . We further assume that the force  $F(x(t))$  acting on the particle is *conservative*, i.e.,  $F(x(t)) = -\nabla V(x(t))$  for some  $V \in C^\infty(\mathbb{R}^d)$  signifying the particle's potential energy. The study of dynamical systems began from the theory of differential equations. In order to describe the dynamics, we pick the *Hamiltonian formalism*, in which one considers a system of 2 first-order ordinary differential equations called *Hamilton's equations of movement* given by,

$$\begin{cases} \frac{dx_j}{dt} = \frac{\partial H}{\partial \xi_j}(x(t), \xi(t)) \\ \frac{d\xi_j}{dt} = -\frac{\partial H}{\partial x_j}(x(t), \xi(t)) \end{cases} \quad (\text{P.1})$$

The function  $H$  solving this system models the total conserved energy of the particle and is called the Hamiltonian of the system. Here the space of all states ("configurations") of the system is given by

$$T^*\mathbb{R}^d = \{(x_1, \dots, x_d, \xi_1, \dots, \xi_d) : x_j, \xi_j \in \mathbb{R}\}$$

with  $x = (x_1, \dots, x_d)$  being position coordinates whereas  $\xi = (\xi_1, \dots, \xi_d)$  being the momentum of the particle.

**Example P.1.1.** *In these settings of a non-relativistic particle with mass  $m$  on which acts a conservative*



force  $F$ , its total energy is given by the Hamiltonian,

$$H(x, \xi) = \frac{\|\xi\|_2^2}{2m} + V(x).$$

The first term is kinetic energy of the particle and the second is the potential energy.

Assume that for every  $(x(0), \xi(0)) \in T^*\mathbb{R}^d$  the system (P.1) admits a solution for every  $t \in \mathbb{R}$ . Then for every value of  $(x(0), \xi(0)) \in T^*\mathbb{R}^d$  one obtains a *trajectory of the dynamics*, being  $((x(t), \xi(t)))_{t \in \mathbb{R}}$ .

We will be interested in studying the properties of the trajectories corresponding to large times (i.e., the "long time evolution of the system"). For that purpose we represent the trajectories through a *flow*,  $\{\Phi^t\}_{t \in \mathbb{R}}$ , a one-parameter family of diffeomorphisms  $\Phi^t : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ , mapping  $(x(0), \xi(0))$  to its state at time  $t$ , i.e.,  $(x(t), \xi(t))$ . We can verify that for every  $t_1, t_2 \in \mathbb{R}$  the flow satisfies

$$\Phi^0 = \text{Id}_{\mathbb{R}^{2d}} \quad \Phi^{t_1+t_2} = \Phi^{t_1} \circ \Phi^{t_2}. \quad (\text{P.2})$$

Let us connect geometrically between the flow and the system. Equip  $\mathbb{R}^{2d}$  with the bi-linear form  $\omega$  called *the symplectic form*, given by

$$\omega(u, v) = u^t \left( \begin{array}{c|c} 0_d & \text{Id}_d \\ \hline -\text{Id}_d & 0_d \end{array} \right) v, \quad u, v \in \mathbb{R}^{2d}$$

or viewed as a differential form,

$$\omega = \sum_{j=1}^d dx_j \wedge d\xi_j.$$

The system above corresponds to  $X_H$ , the unique vector field on  $\mathbb{R}^{2d}$  satisfying for every  $v \in \mathbb{R}^{2d}$

$$\langle \nabla H, v \rangle_{\mathbb{R}^{2d}} = \omega(X_H, v)$$

and *generating the flow*  $\Phi^t$ . In fact,

$$X_H(x, \xi) = \left( \frac{\partial H}{\partial \xi_1}, \dots, \frac{\partial H}{\partial \xi_d}, -\frac{\partial H}{\partial x_1}, \dots, -\frac{\partial H}{\partial x_d} \right) (x, \xi)$$

and is called a *Hamiltonian vector field* and  $\Phi^t$  is called a *Hamiltonian flow*. The level sets of  $H$ , on which the particle has some fixed energy  $E > 0$ , are called *energy shells*,

$$\mathcal{S}(E) := \{(x, \xi) \in \mathbb{R}^{2d} : H(x, \xi) = E\}.$$

It is known that energy shells are preserved by the flow, hence we can restrict the dynamics to some fixed  $\mathcal{S}(E)$ .

The theory of dynamical systems is interested in the long time behavior of the trajectories  $(x(t), \xi(t))_{t \in \mathbb{R}}$  arising from (P.1). We won't restrict our attention only to  $(\mathbb{R}^{2d}, \omega)$  but rather to a smooth symplectic

manifold, i.e., the pair  $(M, \omega)$ , with  $\omega$  being a closed non-degenerate differential 2-form, having local coordinate patches isomorphic to  $(\mathbb{R}^{2d}, \omega)$ . The Hamiltonian flow  $\{\Phi^t\}_{t \in \mathbb{R}}$  might be replaced by a symplectic diffeomorphism  $\Phi : M \rightarrow M$  not necessarily associated to an ordinary differential equation. This map will generate the dynamics through trajectories  $(x(n), \xi(n))_{n \in \mathbb{Z}}$  describing the time evolution of the system. In other words, we distinguish between continuous dynamics, corresponding to a flow  $\{\Phi^t\}$ , and discrete dynamics, induced by iterating the map  $\Phi$  (and thus correspond to discrete times  $n \in \mathbb{Z}$ ).

As we explain in subsection P.1.4 below under some circumstances it is possible to obtain from continuous dynamics, a corresponding "discrete counterpart" through restriction to a low dimensional subspace.

### P.1.1 Ergodicity and mixing

We now define an additional structure on the phase space  $M$ . Consider the triple  $(M, \omega, \Phi)$  for an invertible  $\Phi : M \rightarrow M$ . Defining on  $M$  a  $\sigma$ -algebra  $\mathcal{B}$  and a  $\Phi$ -invariant probability measure  $\mu$  (i.e., for every  $A \in \mathcal{B}$   $\mu(\Phi^{-1}(A)) = \mu(A)$ ), we obtain a *measure space* on which acts the map  $\Phi$ .

**Example P.1.2.** Consider the symplectic manifold  $(\mathbb{R}^{2d}, \omega)$  and equip it with the Liouville measure  $\mu_{\text{Liouville}} = \frac{\omega}{d!}$ . Any symplectic diffeomorphism  $\Phi$  preserves  $\mu_{\text{Liouville}}$  (which means  $\mu_{\text{Liouville}}$  is  $\Phi$ -invariant).

We refer to the quadruple  $(M, \mathcal{B}, \mu, \Phi)$  a *measure preserving transformation*. Given such transformation we would like to understand the long time behavior of  $\Phi$ -orbits.

The first result is due to Poincaré who proved that if  $\mu$  is a probability measure, for every  $E \in \mathcal{B}$  for almost every  $y \in A$  there is a sequence  $\{n_k\}_k$  for which  $\Phi^{n_k}(y) \in A$ .

**Definition P.1.3.** Let  $(\mathbb{R}^{2d}, \mathcal{B}, \mu, \Phi)$  be a measure preserving transformation. We say  $\Phi$  is  $(\mu)$ -ergodic if for every measurable set  $A \in \mathcal{B}$ , if  $\Phi^{-1}(A) = A$  then  $\mu(A) \in \{0, 1\}$ .

**Example P.1.4.** Equip  $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$  with the Euclidean measure  $dx$ . For some  $\alpha \in \mathbb{R}$  consider the circle rotation  $\Phi_\alpha(x) = x + \alpha \pmod{1}$  preserving  $dx$ . It is natural to wonder whether it is ergodic. If  $\alpha = \frac{j}{k}$  for some  $j \in \mathbb{Z}, k \in \mathbb{N}^*$  the set  $A = \bigcup_{m=0}^{k-1} \left[ \frac{m}{k}, \frac{m+\frac{1}{2}}{k} \right]$  is  $dx$ -invariant of measure  $\frac{1}{2}$  and then for these values of  $\alpha$  the map  $\Phi_\alpha$  is not ergodic. On the contrary,  $\Phi_\alpha$  is ergodic if  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ .

Our next goal is to formulate conditions for "asymptotic equidistribution": Beginning with  $A \in \mathcal{B}$ , we would like that  $\Phi^t(A)$  will be "equidistributed" with respect to  $\mu$  as  $t \rightarrow \infty$ . For that purpose we introduce another measure theoretic (chaotic) property called *mixing*,

**Definition P.1.5.** We say  $(\mathbb{R}^{2d}, \mathcal{B}, \mu, \Phi)$  is (strongly) mixing if for every measurable sets  $A, B \subset \mathcal{B}$

$$\lim_{n \rightarrow \infty} \mu(B \cap \Phi^n(A)) = \mu(A) \mu(B),$$

where  $\Phi^n = \underbrace{\Phi \circ \dots \circ \Phi}_{n \text{ times}}$ . We test how much of  $\Phi^n(A)$  intersects with any subset  $B \in \mathcal{B}$ .

**Remark P.1.6.** *Strong mixing implies ergodicity: If  $A \in \mathcal{B}$  is a  $\Phi$ -invariant set, then for every  $n \in \mathbb{Z}$ ,  $\mu(A \cap \Phi^n(A)) = \mu(A)$ . If  $(M, \mathcal{B}, \mu, \Phi)$  is a strong mixing,  $\lim_{n \rightarrow \infty} \mu(A \cap \Phi^n(A)) = \mu^2(A)$ , hence  $\mu(A) \in \{0, 1\}$ .*

Below we will use an equivalent criterion for strongly mixing and ergodicity due to Von Neumann:

**Theorem P.1.7** (Von Neumann, Theorem 2.21 in [EW10]). *Consider the measure preserving transformation  $(\mathbb{R}^{2d}, \mathcal{B}, \mu, \Phi)$ .*

1. *The system is ergodic if and only if for every  $f \in L^2(M, \nu)$*

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{k=0}^{N-1} f \circ \Phi^k - \int_{\mathbb{R}^{2d}} f d\mu \right\|_{L^2(M, \mu)} = 0.$$

2. *It is strongly mixing if and only if for every  $f, g \in L^2(M, \mu)$*

$$\lim_{N \rightarrow \infty} \int_M f \circ \Phi^N \cdot g d\mu = \int_M f d\mu \cdot \int_M g d\mu.$$

**Example P.1.8.** *We will use below the fact that the dynamics on  $M = \mathbb{T}^{2d}$  defined by an automorphism  $\gamma \in SL_d(\mathbb{Z})$  with  $\text{Spec}(\gamma) \cap \mathbb{S}_{\mathbb{C}}^1 = \emptyset$  is strongly mixing (for proof cf. [KH97, Proposition 4.2.12] and Figure P.1).*

An equivalent criterion for ergodicity was given by Birkhoff,

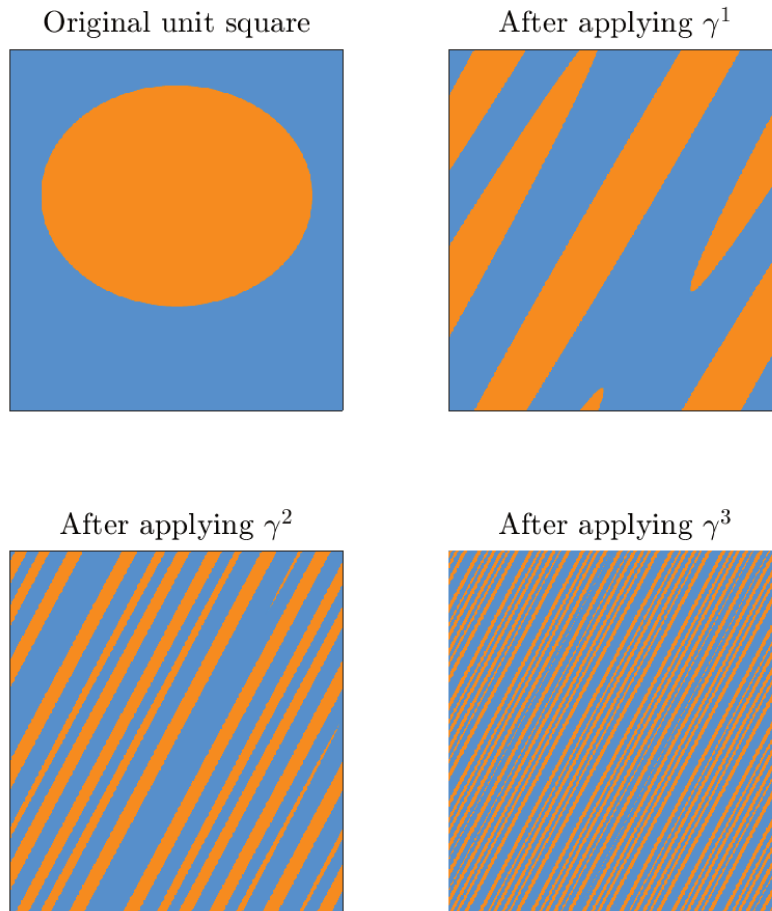
**Theorem P.1.9** (Birkhoff's Pointwise Ergodic Theorem, Section 4.1 in [KH97]). *Let  $(M, \mathcal{B}, \mu, \Phi)$  be a probability preserving transformation. If  $f \in L^1(M, \mu)$  the limit*

$$\bar{f}(y) := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(\Phi^k(y))$$

*exists  $\mu$ -almost everywhere and  $\frac{1}{N} \sum_{k=0}^{N-1} f(\Phi^k(y)) \xrightarrow{L^1} \bar{f}(y)$ . The function  $\bar{f}$  is  $\Phi$ -invariant, absolutely integrable and  $\int \bar{f} d\mu = \int f d\mu$ .*

*If  $\Phi$  is ergodic, then  $\bar{f}(y) = \int f d\mu$  for  $\mu$ -almost every  $y \in M$ . In other words, almost every time average along a trajectory converges to the  $\mu$ -average.*

We remark that even though Birkhoff's and Von Neumann theorems were formulated for discrete time systems we could replace summation with integration to obtain characterizations of ergodic and mixing properties for continuous time systems.



**Figure P.1:** The iterations of a cluster of points  $\Omega$  spread along the unstable branches, "filling"  $\mathbb{T}^2$  for a large time. Consequence:  $\gamma_{DE}$  (see Example P.2.1 below) is mixing and thus ergodic with respect to Liouville measure.

P.1.2 Hyperbolicity, Anosov property and the geodesic flow

Let  $N$  be a compact smooth Riemannian manifold without boundary (that is "a closed manifold"). Henceforth the dynamics we consider will be hyperbolic, a notion we recall below, and  $C^1$  (continuously differentiable). To simplify the presentation we refer to the local coordinates of the co-spherical<sup>1</sup> bundle  $M = S^*N$  as  $y$  and to the local coordinates on  $N$  as  $x$ .

First, let us consider the notion of uniform hyperbolicity ("Anosov") for maps,

**Definition P.1.10** (Anosov diffeomorphism). *Let us assume  $M = SN$  is a Riemannian manifold*

---

<sup>1</sup>We recall that given a Riemannian manifold  $(N, g)$ ,  $g$  lifts to a metric  $\hat{g}$  on  $TN$  called *the Sasaki metric*. In these notations,

$$S^*N := \{(x, \xi) : x \in N, \xi \in S_x^*N\}$$

with  $S_x^*N$  being the dual space of unit length tangent vectors at  $x$ .

with a norm  $\|\cdot\|$ . The map  $\Phi$  is called an Anosov diffeomorphism if there exist  $\Phi$ -invariant sub-bundles  $E^s, E^u \subset TM$  such that for every  $y \in M$

$$T_y M = E^s(y) \oplus E^u(y)$$

such that there exist  $\lambda > 1$  and  $C > 0$  such that for every  $t \geq 0$  if  $v \in E^u(y)$  then  $\|(d\Phi^t(y))v\| \leq C\lambda^t \|v\|$  and if  $v \in E^s(y)$  then  $\|(d\Phi^t(y))v\| \leq C\lambda^{-t} \|v\|$ . In other words, it has an exponentially expanding direction described by the unstable sub-bundle  $E^u$  and exponentially contracting direction given by the stable sub-bundle  $E^s$ .

We can often realize  $E^u$  and  $E^s$  using *unstable and stable manifolds*,

**Definition P.1.11.** Define the stable manifold of  $y \in M$  is the set

$$V^s(y) := \left\{ y' \in M : \lim_{t \rightarrow \infty} d(\Phi^t(y), \Phi^t(y')) = 0 \right\}.$$

and similarly its unstable manifold is the set

$$V^u(y) := \left\{ y' \in M : \lim_{t \rightarrow -\infty} d(\Phi^t(y), \Phi^t(y')) = 0 \right\}.$$

For the classical cat maps defined below (cf. section P.2)  $E^{u/s}(y) = T_y V^{u/s}(y)$ .

The notion of hyperbolicity and Anosov property can also be treated when studying continuous time systems.

**Definition P.1.12** (Hyperbolic and Anosov flow). Let us assume  $M = S^*N$  is a Riemannian manifold and denote the distance induced from the metric by  $d$ . The flow  $\{\Phi^t\}$  is called an Anosov flow if there exist  $\{\Phi^t\}$ -invariant sub-bundles  $E^s, E^u \subset TM$  such that for every  $y \in M$

$$T_y M = E^s(y) \oplus E^u(y) \oplus E^0(y)$$

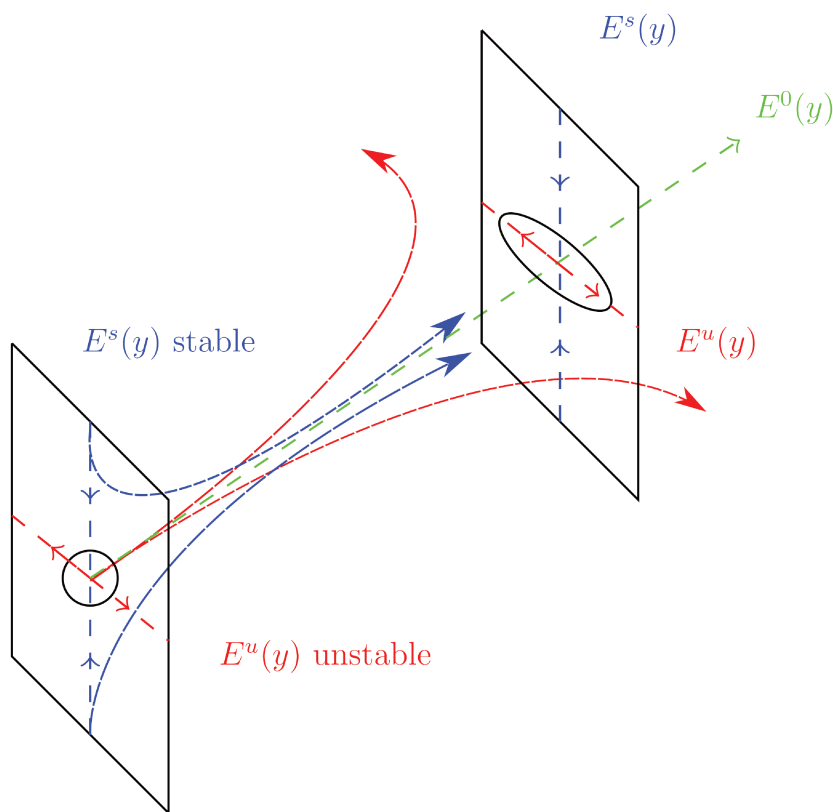
where denoting  $X := \frac{d\Phi^t}{dt}|_{t=0}$ ,  $E^0(x) = \text{Span}(X(y)) = \mathbb{R}X$  and there exist  $\lambda > 1$  and  $C > 0$  such that for every  $t \geq 0$  if  $v \in E^u(y)$  then  $\|(d\Phi^t(y))v\| \leq C\lambda^t \|v\|$  and if  $v \in E^s(y)$  then  $\|(d\Phi^t(y))v\| \leq C\lambda^{-t} \|v\|$ .

Let us now digress and recall the definition of a geodesic flow: Write the local coordinates on  $N$  as  $(x_1, \dots, x_d)$ . Denote the Riemannian metric on  $N$  by  $g(x) = (g_{jk}(x))$  and (using Einstein's summation convention) its inverse matrix by  $g^{-1}(x) = (g^{kn}(x))$ . The *Christoffel symbols* are then defined by

$$\Gamma_{jk}^i(x) = \frac{1}{2} g^{im}(x) \cdot \left( \frac{\partial g_{jm}}{\partial x_k} + \frac{\partial g_{mk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_m} \right) (x).$$

A geodesic is then the  $C^1$ -curve  $c = (c_1, \dots, c_d)$  starting at  $c(0) \in N$  at a non-zero velocity  $c'(0) \in N$  and solving the ordinary differential equation

$$c''_i + \Gamma_{j,k}^i c'_j c'_k = 0.$$



**Figure P.2:** A schematic illustration of an Anosov flow, decomposing the (co-)tangent bundle at every point into an exponentially expanding, exponentially contracting and tangential distributions.

We note that each  $(c(0), c'(0)) \in TN$  defines a unique geodesic. The geodesic flow  $\Phi^t$  maps each such point to  $(c(t), c'(t))$  with  $c(t)$  being the point on  $c$  satisfying  $d(c(0), c(t)) = t \|c'(0)\|_{c(0)} := t \sqrt{g_{c(0)}(c'(0), c'(0))}$ . In fact, a direct calculation shows that  $\|c'(t)\|$ , the speed of any geodesic  $c$ , is constant and we can normalize it, rephrasing  $\{\Phi^t\}$  as a flow on the unit tangent bundle  $SN$  (or rather on  $S^*N$ ). More details on the geodesic flow can be found in textbooks like [Kliin] or [Pat99].

We are interested in a class of manifolds generalizing saddle surfaces<sup>2</sup> in  $\mathbb{R}^d$ . Roughly speaking, when  $(N, g)$  is a surface, at every point  $x$  one can define a matrix called *the second fundamental form* (cf. [Leo6]) whose eigenvalues  $\kappa_1(x), \kappa_2(x)$  describe the minimal and maximal deviation at the point  $x$  from being a plane. The product  $\kappa_1 \kappa_2(x)$  is called the *Gaussian curvature* to which we refer henceforth simply as "the" curvature of a surface. That definition can be generalized to manifolds. As this thesis is not dealing with high dimensional examples we only mention the generalization in a nutshell. When we take  $v \in \mathbb{R}^{2d}$  and "parallel transport" it around a loop ( $C^1$  curve with identical endpoints), it will point again in the original direction when coming back to its initial position. Riemann suggested to

---

<sup>2</sup>e.g.  $f(x_1, x_2) = x_1^2 - x_2^2$

capture the extent of failure of this property on  $M$  by considering *Riemann curvature tensor*,

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w,$$

where  $u, v \in T_y M$ ,  $w$  are tangent vectors and  $[u, v]$  is the Lie bracket of  $u$  and  $v$ . Then one defines the *sectional curvature of  $M$*  by

$$K(u, v) := \frac{\langle R(u, v)v, u \rangle}{\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2}.$$

Returning to our main discussion in lower dimensions, a well studied example of hyperbolic dynamics occurs on manifolds of constant negative curvature -1:

**Theorem P.1.13** (e.g. [KH97, Theorem 17.6.2]). *Suppose  $(N, g)$  is a closed Riemannian manifold of negative sectional curvature, the geodesic flow on  $M = S^*N$  is an Anosov and strongly mixing flow with respect to the natural invariant measure on  $M$  called Liouville measure  $\mu_{\text{Liouville}}$  and defined by  $\mu_{\text{Liouville}} := \det(g_{ij}) dy$ .*

As  $\mu_{\text{Liouville}}|_M$  is finite, we can assume by normalizing that  $\mu_{\text{Liouville}}(M) = 1$ , i.e., it is an invariant probability measure.

We dedicate the next subsection to remind briefly some elementary details concerning hyperbolic surfaces of constant negative curvature.

### P.1.3 Surfaces of constant negative curvature

The dynamics we present below in chapter 1 are a toy model of well studied dynamics we recall briefly in this subsection (for more rigorous treatment cf. for instance [Lan12] or [EW10]): We begin by recalling details on Poincaré upper half plane,

$$\mathbb{H} := \{z = x + iy : y > 0\}.$$

In each point  $z \in \mathbb{H}$  the tangent bundle  $T_z \mathbb{H} = \{z\} \times \mathbb{C} \cong \mathbb{C}$  and we define on it the Riemannian metric

$$d_z(v, w) = \frac{1}{y^2} (\operatorname{Re} v \cdot \operatorname{Re} w + \operatorname{Im} v \operatorname{Im} w)$$

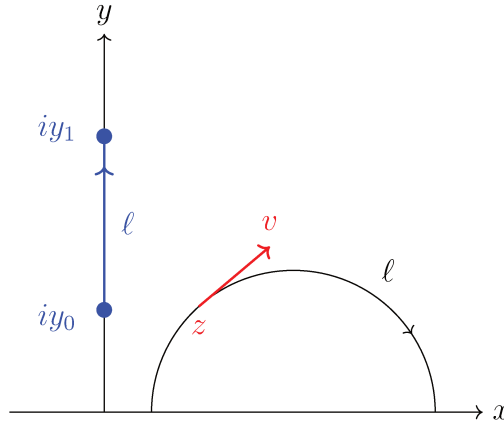
with  $z = x + iy$ . This metric induces a distance on  $\mathbb{H}$  given by

$$d(z_0, z_1) := \cosh^{-1} \left( 1 + \frac{|z_0 - z_1|^2}{2 \operatorname{Im}(z_0) \operatorname{Im}(z_1)} \right).$$

In other words, this is the length of a geodesic connecting  $z_0$  to  $z_1$ . The group action on  $\mathbb{H}$  is given by a transitive action of the group  $\operatorname{PSL}_2(\mathbb{R})$ , namely given a matrix  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{PSL}_2(\mathbb{R})$  its action

---

<sup>3</sup>Denoting  $H(x, \xi) = \frac{\|\xi\|^2}{2}$  this space can be realized as  $H^{-1}(\frac{1}{2})$



**Figure P.3:** The two possible "types" of geodesics in  $\mathbb{H}$ : either lying on the  $y$ -axis or semi-circles. Here  $z$  indicates a complex point on the geodesic while  $v$  is the direction of velocity along it. The stable manifold of  $(z, v)$  is obtained from neighboring vertical geodesics and its unstable from semi-circular geodesics.

on  $z$  is expressed in terms of a Möbius transformation,

$$g.z = \frac{az + b}{cz + d}.$$

We remind that  $g.z \in \mathbb{H}$  since from a direct calculation  $\text{Im}(g.z) = \frac{\text{Im}z}{|cz+d|^2}$ . That allows us by an abuse of notation to associate with each  $g$  a function  $\mathbf{g} : \mathbb{H} \rightarrow \mathbb{H}$  given by  $\mathbf{g}(z) = g.z$ . We denote its derivative with respect to  $z$  by  $\mathbf{g}'(z) = \frac{1}{(cz+d)^2}$ . One can then extend the action to  $S\mathbb{H}$  by

$$g.(z, v) := Dg(z, v) = (\mathbf{g}(z), \mathbf{g}'(z) \cdot v) = \left( \frac{az + b}{cz + d}, \frac{v}{(cz + d)^2} \right).$$

For a fixed  $z \in \mathbb{H}$ , the map  $(Dg)_z := Dg(z, \cdot)$  is a linear map  $(Dg)_z : T_z\mathbb{H} \rightarrow T_{g(z)}\mathbb{H}$  lifting the action from  $\mathbb{H}$  to  $T\mathbb{H}$ . One useful property of this action is that it is an isometry both on  $\mathbb{H}$ , i.e.,  $d(gz_0, gz_1) = d(z_0, z_1)$ , and on  $S\mathbb{H}$  (for an appropriate lifting of  $d$ ). That means one can associate with each geodesic on  $S\mathbb{H}$  a geodesic in  $\mathbb{H}$ , being the image of some fixed "comfortable" curve under an element  $g \in \text{PSL}_2(\mathbb{Z})$ . The geodesics in  $\mathbb{H}$  are semi-circles perpendicular to the real axis. We allow one of the endpoints of the semi-circle to be  $i\infty$ , in which case the geodesic is a straight vertical line parallel to the imaginary axis. The "reference geodesic" is the vertical geodesic going through  $i$  and parameterized by  $\gamma : \mathbb{R}_+ \rightarrow i\mathbb{R}$  given by  $\gamma(t) = ie^t$ . Noting that

$$\text{Stab}_{\text{PSL}_2(\mathbb{R})}(i) = \text{PSO}_2(\mathbb{R})$$

gives an identification  $\mathbb{H} = \text{PSL}_2(\mathbb{R}) / \text{PSO}_2(\mathbb{R})$ . Combined with the transitivity of the action it yields  $S\mathbb{H} = \text{PSL}_2(\mathbb{R})$ . We obtain correspondence between elements of  $S\mathbb{H}$  and unit-speed geodesics given by



$$\mathfrak{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \longrightarrow \mathbf{D}\mathfrak{g}(i, i) := (\mathfrak{g}(i), \mathfrak{g}'(i) \cdot i) = \left( \frac{ai+b}{ci+d}, \frac{i}{(ci+d)^2} \right) = (z, v) \in S\mathbb{H}$$

$$\searrow \qquad \qquad \qquad \downarrow$$

$$\gamma(t) := \mathfrak{g}(ie^{it}) = \frac{aie^{it}+b}{cie^{it}+d}$$

The geodesic flow  $\Phi^t : S\mathbb{H} \rightarrow S\mathbb{H}$  is then defined as the mapping taking  $(z_{t_0}, v_{t_0}) = \mathfrak{g}(ie^{it_0})$  to  $(z_{t+t_0}, v_{t+t_0}) := \mathfrak{g}(ie^{i(t+t_0)})$ . at time  $t$ .

The unstable and stable manifolds are given in terms of corresponding *horocyclic flows*, namely, writing  $(z, v) = \mathbf{D}\mathfrak{g}(i, i)$  for some  $\mathfrak{g} \in \mathrm{PSL}_2(\mathbb{Z})$

$$V^s((z, v)) = \left\{ D \left( \mathfrak{g} \cdot \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \right) (i, i) : t \in \mathbb{R} \right\}, \quad V^u((z, v)) = \left\{ D \left( \mathfrak{g} \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) (i, i) : t \in \mathbb{R} \right\}$$

From a direct calculation,  $\mathbb{H}$  has constant negative curvature  $-1$  and it serves as a model of hyperbolicity: A surface is called *hyperbolic* if every open neighborhood of it is isometric to  $\mathbb{H}$ .

**Example P.1.14.** *Compactifying  $\mathbb{H}$  by considering its quotient under a co-compact subgroup  $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$  we obtain a hyperbolic surface  $X = \mathbb{H}/\Gamma$ . Each geodesic of  $\mathbb{H}$  is wrapped around  $\Gamma$ , giving rise to unstable/stable manifolds and sub-bundles of  $SX$  thus the geodesic flow is Anosov also when considering the quotient.*

#### P.1.4 Passing from continuous to discrete dynamics

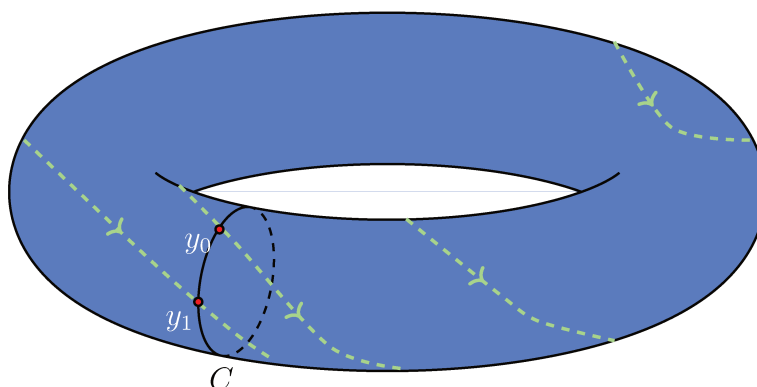
There is a connection between continuous and discrete dynamics. In the beginning of the 20th century Poincaré suggested *the first recurrence map* also known as *Poincaré map* in order to analyze the solutions of ordinary differential equations like (P.1): Let  $X$  be a vector field on  $M$  without fixed points, i.e.,  $X(y) \neq 0$  for every  $y \in M$ . Given the flow  $\{\Phi^t\}$  on  $M$ , generated by it, one may construct a family of sub-manifolds with boundary  $\Sigma_j \subset M = SN$  such that

1.  $\mathrm{co-dim}\Sigma = 1$ .
2. Each of  $\Sigma_j$  is transversal to  $X$ .
3. for every  $y \in M$ , the flow  $\{\Phi^t\}_{t>0}$  crosses  $\Sigma$  for some  $t > 0$ .

Assume there exists  $T > 0$  such that that for every  $y \in M$  we can find  $\tau(y) < T$  for which  $\Phi^{\tau(y)}(y) \in \Sigma := \bigcup_{j=1}^J \Sigma_j$ . Fixing  $y_0 \in M$ , the flow intersects  $\Sigma$  at the first crossing time  $y_1$  i.e.,  $\Phi^t(y_0) = y_1 \in \Sigma \cap M$  (cf. Figure P.4). Iterating the process we obtain a sequence of points  $\{y_j\}_{j \in \mathbb{N}}$ , although  $t_{j+1} - t_j$  might vary, which can be viewed as a trajectory of discrete dynamics beginning from  $y_0$  and which is given by *Poincaré's return map*  $R : \Sigma \rightarrow \Sigma$  given by  $R(y_0) = y_1, \dots, R^n(y_0) = R(y_{n-1}) = y_n$ .

**Example P.1.15.** *Let  $N$  be a close Riemannian manifold of negative curvature with  $\dim N = d$ . Consider the geodesic flow  $\Phi^t$  on its spherical bundle  $SN$ . Then taking  $\Sigma \subset SN$  to be a union of hyper-surfaces  $\Sigma_j$  of dimension  $\dim \Sigma_j = 2d - 2$  we can obtain through  $R : \Sigma \rightarrow \Sigma$  discrete dynamics.*

The discrete dynamics in this example is hyperbolic.



**Figure P.4:** Restriction of continuous dynamics on a torus to a Poincaré section  $C$  generates low-dimensional discrete dynamics

P.1.5 Quantum dynamics

During the 19th and 20th centuries physicists conducted experiments that lead to formulate mechanics in a new "quantic" language: In early 19th century James Clerk Maxwell, Ludwig Boltzmann and others relied on ideas of John Dalton and Amedeo Avogadro to establish the kinetic theory of gases. In 1900 Max Planck introduced the idea that energy is *quantized*. He conjectured that for some types of systems the energy can (theoretically) be divided into discrete "energy elements"  $\varepsilon$ , each of them proportional to a frequency  $\nu$  through the formula  $\varepsilon = 2\pi\nu\hbar$  with  $\hbar \approx 10^{-34} J.s$  called *Planck's constant*. Following ideas presented in the first Solvay Congress of 1911, scientists replaced the functions on the phase space (*classical observables*), by operators and wave functions characterizing the behavior of the system. In fact, in quantum mechanics the state of the system is described in terms of a unit norm *wave-function*  $u(t)$ . The particle is described as a wave whose precise position at any time cannot be determined deterministically but only in terms of a probability density  $|u(x)|^2$  called *the position probability density*. This uncertainty about the position was formulated by Heisenberg in 1918 in one of the pillars of modern quantum mechanics:

**Theorem P.1.16** (Heisenberg's uncertainty principle, 1918). *A wave-function cannot be localized both in position and in momentum. More precisely, let  $u \in \mathcal{S}(\mathbb{R}^d)$  with  $\|u\|_{L^2(\mathbb{R}^d)} = 1$ . Denoting the ( $\hbar$ -rescaled) Fourier transform by*

$$(\mathcal{F}_\hbar u)(\xi) := \int_{\mathbb{R}^d} u(x) e^{-\frac{i}{\hbar}\langle x, \xi \rangle} dx,$$

we have

$$\frac{\hbar}{2} \|u\|_{L^2} \cdot \|\mathcal{F}_\hbar u\|_2 \leq \|x_j u\|_2 \cdot \|\xi_j \mathcal{F}_\hbar u\|_2, \quad \begin{aligned} x &= (x_1, \dots, x_d) \\ \xi &= (\xi_1, \dots, \xi_d) \end{aligned}$$

An immediate question would then be, "what connection can one find between the classical formulation and the quantum one? Are classical properties reflected in the quantum level?". More formally, we would like to understand how the ergodic properties of a classical system dictate the spectral properties

of its quantization. We will in fact build up a *symbol calculus*, meaning systematic rules for manipulating pseudo-differential operators. A process in which we associate to each classical observable (sometimes called a *symbol*)  $a \in C^\infty(\mathbb{R}_{x,\xi}^{2d}, \mathbb{R})$  a pseudo-differential operator  $A : L^2(\mathbb{R}_x^d) \rightarrow L^2(\mathbb{R}_x^d)$  is called *quantization*. As  $A$  can be an unbounded operator, we additionally demand that  $\overline{\text{Dom}(A)} = L^2(\mathbb{R}^d)$ . Even though several similar quantizations exist, below we will consider a fixed quantization procedure useful for our needs we denote by  $a \mapsto A = \text{Op}_h^W(a) = \text{Op}_h(a)$ .

**Example P.1.17.** Recall Example P.1.1 of a classical observable describing the energy of a particle moving in  $\mathbb{R}^d$ ,

$$p(x, \xi) = \frac{\|\xi\|^2}{2} + V(x)$$

with  $V \in C^\infty(\mathbb{R}^d)$ . It induces to the Hamiltonian flow  $\Phi^t = \exp(tH_p)$  with  $H_p$  being the the Hamiltonian vector field generated by  $p$ ; its action on  $g$  is given by  $H_p g := \{p, g\} = \frac{\partial p}{\partial x} \frac{\partial g}{\partial \xi} - \frac{\partial p}{\partial \xi} \frac{\partial g}{\partial x}$ , playing the role of the Hamiltonian vector field of the system. The quantization of the classical Hamiltonian, is called the quantum Hamiltonian of the system is given by

$$P_h = -\frac{\hbar^2 \Delta}{2} + V(x) \tag{P.3}$$

with its domain  $\text{Dom}(P_h)$  being a dense subspace of  $L^2(\mathbb{R}^d)$ . In fact, the dynamics of a massive (non-relativistic) quantum particle is governed by the time-dependent Schrödinger equation on  $L^2(\mathbb{R}^d)$ , i.e.,

$$i\hbar \partial_t u(x, t) = P_h u(x, t).$$

One often puts assumptions on  $V(x)$  so that  $P$  is a self-adjoint operator. Taking  $V$  to be growing polynomially and  $\lim_{|x| \rightarrow \infty} V(x) = \infty$  the operator is self-adjoint and Schrödinger's equation has discrete real unbounded spectrum  $\{E_{\hbar,j}\}$  allowing one to write

$$P_h u_{\hbar,j} = E_{\hbar,j} u_{\hbar,j}$$

with  $u_{\hbar,j}$  being the stationary states of the system and  $E_{\hbar,j}$  the corresponding eigenfrequencies. In that scenario one is interested in spectral data:

- The distribution of the eigenstates in some fixed interval  $I$ .
- The spatial (micro-)localization of the eigenstates  $u_{\hbar,j}$  and their nodal domains.
- The value distribution of  $u_{\hbar,j}$ .

In order to quantize the flow  $\Phi^t$ , we introduce the quantum propagator,  $U(t) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ , a unitary operator defined formally by  $U(t) := \exp\left(-\frac{itP_h}{\hbar}\right)$  or equivalently in the eigenbasis representation,

$$U(t) u_0 := \sum_{j \in \mathbb{N}} e^{-\frac{itE_{\hbar,j}}{\hbar}} \langle u_{\hbar,j}, u_0 \rangle u_{\hbar,j}$$

Replacing  $\hbar$  by a positive  $h \rightarrow 0$  (called *the semiclassical parameter*), we investigate the long-term (high energy) behavior of the pseudo-differential operator (in the previous case of  $-h^2\Delta_{\mathbb{R}^d} + V$ ).

In the past 30 years there has been increasing interest in investigating the properties of quantum systems (in fact those mentioned above) in the *high frequency limit*, which mathematically corresponds to taking  $h \rightarrow 0$ . The field investigating the spectral properties of a chaotic system for these low values of  $h$  is called *quantum chaos*.

**Example P.1.18.** Taking as the phase space  $S^*N$  where  $N = \mathbb{H}/\Gamma$  and  $\Gamma \leq PSL_2(\mathbb{R})$  a co-compact subgroup, the geodesic flow on  $S^*N$  is quantized by the semiclassical Laplace-Beltrami operator  $-\Delta_g$  for  $g$  being the metric on  $N$ . It is well known that its spectrum is discrete, that its eigenvalues satisfy  $\text{Spec}(-\Delta_g) = \{\lambda_j^2\}$  for sequence  $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$  and that the corresponding eigenstates  $\{\varphi_j\}_j$  form an orthonormal basis of  $L^2(N)$ . We often study the eigenvalue problem

$$-h^2\Delta_g\varphi_j = \lambda_j^2\varphi_j$$

in the high frequency limit  $j \rightarrow \infty$ .

### P.1.6 Quantum Ergodicity and Quantum Unique Ergodicity

Let  $(N, g)$  be a compact surface of constant negative curvature. We expand  $L^2(N)$  as a direct sum of eigenspaces arising from a choice of an orthonormal eigenbasis  $\{\varphi_j\}_j$ . The quantum-classical correspondence establishes a link between the classical flow and the quantum operator. This connection is given in terms of the Egorov theorem. The first results on spatial distribution of high frequency eigenmodes are due to Shnirelman, Zelditch and Colin de Verdière ([Shn74],[Zel87],[CdV85]) stating that almost every  $\varphi_j$  is asymptotically equidistributed on  $M$  in the high frequency limit, a property called *quantum ergodicity*. A corollary of it is that there exists a set  $J \subset \mathbb{N}$  of density 1 such that for every  $f \in C(M)$ ,

$$\int_N f(x) |\varphi_j(x)|^2 dx \xrightarrow[j \in J]{j \rightarrow \infty} \int_N f(x) dx. \tag{P.4}$$

Rudnick and Sarnak have conjectured that the eigenmodes of  $-\Delta_g$  satisfy the *quantum unique ergodicity* (QUE) property, meaning that  $J = \mathbb{N}$ , i.e., (P.4) holds for the full sequence  $\{\varphi_j\}$ . Quantum ergodicity and QUE can be both rephrased by lifting the eigenmodes to the distributions  $dW_j \in \mathcal{D}'(T^*M)$  analogous to Wigner distributions on  $T^*M$  which are defined by

$$a \mapsto \langle \text{Op}_h(a) \varphi_j, \varphi_j \rangle, \quad a \in C_c^\infty(T^*M), h = h_j = \lambda_j^{-1}. \tag{P.5}$$

One can always extract a convergent subsequence  $\{dW_{j_k}\}_{k \geq 1}$ , converging in the distributional sense to a probability measure on  $S^*N$ . The weak-\* limits of  $\{dW_{j_k}\}_{k \geq 1}$  are called semiclassical measures, denoted  $\mu_{sc}$ , and are invariant with respect to the geodesic flow. They represent the micro-localization of  $\{\varphi_{j_k}\}_{k \geq 1}$  on the phase space. In these notations quantum ergodicity amounts to the existence of a subsequence  $J \subset \mathbb{N}$  of density 1 such that  $dW_j \xrightarrow[j \in J]{j \rightarrow \infty} \mu_{\text{Liouv}}$  of  $S^*N$  and QUE means that the full sequence weakly-converges to the Liouville measure. We note that since the geodesic flow is Anosov,

there exist many probability measures invariant with respect to it apart from Lebesgue, e.g. measures supported on a fractal invariant set or on closed geodesics.

While QUE remains open, one can wonder which of these myriad probability measures, being invariant under the geodesic flow, can be obtained as weak-\* limits of  $\{dW_j\}_j$ . Given such a measure, one can also wonder where it is localized. There are several results concerning the constraints that semiclassical measures have to satisfy.

Some of these constraints are expressed in terms of the Kolmogorov-Sinai entropy quantifying the information-theoretic complexity of a given invariant measure. This entropy gives information on the localization of the measure: the higher the entropy, the more delocalized the measure. For instance the Liouville measure is the measure of maximal entropy 1 whereas a delta measure on a closed geodesic is of minimal entropy 0. Anantharaman proved in [Ana08] that the Kolmogorov-Sinai entropy of every  $\mu_{sc}$  is positive. Joint with Nonnenmacher they showed in [AN07b] that the entropy is in fact bounded from below by half the maximal entropy. These bounds mean that  $\mu_{sc}$  cannot be "too" localized. Nevertheless there exist flow-invariant measures with high entropy which are supported on invariant proper subsets. A recent result due to Dyatlov, Jin and Nonnenmacher, [DJ18, DJN22], states that for hyperbolic surfaces any semiclassical measure is fully-supported on  $S^*N$ , and moreover that for every open  $\emptyset \neq \Omega \subset S^*N$  there exists a constant  $C_\Omega$  independent of the choice of  $\mu_{sc}$  such that  $\mu_{sc}(\Omega) > C_\Omega$ . Their method of proof relies on the fractal uncertainty principle introduced in [BD18] (see Proposition 2.6.2). In addition, combined with the unique continuation principle they deduce that for every open  $\Omega \neq \emptyset$  there exists a constant  $c_\Omega$  such that for every normalized eigenmode  $\varphi_j$

$$\int_\Omega |\varphi_j|^2 dx \geq c_\Omega \int_N |\varphi_j|^2 dx = c_\Omega. \tag{P.6}$$

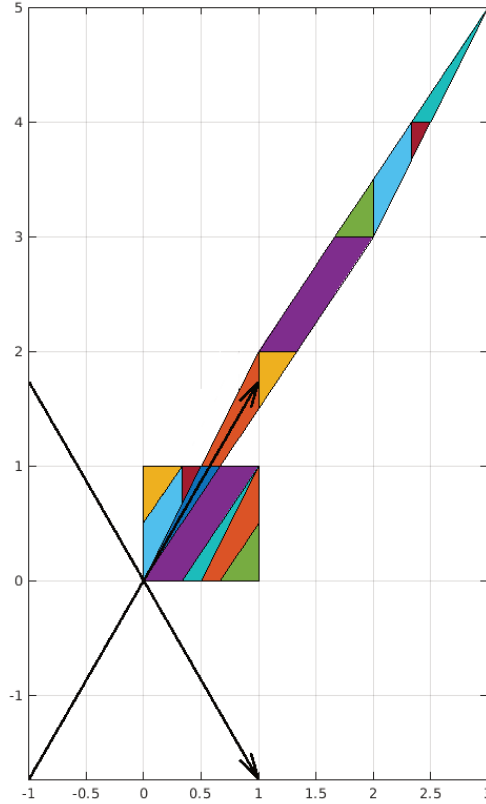
## P.2 Classical Arnold's cat map and the quantum cat map

In the present section we only skim through the central details required for the presentation of results. In chapter 1, we give a more comprehensive treatment of the classical system and its quantization. This thesis is dedicated for the study of the cat map, a toy model of discrete-time dynamics on the 2-dimensional torus  $\mathbb{T}^2$ , having dynamical properties analogous to those of the geodesic flow on negatively curved manifolds. This class of examples was popularized by Arnold in 1967 (cf. [AA67]) and thus is named after him, Arnold's cat maps. These cat maps are Anosov diffeomorphisms acting on  $\mathbb{T}^2$  and preserving its Lebesgue measure<sup>4</sup>. More precisely, each of them is a matrix  $\gamma \in \text{SL}_2(\mathbb{Z})$  with eigenvalues  $\lambda = \lambda_u, \lambda_s = \lambda^{-1}$  such that  $|\lambda| > 1$  having an unstable (expanding) eigendirection  $V_u$  and a stable (contracting) one  $V_s$ . For technical reasons explained below we additionally suppose that the classical dynamics is given by checkerboard matrices mod 2.

**Example P.2.1.** *A canonical numerical example of such matrix was given by Hannay-Berry in [HB80] and later in Keating's and Degli-Esposti's works,  $\gamma_{DE} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ . Note that we cannot use the original example in Arnold's work (cf. [AA67]),  $\gamma_{Arnold} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$  for reasons explained in 1.4.2 below. We illustrate the action of  $\gamma_{DE}$  on the fundamental unit square in Figure P.5. Its eigenvalues are  $\lambda_u = 2 + \sqrt{3}$ ,  $\lambda_s = 2 - \sqrt{3}$ . For Degli—Esposti's cat map the corresponding eigenspaces are given explicitly by parallel trans-*

---

<sup>4</sup>In fact, equipping  $\mathbb{T}^2$  with the symplectic form  $\omega$ , cat maps are symplectomorphisms.



**Figure P.5:** The action of  $\gamma_{DE}$  on  $\mathbb{T}^2$  can be visualized as taking mod 1 its action on  $[0, 1]^2$ . The arrows correspond to its unstable/stable directions.

port of the vectors

$$v_u(\gamma) = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad v_s(\gamma) = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}.$$

Passing to the quantum level, the quantum space is a Hilbert subspace  $\mathcal{H}_N \subset \mathcal{S}'(\mathbb{R})$  isomorphic to  $\mathbb{C}^N$  with  $N = \frac{1}{2\pi h}$ . We view  $N$  as our semiclassical parameter, replacing the limit  $h \rightarrow 0$  by  $N \rightarrow \infty$ . Note that (P.5) gives an operator  $\text{Op}_h(a) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ . Bouzouina and De Bièvre show in [BDB96, Proposition 2.3] that  $L^2(\mathbb{R})$  splits to a direct integral of finite dimensional Hilbert spaces and deduce that so does the  $\text{Op}_h$ -quantization. As  $\mathcal{H}_N$  appears as a direct integrand in the decomposition of  $L^2(\mathbb{R})$ , that means we can construct from every operator  $\text{Op}_h(a)$  a quantization on  $\mathcal{H}_N$ ,  $\text{Op}_N(a) : \mathcal{H}_N \rightarrow \mathcal{H}_N$ : Given a function  $f \in L^2(\mathbb{R})$  whose projection to  $\mathcal{H}_N$  is denoted by  $f_N \in \mathcal{H}_N$ ,  $\text{Op}_N(a) f_N$  is the projection of  $\text{Op}_h(a) f$  to  $\mathcal{H}_N$ ,

$$\begin{array}{ccc}
 f \in L^2(\mathbb{R}) & \hookrightarrow & f_N \in \mathcal{H}_N \\
 \downarrow & & \downarrow \\
 \text{Op}_h(a) f \in L^2(\mathbb{R}) & \hookrightarrow & \text{Op}_N(a) f_N \in (\mathcal{H}_N)
 \end{array}$$

The classical dynamics is quantized by a family of unitary matrices  $\{\mathcal{M}_N(\gamma)\}$  called *the quantum cat map* obtained from projecting the evaluation of a metaplectic representation<sup>5</sup> (cf. [HB80]),  $\mathcal{M}_h(\gamma)$ , to  $\mathcal{H}_N$ . For the standard choice of basis for  $\mathcal{H}_N$  given after Claim 1.2.2 below,

**Example P.2.2.** *In the case of Degli-Esposti's map above, One can express  $\mathcal{M}_N(\gamma) = (\mathcal{M}_N(\gamma)_{jk})_{j,k} \in \mathbb{C}^{N \times N}$  explicitly by number theoretic sums. For instance for  $\gamma_{DE}$ , for every  $N \in 2\mathbb{N} + 1$ , up to some scalar phase factor (its expression as a quotient of Gauss sums is expanded in [KR00] and [DEG03]),*

$$\mathcal{M}_N(\gamma_{DE})_{jk} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} (k^2 - kj + j^2)\right).$$

The quantum cat map satisfies an exact intertwining relation: for every  $a \in C^\infty(\mathbb{T}^2)$  and  $N \in \mathbb{N}^*$

$$\text{Op}_N(a \circ \gamma) = \mathcal{M}_N^{-1}(\gamma) \text{Op}_N(a) \mathcal{M}_N(\gamma).$$

That relation allows to connect between the classical system and its quantum counterpart.

### P.2.1 Quantum periods

One particular property of the matrices  $\mathcal{M}_N$  is their quasi-idempotency: for any  $N$ , there exists a *quantum period*  $P(N)$  such that

$$\mathcal{M}_N^{P(N)} = e^{i\kappa_N} \text{Id}_N, \quad \text{for some } \kappa_N \in [-\pi, \pi].$$

(here  $P(N)$  will be the smallest such integer). As a consequence, the eigenvalues of  $\mathcal{M}_N$  are given by  $P(N)$ -th roots of  $e^{i\kappa_N}$ . If  $P(N) < N$ , this implies some spectral degeneracy of  $\mathcal{M}_N$ . The quantum period behaves quite erratically with  $N$  [Kea91a], and  $\frac{1}{N} \sum_{n=1}^N P(n) \asymp N$ . More precisely, there exists  $C > 0$  such that the period is bounded by

$$2 \frac{\log N}{\log |\lambda|} - C \leq P(N) \leq CN \log \log N, \tag{P.7}$$

and there exists a density one subsequence  $\mathcal{N} \subset \mathbb{N}$  such that the periods  $P(N) \geq N^{1/2} e^{(\log(N)^\delta)}$  [KR01a] (assuming the Generalized Riemann Hypothesis, one can show that  $P(N) \geq N^{1-o(1)}$  along a density one subsequence [Kuro3]).

---

<sup>5</sup>Recall that while  $\mathcal{M}_h(\gamma)$  is initially an operator on  $L^2(\mathbb{R})$ , one can extend it to an operator on  $\mathcal{S}'(\mathbb{R})$  and then can project it to  $\mathcal{H}_N$ .

Bouzouina and De Bièvre proved [BDB96] showed that, along the full sequence  $N \in \mathbb{N}$ , whatever the choice of eigenbases of  $\mathcal{M}_N$ , the quantum map  $(\mathcal{M}_N)_{N \in \mathbb{N}^*}$  satisfies Quantum Ergodicity. Restricting the  $N$  to a specific sequence of primes, Degli Esposti, Graffi, Isola [DEGI95] showed QUE for the  $\mathcal{M}_N$ . Kurlberg-Rudnick [KR01a] and Bourgain [Bou07], showed that QUE holds as soon as one is restricted to the density one subsequence  $\mathcal{N}$  with periods  $P(N) \geq N^\epsilon$ . Kurlberg-Rudnick also showed in [KR00] that, if one imposes, for any  $N \in \mathbb{N}^*$ , the eigenstates of  $\mathcal{M}_N$  to be joint eigenstates of a family of "Hecke" operators, then the corresponding eigenbases satisfy QUE for the full sequence  $N \in \mathbb{N}$  (a result similar to the one of Lindenstrauss on arithmetic surfaces [Lino6]).

At the other extreme of the period spectrum, Bonechi-De Bièvre exhibited in [BDB00] an infinite subsequence  $\mathcal{N}_2 \subset \mathbb{N}$  (of asymptotic zero density) such that the periods are *minimal*, namely saturate the lower bound in (P.7):

$$P(N) = 2 \frac{\log N}{\log |\lambda|} + O(1), \quad \mathcal{N}_2 \ni N \rightarrow \infty.$$

This small period induces huge spectral multiplicities  $\asymp \frac{N}{\log N}$ , which gives a large freedom to choose our eigenstates inside each eigenspace. Taking advantage of this freedom, Faure, Nonnenmacher and De Bièvre [FNDB03] exhibited a sequence of eigenstates of  $(\mathcal{M}_N)_{N \in \mathcal{N}_2}$  which are not asymptotically equidistributed on  $\mathbb{T}^2$ , but rather feature a partial localization (called a "scar") on a given periodic orbit of  $\gamma$ . These eigenstates were the first counter-examples to QUE for an Anosov system. A few years later, Kelmer [Kel10] studied quantized hyperbolic automorphisms of higher dimensional tori  $\mathbb{T}^{2d}$ , and exhibited sequences of eigenstates fully localized on certain co-isotropic subtori of  $\mathbb{T}^{2d}$ , using a different mechanism than the one in [FNDB03].

The baker's map, another well-known example of chaotic map on the 2-dimensional torus, has been quantized by Balasz-Voros [BV89] and much studied numerically and analytically, mostly in the theoretical physics community. In spite of technical complications due to the discontinuity of the map, quantum ergodicity could be proven for this system [DENW06]. An alternative quantization of this map was proposed in [AN07a], in which counter-examples to QUE were given.

Let us also mention the work of Chang, Krüger, Schubert and Troubetzkoy [CKST08], who constructed quantum maps associated with ergodic maps on the interval, which also featured non-equidistributed eigenstates.

### P.3 Statement of results

The thesis is dedicated for proving and presenting three kinds of results, all concerning (sequences of) eigenstates and their delocalization and statistical properties. The first result concerns the full delocalization of such sequences (with no restrictions on the quantum periods), whereas the other two concern quantum maps of large spectral degeneracies and investigate the delocalization and statistical properties of random eigenstates.



P.3.1 The full delocalization of semiclassical measures

Bouzouina and De Bièvre have proved in [BDB96] an analogue of quantum ergodicity for the quantum cat map, i.e., there exists a set  $J \subset \mathbb{N}$  of density 1 such that in the limit  $N \rightarrow \infty$  the Wigner measures of almost all the eigenstates (whose rigorous adaptation to  $\mathcal{H}_N$  is given in ??) weakly converge to  $\mu_{\text{Leb}}$ . We recall a proof of that result in section A.2. However, quantum ergodicity does not prevent exceptional subsequences converging to a  $\gamma$ -invariant measure different from  $\mu_{\text{Liouv}}$  on  $\mathbb{T}^2$ . A work by Kurlberg and Rudnick ([KR00]) has constructed "joint" eigenbases composed of mutual eigenvectors of  $\mathcal{M}_N(\gamma)$  and "Hecke" operators on  $\mathbb{T}^2$  (analogous to those defined for arithmetic surfaces). They proved that these eigenbases satisfy quantum unique ergodicity. In 2003, Faure, Nonnenmacher and De Bièvre ([FNDB03]) demonstrated that for the quantized cat map quantum unique ergodicity does not hold. For every periodic orbit  $\mathcal{P}$  of  $\gamma$ , they found a weak-\* limit of the form  $\mu_{\text{sc}} = \frac{1}{2}\mu_{\text{Liouv}} + \frac{1}{2}\delta_{\mathcal{P}}$ . Faure and Nonnenmacher later showed in [FN04] that for every  $\mu_{\text{sc}}$  of the cat map the weight of its atomic component is less or equal to the weight of its Liouville component, implying that the semiclassical measure cannot be entirely carried on a periodic orbit. Anantharaman and Nonnenmacher have obtained a result on the entropy of semi-classical measures for another toy model on  $\mathbb{T}^2$ , the Walsh-quantized Baker map (cf. [AN07a]), namely that every semiclassical measure has at least half the maximal (i.e., Lebesgue) entropy.

Brooks [Bro10] has given a related characterization of semi-classical measures for the cat map. He interpolated between the constraint from [FN04] and the analogue of the lower bound from [AN07a] for the cat map. He proved that ergodic components of high entropy of the semiclassical measures carry at least as much weight as those of low entropy.

A central portion of the thesis is dedicated to proving a result analogous to [DJ18] for the quantum cat map, that every semi-classical measure  $\mu_{\text{sc}}$  associated with  $\gamma$  is fully supported on  $\mathbb{T}^2$  meaning that all sequences of eigenstates  $\{\varphi_N\}_N$  are fully delocalized on  $\mathbb{T}^2$  for  $N$  large enough,

**Theorem P.3.1.** *Let  $\gamma \in \tilde{\Gamma}(2)$  (where  $\tilde{\Gamma}(2)$  is defined in (1.2)) be a hyperbolic matrix quantized into the family  $\{\mathcal{M}_N(\gamma)\}_N$ . Let  $\mu_{\text{sc}}$  be an associated semiclassical measure. Then for every open  $\emptyset \neq \Omega \subset \mathbb{T}^2$  there exists a constant  $c_\Omega > 0$  independent of  $\mu_{\text{sc}}$  such that  $\mu_{\text{sc}}(\Omega) > c_\Omega$ .*

It can be stated in terms of  $\varphi_N \in \mathbb{C}^N$  through the choice of the canonical basis below 1.4.2 and the identification (1.23) below:

**Theorem P.3.2** (The "delocalization" of the eigenfunctions). *Let  $\gamma \in \tilde{\Gamma}(2)$ . Fix  $0 \leq \alpha_1 < \alpha_2 \leq 1$ . Then there exists a constant  $c_{\alpha_1, \alpha_2} > 0$  and an index  $N_0 \in \mathbb{N}^*$  such that for every  $N > N_0$  and normalized eigenvector  $\varphi_N = (\varphi_{N,k})_{k=1}^N$  of  $\mathcal{M}_N(\gamma)$*

$$\frac{1}{N} \sum_{k \in \llbracket [\alpha_1 N], [\alpha_2 N] \rrbracket} |\varphi_{N,k}|^2 > c_{\alpha_1, \alpha_2}. \tag{P.8}$$

Both theorems are deduced from Theorem 2.1.2 (that we prove in chapter 2).

We mention that one can ask similar questions for higher-dimensional analogues of the cat map hyperbolic dynamics, meaning hyperbolic maps  $\gamma \in \text{Sp}(2d, \mathbb{Z})$ . Rivière (cf. [Riv11]) has generalized the result of [AN07b] for these higher dimensional settings bounding explicitly the entropy of  $\mu_{\text{sc}}$  from

below in terms of the spectrum of  $\gamma$ . In addition Kelmer has obtained in [Kel10] that if  $\gamma$  belongs to a certain subgroup of  $\text{Sp}(2d, \mathbb{Z})$ , one can construct eigenvectors of the corresponding quantum operators  $\mathcal{M}_N(\gamma)$  for which the associated semiclassical measure is the Liouville measure on an invariant Lagrangian subspace of  $\mathbb{T}^{2d}$ , disproving QUE for  $\gamma$  as well as an analogue of Theorem P.3.1 in high dimension.

Recently Dyatlov and Jézéquel extended (cf. [DJ21]) our result to cat maps of arbitrary dimension  $\gamma \in \text{Sp}(2d, \mathbb{Z})$  having a unique simple eigenvalue and whose characteristic polynomial is irreducible over  $\mathbb{Q}$  (making the work result of Kelmer inapplicable). Their method of proof relies as well on the one dimensional fractal uncertainty principle (Proposition 2.6.2 below).

### P.3.2 Probabilistic and small scale quantum (unique) ergodicity

Apart from Quantum Ergodicity theorem, which links the ergodic behavior of a classical (deterministic) dynamics with the equidistribution of eigenstates, another way to construct equidistributed quantum states is to consider systems with large eigenspaces, and by selecting eigenstates at random in those eigenspaces. This has been done primarily on the unit 2-dimensional sphere, on which the space  $V_l \subset L^2(\mathbb{S}^2)$  of spherical harmonics of level  $l \in \mathbb{N}$  (and eigenvalue  $l(l+1)$ ) has dimension  $2l+1$  (the dimension  $\dim V_l \sim \sqrt{l(l+1)}$  is essentially the maximal one permitted by Weyl's spectral asymptotic formula). If we take, for each  $l \geq 0$ , a random orthonormal basis of  $V_l$  (using the natural Haar measure), then Zelditch [Zel92] showed that almost surely, the global random eigenbasis of  $L^2(\mathbb{S}^2)$  satisfies Quantum Ergodicity. Vanderkam improved the result in [Van97], showing that this global random eigenbasis actually satisfies QUE. Note that here we use the notions of QE or QUE, although there is no ergodicity in the classical geodesic flow; one would rather say that the eigenvectors along the random eigenbasis become equidistributed on  $S^*\mathbb{S}^2$  in the high frequency limit. Random spherical harmonics have been shown to exhibit other interesting statistical properties, for instance they statistically converge, in the high frequency limit, to the monochromatic random wave model put forward by Berry to describe chaotic eigenstates [Ber77]. We will come back to his model in subsection P.3.3 below.

Burq and Lebeau generalized the construction to arbitrary compact Riemannian manifolds: for an increasing sequence of frequencies  $\lambda_{j_N} \rightarrow \infty$  they considered spectral intervals  $I_N$  in the spectrum of  $\sqrt{\Delta_N}$ , centered on  $\lambda_{j_N}$ , such that the corresponding spectral subspace  $V_{I_N}$  has growing dimension  $d_N$  as  $N \rightarrow \infty$ . They then constructed random linear combinations of those eigenstates, such as to form a random (normalized) state  $u_N \in V_{I_N}$ . Provided the intervals  $I_N$  are sufficiently wide (they assume that  $|I_{N_n}| \gg 1$  when  $N \rightarrow \infty$ ), they show that with full probability, the sequence  $(u_N)_N$  satisfies QUE [BL13, Thm 3]. Notice that in their construction, the  $u_N$  are not eigenstates of the Laplacian, but rather quasimodes. There is no dynamical assumption on the geodesic flow on  $M$ . The main ingredient of their proof is the local Weyl's law

$$\frac{1}{d_{N_n}} \text{Tr}(\Pi_{N_n} A \Pi_{N_n}) \xrightarrow{n \rightarrow \infty} \frac{1}{\text{Vol}(S^*N)} \int_{S^*N} \sigma(A) d\mu_{\text{Liouv}}, \quad (\text{P.9})$$

where  $\Pi_N$  is the orthogonal projector on  $V_{I_N}$  and  $\sigma(A)$  is the principal symbol of  $A$ .

Zelditch [Zel14] developed this idea in a more general setting. He considered a sequence of vector spaces  $(\mathcal{H}_N \subset L^2(N))_{N \in \mathbb{N}}$  orthogonal to each other, of dimensions  $d_N \rightarrow \infty$ , with the assumption

that for any pseudodifferential operator  $A$  of order zero on  $M$ , the projected operators  $\Pi_N A \Pi_N$  satisfy a generalized Weyl law of the type (P.9), like in the previous settings of Burq-Lebeau. He then proves that, choosing orthonormal bases at random on the  $\mathcal{H}_N$ , the global random basis for  $\bigoplus_N \mathcal{H}_N$  almost surely satisfies QE. Maples [Map13] improved Zelditch's result to an almost sure QUE statement similar to the one in [BL13].

The improvement from probabilistic QE to probabilistic QUE comes from the probabilistic methods used. Zelditch computes the quantum variances associated with the eigenbases  $(u_{N,k})_{k=1,\dots,d_N}$ :

$$\text{Var}_N := \frac{1}{d_N} \sum_{k=1}^N |\langle u_{N,k}, Au_{N,k} \rangle - \omega(A)|^2, \quad \text{where } \omega(A) = \frac{1}{\text{Vol}(S^*M)} \int_{S^*M} \sigma(A) d\mu_{Liouv},$$

and proves QE through the fact that the variances decay to zero when  $N \rightarrow \infty$ . The methods of [Van97, BL13, Map13] show more precise estimates for  $\mathbb{P}(|\langle u_{N,k}, Au_{N,k} \rangle - \omega(A)| \geq \epsilon)$ , allowing to control all states together.

### P.3.2.1 Equidistribution at small scale

Quantum Ergodicity statements use fixed observables to test the equidistribution of the eigenfunctions in the semiclassical limit. In the case of the Laplacian on a Riemannian manifold  $(M, g)$ , a recent strengthening of QE addresses the weights of the eigenfunctions in certain semiclassically small sets on  $M$ , for instance in balls  $B(x, r(\lambda_j))$  of radii  $r(\lambda_j)$  such that  $r(\lambda_j) \searrow 0$  in the semiclassical limit  $\lambda_j \rightarrow \infty$ . Small scale quantum ergodicity (SSQE), with respect to the scales  $(r(\lambda_j))_{\lambda_j}$ , then refers to the fact that, for a subsequence of density one,

$$\int_{B(x, r(\lambda_j))} |\varphi_j(x)|^2 d\text{Vol}_M(x) = \frac{\text{Vol}(B(x, r(\lambda_j)))}{\text{Vol}(M)} + o(r(\lambda_j)^d) \quad \text{when } N \rightarrow \infty.$$

uniformly with respect to the choices  $x_N$  of the centers of the balls.

This form of small scale delocalization was introduced almost simultaneously by Han [Han15] and Hezari-Rivière [HR16] in the context of negatively curved manifolds, with small scales of logarithmic decay  $r(N) \sim |\log \lambda_N|^{-\alpha}$ . Han also considers the microlocalization of  $u_N$  on small phase space regions, using pseudodifferential operators  $A_N$  with symbols  $\sigma(A_N)$  depending explicitly on the semiclassical parameter, such as to be supported on smaller phase space regions as  $N \rightarrow \infty$ .

Hezari-Rivière [HR17] and Lester-Rudnick [LR17] studied the small scale concentration of eigenstates of the Laplacian on the torus  $\mathbb{T}^d$ , reaching algebraic scales  $r(N) \sim \lambda_N^{-\alpha}$  for some  $\alpha > 0$ ; in this geometry, since the geodesic flow preserves the angle, the delocalization only occurs in the position variable, while the eigenstates can be very concentrated in the Fourier space.

In [Han15], Han considered the Laplacian on a  $d$ -dimensional Riemannian manifold on which a group of isometries acts transitively (this generalizes the round spheres, or the flat tori), and considered a subsequence  $\mathcal{N} \subset \mathbb{N}$  such that, along this subsequence the eigenfrequencies  $(\lambda_j)_{j \in \mathcal{N}}$  of the Laplacian

admit "logarithmically large" multiplicities  $m_{\lambda_j}$ :

$$\liminf_{\mathcal{N} \ni j \rightarrow \infty} \frac{m_{\lambda_j}}{\log \lambda_j} > 0.$$

He considered a random eigenbasis on the spectral subspace  $\bigoplus_{j \in \mathcal{N}} V_{\lambda_j}$ , and showed that, for any  $\alpha \in (0, \frac{1}{2d})$ , then almost surely the eigenbasis equidistributes on  $M$ , at the scale  $r(\lambda_j) = m_{\lambda_j}^{-\alpha}$  in the limit  $j \rightarrow \infty$ .

Let us now return to our setting of the quantum cat map on  $\mathbb{T}^2$  briefly described in section P.2. Han recently [Han18] proved several results on the small scale delocalization of the eigenstates of  $\mathcal{M}_N$ . To give a definition of SSQE in this context, one needs to approximate the characteristic function on balls  $B(\rho, r(N)) \subset \mathbb{T}^2$  centered at  $\rho \in \mathbb{T}^2$  and of small radius  $r(N)$ , by smooth functions  $\chi_{\rho, r(N)}$  living in some mildly exotic symbol class on  $\mathbb{T}^2$ . Han approximated  $\mathbb{1}_{B(\rho), r}$  by two family of trigonometric polynomials  $b_{\rho, r}^{\pm}$  whose Fourier coefficients satisfy some control conditions; we will rather use a family of functions  $\chi_{\rho, r}$  of small supports, defined in Equation 3.9 below. This function satisfy important properties: First, they approximate the characteristic function on  $B(\rho, r)$ , and in fact satisfy  $\int_{\mathbb{T}^2} \chi_{\rho, r} d\rho = \pi r(N)^2 + \text{remainder}$ . Second, analogously to Han's functions we have control on their Fourier coefficients (e.g. in (3.10) and (3.11)) which allows us to prove the SSQE.

Once one has fixed the class of smoothed characteristic functions, the definition of small scale Q(U)E reads as follows.

**Definition P.3.3** (Small scale Quantum (Unique) Ergodicity for  $\mathcal{M}_N$ ). *Let  $\mathbf{r} := (r(N))_{N \in \mathbb{N}^*}$  be a sequence of microscopic scales, i.e.,  $\lim_{N \rightarrow \infty} r(N) = 0$  with  $r(N) > N^{-\frac{1}{2}}$ . Fix an infinite subsequence  $\mathcal{N} \subset \mathbb{N}$ , and choose corresponding eigenbases  $\mathcal{B}_N$  for  $(\mathcal{M}_N)_{N \in \mathcal{N}}$ .*

1. *We say that quantum ergodicity at scale  $\mathbf{r}$  holds along the sequence of eigenbases  $(\mathcal{B}_N)_{N \in \mathcal{N}}$ , if for every  $N \in \mathcal{N}$ , there exists a subset  $S(N) \subset \{1, \dots, N\}$  such that  $\frac{\#S(N)}{N} \xrightarrow{\mathcal{N} \ni N \rightarrow \infty} 1$ , and the following estimate holds, uniformly with respect to the choices of  $\rho \in \mathbb{T}^2$ :*

$$\langle \text{Op}_N(\chi_{\rho, r(N)}) \varphi_{j, N}, \varphi_{j, N} \rangle = \pi r^2(N) + o(r^2(N)) \text{ as } \mathcal{N} \ni N \rightarrow \infty, \quad j \in S(N).$$

2. *We say that QUE holds at scale  $\mathbf{r}$  along the sequence of eigenbases  $(\mathcal{B}_N)_{N \in \mathcal{N}}$  if the above estimate holds along the full eigenbases  $(\mathcal{B}_N)_{N \in \mathcal{N}}$ .*
3. *We say that QE (resp. QUE) holds at scale  $\mathbf{r}$  along the sequence  $N \in \mathcal{N}$  if the above estimates hold for any sequence of eigenbases  $(\mathcal{B}_N)_{N \in \mathcal{N}}$  of  $(\mathcal{M}_N)_{N \in \mathcal{N}}$ .*

Here  $\text{Op}_N(\chi)$  is the quantization of the observable  $\chi \in C^\infty(\mathbb{T}^2)$  on the quantum space of level  $N$ , see chapter 1. When discussing small scale Q(U)E we always take  $r(N) \gg N^{-\frac{1}{2}}$  to ensure that  $\chi$  is in a "well-behaved" symbol class.

[Han18] shows several forms of SSQE for the quantum cat map<sup>6</sup>:

1. for the full sequence  $N \in \mathbb{N}^*$  of Planck parameters and all eigenbases, the quantum cat map satisfies QE at scale  $r(N) = |\log N|^{-\alpha}$ . for any  $\alpha \in [0, 1/4)$ .
2. take the sequence  $\mathcal{N} \subset \mathbb{N}$  of density one corresponding to the "long" quantum periods  $P(N) \geq N^{1/2} e^{(\log N)^\delta}$ ; then for any choice of eigenbases of  $(\mathcal{M}_N)_{N \in \mathcal{N}}$ , QE holds at scale  $r(N) = N^{-\alpha}$  for any  $0 \leq \alpha < 1/16$ .
3. consider the full sequence  $N \in \mathbb{N}$  but consider only the joint eigenbases of  $\mathcal{M}_N$  and the Hecke operators; then these Hecke eigenbases satisfy QE to the scale  $r(N) = N^{-\alpha}$  for any  $0 \leq \alpha < 1/12$ .

While the proof of the first part is similar to the one in [Han15, HR16] for manifolds of negative curvature, the last two points are specific to the quantum cat map, and directly use estimates obtained by [KR00, KR01a].

### P.3.2.2 Our probabilistic SSQE results

In the present work we consider the quantum cat map  $\mathcal{M}_N$ , and restrict the values of  $N$  to subsequence featuring very short periods. more precisely we fix an  $\alpha \in [2, 4)$ , and a sequence  $\mathcal{N}_\alpha$  such that for every  $N$  in this sequence

$$P(N) \leq \alpha \frac{\log N}{\log |\lambda|} + O(1) \quad \mathcal{N}_\alpha \ni N \rightarrow \infty. \quad (\text{P.10})$$

Up to the factor  $\alpha$ , those periods are minimal, and any subsequence  $\mathcal{N}_\alpha$  is necessarily of density zero inside  $\mathbb{N}$ .

