



Metastability in slowly time-dependent non-singular or singular stochastic PDEs

Rita Nader

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Institut Denis Poisson

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Metastability in slowly time-dependent
non-singular or singular stochastic PDEs

Métastabilité dans des EDP stochastiques lentement dépendantes du temps non singulières et singulières

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Dédicace

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la personne que je suis aujourd'hui.
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Aperçu de la thèse

La métastabilité est un phénomène observé dans de nombreux modèles physiques tels que la surfusion: un état dans lequel un liquide dont la température est sous le point de solidification demeure à l'état liquide au lieu de passer à l'état solide. En chimie, le diamant est caractérisé par la métastabilité à température ambiante car il se transforme en graphite à une vitesse tellement lente qu'elle est inobservable, et lorsque la température augmente, la transformation s'accélère. La métastabilité décrit alors l'existence des états stables pour un système dans lesquels ce dernier demeure pour une longue durée avant de rejoindre son état d'équilibre sous l'action de perturbations diverses.

Un exemple de système plus proche du contexte de la thèse est de considérer une bille qui glisse dans un potentiel à double puit. Sans perturbation, la dynamique du système emmène la bille dans l'un des puits du potentiel. Lorsque le système est soumis à des forces, la bille pourra effectuer des transitions entre les deux puits selon l'intensité de la perturbation. Et dans ce cas le système va passer un temps relativement long dans le voisinage de l'un des deux minima avant d'effectuer une transition vers l'autre minimum. Un tel système peut être modéliser par une équation différentielle stochastique (EDS).

Cette thèse porte sur l'étude de la métastabilité pour des systèmes modélisés par des équations aux dérivées partielles stochastiques (EPDS) soumis à deux types de perturbations: une force de dérive périodique déterministe et un bruit additif qui modélise une perturbation aléatoire. On est intéressé par ce type d'EPDS

$$\partial_t \phi(t, x) = \Delta \phi(t, x) + F(\varepsilon t, \phi(t, x)) + \sigma \xi(t, x) , \quad (0.0.1)$$

où ϕ est l'état du système, Δ est le Laplacien, F est le terme de dérive périodique, εt est le temps lent, σ est un paramètre qui mesure l'intensité du bruit blanc espace-temps noté par ξ . Les objets modélisant le bruit aléatoire sont connus d'être très irréguliers et cela rend l'étude des équations aux dérivées partielles (EDP) associées sensible lorsque des produits entre deux objets irréguliers apparaissent. Et dans ce cas les équations ne peuvent pas être résolues dans des espaces fonctionnels classiques. En dimension 1, ces équations sont bien posées mais lorsqu'on passe à des dimensions supérieures on ne peut plus définir une notion de solution. On développe dans le Chapitre 5 une approche permettant de contourner ce problème en dimension 2.

Un exemple intéressant d'un système montrant la métastabilité est l'équation d'Allen-Cahn périodiquement forcée

$$\partial_t \phi(t, x) = \Delta \phi(t, x) + \phi(t, x) - \phi(t, x)^3 + A \cos(\varepsilon t) + \sigma \xi(t, x) .$$

Quand $A = 0$ et $\sigma = 0$, l'équation déterministe a été étudiée par W. Cahn et Sam Allen pour modéliser la séparation de phase dans des alliages [1]. Une particularité de cette EDP est qu'elle admet deux solutions stationnaires stables ± 1 et une instable égale à 0 dans laquelle ϕ change de signe. On peut représenter les solutions stables par un mélange de deux fluides différents, tels que l'huile et l'eau, et on interprète le changement de signe comme des interfaces entre ces deux phases. Une question naturelle est de se demander combien de temps il faut pour qu'un système perturbé par un bruit aléatoire passe d'un de ces états stables à l'autre. Des phénomènes intéressants et inattendus peuvent se produire lorsque l'équilibre perd sa stabilité et lorsque le système s'approche d'un point de bifurcation tels que la résonance stochastique et le retard à la bifurcation qu'on discutera plus tard.

Dans cette thèse, on donne des estimées de concentrations sur les solutions des EDPS considérées. Soient E un espace fonctionnel et $\mathcal{A}$ un ouvert borné de $\mathbb{R}^+ \times E$. L'approche est basée sur le premier temps de sortie qui est défini par un temps d'arrêt

$$\tau_{\mathcal{A}} = \inf\{t \geq 0 : (t, \phi(t, \cdot)) \notin \mathcal{A}\}.$$

Les problèmes de sorties ont été initialement étudiés dans la théorie des grandes déviations qui donne les comportements asymptotiques des systèmes métastables. Mais dans la littérature il existe des résultats plus précis utilisant d'autres méthodes, obtenus pour des EDS voir [10]. On généralise ces méthodes pour des systèmes de dimensions infinis modélisés par des EDPS. On montre que sous certaines conditions sur les paramètres en jeu les solutions sont concentrées dans un certain voisinage avec une probabilité exponentiellement petite. Dans le sens, ou pour tout $h > 0$ mesurant la largeur du voisinage, il existe une constante positive κ telle que

$$\mathbb{P}\{\tau_{\mathcal{A}} < t\} \leq C(t, \varepsilon) e^{-\kappa h^2 / \sigma^2},$$

pour tout $t \geq 0$.

Les Chapitres 3 et 4 concernent la résonance stochastique (RS) pour les EDPS sur le tore de dimension 1, $\mathbb{T} = \mathbb{R}/(L\mathbb{Z})$. La RS a été introduite initialement dans le contexte du climat pour expliquer l'apparition presque périodique des périodes glaciaires et depuis, elle est apparu dans de nombreux d'autres systèmes biologiques et physiques y compris les lasers, les circuits électroniques [48], la neuroscience [38]. Ce mécanisme se produit lorsqu'un système bistable ou multistable est forcé périodiquement en temps tout en étant soumis également à un bruit. Lorsque la période du forçage est proche du temps nécessaire au bruit pour faire passer le système d'un état métastable à l'autre, des oscillations périodiques ont lieu. Les principaux résultats sur la RS sont obtenus pour des EDS de la forme suivante

$$dX_t = f(\varepsilon t, X_t) dt + \sigma dW_t, \quad (0.0.2)$$

ou ε détermine la dépendance en temps, W_t est le processus de Wiener standard et f est le terme dérive périodique en temps. Le terme de dérive peut modéliser le mouvement d'une particule dans un potentiel à double puits. On étend l'analyse effectuée pour des EDS de la forme (0.0.2) dans [8] à des EDPS de dimension infinie de la forme (0.0.1). L'exemple de référence qu'on traite dans ce manuscrit est l'équation d'Allen-Cahn qui décrit une situation bistable

$$f(\varepsilon t, \phi(t, x)) = \phi(t, x) - \phi(t, x)^3 + A \cos(\varepsilon t).$$

Dès que A est plus petit qu'une certaine valeur critique A_c , f s'annule en 3 branches, deux parmi eux deviennent proches ou se touchent une fois par période. La distance minimale entre les différentes branches est mesurées par un petit paramètre δ . Toutefois, les résultats obtenus s'appliquent pour des termes de dérives plus généraux qui doivent satisfaire certaines conditions de régularité. La description du comportement des solutions près des branches d'équilibres et les relations entre les différents paramètres assurant les oscillations périodiques entre les puits pour les EDPS sont similaires à celles établies pour les EDS dans [8]. La difficulté de cette généralisation est qu'on travaille avec du bruit blanc espace-temps en dimension infinie, nous empêchant d'utiliser directement les méthodes développées dans [8]. On estime les termes d'erreurs apparaissant dans des espaces de Sobolev. Ces espaces sont liés à la décomposition en Fourier des équations étudiées. Les résultats se résument comme suit:

On effectue tout d'abord une étude près d'une branche d'équilibre stable $\phi^*(t)$ ou on suppose que

$$f(t, \phi^*(t)) = 0 \text{ et } a(t) = \partial_{\phi} f(t, \phi^*(t)) < 0, \forall t \in I = [0, T].$$

- En absence du bruit, quand $\sigma = 0$, le comportement de la solution déterministe $\bar{\phi}(t, x)$ près d'un équilibre stable est décrit par le résultat suivant

$$\|\bar{\phi}(t, \cdot) - \phi^*(t)e_0\|_{H^1} \leq C\varepsilon ,$$

pour tout $t \in I$ et $e_0(x) = 1/\sqrt{L}$. En d'autres termes, la solution déterministe reste proche de l'équilibre stable d'une distance d'ordre ε tout au long de l'intervalle du temps.

- Lorsqu'on ajoute du bruit et lorsque les branches d'équilibre sont bien séparées les solutions de l'EDPS restent concentrées dans un voisinage centré autour de la solution déterministe qui reste proche de l'équilibre stable avec grande probabilité. Ceci est justifié par la borne obtenue dans le Théorème 3.3.1 suivante

$$\mathbb{P}\{\|\phi - \bar{\phi}(t, \cdot)\|_{H^s} \geq h\} \leq C(\kappa, t, \varepsilon, s) \exp\left\{-\kappa \frac{h^2}{\sigma^2} \left[1 - \mathcal{O}\left(\frac{h}{\varepsilon^\nu}\right)\right]\right\} ,$$

for all $t \in I$ et $s < 1/2$.

Quand le système perd sa stabilité, les branches d'équilibres se rapprochent l'une de l'autre engendrant des bifurcations. On décompose la solution $\phi(t, x)$ en une partie constante en espace $\phi_0(t)$ et une partie oscillante $\phi_\perp(t, x)$ ou l'étude de chacune des EDS et EDPS relatives donne les résultats ci-dessous:

- Les solutions de l'EDPS en $\phi_\perp(t, x)$ se comportent comme les solutions de l'EDPS près d'un équilibre stable. Le Théorème 3.3.1 reste valide près des points de bifurcations et on conclut que les solutions restent concentrées près de l'équilibre stable avec grande probabilité.
- Alors que le comportement de $\phi_0(t)$ dépend de l'intensité du bruit σ et on distingue deux régimes de bruit. Pour des intensités de bruit inférieures à $\sigma_c = (\delta \vee \varepsilon)^{3/4}$, les solutions restent proches de l'équilibre stable et la probabilité de passer à l'autre équilibre stable est exponentiellement petite.
- Tandis que pour une intensité de bruit supérieure à σ_c , des transitions entre les différentes branches d'équilibres ont lieu avec grande probabilité au delà d'un temps d'ordre $-\sigma^{2/3}$.

Dans le Chapitre 5, on traite un autre phénomène intéressant le retard à la bifurcation qui a lieu quand les solutions restent proches de l'équilibre instable pendant un certain temps après qu'une bifurcation de fourche s'est produite. Ceci est dû au fait que les solutions deviennent exponentiellement proches de 0 durant la phase stable et un temps d'ordre 1 après la bifurcation est nécessaire pour que la solution atteigne de nouveau un état stable. Une étude de ce phénomène pour des EDS de la forme (0.0.2) a montré que l'ajout du bruit blanc a réduit le retard à la bifurcation d'ordre 1 à d'ordre $(\varepsilon \log(\sigma^{-1}))^{1/2}$, voir [7]. Une extension de ces résultats en dimension infinie pour des EDPS du type (0.0.1) est élaborée dans ce chapitre, sur le tore de dimension 2.

L'EDPS dans ce cas n'est pas bien posée due au fait que le bruit blanc espace temps est plus irrégulier en dimension 2 qu'en dimension 1. On peut le voir comme une distribution aléatoire non pas une fonction aléatoire. L'existence des solutions via le théorème de point fixe de Banach demande une certaine régularité qui n'est pas assurée ici. Cette irrégularité est causée par la convolution stochastique, le terme de l'équation du point fixe faisant intervenir le bruit blanc espace-temps. Ceci induit des produits de termes non définis surtout en analysant le terme non linéaire de l'EDPS évalué en la différence entre ϕ et la convolution stochastique. Giuseppe Da Prato et Arnaud Debussche dans [42] ont introduit une méthode pour contourner ce problème en dimension 2. Elle consiste à soustraire une constante divergente de la partie droite de l'EDPS ou on se ramène à étudier une version renormalisée de l'EDPS

$$d\phi(t, x) = [\Delta\phi(t, x) + :F(\varepsilon t, \phi(t, x)):] dt + \sigma dW(t, x) ,$$

ou $:F:$ est la renormalisation de Wick. Un exemple de non linéarité est de nouveau l'équation d'Allen Cahn renormalisée donnée par

$$:F(\varepsilon t, \phi(t, x)):= a(t)\phi(t, x) - :\phi(t, x)^3:,$$

ou $:\phi(t, x)^3:= \phi(t, x)^3 - 3C_N\phi(t, x)$ et C_N est une constante divergente. Ils ont élaboré ensuite une nouvelle idée qui est basée sur l'équation de la différence ϕ_1 entre ϕ et la convolution stochastique où ϕ_1 appartient à un espace avec une meilleure régularité. On se base sur cette idée pour construire les estimées de concentration dans des espaces de Besov appropriés. Des résultats similaires aux résultats obtenus en dimension 1, sur le comportement des solutions aux voisinages des branches d'équilibre stable et instable sont attendus et se résument de la façon suivante:

- Comme les puissances de Wick de la convolution stochastique sont les termes les plus irréguliers de notre étude, le Théorème 5.3.12 montre qu'elles sont concentrées avec grande probabilité dans un voisinage de la branche d'équilibre stable. Cette distance est mesurée dans des normes de Besov $\|\cdot\|_{\mathcal{B}_{2,\infty}^\alpha}$ pour des $\alpha < 0$. En appliquant la méthode élaborée par Da Prato et Debussche, Théorème 5.4.7 montre que des estimées de concentration reste valide pour ϕ_1 dans un espace de Hölder à indice positive.
- D'une autre part, on montre que même si la convolution stochastique est singulière, on peut motiver ces estimations et voir le côté pratique. Effectivement, les théorèmes 5.5.1 et 5.5.3 concerne une généralisation de l'étude de la dynamique près d'une bifurcation de fourche, étudiée auparavant pour des EDS dans [7]. Suivant une démarche similaire à celle utilisée sur le tore de dimension 1, on divise la solution en une partie constante en espace et une autre oscillante. L'étude de chacune se résume de la façon suivante: La dynamique de la partie oscillante a un comportement analogue à celle près d'un équilibre stable alors que l'autre montre que les solutions quittent un voisinage de l'équilibre instable de taille σ à un temps d'ordre $\sqrt{\varepsilon \log(\sigma^{-1})}$.

Introduction

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1.1 Goal of the thesis

Metastability is a dynamical phenomenon that is observed in a large variety of situations in nature. A classical example is a glass of water, suddenly exposed to an environment of below-freezing temperature: its content may stay liquid for a very long time, unless the glass is shaken, in which the water freezes instantly and we are again in an equilibrium state. It is called the supercooled water. Generally speaking, metastability is a common physical phenomenon in which a system spends a long time in metastable states before reaching its equilibrium.

A metastable state can be illustrated in another example, considering a hill on a golf course. The valley around that hill can be related to the stable states while the top of the hill represents a metastable state. If we hit the ball lightly from the right valley, it remains where it is. If we hit the ball harder, it crosses the hill and reaches the second valley. But if we hit it such that it lands at the top of the hill and stays there, even the slightest perturbations, like the wind, will make the ball fall into one of the valleys. The hill top is the metastable state because, even if the ball can reach this state, it can't stay there infinitely because of the different perturbations to which it is subjected. We know that the ball will eventually fall but we don't know when.

It is then interesting to describe the long time behaviour of such systems. The first mathematical description of metastable systems was introduced by Eyring [22] and Kramer [36] in the context of chemical reactions where they consider a one-dimensional diffusion process in a double-well potential as a model. Then, it set the way for the study of metastability as a phenomenon that occurs in dynamical stochastic models or stochastic processes. This is the framework considered in this thesis.

Many of the systems that display metastability are described by ordinary differential equations (ODEs) or partial differential equations (PDEs). In this manuscript, we are interested in physical systems that can be modeled by PDEs of the form

$$\partial_t \phi(t, x) = \Delta \phi(t, x) + F(\varepsilon t, \phi(t, x)) , \quad (1.1.1)$$

where ϕ is the state of the system, Δ is the Laplacian, εt denotes a slow time and F is a non-linearity depending on ϕ and periodic in the time variable. This model does not produce any interesting dynamics from our point of view: we are interested in systems subjected to two kinds of perturbations, in particular in the combined effect of them. These perturbations are

- A deterministic periodic driving force, which describes the change of an exterior influence, such that a control parameter in an experiment, or the periodic variation of insolation, which depends on the Earth's orbital parameters.
- An additive noise, which models the random influence of internal perturbation in the system such that the thermal fluctuations in laboratory experiments, or the random influence of the weather in climate models.

The perturbation of (1.1.1) by an additive noise can be modeled by a stochastic partial differential equation (SPDE) of the form

$$\partial_t \phi(t, x) = \Delta \phi(t, x) + F(\varepsilon t, \phi(t, x)) + \sigma \xi(t, x), \quad (1.1.2)$$

where σ measures the noise intensity and ξ denotes the space-time white noise that can be understood as a Gaussian random forcing acting independently at different points in space and time. Since it does not belong to any classical functions space because of its roughness, the SPDE is not always well-posed. A way around this difficulty is developed later.

An example of system displaying metastability is the periodically forced Allen–Cahn equation perturbed by a small noise term. It is given by the stochastic PDE

$$\partial_t \phi(t, x) = \Delta \phi(t, x) + \phi(t, x) - \phi(t, x)^3 + A \cos(\varepsilon t) + \sigma \xi(t, x). \quad (1.1.3)$$

For $A = 0$ and $\sigma = 0$, the deterministic version of this equation was studied by W. Cahn and Sam Allen to model phase separation in multi-component alloys in [1] and by Nathaniel Chafee and Ettore Infante in [18] to describe PDEs admitting bifurcations. A particularity of this PDE is that it admits three stationary solutions: two stable ones ± 1 and an unstable equals to 0 in which ϕ changes its sign. Stable solutions can represent a mixture of two different fluids, such as oil and water, and we interpret the change of sign as interfaces between two different phases. It is natural to ask how long a small noise intensity needs to move the system from one of these stable states to the other one. In that case, the equilibrium branch loses its stability and undergoes a bifurcation. Interesting and unexpected phenomena may occur, for instance stochastic resonance and bifurcation delay that we discuss later.

This thesis concerns the sample-path behaviour of stochastic processes, which are solutions of (1.1.2). Our approach is based on the concept of first exit times. Let E be a functional space and $\mathcal{A}$ be a bounded open subset of $\mathbb{R}^+ \times E$. Fix an initial condition $(t_0, \phi_0) \in \mathcal{A}$, the exit problem consists in characterising the laws of first-exit time

$$\tau_{\mathcal{A}} = \inf \{t \geq 0 : (t, \phi(t, \cdot)) \notin \mathcal{A}\}$$

For each realisation w of the noise, $\tau_{\mathcal{A}}(w)$ is the first time where the sample path $\phi(w)$ leaves $\mathcal{A}$. The theory of large deviations is the first step in the study of the exit problems. It gives an exponential asymptotics on the probability of rare events like transitions between potential wells, by minimising the so-called rate function over the set of all possible escape paths. There are more precise mathematical results using other methods, obtained for one-dimensional SDEs, see for instance [10]. A generalisation to the infinite dimensional setting of sample-path method introduced in that work is developed below.

We show that paths under appropriate conditions on the different parameters are concentrated in a certain neighbourhood of the equilibrium branch with an exponentially small probability. In the sense that for any $h > 0$ measuring the width of $\mathcal{A}$, there exists a positive constant κ such that for all $t \geq 0$,

$$\mathbb{P}\{\tau_{\mathcal{A}} < t\} \leq C(t, \varepsilon) e^{-\kappa h^2 / \sigma^2},$$

where σ is the constant in front of the noise in (1.1.2).

1.2 Stochastic resonance

Stochastic resonance (SR) was initially introduced in the context of climate science in order to explain the close-to-periodic appearance of the major Ice Ages [3, 39]. Since then, SR has shown up in a large number of physical and biological systems, including lasers, quantum electronics [48] and neuroscience [38].

It can be illustrated in a simple model: we consider an overdamped particle in a double well potential subjected to two different perturbations: a deterministic periodic driving force as well as an additive noise. The periodic forcing, with its small amplitude, is not sufficient to allow transitions between the wells in the absence of noise. While the additive noise, without the periodic forcing, allows the particle to switch from one potential well to the other at random times. When both perturbations are combined and their amplitudes suitably tuned, the particle will flip back and forth between the wells in a close to periodic way.

Then, SR is a mechanism that occurs when a bistable or multistable dynamical system is forced periodically in time, while also subjected to noise. When the forcing period is close to the typical time needed by the noise to move the system from one metastable state to another one, large-amplitude, periodic oscillations may occur, hence the name of resonance. Even if this resonance condition is not exactly met, the response of the system shows a trace of the periodic forcing in its power spectrum.

The most mathematical results on SR have been obtained for one-dimensional SDEs of the form

$$dX_t = f(\varepsilon t, X_t) dt + \sigma dW_t, \quad (1.2.1)$$

where W_t is a standard Wiener process, and f is a time-periodic bistable drift term. One can model the dynamics of an overdamped particle in a double-well potential by

$$f(\varepsilon t, x) = x - x^3 + A \cos(\varepsilon t) = -\frac{\partial}{\partial x} \left(\frac{1}{4}x^4 - \frac{1}{2}x^2 - A \cos(\varepsilon t)x \right).$$

Whenever A is smaller than a critical value A_c , the drift term vanishes in three different values of x , which correspond to equilibrium states of the system with a frozen value of εt . These states are also critical points of the double-well potential $V(x, \varepsilon t) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - A \cos(\varepsilon t)x$, where the middle point is the unstable saddle, and the two outer points are stable potential minima.

Since the SR involve several parameters such noise intensity σ , amplitude A and frequency ε of the forcing, many of its aspects are not yet fully understood. The first investigations of SR in systems of the form (1.2.1) focused on the case of small amplitude A [24, 25, 35], but many other parameter regimes have been considered as well (see [32] for an overview of mathematical results). We are mainly interested in the case where A is slightly smaller than A_c and for small and finite parameter σ and ε .

A precise description of the behaviour of paths and on the optimal noise intensity as a function of the driving frequency and the minimal barrier height, guaranteeing periodic oscillation between the

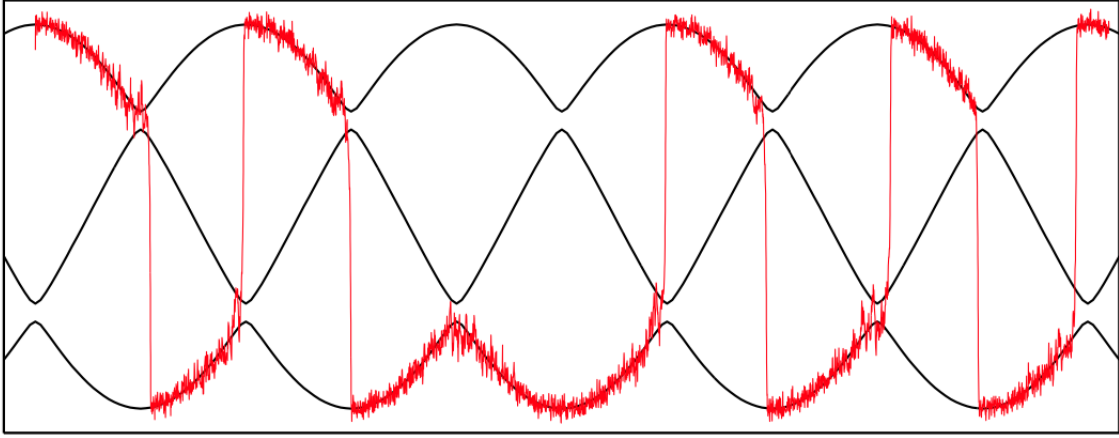


FIGURE 1.1. A solution of (1.2.1) in an asymmetric potential. The upper and lower curves represent the position of the wells, while the middle curve indicate the saddle.

two wells is given in [8] for one-dimensional SDEs of the form (1.2.1). One can study the dynamics near uniformly hyperbolic, stable or unstable equilibrium branches as one can also see other situations, arising when an equilibrium branch undergoes a bifurcation point (loss of hyperbolicity) or an avoided bifurcation in which equilibrium branches come close without actually touching and it can be sufficient to produce transitions. Let us resume their results. Without noise, when $\sigma = 0$, solutions of the PDE (1.2.1) remain in the same potential well and never approach the saddle closer than a distance of order $\sqrt{\varepsilon}$. When adding noise, paths, near a stable equilibrium branch, remain close to the deterministic solutions where transitions between potential wells are very rare. When approaching a transcritical bifurcation or an avoided bifurcation, transitions between the potential wells may occur and depend on the noise intensity. For small values of σ , the probability of overcoming the barrier is exponentially small and have similar behaviours as in a neighbourhood of a stable equilibrium branch. For larger σ , it is very likely that the system goes back and forth between the local minima twice per period. The probability of jumping from the less deep potential well to the deeper one is exponentially close to 1, while paths are unlikely to come back, see Figure 1.1.

Extending the methods developed in [8] to infinite-dimensional SPDEs of the form (1.1.2) is developed later in this thesis. A first step towards this generalisation has been taken in [26]. However, that work considers noise that is coloured in space and white in time, given by a Q -Wiener process with trace class covariance, while we consider a more difficult situation of space-time white noise.

1.3 Bifurcation delay

Another interesting situation arises when a single-well potential transforms into a double well potential as time increases, the overdamped particle will fall into one of the new wells or it remains in the unstable equilibrium, on the top of the barrier, see Figure 1.2. It can be described by an SDE of the form (1.2.1) where f undergoes a pitchfork bifurcation, at $(0, 0)$, where all equilibria meet.

In the deterministic case, when $\sigma = 0$, the system displays what is called bifurcation delay: solutions

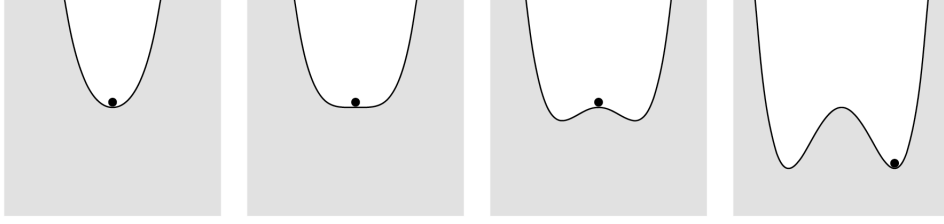
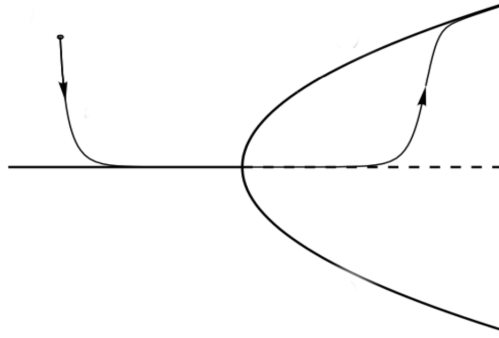


FIGURE 1.2. Overdamped particle in a potential undergoing a pitchfork bifurcation.

FIGURE 1.3. A deterministic solution is attracted by $x = 0$ and stays close to the origin for a finite time after a bifurcation: phenomenon known as bifurcation delay.

attracted by the stable equilibrium branch (the bottom of the well) for $t < t^*$ remain close to 0 for a time of order 1 beyond the bifurcation time t^* , even though the equilibrium branch has become unstable (broken line in the Figure 1.3). This is due to the solution becoming exponentially close to 0 during the stable phase, and a time of order 1 being required for the solution to reach again values of order 1, see Figure 1.3. On the other hand, adding white noise may affect the delayed jump which accompanies the slow passage through a pitchfork bifurcation. In fact, the fluctuations around the saddle push the paths away from it, which decreases the bifurcation delay. In the one-dimensional SDE case, the effect of noise on such system has been studied in [7]. The main result of that work is that sample path remain with high probability at a distance of order $\sigma\varepsilon^{-1/4}$ from zero up to a time $t^* + \mathcal{O}(\varepsilon^{1/2})$, but are unlikely to remain close to 0 after times of order $t^* + \mathcal{O}((\varepsilon \log(\sigma^{-1}))^{1/2})$. The effect of noise is thus to reduce the bifurcation delay from order 1 to order $(\varepsilon \log(\sigma^{-1}))^{1/2}$. We develop in Chapter 5 similar results on bifurcation delay for infinite-dimensional SPDEs of the form (1.1.2).

1.4 Singular SPDEs and renormalisation

We consider in this thesis parabolic SPDEs of the form (1.1.2) that describe the time-evolution of spatially extended systems subjected to random driving. As we already mention, these SPDEs are not always well-posed due to the irregularity of the space-time white noise. A process known as renormalisation is

needed to define a notion of solution. It consists on subtracting an "infinite constant" from the right hand side of (1.1.2) to encounter the singularity. In order to understand the well-posedness problem, we first discuss a classical example: the stochastic heat equation on $\mathbb{R}_+ \times \mathbb{T}^d$

$$\partial_t \phi(t, x) = \Delta \phi(t, x) + \sigma \xi(t, x), \quad (1.4.1)$$

with initial condition $\phi(0, x) = 0$ for all $x \in \mathbb{T}^d$. It can be solved by the method of variation of constant, also known as Duhamel principle in the theory of PDEs:

$$\phi(t, x) = \sigma \int_0^t (e^{(t-s)\Delta} \xi)(s, x) ds = \sigma \int_0^t \int_{\mathbb{T}^2} P(t-s, x-y) \xi(s, y) dy ds,$$

where $P(t, x)$ is a periodicised version of the heat kernel, since we are working in $\mathbb{T}^d = (\mathbb{R}/(L\mathbb{Z}))^d$. In fact, this term is called the stochastic convolution $(P * \xi)(t, x)$, where the star here denotes the space-time convolution. One can see it as applying ξ to a test function, knowing that P is not strictly a test function since it has a singularity at the origin.

One can show that space-time white noise on the d -dimensional torus belongs to $\mathcal{C}^\alpha$ for any $\alpha < -(d+2)/2$, with respect to the parabolic metric $\sqrt{|t-t'|} + |x-x'|$ between space-time points. Then, convolving ξ against the heat kernel should gain two derivatives so we would guess that the stochastic convolution has regularity $(2-d)/2$, see for instance [5]. When $d = 1$, the stochastic convolution belongs to $\mathcal{C}^\alpha$ for any $\alpha < 1/2$ which is well-defined function. While for $d \geq 2$, it belongs to Hölder spaces with negative index, which is distribution-valued.

We turn out to a more general SPDE of the form (1.1.2) which solution is given by the fixed point equation with initial condition $\phi(0, x) = 0$ for all $x \in \mathbb{T}^d$,

$$\phi(t) = \int_0^t e^{(t-s)\Delta} F(\varepsilon s, \phi(s)) ds + \sigma \int_0^t e^{(t-s)\Delta} d\xi(s), \quad (1.4.2)$$

The existence of such solution via Banach's fixed point theorem needs a certain regularity in a sense that the right hand side of (1.4.2) belongs to a functional space. This can be problematic because the stochastic convolution is the most irregular object in (1.4.2) and it is not necessarily a function. Due to the embeddings between Besov, Hölder and Sobolev spaces, we analyse in Chapter 3 the stochastic convolution in fractional Sobolev spaces where we show that it belongs to H^s for any $s < 1/2$. Therefore, on the one-dimensional torus, the stochastic convolution is a well-defined function. Then, on the two-dimensional torus, we analyse it in Besov spaces and one can see it as a distribution in $\mathcal{B}_{2,\infty}^\alpha$ for any $\alpha < 0$. This poses a problem when trying to study the non-linearity F evaluated in the difference between the solution and the stochastic convolution. There is no canonical way of defining the product of distributions. It is known that given two distributions $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, the product fg can be defined as a bilinear and continuous form if and only if $\alpha + \beta > 0$.

A way around this difficulty in dimension 2 was found by Giuseppe Da Prato and Arnaud Debussche in the landmark work [42]. They considered renormalised equation of the form

$$d\phi(t, x) = [\Delta \phi(t, x) + :F(\varepsilon t, \phi(t, x)):] dt + \sigma dW(t, x), \quad (1.4.3)$$

where $:F:$ denotes Wick renormalisation detailed later. An example of F is

$$:F(\varepsilon t, \phi(t, x)):= : \phi(t, x)^3 : = \phi(t, x)^3 - 3C_N \phi(t, x)$$

where C_N is a logarithmically divergent constant depending on a cut-off N .

The main idea of their approach is to write an equation for the difference ϕ_1 between the solution and the stochastic convolution, which solves a linear equation. The observation is that ϕ_1 belongs to a Besov space with a better regularity. This means that the solution of the renormalised equation (1.4.3) differs from the stochastic convolution by a smoother object, see [20]. While the method has been spelled out for time-independent systems, extending it to time-dependent equations is straightforward.

The work [20] has later given rise to far-reaching generalisations, that allow to solve large classes of singular SPDEs. These generalisations include the theory of regularity structures, introduced by Martin Hairer in the work [28] and further developed with Ajay Chandra, Yvain Bruned, Ilya Chevyrev and Lorenzo Zambotti in [16, 17, 19], and the theory of paracontrolled distributions, introduced in [27] by Massimiliano Gubinelli, Peter Imkeller and Nicholas Perkowski. Most of these more general singular SPDEs require more refined renormalisation methods than Wick renormalisation.

For time-independent versions of the equation (1.4.3) on the two-dimensional torus, many results going beyond well-posedness and existence/uniqueness of solutions have been obtained. For instance, the fact that their solutions satisfy the Markov property and are reversible with respect to the Gibbs measure was proved in [44] using Dirichlet forms, while uniqueness of the Gibbs measure and convergence to it were obtained in [43]. The fact that solutions satisfy the strong Feller property and are exponentially mixing was shown in [46] using a dissipative bound, while the strong Feller property was also proved (for more general equations) in [30], using the theory of regularity structures. The work [31] provided a large-deviation principle, valid for a class of two- and three-dimensional singular SPDEs. In the particular case of the Allen–Cahn equation, sharper asymptotics on transition times between metastable states than those provided by large-deviation estimates have been obtained in [6] and [47].

In this thesis, we are interested in obtaining more detailed non-equilibrium properties for time-dependent renormalised SPDEs on the two-dimensional torus similarly to the properties obtained for SPDEs on the one-dimensional torus, see Chapter 3 and 4. We are interested in results concerning the motion near the so-called stable equilibrium branches of the system. These are curves of the form $t \mapsto \phi^*(t, x)$ on which the right-hand side of the equation vanishes in the absence of noise. In the other hand, we want results concerning certain situations involving bifurcations, or avoided bifurcations. These occur when the equilibrium branch $t \mapsto \phi^*(t, x)$ (almost) loses stability at some time, usually because of the presence of a nearby unstable equilibrium branch. This can lead to an interesting phenomena such as stochastic resonance, where solutions of the equation make fast jumps in a close-to-periodic way discussed on the one-dimensional torus in [13]. Or bifurcation delay when solutions keep tracking a branch of unstable equilibria for some time after a bifurcation has occurred and which will be discussed for renormalised SPDEs in Chapter 5.

1.5 Results and perspectives

On the one-dimensional torus $\mathbb{T}$, the results can be summarised as follows. We recall the stochastic Allen–Cahn equation (1.1.3) where the time-periodic drift term describes a bistable situation and vanishes on three branches, two of which come close to each other or meet once per period. The minimal distance between the branches at these close encounters is measured by a small parameter δ , which corresponds to $A_c - A$. The results apply to more general drift terms satisfying certain conditions detailed later. We point out that the system depends on three small parameters ε , σ and δ and the relation between them

determines the transition probability.