As explained in Lemma 3.4.2 such short periods imply huge spectral degeneracies: we will actually show that along this sequence  $\mathcal{N}_\alpha$ , *all* eigenspaces  $V_{\nu,N} = \ker(\mathcal{M}_N - \nu)$  of  $\mathcal{M}_N$  have comparable dimensions

$$d_{\nu,N} := \dim(V_{\nu,N}) \sim \frac{N}{P(N)} \asymp \frac{N}{\log N}.$$

Instead of looking for the most localized eigenstates, as was done in [FNDB03], we will rather consider the typical eigenstates in  $V_{\nu,N}$ , by constructing the ensemble of random orthonormal eigenbases on  $V_{N,\nu}$ , as explained in section 3.2 in various settings. Putting together the random eigenbases of all  $\{\nu \in \text{Spec}(\mathcal{M}_N)\}$  we obtain a probability law  $\mathbb{P}_N$  on the set of eigenbases of  $\mathcal{M}_N$ . Joining these eigenbases for all  $N \in \mathcal{N}_\alpha$ , we obtain a global probability law  $\mathbb{P} = \prod_{N \in \mathcal{N}_\alpha} \mathbb{P}_N$  on the global set of eigenbases of  $(\mathcal{M}_N)_{N \in \mathcal{N}_\alpha}$ .

Our main result states a probabilistic form of QUE for these random eigenbases:

**Theorem P.3.4** (Probabilistic QUE for short-period quantum cat maps). *For  $2 \leq \alpha < 4$ , consider an infinite sequence  $\mathcal{N}_\alpha \subset \mathbb{N}$  with short quantum periods satisfying the bound (P.10), and construct the corresponding ensemble of random eigenbases. Then,*

---

<sup>6</sup>Although Han uses a different family of test functions than our  $\chi_{\rho,r(N)}$ , his proofs can be easily adapted to our test functions.

1. *almost surely the random eigenbases of  $(\mathcal{M}_N)_{N \in \mathcal{N}_\alpha}$  satisfy QUE.*
2. *More precisely, if we take  $0 \leq \delta < 1/2 - \alpha/8$ , then almost surely the random eigenbases of  $(\mathcal{M}_N)_{N \in \mathcal{N}_\alpha}$  satisfy SSQUE at the scale  $r(N) = N^{-\delta}$ .*

Notice that, in the case  $\alpha = 2$ , our sequence  $\mathcal{N}_2$  was constructed by [BDB00], and used in [FNDB03], to exhibit the existence of partially localized (“scarred”) eigenstates. The above theorem shows that typical eigenstates of  $(\mathcal{M}_N)_{N \in \mathcal{N}_2}$  are instead equidistributed down to the scale  $N^{-1/4}$ . The scarring phenomenon described in [FNDB03] is thus very atypical, also inside the eigenspaces.

In view of the “abstract” results of [Map13, Zel14], the main step to show our probabilistic QUE is to prove a Local Weyl law on the eigenspaces  $V_{\nu,N}$  of  $(\mathcal{M}_N)_{N \in \mathcal{N}_\alpha}$ . We prove it in section 3.4 relying on a method by Keating, using heuristics of the Gutzwiller trace formula over classical periodic orbits. This Local Weyl Law shows that the “quantum average” of  $\text{Op}_N(a)$  on the eigenspaces  $V_{\nu,N}$ , converges to the classical average of  $a$  on  $\mathbb{T}^2$ ; in a sense, it says that the eigenspaces  $V_{\nu,N}$  are, globally, quantum ergodic in the limit  $N \rightarrow \infty$ . Using a measure-concentration inequality due to Hanson and Wright (appearing in a modern form in Equation 3.32) we obtain that the probability a matrix element deviates from the corresponding classical average is exponentially small. As we explain in subsection 3.6.3.2 below, in order to end the proof in the small-scale case one has to require additional control conditions on the Fourier series of  $\chi_{\rho,r(N)}$ .

### P.3.3 Statistical properties of individual random eigenstates

In the last part of the thesis we discuss a previously-alluded theme: the statistical characterization of *individual* eigenstates of a quantum chaotic system. In the previous subsections we have had in mind fluctuations of eigenstates either on macroscopic scales (when considering QE and QUE) or on microscopic scales larger than  $\frac{1}{N}$  (when considering SSQUE). When dealing with individual eigenfunctions, one studies the fluctuations on a Planck scale, i.e.,  $\approx \frac{1}{N}$ . In 1977, Berry ([Ber77]) suggested a *Random Wave Model*. Namely, he conjectured that for eigenstates of chaotic systems  $\psi$  should resemble a random combination of plane-waves having the same frequency  $\lambda \asymp N$  yet various arbitrary direction  $\xi_k$ ,

$$\psi(x) \approx \text{Re } \psi_{RW,\lambda}(x) := \sum_{j=1}^{J(\lambda)} \alpha_j e^{i\lambda \xi_k x}$$

where  $\alpha_j$  are independent identically distributed complex random variables. The value distribution of  $\psi_{RW,\lambda}(x)$  should be in this case Gaussian. His work relied on monochromatic Gaussian random fields on  $\mathbb{R}^2$ , defined through the 2-point covariance function  $\mathbb{E}(\psi_{RW,\lambda}(x) \psi_{RW,\lambda}(x + \lambda y)) = J_0(\|y\|)$ , to propose a connection between fluctuations “on Planck’s scale” and the global equidistribution of Wigner’s function (with quantum ergodicity).

Following Berry’s observation there were several numerical discoveries supporting his model and proposing a connection to the theory of random matrix ensembles. The first attempts connected between Gaussian and circular ensembles of matrices and the spectrum of quantum Hamiltonians or Floquet operators (cf. [FM86, HKS87, BGS84]). Later followed by Izrailev’s numeric (cf. [Izr87]) Kuś, Mostowski and Haake (cf. [KMH88]) suggested a relation between circular matrix ensembles (i.e.,

the Circular Orthogonal Ensemble, COE, and the Circular Unitary Ensemble, CUE) and the eigenstates of Floquet operator of periodically kicked tops. They showed that the eigenvector statistics agree with those of Dyson’s circular matrix ensembles.

Berry’s model has been confirmed numerically for several dynamical systems. Eckhardt constructed in [Eck86] explicit eigenstates for the quantized toral automorphisms corresponding to values of  $N$  of low spectral degeneracy and exhibiting random matrix statistics. Bäcker in [Bäc03] arrived to similar findings when perturbing the classical cat map by the (area preserving) ”standard map” and quantizing the new map.

Later several works established weak versions of Berry’s model in spectral windows of varying widths: For stating these results we let  $(M, g)$  be a compact, smooth, Riemannian manifold of dimension  $\dim M \geq 2$  and consider the eigenfunctions of Laplace-Beltrami operator, i.e.,  $-\Delta_g \varphi_j = \lambda_j^2 \varphi_j$ . Canzani and Hanin showed in [CH15, CH18] that under further conditions on the geodesics, fixing some  $R > 0$ , defining

$$\mathcal{H}_N := \bigoplus_{\lambda_k \in [N, N+1]} \ker (\Delta_g - \lambda_k^2), \quad d(N) := \dim \mathcal{H}_N$$

and taking a standard real normal vector  $(\alpha_1, \dots, \alpha_{d(N)}) \sim \mathcal{N}_{\mathbb{R}}(0, \text{Id}_{d(N)})$ , the ”rescaled random waves”,

$$\phi_N := \frac{1}{\sqrt{\dim \mathcal{H}}} \sum_{\lambda_j \in [N, N+1]} \alpha_j \varphi_j$$

satisfy Berry’s model to some weak extent. Han and Tacy [HT20] consider random combinations of eigenstates in spectral windows as well: Under a constraint on the geodesics, taking a window  $[N, N+1]$  having  $d(N)$  eigenvalues inside and  $(\tilde{\alpha}_1, \dots, \tilde{\alpha}_{d(N)})$  being a random variables on  $\mathbb{S}^{d(N)-1}$ , there exists  $C > 0$  such that for every  $x \in M, r > 0$

$$\mathbb{E} \left[ \int_{B(x,r)} |\tilde{\phi}_N|^2 d\text{Vol}_M \right] = C \frac{\text{Vol}(B(x,r))}{\text{Vol}(M)} \quad \text{where } \tilde{\phi}_N = \sum_{\lambda_k \in [N, N+1]} \alpha_k \varphi_k$$

with the remainder being independent of  $x$ . If  $r^{-1} = o(N)$ , the variance is  $o(\text{Vol}(B(x,r))^2)$ .

### P.3.3.1 Our results on statistical properties of eigenstates of the quantum cat map

The three works we have just mentioned relied on a large spectral window of constant size corresponding to the direct sum of many eigenspaces, whereas we focus on ”logarithmically small” windows populated by eigenstates generating a large *single* eigenspace  $V_\lambda$  of  $\dim V_\lambda \asymp \frac{N}{\log N}$  (as we discuss in Proposition 3.4.1 below).

We use the quantum cat map to produce the first example to our knowledge of a family of *true eigenstates* of quantized chaotic system satisfying a toy version of Berry’s model statistics. We consider only a specific sequence of  $N$ s having minimal quantum period. We denote the set of these  $N$ s by  $\mathcal{N}$ . In order to study random eigenstates we begin from a random Gaussian vector  $(\alpha_{1,N}, \dots, \alpha_{N,N}) \sim$

$\mathcal{N}(0, \text{Id}_N)$ . As we explain below we take  $\alpha_N = \alpha_N^{\mathbb{C}} \sim \mathcal{N}_{\mathbb{C}}(0, \text{Id}_N)$  unless  $\gamma$  possesses symmetries, in which case we consider  $\alpha_N = \alpha_N^{\mathbb{R}} \sim \mathcal{N}_{\mathbb{R}}(0, \text{Id}_N)$ . Projecting this random Gaussian vector to  $V_{\nu, N}$ . We denote the random eigenstate obtained by  $v_N$ . In order to simplify the presentation we consider its multiple  $\beta_N = \sqrt{P(N)}v_N := (\beta_{1,N}, \dots, \beta_{N,N})$ . Before stating our results we define the  $\ell^\infty$ -norm of a matrix  $A = (a_{jk})_{j,k \in \llbracket 1, N \rrbracket}$  to be  $\|A\|_{\ell^\infty} = \sup_{j,k} |a_{jk}|$ . As a first proposition we show that  $\beta_N$  statistically resembles a standard random Gaussian vector,

**Theorem.** Fix  $\gamma \in \tilde{\Gamma}(2)$ . Consider the projector  $\Pi_{\nu, N}$  corresponding to the eigenspace  $V_{N, \nu}$  where  $\nu \in \text{Spec}(\mathcal{M}_N)$ ,  $N \in \mathcal{N}$ . Then there exists a sequence of matrices  $(\Delta_N)_N$  with  $\sup_N \|\Delta_N\|_{\ell^\infty} < \infty$  such that

1.

$$\Pi_{\nu, N} = \frac{1}{P(N)} I_N + \frac{1}{\sqrt[4]{N}} \Delta_N.$$

2. Constructing a random eigenstate  $\beta_N$  as above and denoting the covariance matrix  $\text{cov}(\beta_N)$

$$\text{cov}(\beta_N) = I_N + \frac{P(N)}{\sqrt[4]{N}} \Delta_N.$$

The proof relies on observing the matrix elements of  $\Pi_{\nu, N}$  are proportional to those of the covariance matrix and then studying them in Theorem 4.1.1.

Our second result concerns random normalized eigenstates of  $\mathcal{M}_N$  when  $N \in \mathcal{N}$ . We prove in Theorem 4.2.3 that their value distribution converges in probability to the standard normal random variable,

**Theorem.** Let  $\gamma \in \tilde{\Gamma}(2)$ ,  $N \in \mathcal{N}$  and let  $\beta_N$  a random eigenstate of  $\mathcal{M}_{N_k}(\gamma)$ .

1. if  $\beta_N = \beta_N^{\mathbb{C}}$  (i.e., obtained from projecting a complex standard Gaussian), Consider the random normalized vector

$$\tilde{\beta}_N = \left( \tilde{\beta}_{1,N}, \dots, \tilde{\beta}_{N,N} \right) := \frac{\sqrt{2d(\nu, N) P(N)}}{\|\beta_N\|} (\beta_{1,N}, \dots, \beta_{N,N}).$$

Define the empirical measures

$$\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{\beta_{j,N}} \quad \tilde{\mu}_N = \frac{1}{N} \sum_{j=1}^N \delta_{\tilde{\beta}_{j,N}}.$$

(a) There exists a constant  $C_\gamma^{\mathbb{C}} > 0$  such that as  $\mathcal{N} \ni N \rightarrow \infty$ , for every complex rectangle  $R \subset \mathbb{C}$  and  $\varepsilon \in (0, \frac{1}{8})$

$$\mathbb{P} \left( \left| \tilde{\mu}_N(R) - \mu_{\mathcal{N}_{\mathbb{C}}(0,1)}(R) \right| \geq C_\gamma^{\mathbb{C}} N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}. \quad (\text{P.II})$$



(b) There exists a constant  $c_\gamma^{\mathbb{C}} > 0$  such that as  $\mathcal{N} \ni N \rightarrow \infty$ , for every complex rectangle  $R \subset \mathbb{C}$  and  $\varepsilon \in (0, \frac{1}{8})$

$$\mathbb{P} \left( \left| \mu_N(R) - \mu_{\mathcal{M}_{\mathbb{C}}(0,1)}(R) \right| \geq c_\gamma^{\mathbb{C}} N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}. \quad (\text{P.I2})$$

2. if  $\beta_N = \beta_N^{\mathbb{R}}$  (i.e., obtained from projecting a real standard Gaussian), Consider the random normalized vector

$$\tilde{\beta}_N = \left( \tilde{\beta}_{1,N}, \dots, \tilde{\beta}_{N,N} \right) := \frac{\sqrt{d(\nu, N) P(N)}}{\|\beta_N\|} (\beta_{1,N}, \dots, \beta_{N,N}).$$

Define the empirical measures

$$\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{\beta_{j,N}} \quad \tilde{\mu}_N(I) = \frac{1}{N} \sum_{j=1}^N \delta_{\tilde{\beta}_{j,N}}.$$

(a) There exists a constant  $C_\gamma^{\mathbb{R}} > 0$  such that as  $\mathcal{N} \ni N \rightarrow \infty$ , for every interval  $I \subset \mathbb{R}$  and  $\varepsilon \in (0, \frac{1}{8})$

$$\mathbb{P} \left( \left| \tilde{\mu}_N(I) - \mu_{\mathcal{M}_{\mathbb{R}}(0,1)}(I) \right| \geq C_\gamma^{\mathbb{R}} N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}. \quad (\text{P.I3})$$

(b) There exists a constant  $c_\gamma^{\mathbb{R}} > 0$  such that as  $\mathcal{N} \ni N \rightarrow \infty$ , for every interval  $I \subset \mathbb{R}$  and  $\varepsilon \in (0, \frac{1}{8})$

$$\mathbb{P} \left( \left| \mu_N(I) - \mu_{\mathcal{M}_{\mathbb{R}}(0,1)}(I) \right| \geq c_\gamma^{\mathbb{R}} N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}. \quad (\text{P.I4})$$

One deduces the statements for  $\tilde{\beta}_N$ s from showing that the norm of  $\beta_N$  does not fluctuate much and thus the probability that  $\beta_{j,N}$  is in some set is asymptotically identical to the probability  $\tilde{\beta}_{j,N}$  is in it.

We would like to mention a previous result due to Kurlberg and Rudnick in [KR01b] who studied the value distribution of a normalized Hecke eigenstate  $\psi_N = (\psi_{1,N}, \dots, \psi_{N,N})$  of  $\mathcal{M}_N$  whenever  $N$  is a split prime<sup>7</sup>. Namely, for these values of  $N$  they wondered what the limits of the empirical measures of  $\psi_N$

$$\mu_N(I) := \frac{1}{N} \sum_{j=1}^N \mathbb{1}_I(|\psi_{j,N}|), \quad \text{where } I \subset [0, 1]$$

can be. They found that,

**Theorem P.3.5** (Kurlberg-Rudnick, [KR01b]). *Taking a sequence of  $L^2$ -normalized Hecke eigenstates*

---

<sup>7</sup>meaning that  $\gamma$  is diagonalizable mod  $N$  for each of them, allowing to apply methods the theory of exponential sums and characters.

of the quantum cat map  $\psi_{\chi, N}$ , as  $N \rightarrow \infty$  along a sequence of split primes, for every  $I \subset [0, 1]$

$$\frac{1}{2}\mu_N(\psi_{\chi, N}; I) \rightarrow \mu_{s.c.}(I) := \frac{4}{\pi} \int_I \sqrt{1-x^2} dx.$$

# 1

## Quantization of classical observables and anisotropic calculi

This chapter is dedicated for presenting in detail the dynamical and analytical infrastructure for our results. We begin with quantizing the classical hyperbolic toral dynamics and then turn to defining anisotropic symbol calculi on  $\mathbb{T}^2$ . We end this chapter stating our results on the complete delocalization of semiclassical measures associated to the quantum cat map.

### 1.1 Hyperbolic automorphisms on $\mathbb{T}^2$

Let us consider  $\mathbb{R}^2$  equipped with the coordinates  $(y, \eta)$ . We call  $y$  the "position coordinate" and  $\eta$  the "momentum coordinate". We view the 2-dimensional torus as the quotient  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ . The projection map between the two spaces will be denoted by

$$\pi(y, \eta) = (y, \eta) \pmod{\mathbb{Z}^2} \quad \pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2. \quad (1.1)$$

We study the dynamics arising from a class of hyperbolic automorphisms, that is  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  with eigenvalues  $|\lambda_u| > 1, |\lambda_s| < 1$  or equivalently  $|\mathrm{Tr}(\gamma)| > 2$ .

**Lemma 1.1.1.** *Let  $v$  be an eigenvector of  $\gamma$ , then  $v \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ .*

*Proof.* Let us first realize that  $\lambda_u \notin \mathbb{Q}$ : Recall that  $\mathrm{Tr}(\gamma)^2 - 4$  is a square if and only if there exists  $n \in \mathbb{Z}$  such that  $\mathrm{Tr}(\gamma)^2 - n^2 = 4$ , or equivalently if and only if

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{Tr}(\gamma) \\ n \end{pmatrix} \in \left\{ \pm \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \pm \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \pm \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}.$$

Multiplying both sides by  $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ , in the first two cases  $\mathrm{Tr}(\gamma) \notin \mathbb{Z}$  and in the third  $\mathrm{Tr}(\gamma) = \pm 2$ , contradicting the hyperbolicity of  $\gamma$ .

Assume to the contrary that  $v \in \mathbb{Q}^2$  is an eigenvector of  $\gamma$ , without loss of generality in the unstable direction. Then  $\gamma v \in \mathbb{Q}^2$  whereas at least one component of  $\lambda_u v$  is irrational in contradiction.  $\square$

We deduce that the corresponding eigenspaces can be expressed by

$$W_u(\gamma) = \text{span} \left\{ \begin{pmatrix} 1 \\ m_u \end{pmatrix} \right\}, \quad W_s(\gamma) = \text{span} \left\{ \begin{pmatrix} 1 \\ m_s \end{pmatrix} \right\}.$$

with  $m_u, m_s \in \mathbb{R} \setminus \mathbb{Q}$ . Henceforth we refer to  $W_u(\gamma)$  as the unstable space and to  $W_s(\gamma)$  as the stable space. Strictly speaking, we are interested in hyperbolic automorphisms arising from the group<sup>1</sup>

$$\tilde{\Gamma}(2) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a_{11}a_{12} \equiv a_{21}a_{22} \equiv 0 \pmod{2} \right\}. \quad (1.2)$$

We recall several elementary properties of elements in this group:

**Lemma 1.1.2.** *Let  $\gamma \in \tilde{\Gamma}(2)$ , then  $\text{Tr}(\gamma) \in 2\mathbb{Z}$ .*

*Proof.* Let  $\gamma = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ . Assume that  $\text{Tr}(\gamma) = a_{11} + a_{22} \in 2\mathbb{Z} + 1$ . Without loss of generality  $a_{11} \in 2\mathbb{Z} + 1$  and  $a_{22} \in 2\mathbb{Z}$ . Then  $a_{11}a_{22} = \det(\gamma) + a_{12}a_{21} = 1 + a_{12}a_{21} \in 2\mathbb{Z}$ . As follows  $a_{12}, a_{21} \in 2\mathbb{Z} + 1$  and thus  $a_{11}a_{12} \not\equiv 0 \pmod{2}$  contradicting  $\gamma \in \tilde{\Gamma}(2)$ .  $\square$

**Corollary 1.1.3.** *For every hyperbolic matrix  $\gamma \in \tilde{\Gamma}(2)$*

$$\max_{\lambda \in \text{Spec}(\gamma)} |\lambda| = \frac{1}{2} \cdot \max \left\{ \left| \text{Tr}(\gamma) + \sqrt{\text{Tr}(\gamma)^2 - 4} \right|, \left| \text{Tr}(\gamma) - \sqrt{\text{Tr}(\gamma)^2 - 4} \right| \right\} \geq 2 + \sqrt{3}.$$

We end this subsection with a result on the minimality of the associated horocyclic flows: Let us fix an unstable eigenvector  $v_u \in W_u$  and a stable eigenvector,  $v_s \in W_s$ . We define the unstable and the stable horocyclic continuous flows associated to  $\gamma$  by

$$\mathcal{H}_u(v, t) := v + tv_u \pmod{\mathbb{Z}^2} \quad \mathcal{H}_s(v, t) := v + tv_s \pmod{\mathbb{Z}^2} \quad (v, t) \in \mathbb{T}^2 \times \mathbb{R}$$

We can verify by a direct calculation that both flows satisfy an intertwining relation with  $\gamma$ ,

$$\gamma(\mathcal{H}_u(v, t)) = \mathcal{H}_u(\gamma.v, \lambda_u t) \quad \gamma(\mathcal{H}_s(v, t)) = \mathcal{H}_s(\gamma.v, \lambda_s t) \quad (1.3)$$

From Lemma 1.1.1 we deduce that both flows are minimal,

**Lemma 1.1.4.** *Every orbit  $\mathcal{O}_u$  of  $\mathcal{H}_u$  and every orbit  $\mathcal{O}_s$  of  $\mathcal{H}_s$  is dense in  $\mathbb{T}^2$ .*

For sake of completeness we recall the proof.

*Proof.* The proof for orbits of  $\mathcal{H}_s$  is completely analogous thus we only prove the lemma for the orbits of  $\mathcal{H}_u$ . Let  $(y_0, \eta_0) \in \mathbb{T}^2$  and fix an unstable orbit  $\mathcal{O}_1 = \left\{ \begin{pmatrix} y_1 \\ \eta_1 \end{pmatrix} + tv_u \pmod{\mathbb{Z}^2} \right\}$  where  $v_u = \begin{pmatrix} 1 \\ m_u \end{pmatrix}$  as before. Let us consider the set

$$\mathcal{T} := \{t \in \mathbb{R} : t - (y_0 - y_1) \in \mathbb{Z}\}.$$

---

<sup>1</sup>in the present work we quantify only matrices in  $\tilde{\Gamma}(2)$ . However, we could have considered a larger subgroup of  $\text{SL}_2(\mathbb{Z})$  as explained in [GHo6, Section 2]

and the corresponding sequence of points on the orbit  $\{(m_u n + \eta_1 + m_u(y_0 - y_1)) \pmod{\mathbb{Z}^2}\}_{n \in \mathbb{Z}} \subset \mathcal{O}_1 \cap \{y \equiv y_0 - y_1 \pmod{\mathbb{Z}}\}$ . Then,  $\{m_u n \pmod{\mathbb{Z}}\}_{n \in \mathbb{Z}}$  is a dense sequence on  $[0, 1]$  as  $m_u \in \mathbb{R} \setminus \mathbb{Q}$  hence we can extract a sub-sequence of times  $t_{n_k}$  for which  $\{t_{n_k} m_u \pmod{\mathbb{Z}}\} \rightarrow \eta_1$  and thus  $\mathcal{H}_u \left( \begin{pmatrix} y_1 \\ \eta_1 \end{pmatrix}, t_{n_k} \right) \xrightarrow{k \rightarrow \infty} \begin{pmatrix} y_0 \\ \eta_0 \end{pmatrix}$ .  $\square$

We recall below that each  $\gamma \in \tilde{\Gamma}(2)$  is quantized by a family of unitary operators  $\{\mathcal{M}_N(\gamma)\}$  satisfying an exact Egorov property (I.19).

## 1.2 The quantum spaces

Consider the unitary Weyl-Heisenberg operators  $T_{\begin{pmatrix} x^* \\ 0 \end{pmatrix}}^h, T_{\begin{pmatrix} 0 \\ \xi^* \end{pmatrix}}^h : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$  which correspond to  $x^*, \xi^* \in \mathbb{R}$  and are given by

$$\left( T_{\begin{pmatrix} x^* \\ 0 \end{pmatrix}}^h \psi \right) (x) = \psi(x - x^*), \quad \left( T_{\begin{pmatrix} 0 \\ \xi^* \end{pmatrix}}^h \psi \right) (x) = e^{\frac{i\xi^*}{h}x} \psi(x). \quad (I.4)$$

By duality the action of these operators extend naturally to  $\mathcal{S}'(\mathbb{R})$ . The commutation relations between  $T_{\begin{pmatrix} x^* \\ 0 \end{pmatrix}}^h$  to  $T_{\begin{pmatrix} 0 \\ \xi^* \end{pmatrix}}^h$  are non-trivial,

$$T_{\begin{pmatrix} x^* \\ 0 \end{pmatrix}}^h T_{\begin{pmatrix} 0 \\ \xi^* \end{pmatrix}}^h = \exp(ix^* \xi^* / h) T_{\begin{pmatrix} 0 \\ \xi^* \end{pmatrix}}^h T_{\begin{pmatrix} x^* \\ 0 \end{pmatrix}}^h, \quad x^*, \xi^* \in \mathbb{R}. \quad (I.5)$$

The Weyl-Heisenberg operator which corresponds to a pair  $(x^*, \xi^*) \in \mathbb{R}^2$  will be defined as

$$T_{\begin{pmatrix} x^* \\ \xi^* \end{pmatrix}}^h = \exp\left(\frac{\pi i x^* \xi^*}{h}\right) T_{\begin{pmatrix} x^* \\ 0 \end{pmatrix}}^h T_{\begin{pmatrix} 0 \\ \xi^* \end{pmatrix}}^h.$$

The sub-spaces of  $\mathcal{S}'(\mathbb{R})$

$$\mathcal{S}_{h,\kappa} = \left\{ \psi \in \mathcal{S}'(\mathbb{R}) : T_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^h \psi = e^{2\pi i \kappa_1} \psi, T_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^h \psi = e^{2\pi i \kappa_2} \psi \right\} \quad \kappa \in [0, 1]^2,$$

will play a major role in the quantization of toral automorphisms.

**Lemma 1.2.1.**  $\mathcal{S}_{h,\kappa} \neq 0$  if and only if  $h^{-1} \in 2\pi\mathbb{N}$ .

*Proof.* Let  $\psi \in \mathcal{H}_{h,\kappa}$ , then

$$T_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^h T_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^h \psi = T_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^h T_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^h \psi = e^{2\pi i(\kappa_1 + \kappa_2)} \psi.$$

Explicitly,  $\exp(i/h) \exp(ix/h) \psi(x-1) = \exp(ix/h) \psi(x-1)$ . If  $\psi \not\equiv 0$ ,  $\exp(i/h) = 1$  hence for every  $h$  exists  $N \in \mathbb{N}$  such that  $h = \frac{1}{2\pi N}$ .  $\square$

We deduce that

**Claim 1.2.2.**  $\mathcal{S}_{h,\kappa}$  is a finite dimensional space of distributions.

*Proof.* Every  $\psi \in \mathcal{S}_{h,\kappa}$  satisfies  $e^{2\pi i \kappa_2 x} \psi = e^{\frac{2\pi i x}{h}} \psi$ . It means that  $\text{supp } \psi$  is discrete, so  $\psi$  can be expressed as  $\psi = \sum_{k \in \mathbb{Z}} \sum_{\tilde{k} \in \mathbb{N}_0} c_{k,\tilde{k}} \delta_{x_k}^{(\tilde{k})}$  for  $x_k = \frac{\kappa_2 + k}{N} \in \mathbb{R}$ . For every test function  $\eta \in \mathcal{S}(\mathbb{R})$ ,

$$\begin{aligned} e^{2\pi i \kappa_2} \sum_{k \in \mathbb{Z}, \tilde{k} \in \mathbb{N}_0} (-1)^{\tilde{k}} c_{k,\tilde{k}} \eta^{(\tilde{k})}(x_k) &= \left\langle T_{(0,1)}^h \psi, \eta \right\rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} \\ &= \left\langle e^{\frac{2\pi i x}{h}} \psi, \eta \right\rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} \\ &= \left\langle \psi, e^{\frac{2\pi i x}{h}} \eta \right\rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} \\ &= \sum_{k \in \mathbb{Z}, \tilde{k} \in \mathbb{N}_0} (-1)^{\tilde{k}} c_{k,\tilde{k}} \left\langle \delta_{x_k}, \partial_x^{\tilde{k}} \left( e^{\frac{2\pi i x}{h}} \eta \right) \right\rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} \\ &= e^{2\pi i \kappa_2} \sum_{k \in \mathbb{Z}, \tilde{k} \in \mathbb{N}_0} \sum_{p=0}^{\tilde{k}} \binom{\tilde{k}}{p} c_{k,\tilde{k}} \left( \frac{2\pi i}{h} \right)^{\tilde{k}-p} (-1)^{\tilde{k}} \eta^{(p)}(x_k). \end{aligned}$$

Fix  $k \in \mathbb{Z}$ . Comparing the coefficient of  $\delta_{x_k}^{(p)}$  in both sides,

$$\sum_{\tilde{k}=p+1}^{\infty} (-1)^{\tilde{k}} \left( \frac{2\pi i}{h} \right)^{\tilde{k}-p} \binom{\tilde{k}}{p} c_{k,\tilde{k}} = 0.$$

This constraint can be formulated in an (infinite) upper-triangular matrix form,

$$\begin{pmatrix} 0 & (-1)^{p+1} \frac{2\pi i}{h} & (-1)^{p+2} \left( \frac{2\pi i}{h} \right)^2 & (-1)^{p+3} \left( \frac{2\pi i}{h} \right)^3 & \cdots \\ 0 & 0 & 2(-1)^{p+2} \left( \frac{2\pi i}{h} \right)^2 & 3(-1)^{p+3} \left( \frac{2\pi i}{h} \right)^3 & \cdots \\ 0 & 0 & 0 & 4(-1)^{p+3} \left( \frac{2\pi i}{h} \right)^3 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} c_{k,0} \\ c_{k,1} \\ c_{k,2} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

This matrix has a single free variable,  $c_{k,0}$ , and its lines are linearly independent: The  $(\tilde{k} - p)$ -th row is a  $(\tilde{k} - p)$ -right shift of a Vandermonde vector (with a progression  $-\frac{2\pi i}{h}$ ) weighted by

$$(-1)^{p+1} \left( \frac{2\pi i}{h} \right)^{\tilde{k}-p} \left( \binom{p+1}{p} \quad \binom{p+2}{p} \quad \cdots \quad \binom{p+\tilde{j}}{p} \quad \cdots \right)_{\tilde{j} \in \mathbb{N}},$$

hence  $c_{k,\tilde{k}} = 0$  for  $\tilde{k} \geq 1$ .  $T_{(1,0)}^h$  multiplies  $\psi$  by a scalar,

$$\sum_{k \in \mathbb{Z}} e^{2\pi i \kappa_1 k} c_{k,0} \delta_{x_k}(x) = e^{2\pi i \kappa_1} \psi(x) = \left( T_{(1,0)}^h \psi \right)(x) = \psi(x-1) = \sum_{k \in \mathbb{Z}} c_{k-N,0} \delta_{x_k}(x).$$

Comparing the supports of the distributions in both sides implies that  $\psi$  is determined by its values on  $x_k$  for  $k = 0, \dots, N-1$ , and that the coefficients satisfy the recursion relation  $c_{k+N,0} = e^{-2\pi i \kappa_1} c_{k,0}$  for  $k \in \mathbb{Z}$ . For brevity we denote  $c_k = c_{k,0}$ .  $\square$

A possible choice of a base for  $\mathcal{S}_{h,\kappa}$  is  $e_{j,\kappa} = \sum_{n \in \mathbb{Z}} e^{i \kappa_1 n} \delta_{\frac{\kappa_2 + j}{N} + n}$  for  $j = 0, \dots, N-1$ . For later use we recall that

$$\begin{aligned} T_{\left(\frac{k}{N}, 0\right)} e_{j,\kappa}(x) &= \sum_{n \in \mathbb{Z}} e^{i \kappa_1 n} \delta_{\frac{\kappa_2 + j - k}{N} + n}(x), \\ T_{\left(0, \frac{k}{N}\right)} e_{j,\kappa}(x) &= e^{2\pi i k x} \sum_{n \in \mathbb{Z}} e^{i \kappa_1 n} \delta_{\frac{\kappa_2 + j}{N} + n}(x). \end{aligned}$$

We equip  $\mathcal{S}_{h,\kappa}$  with a "natural" inner product: Given  $\psi_1 = \sum_{j=0}^{N-1} c_j e_{j,\kappa}$ ,  $\psi_2 = \sum_{j=0}^{N-1} c'_j e_{j,\kappa}$  their inner product will be defined as

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{H}_{h,\kappa}} = \frac{1}{N} \sum_{j=0}^{N-1} c_j \overline{c'_j}.$$

The resulting Hilbert space will be denoted by  $\mathcal{H}_{N,\kappa} = \left( \mathcal{S}_{h,\kappa}, \langle \cdot, \cdot \rangle_{\mathcal{H}_{h,\kappa}} \right)$ .

### 1.3 Quantization of bi-periodic symbols

The Weyl Quantization of bi-periodic symbols  $f \in C^\infty(\mathbb{R}^2)$  can be expressed as (cf. [BDB96])

$$\text{Op}_h^W(f) = \iint_{\mathbb{R}^2} \check{f}(x^*, \xi^*) T_{\left(\frac{h x^*}{h \xi^*}\right)}^h dx^* d\xi^* \quad \text{Op}_h^W(f) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad (1.6)$$

with  $\check{g}(x, \xi)$  being the Fourier transform of a function  $g \in \mathcal{S}(\mathbb{R}^2)$ , namely,

$$\check{g}(x^*, \xi^*) = \int_{\mathbb{R}^2} g(\tilde{x}, \tilde{\xi}) e^{-2\pi i(x^* \tilde{x} - \xi^* \tilde{\xi})} d\tilde{x} d\tilde{\xi}.$$

**Lemma 1.3.1.** *For every  $n \in \mathbb{Z}$  the operators  $T_{\left(\frac{n}{N}, 0\right)}^h, T_{\left(0, \frac{n}{N}\right)}^h$  preserve  $\mathcal{H}_{N,\kappa}$*

*Proof.* As the Weyl-Heisenberg operators in the same direction are commuting, for every  $f \in \mathcal{H}_{N,\kappa}$ ,

$$T_{\left(\frac{1}{0}\right)}^h T_{\left(\frac{n}{N}\right)}^h f = T_{\left(\frac{n}{N}\right)}^h T_{\left(\frac{1}{0}\right)}^h f = e^{2\pi i \kappa_1} T_{\left(\frac{n}{N}\right)}^h f,$$

$$T_{(0)}^h T_{(\frac{0}{N})}^h f = T_{(\frac{0}{N})}^h T_{(0)}^h f = e^{2\pi i \kappa_2} T_{(\frac{0}{N})}^h f.$$

In addition using (1.5),  $T_{(1)}^h T_{(\frac{0}{N})}^h f = T_{(\frac{0}{N})}^h T_{(1)}^h f$  and  $T_{(0)}^h T_{(\frac{0}{N})}^h f = T_{(\frac{0}{N})}^h T_{(0)}^h f$ .  $\square$

Henceforth we denote the projection of  $\phi \in L^2(\mathbb{R})$  to  $\mathcal{H}_{N,\kappa}$  will be given by

$$\phi_\kappa = \sum_{(l_1, l_2) \in \mathbb{Z}^2} (-1)^{N l_1 l_2} e^{i(\kappa_1 l_1 - \kappa_2 l_2)} T_{(l_1, l_2)}^h \phi.$$

The Weyl-Heisenberg operators induce a direct-integral decomposition on  $L^2(\mathbb{R})$ ,

$$L^2(\mathbb{R}) = \int_{[0,1]^2}^{\oplus} \mathcal{H}_{N,\kappa} d\kappa,$$

with an isometry  $L^2(\mathbb{R}) \rightarrow L^2([0,1]^2, \mathbb{C}^N)$  defined explicitly by mapping any  $\phi \in L^2(\mathbb{R})$  to  $\iota^\phi(\kappa) = \{\langle e_{j,\kappa}, \phi_\kappa \rangle\}_{j=0}^{N-1} \in \mathbb{C}^N \cong \mathcal{H}_{N,\kappa}$ . In light of Lemma 1.3.1, it follows that for a bi-periodic  $f$ ,  $\text{Op}_h^W(f)$  preserves the sub-spaces hence splits,

$$\text{Op}_h^W(f) = \int_{[0,1]^2}^{\oplus} \text{Op}_{h,\kappa}(f) d\kappa. \quad (1.7)$$

**Remark 1.3.2.**  $\text{Op}_{N,\kappa}$  has an explicit formula in terms of the Weyl-Heisenberg operators. Namely, for every  $f \in C^\infty(\mathbb{T}^2)$  having Fourier coefficients

$$\widehat{f}(l_1, l_2) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x, \xi) e^{-2\pi i \langle l, (x, \xi) \rangle} dx d\xi$$

(for  $l = (l_1, l_2) \in \mathbb{Z}^2$ ),

$$\text{Op}_{N,\kappa}(f) = \sum_{(l_1, l_2) \in \mathbb{Z}^2} \widehat{f}(l_1, l_2) T_{(l_1, l_2)}^h \Big|_{\mathcal{H}_{N,\kappa}}. \quad (1.8)$$

This decomposition yields several spectral properties of  $\text{Op}_{N,\kappa}$ .

**Lemma 1.3.3.** Let  $f, g \in C^\infty(\mathbb{T}^2, \mathbb{R})$ , then

1.  $\text{Op}_{N,\kappa}(f)$  is a self-adjoint operator on  $\mathcal{H}_{N,\kappa}$  for every  $\kappa \in [0, 1]^2$ .
2.  $\|\text{Op}_h(f)\|_{\mathcal{B}(L^2(\mathbb{R}))} = \max_\kappa \|\text{Op}_{h,\kappa}(f)\|_{\mathcal{B}(\mathcal{H}_{N,\kappa})}$ .
3.  $\text{Op}_{h,\kappa}(f) \text{Op}_{h,\kappa}(g) = \text{Op}_{h,\kappa}(fg) + h \text{Op}_{h,\kappa}(r)$  where for every  $m \in \mathbb{N}_0$ ,  $r \in O_{S(1),f,g}(1)$  that is  $\|r^{(m)}(h)\| \leq C_{f,g,m}$  for constants  $C_{f,g,m}$  which depends on a finite number of semi-norms of  $f, g$  and on  $m$ . Equivalently  $\|\text{Op}_{h,\kappa}(r)\| \leq C_{f,g}$ .



*Proof.*

- i. Consider the functional  $\varphi_{\kappa,u}(v) = \langle u_{\kappa}, (\text{Op}_{N,\kappa}(f)^* - \text{Op}_{N,\kappa}(f)) v_{\kappa} \rangle$ . Since  $\text{Op}_h^W(f)$  is a self-adjoint operator, applying (1.7), for every  $u, v \in L^2(\mathbb{R})$ ,

$$\begin{aligned} \int_{[0,1]^2}^{\oplus} \langle u_{\kappa}, \text{Op}_{N,\kappa}(f) v_{\kappa} \rangle d\kappa &= \langle u, \text{Op}_h^W(f) v \rangle = \langle \text{Op}_h^W(f) u, v \rangle \\ &= \int_{[0,1]^2}^{\oplus} \langle \text{Op}_{N,\kappa}(f) u_{\kappa}, v_{\kappa} \rangle d\kappa \\ &= \int_{[0,1]^2}^{\oplus} \langle u_{\kappa}, (\text{Op}_{N,\kappa}(f))^* v_{\kappa} \rangle d\kappa. \end{aligned} \quad (1.9)$$

It means that  $\int \varphi_{\kappa,u}(v) d\kappa \equiv 0$ , or equivalently

$$0 = \int 0_{\kappa} d\kappa = \int \varphi_{u,\kappa}(v) d\kappa.$$

As for every  $v \in L^2(\mathbb{R})$  and  $\kappa \in [0, 1]^2$ ,  $\varphi_{\kappa,u}(v) \equiv 0_{\kappa}$ . It implies that  $(\text{Op}_{N,\kappa}(f))^* - \text{Op}_{N,\kappa}(f) \equiv 0$  by Riesz representation theorem.

2. Let  $\kappa_{\max}$  be the value of  $\kappa \in [0, 1]^2$  which maximizes  $\|B_{\kappa}\|_{\mathcal{B}(\mathcal{H}_{N,\kappa})}$ . Consider a sequence of functions  $\{\varphi_m\}_m \in L^2(\mathbb{R})$  such that  $\frac{\|B\varphi_m\|_{L^2(\mathbb{R})}}{\|\varphi_m\|_{L^2(\mathbb{R})}} \rightarrow \|B\|_{\mathcal{B}(L^2(\mathbb{R}))}$ . Applying Cauchy-Schwarz inequality

$$\begin{aligned} \|B\varphi_m\|_{L^2(\mathbb{R})}^2 &= \int_{[0,1]^2} \|B_{\kappa}\varphi_{m,\kappa}\|_{\mathcal{H}_{N,\kappa}}^2 d\kappa \\ &\leq \int_{[0,1]^2} \|B_{\kappa}\|_{\mathcal{B}(\mathcal{H}_{N,\kappa})}^2 \|\varphi_{m,\kappa}\|_{\mathcal{H}_{N,\kappa}}^2 d\kappa \\ &\leq \|B_{\kappa_{\max}}\|_{\mathcal{B}(\mathcal{H}_{N,\kappa_{\max}})}^2 \int \|\varphi_{m,\kappa}\|_{\mathcal{H}_{N,\kappa}}^2 d\kappa \\ &= \|B_{\kappa_{\max}}\|_{\mathcal{B}(\mathcal{H}_{N,\kappa_{\max}})}^2 \|\varphi_m\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Letting  $m \rightarrow \infty$ , amounts to  $\|B\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq \|B_{\kappa_{\max}}\|_{\mathcal{B}(\mathcal{H}_{N,\kappa_{\max}})}$ . On the other direction, consider  $\varphi^{\varepsilon} \in L^2(\mathbb{R})$  such that<sup>2</sup> for every  $\kappa \in \mathbb{B}(\kappa_{\max}, \varepsilon)$ ,  $\varphi_{\kappa}^{\varepsilon} \neq 0$  and  $\|B_{\kappa}\varphi_{\kappa}^{\varepsilon}\|_{\mathcal{H}_{N,\kappa}} = \|B_{\kappa}\|_{\mathcal{B}(\mathcal{H}_{N,\kappa})} \|\varphi_{\kappa}^{\varepsilon}\|_{\mathcal{H}_{N,\kappa}}$ .

<sup>2</sup>We can choose such function since the operator norm can be viewed as a continuous function defined over the unit ball  $\{\varphi_{\kappa}^{\varepsilon} \in \mathcal{H}_{N,\kappa} : \|\varphi_{\kappa}^{\varepsilon}\| \leq 1\}$  and thus it attains maxima inside it

Then,

$$\begin{aligned} \|B\varphi^\varepsilon\|_{L^2(\mathbb{R})}^2 &= \int_{[0,1]^2} \|B_\kappa \varphi_\kappa^\varepsilon\|_{\mathcal{H}_{N,\kappa}}^2 d\kappa \\ &= \int_{\mathbb{B}(\kappa_{\max}, \varepsilon)} \|B_\kappa\|_{\mathcal{B}(\mathcal{H}_{N,\kappa})}^2 \|\varphi_\kappa^\varepsilon\|_{\mathcal{H}_{N,\kappa}}^2 d\kappa \\ &\geq \left( \|B_{\kappa_{\max}}\|_{\mathcal{B}(\mathcal{H}_{N,\kappa})}^2 - \delta \right) \|\varphi\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

for some  $\delta > 0$  by the continuity of  $\|B_\kappa\|$  as a function in  $\kappa$ . Letting  $\varepsilon \rightarrow 0$  means that  $\|B\|_{\mathcal{B}(L^2(\mathbb{R}))} \geq \|B_{\kappa_{\max}}\|_{\mathcal{B}(\mathcal{H}_{N,\kappa})}$ .

3. For smooth symbols  $f, g$  the composition formula for Weyl quantization (theorem 4.11 in [Zwo12]) is given by the Moyal product,  $\text{Op}_h^W(f) \text{Op}_h^W(g) = \text{Op}_h^W(fg) + R_{f,g}^W$  with  $R_{f,g}^W = \text{Op}_h^W(r)$  for  $r = O_{S(1),f,g}(1)$ . Projecting to the  $\kappa$ -fiber it amounts to

$$\begin{aligned} \int_{[0,1]^2}^\oplus \text{Op}_{h,\kappa}(f) \text{Op}_{h,\kappa}(g) d\kappa &= \text{Op}_h^W(f) \text{Op}_h^W(g) = \text{Op}_h^W(fg) + \text{Op}_h^W(r) \\ &= \int_{[0,1]^2}^\oplus \left( \text{Op}_{h,\kappa}(fg) + \text{Op}_{h,\kappa}(r) \right) d\kappa. \end{aligned}$$

Invoking the same argument from 1., shows the equality fiber-wise. From 2.,  $\|\text{Op}_{h,\kappa}(r)\|_{\mathcal{B}(\mathcal{H}_{h,\kappa})} \leq \|\text{Op}_h^W(r)\|_{\mathcal{B}(L^2(\mathbb{R}))} = C_{f,g}h$ .

□

**Remark 1.3.4.** Iterating the argument appearing in the proof of Lemma 1.3.3.ii implies that the statement holds for finite products of pseudo-differential operators: Suppose  $a_1, \dots, a_n \in C^\infty(\mathbb{T}^2)$  then

$$\left\| \prod_{j=1}^n \text{Op}_h(a_j) \right\|_{\mathcal{B}(L^2(\mathbb{R}))} = \sup_{\kappa \in [0,1]^2} \left\| \prod_{j=1}^n \text{Op}_{N,\kappa}(a_j) \right\|_{\mathcal{B}(\mathcal{H}_{N,\kappa})}.$$

## 1.4 Quantizing hyperbolic toral automorphisms

We dedicate this section for quantizing the hyperbolic toral automorphisms introduced previously in section 1.1, In other words we associate to each such automorphism  $A \in \tilde{\Gamma}(2)$  an operator  $\mathcal{M}_N(A)$ . We begin by recalling without proof an important construction for  $\tilde{\Gamma}(2)$ ,

**Lemma 1.4.1.** For every  $A \in \tilde{\Gamma}(2)$ ,  $\kappa \in [0,1]^2$ , there exists an unitary map  $\mathcal{M}_{N,\kappa}(A) : \mathcal{H}_{N,\kappa} \rightarrow \mathcal{H}_{N,\kappa'}$  (for  $\kappa' = {}^t A^{-1}\kappa$  as we explain in the proof of 1.4.2) which intertwines  $T_{\begin{pmatrix} 1 & \\ & N \end{pmatrix}}^h$  and  $T_{A^{-1}\begin{pmatrix} 1 & \\ & N \end{pmatrix}}^h$ , that

is

$$T^h_{\begin{pmatrix} 1 & \\ & \kappa \end{pmatrix}} \mathcal{M}_{N,\kappa}(A) = \mathcal{M}_{N,\kappa}(A) T^h_{A^{-1} \begin{pmatrix} 1 & \\ & \kappa \end{pmatrix}}.$$

From Lemma 1.4.1 let us deduce

**Conclusion 1.4.2.**  $\mathcal{M}_{N,\kappa}(A)$  preserves  $\mathcal{H}_{N,\kappa}$  if and only if  ${}^t A \kappa = \kappa$ . In particular,

$$\mathcal{H}_{N,0} = \left\{ \sum_{k=0}^{N-1} c_k e_{k,0} : c_k \in \mathbb{C} \right\},$$

the space of 1-periodic functions both in position and in momentum is conserved for every  $A \in \tilde{\Gamma}(2)$ .

*Proof.* Given  $\psi \in \mathcal{H}_{N,\kappa}$  and  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}_0(2)$ , applying the lemma,

$$\begin{aligned} T^h_{\begin{pmatrix} 1 & \\ & \kappa \end{pmatrix}} \mathcal{M}_{N,\kappa}(A) \psi &= \mathcal{M}_{N,\kappa}(A) T^h_{A^{-1} \begin{pmatrix} 1 & \\ & \kappa \end{pmatrix}} \psi \\ &= \mathcal{M}_{N,\kappa}(A) T^h_{\begin{pmatrix} dl_1 - bl_2 \\ -cl_1 + al_2 \end{pmatrix}} \psi \end{aligned}$$

which for  $l = (1, 0)$  as  $cd \equiv 0 \pmod{2}$  means that,

$$\begin{aligned} T^h_{\begin{pmatrix} 1 & \\ & \kappa \end{pmatrix}} \mathcal{M}_{N,\kappa}(A) \psi &= \mathcal{M}_{N,\kappa}(A) \exp(-\pi i h c d) T^h_{\begin{pmatrix} 0 \\ -c \end{pmatrix}} T^h_{\begin{pmatrix} a \\ d \end{pmatrix}} \psi \\ &= \exp(2\pi i (d\kappa_1 - c\kappa_2)) \mathcal{M}_{N,\kappa}(A) \psi. \end{aligned}$$

Similarly, for  $l = (0, 1)$ ,

$$T^h_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \mathcal{M}_{N,\kappa}(A) \psi = \exp(2\pi i (-b\kappa_1 + a\kappa_2)) \mathcal{M}_{N,\kappa}(A) \psi,$$

which amounts to the conclusion that  $\mathcal{M}_{N,\kappa}(A) (\mathcal{H}_{N,\kappa}) = \mathcal{H}_{N,{}^t A^{-1} \kappa}$ .  $\square$

The restriction of  $\text{Op}_h^W$  to  $\mathcal{H}_{N,0}$  will be denoted  $\text{Op}_N$ .  $\mathcal{M}_N = \mathcal{M}_{N,0}$  intertwines with the Weyl-Heisenberg operators. Combining this with (1.8) yields an exact Egorov property,

$$\mathcal{M}_N^{-1}(A) \text{Op}_N(f) \mathcal{M}_N(A) = \text{Op}_N(f \circ A), \quad A \in \tilde{\Gamma}_0(2). \quad (1.10)$$

**Definition 1.4.3.** For every toral map  $A \in \tilde{\Gamma}(2)$ , the family of maps  $\{\mathcal{M}_N(A)\}_N$  is the quantum map associated to  $A$ .

In fact,  $\mathcal{M}_N(A)$  is obtained as the restriction of  $\mathcal{M}_h(A)$ , the full metaplectic representation of  $\text{Mp}_2(\mathbb{R})$  from  $L^2(\mathbb{R})$  to  $\mathcal{H}_N$ . From the formula to the full representation we can deduce explicitly  $\mathcal{M}_N(A)$ , cf. section A.1

**Remark 1.4.4.** We note that  $\text{Op}_N$  satisfies an exact Egorov relation with the dynamics (cf. [DEG03]). Namely the restriction of (1.20) to  $\mathcal{H}_N$  means that for every  $\gamma \in \tilde{\Gamma}(2)$ ,

$$\mathcal{M}_N(\gamma)^{-1} \text{Op}_N(a) \mathcal{M}_N(\gamma) = \text{Op}_N(a \circ \gamma), \quad a \in C^\infty(\mathbb{T}^2). \quad (1.11)$$

### 1.5 Anisotropic calculi associated to Lagrangian foliations

Let us recall the semiclassical parameter  $h \in (0, 1)$  and recall the class of symbols

$$S(\mathbb{R}^2) = \{a = a(h) \in C^\infty(\mathbb{R}^2) : |\partial_y^k \partial_\eta^m a(y, \eta; h)| < C_{k,m}(a), \\ C_{k,m}(a) \in \mathbb{R} \text{ depending only on } k, m, a\}.$$

We would like to propagate smooth symbols  $a \in C^\infty(\mathbb{R}^2)$  with respect to the dynamics of  $\gamma$ , up to large semiclassical time, namely twice the Ehrenfest time, that is denoting the semiclassical parameter by  $h > 0$  we propagate up to time  $2 \left\lfloor \frac{\log \frac{1}{h}}{\log \lambda_u} \right\rfloor$ . That results in highly oscillatory symbols lying outside  $S(\mathbb{R}^2)$ , hence one has to define classes of symbols smooth along a 1-dimensional subspace of  $\mathbb{R}^2$  and oscillating along another.

In our settings we consider only linear foliations, that is those having as leaves  $L_{(y_0, \eta_0)}$  lines with a fixed incline passing through  $(y_0, \eta_0)$ . By a direct calculation one dimensional foliations are always Lagrangian.

**Remark 1.5.1.** *Each hyperbolic toral automorphism  $\gamma$  induces two one-dimensional foliations, an unstable  $L^u$  and a stable  $L^s$ , both having linear leaves. Define the global unstable leaf at  $(y_0, \eta_0) \in \mathbb{R}^2$  by  $W_{u, (y_0, \eta_0)}^{\mathbb{R}^2} = \{(y, \eta) \in \mathbb{R}^2 : \eta = m_u y + \eta_0 - m_u y_0\}$  and similarly the global stable leaf by  $W_{s, (y_0, \eta_0)}^{\mathbb{R}^2} = \{(y, \eta) \in \mathbb{R}^2 : \eta = m_s y + \eta_0 - m_s y_0\}$ . These yield the unstable and stable foliations of a hyperbolic  $\gamma \in SL_2(\mathbb{R})$ , namely the foliations  $L^u, L^s$  with leaves which are explicitly given by  $W_{u, (y_0, \eta_0)}^{\mathbb{R}^2}$  and  $W_{s, (y_0, \eta_0)}^{\mathbb{R}^2}$  correspondingly.*

Let  $L$  be a linear foliation and associate to it a vector  $v \in \mathbb{R}^2$  tangent to  $L$ . Fix a  $v^\pitchfork \in \mathbb{R}^2$  transversal to  $v$ . The foliation associated to  $v^\pitchfork$  is denoted by  $L_{\pitchfork}$  and is called a  $L$ -transversal foliation. We remind that a symbol is a function  $a(y, \eta; h) \in C^\infty(\mathbb{R}^2)$  depending on a parameter  $h$ . We use the notation  $|a(x, \xi; h)| \leq h^{-\varepsilon}$  if for every  $\varepsilon > 0$  (independent of  $h$ ),

$$|a(x, \xi; h)| \leq h^{-\varepsilon}.$$

We introduce two classes of symbols independent of the choice of  $v^\pitchfork$ ,

**Definition 1.5.2.** *Let  $L$  be a linear foliation,  $L_{\pitchfork}$  a  $L$ -transverse foliation and  $V_L = {}^t v \cdot (\partial_y, \partial_\eta)$ ,  $V_{L_{\pitchfork}} = {}^t v^\pitchfork \cdot (\partial_y, \partial_\eta)$  vector fields. Fix a  $0 < \rho < 1$ . A symbol  $a \in S_{L, \rho+}(\mathbb{R}^2)$  if  $a : \mathbb{R}^2 \rightarrow \mathbb{R}$  is smooth and for every  $k, m \in \mathbb{N}$  there exists a constant  $C_{k,m,\varepsilon}$  such that for every  $h \in (0, 1)$*

$$\sup_{(y, \eta) \in U} |V_L^m V_{L_{\pitchfork}}^k a(y, \eta; h)| \leq C_{k,m,\varepsilon} h^{-\rho k - \varepsilon}.$$

**Remark 1.5.3.** *We note that these classes are independent of the choice of  $L_{\pitchfork}$ : given two different  $L$ -transverse foliations  $L_{u^\pitchfork}, L_{v^\pitchfork}$ , we can write  $V_{L_{u^\pitchfork}} = C V_L + C' V_{L_{v^\pitchfork}}$  and using the triangle inequality*

$$\left| V_L^m V_{L_{u^\pitchfork}}^k a(y, \eta; h) \right| \leq \sum_{j+j'=k} C_{j,j',u^\pitchfork,v^\pitchfork} \left| V_L^{m+j} V_{L_{v^\pitchfork}}^{j'} a(y, \eta; h) \right| \leq C'_{k,m,\varepsilon} h^{-\rho k - \varepsilon}.$$

A similar argument shows that keeping  $L_{\uparrow}$  while changing  $L$ , does not preserve the symbol class. As follows, we realize symbols in these anisotropic classes have sharp oscillations along any direction transverse to  $L$  and controlled ones along  $L$ .

For future reference let us recall the isotropic classes<sup>3</sup>

$$\begin{aligned} S_{\rho}(\mathbb{R}^2) &= \{a \in C^{\infty}(\mathbb{R}^2) : \forall \kappa \in \mathbb{N}^2, \quad |\partial^{\kappa} a| \leq C_{\kappa} h^{-\rho|\kappa|}\}, \\ S_{\rho+}(\mathbb{R}^2) &= \{a \in C^{\infty}(\mathbb{R}^2) : \forall \kappa \in \mathbb{N}^2, \quad |\partial^{\kappa} a| \leq C_{\kappa} h^{-\rho|\kappa|-}\} \end{aligned}$$

**Example 1.5.4.** Let  $\gamma \in \tilde{\Gamma}(2)$  and denote  $v_u = \begin{pmatrix} 1 \\ m_u \end{pmatrix}$ ,  $v_s = -\begin{pmatrix} 1 \\ m_s \end{pmatrix}$  the vectors corresponding to its unstable and stable foliations (see Remark 1.5.1). Since  $v_s$  is transversal to  $v_u$  we define the symbol classes

$$S_{L^u, \rho+}(\mathbb{R}^2) = \left\{ a \in C^{\infty}(\mathbb{R}^2) : \forall k, m \in \mathbb{N}, \quad \sup_{(y, \eta) \in \mathbb{R}^2} |V_u^m V_s^k a(y, \eta; h)| \leq C_{k, m, \varepsilon} h^{-\rho k-} \right\}, \quad (1.12)$$

$$S_{L^s, \rho+}(\mathbb{R}^2) = \left\{ a \in C^{\infty}(\mathbb{R}^2) : \forall k, m \in \mathbb{N}, \quad \sup_{(y, \eta) \in \mathbb{R}^2} |V_s^m V_u^k a(y, \eta; h)| \leq C_{k, m, \varepsilon} h^{-\rho k-} \right\}, \dots$$

**Definition 1.5.5.** Let us consider analogous symbol classes on  $\mathbb{T}^2$ . Every linear foliation  $L$  can be projected to a foliation  $L_{\mathbb{T}^2}$  on  $\mathbb{T}^2$ . Equivalently  $V_{L_{\mathbb{T}^2}}$  is the vector field on  $\mathbb{T}^2$  acting on  $\mathbb{Z}^2$ -periodic functions  $a \in C^{\infty}(\mathbb{R}^2)$  by

$$V_{L_{\mathbb{T}^2}} a(y, \eta; h) := V_L a(y, \eta; h).$$

Let  $\gamma \in \tilde{\Gamma}(2)$  acting on  $\mathbb{T}^2$  and let  $v_u, v_s$  as above. We denote the associated linear foliations on  $\mathbb{T}^2$  as  $L_{\mathbb{T}^2}^u$  and  $L_{\mathbb{T}^2}^s$ . We denote by  $C^{\infty}(\mathbb{T}^2)$  the class of smooth real-valued functions on  $\mathbb{T}^2$ . The symbol classes on the torus are defined as,

$$S_{L^u, \rho+}(\mathbb{T}^2) = \left\{ a \in C^{\infty}(\mathbb{T}^2) : \forall k, m \in \mathbb{N}, h \in (0, 1) \quad \sup_{(y, \eta) \in \mathbb{T}^2} |V_{L_{\mathbb{T}^2}^u}^m V_{L_{\mathbb{T}^2}^s}^k a(y, \eta; h)| \leq C_{k, m, \varepsilon} h^{-\rho k-} \right\}, \quad (1.13)$$

$$S_{L^s, \rho+}(\mathbb{T}^2) = \left\{ a \in C^{\infty}(\mathbb{T}^2) : \forall k, m \in \mathbb{N}, h \in (0, 1), \quad \sup_{(y, \eta) \in \mathbb{T}^2} |V_{L_{\mathbb{T}^2}^s}^m V_{L_{\mathbb{T}^2}^u}^k a(y, \eta; h)| \leq C_{k, m, \varepsilon} h^{-\rho k-} \right\},$$

with  $V_{L_{\mathbb{T}^2}^u}$  and  $V_{L_{\mathbb{T}^2}^s}$  being the vector fields associated to the linear foliations  $L_{\mathbb{T}^2}^u$  and  $L_{\mathbb{T}^2}^s$ .

The next class allows stronger oscillations along  $L$ ,

**Definition 1.5.6.** Let  $L$  be a linear foliation,  $V_L, V_{L_{\uparrow}}$  vector fields acting by a directional derivative with respect to  $v$  and  $v^{\uparrow}$  correspondingly and let  $\rho, \rho' \in [0, 1)$  satisfying

$$0 \leq \rho' < \rho \quad \rho + \rho' < 1.$$

<sup>3</sup>Note that these classes are well behaved only for  $\rho \in (0, \frac{1}{2})$ . For extended discussion we refer the reader to [Zwo12, Section 4.4]

A symbol  $a \in S_{L,\rho,\rho'}(\mathbb{R}^2)$  if  $a : \mathbb{R}^2 \rightarrow \mathbb{R}$  is smooth and for every  $k, m \in \mathbb{N}$  there exists an  $h$ -independent constant  $C_{k,m}$  such that for every  $h \in (0, 1)$

$$\sup_{(y,\eta) \in \mathbb{R}^2} |V_L^m V_{L_\natural}^k a(y, \eta; h)| \leq C_{k,m} h^{-\rho k - \rho' m}.$$

**Remark 1.5.7.** As in Remark 1.5.3 the class characterizes the direction  $L$  rather than  $L_\natural$ . In fact, symbols in the class can have sharper oscillations along the transversal direction  $L_\natural$ .

**Example 1.5.8.** For  $L = L^\eta$  the vertical foliation in  $\mathbb{R}^2$  fixing as the transversal foliation  $L_{v^\natural} = L^y$  we obtain a model  $\mathbb{R}^2$ -calculus slightly broader compared to the one introduced in [Appendix A.2, [DJ18]] which allowed only compact symbols:

**Lemma 1.5.9.** Let  $0 \leq \rho, \rho' < 1$ . The class of symbols  $S_{L^\eta,\rho,\rho'}(\mathbb{R}^2)$  consists of the functions  $a \in C^\infty(\mathbb{R}^2)$  such that for every  $k, m \in \mathbb{N}$  there exists a constant  $C_{k,m} > 0$  satisfying for every  $h \in (0, 1)$  the derivative bounds

$$\sup_{(y,\eta) \in \mathbb{R}^2} |\partial_y^k \partial_\eta^m a(y, \eta; h)| \leq C_{k,m} h^{-\rho k - \rho' m}. \quad (1.14)$$

*Proof.* Any vector associated to  $L^\eta$  is of the form  $\binom{0}{j}$  for  $j \in \mathbb{R}^*$  and any vector associated to  $L_\natural$  is of the form  $v^\natural = \binom{j'}{0}$  with  $j' \in \mathbb{R}^*$ . Therefore for every  $k, m \in \mathbb{N}$  there is a constant  $C_{k,m}$  satisfying

$$\sup_{(y,\eta) \in \mathbb{R}^2} |\partial_\eta^m \partial_y^k a(y, \eta; h)| = j^{-m} (j')^{-k} |V_{L^\eta}^m V_{L^y}^k a(y, \eta; h)| \leq C_{k,m} h^{-\rho k - \rho' m}.$$

□

We note that for every foliation  $L$

$$S_{L,\rho+}(\mathbb{R}^2) = \bigcap_{\varepsilon > 0} S_{L,\rho+\varepsilon,\varepsilon}(\mathbb{R}^2) \quad (1.15)$$

hence Lemma 1.5.9 implies that

$$S_{L^\eta,\rho+}(\mathbb{R}^2) = \{a \in C^\infty(\mathbb{R}^2) : \text{For every } k, m \in \mathbb{N} \text{ exists a constant } C_{k,m,\varepsilon} \text{ such that} \\ \sup_{(y,\eta) \in \mathbb{R}^2} |\partial_y^k \partial_\eta^m a(y, \eta; h)| \leq C_{k,m,\varepsilon} h^{-\rho k - \varepsilon m}\}.$$

There is an algebraic identity which reduces proofs for these classes of symbols to the model calculus. Recall that

$$m_u = \frac{a_{22} - a_{11} + \sqrt{\text{Tr}^2(\gamma) - 4}}{2a_{12}} \quad m_u = \frac{a_{22} - a_{11} - \sqrt{\text{Tr}^2(\gamma) - 4}}{2a_{12}}.$$

From the the hyperbolicity of  $\gamma$ ,  $m_u \neq m_s$ . We assume henceforth that  $m_u > m_s$  (as in ?? below). Define  $\iota \in \mathrm{SL}_2(\mathbb{R})$  by

$$\iota = \frac{1}{\sqrt{m_u - m_s}} \begin{pmatrix} 1 & -1 \\ m_u & -m_s \end{pmatrix}, \quad \iota^{-1} = \frac{1}{\sqrt{m_u - m_s}} \begin{pmatrix} -m_s & 1 \\ -m_u & 1 \end{pmatrix}. \quad (\text{I.16})$$

Note that the columns of  $\iota$  are obtained by rescaling  $v_u, v_s$ , ensuring  $\iota \in \mathrm{SL}_2(\mathbb{R})$ . This automorphism maps the standard orthonormal basis of  $\mathbb{R}^2$  to an eigenbasis of  $\gamma$ :

$$\iota \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{m_u - m_s}} v_u, \quad \iota \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{m_u - m_s}} v_s.$$

We view  $\iota$  as a coordinate chart  $([0, 1]^2, \iota)$ : It maps every point  $(y, \eta) \in U \subset [0, 1]^2$  with  $U$  being open to  $\begin{pmatrix} x \\ \xi \end{pmatrix} := \iota \begin{pmatrix} y \\ \eta \end{pmatrix} \in \iota U \subset \iota([0, 1]^2)$ , hence  $\iota$  can be viewed as a change from the  $(y, \eta)$  coordinate frame (parametrizing  $\mathbb{T}^2$ ) to the  $(x, \xi)$  coordinate frame (parametrizing  $\iota\mathbb{T}^2$ ). Henceforth for every symbol  $a \in C^\infty(\mathbb{R}^2)$  we define the twisted symbol  $\tilde{a}$  by  $\tilde{a} := a \circ \iota$ .

**Lemma 1.5.10.** *Let  $\gamma \in \tilde{\Gamma}(2)$  and  $a \in S(\mathbb{R}^2)$  then for  $h \in (0, 1)$  and uniformly in  $t \in \left[0, \rho \frac{\log \frac{1}{h}}{\log |\lambda_u|}\right]$*

1.  $\widetilde{a \circ \gamma^t} \in S_{L^{\xi, \rho, 0}}(\mathbb{R}^2)$ .
2.  $\widetilde{a \circ \gamma^{-t}} \in S_{L^x, \rho, 0}(\mathbb{R}^2)$ .
3.  $a \circ \gamma^t \in S_{L^s, \rho, 0}(\mathbb{R}^2)$
4.  $a \circ \gamma^{-t} \in S_{L^u, \rho, 0}(\mathbb{R}^2)$ .

*Proof.* We prove only (i) and (iii) as the others follow from replacing  $\gamma$  by  $\gamma^{-1}$  exchanging between the contracting and expanding eigendirections.

- i. Suppose  $a \in S(\mathbb{R}^2)$  so does  $a \circ \iota$  since

$$|\partial_y^k \partial_\eta^m \tilde{a}(y, \eta)| \leq B(k, m) \|\iota\|^{k+m} \max_{0 \leq k' + m' \leq k+m} |\partial_x^{k'} \partial_\xi^{m'} a(x, \xi; h)| \leq C'_{k, m}(a) \quad (\text{I.17})$$

with  $B(k, m)$  being a combinatorial constant depending on  $k, m$ . We note that  $\iota^{-1}$  diagonalises  $\gamma$  hence

$$\iota^{-1} \gamma^t \iota = D^t = \begin{pmatrix} \lambda_u^t & 0 \\ 0 & \lambda_u^{-t} \end{pmatrix}. \quad (\text{I.18})$$

Multiplying by  $\iota$  from the left and using (I.17) we conclude that for  $0 < t \leq \left\lfloor \rho \frac{\log \frac{1}{h}}{\log |\lambda_u|} \right\rfloor$ ,

$$\sup_{(y, \eta) \in \mathbb{R}^2} \left| \partial_y^k \partial_\eta^m \left( \widetilde{a \circ \gamma^t} \right) (y, \eta; h) \right| = \sup_{(y, \eta) \in \mathbb{R}^2} \left| \partial_y^k \partial_\eta^m \left( \tilde{a}(\lambda_u^t y, \lambda_u^{-t} \eta; h) \right) \right|$$

$$\begin{aligned}
&= |\lambda_u|^{(k-m)t} \sup_{(y,\eta) \in \mathbb{R}^2} \left| \partial_{y'}^k \partial_{\eta'}^m (\tilde{a})(y', \eta'; h) \Big|_{(y', \eta') = (\lambda_u^t y, \lambda_u^{-t} \eta)} \right| \\
&\leq C'_{k,m}(a) |\lambda_u|^{(k-m)t} \\
&\leq C'_{k,m}(a) |\lambda_u|^{kt} \\
&< C'_{k,m}(a) |\lambda_u|^{k\rho \frac{\log \frac{1}{h}}{\log |\lambda_u|}} \\
&= C'_{k,m}(a) h^{-k\rho}.
\end{aligned}$$

Thus,

$$\sup_{(y,\eta) \in \mathbb{R}^2} \left| \partial_x^k \partial_\xi^m \widetilde{a \circ \gamma^t}(y, \eta; h) \right| \leq C_{k,m}(a) h^{-\rho k}.$$

3. We recall that for every symbol  $b \in C^\infty(\mathbb{T}^2)$ ,  $(V_u b) \circ \iota = \partial_y(b \circ \iota)$  and similarly  $(V_s b) \circ \iota = \partial_\eta(b \circ \iota)$  and therefore taking  $b = a \circ \gamma^t$  we obtain from (i)

$$\max_{(y,\eta) \in \mathbb{R}^2} |V_u^k V_s^m (a \circ \gamma^t)| = \max_{(y,\eta) \in \mathbb{R}^2} \left| \partial_y^k \partial_\eta^m (\widetilde{a \circ \gamma^t}) \right| \leq C'_{k,m}(a) h^{-\rho k}.$$

□

## 1.6 The properties of the quantization of anisotropic symbols

We quantize symbols from all classes mentioned above by the Weyl quantization (cf. subsection 4.1.1 in [Zwo12])  $\text{Op}_h^w(\cdot) = \text{Op}_h(\cdot)$ . This choice of quantization is motivated by the exact intertwining relations (1.20) and (1.25) which hold in our settings as we mention below. We recall that for every symbol  $a \in S(\mathbb{R}^2)$  the operators  $\text{Op}_h(a)$  map  $S'(\mathbb{R}^2)$  to itself and are bounded on  $L^2(\mathbb{R})$  uniformly in  $h$ . The same statement holds for  $a \in S_{L,\rho+}(\mathbb{R}^2)$  as we show in Lemma 1.6.1.iii below. By Stone-Weierstrass theorem the Weyl quantization of smooth bounded symbols satisfies an exact Egorov relation (see subsections 1.5 and 4.2 in [Fol89]): for every fixed  $h$  and bounded  $a \in S(\mathbb{R}^2)$  denote by  $\mathcal{M}_h$  the full unitary metaplectic representation of  $\text{Mp}_2(\mathbb{R})$  (playing the role of a double cover of  $\text{SL}_2(\mathbb{R})$ ). Identifying each  $\gamma \in \tilde{\Gamma}(2)$  with an antecedent in  $\text{Mp}_2(\mathbb{R})$ ,

$$\text{Op}_h(a) = \mathcal{M}_h(\gamma) \text{Op}_h(a \circ \gamma) \mathcal{M}_h(\gamma)^* \quad (1.19)$$

and

$$\text{Op}_h(a) = \mathcal{M}_h(\iota) \text{Op}_h(\tilde{a}) \mathcal{M}_h(\iota)^*, \quad (1.20)$$

independently of the size of the derivatives of  $a$ . This intertwining relation between  $\text{Op}_h(a)$  and  $\text{Op}_h(\tilde{a})$  plays a major role in studying the properties of the calculus.

**Lemma 1.6.1.** *Let  $a \in S(\mathbb{R}^2)$  and  $\alpha \in S_{L^s, \rho+}(\mathbb{R}^2)$  be real valued symbols then for every  $0 < t < \rho \frac{\log \frac{1}{h}}{\log |\lambda_u|}$*

1. *Self-adjointness:  $(\text{Op}_h(a))^* = \text{Op}_h(a)$  on  $L^2(\mathbb{R})$ .*



2. *The Moyal product of symbols: if  $d \in S_{L^s, \rho+}(\mathbb{R}^2)$  then  $\text{Op}_h(\alpha) \text{Op}_h(d) = \text{Op}_h(c)$  with  $c = \alpha \cdot d + O_{S_{L^s, \rho+}(\mathbb{R}^2)}(h^{1-\rho-})$ . In fact, recalling Lemma 1.5.10.(iii),  $\text{Op}_h(\alpha \circ \gamma^t) \text{Op}_h(d) = \text{Op}_h(c)$  with  $c = (\alpha \circ \gamma^t) \cdot d + O_{S_{L^s, \rho+}(\mathbb{R}^2)}(h^{1-\rho-})$ .*
3. *The operator  $\text{Op}_h(\alpha)$  is bounded on  $L^2(\mathbb{R})$  uniformly in  $h$ .*
4. *Gårding inequality: For every linear foliation  $L$  if  $d \geq 0$  and  $d \in S_{L, \rho, \rho'}(\mathbb{R}^2)$  then for some constant  $C$  independent of  $h$ , such that for every  $\varphi \in L^2(\mathbb{R})$*

$$\langle \text{Op}_h(d) \varphi, \varphi \rangle \geq -Ch^{1-\rho-\rho'} \|\varphi\|_{L^2(\mathbb{R})}^2.$$

*Proof.*

1. The Weyl quantization satisfies  $(\text{Op}_h(a))^* = \text{Op}_h(a^*)$  and  $a$  is a real symbol.
2. We introduce the unitary rescaling operator

$$T_\rho : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad (T_\rho f)(x') = h^{\frac{\rho}{4}} f\left(h^{\frac{\rho}{2}} x'\right) \quad (1.21)$$

We note that both for symbols  $a \in S(\mathbb{R}^2)$  and for symbols  $\tilde{a} \in S_{L^\xi, \rho, 0}(\mathbb{R}^2)$  conjugation by  $T_\rho$  leads to

$$T_\rho \text{Op}_h(a) T_\rho^{-1} = \text{Op}_h(a^b) \quad (1.22)$$

with  $a^b(x, \xi; h) = a(h^{\frac{\rho}{2}} x, h^{-\frac{\rho}{2}} \xi) \in S_{\frac{\rho}{2}}(\mathbb{R}^2)$  (cf. subsection 4.4 in [Zwo12]) and  $\tilde{a}^b \in S_{\frac{\rho}{2}+}(\mathbb{R}^2)$  since for every  $n = (n_1, n_2) \in \mathbb{N}^2$

$$\sup_{(x, \xi) \in \mathbb{R}^2} |\partial^n a^b| = h^{\frac{\rho}{2}(n_1 - n_2)} \sup_{(x, \xi) \in \mathbb{R}^2} |\partial^n a| \leq C_n h^{-\frac{\rho}{2}|n| - \varepsilon}.$$

A similar calculation shows that  $a^b \in S_{L^s, \rho+}$ . Denoting  $b = \tilde{a} \in S_{L^\xi, \rho+}$  and applying theorem 4.18 from [Zwo12] implies that

$$\text{Op}_h(b^b) \text{Op}_h\left(\left(\tilde{d}\right)^b\right) = \text{Op}_h\left(\left(\tilde{c}\right)^b\right)$$

with  $(\tilde{c})^b = b^b \cdot \left(\tilde{d}\right)^b + O_{S_{\frac{\rho}{2}+}}(h^{1-\rho-})$ . Conjugating  $\text{Op}_h(\cdot)$  by  $\mathcal{M}_h(\iota)$  maps symbols from  $S_{L^s, \rho+}$  to  $S_{L^\xi, \rho+}$ . Then, by the exact Egorov relation (1.20) applied with  $\iota^{-1}$ ,

$$\begin{aligned} \text{Op}_h(\alpha) \text{Op}_h(d) &= \mathcal{M}_h(\iota) \text{Op}_h(b) \mathcal{M}_h(\iota)^* \mathcal{M}_h(\iota) \text{Op}_h\left(\tilde{d}\right) \mathcal{M}_h(\iota)^* \\ &= \mathcal{M}_h(\iota) \text{Op}_h(b) \text{Op}_h\left(\tilde{d}\right) \mathcal{M}_h(\iota)^* \\ &= \mathcal{M}_h(\iota) T_\rho^{-1} \text{Op}_h(b^b) \text{Op}_h\left(\left(\tilde{d}\right)^b\right) T_\rho \mathcal{M}_h(\iota)^* \end{aligned}$$

$$\begin{aligned}
&= \mathcal{M}_h(\iota) \text{Op}_h(\tilde{c}) \mathcal{M}_h(\iota)^* \\
&= \text{Op}_h(c).
\end{aligned}$$

3. Note that  $\text{Op}_h(\alpha) = T_\rho^{-1} \text{Op}_h(\alpha^b) T_\rho$ . Since  $\alpha^b \in S_{\frac{\rho}{2}+}(\mathbb{R}^2)$  the operator  $\text{Op}_h(\alpha^b)$  and thus  $\text{Op}_h(\alpha)$  are both bounded on  $L^2(\mathbb{R})$  uniformly in  $h$ .
4. First, from a direct calculation analogous to the one appearing in Lemma 1.5.10 we deduce that  $\tilde{d} \in S_{L^\varepsilon, \rho, \rho'}(\mathbb{R}^2)$ . Then, we recall the rescaling from [DJ18, Lemma A.2],

$$h^\# := h^{1-\rho-\rho'} \quad \left(\tilde{d}\right)^\#(x, \xi; h) := \tilde{d}(h^\rho x, h^{\rho'} \xi; h) \quad u^\#(x) := h^{\frac{\rho}{2}} u(h^\rho x)$$

for which using the exact intertwining relation in (1.20)

$$\left\langle \text{Op}_h(\tilde{d}) \mathcal{M}_h(\iota) \varphi, \mathcal{M}_h(\iota) \varphi \right\rangle = \left\langle \text{Op}_{h^\#} \left( \left(\tilde{d}\right)^\# \right) (\mathcal{M}_h(\iota) \varphi)^\#, (\mathcal{M}_h(\iota) \varphi)^\# \right\rangle = \left\langle \text{Op}_h(d) \varphi, \varphi \right\rangle$$

and adapting [DJ18, Lemma A.4] we can also verify that  $\left(\tilde{d}\right)^\# \in S(\mathbb{R}^2)$ . Then applying [Zwo12, Theorem 4.32] and using the unitarity of  $\mathcal{M}_h(\iota)$  we deduce the inequality.

□

The same argument used to prove Lemma 1.6.1.ii can be applied to give a Moyal product in  $S_{L, \rho, \rho'}(\mathbb{R}^2)$  as well,

**Lemma 1.6.2** (Moyal product of symbols in  $S_{L, \rho, \rho'}(\mathbb{R}^2)$ ). *Fix a linear foliation  $L$  and let  $\alpha, d \in S_{L, \rho, \rho'}(\mathbb{R}^2)$  be real valued symbols, then  $\text{Op}_h(\alpha) \text{Op}_h(d) = \text{Op}_h(c)$  with  $c = \alpha \cdot d + O_{S_{L, \rho, \rho'}(\mathbb{R}^2)}(h^{1-\rho-\rho'})$ .*

We deduce a bound on  $\|\text{Op}_h(a)\|_{L^2(\mathbb{R})}$  for a bounded  $a \in S_{L, \rho+}(\mathbb{R}^2)$  with  $L$  being any linear foliation,

**Lemma 1.6.3.** *Suppose  $a \in S_{L, \rho+}(\mathbb{R}^2)$ . If  $\sup |a| \leq 1$ ,*

$$\|\text{Op}_h(a)\| \leq 1 + Ch^{1-\rho-}.$$

*Proof.* We note that from Lemma 1.6.1.iv, for every  $\varphi \in L^2(\mathbb{R})$

$$\langle \varphi, \varphi \rangle - \langle \text{Op}_h(|a|^2) \varphi, \varphi \rangle \geq -Ch^{1-\rho-\varepsilon} \|\varphi\|^2,$$

and as  $\text{Op}_h(a) \text{Op}_h(a) = \text{Op}_h(|a|^2) + O_{B(L^2(\mathbb{R}))}(h^{1-\rho-\varepsilon})$

$$\|\text{Op}_h(a) \varphi\|^2 \leq \|\varphi\|^2 + Ch^{1-\rho-\varepsilon} \|\varphi\|^2,$$

implying the lemma. □

### 1.7 The Quantum space of the torus phase space

We come back to our discussion on the quantum spaces and on quantization of smooth periodic observables. We identify  $\mathcal{H}_{N,\kappa}$  and  $\mathbb{C}^N$  through the (non-canonical) map

$$\psi = \sum_{j=0}^{N-1} \sum_{n \in \mathbb{Z}} \psi_j e^{\kappa_1 n} \delta_{\frac{j+\kappa_2}{N} + n} \mapsto (\psi_0, \dots, \psi_{N-1}) \quad (1.23)$$

and equip  $\mathcal{H}_{N,\kappa}$  with the (non-standard) scalar product

$$\langle \phi_N, \psi_N \rangle_{\mathcal{H}_{N,\kappa}} = \frac{1}{N} \sum_{j=1}^N \phi_{N,j} \overline{\psi_{N,j}}, \quad \phi_N = (\phi_{N,j})_{j=0}^{N-1}, \psi_N = (\psi_{N,j})_{j=0}^{N-1}. \quad (1.24)$$

**Remark 1.7.1.** *Note that our choice of basis and inner product differs from the one in [BDB96, Proposition 2.1, Proposition 2.3], in which the authors chose as basis  $\{\frac{1}{\sqrt{N}}\psi_{j,N}\}_{j=0}^{N-1}$ . This change of basis reflects on the inner product (1.24) by multiplication in a factor  $N$ . However, our results Theorem P.3.1, Theorem 2.1.2 do not depend on that choice (and an analogue of Theorem P.3.2 holds as well when using the conventions of [BDB96]).*

**Remark 1.7.2.** *The proof of Lemma 1.3.3.ii implies that the statement holds for finite products of pseudo-differential operators: Suppose  $a_1, \dots, a_n \in C^\infty(\mathbb{T}^2)$  then*

$$\left\| \prod_{j=1}^n \text{Op}_h(a_j) \right\|_{\mathcal{B}(L^2(\mathbb{R}))} = \sup_{\kappa \in [0,1]^2} \left\| \prod_{j=1}^n \text{Op}_{N,\kappa}(a_j) \right\|_{\mathcal{B}(\mathcal{H}_{N,\kappa})}.$$

For every  $\gamma \in \tilde{\Gamma}(2)$  the map  $\mathcal{M}_h(\gamma)$  sends  $\mathcal{H}_{N,\kappa}$  to  $\mathcal{H}_{N,t\gamma\kappa}$  (cf. [DE93]), hence we will only focus on  $\mathcal{H}_{N,0}$  for which  $\mathcal{M}_{N,0} : \mathcal{H}_{N,0} \rightarrow \mathcal{H}_{N,0}$ . This space has the basis  $\{e_j\}_{j=0}^{N-1}$  with  $e_j = \delta_{\frac{j}{N} + \mathbb{Z}}$  and is viewed as a Hilbert space by (1.24). One considers the stable and the unstable foliations on  $\mathbb{T}^2$ : The global leaf at the point  $(y_1, \eta_1) \in \mathbb{T}^2$  is defined by  $W_{u,(y_1,\eta_1)}^{\mathbb{T}^2} = W_{u,(y_0,\eta_0)}^{\mathbb{R}^2} / \mathbb{Z}^2$  and  $W_{s,(y_1,\eta_1)}^{\mathbb{T}^2} = W_{s,(y_0,\eta_0)}^{\mathbb{R}^2} / \mathbb{Z}^2$  for  $(y_0, \eta_0) \equiv (y_1, \eta_1) \pmod{1}$ .

**Remark 1.7.3.** *For every  $A = \text{Op}_h(a)$  with  $a \in S_{L,\rho+}(\mathbb{T}^2)$  we denote for brevity*

$$A_N = \text{Op}_{N,0}(a) = \text{Op}_N(a) \quad \mathcal{M}_{N,0} = \mathcal{M}_N \quad \mathcal{H}_N = \mathcal{H}_{N,0}.$$

*With  $\gamma$  defined above we denote the eigenfunctions of  $\mathcal{M}_N(\gamma)$  by  $\{\varphi_{j,N}\}$ .*

Restricting to a symbol  $a \in S_{L,\rho+}(\mathbb{T}^2)$  analogous properties (proved for sake of completeness)

**Lemma 1.7.4** (Analytical properties of  $\text{Op}_N(a)$ ). *Let  $a \in S_{L,\rho+}(\mathbb{T}^2)$ , then*

1.  $\text{Op}_N(a) : \mathcal{H}_N \rightarrow \mathcal{H}_N$  is bounded uniformly in  $N$ .

2. A Moyal product formula holds: for  $d \in S_{L,\rho^+}(\mathbb{T}^2)$ ,  $\text{Op}_N(a) \text{Op}_N(d) = \text{Op}_N(c)$  with  $c = ad + O_{S_{L,\rho^+}(\mathbb{T}^2)}(h^{1-\rho^-})$ .

*Proof.*

1. Follows immediately from combining Lemma 1.3.3.ii with the boundedness of  $\text{Op}_h(a)$  proved in Lemma 1.6.3 (potentially after rescaling).
2. From Lemma 1.6.1.ii, a Moyal product formula holds when considering the Weyl quantization of the symbols,  $\text{Op}_h(a) \text{Op}_h(d) = \text{Op}_h(c)$  with  $c = ad + O(h^{1-\rho^-})$ . Since  $L^2(\mathbb{R})$  and the quantization split we can write

$$\int_{[0,1]^2}^{\oplus} \text{Op}_{N,\kappa}(a) \text{Op}_{N,\kappa}(d) d\kappa = \text{Op}_h(a) \text{Op}_h(d) = \text{Op}_h(ad) + \text{Op}_h(r) = \int_{[0,1]^2}^{\oplus} \text{Op}_{N,\kappa}(ad) + \text{Op}_{N,\kappa}(r) d\kappa$$

with  $\|\text{Op}_h(r)\|_{\mathcal{B}(L^2(\mathbb{R}))} = O(h^{1-\rho^-})$ . Then

$$\Phi_\phi(\psi) = \langle (\text{Op}_h(ad) + \text{Op}_h(r) - \text{Op}_h(a) \text{Op}_h(d)) \phi, \psi \rangle$$

is the zero functional in  $L^2(\mathbb{R})^*$  and its projection to any  $\kappa$ -fiber is the zero functional in  $\mathcal{H}_{N,\kappa}^*$ .

From Lemma 1.3.3.ii,  $\|\text{Op}_{N,\kappa}(r)\|_{\mathcal{B}(\mathcal{H}_{N,\kappa})} \leq \|\text{Op}_h(r)\|_{\mathcal{B}(L^2(\mathbb{R}))} = O(h^{1-\rho^-})$ .

□

Similar analytical properties hold when  $a \in S_{L,\rho,0}(\mathbb{T}^2)$ . In fact,

**Lemma 1.7.5** (Moyal product for symbols in  $S_{L,\rho,0}(\mathbb{T}^2)$ ). *Let  $a, b \in S_{L,\rho,0}(\mathbb{T}^2)$  then*

$$\text{Op}_N(a) \text{Op}_N(b) = \text{Op}_N(c)$$

where  $c = ab + O_{S_{L,\rho,0}(\mathbb{T}^2)}(h^{1-\rho})$ .

**Remark 1.7.6.** *Both in Lemma 1.7.4.ii and in Lemma 1.7.5 if the supports of  $a$  and  $b$  are disjoint, repeated integration by parts yields that  $\text{Op}_N(c) = O_{\mathcal{B}(\mathcal{H}_N)}(N^{-\infty})$*

We formulate a version of Gårding inequalities holding for  $\text{Op}_N(a)$ .

**Lemma 1.7.7.** *Suppose  $a \in S_{L^s,\rho,\rho'}(\mathbb{T}^2)$  and  $a \geq 0$  then there is some  $C > 0$  and  $h_0 > 0$  such that for  $h \in (0, h_0)$*

$$\langle \text{Op}_N(a) u, u \rangle \geq -Ch^{1-\rho-\rho'} \|u\|_{\mathcal{H}_{N,0}}^2 \quad u \in \mathcal{H}_{N,0}.$$

*Proof.* Let  $a \in S_{L^s,\rho,\rho'}(\mathbb{R}^2)$  such that  $a \geq 0$ . We deduce from Gårding inequality for  $S_{L^s,\rho,\rho'}(\mathbb{R}^2)$  that the spectrum of  $\text{Op}_h(a)$  lies inside the band  $[-Ch^{1-\rho-\rho'}, \|\text{Op}_h(a)\|]$  and

$$-Ch^{1-\rho-\rho'} \text{Id}_{L^2(\mathbb{R})} \leq \text{Op}_h(a) \leq \|\text{Op}_h(a)\| \text{Id}_{L^2(\mathbb{R})}$$

Denoting  $\mathfrak{a} = a - \frac{1}{2} \|\mathrm{Op}_h(a)\|$  the last equation reads

$$-\frac{1}{2} \left( \|\mathrm{Op}_h(a)\| + Ch^{1-\rho-\rho'} \right) \mathrm{Id}_{L^2(\mathbb{R})} \leq \mathrm{Op}_h(\mathfrak{a}) \leq \frac{1}{2} \left( \|\mathrm{Op}_h(a)\| + Ch^{1-\rho-\rho'} \right) \mathrm{Id}_{L^2(\mathbb{R})}$$

hence the self-adjointness of  $\mathrm{Op}_h(a)$  implies  $\|\mathrm{Op}_h(\mathfrak{a})\| \leq \frac{1}{2} \left( \|\mathrm{Op}_h(a)\| + Ch^{1-\rho-\rho'} \right)$ . Then from Lemma 1.3.3.ii also

$$\|\mathrm{Op}_N(\mathfrak{a})\| \leq \frac{1}{2} \left( \|\mathrm{Op}_h(a)\| + Ch^{1-\rho-\rho'} \right)$$

and from the self-adjointness of  $\mathrm{Op}_N(a)$

$$-\frac{1}{2} \left( \|\mathrm{Op}_h(a)\| + Ch^{1-\rho-\rho'} \right) \mathrm{Id}_{\mathcal{H}_{N,0}} \leq \mathrm{Op}_N(\mathfrak{a}) \leq \frac{1}{2} \left( \|\mathrm{Op}_h(a)\| + Ch^{1-\rho-\rho'} \right) \mathrm{Id}_{\mathcal{H}_{N,0}}$$

implying

$$\mathrm{Op}_N(a) \geq -Ch^{1-\rho-\rho'} \mathrm{Id}_{\mathcal{H}_{N,0}}$$

□

As we discussed before  $\mathrm{Op}_N$  satisfies an exact Egorov relation with the dynamics (cf. [DEGo3]). Namely the restriction of (1.20) to  $\mathcal{H}_N$  means that for every  $\gamma \in \tilde{\Gamma}(2)$ ,

$$\mathcal{M}_N(\gamma)^{-1} \mathrm{Op}_N(a) \mathcal{M}_N(\gamma) = \mathrm{Op}_N(a \circ \gamma), \quad a \in C^\infty(\mathbb{T}^2). \quad (1.25)$$

One can express  $\mathcal{M}_N(\gamma) = \left( \mathcal{M}_N(\gamma)_{jk} \right)_{j,k} \in \mathbb{C}^{N \times N}$  explicitly by number theoretic sums. For instance for  $\gamma_{DE}$ , for every  $N \in 2\mathbb{N} + 1$ , up to some scalar phase factor (its expression as a quotient of Gauss sums is expanded in [KR00] and [DEGo3]),

$$\mathcal{M}_N(\gamma_{DE})_{jk} = \frac{1}{\sqrt{N}} \exp \left( \frac{2\pi i}{N} (k^2 - kj + j^2) \right).$$

We note indeed that the quadratic phase of  $\mathcal{M}_h(\gamma)$  evaluated at points  $y, \eta \in \frac{1}{N}\mathbb{Z}$  is exactly the phase of  $\mathcal{M}_N(\gamma)$ .

# 2

## The complete delocalization of the semiclassical measures

Before we dive to the "raison d'être" of this chapter, we would like to restate our first results concerning delocalization of the semiclassical measures associated to the quantum cat map:

### 2.1 Delocalization statements for Wigner distributions

For sake of completeness let us begin by recalling the Wigner distribution associated to each eigenstate of  $\mathcal{M}_N$ : Let  $f, g \in \mathcal{H}_N, l = (l_1, l_2) \in \mathbb{Z}^2$ . Recall the discrete Fourier—Wigner (FW) transform,

$$\mathcal{V}_N(f_N, g_N)(l_1, l_2) = \left\langle T_{\frac{l}{N}} f_N, g_N \right\rangle_{\mathcal{H}_N} = \frac{e^{\frac{i l_1 l_2}{2N}}}{N} \sum_{k=0}^{N-1} e^{-\frac{i l_2 k}{N}} f_{N, k-l_1} \overline{g_{N, k}}, \quad (2.1)$$

with  $T_{\frac{l}{N}} : \mathcal{H}_N \rightarrow \mathcal{H}_N$  being the restriction of the Weyl—Heisenberg operator (1.4) to  $\mathcal{H}_N$ . In order to pass from  $\text{Op}_h(a)$  to  $\text{Op}_N(a)$  one recalls that  $h = \frac{1}{2\pi N}$  and restricts  $(y^*, \eta^*)$  to lie in  $(2\pi\mathbb{Z})^2$ . We will study the localization of eigenfunctions of  $\mathcal{M}_N(\gamma)$  using  $\mathcal{V}_N$ , or more precisely its Fourier transform, which is the discrete Wigner distribution.

Consider the diagonal matrix coefficients  $\langle \text{Op}_N(a) \varphi_{j,N}, \varphi_{j,N} \rangle$ .

**Definition 2.1.1.** *The distribution  $\mathcal{W}_{j,N}$  defined by  $\langle a, \mathcal{W}_{j,N} \rangle_{\mathcal{D}'(\mathbb{T}^2)} := \langle \text{Op}_N(a) \varphi_{j,N}, \varphi_{j,N} \rangle$  is called the diagonal Wigner distribution. It can be expressed explicitly in terms of  $T_{\frac{l}{N}}$  (cf. [BDB96]). Denoting the Fourier coefficients of  $a$  by  $\check{a} := \{\hat{a}(l)\}_{l \in \mathbb{Z}^2}$ ,*

$$\begin{aligned} \langle \text{Op}_N(a) \varphi_{j,N}, \varphi_{j,N} \rangle &= \left\langle \sum_{l \in \mathbb{Z}^2} \hat{a}(l) T_{\frac{l}{N}} \varphi_{j,N}, \varphi_{j,N} \right\rangle_{\mathcal{H}_N} \\ &= \sum_{(l_1, l_2) \in \mathbb{Z}^2} \int_{\mathbb{T}^2} a(x, \xi; h) e^{-2\pi i(x l_2 - \xi l_1)} dx d\xi \cdot \mathcal{V}_N(\varphi_{j,N}, \varphi_{j,N})(l_1, l_2) \\ &= \langle \check{a}, \check{\mathcal{V}}_N(\varphi_{j,N}, \varphi_{j,N}) \rangle_{\mathbb{C}^{\mathbb{Z}^2}} \end{aligned} \quad (2.2)$$

with  $\check{\mathcal{V}}_N(\varphi_{j,N}, \varphi_{j,N}) := \{\mathcal{V}_N(\varphi_{j,N}, \varphi_{j,N})(l)\}_{l \in \mathbb{Z}^2}$  and  $\check{a}(l)$  being the  $l$ -th symplectic Fourier coefficient

cient, i.e.,

$$\check{a}(l) = \int_{\mathbb{T}^2} a(x, \xi; h) e^{-2\pi i(xl_2 - \xi l_1)} dx d\xi.$$

Since  $\mathcal{V}_N$  has moderate growth and the Fourier coefficients of  $a$  are exponentially decaying this expression is well defined thus applying Plancharel's identity we arrive to a distribution  $\mathcal{W}_{j,N} \in \mathcal{D}'(\mathbb{T}^2)$  having Fourier coefficients

$$\{\widehat{\mathcal{W}_{j,N}}(l)\}_{l \in \mathbb{Z}^2} := \check{\mathcal{V}}_N(\varphi_{j,N}, \varphi_{j,N})$$

.

In fact [HB80] and [BDB96] show that  $\mathcal{W}_{j,N}$  is a linear combination of Dirac peaks on a lattice. One can extract from  $\{\varphi_{j,N}\}$  a sub-sequence  $\{\varphi_{j_k, N_k}\}$  such that  $\mathcal{W}_{j_k, N_k} \rightarrow \mu_{sc}$  as  $N_k \rightarrow \infty$ . That is done in [BDB96] by considering a positive quantization replacing  $\mathcal{W}_{j,N}$  by probability measures. Every such limit  $\mu_{sc}$  is called a semi-classical measure and we say  $\{\mathcal{W}_{j_k, N_k}\}$  converges semi-classically to the measure  $\mu_{sc}$ .

Rather than studying  $\mathcal{W}_{j,N}$  directly we give bounds on an asymptotic expression,

$$\langle \text{Op}_N(a) \varphi_{j,N}, \text{Op}_N(a) \varphi_{j,N} \rangle = \langle \text{Op}_N(|a|^2) \varphi_{j,N}, \varphi_{j,N} \rangle + O(N^{-1}), \quad a \in C^\infty(\mathbb{T}^2).$$

We obtain estimates from below on the left hand side. In turn they yield a lower bound on  $\langle a^2, \mathcal{W}_{j,N} \rangle$  and in fact on the limit of its convergent sub-sequences. We adapt the methods introduced in [DJ18], acquiring a quantitative lower bound on  $\|\text{Op}_N(a) u\|$  for every  $u \in \mathcal{H}_N$ .

**Theorem 2.1.2** (The main theorem of this chapter). *Let  $0 \neq a \in C^\infty(\mathbb{T}^2)$  and  $\gamma \in \tilde{\Gamma}(2)$ . Then there exist constants  $C_1(a)$ ,  $N(a)$  depending only on the choice of  $a$  such that for every  $N \geq N(a)$  and  $u \in \mathcal{H}_N$ ,*

$$\|u\|_{\mathcal{H}_N} \leq C_1(a) \left( \|\text{Op}_N(a) u\|_{\mathcal{H}_N} + \log N \min_{z \in \mathbb{C}: |z|=1} \|(\mathcal{M}_N(\gamma) - z) u\|_{\mathcal{H}_N} \right),$$

From this estimate we deduce Theorem P.3.1

*Proof of Theorem P.3.1.* Fix an open set  $\emptyset \neq \Omega \subseteq \mathbb{T}^2$ . Let  $\{\varphi_{N_j}\}_{j \rightarrow \infty}$  be a subsequence of eigenfunctions of  $\mathcal{M}_{N_j}(\gamma)$  converging semi-classically to  $\mu_{sc}$ . Fix a non-vanishing symbol  $a \in C^\infty(\mathbb{T}^2)$  supported inside  $\Omega$ . Applying Theorem 2.1.2, there exists a constant  $C_1(a) \in \mathbb{R}$  such that  $\|\text{Op}_N(a) \varphi_{j,N}\| \geq C_1^{-1}(a)$  for  $N$  large enough. From the semiclassical convergence

$$\|\text{Op}_N(a) \varphi_{j,N}\|^2 \xrightarrow{N \rightarrow \infty} \int_{\Omega} |a|^2(x, \xi) d\mu_{sc},$$

thus  $\int_{\Omega} |a|^2 d\mu_{sc} \geq C_1(a)^{-2} > 0$  and  $\mu_{sc}(\Omega) \geq C_{\Omega} = C_1(a)^{-2} (\max_{\Omega} |a|)^{-1} > 0$ .  $\square$

Another result of the theorem is Theorem P.3.2:

*Proof of Theorem P.3.2.* Fix  $0 \leq \alpha_1 < \alpha_2 \leq 1$ . Let us take a smoothed characteristic function  $a = a(x) \in C^\infty(\mathbb{T}^2)$  supported inside of  $[\alpha_1, \alpha_2] \times \mathbb{T}$ . Fix an eigenstate of  $\mathcal{M}_N(\gamma)$ ,  $\varphi_N = \sum_k \varphi_{N,k} \delta_{\frac{k}{N} + \mathbb{Z}}$ . We note that in this scenario we can write explicitly  $[\text{Op}_N(a) \varphi_N] = \sum_{k \in \llbracket \alpha_1 N, \alpha_2 N \rrbracket} a\left(\frac{k}{N}\right) \varphi_{N,k} \delta_{\frac{k}{N} + \mathbb{Z}}$ . From Theorem 2.1.2 for these choices of a symbol and an eigenstate there exists a  $N_0 \geq N(a) \in \mathbb{N}^*$  such that for every  $N > N_0$

$$1 = \|\varphi_N\|_{\mathcal{H}_N}^2 \leq C_1^2(a) \|\text{Op}_N(a) \varphi_N\|^2$$

meaning

$$1 \leq \frac{C_1^2(a)}{N} \sum_{k \in \llbracket \alpha_1 N, \alpha_2 N \rrbracket} \left| a\left(\frac{k}{N}\right) \varphi_{N,k} \right|^2 \leq \frac{C_1^2(a)}{N} \sum_{k \in \llbracket \alpha_1 N, \alpha_2 N \rrbracket} |\varphi_{N,k}|^2.$$

□

For the rest of the chapter our central goal is to prove Theorem 2.1.2. The central component required for it is established in Proposition 2.1.6. We first present some geometrical construction inducing a pseudo-differential partition of unity. Constructing from it symbolic dynamics we define a control condition on operators  $A_w$  associated to long words  $w$ . On the controlled region  $\mathcal{Y}$  we apply semi-classical techniques to bound the norm of  $\sum_{w \in \mathcal{Y}} A_w$ . Concerning the uncontrolled region we rely on the fractal uncertainty principle to gain an estimate on the norm of each  $A_w$  separately and then apply combinatorial estimate.

### 2.1.1 Geometric construction and propagated operators

Fix  $a \in C^\infty(\mathbb{T}^2)$  having a non-empty support and consider two proper non-empty open subsets  $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{T}^2$  with  $\mathcal{K}_2 \subset \text{supp}(a)$ . Let  $a_1, a_2 \in C^\infty(\mathbb{T}^2)$  be a couple of symbols satisfying (see Figure 2.5)

$$0 \leq a_\epsilon \leq 1, \quad a_1 + a_2 = 1, \quad \text{supp}(a_1) \subset \text{supp}(a) \quad a_1|_{\mathcal{K}_2} \equiv a_2|_{\mathcal{K}_1} \equiv 1$$

with the corresponding operators  $\text{Op}_N(a_1), \text{Op}_N(a_2)$  satisfying

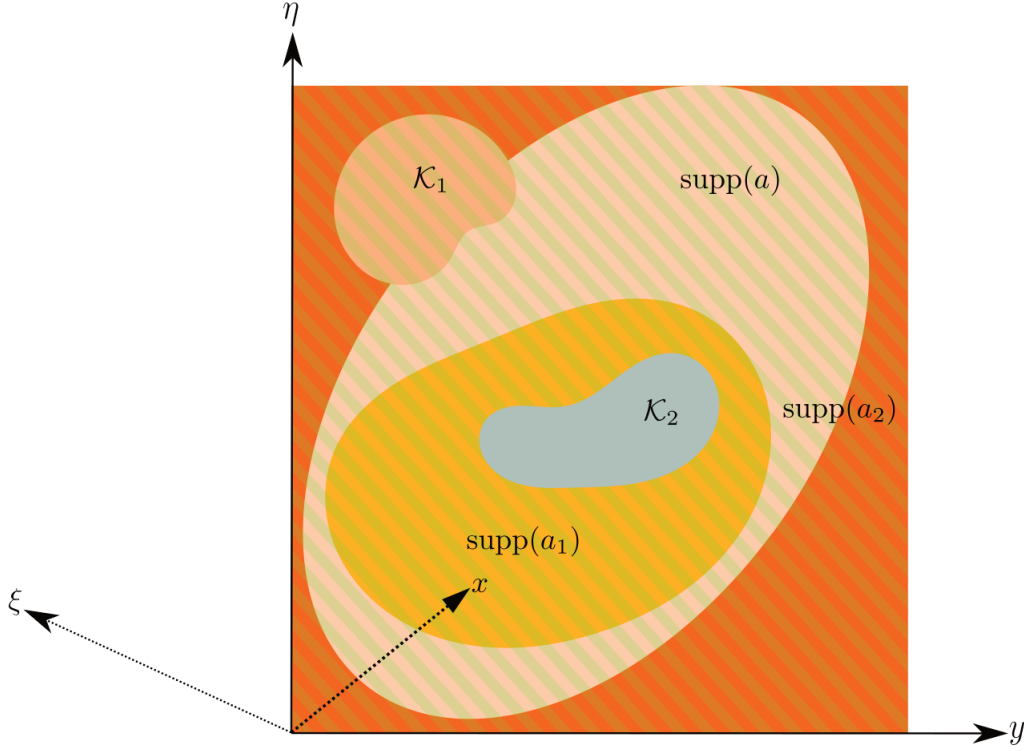
$$A_{1,N} + A_{2,N} = \text{Id}_{\mathcal{H}_N} \quad A_{\epsilon,N} = \text{Op}_N(a_\epsilon). \quad (\star)$$

We can lift our construction to  $\mathbb{R}^2$  by periodization: Quantizing these symbols yields the corresponding pseudo-differential operators  $A_\epsilon = \text{Op}_h(a_\epsilon) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  satisfying

$$a_1 + a_2 = 1, \quad \text{Op}_h(a_1) + \text{Op}_h(a_2) = \text{Id}_{L^2(\mathbb{R})}, \quad a_1|_{\mathcal{K}_2 + \mathbb{Z}^2} \equiv a_2|_{\mathcal{K}_1 + \mathbb{Z}^2} \equiv 1. \quad (\checkmark)$$

By an abuse of notation henceforth we identify  $\mathbb{Z}^2$ -periodic symbols  $a : \mathbb{R}^2 \rightarrow \mathbb{R}$  with their restrictions to  $\mathbb{T}^2$ . Henceforth we fix some  $\rho \in (\frac{1}{2}, 1)$ . Given a pseudo-differential operator  $A = \text{Op}_h(a) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  we use the unitarity of  $\mathcal{M}_N(\gamma)$  in order to propagate with respect to the hyperbolic





**Figure 2.1:** Given a symbol  $a \in C^\infty(\mathbb{T}^2)$  we construct in subsection 2.1.1 a partition of unity on  $\mathbb{T}^2 \cong [0, 1)^2$ .  $\mathbb{R}^2$  is equipped with  $(y, \eta)$  coordinates. The  $(x, \xi)$  coordinates are obtained after applying the coordinate map  $\iota$  to the coordinates  $(y, \eta)$ . The new coordinates describe the decomposition of  $\mathbb{R}^2$  into stable and unstable directions. The yellow domain represents  $\text{supp}(a_1)$  and the domain in green lines represents  $\text{supp}(a_2)$ .

dynamics up to time  $n = 0, \dots, T'$  with

$$T = \left\lfloor \frac{\rho}{4 \log |\lambda_u|} \log \frac{1}{h} \right\rfloor, \quad T' = 4T. \quad (\text{T})$$

Let us denote  $A(n) = \mathcal{M}_h^{-n}(\gamma) A \mathcal{M}_h^n(\gamma)$ . Recalling the exact Egorov relation (1.20),  $A(n) = \text{Op}_h(a \circ \gamma^n)$ . For the restriction  $A_N : \mathcal{H}_N \rightarrow \mathcal{H}_N$  from applying the exact Egorov relation in (1.25),  $A_N(n) = \mathcal{M}_N(\gamma)^{-n} A_N \mathcal{M}_N(\gamma)^n = \text{Op}_N(a \circ \gamma^n)$ . We note that by Lemma 1.5.10  $a \circ \gamma^n \in S_{L^s, \rho^+}(\mathbb{T}^2)$  uniformly in  $0 \leq n < T'$ .

The partition of unity  $(\check{\chi})$  on  $L^2(\mathbb{R})$  together with the evolution through  $M_h(\gamma)$  allows to construct refined partitions of unity, in terms of "word operators": to every word  $w \in \mathcal{W}(n) = \{1, 2\}^n = \{w = w_0 \dots w_{n-1} : w_j \in \{1, 2\}\}$  (where  $0 \leq n \leq T'$ ) corresponds an operator on  $L^2(\mathbb{R})$

$$\begin{aligned} A_w &= A_{w_{n-1}}(n-1) A_{w_{n-2}}(n-2) \cdots A_{w_1}(1) A_{w_0} \\ &= \text{Op}_h(a_{w_{n-1}} \circ \gamma^{n-1}) \cdots \text{Op}_h(a_{w_1} \circ \gamma) \text{Op}_h(a_{w_0}) \end{aligned}$$

as well as an operator on  $\mathcal{H}_N$ ,  $A_{w,N} = (A_w)_N$ . Suppose  $B = \text{Op}_h(b)$  with  $b \in S_{L^s, \rho, 0}(\mathbb{R}^2)$  we denote the "tilted" operator  $\tilde{B} = \text{Op}_h(\tilde{b}) = \mathcal{M}_h^{-1}(\iota) B \mathcal{M}_h(\iota)$  having from Lemma 1.5.10 the "tilted" symbol  $\tilde{b} \in S_{L^s, \rho, 0}(\mathbb{R}^2)$ .

**Lemma 2.1.3.**

1. Let  $0 \leq n \leq 4T$  then  $a_w \in S_{L^s, \rho_+}(\mathbb{R}^2)$  and  $A_w = \text{Op}_h(a_w) + O_{\mathcal{B}(L^2(\mathbb{R}))}(h^{1-\rho_-})$  where  $a_w = \prod_{j=0}^{n-1} a_{w_j} \circ \gamma^j$  uniformly in  $n$  and in  $w \in \mathcal{W}(n)$ .
2. Let  $-4T \leq n \leq 0$  then  $a_w \in S_{L^u, \rho_+}(\mathbb{R}^2)$  and  $A_w = \text{Op}_h(a_w) + O_{\mathcal{B}(L^2(\mathbb{R}))}(h^{1-\rho_-})$  where  $a_w = \prod_{j=n-1}^0 a_{w_j} \circ \gamma^j$  uniformly in  $n$  and in  $w \in \mathcal{W}(n)$ .
3. If  $0 \leq n \leq T$  then  $a_w \in S_{L^s, \frac{\rho}{4}+}(\mathbb{R}^2)$  and  $A_w = \text{Op}_h(a_w) + O_{\mathcal{B}(L^2(\mathbb{R}))}(h^{1-\frac{\rho}{4}-})$  where  $a_w = \prod_{j=0}^{n-1} a_{w_j} \circ \gamma^j$  uniformly in  $n$  and in  $w \in \mathcal{W}(n)$ .
4. If  $-T \leq n \leq 0$  then  $a_w \in S_{L^u, \frac{\rho}{4}+}(\mathbb{R}^2)$  and  $A_w = \text{Op}_h(a_w) + O_{\mathcal{B}(L^2(\mathbb{R}))}(h^{1-\frac{\rho}{4}-})$  where  $a_w = \prod_{j=n-1}^0 a_{w_j} \circ \gamma^j$  uniformly in  $n$  and in  $w \in \mathcal{W}(n)$ .

*Proof.* We prove only the first statement in the lemma as (iii) follows by replacing  $\rho$  by  $\frac{\rho}{4}$  and the others by exchanging the stable and unstable directions. Consider the tilted operator  $\tilde{A}_w$  which from applying (1.20) can be expressed as

$$\tilde{A}_w = \text{Op}_h \left( \widetilde{a_{w_{n-1}} \circ \gamma^{n-1}} \right) \cdots \text{Op}_h \left( \widetilde{a_{w_1} \circ \gamma} \right) \text{Op}_h \left( \widetilde{a_{w_0}} \right).$$

Let us now show that  $\tilde{A}_w$  can be expressed as  $\tilde{A}_w = \text{Op}_h(\tilde{a}_w) + O(h^{1-\rho_-})$  where

$$\tilde{a}_w = \prod_{j=0}^{n-1} b_j := \prod_{j=0}^{n-1} a_{w_j} \circ \gamma^j \circ \iota \in S_{L^s, \rho_+}(\mathbb{R}^2). \quad (2.6)$$

We note that from applying Leibniz's chain rule  $\partial_y^\alpha \partial_\eta^\beta \tilde{a}_w$  is a sum of  $n^{\alpha+\beta} = O(h^-)$  summands, each of the form

$$d(y, \eta) = (\partial_y^{\alpha_0} \partial_\eta^{\beta_0} b_0) \cdots (\partial_y^{\alpha_{n-1}} \partial_\eta^{\beta_{n-1}} b_{n-1})(y, \eta) \quad (2.7)$$

with  $\sum_j \alpha_j = \alpha$ ,  $\sum_k \beta_k = \beta$ . By Lemma 1.5.10  $b_j \in S_{L^s, \rho, 0}(\mathbb{R}^2)$  hence each factor in the product (2.7) is bounded by  $C_{\alpha_j, \beta_j} h^{-\rho \alpha_j}$  uniformly in  $w$  and in  $j$ . However, since the number of derivatives, with respect either to  $y$  or to  $\eta$  does not exceed  $\alpha + \beta$  and since  $|b_j| \leq 1$ ,

$$|d(y, \eta)| \leq \prod_{j: (\alpha_j, \beta_j) \neq 0} C_{\alpha_j, \beta_j} h^{-\rho \sum \alpha_j} \leq \max_j C_{\alpha_j, \beta_j}^{\alpha+\beta} \cdot h^{-\rho \alpha}. \quad (2.8)$$

We remind  $\alpha_j \in \llbracket 0, \alpha \rrbracket$ ,  $\beta_j \in \llbracket 0, \beta \rrbracket$  and that  $C_{\alpha_j, \beta_j}$  depend only on the number of derivatives taken. As follows, for every  $\alpha, \beta$ ,  $\max_j C_{\alpha_j, \beta_j}^{\alpha+\beta}$  is uniformly bounded by some constant  $C$ . Since there are

$O(h^{-\varepsilon})$  terms, each of them bounded by some  $Ch^{-\alpha\rho}$  we deduce that  $|\partial_y^\alpha \partial_\eta^\beta \widetilde{a}_w| \leq Ch^{-\alpha\rho-}$  and  $\widetilde{a}_w \in S_{L^\varepsilon, \rho+}(\mathbb{R}^2)$

We prove that uniformly with respect to the word  $w \in \mathcal{W}(n)$  where  $1 \leq n \leq 4T$ ,

$$\widetilde{A}_w - \text{Op}_h(\widetilde{a}_w) = O_{\mathcal{B}(L^2(\mathbb{R}))}(h^{1-\rho-}). \quad (2.9)$$

It can be verified by induction on  $1 \leq n \leq 4T$  that

$$\widetilde{A}_w - \text{Op}_h(\widetilde{a}_w) = \sum_{k=1}^{n-1} \text{Op}_h(b_{w_{n-1}}) \cdots \text{Op}_h(b_{w_{k+1}}) B_k, \quad (2.10)$$

where

$$B_j = \text{Op}_h(b_{j-1}) \text{Op}_h\left(\prod_{k=0}^{j-2} b_k\right) - \text{Op}_h\left(\prod_{k=0}^{j-1} b_k\right).$$

We remind that  $\prod_{k=0}^{j-2} b_k \in S_{L^\varepsilon, \rho+}$  uniformly with respect to  $j \in \llbracket 0, n-1 \rrbracket$ . Let us show a bound on the norm of this operator: We first note that for every  $1 \leq j \leq 4T-1$ , from the Moyal product in  $S_{L^\varepsilon, \rho+}(\mathbb{R}^2)$  (from Lemma 1.6.1.ii) we deduce that  $\|B_j\|_{\mathcal{B}(L^2(\mathbb{R}))} = O(h^{1-\rho-})$  uniformly in  $j$ . Then following Lemma 1.6.3 we deduce the estimate

$$\|\text{Op}_h(b_{n-1}) \cdots \text{Op}_h(b_{j+1}) B_j\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq C(1 + C''h^{1-\rho})^{C' \log \frac{1}{h}} h^{1-\rho-}$$

for constants  $C, C''$  independent of  $j$  and  $n$ . As  $h \rightarrow 0$

$$\log(1 + C_k h^{1-\rho})^{C' \log \frac{1}{h}} = -C' \log h \cdot \log(1 + C_k h^{1-\rho}) = O(h^{1-\rho} \log h) \rightarrow 0$$

thus  $\frac{\|\text{Op}_h(b_{w_{n-1}}) \cdots \text{Op}_h(b_{w_{k+1}}) B_k\|}{h^{1-\rho-\varepsilon}}$  remains uniformly bounded as  $h \rightarrow 0$ . As the sum is over  $n$  operators of this form, we deduce from the triangle inequality that

$$\|\widetilde{A}_w - \text{Op}_h(\widetilde{a}_w)\| \leq C'' h^{1-\rho-}.$$

With  $C''$  being uniform with respect to  $w$  and  $1 \leq n \leq 4T$ . □

Let us now consider operators which correspond to shorter "words", of length  $T$ . For every function  $c : \mathcal{W}(T) \rightarrow \mathbb{R}$  (with  $T$  defined in (T)) such that  $\sup |c(w)| \leq 1$  corresponds the weighted operator  $A_c = \sum_{w \in \mathcal{W}(T)} c(w) A_w$ . It is a pseudo-differential operator with a principal symbol  $a_c = \sum_{w \in \mathcal{W}(T)} c(w) a_w$  or equivalently,

**Lemma 2.1.4.** *Suppose  $\sup_{w \in \mathcal{W}(T)} |c(w)| \leq 1$  then  $a_c \in S_{L^s, \frac{1}{2}, \frac{1}{4}}(\mathbb{R}^2)$  and*

$$A_c = \text{Op}_h(a_c) + O\left(h^{1-\frac{\rho}{4}} \left(1 + \frac{\log 2}{\log |\lambda u|}\right)^{-\varepsilon}\right).$$

*Proof.* We first observe that  $a_c$  is bounded uniformly

$$|a_c| = \left| \sum_{w \in \mathcal{W}(T)} c(w) a_w \right| \leq \sum_{w \in \mathcal{W}(T)} |c(w)| a_w \leq \sum_{w \in \mathcal{W}(T)} a_w = 1.$$

We estimate  $\left| \partial_x^\alpha \partial_\xi^\beta \tilde{a}_c \right|$  for every  $\alpha, \beta \in \mathbb{N}$ . First, by the second part of Lemma 2.1.3,  $\tilde{a}_w \in S_{L^\varepsilon, \frac{\rho}{4}+}(\mathbb{R}^2)$  and recalling Lemma 1.5.9, if  $(\alpha, \beta) \neq (0, 0)$

$$\begin{aligned} \left| \partial_x^\alpha \partial_\xi^\beta \tilde{a}_c \right| &\leq \sum_{w \in \mathcal{W}(T)} \left| \partial_x^\alpha \partial_\xi^\beta \tilde{a}_w \right| \\ &\leq \sum_{w \in \mathcal{W}(T)} C_{\alpha, \beta} h^{-\frac{\rho}{4}\alpha - \varepsilon} \\ &\leq 2^T C_{\alpha, \beta} h^{-\frac{\rho}{4}\alpha - \varepsilon} \end{aligned}$$

Recalling Lemma 2.1.3,  $C_{\alpha, \beta}$  is uniform. We recall Corollary 1.1.3 implying  $|\lambda_u| \geq 2 + \sqrt{3}$ ,  $2^T \approx h^{-\frac{\rho \log 2}{4 \log |\lambda_u|}} \leq h^{-\frac{\rho \log 2}{4 \log |2 + \sqrt{3}|}} \leq h^{-\frac{\rho}{4} + \rho \varepsilon_0}$  for some  $\varepsilon_0 > 0$  and hence taking  $\varepsilon < \frac{1-\rho}{4}$  and  $\varepsilon_0$  to be small enough

$$\left| \partial_x^\alpha \partial_\xi^\beta \tilde{a}_c \right| \leq C_{\alpha, \beta} h^{-\frac{\rho}{4} - \frac{\rho}{4}\alpha + \rho \varepsilon_0 - \varepsilon} \leq h^{-\rho \frac{\alpha}{2} - \rho \frac{\beta}{4}}$$

If  $\alpha = \beta = 0$ ,  $|\partial_x^\alpha \partial_\xi^\beta \tilde{a}_c| = |\tilde{a}_c| \leq 1$ . We deduce that  $\tilde{a}_c \in S_{L^\varepsilon, \frac{1}{2}, \frac{1}{4}}(\mathbb{R}^2)$  and therefore  $a_c \in S_{L^s, \frac{1}{2}, \frac{1}{4}}(\mathbb{R}^2)$ . The remainder follows from applying Lemma 2.1.3.iii,

$$A_c = \sum_{w \in \mathcal{W}(T)} c(w) A_w = \sum_{w \in \mathcal{W}(T)} c(w) \left( \text{Op}_h(a_w) + O\left(h^{1-\frac{\rho}{4}-\varepsilon}\right) \right).$$

The sum is over  $2^T$  terms thus the reminder is of order  $2^T h^{1-\frac{\rho}{4}-\varepsilon} \leq h^{1-\frac{\rho}{4}(1+\frac{\log 2}{\log |\lambda_u|})-\varepsilon}$   $\square$

Henceforth for every subset  $S \subset \mathcal{W}(T)$  we denote  $A_{S, N} := (A_{\mathbb{1}_S})_N = \sum_{w \in S} A_{w, N}$  where  $\mathbb{1}_S$  is the characteristic function of the set  $S$ . Fix a small  $\delta \in (0, 1)$  that will be specified later in (2.11), and define the control function

$$F : \mathcal{W}(T) \rightarrow \mathbb{R}, \quad F(w) = \frac{\#\{j \in \{0, \dots, T-1\} : w_j = 1\}}{T}. \quad (\text{F})$$

Roughly speaking, in the notations of  $(\star)$ ,  $F$  quantifies the fraction of time a word  $w \in \mathcal{W}(T)$  "spends inside"  $\text{supp}(a_1)$ . Therefore the set

$$\mathcal{Z} = \{w \in \mathcal{W}(T) : F(w) \geq \delta\}$$

contains the "controlled short logarithmic" words, i.e., those who have corresponding symbols spending a fraction  $\delta$  of time supported inside  $\text{supp}(a_1)$ . This definition opens the door for obtaining a

finer pseudo-differential partition of unity. We would like to apply the fractal uncertainty principle (see section 2.6 below) hence we consider a partition arising from  $\mathcal{W}(8T)$  (with  $T$  from (T)), words twice longer than Ehrenfest time  $\frac{\log \frac{1}{h}}{\log |\lambda_u|} \approx T'$ . The set  $\mathcal{W}(8T)$  can be divided according to the control function  $F$  to  $\mathcal{W}(8T) = \mathcal{X} \sqcup \mathcal{Y}$  with

$$\begin{aligned}\mathcal{X} &= \{w^{(1)}w^{(2)} \dots w^{(8)} \in \mathcal{W}(8T) : w^{(j)} \in \mathcal{W}(T) \setminus \mathcal{Z}, \forall j\} \\ \mathcal{Y} &= \{w^{(1)}w^{(2)} \dots w^{(8)} \in \mathcal{W}(8T) : \forall k w^{(k)} \in \mathcal{W}(T) \text{ and } \exists j, w^{(j)} \in \mathcal{Z}\}.\end{aligned}$$

$\mathcal{Y}$  is the set of "long logarithmic controlled words": words such that at least a single subword  $w^{(j)}$  of them is controlled.  $\mathcal{X}$  is the set of uncontrolled words: the time they spend inside this fixed subset of  $\text{supp}(a_1)$  can't be bounded from below.

### 2.1.2 Outline of the proof

The proof of Theorem 2.1.2 is based on two key estimates: First we estimate the mass  $\|A_{\mathcal{Y},Nu}\|$ , i.e., coming from "controlled" words,

**Proposition 2.1.5.** *There are constants  $C, C_1, C_2 \in \mathbb{R}$  satisfying for every  $N \geq 1$  and  $u \in \mathcal{H}_N$ ,*

$$\|A_{\mathcal{Y},Nu}\|_{\mathcal{H}_N} \leq c_\delta(\gamma) \left( C_1 \|\text{Op}_N(a)u\| + \log N \min_{z \in \mathbb{C}: |z|=1} \|(\mathcal{M}_N(\gamma) - z)u\|_{\mathcal{H}_N} + \frac{C_2}{N^{\frac{1}{8}}} \|u\|_{\mathcal{H}_N} \right)$$

with  $c_\delta(\gamma) = \frac{C}{\delta \log |\lambda_u|}$ .

The proof of this proposition is done in subsection 2.1.3 below and is relying on the almost-monotonicity property of the calculus in Lemma 2.1.11. Then we turn to estimate the norm of operators which corresponds to uncontrolled words  $w \in \mathcal{X}$ . We estimate the norm of each  $A_{w,N}u$  (for every  $w \in \mathcal{X}$ ) separately by an argument involving the fractal uncertainty principle from [BD18],

**Proposition 2.1.6.** *There exists  $\beta' > 0$ , a constant  $C > 0$  such that for every  $N \in \mathbb{N}^*$*

$$\sup_{w \in \mathcal{W}(8T)} \|A_{w,N}\|_{\mathcal{B}(\mathcal{H}_N)} \leq \frac{C}{N^{\beta'}},$$

with  $T$  from (T)..

Then, we invoke a combinatorial argument from [DJ18],

**Lemma 2.1.7** (Lemma 3.3 in [DJ18]). *There is some  $C$  (that might depend on  $\delta$ ) such that for  $\delta$  is small enough and for every  $h$ ,  $\#\mathcal{X} \leq Ch^{-4\sqrt{\delta}}$ .*

Note that  $N_0$  appearing in the proof of [DJ18, Lemma 3.3] is larger than  $T$  specified in (T) above, yet one can deduce Lemma 2.1.7 from [DJ18, Lemma 3.3].

*End of the proof of Theorem 2.1.2.* Decomposing  $\text{Id}_{\mathcal{H}_N} = A_{\mathcal{X},N} + A_{\mathcal{Y},N}$  and applying the triangle inequality

$$\|u\|_{\mathcal{H}_N} \leq \|A_{\mathcal{X},N}u\| + \|A_{\mathcal{Y},N}u\|.$$

From Lemma 2.1.7, Lemma 1.3.3.ii and Proposition 2.1.6,  $\|A_{\mathcal{X},N}u\| \leq CN^{-\beta'+4\sqrt{\delta}} \|u\|$ . Combined with Proposition 2.1.5, it yields

$$\|u\| \leq CN^{-\beta'+4\sqrt{\delta}} \|u\| + \frac{C_1}{\delta} \|\text{Op}_N(a)u\| + \frac{C_2}{\delta} \log N \min_{z \in \mathbb{C}: |z|=1} \|(\mathcal{M}_N(\gamma) - z)u\| + \frac{C_3}{N^{\frac{1}{8}}} \|u\|.$$

Choosing  $\delta$  such that

$$\beta' - 4\sqrt{\delta} = \frac{\beta'}{4} \tag{2.11}$$

and picking  $N$  big enough eliminates the first term which in turn implies the result.  $\square$

We dedicate the rest of the chapter for proving Proposition 2.1.5 and Proposition 2.1.6

2.1.3 Obtaining the estimate on the controlled region  $\mathcal{Y}$  from Proposition 2.1.5

We first bound from above  $\|\text{Op}_N(a_1)\|$ .

**Lemma 2.1.8.** *There exists some  $C'_1 \in \mathbb{R}$  such that for every  $u \in \mathcal{H}_N$*

$$\|\text{Op}_N(a_1)u\|_{\mathcal{H}_N} \leq C'_1 \|\text{Op}_N(a)u\|_{\mathcal{H}_N} + O(N^{-\infty}) \|u\|_{\mathcal{H}_N}.$$

*Proof.* First we recall that  $\text{supp}(a_1) \subset \text{supp}(a)$  and thus the function  $\frac{a_1}{a}$  is well defined and one can construct a parametrix recursively. We construct a function  $q \in C^\infty(\mathbb{T}^2)$  satisfying  $q \#_h a = a_1 + O_{C^\infty(\mathbb{T}^2)}(N^{-\infty})$  for every  $(x, \xi) \in (\text{supp}(a))^\circ$ . Expressing by a formal ansatz  $q(x, \xi) = \sum_{j=0}^{\infty} q_j h^j$  and using the expansion of the Moyal product in the left hand side we obtain the values of sequence of functions  $\{q_j\}_{j=0}^{\infty}$ . For every  $j \in \mathbb{N}$  one can write explicitly  $q_j = \frac{p(a, a_1, q_0, \dots, q_{j-1})}{C_j a}$  for a constant  $C_j \in \mathbb{R}$  and  $p \in \mathbb{R}[x_1, \dots, x_{j+1}]$ . Inductively  $a_1, q_0, \dots, q_{j-1}$  are all continuous functions supported inside  $(\text{supp}(a))^\circ$  and thus  $q_j$  is a well defined function on  $\mathbb{T}^2$ . By Borel's lemma one can indeed construct a symbol  $q \sim \sum q_j h^j$  and we obtain

$$\text{Op}_N(a_1) = \text{Op}_N(q) \text{Op}_N(a) + O(N^{-\infty}).$$

We deduce

$$\begin{aligned} \|\text{Op}_N(a_1)u\| &\leq \|\text{Op}_N(q) \text{Op}_N(a)u\| + O(N^{-\infty}) \|u\| \\ &\leq C_1 \|\text{Op}_N(a)u\| + O(N^{-\infty}) \|u\|, \end{aligned}$$

obtaining the lemma.  $\square$

Next, controlling the growth of  $A_N u$  implies control on the growth of  $A_N(m)u$  for  $u \in \mathcal{H}_N$ ,

**Lemma 2.1.9.** *Let  $A_N : \mathcal{H}_N \rightarrow \mathcal{H}_N$  be a bounded operator, then for all  $m \in \mathbb{Z}$ ,*

$$\|A_N(m)u\|_{\mathcal{H}_N} \leq \|A_N u\|_{\mathcal{H}_N} + |m| \|A_N\|_{\mathcal{B}(\mathcal{H}_N)} \min_{z \in \mathbb{C}: |z|=1} \|(\mathcal{M}_N(\gamma) - z)u\|_{\mathcal{H}_N},$$

where  $u \in \mathcal{H}_N$ .

This lemma is the discrete analogue of propagation of singularities for long time (cf. [DZ19, Theorem E.47]).

*Proof.* Without loss of generality assume that  $m \in \mathbb{N}$ . Let  $z \in \mathbb{C}$  having  $|z| = 1$ . From the triangle inequality,

$$\begin{aligned} \|A_N(m)u\|_{\mathcal{H}_N} &= \|A_N \mathcal{M}_N^m(\gamma)u\|_{\mathcal{H}_N} \leq \|A_N u\|_{\mathcal{H}_N} + \|(A_N \mathcal{M}_N^m(\gamma) - z^m A_N)u\| \\ &\leq \|A_N u\|_{\mathcal{H}_N} + \|A_N\|_{\mathcal{B}(\mathcal{H}_N)} \|(\mathcal{M}_N^m(\gamma) - z^m \text{Id}_{\mathcal{H}_N})u\|. \end{aligned}$$

Since  $\mathcal{M}_N^m(\gamma) - z^m \text{Id}_{\mathcal{H}_N} = \sum_{k=1}^m z^k \mathcal{M}_N^{m-k}(\gamma) (\mathcal{M}_N(\gamma) - z \text{Id}_{\mathcal{H}_N})$  and since  $\mathcal{M}_N(\gamma)$  is a unitary operator,

$$\|(\mathcal{M}_N^m(\gamma) - z^m \text{Id}_{\mathcal{H}_N})u\| \leq m \|(\mathcal{M}_N(\gamma) - z)u\|$$

and taking a  $z$  minimizing  $\|(\mathcal{M}_N(\gamma) - z \text{Id}_{\mathcal{H}_N})u\|_{\mathcal{H}_N}$  proves the proposition.  $\square$

Combining Lemma 2.1.8 and Lemma 2.1.9 gives an estimate on the norm of  $A_{1,N}(m) = \text{Op}_N(a_1 \circ \gamma^m)$ ,

**Lemma 2.1.10.** *There exists  $C'_1 \in \mathbb{R}$  such that for every  $m \in \mathbb{Z}$ ,*

$$\|A_{1,N}(m)u\| \leq C'_1 \|A_N u\| + |m| \|A_{1,N}\|_{\mathcal{B}(\mathcal{H}_N)} \min_{z \in \mathbb{C}: |z|=1} \|(\mathcal{M}_N(\gamma) - z)u\|_{\mathcal{H}_N} + O(N^{-\infty}) \|u\|_{\mathcal{H}_N}.$$

$A_{y,N}$  corresponds to a long logarithmic time (precisely  $2T'$  is asymptotically twice the Ehrenfest time), thus the symbols  $a_w$  are no longer in a comfortable symbol class. To solve this issue we first introduce estimates on the norm of  $A_{z,N}$  which is the sum of operators corresponding to the shorter time  $T$ , for which the pseudodifferential calculus still holds. We recall the almost-monotonicity property from [DJ18, lemma 4.5] which holds for operators  $A_{c,N} := (A_c)_N$ ,

**Lemma 2.1.11** (Almost-monotonicity property). *Let  $T$  from (T). Assume that  $c_1, c_2 : \mathcal{W}(T) \rightarrow \mathbb{R}$  and for every  $w \in \mathcal{W}(T)$ ,  $|c_1(w)| \leq c_2(w) \leq 1$ . Then there exists a constant  $C \in \mathbb{R}$  independent of  $N, c_1$  and  $c_2$  satisfying for every  $N \geq 1$  and every  $u \in \mathcal{H}_N$ ,*

$$\|A_{c_1, N} u\|_{\mathcal{H}_N} \leq \|A_{c_2, N} u\|_{\mathcal{H}_N} + C N^{-\frac{1}{8}} \|u\|_{\mathcal{H}_N},$$

with  $C \in \mathbb{R}$  independent of  $c_1, c_2$ .

*Proof.* The proof follows along the lines of lemma 4.5 in [DJ18]. First, note that plugging Corollary 1.1.3 into Lemma 2.1.4, we can write and restricting to  $\mathcal{H}_N$ ,

$$\text{Op}_N(a_{c_1}) = A_{c_1, N} + O\left(h^{\frac{1}{4}}\right) \quad \text{Op}_N(a_{c_2}) = A_{c_2, N} + O\left(h^{\frac{1}{4}}\right).$$

Thus it is sufficient to prove that

$$\|\mathrm{Op}_N(a_{c_1})u\|_{\mathcal{H}_N}^2 \leq \|\mathrm{Op}_N(a_{c_2})u\|_{\mathcal{H}_N}^2 + CN^{-\frac{1}{4}}\|u\|_{\mathcal{H}_N}^2$$

which is equivalent to

$$\langle \mathcal{A}u, u \rangle_{\mathcal{H}_N} \geq -CN^{-\frac{1}{4}}\|u\|^2$$

with

$$\mathcal{A} = \mathrm{Op}_N(a_{c_2})^* \mathrm{Op}_N(a_{c_2}) - \mathrm{Op}_N(a_{c_1})^* \mathrm{Op}_N(a_{c_1}).$$

From Lemma 1.6.2,  $\mathcal{A} = \mathrm{Op}_N(|a_{c_2}|^2 - |a_{c_1}|^2) + O(N^{-\frac{1}{4}})$ . Since  $|c_1(w)| \leq c_2(w)$  we have  $0 \leq |a_{c_2}|^2 - |a_{c_1}|^2 \in S_{L^s, \frac{1}{2}, \frac{1}{4}}$ . Applying Gårding inequality from Lemma 1.7.7,

$$\langle \mathrm{Op}_N(|a_{c_2}|^2 - |a_{c_1}|^2)u, u \rangle \geq -CN^{-\frac{1}{4}}\|u\|$$

from which we deduce the lemma.  $\square$

Using the property we prove

**Proposition 2.1.12.** *There exists constants  $C'_1, C'_2 \in \mathbb{R}$ , independent of  $\delta$ , such that for every  $u \in \mathcal{H}_N$ ,*

$$\|A_{\mathcal{Z}, N}u\|_{\mathcal{H}_N} \leq \frac{C'_1}{\delta \log |\lambda_u|} \|\mathrm{Op}_N(a)u\| + \frac{\log N}{\delta \log |\lambda_u|} \min_{z \in \mathbb{C}: |z|=1} \|(\mathcal{M}_N(\gamma) - z)u\|_{\mathcal{H}_N} + \frac{C'_2}{N^{\frac{1}{8}}\delta} \|u\|_{\mathcal{H}_N}. \quad (2.12)$$

*Proof.* The indicator function  $\mathbb{1}_{\mathcal{Z}}$  satisfies  $0 \leq \delta \mathbb{1}_{\mathcal{Z}} \leq F \leq 1$ , with  $F$  defined in (F), hence from the almost-monotonicity property in Lemma 2.1.11,

$$\delta \|A_{\mathcal{Z}, N}u\|_{\mathcal{H}_N} \leq \|A_{F, N}u\|_{\mathcal{H}_N} + O\left(N^{-\frac{1}{8}}\right) \|u\|_{\mathcal{H}_N}. \quad (2.13)$$

We note that

$$A_{F, N} = \frac{1}{T} \sum_{j=0}^{T-1} \sum_{w \in \mathcal{W}(T), w_j=1} A_{w, N} = \frac{1}{T} \sum_{j=0}^{T-1} A_{1, N}(j), \quad (2.14)$$

From (2.13) and (2.14),

$$\delta \|A_{\mathcal{Z}, N}u\| \leq \|A_{F, N}u\| + O\left(N^{-\frac{1}{8}}\right) \|u\| \leq \frac{1}{T} \sum_{j=0}^{T-1} \|A_{1, N}(j)u\| + O\left(N^{-\frac{1}{8}}\right) \|u\|_{\mathcal{H}_N}$$

which by Lemma 2.1.10 implies (2.12).  $\square$

To finish the proof of Proposition 2.1.5 we connect between short logarithmic times and long logarithmic ones.



*Proof of Proposition 2.1.5.* We observe that  $\mathcal{Y} = \bigsqcup_{j=1}^8 \mathcal{Y}_j$  with

$$\mathcal{Y}_j = \left\{ w = w^{(1)} \dots w^{(8)} \in \mathcal{W}(8T) : \begin{array}{l} w^{(1)}, \dots, w^{(j-1)} \in \mathcal{W}(T) \\ w^{(j)} \in \mathcal{Z} \\ w^{(j+1)}, \dots, w^{(8)} \in \mathcal{W}(T) \setminus \mathcal{Z} \end{array} \right\}.$$

Since  $\sum_{w \in \mathcal{W}(n)} A_{w,N} = \text{Id}_{\mathcal{H}_N}$  for every  $1 \leq n \leq 8T$

$$\begin{aligned} A_{\mathcal{Y}_j,N} &= \sum_{w=w^{(1)} \dots w^{(8)} \in \mathcal{Y}_j} A_{w^{(1)} \dots w^{(8)},N} \\ &= \sum_{w \in \mathcal{Y}_j} A_{w^{(8)},N} (7T) \cdots A_{w^{(j+1)},N} ((j+1)T) A_{w^{(j)},N} (jT) A_{w^{(1)} \dots w^{(j-1)},N} \\ &= A_{\mathcal{W}(T) \setminus \mathcal{Z},N} (7T) \cdots A_{\mathcal{W}(T) \setminus \mathcal{Z},N} ((j+1)T) A_{\mathcal{Z},N} (jT) \sum_{w^{(1)}, \dots, w^{(j-1)} \in \mathcal{W}(T)} A_{w^{(1)} \dots w^{(j-1)},N} \\ &= A_{\mathcal{W}(T) \setminus \mathcal{Z},N} (7T) \cdots A_{\mathcal{W}(T) \setminus \mathcal{Z},N} ((j+1)T) A_{\mathcal{Z},N} (jT). \end{aligned}$$

We note that applying Lemma 2.1.11 there exist constants  $C, C' > 0$ , independent of  $N$ , such that

$$\|A_{\mathcal{W}(T) \setminus \mathcal{Z},N}\| \leq \|\text{Id}_{\mathcal{H}_N}\|_{\mathcal{H}_N} + C' N^{-\frac{1}{8}} \leq C$$

and therefore for every function  $u \in \mathcal{H}_N$ ,

$$\|A_{\mathcal{Y},N}u\|_{\mathcal{H}_N} \leq C \sum_{j=1}^8 \|A_{\mathcal{Y}_j,N}u\|_{\mathcal{H}_N} \leq 8C \|A_{\mathcal{Z},N}u\|_{\mathcal{H}_N}.$$

From Proposition 2.1.12, there are constants  $C_1, C_2 \in \mathbb{R}$  such that

$$\|A_{\mathcal{Y},N}u\| \leq \frac{C_1}{\delta \log |\lambda_u|} \|\text{Op}_N(a)u\| + 16C \frac{\log N}{\delta \log |\lambda_u|} \min_{z \in \mathbb{C}: |z|=1} \|(\mathcal{M}_N(\gamma) - z)u\|_{\mathcal{H}_N} + \frac{C_2}{N^{\frac{1}{8}}} \|u\|_{\mathcal{H}_N}.$$

□

## 2.2 $\nu$ -porous sets used for proving Proposition 2.1.6

The next two sections are dedicated to deducing Proposition 2.1.6 from a version of the fractal uncertainty principle presented in [DJ18]. We first prove a similar result for the Weyl quantization  $\text{Op}_h$  acting on  $L^2(\mathbb{R})$  using the minimality of the horocyclic flow<sup>1</sup> and then pass to  $\text{Op}_N$  in virtue of Lemma 1.3.3.ii. We then deduce the proposition from combining the fractal uncertainty principle Proposition 2.6.2 with a Cotlar-Stein argument.

<sup>1</sup>We supply an additional number theoretic proof for sake of completeness

### 2.3 Introducing a partition of unity by smooth cut-offs on $\mathbb{R}^2$

We equip  $\mathbb{R}^2$  with the  $(y, \eta)$  coordinates. We recall that  $\iota$  is a change of coordinates from the  $(y, \eta)$ -coordinates to  $(x, \xi)$ -coordinates. In order to simplify the proofs we consider "twisted" symbols  $b_{\epsilon, t}$  in (2.17) supported on sets of the form  $\iota^{-1}U$  for  $U \subset \mathbb{R}^2$  being a closed set. We construct a smooth partition of unity on  $\mathbb{R}^2$ : We tile  $\mathbb{R}^2$  by fundamental cells of the lattice  $\iota^{-1}\mathbb{Z}^2$ : Recall Minkowski's first theorem

**Theorem 2.3.1** (Minkowski's first theorem, Theorem 10 in [Sie13]). *Let  $S$  be a convex symmetric body in  $\mathbb{R}^n$  centered at the origin (i.e.,  $x \in S \iff -x \in S$ ) and let  $L$  be a lattice of determinant  $\det(L)$ . If  $\text{Vol}(S) > 2^n \det(L)$  then  $S$  contains a non-zero  $L$ -point.*

Consider the elliptic domain  $E = \{(y, \eta) : \frac{y^2}{4} + \eta^2 \leq 1\}$  being convex and centered at  $O = (0, 0)$ . Since its area is  $2\pi > 2^2 \det(\iota^{-1}) = 4$  we can apply Minkowski's theorem deducing that it contains a non-zero lattice point  $P = (y_P, \eta_P) \in \iota^{-1}\mathbb{Z}^2$ . If  $E$  contains more than a single lattice point we pick one which minimizes the  $\|\cdot\|_2$ -norm and thus one can write  $P = \iota^{-1}\begin{pmatrix} n \\ m \end{pmatrix}$  for two coprime  $n, m \in \mathbb{Z}$ . Then we tile  $\mathbb{R}^2$  by translations of a unit parallelogram: Since  $\gcd(n, m) = 1$  we can find integers  $k, l \in \mathbb{Z}$  such that  $nl - km = 1$ . We pick the parallelogram spanned by  $OP$  and  $OP'$  where  $P' = \iota^{-1}\begin{pmatrix} k \\ l \end{pmatrix}$  which indeed has unit area. In other words we interpret  $OP$  and  $OP'$  as a basis of the lattice  $\iota^{-1}\mathbb{Z}^2$ . That gives us a fundamental cell of the lattice  $\iota^{-1}\mathbb{Z}^2$  which henceforth will be denoted by  $\mathcal{S}^0$  (cf. Figure 2.2). Fixing a small enough  $\kappa > 0$  allows to define  $\mathcal{S}^0(\kappa)$ , a parallelogram (cf. Figure 2.3) satisfying two requirements:

1.  $\overline{\mathcal{S}^0} \subset \mathcal{S}^0(\kappa) \subset \mathcal{S}^0 + \mathbb{B}(0, \kappa)$  with  $\mathbb{B}(0, \kappa) = \{(y, \eta) : y^2 + \eta^2 < \kappa^2\}$ .
2.  $\mathcal{S}^0(\kappa) = \mathcal{S}_{\text{int}}^0 \sqcup \partial_\kappa^0$  where  $\mathcal{S}_{\text{int}}^0$  is the set of points in  $\mathcal{S}^0$  whose distance (with respect to the  $\|\cdot\|_2$ -norm) from  $\partial\mathcal{S}^0$  is strictly larger than  $\kappa$  and  $\partial_\kappa^0 \neq \emptyset$ .

Denote the projections of  $\mathcal{S}^0(\kappa)$  to the coordinate axes by  $\Pi_y(\mathcal{S}^0(\kappa))$ ,  $\Pi_\eta(\mathcal{S}^0(\kappa))$  and the diameters of the projections by

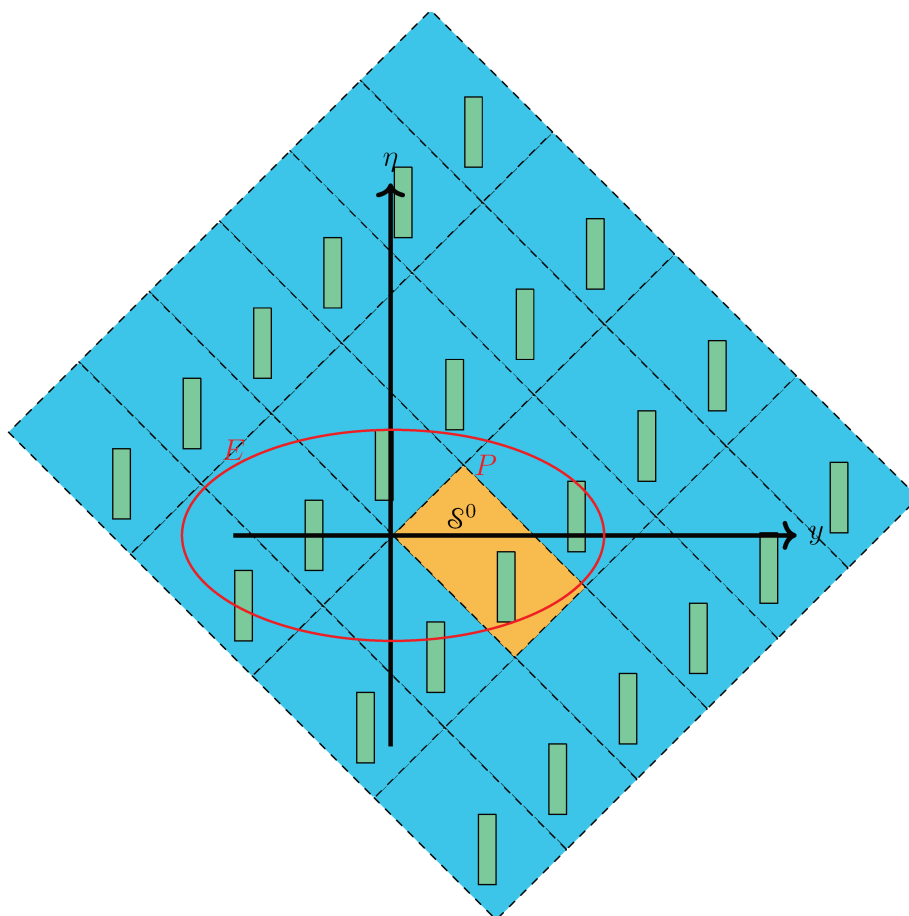
$$\ell_y = \max_{y, y' \in \Pi_y \mathcal{S}^0(\kappa)} |y - y'|, \quad \ell_\eta = \max_{\eta, \eta' \in \Pi_\eta \mathcal{S}^0(\kappa)} |\eta - \eta'|. \quad (2.15)$$

We consider a smooth function  $\mathbb{1}_{\mathcal{S}^0}^\kappa$  supported inside  $\mathcal{S}^0(\kappa)$  and satisfying

1.  $\mathbb{1}_{\mathcal{S}^0}^\kappa(x) \equiv 1$  for a non-empty subset of  $\mathcal{S}^0$ .
2.  $\mathbb{1}_{\mathcal{S}^0}^\kappa \in [0, 1]$  on  $(\partial_\kappa^0)^\circ$ .
3.  $\sum_{m \in \mathbb{Z}^2} \mathbb{1}_{\mathcal{S}^0}^\kappa((y, \eta) - \iota^{-1}m) \equiv \mathbb{1}_{\mathbb{R}^2}$ .