- Theorem 3.3.1 states that as long as the equilibrium branches are well-separated, solutions of the SPDE (1.1.2) are likely to remain close to deterministic solutions tracking the stable branches. Closeness is measured in the H^s Sobolev norm, where s is strictly smaller than $\frac{1}{2}$, but can be arbitrarily close to $\frac{1}{2}$.
- When equilibrium branches become close to each other, we decompose the solution $\phi(t, x)$ into its spatial mean $\phi_0(t)$, and its zero-mean transverse part $\phi_\perp(t, x)$. Theorem 4.3.1 says that the conclusion of Theorem 3.3.1 remains valid at bifurcation points for the transverse part.
- The behaviour of the spatial mean $\phi_0(t)$ depends on the value of the noise intensity σ . Theorem 4.3.4 implies that in the weak-noise regime $\sigma \ll \sigma_c = (\delta \vee \varepsilon)^{3/4}$, sample paths are still likely to remain close to the same stable equilibrium. The probability of making a transition to the other stable equilibrium is exponentially small in σ_c^2/σ^2 .
- In the strong-noise regime $\sigma \geq \sigma_c = (\delta \vee \varepsilon)^{3/4}$, transitions between equilibrium branches become more likely. Theorem 4.3.5 implies that the probability not to make a transition to the other stable equilibrium when approaching an avoided bifurcation point decays roughly like $\exp[-\sigma^{4/3}/(\varepsilon \log(\sigma^{-1}))]$.

Our results thus show that similarly to the one-dimensional situation considered in [8], depending on the noise intensity, transitions between stable equilibria are either exponentially rare, or happen with a probability exponentially close to 1. There are some differences in the error terms, which are due to the fact that we have to deal with the transverse part $\phi_\perp$ of the solution. The main difficulty of the analysis comes from the fact that we work with space-time white noise in an infinite-dimensional situation. This prevents us from applying directly the methods from [9], which work in finite dimension, and include dimension-dependent error terms. These error estimates can be adapted to trace class noise, as was done in [26], but the white noise case needs a different approach, relying on more careful estimates in various Sobolev norms. Key results are an estimate for a linearised equation based on the Fourier decomposition, presented in Section 3.3, and a Schauder estimate given in Lemma 3.3.7.

On the two-dimensional torus $\mathbb{T}^2$, similar non-equilibrium properties are expected for renormalised SPDEs of the form (1.4.3).

- Theorem 5.3.12 shows that the Wick powers of the stochastic convolution remain concentrated, with high probability, in a neighborhood of a stable equilibrium branch $\{\phi^*(t, x)\}_{0 \leq t \leq T}$. The distance is measured in the Besov norm $\|\cdot\|_{\mathcal{B}_{2,\infty}^\alpha}$ for any $\alpha < 0$. Theorem 5.4.7 shows that similar concentration estimates remain valid for the difference ϕ_1 between the solution of the SPDE and the stochastic convolution but in a stronger Hölder norm of positive index.
- Despite this concentration result, one may be concerned that it is of little practical use, because it does only concern the difference between a solution and the more singular stochastic convolution. Theorems 5.5.1 and 5.5.3 show that this is not the case by discussing the particular situation of a dynamic pitchfork bifurcation, which was previously considered in [7] for one-dimensional SDEs. We show that until some time after the bifurcation, the solutions are concentrated in a region of size $\sigma \varepsilon^{-1/4}$ around the bifurcating equilibrium. They leave a neighbourhood of this equilibrium at times of order $\sqrt{\varepsilon \log(\sigma^{-1})}$ and they are likely to stay close to the deterministic solution.

Finally, it would be of interest if one can have answers to other questions that may arise.

- In Chapter 2 and 3, we give concentration estimate in Sobolev spaces. Could we have similar

exponential decays in other spaces like Besov spaces?

- When a system admits a bifurcation point, several phenomena may occur. We studied in this thesis two of them: SR in one-dimensional SPDEs when the system approaches a transcritical bifurcation and bifurcation delay in two dimensional SPDEs where it approaches a pitchfork bifurcation. One can study the effect of noise on dynamical hysteresis appearing in SPDEs where the state of a periodically forced system, for a given value of the forcing can depend on whether the forcing increases or decreases.
- As we have seen, to encounter the problem of well-posedness of two-dimensional SPDEs, we introduce a renormalisation process. If we consider the three dimensional Allen-Cahn equation (1.1.3) where x belongs to $\mathbb{T}^3$, it is known that the deterministic PDE is well-defined. Therefore, the stochastic PDE remains ill-posed, even when the nonlinear term is replaced by its Wick power. This is a consequence of space-time white noise being more singular in three than in two space dimensions. There exist several methods to make sense of the singular equation after renormalisation. One of them is the regularity structures, see [28]. Once the SPDE is well-defined, could we expect concentration estimates like those obtained in dimensions one and two?

1.6 About this thesis

We provide now a more detailed account of the content of each chapter.

- In Chapter 2, we recall some known inequalities which have been used throughout this thesis. We introduce Besov and Sobolev spaces which are important in solving singular SPDEs. We define rigorously the space-time white noise, since it is the most irregular object in the study. Since in dimension greater or equal to 2 equations are not well-posed, they require a renormalisation procedure to make sense. An important tool to understand how it works is Wick calculus for Gaussian random variables and its relation with Hermite polynomials.
- Chapter 3 is dedicated to the one-dimensional Allen-Cahn SPDE. It contains a description on the deterministic dynamics and a precise concentration estimate on the behaviour of the solutions near a stable equilibrium branch. We divide the work in two steps, we start by evaluating the linear version of the equation and then we consider more general drift terms.
- In Chapter 4, we pursue the study started in Chapter 3 with the only difference that we assume that the system admits a bifurcation point or an avoided bifurcation. We describe the effect of the stochastic resonance by describing the paths behaviour. We thus distinguish two different regimes depending on the noise intensity where the dynamics are quite different. A discussion on the optimal parameter one can obtain, ends the study in the one-dimensional setting.
- The final Chapter 5 concerns two-dimensional singular SPDEs. As we have seen previously these equations are not well-posed and require a renormalisation. We give a precise description on the renormalised equation. Then, with the Da Prato-Debussche trick, we obtain a solution in a space with better regularity and we give concentration estimates around a stable equilibrium branch. We end up discussing a case involving a pitchfork bifurcation.

Chapter 5 is completed in Appendix A, that contains technical proofs.

Table 1.1 lists some notations frequently used throughout this manuscript.

Symbol	Meaning
$ x $	absolute value, l^1 norm
$\ x\ $	Euclidean norm
$\mathcal{B}_{p,r}^\alpha$	Besov space
$\mathcal{C}^\alpha$	Hölder space
H^s	fractional Sobolev space
$\mathbb{P}^{t_0, \phi_0}$	probability with initial value
$\mathbb{E}^{t_0, \phi_0}$	expectation with initial value
$\lceil y \rceil$	smallest integer which is greater than or equal to y
$y \vee z$	maximum
$y \wedge z$	minimum
$X \lesssim Y$	bounded up to a constant
$X \asymp Y$	$X \lesssim Y$ and $Y \lesssim X$
$X = \mathcal{O}(Y)$	Landau symbol
$\langle k \rangle^2 = (1 + k^2)$	Japanese bracket's
$1_{\mathcal{D}}(x)$	indicator function

TABLE 1.1. Frequently used notations.

Preliminaries

In this chapter, we collect some classical results and useful tools which we will need in the sequel. It will be important to define the spaces to which belongs our different objects: Besov spaces are the most used when dealing with singular SPDEs. It coincides with Sobolev spaces for particular parameters. Since space-time white noise is an irregular object, we define it properly to understand the reason for the singularity. We recall in the end some properties on Hermite polynomials and its relation with the Wick calculus for Gaussian random variables.

Dans ce chapitre, on rappelle quelques résultats classiques qu'on utilise dans les preuves des résultats évoqués plutard. On définit les différents espaces fonctionnels auxquels appartiennent les différents objets: les espaces de Besov et Sobolev. Ces espaces sont utilisés le plus souvent pour l'étude des EDPS singulières. On donne une définition du bruit blanc espace-temps et quelques propriétés, comme étant l'objet le plus irrégulier dans l'étude. Et enfin, comme les EDPS se sont pas toujours bien posées une renormalisation est demandée. Cette procédure est basée sur les calculs de Wick pour les variables aléatoires gaussiennes qui ont des propriétés similaires aux polynomes d'Hermite qu'on rappelle.

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2.1 Useful inequalities

We recall the following fundamental inequalities which we use constantly whose proof is given in [2].

Lemma 2.1.1 (Hölder's inequality). *Let $p, q, r \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then*

$$\|uv\|_{L^r} \leq \|u\|_{L^p} \|v\|_{L^q} .$$

Lemma 2.1.2 (Young's inequality for convolution). *Let $p, q, r \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then*

$$\|u * v\|_{L^r} \leq \|u\|_{L^p} \|v\|_{L^q} .$$

We recall the following inequality that allows to estimate the probability of a positive submartingale surpassing a level during the time interval $[0, t]$ depending on its expected value at the endpoint, see [10].

Lemma 2.1.3 (Doob's submartingale inequality). *Let $\{M_t\}_{t \geq 0}$ be a positive submartingale with continuous paths. Then, for any $L > 0$ and $t > 0$,*

$$\mathbb{P}\left\{\sup_{0 \leq s \leq t} M_s \geq L\right\} \leq \frac{1}{L} \mathbb{E}[M_t] .$$

A consequence of this result is

Lemma 2.1.4 (Bernstein-type inequality). *Let $\varphi(u)$ be a Borel-measurable deterministic function such that*

$$\Phi(t) = \int_0^t \varphi(u)^2 du$$

exists. Then

$$\mathbb{P}\left\{\sup_{0 \leq s \leq t} \int_0^s \varphi(u) dW_u \geq \delta\right\} \leq \exp\left\{-\frac{\delta^2}{2\Phi(t)}\right\} . \quad (2.1.1)$$

A variant of Gronwall's lemma, not requiring a differentiability is

Lemma 2.1.5 (Gronwall's inequality). *Let $\phi, \psi : [0, \infty) \rightarrow [0, \infty)$ be continuous functions, satisfying*

$$\phi(t) \leq L + K \int_0^t \psi(s) \phi(s) ds$$

for all $t \geq 0$ with positive constants K and L . Then,

$$\phi(t) \leq L \exp\left\{K \int_0^t \psi(s) ds\right\} \quad (2.1.2)$$

for all $t \geq 0$.

2.2 Fractional Sobolev space and inequalities

Let L be a positive constant. We define, on the one-dimensional torus, an orthonormal basis $\{e_k\}_{k \in \mathbb{Z}}$ of $L^2(\mathbb{T}, \mathbb{R})$ by

$$e_k(x) = \begin{cases} \sqrt{\frac{2}{L}} \cos\left(\frac{k\pi x}{L}\right) & \text{if } k > 0, \\ \frac{1}{\sqrt{L}} & \text{if } k = 0, \\ \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi x}{L}\right) & \text{if } k < 0, \end{cases} \quad (2.2.1)$$

where $x \in \mathbb{T} = [0, L]$ with periodic conditions. Note that $\langle e_k, e_l \rangle = \delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$.

Remark 2.2.1. It might seem more convenient to use a complex Fourier basis of the form

$$e_k(x) = \frac{1}{\sqrt{L}} e^{2ik\pi x/L} \quad (2.2.2)$$

when analysing non-linear terms. Real Fourier series are usually used when dealing with linear equations. $\diamond$

When working with complex Fourier basis of the form (2.2.2), we have the following properties

$$\begin{aligned} e_{k_1}(x)e_{k_2}(x) &= \frac{1}{\sqrt{L}} e_{k_1+k_2}(x) \quad \forall k_1, k_2 \in \mathbb{Z}^2, \\ e_k(x_1)e_k(x_2) &= \frac{1}{\sqrt{L}} e_k(x_1 + x_2) \quad \forall k \in \mathbb{Z}^2, \\ \bar{e}_k(x) &= e_{-k}(x) \quad \forall k \in \mathbb{Z}^2. \end{aligned}$$

While on the two-dimensional torus, we define the orthonormal basis $\{e_k\}_{k \in \mathbb{Z}^2}$ of $L^2(\mathbb{T}^2, \mathbb{R})$, by

$$e_k(x) = e_{k_1}(x_1)e_{k_2}(x_2), \quad k = (k_1, k_2) \in \mathbb{Z}^2, \quad (2.2.3)$$

where e_{k_i} are the one-dimensional basis functions defined above by (2.2.1) or (2.2.2) depending on whether we want to work in complex Fourier basis or not. We use the same symbol for one and two-dimensional basis functions to not overload the notation. We write the expansion of $\phi \in L^2(\mathbb{T}^d)$ in a Fourier basis

$$\phi(x) = \sum_{k \in \mathbb{Z}^d} \phi_k e_k(x).$$

Fourier series are intimately related to the scale of fractional Sobolev spaces (also called Bessel potential spaces).

Definition 2.2.2. (Fractional Sobolev spaces). For $s \geq 0$, the fractional Sobolev space $H^s(\mathbb{T}, \mathbb{R})$ is given by the subspace of functions $\phi \in L^2(\mathbb{T}, \mathbb{R})$ such that

$$\|\phi\|_{H^s}^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \phi_k^2 < \infty.$$

Note that H^s is a Hilbert space, and $H^0 = L^2$.

We recall some useful inequalities on Sobolev spaces $H^s(\mathbb{T}, \mathbb{R}) = H^s$:

Proposition 2.2.3 (Sobolev embedding). For any $p \geq 2$ and $s > \frac{1}{2} - \frac{1}{p}$, there exists a finite constant $C_{Sob}(s, p)$ such that

$$\|\psi\|_{L^p} \leq C_{Sob}(s, p) \|\psi\|_{H^s}. \quad (2.2.4)$$

An estimate on the product of two functions in Sobolev spaces is given by the following lemma. A concise proof can be found in [14, Théorème 7].

Lemma 2.2.4 (Product in Sobolev spaces). If $s > \frac{1}{2}$ then there is a bilinear application

$$\begin{aligned} H^s \times H^s &\longrightarrow H^s \\ (\psi, \phi) &\longmapsto \psi\phi, \end{aligned}$$

which coincides with the pointwise product and satisfies the estimate

$$\|\psi\phi\|_{H^s} \leq C \|\psi\|_{H^s} \|\phi\|_{H^s}$$

for some finite constant $C = C(s)$.

While the above result does not hold if $s \leq \frac{1}{2}$, we have the following consequence of Young's inequality, a proof of which can be found, for instance, in [11, Lemma 4.3].

Lemma 2.2.5 (Young-type inequality). *Let $r, s, t \in (0, \frac{1}{2})$ be such that $t < r + s - \frac{1}{2}$. Then there exists a finite constant $C = C(r, s, t)$ such that*

$$\|\psi * \phi\|_{H^t} \leq C \|\psi\|_{H^r} \|\phi\|_{H^s} < \infty. \quad (2.2.5)$$

2.3 Besov spaces and inequalities

Measuring the size of each Fourier coefficient separately does not provide enough information. Instead it is more useful to group the different frequency ranges into blocks, known as dyadic Fourier blocks.

Definition 2.3.1 (Besov spaces). *Let ϕ admit the Fourier series (2.2.1). We define a collection of annuli by setting $\mathcal{A}_0 = \{(0, 0)\}$ and $\mathcal{A}_q = \{k \in \mathbb{Z}^2 : 2^{q-1} \leq |k| < 2^q\}$ for any $q \in \mathbb{N}$. The projection of ϕ on $\mathcal{A}_q$ is defined by*

$$\delta_q \phi(x) := \sum_{k \in \mathcal{A}_q} \phi_k e_k(x).$$

For $\alpha \in \mathbb{R}$ and $p, r \in [1, \infty]$, define the norm

$$\begin{aligned} \|\phi\|_{\mathcal{B}_{p,r}^\alpha} &:= \left\| \left\{ 2^{rq\alpha} \|\delta_q \phi\|_{L^p} \right\}_{q \geq -1} \right\|_{\ell^r} \\ &:= \begin{cases} \left(\sum_{q \geq 0} 2^{rq\alpha} \|\delta_q \phi\|_{L^p} \right)^{1/r} & \text{if } 1 \leq r < \infty, \\ \sup_{q \geq 0} 2^{q\alpha} \|\delta_q \phi\|_{L^p} & \text{if } r = \infty. \end{cases} \end{aligned} \quad (2.3.1)$$

Then the Besov space $\mathcal{B}_{p,r}^\alpha = \mathcal{B}_{p,r}^\alpha(\mathbb{T}^2)$ is defined as the set of all ϕ such that $\|\phi\|_{\mathcal{B}_{p,r}^\alpha} < \infty$.

The Besov space $\mathcal{B}_{p,r}^\alpha$ is a Banach space for all $\alpha \in \mathbb{R}$ and $p, r \in [1, \infty]$. In particular,

$$\mathcal{C}^\alpha := \mathcal{B}_{\infty,\infty}^\alpha \quad \text{and} \quad H^\alpha := \mathcal{B}_{2,2}^\alpha$$

coincide with the usual Hölder and (fractional) Sobolev spaces respectively.

We recall the classical Besov embeddings, see for instance [37].

Proposition 2.3.2. *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Then B_{p_1,q_1}^α is continuously embedded in $B_{p_2,q_2}^{\alpha - d(\frac{1}{p_1} - \frac{1}{p_2})}$.*

Proposition 2.3.3. *For any $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 < \alpha_2$, any $p, q \in [1, \infty]$ the Besov space $B_{p,q}^{\alpha_1}$ is compactly embedded into $B_{p,q}^{\alpha_2}$.*

The following lemma shows that the product of two functions in Besov spaces is not always well-defined.

Lemma 2.3.4 (Product in Besov spaces). *Let $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha + \beta > 0$. Then there exists a bilinear map $B : \mathcal{C}^\alpha \times \mathcal{C}^\beta \longrightarrow \mathcal{C}^{\alpha \wedge \beta}$ with the following properties:*

1. *If $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$ are continuous functions, then*

$$B(f, g)(z) = f(z)g(z).$$

2. *For arbitrary $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, one has the bound*

$$\|B(f, g)\|_{\mathcal{C}^{\alpha \wedge \beta}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta}.$$

If $\alpha + \beta \leq 0$, then no bilinear map satisfying these two properties exists.

2.4 Space-time white noise and stochastic convolution

We turn now to the precise definition of the space-time white noise process ξ formally introduced in (1.1.2). It should have the following properties:

- Each $\xi(t, x)$ should be a Gaussian random variable.
- Each $\xi(t, x)$ should be centered.
- The values of ξ at different space-time points should be independent, and thus uncorrelated, which is sometimes written informally as

$$\mathbb{E}[\xi(t, x)\xi(s, y)] = \delta(t - s)\delta(x - y).$$

It turns out that there is no random function having the above properties but there exists a random Schwartz distribution instead. One can consider ξ as a random linear functional acting on test functions. We denote by $\mathcal{H}$ the Hilbert space $L^2(\mathbb{R} \times \mathbb{T}^d)$. Let $\mathcal{S}'(\mathcal{H})$ be the space of Schwartz distributions, and denote by $\langle \xi, \varphi \rangle$ the duality pairing between a distribution $\xi \in \mathcal{S}'(\mathcal{H})$ and a test function $\varphi \in \mathcal{H}$.

Definition 2.4.1 (Space-time white noise). *Space-time white noise on $\mathbb{R} \times \mathbb{T}^d$ is a random distribution ξ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any smooth test function $\varphi \in \mathcal{H}$, $\langle \xi, \varphi \rangle$ is a centered Gaussian random variable of variance $\|\varphi\|_{\mathcal{H}}^2$ and the covariance is given by*

$$\mathbb{E}[\langle \xi, \varphi_1 \rangle \langle \xi, \varphi_2 \rangle] = \langle \varphi_1, \varphi_2 \rangle$$

for any two smooth test functions $\varphi_1, \varphi_2 \in \mathcal{H}$.

Since for Gaussian random variables, second moments determine all other moments. A consequence of this property is given by the following important result.

Theorem 2.4.2 (Regularity of space-time white noise). *We have $\xi \in \mathcal{C}^{-(d+2)/2-\kappa}$ for any $\kappa > 0$.*

Proof. See Theorem 2.2.8 in [5] for $d = 1$ and by making the necessary changes in that proof we find the result for higher dimensions. $\square$

We define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, generated by a stochastic process $W(t, x)$, known as cylindrical Wiener process. It sometimes denotes space-time noise, in the following sense

$$\sigma \xi(t, x) dt = \sigma dW(t, x) ,$$

with $\xi \sim \mathcal{N}(0, 1)$. The projection of the stochastic heat equation (1.4.1) on the k th basis vector gives

$$d\phi_k(t) = -\mu_k \phi_k(t) dt + \sigma dW_k(t),$$

where μ_k are the eigenvalue of the Laplacian and $W_k(t)$ is an independent Wiener process. Its solution is an Ornstein-Uhlenbeck process and it is given by

$$\phi_k(t) = \sigma \int_0^t e^{-\mu_k(t-s)} dW_k(s) .$$

We recall a periodicised version of the heat kernel on the one-dimensional torus

$$P(t, x) := \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{4\pi t}} e^{-(x-kL)^2/(4t)} \mathbf{1}_{\{t \geq 0\}}$$

Therefore, the stochastic convolution can be written as

$$(P * \xi)(t, x) = \sigma \sum_{k \in \mathbb{Z}} \int_0^t e^{-\mu_k(t-s)} dW_s^{(k)} e_k(x) ,$$

and its variance is given by

$$\begin{aligned} \mathbb{E}[(P * \xi)(t, x)^2] &= \sigma^2 \sum_{k \in \mathbb{Z}} e^{-2\mu_k t} \mathbb{E} \left[\left(\int_0^t e^{\mu_k s} dW_s^{(k)} \right)^2 \right] e_k(x)^2 \\ &= \sigma^2 \sum_{k \in \mathbb{Z}} e^{-2\mu_k t} \left(\int_0^t e^{2\mu_k s} ds \right) e_k(x)^2 \\ &= \sigma^2 \int_0^t e^{2(t-s)\Delta} ds , \end{aligned}$$

owing to Ito's isometry.

A proof of the following theorem can be found in [5].

Theorem 2.4.3 (Regularity of the stochastic convolution). *The stochastic convolution $(P * \xi)(t, \cdot)$ belongs to $\mathcal{C}^{(2-d)/2-\kappa}$ for any $\kappa > 0$.*

Remark 2.4.4. The Besov embeddings introduced above allow to extend the result on the regularity of the stochastic convolution to Sobolev spaces or to more general spaces. $\diamond$

2.5 Hermite polynomials and Wick calculus

We recall some properties of Hermite polynomials and Wick calculus needed in Chapter 5. Proofs of these properties can be found, for instance, in the monographs [40, 41], the lecture notes [29], and Section 4.2.2 and Appendix D of [5].

Hermite polynomials are defined recursively by setting

$$\begin{cases} H_0 = 1 \\ H_n = xH_{n-1} - C\partial_x H_{n-1} \quad n \in \mathbb{N} \end{cases}$$

The first few of them are given by

$$\begin{aligned} H_1(x; C) &= x, \\ H_2(x; C) &= x^2 - C, \\ H_3(x; C) &= x^3 - 3Cx, \\ H_4(x; C) &= x^4 - 6Cx^2 + 3C^2. \end{aligned}$$

We consider here Hermite polynomials with variance C . Some of the above references consider the special case $C = 1$, but results for that case can easily be converted into results for the general case by using the scaling property

$$H_n(x; C) = C^{n/2} H_n(C^{-1/2}x).$$

The first n Hermite polynomials and the monomials $1, \dots, x^n$ both form a basis of the vector space of polynomials of degree n , where the change of basis is given by the formulas

$$\begin{aligned} H_n(x; C) &= \sum_{\ell=0}^{\lfloor n/2 \rfloor} a_{n\ell} C^\ell x^{n-2\ell}, & a_{n\ell} &= \frac{(-1)^\ell n!}{2^\ell \ell! (n-2\ell)!}, \\ x^n &= \sum_{\ell=0}^{\lfloor n/2 \rfloor} b_{n\ell} C^\ell H_{n-2\ell}(x; C), & b_{n\ell} &= \frac{n!}{2^\ell \ell! (n-2\ell)!} = |a_{n\ell}|. \end{aligned} \quad (2.5.1)$$

The Hermite polynomials admit the generating function

$$G(t, x; C) := e^{tx - Ct^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x; C), \quad (2.5.2)$$

which can be used to establish the following identity.

Lemma 2.5.1 (Expectation of products of Wick powers). *Let X and Y be jointly Gaussian centered random variables, of respective variance C_1 and C_2 . Then for any $n, m \geq 0$, one has*

$$\mathbb{E}[H_n(X; C_1) H_m(Y; C_2)] = \begin{cases} n! \mathbb{E}[XY]^n & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

Another consequence of the expression (2.5.2) of the generating function is the following binomial formula.

Theorem 2.5.2 (Binomial formula for Hermite polynomials). *For any $x, y \in \mathbb{R}$, $C_1, C_2 \geq 0$ and $n \in \mathbb{N}_0$,*

$$H_n(x + y, C_1 + C_2) = \sum_{m=0}^n \binom{n}{m} H_m(x, C_1) H_{n-m}(y, C_2). \quad (2.5.3)$$

A generalisation of the binomial formula (2.5.3) is obtained by induction.

Lemma 2.5.3 (Multinomial formula for Hermite polynomials). *Let $(a_q)_{q \geq 0}$ be a sequence of real numbers in ℓ^2 . Then for any convergent sequence $(x_q)_{q \geq 0}$, one has*

$$H_m \left(\sum_{q \geq 0} x_q; \sum_{q \geq 0} a_q^2 \right) = \sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \prod_{q \geq 0} H_{\mathbf{n}_q}(x_q; a_q^2),$$

where the sum runs over all $\mathbf{n} \in \mathbb{N}_0^{\mathbb{N}_0}$ such that $|\mathbf{n}| = \sum_{q \geq 0} \mathbf{n}_q = m$, and $\mathbf{n}! := \prod_{q \geq 0} \mathbf{n}_q!$.

Given a set $\{\psi_q\}_q$ of independent centered Gaussian random variables, one defines the m th homogeneous Wiener chaos $\mathcal{H}_m$ as the vector space generated by all Wick powers of the ψ_q of total degree m , that is, all

$$\Phi_{\mathbf{n}} = \prod_{q \geq 0} H_{\mathbf{n}_q}(\psi_q; \text{Var}(\psi_q))$$

with $|\mathbf{n}| = m$. Then one has the following result on equivalence of norms, which is a consequence of hypercontractivity of the Ornstein–Uhlenbeck semigroup. See for instance [21, Theorem 4.1] or [40, Theorem 1.4.1].

Lemma 2.5.4 (Equivalence of moments). *Let X be a random variable, belonging to the m -th homogeneous Wiener chaos. Then for any $p \geq 1$ one has*

$$\mathbb{E}[X^{2p}] \leq (2p-1)^{mp} \mathbb{E}[X^2]^p. \quad (2.5.4)$$

Part I

Analysis on the one-dimensional torus

The stable case

The description of paths behaviour of systems described by 1-dimensional SDEs has been studied in [8]. In this chapter, we extend the methods developed in that work to infinite dimensional SPDEs. We are interested, here, in dynamics near a stable equilibrium branch. First, we set up the problem and give some assumptions on the periodic drift term f . Then, it is useful to understand the deterministic dynamics without noise before turning on to the stochastic dynamics, when a noise is added. We give concentration estimates of solutions in Sobolev space, for a linear version of f and then for more general drift terms where we have to handle in addition nonlinear terms.

Dans ce chapitre, on généralise les méthodes développées pour la description du comportement des solutions d'un système décrit par une EDS en une dimension, pour des EDPS en dimension infinie. On s'intéresse à la dynamique dans un voisinage d'une branche d'équilibre stable. On commence par présenter le problème et donner les différentes hypothèses sur le terme de dérive périodique en temps. On étudie la dynamique déterministe sans bruit pour ensuite décrire la dynamique stochastique dans des espaces de Sobolev.

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3.1 Set-up

Let $L, T > 0$ be real parameters. We consider time-dependent SPDEs on the torus $\mathbb{T} = \mathbb{R}/L\mathbb{Z}$ of the form

$$d\phi(t, x) = [\Delta\phi(t, x) + f(\varepsilon t, \phi(t, x))] dt + \sigma dW(t, x),$$

for the unknown $\phi : I \times \mathbb{T} \rightarrow \mathbb{R}$, where $I = [0, T]$. Here

- $\varepsilon > 0$ is a small parameter quantifying the slow time dependence;
- $\sigma > 0$ is a small parameter measuring the noise intensity;

- $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a forcing term satisfying a number of assumptions given below;
- $\Delta\phi$ denotes the one-dimensional Laplacian, ∂_{xx} ;
- $dW(t, x)$ denotes space-time white noise on $\mathbb{R}_+ \times \mathbb{T}$, given by a cylindrical Wiener process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Notice that interesting dynamics can only be expected when εt varies by an amount of order 1 and the system has to be considered on the time scale ε^{-1} . This is done, by introducing the slow time $s = \varepsilon t$ which yields the equation

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + f(t, \phi(t, x))] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x), \quad (3.1.1)$$

where we have replaced s by t to avoid confusions. The choice of an initial condition ϕ_0 is always assumed to be independent of $W(t, x)$ for all $t \geq 0$ and $x \in \mathbb{T}$.

The drift term f is periodic in the time variable with period T and describes a bistable situation. We consider throughout this thesis that f is the periodically forced Allen-Cahn equation

$$f(t, \phi) = \phi - \phi^3 + A \cos(t). \quad (3.1.2)$$

Our results apply, however, to more general drift terms f , that only need to satisfy a number of regularity and growth conditions. One can extend naturally the drift term to a more general case where $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. It will sometimes be useful to work with a potential U associated with f , satisfying

$$f(t, \phi) = -\partial_\phi U(t, \phi). \quad (3.1.3)$$

The following assumption on the behaviour of U for large values of ϕ will be assumed to hold all over this study.

Assumption 3.1.1 (Global behaviour of the drift term). *The potential U admits, for all $(t, \phi) \in I \times \mathbb{R}$, a decomposition*

$$U(t, \phi) = P(t, \phi) + g(t, \phi)$$

into a polynomial part and a bounded part. More precisely,

- *there exists an integer $p_0 \geq 1$ such that the map $\phi \mapsto P(t, \phi)$ is a polynomial of degree $2p_0$, of the form*

$$P(t, \phi) = \sum_{j=0}^{2p_0} A_j(t) \phi^j$$

with coefficients $A_j \in C^1(I, \mathbb{R})$ such that $|A_j(t)|$ and $|A'_j(t)|$ are bounded uniformly, and $A_{2p_0}(t) > 0$ for all $t \in I$;

- *the function $g \in C^2(I \times \mathbb{R}, \mathbb{R})$ satisfies*

$$|g(t, \phi)\phi^{-1}|, |\partial_\phi g(t, \phi)|, |\partial_{\phi\phi} g(t, \phi)|, |\partial_t g(t, \phi)| \leq M$$

for all $(t, \phi) \in I \times \mathbb{R}$ and some constant $M > 0$.

We consider the case where the potential $U(t, \phi)$ admits a strict local minimum at all times t , in a way that the drift term f admits a stable equilibrium branch. More precisely, we will require the following.

Assumption 3.1.2 (Stable case). *There exists a map $\phi^* : I \rightarrow \mathbb{R}$ such that*

$$f(t, \phi^*(t)) = 0 \quad \forall t \in I .$$

Furthermore, the linearisation $a(t) = \partial_\phi f(t, \phi^(t))$ satisfies*

$$-a_+ \leq a(t) \leq -a_- \quad \forall t \in I$$

for some constant $a_\pm > 0$.

We notice that assumption 3.1.1 and 3.1.2 implies that $\phi^* \in \mathcal{C}^1(I, \mathbb{R})$ due to the implicit function theorem.

3.2 Deterministic dynamics

We start by analysing (3.1.1) in the deterministic case when $\sigma = 0$. It takes the form of the PDE

$$\varepsilon \partial_t \phi(t, x) = \Delta \phi(t, x) + f(t, \phi(t, x)) , \quad (3.2.1)$$

where $t \in I$ and $x \in \mathbb{T}$ implying implicitly periodic boundary conditions. The drift term f satisfies Assumptions 3.1.1 and 3.1.2.

We are interested in the deviation from the equilibrium branch, given by the difference $\psi(t, \cdot) = \phi(t, \cdot) - \phi^*(t)e_0$. Using Taylor's formula to expand $f(t, \phi^*(t)e_0 + \psi)$, we obtain that ψ satisfies the equation

$$\varepsilon \partial_t \psi(t, x) = \Delta \psi(t, x) + a(t) \psi(t, x) + b(t, \psi(t, x)) - \varepsilon \frac{d}{dt} \phi^*(t) e_0(x) ,$$

where

$$\begin{aligned} a(t) &= \partial_\phi f(t, \phi^*(t)e_0) , \\ b(t, \psi) &= \frac{1}{2} \partial_\phi^2 f(t, \phi^*(t)e_0 + \theta \psi) \psi^2 \quad \text{for some } \theta \in [0, 1] . \end{aligned}$$

This shows in particular that there exist constants $d, M > 0$ such that

$$|b(t, \psi)| \leq M \psi^2 , \quad |\partial_\psi b(t, \psi)| \leq M |\psi| \quad (3.2.2)$$

for all $t \in I$ and all $\psi \in \mathbb{R}$ such that $|\psi| < d$.

There are well-known results in finite dimension on the theory of singular perturbation, see in particular [23, 45]. The deterministic dynamics in infinite-dimension is given by the following proposition. It describes the behaviour of the deterministic solution near the stable equilibrium branch.