Similarly one defines  $\mathbb{1}_{\mathcal{S}^m}^\kappa$  by  $\mathbb{1}_{\mathcal{S}^m}^\kappa(y, \eta) = \mathbb{1}_{\mathcal{S}^0}^\kappa((y, \eta) - \iota^{-1}m)$ . For every  $a \in C^\infty(\mathbb{R}^2)$  we define its truncation on  $\mathcal{S}^m(\kappa) = \mathcal{S}^0(\kappa) + \iota^{-1}m$  by

$${}_m a = a \cdot \mathbb{1}_{\mathcal{S}^m}^\kappa. \quad (2.16)$$



**Figure 2.2:** Tiling  $\mathbb{R}^2$  by translations of  $\mathcal{S}^0$  (the orange tile). The green rectangles correspond to  $\mathcal{R}$  defined in section 2.5.

We recall the notion of  $\nu$ -porous subsets of  $\mathbb{R}$ :

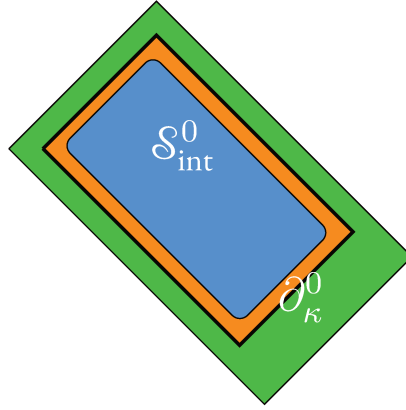
**Definition 2.3.2.** Let  $\nu \in (0, 1)$  and  $0 \leq \tau_0 \leq \tau_1$ . A subset  $\Omega \subset \mathbb{R}$  is  $\nu$ -porous on scales  $[\tau_0, \tau_1]$  if for each interval  $I$  of Lebesgue measure  $|I| \in [\tau_0, \tau_1]$  there exists a sub-interval  $J \subset I$  such that  $|J| = \nu|I|$  and  $J \cap \Omega = \emptyset$ .

Recalling (2.6) and  $\epsilon \in \{1, 2\}$  we denote the twist of the symbol  $a_\epsilon$  evolved at time  $t$  by

$$b_{\epsilon,t} = a_\epsilon \circ \gamma^t \circ \iota. \quad (2.17)$$

Let us recall that  $T' = 4T$  (where  $T$  is defined in (T)) and consider long words  $w \in \mathcal{W}(2T')$ . Every such word can be written as concatenation of two words of length  $T'$ , that is  $w = w^+ w^-$  with  $w^\pm \in \mathcal{W}(T')$ ,

$$w^+ = w_{T'}^+ \dots w_1^+ \quad w^- = w_0^- \dots w_{-T'+1}^- \quad w_k^\pm = w_{k+T'-1} \quad -T' + 1 \leq k \leq T', \quad (2.18)$$



**Figure 2.3:** The parallelogram  $\mathcal{S}^0(\kappa)$  which contain  $\overline{\mathcal{S}^0}$  (the orange rectangle). It is decomposed as the union of  $\mathcal{S}^0_{\text{int}}$  (the blue domain) with  $\partial^0_{\kappa}$ .

and the associated symbols

$$b_w^+ = \prod_{k=1}^{T'} b_{w_k^+, k} = \prod_{k=1}^{T'} a_{w_k^+} \circ \gamma^k \circ \iota, \quad b_w^- = \prod_{k=-T'+1}^0 b_{w_k^-, k} = \prod_{k=-T'+1}^0 a_{w_k^-} \circ \gamma^k \circ \iota.$$

We recall that  ${}_m b_w^\pm$  is the truncation of  $b_w^\pm$  in  $\mathcal{S}^m(\kappa)$  (see (2.16)), and we define the associated projections by

$${}_m \Omega_w^+ = \Pi_y(\text{supp } {}_m b_w^+), \quad {}_m \Omega_w^- = \Pi_\eta(\text{supp } {}_m b_w^-). \quad (2.19)$$

Let us give two analogous manners of proving the  $\nu$ -porosity of these sets. The first, will rely on the minimality of the unstable flow. The latter will rely on number theoretic principles and will give an exact way of computing the porosity parameters explicitly.

#### 2.4 A Proof relying on the minimality of the horocyclic flow

Recalling that the  $y$ -axis corresponds to the unstable direction of  $\iota^{-1} \circ \gamma \circ \iota$  and that the  $\eta$ -axis corresponds to the stable direction of it we prove the following central lemma:

**Proposition 2.4.1.** *There exist  $K > 0$  and  $\nu \in (0, 1)$ , both independent of  $\rho$ , such that the sets  ${}_m \Omega_w^\pm$  are  $\nu$ -porous on scales  $[Kh^\rho, 1]$ .*

*Proof.* We prove the porosity statement only for  ${}_m \Omega_w^+$  as the proof for  ${}_m \Omega_w^-$  follows along the same lines. For simplicity we take  $m = 0$  as the proof for the general case is analogous and follows by translation. The proof relies on the correspondence between finite intervals in  $\mathbb{R}$  and unstable sub-orbits of  $\gamma$ .

1. For every fixed  $(v_0, t_0) \in \mathbb{T}^2 \times \mathbb{R}$  we define the truncated unstable orbit of length  $\tau$  by

$$\mathcal{O}_{v_0, t_0, \tau} := \{\mathcal{H}_u(v_0, t) : t \in [t_0, t_0 + \tau]\}. \quad (2.20)$$

We denote for brevity  $\mathcal{O}_{v_0, \tau} := \mathcal{O}_{v_0, 0, \tau}$ . Denote the ball of radius  $r$  centered at  $(y, \eta)$  by  $\mathbb{B}((y, \eta), r)$ . Let us recall from the construction above that (cf. Figure 2.5)

$$\mathcal{K}_1 \subset \mathbb{T}^2 \setminus \text{supp}(a_1), \quad \mathcal{K}_2 \subset \mathbb{T}^2 \setminus \text{supp}(a_2) \quad (2.21)$$

are both open hence there exists  $\kappa' > 0$  and balls  $\mathbb{B}_1((y_1, \eta_1), \kappa'), \mathbb{B}_2((y_2, \eta_2), \kappa') \in \mathbb{T}^2$  such that

$$\mathbb{B}_1((y_1, \eta_1), \kappa') \subset \mathcal{K}_1, \quad \mathbb{B}_2((y_2, \eta_2), \kappa') \subset \mathcal{K}_2.$$

We note that each ball  $\mathbb{B}_\epsilon((y_\epsilon, \eta_\epsilon), \kappa')$  contains a smaller ball  $\mathcal{K}'_\epsilon := \mathbb{B}((y_\epsilon, \eta_\epsilon), \frac{\kappa'}{4})$ . Henceforth let us denote the Euclidean distance (both in  $\mathbb{R}^2$  and in  $\mathbb{T}^2$ ) by  $d_{Euc}$ . From the construction

$$d_{Euc}(\partial \mathcal{K}_\epsilon, \mathcal{K}'_\epsilon) \geq \frac{3}{4} \kappa'. \quad (2.22)$$

2. Let us prove that if a truncated unstable orbit is long enough it has to pass through both sets  $\mathcal{K}'_\epsilon$  and moreover that the lengths of the truncated orbits lying inside  $\mathcal{K}'_\epsilon$  are uniformly bounded from below. In other words, we show that:

There exist some  $(L, \ell) \in (\mathbb{R}_+^*)^2$  with  $L > \ell$  such that for every  $v_0$  the truncated unstable orbit  $\mathcal{O}_{v_0, L}$  contains sub-orbits  $\mathcal{O}_{v_0, t_1, \ell}, \mathcal{O}_{v_0, t_2, \ell}$  satisfying  $\mathcal{O}_{v_0, t_j, \ell} \subset \mathcal{K}'_j$ .

Assume to the contrary that for every  $(L, \ell)$  there exists an orbit  $\mathcal{O}_{v_0, L}$  which does not contain either a sub-orbit  $\mathcal{O}_{v_0, t_1, \ell} \subset \mathcal{K}'_1$  or a sub-orbit  $\mathcal{O}_{v_0, t_2, \ell} \subset \mathcal{K}'_2$ . Let us consider the sequences  $L_n = n$  and  $\ell_n = \frac{1}{n}$  and construct a sequence of sub-orbits of lengths  $L_n$ : for every  $n$  there exists a vector  $v_n \in \mathbb{T}^2$  such that the orbit  $\mathcal{O}_{v_n, n}$  does not contain a sub-orbit  $\mathcal{O}_{v_n, t_n, \frac{1}{n}} \subset \mathcal{K}'_\epsilon$ . Since  $\mathbb{T}^2$  is compact there exist a convergent sub-sequence of vectors  $v_{n_k} \rightarrow v_\infty \in \mathbb{T}^2$ . Applying the assumption, for every  $k \in \mathbb{N}$  there exists  $\epsilon_k \in \{1, 2\}$  such that the orbit  $\mathcal{O}_{v_{n_k}, n_k}$  does not contain a sub-orbit  $\mathcal{O}_{v_{n_k}, t_{\epsilon_k}, n_k^{-1}} \subset \mathcal{K}'_{\epsilon_k}$ . Possibly extracting a sub-sequence, we can ensure that for every  $k$   $\epsilon_k = \epsilon$ . Letting  $k \rightarrow \infty$  the sub-orbits  $\{\mathcal{O}_{v_{n_k}, n_k}\}_k$  converge to a full orbit  $\mathcal{O}_{v_\infty}$  which does not intersect  $\mathcal{K}'_\epsilon$ . Since the latter is a proper open subset of  $\mathbb{T}^2$ , it is a contradiction to the minimality of the horocyclic flow we proved in Lemma 1.1.4 and thus we have proved the claim.

3. Fix some big constant  $K > 0$  and let  $I = I(h) \subset \mathbb{R}$  be an interval with  $|I| \in [Kh^\rho, 1]$  and let

$$k_0 = \left\lceil \frac{\log(4\sqrt{m_u - m_s}(1 + \kappa)) - \log(3\kappa')}{\log|\lambda_u|} \right\rceil. \quad (2.23)$$

Recall (T) and define

$$k = \min_{k_0 < j < T'} \{j : |\lambda_u|^j |I| \geq L\}. \quad (2.24)$$

4. Let us associate to  $I$  a truncated orbit: Denoting the projection from  $\mathbb{R}^2$  onto  $\mathbb{T}^2$  by  $\pi(y, \eta) = (y, \eta) \pmod{\mathbb{Z}^2}$  we consider the set

$$\mathcal{O}_I := \left\{ \pi \circ \iota \begin{pmatrix} y \\ 0 \end{pmatrix} : y \in I \right\} = \{\pi(y.v_u) : y \in I\} \subset \mathbb{T}^2. \quad (2.25)$$

From Lemma 1.1.1 it is indeed a truncated unstable orbit of length  $|I|$ .

5. Recalling (1.3) implies that  $\gamma^k \mathcal{O}_I$  is an orbit of length  $|\lambda_u|^k |I| \geq L$ . From part 2 and part 4 there exists a sub-orbit  $\mathcal{O}'_I \subset \gamma^k \mathcal{O}_I \cap \mathcal{K}'_{w_k^+}$  of length  $\ell$ . Then there exists a sub-interval  $J \subset I$  such that  $\gamma^{-k} \mathcal{O}'_I = \{\pi \circ \iota(y) : y \in J\}$  and  $|J| = |\lambda_u|^{-k} \ell$ . The interval  $J$  will be our candidate for a subinterval of  $I$  not intersecting  ${}_0\Omega_w^+$  (and thus establishing the porosity property).
6. For establishing the porosity let us first show that there exists some  $\nu \in (0, 1)$  such that  $\frac{|J|}{|I|} \geq \nu$ . If  $k > k_0 + 1$ , using the minimality of  $k$

$$|I| |\lambda_u|^k < |\lambda_u| L,$$

which implies

$$\frac{\ell}{L} \geq \frac{|J|}{|I|} = \frac{\ell |\lambda_u|^{-k}}{|I|} \geq \frac{\ell}{|\lambda_u| L}. \quad (2.26)$$

Similarly if  $k = k_0 + 1$ ,

$$\frac{\ell}{L} \geq \frac{|J|}{|I|} = \frac{\ell |\lambda_u|^{-k_0-1}}{|I|} \geq \frac{\ell}{|\lambda_u|^{k_0+1} L}. \quad (2.27)$$

As follows, taking  $\nu = \frac{\ell}{|\lambda_u|^{k_0+1} L}$  both (2.26) and (2.27) are satisfied.

7. For demonstrating the  $\nu$ -porosity of  ${}_0\Omega_w^+$  let us prove that for the choice of  $k_0$  in (2.23)  $J$  does not intersect  ${}_0\Omega_w^+$ . Assume to the contrary there is some  $y \in J \cap {}_0\Omega_w^+$  then it can be lifted to a point  $(y, \eta) \in \text{supp } {}_0b_w^+$  for some  $\eta \in \mathbb{R}$ . Let us write,

$$\gamma^k \circ \pi \circ \iota \begin{pmatrix} y \\ \eta \end{pmatrix} = \gamma^k \circ \pi \circ \iota \begin{pmatrix} y \\ 0 \end{pmatrix} + \gamma^k \circ \pi \circ \iota \begin{pmatrix} 0 \\ \eta \end{pmatrix}. \quad (2.28)$$

From the construction of  $J$  we deduce that the first summand is in  $\mathcal{K}'_{w_k^+}$ . Since  $(y, \eta) \in S^0(\kappa)$  we can bound  $|\eta| \leq \ell_\eta$ . Let us bound the second summand as well: We recall that  $\iota^{-1}$  maps any truncated stable orbit  $\mathcal{O}_{v_0, t_0, \tau}^s := \{\mathcal{H}_s(v, t) : t \in I' = [t_0, t_0 + \tau]\}$  to an interval  $I'' \subset \{c\} \times \mathbb{R}$  (for some  $c \in \mathbb{R}$ ) and that

$$d_{\text{Euc}}(\iota^{-1} \mathcal{H}_s(v_0, t_0), \iota^{-1} \mathcal{H}_s(v_0, t_0 + \tau)) = \|\iota^{-1} v_s\| \tau.$$

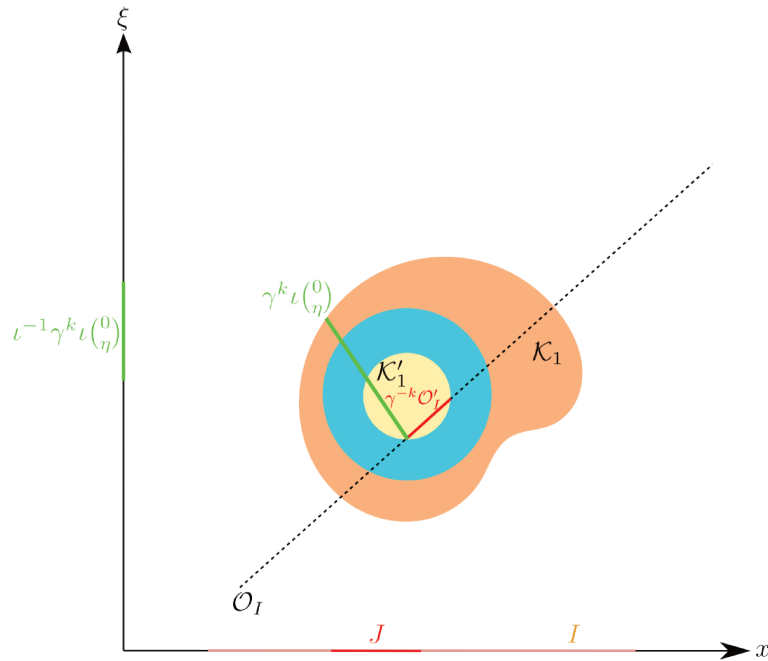
We note that

$$\left| \left\{ \gamma^k \circ \pi \circ \iota \begin{pmatrix} 0 \\ \eta \end{pmatrix} : |\eta| \leq \ell_\eta \right\} \right| = 2\ell_\eta |\lambda_u|^{-k} \|v_s\|$$

and

$$\left| \iota^{-1} \circ \gamma^k \circ \iota \begin{pmatrix} 0 \\ \eta \end{pmatrix} \right| \leq 2 \frac{\ell_\eta}{\sqrt{m_u - m_s}} |\iota^{-1} \circ \gamma^k v_s| = 2\ell_\eta |\lambda_u|^{-k} \|v_s\|.$$

As follows the length of the truncated stable orbit  $\{\gamma^k \circ \pi \circ \iota \begin{pmatrix} 0 \\ \eta \end{pmatrix} : |\eta| \leq \ell_\eta\}$  is bounded from above by  $2\ell_\eta |\lambda_u|^{-k_0} \|v_s\|$ . From the choice of  $k_0$  in (2.23) the second summand in (2.28) is bounded from



**Figure 2.4:** The ball of radius smaller than  $\frac{3}{4}\kappa'$  centered at  $\gamma^k \circ \pi \circ \iota(y, \eta)$  from (2.28), appearing in the proof of Proposition 2.4.1, is contained inside  $\mathcal{K}_{w_k^+}$ . In the figure  $w_k^+ = 1$ .

above by  $\frac{3}{4}\kappa'$  and from (2.22) (see Figure 2.4) adding it to the first summand,  $\gamma^k \circ \pi \circ \iota(y, \eta) \in \mathcal{K}_{w_k^+} \subset \mathbb{C}\text{supp} (a_{w_k^+})$  or equivalently  $(y, \eta) \notin \text{supp} (b_w^+)$  which is a contradiction.

□

## 2.5 A Proof relying on lattice-points counting

We present below a second proof for the porosity of the projections. The proof presented below relies on elementary number theoretic principles instead on the minimality of the horocyclic flow. Unlike the proof presented in the previous section this approach allows to obtain numeric estimates on the porosity of the symbols. In section A.4 we rely on these estimates to obtain  $\nu, \beta$  numerically.

### 2.5.1 Alternative geometric construction

For the purpose of this subsection we will have to consider a geometric construction replacing the one from subsection 2.1.1 above.

Fix  $a \in C^\infty(\mathbb{T}^2)$  having a non-empty support. We construct a pair of parallelograms lying inside  $\text{supp} (a)$ : Consider a rectangle  $\mathcal{R} \subsetneq [0, 1]^2$  aligned along the  $y - \eta$  coordinates system with  $\mathcal{R} = [y_{\mathcal{R}}, y_{\mathcal{R}} + w_{\mathcal{R}}] \times [\eta_{\mathcal{R}}, \eta_{\mathcal{R}} + h_{\mathcal{R}}]$  satisfying  $\iota\mathcal{R} \subset \text{supp} (a)$  and  $|a| \geq c$  on  $\iota\mathcal{R}$  for some  $c > 0$ . We

assume further that

$$w_{\mathcal{R}} < \mathcal{C}_{w(\mathcal{R})}, \quad \hbar_{\mathcal{R}} < \mathcal{C}_{\hbar(\mathcal{R})} \quad (2.29)$$

with  $\mathcal{C}_{w(\mathcal{R})}$  specified later in (2.42) and  $\mathcal{C}_{\hbar(\mathcal{R})}$  defined analogously. We remark that the sides of  $\iota\mathcal{R}$  and  $\iota\mathcal{U}$  are pieces of stable and unstable leaves. Similarly fixing  $\hbar_{\mathcal{U}}, w_{\mathcal{U}} > 0$  we consider a rectangle  $\mathcal{U} = [y_{\mathcal{U}}, y_{\mathcal{U}} + w_{\mathcal{U}}] \times [\eta_{\mathcal{U}}, \eta_{\mathcal{U}} + \hbar_{\mathcal{U}}]$  satisfying  $\iota\mathcal{U} \subset \iota\mathcal{R}$ . The sides of  $\iota\mathcal{R}$  and  $\iota\mathcal{U}$  are pieces of stable and unstable leaves. Let  $a_1, a_2 \in C^\infty(\mathbb{T}^2)$  be a couple of symbols satisfying (while identifying  $\iota\mathcal{R}, \iota\mathcal{U}$  with their projections to  $\mathbb{T}^2$ , see Figure 2.5)

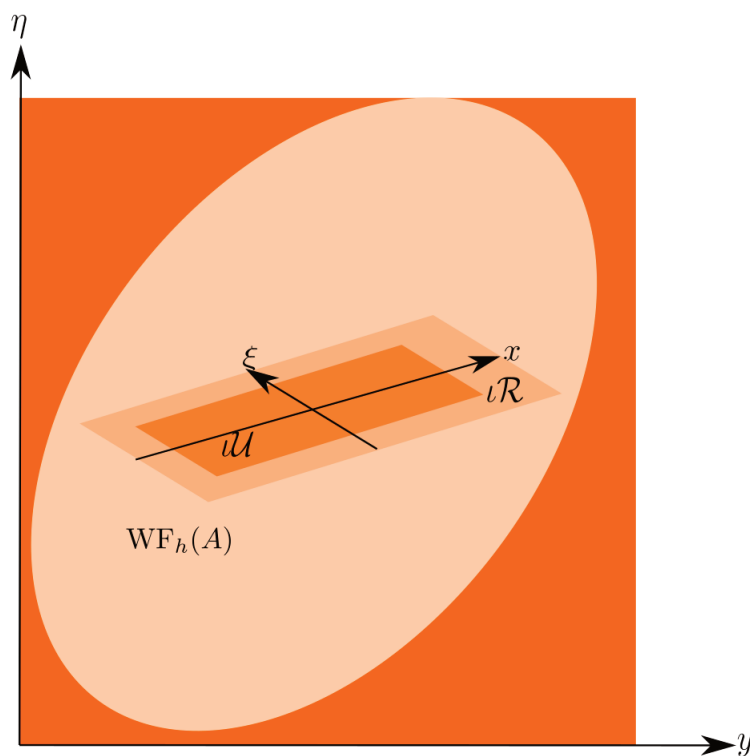
$$0 \leq a_\epsilon \leq 1, \quad a_1 + a_2 = 1, \quad \text{supp}(a_1) \subset \iota\mathcal{R}, \quad \text{supp}(a_2) \subset \mathbb{C}(\iota\mathcal{U}) \quad (2.30)$$

$$a_1|_{\iota\mathcal{U}} \equiv 1$$

with the corresponding operators  $\text{Op}_N(a_1), \text{Op}_N(a_2)$  satisfying

$$A_{1,N} + A_{2,N} = \text{Id}_{\mathcal{H}_N} \quad A_{\epsilon,N} = \text{Op}_N(a_\epsilon). \quad (*)$$

We can lift our construction to  $\mathbb{R}^2$  by periodization: Pick  $L_{\mathbb{R}^2}^u$  to be the Lagrangian foliation whose



**Figure 2.5:** Given a symbol  $a \in C^\infty(\mathbb{T}^2)$  we construct a partition of unity on  $\mathbb{T}^2 \cong [0, 1]^2 \subset \mathbb{R}^2$ .  $\mathbb{R}^2$  is equipped with  $(y, \eta)$  coordinates. The  $(x, \xi)$  coordinates are obtained after applying the coordinate map  $\iota$  to the coordinates  $(y, \eta)$ . The new coordinates describe the decomposition of  $\mathbb{R}^2$  into stable and unstable directions

projection is  $L^u$ . We can lift  $a_\epsilon$  to  $\mathbb{R}^2$  by periodization obtaining symbols  $\check{a}_\epsilon$ . Quantizing these symbols



yields the corresponding pseudo-differential operators  $A_\epsilon = \text{Op}_h(\check{a}_\epsilon)$  satisfying

$$\check{a}_1 + \check{a}_2 = 1, \quad \text{Op}_h(\check{a}_1) + \text{Op}_h(\check{a}_2) = \text{Id}_{L^2(\mathbb{R})}, \quad \text{supp}(\check{a}_1) \subset \iota\mathcal{R} + \mathbb{Z}^2, \quad \check{a}_1|_{\iota\mathcal{U} + \mathbb{Z}^2} \equiv 1. \quad (\check{\star})$$

The lemmas proved previously in chapter 1 hold as well for symbols arising from this construction.

### 2.5.2 Geometric notions

Let us present two geometric notions, depending on  $\gamma$  and on the choice of  $\iota$ , that will be employed during the proof of the important Lemma 2.5.4 below

#### 2.5.2.1 Distances

We begin by defining between parallelograms a pair of distances which will be used in Lemma 2.5.4 below. As before the rectangles we consider are all of the form

$$Q = [y_1, y_2] \times [\eta_1, \eta_2] \quad (2.33)$$

where  $y_2 > y_1, \eta_2 > \eta_1$ . We define the  $\Pi_y$ -diameter of a rectangle as  $\text{diam}_y(Q) = y_2 - y_1$  and the  $\Pi_\eta$ -diameter by  $\text{diam}_\eta(Q) = \eta_2 - \eta_1$ . We introduce horizontal and vertical "distances" between parallelograms  $\mathcal{P} \subset \mathbb{R}^2$  centered at  $(y_{\mathcal{P}}, \eta_{\mathcal{P}})$ , namely,

$$d^y(\mathcal{P}_1, \mathcal{P}_2) = |y_{\mathcal{P}_1} - y_{\mathcal{P}_2}| \quad d^\eta(\mathcal{P}_1, \mathcal{P}_2) = |\eta_{\mathcal{P}_1} - \eta_{\mathcal{P}_2}|. \quad (2.34)$$

#### 2.5.2.2 Gauss rectangle problem

Let us prove an analogue of the Gauss circle problem to a rectangle having a bounded aspect ratio.

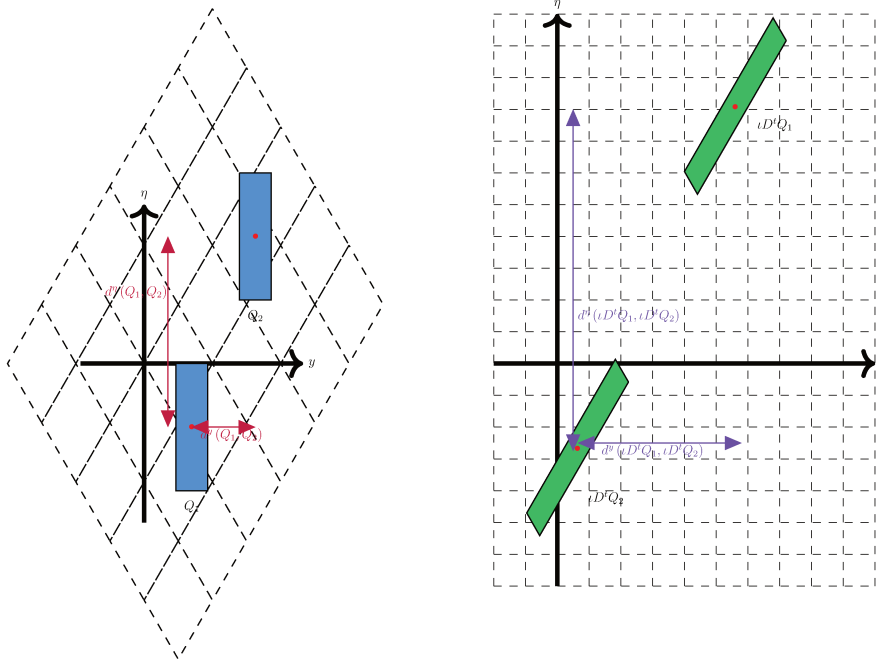
**Lemma 2.5.1.** *Fix  $c \in \mathbb{R}_+^*$  and a map  $\alpha \in SL_2(\mathbb{R})$ . Let  $Q_R$  be a rectangle of width  $R$  and height  $cR$ . Denote the number of  $\alpha\mathbb{Z}^2$ -points inside and on  $Q_R$  by  $N(Q_R)$  and the diameters of the projections of a fundamental cell of  $\alpha$  by  $\text{Diam}_y(\alpha), \text{Diam}_\eta(\alpha)$ . Then for large  $R \gg \max\{2\text{Diam}_\eta(\alpha), 2c^{-1}\text{Diam}_y(\alpha)\}$ ,*

$$N(Q_R) = cR^2 + F_\alpha(Q_R)R + 4\text{Diam}_y(\alpha) \cdot \text{Diam}_\eta(\alpha),$$

with  $|F_\alpha(Q_R)| \leq 2(\text{Diam}_y(\alpha) + c\text{Diam}_\eta(\alpha))$ .

*Proof.* Let us construct fundamental cells  $\mathcal{S}^{(p,q)}$  using Minkowski's theorem (as we did in the beginning of section 2.3). Each cell is of unit area and hence  $N(Q_R)$  equals to sum of the areas of cells  $\mathcal{S}^{(p,q)}$  corresponding to lattice points in  $Q_R$ . The union of these cells is contained in a larger rectangle of width  $R + 2\text{Diam}_\eta(\alpha)$  and height  $cR + 2\text{Diam}_y(\alpha)$ . We deduce that

$$N(Q_R) \leq (R + 2\text{Diam}_\eta(\alpha))(cR + 2\text{Diam}_y(\alpha)).$$



**Figure 2.6:** The distances between rectangles and their images under  $\iota D^t$  in the  $y - \eta$  plane

Similarly a rectangle of width  $R - 2\text{Diam}_\eta(\alpha)$  and height  $cR - 2\text{Diam}_y(\alpha)$  is contained in the union of cells  $S^{(p,q)}$  and we conclude that

$$(R - 2\text{Diam}_\eta(\alpha))(cR - 2\text{Diam}_y(\alpha)) \leq N(Q_R) \leq (R + 2\text{Diam}_\eta(\alpha))(cR + 2\text{Diam}_y(\alpha)).$$

□

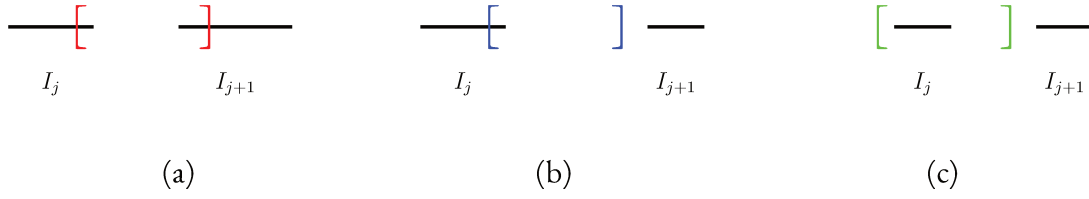
### 2.5.3 $\nu$ -porous sets

**Remark 2.5.2** (Rescaling by a scalar factor). *Let  $\Omega \subset \mathbb{R}$  be a  $\nu$ -porous set on scales  $[\tau_0, \tau_1]$ , then for every constant  $c \in (0, 1]$  such that  $c\nu < 1$ ,  $\Omega$  is a  $c\nu$ -porous set on scales  $[c^{-1}\tau_0, c^{-1}\tau_1]$ .*

The first evident example of a  $\nu$ -porous set is a finite union of intervals,

**Lemma 2.5.3.** *Let  $\Omega = \bigsqcup_{j=1}^J I_j$  a union of disjoint intervals  $I_j \subset \mathbb{R}$  with  $\max I_1 < \min I_2 < \dots < \max I_J$  and  $|I_j| \leq \ell$  for every  $1 \leq j \leq J$ . Denote the minimal distance between any two neighbouring intervals  $I_j$  and  $I_{j+1}$  by  $g$ . For every  $g' \in (0, g)$ , the set  $\Omega$  is a  $\nu$ -porous set on the scale  $(\ell + g')$  with the factor  $\nu = \frac{g'}{2\ell + 2g'}$ .*

*Proof.* Fix some  $g' \in (0, g)$  and let  $I' = [y_0, y_1] \subset \mathbb{R}$  with  $|I'| = \ell + g'$ . We will show it contains a sub-interval  $J \subset I' \cap \mathbb{C}\Omega$  with  $|J| = \nu|I'|$ . Since  $|I_j| \leq \ell$  the endpoints of  $I'$  cannot lie on the same interval  $I_j$ . Let us split the proof into cases based on the position of the endpoints of  $I'$  (see Figure 2.7):



**Figure 2.7:** There are 3 different cases for the positions of the endpoints of  $I'$  in the proof of Lemma 2.5.3: (a) both endpoints are on  $\Omega$ , (b) only a single endpoint is on  $\Omega$ , (c) neither of the endpoints is on  $\Omega$

- (a) If there exists a  $1 \leq j \leq J$  such that  $y_0 \in I_j, y_1 \in I_{j+1}$  then there exists a sub-interval  $I'' \subset I'$  of length  $|I''| \geq g' > \frac{g'}{2} = \nu|I'|$  such that  $I'' \cap \Omega = \emptyset$ .
- (b) If there exists a  $1 \leq j \leq J$  such that  $y_0 \in I_j, y_1 \notin \Omega$ , then  $I' = I_{j'} \sqcup I''$  with  $I_{j'} \subset I_j$  and  $I'' \subset \mathbb{C}\Omega$ .  $|I_{j'}| \leq \ell$  hence  $|I''| \geq g' > \nu|I'|$  satisfies  $I'' \cap \Omega = \emptyset$ .
- (c) If  $y_0, y_1 \notin \Omega$ , such that  $I' = I'_0 \sqcup I'_\ell \sqcup I'_1$  where  $I'_0 \cap \Omega = I'_1 \cap \Omega = \emptyset, I'_\ell \subset \Omega$  and  $|I'_\ell| \leq \ell$ . As follows either  $|I'_0| \geq \frac{g'}{2} = \nu|I'|$  or  $|I'_1| \geq \frac{g'}{2} = \nu|I'|$ .

□

Recalling the notations from (2.16) for the truncated symbols we will demonstrate below that

$${}_m\Omega_{\epsilon,t}^+ = \Pi_y \left( \text{supp } {}_m\check{b}_{\epsilon,t} \right) \quad {}_m\Omega_{\epsilon,t}^- = \Pi_\eta \left( \text{supp } {}_m\check{b}_{\epsilon,t} \right) \quad (2.35)$$

are  $\nu_{\epsilon,\pm}$ -porous sets on certain  $t$ -dependent scales. Recalling that the  $y$ -axis corresponds to the unstable direction of  $\iota^{-1} \circ \gamma \circ \iota$  and that the  $\eta$ -axis corresponds to the stable direction of it we prove the following central lemma:

**Lemma 2.5.4.** *There exist constants  $t_\pm \in \pm\mathbb{N}^*, \nu_{\epsilon,\pm}, C_{\epsilon,\pm} \in \mathbb{R}$  which do not depend on  $t$  or on  $T'$  such that*

1. If  $t > t_+$ ,  ${}_m\Omega_{\epsilon,t}^+$  is  $\nu_{\epsilon,+}$ -porous on the scale  $\tau_{\epsilon,t} = C_{\epsilon,+}\lambda_u^{-t}$ .
2. If  $t < t_-$ ,  ${}_m\Omega_{\epsilon,t}^-$  is  $\nu_{\epsilon,-}$ -porous on the scale  $\tau_{\epsilon,t} = C_{\epsilon,-}\lambda_u^t$ .

*proof of Lemma 2.5.4.* We only prove in details the first part as the second follows by exchanging the expanding and contracting directions. Without loss of generality we assume that  $m = 0$  as  $\check{a}_\epsilon$  are  $\mathbb{Z}^2$ -periodic symbols.

1. Let us first discuss the structure of the set  ${}_0\Omega_{1,t}^+$ , i.e., when  $\epsilon = 1$ : In virtue of (2.30) for every  $t \in \mathbb{N}^*$ ,  $(y, \eta) \in \text{supp } {}_0\check{b}_{1,t}$  if and only if

$$(y, \eta) \in (\iota^{-1}\gamma^{-t}\text{supp } \check{a}_1) \cap S^0(\kappa) = (D^{-t}\iota^{-1}\text{supp } \check{a}_1) \cap S^0(\kappa)$$

$$\begin{aligned} &\subset (D^{-t}\iota^{-1}(\iota\mathcal{R} + \mathbb{Z}^2)) \cap \mathcal{S}^0(\kappa) \\ &= (D^{-t}\mathcal{R} + \iota^{-1}\mathbb{Z}^2) \cap \mathcal{S}^0(\kappa) \end{aligned}$$

with  $D$  the dilation defined in (1.18). In other words  $\text{supp } {}_0\check{b}_{1,t}$  is a union of intersections between  $\mathcal{S}^0(\kappa)$  and  $\iota^{-1}\mathbb{Z}^2$ -shifts of a rectangle elongated along the vertical direction. For every  $(p, q) \in \mathbb{Z}^2$  we define such a rectangle by

$$R_{(p,q),t} := D^{-t}\mathcal{R} + \iota^{-1}\begin{pmatrix} p \\ q \end{pmatrix}. \quad (2.36)$$

This notation allows us to represent  $\text{supp } {}_0\check{b}_{1,t} \subset \mathcal{R}_t \cap \mathcal{S}^0(\kappa)$ , with

$$\mathcal{R}_t = \bigcup_{R_{(p,q),t} \cap \mathcal{S}^0(\kappa) \neq \emptyset} R_{(p,q),t}.$$

2. Let us estimate  $J$ , the number of connected components of  $\mathcal{R}_t$ : There is a bijection between the rectangles and their centers thus it is sufficient to estimate the number of centers of rectangles intersecting  $\mathcal{S}^0(\kappa)$ . For every rectangle  $R_{(p_j,q_j),t} \subset \mathcal{R}_t$ ,  $R_{(p_j,q_j),t} \cap \mathcal{S}^0(\kappa)$  is a polygon.  $R_{(p_j,q_j),t} \cap (\mathcal{S}^0(\kappa))$  is a point (i.e., a degenerate polygon) in two cases

- there exists a vertex of  $R_{(p_j,q_j),t}$  lying on a side of  $\mathcal{S}^0(\kappa)$ .
- there exists a vertex of  $\mathcal{S}^0(\kappa)$  lying on a side of  $R_{(p_j,q_j),t}$ .

The centers of the rectangles satisfying either of the requirements lie on a union of intervals forming a closed plane curve  $\mathcal{o}$  in  $\mathbb{R}^2$ . The centers of rectangles having a polygonal intersection with  $\mathcal{S}^0(\kappa)$  are obtained by  $\iota^{-1}\mathbb{Z}^2$ -translations in the interior of the domain bounded by  $\mathcal{o}$ . We obtain a closed octagonal set  $\mathcal{O}$  having  $\partial\mathcal{O} = \mathcal{o}$  (see Figure 2.8).

We can now estimate the number of lattice points inside  $\mathcal{O}$ : Let us consider  $\mathcal{O}$ , the smallest rectangle containing (entirely)  $\mathcal{O}$ ,

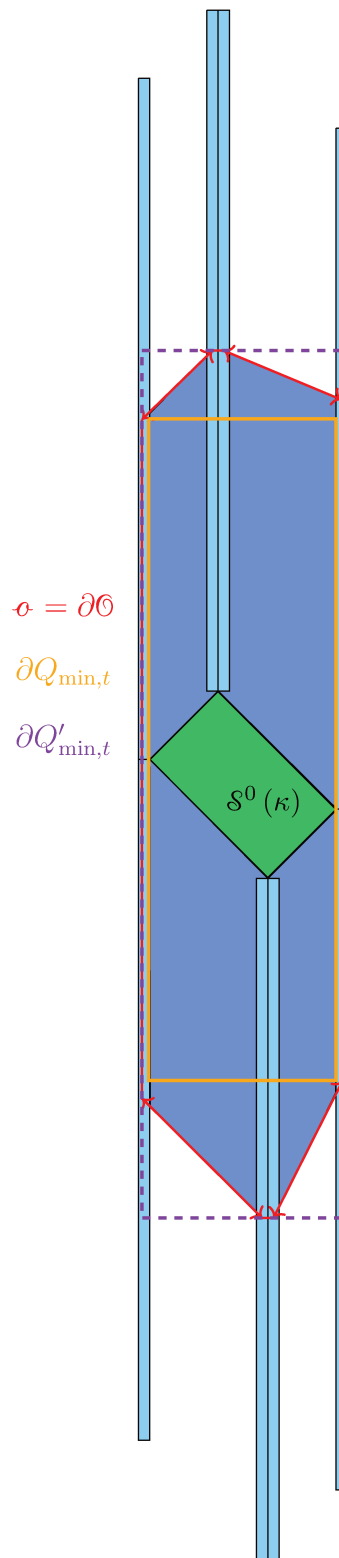
$$Q'_{\min,t} = \left[ y_{\min} - \frac{w_{\mathcal{R}}}{2}\lambda_u^{-t}, y_{\min} + \ell_y + \frac{w_{\mathcal{R}}}{2}\lambda_u^{-t} \right] \times \left[ -\frac{h_{\mathcal{R}}}{2}\lambda_u^t + \eta_{\min}, \frac{h_{\mathcal{R}}}{2}\lambda_u^t + \eta_{\min} + \ell_\eta \right] \subset \mathcal{O},$$

where

$$\begin{aligned} y_{\min} &= \min\{y_0 : (y_0, \eta_0) \text{ is a vertex of } \mathcal{S}^0(\kappa)\} \\ \eta_{\min} &= \min\{\eta_0 : (y_0, \eta_0) \text{ is a vertex of } \mathcal{S}^0(\kappa)\}. \end{aligned} \quad (2.37)$$

We note that the diameters of the projections of  $Q'_{\min,t} \setminus \mathcal{O}$  to the coordinate axes are  $O(1)$  and thus the number of lattice points there is bounded independently of  $t$ . It remains to estimate the number of lattice points in  $Q'_{\min,t}$ . Let us rather approximate the number of  $\iota^{-1}\mathbb{Z}^2$  points in

$$Q_{\min,t} = [y_{\min}, y_{\min} + \ell_y] \times \left[ -\frac{h_{\mathcal{R}}}{2}\lambda_u^t, \frac{h_{\mathcal{R}}}{2}\lambda_u^t \right].$$



**Figure 2.8:** The domain (bounded by the red path) in which centers of rectangles lie is a union of 4 hexagons and  $S^0(\kappa)$ . The orange dashed line bounds  $Q_{\min,t}$  and the purple one bounds  $Q'_{\min,t}$ .

$Q'_{\min,t} \setminus Q_{\min,t}$  is a union of 4 rectangles, 2 of area  $O(\lambda_u^{-t})$  and 2 of area bounded uniformly in  $t$  hence the number of lattice points in  $Q'_{\min,t} \setminus Q_{\min,t}$  is bounded as well uniformly in  $t \in \mathbb{N}$ . We observe that  $Q_{\min,t}$  is a rectangle with height  $\hbar_{\mathcal{R}} \lambda_u^t$  and width  $\ell_y$ .

We recall that since  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  and  $\iota$  diagonalizes  $\gamma$ , the dilation  $D^t$  preserves the lattice  $\iota^{-1}\mathbb{Z}^2$ . Therefore the number of  $\iota^{-1}\mathbb{Z}^2$ -points in  $Q_{\min}$  equals to the number of  $\iota^{-1}\mathbb{Z}^2$ -points in

$$Q_{\min,t}^{\flat} = \begin{cases} D^{\frac{t}{2}} Q_{\min,t} & t \in 2\mathbb{Z} \\ D^{\frac{t+1}{2}} Q_{\min,t} & t \in 2\mathbb{Z} + 1. \end{cases}$$

We note that  $Q_{\min,t}^{\flat}$  is a rectangle of width  $\ell_y \lambda_u^{\frac{t}{2}}$  and height  $\hbar_{\mathcal{R}} \lambda_u^{\frac{t}{2}}$  for  $t \in 2\mathbb{Z}$  and of width  $\ell_y \lambda_u^{\frac{t+1}{2}}$  and height  $\hbar_{\mathcal{R}} \lambda_u^{\frac{t-1}{2}}$  for  $t \in 2\mathbb{Z} + 1$ . Then the aspect ratio of  $Q_{\min,t}^{\flat}$  is  $\hbar_{\mathcal{R}}^{-1} \ell_y$  for  $t \in 2\mathbb{Z}$  and  $\lambda_u \hbar_{\mathcal{R}}^{-1} \ell_y$  for  $t \in 2\mathbb{Z} + 1$  which are both  $t$ -independent constants and hence applying Lemma 2.5.1, the number of lattice points in  $Q_{\min,t}^{\flat}$  and thus in  $\mathcal{O}$  is

$$N(\mathcal{O}) = \begin{cases} \hbar_{\mathcal{R}} \ell_y \lambda_u^t + O\left(\lambda_u^{\frac{t}{2}}\right) & t \in 2\mathbb{Z} \\ \lambda_u \hbar_{\mathcal{R}} \ell_y \lambda_u^t + O\left(\lambda_u^{\frac{t}{2}}\right) & t \in 2\mathbb{Z} + 1 \end{cases}.$$

We deduce that  $J = C_t \lambda_u^t + O\left(\lambda_u^{\frac{t}{2}}\right)$  with  $C_t$  being a scalar dependent on the parity of  $t$  and the choices of  $\iota$  and  $\mathcal{R}$ .

3. Let us now give a choice of the limiting length  $\mathcal{C}_w(\mathcal{R})$  from (2.29) so disjoint rectangles have disjoint projections to the  $y$  axis, i.e. every two distinct rectangles  $R_{(p_j, q_j), t}, R_{(p_{j'}, q_{j'}), t} \subset \mathcal{R}_t$  satisfy

$$\forall 1 \leq j \neq j' \leq J, \quad \Pi_y \left( R_{(p_{j'}, q_{j'}), t} \right) \cap \Pi_y \left( R_{(p_j, q_j), t} \right) = \emptyset. \quad (2.38)$$

Let  $R_{(p_j, q_j), t}, R_{(p_{j'}, q_{j'}), t} \subset \mathcal{R}_t$ . Since they both have to intersect  $\mathcal{S}^0(\kappa)$ ,

$$d^n \left( R_{(p_j, q_j), t}, R_{(p_{j'}, q_{j'}), t} \right) \leq \ell_\eta + \hbar_{\mathcal{R}} \lambda_u^t, \quad d^y \left( R_{(p_j, q_j), t}, R_{(p_{j'}, q_{j'}), t} \right) \leq \ell_y + w_{\mathcal{R}} \lambda_u^{-t}.$$

Then there exists a constant  $c_{\mathcal{R}_t, t}$  such that for every  $R_{(p_j, q_j), t}, R_{(p_{j'}, q_{j'}), t} \subset \mathcal{R}_t$

$$\begin{aligned} |p_j - p_{j'}| &= d^y \left( \iota D^t R_{(p_j, q_j), t}, \iota D^t R_{(p_{j'}, q_{j'}), t} \right) \\ &\leq \lambda_u^t d^y \left( R_{(p_j, q_j), t}, R_{(p_{j'}, q_{j'}), t} \right) + \lambda_u^{-t} d^n \left( R_{(p_j, q_j), t}, R_{(p_{j'}, q_{j'}), t} \right) \\ &\leq c_{\mathcal{R}_t, t} \lambda_u^t, \end{aligned} \quad (2.39)$$

where we denote  $c_{\mathcal{R}_t, t} = \ell_\eta + \ell_y + \hbar_{\mathcal{R}} + w_{\mathcal{R}}$ .  $m_s$  is a quadratic irrational which means it is badly-

approximable. As follows there exists some  $\omega > 0$  such that for every  $p \in \mathbb{N}^*$ ,  $q \in \mathbb{Z}$ ,

$$|pm_s - q| \geq \frac{\omega}{p}. \quad (2.40)$$

4. Henceforth we denote for every  $p_j, q_j \in \mathbb{Z}$

$$\begin{pmatrix} y_j \\ \eta_j \end{pmatrix} := \iota^{-1} \begin{pmatrix} p_j \\ q_j \end{pmatrix} \quad (2.41)$$

Let us pick

$$\mathfrak{C}_{w(\mathcal{R})} := \frac{\omega}{c_{\mathcal{R},t} \sqrt{m_u - m_s}} \quad (2.42)$$

and verify that the choice of the constant implies that (2.38) is satisfied: Let  $j \neq j'$ . Let us first bound the distance when  $p_j \neq p_{j'}$ .  $R_{(p_j, q_j), t}$  and  $R_{(p_{j'}, q_{j'}), t}$  are  $\iota^{-1}\mathbb{Z}^2$ -translations of  $D^{-t}\mathcal{R}$  and hence, recalling the choice of  $w_{\mathcal{R}}$  in (2.29) and assuming  $p_j \neq p_{j'}$ , the gap between their  $\Pi_y$ -projections can be expressed as

$$\begin{aligned} |y_j - y_{j'}| &= \frac{|m_s(p_j - p_{j'}) - (q_j - q_{j'})|}{\sqrt{m_u - m_s}} \\ &\geq \frac{\omega}{|p_j - p_{j'}| \sqrt{m_u - m_s}} \\ &\geq \mathfrak{C}_{w(\mathcal{R})} \lambda_u^{-t} \\ &> w_{\mathcal{R}} \lambda_u^{-t}. \end{aligned}$$

with the last inequality following from (2.42). If  $p_j = p_{j'}$ , we take

$$t > \left\lceil \left| \frac{\log(\sqrt{m_u - m_s} \mathfrak{C}_{w(\mathcal{R})})}{\log|\lambda_u|} \right| \right\rceil \quad (2.43)$$

and then the gap is

$$|y_j - y_{j'}| = \frac{|q_j - q_{j'}|}{\sqrt{m_u - m_s}} \geq \frac{1}{\sqrt{m_u - m_s}} > \mathfrak{C}_{w(\mathcal{R})} \lambda_u^{-t} \geq w_{\mathcal{R}} \lambda_u^{-t}.$$

We conclude that for every  $j \neq j'$  such that  $R_{(p_j, q_j), t} \neq R_{(p_{j'}, q_{j'}), t} \subset \mathcal{R}_t$ ,

$$|y_j - y_{j'}| \geq \mathfrak{C}_{w(\mathcal{R})} \lambda_u^{-t}. \quad (2.44)$$

5. Each  $R_{(p_j, q_j), t}$  has width  $w_{\mathcal{R}} \lambda_u^{-t}$  hence we deduce that the projections of any two neighboring rectangles to the  $y$ -axis do not intersect and furthermore there is a gap between them of size

$$g_{1,t} \geq (\mathfrak{C}_{w(\mathcal{R})} - w_{\mathcal{R}}) \lambda_u^{-t}.$$

Taking  $\Pi_y(\mathcal{R}_t \cap \mathcal{S}^0(\kappa))$  we obtain a union of  $J \asymp \lambda_u^t$  intervals. The length of each such interval is at

most  $w_{\mathcal{R}}\lambda_u^{-t}$  and the gap between every neighboring intervals is at least  $g_{1,t}$ . From Lemma 2.5.3,  ${}_0\Omega_{1,t}^+$  is a  $\nu_{1,+}$ -porous set on the scale  $\tau_{1,t}$  where

$$\nu_{1,+} = \frac{\mathcal{C}_{w(\mathcal{R})} - w_{\mathcal{R}}}{2(\mathcal{C}_{w(\mathcal{R})} + w_{\mathcal{R}})}, \quad \tau_{1,t} = \frac{\mathcal{C}_{w(\mathcal{R})} + w_{\mathcal{R}}}{2}\lambda_u^{-t}. \quad (2.45)$$

6. We consider now the structure of  ${}_0\Omega_{2,t}^+$ : We first recall from (✱) that

$$\begin{aligned} (y, \eta) \in \text{supp } {}_0\check{b}_{2,t} &\iff (y, \eta) \in \mathbb{C} \left( \{(y, \eta) \in \mathbb{R}^2 : {}_0\check{b}_{1,t}(y, \eta) \equiv 1\} \right) \cap \mathcal{S}^0(\kappa) \\ &\subset \mathbb{C} \left( D^{-t}\iota^{-1}(\mathcal{U} + \mathbb{Z}^2) \right) \cap \mathcal{S}^0(\kappa) \\ &= \mathbb{C} \left( D^{-t}\mathcal{U} + \iota^{-1}\mathbb{Z}^2 \right) \cap \mathcal{S}^0(\kappa). \end{aligned} \quad (2.46)$$

For analyzing  $\text{supp } {}_0b_{2,t}$  we have to consider the complement of the set  $D^{-t}\mathcal{U} + \iota^{-1}\mathbb{Z}^2$  inside  $\mathcal{S}^0(\kappa)$ . We introduce the concept of  $y$ -crossing which will be used in the rest of the proof: We express the conditions for a rectangle  $Q = [y_1, \eta_1] \times [y_2, \eta_2] \subset \mathbb{R}^2$  intersecting with  $\mathcal{S}^0(\kappa)$  to  $y$ -cross it (cf. Figure 2.9) ,and in fact to satisfy

$$|\Pi_y(Q) \cap \Pi_y(\mathcal{S}^m(\kappa) \setminus Q)| = 0.$$

Henceforth we denote

$$\eta_{\min} = \inf \Pi_{\eta}(\mathcal{S}^m(\kappa)), \eta_{\max} = \sup \Pi_{\eta}(\mathcal{S}^m(\kappa)), y_{\min} = \inf \Pi_y(\mathcal{S}^m(\kappa)), y_{\max} = \sup \Pi_y(\mathcal{S}^m(\kappa)).$$

**Definition 2.5.5.** *Let  $Q = [y_1, y_2] \times [\eta_1, \eta_2] \subset \mathbb{R}^2$  be a rectangle. We say  $Q$   $y$ -crosses  $\mathcal{S}^m(\kappa)$  if the following conditions are all satisfied,*

- (a)  $\eta_1 < \eta_{\min}$
- (b)  $\eta_2 > \eta_{\max}$
- (c)  $|Q \cap \mathcal{S}^m(\kappa)| > 0$ .

*These conditions can be rephrased as limitations on the position of the center of  $Q$ , i.e., on  $(\frac{y_1+y_2}{2}, \frac{\eta_1+\eta_2}{2})$ : Conditions (a),(b) limit  $\frac{\eta_1+\eta_2}{2}$  whereas (c) restricts  $\frac{y_1+y_2}{2}$ .*

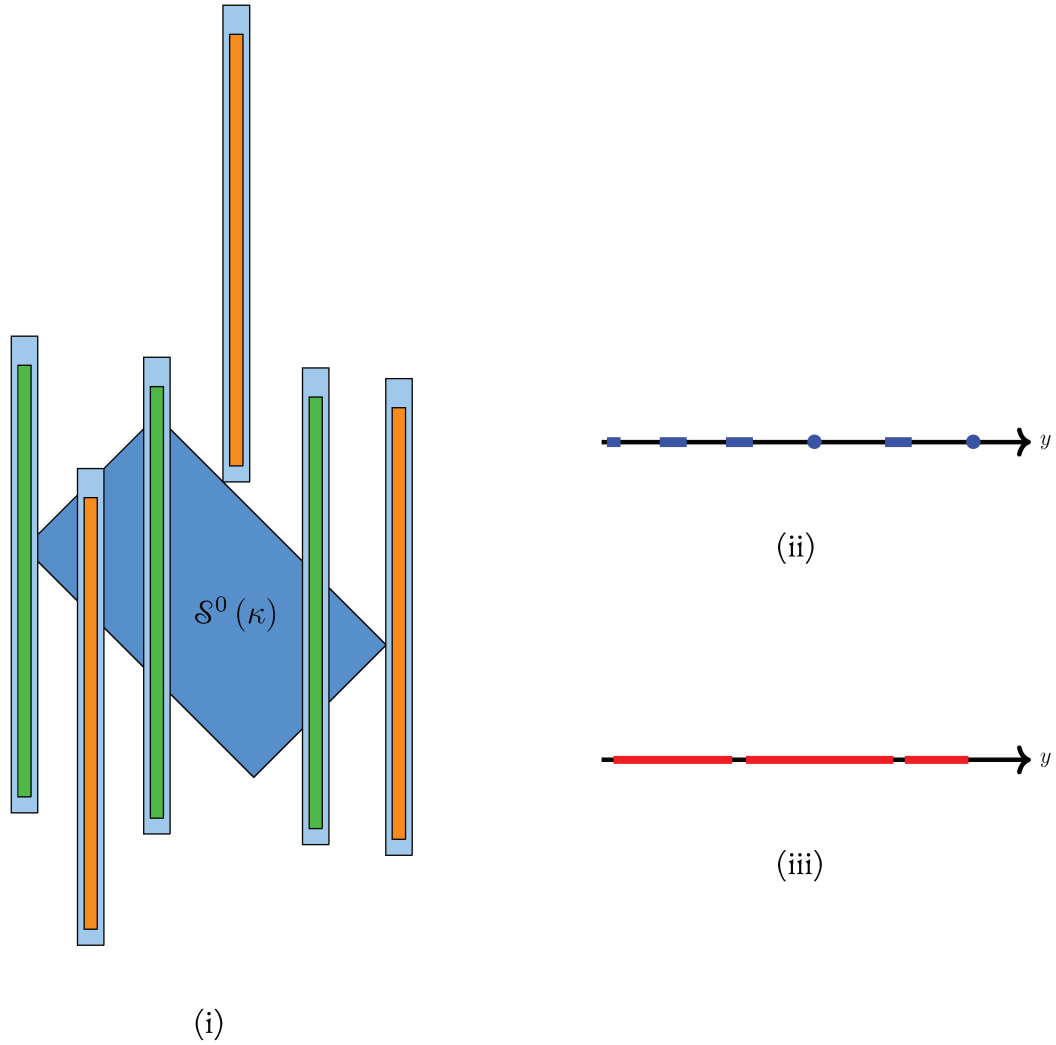
Denoting for every  $(p, q) \in \mathbb{Z}^2$ ,

$$U_{(p,q),t} = D^{-t}\mathcal{U} + \iota^{-1} \begin{pmatrix} p \\ q \end{pmatrix},$$

we write  $(D^{-t}\mathcal{U} + \iota^{-1}\mathbb{Z}^2) \cap \mathcal{S}^0(\kappa) = \mathcal{U}'_t \cap \mathcal{S}^0(\kappa)$  with

$$\mathcal{U}'_t = \bigcup_{U_{(p,q),t} \cap \mathcal{S}^0(\kappa) \neq \emptyset} U_{(p,q),t} \subset \mathcal{R}_t.$$





**Figure 2.9:** Projecting to the  $y$  axis the rectangles on which  $b_{\epsilon,t}^+$  are supported and intersect with  $S^0(\kappa)$  give the sets  ${}_0\Omega_{1,t}^+$  and  $\mathcal{C}({}_0\Omega_{2,t}^+) \cap \Pi_y(S^0(\kappa))$ . (i) The fundamental cell  $S^0(\kappa)$  with several  $\epsilon^{-1}\mathbb{Z}^2$ -translations of  $D^{-t}\mathcal{R}$ . Each such translation contains a shift of  $D^{-t}\mathcal{U}$ . Some of the shifts of  $D^{-t}\mathcal{U}$   $y$ -cross  $S^0(\kappa)$  (in green) and some do not (in orange). (ii) Projecting the  $\mathcal{R}$ -rectangles intersecting  $S^0(\kappa)$  in (i) we obtain the set  ${}_0\Omega_{1,t}^+$ . (iii) Projecting  $\mathcal{C}(D^{-t}\mathcal{U}) \cap S^0(\kappa)$  to the  $y$ -axis in (i) we obtain the set  ${}_0\Omega_{2,t}^+$ .

The gaps between the intervals of  ${}_m\Omega_{2,t}^+$  are obtained by projecting  $\mathcal{U}_t \subset \mathcal{U}'_t$ , the set of rectangles  $U_{(p,q),t}$   $y$ -crossing  $S^0$ ,

$$\mathcal{U}_t = \bigcup_{U_{(p,q),t} \text{ y-crosses } S^0(\kappa)} U_{(p,q),t},$$

to the  $y$ -axis. As  $\mathcal{U}_t$  might be empty we restrict ourselves to  $t > t_+$  where

$$t_+ = \min \{ t \in \mathbb{N}^* : \exists (p, q) \in \mathbb{Z}^2 \text{ s.t. } U_{(p,q),t} \text{ y-crosses } S^0(\kappa) \}. \quad (2.47)$$

In fact, we can bound  $t_+$  naïvely from below: Denote the center of  $\mathcal{U}$  by  $(y_{\mathcal{U}}, \eta_{\mathcal{U}})$ . Since  $\mathcal{S}^0$  is a fundamental cell of  $\iota^{-1}$  it necessarily contains a lattice point  $(y_0, \eta_0) := (y_{(p_0, q_0)}, \eta_{(p_0, q_0)}) \in D^{-t}(y_{\mathcal{U}}, \eta_{\mathcal{U}}) + \iota^{-1}\mathbb{Z}^2$  corresponding to a rectangle  $U_{(p_0, q_0), t} \in \mathcal{U}_t$  having height  $\hbar_{\mathcal{U}}\lambda_u^t$  and width  $w_{\mathcal{U}}\lambda_u^{-t}$ . For  $U_{(p_0, q_0), t}$  to be  $y$ -crossing  $\mathcal{S}^0(\kappa)$  we demand

$$\eta_0 + \frac{1}{2}\hbar_{\mathcal{U}}\lambda_u^t > \eta_{\max} \text{ and } \eta_0 - \frac{1}{2}\hbar_{\mathcal{U}}\lambda_u^t < \eta_{\min}.$$

Since  $\eta_0 \in [\eta_{\min}, \eta_{\max}]$  it is sufficient to demand that  $\lambda_u^t > \frac{2}{\hbar_{\mathcal{U}}}\ell_{\eta}$  that is

$$t_+ \geq \left\lceil \left\lceil \frac{\log(2\ell_{\eta}) - \log \hbar_{\mathcal{U}}}{\log |\lambda_u|} \right\rceil \right\rceil. \quad (2.48)$$

For reasons explained below we choose  $t_+$  satisfying (2.43), (2.48) and for which  $\tau_{1, t_+}, \tau_{2, t_+} < 1$  with  $\tau_{2, t_+}$  defined in (2.53) below.

7. We now give an upper bound on the signed horizontal gaps between rectangles in  $\mathcal{U}_t$ . From (2.46) one can rephrase these horizontal gaps as lengths of intervals containing  ${}_0\Omega_{2, t}^+$ . An upper bound on these gaps would then imply an upper bound on the lengths of the intervals composing  ${}_0\Omega_{2, t}^+$ . We begin by making a technical assumption: We say that a rectangle  $U_{(p_j, q_j), t} \subset \mathcal{U}_t$  is  $C$ -internal if

$$y_j \in \left[ y_{\min} + \left( C + \frac{1}{2}w_{\mathcal{U}} \right) \lambda_u^{-t}, y_{\max} - \left( C + \frac{1}{2}w_{\mathcal{U}} \right) \lambda_u^{-t} \right] \quad (2.49)$$

Let us first bound the horizontal gaps only between  $C_{\iota, \mathcal{R}, \mathcal{U}}^+$ -internal rectangles, with  $C_{\iota, \mathcal{R}, \mathcal{U}}^+$  defined in (A.21) below, and then deduce a general bound. Given a  $C_{\iota, \mathcal{R}, \mathcal{U}}^+$ -internal rectangle  $U_{(p_j, q_j), t} \in \mathcal{U}_t$  we find for it a neighboring rectangle  $U_{(p_{j'}, q_{j'}), t} \subset \mathcal{U}_t$  to its right, i.e., such that

$$0 < y_{j'} - y_j < C_{\iota, \mathcal{R}, \mathcal{U}}^+ \lambda_u^{-t} \quad (2.50)$$

for a constant  $C_{\iota, \mathcal{R}, \mathcal{U}}^+ \in \mathbb{R}_+$  (specified in (A.21)) depending on  $\iota, \mathcal{R}$  and  $\mathcal{U}$  but independent of  $t$ . One can show similarly that every  $C_{\iota, \mathcal{R}, \mathcal{U}}^+$ -internal rectangle has a left neighbor as well, i.e.,  $U_{(p_{j''}, q_{j''}), t} \subset \mathcal{U}_t$  satisfying

$$-C_{\iota, \mathcal{R}, \mathcal{U}}^+ \lambda_u^{-t} < y_{j''} - y_j < 0. \quad (2.51)$$

We remind that from the bad-approximability of  $m_s$  from (2.44), both (2.50) and (2.51) imply that

$$y_{j'} - y_j \asymp \lambda_u^{-t}. \quad (2.52)$$

Roughly speaking it means that neighboring rectangles in  $\mathcal{U}_t$  cannot "accumulate" or to be very far from each other.

8. Let us give a choice of  $C_{\iota, \mathcal{R}, \mathcal{U}}^+$  satisfying (2.50). Every rectangle  $U_{(p_{j'}, q_{j'}), t} \subset \mathcal{U}_t$  neighboring from the right to a fixed rectangle  $U_{(p_j, q_j), t} \subset \mathcal{U}_t$  has to satisfy linear constraints in variables  $p = p_{j'} - p_j, q = q_{j'} - q_j \in \mathbb{Z}$  in both coordinates:

- A relative constraint in the  $\eta$ -direction: In order to  $y$ -cross  $\mathcal{S}^0(\kappa)$  the rectangle  $U_{(p_j, q_j), t}$  has to satisfy both condition (a) and condition (b) in Definition 2.5.5, which expressed in terms of  $(p, q)$  read

$$\begin{cases} \eta_U \lambda_u^t + \eta_j + \frac{q - m_u p}{\sqrt{m_u - m_s}} + \frac{1}{2} \hbar_U \lambda_u^t > \eta_{\max} \\ \eta_U \lambda_u^t + \eta_j + \frac{q - m_u p}{\sqrt{m_u - m_s}} - \frac{1}{2} \hbar_U \lambda_u^t < \eta_{\min}. \end{cases}$$

This constraint on  $(p, q) \in \mathbb{Z}^2 \setminus 0$  is geometrically an infinite strip in the  $x - \xi$  plane bordered by the two lines

$$\begin{aligned} \mathcal{L}_1 &= \left\{ \xi = m_u x + \sqrt{m_u - m_s} \eta_{\max} + m_u p_j - q_j - \sqrt{m_u - m_s} \left( \eta_U + \frac{\hbar_U}{2} \right) \lambda_u^t \right\} \\ \mathcal{L}_2 &= \left\{ \xi = m_u x + \sqrt{m_u - m_s} \eta_{\min} + m_u p_j - q_j - \sqrt{m_u - m_s} \left( \eta_U - \frac{\hbar_U}{2} \right) \lambda_u^t \right\}. \end{aligned}$$

Let us note that the Euclidean distance between them is  $\frac{(\hbar_U \lambda_u^t - \ell_\eta) \sqrt{m_u - m_s}}{\sqrt{m_u^2 + 1}}$  which is  $\asymp \lambda_u^t$ .

- An absolute constraint in the  $y$ -direction: The constraint in (2.50) which is satisfied in the infinite strip bordered by

$$\begin{aligned} \mathcal{L}_3 &= \{ \xi = m_s x \} \\ \mathcal{L}_4 &= \{ \xi = m_s x + \sqrt{m_u - m_s} C_{i, \mathcal{R}, \mathcal{U}}^+ \lambda_u^{-t} \} \end{aligned}$$

which is of width  $\asymp C_{i, \mathcal{R}, \mathcal{U}}^+ \lambda_u^{-t}$ .

The domain in which both constraints are satisfied is a parallelogram  $\mathcal{D}$  in the  $x - \xi$  plane aligned along the stable and unstable directions of  $\gamma$ . We look for  $(p, q) \in \mathbb{Z}^2 \setminus 0$  in  $\mathcal{D}$ . Since  $\gamma$  preserves  $\mathbb{Z}^2$ -points  $\mathcal{D}$  contains a non-zero integer point if and only if  $\mathcal{D}^\sharp = \gamma^n \mathcal{D}$  contains one for every integer  $n \in \mathbb{Z}$ .

9. Let us specify an explicit choice of  $n$  and show it implies  $\mathcal{D}^\sharp$  contains a  $\mathbb{Z}^2 \setminus 0$ -point. Let us take

$$n(\hbar_U, \iota) = t - t_0^+$$

for  $t_0^+ \in \mathbb{N}$  specified explicitly in (A.20). The distance between the unstable sides of  $\mathcal{D}^\sharp$  is  $\asymp \lambda_u^{t_0^+}$  and between the unstable sides is  $\asymp C_{i, \mathcal{R}, \mathcal{U}}^+ \lambda_u^{-t_0^+}$ . Let us denote the center of  $\mathcal{D}^\sharp$  by  $M$ . We consider  $\mathcal{S}_{t, \iota, \mathcal{U}}$ , the maximal square such that (see Figure 2.10)

- $M$  is one of its vertices
- it is properly contained inside  $\mathcal{D}^\sharp$
- Each of its sides is parallel to an axis.

By a direct calculation, the side of  $\mathcal{S}_{t, \iota, \mathcal{U}}$  is  $\ell = \min\{C(t_0^+, C_{i, \mathcal{R}, \mathcal{U}}^+) \lambda_u^{-t_0^+}, C'(t_0^+, C_{i, \mathcal{R}, \mathcal{U}}^+) \lambda_u^{t_0^+}\}$  for positive constants  $C(t_0^+, C_{i, \mathcal{R}, \mathcal{U}}^+), C'(t_0^+, C_{i, \mathcal{R}, \mathcal{U}}^+) \in \mathbb{R}$  dependent on the choice of  $C_{i, \mathcal{R}, \mathcal{U}}^+$  and  $t_0^+$ .

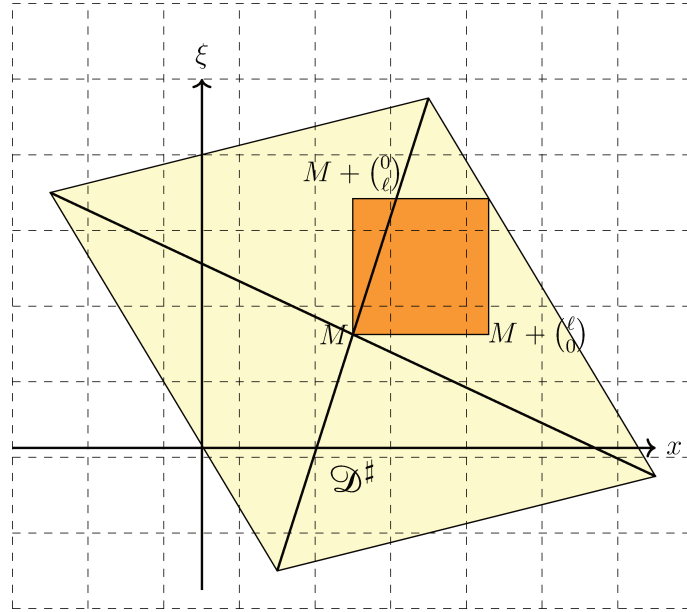
Choosing first  $t_0^+$  and then  $C_{\iota, \mathcal{R}, \mathcal{U}}^+$  to be large enough  $\mathcal{D}^\sharp$  contains a square of side  $\ell > 2$ . We refer to section A.3 for the explicit expressions for  $t_0^+$  and  $C_{\iota, \mathcal{R}, \mathcal{U}}^+$ . Then, both  $\Pi_x \mathcal{S}_{t, \iota, \mathcal{U}}$  and  $\Pi_\xi \mathcal{S}_{t, \iota, \mathcal{U}}$  are intervals of length bounded from below by 2, thus

$$[\min \Pi_x \mathcal{S}_{t, \iota, \mathcal{U}}, \min \Pi_x \mathcal{S}_{t, \iota, \mathcal{U}} + 2] \subseteq \Pi_x \mathcal{S}_{t, \iota, \mathcal{U}}, \quad [\min \Pi_\xi \mathcal{S}_{t, \iota, \mathcal{U}}, \min \Pi_\xi \mathcal{S}_{t, \iota, \mathcal{U}} + 2] \subseteq \Pi_\xi \mathcal{S}_{t, \iota, \mathcal{U}}.$$

We deduce that

$$\{ \lceil \min \Pi_x \mathcal{S}_{t, \iota, \mathcal{U}} \rceil, \lceil \min \Pi_x \mathcal{S}_{t, \iota, \mathcal{U}} \rceil + 1 \} \times \{ \lceil \min \Pi_\xi \mathcal{S}_{t, \iota, \mathcal{U}} \rceil, \lceil \min \Pi_\xi \mathcal{S}_{t, \iota, \mathcal{U}} \rceil + 1 \} \subset \mathcal{S}_{t, \iota, \mathcal{U}}$$

and in fact  $\mathcal{S}_{t, \iota, \mathcal{U}}$  contains a  $\mathbb{Z}^2 \setminus 0$ -point  $(p, q)$ , satisfying both constraints and giving rise to a right neighbor of  $U_{(p_j, q_j)}$ .



**Figure 2.10:** For large enough  $t_0, C_{\iota, \mathcal{R}, \mathcal{U}}^+ > 0$  specified in (A.20),(A.21) the parallelogram  $\mathcal{D}^\sharp$  contains a square with sides parallel to the axes and of length 2.

10. It remains to consider "external", i.e., non-internal, rectangles. From the construction of  $\mathcal{U}$ , any external  $y$ -crossing rectangle satisfies exactly one of the requirements in (2.49). Using the method above, we can find for every external rectangle a neighboring rectangle in a single direction. In the other direction, its distance from a vertex of  $\mathcal{S}^0(\kappa)$  is bounded from above by  $C_{\iota, \mathcal{R}, \mathcal{U}}^+ \lambda_u^{-t}$  so any  $y$ -crossing rectangle neighboring to it from this direction will satisfy either (2.50) or (2.51). Then we conclude that the distance between any two  $y$ -crossing shifts of  $D^{-t}\mathcal{U}$  is at most  $C_{\iota, \mathcal{R}, \mathcal{U}}^+ \lambda_u^{-t}$ .

11. Then, we conclude that  ${}_0\Omega_{2,t}^+ \subset \mathbb{C}\Pi_y(\mathcal{U}_t)$  is a union of intervals of lengths bounded from above by  $C_{\iota, \mathcal{R}, \mathcal{U}}^+ \lambda_u^{-t}$  (with  $C_{\iota, \mathcal{R}, \mathcal{U}}^+$  defined in (A.21)) and between which the gap is  $w_{\mathcal{U}} \lambda_u^{-t}$ . Applying Lemma 2.5.3,

${}_0\Omega_{2,t}^+$  is a  $\nu_{2,+}$ -porous set on a scale  $\tau_{2,t}$  where

$$\nu_{2,+} = \frac{w_{\mathcal{U}}}{4C_{\iota,\mathcal{R},\mathcal{U}}^+ + 2w_{\mathcal{U}}}, \quad \tau_{2,t} = \left( \frac{w_{\mathcal{U}}}{2} + C_{\iota,\mathcal{R},\mathcal{U}}^+ \right) \lambda_u^{-t}. \quad (2.53)$$

We conclude that we may take the following constants in Lemma 2.5.4

$$C_{\epsilon,+} = \begin{cases} \frac{\mathcal{C}_{w(\mathcal{R})+w_{\mathcal{R}}}}{2} & \epsilon = 1 \\ \left( \frac{w_{\mathcal{U}}}{2} + C_{\iota,\mathcal{R},\mathcal{U}}^+ \right) & \epsilon = 2. \end{cases}$$

12. For negative times one can argue similarly. First for every  $t < 0$ , denoting

$$|q_j - q_{j'}| = d^n \left( \iota D^t R_{(p_j, q_j), t}, \iota D^t R_{(p_{j'}, q_{j'}), t} \right) \leq \max\{|m_u|, |m_s|\} c_{\mathcal{R}t, t} \lambda_u^{-t}$$

with  $c_{\mathcal{R}t, t}$  from (2.39). Then, denoting the constant of bad-approximability of  $m_u$  by  $\omega'$  we set

$$\mathcal{C}_{\hbar(\mathcal{R})} = \frac{\omega'}{\max\{|m_u|, |m_s|\} c_{\mathcal{R}t, t} \sqrt{|m_u - m_s|}}.$$

We choose  $t_- \leq \min \left\{ - \left\lfloor \left| \frac{\log(\sqrt{|m_u - m_s|} \mathcal{C}_{\hbar(\mathcal{R})})}{\log|\lambda_u|} \right| \right\rfloor, - \left\lfloor \left| \frac{\log 2 \frac{\ell_y}{w_{\mathcal{U}}}}{\log|\lambda_u|} \right| \right\rfloor, t_0^- \right\}$  so for every  $t < t_-$ ,  $\tau_{\epsilon, t} < 1$  with  $\tau_{\epsilon, t}$  specified below and  $t_0^-$  defined in a completely analogous manner to  $t_0^+$ . Imitating the proof for positive times we obtain that for every  $t < t_-$   ${}_0\Omega_{1,t}^- = \Pi_{\eta} \left( \text{supp } {}_0\check{b}_{1,t} \right)$  is a  $\nu_{1,-} = \frac{\mathcal{C}_{\hbar(\mathcal{R}) - \hbar_{\mathcal{R}}}}{2(\mathcal{C}_{\hbar(\mathcal{R})} + \hbar_{\mathcal{R}})}$ -porous set on a scale  $\tau_{1,t} = \frac{\mathcal{C}_{\hbar(\mathcal{R})} + \hbar_{\mathcal{R}}}{2} \lambda_u^t$  and that  ${}_0\Omega_{2,t}^-$  is a  $\nu_{2,-}$ -porous set on a scale  $\tau_{2,t} = \left( \frac{\hbar_{\mathcal{U}}}{2} + C_{\iota,\mathcal{R},\mathcal{U}}^- \right) \lambda_u^t$  implying

$$C_{\epsilon,-} = \begin{cases} \frac{\mathcal{C}_{\hbar(\mathcal{R})} + \hbar_{\mathcal{R}}}{2} & \epsilon = 1 \\ \left( \frac{\hbar_{\mathcal{U}}}{2} + C_{\iota,\mathcal{R},\mathcal{U}}^- \right) & \epsilon = 2. \end{cases}$$

□

#### 2.5.4 Deducing a porosity estimate on words from the estimate on letters

We deduce from Lemma 2.5.4 that projecting the supports of symbols which correspond to words of length  $4T$  results in sets which are  $\nu$ -porous on a continuous set of scales  $[\tau_{\min, \pm}, \tau_{\max, \pm}]$  defined in (2.55) and (2.56) below. Let us recall that  $T' = 4T$  (where  $T$  is defined in (T)) and consider long words  $w \in \mathcal{W}(2T')$ . Every such word can be written as concatenation of two words of length  $T'$ , that is  $w = w^+ w^-$  with  $w^{\pm} \in \mathcal{W}(T')$ ,

$$w^+ = w_{T'}^+ \dots w_1^+ \quad w^- = w_0^- \dots w_{-T'+1}^- \quad w_k^{\pm} = w_{k+T'-1} \quad -T' + 1 \leq k \leq T', \quad (2.54)$$

and the associated symbols

$$\check{b}_w^+ = \prod_{k=1}^{T'} \check{b}_{w_k^+,k} = \prod_{k=1}^{T'} \check{a}_{w_k^+} \circ \gamma^k \circ \iota, \quad \check{b}_w^- = \prod_{k=-T'+1}^0 \check{b}_{w_k^-,k} = \prod_{k=-T'+1}^0 \check{a}_{w_k^-} \circ \gamma^k \circ \iota.$$

We recall that  ${}_m\check{b}_w^\pm$  is the truncation of  $\check{b}_w^\pm$  in  $\mathcal{S}^m(\kappa)$  (see (2.16)), and we define the associated projections by

$${}_m\Omega_w^+ = \Pi_y \left( \text{supp } {}_m\check{b}_w^+ \right), \quad {}_m\Omega_w^- = \Pi_\eta \left( \text{supp } {}_m\check{b}_w^- \right).$$

Let us now use Lemma 2.5.4 in order to prove that  ${}_m\Omega_w^\pm$  are  $\lambda_u^{-1} \min_\epsilon \nu_{\epsilon,\pm}$ -porous sets uniformly in the choice of  $w^\pm$ . We first recall that

$${}_m\Omega_w^+ = \bigcap_{k=1}^{T'} {}_m\Omega_{w_k^+,k}^+ \quad {}_m\Omega_w^- = \bigcap_{k=-T'+1}^0 {}_m\Omega_{w_k^-,k}^-$$

and as follows  ${}_m\Omega_w^+$  are  $\nu_{w_k^+,+}$  sets on scales  $\tau_{w_k^+,k}$  for every  $k > t_+$  and  ${}_m\Omega_w^-$  are  $\nu_{w_k^-, -}$  sets on scales  $\tau_{w_k^-,k}$  for every  $k < t_-$ .

**Corollary 2.5.6.** *For every  $m \in \mathbb{Z}^2$  and words  $w^\pm \in \mathcal{W}(T')$   ${}_m\Omega_w^+$  is a  $\nu_+ = \lambda_u^{-1} \min_\epsilon \nu_{\epsilon,+}$ -porous set on scales  $I_+ = [\tau_{\min,+}, \tau_{\max,+}]$  where*

$$\tau_{\min,+} = \max_\epsilon \tau_{\epsilon,T'}, \quad \tau_{\max,+} = \min_\epsilon \tau_{\epsilon,t_++1}. \quad (2.55)$$

Analogously  ${}_m\Omega_w^-$  is a  $\lambda_u^{-1} \min_\epsilon \nu_{\epsilon,-}$ -porous set on scales  $I_- = [\tau_{\min,-}, \tau_{\max,-}]$  where

$$\tau_{\min,-} = \max_\epsilon \tau_{\epsilon,-T'+1}, \quad \tau_{\max,-} = \min_\epsilon \tau_{\epsilon,t_- - 1}. \quad (2.56)$$

*Proof.* We prove only the first statement as the proof of the second one follows by exchanging between the stable and the unstable directions. We let  $\sigma \in \text{Sym}(\{t_+ + 1, \dots, T'\})$  be the permutation for which the finite sequence  $\left\{ \tau_{w_{\sigma(t)},\sigma(t)}^+ \right\}_t$  is decreasing in  $t$ . Let  $\tau \in [\tau_{\min,+}, \tau_{\max,+}]$ . It belongs to a minimal sub-interval of the form

$$\left[ \tau_{w_{\sigma(t+1)},\sigma(t+1)}^+, \tau_{w_{\sigma(t)},\sigma(t)}^+ \right] \subset [\tau_{\min,+}, \tau_{\max,+}]$$

for some  $t_+ \leq t \leq T' - 1$ . Let us study the ratio between the endpoints of this interval. If  $w_{\sigma(t)}^+ = w_{\sigma(t+1)}^+$ ,

$$\frac{\tau_{w_{\sigma(t+1)},\sigma(t+1)}^+}{\tau_{w_{\sigma(t)},\sigma(t)}^+} = \lambda_u^{-1}. \quad (2.57)$$

If  $w_{\sigma(t)}^+ \neq w_{\sigma(t+1)}^+$ , we deduce from the minimality of the interval that

$$\tau_{w_{\sigma(t)},\sigma(t)}^+ \in \left[ \tau_{w_{\sigma(t+1)},\sigma(t+1)}^+, \lambda_u \tau_{w_{\sigma(t+1)},\sigma(t+1)}^+ \right]$$

which means

$$\frac{\tau_{w_{\sigma(t+1),\sigma(t+1)}^+}}{\tau_{w_{\sigma(t),\sigma(t)}^+}} \geq \frac{\tau_{w_{\sigma(t+1),\sigma(t+1)}^+}}{\lambda_u \tau_{w_{\sigma(t+1),\sigma(t+1)}^+}} = \lambda_u^{-1}. \quad (2.58)$$

From (2.57) and (2.58) we conclude that,

$$\lambda_u^{-1} \leq \frac{\tau_{w_{\sigma(t+1),\sigma(t+1)}^+}}{\tau_{w_{\sigma(t),\sigma(t)}^+}} \leq \frac{\tau_{w_{\sigma(t+1),\sigma(t+1)}^+}}{\tau} \leq \frac{\tau_{w_{\sigma(t+1),\sigma(t+1)}^+}}{\tau_{w_{\sigma(t+1),\sigma(t+1)}^+}} = 1. \quad (2.59)$$

We recall that for every  $t_+ \leq t \leq T' - 1$  the set  ${}_m\Omega_w^+$  is  $\min_{\epsilon} \nu_{\epsilon,+}$ -porous on a scale  $\tau_{w_{\sigma(t+1),\sigma(t+1)}^+}$ .

Remark 2.5.2 and (2.59) mean that  ${}_m\Omega_w^+$  is a  $\frac{\tau_{w_{\sigma(t+1),\sigma(t+1)}^+}}{\tau} \min_{\epsilon} \nu_{\epsilon,+}$ -porous set on a scale  $\tau$  and in fact a  $\lambda_u^{-1} \min_{\epsilon} \nu_{\epsilon,+}$ -porous set on a scale  $\tau$  for every  $\tau \in [\tau_{\min,+}, \tau_{\max,+}]$ .  $\square$

**Corollary 2.5.7.** *Recalling (T), the construction of  $t_{\pm}$  above and Corollary 2.5.6 (see Remark 2.5.2) means that  ${}_m\Omega_w^+$  is  $\tau_{\max,+} \lambda_u^{-1} \min_{\epsilon} \nu_{\epsilon,+}$ -porous on scales  $[K_+ h^{\rho}, 1]$  and  ${}_m\Omega_w^-$  is  $\tau_{\max,-} \lambda_u^{-1} \min_{\epsilon} \nu_{\epsilon,-}$ -porous on scales  $[K_- h^{\rho}, 1]$  for the constants*

$$K_+ = \frac{\max_{\epsilon} C_{\epsilon,+}}{\tau_{\max,+}}, K_- = \frac{\max_{\epsilon} C_{\epsilon,-}}{\tau_{\max,-}},$$

$\tau_{\max,\pm}$  defined in (2.55) and (2.56) and  $C_{\epsilon,\pm}$  defined in Lemma 2.5.4. We denote henceforth

$$K = \max\{K_+, K_-\}$$

deducing that  ${}_m\Omega_w^{\pm}$  are  $\nu$ -porous sets on scales  $[K h^{\rho}, 1]$  where

$$\nu = \lambda_u^{-1} \min\left\{\tau_{\max,-} \min_{\epsilon} \nu_{\epsilon,-}, \tau_{\max,+} \min_{\epsilon} \nu_{\epsilon,+}\right\}.$$

## 2.6 Estimates for the uncontrolled region

We begin this section by connecting between the operator  $A_w$  and  $\text{Op}_h(b_w^{\pm})$ : The quantization of  $b_w^{\pm}$  satisfies a central role in estimating  $\|A_w\|_{\mathcal{B}(L^2(\mathbb{R}))}$  for  $w \in \mathcal{W}(2T')$ . Since  $2T'$  is bigger than Egorov time  $\frac{\log \frac{1}{h}}{\log |\lambda_u|}$ , we cannot view  $A_w$  as a pseudo-differential operator having a symbol in the previously defined calculi. Nevertheless we show that  $\|A_w\|$  equals to the norm of a product of two pseudo-differential operators up to some reminder. Recalling Remark 1.7.2, we will deduce an estimate on  $\|A_{w,N}\|_{\mathcal{B}(\mathcal{H}_N)} \leq \|A_w\|_{\mathcal{B}(L^2(\mathbb{R}))}$ , proving Proposition 2.1.6. Using the unitarity of  $\mathcal{M}_h(\gamma^{T'-1})$  and Lemma 2.1.3

$$\begin{aligned} \|A_w\|_{\mathcal{B}(L^2(\mathbb{R}))} &= \|A_{w_{2T'-1}}(2T'-1) A_{w_{2T'-2}}(2T'-2) \cdots A_{w_{T'}}(T') \cdots A_{w_1}(1) A_{w_0}\|_{\mathcal{B}(L^2(\mathbb{R}))} \\ &= \left\| \mathcal{M}_h(\gamma^{T'-1}) A_{w_{2T'-1}}(2T'-1) \cdots A_{w_1}(1) A_{w_0} \mathcal{M}_h(\gamma^{-T'+1}) \right\|_{\mathcal{B}(L^2(\mathbb{R}))} \end{aligned} \quad (2.60)$$

$$\begin{aligned}
&= \left\| A_{w_{2T'-1}}(T') \cdots A_{w_{T'}}(1) \cdots A_{w_1}(-T'+2) A_{w_0}(-T'+1) \right\|_{\mathcal{B}(L^2(\mathbb{R}))} \\
&= \left\| A_{w_{T'}^+}(T') \cdots A_{w_1^+}(1) A_{w_0^-}(0) \cdots A_{w_{-T'+2}^-}(-T'+2) A_{w_{-T'+1}^-}(-T'+1) \right\|_{\mathcal{B}(L^2(\mathbb{R}))}
\end{aligned}$$

Invoking the unitary of  $\mathcal{M}_h(\iota)$ , and writing the product of  $T'$  operators as the quantization of a bounded symbol (as in Lemma 2.1.3),

$$\begin{aligned}
&= \left\| \mathcal{M}_h(\iota)^{-1} A_{w_{T'}^+}(T') \cdots A_{w_1^+}(1) \cdots A_{w_{-T'+2}^-}(-T'+2) A_{w_{-T'+1}^-}(-T'+1) \mathcal{M}_h(\iota) \right\|_{\mathcal{B}(L^2(\mathbb{R}))} \\
&= \left\| (\text{Op}_h(b_w^+) + O(h^{1-\rho-\varepsilon})) (\text{Op}_h(b_w^-) + O(h^{1-\rho-\varepsilon})) \right\|_{\mathcal{B}(L^2(\mathbb{R}))} \\
&\leq \left\| \text{Op}_h(b_w^+) \text{Op}_h(b_w^-) \right\|_{\mathcal{B}(L^2(\mathbb{R}))} + O(h^{1-\rho-\varepsilon}),
\end{aligned} \tag{2.61}$$

### 2.6.1 Proof of Proposition 2.1.6

In this subsection we combine the porosity of the sets  ${}_m\Omega_w^+, {}_m\Omega_w^-$  proved in Proposition 2.4.1 with the fractal uncertainty principle in order to deduce Proposition 2.1.6. We will apply the Cotlar-Stein theorem (cf. Theorem 2.6.7 below) in order to prove that

**Proposition 2.6.1.** *There exists  $\beta' > 0$  such that for every two words  $w^\pm \in \mathcal{W}(T')$*

$$\left\| \text{Op}_h(b_w^+) \text{Op}_h(b_w^-) \right\|_{\mathcal{B}(L^2(\mathbb{R}))} = O(h^{\beta'}).$$

Combining Proposition 2.6.1,(2.61), the unitarity of  $\mathcal{M}_h(\iota)$  and Lemma 1.3.3.ii we deduce Proposition 2.1.6 for every  $w \in \mathcal{X}$

$$\|A_{w,N}\|_{\mathcal{B}(\mathcal{H}_N)} \leq \|A_w\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq Ch^{\beta'}.$$

where  $\beta' = \min\{\beta, 1 - \rho\}$  for  $\beta$  defined in Proposition 2.6.2 below. For the rest of the section we prove Proposition 2.6.1. Let us denote

$$\ell(\kappa) = \max\{\ell_y, \ell_\eta\}. \tag{2.62}$$

We begin by obtaining local estimates. Let us define for every  $m, m' \in \mathbb{Z}^2$

$${}_m B_- = \text{Op}_h({}_m b_w^-), \quad {}_m B_+ = \text{Op}_h({}_m b_w^+).$$

We will provide two distinct estimates on  $\|B_{(m,m')}\|_{L^2(\mathbb{R})}$  based on the distance between  $m$  and  $m'$ . First we show that when  $m$  is in the vicinity  $m'$  this norm is  $O(h^\beta)$ .



## 2.6.1.1 Local estimates corresponding to nearby fundamental cells

We recall the concept of smooth cutoffs on a compact  $I \subset \mathbb{R}$ . Consider the  $K$ -thickened set  $I(K) = I + [-K, K]$ . We consider the smooth cut-off  $\chi_I^K(x) \in C_c^\infty(\mathbb{R})$  supported on  $I(2K)$ , equal to unity in  $I(K)$  and  $\chi_I^K|_{(I(2K)) \setminus I(K)}(x) \in [0, 1]$ . The non-smooth characteristic function of a compact  $I \subset \mathbb{R}$  will be denoted by  $\mathbb{1}_I$ . We denote the *semi-classical Fourier transform*  $\mathcal{F}_h$  by

$$(\mathcal{F}_h f)(\eta) := \int_{\mathbb{R}} e^{-2\pi i N \langle y, \eta \rangle} dy$$

and its inverse by

$$(\mathcal{F}_h^{-1} f)(y) := N \int_{\mathbb{R}} e^{2\pi i N \langle y, \eta \rangle} f(\eta) d\eta.$$

For the class of  $\nu$ -porous sets in  $\mathbb{R}$  Dyatlov and Jin have established in [DJ18] (based on preceding [BD18]) an extension of the "original" uncertainty principle. We will use the following version of it,

**Proposition 2.6.2** (Fractal uncertainty principle, [DJN22, Proposition 2.10]). *Fix some  $K > 0, \rho \in (\frac{1}{2}, 1)$  and  $\nu \in (0, 1)$ . Then there exist  $\beta > 0$  depending on  $\nu, \rho$  and a constant  $C$  depending on  $\nu, K$  such that for every  $\nu$ -porous sets  $X, Y \subset \mathbb{R}$  on scales  $Kh^\rho$  to 1,*

$$\left\| \mathbb{1}_{Y(2Kh^\rho)} \mathcal{F}_h^{-1} \mathbb{1}_{X(2Kh^\rho)} \right\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq Ch^\beta \quad h \in (0, 1),$$

where  $X(2Kh^\rho) = X + [-2Kh^\rho, 2Kh^\rho], Y(2Kh^\rho) = Y + [-2Kh^\rho, 2Kh^\rho]$ .

The connection between  $\beta$  to  $\rho$  is expressed in [DJN22] as  $\beta = \beta_0(2\rho - 1)$  for some constant  $\beta_0 > 0$ .

**Remark 2.6.3.** *The sharp cutoffs  $\mathbb{1}_{Y(2Kh^\rho)}$  and  $\mathbb{1}_{X(2Kh^\rho)}$  in Proposition 2.6.2 can be replaced by their smooth counterparts  $\chi_Y^{Kh^\rho}$  and  $\chi_X^{Kh^\rho}$ . Indeed,*

$$\begin{aligned} \left\| \chi_Y^{Kh^\rho} \mathcal{F}_h^{-1} \chi_X^{Kh^\rho} \right\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} &= \left\| \chi_Y^{Kh^\rho} \mathbb{1}_{Y(2Kh^\rho)} \mathcal{F}_h^{-1} \mathbb{1}_{X(2Kh^\rho)} \chi_X^{Kh^\rho} \right\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \\ &\leq \left\| \mathbb{1}_{Y(2Kh^\rho)} \mathcal{F}_h^{-1} \mathbb{1}_{X(2Kh^\rho)} \right\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq Ch^\beta \end{aligned}$$

Suppose the intersection of the fundamental cells is non-empty we deduce a bound on the norm  $\|{}_m B_{+m'} B_{-}\|$  in virtue of the fractal uncertainty principle, or rather its formulation in Remark 2.6.3 above,

**Lemma 2.6.4.** *Let  $m, m' \in \mathbb{Z}^2$  then there exists  $\beta > 0$  and a constant  $C' > 0$  such that for every  $h \in (0, 1]$*

$$\sup_{w=w_+ w_- \in \mathcal{W}(2T')} \|{}_m B_{+m'} B_{-}\| \leq C' h^\beta$$

with  $T'$  specified in (T),  ${}_m B_{+} = \text{Op}_h({}_m b_w^+)$  and  ${}_{m'} B_{-} = \text{Op}_h({}_{m'} b_w^-)$ , the operators corresponding to halves of a word  $w \in \mathcal{W}(2T')$  (see (2.54)).

*Proof.* Recall that  $\chi_{m\Omega_w^+}^{Kh\rho} \in C_c^\infty(\mathbb{R})$  is unity on  $m\Omega_w^+(Kh\rho)$  and that  $\Pi_y \text{supp}(m b_+) \subset m\Omega_w^+$ . Then since  $\chi_{m\Omega_w^+}^{Kh\rho} = \text{Op}_h(\chi_{m\Omega_w^+}^{Kh\rho})$  and since we can take  $\chi_{m\Omega_w^+}^{Kh\rho} \in S_{L^\xi, \rho, 0}(\mathbb{R}^2)$ , from the Moyal product in the class  $S_{L^\xi, \rho, 0}$ ,

$${}_m B_+ \chi_{m\Omega_w^+}^{Kh\rho} = {}_m B_+ + \text{Op}_h(r_+).$$

In fact, since all terms of the Moyal product of  ${}_m B_+ \chi_{m\Omega_w^+}^{Kh\rho}$  except the first vanish repeated integration by parts as in Remark 1.7.6 implies that the norm of the reminder is  $\|\text{Op}_h(r_+)\| = O(h^\infty)$ . Analogously, since  $\chi_{m'\Omega_-}^{Kh\rho} \in C_c^\infty(\mathbb{R})$  and  $\Pi_\eta \text{supp}(m' b_-) = m'\Omega_-$

$$\mathcal{F}_h^{-1} \chi_{m'\Omega_-}^{Kh\rho} \mathcal{F}_h m' B_- = m' B_- + O_{\mathcal{B}(L^2(\mathbb{R}))}(h^\infty).$$

As a result,

$$\begin{aligned} \|{}_m B_+ m' B_-\| &\leq \left\| {}_m B_+ \chi_{m\Omega_w^+}^{Kh\rho} \mathcal{F}_h^{-1} \chi_{m'\Omega_w^-}^{Kh\rho} \mathcal{F}_h m' B_- \right\| + O(h^\infty) \\ &\leq \|{}_m B_+\| \|m' B_-\| \cdot \left\| \chi_{m\Omega_w^+}^{Kh\rho} \mathcal{F}_h^{-1} \chi_{m'\Omega_w^-}^{Kh\rho} \right\| + O(h^\infty). \end{aligned}$$

There exists  $\nu \in (0, 1)$  and a large  $K$  such that  $m\Omega_w^+$  and  $m'\Omega_w^-$  are  $\nu$ -porous sets on scales  $[Kh\rho, 1]$ . From Lemma 1.6.3 the norms of  ${}_m B_\pm$  are uniformly bounded. Applying Remark 2.6.3 there is some  $\beta > 0$  such that

$$\|{}_m B_+ m' B_-\| \leq C'' \left\| \chi_{m\Omega_w^+}^{Kh\rho} \mathcal{F}_h^{-1} \chi_{m'\Omega_w^-}^{Kh\rho} \right\| + O(h^\infty) \leq C' h^\beta.$$

□

### 2.6.1.2 Local estimate for distant fundamental cells

We next provide a local estimate when the two cutoffs are disjoint:

**Lemma 2.6.5.** *Let be a large integer,  $\rho \in [0, 1]$  and  $m, m' \in \mathbb{Z}^2$  such that  $\|\iota^{-1}(m - m')\| \geq 10\ell(\kappa)$  for  $\ell(\kappa)$  from (2.62). Then*

*there exists a scalar  $C$  such that for every  $\alpha, \mathfrak{b} \in C^\infty(\mathbb{R}^2)$  satisfying  $\alpha, \mathfrak{b} \in S_\rho(\mathbb{R}^2)$*

$$\|\text{Op}_h(m\alpha) \text{Op}_h(m'\mathfrak{b})\| \leq h^k \frac{C}{\|m - m'\|^k},$$

*and  $C_k$  depending on the first  $k$  derivatives of  $\alpha$  and  $\mathfrak{b}$ .*

*Proof.* We prove the lemma by adapting the argument of theorem 4.22 in [Zwo12] relying on the disjoint support properties.

1. Let us denote the symplectic form on  $\mathbb{R}^2$  by  $\sigma$ . We will pass from estimates on the derivatives of some smooth function  ${}_{m, m'} c$  to estimates on  $\|\text{Op}_h(m\alpha) \text{Op}_h(m'\mathfrak{b})\|$ . For estimating it we first

write an integral representation of  $\text{Op}_h(m\mathfrak{a})\text{Op}_h(m'\mathfrak{b})$ : For every  $\Sigma = (y_\Sigma, \eta_\Sigma) \in \mathbb{R}^2$  in the dual Fourier phase space, we consider the phase-space Fourier mode

$$e_\Sigma(y, \eta) = \exp(i\sigma(\Sigma, (y, \eta))).$$

We recall the Fourier decomposition of the Weyl quantization from (1.6),

$$\text{Op}_h(m\mathfrak{a})\text{Op}_h(m'\mathfrak{b}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\frac{i}{2h}\sigma(\Sigma_+, \Sigma_-)} \widehat{m\mathfrak{a}}(\Sigma_+) \widehat{m'\mathfrak{b}}(\Sigma_-) T_{h(\Sigma_+ + \Sigma_-)} d\Sigma_+ d\Sigma_- \quad (2.63)$$

$$\begin{aligned} &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{m, m'\mathfrak{c}}(\Sigma) T_{h\Sigma} d\Sigma \\ &= \text{Op}_h(m, m'\mathfrak{c}). \end{aligned} \quad (2.64)$$

Using the quantization of quadratic exponentials (cf. theorem 4.11.ii in [Zwo12]) we express the symbol  $m, m'\mathfrak{c}$  as

$$m, m'\mathfrak{c}(Z) = \mathcal{I}(Z) = \frac{1}{(\pi h)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\frac{2i}{h}\sigma(z_1, z_2)} m\mathfrak{a}(Z - z_1) m'\mathfrak{b}(Z - z_2) dz_1 dz_2. \quad (2.65)$$

2. Let us write  $z = (z_1, z_2) = (y_1, \eta_1, y_2, \eta_2) \in \mathbb{R}^4$ . Let us bound the derivatives  $\partial_Z^\gamma m, m'\mathfrak{c}$ . For that purpose we write the integral representation in (2.65) and estimate it by performing a formal integration by parts. Let us first show that  $\|z\|_2$  is bounded from below: (2.65) implies that exist  $c_1, c_2 \in \mathcal{S}^0(\kappa)$  such that

$$-z_1 = \iota^{-1}m - Z + c_1, \quad -z_2 = \iota^{-1}m' - Z + c_2.$$

Then, using the bound

$$\begin{aligned} \|z_1 - z_2\| &\geq -\|c_1 - c_2\| + \|z_1 - z_2 - c_1 + c_2\| \\ &= -\|c_1 - c_2\| + \|\iota^{-1}(m - m')\| \\ &\geq 8\ell(\kappa) \end{aligned}$$

combined with the Pythagorean and parallelogram laws

$$\|z\|_2^2 = \|z_1\|_2^2 + \|z_2\|_2^2 = \frac{1}{2} (\|z_1 + z_2\|_2^2 + \|z_1 - z_2\|_2^2) \geq \frac{1}{2} \|z_1 - z_2\|_2^2 \geq 32\ell(\kappa)^2. \quad (2.66)$$

3. We use (2.66) for applying the method of non-stationary phase. For performing a formal integration by parts we define the differential operator

$$\mathcal{D}(a) = -\frac{h}{2i} \frac{\langle \nabla \sigma, \nabla a \rangle}{\|\nabla \sigma\|^2}, \quad \mathcal{D} : C^\infty(\mathbb{R}^4) \rightarrow C^\infty(\mathbb{R}^4) \quad (2.67)$$

We note that  $\|\nabla \sigma\|$  does not vanish anywhere on its support: The integration is performed de-facto

on a compact domain, on which in light of (2.66)

$$\|\nabla\sigma\|_2^2 = \|(\eta_2, -y_2, -\eta_1, y_1)\|_2^2 = \|z\|_2^2 \geq 32\ell(\kappa)^2.$$

Then recalling [Hör15, theorem 7.7.1] for every  $k \in \mathbb{N}^*$ ,

$$|({}^t\mathcal{D}^k a)(z)| \leq Ch^k \sum_{j=0}^k \frac{\|\partial_z^j a\|}{\|z\|^{2k-j}}.$$

4. Let us estimate the integral by applying the method of non-stationary phase: for every  $k \in \mathbb{N}^*$  we recall that since  $\mathfrak{a}, \mathfrak{b} \in S_\rho(\mathbb{R}^2)$ ,  $\|\partial_{z_m}^j \mathfrak{a}(Z - z_1) {}_{m'}\mathfrak{b}(Z - z_2)\| \leq C_j h^{-\rho j}$  thus

$$|({}^t\mathcal{D}^k)({}_m\mathfrak{a}(Z - z_1) {}_{m'}\mathfrak{b}(Z - z_2))| \leq Ch^k \left( \sum_{j=0}^k \frac{\|\partial^j({}_m\mathfrak{a} {}_{m'}\mathfrak{b})\|}{\|z\|^{2k-j}} \right) \leq C_k \frac{h^{k(1-\rho)}}{\|z\|^k}. \quad (2.68)$$

We observe that the integral is supported on a product of two compact domains thus using the Pythagorean law,

$$\begin{aligned} |({}^t\mathcal{D}^k)({}_m\mathfrak{a}(Z - z_1) {}_{m'}\mathfrak{b}(Z - z_2))| &\leq C \frac{h^{k(1-\rho)}}{(\|z_1\|^2 + \|z_2\|^2)^{\frac{k}{2}}} \\ &= C \frac{h^{k(1-\rho)}}{(\|\iota^{-1}m - Z + c_1\|^2 + \|\iota^{-1}m' - Z + c_2\|^2)^{\frac{k}{2}}}. \end{aligned}$$

Applying the parallelogram law (i.e.,  $\|z\|^2 + \|z'\|^2 = \frac{1}{2}(\|z + z'\|^2 + \|z - z'\|^2)$ ) for every  $z, z' \in \mathbb{R}^2$ ) and then the reverse triangle inequality

$$\begin{aligned} \|\iota^{-1}m - Z + c_1\|^2 + \|\iota^{-1}m' - Z + c_2\|^2 &= \frac{1}{2} \left( \|\iota^{-1}(m - m') + c_1 - c_2\|^2 \right. \\ &\quad \left. + 4 \left\| Z - \frac{1}{2}(\iota^{-1}(m + m') + c_1 + c_2) \right\|^2 \right) \\ &\geq (\|\iota^{-1}(m - m')\| - 2\ell(\kappa))^2 + \\ &\quad 4 \left( \left\| Z - \frac{1}{2}\iota^{-1}(m + m') \right\| - 2\ell(\kappa) \right)^2. \end{aligned}$$

As follows,

$$|({}^t\mathcal{D}^k)({}_m\mathfrak{a}(Z - z_1) {}_{m'}\mathfrak{b}(Z - z_2))| \leq C_k \frac{h^{k(1-\rho)} N_k(a) N_k(b)}{(\|z_1\|^2 + \|z_2\|^2)^{\frac{k}{2}}} \quad (2.69)$$

$$\begin{aligned}
&= C_k \frac{h^{k(1-\rho)} N_k(a) N_k(b)}{\left(\|\iota^{-1}m - Z + c_1\|^2 + \|\iota^{-1}m' - Z + c_2\|^2\right)^{\frac{k}{2}}} \\
&\leq \frac{C'}{\left(\left(\|\iota^{-1}(m - m')\| - 2\ell(\kappa)\right)^2 + 4\left(\|Z - \frac{1}{2}\iota^{-1}(m + m')\| - 2\ell(\kappa)\right)^2\right)^{\frac{k}{2}}} h^{k(1-\rho)}
\end{aligned}$$

and

$$|\mathcal{I}(Z)| \leq \frac{C''(N_k(a), N_k(b)) \text{Vol}(S^0(\kappa))^2}{\left(\left(\|\iota^{-1}(m - m')\| - 2\ell(\kappa)\right)^2 + 4\left(\|Z - \frac{1}{2}\iota^{-1}(m + m')\| - 2\ell(\kappa)\right)^2\right)^{\frac{k}{2}}} h^{k(1-\rho)-2}.$$

Denoting  $C_{m,m',\ell(\kappa)} = \left(\|\iota^{-1}(m - m')\| - 2\ell(\kappa)\right)^2$  analogously for every multi-index  $\gamma \in \mathbb{N}^2$ ,

$$|\partial_Z^\gamma \mathcal{I}(Z)| \leq \frac{C'_k N_{k+|\gamma|}(a) N_{k+|\gamma|}(b)}{\left(C_{m,m',\ell(\kappa)}^2 + \left(\|Z - \frac{1}{2}\iota^{-1}(m + m')\| - 2\ell\right)^2\right)^{\frac{k}{2}}} h^{k(1-\rho)-|\gamma|\rho-2}, \quad (2.70)$$

5. The estimates on the derivatives of  ${}_{m,m'}c$  obtained above yield an upper bound the norm  $\|\text{Op}_h({}_m a) \text{Op}_h({}_{m'} b)\|_{\mathcal{B}(L^2(\mathbb{R}))}$ . We recall from [Zwo12, Theorem 5.1] that for every  $u \in L^2(\mathbb{R})$

$$\text{Op}_h(c)u = \text{Op}_1(\check{c})\check{u}, \quad \check{u}(\check{x}) = h^{\frac{1}{4}}\check{u}(\sqrt{h}\check{x}), \quad \check{c}(\check{y}, \check{\eta}) = c(\sqrt{h}y, \sqrt{h}\eta)$$

and then  $\check{c} \in S_{\frac{\rho}{2}}$  which from [Zwo12, Theorem 4.23.ii] means there exists some universal  $M$  such that

$$\begin{aligned}
\|\text{Op}_h(c)\|_{\mathcal{B}(L^2(\mathbb{R}))} &= \|\text{Op}_1(\check{c})\|_{\mathcal{B}(L^2(\mathbb{R}))} \\
&\leq C \sum_{|\gamma| \leq M} h^{\frac{|\gamma|\rho}{2}} \|\partial^\gamma \check{c}\|_\infty \\
&\leq C''_k \sum_{|\gamma| \leq M} \sup_Z \frac{N_{k+|\gamma|}(a) N_{k+|\gamma|}(b)}{\left(C_{m,m',\ell(\kappa)}^2 + \left(\|\sqrt{h}Z - \frac{1}{2}\iota^{-1}(m + m')\| - 2\ell\right)^2\right)^{\frac{k}{2}}} h^{k(1-\rho)-\frac{|\gamma|\rho}{2}-2}
\end{aligned}$$

We note that from the definition of  $\mathcal{I}(Z)$  in (2.65),  $Z$  varies inside a compact domain,

$$\|\text{Op}_h(c)\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq C'''_k \frac{N_{k+|\gamma|}(a) N_{k+|\gamma|}(b)}{\|m - m'\|^k} h^{k(1-\rho)-\frac{M\rho}{2}-2}.$$

Taking  $k$  large enough yields the lemma. □

**Remark 2.6.6.** Since  $S_{L^x, \rho, 0}(\mathbb{R}^2), S_{L^\xi, \rho, 0}(\mathbb{R}^2) \subset S_\rho(\mathbb{R}^2)$  we deduce that  ${}_m b_w^+, {}_{m'} b_w^- \in S_\rho(\mathbb{R}^2)$ . For every  $m, m' \in \mathbb{Z}^2$   ${}_m b_w^\pm, {}_{m'} b_w^\pm \in S_\rho(\mathbb{R}^2)$  and for every  $k \in \mathbb{N}^* \setminus \{1, 2\}$  there exists a constant  $C_k$  for which if  $\|\iota^{-1}(m - m')\| > 10\ell(\kappa)$

$$\begin{aligned} \|{}_m B_\pm^* {}_{m'} B_\mp\| &\leq h^k \frac{C_k}{\|m - m'\|^k} \\ \|{}_m B_\pm^* {}_{m'} B_\pm\| &\leq h^k \frac{C_k}{\|m - m'\|^k}, \end{aligned}$$

where  ${}_m B_\pm = \text{Op}_h({}_m b_w^\pm)$ ,  ${}_{m'} B_\pm = \text{Op}_h({}_{m'} b_w^\pm)$  are the operators obtained from quantifying the product of  $\mathbb{1}_{S_m^K}$  with a symbol corresponding to a half of a long word  $w \in \mathcal{W}(2T')$ .

For every  $m, m' \in \mathbb{Z}^2$  let us consider

$$B_{(m, m')} = \text{Op}_h({}_m b_w^+) \text{Op}_h({}_{m'} b_w^-).$$

We deduce now Proposition 2.6.1 from the estimates above,

*Proof of Proposition 2.6.1.* Let us begin by recalling Cotlar-Stein theorem

**Theorem 2.6.7** (Cotlar-Stein theorem, §3 in [Cot55], §VII.2.2 in [SM93]). *Let  $\{A_j\}_{j \in \mathbb{N}}$  be a family of bounded operators on some Hilbert space  $\mathcal{H}$ . Suppose the bounds*

$$\sup_j \sum_k \|A_j^* A_k\|_{\mathcal{H} \rightarrow \mathcal{H}}^{\frac{1}{2}} \leq C, \text{ and } \sup_j \sum_k \|A_j A_k^*\|_{\mathcal{H} \rightarrow \mathcal{H}}^{\frac{1}{2}} \leq C$$

hold then  $\sum_j A_j$  converges, in the strong operator topology, to an operator  $A$  satisfying  $\|A\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq C$ .

We will apply this theorem on the family  $\{B_{(n, n')}\}_{n, n' \in \mathbb{Z}^2}$ . From the definition of  $b_w^\pm$  in (2.54) we note that  $\|\text{Op}_h({}_n b_\pm)\| \leq C$  uniformly in  $n \in \mathbb{Z}^2$  hence this family is bounded. We would like to show the boundedness of the sums. For that purpose let us define the relation  $\sim$  on  $\mathbb{Z}^2 \times \mathbb{Z}^2$  by

$$n \sim n' \iff \|\iota^{-1}(n - n')\| \leq 10\ell(\kappa).$$

Fix  $(q, q') \in \mathbb{Z}^4$  and consider the sum  $\sum_{n, n' \in \mathbb{Z}^2} \|B_{(q, q')}^* B_{(n, n')}\|^{\frac{1}{2}}$ . Splitting the summation,

$$\sum_{n, n' \in \mathbb{Z}^2} \|B_{(q, q')}^* B_{(n, n')}\|^{\frac{1}{2}} = \sum_{\substack{n, n': n \sim n' \\ n' \sim q'}} \|B_{(q, q')}^* B_{(n, n')}\|^{\frac{1}{2}} + \sum_{n, n': n \not\sim n' \text{ or } n' \not\sim q'} \|B_{(q, q')}^* B_{(n, n')}\|^{\frac{1}{2}}.$$

Since  $q \in \mathbb{Z}^2$  is fixed the first term is a finite summation on which we can apply Lemma 2.6.4 thus

$$\sum_{\substack{n' \sim n \\ n' \sim q'}} \|B_{(q, q')}^* B_{(n, n')}\|^{\frac{1}{2}} = O(h^\beta).$$

Let us now consider the second summation. We split it as

$$\begin{aligned} \sum_{n,n': n \not\sim n' \text{ or } n' \not\sim q'} \|B_{(q,q')}^* B_{(n,n')}\|^{\frac{1}{2}} &= \sum_{n,n': \substack{n' \not\sim n \\ n' \not\sim q'}} \|B_{(q,q')}^* B_{(n,n')}\|^{\frac{1}{2}} + \sum_{n,n': \substack{n' \sim n \\ n' \not\sim q'}} \|B_{(q,q')}^* B_{(n,n')}\|^{\frac{1}{2}} \\ &+ \sum_{n,n': \substack{n' \not\sim n \\ n' \sim q'}} \|B_{(q,q')}^* B_{(n,n')}\|^{\frac{1}{2}}. \end{aligned} \quad (2.71)$$

In virtue of Lemma 2.6.5 applied both for  $\|{}_q B_{+n}^* B_+\|$  and for  $\|{}_n B_{+n'} B_-\|$  and where  $k > 3$ ,

$$\begin{aligned} \sum_{n,n': \substack{n' \not\sim n \\ n' \not\sim q'}} \|B_{(q,q')}^* B_{(n,n')}\|^{\frac{1}{2}} &\leq C_k h^{\frac{k}{2}(1-\rho) - \frac{3}{2}\rho - 1} \sum_{n,n': \substack{n' \not\sim n \\ n' \not\sim q'}} \min \left\{ \|\iota^{-1}(n-q)\|^{\frac{1-k}{2}}, \|\iota^{-1}(n-n')\|^{\frac{1-k}{2}} \right\} \\ &\leq C_k h^{\frac{k}{2}(1-\rho) - \frac{3}{2}\rho - 1} \int_{\mathbb{R}^2 \setminus \mathbb{B}(q', 10\ell(\kappa))} \int_{\mathbb{R}^2 \setminus \mathbb{B}(x, 10\ell(\kappa))} \min \left\{ \|\iota^{-1}(x-q)\|^{\frac{1-k}{2}}, \|\iota^{-1}(x-\xi)\|^{\frac{1-k}{2}} \right\} d\xi dx \\ &= C_k h^{\frac{k}{2}(1-\rho) - \frac{3}{2}\rho - 1} \iint_{(\mathbb{R}^2 \setminus \mathbb{B}(0, 10\ell(\kappa)))^2} \min \left\{ \|\iota^{-1}x\|^{\frac{1-k}{2}}, \|\iota^{-1}\xi\|^{\frac{1-k}{2}} \right\} d\xi dx \\ &= O(h^\infty). \end{aligned}$$

The other sums in the right hand side of (2.71) are in a single index,

$$\begin{aligned} \sum_{n,n': \substack{n \sim n' \\ n' \not\sim q'}} \|B_{(q,q')}^* B_{(n,n')}\|^{\frac{1}{2}} &\leq C \sum_{n' \not\sim q'} \|{}_q B_{+n}^* B_+\|^{\frac{1}{2}} \\ &\leq C_k h^{\frac{k}{2}(1-\rho) - \frac{3}{2}\rho - 1} \sum_{n' \not\sim q'} \|\iota^{-1}(n-q)\|^{\frac{1-k}{2}} = O(h^\infty). \\ \sum_{n,n': \substack{n \not\sim n' \\ n' \sim q'}} \|B_{(q,q')}^* B_{(n,n')}\|^{\frac{1}{2}} &\leq C \sum_{n \not\sim n'} \|{}_n B_{+n'}^* B_-\|^{\frac{1}{2}} \\ &\leq C_k h^{\frac{k}{2}(1-\rho) - \frac{3}{2}\rho - 1} \sum_{n \not\sim n'} \|\iota^{-1}(n-n')\|^{\frac{1-k}{2}} = O(h^\infty). \end{aligned}$$

The estimates above yield

$$\sup_{(q,q') \in \mathbb{Z}^4} \sum_{(n,n') \in \mathbb{Z}^4} \|B_{(q,q')}^* (B_{(n,n')})\|^{\frac{1}{2}} = O(h^\beta).$$

By analogous methods we obtain as well that

$$\sup_{(q,q') \in \mathbb{Z}^4} \sum_{(n,n') \in \mathbb{Z}^4} \|(B_{(q,q')}) B_{(n,n')}^*\|^{\frac{1}{2}} = O(h^\beta).$$

From applying Cotlar-Stein theorem and recalling  $\sum_{m \in \mathbb{Z}^2} \mathbb{1}_{S^m}^\kappa = 1$

$$\sum_{n, n' \in \mathbb{Z}^2: \|n\|_\infty, \|n'\|_\infty < M} B_{(n, n')} \xrightarrow{M \rightarrow \infty} \text{Op}_h(b_w^+) \text{Op}_h(b_w^-)$$

in the strong operator topology and

$$\|\text{Op}_h(b_w^+) \text{Op}_h(b_w^-)\|_{\mathcal{B}(L^2(\mathbb{R}))} = O(h^\beta). \quad (2.72)$$

□



# 3

## Small scale Probabilistic Quantum Unique Ergodicity

The present chapter is dedicated for presenting and proving rigorously our results on the probabilistic small scale QUE of random eigenstates. We begin by recalling the quantum periods. Then we construct the random ensemble of eigenbases of  $\mathcal{M}_N$ , by using Haar-measure random rotations in each eigenspace  $V_{\nu,N}$  whose dimension is  $d(\nu, N)$ . The discussion in the following two chapters will deal only with *large-dimensional* eigenspaces having dimension  $\asymp \frac{N}{\log N}$ .

In section 3.4 we compute the Local Weyl Law in the  $V_{\nu,N}$  for the Fourier modes on  $\mathbb{T}^2$ , namely the observables  $e_k(x, \xi) = e^{2\pi i(xk_2 - \xi k_1)}$ , for  $k \in \mathbb{N}^2$ . That is, we will compute the traces  $\text{Tr}(\text{Op}_N(e_k) \Pi_{\nu,N})$ , where  $\Pi_{\nu,N}$  are the spectral projectors on the eigenspaces  $V_{\nu,N}$ . For this aim, we will express the projectors as sums of powers of the propagator,  $\mathcal{M}_N^\tau$ , and compute the traces  $\text{Tr}(\mathcal{M}_N^\tau \text{Op}_N(e_k))$ . This will be done by adapting the trace formula derived by Keating [Kea91b] to compute the  $\text{Tr}(\mathcal{M}_N^\tau)$ .

Expanding an arbitrary observable  $a \in C^\infty(\mathbb{T}^2)$  in Fourier series, we deduce a Local Weyl Law. The proof can be extended adapted to  $N$ -dependent symbols  $a \in S_{\delta+}(\mathbb{T}^2)$ .

In a second step, we apply a version of the Hanson-Wright inequality derived by Chatterjee-Galkowski [CG18], to show probabilistic estimates for the deviation of  $\langle \psi_{j,N}, \text{Op}_N(a) \psi_{j,N} \rangle$  from the trace

$$\frac{1}{d(\nu, N)} \text{Tr}(\Pi_{\nu,N} \text{Op}_N(a))$$

of the mean value  $\int_{\mathbb{T}^2} a(\rho) d\rho$ , where  $\psi_{j,N}$  are the elements of the random eigenbasis of  $\mathcal{M}_N$ . These estimates will allow us to conclude our main theorem.

**Notation 3.0.1.** *For the next two chapters we adapt the "canonical notations" from results on small scale Q(U)E: we denote a point on  $\mathbb{T}^2$  by  $\rho$  and the letter  $\delta$  is reserved for the required symbol classes.*

### Recalling the quantum map

Due to the rich group theoretic structure underlying its construction, one can express the matrices  $\mathcal{M}_N(\gamma)$  explicitly ([HB80] and section A.1).

**Example 3.0.2.** *In the case of the Degli Esposti automorphism  $\gamma_{DE}$  discussed in Example P.2.1, for every odd prime  $N$ , as we discuss in section A.1 the matrix  $\mathcal{M}_N$  is given, up to a scalar phase factor [KR00,*

*Mez02*], by:

$$\mathcal{M}_N(\gamma_{DE})_{q_2q_1} = \frac{1}{\sqrt{iN}} \exp\left(\frac{2\pi i}{N} (q_1^2 - q_2q_1 + q_2^2)\right).$$

A general formula for  $\mathcal{M}_N(\gamma)$  was given in [HB80], using the notation

$$\langle f(k) \rangle_k := \frac{1}{|\gamma_{12}|} \sum_{k=0}^{|\gamma_{12}|-1} f(k) \quad (3.1)$$

for the averaging of an  $|\gamma_{12}|$ -periodic function  $f: \mathbb{Z} \rightarrow \mathbb{C}$ .

For any  $\gamma \in \tilde{\Gamma}(2)$  such that  $\gamma_{12} \neq 0$  (this is the case for all hyperbolic matrices), the elements of  $\mathcal{M}_N(\gamma)$  are then given by:

$$(\mathcal{M}_N(\gamma))_{q_2q_1} = \sqrt{\frac{\gamma_{12}}{iN}} \left\langle \exp\left(\frac{\pi i}{\gamma_{12}N} (\gamma_{11}(q_1 + Nk)^2 - 2q_2(q_1 + Nk) + \gamma_{22}q_2^2)\right) \right\rangle_k. \quad (3.2)$$

Henceforth we fix our hyperbolic matrix  $\gamma \in \tilde{\Gamma}(2)$ , and write for brevity  $\mathcal{M}_N := \mathcal{M}_N(\gamma)$ .

### 3.1 Quantum periods

Due to the group structure underlying the construction of  $\mathcal{M}_N$ , each matrix  $\mathcal{M}_N$  is quasi-idempotent [HB80, KR00], i.e., there exists some power  $p > 0$  such that

$$\mathcal{M}_N^p = e^{\kappa i} \text{Id}_{\mathcal{H}_N} \quad \text{for some } N\text{-dependent } \kappa \in [-\pi, \pi]. \quad (3.3)$$

The minimal  $p$  is called the quantum period of  $\mathcal{M}_N$  and we denote it by  $P(N)$ . Equivalently one can study the classical period, i.e., the minimal power  $P_{\text{class}}(N)$  such that

$$\gamma^{P_{\text{class}}(N)} = I_2 + N \begin{pmatrix} r_{11,N} & r_{12,N} \\ r_{21,N} & r_{22,N} \end{pmatrix}. \quad (3.4)$$

In these notations, we get [HB80, Section 3] the following relation between quantum and classical periods:

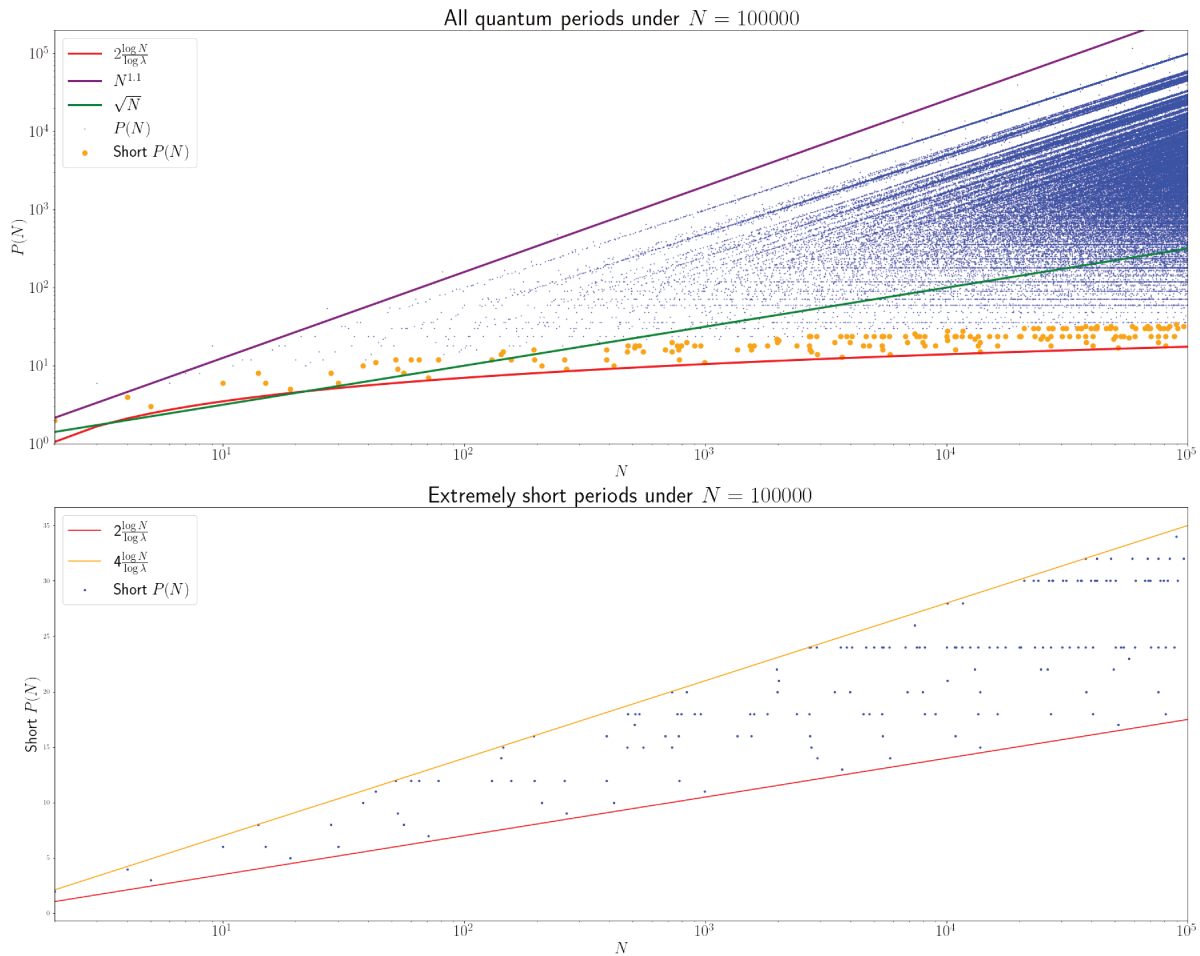
$$P(N) = \begin{cases} P_{\text{class}}(N) & N \in 2\mathbb{N} + 1 \\ P_{\text{class}}(N) & N \in 2\mathbb{N} \text{ and } r_{12,N}, r_{21,N} \in 2\mathbb{Z} \\ 2P_{\text{class}}(N) & \text{Otherwise.} \end{cases} \quad (3.5)$$

It has been shown by Keating in [Kea91a] that  $P(N)$  satisfies for every  $N \in \mathbb{N}^*$  the general bounds (see Figure 3.1)

$$2 \frac{\log N}{\log |\lambda|} - C \leq P(N) \leq CN \log \log N \quad (3.6)$$

for some  $C > 0$ .

Nevertheless the quantum periods tend to accumulate close to the upper bound: Rudnick and Kurlberg showed [KR01a] that for a density-1 set of  $N$ s,  $P(N) \geq \sqrt{N} e^{(\log N)^\delta}$ . Kurlberg sharpened this



**Figure 3.1:** The quantum periods of  $\mathcal{M}_N(\gamma_{DE})$ . Above: The periods for  $2 \leq N \leq 100000$  satisfy the bounds found by Keating in [Kea91a]. Below: Only 168 values of  $N \leq 100000$  are highly degenerate, i.e., in a set  $\mathcal{N}_\alpha$  defined in (3.7) below.

bound further, assuming the General Riemann’s hypothesis (GRH) [Kuro3]: he showed that there exist constants  $C, \varepsilon > 0$  such that for a subset of density  $\varepsilon$  inside  $\mathbb{N}$ ,

$$CN^{1-\varepsilon} \leq P(N) \leq CN^{1+\varepsilon}.$$

Using a number theoretic approach (cf. [KRoo, Kelo8, Kelo]) the relation in (3.4) implies that the concentration of  $P(N)$  can also be studied by considering the order of  $\lambda$  inside  $\mathbb{Z}_N[\lambda]$ . Roskam shows in [Ros00] that for some positive density of primes this order is maximal (other similar results appear in [Wei73, CW75, Len77]). The proofs of these results rely on density type theorems holding under GRH (e.g. Artin’s conjecture or Serre-Chebotarev density theorem).

We will consider a rare opposite situation occurring for a sequence of ”highly degenerate” values of  $N$ , for which the period is very small (see section 3.2 below). We can already deduce that some eigenspace

of  $\mathcal{M}_N$  must be large.

**Lemma 3.1.1.** *For every  $N \in \mathbb{N}^*$  the quantum map  $\mathcal{M}_N$  admits an eigenspace  $V_{\nu,N}$  with dimension  $d(\nu, N) := \dim V_{\nu,N} \geq \frac{N}{P(N)}$ .*

*Proof.* We recall that for every  $\nu \in \text{Spec}(\mathcal{M}_N)$  and  $\psi \in V_{\nu,N}$ ,

$$\mathcal{M}_N^{P(N)}\psi = \nu^{P(N)}\psi = e^{\kappa i}\psi,$$

Hence the eigenvalue  $\nu = e^{\frac{\kappa}{P(N)}i + \frac{2\pi j i}{P(N)}}$  for some  $j \in \llbracket 0, P(N) - 1 \rrbracket$ . Since  $\mathcal{M}_N$  is unitary it is diagonalisable which means that counting the distinct elements in the spectrum  $\#\text{Spec}(\mathcal{M}_N) \leq P(N)$ . Since  $\sum_{\nu \in \text{Spec}(\mathcal{M}_N)} \dim V_{\nu,N} = N$  by the pigeonhole principle at least one of the eigenspaces  $V_{\nu,N}$  is of dimension  $d(\nu, N) \geq \frac{N}{P(N)}$ .  $\square$

### 3.2 Construction of random eigenbases

Henceforth we will consider only quantum maps  $\mathcal{M}_N$  with short quantum period (see also the bottom part of Figure 3.1 for all very degenerate  $N$  under  $N = 100000$ , for the map  $\gamma_{DE}$ ),

**Definition 3.2.1.** *Let  $\alpha, C > 0$ .*

*The map  $\mathcal{M}_N$  has an  $(\alpha, C)$ -short quantum period if*

$$N \in \mathcal{N}_{\alpha,C} := \left\{ N \in \mathbb{N}^* : P(N) \leq \alpha \frac{\log N}{\log \lambda} + C \right\}. \quad (3.7)$$

**Remark 3.2.2.** *Bonechi-De Bièvre [BDB00] showed that, for some  $C_0 > 0$ , the sequence  $\mathcal{N}_{2,C_0}$  is infinite, namely there exists infinitely many  $N$  which saturate the lower bound in (3.6).*

*We note that if  $\alpha_1 \leq \alpha_2$  then  $\mathcal{N}_{\alpha_1,C} \subset \mathcal{N}_{\alpha_2,C}$  and that if  $C_1 \leq C_2$  then  $\mathcal{N}_{\alpha,C_1} \subset \mathcal{N}_{\alpha,C_2}$ . From these monotonicities, for every  $\alpha \geq 2$  and  $C > C_0$  the set  $\mathcal{N}_{\alpha,C}$  is infinite. The numerical computation in Figure 3.1 hints at the fact that the inclusion might be strict for some values of  $\alpha$ .*

We fix some  $\alpha \in [2, 4)$  and  $C \geq C_0$  (the reason to restrict ourselves to  $\alpha < 4$  will be clear below). In the following we will drop the subscript  $C$  in the notation, and call  $\mathcal{N}_\alpha = \mathcal{N}_{\alpha,C}$ .

For  $N$  in this sequence, we will be able to improve the pigeonhole principle of Lemma 3.1.1, and show that the  $P(N)$  eigenspaces of  $\mathcal{M}_N$  actually have very comparable dimensions. Namely, in Proposition 3.4.1.i we will show that, for any  $\alpha \in [2, 4)$ , for  $N$  along the sequence  $\mathcal{N}_\alpha$ , all the eigenspaces  $V_{\nu,N}$  of  $\mathcal{M}_N$  have dimensions satisfying (*asymptotically*)

$$\dim V_{\nu,N} \sim \frac{N}{P(N)}, \quad \text{as } N \rightarrow \infty,$$

uniformly with respect to the eigenvalue  $\nu$  of  $\mathcal{M}_N$ .

Consider one eigenspace  $V_{\nu,N} = \ker(\mathcal{M}_N - \nu)$ , of dimension  $d(\nu, N)$ , and choose one orthonormal basis  $\{\varphi_{j,N}\}_{j=1}^{d(\nu,N)}$  on that space. We will now construct a random basis of  $V_{\nu,N}$ , by "rotating" this reference basis. For this, we consider a random unitary matrix  $X_{\nu,N} \sim (U(d(\nu, N)), \mathbb{P}_{\nu,N})$ , where  $\mathbb{P}_{\nu,N} = d\text{vol}_{U(d(\nu,N))}$  is the *normalized* Haar measure on the unitary group  $U(d(\nu, N))$ .

We then define, for each  $j = 1, \dots, d(\nu, N)$ , the state  $\psi_{j,N}$  through the linear combination

$$\psi_{j,N} = \sum_{k=1}^{d(\nu,N)} x_{jk} \varphi_{k,N} \quad X_{\nu,N} = (x_{jk})_{j,k}.$$

Elementary linear algebra shows that  $\{\psi_{j,N}\}$  forms an orthonormal basis of  $V_{\nu,N}$ , and all orthonormal bases can be obtained this way.

Let us call the set of orthonormal bases of  $V_{\nu,N}$  by  $\mathcal{O}_{\nu,N}$ ; the above construction equips this set with a probability measure  $\mathbb{P}_{\nu,N}$ . Making this construction independently for all eigenvalues  $\nu$  of  $\mathcal{M}_N$ , we obtain a probability measure  $\mathbb{P}_N = \prod_{\nu} \mathbb{P}_{\nu,N}$  on  $\mathcal{O}_N$ , the set of all orthonormal eigenbases of  $\mathcal{M}_N$ .

Performing this construction along the full sequence  $N \in \mathcal{N}_{\alpha}$ , we equip the global set of eigenbases

$$\mathcal{O} = \prod_{N \in \mathcal{N}_{\alpha}} \mathcal{O}_N$$

with the product probability measure  $\mathbb{P} = \prod_{N \in \mathcal{N}_{\alpha}} \mathbb{P}_N$ . A random eigenbasis of  $V_{\nu,N}$ , resp. of  $\mathcal{M}_N$ , resp. of  $(\mathcal{M}_N)_{N \in \mathcal{N}_{\alpha}}$  will be an element of each of these probability spaces.

Our first goal is to show that,  $\mathbb{P}$ -almost surely, the eigenstates in random eigenbasis of  $(\mathcal{M}_N)_{N \in \mathcal{N}_{\alpha}}$  asymptotically equidistribute on the torus, as  $N \rightarrow \infty$ .

**Remark 3.2.3.** *The Quantum Ergodicity result of [BDB96] requires to extract, for each  $N$ , a subset  $S(N) \subset \llbracket 1, N \rrbracket$  of asymptotic density 1, so that the eigenstates  $(\psi_{n,N})_{j \in S(N)}$  become equidistributed as  $N \rightarrow \infty$ . But the rest of the eigenbasis is ignored.*

*Restricting ourselves to longer periods, we know from [KR01a, Bou07] that the sequence  $\{N \in \mathbb{N} : P(N) \geq N^{\varepsilon}\}$  is of full density inside  $\mathbb{N}$ , and that any family of eigenbases along that sequence satisfies QUE. Our sequence  $\mathcal{N}_{\alpha}$  is essentially disjoint from the latter, so it must be of asymptotic density 0. We know from [FNDB03] that some  $\mathcal{M}_N$  along that sequence admit non-ergodic eigenstates, so the process of searching for a typical eigenbasis is necessary to ensure QUE.*

### 3.3 Small scale symbols

As explained in the introduction, we will be able to address the localization of the eigenstates of  $\mathcal{M}_N$  on macroscopic scales (which is the standard framework for QE or QUE), but also on certain microscopic scales. For this aim, we need to test the eigenstates using test functions localized on small balls of phase space, that is balls whose radius  $r = r(N)$  depends on  $N$ , and decays to zero as  $N \rightarrow \infty$  at a certain rate. From now on, we will always assume that these scales do not decay faster than algebraically: for

some  $\delta > 0$  to be made precise below, the scales will satisfy

$$r(N) \geq N^{-\delta}, \quad \forall N \in \mathbb{N}^*.$$

To be quantizable, our symbols must be smooth, but they should approximate a characteristic function on a small ball  $B(\rho, r(N))$ , we will call  $\chi_{\rho, r(N)}$  such a symbol (here  $\rho \in \mathbb{T}^2$  is the center of the ball).

More precisely, we wish our approximate characteristic function  $\chi_{\rho_0, r}$  to have the following properties:

- (i)  $\chi_{\rho_0, r}(\rho) = 1$  for  $\rho \in B(\rho_0, r)$ ;
- (ii)  $\chi_{\rho_0, r}(\rho) = 0$  for  $\rho \notin B(\rho_0, r(1 + \varepsilon))$ , where  $\varepsilon > 0$  is small.

Instead of dragging an extra small parameter  $\varepsilon > 0$ , it will turn easier to replace  $\varepsilon$  by a slowly decreasing function of  $N$ , e.g. by  $\frac{1}{\log N}$ . We thus obtain a modified condition

$$(ii') \quad \chi_{\rho_0, r}(\rho) = 0 \text{ for } \rho \notin B(\rho_0, r(1 + (\log N)^{-1})).$$

These conditions will have the effect that these symbols satisfy

$$\int_{\mathbb{T}^2} \chi_{\rho_0, r(N)}(\rho) d\rho = \pi r(N)^2 (1 + o(1)), \quad N \rightarrow \infty.$$

uniformly with respect to the position of  $\rho$ .

Let us show how to

**Lemma 3.3.1.** *Fix a decaying set of scales  $r = (r(N))_{N \in \mathbb{N}^*}$ . Then, for any  $\rho \in \mathbb{T}^2$ , one can construct a family of symbols  $\chi_{\rho, r(N)}$  satisfying the conditions Condition (i), Condition (ii'), and the following derivative estimates:*

$$\forall \alpha \in \mathbb{N}^2, \exists C_\alpha, \quad \|\partial^\alpha \chi_{\rho, r(N)}\|_\infty \leq C_\alpha \left( \frac{\log N}{r(N)} \right)^{-|\alpha|}. \quad (3.8)$$

*Proof.* One can construct a radially symmetric function  $\chi_0 \in C^\infty(\mathbb{R}^2, [0, 1])$  such that  $\chi_0 = 1$  inside the unit disk, while  $\chi_0 = 0$  outside  $B(0, 2)$ . Re-scaling the radial direction by a factor  $1/L$  outside the unit disk and denoting the polar coordinates on  $\mathbb{R}^2$  by  $(R, \theta)$ , we define

$$\chi_L(R, \theta) = \begin{cases} 1, & |R| \leq 1, \\ \chi_0(L(R-1)+1), & |R| \geq 1, \end{cases}$$

which vanishes outside  $|R| \leq 1 + 1/L$ . Finally, we define  $\chi_{r, L}$  by a global re-scaling by a factor  $r$ :

$$\chi_{r, L}(R, \theta) = \chi_L\left(\frac{R}{r}, \theta\right).$$

The chain rule when differentiating, with respect to  $(x, \xi)$ , shows the global bounds

$$\|\partial^\alpha \chi_{r, L}\|_\infty \leq C_\alpha \left( \frac{L}{r} \right)^{-|\alpha|}.$$

Let us choose  $L = \log N$ , allowing us to remove  $L$  from the indexing of  $\chi$ . Periodizing this function, translating it by  $R$ , we obtain a function  $\chi_{\rho,r}$  satisfying all required properties.  $\square$

The estimates in Lemma 3.3.1 and the bounds  $r(N) \geq N^{-\delta} \sim h^\delta$  for some  $\delta < \frac{1}{2}$  imply that the functions  $\chi_{\rho,r(N)}$  belong to a mildly exotic symbol class. We first recall the symbol class [Zwo12, §4.4]

$$S_\delta(\mathbb{T}^2) := \{a \in C^\infty(\mathbb{T}^2 \times (0, 1]_h) : \forall \alpha \in \mathbb{N}^2, \exists C_\alpha > 0, \|\partial_\rho^\alpha a(\rho; h)\| \leq C_\alpha h^{-\delta|\alpha|}\},$$

which comprises symbols which oscillate on scales  $\geq h^\delta$ . Due to our extra logarithmic factor in (3.8), our function  $\chi_{\rho,r(N)}$  barely belongs to this class. We therefore consider the slight broadening of this class:

$$S_{\delta+}(\mathbb{T}^2) := \bigcap_{\varepsilon \in (0, 1]} S_{\delta+\varepsilon}(\mathbb{T}^2).$$

One checks that when  $r(N) \geq N^{-\delta}$ , the symbols  $\chi_{\rho,r(N)}$  constructed in the Lemma belong to the class  $S_{\delta+}(\mathbb{T}^2)$ .

**Remark 3.3.2.** *The notation  $S_{\delta+}$  comes with the following piece of notation. Since for all  $\varepsilon > 0$  one has  $\log N \leq C_\varepsilon N^\varepsilon$  for some  $C_\varepsilon$ , we may replace estimates of the type  $f(N) \leq CN^\delta \log N$  by the more "economical"  $f(N) \leq C_\varepsilon N^{\delta+\varepsilon}$ . We will then use the following shortened notation:*

$$f(N) = O(N^{\delta+}) \text{ iff } \forall \varepsilon > 0, \exists C_\varepsilon, \quad |f(N)| \leq C_\varepsilon N^{\delta+\varepsilon}.$$

We will not necessarily restrict ourselves to the cutoffs constructed in the Lemma. Still assuming that our set of scales  $\mathbf{r} = (r(N))_N$  satisfies  $r(N) \geq N^{-\delta}$ , we will allow a more general class of  $N$ -dependent symbols, and consider the following family of symbols:

$$\mathfrak{X}(\rho, \mathbf{r}) = \left\{ \chi \in S_{\delta+}(\mathbb{T}^2) : 0 \leq \chi \leq 1, \mathbb{1}_{B(\rho, r(N))} \leq \chi \leq \mathbb{1}_{B(\rho, r(N)(1 + \frac{1}{\log N}))} \right\}. \quad (3.9)$$

This is the symbol class we used in our definition of small scale Q(U)E in Definition P.3.3.

An important property of these symbols will be the control of their Fourier series.

**Lemma 3.3.3.** *For every  $j \in \mathbb{Z}^2$*

$$|\widehat{\chi_{\rho, r(N)}}(j)| \leq Cr^2. \quad (3.10)$$

*If  $|j| \gg N^{\delta+\varepsilon}$  for some  $\varepsilon > 0$  this estimate can be improved: for every  $m \in \mathbb{N}^*$  there exists a constant  $C_{\chi_{\rho, r(N)}, m, \varepsilon}$  such that*

$$|\widehat{\chi_{\rho, r(N)}}(j)| \leq C_{\chi_{\rho, r(N)}, m, \varepsilon} r^2 \frac{N^{(\delta+\varepsilon)m}}{(1 + |j|)^m}. \quad (3.11)$$

*Proof.* Bounding naively the integral defining  $\widehat{\chi_{\rho, r(N)}}$  and using the condition  $\chi_{\rho, r(N)} \leq \mathbb{1}_{B(\rho, r(N)(1 + \frac{1}{\log N}))}$ ,

$$|\widehat{\chi_{\rho, r}}(j)| \leq \|\chi_{\rho, r}\|_\infty \pi r^2 \left(1 + \frac{1}{\log N}\right)^2 \leq Cr^2, \quad (3.12)$$

If  $|j| \gg N^{\delta+\varepsilon}$ , recalling that  $\chi_{\rho,r(N)} \in S_{\delta+}$  and performing integration by parts repeatedly  $m$  times there exists a constant  $C_{\chi_{\rho,r(N)},m\varepsilon}$

$$|\widehat{\chi_{\rho,r}}(j)| \leq C_{\chi_{\rho,r(N)},m,\varepsilon} r^2 \frac{N^{(\delta+\varepsilon)m}}{(1+|j|)^m}. \quad (3.13)$$

□

**Remark 3.3.4.** *Han's small scale equidistribution on  $\mathbb{T}^2$  [Han18] involved a slightly different family of approximants of  $\mathbb{1}_{B(\rho,r)}$ , the so-called Beurling-Selberg trigonometric polynomials  $b_{\rho,r}^{\pm}$ , which are not of small support, but whose Fourier coefficients are easier to control than the ones of our functions  $\chi_{\rho,r}$  [Har98, Lemma 4]. Our probabilistic QUE result would hold as well using these functions, whose Fourier coefficients have similar properties as those of our  $\chi_{\rho,r(N)}$ .*

### 3.4 A local Weyl law on the eigenspaces

Like in the references [Zel92, Van97, Map13, BL13, Zel14], the first ingredient in the proof of the probabilistic Q(U)E is an asymptotic estimate for the trace of the restriction of quantum observables on the subspaces on which the averaging is performed. In our case, those subspaces are the eigenspaces  $V_{\nu,N}$  of  $\mathcal{M}_N$ . We recall that  $\Pi_{\nu,N} : \mathcal{H}_N \rightarrow V_{\nu,N}$  is the orthogonal projector on the eigenspace  $V_{\nu,N}$ .

The local Weyl law is a comparison between the classical average of an observable  $a \in C^\infty(\mathbb{T}^2)$ , which we may denote by

$$\omega_\infty(a) := \int_{\mathbb{T}^2} a(x, \xi) dx d\xi,$$

and the *global quantum average* of  $\text{Op}_N(a)$  on the subspace  $V_{\nu,N} \subset \mathcal{H}_N$ , which we may denote by

$$\omega_{\nu,N}(a) := \frac{1}{d(\nu, N)} \text{Tr}(\Pi_{\nu,N} \text{Op}_N(a) \Pi_{\nu,N}) = \frac{1}{d(\nu, N)} \text{Tr}(\text{Op}_N(a) \Pi_{\nu,N}).$$

(here we borrow notations often used by Zelditch, e.g. in [Zel14]).

**Proposition 3.4.1.** *Choose some  $\alpha \in [2, 4)$  and  $\delta < \frac{1}{2} - \frac{\alpha}{8}$ , consider an infinite subsequence  $N \in \mathcal{N}_\alpha$ . For each  $N \in \mathcal{N}_\alpha$  we consider any eigenvalue  $\nu = \nu_N$  of  $\mathcal{M}_N$ . We have the following asymptotics as  $N \rightarrow \infty$ :*

1.  $\dim V_{\nu,N} = \frac{N}{P(N)} + O(N^{\frac{\alpha}{4}}) \asymp \frac{N}{\log N}$ .
2. for every symbol  $a \in C^\infty(\mathbb{T}^2)$ , possibly depending on  $N$ ,

$$\left| \frac{1}{d(\nu, N)} \text{Tr}(\text{Op}_N(a) \Pi_{\nu,N}) - \int_{\mathbb{T}^2} a(x, \xi) dx d\xi \right| \leq C \left( \sum_{n \neq 0} |\hat{a}(n)| N^{\frac{\alpha}{4}-1} + \sum_{k \neq 0} |\hat{a}(Nk)| \right), \quad (3.14)$$

where  $C > 0$  is a constant independent of the symbol  $a$ .