Proposition 3.2.1 (Deterministic dynamics in the stable case). *There exist constants $C, \varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the equation (3.2.1) admits a particular solution $\bar{\phi}(t, x)$ satisfying*

$$\|\bar{\phi}(t, \cdot) - \phi^*(t)e_0\|_{H^1} \leq C\varepsilon \quad \forall t \in I \text{ and } e_0(x) = \frac{1}{\sqrt{L}} .$$

Proof of Proposition 3.2.1. Following the main idea of the proof in [45] in the finite-dimensional case, we define a Lyapunov function

$$V(\psi) = \frac{1}{2} \|\psi\|_{H^1}^2 = \frac{1}{2} \|\psi\|_{L^2}^2 + \frac{L^2}{2\pi^2} \|\nabla\psi\|_{L^2}^2 .$$

Let $\langle \cdot, \cdot \rangle$ denote the L^2 inner product. Observing that $\|\nabla\psi\|_{L^2}^2 = \langle \nabla\psi, \nabla\psi \rangle = -\langle \psi, \Delta\psi \rangle$, and using self-adjointness of the Laplacian, we obtain that the time derivative of the Lyapunov function along a solution of (3.2.1) satisfies

$$\begin{aligned} \varepsilon \frac{d}{dt} V(\psi(t, \cdot)) &= \langle \psi, \varepsilon \partial_t \psi \rangle - \frac{L^2}{\pi^2} \langle \Delta\psi, \varepsilon \partial_t \psi \rangle \\ &= \langle \psi, \Delta\psi \rangle + a(t) \|\psi\|_{L^2}^2 + \langle \psi, b(t, \psi) \rangle - \varepsilon \frac{d}{dt} \phi^*(t) \langle \psi, e_0 \rangle \\ &\quad - \frac{L^2}{\pi^2} \left[\|\Delta\psi\|_{L^2}^2 + a(t) \langle \Delta\psi, \psi \rangle + \langle \Delta\psi, b(t, \psi) \rangle \right] . \end{aligned}$$

In the last line, we have used the fact that $\langle \Delta\psi, e_0 \rangle = 0$ (here and below, we sometimes write ψ instead of $\psi(t, \cdot)$ in order not to overload the notation). Regrouping terms, and bounding some obviously negative terms above by zero, we get

$$\varepsilon \frac{d}{dt} V(\psi) \leq 2a(t)V(\psi) + \langle \psi, b(t, \psi) \rangle - \frac{L^2}{\pi^2} \langle \Delta\psi, b(t, \psi) \rangle - \varepsilon \frac{d}{dt} \phi^*(t) \langle \psi, e_0 \rangle . \quad (3.2.3)$$

Let $C_0 > 0$ be a constant to be fixed below. Assume that $\|\psi(0, \cdot)\|_{H^1} < C_0$, and define the first-exit time

$$\bar{\tau} = \inf \{ t > 0 : \|\psi(t, \cdot)\|_{H^1} \geq C_0 \} .$$

By convention, we set $\bar{\tau} = \infty$ whenever $\|\psi(t, \cdot)\|_{H^1} < C_0$ for all $t \in I$. Thus, for all $t \leq \bar{\tau}$ in I , we have $\|\psi(t, \cdot)\|_{H^1} < C_0$. By Sobolev's inequality (2.2.4), this implies that for these t , one has

$$|\psi(t, x)| \leq \|\psi(t, \cdot)\|_{L^\infty} \leq C_{\text{Sob}} \|\psi(t, \cdot)\|_{H^1} \leq C_{\text{Sob}} C_0$$

for all $x \in \mathbb{R}$ and some numerical constant C_{Sob} . By (3.2.2), provided $C_{\text{Sob}} C_0 \leq d$, it follows that

$$|b(t, \psi(t, x))| \leq C_{\text{Sob}}^2 M \|\psi(t, \cdot)\|_{H^1}^2 ,$$

and thus

$$\|b(t, \psi(t, \cdot))\|_{L^2}^2 \leq C_{\text{Sob}}^4 M^2 L \|\psi(t, \cdot)\|_{H^1}^4 .$$

By the Cauchy–Schwarz inequality, we get

$$|\langle \psi(t, \cdot), b(t, \psi(t, \cdot)) \rangle| \leq \|\psi(t, \cdot)\|_{L^2} \|b(t, \psi(t, \cdot))\|_{L^2} \leq C_{\text{Sob}}^2 M L^{1/2} \|\psi(t, \cdot)\|_{H^1}^3 .$$

Furthermore, integration by parts and (3.2.2) yield

$$\begin{aligned} |\langle \Delta\psi(t, \cdot), b(t, \psi(t, \cdot)) \rangle| &= \left| \int_0^L \nabla\psi(t, x)^2 \partial_\psi b(t, \psi(t, x)) dx \right| \\ &\leq M \|\psi(t, \cdot)\|_{L^\infty} \|\nabla\psi(t, \cdot)\|_{L^2}^2 \\ &\leq C_{\text{Sob}} M \|\psi(t, \cdot)\|_{H^1}^3 . \end{aligned}$$

Finally, owing to the implicit function theorem and Assumption 3.1.2, the derivative of $\phi^*(t)$ is bounded by a constant c , so that

$$\left| \frac{d}{dt} \phi^*(t) \langle \psi(t, \cdot), e_0 \rangle \right| \leq c \|\psi(t, \cdot)\|_{L^2} \leq c \|\psi(t, \cdot)\|_{H^1} .$$

Plugging the last three estimates in (5.4.2), since $a(t)$ is negative and bounded away from zero by Assumption 3.1.2, we obtain that $V(t) = V(\psi(t, \cdot))$ satisfies

$$\begin{aligned} \varepsilon \dot{V} &\leq -C_1 V + C_2 V^{3/2} + \varepsilon C_3 V^{1/2} \\ &\leq -C_1 \left[1 - \frac{C_0^{1/2} C_2}{C_1} \right] V + \varepsilon C_3 V^{1/2} \end{aligned} \quad (3.2.4)$$

for all $t \leq \bar{\tau}$, and some constants $C_1, C_2, C_3 > 0$. Choosing C_0 such that $C_0^{1/2} \leq \frac{C_1}{2C_2}$, we obtain

$$\varepsilon \dot{V} \leq -\frac{1}{2} C_1 V + \varepsilon C_3 V^{1/2}$$

for all $t \leq \bar{\tau}$. Setting $V(t) = Z(t)^2$ and dividing by $2Z(t)$, we get

$$\varepsilon \dot{Z} \leq -\frac{1}{4} C_1 Z + \frac{1}{2} C_3 \varepsilon .$$

Since the variable $W = Z - \frac{2C_3}{C_1} \varepsilon$ satisfies $\varepsilon \dot{W} \leq -\frac{1}{4} C_1 W$, Gronwall's inequality (2.1.2) yields

$$W(t) \leq W(0) e^{-C_1 t / (4\varepsilon)}$$

for all $t \leq \bar{\tau}$. Thus for any $W(0)$ of order ε , we find that $Z(t)$ remains of order ε for all $t < \bar{\tau}$, and thus $V(t)$ remains of order ε^2 . Choosing ε_0 small enough and $0 < \varepsilon < \varepsilon_0$, we obtain in particular that $V(t) < C_0$ for all $t < \bar{\tau}$, so that assuming $\bar{\tau} < T$ would lead to a contradiction. We conclude that $\bar{\tau} \geq T$, showing that $V(t) = \mathcal{O}(\varepsilon^2)$ for all $t \in I$, which is the claimed result. $\square$

Remark 3.2.2. Another choice of Lyapunov function would have been

$$V(t, \psi) = \|\psi\|_{H^1}^2 + \int_0^L U_1(t, \psi(x)) dx ,$$

where $U_1(t, \psi) = U(t, \phi^*(t)e_0 + \psi)$ is a shifted version of the potential U introduced in (3.1.3). This function is useful to control the behaviour of solutions of large H^1 -norm. Indeed, one can show that there exist constants $M_1, M_2 > 0$ such that

$$-M_1 \leq V(t, \psi) \leq M_2 \left(1 + \|\psi\|_{H^1}^{2p_0} \right) \quad \forall t \in I$$

for all $t \in I$, and that $V(t, \psi(t, \cdot))$ is decreasing at least exponentially fast when it is large. $\diamond$

3.3 Stochastic dynamics

We return now to the SPDE (3.1.1) with $\sigma > 0$. We are interested in the stochastic process $\psi(t, x) = \phi(t, x) - \bar{\phi}(t, x)$, which describes the deviation due to noise from the deterministic solution tracking the stable equilibrium branch $\phi^*(t)e_0$. It obeys the SPDE

$$d\psi(t, x) = \frac{1}{\varepsilon} [\Delta\psi(t, x) + a(t)\psi(t, x) + b(t, \psi(t, x))] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x), \quad (3.3.1)$$

where

$$a(t) = \partial_\phi f(t, \phi^*(t)e_0)$$

and

$$b(t, \psi) = a_1(t, \cdot)\psi + b_1(t, \psi).$$

Proposition 3.2.1 shows that $\|a_1(t, \cdot)\|_{H^1} = \mathcal{O}(\varepsilon)$ while $b_1(t, \psi)$ denotes a nonlinear term, satisfying bounds analogous to (3.2.2). In the finite-dimensional case, it is known (see [7, Theorem 2.4]) that solutions of the stochastic equation (3.1.1), starting near the equilibrium branch ϕ^* , remain close to that branch with high probability. To quantify this in our infinite-dimensional situation: for a given $s > 0$, we define for any $h > 0$ the following set, centered around $\bar{\phi}(t, \cdot)$,

$$\mathcal{B}(h) = \left\{ (t, \phi) : t \in I, \|\phi - \bar{\phi}(t, \cdot)\|_{H^s} < h \right\}.$$

Given an initial condition $(0, \phi_0)$ in $\mathcal{B}(h)$, the first-exit time from $\mathcal{B}(h)$ is the stopping time

$$\begin{aligned} \tau_{\mathcal{B}(h)} &= \inf \{ t > 0 : (t, \phi(t, \cdot)) \notin \mathcal{B}(h) \} \\ &= \inf \{ t > 0 : \|\phi - \bar{\phi}(t, \cdot)\|_{H^s} \geq h \}. \end{aligned}$$

By convention, we set $\tau_{\mathcal{B}(h)} = +\infty$ whenever $(t, \phi(t, \cdot)) \in \mathcal{B}(h)$ for all $t \in I$.

The following theorem describes the dynamics in a neighborhood of the deterministic solution, $\bar{\phi}(t, \cdot)$.

Theorem 3.3.1 (Stochastic dynamics in the stable case). *For any $s \in (0, \frac{1}{2})$ and any $\nu > 0$, there exist constants $\kappa = \kappa(s), \varepsilon_0, h_0$ and $C(\kappa, t, \varepsilon, s) > 0$ such that, whenever $0 < \varepsilon \leq \varepsilon_0$ and $0 < h \leq h_0 \varepsilon^\nu$, the solution of (3.1.1) with initial condition $\phi(0, \cdot) = \bar{\phi}(0, \cdot)$ satisfies*

$$\mathbb{P}\{\tau_{\mathcal{B}(h)} < t\} \leq C(\kappa, t, \varepsilon, s) \exp \left\{ -\kappa \frac{h^2}{\sigma^2} \left[1 - \mathcal{O}\left(\frac{h}{\varepsilon^\nu}\right) \right] \right\},$$

for all $t \in I$.

Remark 3.3.2. The exponential factor in the bound is very small as soon as h is significantly larger than σ , so it is unlikely to leave the set $\mathcal{B}(h)$. The proof yields explicit bounds on $C(\kappa, t, \varepsilon, s)$. In particular, this quantity can be taken proportional to t/ε , while its dependence on κ and s is more complicated. $\diamond$

Remark 3.3.3. The result also holds for general initial conditions $\phi(0, \cdot)$ in an H^s -neighbourhood of order 1 of $\bar{\phi}(0, \cdot)$, provided one only considers the probability of leaving $\mathcal{B}(h)$ after a time of order $\varepsilon \log(\|\phi(0, \cdot)\|_{H^s} h^{-1})$, since solutions need a time of that order to reach $\mathcal{B}(h)$. See [10, Theorem 5.1.6] for a precise formulation, which can be adapted to the present situation by a similar argument. $\diamond$

We perform the proof of Theorem 3.3.1 in two steps: we study first the linear case where b vanishes and second we consider a more general case where the drift term is nonlinear.

3.3.1 Linear case

We consider the linearised version of (3.3.1) given by

$$d\psi(t, x) = \frac{1}{\varepsilon} [\Delta\psi(t, x) + a(t)\psi(t, x)] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x). \quad (3.3.2)$$

We recall the Fourier basis $(e_k)_{k \in \mathbb{Z}}$, defined in (2.2.1). These basis satisfy the eigenvalue problem

$$\Delta e_k = -\mu_k e_k, \quad \mu_k = \frac{k^2 \pi^2}{L^2}. \quad (3.3.3)$$

Projecting (3.3.2) on the k -th basis vector e_k , we obtain

$$d\psi_k(t) = \frac{1}{\varepsilon} a_k(t) \psi_k(t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_k(t), \quad (3.3.4)$$

where $a_k(t) = -\mu_k + a(t)$ and the $\{W_k(t)\}_{t \geq 0}$ are independent Wiener processes (see for instance [34]). The solution of (3.3.4) is a Gaussian process and can be represented by the Ito integral (cf. Duhamel's principle)

$$\psi_k(t) = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha_k(t, t_1)/\varepsilon} dW_k(t_1),$$

where $\alpha_k(t, t_1) = \int_{t_1}^t a_k(t_2) dt_2$. Thus, for each time t , $\psi_k(t)$ is characterised by its mean being zero and its variance given by

$$\text{Var} \{\psi_k(t)\} = \frac{\sigma^2}{\varepsilon} \int_0^t e^{2\alpha_k(t, t_1)/\varepsilon} dt_1.$$

3.3.1.1 Some useful bounds

In order to bound the probability of leaving $\mathcal{B}(h)$, we provide the following lemmas. We assume that there are positive constants $c_0^\pm$ such that for all $t \in I$

$$c_0^- \langle k \rangle^2 \leq \mu_k + a_- \leq |a_k(t)| \leq \mu_k + a_+ \leq c_0^+ \langle k \rangle^2.$$

Due to the implicit function theorem, we also have the existence of a constant C such that

$$|a(t)'| \leq C \quad \forall t \in I = [0, T].$$

The following lemma gives a bound on the variance which is decreasing when k increases.

Lemma 3.3.4 (Bound on the variance). *There exists a constant $C_0 > 0$ such that the variance satisfies the bound*

$$\text{Var} \{\psi_k(t)\} \leq C_0 \frac{\sigma^2}{\langle k \rangle^2} \quad \forall t \in I.$$

Proof. Using integration by parts, we obtain

$$\begin{aligned} \frac{\text{Var} \{\psi_k(t)\}}{\sigma^2} &= \int_0^t \frac{1}{2a_k(s)} \frac{2a_k(s)}{\varepsilon} e^{2\alpha_k(t, s)/\varepsilon} ds \\ &= \frac{1}{2a_k(t)} - \frac{1}{2a_k(0)} e^{2\alpha_k(t, 0)/\varepsilon} + \frac{1}{2} \int_0^t \frac{a_k'(s)}{a_k(s)^2} e^{2\alpha_k(t, s)/\varepsilon} ds, \end{aligned}$$

where we write $\alpha_k(t, 0) = \alpha_k(t)$ for brevity. The absolute value of integral can be bounded by

$$\int_0^t \frac{|a'_k(s)|}{(-\mu_k - a_-)^2} e^{-2(\mu_k + a_-)(t-s)/\varepsilon} ds \leq \frac{C}{2(\mu_k + a_-)^3} \varepsilon.$$

Therefore,

$$\frac{\text{Var} \{\psi_k(t)\}}{\sigma^2} \leq \frac{1 + \mathcal{O}(\varepsilon)}{2(\mu_k + a_-)} \leq C_0 \frac{1}{\langle k \rangle^2},$$

as claimed. $\square$

To investigate $\psi_k(t)$ which is a one-dimensional process, we need to estimate the stochastic integral $\int_0^t e^{\alpha_k(t, t_1)/\varepsilon} dW_k(t_1)$. Applying the Bernstein-type inequality (2.1.1) on a partition of $[0, T]$, we can easily adapt Theorem 2.4 in [7] to obtain the following estimate.

Lemma 3.3.5. *Fix $\gamma > 0$. Then for any $k \in \mathbb{Z}$, we have the bound*

$$\mathbb{P} \left\{ \sup_{t \in I} |\psi_k(t)| \geq h \right\} \leq C_k(T, \varepsilon) \exp \left\{ -\kappa \langle k \rangle^2 \frac{h^2}{\sigma^2} \right\},$$

where $C_k(T, \varepsilon) = \frac{2c_0^- \langle k \rangle^2}{\gamma \varepsilon} T$ and $\kappa = \frac{e^{-2\gamma}}{2C_0}$.

Proof. As in [7, Theorem 2.4], we introduce a partition $0 = u_0 < u_1 < \dots < u_N = T$ of $[0, T]$ by requiring $\alpha_k(u_{l+1}, u_l) = -\gamma \varepsilon$ for $1 \leq l \leq N = \lfloor c_0^- \langle k \rangle^2 T / (\gamma \varepsilon) \rfloor$. The proof then follows by approximating the process by a martingale on each interval $[u_l, u_{l+1}]$ and using a Bernstein-type inequality that follows directly from Doob's submartingale inequality. $\square$

3.3.1.2 Proof of Theorem 3.3.1 in the linear case

Having all the needed tools, we can develop an upper bound on the probability of escaping the neighborhood of $\bar{\phi}(t, x)$ before time t .

Proof of Theorem 3.3.1 in the linear case. Fix constants $\eta, \rho > 0$ and $s \in (0, \frac{1}{2})$ such that $s = \frac{1}{2} - \rho$. For every decomposition $h^2 = \sum_{k \in \mathbb{Z}} h_k^2$ one has

$$\begin{aligned} \mathbb{P} \{ \tau_{B(h)} < t \} &= \mathbb{P} \left\{ \sup_{t \in I} \|\psi(t, \cdot)\|_{H^s}^2 \geq h^2 \right\} \\ &= \mathbb{P} \left\{ \sup_{t \in I} \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\psi_k(t)|^2 \geq h^2 \right\} \\ &\leq \sum_{k \in \mathbb{Z}} \mathbb{P} \left\{ \sup_{t \in I} |\psi_k(t)|^2 \geq h_k^2 \langle k \rangle^{-2s} \right\} \\ &\leq \sum_{k \in \mathbb{Z}} C_k(T, \varepsilon) \exp \left\{ -\kappa \frac{h_k^2}{\sigma^2} \langle k \rangle^{2-2s} \right\}. \end{aligned}$$

Choosing

$$h_k^2 = C(\eta, s) h^2 \langle k \rangle^{-2+2s+\eta}, \quad (3.3.5)$$

the condition $h^2 = \sum_{k \in \mathbb{Z}} h_k^2$ yields

$$C(\eta, s) = \frac{1}{\sum_{k \in \mathbb{Z}} \langle k \rangle^{-2+2s+\eta}} .$$

Since the Riemann zeta function $\zeta(v) = \sum_{n \geq 1} n^{-v}$ converges for $v > 1$, we get

$$\sum_{k \in \mathbb{Z}} \frac{1}{\langle k \rangle^{2-2s-\eta}} \leq 1 + 2 \sum_{k=1}^{\infty} \frac{1}{k^{2-2s-\eta}} = 1 + 2\zeta(2-2s-\eta) < \infty \quad \forall 0 < \eta < 2\rho . \quad (3.3.6)$$

With h_k given by (3.3.5) and η satisfying this condition, we get

$$\begin{aligned} \mathbb{P} \left\{ \sup_{t \in I} \|\psi(t, \cdot)\|_{H^s}^2 \geq h^2 \right\} &\leq \sum_{k \in \mathbb{Z}} C_k(T, \varepsilon) \exp \left\{ -\kappa C(\eta, s) \frac{h^2}{\sigma^2} \langle k \rangle^\eta \right\} \\ &= \alpha_T \sum_{k \in \mathbb{Z}} \langle k \rangle^2 e^{-\beta \langle k \rangle^\eta} , \end{aligned}$$

where we write $\alpha_T = \frac{2c_0^-}{\gamma\varepsilon} T$ and $\beta = \kappa C(\eta, s) \frac{h^2}{\sigma^2}$ for simplicity. In order to bound the sum, we write

$$f(x) = (1 + x^2) e^{-\beta(1+x^2)^{\eta/2}} .$$

Note that we may assume that f is decreasing by taking h/σ larger than an η -dependent constant of order 1 (which we may do, because otherwise the result is trivially true). Therefore, we obtain

$$\sum_{k \in \mathbb{Z}} f(k) = f(0) + 2 \sum_{k=1}^{\infty} f(k) \leq e^{-\beta} + 2 \int_0^{\infty} f(x) dx .$$

In what follows, we show that the integral

$$I = \int_0^{\infty} f(x) dx = \int_0^{\infty} (1 + x^2) e^{-\beta(1+x^2)^{\eta/2}} dx$$

is finite, and, more precisely, has order $\beta^{-1/2} e^{-\beta}$. We first make the change of variable $y = \beta(1+x^2)^{\eta/2}$, yielding

$$I = \frac{1}{\eta \beta^{4/\eta}} \int_{\beta}^{\infty} e^{-y} \frac{y^{4/\eta-1}}{\sqrt{\left(\frac{y}{\beta}\right)^{2/\eta} - 1}} dy .$$

The further change of variable $y = \beta + z$ gives

$$I = \frac{e^{-\beta}}{\eta \beta^{4/\eta}} \int_0^{\infty} e^{-z} \frac{(\beta + z)^{4/\eta-1}}{\sqrt{\left(1 + \frac{z}{\beta}\right)^{2/\eta} - 1}} dz .$$

Using Taylor's formula, we get the lower bound

$$\left(1 + \frac{z}{\beta}\right)^{2/\eta} - 1 \geq \frac{2}{\eta} \frac{z}{\beta} .$$

Therefore,

$$\begin{aligned}
I &\leq \frac{e^{-\beta}}{\eta\beta^{4/\eta}} \sqrt{\frac{\beta\eta}{2}} \int_0^\infty \frac{e^{-z}}{\sqrt{z}} (\beta+z)^{4/\eta-1} dz \\
&= \frac{e^{-\beta}}{\eta\beta^{4/\eta}} \sqrt{\frac{\beta\eta}{2}} \left[\int_0^\beta \frac{e^{-z}}{\sqrt{z}} (\beta+z)^{4/\eta-1} dz + \int_\beta^\infty \frac{e^{-z}}{\sqrt{z}} (\beta+z)^{4/\eta-1} dz \right] \\
&\leq \frac{e^{-\beta}}{\sqrt{2}\eta\beta^{4/\eta-1/2}} (2\beta)^{4/\eta-1} \int_0^\beta \frac{e^{-z}}{\sqrt{z}} dz + \frac{e^{-\beta}}{\sqrt{2}\eta\beta^{4/\eta-1/2}} 2^{4/\eta-1} \int_\beta^\infty \frac{e^{-z}}{\sqrt{z}} z^{4/\eta-1} dz \\
&\leq c_1(\eta) \frac{e^{-\beta}}{\sqrt{\beta}} + c_2(\eta) \frac{e^{-\beta}}{\beta^{4/\eta-1/2}},
\end{aligned}$$

where $c_1(\eta)$ and $c_2(\eta)$ are bounded uniformly in β , provided $\eta < 8$. It follows that

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \langle k \rangle^2 e^{-\beta \langle k \rangle^\eta} &\leq e^{-\beta} + 2c_1(\eta) \frac{e^{-\beta}}{\sqrt{\beta}} + 2c_2(\eta) \frac{e^{-\beta}}{\beta^{4/\eta-1/2}} \\
&= e^{-\kappa C(\eta,s)h^2/\sigma^2} \left[1 + \bar{c}_1(\eta) \frac{\sigma}{h} + \bar{c}_2(\eta) \left(\frac{\sigma^2}{h^2} \right)^{4/\eta-1/2} \right].
\end{aligned}$$

We thus conclude that

$$\begin{aligned}
\mathbb{P} \left\{ \sup_{t \in I} \|\psi(t, \cdot)\|_{H^s}^2 \geq h^2 \right\} &\leq \alpha_T \left[1 + \bar{c}_1(\eta) \frac{\sigma}{h} + \bar{c}_2(\eta) \left(\frac{\sigma^2}{h^2} \right)^{4/\eta-1/2} \right] e^{-\kappa C(\eta,s)h^2/\sigma^2} \\
&=: C(\gamma, T, \varepsilon, s) e^{-\kappa C(\eta,s)h^2/\sigma^2},
\end{aligned} \tag{3.3.7}$$

where we can fix, for instance, $\eta = \rho = \frac{1}{2} - s$, which yields $C(\eta, s) = [1 + 2\zeta(\frac{3}{2} - s)]^{-1}$ by (3.3.6). $\square$

3.3.2 Nonlinear case

We return now to the general nonlinear equation (3.3.1). By Duhamel's principle, its solution satisfies the equation

$$\begin{aligned}
\psi(t, \cdot) &= \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha(t,t_1)/\varepsilon} e^{[(t-t_1)/\varepsilon]\Delta} dW(t_1, \cdot) + \frac{1}{\varepsilon} \int_0^t e^{\alpha(t,t_1)/\varepsilon} e^{[(t-t_1)/\varepsilon]\Delta} b(t_1, \psi(t_1, \cdot)) dt_1, \\
&= \psi^0(t, \cdot) + \psi^1(t, \cdot).
\end{aligned} \tag{3.3.8}$$

Here $\alpha(t, t_1) = \int_{t_1}^t a(u) du$, and $e^{t\Delta}$ denotes the heat kernel. We notice that $\psi^0(t, x)$ is the solution of the linear equation (3.3.2), and therefore satisfies the estimate (3.3.7).

3.3.2.1 Some technical results

We give some technical results that will be needed to show that $\psi^1(t, \cdot)$ belongs to a certain Sobolev space included in H^s .

Lemma 3.3.6. *Let the potential $U(t, \phi)$ satisfy Assumption 3.1.1, and assume $\psi(t, \cdot) \in H^s$ for all $0 < s < \frac{1}{2}$. Then*

$$\beta_1(t) = b_1(t, \psi(t, \cdot)).$$

belongs to H^r for all $r < \frac{1}{2}$. Furthermore, for all $r < \frac{1}{2} - (2p_0 + 1)(\frac{1}{2} - s)$, there exists $C(r, s) < \infty$ such that

$$\|\beta_1(t)\|_{H^r} \leq C(r, s) \max\{\|\psi\|_{H^s}^2, \|\psi\|_{H^s}^{2p_0+1}\}. \quad (3.3.9)$$

Proof. Consider first the case where U is a polynomial in ψ of degree $2p_0$. Then $f(t, \psi)$ and $\beta_1(t)$ are polynomials of degree $2p_0 + 1$. Applying Young's inequality (2.2.5), we obtain by induction that if $\psi(t, \cdot) \in H^{\frac{1}{2}-\kappa}$ for a $\kappa > 0$, then for any $k \geq 2$, $\psi(t, \cdot)^k \in H^r$ for any $r < \frac{1}{2} - k\kappa$. It follows that $\beta_1(t) \in H^r$ for all $r < \frac{1}{2} - (2p_0 + 1)\kappa$. Since $\kappa > 0$ is arbitrary, we conclude that indeed $\beta_1(t) \in H^r$ for all $r < \frac{1}{2}$. The bound (3.3.9) is then a consequence of Young's inequality (2.2.5), the bound (3.2.2) on $b_1(t, \psi)$ for small ψ , and the fact that $\beta_1(t)$ is a polynomial of degree $2p_0 + 1$. Consider now the general case. By Assumption 3.1.1, $f(t, \psi)$ and $\beta_1(t)$ are each the sum of a polynomial of degree $2p_0 + 1$ and a bounded function $g(t, \psi)$. For $0 < s < 2$ and $1 \leq p, q \leq \infty$, consider the Besov space $\mathcal{B}_p^{s,q}$. Then, [15] shows that there exists a constant $R(p, q, s, M) > 0$ such that for all ψ in the positive cone, $(\mathcal{B}_p^{s,q})^+$, we have

$$\|g \circ \psi\|_{\mathcal{B}_p^{s,q}} \leq R(p, q, s, M) \|\psi\|_{\mathcal{B}_p^{s,q}}.$$

In particular, whenever $p = q = 2$ the Besov space $\mathcal{B}_2^{s,2}$ is nothing but the Sobolev space H^s . Thus, if $\psi(t, \cdot) \in H^s$, then $g \circ \psi \in H^s$ and $\beta_1(t) \in H^r$ for all $r < \frac{1}{2}$. $\square$

Then,

$$\beta(t) = a_1(t, \cdot)\psi(t, \cdot) + \beta_1(t)$$

belongs to H^r for all $r < \frac{1}{2}$. By the triangle inequality and the previous lemma, we get

$$\|\beta(t)\|_{H^r} \leq \|a_1(t, \cdot)\psi(t, \cdot)\|_{H^r} + \|\beta_1(t)\|_{H^r} \leq \varepsilon \|\psi\|_{H^s} + C(r, s) \max\{\|\psi\|_{H^s}^2, \|\psi\|_{H^s}^{2p_0+1}\}.$$

A way to analyse the stochastic convolution is based on Schauder estimate. Here is a variant of it.

Lemma 3.3.7 (Schauder-type estimate). *Assume $\beta \in H^r$ for some $r \in (0, \frac{1}{2})$. Then for all $q < r + 2$, there exists a constant $M(q, r) < \infty$ such that*

$$\|e^{t\Delta} \beta\|_{H^q} \leq M(q, r) t^{-\frac{q-r}{2}} \|\beta\|_{H^r}$$

for all $t > 0$.

Proof. Let $\gamma = \frac{q-r}{2}$. Writing the Fourier expansion of β as $\beta(x) = \sum_{k \in \mathbb{Z}} \beta_k e_k(x)$, we have

$$e^{t\Delta} \beta(x) = \sum_{k \in \mathbb{Z}} e^{-\mu_k t} \beta_k e_k(x),$$

where the $-\mu_k$ are the eigenvalues of the Laplacian, cf. (3.3.3). By definition of the fractional Sobolev norm, we obtain

$$\begin{aligned} \|t^\gamma e^{t\Delta} \beta\|_{H^q}^2 &= \sum_k \langle k \rangle^{2q} t^{2\gamma} e^{-2\mu_k t} \beta_k^2 \\ &\leq \sum_k [\langle k \rangle^2 t]^{q-r} e^{-c_0^- \langle k \rangle^2 t} \langle k \rangle^{2r} \beta_k^2 \\ &= \sum_k H(\langle k \rangle^2 t) \langle k \rangle^{2r} \beta_k^2, \end{aligned}$$

where $H(z) = z^{q-r} e^{-c_0^- z}$ reaches its maximum at $z^* = \frac{q-r}{c_0^-}$. Therefore,

$$0 \leq H(z) \leq M(q, r)^2 = H(z^*) = \left(\frac{q-r}{c_0^-} \right)^{q-r} e^{-(q-r)}$$

for all $z \geq 0$. We conclude that for all $t \in I$,

$$\|t^{\frac{q-r}{2}} e^{t\Delta} \beta\|_{H^q}^2 \leq \sum_k M(q, r)^2 \langle k \rangle^{2r} \beta_k^2 = M(q, r)^2 \|\beta\|_{H^r}^2$$

as claimed. $\square$

Applying the previous lemma appropriately to $\psi^1(t, \cdot)$ defined in (3.3.8), we obtain the following key estimate.

Corollary 3.3.8. *Assume there exists $r \in (0, \frac{1}{2})$ such that $\beta(t) \in H^r$ for all $t \in I$. Then for all $q < r + 2$, there exists a constant $M'(q, r) < \infty$ such that for all $t \in I$, one has $\psi^1(t, \cdot) \in H^q$ and*

$$\|\psi^1(t, \cdot)\|_{H^q} \leq M'(q, r) \varepsilon^{\frac{q-r}{2}-1} \sup_{0 \leq t_1 \leq t} \|\beta(t_1)\|_{H^r}.$$

Proof. Note that $\alpha(t, t_1) \leq -\frac{c_0^-}{2}(t - t_1)$ whenever $t_1 \leq t$. Furthermore, the previous result implies that for any $q < r + 2$, one has

$$\|e^{(t/\varepsilon)\Delta} \beta(t)\|_{H^q} \leq M(q, r) \left(\frac{\varepsilon}{t} \right)^{\frac{q-r}{2}} \|\beta(t)\|_{H^r}.$$

Therefore

$$\begin{aligned} \|\psi^1(t, x)\|_{H^q} &\leq \frac{1}{\varepsilon} \int_0^t e^{-c_0^-(t-t_1)/(2\varepsilon)} \|e^{[(t-t_1)/\varepsilon]\Delta} \beta(t_1)\|_{H^q} dt_1 \\ &\leq M(q, r) \varepsilon^{\frac{q-r}{2}-1} \sup_{0 \leq t_1 \leq t} \|\beta(t_1)\|_{H^r} \int_0^t (t-t_1)^{-\frac{q-r}{2}} dt_1, \end{aligned}$$

and the integral over t_1 is bounded whenever $q - r < 2$. $\square$

3.3.2.2 Proof of Theorem 3.3.1

Now, if $s \leq q$ then $H^q \subset H^s$ and thus $\psi^1(t, \cdot) \in H^s$ whenever $\psi(t, \cdot) \in H^s$. With these results, we can now prove immediately Theorem 3.3.1 in a general framework.

Proof of Theorem 3.3.1. For every decomposition $h = h_0 + h_1$ with $h_0, h_1 > 0$, one has

$$\begin{aligned} \mathbb{P}\{\tau_{\mathcal{B}(h)} < t\} &= \mathbb{P}\left\{ \sup_{0 \leq t \leq T \wedge \tau_{\mathcal{B}(h)}} \|\psi(t, \cdot)\|_{H^s} > h \right\} \\ &\leq \mathbb{P}\left\{ \sup_{0 \leq t \leq T \wedge \tau_{\mathcal{B}(h)}} \|\psi^1(t, \cdot)\|_{H^s} + \|\psi^0(t, \cdot)\|_{H^s} > h \right\} \\ &\leq \mathbb{P}\left\{ \sup_{0 \leq t \leq T} \|\psi^0(t, \cdot)\|_{H^s} > h_0 \right\} \\ &\quad + \mathbb{P}\left\{ \sup_{0 \leq t \leq T \wedge \tau_{\mathcal{B}(h)}} \|\psi^1(t, \cdot)\|_{H^s} > h_1, \sup_{0 \leq t \leq T} \|\psi^0(t, \cdot)\|_{H^s} \leq h_0 \right\}. \end{aligned}$$

The first term on the right-hand side can be estimated by (3.3.7). Furthermore, for all $t < \tau_{\mathcal{B}(h)}$, we have $\|\beta(t)\|_{H^r} \leq \varepsilon \|\psi(t, \cdot)\|_{H^s} + M \|\psi(t, \cdot)\|_{H^s}^2 \leq \varepsilon h + Mh^2$, so that

$$\|\psi^1(t, x)\|_{H^q} \leq M'(q, r) \varepsilon^{\frac{q-r}{2}-1} (\varepsilon h + Mh^2) .$$

Choosing $h_1 = M'(q, r) \varepsilon^{\frac{q-r}{2}-1} (\varepsilon h + Mh^2)$, we get

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T \wedge \tau_{\mathcal{B}(h)}} \|\psi^1(t, x)\|_{H^s} > h_1, \sup_{0 \leq t \leq T} \|\psi^0(t, x)\|_{H^s} \leq h_0 \right\} = 0 .$$

We thus obtain the result by choosing $h_0 = h - h_1 = h - M'(q, r) \varepsilon^{\frac{q-r}{2}-1} (\varepsilon h + Mh^2) = h(1 - \mathcal{O}(h/\varepsilon^\nu))$ and $\nu = 1 - \frac{q-r}{2}$. $\square$

The case of bifurcations

When solutions are close to a neighbourhood of the equilibrium branch, noise term may help sample paths to escape from the deterministic solution and to reach the unstable equilibrium branch. Bifurcations or avoided bifurcations may occur during this process. First, we define the dynamic transcritical bifurcation and give some useful bounds on the non linear terms. Then, we describe the dynamics without noise near an unstable equilibrium. After adding noise, we distinguish two qualitatively different regimes that depend on the noise intensity and in which we describe the behaviour of sample paths. Finally, we discuss the relation between the different parameters.

Dans ce chapitre, on traite le cas où le système perd sa stabilité et s'approche d'une bifurcation transcritique. On donne des estimées sur les termes non linéaires ainsi qu'on décrit la dynamique déterministe dans un voisinage d'une branche d'équilibre instable. Lorsque'on ajoute du bruit, on distingue deux régimes de bruit qualitativement différents dépendants de l'intensité du bruit. On décrit les comportements des solutions dans chacun des régimes. On termine l'étude des EDPS d'une dimension par une discussion sur les relations entre les différents paramètres en jeu.

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4.1 Coupled SDE-SPDE system

We consider again the periodically forced Allen-Cahn equation

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + \phi(t, x) - \phi(t, x)^3 + A \cos(t)] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x) .$$

4.1.1 Bifurcations or avoided bifurcations

Let $A_c = \frac{2}{3\sqrt{3}}$ be the critical driving amplitude above which the potential ceases always to have two wells. When $\sigma = 0$ and whenever $A < A_c$, transitions between potential wells are impossible. The equation

$$\phi - \phi^3 + A \cos(t) = 0$$

has exactly three solutions

$$\phi_1^*(t) < \phi_2^*(t) < \phi_3^*(t) .$$

If t is replaced by a fixed parameter t_0 , the equilibrium branches $\phi_1^*(t_0)$ and $\phi_3^*(t_0)$ are stable for the deterministic fast system

$$\partial_t \phi(t, x) = \Delta\phi(t, x) + \phi(t, x) - \phi(t, x)^3 + A \cos(t_0) ,$$

while $\phi_2^*(t_0)$ is unstable. If $A = A_c$, a stable branch and the unstable branch meet a transcritical bifurcation point whenever t is a multiple of π . If A is slightly smaller than A_c , the branches approach each other without quite touching.

If $\sigma > 0$ and $A = 0$, noise will cause the particle to jump from one well to the other one at random times. If both parameters A and σ are positive, however, and suitably tuned, transitions between the branches likely occur, which is one of the basic mechanisms responsible for stochastic resonance.

In what follows, we are mainly interested in the case where A is slightly smaller than A_c which was analysed in the one-dimensional setting in the work [8]. And we consider more general equations of the form (3.1.1), assuming that the drift term $f(t, \phi)$ vanishes on three equilibrium branches, two of which come close to each other at particular times. Whenever the three branches are well-separated, we can describe the dynamics near stable branches by adapting previous results. It is thus sufficient to describe the dynamics near times of bifurcation, or avoided bifurcation.

By an affine change of variables, it is possible to translate these (avoided) bifurcation points to the origin $(t, \phi) = (0, 0)$. We then make the following assumptions.

Assumption 4.1.1 (Bifurcation point near $t = 0$). *The drift term f is of class $\mathcal{C}^3$, and satisfies*

$$\begin{aligned} f(t, 0) &= \delta + a_1 t^2 + \mathcal{O}(t^3) , \\ \partial_\phi f(t, 0) &= \mathcal{O}(t^2) , \\ \partial_{\phi\phi} f(0, 0) &< 0 \end{aligned}$$

for constants $\delta \geq 0$ and $a_1 > 0$.

We can always assume that $a_1 = 1$ due to scaling properties. Under Assumption 4.1.1, one can check (see [8, Section 4]) that in a neighbourhood of $(0, 0)$, the drift term $f(t, \phi)$ vanishes only on two branches $\phi_\pm^*(t)$, satisfying

$$\begin{aligned}\phi_\pm^*(t) &\asymp \pm(\sqrt{\delta} + |t|) , \\ a_\pm(t) = \partial_\phi f(t, \phi_\pm^*(t)) &\asymp \mp(\sqrt{\delta} + |t|) .\end{aligned}$$

In particular, ϕ_+^* is stable, while ϕ_-^* is unstable, unless $\delta = 0$ and $t = 0$, when there is a transcritical bifurcation (Figure 4.1). We rewrite the SPDE (3.1.1) in the form

$$d\phi(t, x) = \frac{1}{\varepsilon} \left[\Delta\phi(t, x) + g(t) - \phi(t, x)^2 - b(t, \phi(t, x)) \right] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x) , \quad (4.1.1)$$

where

$$\begin{aligned}g(t) &= \delta + t^2 + \mathcal{O}(t^3) , \\ b(t, \phi) &= \mathcal{O}(\phi^3) + \mathcal{O}(t\phi^2) + \mathcal{O}(t^2\phi) .\end{aligned}$$

4.1.2 Spatial mean and oscillating part

It will be convenient to decompose the solution of (4.1.1) into its spatial mean and oscillating part, by writing

$$\phi(t, x) = \phi_0(t)e_0(x) + \phi_\perp(t, x) , \quad \int_{\mathbb{T}} \phi_\perp(t, x) dx = 0 .$$

Let us start with a preliminary change of variables. Let $\alpha, \beta, \gamma \in \mathbb{R}$. Using the scaling $t = \alpha\bar{t}$, $x = \beta\bar{x}$ and $\phi = \gamma\bar{\phi}$ in (4.1.1), we obtain the following SPDE. For all $\bar{x} \in [0, \bar{L} = \frac{L}{\beta}]$, one has

$$d\bar{\phi}(\bar{t}, \bar{x}) = \frac{1}{\bar{\varepsilon}} \left[\Delta\bar{\phi}(\bar{t}, \bar{x}) + \bar{\alpha}g(\bar{t}) - \bar{\beta}\bar{\phi}(\bar{t}, \bar{x})^2 - \bar{\gamma}b(\bar{t}, \bar{\phi}(\bar{t}, \bar{x})) \right] d\bar{t} + \frac{\bar{\sigma}}{\sqrt{\bar{\varepsilon}}} dW(\bar{t}, \bar{x}) ,$$

where $\bar{\varepsilon} = \frac{\beta^2}{\alpha}\varepsilon$, $\bar{\alpha} = \frac{\alpha^2\beta^2}{\gamma}$, $\bar{\beta} = \gamma\beta^2$, $\bar{\gamma} = \gamma^2\beta^2$ and $\bar{\sigma} = \frac{\sqrt{\beta}}{\alpha\gamma}\sigma$ (below, we drop the bars in order not to overload the notation). Taylor's formula yields

$$b(t, \phi_0 e_0 + \phi_\perp) = b(t, \phi_0 e_0) + \partial_\phi b(t, \phi_0 e_0) \phi_\perp + \frac{1}{2} \partial_\phi^2 b(t, \phi_0 e_0) \phi_\perp^2 + R(t, \phi_0 e_0, \phi_\perp) ,$$

where

$$R(t, \phi_0 e_0, \phi_\perp) = \frac{1}{6} \partial_\phi^3 b(t, \phi_0 e_0 + \theta \phi_\perp) \phi_\perp^3$$

for some $\theta \in [0, 1]$. Therefore, the spatially constant part $\phi_0(t)$ of the solution $\phi(t, x)$ satisfies the equation

$$\begin{aligned}d\phi_0(t) &= \frac{1}{\varepsilon} \langle e_0, \Delta\phi(t, \cdot) + \alpha g(t) - \beta \phi(t, \cdot)^2 - \gamma b(t, \phi(t, \cdot)) \rangle dt + \frac{\sigma}{\sqrt{\varepsilon}} \langle e_0, dW(t, \cdot) \rangle \\ &= \frac{1}{\varepsilon} \left[\alpha \sqrt{L} g(t) - \frac{\beta}{\sqrt{L}} \phi_0(t)^2 - \frac{\beta}{\sqrt{L}} \|\phi_\perp\|_{L^2}^2 - \gamma \sqrt{L} b(t, \phi_0(t) e_0) \right. \\ &\quad \left. - \frac{\gamma}{2\sqrt{L}} \partial_\phi^2 b(t, \phi_0(t) e_0) \|\phi_\perp(t, \cdot)\|_{L^2}^2 - \gamma \langle e_0, R(t, \phi_0(t) e_0, \phi_\perp(t, \cdot)) \rangle \right] dt \\ &\quad + \frac{\sigma}{\sqrt{\varepsilon}} dW_0(t) .\end{aligned}$$

On the other hand, the mean zero part $\phi_\perp(t, x) = \phi(t, x) - \phi_0(t)e_0(x)$ satisfies

$$\begin{aligned} d\phi_\perp(t, x) &= d\phi(t, x) - d\phi_0(t)e_0(x) \\ &= \frac{1}{\varepsilon} \left[\Delta\phi_\perp(t, x) - \left(2\frac{\beta}{\sqrt{L}}\phi_0(t) + \gamma\partial_\phi b(t, \phi_0(t)e_0(x)) \right) \phi_\perp(t, x) \right. \\ &\quad - \left(\beta + \frac{\gamma}{2}\partial_\phi^2 b(t, \phi_0(t)e_0(x)) \right) \left(\phi_\perp(t, x)^2 - \frac{1}{L} \|\phi_\perp(t, \cdot)\|_{L^2}^2 \right) \\ &\quad \left. - \gamma \left[R(t, \phi_0(t)e_0(x), \phi_\perp(t, x)) - \frac{1}{\sqrt{L}} \langle e_0, R(t, \phi_0(t)e_0, \phi_\perp(t, \cdot)) \rangle \right] \right] dt \\ &\quad + \frac{\sigma}{\sqrt{\varepsilon}} dW_\perp(t, x). \end{aligned}$$

Choosing $\alpha = \frac{1}{\sqrt{L}}$, $\beta = \sqrt{L}$ and $\gamma = \frac{1}{\sqrt{L}}$ yields the coupled SDE-SPDE system

$$\begin{aligned} d\phi_0(t) &= \frac{1}{\varepsilon} \left[g(t) - \phi_0(t)^2 - b(t, \phi_0(t)e_0) + b_0(t, \phi_0(t), \phi_\perp(t, \cdot)) \right] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_0(t), \\ d\phi_\perp(t, x) &= \frac{1}{\varepsilon} \left[\Delta\phi_\perp(t, x) + a(t, \phi_0(t))\phi_\perp(t, x) + b_\perp(t, \phi_0(t), \phi_\perp(t, \cdot)) \right] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_\perp(t, x), \end{aligned} \quad (4.1.2)$$

with

$$\begin{aligned} b_0(t, \phi_0, \phi_\perp) &= - \left(1 + \frac{1}{2L} \partial_\phi^2 b(t, \phi_0 e_0) \right) \|\phi_\perp\|_{L^2}^2 - \frac{1}{\sqrt{L}} \langle e_0, R(t, \phi_0 e_0, \phi_\perp) \rangle, \\ a(t, \phi_0) &= - 2\phi_0 - \frac{1}{\sqrt{L}} \partial_\phi b(t, \phi_0 e_0), \\ b_\perp(t, \phi_0, \phi_\perp) &= - \sqrt{L} \left(1 + \frac{1}{2L} \partial_\phi^2 b(t, \phi_0 e_0) \right) \left(\phi_\perp(\cdot)^2 - \frac{1}{L} \|\phi_\perp\|_{L^2}^2 \right) \\ &\quad - \frac{1}{\sqrt{L}} R(t, \phi_0 e_0, \phi_\perp) + \frac{1}{L} \langle e_0, R(t, \phi_0 e_0, \phi_\perp) \rangle. \end{aligned} \quad (4.1.3)$$

Note that $W_0(t)$ is a standard Brownian motion while $W_\perp(t, x)$ is an independent zero-mean space-time white noise. The terms b_0 , $b_\perp$ are no longer local non-linearities, since they involve integrals over the whole torus. This remains, however, a relatively harmless non-locality, that will not cause any problems.

4.1.3 Bounds on the remainder terms

We give some estimates on the remainders which will be useful for what follows. By similar arguments as in the proof of Proposition 3.2.1, there exist constants $d, \bar{d} > 0$ such that whenever $|\phi_0| < \bar{d}$ and $\|\phi_\perp\|_{H^1} < d$, one has

$$|R(t, \phi_0 e_0, \phi_\perp(x))| \leq M |\phi_\perp(x)|^3 \leq MC_{\text{Sob}} \|\phi_\perp\|_{H^1}^3$$

for some finite constant M . Therefore, under these conditions on ϕ_0 and $\phi_\perp$, we obtain

$$|b_\perp(t, \phi_0, \phi_\perp(x))| \leq M_1 \|\phi_\perp\|_{H^1}^3 \quad (4.1.4)$$

for some constant M_1 . Furthermore, the same argument as in Lemma 3.3.6 shows that for all $r < \frac{1}{2} - (2p_0 + 1)(\frac{1}{2} - s)$, there exists $C(r, s) < \infty$ such that

$$\|R(t, \phi_0 e_0, \phi_\perp)\|_{H^r} \leq C(r, s) \max\{\|\phi_\perp\|_{H^s}^3, \|\phi_\perp\|_{H^s}^{2p_0-1}\}.$$

Combining this with the Cauchy–Schwarz inequality, we obtain the existence of a constant M_2 such that the bounds

$$\begin{aligned} |b_0(t, \phi_0, \phi_\perp)| &\leq M_2 \max\{\|\phi_\perp\|_{H^s}^2, \|\phi_\perp\|_{H^s}^{2p_0-1}\}, \\ \|b_\perp(t, \phi_0, \phi_\perp)\|_{H^r} &\leq M_2 C(r, s) \max\{\|\phi_\perp\|_{H^s}^2, \|\phi_\perp\|_{H^s}^{2p_0-1}\} \end{aligned} \quad (4.1.5)$$

hold for all $\phi_0 \in \mathbb{R}$ such that $|\phi_0| < \bar{d}$.

4.2 Deterministic dynamics

The deterministic behaviour of the solution $(\phi_0(t), \phi_\perp(t, \cdot))$ is described by the following result.

Proposition 4.2.1 (Deterministic dynamics near the origin). *The deterministic system given by (4.1.2) with $\sigma = 0$ admits a particular solution satisfying $\phi_\perp(t, x) = 0$, while ϕ_0 obeys the ordinary differential equation*

$$\varepsilon \dot{\phi}_0(t) = g(t) - \phi_0(t)^2 - b(t, \phi_0(t)e_0). \quad (4.2.1)$$

The equation (4.2.1) for $\phi_0(t)$ is exactly of the form previously analysed in the work [8]. In particular, Theorem 2.5 in that article states that there exists a particular solution $\bar{\phi}_0(t)$ tracking $\phi_+^*(t)$, in the sense that there are constants $T_0, c_0 > 0$ such that

$$\bar{\phi}_0(t) - \phi_+^*(t) \asymp \begin{cases} \frac{\varepsilon}{|t|} & \text{for } -T_0 \leq t \leq -c_0(\sqrt{\delta} \vee \varepsilon), \\ -\frac{\varepsilon}{|t|} & \text{for } c_0\sqrt{\delta} \vee \varepsilon \leq t \leq T_0 \end{cases}$$

(Figure 4.1). Furthermore, one has

$$\bar{\phi}_0(t) \asymp \sqrt{\delta} \vee \varepsilon \quad \text{for } |t| \leq c_0\sqrt{\delta} \vee \varepsilon.$$

As a consequence, the linearisation

$$\begin{aligned} \bar{a}(t, \bar{\phi}_0(t)) &= \partial_\phi [g(t) - \phi^2 - b(t, \phi)] \Big|_{\phi=\bar{\phi}_0(t)} \\ &= -2\bar{\phi}_0(t) - \partial_\phi b(t, \bar{\phi}_0(t)) \end{aligned}$$

satisfies

$$\bar{a}(t, \bar{\phi}_0(t)) \asymp -(|t| \vee \sqrt{\delta} \vee \varepsilon) \quad (4.2.2)$$

for all $t \in [-T_0, T_0]$. By a symmetry argument, similar results, with some signs reversed, hold for a particular solutions $\hat{\phi}_0(t)$ tracking the unstable equilibrium branch $\phi_-^*(t)$.

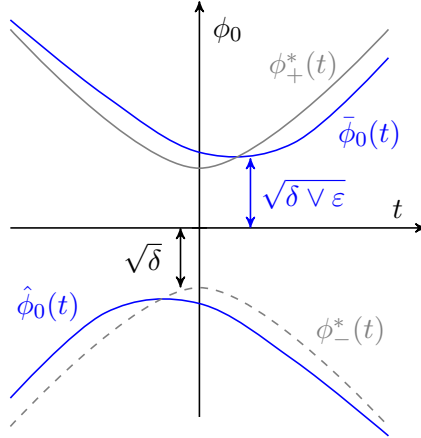


FIGURE 4.1. Equilibrium branches and associated adiabatic solutions near the avoided bifurcation point $(0, 0)$.

Proof of Proposition 4.2.1. The proof for the oscillating part $\phi_\perp$ is almost the same as the proof of Proposition 3.2.1, so that we only comment on the differences. We recall the deterministic equation for $\phi_\perp$

$$d\phi_\perp(t, x) = \frac{1}{\varepsilon} \left[\Delta\phi_\perp(t, x) + a(t, \phi_0(t))\phi_\perp(t, x) + b_\perp(t, \phi_0(t), \phi_\perp(t, \cdot)) \right] dt.$$

We define the Lyapunov function

$$V(\phi_\perp) = \frac{1}{2} \|\phi_\perp\|_{H^1}^2 = \frac{1}{2} \|\phi_\perp\|_{L^2}^2 + \frac{L^2}{2\pi^2} \|\nabla\phi_\perp\|_{L^2}^2.$$

Its time derivative satisfies

$$\begin{aligned} \varepsilon \frac{d}{dt} V(\phi_\perp(t, \cdot)) &= \langle \phi_\perp, \Delta\phi_\perp \rangle + a(t, \phi_0) \|\phi_\perp\|_{L^2}^2 + \langle \phi_\perp, b_\perp(t, \phi_0, \phi_\perp) \rangle \\ &\quad - \frac{L^2}{\pi^2} \left[\|\Delta\phi_\perp\|_{L^2}^2 + a(t, \phi_0) \langle \Delta\phi_\perp, \phi_\perp \rangle + \langle \Delta\phi_\perp, b_\perp(t, \phi_0, \phi_\perp) \rangle \right] \\ &\leq 2a(t, \phi_0) V(\phi_\perp) + \langle \phi_\perp, b_\perp(t, \phi_0, \phi_\perp) \rangle - \frac{L^2}{\pi^2} \langle \Delta\phi_\perp, b_\perp(t, \phi_0, \phi_\perp) \rangle. \end{aligned}$$

Using (4.1.4) and the Cauchy–Schwarz inequality, we obtain that for ϕ_0 and $\|\phi_\perp\|_{H^1}$ small enough, the term $\langle \phi_\perp, b_\perp(t, \phi_0, \phi_\perp) \rangle$ has order $\|\phi_\perp\|_{H^1}^3$. As for the last term, it follows from the expression (4.1.3) of $b_\perp$ that it has the form

$$\begin{aligned} \langle \Delta\phi_\perp, b_\perp(t, \phi_0, \phi_\perp) \rangle &= A(t) \langle \Delta\phi_\perp, \phi_\perp^2 \rangle + B(t) \langle \Delta\phi_\perp, 1 \rangle \|\phi_\perp\|_{L^2}^2 \\ &\quad - \frac{1}{\sqrt{L}} \langle \Delta\phi_\perp, R(t, \phi_0 e_0, \phi_\perp) \rangle + \frac{1}{L} \langle \Delta\phi_\perp, 1 \rangle \langle e_0, R(t, \phi_0 e_0, \phi_\perp) \rangle \end{aligned}$$

for some bounded functions A and B . The first term on the right-hand side can be bounded using integration by parts. The third one has order $\|\phi_\perp\|_{H^1}^2 \|\phi_\perp\|_{L^\infty}^2$, and the other two terms vanish because $\langle \Delta\phi_\perp, 1 \rangle = 0$. It follows that $\langle \Delta\phi_\perp, b_\perp(t, \phi_0, \phi_\perp) \rangle$ has also order $\|\phi_\perp\|_{H^1}^3$, provided ϕ_0 and $\|\phi_\perp\|_{H^1}$ are small enough. Writing as before $\bar{\tau}$ for the first-exit time from the set $\{V(\phi_\perp(t, \cdot)) \leq C_0\}$, we obtain

$$\varepsilon \dot{V} \leq -C_1 V + C_2 V^{3/2} \leq -C_1 \left[1 - \frac{C_0^{1/2} C_2}{C_1} \right] V$$

for all $t \leq \bar{\tau}$, and some constants $C_1, C_2 > 0$. Choosing C_0 such that $C_0^{1/2} \leq \frac{C_1}{2C_2}$, we obtain

$$\varepsilon \dot{V} \leq -\frac{1}{2}C_1 V ,$$

which allows to show that there exists a particular solution satisfying $V(t) = 0$ for all $t \in I$. As for $\phi_0(t)$, it obeys the ODE

$$\varepsilon \dot{\phi}_0(t) = g(t) - \phi_0(t)^2 - b(t, \phi_0(t)e_0) ,$$

which can be analysed in exactly the same way as in [8], concluding the proof. $\square$

4.3 Stochastic dynamics

We return now to the coupled SDE–SPDE system (4.1.2) with $\sigma > 0$. In this section, we analyse separately the dynamics for the spatial mean solution and the one for the oscillating part.

Let $\zeta(t)$ be the solution of

$$\varepsilon \dot{\zeta}(t) = 2\bar{a}(t, \bar{\phi}_0(t))\zeta(t) + 1$$

with initial condition $\zeta(-T_0) = (2|\bar{a}(-T_0, \bar{\phi}_0(-T_0))|)^{-1}$. This function is related to the variance of the linearisation around $\bar{\phi}_0(t)$ of the equation for ϕ_0 . It can be written explicitly as

$$\zeta(t) = \frac{1}{2|\bar{a}(-T_0, \bar{\phi}_0(-T_0))|} e^{2\bar{\alpha}(t, -T_0)/\varepsilon} + \frac{1}{\varepsilon} \int_{-T_0}^t e^{2\bar{\alpha}(t, t_1)/\varepsilon} dt_1 ,$$

where $\bar{\alpha}(t, t_1) = \int_{t_1}^t \bar{a}(u, \bar{\phi}_0(u)) du$ is the curvature accumulated between times t_1 and t . Using (4.2.2), one obtains

$$\zeta(t) \asymp \frac{1}{|\bar{a}(t, \bar{\phi}_0(t))|} \asymp \frac{1}{|t| \vee \sqrt{\delta} \vee \varepsilon} \quad \forall t \in [-T_0, T_0] , \quad (4.3.1)$$

see [8, Equation (4.18)]. With these notations in place, we are able to define the sets

$$\begin{aligned} \mathcal{B}_0(h) &= \left\{ (t, \phi_0) : t \in [-T_0, T_0], |\phi_0 - \bar{\phi}_0(t)| < h\sqrt{\zeta(t)} \right\} , \\ \mathcal{B}_\perp(h_\perp) &= \left\{ (t, \phi_\perp) : t \in [-T_0, T_0], \|\phi_\perp\|_{H^s} < h_\perp \right\} , \end{aligned}$$

where $s \in (0, \frac{1}{2})$, and $h, h_\perp > 0$. Denote by $\tau_{\mathcal{B}_0(h)}$ and $\tau_{\mathcal{B}_\perp(h_\perp)}$ the first exit times of ϕ_0 from $\mathcal{B}_0(h)$ and $\phi_\perp$ from $\mathcal{B}_\perp(h_\perp)$ respectively.

4.3.1 Transverse stochastic dynamics for $\phi_\perp$

For a given realisation of $\phi_0(t)$, we recall the equation for $\phi_\perp$

$$d\phi_\perp(t, x) = \frac{1}{\varepsilon} \left[\Delta \phi_\perp(t, x) + a(t, \phi_0(t))\phi_\perp(t, x) + b_\perp(t, \phi_0(t), \phi_\perp(t, x)) \right] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_\perp(t, x) .$$

As in Subsection 3.3.2, it admits a solution given by

$$\begin{aligned} \phi_\perp(t, \cdot) &= \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha(t, t_1)/\varepsilon} e^{[(t-t_1)/\varepsilon]\Delta} dW(t_1, \cdot) \\ &\quad + \frac{1}{\varepsilon} \int_0^t e^{\alpha(t, t_1)/\varepsilon} e^{[(t-t_1)/\varepsilon]\Delta} b_\perp(t_1, \phi_0(t_1), \phi_\perp(t_1, \cdot)) dt_1 , \end{aligned}$$

where $\alpha(t, t_1) = \int_{t_1}^t a(u, \phi_0(u)) du$. The exit from $\mathcal{B}_\perp(h_\perp)$ is described by the following analogue of Theorem 3.3.1 and it has similar consequences only with different values of exponents. Thus, the probability of leaving $\mathcal{B}_\perp(h_\perp)$, or making a transition to the other stable equilibrium branch is exponentially small.

Theorem 4.3.1 (Transverse stochastic dynamics for $\phi_\perp$). *If T_0 is sufficiently small, then for any $s \in (0, \frac{1}{2})$ and any $\nu > 0$, there exist constants $\kappa = \kappa(s), \varepsilon_0, h_\perp^0$ and $C(\kappa, t, \varepsilon, s) > 0$ such that, whenever $0 < \varepsilon \leq \varepsilon_0$ and $0 < h_\perp \leq h_\perp^0 \varepsilon^\nu$, the solution of (4.1.1) with initial condition $\phi(-T_0, \cdot) = \bar{\phi}_0(-T_0)e_0$ satisfies*

$$\mathbb{P}\{\tau_{\mathcal{B}_\perp(h_\perp)} < t \wedge \tau_{\mathcal{B}_0(h)}\} \leq C(\kappa, t, \varepsilon, s) \exp\left\{-\kappa \frac{h_\perp^2}{2\sigma^2} \left[1 - \mathcal{O}\left(\frac{h_\perp}{\varepsilon^\nu}\right)\right]\right\},$$

for all $t \in [-T_0, T_0]$.

Proof of Theorem 4.3.1. The proof is virtually the same as the proof of Theorem 3.3.1, the only difference being that we use here the fact that $\phi_0(t)$ is bounded by a constant of order T_0 , owing to the definition of $\mathcal{B}_0(h)$. Therefore, $a(t, \phi_0)$ is bounded above by a constant of order T_0 . Since the largest eigenvalue of the Laplacian acting on mean-zero functions $\phi_\perp$ is equal to $-\pi^2/L^2$, taking T_0 small enough we obtain again a bound of the form (3.2.4) for the Lyapunov function $V = \|\phi_\perp\|_{H^1}^2$. $\square$

Remark 4.3.2. The result remains true when $\tau_{\mathcal{B}_0(h)}$ is replaced by $\inf\{t \in [-T_0, T_0] : |\phi_0(t)| > d\}$ for any sufficiently small d of order 1. $\diamond$

Remark 4.3.3. As before, the result also holds for initial conditions with a transverse part $\phi_\perp(-T_0, \cdot)$ having H^s norm up to order 1, provided one considers the probability of leaving $\mathcal{B}_\perp(h_\perp)$ after a time of order $\varepsilon \log(\|\phi_\perp(-T_0, \cdot)\|_{H^1} h_\perp^{-1})$. $\diamond$

4.3.2 Stochastic dynamics for $\phi_0(t)$

In this section, we discuss the exit or not from a neighborhood of $\bar{\phi}_0(t)$, tracking the stable equilibrium branch ϕ_+^* . We fix now a realisation $\phi_\perp(t)$.

4.3.2.1 Before the jump

We introduce the difference $\psi_0(t) = \phi_0(t) - \bar{\phi}_0(t)$, which satisfies the SDE

$$d\psi_0(t) = \frac{1}{\varepsilon} \left[\bar{a}(t, \bar{\phi}_0(t)) \psi_0(t) + \bar{b}(t, \psi_0(t)) \right] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_0(t), \quad (4.3.2)$$

where Taylor formula yields the relations

$$\bar{a}(t, \bar{\phi}_0) = -2\bar{\phi}_0 - \partial_{\phi_0} b(t, \bar{\phi}_0 e_0),$$

and

$$\begin{aligned} \bar{b}(t, \psi_0) = & - \left(1 + \frac{1}{2} \partial_{\phi_0} b(t, \bar{\phi}_0 e_0 + \theta \psi_0 e_0) \right) \psi_0^2 \\ & + b_0(t, \bar{\phi}_0(t) + \psi_0, \phi_\perp(t, \cdot)) - b_0(t, \bar{\phi}_0(t), \phi_\perp(t, \cdot)) \end{aligned}$$

for some $\theta \in (0, 1)$. By (4.1.5), there is a constant $M > 0$ such that $\bar{b}(t, \psi_0(t))$ satisfies

$$|\bar{b}(t, \psi_0(t))| \leq M\psi_0(t)^2 + 2M_2\|\phi_\perp\|_{H^s}^2 \leq M\psi_0(t)^2 + 2M_2h_\perp^2 \quad \forall t < \tau_{\mathcal{B}_\perp(h_\perp)}.$$

A solution of (4.3.2) is given by

$$\psi_0(t) = \psi_0^0(t) + \psi_0^1(t),$$

where $\psi_0^0(t)$ is the solution of the linearisation of (4.3.2), and

$$\psi_0^1(t) = \frac{1}{\varepsilon} \int_{-T_0}^t e^{\bar{\alpha}(t, t_1)/\varepsilon} \bar{b}(t_1, \psi_0(t_1)) dt_1, \quad (4.3.3)$$

where $\bar{\alpha}(t, t_1) = \int_{t_1}^t \bar{a}(t_2, \bar{\phi}_0(t_2)) dt_2$.

The stochastic dynamics near $\bar{\phi}_0(t)$ is described by the following theorem.

Theorem 4.3.4 (Stochastic dynamics near $\bar{\phi}_0(t)$). *For any $t \in [-T_0, T_0]$, let*

$$\hat{\zeta}(t) = \sup_{-T_0 \leq s \leq t} \zeta(s) \asymp \zeta(t).$$

Then there exist constants $\varepsilon_0, h_0, c_\perp, \kappa > 0$ such that, whenever $0 < \varepsilon \leq \varepsilon_0$, $0 < h \leq h_0 \hat{\zeta}(t)^{-3/2}$ and $0 < h_\perp < c_\perp h \hat{\zeta}(t)^{1/2}$, the solution of (4.1.1) with initial condition $\phi(-T_0, \cdot) = \bar{\phi}_0(-T_0)e_0$ satisfies

$$\mathbb{P}\{\tau_{\mathcal{B}_0(h)} < t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} \leq C(t, \varepsilon) \exp\left\{-\kappa \frac{h^2}{2\sigma^2}\right\},$$

where $\kappa = 1 - \mathcal{O}(h \hat{\zeta}(t)^{3/2})$ and $C(t, \varepsilon) = \frac{\bar{\alpha}(t, -T_0)}{\varepsilon^2} + 2$. As before, the bound extends to general initial conditions in $\mathcal{B}_\perp(h_\perp)$ with $(-T_0, \phi_0(-T_0))$ in $\mathcal{B}_0(h)$.

Notice that $\hat{\zeta}(t)$ has similar asymptotic behaviour as $\zeta(t)$ in (4.3.1). Then, a consequence is that there are two qualitatively different regimes, depending on the noise intensity:

- *Weak-noise regime:* if $\sigma \ll (\delta \vee \varepsilon)^{3/4}$, Theorem 4.3.4 can be applied for any $t \in [-T_0, T_0]$, and shows that $\phi_0(t)$ remains close to $\bar{\phi}_0(t)$ with high probability during the whole time interval (Figure 4.2).
- *Strong-noise regime:* if $\sigma \gg (\delta \vee \varepsilon)^{3/4}$, Theorem 4.3.4 can only be applied up to times t of order $-\sigma^{2/3}$, showing that $\phi_0(t)$ is unlikely to become negative up to times of that order (Figure 4.3).

In weak noise regime, the probability of leaving either $\mathcal{B}_0(h)$ or $\mathcal{B}_\perp(h_\perp)$ before time t is given by

$$\begin{aligned} & \mathbb{P}\{\tau_{\mathcal{B}_0(h)} \wedge \tau_{\mathcal{B}_\perp(h_\perp)} < t\} \\ &= \mathbb{P}\{\tau_{\mathcal{B}_0(h)} \wedge \tau_{\mathcal{B}_\perp(h_\perp)} < t, \tau_{\mathcal{B}_0(h)} < \tau_{\mathcal{B}_\perp(h_\perp)}\} + \mathbb{P}\{\tau_{\mathcal{B}_0(h)} \wedge \tau_{\mathcal{B}_\perp(h_\perp)} < t, \tau_{\mathcal{B}_\perp(h_\perp)} \leq \tau_{\mathcal{B}_0(h)}\} \\ &= \mathbb{P}\{\tau_{\mathcal{B}_0(h)} < t, \tau_{\mathcal{B}_0(h)} < \tau_{\mathcal{B}_\perp(h_\perp)}\} + \mathbb{P}\{\tau_{\mathcal{B}_\perp(h_\perp)} < t, \tau_{\mathcal{B}_\perp(h_\perp)} \leq \tau_{\mathcal{B}_0(h)}\} \\ &= \mathbb{P}\{\tau_{\mathcal{B}_0(h)} < t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} + \mathbb{P}\{\tau_{\mathcal{B}_\perp(h_\perp)} < t \wedge \tau_{\mathcal{B}_0(h)}\}, \end{aligned}$$

where the first probability on the right-hand side is bounded by Theorem 4.3.4 and the second one by Theorem 4.3.1. Thus, we conclude that the behaviour of $\phi_0(t)$ in this regime does not differ much from the behaviour of the deterministic solution $\bar{\phi}_0(t)$ during the whole time interval $[-T_0, T_0]$. However, in the strong-noise regime, the situation is different.

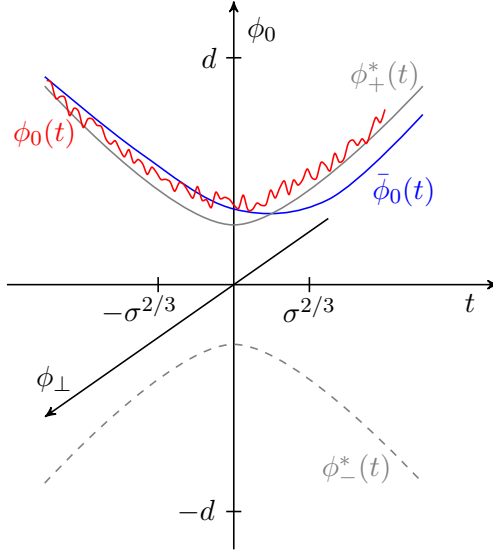


FIGURE 4.2. Weak noise regime $\sigma \ll (\delta \vee \varepsilon)^{3/4}$. The equilibrium branches $\phi_{\pm}^*(t)$, as well as the deterministic solution $\bar{\phi}_0(t)$, belong to the hyperplane $\{\phi_{\perp} = 0\}$, while $\phi_0(t)$ denotes the projection of the solution $\phi(t, x)$ on this hyperplane.

4.3.2.2 Transition regime

We consider now that σ is sufficiently large for a transition to take place i.e. $\sigma \geq (\varepsilon \vee \delta)^{3/4}$. Theorem 4.3.4 shows that sample paths are concentrated near the adiabatic solution tracking the stable potential well at ϕ_+^* up to times of order $-\sigma^{2/3}$. The following theorem describes the behaviour of sample paths as time increases. We recall the linearisation of f at $\hat{\phi}_0$ satisfies

$$\hat{a}(t, \hat{\phi}_0(t)) \asymp (|t| \vee \sqrt{\delta \vee \varepsilon}) \asymp |\bar{a}(t, \bar{\phi}_0(t))| \asymp \frac{1}{\zeta(t)}. \quad (4.3.4)$$

Theorem 4.3.5 (Strong-noise regime). *Fix sufficiently small constants $d, c_1 > 0$. Let $h > 0$ be such that*

$$\bar{\phi}_0(t) + h\sqrt{\zeta(t)} \leq d \quad \forall t \in [-c_1\sigma^{2/3}, c_1\sigma^{2/3}]. \quad (4.3.5)$$

Then there exist constants $\kappa, \bar{c}_{\perp} > 0$ such that for

$$0 < h_{\perp} < \bar{c}_{\perp} \left[\sigma^{2/3} \wedge \sqrt{h} \hat{\zeta}(t)^{-1/4} \right],$$

any solution of (4.1.1) starting at time $-c_1\sigma^{2/3}$ with an initial value ϕ_0 belonging to the interval $(-d, \bar{\phi}_0(-c_1\sigma^{2/3} + \frac{1}{2}h\sqrt{\zeta(-c_1\sigma^{2/3})})]$ satisfies

$$\begin{aligned} \mathbb{P}\left\{ \phi_0(t_1) > -d \forall t_1 \in [-c_1\sigma^{2/3}, t \wedge \tau_{\mathcal{B}_{\perp}(h_{\perp})} \right\} &\leq \frac{3}{2} \exp\left\{ -\kappa \frac{\hat{\alpha}(t, -c_1\sigma^{2/3})}{\varepsilon \log(\sigma^{-1})} \right\} \\ &\quad + C(t, \varepsilon) e^{-\kappa h^2/\sigma^2} \end{aligned} \quad (4.3.6)$$

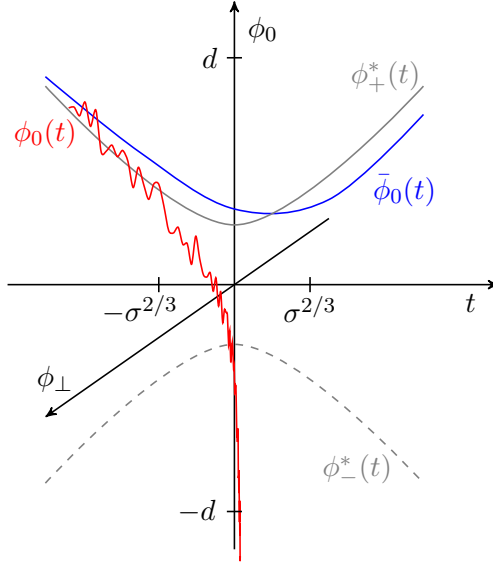


FIGURE 4.3. Strong noise regime $\sigma \gg (\delta \vee \varepsilon)^{3/4}$. Solutions are likely to cross the unstable equilibrium branch $\phi_-^*(t)$.

for all $t \in [-c_1\sigma^{2/3}, c_1\sigma^{2/3}]$, where

$$\hat{\alpha}(t, t_1) = \int_{t_1}^t \hat{a}(t_2, \hat{\phi}_0(t_2)) dt_2, \quad \hat{a}(t, \hat{\phi}_0(t)) = \partial_\phi [g(t) - \phi^2 + b(t, \phi)] \Big|_{\phi=\hat{\phi}_0(t)}$$

$$\text{and } C(t, \varepsilon) = \frac{|\bar{\alpha}(t, -c_1\sigma^{2/3})|}{\varepsilon^2} + 2.$$

As time increases, it quickly becomes very unlikely not to reach and overcome the unstable solution $\hat{\phi}_0(t)$ tracking ϕ_-^* .