3. As a consequence of this estimate, we obtain the Local Weyl Law for any  $a \in C^\infty(\mathbb{T}^2)$ :

$$\lim_{N_\alpha \ni N \rightarrow \infty} \frac{1}{d(\nu, N)} \operatorname{Tr}(\operatorname{Op}_N(a) \Pi_{\nu, N}) = \int_{\mathbb{T}^2} a(x, \xi) dx d\xi.$$

Using the notations introduced above, this reads

$$\lim_{N_\alpha \ni N \rightarrow \infty} \omega_{\nu, N}(a) = \omega_\infty(a).$$

For a symbol  $a \in S_{\delta+}$ ,

$$\omega_{\nu, N}(a) = \omega_\infty(a) + O(N^{-\beta+})$$

where  $\beta = 1 - \frac{\alpha}{4} - 2\delta$ .

Note that the proposition implies quantum ergodicity on each eigenspace. The proof of this proposition proceeds by computing the traces in the case of the Fourier modes  $e_j(x, \xi) = e^{ij_2x - j_1\xi}$ , for any  $j \in \mathbb{Z}^2$ . As explained before, the quantization of these modes on  $\mathcal{H}_N$  give the unitary Weyl-Heisenberg operators  $\operatorname{Op}_N(e_j) = T_{\left(\frac{j_1}{N}, \frac{j_2}{N}\right)}$ . We will thus prove the following

**Lemma 3.4.2.** *Following the assumptions of Proposition 3.4.1, there exists  $C_\alpha > 0$  depending on the sequence  $N_\alpha$ , such that such that, for  $N \in N_\alpha$ :*

1.  $\left| \operatorname{Tr}(\Pi_{\nu, N}) - \frac{N}{P(N)} \right| \leq C_\alpha \frac{N^{\frac{\alpha}{4}}}{\log N}$ . Due to the fact that the Weyl operators  $T_n$  for  $n \in \mathbb{Z}^2$  are equal (up to a sign) to the identity on  $\mathcal{H}_N$ , this formula also settles the case of  $j \in (N\mathbb{Z})^2$ .
2. for any  $j \in \mathbb{Z}^2 \setminus (N\mathbb{Z})^2$ ,  $\left| \operatorname{Tr}\left(T_{\frac{j}{N}} \Pi_{\nu, N}\right) \right| \leq C_\alpha \frac{N^{\frac{\alpha}{4}}}{\log N}$ .

**Lemma 3.4.3.** *The very particular property of quasi-idempotence (3.3) of the quantum cat map implies that each spectral projector  $\Pi_{\nu, N}$  can be expanded as a finite sum over the powers  $\mathcal{M}_N^\tau$  of the propagator:*

$$\Pi_{\nu, N} = \frac{1}{P(N)} \sum_{\tau=0}^{P(N)-1} e^{-i\nu\tau} \mathcal{M}_N^\tau. \quad (3.15)$$

*Proof of Lemma 3.4.3.* Note that from (3.3) for every  $v \in \mathcal{H}_N$  writing  $\nu = \frac{\kappa+2\pi j}{P(N)}$ ,

$$\begin{aligned} \mathcal{M}_N \left( \sum_{\tau=0}^{P(N)-1} e^{-i\nu\tau} \mathcal{M}_N^\tau v \right) &= \sum_{\tau=0}^{P(N)-1} e^{-i\nu\tau} \mathcal{M}_N^{\tau+1} v \\ &= \sum_{\tau=0}^{P(N)-1} e^{-i\nu\tau} \mathcal{M}_N^{\tau-P(N)+1} \mathcal{M}_N^{P(N)} v \\ &= \sum_{\tau=0}^{P(N)-1} e^{-i \frac{(\tau-P(N))}{P(N)} \kappa - i \frac{2\pi j \tau}{P(N)}} \mathcal{M}_N^{\tau-P(N)+1} v \end{aligned}$$

$$= e^{i\frac{\kappa+2\pi j}{P(N)}} \left( \sum_{1-P(N)}^1 e^{-i\nu\tau} \mathcal{M}_N^\tau \right).$$

As The identity (3.3) allows to shift the summation indices and rescaling by  $\frac{1}{P(N)}$  we deduce the lemma.  $\square$

Using (3.3) for "centering" the sum:

$$\Pi_{\nu,N} = \begin{cases} \frac{1}{P(N)} \sum_{\tau=-\lfloor \frac{P(N)}{2} \rfloor}^{\lfloor \frac{P(N)}{2} \rfloor} e^{-i\nu\tau} \mathcal{M}_N^\tau, & P(N) \text{ odd,} \\ \frac{1}{P(N)} \sum_{\tau=-\frac{P(N)}{2}}^{\frac{P(N)}{2}-1} e^{-i\nu\tau} \mathcal{M}_N^\tau, & P(N) \text{ even.} \end{cases} \quad (3.16)$$

The estimates in the above Lemma will directly follow from explicit expressions one can obtain for the traces  $\text{Tr} \left( T_{\frac{j}{N}} \mathcal{M}_N^\tau \right)$ , for any  $\tau \in \mathbb{Z}$ . Expressions for the case  $j = 0$ , that is the traces of  $\mathcal{M}_N^\tau$ , were derived by Keating [Kea91b], using heuristics of the Gutzwiller trace formula over classical periodic orbits. We will use the same method to generalize his computation to the "dressed" traces  $\text{Tr} \left( T_{\frac{j}{N}} \mathcal{M}_N^\tau \right)$ .

**Lemma 3.4.4.** *For every  $N \in \mathbb{N}$ ,  $j_1, j_2 \in \llbracket 0, N-1 \rrbracket$  and for every  $\tau \in \mathbb{Z}^*$ , the trace of  $T_{(\frac{j_1}{N}, 0)} T_{(0, \frac{j_2}{N})} \mathcal{M}_N^\tau$  takes the following form:*

$$\text{Tr} \left( T_{(\frac{j_1}{N}, 0)} T_{(0, \frac{j_2}{N})} \mathcal{M}_N^\tau \right) = \frac{1}{\sqrt{|\text{Tr}(\gamma^\tau) - 2|}} \sum_{(n,m) \in {}^t \square_\tau} \exp \left( \frac{\pi i}{N \gamma_{12,\tau} |\text{Tr}(\gamma^\tau) - 2|} Q(\gamma^\tau, n, m, j_1, j_2) \right).$$

Here  ${}^t \square_\tau$  is a certain parallelogram in  $\mathbb{R}^2$ , of area  $|\text{Tr}(\gamma^\tau) - 2|$ , while  $Q$  is a bilinear form of  $(n, m, j_1, j_2)$  with integer coefficients.

Let us mention that, in the specific case where  $N$  is the power of an "inert prime" ( $\mathcal{M}_N$  not necessarily having short quantum period), somewhat simpler expressions were obtained by Kelmer for  $\text{Tr} \left( T_{\frac{j}{N}} \mathcal{M}_N^\tau \right)$  [Kelo8, Proposition 3.4]. These expressions should coincide with ours when  $N$  has a short period and is a power of an inert prime.

**Remark 3.4.5.** *For the "time"  $\tau = 0$ , we have  $\mathcal{M}_N^0 = I_N$ . The cyclic actions of  $T_{(\frac{1}{N}, 0)}$  and  $T_{(0, \frac{1}{N})}$  on  $\mathcal{H}_N$  (see the matrix representation (3.18) below) easily produce the following values:*

$$\text{Tr} \left( T_{(\frac{j_1}{N}, 0)} T_{(0, \frac{j_2}{N})} \mathcal{M}_N^0 \right) = \begin{cases} N, & (j_1, j_2) \in N\mathbb{Z}^2 \\ 0, & \text{otherwise} \end{cases} \quad (3.17)$$

*Proof of Lemma 3.4.4.* The Weyl-Heisenberg operators from (1.4) amount to cyclic translations in the position and momentum hence their matrix representations with respect to the canonical basis introduced in section 1.2 are

$$\left[ T_{(\frac{1}{N}, 0)} |_{\mathcal{H}_N} \right] = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \end{pmatrix}, \quad \left[ T_{(0, \frac{1}{N})} |_{\mathcal{H}_N} \right] = \begin{pmatrix} e^{\frac{2\pi i}{N}} & 0 & \cdots & 0 \\ 0 & e^{\frac{4\pi i}{N}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \quad (3.18)$$

We begin by calculating  $\text{Tr} \left( T_{(\frac{j_1}{N}, 0)} T_{(0, \frac{j_2}{N})} \mathcal{M}_N \right)$ , adapting the argument of [Kea91b]. From the explicit expression (3.2) for the matrix elements of  $\mathcal{M}_N$  and of the Weyl-Heisenberg operators we compute for any  $p, q \in \llbracket 0, N-1 \rrbracket$  the matrix coefficients. First, applying  $T_{(0, \frac{j_2}{N})}$  from the left multiplies the  $p$ -th line by  $\exp \left( \frac{2\pi i j_2 p}{N} \right)$

$$\begin{aligned} & \left( T_{(0, \frac{j_2}{N})} \mathcal{M}_N \right)_{pq} \\ &= \sqrt{\frac{\gamma_{12}}{iN}} \left\langle \exp \left( \frac{\pi i}{N\gamma_{12}} \left( \gamma_{22} p^2 + 2p(\gamma_{12} j_2 - q - Nm) + \gamma_{11} (q + Nm)^2 \right) \right) \right\rangle_m. \end{aligned} \quad (3.19)$$

Applying  $T_{(\frac{j_1}{N}, 0)}$  amounts for the permutation  $p \mapsto p - j_1 \pmod N$ , i.e.,

$$\begin{aligned} & \left( T_{(\frac{j_1}{N}, 0)} T_{(0, \frac{j_2}{N})} \mathcal{M}_N \right)_{pq} \\ &= \sqrt{\frac{\gamma_{12}}{iN}} \left\langle \exp \left( \frac{\pi i}{N\gamma_{12}} \left( \gamma_{22} (p - j_1)^2 + 2(p - j_1)(\gamma_{12} j_2 - q - Nm) + \gamma_{11} (q + Nm)^2 \right) \right) \right\rangle_m, \end{aligned} \quad (3.20)$$

Recall that in both formulae the bracket indicates an average in the sense of (3.1) over  $m \in \llbracket 0, \gamma_{12} - 1 \rrbracket$  (one can check that the expression is  $\gamma_{12}$ -periodic). The trace can be written explicitly:

$$\begin{aligned} & \text{Tr} \left( T_{(\frac{j_1}{N}, 0)} T_{(0, \frac{j_2}{N})} \mathcal{M}_N \right) \\ &= \sqrt{\frac{\gamma_{12}}{iN}} \left\langle \sum_{p=1}^N \exp \left( \frac{\pi i}{N\gamma_{12}} \left( \gamma_{22} (p - j_1)^2 + 2(p - j_1)(\gamma_{12} j_2 - p - mN) + \gamma_{11} (p + mN)^2 \right) \right) \right\rangle_m \\ &= \sqrt{\frac{\gamma_{12}}{iN}} \left\langle \exp \left( \frac{\pi i}{N\gamma_{12}} g_N(\gamma, j_1, j_2, m) \right) \sum_{p=1}^N \exp \left( \frac{\pi i}{N\gamma_{12}} \left( -\det_1 p^2 + 2s_N(\gamma, j_1, j_2, m) p \right) \right) \right\rangle_m \end{aligned} \quad (3.21)$$

with the notations:

$$\begin{aligned}\det_1 &:= \det(\gamma - I) = 2 - \text{Tr}(\gamma) \\ s_N &:= s_N(\gamma, j_1, j_2, m) = -(\gamma_{22} - 1)j_1 + \gamma_{12}j_2 + mN(\gamma_{11} - 1) \\ g_N &:= g_N(\gamma, j_1, j_2, m) = \gamma_{22}j_1^2 - 2(\gamma_{12}j_2 - mN)j_1 + m^2N^2\gamma_{11}.\end{aligned}$$

Fix  $\kappa \in (0, 1)$  and define the function  $f_m : [1 - \kappa, N + 1 - \kappa] \rightarrow \mathbb{C}$  by

$$f_m(x) = \exp\left(\frac{\pi i}{N\gamma_{12}}(-\det_1 x^2 + 2s_N(\gamma, j_1, j_2, m)x)\right).$$

From a direct consequence of the Poisson summation formula (see A.5 for details), we perform the sum over  $p$  in (3.21), to get:

$$\begin{aligned}\text{Tr}\left(T_{\left(\frac{j_1}{N}, 0\right)}T_{\left(0, \frac{j_2}{N}\right)}\mathcal{M}_N\right) & \tag{3.22} \\ &= \sqrt{\frac{\gamma_{12}}{iN}} \left\langle \lim_{L \rightarrow \infty} \sum_{n \in \mathbb{Z}} \exp\left(\frac{\pi i}{N\gamma_{12}}g_N\right) \int_{-\kappa}^{N-\kappa} \exp\left(\frac{\pi i}{N\gamma_{12}}(-\det_1 \xi^2 + 2(s_N - n\gamma_{12}N)\xi)\right) d\xi \right\rangle_m \\ &= \sqrt{\frac{\gamma_{12}}{iN}} \left\langle \sum_{n \in \mathbb{Z}} \exp\left(\frac{\pi i}{N\gamma_{12}}\left(g_N + \frac{(s_N - n\gamma_{12}N)^2}{\det_1}\right)\right) \right. \\ & \quad \left. \times \int_{-\kappa}^{N-\kappa} \exp\left[-\frac{\pi i N \det_1}{\gamma_{12}}\left(\frac{\xi}{N} - \frac{s_N - n\gamma_{12}N}{N \det_1}\right)^2\right] d\xi \right\rangle_m.\end{aligned}$$

We remark that in (3.22) and for the rest of the proof we rely on Fourier series of a piecewise smooth function and hence infinite summations on  $\mathbb{Z}$  are realized as limits of the form  $\lim_{L \rightarrow \infty} \sum_{n \in [-L, L]}$ . Denoting

$$\tilde{s}_N(\gamma, j_1, j_2, m, n) := s_N - n\gamma_{12}N = (1 - \gamma_{22})j_1 + \gamma_{12}j_2 + ((\gamma_{11} - 1)m - n\gamma_{12})N$$

and applying in the integral the change of variable  $y = \frac{\xi}{N} - \frac{\tilde{s}_N}{N \det_1}$ , we get

$$\begin{aligned}\text{Tr}\left(T_{\left(\frac{j_1}{N}, 0\right)}T_{\left(0, \frac{j_2}{N}\right)}\mathcal{M}_N\right) & \tag{3.23} \\ &= \sqrt{\frac{N\gamma_{12}}{i}} \left\langle \sum_{n \in \mathbb{Z}} \exp\left(\frac{\pi i}{N\gamma_{12}}\left(g_N + \frac{\tilde{s}_N^2}{\det_1}\right)\right) \int_{-N^{-1}\tilde{s}_N \det_1^{-1} - \frac{\kappa}{N}}^{1 - N^{-1}\tilde{s}_N \det_1^{-1} - \frac{\kappa}{N}} \exp\left[-\frac{\pi i N \det_1}{\gamma_{12}}y^2\right] dy \right\rangle_m \\ &= \sqrt{\frac{N}{i\gamma_{12}}} \sum_{m=0}^{|\gamma_{12}|-1} \sum_{n \in \mathbb{Z}} \exp\left(\frac{\pi i}{N\gamma_{12}}\left(g_N + \frac{\tilde{s}_N^2}{\det_1}\right)\right) \int_{-N^{-1}\tilde{s}_N \det_1^{-1} - \frac{\kappa}{N}}^{1 - N^{-1}\tilde{s}_N \det_1^{-1} - \frac{\kappa}{N}} \exp\left[-\frac{\pi i N \det_1}{\gamma_{12}}y^2\right] dy. \\ &= \sqrt{\frac{N}{i\gamma_{12}}} \lim_{\mathcal{L} \rightarrow \infty} \sum_{(n, m) \in [-\mathcal{L}, \mathcal{L}] \times [0, |\gamma_{12}|-1]} \exp\left(\frac{\pi i}{N\gamma_{12}}\left(g_N + \frac{\tilde{s}_N^2}{\det_1}\right)\right) \times & \tag{3.24}\end{aligned}$$

$$\int_{-N^{-1}\tilde{s}_N \det_1^{-1} - \frac{\kappa}{N}}^{1-N^{-1}\tilde{s}_N \det_1^{-1} - \frac{\kappa}{N}} \exp \left[ -\frac{\pi i N \det_1}{\gamma_{12}} y^2 \right] dy.$$

Let us denote by  $\diamond_1$  a fundamental cell of the lattice  $\Lambda = \left\{ \begin{pmatrix} \ell(\gamma) \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma_{11} - 1 \\ \gamma_{12} \end{pmatrix} \right\}$ , where

$$\ell(\gamma) = \begin{cases} |\gamma_{21} - \gamma_{11} + 1|, & \gamma_{12} \geq \gamma_{22} - 1 \\ \left\lfloor \left\lfloor \frac{\gamma_{22}-1}{\gamma_{12}} \right\rfloor \right\rfloor |\gamma_{21} - \gamma_{11} + 1|, & \gamma_{12} < \gamma_{22} - 1. \end{cases}$$

Each finite strip  $[-\mathcal{L}, \mathcal{L}] \times [0, |\gamma_{12}| - 1]$  can be tessellated by "horizontal translations", i.e.,  $\begin{pmatrix} \ell(\gamma) \\ 0 \end{pmatrix} \mathbb{Z}$ , of  $\diamond_1$ . Since  $\gamma \in \tilde{\Gamma}(2)$ ,  $\gamma_{11}\gamma_{12} \in 2\mathbb{Z}$  the function  $\tilde{s}_N$  is invariant with respect to  $\begin{pmatrix} \gamma_{11}-1 \\ \gamma_{12} \end{pmatrix} \mathbb{Z}$ -translations, and the prefactor  $e^{i\pi \frac{g_N}{N\gamma_{12}}}$  is invariant too. Applying a "cut and glue" process described explicitly in Figure 3.2, the summation on each horizontal translation of  $\diamond_1$  is equivalent to a summation on lattice points arising from  $\begin{pmatrix} \gamma_{21} \\ \gamma_{22}-1 \end{pmatrix} \mathbb{Z}$  translations of  ${}^t \square_1$ , the fundamental cell of  $({}^t \gamma - I)$ . Altogether, we arrive to the conclusion that summing over  $\mathbb{Z}^2$ -points in a truncated horizontal strip  $[-\mathcal{L}_\gamma, \mathcal{L}] \times [0, |\gamma_{12}| - 1]$  amounts to summing over  $\mathbb{Z}^2$ -points lying inside a truncation of the strip  $S_\gamma = {}^t \square_1 + \begin{pmatrix} \gamma_{21} \\ \gamma_{22}-1 \end{pmatrix} \mathbb{Z}$  (although not covering it). Letting  $\mathcal{L} \rightarrow \infty$  we conclude that the summation over integers inside the the entire horizontal strip is equivalent to summing over lattice point inside  $S_\gamma$  (cf. Figure 3.3). We can write

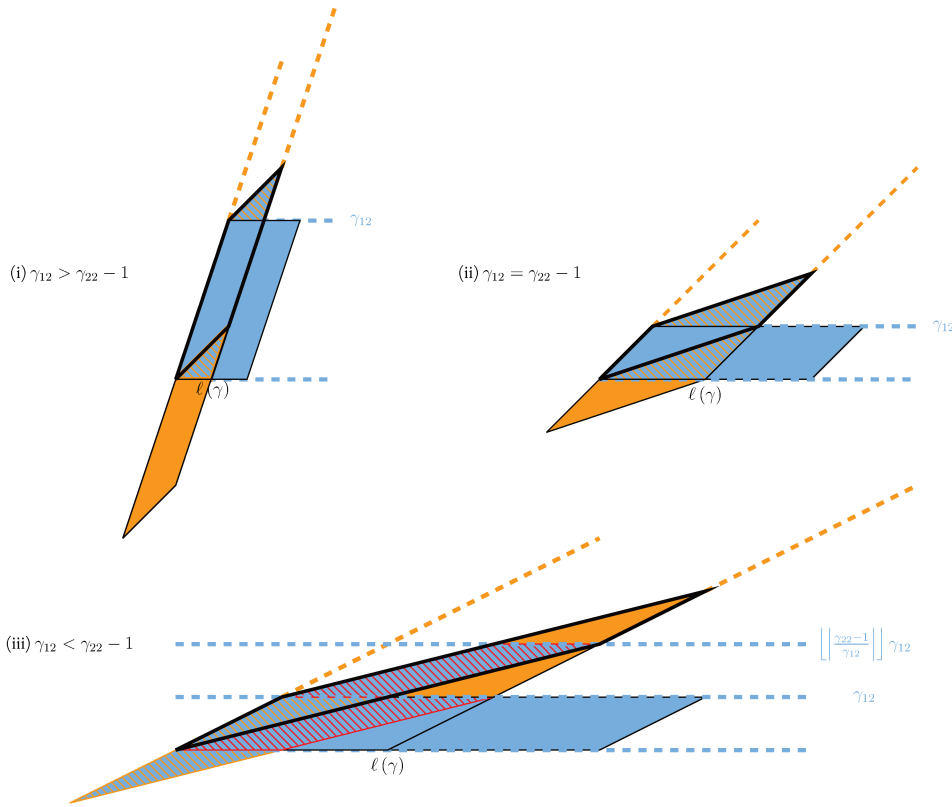
$$\begin{aligned} \text{Tr} \left( T_{\begin{pmatrix} j_1 \\ N, 0 \end{pmatrix}} T_{\begin{pmatrix} j_2 \\ 0, N \end{pmatrix}} \mathcal{M}_N \right) &= \lim_{L' \rightarrow \infty} \sqrt{\frac{N}{i\gamma_{12}}} \sum_{(n,m) \in [-L', L']^2 \cap S_\gamma} \exp \left( \frac{\pi i}{N\gamma_{12}} \left( g_N + \frac{\tilde{s}_N^2}{\det_1} \right) \right) \\ &\quad \times \int_{-N^{-1}\tilde{s}_N \det_1^{-1} - \frac{\kappa}{N}}^{1-N^{-1}\tilde{s}_N \det_1^{-1} - \frac{\kappa}{N}} \exp \left[ -\frac{\pi i N \det_1}{\gamma_{12}} y^2 \right] dy. \end{aligned} \quad (3.25)$$

The function  $\tilde{s}_N$  transforms through the shift  $\begin{pmatrix} n \\ m \end{pmatrix} \mapsto \begin{pmatrix} n \\ m \end{pmatrix} + k \begin{pmatrix} \gamma_{21} \\ \gamma_{22}-1 \end{pmatrix}$ ,  $k \in \mathbb{Z}$  by

$$\tilde{s}_N(\gamma, j_1, j_2, n + \gamma_{12}k, m + (\gamma_{22} - 1)k) = s_N(\gamma, j_1, j_2, n, m) + k \det_1 N$$

and the prefactor is invariant under this shift. In the integral, this shift amounts to translating the domain of integration by  $k$ . As follows, we can simplify the sum, only summing over lattice points inside  ${}^t \square_1$ . Summing over these points, we sum over several finite integrals like the one in (3.25). As  $L' \rightarrow \infty$  the order of summation dictated by the outer sum, yields a principal value integral we will denote  $f_{{}^t \square_1}$

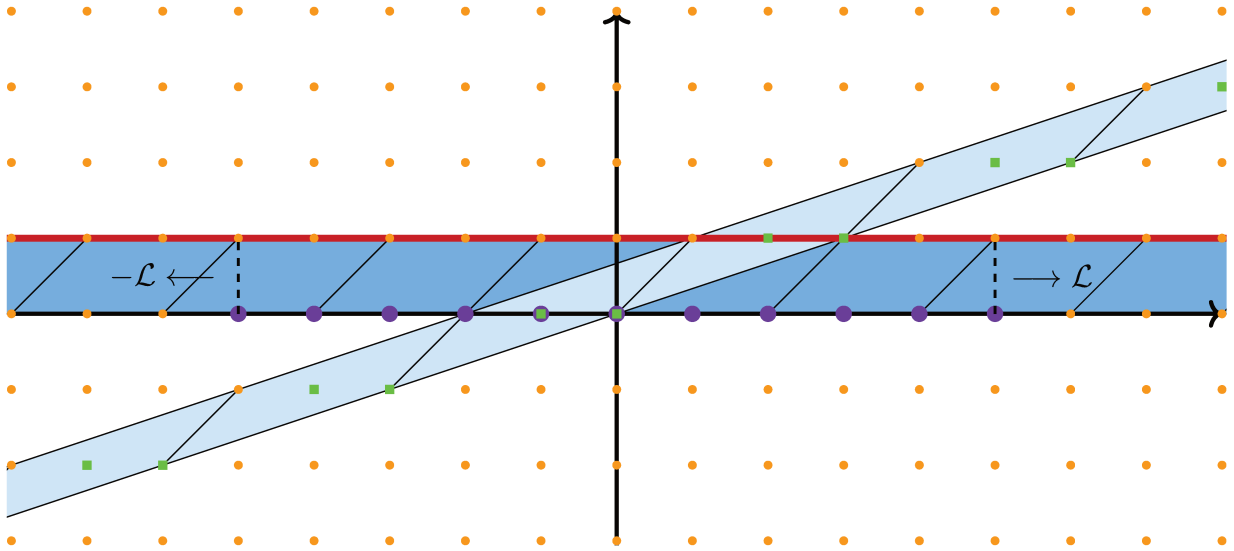
$$\text{Tr} \left( T_{\begin{pmatrix} j_1 \\ N, 0 \end{pmatrix}} T_{\begin{pmatrix} j_2 \\ 0, N \end{pmatrix}} \mathcal{M}_N \right) = \sqrt{-\frac{Ni}{\gamma_{12}}} \sum_{(n,m) \in {}^t \square_1} \exp \left( \frac{\pi i}{N\gamma_{12}} \left( g_N + \frac{\tilde{s}_N^2}{\det_1} \right) \right) f_{{}^t \square_1} \exp \left[ -\frac{\pi i N \det_1}{\gamma_{12}} y^2 \right] dy$$



**Figure 3.2:** We obtain from  $\diamond_1$  through cut and glue procedure the fundamental cell  ${}^t \square_1$ . Summing over integers from one equals to summing over integers from the other. The blue horizontal cell is  $\diamond_1$ . The parallelogram bordered by thick black line is  ${}^t \square_1$ . The hatched areas signify the domains cut from  $\diamond_1$  to create  ${}^t \square_1$  (The red ones are parallelograms and the orange ones are triangles). Subfigure (i):  ${}^t \square_1$  is the fundamental cell of  $\gamma_{DE} - I = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$ . In order to get it from  $\diamond_1$ , one shifts the orange triangle by  $\begin{pmatrix} \gamma_{11}^{-1} \\ \gamma_{12} \end{pmatrix} \mathbb{Z}$ . Subfigure (ii):  ${}^t \square_1$  is the fundamental cell of  ${}^t \gamma_{DE} - I = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ . Shifting the orange triangle one maps  $\diamond_1$  to  ${}^t \square_1$ . Subfigure (iii):  ${}^t \square_1$  is the fundamental cell of  ${}^t \gamma - I$  with  $\gamma = \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix}$ . One can represent  ${}^t \square_1$  as a union of parallelograms with a triangle: Between every two vertical lines under  $\xi_{\text{top}} = \left\lfloor \left\lfloor \frac{\gamma_{22}-1}{\gamma_{12}} \right\rfloor \right\rfloor |\gamma_{12}|$  there is a parallelogram (in red) and above  $\xi_{\text{top}}$  there is a triangle. Gluing first the parallelograms next to each other and then the triangle, one can map  ${}^t \square_1$  to  $\diamond_1$  and vice versa.

$$= \frac{1}{\sqrt{-\det_1}} \sum_{(n,m) \in {}^t \square_1} \exp \left( \frac{\pi i}{N \gamma_{12}} \left( g_N + \frac{\tilde{s}_N^2}{\det_1} \right) \right).$$

We note that as the phase of the integrand is exponentially decaying we could replace the principal value integral with a standard one which can be calculated using techniques from complex analysis. One checks that the exponent has the form indicated in the Lemma, which proves the expression



**Figure 3.3:** Summing over the purple integer points in the truncated horizontal strip to summing over the square green integer points in the sub-strip of  $S_\gamma$ . Letting  $\mathcal{L} \rightarrow \infty$  we sum over integers in  $(\frac{\gamma_{21}}{\gamma_{22}-1})\mathbb{Z}$ -shifts of  ${}^t \square_1$ .

We now turn to calculate  $\text{Tr} \left( T_{\left(\frac{j_1}{N}, \frac{j_2}{N}\right)} \mathcal{M}_N^\tau \right)$  for  $\tau \in \mathbb{Z}$ . Since  $T_{\left(\frac{j_1}{N}, \frac{j_2}{N}\right)}$  is proportional to  $T_{\left(\frac{j_1}{N}, 0\right)} T_{\left(0, \frac{j_2}{N}\right)}$ , it is sufficient to calculate  $\text{Tr} \left( T_{\left(\frac{j_1}{N}, 0\right)} T_{\left(0, \frac{j_2}{N}\right)} \mathcal{M}_N^\tau \right)$ . Using the group property  $\mathcal{M}_N^\tau(\gamma) = \mathcal{M}_N(\gamma^\tau)$  for any  $\tau \in \mathbb{Z}$ , we may apply the same calculations for the transformation  $\gamma^\tau$  for all integers  $\tau \neq 0$  and one deduces an analogous formula for every  $\tau \in \mathbb{Z}^*$ ,

$$\begin{aligned} \text{Tr} \left( T_{\left(\frac{j_1}{N}, 0\right)} T_{\left(0, \frac{j_2}{N}\right)} \mathcal{M}_N^\tau \right) &= \frac{1}{\sqrt{-\det_\tau}} \sum_{(n,m) \in {}^t \square_\tau} \exp \left( \frac{\pi i}{\gamma_{12,\tau} N} \left( g_N(\gamma^\tau, j_1, j_2, m) + \frac{\tilde{s}_N^2(\gamma, j_1, j_2, m)}{\det_\tau} \right) \right) \\ &= \frac{1}{\sqrt{-\det_\tau}} \sum_{(n,m) \in {}^t \square_\tau} \exp \left( \frac{\pi i}{N \gamma_{12,\tau} \det_\tau} Q(\gamma^\tau, n, m, j_1, j_2) \right), \end{aligned}$$

with

$$\det_\tau = \det(\gamma^\tau - I), \quad \gamma^\tau = \begin{pmatrix} \gamma_{11,\tau} & \gamma_{12,\tau} \\ \gamma_{21,\tau} & \gamma_{22,\tau} \end{pmatrix}$$

,  ${}^t \square_\tau$  being the fundamental cell of  $\gamma^\tau - I$ , containing  $\det_\tau$  integer points, and  $Q$  being some bilinear form in  $(n, m, j_1, j_2)$ , with integer coefficients depending on  $\gamma^\tau$ .  $\square$

These rather explicit expressions allow to easily prove the bounds in Lemma 3.4.2.

*Proof of Lemma 3.4.2.* For every  $\tau \neq 0$ , there are exactly  $|\det_\tau|$  integer lattice point in the parallelogram  ${}^t \square_\tau$ , each term being a pure phase. Taking into account the prefactor, the triangle inequality

provides the following bound for any  $\tau \in \mathbb{Z}^*$ :

$$\left| \operatorname{Tr} \left( T_{\left(\frac{j_1}{N}, 0\right)} T_{\left(0, \frac{j_2}{N}\right)} \mathcal{M}_N^\tau \right) \right| \leq \sqrt{|\det \tau|} = |\lambda^\tau + \lambda^{-\tau} - 2| \leq C |\lambda|^{\frac{|\tau|}{2}}, \quad (3.26)$$

for some  $C = C(\lambda) > 1$ . We deduce an immediate bound on  $\left| \operatorname{Tr} \left( T_{\left(\frac{j_1}{N}, \frac{j_2}{N}\right)} \Pi_{\nu, N} \right) \right|$ .

- i. First assume that  $(j_1, j_2) \notin N\mathbb{Z}^2$ . Using the above expressions and 3.17, we find (in the case of even  $P(N)$ , the odd case being similar):

$$\begin{aligned} \left| \operatorname{Tr} \left( T_{\left(\frac{j_1}{N}, \frac{j_2}{N}\right)} \Pi_{\nu, N} \right) \right| &= \left| \frac{1}{P(N)} \sum_{\tau = -\frac{P(N)}{2}}^{\frac{P(N)}{2}-1} e^{i\nu_j \tau} \operatorname{Tr} \left( T_{\left(\frac{j_1}{N}, \frac{j_2}{N}\right)} \mathcal{M}_N^\tau \right) \right| \\ &\leq \frac{1}{P(N)} \sum_{\tau = -\frac{P(N)}{2}}^{\frac{P(N)}{2}-1} \left| \operatorname{Tr} \left( T_{\left(\frac{j_1}{N}, \frac{j_2}{N}\right)} \mathcal{M}_N^\tau \right) \right| \\ &\leq \frac{C_\lambda}{P(N)} \sum_{\tau = -\frac{P(N)}{2}}^{\frac{P(N)}{2}-1} |\lambda|^{\frac{|\tau|}{2}} \\ &\leq C'_\lambda \frac{|\lambda|^{\frac{P(N)}{4}}}{P(N)} \end{aligned} \quad (3.27)$$

We now take into account our assumption  $P(N) \leq \alpha \frac{\log N}{\log |\lambda|} + C_\alpha$ , and obtain

$$\left| \operatorname{Tr} \left( T_{\left(\frac{j_1}{N_j}, \frac{j_2}{N_j}\right)} \Pi_{\nu, N} \right) \right| \leq C_{\alpha, \gamma} \frac{N^{\frac{\alpha}{4}}}{\log N},$$

where the constant depends on  $C_\alpha$  and  $\lambda$ .

2. In the case  $(j_1, j_2) = (0, 0)$ , relying on our estimate in (3.17) and (3.26) above, we find (for  $P(N)$  even)

$$\begin{aligned} \dim V_{\nu, N} &= \operatorname{Tr} (\Pi_{\nu, N}) \\ &= \frac{\operatorname{Tr} \operatorname{Id}_{\mathcal{H}_N}}{P(N)} + \frac{1}{P(N)} \sum_{\substack{\tau = -\frac{P(N)}{2} \\ 0 \neq \tau}}^{\frac{P(N)}{2}-1} e^{i\nu t} \operatorname{Tr} \mathcal{M}_N^\tau \\ &= \frac{N}{P(N)} + O \left( \frac{|\lambda|^{\frac{P(N)}{4}}}{P(N)} \right) \\ &= \frac{N}{P(N)} + O \left( \frac{N^{\frac{\alpha}{4}}}{\log N} \right), \end{aligned}$$



where the implied constant depends on  $\lambda$  and  $C_\alpha$ . This proves the two statements of 3.4.2.

□

### 3.5 End of proof of Proposition 3.4.1

The proof of Proposition 3.4.1 now easily follows. Proposition 3.4.1.i is already contained in Lemma 3.4.2. For Proposition 3.4.1.ii, we recall that for any symbol  $a \in C^\infty(\mathbb{T}^2)$ , the Fourier expansion

$$a = \sum_{j \in \mathbb{Z}^2} \hat{a}(j) e_j$$

leads to the quantization on  $\mathcal{H}_N$ :

$$\text{Op}_N(a) = \sum_{j \in \mathbb{Z}^2} \hat{a}(j) T_{\frac{j}{N}}$$

We therefore obtain, using the linearity of the quantization and the fact that the Fourier series converges absolutely:

$$\begin{aligned} \text{Tr}(\text{Op}_N(a) \Pi_{\nu, N}) &= \sum_{j \in \mathbb{Z}^2} \hat{a}(j) \text{Tr}(T_{\frac{j}{N}} \Pi_{\nu, N}) \\ &= \hat{a}(0) \text{Tr}(\Pi_{\nu, N}) + \sum_{j \in \mathbb{Z}^2 \setminus N\mathbb{Z}^2} \hat{a}(j) \text{Tr}(T_{\frac{j}{N}} \Pi_{\nu, N}) + \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \hat{a}(Nk) \text{Tr}(T_k \Pi_{\nu, N}). \end{aligned}$$

The uniform estimates we obtained in Lemma 3.4.2 for  $j \in \mathbb{Z}^2 \setminus N\mathbb{Z}^2$  show that the norm of the second term can be bounded above by  $C \frac{N^{\frac{q}{4}}}{\log N} \|\hat{a}\|_{\ell^1}$ , while the norm of the last term is bounded above by  $C \sum_{k \neq 0} |\hat{a}(Nk)|$ . Dividing the whole expression by  $d(\nu, N)$ , we obtain the second statement of the Proposition.

For the third statement, Proposition 3.4.1.iii, we assume that the symbol  $a = (a_N)_{N \in \mathbb{N}^*}$  does not fluctuate too fast in the limit  $N \rightarrow \infty$ , by imposing that  $a$  belongs to the symbol class  $S_{\delta+}$ . This assumption has a direct consequence on the behavior of the Fourier coefficients. Integrating by parts the expression

$$\hat{a}(j) = \int_{\mathbb{T}^2} a(x, \xi) e^{2i\pi(j_1\xi - j_2x)} dx d\xi,$$

we obtain, for each  $m \geq 1$  and  $j \in \mathbb{Z}^2$ :

$$|\hat{a}(j)| \leq C_m \frac{\|a\|_{C^m}}{(1 + |j|)^m}. \quad (3.28)$$

In view of the bounds  $\|a\|_{C^m} \leq C_{a,m,\varepsilon} N^{(\delta+\varepsilon)m}$ , we see that, as long as  $|j| \leq N^{(\delta+\varepsilon)}$ , the above estimate is worse than the trivial bound  $|\hat{a}(j)| \leq \|a\|_\infty$ . The number of coefficients for which this is

the case is  $\sim N^{2(\delta+\varepsilon)}$ , so the sum over these coefficients gives

$$\sum_{|j| \leq N^{\delta+\varepsilon}} |\hat{a}(j)| \leq C \|a\|_\infty N^{2(\delta+\varepsilon)}$$

On the opposite, for  $|j| \geq N^{(\delta+\varepsilon)}$ , the bound (3.28) shows that  $\hat{a}(j)$  decays fast if  $m$  is large. Taking  $m \geq 3$ , one can sum over all those coefficients, and obtain a result of the same order:

$$\sum_{|j| > N^{\delta+\varepsilon}} |\hat{a}(j)| \leq C_a N^{2(\delta+\varepsilon)},$$

so that we globally have

$$\sum_{j \in \mathbb{Z}^2} |\hat{a}(j)| \leq C_a N^{2(\delta+\varepsilon)}.$$

Finally, since all modes  $\hat{a}(kN)$ ,  $k \neq 0$ , lie in the "decay region", one easily shows that for any  $m \geq 3$ ,

$$\sum_{k \in \mathbb{Z}^2} |\hat{a}(kN)| \leq C_{a,m} N^{-(1-2\delta-\varepsilon)m}.$$

We can obtain similar estimates for  $\chi_{\rho,r} \in S_{\delta+}$ : Recalling our bounds on the Fourier coefficients in Lemma 3.3.3, the sum over all coefficients  $|j_1|, |j_2| \leq N/2$  then gives:

$$\sum_{j_1, j_2 = -N/2}^{N/2-1} |\hat{\chi}_{\rho,r}(j)| \leq C N^{2(\delta+\varepsilon)} r(N)^2, \quad (3.29)$$

while the "outer sum"

$$\sum_{|j| \geq N/2} |\hat{\chi}_{\rho,r}(j)| = O(N^{-\infty}).$$

If  $r(N) = N^{-\delta}$  as in the assumptions of the Theorem, the factor  $r(N)^2$  in (3.29) almost balances the factor  $N^{2(\delta+\varepsilon)}$ , producing a net contribution

$$\sum_{j_1, j_2 = -N/2}^{N/2-1} |\hat{\chi}_{\rho,r}(j)| \leq C N^{2\varepsilon}. \quad (3.30)$$

Equipped of these estimates, we may inspect the various terms in Equation 3.14. The assumption  $\delta < \frac{1}{2} - \frac{\alpha}{8}$ , or equivalently

$$\beta := 1 - \frac{\alpha}{4} - 2\delta > 0,$$

ensures that the second term in Equation 3.14 is of order  $O(N^{-\beta+})$ , while the last term is  $O(N^{-\infty})$ . The only nondecaying term is therefore the first one, which gives the limit.

This ends the proof of the Proposition.  $\square$

### 3.6 From Hanson-Wright inequality to probabilistic QUE

#### 3.6.1 The Hanson-Wright concentration inequality

The proof of our almost sure QUE relies on concentration inequalities for the random variables  $\langle \psi_{j,N}, \text{Op}_N(a) \psi_{j,N} \rangle$  (here we remind that  $\psi_{j,N}$  is a random normalized vector in one eigenspace  $V_{\nu,N}$  of  $\mathcal{M}_N$ ). This random variable will be, with high probability, close to the global quantum average over  $V_{\nu,N}$ , that is the normalized trace  $\frac{1}{d(\nu,N)} \text{Tr} (\text{Op}_N(a) \Pi_{\nu,N})$ . This closeness will be expressed using the so-called Hanson-Wright inequalities. We will use the version of these inequalities derived by Rudelson-Vershynin [RV13], and used by Chatterjee-Galkowski [CG18] in their proof of QUE for a randomly perturbed Laplacian.

For our aims, we will use the concentration inequality from the following theorem.

**Theorem 3.6.1** (Theorem 4.1 in [CG18]). *Let  $M \in \mathbb{C}^{d \times d}$  and  $X_d \sim (U(d), \mathbb{P}_d)$  a random unitary matrix in dimension  $d \geq 1$ , where  $\mathbb{P}_d$  is the normalized Haar measure on the unitary group  $U(d)$ . Denote the standard orthonormal basis of  $\mathbb{C}^d$  by  $e_1, \dots, e_d \in \mathbb{C}^d$  and consider the random orthonormal basis  $\{v_j\}_{j=1}^d$  defined by  $v_j = X_d e_j$ ,  $j = 1, \dots, d$ . Then there exist absolute constants  $C, C' > 0$  (independent on  $d$  or  $M$ ), such that for any parameter  $t > 0$  and any  $k = 1, \dots, d$ ,*

$$\mathbb{P}_d \left( \left| \langle M v_k, v_k \rangle - \frac{1}{d} \text{Tr} M \right| \geq t \right) \leq C \exp \left( -C' d \min \{ \|M\|^{-2}, \|M\|^{-1} \} \min \{ t, t^2 \} \right). \quad (3.31)$$

We briefly explain the required modifications in the statement and proof of [CG18, Theorem 4.1], in order to apply it to our scenario. [CG18] relied on the version of the Hanson-Wright inequality appearing in [RV13]:

**Theorem 3.6.2** (Rudelson, Vershynin [RV13], Hanson-Wright, [HW71]). *Let  $X_1, \dots, X_d$  independent centered random variables satisfying for some fixed  $K > 0$  and for every  $j \in \llbracket 1, d \rrbracket$*

$$\|X_j\|_{\psi_2} = \sup_{p \geq 2} p^{-\frac{1}{2}} \sqrt{\mathbb{E}[|X_j|^p]} \leq K.$$

*Denote  $M = (m_{jk}) \in \mathbb{C}^{d \times d}$  and  $R = \sum_{j,k} m_{jk} X_j X_k$ . Then for every  $t \geq 0$*

$$\mathbb{P} (|R - \mathbb{E}[R]| \geq t) \leq 2 \exp \left( -C \min \left\{ \frac{t^2}{K^4 \|M\|_{HS}^2}, \frac{t}{K^2 \|M\|} \right\} \right) \quad (3.32)$$

Let us show how to deduce Theorem 3.6.1 from the above theorem.

*Proof of Theorem 3.6.1.* Fix a  $1 \leq k \leq d$  and define a random variable  $Y_k := \langle M v_k, v_k \rangle$ . Denote

$$X = \begin{pmatrix} | & & | \\ x_1 & \dots & x_d \\ | & & | \end{pmatrix} = (x_{nm})_{1 \leq n, m \leq d}.$$

Let us recall that since  ${}^t(XX^*) = \overline{X} \cdot {}^tX = \text{Id}_d$

$$\delta_n(m) = \mathbb{E}[(\overline{X} \cdot {}^tX)_{n,m}] = \sum_{k=1}^d \mathbb{E}[x_{k,n}\overline{x_{k,m}}].$$

The distribution of  $X$  is invariant under complex rotations thus for every  $1 \leq m, k \leq d$ ,

$$\mathbb{E}[x_{1,k}\overline{x_{1,m}}] = \cdots = \mathbb{E}[x_{d,k}\overline{x_{d,m}}].$$

Then for every  $1 \leq k \leq d$   $\mathbb{E}[x_{k,n}\overline{x_{k,m}}] = \frac{1}{d}\delta_n(m)$  and we can express  $\mathbb{E}[Y_k]$

$$\mathbb{E}[Y_k] = \sum_{n,m=1}^d \mathbb{E}[\langle Mx_{k,n}e_n, x_{k,m}e_m \rangle] = \sum_{n,m=1}^d \mathbb{E}[x_{k,n}\overline{x_{k,m}}] \langle Me_n, e_m \rangle = \frac{1}{d} \sum_{n=1}^d \langle Me_n, e_n \rangle = \frac{1}{d} \text{Tr}(M\Pi).$$

As  $X \sim (U(d), d\text{Vol})$  each of its columns can be viewed as  $x_m \sim \text{Unif}(\mathbb{S}_{\mathbb{C}}^{d-1})$ . Let  $z \sim \mathcal{N}_{\mathbb{C}}(\vec{0}, \text{Id}_d)$  with  $\vec{0} = (0, \dots, 0)$ . Since  $\frac{z}{\|z\|} \sim \text{Unif}(\mathbb{S}_{\mathbb{C}}^{d-1})$  letting  $r_m$  be a random variable distributed like  $\|z\|$  and independent of  $x_m$ ,  $w_m = x_m r_m \sim \mathcal{N}_{\mathbb{C}}(\vec{0}, \text{Id}_d)$ . We obtain independent random variables  $w_{k,m} = w_m x_{k,m} \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ . Denoting the matrix  $H = (h_{n,m})_{1 \leq n, m \leq d} = (\langle Me_n, e_m \rangle)_{n,m}$  we rephrase  $Y_k$  as

$$Y_k = \sum_{n,m=1}^d x_{k,n}\overline{x_{k,m}}h_{n,m}.$$

We estimate the norms of  $H$  by those of  $M$ : First

$$\|H\| = \|\Pi M \Pi\| \leq \|M\| \quad (3.33)$$

where  $\Pi \in \mathbb{C}^{d \times d}$  is the spectral projector to  $\text{Span}(e_1, \dots, e_d)$ . Moreover,

$$\|H\|_{\text{HS}} = \sqrt{\text{Tr}(H^*H)} = \sqrt{\text{Tr}(\Pi^*M^*M\Pi)} \leq \sqrt{\text{Tr}(M^*M)}\sqrt{\text{Tr}(\Pi^*\Pi)} = \|M\|_{\text{HS}} \leq \sqrt{d}\|M\|. \quad (3.34)$$

Applying Hanson-Wright inequality from (3.32) on  $Y'_k = r_k^2 Y_k = \sum_{n,m} h_{n,m} w_{k,n} \overline{w_{k,m}}$  we conclude from (3.33), (3.34) that there exists a constant  $C > 0$  such that

$$\mathbb{P}(|Y'_k - \mathbb{E}[Y'_k]| \geq t) \leq 2 \exp\left(-C \min\{\|M\|^{-2}, \|M\|^{-1}\} \min\left\{\frac{t^2}{d}, t\right\}\right) \quad (3.35)$$

One can apply again the inequality, but on the variable  $r_k^2$ , deducing there exists  $C' > 0$  such that

$$\mathbb{P}(|r_k^2 - \mathbb{E}[r_k^2]| \geq t) \leq 2 \exp\left(-C' \min\left\{\frac{t^2}{d}, t\right\}\right). \quad (3.36)$$

Then,

$$|Y'_k| \leq \|M\| \|v_k\|_2^2 = \|M\| \sum_{j=1}^d x_{kj}^2 = \|M\| \quad (3.37)$$

and from the definition of  $Y'_k$

$$\mathbb{E}[Y'_k] = d \cdot \mathbb{E}[Y_k]. \quad (3.38)$$

Combining (3.35)-(3.38) we deduce that

$$\begin{aligned} \mathbb{P}(|Y_k - \mathbb{E}[Y_k]| \geq t) &\leq \mathbb{P}\left(|d \cdot Y_k - Y'_k| \geq \frac{dt}{2}\right) + \mathbb{P}\left(|Y'_k - \mathbb{E}[Y'_k]| \geq \frac{dt}{2}\right) \\ &\leq \mathbb{P}\left(|(r_k^2 - d) Y_k| \geq \frac{dt}{2}\right) + 2 \exp(-Cd \min\{\|M\|^{-2}, \|M\|^{-1}\} \min\{t^2, t\}) \\ &\leq \mathbb{P}\left(|r_k^2 - d| \geq \frac{dt}{2|Y_k|}\right) + 2 \exp(-Cd \min\{\|M\|^{-2}, \|M\|^{-1}\} \min\{t^2, t\}) \\ &\leq C'' \exp(-C''' d \min\{\|M\|^{-2}, \|M\|^{-1}\} \min\{t^2, t\}) \end{aligned}$$

□

### 3.6.2 Applying the concentration inequality to our eigenstates

#### 3.6.2.1 Deviations from the quantum global average

We will start to apply the estimate (3.31) to an individual random eigenstate in some eigenspace  $V_{\nu,N} \in \mathcal{H}_N$ ; without loss of generality we can take the eigenstate  $\psi_{1,N}$ . Let us also take for observable a single Fourier mode  $e_j, j \in \mathbb{Z}^2$ ; in that case, the operator  $\text{Op}_N(e_j)$  has norm unity.

To show the macroscopic equidistribution of the state  $\psi_{1,N}$ , we need to show that for each fixed Fourier mode  $j \in \mathbb{Z}^2$ , the matrix element  $\langle \psi_{1,N}, T_{\frac{j}{N}} \Pi_{\nu,N} \psi_{1,N} \rangle$  is close to the classical average

$$\int e_j(\rho) d\rho = \delta_{j,(0,0)}.$$

Here “close” means that the difference should be an  $o(1)$  when  $N \rightarrow \infty$ .

For this aim, we should investigate the probability that the deviation between the two quantities is greater than  $t$ , for some  $t = t(N) > 0$  decaying to zero when  $N \rightarrow \infty$ . In particular, we will only consider “small” values of  $t > 0$ .

Before comparing these two quantities, the bound (3.31) allows to compare the matrix element  $\langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle$ , with the global quantum average of  $T_{\frac{j}{N}}$  on  $V_{\nu,N}$ , namely  $\omega_{\nu,N}(e_j) = \frac{1}{d(\nu,N)} \text{Tr}(T_{\frac{j}{N}} \Pi_{\nu,N})$ . One obtains, for any  $t \in (0, 1]$ :

$$\mathbb{P}_{\nu,N}\left(\left|\langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle - \omega_{\nu,N}(e_j)\right| > t\right) \leq C e^{-C' d(\nu,N) t^2}. \quad (3.39)$$

Since  $d(\nu, N) \asymp \frac{N}{\log N}$  along the sequence  $\mathcal{N}_\alpha$ , we could take  $t$  as small as  $N^{-1/2+\varepsilon}$  for some  $\varepsilon > 0$ , and still get an exponentially small probability. This shows the strength of this concentration estimate.

### 3.6.2.2 Deviation from the classical average

However, we actually want to estimate the deviation between the “matrix element”  $\langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle$  and the classical average  $\omega_\infty(e_j)$ . The local Weyl law in Lemma 3.4.2 can be rewritten as

$$|\omega_{\nu,N}(e_j) - \omega_\infty(e_j)| \leq C_\alpha N^{\frac{\alpha}{4}-1},$$

where the constant is uniform for  $j \in \{(0,0)\} \cup \mathbb{Z}^2 \setminus N\mathbb{Z}^2$ .

From the reverse triangle inequality, we draw that

$$\left| \langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle - \omega_\infty(e_j) \right| > t' \implies \left| \langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle - \omega_{\nu,N}(e_j) \right| > t' - C_\alpha N^{\frac{\alpha}{4}-1}.$$

In other words, the event

$$\left\{ \left| \langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle - \omega_\infty(e_j) \right| > t' \right\}$$

is contained in the event

$$\left\{ \left| \langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle - \omega_{\nu,N}(e_j) \right| > t' - C_\alpha N^{\frac{\alpha}{4}-1} \right\}.$$

In terms of probabilities, this shows that

$$\mathbb{P}_{\nu,N} \left( \left| \langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle - \omega_\infty(e_j) \right| > t' \right) \leq \mathbb{P}_{\nu,N} \left( \left| \langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle - \omega_{\nu,N}(e_j) \right| > t' - C_\alpha N^{\frac{\alpha}{4}-1} \right).$$

For the latter probability to be small, we need that  $t := t' - C_\alpha N^{\frac{\alpha}{4}-1} > 0$ . So the smallest deviation from  $\omega_\infty(e_j)$  we may test is of size  $\asymp N^{\frac{\alpha}{4}-1}$ . For instance, taking  $t' = 2C_\alpha N^{\frac{\alpha}{4}-1}$  and applying (3.39), we find

$$\mathbb{P}_{\nu,N} \left( \left| \langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle - \omega_\infty(e_j) \right| > 2C_\alpha N^{\frac{\alpha}{4}-1} \right) \leq C e^{-C'' \frac{N^{\alpha/2-1}}{\log N}}.$$

The right hand side is not small in the limiting case  $\alpha = 2$ , but it is for all  $\alpha \in (2, 4)$ . In order to have an exponentially small bound in all cases, we rather take

$$t' = N^{-\beta} \quad \text{for some } 0 < \beta < 1 - \frac{\alpha}{4} \leq \frac{1}{2}. \quad (3.40)$$

The upper bound implies the exponential bound  $e^{-CN^\varepsilon}$  for some  $\varepsilon > 0$ , while the lower bound is imposed by the condition that our threshold  $t'$  is semiclassically small. For  $N$  large enough, this threshold  $t'$  will be much larger than  $C_\alpha N^{\frac{\alpha}{4}-1}$ .

With the above choice, our probability of deviation reads (for  $N$  large enough)

$$\mathbb{P}_{\nu,N} \left( \left| \langle \psi_{1,N}, T_{\frac{j}{N}} \psi_{1,N} \rangle - \omega_\infty(e_j) \right| > N^{-\beta} \right) \leq C e^{-C'' \frac{N^{1-2\beta}}{\log N}} \leq C e^{-C'' N^{1-2\beta-}}. \quad (3.41)$$

## 3.6.2.3 Controlling all eigenstates and all Fourier modes simultaneously

So far we have controlled the deviations between quantum and classical averages for a single eigenstate, and a single Fourier mode  $e_j$ . To impose a QUE on the random eigenbasis, we should first impose a small deviation to all the eigenstates  $\{\psi_{k,N}\}_{k=1}^{d(\nu,N)}$  of the random eigenbasis. Since the random eigenstates all have the same distribution, each  $\psi_{k,N}$  satisfies the same individual bound (3.41). The random variables  $\psi_{1,N}, T_{\frac{j}{N}}\psi_{1,N}$  are not independent of each other, but we may apply a simple union bound to the  $d(\nu, N)$  events  $\left\{ \left| \langle \psi_{k,N}, T_{\frac{j}{N}}\psi_{k,N} \rangle - \omega_\infty(e_j) \right| > t' \right\}$ , and get, for  $t'$  given by (3.40):

$$\mathbb{P}_{\nu,N} \left( \exists k = 1, \dots, d(\nu, N) : \left| \langle \psi_{k,N}, T_{\frac{j}{N}}\psi_{k,N} \rangle - \omega_\infty(e_j) \right| > N^{-\beta} \right) \leq Cd(\nu, N) e^{-C''N^{1-2\beta-}}. \quad (3.42)$$

We may also take into account simultaneously the different eigenspaces of  $\mathcal{M}_N$ . Although the events corresponding to different eigenspaces are independent of one another, the union bound will still give us a good bound:

$$\mathbb{P}_N \left( \exists k = 1, \dots, N : \left| \langle \psi_{k,N}, T_{\frac{j}{N}}\psi_{k,N} \rangle - \omega_\infty(e_j) \right| > N^{-\beta} \right) \leq CN e^{-C''N^{1-2\beta-}}. \quad (3.43)$$

This bound shows that, with very high probability as  $\mathcal{N}_\alpha \ni N \rightarrow \infty$ , all the matrix elements  $\langle \psi_{k,N}, T_{\frac{j}{N}}\psi_{k,N} \rangle$  are simultaneously very close to the classical average  $\omega_\infty(e_j)$ .

In order to prove QUE, we need to control the matrix elements of many Fourier modes simultaneously. The local Weyl law for the Fourier modes, Lemma 3.4.2, is uniform for all Fourier indices  $j \in \{(0, 0)\} \cup \mathbb{Z}^2 \setminus N\mathbb{Z}^2$ , yet only  $N^2$  operators  $T_{\frac{j}{N}}$  are independent of one another, namely the ones for  $j$  running in a fundamental cell of  $N\mathbb{Z}^2$ , for instance  $\llbracket -N/2, N/2 - 1 \rrbracket^2$  if  $N$  is even.

The events corresponding to different  $e_j$ 's are not independent, but we still may apply a union bound encompassing the  $N^2$  Fourier modes, to obtain our main quantitative probabilistic result:

**Proposition 3.6.3.** *Consider for a given  $\alpha \in [2, 4)$ , a sequence  $\mathcal{N}_\alpha$  satisfying (P.10). For each  $N \in \mathcal{N}_\alpha$  we construct random eigenbases of  $\mathcal{H}_N$ . Then, this random eigenbasis is sharply equidistributed, as shown by the quantitative probabilistic estimate:*

$$\mathbb{P}_N \left( \exists k = 1, \dots, N, \exists j \in \llbracket -N/2, N/2 - 1 \rrbracket^2 : \left| \langle \psi_{k,N}, T_{\frac{j}{N}}\psi_{k,N} \rangle - \omega_\infty(e_j) \right| > N^{-\beta} \right) \leq CN^3 e^{-C''N^{1-2\beta-}}. \quad (3.44)$$

It is finally time to group the different  $N$  together. Let us denote the event in the above probability by  $B_N$  (this is a “bad” event, from the point of view of QUE). We can view this event as an event in the global probability space

$$(O, \mathbb{P}) = \prod_{N \in \mathbb{N}^*} (O_N, \mathbb{P}_N),$$

of all eigenbases for all  $N$ , and thus replace  $\mathbb{P}_N$  by  $\mathbb{P}$  in (3.44). We then notice that the probabilities

$\mathbb{P}(B_N) = \mathbb{P}_N(B_N)$  are summable:

$$\sum_{N \in \mathcal{N}_\alpha} \mathbb{P}(B_N) < \infty.$$

Hence, we may apply the Borel-Cantelli Lemma, and state that,  $\mathbb{P}$ -almost surely, a sequence of random bases will eventually be outside of the “bad” events  $B_N$ . This means that, for  $N$  large enough (depending on the sampled sequence), the eigenstates  $\{\psi_{j,N}\}$  all satisfy

$$\forall k = 1, \dots, N, \quad \forall j \in \llbracket -N/2, N/2 - 1 \rrbracket^2, \quad \left| \langle \psi_{k,N}, T_{\frac{j}{N}} \psi_{k,N} \rangle - \omega_\infty(e_j) \right| \leq N^{-\beta}. \quad (3.45)$$

### 3.6.3 Ending the proof of the probabilistic QUE (Theorem P.3.4)

Let us now choose a symbol  $a \in C^\infty(\mathbb{T}^2)$ , which may be  $N$ -dependent.

#### 3.6.3.1 Macroscopic QUE

Injecting the “good” estimates of (3.45) in the Fourier series of  $a$ , we find that, for almost every sequence of eigenbases, for  $N \in \mathcal{N}_\alpha$  large enough, we have:

$$\forall k = 1, \dots, N, \quad \left| \langle \psi_{k,N}, \text{Op}_N(a) \psi_{k,N} \rangle - \hat{a}(0) \right| \leq N^{-\beta} \sum_{j_1, j_2 = -N/2}^{N/2-1} |\hat{a}(j)| + \sum_{|j| \geq N/2} |\hat{a}(j)|. \quad (3.46)$$

If the symbol  $a$  is  $N$ -independent, the first Fourier sum is  $O(1)$ , while the second sum is  $O(N^{-\infty})$ . We thus get:

$$\forall k = 1, \dots, N, \quad \left| \langle \psi_{k,N}, \text{Op}_N(a) \psi_{k,N} \rangle - \omega_\infty(a) \right| \leq C_a N^{-\beta}.$$

This ends the proof of the probabilistic QUE at the macroscopic scale, Theorem P.3.4.i.

#### 3.6.3.2 Small scale QUE

We would like to consider the time-dependent symbols  $\chi_{\rho, r(N)}$  in the family described in (3.9). These functions are in the class  $S_{\delta+}(\mathbb{T}^2)$ , but they have the particularity to be supported in a disk of area  $\sim \pi N^{-2\delta}$ . This small support impacts the computation of the Fourier coefficients of  $\chi_{\rho, r(N)}$  as we have already witnessed in Lemma 3.3.3 which implied the estimates in (3.30). If we inject these estimates in (3.46), we find

$$\forall k = 1, \dots, N, \quad \left| \langle \psi_{k,N}, \text{Op}_N(\chi_{\rho, N^{-\delta}}) \psi_{k,N} \rangle - \int_{\mathbb{T}^2} \chi_{\rho, N^{-\delta}} \right| \leq C N^{-\beta+2\epsilon} + O(N^{-\infty}). \quad (3.47)$$



The integral of  $\chi_{\rho, N^{-\delta}}$  satisfies

$$\int_{\mathbb{T}^2} \chi_{\rho, N^{-\delta}} d\rho = \pi N^{-2\delta} (1 + O(1/\log N)) .$$

In order for the remainder in (3.47) to be smaller than this integral, we need to assume that  $\beta - 2\varepsilon > 2\delta$ . Since  $\varepsilon > 0$  can be chosen arbitrary small, it suffices to take

$$\beta > 2\delta .$$

This choice of  $\beta$  is possible, since from (3.40)  $\delta$  satisfies the condition  $\delta < \frac{1}{2} - \frac{\alpha}{8}$ , announced in the main theorem. In the above equations, the ordering of the parameters should be:

$$0 < 2\delta < \beta - 2\varepsilon < \beta < 1 - \frac{\alpha}{4} .$$

It ensures that, almost surely, and for  $N$  large enough,

$$\forall k = 1, \dots, N, \quad \left| \langle \psi_{k,N}, \mathbf{Op}_N(\chi_{\rho, N^{-\delta}}) \psi_{k,N} \rangle - \pi N^{-2\delta} \right| \leq C \frac{N^{-2\delta}}{\log N}, \quad (3.48)$$

the bound being uniform with respect to  $\rho \in \mathbb{T}^2$ .

This ends the proof of the small scale probabilistic QUE. □

# 4

## The statistical distribution of random eigenstates of the quantum cat map

We dedicate this chapter for studying the statistics of eigenstates of  $\mathcal{M}_N$  in a unique scenario: Taking  $N \in \mathbb{N}$  in the sequence described in [FNDB03] having short quantum period<sup>1</sup>, we show a "typical" eigenstate distributes like a standard Gaussian random vector on  $\mathbb{C}^N$ .

As we explain below we distinguish between quantum maps invariant under some (anti-) canonical symmetry and those who are not preserved under any. The matrix  $\gamma_{DE}$  gives an example for matrices of the first type: For instance,  $\gamma_{DE}$  admits as an anti-symmetry the time involution  $\iota = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

Matrices of the second type are highly rare. Examples for such matrices arise in works of Voros with Nonnenmacher (in context of  $\mathrm{SL}_2(\mathbb{Z})$  (anti-)symmetries, cf. [NV98]) and those of Baake with Roberts (concerning  $\mathrm{GL}_2(\mathbb{Z})$  and  $\mathrm{PGL}_2(\mathbb{Z})$  symmetries). Later Mezzadri with Keating showed perturbing these matrices does not preserve the statistics of eigenvalues. For the numerical simulations below we quantize Baake Roberts example,  $\gamma_{BR} = \begin{pmatrix} 4 & 9 \\ 7 & 16 \end{pmatrix} \in \tilde{\Gamma}_2$  as it has the smallest Frobenius norm possible for a matrix with no  $\mathrm{GL}_2(\mathbb{Z})$ -symmetries.

**Assumption 4.0.1.** *To simplify the presentation of our result we fix for the rest of the chapter an eigenvalue  $\nu := \nu_j \in \mathrm{Spec}(\mathcal{M}_N)$  and assume that asymptotically  $\dim V_{\nu_j} \sim \frac{N}{\log N}$ . We denote  $\nu = e^{i\varphi}$  for a phase  $\varphi \in [0, 2\pi]$  and the dimension by  $d(\nu, N)$ .*

**Assumption 4.0.2.** *For reasons explained below we focus in the present result only on a specific sequence of odd integers  $N \in \mathbb{N}^*$  featuring maximal quantum periods constructed in [BDB00],*

$$\mathcal{N} = \{N_{2m+1} = \mathrm{gcd}(p(2m+1), p(2m)+1) : m \in \mathbb{N}^*\} \quad (4.1)$$

where  $p : \mathbb{Z} \rightarrow \mathbb{R}_+$  is the sequence satisfying for every  $k \in \mathbb{N}^*$  either of the following recursive relations

$$p(0) = 0, \gamma^k = p(k)\gamma - p(k-1), \quad \begin{cases} p(k+1) = \mathrm{Tr}(\gamma)p(k) - p(k-1) \\ p(0) = 0 \\ p(1) = 1 \end{cases} . \quad (4.2)$$

This sequence attains a closed formula as well,

$$p(t) = \begin{cases} \frac{\lambda^{|t|} - \lambda^{-|t|}}{\lambda - \lambda^{-1}} & t \in \mathbb{Z}^* \\ 0 & t = 0 \end{cases} .$$

<sup>1</sup>In fact, the construction in [FNDB03] gives a sequence  $\{N_k\}$  for which  $P(N_k) = k$ .

Indeed in [BDB00] the authors prove that for these values of  $N$ ,

$$2m + 1 = 2 \frac{\log N_{2m+1}}{\log |\lambda|} + O(1), \gamma^{2m+1} \equiv I_2 \pmod{N_{2m+1}}$$

which means from (3.4) the quantum map  $\mathcal{M}_{2m+1}$  attains minimal period.

Notations

The  $\ell^\infty$  norm of a matrix  $A = (a_{j,k})_{j,k} \in \mathbb{C}^N$  is defined as  $\|A_N\|_{\ell^\infty} := \sup_{j,k} |a_{j,k}|_{1 \leq j,k \leq N}$ .

### 4.1 Random eigenstates for quantum cat maps

Below we study individual random eigenstates of  $\mathcal{M}_N$ . Let us explain how we obtain them. In fact, our choice of a random state depends on the classical and quantum dynamics.

- ("CUE" case) for every choice of (quantizable)  $\gamma$  and  $N \in \mathbb{N}^*$  we can construct random eigenbases of  $\mathcal{M}_N$ . First of all, fix for every eigenspace  $V_{\nu,N}$  an eigenbasis  $\{u_{1;\nu}, \dots, u_{d(\nu,N);\nu}\}$ . Let us equip the group of unitary matrices,  $\mathcal{U}(N)$ , with the Haar measure  $d\text{Vol}$ . Conjugating  $\mathcal{M}_N$  by  $U \in \mathcal{U}(N)$  preserves the eigen-decomposition yet each eigenbasis is "rotated" by  $U^* = U^{-1}$ : for every  $j \in \llbracket 1, d(\nu, N) \rrbracket$

$$(U^* \mathcal{M}_N U) (U^* u_{j;\nu}) = U^* \mathcal{M}_N u_{j;\nu} = \nu U^* u_{j;\nu}.$$

Applying  $U^*$  on each  $u_{j;\nu}$  yields a random basis of  $V_{\nu,N}$ ,  $\{v_{1;\nu}, \dots, v_{d(\nu,N);\nu}\}$ . One manner of obtaining an individual random eigenstate is to take a vector from this random eigenbasis. For the calculations below we apply an equivalent approach: let  $(\alpha_{1,N}^{\mathbb{C}}, \dots, \alpha_{N,N}^{\mathbb{C}}) \in \mathcal{N}_{\mathbb{C}}(0, \text{Id}_N)$  be a vector of *complex* Gaussians with uncorrelated entries satisfying

$$\begin{aligned} \mathbb{E}[\alpha_{j,N}] &= 0 \\ \mathbb{E}[\alpha_{j,N} \alpha_{j',N}] &= 0 \\ \mathbb{E}[\alpha_{j,N} \overline{\alpha_{j',N}}] &= \delta_j(j'), \end{aligned}$$

and let<sup>2</sup>

$$\alpha_N^{\mathbb{C}} = \sum_{j=1}^N \alpha_{j,N}^{\mathbb{C}} e_j, \quad e_j := \delta_{\frac{j}{N} + \mathbb{Z}}$$

be a complex random Gaussian state in  $\mathcal{H}_N$ . Its projection to  $V_{\nu,N}$  denoted by  $v_N^{\mathbb{C}} = \Pi_{\nu,N} \alpha_N^{\mathbb{C}}$  with  $\Pi_{\nu,N} = (\pi_{j,k})_{j,k}$  gives a random Gaussian state in  $V_{\nu,N}$ . We denote henceforth

$$\beta_N^{\mathbb{C}} = \sqrt{P(N)} v_N^{\mathbb{C}}, \quad \beta_N^{\mathbb{C}} = \sum_{n=1}^N \beta_{n,N}^{\mathbb{C}} e_n. \tag{4.3}$$

---

<sup>2</sup>Recall that  $\{e_j\}$  is the standard basis of the quantum space  $\mathcal{H}_N$  defined in chapter 1.

- (“COE” scenario) If  $\mathcal{M}_N$  is symmetric we discuss *real* random eigenstates. We restrict ourselves to matrices  $\gamma$  invariant by time-reversal symmetry, namely  $\gamma_{11} = \gamma_{22}$ . For example, given  $N \in \mathbb{N}^*$  the formulae from [KR01a, HB80] imply that  $\mathcal{M}_N(\gamma_{DE})$  is symmetric. In that scenario each eigenspace is spanned by real eigenstates since for every eigenstate  $u \in V_{\nu, N}$ ,

$$\mathcal{M}_N \bar{u} = \overline{\mathcal{M}_N u} = \overline{\mathcal{M}_N^* u} = \overline{\mathcal{M}_N^{-1} u} = \overline{\nu^{-1} u} = \nu \bar{u}$$

(so that  $\operatorname{Re}(u)$ ,  $\operatorname{Im}(u)$  are also eigenstates in  $V_{\nu, N}$ ). This quantum symmetry under complex conjugation corresponds to classical symmetry under the involution  $\iota = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . In order to obtain a real random eigenstate we can either rotate  $\{u_{1;\nu}, \dots, u_{d(\nu, N);\nu}\}$  a fixed *real* eigenbasis of  $V_{\nu, N}$  by a random orthogonal matrix  $O \in (O(N), \operatorname{dVol})$  or project a real Gaussian vector  $(\alpha_{1, N}^{\mathbb{R}}, \dots, \alpha_{N, N}^{\mathbb{R}}) \sim \mathcal{N}_{\mathbb{R}}(0, \operatorname{Id}_N)$  to  $V_{\nu, N}$ . We denote the projection in this case by  $v_N^{\mathbb{R}}$  and similarly  $\beta_N^{\mathbb{R}} = P(N) v_n^{\mathbb{R}}$ .

In order to simplify notations along this chapter we refer to our random eigenstate by  $\beta_N$  and to a standard Gaussian vector by  $\alpha_N$ . In the formulations of theorems we always indicate whether  $\beta_N = \beta_N^{\mathbb{C}}$  or  $\beta_N = \beta_N^{\mathbb{R}}$ .