Remark 4.3.6. The condition (4.3.5) is required since we did not make any assumptions on the behaviour of f for $x \geq d$. For instance, our results apply if there exist more equilibrium branches above d . If, however, there are no such branches, as in the case of the Allen–Cahn equation with drift term (3.1.2), this condition can probably be relaxed. $\diamond$

One can show also that if ϕ_0 reaches ϕ_-^* , then it is likely to reach a neighbourhood of the next stable equilibrium branch as well, see Proposition 4.7 in [8]. The following result finishes the description of the dynamics.

Proposition 4.3.7 (Reaching level $-d_0 < -d$). *There exists a constant $M > 0$ such that if the drift term f satisfies*

$$f(t, \phi) \leq -f_0 - Mh_\perp^2 \quad \forall (t, x) \in [-T_0, T_0] \times [-d_0, -d + \rho]$$

for some constants $d_0, f_0 > 0$ and $\rho \in (0, d)$, then there exist constant $\tilde{c}, \tilde{\kappa} > 0$ such that for all $t_0 \in [-T_0, T_0 - \tilde{c}\varepsilon]$, the solution of (4.1.1) with initial condition $\phi_0(t_0) = -d$ satisfies

$$\mathbb{P}\{\phi_0(t_1) > -d_0 \ \forall t_1 \in [t_0, (t_0 + \tilde{c}\varepsilon) \wedge \tau_{B_\perp}(h_\perp)]\} \leq e^{-\tilde{\kappa}/\sigma^2}.$$

We notice that once we reach a neighbourhood of the next stable equilibrium branch ϕ^* , one can apply Theorem 3.3.1 to describe the dynamics up to the next (avoided) bifurcation point (Figure 4.4).

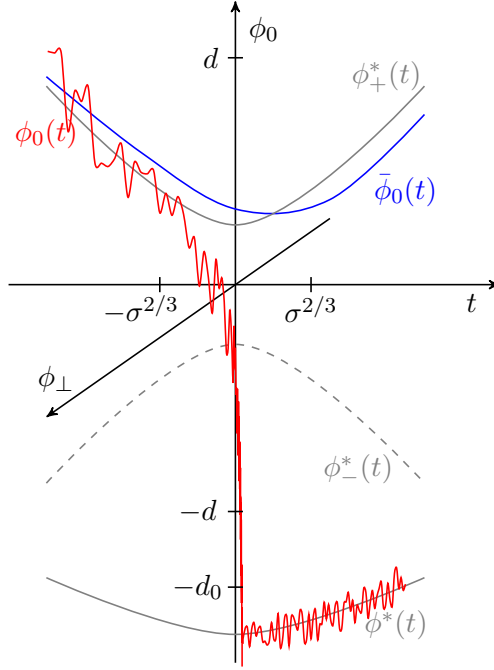


FIGURE 4.4. Strong noise regime, behaviour after reaching level $-d$. If the drift term is negative, bounded away from zero, in an interval $[-d_0, -d]$, solutions are likely to reach $-d_0$ after another time of order ε .

4.3.2.3 Proof of Theorem 4.3.4

According to [8, Proposition 3.8], $\psi_0^0(t)$ is likely to remain in a strip of width proportional to $\sqrt{\zeta(t)}$. More precisely,

$$\mathbb{P}\left\{\sup_{-T_0 \leq t_1 \leq t} \frac{|\psi_0^0(t_1)|}{\sqrt{\zeta(t_1)}} \geq h\right\} \leq C(t, \varepsilon) \exp\left\{-\frac{h^2}{2\sigma^2}(1 - \mathcal{O}(\varepsilon))\right\}, \quad (4.3.7)$$

where

$$C(t, \varepsilon) = \frac{|\bar{\alpha}(t, -T_0)|}{\varepsilon^2} + 2.$$

We now use this estimate to prove Theorem 4.3.4.

Proof of Theorem 4.3.4. For any decomposition $h = h_0 + h_1$ with $h_0, h_1 > 0$, one has

$$\begin{aligned} \mathbb{P}\{\tau_{\mathcal{B}_0(h)} < t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} &= \mathbb{P}\{\tau_{\mathcal{B}_0(h)} < t, \tau_{\mathcal{B}_0(h)} < \tau_{\mathcal{B}_\perp(h_\perp)}\} \\ &\leq \mathbb{P}\left\{\sup_{-T_0 \leq t_1 \leq t \wedge \tau_{\mathcal{B}_0(h)}} \frac{|\psi_0^0(t_1)|}{\sqrt{\zeta(t_1)}} \geq h, \tau_{\mathcal{B}_0(h)} < \tau_{\mathcal{B}_\perp(h_\perp)}\right\} \\ &\leq \mathbb{P}\left\{\sup_{-T_0 \leq t_1 \leq t} \frac{|\psi_0^0(t_1)|}{\sqrt{\zeta(t_1)}} \geq h_0\right\} + \mathbb{P}\left\{\sup_{-T_0 \leq t_1 \leq t \wedge \tau_{\mathcal{B}_0(h)}} \frac{|\psi_0^1(t_1)|}{\sqrt{\zeta(t_1)}} \geq h_1, \tau_{\mathcal{B}_0(h)} < \tau_{\mathcal{B}_\perp(h_\perp)}\right\} \end{aligned}$$

The first probability satisfies the bound (4.3.7), so that it remains to control the second one. By (4.3.3)

and for all $t_1 \leq t \wedge \tau_{\mathcal{B}_0(h)} < \tau_{\mathcal{B}_\perp(h_\perp)}$, as in [8, Proposition 3.10], we have the bound

$$\begin{aligned} \frac{|\psi_0^1(t_1)|}{\sqrt{\zeta(t_1)}} &\leq \frac{(Mh^2\zeta(t_1) + M_2h_\perp^2)}{\sqrt{\zeta(t_1)}} \frac{1}{\varepsilon} \int_{-T_0}^{t_1} e^{\bar{\alpha}(t_1, t_2)/\varepsilon} dt_2 \\ &\leq Mh^2\hat{\zeta}(t)^{3/2} + M_2h_\perp^2\hat{\zeta}(t)^{1/2}. \end{aligned}$$

Choosing $h_\perp^2 \leq \frac{M}{M_2}h^2\hat{\zeta}(t)$ and $h_1 = \text{const } h^2\hat{\zeta}(t)^{3/2}$, we get

$$\mathbb{P}\left\{\sup_{-T_0 \leq t_1 \leq t \wedge \tau_{\mathcal{B}_0(h)}} \frac{|\psi_0^1(t_1)|}{\sqrt{\zeta(t_1)}} \geq h_1, \tau_{\mathcal{B}_0(h)} < \tau_{\mathcal{B}_\perp(h_\perp)}\right\} = 0.$$

Therefore,

$$\mathbb{P}\{\tau_{\mathcal{B}_0(h)} < t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} \leq C(t, \varepsilon) \exp\left\{-\frac{h_0^2}{2\sigma^2}(1 - \mathcal{O}(\varepsilon))\right\}.$$

We thus obtain the result by choosing $h_0 = h - h_1 = h - \mathcal{O}(h^2\hat{\zeta}(t)^{3/2}) = h(1 - \mathcal{O}(h\hat{\zeta}(t)^{3/2}))$. $\square$

4.3.2.4 Proof of Theorem 4.3.5

In what follows we prove Theorem 4.3.5, where the two terms on the right-hand side of (4.3.6) bound, respectively, the probability that ϕ_0 does not reach $-d$ before time t , while staying below $\bar{\phi}_0 + h\sqrt{\zeta}$, and the probability that ϕ_0 crosses the level $\bar{\phi}_0 + h\sqrt{\zeta}$ before time t .

Proof of Theorem 4.3.5. Let h be such that $\bar{\phi}_0(t) + h\sqrt{\zeta(t)} \leq d$ for all $t \in [-c_1\sigma^{2/3}, c_1\sigma^{2/3}]$. We introduce the stopping times

$$\begin{aligned} \tau_+ &= \inf\left\{t_1 \in [-c_1\sigma^{2/3}, T_0]: \frac{\phi_0(t_1) - \bar{\phi}_0(t_1)}{\sqrt{\zeta(t_1)}} > h\right\}, \\ \tau_- &= \inf\{t_1 \in [-c_1\sigma^{2/3}, T_0]: \phi_0(t_1) < -d\}. \end{aligned}$$

Then, the probability that ϕ_0 does not reach $-d$ while $\phi_\perp$ remains in $\mathcal{B}_\perp(h_\perp)$ is given by

$$\begin{aligned} &\mathbb{P}\{\tau_- > t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} \\ &= \mathbb{P}\{\tau_- > t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}, \tau_+ \leq t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} + \mathbb{P}\{\tau_- > t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}, \tau_+ > t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} \\ &\leq \mathbb{P}\{\tau_+ \leq t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} + \mathbb{P}\{\tau_- \wedge \tau_+ > t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} \end{aligned}$$

We estimate these two terms separately and the crucial term is the second one. Since we are going to use the Markov property and restart the process at certain times, we recall the notation $\mathbb{P}^{t_0, \phi_0}$ for the law of the process started at time t_0 in ϕ_0 whenever necessary.

Proposition 4.3.8. *Under the assumptions of Theorem 4.3.5, there exist constants $\kappa_1, M_3 > 0$ such that whenever $(-c_1\sigma^{2/3}, \phi_{0,0}) \in \mathcal{B}_0(h/2)$, one has*

$$\mathbb{P}^{-c_1\sigma^{2/3}, \phi_{0,0}}\{\tau_+ \leq t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} \leq C(t, \varepsilon) \exp\left\{-\frac{\kappa_1}{2\sigma^2}\left(h - M_3h_\perp^2\sqrt{\hat{\zeta}(t)}\right)^2\right\},$$

for all $t \in [-c_1\sigma^{2/3}, T_0]$, where $C(t, \varepsilon) = \frac{|\bar{\alpha}(t, -c_1\sigma^{2/3})|}{\varepsilon^2} + 2$.

Proof. The solution of (4.3.2) is given by

$$\psi_0(t) = \psi_0^0(t) + \frac{1}{\varepsilon} \int_{-c_1 \sigma^{2/3}}^t e^{\bar{\alpha}(t, t_1)/\varepsilon} \bar{b}(t_1, \psi_0(t_1)) dt_1.$$

We define a partition $-c_1 \sigma^{2/3} = u_0 < u_1 < \dots < u_K = t$ of $[-c_1 \sigma^{2/3}, t]$ by

$$\bar{\alpha}(u_k, u_{k-1}) = \varepsilon \quad \text{for } 1 \leq k \leq K = \left\lceil \frac{\bar{\alpha}(t, -c_1 \sigma^{2/3})}{\varepsilon} \right\rceil.$$

We also introduce the notation $\rho_k = \frac{1}{2}h\sqrt{\zeta(u_k)}$. As shown in [8, Proposition 3.12], the Markov property implies

$$\begin{aligned} \mathbb{P}^{-c_1 \sigma^{2/3}, \phi_{0,0}} \{ \tau_+ < t \wedge \tau_{\mathcal{B}_\perp(h_\perp)} \} &= \mathbb{P}^{-c_1 \sigma^{2/3}, \phi_{0,0}} \left\{ \sup_{-c_1 \sigma^{2/3} \leq t_1 \leq t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}} \frac{\psi_0(t_1)}{\sqrt{\zeta(t_1)}} > h \right\} \\ &\leq \sum_{k=0}^{K-1} Q_k, \end{aligned}$$

where

$$\begin{aligned} Q_k &= \sup_{\psi_0(u_k) \leq \rho_k} \left[\mathbb{P}^{u_k, \psi_0(u_k)} \left\{ \sup_{u_k \leq t_1 \leq u_{k+1}} \frac{\psi_0(t_1)}{\sqrt{\zeta(t_1)}} > h \right\} \right. \\ &\quad \left. + \mathbb{P}^{u_k, \psi_0(u_k)} \left\{ \sup_{u_k \leq t_1 \leq u_{k+1}} \frac{\psi_0(t_1)}{\sqrt{\zeta(t_1)}} \leq h, \psi_0(u_{k+1}) > \rho_{k+1} \right\} \right]. \end{aligned}$$

For h smaller than a constant of order 1 and $t_1 \leq \tau_{\mathcal{B}_\perp(h_\perp)}$, (4.1.5) shows that $\bar{b}(t_1, \psi_0(t_1))$ is bounded by $M_2 h_\perp^2$. It follows that for any $t_1 \in [u_k, u_{k+1}]$, one has

$$\begin{aligned} \psi_0^1(t_1) &\leq M_2 h_\perp^2 \int_{-c_1 \sigma^{2/3}}^t \frac{1}{-\bar{a}(t_1, \bar{\phi}_0(t_1))} \frac{-\bar{a}(t_1, \bar{\phi}_0(t_1))}{\varepsilon} e^{\bar{\alpha}(t, t_1)/\varepsilon} dt_1 \\ &\leq M_2 h_\perp^2 \sup_{u \in [u_k, u_{k+1}]} \frac{1}{|\bar{a}(u, \bar{\phi}_0(u))|}. \end{aligned} \tag{4.3.8}$$

Therefore, there is a constant M_3 such that for any $t_1 \in [u_k, u_{k+1}]$ one has

$$\frac{\psi_0^1(t_1)}{\sqrt{\zeta(t_1)}} \leq M_3 h_\perp^2 \sqrt{\hat{\zeta}(u_{k+1})}.$$

Proceeding as in the proof of [8, Proposition 3.12], but with a shifted value of h , one obtains

$$P_k \leq \exp \left\{ -\frac{\kappa_1}{\sigma^2} \left(h - M_3 h_\perp^2 \sqrt{\hat{\zeta}(u_{k+1})} \right)^2 \right\}$$

for some $\kappa_1 > 0$, which implies the claimed result. $\square$

The main part of the proof is contained in the following estimate, whose proof is very close in spirit to the proof of [8, Proposition 4.6], but with some changes due to the zero-mean part $\phi_\perp$ of the field.

Proposition 4.3.9. *Under the assumptions of Theorem 4.3.5, there exists a choice of $c_1 > 0$ and constants $\bar{c}_\perp$ and $\kappa_2 > 0$ such that for $0 < h_\perp < \bar{c}_\perp \sigma^{2/3}$, and all initial conditions $\phi_{0,0}$ in the interval $(-d, \bar{\phi}_0(-c_1 \sigma^{2/3}) + h\sqrt{\zeta(-c_1 \sigma^{2/3})}]$, one has*

$$\begin{aligned} & \mathbb{P}^{-c_1 \sigma^{2/3}, \phi_{0,0}} \{ \tau_- \wedge \tau_+ > t \wedge \tau_{\mathcal{B}_\perp(h_\perp)} \} \\ &= \mathbb{P}^{-c_1 \sigma^{2/3}, \phi_{0,0}} \left\{ -d < \phi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \quad \forall t_1 \in [-c_1 \sigma^{2/3}, t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}] \right\} \\ &\leq \frac{3}{2} \exp \left\{ -\kappa_2 \frac{\hat{\alpha}(t, -c_1 \sigma^{2/3})}{\log(\sigma^{-1})\varepsilon} \right\}. \end{aligned}$$

Proof. Let $\varrho \geq 1$ and define a partition $-c_1 \sigma^{2/3} = u_0 < u_1 < \dots < u_K = t$ by

$$\hat{\alpha}(u_k, u_{k-1}) = \varrho \varepsilon \quad \text{for } 1 \leq k \leq K = \left\lceil \frac{\hat{\alpha}(t, -c_1 \sigma^{2/3})}{\varrho \varepsilon} \right\rceil. \quad (4.3.9)$$

Writing

$$Q_k = \sup_{\phi_0(u_k) \in (-d, \bar{\phi}_0(u_k) + h\sqrt{\zeta(u_k)})} \mathbb{P}^{u_k, \phi_0(u_k)} \left\{ -d < \phi_0 \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \quad \forall t_1 \in [u_k, u_{k+1}] \right\},$$

we have, by the Markov property,

$$\begin{aligned} & \mathbb{P}^{-c_1 \sigma^{2/3}, \phi_{0,0}} \left\{ -d < \phi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \quad \forall t_1 \in [-c_1 \sigma^{2/3}, t] \right\} \\ &= \mathbb{E}^{-c_1 \sigma^{2/3}, \phi_{0,0}} \left\{ 1_{\{-d < \phi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \quad \forall t_1 \in [-c_1 \sigma^{2/3}, u_{K-1}]\}} \right. \\ &\quad \times \mathbb{P}^{u_{K-1}, \phi_0(u_{K-1})} \left\{ -d < \phi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \quad \forall t_1 \in [u_{K-1}, u_K] \right\} \Big\} \\ &\leq Q_{K-1} \mathbb{P}^{-c_1 \sigma^{2/3}, \phi_{0,0}} \left\{ -d < \phi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \quad \forall t_1 \in [-c_1 \sigma^{2/3}, u_{K-1}] \right\} \\ &\leq \dots \leq \prod_{k=0}^{K-1} Q_k. \end{aligned}$$

Our plan is to show that for an appropriate choice of ϱ , Q_k is bounded away from 1 for $k = 0, \dots, K-1$. In order to estimate Q_k we shall distinguish three cases corresponding to ϕ_0 crossing the levels $\bar{\phi}_0$ and $\hat{\phi}_0$ before reaching $-d$. We set

$$\overline{M}_k = M_2 h_\perp^2 \sup_{u \in [u_k, u_{k+1}]} \frac{1}{|\bar{a}(u, \bar{\phi}_0(u))|}, \quad \widehat{M}_k = M_2 h_\perp^2 \sup_{u \in [u_k, u_{k+1}]} \frac{1}{\hat{a}(u, \hat{\phi}_0(u))},$$

and introduce a further subdivision $u_k < \tilde{u}_{k,1} < \tilde{u}_{k,2} < u_{k+1}$ defined by

$$\hat{\alpha}(\tilde{u}_{k,1}, u_k) = \frac{1}{3} \varrho \varepsilon, \quad \hat{\alpha}(\tilde{u}_{k,2}, u_k) = \frac{2}{3} \varrho \varepsilon.$$

Define the stopping times

$$\begin{aligned} \tau_{k,1} &= \inf \left\{ t_1 \in [u_k, \tilde{u}_{k,1}] : \phi_0(t_1) \leq \bar{\phi}_0(t_1) + \overline{M}_k \right\}, \\ \tau_{k,2} &= \inf \left\{ t_1 \in [u_k, \tilde{u}_{k,2}] : \phi_0(t_1) \leq \hat{\phi}_0(t_1) + \widehat{M}_k \right\}. \end{aligned}$$

Then we can write

$$\begin{aligned}
& \mathbb{P}^{u_k, \phi_0(u_k)} \{ -d < \phi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \forall t_1 \in [u_k, u_{k+1}] \} \\
& \leq \mathbb{P}^{u_k, \phi_0(u_k)} \left\{ \bar{\phi}_0(t_1) + \bar{M}_k < \phi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \forall t_1 \in [u_k, \tilde{u}_{k,1}] \right\} \\
& \quad + \mathbb{E}^{u_k, \phi_0(u_k)} \left\{ 1_{\{\tau_{k,1} < \tilde{u}_{k,1}\}} \right. \\
& \quad \times \mathbb{P}^{\tau_{k,1}, \phi_0(\tau_{k,1})} \{ -d < \phi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \forall t_1 \in [\tau_{k,1}, u_{k+1}] \} \left. \right\}. \tag{4.3.10}
\end{aligned}$$

We start by bounding the first term on the right-hand side. Let

$$\psi_0^{(k)}(t_1) = \psi_0(u_k) e^{\bar{\alpha}(t_1, u_k)/\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}} \int_{u_k}^{t_1} e^{\bar{\alpha}(t_1, v)/\varepsilon} dW_0(v)$$

be the solution of the equation linearised around $\bar{\phi}_0(t)$, starting in $\psi_0(u_k) = \phi_0(u_k) - \bar{\phi}_0(u_k)$. Then it follows from (4.3.8) that

$$\psi_0(t) \leq \psi_0^{(k)}(t) + \bar{M}_k \quad \forall t_1 \in [u_k, u_{k+1}].$$

Note that $\psi_0^{(k)}(\tilde{u}_{k,1})$ is a normal random variable with parameters

$$\begin{aligned}
\mathbb{E}[\psi_0^{(k)}(\tilde{u}_{k,1})] &= \psi_0(u_k) e^{\bar{\alpha}(\tilde{u}_{k,1}, u_k)/\varepsilon} \leq \psi_0(u_k) e^{-\varrho/3R} \\
\text{Var}(\psi_0^{(k)}(\tilde{u}_{k,1})) &= \frac{\sigma^2}{\varepsilon} \int_{u_k}^{\tilde{u}_{k,1}} e^{2\bar{\alpha}(t_1, v)/\varepsilon} dv \\
&\geq \frac{\sigma^2}{2} \inf_{u_k \leq t_1 \leq u_{k+1}} \frac{1}{|\bar{a}(u, \bar{\phi}_0(u))|} [1 - e^{-2\varrho/3R}],
\end{aligned}$$

where $R > 0$ is a constant such that $\hat{a}(t_1, \hat{\phi}_0(t_1)) \leq R|\bar{a}(t_1, \bar{\phi}_0(t_1))|$ for all $t_1 \in [-c_1\sigma^{2/3}, t]$. Then André's reflection principle shows that the first term on the right-hand side of (4.3.10) is bounded above by

$$\begin{aligned}
\mathbb{P}^{u_k, \phi_0(u_k)} \{ \psi_0^{(k)}(t_1) > 0 \forall t_1 \in [u_k, \tilde{u}_{k,1}] \} &= 1 - 2\mathbb{P}^{u_k, \phi_0(u_k)} \{ \psi_0^{(k)}(t_1) \leq 0 \} \\
&= 2\mathbb{P}^{u_k, \phi_0(u_k)} \{ \psi_0^{(k)}(t_1) > 0 \} - 1 \\
&\leq \frac{2}{\sqrt{\pi}} \frac{h}{\sigma} C_1(k) \frac{e^{-\varrho/3R}}{\sqrt{1 - e^{-2\varrho/3R}}}, \tag{4.3.11}
\end{aligned}$$

where

$$C_1(k) = \sup_{u_k \leq t_1 \leq u_{k+1}} \sqrt{|\bar{a}(u_k, \bar{\phi}_0)|} \sqrt{\zeta(u_k)}$$

is a constant of order 1, owing to (4.3.4).

In order to bound the second term on the right-hand side of (4.3.10), we set $\varphi_0(t) = \phi_0(t) - \hat{\phi}_0(t)$, where we recall that $\hat{\phi}_0(t)$ is the deterministic solution tracking ϕ_-^* . Observe that if $\tau_{k,1} < \tilde{u}_{k,1}$, we also

have

$$\begin{aligned}
& \mathbb{P}^{\tau_{k,1}, \varphi_0(\tau_{k,1})} \left\{ -d < \hat{\phi}_0(t_1) + \varphi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \quad \forall t_1 \in [\tau_{k,1}, u_{k+1}] \right\} \\
& \leq \mathbb{P}^{\tau_{k,1}, \varphi_0(\tau_{k,1})} \left\{ \widehat{M}_k < \varphi_0(t_1) \leq \bar{\phi}_0(t_1) - \hat{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \quad \forall t_1 \in [\tau_{k,1}, \tilde{u}_{k,2}] \right\} \\
& \quad + \mathbb{E}^{u_k, \varphi_0(u_k)} \left\{ 1_{\{\tau_{k,2} < \tilde{u}_{k,2}\}} \right. \\
& \quad \times \mathbb{P}^{\tau_{k,2}, \varphi_0(\tau_{k,2})} \left\{ -d < \hat{\phi}_0(t_1) + \varphi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \quad \forall t_1 \in [\tau_{k,2}, u_{k+1}] \right\} \left. \right\}.
\end{aligned} \tag{4.3.12}$$

To bound the first term on the right-hand side, we introduce the linear process

$$\varphi_0^{(k)}(t_1) = \varphi_0(\tilde{u}_{k,1}) e^{\hat{\alpha}(t_1, \tilde{u}_{k,1})/\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}} \int_{\tilde{u}_{k,1}}^{t_1} e^{\hat{\alpha}(t_1, v)/\varepsilon} dW_0(v)$$

which satisfies

$$\varphi_0(t) \leq \varphi_0^{(k)}(t) + \widehat{M}_k \quad \forall t_1 \in [\tilde{u}_{k,1}, u_{k+1}].$$

Then we have the estimates

$$\begin{aligned}
\mathbb{E}[\varphi_0^{(k)}(\tilde{u}_{k,2})] &= \varphi_0(\tau_{k,1}) e^{\hat{\alpha}(\tilde{u}_{k,2}, \tau_{k,1})/\varepsilon} \\
&\leq [\bar{\phi}_0(\tau_{k,1}) + \overline{M}_k - \hat{\phi}_0(\tau_{k,1})] e^{\varrho/3}, \\
e^{-2\hat{\alpha}(\tilde{u}_{k,2}, \tau_{k,1})/\varepsilon} \text{Var}(\varphi_0^{(k)}(\tilde{u}_{k,2})) &\geq \inf_{u_k \leq t_1 \leq u_{k+1}} \frac{\sigma^2}{2\hat{\alpha}(t_1, \hat{\phi}_0(t_1))} [1 - e^{-2\varrho/3}].
\end{aligned}$$

The first term on the right-hand side of (4.3.12) can then be bounded by

$$\frac{2}{\sqrt{\pi}} \frac{1}{\sigma} C_2(k) \frac{1}{\sqrt{1 - e^{-2\varrho/3}}}, \tag{4.3.13}$$

where

$$C_2(k) = \sup_{u_k \leq t_1 \leq u_{k+1}} \sqrt{\hat{\alpha}(t_1, \hat{\phi}_0(t_1))} \sup_{u_k \leq t_1 \leq u_{k+1}} (\bar{\phi}_0(t_1) + \overline{M}_k - \hat{\phi}_0(t_1)).$$

Finally, in order to estimate the second summand in (4.3.12), we use the end point estimate

$$\begin{aligned}
& \mathbb{P}^{\tau_{k,2}, \varphi_0(\tau_{k,2})} \left\{ -d < \hat{\phi}_0(t_1) + \varphi_0(t_1) \leq \bar{\phi}_0(t_1) + h\sqrt{\zeta(t_1)} \quad \forall t_1 \in [\tau_{k,2}, u_{k+1}] \right\} \\
& \leq \mathbb{P}^{\tau_{k,2}, \varphi_0(\tau_{k,2})} \left\{ -d < \hat{\phi}_0(u_{k+1}) + \varphi_0(u_{k+1}) \right\} \\
& \leq \frac{1}{2} + \frac{1}{\sqrt{\pi}} \frac{1}{\sigma} C_3(k) \left[d + \hat{\phi}_0(u_{k+1}) + \widehat{M}_k \right] \frac{e^{-\varrho/3}}{\sqrt{1 - e^{-2\varrho/3}}},
\end{aligned} \tag{4.3.14}$$

where

$$C_3(k) = \sup_{u_k \leq t_1 \leq u_{k+1}} \sqrt{\hat{\alpha}(t_1, \hat{\phi}_0(t_1))}.$$

Summing (4.3.11), (4.3.13) and (4.3.14) we get the existence of a constant $C_0 > 0$ such that

$$Q_k \leq \frac{1}{2} + C_0 \left[\frac{h}{\sigma} e^{-\varrho/3R} C_1(k) + \frac{1}{\sigma} C_2(k) + \frac{1}{\sigma} e^{-\varrho/3} C_3(k) (1 + \widehat{M}_k) \right].$$

Since

$$\begin{aligned} |t_1| &\leq c_1 \sigma^{2/3}, \\ \hat{a}(t_1, \hat{\phi}_0(t_1)) &\asymp |t_1| \vee \sqrt{\delta \vee \varepsilon}, \\ \bar{\phi}_0(t_1) - \hat{\phi}_0(t_1) &\leq \bar{c}_1 |t_1|, \end{aligned}$$

where $\bar{c}_1$ is proportional to c_1 and $\varrho \geq 1$, there exists another constant C_4 such that

$$Q_k \leq \frac{1}{2} + C_4 \left[\frac{h}{\sigma} e^{-\varrho/3R} + \bar{c}_1^{3/2} + \frac{h_{\perp}^2}{\sigma^{4/3}} (1 + e^{-\varrho/3}) + \frac{1}{\sigma^{2/3}} e^{-\varrho/3} \right].$$

Choosing $h_{\perp} \leq \bar{c}_{\perp} \sigma^{2/3}$ we get

$$Q_k \leq \frac{1}{2} + C_4 \left[\frac{h}{\sigma} e^{-\varrho/3R} + \bar{c}_1^{3/2} + 2\bar{c}_{\perp}^2 + \frac{1}{\sigma^{2/3}} e^{-\varrho/3} \right].$$

For c_1 such that $\bar{c}_1^{3/2} = 2\bar{c}_{\perp}^2 = \frac{1}{24C_4}$ and

$$\varrho = 3R \log \left(36C_4 \frac{h}{\sigma} \right) \vee 3 \log \left(\frac{18C_4}{c_1 \sigma^{2/3}} \right) \vee 1,$$

Q_k is bounded by $\frac{2}{3}$ for $k = 0, \dots, K-1$. We conclude that with this choice of ϱ , we have

$$\begin{aligned} \mathbb{P}^{-c_1 \sigma^{2/3}, \phi_0} \left\{ -d < \phi_0(t_1) \leq \bar{\phi}_0(t_1) + h \sqrt{\zeta(t_1)} \forall t_1 \in [-c_1 \sigma^{2/3}, t] \right\} \\ \leq \left(\frac{2}{3} \right)^{K-1} = \frac{3}{2} \exp \left\{ -K \log \left(\frac{3}{2} \right) \right\} \end{aligned}$$

which yields the claimed result, owing to our choice (4.3.9) of K , and the fact that ϱ has order $\log(\sigma^{-1})$. $\square$

The conclusion of Theorem 4.3.5 now follows immediately by combining the last two propositions. $\square$

4.3.2.5 Proof of Proposition 4.3.7

By continuity of the drift term, we can assume that f is bounded away from zero on an interval slightly larger than $[-d_0, -d]$. With this additional assumption, we are able to prove the Proposition 4.3.7.

Proof of Proposition 4.3.7. We introduce the stopping times

$$\begin{aligned} \tau_+ &= \inf \{ t_1 \in [t_0, t_0 + \tilde{c}\varepsilon] : \phi_0(t_1) > -d + \rho \}, \\ \tau_- &= \inf \{ t_1 \in [t_0, t_0 + \tilde{c}\varepsilon] : \phi_0(t_1) < -d_0 \}, \end{aligned}$$

and the process

$$\tilde{\phi}_0(t) = -d - \frac{1}{\varepsilon} f_0(t - t_0) + \frac{\sigma}{\sqrt{\varepsilon}} W_{t-t_0}.$$

Taking the constant M in the statement of the proposition equal to M_2 , one can check, in a similar way as before, that $\phi_0(t_1) \leq \tilde{\phi}_0(t_1)$ for all $t_1 \leq \tau_- \wedge \tau_+ \wedge \tau_{\mathcal{B}_\perp(h_\perp)}$. Now we observe that

$$\begin{aligned} & \mathbb{P}\{\tilde{\phi}_0(t) \geq -d_0 \ \forall t \in [t_0, t_0 + \tilde{c}\varepsilon]\} \\ & \leq \mathbb{P}\left\{\sup_{t \in [t_0, t_0 + \tilde{c}\varepsilon]} \left[\tilde{\phi}_0(t) + \frac{1}{\varepsilon}f_0(t - t_0)\right] > -d + \rho\right\} \\ & \quad + \mathbb{P}\left\{-d_0 \leq \tilde{\phi}_0(t) \leq -d + \rho - \frac{1}{\varepsilon}f_0(t - t_0) \ \forall t \in [t_0, t_0 + \tilde{c}\varepsilon]\right\}. \end{aligned}$$

The second term on the right-hand side vanishes as soon as we take $\tilde{c} > (d_0 - d + \rho)/f_0$, while the first one is equal to

$$\mathbb{P}\left\{\sup_{t \in [t_0, t_0 + \tilde{c}\varepsilon]} \frac{\sigma}{\sqrt{\varepsilon}} W_{t-t_0} > \rho\right\} \leq e^{-\rho^2/(2\tilde{c}\sigma^2)}$$

by a Bernstein-type inequality. Now we note that for any $t \in [t_0, t_0 + \tilde{c}\varepsilon]$, we have

$$\begin{aligned} \mathbb{P}\{\tau_- > t\} &= \mathbb{P}\{\tau_- > t, t \leq \tau_- \wedge \tau_+ \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} + \mathbb{P}\{\tau_+ \wedge \tau_{\mathcal{B}_\perp(h_\perp)} < t < \tau_-\} \\ &\leq \mathbb{P}\{\tilde{\phi}_0(t) \geq -d_0 \ \forall t \in [t_0, t_0 + \tilde{c}\varepsilon]\} + \mathbb{P}\{\tau_+ \wedge \tau_{\mathcal{B}_\perp(h_\perp)} < t \wedge \tau_-\}. \end{aligned}$$

We have already shown that the first term on the right-hand side is exponentially small, and the second term can be controlled as in the preceding results. $\square$

4.4 Discussion

Let us first consider the weak-noise regime $\sigma \ll (\delta \vee \varepsilon)^{3/4}$. For Theorems 4.3.1 and 4.3.4 to yield useful results, we need

$$h \gg \sigma, \quad h_\perp \gg \sigma. \quad (4.4.1)$$

If $t \geq 0$, then $\hat{\zeta}(t) \asymp (\delta \vee \varepsilon)^{-1/2}$. For the theorems to be applicable, we then need the conditions

$$h \lesssim (\delta \vee \varepsilon)^{3/4}, \quad h_\perp \lesssim h(\delta \vee \varepsilon)^{-1/4} \wedge \varepsilon^\nu,$$

where $\nu > 0$ can be chosen arbitrarily small. The weak-noise condition implies that all conditions on h and $h_\perp$ can indeed be met simultaneously. In particular, since the minimal value of $\bar{\phi}_0(t)$ has order $(\delta \vee \varepsilon)^{1/2}$, we can take h of order $(\delta \vee \varepsilon)^{3/4}$, and $h_\perp$ of order $(\delta \vee \varepsilon)^{1/2} \wedge \varepsilon^\nu$. We thus obtain

$$\begin{aligned} \mathbb{P}\{\exists t \in [-T_0, T_0]: \phi_0(t) < 0\} &\leq \mathbb{P}\{\tau_{\mathcal{B}_0(h)} \wedge \tau_{\mathcal{B}_\perp(h_\perp)} < T_0\} \\ &= \mathbb{P}\{\tau_{\mathcal{B}_\perp(h_\perp)} < T_0 \wedge \tau_{\mathcal{B}_0(h)}\} + \mathbb{P}\{\tau_{\mathcal{B}_0(h)} < T_0 \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\} \\ &\leq C_1(\varepsilon) \exp\left\{-\kappa \frac{(\delta \vee \varepsilon)^{3/2} \wedge \varepsilon^{2\nu}}{\sigma^2}\right\}. \end{aligned} \quad (4.4.2)$$

The term $\varepsilon^{2\nu}$ can be disregarded as soon as δ is sufficiently small. In other words, the probability of making a transition from a neighbourhood of the stable branch ϕ_+^* to the unstable branch ϕ_-^* or to the other stable branch is exponentially small, with a parameter of order $(\delta \vee \varepsilon)^{3/2}/\sigma^2$.

Consider now the strong-noise regime $\sigma \geq (\delta \vee \varepsilon)^{3/4}$. We still require the conditions (4.4.1) to hold, but modify the upper bounds on h and $h_\perp$. As long as $t < -c_1 \sigma^{2/3}$, Theorem 4.3.4 can be applied with $h \lesssim |t|^{3/2}$, yielding

$$\mathbb{P}\{\tau_{\mathcal{B}_0(h)} \wedge \tau_{\mathcal{B}_\perp(h_\perp)} < t\} \leq C(t, \varepsilon) \exp\left\{-\kappa' \frac{|t|^3}{\sigma^2}\right\}$$

for some $\kappa' > 0$. This shows in particular that ϕ_0 is unlikely to reach 0 before times of order $-\sigma^{2/3}$.

To see what happens for larger times, we do no longer use Theorem 4.3.4, but only Theorems 4.3.1 and 4.3.5, applied to an interval of the form $[-c_1 \sigma^{2/3}, -c_2 \sigma^{2/3}]$. Then $\hat{\alpha}(-c_2 \sigma^{2/3}, -c_1 \sigma^{2/3})$ has order $\sigma^{4/3}$ and the conditions on h and $h_\perp$ can be summarised as

$$h \lesssim \sigma^{1/3}, \quad h_\perp \lesssim \sqrt{h} \sigma^{-1/3} \wedge \sigma^{2/3} \wedge \varepsilon^\nu.$$

In particular, it is possible to take h of order $\sigma^{1/3}$ and $h_\perp$ of order $\sigma^{2/3} \wedge \varepsilon^\nu$. This yields

$$\mathbb{P}\{\phi_0(t) > -d_0 \ \forall t \in [-c_1 \sigma^{2/3}, -c_2 \sigma^{2/3}]\} \leq \frac{3}{2} \exp\left\{-\kappa' \frac{\sigma^{4/3}}{\varepsilon \log(\sigma^{-1})}\right\} + \mathcal{O}\left(\frac{\sigma^{4/3}}{\varepsilon^2} e^{-\kappa'/\sigma^{4/3}}\right). \quad (4.4.3)$$

To summarise, we have thus obtained that with a probability exponentially close to 1, the transverse component $\phi_\perp$ of the solution remains small in H^1 norm, while the spatial mean ϕ_0 behaves in the same way as the solution of the one-dimensional SDE studied in [8]. In particular, there exist a weak-noise regime in which transitions between stable equilibria are very unlikely, cf. (4.4.2), and a strong-noise regime, in which transitions are very likely, see (4.4.3).

An interesting question that remains open so far, is what can be said on regimes where the periodic forcing has a smaller amplitude, so that one stays in the weak-noise regime, but transitions still become likely over very long time spans. In the one-dimensional case, very precise results on the distribution of transition times have been obtained, for instance, in [4, 12]. Generalising these results to the infinite-dimensional situation would require a good understanding of the effect of the dynamics of $\phi_\perp$ on transition times.

Part II

Analysis on the two-dimensional torus

Concentration estimates for slowly time-dependent singular SPDEs

The two-dimensional SPDEs, introduced previously are no longer well posed due to the fact that space-time white noise is more irregular in dimension 2 than in dimension 1. Previous arguments and estimates do not hold in higher dimensions, other methods are developed in this chapter. We introduce first the Wick-renormalised SPDE and we discuss the Da Prato-Debussche argument, see [20]. Since the stochastic convolution, solution of the linear equation is the most irregular object in the study, we start by controlling its Wick's power. We give after that concentration estimates around a stable equilibrium branch and in a neighbourhood of pitchfork bifurcation.

Dans ce chapitre, on developpe de nouvelle méthode pour étudier les EDPS singulières sur le tore de dimension deux. Les méthodes utilisées dans les chapitres précédents ne s'appliquent pas dans ce cas car le bruit blanc espace-temps est plus irrégulier en dimension deux qu'en dimension une. On présente le problème renormalisé et l'argument de Da Prato-Debussche. Cet argument permet de définir une solution de l'EDPS dans un espace fonctionnel et ainsi construire des estimées de concentration. Comme la convolution stochastique et ses puissances de Wick sont les termes les plus irréguliers, on commence par contrôler leur comportements. Pour ensuite établir des estimées de concentration dans un voisinage d'une branche d'équilibre stable. Et on motive les résultats par une étude près d'une bifurcation de fourche où un retard à la bifurcation a lieu.

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5.1 Set-up

We are interested in renormalised versions of the SPDE

$$d\phi(t, x) = [\Delta\phi(t, x) + F(\varepsilon t, \phi(t, x))] dt + \sigma dW(t, x) , \quad (5.1.1)$$

where time t belongs to an interval $I = [0, T] \subset \mathbb{R}_+$, the spatial variable x belongs to the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$, and the solution $\phi(t, x)$ is real-valued. In addition, we assume that

- $\varepsilon > 0$ and $\sigma \geq 0$ are small positive parameters;
- $\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2}$ is the Laplacian acting on both components of x ;
- F is polynomial, of the form

$$F(t, \phi) = \sum_{j=0}^n A_j(t) \phi^j \quad (5.1.2)$$

for some odd $n \geq 3$, where the coefficients $A_j : I \rightarrow \mathbb{R}$ are of class $\mathcal{C}^1$, and the leading coefficient $A_n(t)$ is strictly negative for all $t \in I$, to avoid blow-up of solutions;

- $dW(t, x)$ denotes space-time white noise on $I \times \mathbb{T}^2$.

It is well-known (see for instance [20]) that the SPDE (5.1.1) is not well-posed, and that a renormalisation procedure is required to define a notion of solution. It consists on subtracting an "infinite constant" from the right hand side of (5.1.1).

We recall the complex Fourier basis of $L^2(\mathbb{T}^2)$ introduced in (2.2.3)

$$\{e_k(x) = e^{2\pi i k \cdot x}\}_{k \in \mathbb{Z}^2} ,$$

where we assumed that $L = 1$ and by rescaling, the result will be true for every $L > 0$. We write any $\phi \in L^2(\mathbb{T}^2)$ as

$$\phi(x) = \sum_{k \in \mathbb{Z}^2} \phi_k e_k(x) .$$

For our purposes, it will be convenient to work with a finite-dimensional approximation of the infinite-dimensional system and then pass to the limit. We use a spectral Galerkin approximation, which has the advantage that the relevant eigenvalues are independent of any cut-off $N \in \mathbb{N}$. For $\phi \in L^2(\mathbb{T}^2)$, we define the spectral Galerkin approximation at order N of ϕ by

$$\phi_N(x) = (P_N \phi)(x) := \sum_{k \in \mathbb{Z}^2 : |k| \leq N} \phi_k e_k(x) ,$$

where $|k| = |k_1| + |k_2|$. We denote the eigenvalues of the Laplacian on $\mathbb{T}^2$ by $-\mu_k$, where

$$\mu_k := (2\pi)^2 \|k\|^2 , \quad k \in \mathbb{Z}^2 , \quad (5.1.3)$$

where $\|k\|$ denotes the Euclidean norm of k and define the renormalisation constant

$$C_N = \sigma^2 \text{Tr}([-P_N \Delta + 1]^{-1}) = \sigma^2 \sum_{k \in \mathbb{Z}^2 : |k| \leq N} \frac{1}{\mu_k + 1} . \quad (5.1.4)$$

Lemma 5.1.1. C_N diverges like $\sigma^2 \log N / (2\pi)$ as $N \rightarrow \infty$.

Proof. One can view C_N as a Riemann sum then we integrate using Polar coordinates:

$$\begin{aligned} \sum_{k \in \mathbb{Z}^2: |k| \leq N} \frac{1}{\mu_k + 1} &= \sum_{k \in \mathbb{Z}^2: |k| \leq N} \frac{1}{(2\pi)^2 \|k\|^2 + 1} \leq \int_0^{2\pi} \int_0^N \frac{1}{(2\pi)^2 r^2 + 1} r \, dr \, d\theta \\ &= \frac{2\pi}{2(2\pi)^2} \int_0^N \frac{2(2\pi)^2 r}{(2\pi)^2 r^2 + 1} \, dr = \frac{\log(N)}{2\pi} + \mathcal{O}(1). \end{aligned}$$

□

Remark 5.1.2. Note that the shift $+1$ in the definition (5.1.4) of C_N is only there to avoid problems with the $k = 0$ mode, and can be replaced by any other strictly positive constant. ◇

We recall the m th Wick power of ϕ_N defined by

$$:\phi_N^m: = :\phi_N^m:_{C_N} := H_m(\phi_N; C_N),$$

where $H_m(x; C_N)$ is the Hermite polynomials with variance C_N , see Section 2.5. The renormalised version of (5.1.1) that we want to study is given by the limit, as $N \rightarrow \infty$, of

$$d\phi_N(t, x) = [\Delta\phi_N(t, x) + :F(\varepsilon t, \phi_N(t, x)):_{C_N}] \, dt + \sigma \, dW_N(t, x), \quad (5.1.5)$$

where $dW_N = P_N \, dW$, and

$$:F(t, \phi):_{C_N} := \sum_{j=0}^n A_j(t) : \phi^j :_{C_N}.$$

As proved in [20], solutions of the renormalised equation (5.1.5) do admit a well-defined limit as $N \rightarrow \infty$, in an appropriate Besov spaces that we precise below. The limiting equation is denoted by

$$d\phi(t, x) = [\Delta\phi(t, x) + :F(\varepsilon t, \phi(t, x)):] \, dt + \sigma \, dW(t, x).$$

In what follows, it will be convenient to rescale time by a factor ε , which results in the SPDE

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + :F(t, \phi(t, x)):] \, dt + \frac{\sigma}{\sqrt{\varepsilon}} \, dW(t, x).$$

An example of renormalised SPDEs is the renormalised Allen-Cahn equation

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + A \cos(t) + \phi(t, x) - :\phi(t, x)^3:_{C_N}] \, dt + \frac{\sigma}{\sqrt{\varepsilon}} \, dW(t, x),$$

where $:\phi^3:_{C_N} = \phi^3 - 3C_N\phi$ is the third Wick power of ϕ .

5.2 Da Prato-Debussche trick

In this section, we discuss briefly the idea exploited by Da Prato and Debussche in [20]. The results that we will discuss later are mainly based on this idea. Let ψ denote the stochastic convolution, that is, the solution of the linear equation

$$d\psi(t, x) = \Delta\psi(t, x) \, dt + \sigma \, dW(t, x)$$

with initial condition $\psi(0, x) = 0$. Since the stochastic convolution belongs to all Besov spaces with *negative* regularity α , but not with positive α , this means that ψ is a distribution, but not a function, see for instance Lemma 3.2 in [20]. The trick is to write ϕ as a sum of ϕ_1 and ψ . Then, the observation is that ϕ_1 is much smoother than ϕ and one writes the problem in terms of ϕ_1 where the nonlinear term is a continuous function with respect to ϕ . So, ϕ_1 enjoys much better regularity properties. The price to pay is to work with Besov spaces which are well suited to define the product of distributions. The main result in this paper is the following.

Theorem 5.2.1 ([20, Theorem 4.2]). *For any $p > n$ and $r \geq 1$, let α and s satisfy*

$$0 > \alpha > \max \left\{ -\frac{2}{p(n+1)}, -\frac{1}{n-1} \left(1 - \frac{n}{p} \right) \right\}, \quad s = \frac{2}{p} + 2\alpha.$$

Then, for almost any initial condition (with respect to a natural probability measure), the renormalised SPDE admits for any $T \geq 0$ a unique solution ϕ such that

$$\phi - \psi \in \mathcal{C}([0, T], \mathcal{B}_{p,r}^\alpha) \cap L^p([0, T], \mathcal{B}_{p,r}^s).$$

Note in particular that $s > 0$, implying that the difference $\phi - \psi$ takes values in the space of functions $\mathcal{B}_{p,r}^s$, which have some Hölder regularity in space.

5.3 Wick powers of the stochastic convolution

This section concerns the most irregular terms in the study: the stochastic convolution and its Wick powers. Let $a : I \rightarrow \mathbb{R}$ be a continuously differentiable function satisfying

$$-a_+ < a(t) < -a_- \quad \forall t \in I \quad (5.3.1)$$

for some constants $a_+ > a_- > 0$. The stochastic convolution is defined as the solution of the linear equation

$$d\psi(t, x) = \frac{1}{\varepsilon} [\Delta \psi(t, x) + a(t)\psi(t, x)] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x) \quad (5.3.2)$$

with initial condition $\psi(0, x) = 0 \forall x \in \mathbb{T}^2$. Its projection on the k th basis vector e_k , gives

$$d\psi_k(t) = \frac{1}{\varepsilon} a_k(t) \psi_k(t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_k(t), \quad (5.3.3)$$

where $a_k(t) = -\mu_k + a(t)$ and the $\{W_k(t)\}_{t \geq 0}$ are independent Wiener processes. The solution of (5.3.3) is a Gaussian process and can be represented using Duhamel's principle by the Ito integral

$$\psi_k(t) = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha_k(t, t_1)/\varepsilon} dW_k(t_1), \quad (5.3.4)$$

where $\alpha_k(t, t_1) = \int_{t_1}^t a_k(t_2) dt_2$. Thus, for each time t , $\psi_k(t)$ is characterised by its mean being zero and its variance given by

$$\text{Var}\{\psi_k(t)\} = \frac{\sigma^2}{\varepsilon} \int_0^t e^{2\alpha_k(t, t_1)/\varepsilon} dW_k(t_1).$$

One can bound the variance similarly to Lemma 3.3.4

$$v_k := \text{Var}\{\psi_k(t)\} \lesssim \frac{\sigma^2}{1 + \|k\|^2} \quad \forall t \in I. \quad (5.3.5)$$

Remark 5.3.1. For any N and $x \in \mathbb{T}^2$, we consider the truncated Gaussian free field at any fixed time t

$$\psi(t, \cdot) = \sum_{k \in \mathbb{Z}^2: |k| \leq N} \frac{\xi_k}{\sqrt{1 + \mu_k}} e_k(x) = \sum_{k \in \mathbb{Z}^2: |k| \leq N} \psi_k e_k(x),$$

where ξ_k are i.i.d. standard normal random variables. The ψ_k are complex-valued Gaussian random variables which satisfy

$$\mathbb{E}[\psi_k \psi_{-l}] = \begin{cases} v_k & \text{if } k = l \\ 0 & \text{else.} \end{cases}$$

◇

It is known that the Wick powers of the stochastic convolution $:\psi^m:$ belong to any Besov space $\mathcal{B}_{p,r}^\alpha$ for any $m \in \mathbb{N}$, $p, r \geq 1$ and $\alpha < 0$. The proof exists in [20, Lemma 3.2]. We adapt it in our context. In particular, we prove the following result for $m = 1$ and for any $m \in \mathbb{N}$ separately.

Lemma 5.3.2. *Let $\alpha < 0$. Then for any $m \in \mathbb{N}$, $:\psi^m:$ belongs to $\mathcal{B}_{2,\infty}^\alpha(\mathbb{T}^2)$.*

Proof of Lemma 5.3.2 for $m = 1$. For any $\alpha < 0$, $p = 2$ and $r = \infty$, Definition 2.3.1 gives

$$\|\psi\|_{\mathcal{B}_{2,\infty}^\alpha} = \sup_{q \geq 0} 2^{-|\alpha|q} \|\delta_q \psi\|_{L^2}.$$

$$\mathbb{E}[\|\psi\|_{\mathcal{B}_{2,\infty}^\alpha}] = \mathbb{E}\left[\sup_{q \geq 0} 2^{-|\alpha|q} \|\delta_q \psi\|_{L^2}\right] \leq \sum_{q \geq 0} 2^{-|\alpha|q} \mathbb{E}[\|\delta_q \psi\|_{L^2}].$$

By Cauchy Schwarz's inequality, we get

$$\mathbb{E}[\|\psi\|_{\mathcal{B}_{2,\infty}^\alpha}] \leq \sum_{q \geq 0} 2^{-|\alpha|q} \mathbb{E}[\|\delta_q \psi\|_{L^2}^2]^{1/2}.$$

We recall that

$$\delta_q \psi(x) = \sum_{k \in \mathcal{A}_q} \psi_k e_k(x),$$

its L^2 -norm is given by

$$\begin{aligned} \|\delta_q \psi(\cdot)\|_{L^2}^2 &= \int_{\mathbb{T}^2} |\delta_q \psi(x)|^2 dx \\ &= \sum_{k_1, k_2 \in \mathcal{A}_q} \psi_{k_1} \bar{\psi}_{k_2} \int_{\mathbb{T}^2} e_{k_1}(x) \bar{e}_{k_2}(x) dx \\ &= \sum_{k_1, k_2 \in \mathcal{A}_q} \psi_{k_1} \bar{\psi}_{k_2} \delta_{k_1 k_2} \\ &= \sum_{k_1 \in \mathcal{A}_q} \psi_{k_1} \psi_{-k_1} = \sum_{k_1 \in \mathcal{A}_q} |\psi_{k_1}|^2. \end{aligned}$$

It follows that

$$\mathbb{E}[\|\psi\|_{\mathcal{B}_{2,\infty}^\alpha}] \leq \sum_{q \geq 0} 2^{-|\alpha|q} \left(\sum_{k_1 \in \mathcal{A}_q} \mathbb{E}[|\psi_{k_1}|^2] \right)^{1/2} = \sum_{q \geq 0} 2^{-|\alpha|q} \left(\sum_{k_1 \in \mathcal{A}_q} v_{k_1} \right)^{1/2}.$$

We notice that the sum over k_1 in the annulus $\mathcal{A}_q$ is of $\mathcal{O}(\sigma^2)$. In fact, by using polar coordinates we get

$$\sum_{k_1 \in \mathcal{A}_q} v_{k_1} \lesssim \sum_{2^{q-1} < \|k_1\| \leq 2^q} \frac{\sigma^2}{1 + \|k_1\|^2} \lesssim \sigma^2 \int_{2^{q-1}}^{2^q} \frac{r \, dr}{1 + r^2} = \frac{\sigma^2}{2} \log \left(\frac{1 + 2^{2q}}{1 + 2^{2(q-1)}} \right) \leq 2\sigma^2 \log(2). \quad (5.3.6)$$

Therefore,

$$\mathbb{E} \left[\|\psi\|_{\mathcal{B}_{2,\infty}^\alpha} \right] \lesssim \sigma \sum_{q \geq 0} 2^{-|\alpha|q} \leq \frac{\sigma}{1 - 2^{-|\alpha|}} < \infty.$$

□

Proof of Lemma 5.3.2 for any $m \in \mathbb{N}$. For any $p, r \geq 1$, one can see the proof in [20]. In our case, we take $l \geq 1, p = 2$ and $r = \infty$ and we obtain

$$\mathbb{E} \left[\|\psi^m\|_{\mathcal{B}_{2,\infty}^\alpha}^l \right] = \mathbb{E} \left[\left(\sup_{q \geq 0} 2^{q\alpha} \|\delta_q : \psi^m : \|_{L^2} \right)^l \right] \leq \mathbb{E} \left[\left(\sum_{q \geq 0} 2^{q\alpha} \|\delta_q : \psi^m : \|_{L^2} \right)^l \right].$$

By Hölder's inequality with $\frac{1}{p} = \frac{l-1}{l}, \frac{1}{q} = \frac{1}{l}$, we get

$$\begin{aligned} \mathbb{E} \left[\|\psi^m\|_{\mathcal{B}_{2,\infty}^\alpha}^l \right] &\leq \mathbb{E} \left[\left(\sum_{q \geq 0} 2^{ql\alpha/2(l-1)} \right)^{l-1} \left(\sum_{q \geq 0} 2^{ql\alpha/2} \|\delta_q : \psi^m : \|_{L^2}^l \right) \right] \\ &= \left(\sum_{q \geq 0} 2^{ql\alpha/2(l-1)} \right)^{l-1} \sum_{q \geq 0} 2^{ql\alpha/2} \mathbb{E} \left[\|\delta_q : \psi^m : \|_{L^2}^l \right]. \end{aligned}$$

Since $\alpha < 0$, one can bound the following geometric series,

$$\sum_{q \geq 0} 2^{-ql|\alpha|/2(l-1)} \leq \frac{1}{1 - 2^{-|\alpha|/2}} = c(\alpha).$$

Again by Hölder's inequality with $\frac{1}{p} = \frac{2}{l}, \frac{1}{q} = \frac{l-2}{l}$, we obtain

$$\mathbb{E} \left[\|\delta_q : \psi^m : \|_{L^2}^l \right] = \mathbb{E} \left[\left(\int_{\mathbb{T}^2} |\delta_q : \psi^m :|^2 \, dx \right)^{l/2} \right] \leq \int_{\mathbb{T}^2} \mathbb{E} \left[|\delta_q : \psi^m :|^l \right] \, dx.$$

Since the $\delta_q : \psi^m :$ belongs to the m th Wiener chaos, we can use the hypercontractivity argument (2.5.4) to obtain the bound

$$\mathbb{E} \left[|\delta_q : \psi^m :|^l \right] \leq (l-1)^{ml/2} \mathbb{E} \left[|\delta_q : \psi^m :|^2 \right]^{l/2}.$$

Therefore,

$$\begin{aligned} \mathbb{E} \left[|\delta_q : \psi^m :|^2 \right] &= \mathbb{E} \left[\left| \sum_{k \in \mathcal{A}_q} \langle : \psi^m : , e_k \rangle e_k \right|^2 \right] \\ &= \sum_{k, l \in \mathcal{A}_q} \mathbb{E} \left[\int_{\mathbb{T}^2} \int_{\mathbb{T}^2} : \psi^m(x_1) : : \psi^m(x_2) : e_{-k}(x_1) e_l(x_2) e_k(x) e_{-l}(x) \, dx_1 \, dx_2 \right] \\ &= \sum_{k, l \in \mathcal{A}_q} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \mathbb{E} \left[: \psi^m(x_1) : : \psi^m(x_2) : \right] e_{-k}(x_1) e_l(x_2) e_k(x) e_{-l}(x) \, dx_1 \, dx_2. \end{aligned}$$

By Wick's theorem, see Theorem 2.5.1 we have

$$\mathbb{E} \left[: \psi^m(x_1) : : \psi^m(x_2) : \right] = m! \mathbb{E} \left[\psi(x_1) \psi(x_2) \right]^m ,$$

where

$$\begin{aligned} \mathbb{E} \left[\psi(x_1) \psi(x_2) \right] &= \sum_{k_1, k_2 \in \mathbb{Z}^2} \mathbb{E} \left[\psi_{k_1} \psi_{k_2} \right] e_{k_1}(x_1) e_{k_2}(x_2) \\ &= \sum_{k_1, k_2 \in \mathbb{Z}^2} v_{k_1} \delta_{k_1, -k_2} e_{k_1}(x_1) e_{k_2}(x_2) \\ &= \sum_{k_1 \in \mathbb{Z}^2} v_{k_1} e_{k_1}(x_1 - x_2) . \end{aligned}$$

We define

$$\gamma^m(x) = \left(\sum_{k_1 \in \mathbb{Z}^2} \frac{1}{1 + \|k_1\|^2} e_{k_1}(x) \right)^m = \sum_{k_1 \in \mathbb{Z}^2} \beta^m(N) e_{k_1}(x) .$$

Thus

$$\begin{aligned} &\mathbb{E} \left[|\delta_q : \psi^m :|^2 \right] \\ &= m! \sigma^{2m} \sum_{k, l \in \mathcal{A}_q} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \sum_{k_1 \in \mathbb{Z}^2} \beta^m(N) e_{k_1}(x_1 - x_2) e_{-k}(x_1) e_l(x_2) e_k(x) e_{-l}(x) dx_1 dx_2 \\ &= m! \sigma^{2m} \sum_{k, l \in \mathcal{A}_q} \sum_{k_1 \in \mathbb{Z}^2} \beta^m(N) \delta_{k_1 k} \delta_{k_1 l} . \end{aligned}$$

Then,

$$\mathbb{E} \left[\|\delta_q : \psi^m : \|_{L^2}^l \right] \leq c'(m, l) \left(\sum_{k \in \mathcal{A}_q} \beta^m(N) \right)^{l/2} .$$

By Cauchy Schwarz's inequality, we get

$$\mathbb{E} \left[\|\delta_q : \psi^m : \|_{L^2}^l \right] \leq c'(m, l) 2^{lq/2} \left(\sum_{k \in \mathcal{A}_q} \beta^m(N)^2 \right)^{l/4} .$$

We fix a $\tilde{s} < 0$. Then, we have

$$\sum_{k \in \mathcal{A}_q} \beta^m(N)^2 \leq 2^{-2(q-1)(1+\tilde{s})} \sum_{k \in \mathbb{Z}^2} \|k\|^{2(1+\tilde{s})} \beta^m(N)^2 \leq 2^{-2(q-1)(1+\tilde{s})} \|\gamma^m\|_{H^{1+\tilde{s}}}^2 .$$

With similar argument as [20, Lemma 3.2], we get

$$\|\gamma^m\|_{H^{1+\tilde{s}}}^2 \leq c(m, \tilde{s}) .$$

For $\tilde{s} = 2s/l$, it follows that

$$\mathbb{E} \left[\|\delta_q : \psi^m : \|_{L^2}^l \right] \leq c(m, l, s) 2^{-qs} ,$$

for all $s < 0$. Choosing $s = l\alpha/4$

$$\mathbb{E} \left[\|\psi^m : \mathcal{B}_{2,\infty}^\alpha\|^l \right] = C(\alpha, l, m) \sum_{q \geq 0} 2^{ql\alpha/4} < \infty ,$$

where $\sum_{q \geq 0} 2^{ql\alpha/4}$ is a convergent geometric series for any $\alpha < 0$. □

Remark 5.3.3. For instance, we have

$$\mathbb{E} \left[\|\psi(t, \cdot)\|_{\mathcal{B}_{\infty,\infty}^\alpha} \right] = \sup_{q \geq -1} 2^{q\alpha} \sum_{k \in \mathcal{A}_q} \mathbb{E} [|\psi_k(t)|] .$$

Since the random variables $\psi_k(t)$ follow centred normal distributions of variance of order $\|k\|^{-2}$, the sum over $k \in \mathcal{A}_q$ of the expectations of $|\psi_k(t)|$ has order 2^q . Therefore, the expectation of $\|\psi(t, \cdot)\|_{\mathcal{B}_{\infty,\infty}^\alpha}$ diverges with the cut-off N as $N^{\alpha+1}$ if $\alpha > -1$. ◇

5.3.1 Tail estimates for a fixed time $t \in I$

With the tools put in place above, we can now establish tail estimates on Wick powers of the stochastic convolution. We first construct concentration estimates for any fixed time $t \in I$. It will help us to understand the main result of this section: the behaviour of the probability when adding a supremum over all $t \in I$. We consider that $\psi(t, \cdot) = \psi(\cdot)$ for any $t \in I$ and we have the following result.

Theorem 5.3.4. *For any $\alpha < 0$ and for any $m \in \mathbb{N}$, there exist constants $C_m(\alpha)$ and $\kappa_m(\alpha)$, independent of the cut-off N , such that*

$$\mathbb{P} \left\{ \|\psi(\cdot)^m : \mathcal{B}_{2,\infty}^\alpha\| > h^m \right\} \leq C_m(\alpha) e^{-\kappa_m(\alpha) h^2 / \sigma^2}$$

holds for all $h > 0$.

Remark 5.3.5. The proof yields explicit bounds on $C_m(\alpha)$ and $\kappa_m(\alpha)$. ◇

For a better understanding of the proof, it is helpful to start with proving the particular case $m = 1$. The case $m = 2$ is proved in [Appendix A](#).

5.3.1.1 Proof of Theorem 5.3.4 for $m = 1$

Proof of Theorem 5.3.4 for $m = 1$. Fix $\alpha < 0$. By Definition 2.3.1,

$$\begin{aligned} \mathbb{P} \left\{ \|\psi(\cdot)\|_{\mathcal{B}_{2,\infty}^\alpha} > h \right\} &= \mathbb{P} \left\{ \sup_{q \geq 0} 2^{-|\alpha|q} \|\delta_q \psi(\cdot)\|_{L^2} > h \right\} \\ &= \mathbb{P} \left\{ \exists q \geq 0 : \|\delta_q \psi(\cdot)\|_{L^2} > h 2^{|\alpha|q} \right\} \\ &\leq \sum_{q \geq 0} \mathbb{P} \left\{ \|\delta_q \psi(\cdot)\|_{L^2} > h 2^{|\alpha|q} \right\} . \end{aligned}$$

Fix $\gamma \in \mathbb{R}$ and let $\hat{\gamma} = \gamma/\sigma^2$. By Markov's inequality, we get

$$\begin{aligned} \mathbb{P}\left\{\|\psi(\cdot)\|_{\mathcal{B}_{2,\infty}^\alpha} > h\right\} &\leq \sum_{q \geq 0} \mathbb{P}\left\{e^{\hat{\gamma}\|\delta_q \psi(\cdot)\|_{L^2}^2} > e^{\hat{\gamma}h^2 2^{2|\alpha|q}}\right\} \\ &\leq \sum_{q \geq 0} e^{-\hat{\gamma}h^2 2^{2|\alpha|q}} \mathbb{E}\left[e^{\hat{\gamma}\sum_{k \in \mathcal{A}_q} |\psi_k|^2}\right]. \end{aligned} \quad (5.3.7)$$

Since $\{\psi_k\}_{k \in \mathbb{Z}^2}$ are zero-mean gaussian random variables i.i.d of variance v_k , we have

$$\mathbb{E}\left[e^{\hat{\gamma}\sum_{k \in \mathcal{A}_q} |\psi_k|^2}\right] = \prod_{k \in \mathcal{A}_q} \mathbb{E}\left[e^{\hat{\gamma}|\psi_k|^2}\right]$$

where

$$\begin{aligned} \mathbb{E}\left[e^{\hat{\gamma}|\psi_k|^2}\right] &= \int_{-\infty}^{+\infty} e^{\hat{\gamma}z^2} \frac{1}{\sqrt{2\pi v_k}} e^{-z^2/2v_k} dz \\ &= \frac{1}{\sqrt{2\pi v_k}} \int_{-\infty}^{+\infty} e^{-z^2(1-2\hat{\gamma}v_k)/2v_k} dz. \end{aligned}$$

The bound (5.3.5) on v_k implies that $1 - 2\hat{\gamma}v_k > 0$ for any $\gamma < 1/2$. We can thus make the change of variable $u = z\sqrt{1 - 2\hat{\gamma}v_k}$, yielding the estimate

$$\mathbb{E}\left[e^{\hat{\gamma}|\psi_k|^2}\right] = \frac{1}{\sqrt{1 - 2\hat{\gamma}v_k}}.$$

We can thus bound the product of expectations by

$$\begin{aligned} \prod_{k \in \mathcal{A}_q} \mathbb{E}\left[e^{\hat{\gamma}|\psi_k|^2}\right] &= \prod_{k \in \mathcal{A}_q} \frac{1}{\sqrt{1 - 2\hat{\gamma}v_k}} \\ &= \exp\left\{-\frac{1}{2} \sum_{k \in \mathcal{A}_q} \ln(1 - 2\hat{\gamma}v_k)\right\}. \end{aligned}$$

There exists a positive constant c such that $\ln(1 - x) \geq -cx$ for $0 \leq x \leq \frac{1}{2}$. Therefore,

$$\prod_{k \in \mathcal{A}_q} \mathbb{E}\left[e^{\hat{\gamma}|\psi_k|^2}\right] \leq \exp\left\{c\hat{\gamma} \sum_{k \in \mathcal{A}_q} v_k\right\}. \quad (5.3.8)$$

The sum over k is of order σ^2 , see (5.3.6). Replacing (5.3.8) in (5.3.7), we get

$$\mathbb{P}\left\{\|\psi(\cdot)\|_{\mathcal{B}_{2,\infty}^\alpha} > h\right\} \leq e^{c_0\gamma} \sum_{q \geq 0} e^{-\gamma 2^{2|\alpha|q} h^2 / \sigma^2}.$$

It remains to bound the sum over q . Choosing $\gamma < 1/4$ and let $f(x) = e^{-\beta 2^{2|\alpha|x}}$, where $\beta = h^2/(4\sigma^2)$. One checks that f is decreasing and bounded

$$\sum_{q \geq 0} f(q) = f(0) + f(1) + \sum_{q=2}^{\infty} f(q) \leq e^{-\beta} + e^{-\beta 2^{2|\alpha|}} + \int_1^{\infty} f(x) dx.$$

To compute the integral, we give the following bound $2^{2|\alpha|x} = e^{2|\alpha|x \ln(2)} \geq 2|\alpha|x \ln(2)$. Then, we obtain

$$\int_1^\infty f(x) dx \leq \int_1^\infty e^{-2\beta|\alpha|x \ln(2)} dx = \frac{1}{2\beta|\alpha| \ln(2)} e^{-2\beta|\alpha| \ln(2)} .$$

We conclude that

$$\mathbb{P}\left\{\|\psi\|_{\mathcal{B}_{2,\infty}^\alpha} > h\right\} \leq C(\alpha) e^{-\kappa(\alpha)h^2/\sigma^2} . \quad (5.3.9)$$

□

5.3.1.2 Proof of Theorem 5.3.4 for any $\mathbf{m}$

In what follows, it will be convenient to introduce multiindex notations. For any $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$ with finitely many nonzero components, we write

$$|\mathbf{n}| = \sum_{q \geq 0} \mathbf{n}_q \quad \text{and} \quad \mathbf{n}! := \prod_{q \geq 0} \mathbf{n}_q! .$$

Since $\mathbf{n}$ has finitely many nonzero components, these quantities are indeed well-defined. Let

$$[\mathbf{n}] = \#\{q : \mathbf{n}_q > 0\}$$

be the number of these nonzero components. We can order them as $q_1 < q_2 < \dots < q_{[\mathbf{n}]}$, where

$$q_{[\mathbf{n}]} = \max\{q : \mathbf{n}_q > 0\}$$

is the index of the largest nonzero entry of $\mathbf{n}$. In what follows, we will always assume that $|\mathbf{n}| = m$. We notice that this implies $[\mathbf{n}] \leq m$.

As we have already seen, the projection of $\psi(t, \cdot)$ on the annulus $\mathcal{A}_q$ has constant variance

$$c_q := \mathbb{E}\left[\|\delta_q \psi(t, \cdot)\|_{L^2}^2\right] = \sum_{k \in \mathcal{A}_q} v_k . \quad (5.3.10)$$

In fact, $\delta_q \psi(t, x)$ has variance c_q for all $x \in \mathbb{T}^2$. We recall the important feature of this projection

$$c_q \leq \sigma^2 \log 2 ,$$

for all q , see (5.3.6). In addition, at any fixed time t , the law of $\psi(t, \cdot)$ is that of the truncated Gaussian free field, with variance

$$\sum_{k \in \mathbb{Z}^2 : |k| \leq N} v_k = C_N ,$$

which diverges like $\sigma^2 \log(N)/(2\pi)$, as mentioned in (5.1.4). Finally note that the cut-off condition $|k| \leq N$ implies $q \leq \ell_N = \lfloor \log_2 N \rfloor$, and that

$$C_N = \sum_{q=0}^{\ell_N} c_q .$$

With these notations in place, we can introduce the binomial formula for Hermite polynomials, see Lemma 2.5.3 in Section 2.5 and give a proof for Theorem 5.3.4 for any $m \in \mathbb{N}$.