Our first result concerns the covariance matrix<sup>3</sup> of  $\beta_N = \beta_N^{\mathbb{C}}$ . It can be obtained from a direct calculation. We prove this result only in the CUE case as the calculation for the COE case is completely analogous. Realizing  $\beta_{n, N} = \sqrt{P(N)} \sum_{k=1}^N \pi_{nk} \alpha_{k, N}$ , we notice that from the properties of  $\{\alpha_{k, N}\}$  and since  $\Pi_{\nu, N}$  is an orthogonal spectral projector

$$\begin{aligned} \frac{1}{P(N)} \operatorname{cov}(\beta_{n, N}, \overline{\beta_{m, N}}) &= \sum_{j, k} \pi_{n, j} \overline{\pi_{m, k}} \operatorname{cov}(\alpha_{j, N}, \overline{\alpha_{k, N}}) \\ &= \sum_{j=1}^N \pi_{n, j} \overline{\pi_{m, j}} \\ &= \sum_{j=1}^N \pi_{n, j} \pi_{j, m} \\ &= (\Pi_{\nu, N}^2)_{n, m} = \pi_{n, m}. \end{aligned} \tag{4.4}$$

In particular,

$$\frac{1}{P(N)} \operatorname{Var}(\beta_{n, N}) = \frac{1}{P(N)} \operatorname{cov}(\beta_{n, N}, \overline{\beta_{n, N}}) = \pi_{n, n}. \tag{4.5}$$

#### 4.1.1 Calculation of $\Pi_{\nu, N}$ for highly degenerate $N$ s

Henceforth we focus on odd values of the highly degenerate  $N$ s contained inside

$$\mathcal{N}_2 \cap (2\mathbb{N} + 1) \tag{4.6}$$

---

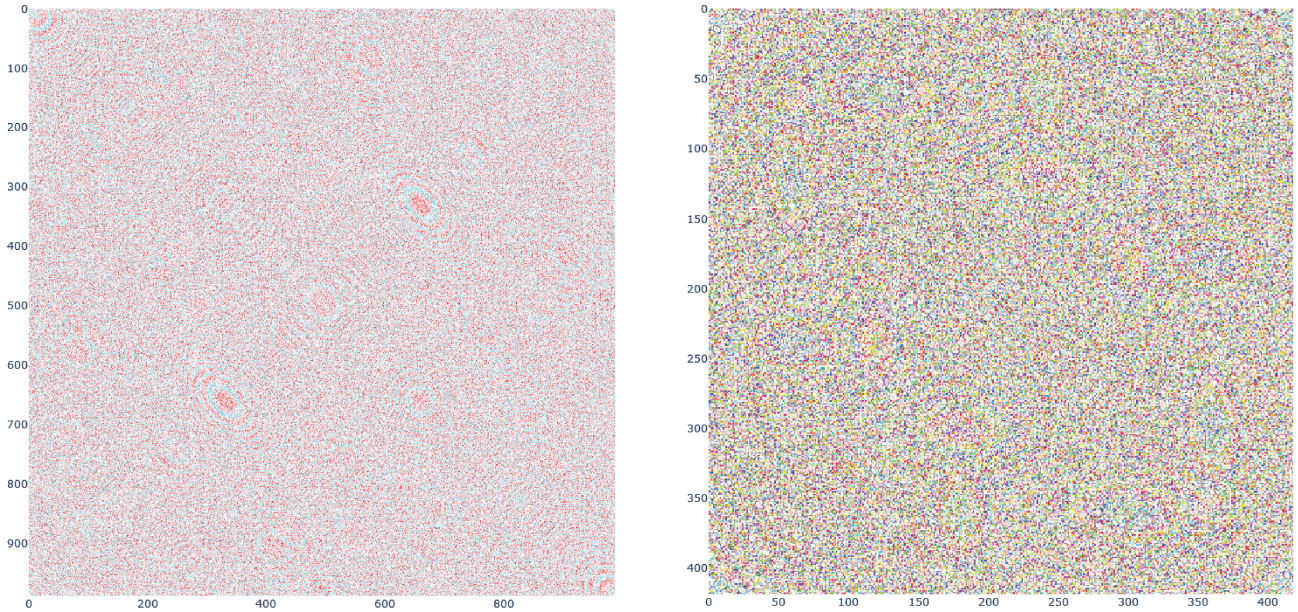
<sup>3</sup>The proof for real  $\beta_N^{\mathbb{R}}$  is completely analogous

with  $\mathcal{N}_2$  defined previously in Definition 3.2.1. As we mentioned in the introduction to this chapter we only deal with  $N$ s in  $\mathcal{N}$ . Whenever  $N \in \mathcal{N}$  we are able to obtain an explicit formula for the projection to  $V_{\nu,N}$ . The result we are about to prove holds both for the CUE and COE cases. In fact, we remind that in the CUE case,  $\beta_{k,N}, \beta_{k',N}$  are complex random variables thus their covariance is given by

$$\text{cov}(\beta_{k,N}, \overline{\beta_{k',N}}) = \mathbb{E}[\beta_{k,N} \overline{\beta_{k',N}}] - \mathbb{E}[\beta_{k,N}] \mathbb{E}[\overline{\beta_{k',N}}] = \mathbb{E}[\beta_{k,N} \overline{\beta_{k',N}}].$$

**Theorem 4.1.1.** Fix  $\gamma \in \tilde{\Gamma}(2)$  and fix a sequence of  $N$ s in  $\mathcal{N}$  (where  $\mathcal{N}$  is defined in (4.1)). Consider the projector matrix  $\Pi_{\nu,N} = (\pi_{j,k})_{j,k}$  corresponding to an eigenspace  $V_{\nu,N}$  where  $\nu \in \text{Spec}(\mathcal{M}_N)$ . Then there exists a sequence of matrices  $(\Delta_N)_N = \left( (\delta_{j,k})_{j,k \in \llbracket 1, N \rrbracket} \right)_N$  with  $\|\Delta_N\|_{\ell^\infty}$  bounded as  $N \rightarrow \infty$  such that  $\Pi_{\nu,N} = \frac{1}{P(N)} I_N + \frac{1}{\sqrt{4N}} \Delta_N$ . As a consequence of (4.4) this means

$$\text{cov}(\beta_N) = I_N + \frac{1}{\sqrt{4N}} \Delta_N.$$



**Figure 4.1:** Hue, Saturation, Value (HSV) maps of  $\Delta_N$  in the COE and CUE cases. Brighter colors correspond to lower values. The hue of each pixel represents the phase of the matrix entry. Left: The matrix  $\Delta_{989}$  corresponding to  $\gamma_{DE}$ . The imaginary part of each element is negligible compared to the real part. Right: The map  $\Delta_{419}$  corresponding to  $\gamma_{BR}$  has complex entries.

**Remark 4.1.2.**

1. The proof of the theorem does not involve probabilistic methods – it roughly tells that for the choice of  $N$ s inside  $\mathcal{N}$  the spectral projector  $\Pi_{\nu,N} \approx \frac{1}{P(N)} I_N$ .

2. The method used in the proof of Theorem 4.1.1 does not allow to study the properties of the remainder matrices  $\Delta_N$  in great details. In the COE case,  $(\Delta_N)$  is a sequence of real symmetric matrices while in the CUE case it is a sequence of Hermitian matrices (cf. Figure 4.1).
3. Following the numerical observations in Figure 4.2, Figure 4.3, we conjecture the remainder we find in the theorem is not sharp and is in fact  $O_{\ell^\infty} \left( \frac{1}{N^{\frac{1}{2}-\varepsilon}} \right)$ .
4. An analogous asymptotic expansion does not hold for  $N \in 2\mathbb{N}^*$ : for instance in subfigure Figure 4.2b we see large off-diagonal entries of the projector. A close numerical inspection shows that the difference between odds and evens arrive from the terms corresponding to powers  $\mathcal{M}_N^t$  (where  $|t| = \lfloor \frac{P(N)}{2} \rfloor$ ): while for even entries  $\mathcal{M}_N^t$  can be a permutation matrix, for odd  $N$ s the difference between entries is small (see Figure 4.5)
5. Note that a similar result holds also for sub-tuples of  $\beta_N$ : for every  $N \in \mathcal{N}$  fix  $m(N) \in \llbracket 1, N \rrbracket$ . Denote by  $f_{j_1, \dots, j_{m(N)}} : \mathbb{C}^N \rightarrow \mathbb{C}^{m(N)}$  the restriction defined by

$$f_N(x_1, \dots, x_N) = (x_{j_1}, \dots, x_{j_{m(N)}})$$

. There exists a sequence of matrices  $(\Delta_{m(N)})_N$  with  $\|\Delta_{m(N)}\|_{\ell^\infty}$  bounded as  $N \rightarrow \infty$  such that the vector  $f_{j_1, \dots, j_{m(N)}} \beta_N$  satisfies

$$\text{cov} \left( f_{j_1, \dots, j_{m(N)}} \beta_N \right) = I_{m(N)} + \frac{1}{\sqrt[4]{m(N)}} \Delta_{m(N)}''.$$

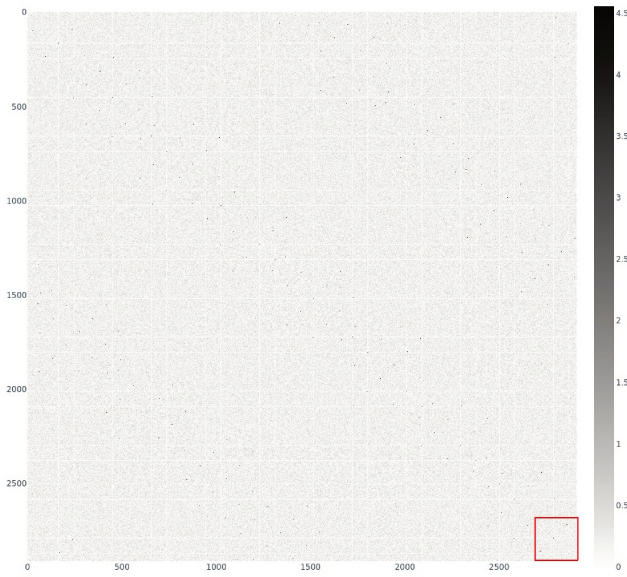
The result follows by imitating the argument used to prove Theorem 4.1.1.

*Proof of Theorem 4.1.1.* Let  $t \in \mathbb{Z}^*$  and assume for simplicity that  $b > 0$ . Following (4.2), denoting the  $t$ -th power of a matrix  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}(2)$  by  $\gamma^t = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}$ ,

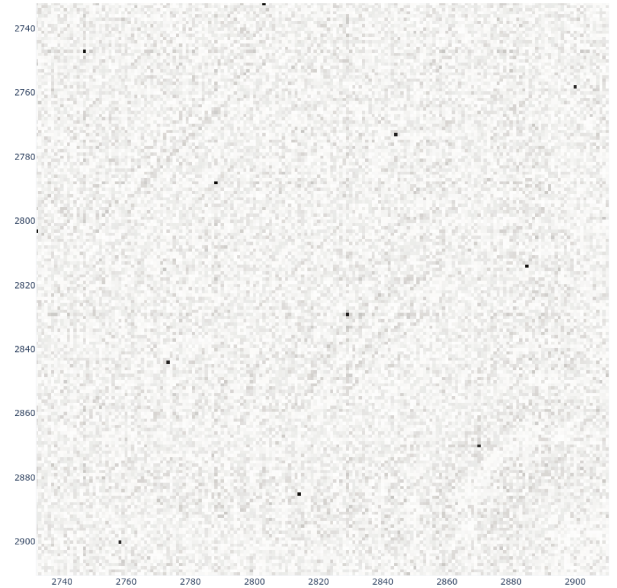
$$\begin{aligned} b_t &= p(t) \operatorname{sgn}(t) b \\ a_t &= \begin{cases} p(t) a - p(t-1) & t > 0 \\ p(t) d - p(t+1) & t < 0 \end{cases} \\ d_t &= \begin{cases} p(t) d - p(t-1) & t > 0 \\ p(t) a - p(t+1) & t < 0 \end{cases}. \end{aligned} \tag{4.7}$$

Substituting in the formula for  $\mathcal{M}_N(\gamma^t)$  appearing in [HB80, Equation 16], for every  $t \in \mathbb{Z}^*$

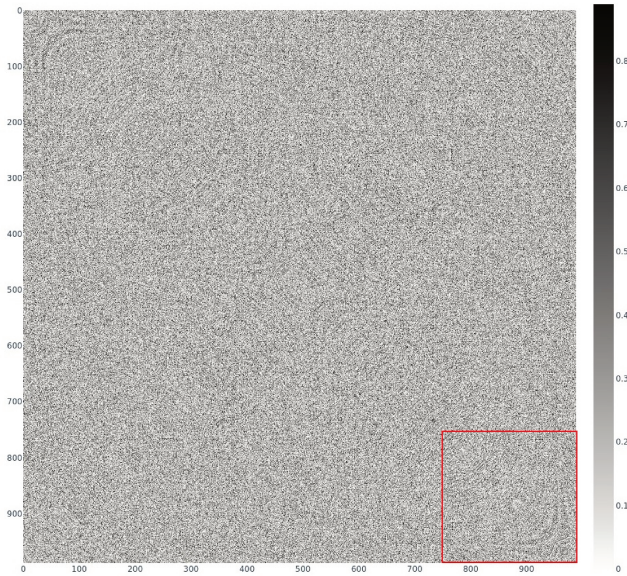
$$\begin{aligned} (\mathcal{M}_N(\gamma^t))_{j,k} &= \sqrt{\frac{i}{Nbp(t)}} \exp \left[ \frac{\pi i}{Nb_t} (a_t k^2 - 2jk + d_t j^2) \right] \times \\ &\quad \sum_{m=1}^{bp(t)} \exp \left( \frac{\pi i}{b_t} (Na_t m^2 + 2(ka_t - j)m) \right). \end{aligned}$$



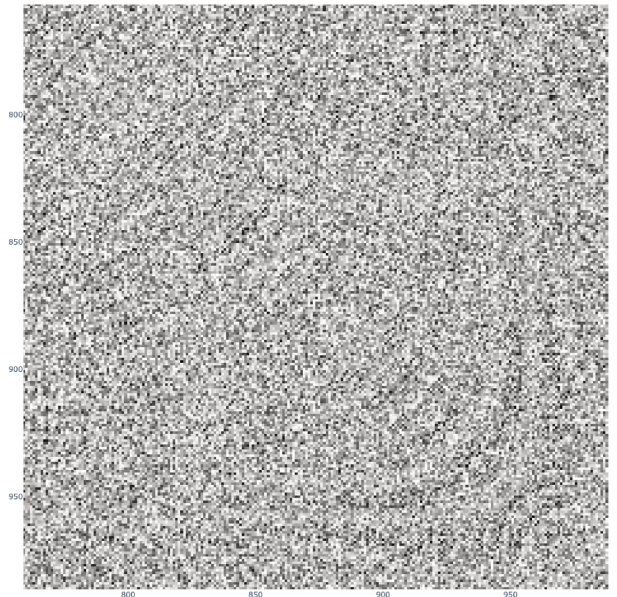
(a) The matrix  $\sqrt{N} \left| \Pi_{\nu, N} - \frac{I_N}{P(N)} \right|$  for  $N = 2911, P(N) = 14$



(b) The  $180 \times 180$  last entries for  $N = 2911$



(c) The matrix  $\sqrt{N} \left| \Pi_{\nu, N} - \frac{I_N}{P(N)} \right|$  for  $N = 989, P(N) = 11$



(d) The  $180 \times 180$  last entries for  $N = 989$

**Figure 4.2:** (A,C) Heatmaps of  $\sqrt{N} \left| \Pi_{\nu, N} - \frac{1}{P(N)} I_N \right|$  (lighter colors correspond to smaller values) where  $N$  is an element of the sequence in [FNDB03] for which  $\mathcal{M}_N$  has a minimal quantum period  $P(N) \approx 2 \frac{\log N}{\log \lambda}$ . (B,D) zooming on the last 180 entries of each of the matrices to the left (marked in red squares) the value of entries vary.

From the standard theory of quadratic Gauss sums (for a proof using Poisson averaging we refer the reader to [HB80, Appendix A, pages 287-288]) the (Gauss) sum appearing in the last equation does not vanish if and only if

$$\frac{1}{\gcd(N, b_t)} \left[ \frac{Na_t b_t}{\gcd(N, b_t)} + 2(a_t k - j) \right] \in 2\mathbb{Z}.$$

As follows, from [HB80, Equation 14], the entry can be expressed as

$$(\mathcal{M}_N(\gamma^t))_{j,k} = \sqrt{\frac{i \gcd(N, b_t)}{N}} \exp(\Phi_{j,k}(\lambda, N; t)) \times \delta_{2\mathbb{Z}} \left( \frac{1}{\gcd(N, b_t)} \left[ \frac{Na_t b_t}{\gcd(N, b_t)} + 2(a_t k - j) \right] \right) \quad (4.8)$$

for some phase  $\Phi_{j,k}(\lambda, N; t)$  and  $\delta_{2\mathbb{Z}}(x) = \sum_{k \in 2\mathbb{Z}} \delta_k(x)$ . We remind remark (iii) from [HB80]: In every column or row of  $\mathcal{M}_N(\gamma^t)$  there are only  $\frac{N}{\gcd(N, b_t)}$  non-zero terms in equal spacing of  $\gcd(N, b_t)$  and each has modulus  $\frac{\gcd(N, b_t)}{N}$ . Henceforth let us focus only on  $N = N_{2m+1} \in \mathcal{N}$ , having extremely short odd period. We then split the entries of  $\Pi_{\nu, N}$  to three temporal components corresponding to positive times  $t \in [1, \frac{P(N)-1}{2}]$ , negative times  $t \in [-\frac{P(N)-1}{2}, -1]$  and  $t = 0$ , using the formula in (3.16). The central goal is going to be estimating the factors  $\gcd(N, b_t)$  for positive times (recall that  $b_{-t} = -b_t$  hence the estimate for negative times is identical).

**Lemma 4.1.3.** For  $t \in [1, \frac{P(N)-1}{2}]$  and  $N \in \mathcal{N}$ ,

$$\gcd(N_{2m+1}, b_t) = \min \{p(m-t+1) + p(m-t), p(t)\} = O\left(\sqrt{N_{2m+1}}\right)$$

*Proof.* First, recall that in [FNDB03]  $N_{2m+1}$ , which is shown to be the largest  $N$  for which  $\mathcal{M}_N$  has quantum period  $2m+1$ , can also be expressed in terms of  $p(\cdot)$  as

$$N_{2m+1} = p(m+1) + p(m) \quad (4.9)$$

and denoting  $\tau_\gamma := \text{Tr}(\gamma)$  and from the recurrence relation appearing previously and in [BDB00] for every  $m \in \mathbb{N}^*$

$$p(m+1) = \tau_\gamma p(m) - p(m-1). \quad (4.10)$$

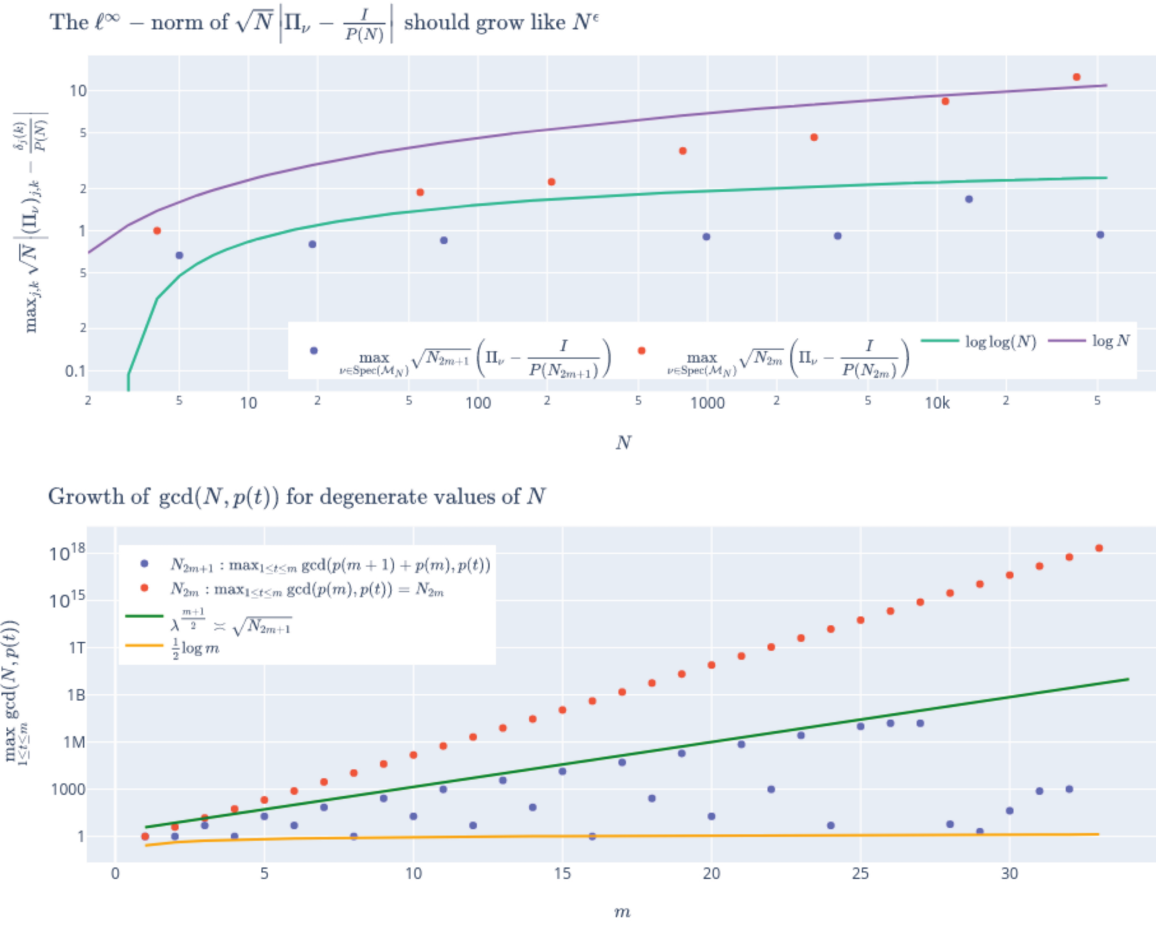
We remark that the recursive relation of  $p$  can be expressed in a matrix form: for every  $m \in \mathbb{N}^*$

$$\begin{pmatrix} p(m+1) \\ p(m) \end{pmatrix} = \begin{pmatrix} \tau_\gamma & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p(m) \\ p(m-1) \end{pmatrix} = \begin{pmatrix} \tau_\gamma & -1 \\ 1 & 0 \end{pmatrix}^{m-t} \begin{pmatrix} p(t+1) \\ p(t) \end{pmatrix}$$

thus diagonalizing the matrix and noting its spectrum is identical to  $\text{Spec}(\gamma)$

$$\begin{aligned} p(m+1) &= \frac{1}{\lambda - \lambda^{-1}} [(\lambda^{m+1-t} - \lambda^{t-m-1}) p(t+1) + (\lambda^{m-t} - \lambda^{t-m}) p(t)] \\ &= p(t+1) p(m-t+1) + p(t) p(m-t) \end{aligned} \quad (4.11)$$

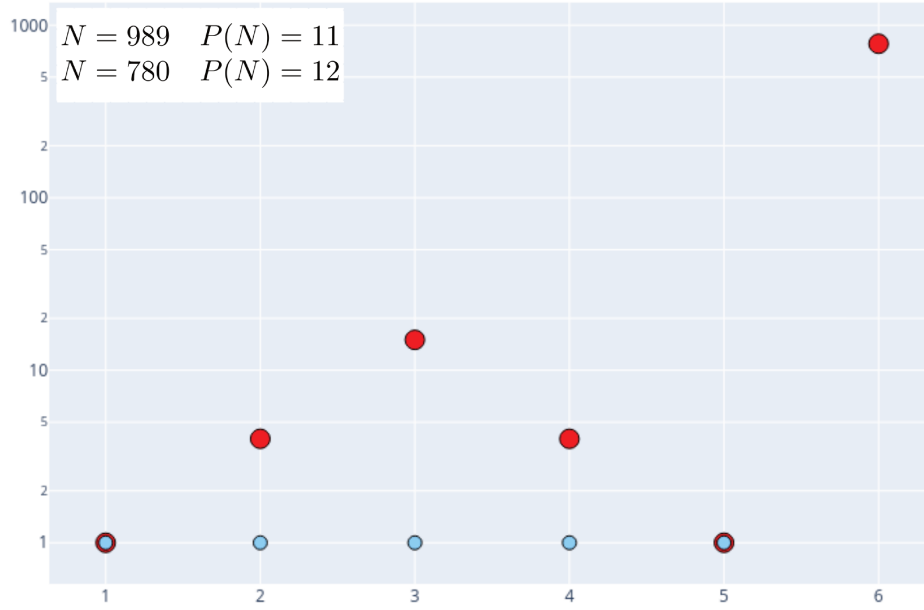




**Figure 4.3:** Above: Numerical computation shows that  $\max_{\nu, j, k} \left| \left( \Pi_{\nu, N} - \frac{1}{P(N)} \text{Id}_N \right)_{j, k} \right| = O\left(N^{\epsilon - \frac{1}{2}}\right)$ . Below: The estimates on the  $\gcd$ s in the proof of Theorem 4.1.1 are tight and do not apply for  $N_{2m}$ . For  $N_{2m}$  we observe the maximal value of  $\gcd(p(m), p(t))$  is  $p\left(\frac{P(N_{2m})}{2}\right) = p(m) = N_{2m}$  while for  $N$ s with odd period it is  $O\left(\lambda^{\frac{m+1}{2}}\right) = O\left(\sqrt{N_{2m+1}}\right)$ .

and in fact  $p(m+1)$  is a linear combination with integer coefficients of  $p(t)$  and  $p(t+1)$ . Let us take  $2m+1 \in 2\mathbb{N}^* + 1, t \in \llbracket 1, m \rrbracket$ . From the relation in (4.11) we deduce

$$\begin{aligned}
 \gcd(N_{2m+1}, b_t) &= \gcd(p(m+1) + p(m), bp(t)) & (4.12) \\
 &= \frac{\gcd(p(m+1) + p(m), p(t)) \cdot \gcd(p(m+1) + p(m), b)}{\gcd(p(m+1) + p(m), \gcd(b, p(t)))} \\
 &\leq b \gcd(p(m+1) + p(m), p(t)) \\
 &= b \gcd(p(t+1) [p(m-t+1) + p(m-t)] + p(t) [p(m-t) + p(m-1-t)], p(t))
 \end{aligned}$$



**Figure 4.4:** Observing  $\gcd(N_{11}, p(t))$  and  $\gcd(N_{12}, p(t))$  arising from (4.12) alludes to the difference between "highly degenerate"  $N$ s with odd and even periods. When  $t = \frac{P(N)}{2}$ , appearing only when  $P(N) \in 2\mathbb{N}^*$ , we add to the sum a term of magnitude  $N_{2m}$ .

$$= b \gcd(p(t+1) [p(m-t+1) + p(m-t)], p(t))$$

Invoking the recursive relation in (4.10) we obtain for every  $t$  that consecutive elements of  $p(\cdot)$  are co-prime,

$$\begin{aligned} \gcd(p(t+1), p(t)) &= \gcd(\tau_\gamma p(t) + p(t-1), p(t)) \\ &= \gcd(p(t-1), p(t)) \\ &= \cdots = \gcd(p(0), p(1)) = 1 \end{aligned}$$

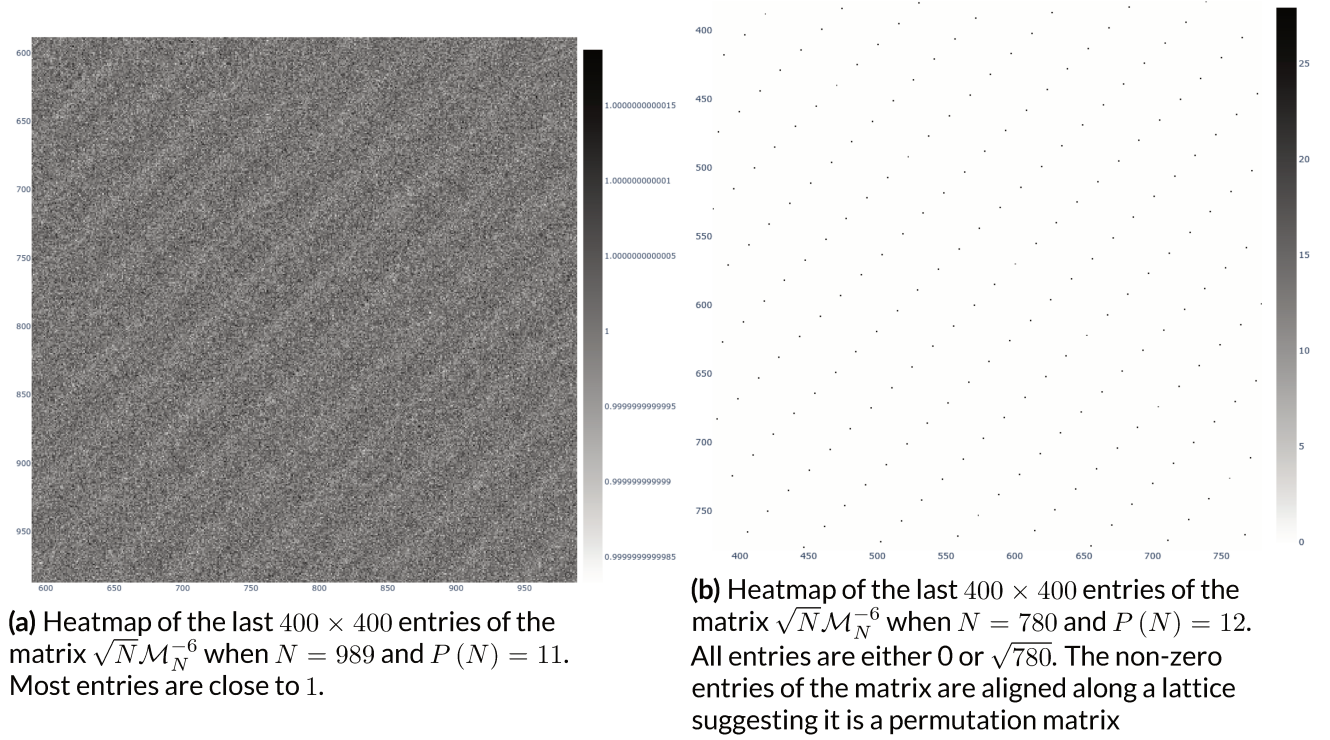
hence

$$\begin{aligned} \gcd(N_{2m+1}, p(t)) &\leq b \frac{\gcd(p(t+1), p(t)) \cdot \gcd(p(m-t+1) + p(m-t), p(t))}{\gcd(p(m-t+1) + p(m-t), \gcd(p(t+1), p(t)))} \\ &= b \gcd(p(m-t) + p(m+1-t), p(t)) \end{aligned}$$

Let us view the right hand side as a function in  $t$ . Noting that  $p(m-t+1), p(m-t) \asymp \lambda^{m-t}$  and  $p(t) \asymp \lambda^t$ , the right hand side is approximately maximized when  $t = \lfloor \frac{m}{2} \rfloor$  (see Figure 4.4) and as follows

$$\gcd(p(m+1) + p(m), p(t)) \leq C' \lambda^{\frac{m+1}{2}} \approx C'' \sqrt{N_{2m+1}}.$$

□



**Figure 4.5:** Matrices contributing to the sum making  $\Pi_{\nu,N}$  differ depending on the parity of  $N$

Altogether bounding the sum of exponents naively we deduce that for the sub-sequence of odd  $N_{2m+1}$ s from [FNDBo3]

$$\left| \sum_{t=1}^{\frac{P(N)-1}{2}} \sqrt{\gcd(N_{2m+1}, b_t)} \exp(\Phi_{j,k}(\lambda, N_{2m+1}; t) - i\varphi t) \times \delta_{2\mathbb{Z}} \left( \frac{1}{\gcd(N_{2m+1}, b_t)} \left[ \frac{N_{2m+1} a_t b_t}{\gcd(N_{2m+1}, b_t)} + 2(a_t k - j) \right] \right) \right| \leq C'' \sqrt[4]{N}.$$

We deduce that

$$|\pi_{j,j}| = \frac{1}{P(N)} \left( 1 + O\left(N^{-\frac{1}{4}}\right) \right)$$

while if  $j \neq k$ ,  $|\pi_{j,k}| = O\left(N^{-\frac{1}{4}} P^{-1}(N)\right)$ . We note that in both cases the implied constant in the asymptotic expansion is uniform with respect to the choice of the entries  $j$  and  $k$ .  $\square$

We deduce from Theorem 4.1.1 a result on the correlation matrix,

**Proposition 4.1.4.** *Let  $\gamma \in \tilde{\Gamma}(2)$  be the classical map quantized by  $\{\mathcal{M}_N(\gamma)\}_N$ . For every  $N \in \mathcal{N}$  and  $1 \leq k < k' \leq N$  the random variables  $\beta_{k,N}$  and  $\beta_{k',N}$  are essentially uncorrelated: for every such*

$\gamma$  there exists a sequence of matrices  $(\Delta'_N)_N = \left( (\delta'_{j,k})_{j,k} \right)_N$  with  $\sup_N \|\Delta'_N\|_{\ell^\infty} = O(1)$  such that

$$\text{Corr}(\beta_N) := (\text{Corr}(\beta_{n,N}, \beta_{m,N}))_{n,m} = I_N + \frac{P(N)}{N^{\frac{1}{4}}} \Delta'_N$$

We notice that for every  $n, m \in \llbracket 1, N \rrbracket$

$$\text{Corr}(\beta_{n,N}, \beta_{m,N}) = \frac{\text{cov}(\beta_{n,N}, \overline{\beta_{m,N}})}{\sigma_{\beta_{n,N}} \cdot \sigma_{\beta_{m,N}}}$$

where  $\sigma_{\beta_{n,N}} = \sqrt{\text{Var}(\beta_{n,N})}$  for every  $n \in \llbracket 1, N \rrbracket$ .

*Proof of Proposition 4.1.4:* From Theorem 4.1.1 joint with (4.4) and (4.5),

for every  $j \neq k$

$$(\text{Corr}(\beta_N))_{j,k} = \frac{\delta_j(k) + O\left(\frac{1}{\sqrt[4]{N}}\right)}{\sqrt{\left(1 + O\left(\frac{1}{\sqrt[4]{N}}\right)\right) \left(1 + O\left(P(N)\sqrt[4]{N}\right)\right)}} = \delta_j(k) + O\left(P(N)N^{-\frac{1}{4}}\right)$$

with the constant in the remainder being uniform with respect to the choice of entries  $j, k$ . The matrix of remainders  $\Delta'_N$  (i.e.,  $(\Delta'_N)_{j,k} = (\text{Corr}(\beta_N))_{j,k} - \delta_j(k)$ ) has uniformly bounded  $\ell^\infty$ -norm as  $N \rightarrow \infty$ .  $\square$

#### 4.1.2 The norm of $\beta_N$

The proof appearing in previous section did not rely on the explicit distribution of  $\alpha_N$ . For the rest of this chapter we restrict ourselves only to a Gaussian  $\alpha_N$ s, obtaining results on random eigenstates in  $\mathcal{H}_N$ . Such results will rely on our explicit expression in Theorem 4.1.1 for the projector. Before we prove these results we give several details on the norm of  $\beta_N$  when  $\alpha_N$  is a random Gaussian vector<sup>5</sup>. Beginning from a random *Gaussian* vector  $\alpha_N$  its projection to  $V_{\nu,N}$  is itself a Gaussian vector (since projection is a linear transformation), which is a key property in the proofs below. Moreover, applying rotations on a Gaussian vector yields another Gaussian vector. In that scenario, we realize that one can apply on  $\Pi_{\nu,N}$  a unitary transformation mapping the basis  $\{\Pi_{\nu,N}e_j\}_{j=1}^N$  to  $\{e_1, \dots, e_{d(\nu,N)}, 0, \dots, 0\}$ . Applying this unitary transformation on  $\langle \beta_N^{\mathbb{C}}, \beta_N^{\mathbb{C}} \rangle$ ,

$$\|\beta_N^{\mathbb{C}}\|_{\mathcal{H}_N}^2 = P(N) \langle \Pi_{\nu,N} \alpha_N^{\mathbb{C}}, \Pi_{\nu,N} \alpha_N^{\mathbb{C}} \rangle = P(N) \sum_{k=1}^{d(\nu,N)} |\alpha_{k,N}^{\mathbb{C}}|^2. \quad (4.13)$$

If  $\alpha_{1,N}^{\mathbb{C}}, \dots, \alpha_{N,N}^{\mathbb{C}} \sim \mathcal{N}_{\mathbb{C}}(0, 1)$  and independent, then

<sup>5</sup>We prove that the concentration of the norm occurs also when beginning from any random vector whose components are independent identically distributed in section A.6

1. For every  $k$ ,  $|\alpha_{k,N}^{\mathbb{C}}|^2 \sim \chi^2(2)$  where  $\chi^2(k)$  is *chi-squared probability distribution of  $k$  degrees of freedom*<sup>6</sup> and

$$\mathbb{E}[|\alpha_{k,N}^{\mathbb{C}}|^2] = 2, \text{Var}[|\alpha_{k,N}^{\mathbb{C}}|^2] = 4.$$

2. For every  $N \in \mathbb{N}^*$ ,  $\frac{1}{P(N)} \|\beta_N^{\mathbb{C}}\|^2 \sim \chi^2(2d(\nu, N))$ . From central limit theorem

$$\sqrt{2d(\nu, N)} \left( \frac{1}{2d(\nu, N)} \sum_{k=1}^{d(\nu, N)} |\alpha_{k,N}^{\mathbb{C}}|^2 - 1 \right) \xrightarrow{d} \mathcal{N}_{\mathbb{R}}(0, 2)$$

and from the delta method (cf. for instance [CB21, Theorem 5.5.24]),

$$\sqrt{2d(\nu, N)} \left( \frac{\|\beta_N^{\mathbb{C}}\|_{\mathcal{H}_N}}{\sqrt{2d(\nu, N)P(N)}} - 1 \right) \xrightarrow{d} \mathcal{N}_{\mathbb{R}}\left(0, \frac{1}{2}\right),$$

that is, the norm of  $\beta_N^{\mathbb{C}}$  is concentrated around  $\sqrt{2P(N)d(\nu, N)} \approx \sqrt{N}$ .

Below we use a more precise formulation of this idea; let us denote

$$Y_{d(\nu, N)} := \frac{1}{2\sqrt{d(\nu, N)}} \sum_{k=1}^{d(\nu, N)} |\alpha_{k,N}^{\mathbb{C}}|^2 - \sqrt{d(\nu, N)}.$$

Berry-Esseen theorem [Ber41, Ess42] says that there exists a function  $\varepsilon_{d(\nu, N)}$  satisfying  $|\varepsilon_{d(\nu, N)}| = O\left(\frac{1}{\sqrt{d(\nu, N)}}\right)$  such that for every  $x \in \mathbb{R}$

$$\mathbb{P}(Y_{d(\nu, N)} \leq x) = F_{\mathcal{N}_{\mathbb{R}}(0,1)}(x) + \varepsilon_{d(\nu, N)},$$

where  $F_{\mathcal{N}_{\mathbb{R}}(0,1)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-y^2/2) dy$  is the cumulative density of a standard normal variable. We deduce

$$\mathbb{P}(Y_d \leq x) = \mathbb{P}\left(\|\beta_N^{\mathbb{C}}\|^2 \leq 2\sqrt{d(\nu, N)P(N)}\left(x + \sqrt{d(\nu, N)}\right)\right)$$

which after a change of variable reads

$$\begin{aligned} \mathbb{P}(\|\beta_N^{\mathbb{C}}\| \leq z) &= \mathbb{P}\left(Y_d \leq \frac{z^2 - 2P(N)d(\nu, N)}{2P(N)\sqrt{d(\nu, N)}}\right) \\ &= F_{\mathcal{N}_{\mathbb{R}}(0,1)}\left(\frac{z^2 - 2P(N)d(\nu, N)}{2P(N)\sqrt{d(\nu, N)}}\right) + \varepsilon_{d(\nu, N)}\left(\frac{z^2 - 2P(N)d(\nu, N)}{2P(N)\sqrt{d(\nu, N)}}\right). \end{aligned}$$

---

<sup>6</sup>which indeed describes the distribution of sum of  $k$  squares of standard Gaussians

If  $\alpha_{1,N}^{\mathbb{R}}, \dots, \alpha_{N,N}^{\mathbb{R}} \sim \mathcal{N}_{\mathbb{R}}(0, 1)$  and independent similar conclusions hold, replacing  $2d(\nu, N)$  with  $d(\nu, N)$  as  $\|\alpha_N^{\mathbb{R}}\|^2$  in that case is a sum of  $d(\nu, N)$  independent standard Gaussians (and not of  $2d(\nu, N)$  as in the complex case). Then,

$$\mathbb{P}(\|\beta_N^{\mathbb{R}}\| \leq z) = F_{\mathcal{N}_{\mathbb{R}}(0,1)}\left(\frac{z^2 - P(N)d(\nu, N)}{P(N)\sqrt{2d(\nu, N)}}\right) + \varepsilon_{d(\nu, N)}\left(\frac{z^2 - P(N)d(\nu, N)}{P(N)\sqrt{2d(\nu, N)}}\right). \quad (4.14)$$

## 4.2 The value distribution of random vectors

Our goal in this section will be to examine the analogous counting measures for  $\beta_N$  and their normalized counterpart  $\frac{\beta_N}{\|\beta_N\|}$ . Our strategy will be to use results on Gaussian random eigenstates to study normalized eigenstates chosen with law  $(\mathbb{S}^{d(\nu, N)-1}, d\text{Vol})$ .

### Remark 4.2.1.

1. Below we give statements only for maps admitting quantum symmetries ("COE") whose eigenspaces as we mention above have bases of real eigenstates. Nevertheless, statements for the CUE case can be obtained directly from our results below: Given a complex random variable  $\beta_N^{\mathbb{C}} = \text{Re}(\beta_N^{\mathbb{C}}) + i \text{Im}(\beta_N^{\mathbb{C}})$  we can apply the statements below on its real and imaginary parts separately. Combining them we can deduce analogous statements on the value distribution on complex squares  $J \cong I \times I' \subset \mathbb{R}^2$ . (cf. Figure 4.7 for illustration of the complex analogue of Theorem 4.2.3).
2. Although we focus below only on  $N \in \mathcal{N}$  and not on the full sequence presented in [FNDB03], we conjecture that the value distribution for  $N$ s of even period might exhibit similar phenomenon (for instance cf. Figure 4.6b portraying the value distribution for  $\mathcal{M}_{2911}$  for which  $P(N) = 14$ ).
3. In [KR01b] the authors show that when  $N$  is a split prime for  $\gamma$  the value distribution of Hecke eigenstates is given by the Sato-Tate law. When the prime decomposition of  $N$  includes only powers of split primes, in light of the Chinese remainder theorem, we might have expected to get (up to a sign) a log-normal law. However along our sparse subsequence, the high degeneracy of the spectrum "overcomes" this log-normality phenomenon and we obtain a Gaussian law.

### 4.2.1 Value distribution of Gaussian random eigenstates

As a first result we prove that

**Proposition 4.2.2.** *Let  $\gamma \in \tilde{\Gamma}_2$  quantized by  $\{\mathcal{M}_N\}$ . Let  $N \in \mathcal{N}$  (where  $\mathcal{N}$  defined in (4.1)). Define the empirical measure on  $\mathbb{R}$*

$$\mu_N(I) := \frac{1}{N} \sum_{j=1}^N \mathbb{1}_I(\beta_{j,N}).$$

There exists a constant  $C_\gamma > 0$  such that as  $N \ni N \rightarrow \infty$ , for every  $I \subset \mathbb{R}$  and  $\varepsilon \in (0, \frac{1}{8})$

$$\mathbb{P} \left( \left| \mu_N(I) - \mu_{N(0,1)}(I) \right| \geq C_\gamma N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}$$

We remind that  $\mu_N$  is the empirical measure associated with the entries of the eigenvector  $\beta_N$ .

*proof of Proposition 4.2.2.* Fix an interval  $I = [\zeta_0, \zeta_1] \subset \mathbb{R}$  and consider the appropriate counting random variable

$$\mathcal{Z}_N := \mu_N(I) = \frac{1}{N} \sum_{j=1}^N \mathbb{1}_I(\beta_{j,N}).$$

We can calculate directly the expected value of  $\mathcal{Z}_N$ , recalling the notation  $P(N) \Pi_{\nu,N} - I_N = \Delta_N := (\delta_{j,k})_{j,k}$

$$\begin{aligned} \mathbb{E}[\mathcal{Z}_N] &= \frac{1}{N} \sum_{j=1}^N \mathbb{E}[\mathbb{1}_I(\beta_{j,N})] = \frac{1}{N} \sum_{j=1}^N \mathbb{P}(\beta_{j,N} \in I) \\ &= \sum_{j=1}^N \frac{1}{N \sqrt{2\pi P(N) \pi_{j,j}}} \int_I \exp\left(-\frac{1}{2} \left(\frac{x}{\sqrt{P(N) \pi_{j,j}}}\right)^2\right) dx \\ &= \sum_{j=1}^N \frac{1}{\sqrt{2\pi N} \sqrt{1 + \delta_{j,j} N^{-\frac{1}{4}}}} \int_I \exp\left(-\frac{1}{2} \left(\frac{x}{\sqrt{1 + \delta_{j,j} N^{-\frac{1}{4}}}}\right)^2\right) dx \end{aligned} \quad (4.15)$$

Let us denote  $\mu_{N(0,1)}(I) := \frac{1}{\sqrt{2\pi}} \int_I \exp\left(-\frac{x^2}{2}\right) dx$ . Re-scaling by the standard deviation each of the integrals in the sum, expand each as

$$\begin{aligned} \int_I \exp\left(-\frac{1}{2} \left(\frac{x}{\sqrt{1 + \delta_{j,j} N^{-\frac{1}{4}}}}\right)^2\right) dx &= \mu_{N(0,1)}(I) - \\ &\quad \operatorname{sgn}(\delta_{j,j}) \left( \int_{\frac{\zeta_0}{\sqrt{1 + \delta_{j,j} N^{-\frac{1}{4}}}}^{\zeta_0} \exp\left(-\frac{1}{2} y^2\right) dy + \int_{\frac{\zeta_1}{\sqrt{1 + \delta_{j,j} N^{-\frac{1}{4}}}}^{\zeta_1} \exp\left(-\frac{1}{2} y^2\right) dy \right). \end{aligned}$$

Fix some big  $C > 3$ . We can assume that both ends points of  $I$  are inside  $[-C, C]$  or both are outside of it (in the remaining case, choose larger  $C$  and thus  $I \subset [-C, C]$ ). In the first case  $\mu_{\text{Lebesgue}}(I) \leq 2C$

and we can bound the integral naively,

$$|\mathbb{P}(\beta_{j,N} \in I) - \mu_{\mathcal{N}(0,1)}(I)| \leq C' \left( 1 - \frac{1}{\sqrt{1 + \delta_{j,j} N^{-\frac{1}{4}}}} \right). \quad (4.16)$$

Since  $\delta_{j,j}$  are uniformly bounded as  $\mathcal{N} \ni N \rightarrow \infty$ , taking minimum over  $j$  on the right side,

$$|\mathbb{P}(\beta_{j,N} \in I) - \mu_{\mathcal{N}(0,1)}(I)| = O\left(N^{-\frac{1}{4}}\right).$$

In the second case we remind that denoting  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$ , for large  $x$   $|\operatorname{erf}(x)| \sim \left| 1 - \frac{\exp(-x^2)}{x} \right|$  hence

$$|\mathbb{P}(\beta_{j,N} \in I) - \mu_{\mathcal{N}(0,1)}(I)| \leq \frac{C''}{C} \exp\left(-\frac{C^2}{1 + \delta_{j,j} N^{-\frac{1}{4}}}\right) \cdot \sqrt{1 + \delta_{j,j} N^{-\frac{1}{4}}} N^{-\frac{1}{4}} \quad (4.17)$$

which is independent of the choice of  $I$  and  $O\left(N^{-\frac{1}{4}}\right)$  as well. Since from Theorem 4.1.1  $\delta_{j,j}$  is uniformly bounded as  $N \rightarrow \infty$ ,

$$\mathbb{E}[\mathcal{Z}_N] = \mu_{\mathcal{N}(0,1)}(I) + O\left(N^{-\frac{1}{4}}\right) \quad (4.18)$$

Let us compute the variance of  $\mathcal{Z}_N$  in a similar fashion. For that purpose we consider for every two  $j \neq j' \in \Sigma_k$  the bi-variate Gaussian vector  $(\beta_{j,N}, \beta_{j',N}) \sim \mathcal{N}_{\mathbb{R}}(0, \mathbf{M}_k(j, j'))$  with  $\mathbf{M}_k(j, j')$  being the corresponding sub-covariance matrix obtained from  $\Pi_N$  which recalling the notation  $P(N) \Delta_N - I_N = (\delta_{j,k})_{j,k}$  is given by

$$\begin{aligned} \mathbf{M}_k(j, j') &= P(N) \begin{pmatrix} \operatorname{Var}(\beta_{j,N}) & \operatorname{cov}(\beta_{j,N}, \beta_{j',N}) \\ \operatorname{cov}(\beta_{j,N}, \beta_{j',N}) & \operatorname{Var}(\beta_{j',N}) \end{pmatrix} \\ &= \begin{pmatrix} 1 + \delta_{j,j} N^{-\frac{1}{4}} & \delta_{j,j'} N^{-\frac{1}{4}} \\ \delta_{j,j'} N^{-\frac{1}{4}} & 1 + \delta_{j',j'} N^{-\frac{1}{4}} \end{pmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} \operatorname{Var}(\mathcal{Z}_N) &= \mathbb{E}[\mathcal{Z}_N^2] - \mathbb{E}[\mathcal{Z}_N]^2 \quad (4.19) \\ &= \frac{1}{N^2} \left[ 2 \sum_{1 \leq j < j' \leq N} \mathbb{E}[\mathbb{1}_I(\beta_{j,N}) \mathbb{1}_I(\beta_{j',N})] + \sum_{j=1}^N \mathbb{E}[\mathbb{1}_I(\beta_{j,N})] \right] - \mathbb{E}^2[\mathcal{Z}_N] \\ &= \frac{2}{N^2} \sum_{1 \leq j < j' \leq N} \mathbb{E}[\mathbb{1}_I(\beta_{j,N}) \mathbb{1}_I(\beta_{j',N})] + \frac{1}{N} \mathbb{E}[\mathcal{Z}_N] - \mathbb{E}^2[\mathcal{Z}_N]. \end{aligned}$$



Let us give an asymptotic expansion of the first sum in (4.19)

$$\begin{aligned} \frac{2}{N^2} \sum_{1 \leq j < j' \leq N} \mathbb{E} [\mathbb{1}_I(\beta_{j,N}) \mathbb{1}_I(\beta_{j',N})] &= \frac{2}{N^2} \sum_{1 \leq j < j' \leq N} \mathbb{P}((\beta_{j,N}, \beta_{j',N}) \in I^2) \\ &= \frac{1}{\pi N^2} \sum_{1 \leq j < j' \leq N} \frac{1}{\sqrt{\det \mathbf{M}_k(j, j')}} \iint_{I^2} \exp\left(-\frac{1}{2} \langle \varrho, \mathbf{M}_k^{-1}(j, j') \varrho \rangle\right) d\varrho \\ &= \sum_{1 \leq j < j' \leq N} \frac{1}{\pi N^2 \sqrt{\det(\text{Id}_2 + R_{j,j'} N^{\varepsilon-\frac{1}{4}})}} \iint_{I^2} \exp\left(-\frac{1}{2} \langle \varrho, (\text{Id}_2 + R'_{j,j'} N^{\varepsilon-\frac{1}{4}}) \varrho \rangle\right) d\varrho \end{aligned}$$

for some matrices  $R_{j,j'}, R'_{j,j'} = O_{\ell^\infty}(1)$  satisfying

$$R_{j,j'} + R'_{j,j'} = -N^{\varepsilon-\frac{1}{4}} R_{j,j'} R'_{j,j'}$$

obtained from restricting  $P(N) \Delta_N$ . Realizing

$$\det(\text{Id}_2 + N^{\varepsilon-\frac{1}{4}} R_{j,j'}) = 1 + N^{\varepsilon-\frac{1}{4}} \text{Tr}(R_{j,j'}) + N^{2\varepsilon-\frac{1}{2}} \det(R_{j,j'})$$

we deduce  $\det \mathbf{M}_k(j, j') = 1 + O(N^{\varepsilon-\frac{1}{4}})$ . Decomposing the exponent to

$$\begin{aligned} \exp\left(-\frac{1}{2} \langle \varrho, (\text{Id}_2 + N^{\varepsilon-\frac{1}{4}} R'_{j,j'}) \varrho \rangle\right) &= \exp\left(-\frac{1}{2} (x^2 + y^2)\right) \cdot \exp\left(-\frac{N^{\varepsilon-\frac{1}{4}}}{2} \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, R'_{j,j'} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle\right) \\ &= \exp\left(-\frac{1}{2} (x^2 + y^2)\right) \cdot \left(1 + O\left(N^{\varepsilon-\frac{1}{4}} \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, R'_{j,j'} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle\right)\right) \end{aligned}$$

we can write

$$\begin{aligned} &\left| \iint_{I^2} \exp\left(-\frac{1}{2} \langle \varrho, (\text{Id}_2 + R'_{j,j'} N^{\varepsilon-\frac{1}{4}}) \varrho \rangle\right) d\varrho - \mu_{\mathcal{N}(0,1)}^2(I) \right| = \\ &O\left(N^{\varepsilon-\frac{1}{4}} \int_{I^2} \exp\left(-\frac{1}{2} (x^2 + y^2)\right) \cdot \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, R'_{j,j'} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle d\rho\right). \end{aligned}$$

Recalling the entries of  $R_{j,j'}$  are uniformly bounded as  $\mathcal{N} \ni N \rightarrow \infty$  the integral on the right hand side is sum of terms, each is proportional to

$$N^{\varepsilon-\frac{1}{4}} \iint_{I^2} xy \exp\left(-\frac{1}{2} (x^2 - y^2)\right) dx dy = N^{-\frac{1}{4}} \left(\exp\left(-\frac{1}{2} \zeta_1^2\right) - \exp\left(-\frac{1}{2} \zeta_0^2\right)\right)^2 = O\left(N^{-\frac{1}{4}}\right)$$

or to

$$N^{\varepsilon-\frac{1}{4}} \int_I \exp(-y^2) dy \cdot \int_I x^2 \exp(-x^2) dx = \frac{\sqrt{\pi}}{2} N^{\varepsilon-\frac{1}{4}} [\text{erf}(b) - \text{erf}(a)].$$

$$\begin{aligned} & \left[ \sqrt{\frac{\pi}{2}} \left( \operatorname{erf} \left( \frac{b}{\sqrt{2}} \right) - \operatorname{erf} \left( \frac{a}{\sqrt{2}} \right) \right) - x \exp \left( -\frac{x^2}{2} \right) \right] \Big|_a^b \\ &= O \left( N^{-\frac{1}{4}} \right) \end{aligned}$$

since each term in the product is bounded from (4.16) and (4.17). Altogether,

$$\iint_{I^2} \exp \left( -\frac{1}{2} \langle \varrho, (\operatorname{Id}_2 + R'_{j,j'}) \varrho \rangle \right) d\varrho = \mu_{N(0,1)}^2(I) + O \left( N^{-\frac{1}{4}} \right).$$

Since  $\det \left( \operatorname{Id} + N^{-\frac{1}{4}} R_{j,j'} \right) = 1 + O \left( N^{-\frac{1}{4}} \right)$  and the high order terms are bounded independently of  $I$  the first sum in (4.19) is asymptotic to

$$\begin{aligned} \frac{2}{N^2} \sum_{1 \leq j < j' \leq N} \mathbb{E} [\mathbb{1}_I(\beta_{j,N}) \mathbb{1}_I(\beta_{j',N})] &= \frac{N-1}{N} \cdot \left( \mu_{N(0,1)}^2(I) + O \left( N^{-\frac{1}{4}} \right) \right) \\ &= \left( 1 - \frac{1}{N} \right) \mu_{N(0,1)}^2(I) + O \left( N^{-\frac{1}{4}} \right). \end{aligned} \quad (4.20)$$

We conclude that

$$\begin{aligned} \operatorname{Var}(\mathcal{Z}_N) &= \left( 1 - \frac{1}{N} \right) \cdot \frac{\mu_{N(0,1)}^2(I)}{2\pi} + O \left( N^{-\frac{1}{4}} \right) + O \left( N^{-1} \right) - \left( \frac{1}{2\pi} \mu_{N(0,1)}(I) + O \left( N^{-\frac{1}{4}} \right) \right)^2 \\ &= O \left( N^{-\frac{1}{4}} \right). \end{aligned}$$

Applying Chebyshev inequality, there exists a constant  $C''_\gamma > 0$  such that for every  $n > 0$

$$\mathbb{P} \left( |\mathcal{Z}_N - \mathbb{E}[\mathcal{Z}_N]| < C''_\gamma N^{-\frac{1}{8}} n \right) \geq 1 - \frac{1}{n^2}.$$

We note that since  $|\mathbb{E}[\mathcal{Z}_N] - \mu_{N(0,1)}(I)| = O \left( N^{\varepsilon - \frac{1}{8}} \right)$ , this inequality means that there exists a constant  $C_\gamma > 0$  such that for every  $n > 1$

$$\mathbb{P} \left( |\mathcal{Z}_N - \mu_{N(0,1)}(I)| < C_\gamma N^{-\frac{1}{8}} n \right) \geq 1 - \frac{1}{n^2}.$$

Taking  $n = N^\delta$  for some  $\delta \in \left( 0, \frac{1}{8} \right)$  ends the proof.  $\square$

#### 4.2.2 Normalized random eigenstates

Let us deduce from Proposition 4.2.2 on the value distribution of  $\beta_N$  <sup>7</sup> an analogous result for  $\tilde{\beta}_N$  (cf. Figure 4.6a for a numerical example).

<sup>7</sup>Note that according to the discussion we had in subsection 4.1.2 the norm is centered around  $\sqrt{N}$

**Theorem 4.2.3.** Fix  $\gamma \in \tilde{\Gamma}_2$  quantized by  $\{\mathcal{M}_N\}$ . Let  $\beta_N$  real valued Gaussian eigenstate and let  $N$  be any increasing sequence in  $\mathcal{N}$  (where  $\mathcal{N}$  defined in (4.1)) and consider the random normalized vector of  $\mathcal{M}_N$

$$\tilde{\beta}_N = (\tilde{\beta}_{1,N}, \dots, \tilde{\beta}_{N,N}) := \frac{\sqrt{d(\nu, N) P(N)}}{\|\beta_N\|} (\beta_{1,N}, \dots, \beta_{N,N}).$$

Define the empirical measure  $\tilde{\mu}_N = \frac{1}{N} \sum_{j=1}^N \delta_{\tilde{\beta}_{j,N}}$ . There exists a constant  $C_\gamma > 0$  such that as  $N \ni N \rightarrow \infty$ , for every interval  $I \subset \mathbb{R}$  and  $\varepsilon \in (0, \frac{1}{8})$

$$\mathbb{P} \left( |\tilde{\mu}_N(I) - \mu_{\mathcal{N}(0,1)}(I)| \geq C_\gamma N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}. \tag{4.21}$$

The proof we present relies on the connection between standard Gaussian vectors and random vectors on the unit sphere: Every random variable  $X \sim \mathcal{N}_{\mathbb{R}} \left( 0, \begin{pmatrix} \text{Id}_M & 0_{N-M} \\ 0_M & 0_{N-M} \end{pmatrix} \right)$  can be expressed as  $X = \|X\| \cdot \tilde{X}$  with  $\tilde{X} \sim \text{Unif}(\mathbb{S}_{\mathbb{R}}^M)$  uniformly distributed on the sphere such that  $\tilde{X}$  and  $\|X\|$  are independent. In our case of interest we take  $M = d(\nu, N)$ .

As we mentioned in Remark 4.2.1 above, the theorem can be adapted to complex random eigenstates as well,

**Proposition 4.2.4.** If  $\beta_N$  is a complex Gaussian eigenstate as in (4.3) and let  $N \in \mathcal{N}$  and define  $\tilde{\beta}_N$  as in the real case. There exists a constant  $C'_\gamma > 0$  such that for every  $\varepsilon \in (0, \frac{1}{8})$  and  $(\zeta_0, \zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^4$  inducing  $I = \{z = a + bi : a \in [\zeta_0, \zeta_1], b \in [\zeta_2, \zeta_3]\} \subset \mathbb{C}$  as  $\mathcal{N} \ni N \rightarrow \infty$ ,

$$\mathbb{P} \left( |\tilde{\mu}_N(I) - \mu_{\mathcal{N}_{\mathbb{C}}(0,1)}(I)| \geq C'_\gamma N^{\varepsilon - \frac{1}{8}} \right) \leq \frac{1}{N^{2\varepsilon}}. \tag{4.22}$$

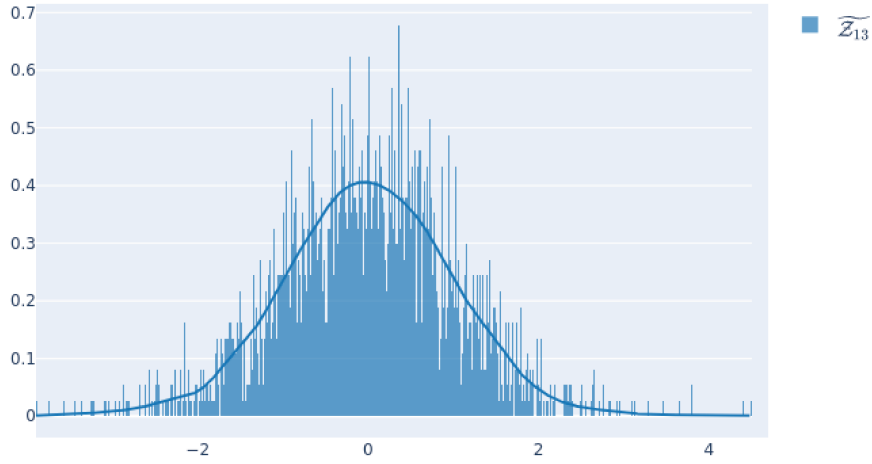
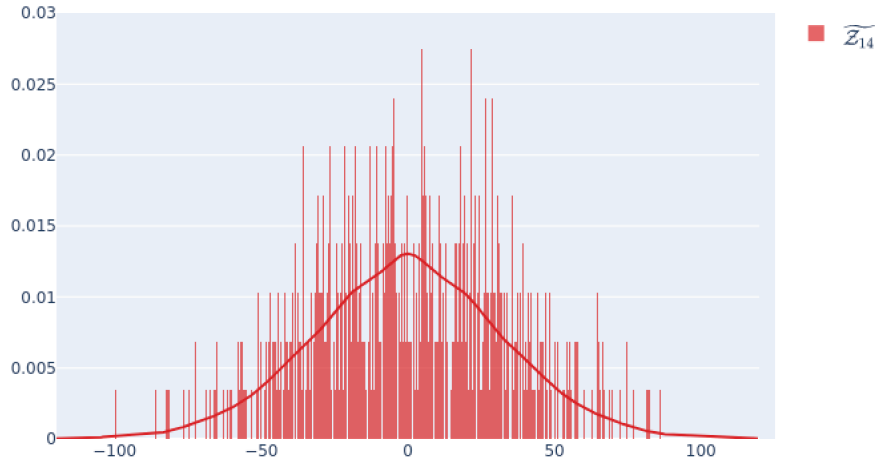
The proof of this generalization is follows from the proof of Theorem 4.2.3 applied on the real and complex parts of  $I$  separately and then using the triangle inequality.

*Proof of Theorem 4.2.3.* Let us note that for every  $N \in \mathcal{N}, j \in \llbracket 1, N \rrbracket$  and  $I = [a, b]$

$$\begin{aligned} \mathbb{P} \left( \tilde{\beta}_{j,N} \in I \right) &= \mathbb{P} \left( \left\{ \tilde{\beta}_{j,N} \in I \right\} \cap \left\{ \left| \frac{\|\beta_N\|^2}{d(\nu, N) P(N)} - 1 \right| \leq (d(\nu, N) P(N))^{-\frac{1}{4}} \right\} \right) \\ &\quad + \mathbb{P} \left( \left\{ \tilde{\beta}_{j,N} \in I \right\} \cap \left\{ \left| \frac{\|\beta_N\|^2}{d(\nu, N) P(N)} - 1 \right| > (d(\nu, N) P(N))^{-\frac{1}{4}} \right\} \right). \end{aligned}$$

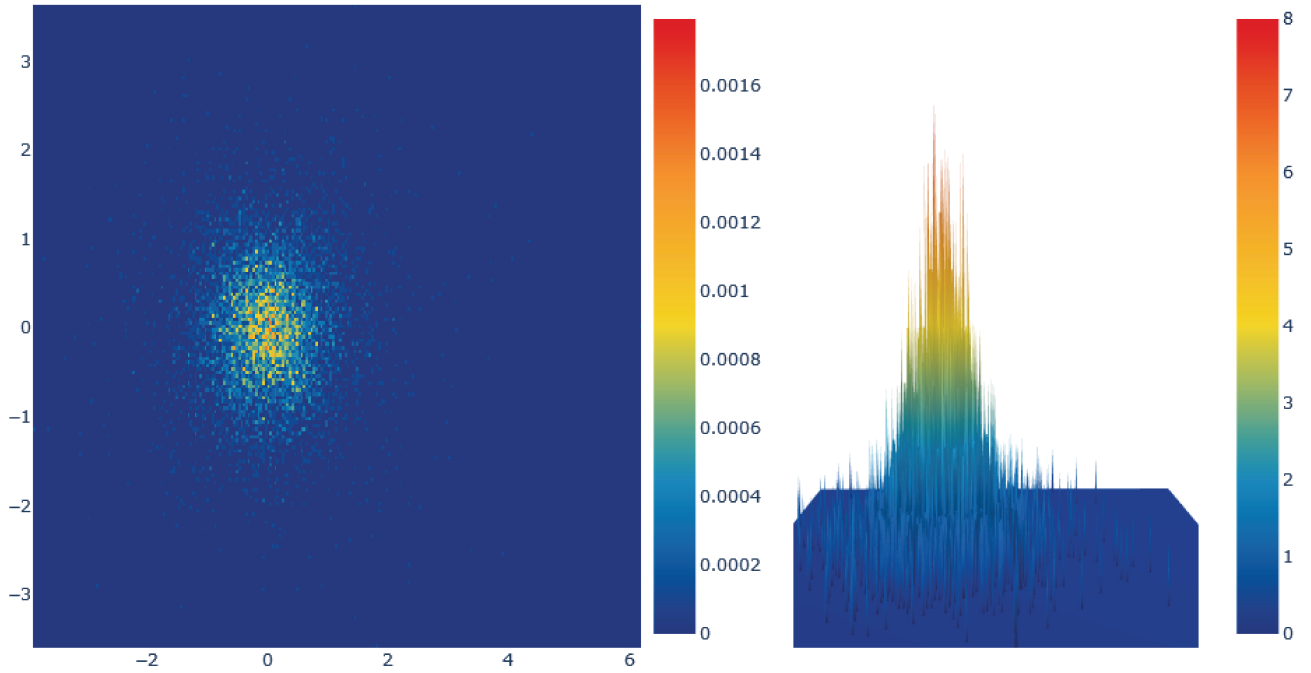
We remind that from (4.14)

$$\begin{aligned} &\mathbb{P} \left( \left| \frac{\|\beta_N\|^2}{d(\nu, N) P(N)} - 1 \right| \leq (d(\nu, N) P(N))^{-\frac{1}{4}} \right) \\ &= \mathbb{P} \left( \sqrt{d(\nu, N) P(N) - (d(\nu, N) P(N))^{\frac{3}{4}}} \leq \|\beta_N\| \leq \sqrt{d(\nu, N) P(N) + (d(\nu, N) P(N))^{\frac{3}{4}}} \right) \end{aligned}$$

(a)  $\gamma_{DE} : N = 3691, P(N) = 13$ (b)  $\gamma_{DE} : N = 2911, P(N) = 14$ 

**Figure 4.6:** The empirical measure obtained when sampling over 250 intervals of same length. (A) The value distribution of a normalized random eigenstate of  $\mathcal{M}_{3691}(\gamma_{DE})$  is asymptotically normal. (B) The value distribution of a normalized random eigenstate may converge to Gaussian when  $N$  has short even period (here we plot it for  $\mathcal{M}_{2911}(\gamma_{DE})$  having  $P(N) = 14$ ).

$$= F_{\mathcal{N}_{\mathbb{R}}(0,1)} \left( \sqrt[4]{\frac{d(\nu, N)}{P(N)}} \right) - F_{\mathcal{N}_{\mathbb{R}}(0,1)} \left( -\sqrt[4]{\frac{d(\nu, N)}{P(N)}} \right) + O \left( \sqrt{\frac{1}{d(\nu, N)}} \right).$$



**Figure 4.7:** Analogous results on the value distribution hold when the quantum map has no quantum symmetries, replacing real intervals by complex boxes. We consider  $\mathcal{M}_{8359}(\gamma_{BR})$  having  $P(N) = 7$  with  $\gamma_{BR} = \begin{pmatrix} 4 & 9 \\ 7 & 16 \end{pmatrix}$ . Left: the probability that  $\tilde{Z}_7$  attains values in (complex) 250 squares of equal sides. Right: A 3D histogram counting how many of the entries of the random vector are in any such square.