Lemma 5.3.6. *For any $m \in \mathbb{N}$, we have*

$$H_m(\psi(t, \cdot), C_N) = \sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \psi(t, \cdot), c_q).$$

Proof of Theorem 5.3.4 for any $m \in \mathbb{N}$. Fix $\alpha < 0$, Definition 2.3.1 gives

$$\begin{aligned} P_m(h) &:= \mathbb{P}\left\{\|\psi(\cdot)^m\|_{\mathcal{B}_{2,\infty}^\alpha} > h^m\right\} = \mathbb{P}\left\{\sup_{q_0 \geq 0} 2^{-|\alpha|q_0} \|\delta_{q_0}(\psi(\cdot)^m)\|_{L^2} > h^m\right\} \\ &= \mathbb{P}\left\{\exists q_0 \geq 0; \|\delta_{q_0}(\psi(\cdot)^m)\|_{L^2} > h^m 2^{|\alpha|q_0}\right\} \\ &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\|\delta_{q_0}(\psi(\cdot)^m)\|_{L^2} > h^m 2^{|\alpha|q_0}\right\}. \end{aligned}$$

Binomial's formula for Hermite polynomials given in Lemma 5.3.6 yields

$$\begin{aligned} P_m(h) &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\|\delta_{q_0}(\sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \prod_{q \geq 0} \delta_q \psi(t, \cdot)^{\mathbf{n}_q})\|_{L^2} > h^m 2^{|\alpha|q_0}\right\} \\ &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \|\delta_{q_0}(\prod_{q \geq 0} \delta_q \psi(t, \cdot)^{\mathbf{n}_q})\|_{L^2} > h^m 2^{|\alpha|q_0}\right\}. \end{aligned}$$

Remark 5.3.7. Note that for any $q_1, q_2 \geq 0$, one has $2^{q_1} + 2^{q_2} \leq 2^{\max\{q_1, q_2\}+1}$. Therefore,

$$\delta_{q_0}(\prod_{q \geq 0} \delta_q \psi(t, \cdot)^{\mathbf{n}_q}) \neq 0 \quad \Rightarrow \quad q_0 \leq \max_{i \leq q[\mathbf{n}]} \{q_i + \mathbf{n}_{q_i}\} \leq q[\mathbf{n}] + \mathbf{n}_{q[\mathbf{n}]}$$

for any $\mathbf{n}$, which will be useful in restricting the domains of the sums. $\diamond$

Then, by the equivalence of l^p -norms we get

$$\begin{aligned} P_m(h) &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\left(\sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \|\delta_{q_0}(\prod_{q \geq 0} \delta_q \psi(t, \cdot)^{\mathbf{n}_q})\|_{L^2}\right)^{2/m} > h^2 2^{2|\alpha|q_0/m}\right\} \\ &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\sum_{|\mathbf{n}|=m} \left(\frac{m!}{\mathbf{n}!}\right)^{2/m} \|\delta_{q_0}(\prod_{q \geq 0} \delta_q \psi(t, \cdot)^{\mathbf{n}_q})\|_{L^2}^{2/m} > h^2 2^{2|\alpha|q_0/m}\right\}. \end{aligned} \quad (5.3.11)$$

Fix $\gamma \in \mathbb{R}$ such that $\hat{\gamma} = \gamma/\sigma^2$. By Markov's inequality, we get

$$\begin{aligned} P_m(h) &\leq \sum_{q_0 \geq 0} \exp\{-\hat{\gamma} 2^{2|\alpha|q_0/m} h^2\} \\ &\quad \times \mathbb{E}\left[\exp\left\{\hat{\gamma} \sum_{|\mathbf{n}|=m} \left(\frac{m!}{\mathbf{n}!}\right)^{2/m} \|\delta_{q_0}(\prod_{q \geq 0} \delta_q \psi(t, \cdot)^{\mathbf{n}_q})\|_{L^2}^{2/m}\right\}\right] \\ &= \sum_{q_0 \geq 0} \exp\{-\hat{\gamma} 2^{2|\alpha|q_0/m} h^2\} \\ &\quad \times \prod_{|\mathbf{n}|=m} \mathbb{E}\left[\exp\left\{\hat{\gamma} \left(\frac{m!}{\mathbf{n}!}\right)^{2/m} \|\delta_{q_0}(\prod_{q \geq 0} \delta_q \psi(t, \cdot)^{\mathbf{n}_q})\|_{L^2}^{2/m}\right\}\right], \end{aligned} \quad (5.3.12)$$

the last operation is due to the independence of the projections.

Remark 5.3.8. Corollary 6.13 in [33] implies that if X is a polynomial of degree m in the field, then $\mathbb{E}[e^{t|X|}]$ is finite for $m = 2$, and is in general infinite if $m \geq 3$. This explains the m th root in (5.3.11). $\diamond$

In order to bound the exponential moment in (5.3.12), we provide the following technical lemma, whose proof is postponed to Appendix A.

Lemma 5.3.9. *There exists a numerical constant $C_0 \geq 0$ such that for any l , one has*

$$\mathbb{E} \left[\left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^2 \right] \leq C_m \sigma^{2m} \frac{2^{2q_0}}{2^{2q_{[\mathbf{n}]}}}$$

where $C_m = C_0^m m!$.

With this bound, we get the following result.

Proposition 5.3.10. *For any $\gamma < (2C'_m)^{-1} 2^{2(q_{[\mathbf{n}]} - q_0)/m}$, one has*

$$\mathbb{E} \left[\exp \left\{ \hat{\gamma} \left(\frac{m!}{\mathbf{n}!} \right)^{2/m} \left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^{2/m} \right\} \right] \leq \frac{1}{1 - 2\gamma C'_m 2^{2(q_0 - q_{[\mathbf{n}])}/m}},$$

where $C'_m = C_0 e((m!)^3 / (n!)^2)^{1/m}$.

Proof. Expanding the exponential, we get

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \hat{\gamma} \left(\frac{m!}{\mathbf{n}!} \right)^{2/m} \left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^{2/m} \right\} \right] \\ &= \sum_{p \geq 0} \frac{\gamma^p}{\sigma^{2p} p!} \left(\frac{m!}{\mathbf{n}!} \right)^{2p/m} \mathbb{E} \left[\left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^{2p/m} \right]. \end{aligned}$$

By Hölder's inequality with $p = m, q = \frac{m}{m-1}$, we obtain

$$\mathbb{E} \left[\left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^{2p/m} \right] \leq \mathbb{E} \left[\left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^{2p} \right]^{1/m}.$$

Since the square of the L^2 -norm of the projection belongs to the $2m$ th Wiener chaos, we get by the hypercontractivity inequality (2.5.4) the following bound

$$\mathbb{E} \left[\left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^{2p} \right] \leq (2p-1)^{pm} \mathbb{E} \left[\left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^2 \right]^p.$$

By Lemma 5.3.9, we have

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \hat{\gamma} \left(\frac{m!}{\mathbf{n}!} \right)^{2/m} \left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^{2/m} \right\} \right] \\ &= \sum_{p \geq 0} \frac{\gamma^p}{\sigma^{2p} p!} \left(\frac{m!}{\mathbf{n}!} \right)^{2p/m} (2p-1)^p C_m^{p/m} \sigma^{2p} 2^{2(q_0 - q_{[\mathbf{n}])p/m}. \end{aligned}$$

Stirling's formula yields $p^p/p! \leq e^p$. Then the result follows by summing a geometric series

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ \hat{\gamma} \left(\frac{m!}{\mathbf{n}!} \right)^{2/m} \left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^{2/m} \right\} \right] &\leq \sum_{p \geq 0} (2e\gamma)^p \left(\frac{m!}{\mathbf{n}!} \right)^{2p/m} C_m^{p/m} 2^{2(q_0 - q_{[\mathbf{n}]})p/m} \\ &= \frac{1}{1 - 2\gamma C'_m 2^{2(q_0 - q_{[\mathbf{n}]})/m}}, \end{aligned}$$

for any $\gamma < \frac{1}{2C'_m} 2^{2(q_{[\mathbf{n}]} - q_0)/m}$ and $C'_m = e \left(\frac{m!}{\mathbf{n}!} \right)^{2/m} C_m^{1/m}$. Note that in what follows, we choose a numerical γ independent of q_0 and $q_{[\mathbf{n}]}$. $\square$

We can thus bound the product of expectations

$$\begin{aligned} \prod_{|\mathbf{n}|=m} E \left[\exp \left\{ \hat{\gamma} \left(\frac{m!}{\mathbf{n}!} \right)^{2/m} \left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^{2/m} \right\} \right] &\leq \prod_{|\mathbf{n}|=m} \frac{1}{1 - 2\gamma C'_m 2^{2(q_0 - q_{[\mathbf{n}]})/m}} \\ &= \exp \left\{ - \sum_{|\mathbf{n}|=m} \ln(1 - 2\gamma C'_m 2^{2(q_0 - q_{[\mathbf{n}]})/m}) \right\}. \end{aligned}$$

Using the existence of $c > 0$ such that $\ln(1 - x) \geq -cx$ for $0 \leq x \leq \frac{1}{2}$ yields

$$\sum_{|\mathbf{n}|=m} \ln(1 - 2\gamma C'_m 2^{2(q_0 - q_{[\mathbf{n}]})/m}) \geq -2c\gamma C'_m \sum_{|\mathbf{n}|=m} 2^{2(q_0 - q_{[\mathbf{n}]})/m}.$$

The proof of the following Lemma is given in Appendix A.

Lemma 5.3.11. *There exist numerical constants $c_0, c_1 > 0$ such that*

$$\sum_{|\mathbf{n}|=m} \frac{1}{2^{2(q_{[\mathbf{n}]} - q_0)/m}} \leq c_0 m! (q_0 + c_1)^m. \quad (5.3.13)$$

It follows that

$$P_m(h) \leq \sum_{q_0 \geq 0} \exp \left\{ -\gamma 2^{2|\alpha|q_0/m} \frac{h^2}{\sigma^2} \right\} \exp \left\{ \gamma \bar{C}_m (q_0 + c_1)^m \right\}. \quad (5.3.14)$$

Choosing $\gamma = 1/(4C'_m)$, we compute the following sum

$$\sum_{q_0 \geq 0} f(q_0), \quad f(x) = e^{\beta_1(x+c_1)^m} e^{-\beta_2 2^{2|\alpha|x/m}}, \quad \beta_1 = \frac{\bar{C}_m}{4C'_m}, \quad \beta_2 = \frac{h^2}{4C'_m \sigma^2}.$$

One can check that f is decreasing, so that one has the upper bound

$$\sum_{q_0 \geq 0} f(q_0) = f(0) + f(1) + \sum_{q_0 \geq 2} f(q_0) \leq e^{\beta_1 c_1^m} e^{-\beta_2} + e^{\beta_1(1+c_1)^m} e^{-\beta_2 2^{2|\alpha|/m}} + \int_1^\infty f(x) dx.$$

We notice that

$$(q_0 + c_1)^m \leq c_m(\alpha) 2^{2|\alpha|q_0/m}, \quad c_m(\alpha) = \mathcal{O} \left(2^{2c_1|\alpha|/m} \frac{|\alpha|}{m^2} \right).$$

It follows that

$$\begin{aligned} \int_1^\infty f(x) dx &\leq \int_1^\infty e^{-(\beta_2 2^{2|\alpha|x/m} - \beta_1 c_m(\alpha) 2^{2|\alpha|x/m})} dx \\ &\leq \int_1^\infty e^{-2|\alpha| \ln(2)(\beta_2 - \beta_1 c_m(\alpha))x/m} dx \\ &= \frac{m 2^{2|\alpha|\beta_1 c_m(\alpha)/m}}{(\beta_2 - \beta_1 c_m(\alpha)) 2|\alpha| \ln(2)} e^{-2|\alpha| \ln(2)\beta_2/m}. \end{aligned}$$

Replacing this in (5.3.14) yields a proof on $P_m(h)$, completing the proof of Theorem 5.3.4. $\square$

5.3.2 Tail estimates for all time $t \in I$

Now, as we understood the bound for a fixed $t \in I$, we can expect similar bounds when replacing $\|\psi(t, x)^m\|_{\mathcal{B}_{2,\infty}^\alpha}$ by $\sup_{0 \leq t \leq T} \|\psi(t, x)^m\|_{\mathcal{B}_{2,\infty}^\alpha}$ for all $x \in \mathbb{T}^2$. Similar approach as in Subsection 5.3.1 will be used to get the following result. We give below a particular case of the proof when $m = 1$ then we turn on to a generalised proof for any $m \in \mathbb{N}$.

Theorem 5.3.12 (Tail estimates on Wick powers of the stochastic convolution). *For any $\alpha < 0$ and for any $m \in \mathbb{N}$, there exist constants $C_m(T, \varepsilon, \alpha)$ and $\kappa_m(\alpha)$, independent of the cut-off N , such that*

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\psi(t, \cdot)^m\|_{\mathcal{B}_{2,\infty}^\alpha} > h^m\right\} \leq C_m(T, \varepsilon, \alpha) e^{-\kappa_m(\alpha) h^2 / \sigma^2}$$

holds for all $h > 0$. Furthermore, there are constants c_0, c_1 , uniform in m, α, T and ε , such that

$$\kappa_m(\alpha) \geq c_0 \frac{\alpha^2}{m^7}, \quad C_m(T, \varepsilon, \alpha) \leq c_1 \frac{T m^{3/2} e^m m^m}{\varepsilon |\alpha|}.$$

Remark 5.3.13. Comparable results cannot be expected to hold in any $\mathcal{B}_{p,\infty}^\alpha$. Since the limiting random variable does not admit a first moment, its tail probabilities have to decay more slowly than $1/h$. $\diamond$

The following observation may provide some intuition on what it means for a distribution to be concentrated in a ball in the Besov space $\mathcal{B}_{2,\infty}^\alpha$.

Proposition 5.3.14. *Let $\eta : \mathbb{T}^2 \rightarrow \mathbb{R}$ be a compactly supported function of class $\mathcal{C}^1$, with $\|\eta\|_{\mathcal{C}^1} = 1$. For any $p \in [2, \infty]$ and any $\rho \in (0, 1]$, let*

$$\eta_\rho^{(p)}(x) = \frac{1}{\rho^{2(1-1/p)}} \eta\left(\frac{x}{\rho}\right).$$

Then $\|\eta_\rho^{(p)}\|_{L^r} = \|\eta\|_{L^r}$ for all $\rho \in (0, 1]$, where r is the Hölder conjugate of p . Moreover, for any $\psi \in \mathcal{B}_{p,\infty}^\alpha$ with $\alpha \in (-1, 0)$, and any $q \in \mathbb{N}_0$, one has

$$|\langle \psi, \eta_{2^{-q_0}}^{(p)} \rangle| \lesssim 2^{|\alpha|q_0} \|\psi\|_{\mathcal{B}_{p,\infty}^\alpha}.$$

Proof. We have

$$\|\eta_\rho^{(p)}\|_{L^r}^r = \frac{1}{\rho^2} \int_{\mathbb{T}^2} \eta\left(\frac{x}{\rho}\right)^r dx = \int_{\mathbb{T}^2} \eta(y)^r dy = \|\eta\|_{L^r}^r,$$

where we have used the change of variables $x = \rho y$, and the fact that the integration domain does not change because η is compactly supported. For the same reason, we have

$$|\langle e_k, \eta_\rho^{(p)} \rangle| = \rho^{2/p} \left| \int_{\mathbb{T}^2} e^{-i\rho k \cdot y} \eta(y) dy \right| \leq \frac{\rho^{2/p}}{1 \vee \rho^2 |k_1 k_2|},$$

where we have used one integration by parts if $\rho^2 |k_1 k_2| \leq 1$. In particular, for $k \in \mathcal{A}_q$ and $\rho = 2^{-q_0}$, this yields

$$|\langle e_k, \eta_{2^{-q_0}}^{(p)} \rangle| \leq \frac{2^{-2q_0/p}}{1 \vee 2^{2(q-q_0)}}.$$

Using Hölder's inequality, we obtain

$$\begin{aligned} |\langle \delta_q \psi, \eta_{2^{-q_0}}^{(p)} \rangle| &= |\langle \delta_q \psi, \delta_q \eta_{2^{-q_0}}^{(p)} \rangle| \leq \|\delta_q \psi\|_{L^p} \|\delta_q \eta_{2^{-q_0}}^{(p)}\|_{L^r} \\ &\leq \|\delta_q \psi\|_{L^p} \left(\sum_{k \in \mathcal{A}_q} |\langle e_k, \eta_{2^{-q_0}}^{(p)} \rangle|^p \right)^{1/p} \\ &\leq 2^{|\alpha|q} \|\psi\|_{\mathcal{B}_{p,\infty}^\alpha} \frac{2^{2(q-q_0)/p}}{1 \vee 2^{2(q-q_0)}}. \end{aligned}$$

The result then follows by summing over all $q \in \mathbb{N}_0$, noticing that this sum is dominated by the term $q = q_0$. $\square$

So that, Theorem 5.3.12 implies

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \langle : \psi(t)^m :, \eta_{2^{-q_0}} \rangle > h^m \right\} \leq C_m(T, \varepsilon) \exp \left\{ -\kappa_m 2^{-2|\alpha|q_0/m} \frac{h^2}{\sigma^2} \right\}$$

for any $m \in \mathbb{N}$ and any $q_0 \in \mathbb{N}_0$. This shows that sample paths of $\langle : \psi(t)^m :, \eta_{2^{-q_0}} \rangle$ are concentrated in a strip of width $\sigma^m 2^{|\alpha|q_0}$. The same holds of course for $\eta_\rho(x - x_0)$, for any $x_0 \in \mathbb{T}^2$.

5.3.2.1 Proof of Theorem 5.3.12 for $m = 1$

Proof of Theorem 5.3.12 for $m = 1$. Fix $\alpha < 0$. Definition 2.3.1 of Besov norms, gives

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\psi(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\alpha} > h \right\} &= \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \sup_{q \geq 0} 2^{-|\alpha|q} \|\delta_q \psi(t, \cdot)\|_{L^2} > h \right\} \\ &= \mathbb{P} \left\{ \exists q \geq 0; \sup_{0 \leq t \leq T} \|\delta_q \psi(t, \cdot)\|_{L^2} > h 2^{|\alpha|q} \right\} \\ &\leq \sum_{q \geq 0} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\delta_q \psi(t, \cdot)\|_{L^2} > h 2^{|\alpha|q} \right\}. \end{aligned} \tag{5.3.15}$$

We recall that

$$\delta_q \psi(t, x) = \sum_{k \in \mathcal{A}_q} \psi_k(t) e_k(x)$$

and its variance is given by (5.3.10). For a fixed $q \geq 0$, we estimate one term of the sum (5.3.15)

$$P_q(h) = \mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\delta_q \psi(t, \cdot)\|_{L^2} > h 2^{|\alpha|q}\right\} = \mathbb{P}\left\{\sup_{0 \leq t \leq T} \sum_{k \in \mathcal{A}_q} |\psi_k(t)|^2 > h^2 2^{2|\alpha|q}\right\}.$$

We notice that the stochastic integral $\psi_k(t)$ is not a martingale. However,

$$e^{-\alpha_k(t)/\varepsilon} \psi_k(t) = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{-\alpha_k(t_1)/\varepsilon} dW_k(t_1) \quad (5.3.16)$$

is a martingale. We approach these processes by introducing a partition $0 = u_0 < u_1 < \dots < u_L = T$ of $[0, T]$. Given $\gamma_0 > 0$ and any $|k_0| = 2^q$ we define the partition by

$$\alpha_{k_0}(u_{l+1}, u_l) = -\gamma_0 \varepsilon \quad \text{for } 1 \leq l \leq L = \left\lfloor \frac{((2\pi)^2 \|k_0\|^2 + 1)T}{\gamma_0 \varepsilon} \right\rfloor,$$

and write $I_l = [u_l, u_{l+1}]$. Multiplying (5.3.16) by $e^{\alpha_k(u_{l+1})/\varepsilon}$, we obtain the martingale

$$\hat{\psi}_k(t) := e^{\alpha_k(u_{l+1}, t)/\varepsilon} \psi_k(t) = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha_k(u_{l+1}, t_1)/\varepsilon} dW_k(t_1).$$

Applying Hölder's inequality with $p = 1$, $q = \infty$, we get

$$\sum_{k \in \mathcal{A}_q} |\psi_k(t)|^2 \leq \sup_{k \in \mathcal{A}_q} e^{-2\alpha_k(u_{l+1}, t)/\varepsilon} \sum_{k \in \mathcal{A}_q} |\hat{\psi}_k(t)|^2.$$

The process $(\sum_{k \in \mathcal{A}_q} |\hat{\psi}_k(t)|^2)_{t \in I_l}$ is a submartingale, because it is a sum of squares of independent martingales. It follows that

$$\begin{aligned} P_q(h) &\leq \mathbb{P}\left\{\exists l \in \{1, \dots, L\}, \sup_{t \in I_l} \sum_{k \in \mathcal{A}_q} |\psi_k(t)|^2 > h^2 2^{2|\alpha|q}\right\} \\ &\leq 2 \sum_{l=1}^L \mathbb{P}\left\{\sup_{t \in I_l} \sum_{k \in \mathcal{A}_q} |\hat{\psi}_k(t)|^2 > h^2 2^{2|\alpha|q} \inf_{u_l \leq t \leq u_{l+1}} \frac{1}{\sup_{k \in \mathcal{A}_q} e^{-2\alpha_k(u_{l+1}, t)/\varepsilon}}\right\} \\ &\leq 2 \sum_{l=1}^L \mathbb{P}\left\{\sup_{t \in I_l} \sum_{k \in \mathcal{A}_q} |\hat{\psi}_k(t)|^2 > h^2 2^{2|\alpha|q} \inf_{u_l \leq t \leq u_{l+1}} e^{2\alpha_{k_0}(u_{l+1}, t)/\varepsilon}\right\} \\ &\leq 2 \sum_{l=1}^L \mathbb{P}\left\{\sup_{t \in I_l} \sum_{k \in \mathcal{A}_q} |\hat{\psi}_k(t)|^2 > e^{-2\gamma_0} h^2 2^{2|\alpha|q}\right\}. \end{aligned}$$

Fix $\gamma \in \mathbb{R}$. By Doob's submartingale inequality, we get

$$P_q(h) \leq 2 \sum_{l=1}^L \exp\left\{-\gamma \frac{h^2}{\sigma^2} 2^{2|\alpha|q} e^{-2\gamma_0}\right\} \mathbb{E}\left[\exp\left\{\frac{\gamma}{\sigma^2} \sum_{k \in \mathcal{A}_q} |\hat{\psi}_k(u_{l+1})|^2\right\}\right]. \quad (5.3.17)$$

Since $\{\hat{\psi}_k(t)\}_{k \in \mathbb{Z}^2}$ are zero-mean gaussian random variables i.i.d with variance $\hat{v}_k(t) = v_k e^{2\alpha_k(u_{l+1}, t)/\varepsilon}$, we have

$$\mathbb{E} \left[\exp \left\{ \frac{\gamma}{\sigma^2} \sum_{k \in \mathcal{A}_q} |\hat{\psi}_k(u_{l+1})|^2 \right\} \right] = \prod_{k \in \mathcal{A}_q} \mathbb{E} \left[\exp \left\{ \frac{\gamma}{\sigma^2} |\hat{\psi}_k(u_{l+1})|^2 \right\} \right].$$

We bound the expectation

$$\mathbb{E} \left[\exp \left\{ \frac{\gamma}{\sigma^2} |\hat{\psi}_k(u_{l+1})|^2 \right\} \right] = \frac{1}{\sqrt{2\pi\hat{v}_k}} \int_{-\infty}^{+\infty} \exp \left\{ -\frac{z^2}{2\hat{v}_k} (1 - 2\gamma\hat{v}_k/\sigma^2) \right\} dz.$$

The bound (5.3.5) on $\hat{v}_k$ implies that $1 - 2\gamma\hat{v}_k/\sigma^2 > 0$ for any $\gamma < 1/2$. We can thus make the change of variable $u = z\sqrt{1 - 2\gamma\hat{v}_k/\sigma^2}$, yielding the estimate

$$\mathbb{E} \left[\exp \left\{ \frac{\gamma}{\sigma^2} |\hat{\psi}_k(u_{l+1})|^2 \right\} \right] = \frac{1}{\sqrt{1 - 2\gamma\hat{v}_k/\sigma^2}}.$$

We can thus bound the product of expectations by

$$\begin{aligned} \prod_{k \in \mathcal{A}_q} \mathbb{E} \left[\exp \left\{ \frac{\gamma}{\sigma^2} |\hat{\psi}_k(u_{l+1})|^2 \right\} \right] &= \prod_{k \in \mathcal{A}_q} \frac{1}{\sqrt{1 - 2\gamma\hat{v}_k/\sigma^2}} \\ &= \exp \left\{ -\frac{1}{2} \sum_{k \in \mathcal{A}_q} \ln(1 - 2\gamma\hat{v}_k/\sigma^2) \right\}. \end{aligned}$$

There exists a positive constant c such that $\ln(1 - x) \geq -cx$ for $0 \leq x \leq \frac{1}{2}$. Therefore,

$$\prod_{k \in \mathcal{A}_q} \mathbb{E} \left[\exp \left\{ \frac{\gamma}{\sigma^2} |\hat{\psi}_k(u_{l+1})|^2 \right\} \right] \leq \exp \left\{ c \frac{\gamma}{\sigma^2} \sum_{k \in \mathcal{A}_q} \hat{v}_k \right\} \leq \exp \{ c' \gamma \}, \quad (5.3.18)$$

where the sum is bounded exactly in the same way as v_k in (5.3.6) since $\alpha_k(u_{l+1}, t)$ is negative whenever $u_{l+1} \geq t$. Replacing this bound (5.3.18) in (5.3.17), we get

$$\begin{aligned} P_q(h) &\leq 2 \sum_{l=1}^L \exp \{ c' \gamma \} \exp \left\{ -\gamma \frac{h^2}{\sigma^2} 2^{2|\alpha|q} e^{-2\gamma_0} \right\} \\ &\leq 2 \left\lfloor \frac{((2\pi)^2 \|k_0\|^2 + 1)T}{\gamma_0 \varepsilon} \right\rfloor \exp \{ c' \gamma \} \exp \left\{ -\gamma \frac{h^2}{\sigma^2} 2^{2|\alpha|q} e^{-2\gamma_0} \right\} \end{aligned}$$

Thus, choosing $\gamma = \frac{1}{4}$ we get

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\psi(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\alpha} > h \right\} \leq c(T, \varepsilon) \sum_{q \geq 0} 2^{2q} \exp \left\{ -e^{-2\gamma_0} \frac{h^2}{4\sigma^2} 2^{2|\alpha|q} \right\}.$$

The sum we have to compute is of the form

$$\sum_{q \geq 0} f(q), \quad f(x) = 2^{2x} e^{-\beta 2^{2|\alpha|x}}, \quad \beta = e^{-2\gamma_0} \frac{h^2}{4\sigma^2}.$$

One checks that f is decreasing, so that one has the upper bound

$$\begin{aligned} \sum_{q \geq 0} f(q) &= f(0) + f(1) + \int_1^\infty f(x) dx \\ &= e^{-\beta} + 4e^{-\beta 2^{2|\alpha|}} + \int_1^\infty f(x) dx \end{aligned}$$

To evaluate the integral, we bound $2^{2|\alpha|x} \geq 2 \ln(2)|\alpha|x$ and we get

$$\begin{aligned} \int_1^\infty f(x) dx &\leq \int_1^\infty e^{-(2 \ln(2)\beta|\alpha| - 2 \ln(2))x} dx \\ &= \frac{4}{2 \ln(2)\beta|\alpha| - 2 \ln(2)} e^{-2 \ln(2)\beta|\alpha|} . \end{aligned}$$

It follows that

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\psi(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\alpha} > h\right\} \leq C(T, \varepsilon, \alpha) e^{-\kappa(\alpha)h^2/\sigma^2} .$$

□

5.3.2.2 Proof of Theorem 5.3.12 for any m

We first give a proof of Theorem 5.3.12 in the particular case where $a(t) = -1$ for all t . That is, we consider the linear equation

$$d\psi(t, x) = \frac{1}{\varepsilon} [\Delta\psi(t, x) - \psi(t, x)] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x) .$$

Its projection on the k th basis vector e_k is given by

$$d\psi_k(t) = -\frac{1}{\varepsilon}(\mu_k + 1)\psi_k(t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_k(t) , \quad (5.3.19)$$

where the μ_k are the eigenvalues (5.1.3) of the Laplacian, and the $\{W_k(t)\}_{t \geq 0}$ are independent Wiener processes. We write $\alpha_k(t, t_1) = -(\mu_k + 1)(t - t_1)$ and $\alpha_k(t, 0) = \alpha_k(t)$ for brevity. The solution of (5.3.19) is an Ornstein–Uhlenbeck process, which can be represented using Duhamel’s principle by the Ito integral

$$\psi_k(t) = e^{\alpha_k(t)/\varepsilon} \psi_k(0) + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha_k(t, t_1)/\varepsilon} dW_k(t_1) . \quad (5.3.20)$$

At any time $t \geq 0$, $\psi_k(t)$ is a zero-mean Gaussian random variable of variance

$$\begin{aligned} v_k(t) &= \text{Var}(\psi_k(0)) e^{2\alpha_k(t)/\varepsilon} + \frac{\sigma^2}{\varepsilon} \int_0^t e^{2\alpha_k(t, t_1)/\varepsilon} dt_1 \\ &= \text{Var}(\psi_k(0)) e^{2\alpha_k(t)/\varepsilon} + \frac{\sigma^2}{2(\mu_k + 1)} [1 - e^{2\alpha_k(t)/\varepsilon}] . \end{aligned}$$

In order to obtain a stationary process, we assume that the initial conditions $\psi_k(0)$ follow centred normal distributions with variance $v_k = \sigma^2/[2(\mu_k + 1)]$, which are mutually independent, and independent of the Wiener processes. In this way, we have $v_k(t) = v_k$ for all t .

Fix $\alpha < 0$. By Definition 2.3.1 of Besov norms,

$$\begin{aligned} P_m(h) &:= \mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\psi(t, \cdot)^m\|_{\mathcal{B}_{2,\infty}^\alpha} > h^m\right\} \\ &= \mathbb{P}\left\{\sup_{0 \leq t \leq T} \sup_{q_0 \geq 0} 2^{-|\alpha|q_0} \|\delta_{q_0}(\psi(t, \cdot)^m)\|_{L^2} > h^m\right\} \\ &= \mathbb{P}\left\{\exists q_0 \geq 0; \sup_{0 \leq t \leq T} \|\delta_{q_0}(\psi(t, \cdot)^m)\|_{L^2} > h^m 2^{|\alpha|q_0}\right\} \\ &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\delta_{q_0}(\psi(t, \cdot)^m)\|_{L^2} > h^m 2^{|\alpha|q_0}\right\}. \end{aligned}$$

By the multiindex notations introduced in the beginning of subsection 5.3.1.2 and the multinomial formula for Hermite polynomials given in Lemma 5.3.6, it follows that

$$\begin{aligned} P_m(h) &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\delta_{q_0}(\sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} :)\|_{L^2} > h^m 2^{|\alpha|q_0}\right\} \\ &= \sum_{q_0 \geq 0} \mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \delta_{q_0}(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} :)\|_{L^2} > h^m 2^{|\alpha|q_0}\right\} \\ &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\sup_{0 \leq t \leq T} \sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \|\delta_{q_0}(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} :)\|_{L^2} > h^m 2^{|\alpha|q_0}\right\}. \end{aligned}$$

We recall Remark 5.3.7 which is useful in restricting the domains of the sums for any $\mathbf{n}$. For any decomposition $h^m = \sum_{|\mathbf{n}|=m} h_{\mathbf{n}}^m$, one has

$$P_m(h) \leq \sum_{q_0 \geq 0} \sum_{|\mathbf{n}|=m} \mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\delta_{q_0}(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} :)\|_{L^2} > \frac{\mathbf{n}!}{m!} h_{\mathbf{n}}^m 2^{|\alpha|q_0}\right\}. \quad (5.3.21)$$

From now on, we fix $q_0 \geq 0$ and $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$ with $|\mathbf{n}| = m$. Our aim is to estimate one term in the double sum (5.3.21). We notice that the stochastic integral $\psi_k(t)$ is not a martingale. However,

$$e^{-\alpha_k(t)/\varepsilon} \psi_k(t) = \psi_k(0) + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{-\alpha_k(t_1)/\varepsilon} dW_k(t_1) \quad (5.3.22)$$

is a martingale of variance $e^{-2\alpha_k(t)/\varepsilon} v_k$. The variances of $\psi_k(t)$ and $\hat{\psi}_k(t)$ are too different on the whole time interval $[0, T]$ to allow a useful comparison of the two processes. This is why we introduce a partition $0 = u_0 \leq u_1 < \dots < u_L = T$ of this interval. Given $\gamma_0 > 0$ and any $k_0 \in \mathbb{Z}^2$ such that $|k_0| = 2^{q[\mathbf{n}]}$, we define the partition by

$$\alpha_{k_0}(u_{l+1}, u_l) = -\gamma_0 \varepsilon \quad \text{for } 1 \leq l \leq L = \left\lfloor \frac{((2\pi)^2 \|k_0\|^2 + 1)T}{\gamma_0 \varepsilon} \right\rfloor, \quad (5.3.23)$$

and write $I_l = [u_l, u_{l+1}]$. Multiplying (5.3.22) by $e^{\alpha_k(u_{l+1})/\varepsilon}$, we obtain the martingale

$$\hat{\psi}_k(t) := e^{\alpha_k(u_{l+1}, t)/\varepsilon} \psi_k(t) = e^{\alpha_k(u_{l+1})/\varepsilon} \psi_k(0) + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha_k(u_{l+1}, t_1)/\varepsilon} dW_k(t_1) ,$$

where we do not indicate the l -dependence of $\hat{\psi}_k(t)$ in order not to overload the notations. The variance of $\hat{\psi}_k(t)$ is

$$\hat{v}_k(t) = v_k e^{2\alpha_k(u_{l+1}, t)/\varepsilon} .$$

The key observation is the following property of Hermite polynomials.

Lemma 5.3.15. *For any $m \geq 1$, and $\{H_m(\hat{\psi}_k(t); \hat{v}_k(t))\}_{t \geq 0}$ is a martingale with respect to the canonical filtration $\mathcal{F}_t$ of the Wiener process $(W_k(t))_{t \geq 0}$.*

Proof. We write $H_m(\hat{\psi}_k(t); v_k(t)) = H_m(\hat{\psi}_k(t))$ in order not to overload the notation. For any $0 \leq s < t$, we have

$$\mathbb{E}[H_m(\hat{\psi}_k(t)) \mid \mathcal{F}_s] = \mathbb{E}[H_m(\hat{\psi}_k(s) + (\hat{\psi}_k(t) - \hat{\psi}_k(s))) \mid \mathcal{F}_s] . \quad (5.3.24)$$

By the binomial formula (2.5.3),

$$H_m(\hat{\psi}_k(s) + (\hat{\psi}_k(t) - \hat{\psi}_k(s))) = \sum_{n=0}^m \binom{m}{n} H_n(\hat{\psi}_k(s)) H_{m-n}(\hat{\psi}_k(t) - \hat{\psi}_k(s)) .$$

We replace this expression in (5.3.24). Since $H_n(\hat{\psi}_k(s))$ is $\mathcal{F}_s$ -measurable and $H_{m-n}(\hat{\psi}_k(t) - \hat{\psi}_k(s))$ is independent of $\mathcal{F}_s$ we obtain

$$\begin{aligned} \mathbb{E}[H_m(\hat{\psi}_k(t)) \mid \mathcal{F}_s] &= \sum_{n=0}^m \binom{m}{n} \mathbb{E}[H_n(\hat{\psi}_k(s)) H_{m-n}(\hat{\psi}_k(t) - \hat{\psi}_k(s)) \mid \mathcal{F}_s] \\ &= \sum_{n=0}^m \binom{m}{n} H_n(\hat{\psi}_k(s)) \mathbb{E}[H_{m-n}(\hat{\psi}_k(t) - \hat{\psi}_k(s))] \\ &= H_m(\hat{\psi}_k(s)) . \end{aligned}$$

The last equality is due to the fact that m th Hermite polynomials are centred variables for $m \geq 1$ and for $m = n$, $H_0(\hat{\psi}_k(t) - \hat{\psi}_k(s)) = 1$. $\square$

This observation will allow us to deal with the supremum over times in (5.3.21), by using Doob's submartingale inequality. We will thus be interested in the martingales

$$\delta_q \hat{\psi}(t, x) = \sum_{k \in \mathcal{A}_q} \hat{\psi}_k(t) e_k(x) ,$$

as well as of the related quantities

$$X_{\mathbf{n}}^2(t) = \left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \hat{\psi}(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2}^2 .$$

Later on, we will extend the obtained bounds to functions of $\delta_q \psi(t, x)$.