Recalling that for large  $x$ ,  $1 - \frac{1}{\sqrt{2\pi}} \int_{-x}^x \exp\left(-\frac{y^2}{2}\right) dy = O(\exp(-x^2)/x)$  yields

$$\mathbb{P}\left(\left|\frac{\|\beta_N\|^2}{d(\nu, N)P(N)} - 1\right| > (d(\nu, N)P(N))^{-\frac{1}{4}}\right) = O\left(\sqrt{\frac{1}{d(\nu, N)}}\right).$$

Then,

$$\begin{aligned} \mathbb{P}\left(\tilde{\beta}_{j,N} \in I\right) &= \mathbb{P}\left(\left\{\tilde{\beta}_{j,N} \in I\right\} \cap \left\{\left|\frac{\|\beta_N\|^2}{d(\nu, N)P(N)} - 1\right| \leq (d(\nu, N)P(N))^{-\frac{1}{4}}\right\}\right) + O\left(\frac{1}{\sqrt{d(\nu, N)}}\right) \\ &= \mathbb{P}\left(\left\{\beta_{j,N} \in \frac{\|\beta_N\|}{\sqrt{d(\nu, N)P(N)}}I\right\} \cap \left\{\left|\frac{\|\beta_N\|^2}{d(\nu, N)P(N)} - 1\right| \leq (d(\nu, N)P(N))^{-\frac{1}{4}}\right\}\right) \\ &\quad + O\left(\frac{1}{\sqrt{d(\nu, N)}}\right). \end{aligned}$$

Recalling subsection 4.1.2 and denoting  $\sigma_j := \sqrt{1 + \delta_{j,j}N^{\varepsilon - \frac{1}{4}}}$ ,

$$\begin{aligned} & \mathbb{P} \left( \left\{ \beta_{j,N} \in \frac{1}{\sqrt{P(N) d(\nu, N)}} \|\beta_N\| I \right\} \cap \left\{ \left| \frac{\|\beta_N\|^2}{d(\nu, N) P(N)} - 1 \right| \leq (d(\nu, N) P(N))^{-\frac{1}{4}} \right\} \right) \\ &= \frac{1}{2\pi\sigma_j} \int_{-\sqrt[4]{\frac{d(\nu, N)}{P(N)}}}^{\sqrt[4]{\frac{d(\nu, N)}{P(N)}}} e^{-\frac{1}{2}\xi^2} \left( \int_{\sqrt{1-\sqrt{\frac{2}{d(\nu, N)}}}\xi\zeta_0}^{\sqrt{1+\sqrt{\frac{2}{d(\nu, N)}}}\xi\zeta_1} e^{-\frac{y^2}{2\sigma_j^2}} dy \right) d\xi + O \left( \frac{1}{\sqrt{d(\nu, N)}} \right) \\ &= \frac{1}{2\pi\sigma_j} \int_{-\sqrt[4]{\frac{d(\nu, N)}{P(N)}}}^{\sqrt[4]{\frac{d(\nu, N)}{P(N)}}} e^{-\frac{1}{2}\xi^2} \left( \mathbb{P}(\beta_{j,N} \in I) + \operatorname{sgn}(\xi\zeta_0) \int_{\zeta_0\sqrt{1-\sqrt{\frac{2}{d(\nu, N)}}}\xi}^{\zeta_0} e^{-\frac{1}{2}\left(\frac{y}{\sigma_j}\right)^2} dy \right. \\ & \quad \left. - \operatorname{sgn}(\xi\zeta_1) \int_{\zeta_1}^{\zeta_1\sqrt{1+\sqrt{\frac{2}{d(\nu, N)}}}\xi} e^{-\frac{1}{2}\left(\frac{y}{\sigma_j}\right)^2} dy \right) d\xi + O \left( \frac{1}{\sqrt{d(\nu, N)}} \right) \end{aligned}$$

Recalling the asymptotics of erf for large values, the first term equals

$$\left( 1 - O \left( \frac{\sqrt[4]{P(N)} \exp \left( -\sqrt{\frac{d(\nu, N)}{P(N)}} \right)}{\sqrt[4]{d(\nu, N)}} \right) \right) \mathbb{P}(\beta_{j,N} \in I)$$

and the other terms are  $O \left( \frac{1}{\sqrt[4]{d(\nu, N)}} \right)$  following the arguments from (4.16) and (4.17). We conclude that

$$\mathbb{P}(\tilde{\beta}_{j,N} \in I) = \mathbb{P}(\beta_{j,N} \in I) + O \left( \frac{1}{\sqrt[4]{d(\nu, N)}} \right).$$

Let us define

$$\tilde{\mathcal{Z}}_N := \frac{1}{N} \sum_{j=1}^N \mathbb{1}_I(\tilde{\beta}_{j,N}).$$

Recalling (4.15), we deduce

$$\begin{aligned} \mathbb{E}[\tilde{\mathcal{Z}}_N] &= \frac{1}{N} \sum_{j=1}^N \mathbb{P}(\tilde{\beta}_{j,N} \in I) \\ &= \frac{1}{N} \sum_{j=1}^N \mathbb{P}(\beta_{j,N} \in I) + O \left( d(\nu, N)^{-\frac{1}{4}} \right) \\ &= \mu_{N(0,1)}(I) + O \left( d(\nu, N)^{-\frac{1}{4}} \right). \end{aligned} \tag{4.23}$$

We can adapt the same argument to express  $\operatorname{Var}(\tilde{\mathcal{Z}}_N)$  and show it equals  $\operatorname{Var}(\mathcal{Z}_N)$  asymptotically: for every  $1 \leq j < j' \leq N$  the joint probability distribution of  $\beta_{j,N}, \beta_{j',N}$  is concentrated “close to the

$$\text{event } \left\{ \left| \frac{\|\beta_N\|^2}{d(\nu, N)P(N)} - 1 \right| > (d(\nu, N)P(N))^{-\frac{1}{4}} \right\},$$

$$\mathbb{P} \left( (\tilde{\beta}_{j,N}, \tilde{\beta}_{j',N}) \in I^2 \right) =$$

$$\mathbb{P} \left( \left\{ (\beta_{j,N}, \beta_{j',N}) \in \left( \frac{\|\beta_N\|}{\sqrt{d(\nu, N)P(N)}} I \right)^2 \right\} \cap \left\{ \left| \frac{\|\beta_N\|^2}{d(\nu, N)P(N)} - 1 \right| \leq (d(\nu, N)P(N))^{-\frac{1}{4}} \right\} \right) + O \left( \frac{1}{\sqrt{d(\nu, N)}} \right).$$

Note that for  $I = [\zeta_0, \zeta_1]$ ,

$$\begin{aligned} & \mathbb{P} \left( \left\{ (\beta_{j,N}, \beta_{j',N}) \in \left( \frac{\|\beta_N\|}{\sqrt{d(\nu, N)P(N)}} I \right)^2 \right\} \cap \left\{ \left| \frac{\|\beta_N\|^2}{d(\nu, N)P(N)} - 1 \right| \leq (d(\nu, N)P(N))^{-\frac{1}{4}} \right\} \right) \\ &= \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_x \sqrt{\det(\mathbf{M}_k(j, j')^{-1})}} \int_{-\sqrt[4]{\frac{d(\nu, N)}{P(N)}}}^{\sqrt[4]{\frac{d(\nu, N)}{P(N)}}} e^{-\frac{1}{2}\xi^2} \iint \left( \left[ \sqrt{1 - \sqrt{\frac{2}{d(\nu, N)}} \xi \zeta_0}, \sqrt{1 + \sqrt{\frac{2}{d(\nu, N)}} \xi \zeta_1} \right] \right)^2 e^{-\frac{1}{2}\langle \rho, \mathbf{M}_k(j, j') \rho \rangle} d\rho d\xi \\ & \quad + O \left( \frac{1}{\sqrt{d(\nu, N)}} \right) \\ &= \frac{1}{(2\pi)^{\frac{3}{2}} \sqrt{\det(\mathbf{M}_k(j, j')^{-1})}} \int_{-\sqrt[4]{\frac{d(\nu, N)}{P(N)}}}^{\sqrt[4]{\frac{d(\nu, N)}{P(N)}}} e^{-\frac{1}{2}\xi^2} \left( \iint_{I^2} e^{-\frac{1}{2}\langle \rho, \mathbf{M}_k(j, j') \rho \rangle} d\rho \right. \\ & \quad \left. + \text{sgn}(\xi) \iint \left( \left[ \sqrt{1 - \sqrt{\frac{2}{d(\nu, N)}} \xi \zeta_0}, \sqrt{1 + \sqrt{\frac{2}{d(\nu, N)}} \xi \zeta_1} \right] \right)^2 \Delta I^2 e^{-\frac{1}{2}\langle \rho, \mathbf{M}_k(j, j') \rho \rangle} d\rho \right) d\xi \\ & \quad + O \left( \frac{1}{\sqrt{d(\nu, N)}} \right). \end{aligned}$$

The domain  $\left( \left[ \sqrt{1 - \sqrt{\frac{2}{d(\nu, N)}} \xi \zeta_0}, \sqrt{1 + \sqrt{\frac{2}{d(\nu, N)}} \xi \zeta_1} \right] \right)^2 \Delta I^2$  can be decomposed into the union of 4 rectangles, each having a side of length  $O \left( \sqrt[4]{\frac{1}{d(\nu, N)}} \right)$  and a side of length  $O(1)$ . The integral along the short side is bounded by  $O \left( \sqrt[4]{\frac{1}{d(\nu, N)}} \right)$ . The integration along the other side can be bounded by splitting the domain of integration as in (4.16) and (4.17), yielding a bounded term. Combining these

estimates with the asymptotics of erf,

$$\begin{aligned} & \mathbb{P} \left( \left\{ (\beta_{j,N}, \beta_{j',N}) \in \left( \frac{\|\beta_N\|}{\sqrt{d(\nu, N) P(N)}} I \right)^2 \right\} \cap \left\{ \left| \frac{\|\beta_N\|^2}{d(\nu, N) P(N)} - 1 \right| \leq (d(\nu, N) P(N))^{-\frac{1}{4}} \right\} \right) \\ &= \left( 1 - O \left( \frac{\sqrt[4]{P(N)} \exp \left( -\sqrt{\frac{d(\nu, N)}{P(N)}} \right)}{\sqrt[4]{d(\nu, N)}} \right) \right) \mathbb{P} \left( (\beta_{j,N}, \beta_{j',N}) \in I^2 \right) + O \left( \frac{1}{\sqrt[4]{d(\nu, N)}} \right). \end{aligned}$$

hence

$$\mathbb{P} \left( (\tilde{\beta}_{j,N}, \tilde{\beta}_{j',N}) \in I^2 \right) = \mathbb{P} \left( (\beta_{j,N}, \beta_{j',N}) \in I^2 \right) + O \left( \frac{1}{\sqrt[4]{d(\nu, N)}} \right).$$

As follows,

$$\begin{aligned} \text{Var} \left( \tilde{\mathcal{Z}}_N \right) &= \frac{2}{N^2} \sum_{j < j'} \mathbb{E} \left[ \mathbb{1}_I (|\beta_{j,N}|) \mathbb{1}_I (|\beta_{j',N}|) \right] + \frac{1}{N} \mathbb{E} [|\beta_{1,N}|] - \mathbb{E}^2 [|\beta_{1,N}|] + O \left( \frac{1}{\sqrt{d(\nu, N)}} \right) \\ &= \left( 1 - \frac{1}{N} \right) \cdot \mu_{\mathcal{N}(0,1)}^2 (I) + O \left( \frac{1}{\sqrt[4]{d(\nu, N)}} \right) + \frac{1}{N} \mathbb{E} [|\beta_{1,N}|] - \left( \mu_{\mathcal{N}(0,1)} (I) + O \left( \frac{1}{\sqrt[4]{d(\nu, N)}} \right) \right)^2 \\ &= O \left( \frac{1}{\sqrt[4]{d(\nu, N)}} \right) \end{aligned}$$

which resembles  $\text{Var}(\mathcal{Z}_k)$ . Applying Chebyshev inequality and recalling  $\left| \mathbb{E}[\tilde{\mathcal{Z}}_N] - \mu_{\mathcal{N}(0,1)}(I) \right| = O(N^{-\frac{1}{4}})$  there exists  $C_\gamma > 0$  such that for every  $n \in \mathbb{N}^*$

$$\begin{aligned} \mathbb{P} \left( \left| \tilde{\mathcal{Z}}_N - \mu_{\mathcal{N}(0,1)}(I) \right| < C_\gamma N^{-\frac{1}{8}} n \right) &\geq \mathbb{P} \left( \left| \tilde{\mathcal{Z}}_N - \mathbb{E}[\tilde{\mathcal{Z}}_N] \right| < C'_\gamma N^{-\frac{1}{8}} n \right) \\ &\geq 1 - \frac{1}{n^2}. \end{aligned}$$

Taking  $n = N^\delta$  with  $\delta \in (0, \frac{1}{8})$  implies (4.21) □





# Appendices and additions

## A.1 Deriving an explicit formula for $\mathcal{M}_N(A)$

Let us denote

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

The metaplectic representation at the matrix  $\alpha$ ,  $\mathcal{M}_h(\alpha) : \mathcal{S}'(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$ , is given by (cf. [Fol89])

$$\mathcal{M}_h(\alpha) f(x) = \sqrt{\frac{iN}{\alpha_{12}}} \int_{\mathbb{R}} \exp\left(\frac{\pi i N}{\alpha_{12}} (\alpha_{11} y^2 - 2xy + \alpha_{22} x^2)\right) f(y) dy.$$

We obtain  $\mathcal{M}_N \in U(N)$  from  $\mathcal{M}_h(\alpha)$  relying on the fact that  $\mathcal{M}_h(\alpha)$  maps  $\mathcal{H}_N$  to itself. Let us pick for  $\mathcal{H}_N$  the basis  $\{\delta_{\frac{p}{N} + \mathbb{Z}} : p = 0, \dots, N-1\}$  and consider the distributional pairing  $\langle \mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}}, v \rangle := \langle \delta_{\frac{p}{N} + \mathbb{Z}}, {}^t \mathcal{M}_h(\alpha) v \rangle$  for some Schwartz function  $v \in \mathcal{S}(\mathbb{R})$  and a fixed  $p \in \llbracket 0, N-1 \rrbracket$ . Let us denote  $v_\alpha(x) := \exp(\alpha_{12} \alpha_{22} \pi i N x^2) v(\alpha_{12} x)$  and recall the semiclassical Fourier transform  $\mathcal{F}_h : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$  from [Zwo12] which in our notations reads

$$(\mathcal{F}_h f)(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i N x \xi} dx.$$

${}^t \mathcal{M}_h(\alpha)$  can be expressed in terms of  $\mathcal{F}_h$  and  $v_\alpha$  after performing a change of variable

$$\begin{aligned} ({}^t \mathcal{M}_h(\alpha) v)(x) &= \sqrt{\frac{N}{i\alpha_{12}}} \int_{\mathbb{R}} \exp\left(\frac{\pi i N}{\alpha_{12}} (\alpha_{11} x^2 - 2xy + \alpha_{22} y^2)\right) f(y) dy \\ &= \sqrt{\frac{\alpha_{12} N}{i}} \exp\left(\frac{\pi i N \alpha_{11}}{\alpha_{12}} x^2\right) (\mathcal{F}_h v_\alpha)(x). \end{aligned}$$

Since  $\exp\left(\frac{\pi i N \alpha_{11}}{\alpha_{12}} x^2\right)$  is a  $\alpha_{12}$ -periodic function we decompose  $\delta_{\frac{p}{N} + \mathbb{Z}}$  as a finite sum of delta combs, each having peaks on a coset  $\alpha_{12} \mathbb{Z} + k$  for  $k \in \{0, \dots, \alpha_{12} - 1\}$ ,

$$\langle \mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}}, v \rangle = \sqrt{\frac{\alpha_{12} N}{i}} \left\langle \delta_{\frac{p}{N} + \mathbb{Z}}, \exp\left(\frac{\pi i N \alpha_{11}}{\alpha_{12}} x^2\right) \mathcal{F}_h v_\alpha \right\rangle$$

$$= \sqrt{\frac{\alpha_{12}N}{i}} \sum_{k=0}^{|\alpha_{12}|-1} \left( \exp\left(\frac{\pi i N \alpha_{11}}{\alpha_{12}} \left(\frac{p}{N} + k\right)^2\right) \sum_{n \in \mathbb{Z}} \langle \delta_{\frac{p}{N} + \alpha_{12}n + k}, \mathcal{F}_h v_\alpha \rangle \right)$$

Since  ${}^t \mathcal{F}_h = \mathcal{F}_h$  we deduce that

$$\langle \mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}}, v \rangle = \sqrt{\frac{\alpha_{12}N}{i}} \sum_{k=0}^{|\alpha_{12}|-1} \left( \exp\left(\frac{\pi i N \alpha_{11}}{\alpha_{12}} \left(\frac{p}{N} + k\right)^2\right) \sum_{n \in \mathbb{Z}} \langle \mathcal{F}_h \delta_{\frac{p}{N} + \alpha_{12}n + k}, v_\alpha \rangle \right)$$

The inner sum can be expressed using semiclassical Poisson's summation formula (for sake of completeness we recall it in section A.5),

$$\begin{aligned} \langle \mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}}, v \rangle &= \sqrt{\frac{1}{i\alpha_{12}N}} \sum_{k=0}^{|\alpha_{12}|-1} \left( \exp\left(\frac{\pi i N \alpha_{11}}{\alpha_{12}} \left(\frac{p}{N} + k\right)^2\right) \sum_{m \in \mathbb{Z}} \exp\left(-\frac{2\pi i \left(\frac{p}{N} + k\right) m}{\alpha_{12}}\right) \langle \delta_{\frac{m}{\alpha_{12}N}}, v_\alpha \rangle \right) \\ &= \sqrt{\frac{1}{i\alpha_{12}N}} \sum_{m \in \mathbb{Z}} \sum_{k=0}^{|\alpha_{12}|-1} \exp\left(\frac{\pi i}{\alpha_{12}N} (\alpha_{11} (p + Nk)^2 - 2m(p + Nk) + \alpha_{22}m^2)\right) v\left(\frac{m}{N}\right) \\ &= \sqrt{\frac{1}{i\alpha_{12}N}} \sum_{m \in \mathbb{Z}} \left( \sum_{k=0}^{|\alpha_{12}|-1} \exp\left(\frac{\pi i}{\alpha_{12}} (N\alpha_{11}k^2 + 2(\alpha_{11}p - m)k)\right) \right) \quad (\text{A.1}) \\ &\quad \cdot \exp\left(\frac{\pi i}{\alpha_{12}N} (\alpha_{11}p^2 - 2mp + \alpha_{22}m^2)\right) v\left(\frac{m}{N}\right) \end{aligned}$$

The inner sum on  $k$  can be calculated explicitly using Poisson's summation formula (cf. [HB80, appendix A]). Let us denote

$$g = \gcd(N, \alpha_{12}), \quad N_g = \frac{N}{g}, \quad \alpha_{12,g} = \frac{\alpha_{12}}{g}.$$

Let us introduce the finite commutative ring  $\mathbb{Z} / \alpha_{12,g}\mathbb{Z} = \{\overline{0}, \dots, \overline{\alpha_{12,g} - 1}\}$ . Since  $\gcd(N_g, \alpha_{12,g}) = 1$ ,  $N_g$  admits a multiplicative inverse in the ring. Since  $\det(\alpha) = 1$

$$1 = \det(\alpha) = \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12} = \alpha_{11}\alpha_{22} - \alpha_{21}g\alpha_{12,g}.$$

Viewing the equation mod  $\alpha_{12,g}$  we obtain that  $\overline{\alpha_{11}} \cdot \overline{\alpha_{22}} \equiv 1 \pmod{\alpha_{12,g}}$  from which we deduce in fact that there exists some element

$$\bar{\vartheta} = \overline{\vartheta' \cdot \alpha_{22}} \in \mathbb{Z} / \alpha_{12,g}\mathbb{Z}$$

for which

$$\overline{N_g \cdot \vartheta'} \equiv 1 \pmod{\alpha_{12,g}} \quad \text{and} \quad \overline{\alpha_{11} N_g \vartheta} \equiv 1 \pmod{\alpha_{12,g}}. \quad (\text{A.2})$$

If  $\alpha_{12,g} \in 2\mathbb{Z} + 1$  there exists some  $\bar{\vartheta}'$  satisfying

$$\overline{4N_g\alpha_{11}} \cdot \bar{\vartheta}' \equiv 1 \pmod{\alpha_{12,g}}. \quad (\text{A.3})$$

Then (A.1) can be reformulated as

$$\begin{aligned} \left\langle \mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}}, v \right\rangle &= \sqrt{\frac{1}{ig^2\alpha_{12,g}N_g}} \sum_{m \in \mathbb{Z}} \left( \sum_{k=0}^{|\alpha_{12}|-1} \exp\left(\frac{\pi i}{\alpha_{12,g}} (N_g\alpha_{11}k^2 + 2g^{-1}(\alpha_{11}p - m)k)\right) \right) \\ &\quad \cdot \exp\left(\frac{\pi i}{\alpha_{12}N} (\alpha_{11}p^2 - 2mp + \alpha_{22}m^2)\right) v\left(\frac{m}{N}\right) \end{aligned} \quad (\text{A.4})$$

The sum in index  $k$  from (A.4) does not vanish only if  $\alpha_{11}\alpha_{12,g}N_g + 2g^{-1}(\alpha_{11}p - m) \in 2\mathbb{Z}$ . That happens only in two situations:

1. Either

$$2\frac{\alpha_{11}p - m}{g} \in 2\mathbb{Z}, \quad \alpha_{11}\alpha_{12,g}N_g \in 2\mathbb{Z}. \quad (\text{A.5})$$

2. Alternatively

$$2\frac{\alpha_{11}p - m}{g} \in 2\mathbb{Z} + 1, \quad \alpha_{11}\alpha_{12,g}N_g \in 2\mathbb{Z} + 1. \quad (\text{A.6})$$

In both cases  $m$  admits the general form

$$m = \alpha_{11}p + \frac{g}{2}(\alpha_{11}\alpha_{12,g}N_g - 2l) = \alpha_{11}p - lg + \frac{\alpha_{11}\alpha_{12}}{2} \cdot N_g. \quad (\text{A.7})$$

for some  $l \in \mathbb{Z}$ . It means that only for  $m \in g\mathbb{Z} + \alpha_{11}p + \frac{\alpha_{11}\alpha_{12}}{2} \cdot N_g$  the  $m$ -th summand does not vanish.

Then (A.4) reads

$$\left\langle \mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}}, v \right\rangle = C_{\alpha,N} \sum_{m \in g\mathbb{Z} + \alpha_{11}p + \frac{\alpha_{11}\alpha_{12}}{2} N_g} \exp\left(\frac{\pi i}{\alpha_{12}N} (\alpha_{11}p^2 - 2mp + \alpha_{22}m^2)\right) e_{\alpha,N,p}(m) v\left(\frac{m}{N}\right) \quad (\text{A.8})$$

with

$$C_{\alpha,N} = \begin{cases} \sqrt{\frac{\alpha_{12,g}}{iN_g}} \left(\frac{\alpha_{11}N_g}{\alpha_{12,g}}\right) \exp\left(-\frac{\pi i}{4}(\alpha_{12,g} - 1)\right) & \text{Either } \alpha_{11}N_g, 2\frac{\alpha_{11}p-m}{g} \in 2\mathbb{Z}, \quad \alpha_{12,g} \in 2\mathbb{Z} + 1 \\ & \text{or } 2\frac{\alpha_{11}p-m}{g}, \alpha_{11}N_g, \alpha_{12,g} \in 2\mathbb{Z} + 1 \\ \sqrt{\frac{\alpha_{12,g}}{iN_g}} \left(\frac{\alpha_{12,g}}{\alpha_{11}N_g}\right) \exp\left(\frac{\pi i}{4}\alpha_{11}N_g\right) & \alpha_{11}N_g \in 2\mathbb{Z} + 1, \quad \alpha_{12,g}, 2\frac{\alpha_{11}p-m}{g} \in 2\mathbb{Z} \end{cases} \quad (\text{A.9})$$

where  $(\cdot)$  signifies the Jacobi symbol and recalling (A.2) and (A.3)

$$e_{\alpha,N,p}(m) = \begin{cases} \exp\left(-\frac{\pi i \alpha_{11} N_g}{\alpha_{12,g}} \left(\frac{\alpha_{11} p - m}{g}\right)^2 \frac{1}{\vartheta^2}\right) & 2\frac{\alpha_{11} p - m}{g} \in 2\mathbb{Z} \\ \exp\left(-\frac{4\pi i \alpha_{11} N_g}{\alpha_{12,g}} \left(\frac{2(\alpha_{11} p - m)}{g}\right)^2 \frac{1}{\vartheta'^2}\right) & 2\frac{\alpha_{11} p - m}{g} \in 2\mathbb{Z} + 1 \end{cases}. \quad (\text{A.10})$$

$e_{\alpha,N}(m)$  is well defined independently of the choice of a representative of  $\overline{\vartheta}$ : indeed

1. If  $2\frac{\alpha_{11} p - m}{g} \in 2\mathbb{Z}$ , then for every  $k\alpha_{12} + r \in \mathbb{Z}$  with  $r \equiv \overline{N_g^{-1}\alpha_{22}} \pmod{\alpha_{12,g}}$

$$\begin{aligned} & \exp\left(-\frac{\pi i \alpha_{11} N_g}{\alpha_{12,g}} \left(\frac{\alpha_{11} p - m}{g}\right)^2 (k\alpha_{12,g} + r)^2\right) \\ &= \exp\left(-\frac{\pi i \alpha_{11} N_g}{\alpha_{12,g}} \left(\frac{\alpha_{11} p - m}{g}\right)^2 r^2\right) \exp\left(-\pi i \alpha_{11} N_g \left(\frac{\alpha_{11} p - m}{g}\right)^2 (2kr + k^2\alpha_{12,g})\right). \end{aligned}$$

The second factor is of the form  $\exp(4\pi i \kappa)$  for some  $\kappa \in \mathbb{Z}$  hence it equals 1.

2. Alternatively  $2\frac{\alpha_{11} p - m}{g} \in 2\mathbb{Z} + 1$  and then for every  $k\alpha_{12} + r \in \mathbb{Z}$  with  $r \equiv \overline{4N_g^{-1}\alpha_{22}} \pmod{\alpha_{12,g}}$  writing

$$\begin{aligned} & \exp\left(-\frac{4\pi i \alpha_{11} N_g}{\alpha_{12,g}} \left(\frac{2(\alpha_{11} p - m)}{g}\right)^2 (k\alpha_{12,g} + r)^2\right) \\ &= \exp\left(-\frac{4\pi i \alpha_{11} N_g}{\alpha_{12,g}} \left(\frac{2(\alpha_{11} p - m)}{g}\right)^2 r^2\right) \cdot \exp\left(-4\pi i \alpha_{11} N_g \left(\frac{2(\alpha_{11} p - m)}{g}\right)^2 (2kr + \alpha_{12,g} k^2)\right) \end{aligned}$$

The second factor is of the form  $\exp(4\pi i \kappa')$  for some  $\kappa' \in \mathbb{Z}$  hence it equals 1.

We obtained that  $\mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}}$  is a weighted Dirac comb with peaks on  $\frac{1}{N}\mathbb{Z}$ . In order to prove that in fact  $\mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}}$  maps  $\mathcal{H}_N$  to itself we show that the sum is  $N$ -periodic in  $m$ , from which we deduce that  $\mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}} = \sum_{q=0}^{N-1} w_{p,q} \delta_{\frac{q}{N} + \mathbb{Z}}$  for some finite collection of weights  $w_{p,q} \in \mathbb{R}$ . Then we will be able to view  $\mathcal{M}_N = \mathcal{M}_h(\alpha) |_{\mathcal{H}_N}$  as the matrix  $(\mathcal{M}_N)_{p,q} = (w_{pq})_{1 \leq p,q \leq N} \in \mathbb{C}^{N \times N}$ .

The quotient between the  $(m+N)$ -th factor of (A.8) and its  $m$ -th can be simplified by recalling (A.7) and is then given by

$$\exp\left(\frac{\pi i}{\alpha_{12}} (-2p + 2\alpha_{22}m + \alpha_{22}N)\right) \frac{e_{\alpha,N,p}(m+N)}{e_{\alpha,N,p}(m)}.$$

i. If  $2\frac{\alpha_{11}p-m}{g} \in 2\mathbb{Z}$  the expanded quotient reads

$$\begin{aligned} & \exp\left(\frac{\pi i}{\alpha_{12}} \left[2\alpha_{22} \left(-lg + \frac{\alpha_{11}\alpha_{12}N_g}{2}\right) + \alpha_{22}N\right]\right) \\ & \quad \cdot \exp\left(-\frac{\pi i\alpha_{11}N_g\bar{\vartheta}^2}{\alpha_{12,g}g^2} \left(2N \left(-lg + \frac{\alpha_{11}\alpha_{12}N_g}{2}\right) + N^2\right)\right) \\ & = \exp\left(\frac{\pi i}{\alpha_{12}} (N - 2lg + \alpha_{12}\alpha_{11}N_g) (\alpha_{22} - \alpha_{11}N_g^2\bar{\vartheta}^2)\right) \\ & = \exp\left(\frac{\pi i}{\alpha_{12,g}} (N_g - 2l + \alpha_{12,g}\alpha_{11}N_g) (\alpha_{22} - \alpha_{11}N_g^2\bar{\vartheta}^2)\right). \end{aligned}$$

Let us show that  $\frac{\alpha_{22}-\alpha_{11}N_g^2\bar{\vartheta}^2}{\alpha_{12,g}} \in \mathbb{Z}$ : Indeed from (A.2), there exist  $j, j'$  such that

$$\begin{aligned} \frac{\alpha_{22} - \alpha_{11}N_g^2\bar{\vartheta}^2}{\alpha_{12,g}} &= \frac{\alpha_{22} - \alpha_{11}(1 + j'\alpha_{12,g})^2(\alpha_{22} + j\alpha_{12,g})^2}{\alpha_{12,g}} & \text{(A.II)} \\ &= (1 + j'\alpha_{12,g})^2(-2j\alpha_{11} - j^2\alpha_{11}\alpha_{12,g}) + \frac{\alpha_{22}(1 - \alpha_{11}\alpha_{22}(1 + j'\alpha_{12,g})^2)}{\alpha_{12,g}}. \end{aligned}$$

Using the unimodularity of  $\alpha$ ,

$$\begin{aligned} \frac{\alpha_{22}(1 - \alpha_{11}\alpha_{22}(1 + j'\alpha_{12,g})^2)}{\alpha_{12,g}} &= \frac{\alpha_{22}(-\alpha_{12}\alpha_{21} - 2j'\alpha_{11}\alpha_{22}\alpha_{12,g} - \alpha_{11}\alpha_{22}j'^2\alpha_{12,g}^2)}{\alpha_{12,g}} & \text{(A.I2)} \\ &= -\alpha_{21}\alpha_{22} - 2\alpha_{11}\alpha_{22}^2j' - \alpha_{11}\alpha_{22}j'^2\alpha_{12,g}. \end{aligned}$$

In fact if  $\alpha_{11}\alpha_{12,g} \in 2\mathbb{Z}$ , (A.II) and (A.I2) imply that  $\frac{\alpha_{22}-\alpha_{11}N_g^2\bar{\vartheta}^2}{\alpha_{12,g}} \in 2\mathbb{Z}$ . If  $\alpha_{11}\alpha_{12,g} \in 2\mathbb{Z} + 1$ ,

$$N_g - 2l + \alpha_{12,g}\alpha_{11}N_g = N_g(1 + \alpha_{12,g}\alpha_{11}) - 2l \in 2\mathbb{Z}$$

and we conclude that in both cases the sum is  $N$  periodic.

2. If  $2\frac{\alpha_{11}p-m}{g} \in 2\mathbb{Z} + 1$  we argue similarly that

$$\begin{aligned} & \exp\left(\frac{\pi i}{\alpha_{12}} \left[2\alpha_{22} \left(-lg + \frac{\alpha_{11}\alpha_{12}N_g}{2}\right) + \alpha_{22}N\right]\right) \\ & \quad \cdot \exp\left(-\frac{\pi i4\alpha_{11}N_g\bar{\vartheta}^2}{\alpha_{12,g}} \cdot \left(\frac{2}{g}\right)^2 \cdot \left(2N \left(-lg + \frac{\alpha_{11}\alpha_{12}N_g}{2}\right) + N^2\right)\right) \\ & = \exp\left(\frac{\pi i}{\alpha_{12,g}} (N_g - 2l + \alpha_{12,g}\alpha_{11}N_g) (\alpha_{22} - 16\alpha_{11}N_g^2\bar{\vartheta}^2)\right). \end{aligned}$$

The factor  $-2l + (1 + \alpha_{11}\alpha_{12,g}) N_g \in 2\mathbb{Z}$  from the assumptions of this case and then we only need to prove that  $\alpha_{12,g} \mid (\alpha_{22} - 16\alpha_{11}N_g^2\bar{\vartheta}^2)$ . There exist  $j, j', j'' \in \mathbb{Z}$  such that, using the unimodularity,

$$\begin{aligned} \alpha_{22} - 16\alpha_{11}N_g^2\bar{\vartheta}^2 &= \alpha_{22} - \alpha_{11} (1 + j\alpha_{12,g})^2 (\alpha_{22} + j'\alpha_{12,g})^2 \\ &= j''\alpha_{12,g} + \alpha_{22} - \alpha_{11}\alpha_{22}^2 \\ &= j''\alpha_{12,g} - \alpha_{22}\alpha_{12}\alpha_{21} \end{aligned}$$

which is divisible by  $\alpha_{12,g}$ .

In both cases we proved the  $N$ -periodicity in  $m$ . We arrive to the distribution  $\mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}}$  acting on  $\mathcal{S}(\mathbb{R})$ , which after denoting  $q(m') = gm' + \alpha_{11}p + \frac{1}{2}N_g\alpha_{11}\alpha_{12}$  is defined by

$$\begin{aligned} \mathcal{M}_h(\alpha) \delta_{\frac{p}{N} + \mathbb{Z}} &= \\ C_{\alpha,N} \sum_{m'=0}^{N_g-1} \exp\left(\frac{\pi i}{\alpha_{12}N} (\alpha_{11}p^2 - 2q(m')p + \alpha_{22}q(m')^2)\right) e_{\alpha,N,p}(q(m')) \delta_{\frac{q(m')}{N} + \mathbb{Z}}. \end{aligned}$$

From this expression we obtain  $\mathcal{M}_N \in \mathbb{C}^{N \times N}$  defined by

$$(\mathcal{M}_N)_{p,q} = \begin{cases} C_{\alpha,N} \exp\left(\frac{\pi i}{\alpha_{12}N} (\alpha_{11}p^2 - 2qp + \alpha_{22}q^2)\right) e_{\alpha,N,p}(q) & \exists m' \in \llbracket 0, N_g - 1 \rrbracket : q \equiv q(m') \pmod{N} \\ 0 & \text{Otherwise} \end{cases} \quad (\text{A.13})$$

with  $C_{\alpha,N}$  defined in (A.9) and  $e_{\alpha,N,p}$  in (A.10).

## A.2 The quantum cat maps are quantum ergodic

In this section we give a proof for the quantum ergodicity of  $\mathcal{M}_N(\gamma)$  with  $\gamma \in \tilde{\Gamma}(2)$ . That We recall that an automorphism  $A : (\mathbb{T}^2, dm) \rightarrow (\mathbb{T}^2, dm)$  is *ergodic* if the only  $L^2$  function conserved almost everywhere under it is the constant function.  $A$  is Ergodic if and only if it satisfies the Von-Neumann property,

$$\lim_{\tau \rightarrow \infty} \int_{\mathbb{T}^2} \left| \frac{1}{\tau} \sum_{k=0}^{\tau-1} f\left(A^k \begin{pmatrix} x \\ \xi \end{pmatrix}\right) - \int_{\mathbb{T}^2} f(\tilde{x}, \tilde{\xi}) d\tilde{x}d\tilde{\xi} \right|^2 dx d\xi = 0. \quad (\text{A.14})$$

**Lemma A.2.1.** *Every hyperbolic toral automorphism  $A \in SL_2(\mathbb{Z})$  is Ergodic.*

*Proof.* Let  $f \in L^2(\mathbb{T}^2)$  satisfying  $f \circ A \stackrel{a.e.}{=} f$ . Expanding  $f$  in a Fourier series,

$$f(A\eta) = \sum_{l \in \mathbb{Z}^2} \widehat{f \circ A}(l) e^{2\pi i \langle \eta, l \rangle} = \sum_{l \in \mathbb{Z}^2} \widehat{f}({}^t A^{-1}l) e^{2\pi i \langle \eta, l \rangle}$$

, hence for every  $l \in \mathbb{Z}^2$ ,  $\hat{f}({}^t A^{-1}l) = \hat{f}(l)$ . Putting it in another way, the Fourier coefficients are constant along orbits of  ${}^t A^{-1}$ . As the action of  ${}^t A^{-1}$  on the lattice  $\mathbb{Z}^2$  has a fixed point only at 0,  $\hat{f}(l) \equiv 0$  for  $l \neq 0$ : Otherwise there are infinitely many constant non-zero coefficients along some orbit of the  $A$ -action, which contradicts  $f \in L^2(\mathbb{T}^2)$  (by the Riemann-Lebesgue lemma).  $\square$

We recall that the upper density of a collection of sets  $J = \{J(N)\}_{N=1}^\infty$  is  $\bar{d}(J) = \overline{\lim}_{N \rightarrow \infty} \frac{\#J(N)}{N}$ . For every hyperbolic toral map  $A$ . If the lower density  $\underline{d}(J) = \underline{\lim}_{N \rightarrow \infty} \frac{\#J(N)}{N}$  equals  $\bar{d}(J) = d$ , we say that  $J$  has density  $d$ .

**Theorem A.2.2.** *Let  $\{\varphi_{j,N}\}_{j=0}^{N-1}$  be an orthonormal basis of eigenfunctions of  $\mathcal{M}_N(A)$ , then there exists a collection of sets  $J = \{J(N)\}_{N=1}^\infty$  with  $J(N) \subset \{0, \dots, N-1\}$  of density 1 such that for every  $f \in C^\infty(\mathbb{T}^2)$*

$$\lim_{N \rightarrow \infty} \max_{j \in J(N)} \left| \langle \varphi_{j,N}, \text{Op}_N(f) \varphi_{j,N} \rangle - \int_{\mathbb{T}^2} f(x, \xi) dx d\xi \right| = 0. \quad (\text{A.15})$$

*Proof.* The proof will follow in several steps. Let us denote the Birkhoff averages by  $f_\tau(x, \xi) = \frac{1}{\tau} \sum_{k=0}^{\tau-1} f(A^k(x, \xi))$  and the phase space average by  $\mu(f) = \int_{\mathbb{T}^2} f(x, \xi) dx d\xi$ .

**Lemma A.2.3.** *For  $f \in C^\infty(\mathbb{T}^2)$ ,  $\lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr}(\text{Op}_N(f)) = \mu(f)$ .*

*Proof.* Decompose  $l_k = q_k + r_k N$  with  $q_k \in \{0, \dots, N-1\}$ ,  $r_k \in \mathbb{Z}$  for  $k = 1, 2$ . Then, using (1.8) and the orthonormality of  $\{e_{j,0}\}_{j=0}^{N-1}$ ,

$$\begin{aligned} \frac{1}{N} \text{Tr}(\text{Op}_N(f)) &= \frac{1}{N} \sum_{j=0}^{N-1} \sum_{(l_1, l_2) \in \mathbb{Z}^2} \hat{f}(l_1, l_2) \left\langle e_{j,0}, T^h \left( \frac{l_1}{N}, \frac{l_2}{N} \right) e_{j,0} \right\rangle \\ &= \frac{1}{N} \sum_{\substack{q_1, q_2, j \in \{0, \dots, N-1\} \\ r_1, r_2 \in \mathbb{Z}}} \hat{f}(q_1 + r_1 N, q_2 + r_2 N) \\ &\quad \cdot e^{-\pi i \frac{(q_1 + r_1 N)(q_2 + r_2 N)}{N}} e^{-\frac{2\pi i (q_2 + r_2 N)j}{N}} \langle e_{j,0}, e_{j+q_1,0} \rangle \\ &= \frac{1}{N} \sum_{\substack{q_2 \in \{0, \dots, N-1\} \\ r_1, r_2 \in \mathbb{Z}}} \hat{f}(r_1 N, q_2 + r_2 N) \left( \sum_{j=0}^{N-1} e^{-2\pi i \frac{q_2 j}{N}} \right) \langle e_{j,0}, e_{j,0} \rangle \\ &= \hat{f}(0, 0) + \sum_{r_1, r_2 \in \mathbb{Z} \setminus \{0\}} \hat{f}(r_1 N, r_2 N) \end{aligned}$$

Since  $f \in C^\infty(\mathbb{T}^2)$ ,  $|\hat{f}(l_1, l_2)| \leq \frac{C_{l_1, l_2}}{|l_1^n + l_2^n|}$  for every  $n \in \mathbb{N}$  and  $(l_1, l_2) \neq 0$ . The sum to the right is

$O(N^{-\infty})$  since for every  $n \in \mathbb{N}$ ,

$$\left| \sum_{(r_1, r_2) \in \mathbb{Z}^2 \setminus 0} \hat{f}(Nr_1, Nr_2) \right| \leq \sum_{(r_1, r_2) \in \mathbb{Z}^2 \setminus 0} \left| \hat{f}(Nr_1, Nr_2) \right| \leq N^{-n} C \sum_{(r_1, r_2) \in \mathbb{Z}^2 \setminus 0} \frac{1}{r_1^n + r_2^n}.$$

Letting  $N \rightarrow \infty$  implies that  $\frac{1}{N} \text{Tr}(\text{Op}_N(f)) \rightarrow \hat{f}(0, 0) = \mu(f)$ .  $\square$

**Remark A.2.4.** For  $f \in C^\infty(\mathbb{T}^2)$  such that  $f = O_{S(1)}(h)$ ,  $\text{Tr}(\text{Op}_N(f)) = O(1)$  since

$$\begin{aligned} |\text{Tr}(\text{Op}_N(f))| &\leq \sum_{j=0}^{N-1} |\langle e_{j,0}, \text{Op}_N(f) e_{j,0} \rangle| \\ &\leq N \|\text{Op}_N(f)\| \\ &\leq N \left( \hat{r}(0) + \sum_{(l_1, l_2) \in \mathbb{Z}^2 \setminus 0} \left| \hat{f}(l_1, l_2) \right| \right) \\ &\quad + \int N f(x, \xi) dx d\xi + O(N^{-\infty}) \leq C_f. \end{aligned}$$

Using this lemma, one can find a collection of subsets  $J_f = \{J_f(N)\}_N$  which satisfies (A.15) though depends on  $f$ .

**Lemma A.2.5.** Let  $f \in C^\infty(\mathbb{T}^2)$ , then there exists  $J_f = \{J_f(N)\}_N$  with  $J_f(N) \subset \{0, \dots, N-1\}$  of density 1 satisfying (A.15).

*Proof.* From the unitarity of  $\mathcal{M}_N(A)$  and from the Egorov property in (1.10),

$$\begin{aligned} \langle \varphi_{j,N}, \text{Op}_N(f) \varphi_{j,N} \rangle &= \frac{1}{\tau} \sum_{k=0}^{\tau-1} \langle \varphi_{j,N}, \text{Op}_N(f) \varphi_{j,N} \rangle \tag{A.16} \\ &= \frac{1}{\tau} \sum_{k=0}^{\tau-1} \langle \varphi_{j,N}, \mathcal{M}_N(A)^{-k} \text{Op}_N(f) \mathcal{M}_N(A)^k \varphi_{j,N} \rangle \\ &= \langle \varphi_{j,N}, \text{Op}_N(f_\tau) \varphi_{j,N} \rangle. \end{aligned}$$

For a fixed  $f \in C^\infty(\mathbb{T}^2)$  consider  $\check{f} = f - \mu(f)$ . Consider the sequence of discrete random variables  $X_N$  taking the values  $\langle \varphi_{j,N}, \text{Op}_N(\check{f}) \varphi_{j,N} \rangle$  (for  $j = 0, \dots, N-1$ ) with probability  $\frac{1}{N}$ . Since  $\text{Op}_N(\check{f}^*) = \left(\text{Op}_N(\check{f})\right)^*$  and from (A.16),

$$\mathbb{E}(X_N) = \frac{1}{N} \sum_{j=0}^{N-1} \langle \varphi_{j,N}, \text{Op}_N(\check{f}) \varphi_{j,N} \rangle$$



$$\begin{aligned} &= \frac{1}{N} \sum_{j=0}^{N-1} \left\langle \varphi_{j,N}, \text{Op}_N \left( \check{f}_\tau \right) \varphi_{j,N} \right\rangle \\ &= \frac{1}{N} \text{Tr} \left( \text{Op}_N \left( \check{f}_\tau \right) \right). \end{aligned}$$

hence, applying Lemma A.2.3,

$$\lim_{N \rightarrow \infty} \mathbb{E}[X_N] = \mu \left( \check{f}_\tau \right) = \int_{\mathbb{T}^2} \left( \frac{1}{\tau} \sum_{k=0}^{\tau-1} f \circ A^k (x, \xi) - \mu(f) \right) dx d\xi = 0.$$

The second moment of this random variable is given by

$$\begin{aligned} \mathbb{M}_2[X_N] &= \frac{1}{N} \sum_{j=0}^{N-1} \left| \left\langle \varphi_{j,N}, \text{Op}_N \left( \check{f} \right) \varphi_{j,N} \right\rangle \right|^2 \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \left| \left\langle \varphi_{j,N}, \text{Op}_N \left( \check{f}_\tau \right) \varphi_{j,N} \right\rangle \right|^2. \end{aligned}$$

In virtue of Cauchy-Schwarz inequality,

$$\begin{aligned} &\leq \frac{1}{N} \sum_j \|\varphi_{j,N}\|_{\mathcal{H}_{N,0}}^2 \left\| \text{Op}_N \left( \check{f}_\tau \right) \varphi_{j,N} \right\|_{\mathcal{H}_{N,0}}^2 \\ &= \frac{1}{N} \sum_j \left\langle \varphi_{j,N}, \text{Op}_N \left( \check{f}_\tau^* \right) \text{Op}_N \left( \check{f}_\tau \right) \varphi_{j,N} \right\rangle \\ &= \frac{1}{N} \sum_j \left[ \left\langle \varphi_{j,N}, \text{Op}_N \left( \check{f}_\tau \check{f}_\tau^* \right) \varphi_{j,N} \right\rangle + \left\langle \varphi_{j,N}, \text{Op}_N (r_\tau) \varphi_{j,N} \right\rangle \right] \\ &= \frac{1}{N} \left[ \text{Tr} \left( \text{Op}_N \left( \check{f}_\tau \check{f}_\tau^* \right) \right) + \text{Tr} \left( \text{Op}_N (r) \right) \right]. \end{aligned}$$

$r_\tau$  is the symbol of the reminder operator. As follows,  $r = O(h)$  which in turn implies (from Remark A.2.4) that  $\text{Tr} \left( \text{Op}_N (r) \right) = O(1)$ . Letting  $N \rightarrow \infty$ , by Lemma A.2.3,

$$\mathbb{M}_2[X_N] \leq \frac{\text{Tr} \left( \text{Op}_N \left( \check{f}_\tau \check{f}_\tau^* \right) \right) + \text{Tr} \left( \text{Op}_N (r) \right)}{N} \xrightarrow{N \rightarrow \infty} \int_{\mathbb{T}^2} \left| \left( \check{f}_\tau \right) (x, \xi) \right|^2 dx d\xi.$$

Then letting  $\tau \rightarrow \infty$ , since  $A$  is an ergodic map (from (A.14)),

$$0 \leq \lim_{\tau \rightarrow \infty} \overline{\lim}_{N \rightarrow \infty} \mathbb{M}_2[X_N] \leq \lim_{\tau \rightarrow \infty} \int_{\mathbb{T}^2} \left| \check{f}_\tau (x, \xi) \right|^2 dx d\xi = 0.$$

As  $\lim_{\tau \rightarrow \infty} \lim_{N \rightarrow \infty} \text{Var}[X_N] = \lim_{N \rightarrow \infty} \mathbb{E}(X_N) = 0$ , one can construct explicitly a collection of sets,

$\{J_f(N)\}_{N=1}^\infty$ , of density 1: Let  $\{k_N\}_N$  be a sequence such that<sup>1</sup>  $k_N > 1$ ,  $k'_N = k_N \sqrt{\text{Var}[X_N]} \rightarrow 0$  and  $k_N \rightarrow \infty$  as  $N \rightarrow \infty$ . By Chebyshev inequality,

$$\begin{aligned} 1 - \frac{1}{k_N^2} &\leq \mathbb{P}(|X_N - \mathbb{E}[X_N]| < k'_N) \\ &= \mathbb{P}\left(\left|\langle \varphi_{j,N}, \text{Op}_N(f) \varphi_{j,N} \rangle - \mu(f) - \left(\frac{1}{N} \sum_k \langle \varphi_{k,N}, \text{Op}_N(f) \varphi_{k,N} \rangle - \mu(f)\right)\right| < k'_N\right) \\ &\leq \mathbb{P}\left(\left|\langle \varphi_{j,N}, \text{Op}_N(f) \varphi_{j,N} \rangle - \mu(f)\right| - \left|\frac{1}{N} \sum_k \langle \varphi_{k,N}, \text{Op}_N(f) \varphi_{k,N} \rangle - \mu(f)\right| < k'_N\right). \end{aligned}$$

As  $A \in \tilde{\Gamma}_0(2)$ , one can find large enough  $N$  that satisfies

$$\left|\frac{1}{N} \sum_j \langle \varphi_{j,N}, \text{Op}_N(f) \varphi_{j,N} \rangle - \mu(f)\right| \leq c_N k'_N$$

with  $c_N k'_N \rightarrow 0$  hence

$$\begin{aligned} &\frac{\#\{j \in \{0, \dots, N-1\} : |\langle \varphi_{j,N}, \text{Op}_N(f) \varphi_{j,N} \rangle - \mu(f)| < (1+c_N)k'_N\}}{N} \\ &= \mathbb{P}(|\langle \varphi_{j,N}, \text{Op}_N(f) \varphi_{j,N} \rangle - \mu(f)| < (1+c_N)k'_N) \geq 1 - \frac{1}{k_N^2}. \end{aligned}$$

hence for big enough  $N \in \mathbb{N}$ , one can find a sequence of nonempty sets

$$J_f(N) = \{j \in \{0, \dots, N-1\} : |\langle \varphi_{j,N}, \text{Op}_N(f) \varphi_{j,N} \rangle - \mu(f)| < (1+c_N)k'_N\}.$$

The collection of sets  $J_f = \{J_f(N)\}_{N=1}^\infty$  is of density 1 and satisfies (A.15).  $\square$

We construct a set which is independent in  $f$ :

**Remark A.2.6.** For any two collections of sets  $J = \{J(N)\}_{N=1}^\infty$  and  $J' = \{J'(N)\}_{N=1}^\infty$ , set-theoretic actions are defined with respect to the  $N$ -th level. Explicitly,  $J \subset J'$  means  $J(N) \subset J'(N)$  for every  $N \in \mathbb{N}$ ,  $J \cap J' = \{J(N) \cap J'(N)\}_N$  and  $J \cup J' = \{J(N) \cup J'(N)\}_N$ .

We recall that

**Remark A.2.7.** For every two collection sets  $J, J'$  of density 1,  $d(J^c) = d(J'^c) = 0$  hence  $d((J \cap J')^c) = d(J^c \cup J'^c) = 0$ , which implies that  $d(J \cap J') = 1$ . It follows that finite intersections of collections of sets are of full density.

For every  $l = (l_1, l_2) \in \mathbb{Z}^2$ , one can find a collection of sets,  $J^{(l)} = \{J^{(l)}(N)\}_N$ , corresponding to the trigonometric monomial  $f_l(x, \xi) = e^{2\pi i(l_1 x + l_2 \xi)}$ . Consider  $F^{(P)} = \bigcap_{|l| \leq P} J^{(l)}$ . This set satisfies

<sup>1</sup>It tends to  $\infty$  slower than  $\text{Var}[X_N]$  goes to 0.

(A.15) by linearity, that is for every  $f = \sum_{|l| \leq P} c_l f_l$ ,

$$\lim_{N \rightarrow \infty} \max_{j \in F^{(P)}(N)} \left| \left\langle \varphi_j, \text{Op}_N \left( \sum_{|l| \leq P} c_l f_l \right) \varphi_j \right\rangle - \sum_{|l| \leq P} c_l \int_{\mathbb{T}^2} f_l(x, \xi) dx d\xi \right| = 0.$$

Using Remark A.2.7,  $d(F^{(P)}) = 1$ , therefore for every  $P \in \mathbb{N}$ , exists an index  $N_P \in \mathbb{N}$  such that,

$$\frac{\#F^{(P)}(N)}{N} \geq 1 - \frac{1}{P}. \quad \text{for every } N > N_P \quad (\text{A.17})$$

One can construct  $\{N_P\}_{P=0}^\infty$  as a strictly increasing sequence. We define the set  $J$  level-wise, extracting sets from  $F^{(k)}$  diagonally:

$$J(N) = \begin{cases} \emptyset & N < N_0 \\ F^{(0)}(N) & N \in [N_0, N_1) \\ F^{(1)}(N) & N \in [N_1, N_2) \\ \vdots & \end{cases} \quad (\text{A.18})$$

From (A.17),  $d(J) = 1$  and as  $F^{(P)} \supseteq F^{(P+1)}$  decreases in  $P$ ,  $J(N) \subset F^{(P)}(N)$  for  $N \geq N_P$ . It means that (A.15) holds for  $J$  and for any trigonometric polynomial of an arbitrary degree  $P$ . We have shown the theorem for the set of trigonometric polynomials. For every  $g \in L^2(\mathbb{T}^2)$ , we note that for every  $M \in J(N)$

$$\begin{aligned} \left| \left\langle \varphi_{j,M}, \text{Op}_M \left( g - \sum_{|l| \leq P} \hat{g}(l) f_l \right) \varphi_{j,M} \right\rangle \right| &\leq \|\varphi_{j,M}\| \left\| \text{Op}_M \left( g - \sum_{|l| \leq P} \hat{g}(l) f_l \right) \varphi_{j,M} \right\| \\ &\leq \sum_{(l_1, l_2) > P} |\hat{g}(l)| \left\| T_{\binom{l_1/N}{l_2/N}}^h \varphi_{j,M} \right\| \\ &\leq N^{-n} \sum_{|l| > P} C_l \end{aligned}$$

for every  $n \in \mathbb{N}$  which decays as  $N \rightarrow \infty$  and implies the theorem for every such  $g$ . □

### A.3 Expressing explicitly the values of $t_0, C_{\nu, \mathcal{R}, \mathcal{U}}^+$ from Lemma 2.5.4

We dedicate this appendix for computing exactly the side of the square appearing in the proof of Lemma 2.5.4 (i.e., the orange square in Figure 2.10). First one can express explicitly  $\ell$  from the diagonal of the maximal

square, and it is given by

$$\ell = \frac{C_{\iota, \mathcal{R}, \mathcal{U}}^+ \lambda_u^{-t_0^+}}{2\sqrt{2}} \sqrt{1 + \left(\frac{m_s + 1}{m_s - 1}\right)^2}.$$

For the square to be fully contained in  $\mathcal{D}^\sharp$ , its sides have to be contained inside the parallelogram. As the distance between the unstable sides of  $\mathcal{D}^\sharp$  is  $\asymp \lambda_u^{t_0^+}$  choosing  $t_0^+$  large enough yields control on the distance between the unstable sides. In terms of plane geometry, we require that the horizontal side of the square lies inside  $\mathcal{D}^\sharp$ , for which a sufficient condition using (2.48) is

$$\ell \leq \frac{\sqrt{m_u - m_s}}{4} \hbar_{\mathcal{U}} \lambda_u^{t_0^+}. \quad (\text{A.19})$$

This inequality is satisfied when

$$t_0^+ \geq \left\lceil \frac{1}{2} \left| \frac{\log \left( 2C_{\iota, \mathcal{R}, \mathcal{U}}^+ \sqrt{m_s^2 + 1} \right) - \log \left( \hbar_{\mathcal{U}} (1 - m_s) \sqrt{m_u - m_s} \right)}{\log \lambda_u} \right| \right\rceil. \quad (\text{A.20})$$

For the square to be large enough it is then sufficient to demand that  $\ell \geq 2$ , that is

$$C_{\iota, \mathcal{R}, \mathcal{U}}^+ \lambda_u^{-t_0^+} > 4 \frac{1 - m_s}{\sqrt{1 + m_s^2}}.$$

For  $t_0^+$  satisfying (A.20), choosing

$$C_{\iota, \mathcal{R}, \mathcal{U}}^+ > \frac{32(1 - m_s)}{\sqrt{(1 + m_s^2)(m_u - m_s)}} \quad (\text{A.21})$$

both (A.19) and  $\ell \geq 2$  hold and one can find a square containing a non-trivial lattice point inside  $\mathcal{D}^\sharp$ .

#### A.4 Expressing numerically the porosity parameters of Degli-Esposti's cat map

We calculate the porosity parameters  $\nu$ ,  $K$  (from Lemma 2.5.4, Proposition 2.4.1, (2.55) and (2.56)) for the example of map  $\gamma = \gamma_{DE}$ . We remind that in the same subsection we choose a diagonalizing matrix  $\iota \in \text{SL}_2(\mathbb{R})$  yielding

$$\ell_y = \ell_\eta = \frac{1}{\sqrt{2}\sqrt[4]{3}} (\sqrt{3} + 1).$$

Recalling that  $m_u = -m_s = \sqrt{3}$  their constants of bad approximability  $\omega$  (defined in (2.40)) are equal. From the theory of continued fractions (cf. for instance theorem 27 in [KE97]) we can take  $\omega = 2 - \sqrt{3}$ . Then (2.40) reads

$$\left| p\sqrt{3} - q \right| \geq \frac{2 - \sqrt{3}}{p},$$

for every choice of  $p \in \mathbb{N}^*$ ,  $q \in \mathbb{Z}$ . One can choose  $w_{\mathcal{R}} = \hbar_{\mathcal{R}} = \frac{1}{100}$ ,  $w_{\mathcal{U}} = \hbar_{\mathcal{U}} = \frac{1}{200}$  and verify that they satisfy (2.29) (where  $\mathcal{C}_{w(\mathcal{R})}$  can be computed directly from (2.42) and  $\mathcal{C}_{w(\mathcal{R})} = \sqrt{3}\mathcal{C}_{\hbar(\mathcal{R})}$  from the choice of  $\iota$ ). Substituting the values we assigned to every constant in (2.45) and (2.53), Lemma 2.5.4 means that for every  $t \geq 5$  and  $m \in \mathbb{Z}^2$ ,  ${}_m\Omega_{\epsilon,t}^+$  is a  $\nu_{\epsilon,+}$ -porous set on scales  $C_{\epsilon,+} \cdot (2 + \sqrt{3})^{-t}$  where

$$\nu_{\epsilon,+} = \begin{cases} 0.3297 & \epsilon = 1 \\ 5.1 \cdot 10^{-5} & \epsilon = 2 \end{cases}, \quad C_{\epsilon,+} = \begin{cases} 0.0294 & \epsilon = 1 \\ 24.4887 & \epsilon = 2 \end{cases}. \quad (\text{A.22})$$

Similarly, we obtain that for every  $t \leq -5$  and  $m \in \mathbb{Z}^2$ ,  ${}_m\Omega_{\epsilon,t}^-$  is a  $\nu_{\epsilon,-}$ -porous set on scales  $C_{\epsilon,-} \cdot (2 + \sqrt{3})^t$  with

$$\nu_{\epsilon,-} = \begin{cases} 0.2377 & \epsilon = 1 \\ 5.1 \cdot 10^{-5} & \epsilon = 2 \end{cases}, \quad C_{\epsilon,-} = \begin{cases} 0.0191 & \epsilon = 1 \\ 24.4887 & \epsilon = 2 \end{cases}. \quad (\text{A.23})$$

We conclude from Proposition 2.4.1 that  ${}_m\Omega_{\pm}$  is a  $1.24 \cdot 10^{-7}$ -porous set on scales  $[2702h^\rho, 1]$ .

To conclude this appendix, let us obtain an estimate on the exponent  $\beta$  appearing in Lemma 2.6.4. Jin and Zhang have obtained an estimate on  $\beta$  in [JZ19] for a related class of sets (see [BD18] for the precise definitions and the connection to  $\nu$ -porous sets). Using lemma 5.4 in [DJ18], one can rephrase the result in [JZ19] in terms of  $\nu$ -porous sets,

**Theorem A.4.1** (Adaptation of theorem 5.1 in [JZ19]). *Let  $X, Y$  be  $\nu$ -porous sets on scales  $[\alpha(h), 1]$  for some  $\alpha(h) \in (0, 1]$ . Then for every  $h \in (0, 1)$*

$$\|\mathbb{1}_{Y(2h^\rho)} \mathcal{F}_h^{-1} \mathbb{1}_{X(2h^\rho)}\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq Ch^\beta.$$

with

$$\beta = \exp\left(-K \left(\frac{C}{\delta(1-\delta)}\right)^{\frac{K}{(1-\delta)^3}}\right), \quad \delta = \frac{\log\left(\left\lceil \frac{2}{\nu} \right\rceil - 1\right)}{\log\left\lceil \frac{2}{\nu} \right\rceil},$$

$\rho = 1 - \frac{\beta}{2}$  and  $K$  a universal constant independent of  $X, Y$  and bounded from below.

Plugging our constants in this numerically-explicit version of uncertainty principle we deduce that for Degli-Esposti's cat map,

$$\beta = \exp\left[-KC^{1.92 \cdot 10^{25}K} \cdot (2.68 \cdot 10^8)^{1.92 \cdot 10^{25}K}\right].$$

### A.5 Averaging periodizations over lattice points

We remind an analogue of Poisson's summation formula, namely

**Lemma.** *Let  $P \in \mathbb{N}^*$ ,  $\kappa \in (0, 1)$  and  $f : [-\kappa, P - \kappa] \rightarrow \mathbb{C}$  be a smooth function. Then,*

$$\sum_{p=0}^{P-1} f(p) = P \sum_{n \in \mathbb{Z}} \widehat{f}_{nP}$$

where  $\widehat{f}_j$  is the  $j$ -th Fourier coefficient of the  $P$ -periodic function  $f_{\mathbb{R}}$ ,

$$\widehat{f}_j = \frac{1}{P} \int_0^P f(t) e^{-\frac{2i\pi jt}{P}} dt.$$

*Proof.*  $f_{\mathbb{R}}$  is a piecewise smooth function on  $\mathbb{R}$  discontinuous only on the lattice points  $-\kappa + P\mathbb{Z}$ . Since  $\kappa \notin \mathbb{Z}$ , from Dirichlet's theorem  $f_{\mathbb{R}}$  is equal to its Fourier series except on its discontinuity points, in particular  $\sum_{n \in \mathbb{Z}} \widehat{f}_n \exp\left(\frac{2\pi in}{P}t\right)$  converges to  $f_{\mathbb{R}}(t)$  on integral points. As a result,

$$\sum_{p=0}^{P-1} f(p) = \sum_{p=0}^{P-1} \sum_{n \in \mathbb{Z}} \widehat{f}_n e^{\frac{2\pi in}{P}p} = P \sum_{m \in \mathbb{Z}} \widehat{f}_{Pm} + \sum_{n \notin P\mathbb{Z}} \widehat{f}_n \sum_{p=0}^{P-1} e^{\frac{2\pi in p}{P}}.$$

The last sum on  $p$  vanishes, and we deduce the lemma.  $\square$

### A.6 Concentration of the norm of projections to $V_{N,\nu}$

We dedicate this appendix to show that the norm is concentrated asymptotically around the same mean in any random model as long as the vector projected to  $V_{N,\nu}$  has independent identically distributed entries. More precisely,

**Corollary A.6.1.** *let  $(a_{1,N}, \dots, a_{N,N})$  be an independent identically distributed random vector satisfying  $\mathbb{E}(a_{j,N}) = 0$ ,  $\text{Var}(a_{j,N}) = 1$  and denote by  $b_N = \Pi_{\nu,N} a_N$ . Consider the random variable  $\mathcal{L}_N := \|b_N\|_2^2$ . If  $N \in \mathcal{N}$ ,  $\mathcal{L}_N$  is centered around*

$$\mathbb{E}[\mathcal{L}_N] = \frac{1}{P(N)} + O\left(\frac{1}{\sqrt[4]{N}}\right)$$

and its variance satisfies

$$|\text{Var}[\mathcal{L}_N]| \leq \frac{1}{P(N) \sqrt[4]{N}}.$$

*Proof.* The proof follows from a direct calculation: Since

$$\mathcal{L}_N = \langle \Pi_{\nu,N} a_N, \Pi_{\nu,N} a_N \rangle = \langle \Pi_{\nu,N} a_N, a_N \rangle = \sum_{j,k=1}^N \pi_{j,k} a_{k,N} \overline{a_{j,N}}$$

and since  $\{a_{j,N}\}_j$  are independent

$$\begin{aligned} \mathbb{E}[\mathcal{L}_N] &= \sum_{j,k=1}^N \mathbb{E}[\pi_{j,k} a_{k,N} \overline{a_{j,N}}] \\ &= \sum_{j=1}^N \left( \frac{1}{P(N)} + \frac{C_{jj}}{\sqrt[4]{N}} \right) \mathbb{E}[|a_{j,N}|^2] + \frac{1}{\sqrt[4]{N}} \sum_{j \neq k=1}^N C_{j,k} \mathbb{E}[a_{k,N} \overline{a_{j,N}}]. \end{aligned} \tag{A.24}$$

$$= \frac{1}{P(N)} + \frac{C}{\sqrt[4]{N}}$$

The variance can be computed similarly: Since  $f(z) = |z|^2$  is a measurable function, the variables  $|a_{j,N}|^2, |a_{k,N}|^2$  are independent as well and thus

$$\begin{aligned} \sum_{j,j'=1}^N \mathbb{E}[\pi_{j,j'} \pi_{j',j'} |a_{j,N}|^2 |a_{j',N}|^2] &= \frac{1}{P^2(N)} \sum_{j,j'} \mathbb{E}[|a_{j,N}|^2 |a_{j',N}|^2] \\ &\quad + \sum_{k \neq k'} \left( \frac{C_{k,k} + C_{k',k'}}{\sqrt[4]{N} P(N)} + \frac{C_{k,k} C_{k',k'}}{\sqrt{N}} \right) \mathbb{E}[|a_{k,N}|^2 |a_{k',N}|^2] \\ &= \frac{1}{P^2(N)} + O\left(\frac{1}{P(N) \sqrt[4]{N}}\right). \end{aligned}$$

From Cauchy-Schwarz inequality

$$\begin{aligned} \left| \sum_{\substack{j \neq k=1 \\ j' \neq k'=1}}^N \mathbb{E}[\pi_{j,k} \pi_{j',k'} a_{k,N} \overline{a_{j,N}} a_{k',N} \overline{a_{j',N}}] \right| &\leq \frac{C}{\sqrt[4]{N} P(N)} \sum |\mathbb{E}[a_{k,N} a_{k',N} \overline{a_{j,N}} \overline{a_{j',N}}]| \\ &\leq \frac{C}{\sqrt[4]{N} P(N)} \sum \sqrt{\mathbb{E}[|a_{k,N} a_{k',N}|^2] \mathbb{E}[|a_{j,N} a_{j',N}|^2]}. \end{aligned}$$

The only non-zero term in the summation occurs for  $k = k', j = j'$  thus,

$$\left| \sum_{\substack{j \neq k=1 \\ j' \neq k'=1}}^N \mathbb{E}[\pi_{j,k} \pi_{j',k'} a_{k,N} \overline{a_{j,N}} a_{k',N} \overline{a_{j',N}}] \right| = O\left(\frac{1}{P(N) N^{\frac{5}{4}}}\right).$$

Altogether,

$$\begin{aligned} \left| \mathbb{E}[\mathcal{L}_N^2] - \frac{1}{P^2(N)} \right| &\leq \left| \sum_{\substack{j \neq k=1 \\ j' \neq k'=1}}^N \mathbb{E}[\pi_{j,k} \pi_{j',k'} a_{k,N} \overline{a_{j,N}} a_{k',N} \overline{a_{j',N}}] \right| + \left| \sum_{j,j'=1}^N \mathbb{E}[\pi_{j,j'} \pi_{j',j'} |a_{j,N}|^2 |a_{j',N}|^2] - \frac{1}{P^2(N)} \right| \\ &= O\left(\frac{1}{P(N) \sqrt[4]{N}}\right). \end{aligned}$$

Since  $\mathcal{L}_N^2$  is non-negative its expected value is non-negative as well (since  $\mathbb{E}[\mathcal{L}_N^2] = \int_0^\infty (1 - F_{\mathcal{L}_N^2}(x)) dx$ )

and we deduce,

$$|\text{Var}[\mathcal{L}_N]| = |\mathbb{E}[\mathcal{L}_N^2] - \mathbb{E}^2[\mathcal{L}_N]| = O\left(\frac{1}{P(N)\sqrt{N}}\right).$$

□



# References

- [AA67] Vladimir Igorevich Arnold and André Avez. *Problèmes ergodiques de la mécanique classique*. Gauthier-Villars, 1967.
- [ANo7a] Nalini Anantharaman and Stéphane Nonnenmacher. Entropy of semiclassical measures of the Walsh-quantized baker’s map. In *Annales Henri Poincaré*, volume 8, pages 37–74. Springer, 2007.
- [ANo7b] Nalini Anantharaman and Stéphane Nonnenmacher. Half-delocalization of eigenfunctions for the laplacian on an Anosov manifold. In *Annales de l’institut Fourier*, volume 57, pages 2465–2523, 2007.
- [Anao8] Nalini Anantharaman. Entropy and the localization of eigenfunctions. *Annals of Mathematics*, 168(2):435–475, 2008.
- [Bäco3] Arnd Bäcker. Numerical aspects of eigenvalue and eigenfunction computations for chaotic quantum systems. In *The mathematical aspects of quantum maps*, pages 91–144. Springer, 2003.
- [BD18] Jean Bourgain and Semyon Dyatlov. Spectral gaps without the pressure condition. *Annals of Mathematics*, 187(3):825–867, 2018.
- [BDB96] Abdelkader Bouzouina and Stephan De Bièvre. Equipartition of the eigenfunctions of quantized ergodic maps on the torus. *Communications in mathematical physics*, 178(1):83–105, 1996.
- [BDB00] Francesco Bonechi and Stephan De Bièvre. Exponential mixing and time scales  $p$  in quantized hyperbolic maps on the torus. *Communications in Mathematical Physics*, 211(3):659–686, 2000.
- [Ber41] Andrew C Berry. The accuracy of the gaussian approximation to the sum of independent variates. *Transactions of the american mathematical society*, 49(1):122–136, 1941.
- [Ber77] Michael V Berry. Regular and irregular semiclassical wavefunctions. *Journal of Physics A: Mathematical and General*, 10(12):2083, 1977.
- [BGS84] Oriol Bohigas, Marie-Joya Giannoni, and Charles Schmit. Characterization of chaotic quantum spectra and universality of level fluctuation laws. *Physical review letters*, 52(1):1, 1984.
- [BL13] Nicolas Burq and Gilles Lebeau. Injections de sobolev probabilistes et applications. *Ann. Scient. Ec. Norm. Sup.*, 46(7):917–962, 2013.
- [Bou07] Jean Bourgain. A remark on quantum ergodicity for cat maps. *Lecture Notes in Mathematics* 1910:89–98, 2007.
- [Bro10] Shimon Brooks. On the entropy of quantum limits for 2-dimensional cat maps. *Communications in Mathematical Physics*, 293(1):231, 2010.
- [BV89] Nandor Balasz and André Voros. The quantized baker’s transformation. *Ann. Phys. (NY)*, 190(3):1–31, 1989.

- [CB21] George Casella and Roger L Berger. *Statistical inference*. Cengage Learning, 2nd edition, 2021.
- [CdV85] Yves Colin de Verdiere. Ergodicité et fonctions propres du laplacien. *Communications in Mathematical Physics*, 102(3):497–502, 1985.
- [CG18] Sourav Chatterjee and Jeffrey Galkowski. Arbitrarily small perturbations of dirichlet laplacians are quantum unique ergodic. *Journal of Spectral Theory*, 8(3):909–947, 2018.
- [CH15] Yaiza Canzani and Boris Hanin. Scaling limit for the kernel of the spectral projector and remainder estimates in the pointwise weyl law. *Analysis & PDE*, 8(7):1707–1731, 2015.
- [CH18] Yaiza Canzani and Boris Hanin.  $c^\infty$ -scaling asymptotics for the spectral projector of the laplacian. *The Journal of Geometric Analysis*, 28(1):111–122, 2018.
- [CKST08] Cheng-Hung Chang, Tyll Krüger, Roman Schubert, and Serge Troubetzkoy. Quantisations of piecewise parabolic maps on the torus and their quantum limits. *Communications in Mathematical Physics*, 282(1):395–418, 2008.
- [Cot55] M Cotlar. A combinatorial inequality and its applications to  $L^2$ -spaces. *Rev. Mat. Cuyana*, 1:41–55, 1955. <http://cms.dm.uba.ar/depto/public/Cuyana/vol1-41-167.pdf>.
- [CW75] George Cooke and Peter J Weinberger. On the construction of division chains in algebraic number rings, with applications to  $sl_2$ . *Communications in Algebra*, 3(6):481–524, 1975.
- [DE93] Mirko Degli Esposti. Quantization of the orientation preserving automorphisms of the torus. *Annales de l’IHP Physique théorique*, 58(3):323–341, 1993.
- [DEG03] Mirko Degli Esposti and Sandro Graffi. Mathematical aspects of quantum maps. In *The mathematical aspects of quantum maps*, pages 49–90. Springer, 2003.
- [DEGI95] Mirko Degli Esposti, Sandro Graffi, and Stefano Isola. Classical limit of the quantized hyperbolic toral automorphisms. *Communications in Mathematical Physics*, 167(3):471–507, 1995.
- [DENW06] Mirko Degli Esposti, Stéphane Nonnenmacher, and Brian Winn. Quantum variance and ergodicity for the baker’s map. *Communications in Mathematical Physics*, 263(1):325–352, 2006.
- [DJ18] Semyon Dyatlov and Long Jin. Semiclassical measures on hyperbolic surfaces have full support. *Acta Mathematica*, 220(2):297–339, 2018.
- [DJ21] Semyon Dyatlov and Malo Jézéquel. Semiclassical measures for higher dimensional quantum cat maps. *arXiv preprint arXiv:2108.10463*, 2021.
- [DJN22] Semyon Dyatlov, Long Jin, and Stéphane Nonnenmacher. Control of eigenfunctions on surfaces of variable curvature. *Journal of the American Mathematical Society*, 35(2):361–465, 2022.
- [DZ19] Semyon Dyatlov and Maciej Zworski. *Mathematical theory of scattering resonances*, volume 200. American Mathematical Soc., 2019.

- [Eck86] Bruno Eckhardt. Exact eigenfunctions for a quantised map. *Journal of Physics A: Mathematical and General*, 19(10):1823, 1986.
- [Ess42] C.G. Esseen. On the liapunoff limit of error in the theory of probability. *Arkiv för matematik, astronomi och fysik*, 28A(2):1–19, 1942.
- [EW10] Manfred Einsiedler and Thomas Ward. *Ergodic theory with a view towards number theory, volume 259 of*. Number 259 in Graduate Texts in Mathematics. Springer-Varleg, 2010.
- [FM86] H Frahm and HJ Mikeska. Levelstatistics and stochasticity in a driven quantum system. *Zeitschrift für Physik B Condensed Matter*, 65(2):249–253, 1986.
- [FN04] Frédéric Faure and Stéphane Nonnenmacher. On the maximal scarring for quantum cat map eigenstates. *Communications in mathematical physics*, 245(1):201–214, 2004.
- [FNDB03] Frédéric Faure, Stéphane Nonnenmacher, and Stephan De Bièvre. Scarred eigenstates for quantum cat maps of minimal periods. *Communications in Mathematical Physics*, 239(3):449–492, 2003.
- [Fol89] Gerald B Folland. *Harmonic analysis in phase space*. Princeton university press, 1989.
- [GHO6] Shagmar Gurevich and Ronny Hadani. Proof of the kurlberg-rudnick rate conjecture. In *AIP Conference Proceedings*, volume 826, pages 74–80. American Institute of Physics, 2006.
- [Han15] Xiaolong Han. Small scale quantum ergodicity in negatively curved manifolds. *Nonlinearity*, 28(9):3263, 2015.
- [Han18] Xiaolong Han. Small scale quantum ergodicity in cat maps. i. *arXiv preprint arXiv:1810.11949*, 2018.
- [Har98] Glyn Harman. On the erdős-turán inequality for balls. *ACTA ARITHMETICA-WARSZAWA*, 85:389–396, 1998.
- [HB80] JH Hannay and Michael V Berry. Quantization of linear maps on a torus-Fresnel diffraction by a periodic grating. *Physica D: Nonlinear Phenomena*, 1(3):267–290, 1980.
- [HKS87] Fritz Haake, M Kuś, and Rainer Scharf. Classical and quantum chaos for a kicked top. *Zeitschrift für Physik B Condensed Matter*, 65(3):381–395, 1987.
- [Hör15] Lars Hörmander. *The analysis of linear partial differential operators I: Distribution theory and Fourier analysis*. Springer, 2015.
- [HR16] Hamid Hezari and Gabriel Rivière.  $L^p$  norms, nodal sets, and quantum ergodicity. *Advances in Mathematics*, 290:938–966, 2016.
- [HR17] Hamid Hezari and Gabriel Riviere. Quantitative equidistribution properties of toral eigenfunctions. *Journal of Spectral Theory*, 7(2):471–485, 2017.
- [HT20] Xiaolong Han and Melissa Tacy. Equidistribution of random waves on small balls. *Communications in Mathematical Physics*, 376(3):2351–2377, 2020.

- [HW71] David Lee Hanson and Farroll Tim Wright. A bound on tail probabilities for quadratic forms in independent random variables. *The Annals of Mathematical Statistics*, 42(3):1079–1083, 1971.
- [Izr87] FM Izrailev. Chaotic structure of eigenfunctions in systems with maximal quantum chaos. *Physics Letters A*, 125(5):250–252, 1987.
- [JZ19] Long Jin and Ruixiang Zhang. Fractal uncertainty principle with explicit exponent. *Mathematische Annalen*, pages 1–27, 2019.
- [KE97] A Khinchin and H Eagle. Continued fractions. dover books on mathematics, 1997.
- [Kea91a] Jonathan Keating. Asymptotic properties of the periodic orbits of the cat maps. *Nonlinearity*, 4(2):277, 1991.
- [Kea91b] Jonathan Keating. The cat maps: quantum mechanics and classical motion. *Nonlinearity*, 4(2):309, 1991.
- [Kelo8] Dubi Kelmer. On matrix elements for the quantized cat map modulo prime powers. In *Annales Henri Poincaré*, volume 9, pages 1479–1501. Springer, 2008.
- [Kel10] Dubi Kelmer. Arithmetic quantum unique ergodicity for symplectic linear maps of the multidimensional torus. *Annals of mathematics*, pages 815–879, 2010.
- [KH97] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*. Number 54. Cambridge university press, 1997.
- [Kli11] Wilhelm PA Klingenberg. *Riemannian geometry*, volume 1. Walter de Gruyter, 2011.
- [KMH88] M Kuś, J Mostowski, and F Haake. Universality of eigenvector statistics of kicked tops of different symmetries. *Journal of Physics A: Mathematical and General*, 21(22):L1073, 1988.
- [KR00] Pär Kurlberg and Zeév Rudnick. Hecke theory and equidistribution for the quantization of linear maps of the torus. *Duke Math. J.*, 103(1):47–77, 2000.
- [KR01a] Pär Kurlberg and Zeév Rudnick. On quantum ergodicity for linear maps of the torus. *Communications in Mathematical Physics*, 222(1):201–227, 2001.
- [KR01b] Pär Kurlberg and Zeév Rudnick. Value distribution for eigenfunctions of desymmetrized quantum maps. *International mathematics research notices*, 2001(18):985–1002, 2001.
- [Kuro3] Pär Kurlberg. On the order of unimodular matrices modulo integers. *Acta Arithmetica*, 110(2):141–151, 2003.
- [Lan12] Serge Lang.  $SL_2(\mathbb{R})$ , volume 105. Springer Science & Business Media, 2012.
- [Lee06] John M Lee. *Riemannian manifolds: an introduction to curvature*, volume 176. Springer Science & Business Media, 2006.
- [Len77] Hendrik W Lenstra. On Artin’s conjecture and Euclid’s algorithm in global fields. 1977.

- [Lino6] Elon Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Annals of Mathematics*, pages 165–219, 2006.
- [LR17] Stephen Lester and Zeév Rudnick. Small scale equidistribution of eigenfunctions on the torus. *Communications in Mathematical Physics*, 350(1):279–300, 2017.
- [Map13] Kenneth Maples. Quantum unique ergodicity for random bases of spectral projections. *Mathematical Research Letters*, 20(6):1115–1124, 2013.
- [Mezo2] Francesco Mezzadri. On the multiplicativity of quantum cat maps. *Nonlinearity*, 15(3):905, 2002.
- [NV98] S Nonnenmacher and A Voros. Chaotic eigenfunctions in phase space. *Journal of Statistical Physics*, 92(3):431–518, 1998.
- [Pat99] Gabriel P Paternain. Introduction to geodesic flows. In *Geodesic Flows*, pages 7–29. Springer, 1999.
- [Riv11] Gabriel Rivière. Entropy of semiclassical measures for symplectic linear maps of the multidimensional torus. *International Mathematics Research Notices*, 2011(11):2396–2443, 2011.
- [Ros00] Hans Roskam. A quadratic analogue of Artin’s conjecture on primitive roots. *Journal of Number Theory*, 81(1):93–109, 2000.
- [RV13] Mark Rudelson and Roman Vershynin. Hanson-wright inequality and sub-gaussian concentration. *Electronic Communications in Probability*, 18:1–9, 2013.
- [Shn74] Alexander I Shnirelman. Ergodic properties of eigenfunctions. *Uspekhi Matematicheskikh Nauk*, 29(6):181–182, 1974.
- [Sie13] Carl Ludwig Siegel. *Lectures on the Geometry of Numbers*. Springer Science & Business Media, 2013.
- [SM93] Elias M Stein and Timothy S Murphy. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 3. Princeton University Press, 1993.
- [Van97] Jeffrey VanderKam.  $l^\infty$  norms and quantum ergodicity on the sphere. *Internat. Math. Res. Notices*, 1997(7):329–347, 1997.
- [Wei73] Peter J Weinberger. On euclidean rings of algebraic integers. In *Proc. Symp. Pure Math*, volume 24, pages 321–332, 1973.
- [Zel87] Steven Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke mathematical journal*, 55(4):919–941, 1987.
- [Zel92] Steven Zelditch. Quantum ergodicity on the sphere. *Communications in mathematical physics*, 146(1):61–71, 1992.
- [Zel14] Steven Zelditch. Quantum ergodicity of random orthonormal bases of spaces of high dimension. *Phil.Trans.R.Soc. A*, 372(3), 2014. Article ID: 20120511.
- [Zwo12] Maciej Zworski. *Semiclassical analysis*, volume 138. American Mathematical Soc., 2012.