Proposition 5.3.16. *Fix a constant $\gamma_{q_0, q_{[n]}} \in \mathbb{R}$ and $l \in \{0, \dots, L\}$. Then the bound*

$$\mathbb{P}\left\{\sup_{t \in I_l} X_{\mathbf{n}}^2(t) > \left(\frac{\mathbf{n}!}{m!} h_{\mathbf{n}}^m 2^{|\alpha|q_0}\right)^2\right\} \leq C_{\mathbf{n}}(l, \varepsilon) \exp\left\{-\frac{H_{\mathbf{n}}(q_0, l)}{\sigma^2}\right\}$$

holds, where

$$C_{\mathbf{n}}(l, \varepsilon) = e^{m-1} + \mathbb{E}\left[\exp\left\{\frac{\gamma_{q_0, q_{[n]}}}{\sigma^2} [X_{\mathbf{n}}^2(u_{l+1})]^{1/m}\right\}\right], \quad (5.3.25)$$

$$H_{\mathbf{n}}(q_0, l) = \gamma_{q_0, q_{[n]}} \left(\frac{\mathbf{n}!}{m!} 2^{|\alpha|q_0} h_{\mathbf{n}}^m\right)^{2/m}.$$

Proof. The process $(X_{\mathbf{n}}^2(t))_{t \in I_l}$ is a submartingale, because it is the projection of a sum of squares of independent martingales. We note that the function $f_{\gamma} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$f_{\gamma}(x) = \max\{e^{m-1}, e^{\gamma x^{1/m}}\} = \begin{cases} e^{m-1} & \text{if } x \leq \frac{(m-1)^m}{\gamma} \\ e^{\gamma x^{1/m}} & \text{if } x > \frac{(m-1)^m}{\gamma} \end{cases},$$

is non-decreasing and convex. By Doob's submartingale inequality, we get

$$\begin{aligned} \mathbb{P}\left\{\sup_{t \in I_l} X_{\mathbf{n}}^2(t) > \left(\frac{\mathbf{n}!}{m!} h_{\mathbf{n}}^m 2^{|\alpha|q_0}\right)^2\right\} &= \mathbb{P}\left\{\sup_{t \in I_l} f_{\gamma}(X_{\mathbf{n}}^2(t)) > f_{\gamma}\left(\left(\frac{\mathbf{n}!}{m!} h_{\mathbf{n}}^m 2^{|\alpha|q_0}\right)^2\right)\right\} \\ &\leq \frac{1}{f_{\gamma}\left(\left(\frac{\mathbf{n}!}{m!} h_{\mathbf{n}}^m 2^{|\alpha|q_0}\right)^2\right)} \mathbb{E}\left[f_{\gamma}(X_{\mathbf{n}}^2(u_{l+1}))\right]. \end{aligned}$$

In the denominator, we bound $f_{\gamma}(x)$ below by $e^{\gamma x^{1/m}}$. In the expectation, we bound the maximum defining f_{γ} above by the sum. Setting $\gamma = \gamma_{q_0, q_{[n]}}/\sigma^2$ yields the result. $\square$

Following the same reasoning of Remark 5.3.8 we explain the m th root in Proposition 5.3.16. In order to bound the exponential moment in (5.3.25), we recall the following technical lemma introduced before and whose proof is given in Appendix A.

Lemma 5.3.17. *There exists a numerical constant $C_0 \geq 0$ such that for any l , one has*

$$\mathbb{E}[X_{\mathbf{n}}^2(u_{l+1})] \leq C_m \sigma^{2m} \frac{2^{2q_0}}{2^{2q_{[n]}}}$$

where $C_m = C_0^m m!$.

This bound says that although high frequency modes, of order $2_{[n]}^q$, have some influence on lower modes of order 2^{q_0} , this influence decreases exponentially in their ratio.

Proposition 5.3.18. *For any $\gamma_{q_0, q_{[n]}} \leq e^{-1}(C_m)^{-1/m} 2^{2(q_{[n]} - q_0)/m}$, one has*

$$\mathbb{E}\left[\exp\left\{\frac{\gamma_{q_0, q_{[n]}}}{\sigma^2} [X_{\mathbf{n}}^2(u_{l+1})]^{1/m}\right\}\right] < \frac{1}{1 - \gamma_{q_0, q_{[n]}} e^{C_m^{1/m} 2^{2(q_0 - q_{[n]})/m}}}.$$

Proof. Expanding the exponential, we get

$$\mathbb{E} \left[\exp \left\{ \frac{\gamma_{q_0, q_{[n]}}}{\sigma^2} [X_{\mathbf{n}}^2(u_{l+1})]^{1/m} \right\} \right] = \sum_{p \geq 0} \frac{\gamma_{q_0, q_{[n]}}^p}{\sigma^{2p} p!} \mathbb{E} \left[(X_{\mathbf{n}}^2(u_{l+1}))^{p/m} \right].$$

By Jensen's inequality (or Hölder's inequality with conjugates m and $\frac{m}{m-1}$), we have

$$\mathbb{E} \left[(X_{\mathbf{n}}^2(u_{l+1}))^{p/m} \right] \leq \mathbb{E} \left[(X_{\mathbf{n}}^2(u_{l+1}))^p \right]^{1/m}.$$

Since $X_{\mathbf{n}}^2(u_{l+1})$ belongs to the $2m$ th Wiener chaos, we can use for even p equivalence of norms (see Lemma 2.5.4) to obtain the bound

$$\mathbb{E} \left[(X_{\mathbf{n}}^2(u_{l+1}))^p \right] \leq (p-1)^{mp} \mathbb{E} [X_{\mathbf{n}}^2(u_{l+1})]^p \leq \left[(p-1)^m C_m \sigma^{2m} 2^{2(q_0 - q_{[n]})} \right]^p,$$

where we have used Lemma 5.3.9 in the last inequality. A similar bound follows for odd p by the Cauchy–Schwarz inequality. Combining these inequalities, we get

$$\mathbb{E} \left[\exp \left\{ \frac{\gamma_{q_0, q_{[n]}}}{\sigma^2} [X_{\mathbf{n}}^2(u_{l+1})]^{1/m} \right\} \right] \leq \sum_{p \geq 0} \frac{(p-1)^p}{p!} \left[\gamma_{q_0, q_{[n]}} C_m^{1/m} 2^{2(q_0 - q_{[n]})/m} \right]^p.$$

Stirling's formula yields $p^p/p! \leq e^p$. The result follows by summing a geometric series. $\square$

Choosing $\gamma_{q_0, q_{[n]}} = (2e C_m^{1/m})^{-1} 2^{2(q_{[n]} - q_0)/m}$, we obtain

$$C_{\mathbf{n}}(l, \varepsilon) \leq 2 + e^{m-1},$$

$$H_{\mathbf{n}}(q_0, l) = \frac{1}{2e C_m^{1/m}} \left(\frac{\mathbf{n}!}{m!} 2^{|\alpha|q_0} 2^{q_{[n]} - q_0} h_{\mathbf{n}}^m \right)^{2/m}.$$

This motivates the choice

$$h_{\mathbf{n}}^m = \frac{1}{K_m(q_0)} h^m \frac{m!}{\mathbf{n}!} \frac{1}{2^{(q_{[n]} - q_0)/2}} 1_{\{q_{[n]} + \mathbf{n}_{q_{[n]}} \geq q_0\}},$$

where the indicator is due to Remark 5.3.7, which yields

$$\mathbb{P} \left\{ \sup_{t \in I_t} X_{\mathbf{n}}^2(t) > \left(\frac{\mathbf{n}!}{m!} h_{\mathbf{n}}^m 2^{|\alpha|q_0} \right)^2 \right\} \leq (2 + e^{m-1}) \exp \left\{ -\frac{h^2}{2e\sigma^2} \left(\frac{2^{(q_{[n]} - q_0)/2} 2^{|\alpha|q_0}}{K_m(q_0) C_m^{1/2}} \right)^{2/m} \right\}. \quad (5.3.26)$$

The condition $h^m = \sum_{|\mathbf{n}|=m} h_{\mathbf{n}}^m$ imposes

$$K_m(q_0) = \sum_{\substack{|\mathbf{n}|=m \\ q_{[n]} + \mathbf{n}_{q_{[n]}} \geq q_0}} \frac{m!}{\mathbf{n}!} \frac{1}{2^{(q_{[n]} - q_0)/2}}. \quad (5.3.27)$$

The proof of the following bound is postponed to Appendix A.

Lemma 5.3.19. *There exist numerical constants $c_0, c_1, c_2 > 0$ such that*

$$K_m(q_0) \leq c_0 m! (m + c_2)^m (q_0 + c_1)^m .$$

Substituting in (5.3.26) yields the bound

$$\mathbb{P} \left\{ \sup_{t \in I_t} X_{\mathbf{n}}^2(t) > \left(\frac{h^m 2^{|\alpha|q_0}}{K_m(q_0) 2^{(q_{[\mathbf{n}]} - q_0)/2}} \right)^2 \right\} \leq (2 + e^{m-1}) \exp \left\{ -\kappa_m \frac{2^{(q_{[\mathbf{n}]} - q_0)/m} 2^{2|\alpha|q_0/m} h^2}{(q_0 + c_1)^2 \sigma^2} \right\} . \quad (5.3.28)$$

where

$$\kappa_m = \frac{1}{2e c_0^{2/m} (C_m(m!)^2)^{1/m} (m + c_2)^2} = \mathcal{O} \left(\frac{1}{m^5} \right) .$$

We now have to convert the estimate (5.3.28) into an estimate involving Wick powers of $\delta_q \psi(t, \cdot)$ instead of $\delta_q \hat{\psi}(t, \cdot)$. For that, we are going to use the following, rather rough bound. For any $\mathbf{l} \in \mathbb{N}^{\mathbb{N}}$ with finitely many nonzero components, we write

$$|\mathbf{l}| := \sum_{q \geq 0} \mathbf{l}_q , \quad \mathbf{l}! := \prod_{q \geq 0} \mathbf{l}_q! , \quad \text{and} \quad \mathbf{l} \leq \lfloor \frac{\mathbf{n}}{2} \rfloor \Leftrightarrow \mathbf{l}_q \leq \lfloor \frac{\mathbf{n}_q}{2} \rfloor \quad \forall q \geq 0 .$$

We introduce the shorthands

$$\begin{aligned} \varphi(t, \cdot) &= \prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \psi(t, \cdot); c_q) , \\ \hat{\varphi}(t, \cdot) &= \prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \hat{\psi}(t, \cdot); \hat{c}_q(t)) . \end{aligned}$$

The proof of the following result is postponed to Appendix A.

Proposition 5.3.20. *There is a numerical constant c_1 such that for all $t \in I_t$, one has*

$$\begin{aligned} &\| \delta_{q_0}(\varphi(t, \cdot) - \hat{\varphi}(t, \cdot)) \|_{L^2} \\ &\leq 2^{q_0} (c_1 \gamma_0)^{[\mathbf{n}]} \left(\prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \mathbf{n}_q \right) \sum_{\mathbf{l}: \mathbf{l} \leq \lfloor \mathbf{n}/2 \rfloor} A_{\mathbf{n}\mathbf{l}} \hat{c}(t)^{|\mathbf{l}|} \prod_{q \geq 0} \| \delta_q \hat{\psi}(t, \cdot) \|_{L^2}^{\mathbf{n}_q - 2\mathbf{l}_q} , \end{aligned} \quad (5.3.29)$$

where

$$\hat{c}(t) = \sup_{q \geq 0} \hat{c}_q(t) \quad \text{and} \quad A_{\mathbf{n}\mathbf{l}} = \frac{\mathbf{n}!}{2^{|\mathbf{l}|} \mathbf{l}! (\mathbf{n} - 2\mathbf{l})!} 2^{2q_{[\mathbf{n}]}(|\mathbf{n}| - 2|\mathbf{l}|)} .$$

Note that the first product over q in (5.3.29) can be bounded above by $m^{[\mathbf{n}]}$.

We can now derive the main estimate of this section.

Proposition 5.3.21. *There is a constant $Q_m = \mathcal{O}(m^{-1/2})$ such that, if one chooses γ_0 of order $q_0^m Q_m 2^{-(m+1)q_{[\mathbf{n}]}}$, there exists a constant $\bar{\kappa}_m$, comparable to κ_m , such that*

$$\mathbb{P} \left\{ \sup_{t \in I_t} \| \delta_{q_0} \varphi(t, \cdot) \|_{L^2} > \frac{2^{|\alpha|q_0} h^m}{K_m(q_0) 2^{(q_{[\mathbf{n}]} - q_0)/2}} \right\} \leq \bar{C}_m(\mathbf{n}) \exp \left\{ -\bar{\kappa}_m \frac{2^{(q_{[\mathbf{n}]} - q_0)/m} 2^{2|\alpha|q_0/m} h^2}{(q_0 + c_1)^2 \sigma^2} \right\}$$

holds for all $h \geq \sigma$, where $\bar{C}_m(\mathbf{n}) = 2 + e^{m-1} + c_0(q_{[\mathbf{n}]} + m)$ for a numerical constant c_0 .

Proof. The argument is essentially deterministic. We introduce the two events

$$\Omega_1(\tilde{h}) = \left\{ \forall q \leq q_{[\mathbf{n}]} + \mathbf{n}_{q_{[\mathbf{n}]}} , \sup_{t \in I_\ell} \|\delta_q \hat{\psi}(t, \cdot)\|_{L^2} \leq \tilde{h} \right\} ,$$

$$\Omega_2(h, q_0) = \left\{ \sup_{t \in I_\ell} \|\delta_{q_0} \hat{\varphi}(t, \cdot)\|_{L^2} \leq \frac{1}{2} \frac{2^{|\alpha|q_0} h^m}{K_m(q_0) 2^{(q_{[\mathbf{n}]} - q_0)/2}} \right\} .$$

The estimate (5.3.28) provides an upper bound on $\mathbb{P}(\Omega_2(h, q_0)^c)$. As for $\Omega_1(\tilde{h})$, the bound

$$\mathbb{P}(\Omega_1(\tilde{h})^c) \leq c_0(q_{[\mathbf{n}]} + \mathbf{n}_{q_{[\mathbf{n}]}}) e^{-\kappa_0 \tilde{h}^2 / \sigma^2}$$

follows as a particular case of (5.3.28), applied separately to all $\mathbf{n}$ of size $|\mathbf{n}| = 1$. We now choose $\tilde{h}$ is such a way that

$$\kappa_0 \tilde{h}^2 = \kappa_m \frac{2^{(q_{[\mathbf{n}]} - q_0)/m} 2^{2|\alpha|q_0/m}}{(q_0 + c_1)^2} h^2 ,$$

so that $\mathbb{P}(\Omega_1(\tilde{h})^c)$ and $\mathbb{P}(\Omega_2(h, q_0)^c)$ are of comparable size. This allows us to bound the quantity

$$\sup_{t \in I_\ell} \|\delta_{q_0} \varphi(t, \cdot)\|_{L^2} \leq \sup_{t \in I_\ell} \|\delta_{q_0} \hat{\varphi}(t, \cdot)\|_{L^2} + \sup_{t \in I_\ell} \|\delta_{q_0} \varphi(t, \cdot) - \delta_{q_0} \hat{\varphi}(t, \cdot)\|_{L^2} .$$

on $\Omega_1(\tilde{h}) \cap \Omega_2(h, q_0)$. By Proposition 5.3.20, we have

$$\sup_{t \in I_\ell} \|\delta_{q_0} \varphi(t, \cdot) - \delta_{q_0} \hat{\varphi}(t, \cdot)\|_{L^2} \leq 2^{q_0} (c_1 \gamma_0 m)^{[\mathbf{n}]} \sum_{\mathbf{l}: \mathbf{l} \leq [\mathbf{n}/2]} \frac{\mathbf{n}!}{\mathbf{l}!(\mathbf{n} - 2\mathbf{l})!} \left(\frac{\hat{c}(t)}{2} \right)^{|\mathbf{l}|} (2^{2q_{[\mathbf{n}]}} \tilde{h})^{|\mathbf{n}| - 2|\mathbf{l}|} .$$

Note the relation

$$\begin{aligned} \sum_{\mathbf{l}: \mathbf{l} \leq [\mathbf{n}/2]} \frac{\mathbf{n}!}{\mathbf{l}!(\mathbf{n} - 2\mathbf{l})!} a^{\mathbf{l}} b^{\mathbf{n} - 2\mathbf{l}} &= \prod_{q \geq 0} \sum_{\mathbf{l}_q \leq [\mathbf{n}_q/2]} \frac{\mathbf{n}_q!}{\mathbf{l}_q!(\mathbf{n}_q - 2\mathbf{l}_q)!} a^{\mathbf{l}_q} b^{\mathbf{n}_q - 2\mathbf{l}_q} \\ &\leq \prod_{q \geq 0} \frac{\mathbf{n}_q!}{(\mathbf{n}_q/2)!} (\sqrt{a} + b)^{\mathbf{n}_q} \leq 2^{|\mathbf{n}|} (\sqrt{a} + b)^{|\mathbf{n}|} . \end{aligned}$$

Since $|\mathbf{n}| = m$, it follows that

$$\sup_{t \in I_\ell} \|\delta_{q_0} \varphi(t, \cdot)\|_{L^2} \leq \frac{1}{2} \frac{2^{|\alpha|q_0} h^m}{K_m(q_0) 2^{(q_{[\mathbf{n}]} - q_0)/2}} + 2^{q_0} (c_1 \gamma_0 m)^{[\mathbf{n}]} 2^m (\hat{c}(t) + 2^{2q_{[\mathbf{n}]}} \tilde{h})^m$$

holds on $\Omega_1(\tilde{h}) \cap \Omega_2(h, q_0)$. Choosing γ_0 such that both summands are equal yields the result. $\square$

Corollary 5.3.22. *We have*

$$\begin{aligned} \mathbb{P} \left\{ \sup_{t \in I} \left\| \delta_{q_0} \left(\prod_{q \geq 0} : \delta_q \psi(t, \cdot)^{\mathbf{n}_q} : \right) \right\|_{L^2} > \frac{\mathbf{n}!}{m!} h^m 2^{|\alpha|q_0} \right\} \\ \leq \frac{T}{\varepsilon} \tilde{C}_m(\mathbf{n}) q_0^{-m} 2^{(m+3)q_{[\mathbf{n}]}} \exp \left\{ -\bar{\kappa}_m \frac{2^{(q_{[\mathbf{n}]} - q_0)/m} 2^{2|\alpha|q_0/m} h^2}{(q_0 + c_1)^2 \sigma^2} \right\} , \end{aligned}$$

where $\tilde{C}_m(\mathbf{n}) = Q_m^{-1} \bar{C}_m(\mathbf{n})$.

Proof. It suffices to sum the previous estimate over all $\ell \in \{1, \dots, L\}$, where L has been introduced in (5.3.23). $\square$

Replacing the bound obtained in Corollary 5.3.22 in (5.3.21), we get

$$P_m(h) \leq \frac{T}{\varepsilon} \sum_{q_0 \geq 0} q_0^{-m} \sum_{|\mathbf{n}|=m} \tilde{C}_m(\mathbf{n}) 2^{(m+3)q_{[\mathbf{n}]}} \exp \left\{ -\beta(m, q_0) 2^{(q_{[\mathbf{n}]} - q_0)/m} \right\},$$

where

$$\beta(m, q_0) = \bar{\kappa}_m \frac{2^{2|\alpha|q_0/m} h^2}{(q_0 + c_1)^2 \sigma^2}.$$

We will first perform the sum over $\mathbf{n}$. To this end, we write

$$\bar{K}_{m,b}(q_0) = \sum_{|\mathbf{n}|=m} q_{[\mathbf{n}]}^b 2^{(m+3)q_{[\mathbf{n}]}} \exp \left\{ -\beta(m, q_0) 2^{(q_{[\mathbf{n}]} - q_0)/m} \right\},$$

The following lemma is obtained in a similar way as Lemma 5.3.23. We give its proof in Appendix A.

Lemma 5.3.23. *There are numerical constants c_1, β_0 such that for all $\beta(m, q_0) \geq \beta_0$, one has the bound*

$$\bar{K}_{m,b}(q_0) \leq c_1 q_0^{m+b} m^m 2^{(m+3)q_0} e^{-\beta(m, q_0)}.$$

Using the expression for $\bar{C}_m(\mathbf{n})$ given in Proposition 5.3.21, we thus obtain

$$\begin{aligned} P_m(h) &\leq \frac{T}{\varepsilon} Q_m^{-1} \sum_{q_0 \geq 0} q_0^{-m} \left[(2 + e^{m-1} + c_0 m) \bar{K}_{m,0}(q_0) + \bar{K}_{m,1}(q_0) \right] \\ &\leq \frac{T}{\varepsilon} c_1 m^m Q_m^{-1} \sum_{q_0 \geq 0} \left[2 + e^{m-1} + c_0 m + c_0 q_0 \right] 2^{(m+3)q_0} e^{-\beta(m, q_0)}. \end{aligned} \quad (5.3.30)$$

It remains to perform the sums over q_0 . These are of the form

$$\sum_{q_0 \geq 0} f(q_0), \quad f(x) = x^b 2^{ax} \exp \left\{ -\gamma \frac{2^{|\alpha|x/m}}{(x + c_1)^2} \right\}, \quad a = m + 3, \quad \gamma = \bar{\kappa}_m \frac{h^2}{\sigma^2}.$$

One checks that f is decreasing, so that one has the upper bound

$$\sum_{q_0 \geq 0} f(q_0) \leq f(0) + f(1) + \int_1^\infty f(x) dx.$$

The terms $f(0)$ and $f(1)$ are both exponentially small in h^2/σ^2 . To evaluate the integral, we can absorb the constant c_1 in γ , and the term x^b into 2^{ax} , by changing slightly the definitions of γ and a . We first consider the case where the term x^2 in the denominator is absent, where the changes of variables $y = 2^{|\alpha|x/m}$ and $z = \gamma y$ yield

$$\begin{aligned} \int_1^\infty 2^{ax} e^{-\gamma 2^{|\alpha|x/m}} dx &= \frac{m}{|\alpha| \log 2} \int_{2^{|\alpha|/m}}^\infty y^{am/|\alpha|-1} e^{-\gamma y} dy \\ &\leq \frac{m}{|\alpha| \log 2} \frac{1}{\gamma^{\lambda+1}} \int_\gamma^\infty z^\lambda e^{-z} dz, \end{aligned} \quad (5.3.31)$$

where $\lambda = am/(2|\alpha|)$. The asymptotics of the incomplete Gamma function shows that

$$\int_1^\infty 2^{ax} e^{-\gamma 2^{|\alpha|x/m}} dx \lesssim \frac{m}{|\alpha|} \frac{e^{-\gamma}}{\gamma} \lesssim \frac{m}{|\alpha|} e^{-\gamma/2}.$$

In order to incorporate the effect of the denominator x^2 , we use the upper bound

$$x^2 2^{-|\alpha|x/m} \leq \frac{4e^{-2}}{(\log 2)^2} \frac{m^2}{|\alpha|^2}$$

We can thus bound the integral of f by the integral (5.3.31), with γ multiplied by a constant times $|\alpha|^2/m^2$, and α divided by 2. In other words, we get

$$\int_1^\infty f(x) dx \leq c \frac{m}{|\alpha|} e^{-\kappa |\alpha|^2 \gamma / m^2}.$$

Replacing this in (5.3.30) yields a bound on $P_m(h)$, completing the proof of Theorem 5.3.12 in the case of a constant linearisation $a(t)$.

Recall that we actually want to consider the more general linear equation (5.3.2)

$$d\tilde{\psi}(t, x) = \frac{1}{\varepsilon} [\Delta \tilde{\psi}(t, x) + a(t) \tilde{\psi}(t, x)] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x),$$

where here $a(t)$ satisfies (5.3.1), and we write $\psi = \tilde{\psi}$ to avoid confusion in the notations. Projecting (5.3.32) on the k th basis vector e_k , we obtain

$$d\tilde{\psi}_k(t) = \frac{1}{\varepsilon} a_k(t) \tilde{\psi}_k(t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_k(t), \quad (5.3.32)$$

where $a_k(t) = -\mu_k + a(t)$ and the $\{W_k(t)\}_{t \geq 0}$ are the same independent Wiener processes as before. The solution of (5.3.32) with the same initial condition $\psi_k(0)$ as in (5.3.20), is given by

$$\tilde{\psi}_k(t) = e^{\tilde{\alpha}_k(t)/\varepsilon} \psi_k(0) + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\tilde{\alpha}_k(t, t_1)/\varepsilon} dW_k(t_1),$$

where

$$\tilde{\alpha}_k(t, t_1) = \alpha_k(t, t_1) + \int_{t_1}^t (1 + a(t_2)) dt_2 = \alpha_k(t, t_1) + \mathcal{O}(|t - t_1|).$$

For given q_0 , we use the same partition of $[0, T]$ into intervals I_l as before. On each interval, we can write

$$\hat{\psi}_k(t) := e^{\tilde{\alpha}_k(u_{l+1}, t)/\varepsilon} \tilde{\psi}_k(t) = e^{\tilde{\alpha}_k(u_{l+1})/\varepsilon} \psi_k(0) + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\tilde{\alpha}_k(u_{l+1}, t_1)/\varepsilon} dW_k(t_1).$$

$\hat{\psi}_k(t)$ is again a martingale, so that its supremum over the interval I_l can be estimated as before. Note however that the variance of the associated sums over $k \in \mathcal{A}_q$ is not exactly equal to $\hat{c}_q(t)$: it is rather of the form

$$\hat{V}_q(t) = \hat{c}_q(t) [1 + \mathcal{O}(\gamma_0 2^{-2(q[\mathbf{n}] - q)})].$$

Therefore, the Wick powers of this martingale with respect to $\hat{c}_q(t)$ are not martingales. One can however estimate the supremum of the Wick powers with the correct variance as before, and then compare the two types of Wick powers. In fact, we want to bound the supremum over $t \in I_l$ of

$$\left\| \delta_{q_0} \left(\prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \tilde{\psi}(t, \cdot), c_q) \right) \right\|_{L^2}.$$

By the binomial formula introduced in Lemma 2.5.2, we obtain

$$\begin{aligned} & \prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \tilde{\psi}(t, x), c_q) \\ &= \prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \hat{\psi}(t, x) + \delta_q \tilde{\psi}(t, x) - \delta_q \hat{\psi}(t, x), \hat{V}_q(t) + c_q - \hat{V}_q(t)) \\ &= \sum_{0 \leq |\mathbf{l}| \leq |\mathbf{n}|} \binom{\mathbf{n}}{\mathbf{l}} \prod_{q \geq 0} H_{\mathbf{l}_q}(\delta_q \tilde{\psi}(t, x) - \delta_q \hat{\psi}(t, x), c_q - \hat{V}_q(t)) H_{\mathbf{n}_q - \mathbf{l}_q}(\delta_q \hat{\psi}(t, x), \hat{V}_q(t)) \\ &= \prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \hat{\psi}(t, x), \hat{V}_q(t)) \\ &\quad + \sum_{1 \leq |\mathbf{l}| \leq |\mathbf{n}|} \binom{\mathbf{n}}{\mathbf{l}} \prod_{q \geq 0} H_{\mathbf{l}_q}(\delta_q \tilde{\psi}(t, x) - \delta_q \hat{\psi}(t, x), c_q - \hat{V}_q(t)) H_{\mathbf{n}_q - \mathbf{l}_q}(\delta_q \hat{\psi}(t, x), \hat{V}_q(t)). \end{aligned}$$

Therefore, by the triangle inequality we get

$$\begin{aligned} & \left\| \delta_{q_0} \left(\prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \tilde{\psi}(t, \cdot), c_q) \right) \right\|_{L^2} \\ & \leq \left\| \delta_{q_0} \left(\prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \hat{\psi}(t, \cdot), \hat{V}_q(t)) \right) \right\|_{L^2} \\ & \quad + \sum_{1 \leq |\mathbf{l}| \leq |\mathbf{n}|} \binom{\mathbf{n}}{\mathbf{l}} 2^{q_0} \left\| \prod_{q \geq 0} H_{\mathbf{l}_q}(\delta_q \tilde{\psi}(t, \cdot) - \delta_q \hat{\psi}(t, \cdot), c_q - \hat{V}_q(t)) \right\|_{L^2} \\ & \quad \times \left\| \prod_{q \geq 0} H_{\mathbf{n}_q - \mathbf{l}_q}(\delta_q \hat{\psi}(t, \cdot), \hat{V}_q(t)) \right\|_{L^2}, \end{aligned}$$

where the last inequality is a rough bound, obtained by Cauchy–Schwarz’s inequality. The first norm can be bounded as in (5.3.28), and one can see the last norm as a particular case of it. In the same spirit as in the proof of Proposition 5.3.20, for $|\mathbf{l}| \geq 1$ one can bound

$$\left\| \prod_{q \geq 0} H_{\mathbf{l}_q}(\delta_q \tilde{\psi}(t) - \delta_q \hat{\psi}(t), c_q - \hat{V}_q(t)) \right\|_{L^2} \leq (c_1 \gamma_0)^{|\mathbf{l}|/2} \sum_{\mathbf{p}: \mathbf{p} \leq [\mathbf{l}/2]} A_{\mathbf{l}\mathbf{p}} \hat{c}(t)^{|\mathbf{p}|} \prod_{q \geq 0} \left\| \delta_q \tilde{\psi}(t, \cdot) \right\|_{L^2}^{1_q - 2\mathbf{p}_q},$$

with

$$\hat{c}(t) = \sup_{q \geq 0} \hat{c}_q(t) \quad \text{and} \quad A_{\mathbf{l}\mathbf{p}} = \frac{\mathbf{l}!}{2^{|\mathbf{p}|} \mathbf{p}! (1 - 2\mathbf{p})!} 2^{2q_{[\mathbf{n}]}(|\mathbf{l}| - 2|\mathbf{p}|)}.$$

Notice that

$$|c_q - \hat{V}_q(t)| = |\hat{c}_q(t)| \left| e^{-2\alpha_{k(q)}(u_{l+1}, t)/\varepsilon} - 1 - \mathcal{O}(\gamma_0 2^{-2(q_{[\mathbf{n}]} - q)}) \right| \leq c_0 \gamma_0.$$

Choosing γ_0 small enough in the sense taken in Proposition 5.3.21 yields the result.

Finally, similar results holds for processes $\tilde{\psi}(t, x)$ with another initial condition. In particular, when it is equal to 0, one can proceed exactly in the same way and bound the variance of the associated martingale by $\hat{V}_q(t)$.

5.4 Concentration around stable equilibrium

We recall the renormalised SPDE that interests us

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + :F(t, \phi(t, x)):] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x) .$$

We assume that F admits a non-bifurcating equilibrium branch ϕ^* in the following sense

Assumption 5.4.1 (Stable case). *There exists a map $\phi^* : I \rightarrow \mathbb{R}$ such that*

$$F(t, \phi^*(t)) = 0 \quad \forall t \in I .$$

Furthermore, the linearisation $a(t) = \partial_\phi F(t, \phi^(t))$ satisfies*

$$-a_+ \leq a(t) \leq -a_- \quad \forall t \in I$$

for some constant $a_\pm > 0$.

5.4.1 Deterministic dynamics

Consider the deterministic equation

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + F(t, \phi(t, x))] dt , \quad (5.4.1)$$

where the renormalisation counterterm C_N vanishes for $\sigma = 0$. This PDE is perfectly well-defined. We then have the following generalisation of Tihonov's theorem (cf. [45]).

Proposition 5.4.2 (Deterministic case). *There exist constants $\varepsilon_0, C > 0$ such that, when $0 < \varepsilon < \varepsilon_0$, (5.4.1) admits a particular solution $\bar{\phi}(t)$ satisfying*

$$\|\bar{\phi}(t, \cdot) - \phi^*(t)e_0\|_{H^1} \leq C\varepsilon \quad \forall t \in I .$$

Proof of Proposition 5.4.2. The proof of Proposition 5.4.2 is almost the same as the proof of Proposition 3.2.1, the only difference being that x belongs to the two-dimensional torus. We give here only a few hints of the proof. Recall that the drift terms F satisfies (5.1.2) and Assumption 5.4.1. We consider the difference $\psi(t) = \phi(t) - \phi^*(t)e_0$. Using Taylor's formula to expand $F(t, \psi(t) + \phi^*(t)e_0)$, we obtain that ψ satisfies the equation

$$\varepsilon \partial_t \psi(t, x) = \Delta \psi(t, x) + a(t) \psi(t, x) + b(t, \psi(t, x)) - \varepsilon \frac{d}{dt} \phi^*(t) e_0(x) ,$$

where

$$\begin{aligned} a(t) &= \partial_\phi F(t, \phi^*(t)e_0) , \\ b(t, \psi) &= \frac{1}{2} \partial_\phi^2 F(t, \phi^*(t) + \theta \psi) \psi^2 \quad \text{for some } \theta \in [0, 1] . \end{aligned}$$

We define the Lyapounov function

$$V(\psi) = \frac{1}{2} \|\psi\|_{H^1}^2 = \frac{1}{2} \|\psi\|_{L^2}^2 + \frac{L^2}{2\pi^2} \|\nabla \psi\|_{L^2}^2 .$$

Its time derivative satisfies

$$\varepsilon \frac{d}{dt} V(\psi) \leq 2a(t)V(\psi) + \langle \psi, b(t, \psi) \rangle - \frac{L^2}{\pi^2} \langle \Delta \psi, b(t, \psi) \rangle - \varepsilon \frac{d}{dt} \phi^*(t) \langle \psi, e_0 \rangle . \quad (5.4.2)$$

We introduce for a fixed $C_0 > 0$, $\bar{\tau}$, the first exist-time from the set $\{V(\psi(t, \cdot)) \leq C_0\}$. Then, we bound the different terms in (5.4.2) similarly to the proof in [13]. We arrive at the relation

$$\varepsilon \dot{V} \leq -\frac{1}{2} C_1 V + \varepsilon C_2 V^{1/2}$$

for all $t \leq \bar{\tau}$, and some constants $C_1, C_2 > 0$. Using Gronwall's inequality, we find that there exists a particular solution satisfying $V(\psi(t, \cdot)) = \mathcal{O}(\varepsilon^2)$ for all $t \leq \bar{\tau}$. The result extends for all $t \in I$. $\square$

5.4.2 Stochastic dynamics

For $\sigma > 0$, we are interested in the deviation from the deterministic solution given by the difference $\phi_0 = \phi - \bar{\phi}$ which satisfies the SPDE

$$d\phi_0(t, x) = \frac{1}{\varepsilon} [\Delta \phi_0(t, x) + :F_0(t, x, \phi_0(t, x)):] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x) , \quad (5.4.3)$$

where

$$:F_0(t, x, \phi_0(t, x)): = :F(t, \bar{\phi}(t, x) + \phi_0(t, x)): - F(t, \bar{\phi}(t, x))$$

has similar properties as F , and satisfies in addition $F_0(t, x, 0) = 0$ for all $t \in I$ and all $x \in \mathbb{T}^2$. More precisely, we have the following result.

Lemma 5.4.3. *The renormalised forcing term is given by*

$$:F_0(t, x, \phi_0(t, x)): = a(t)\phi_0(t, x) + \sum_{j=1}^n \hat{A}_j(t, x) : \phi_0(t, x)^j : , \quad (5.4.4)$$

where the $\hat{A}_j(t, \cdot)$ belong to H^1 (which is embedded in $\mathcal{B}_{2,\infty}^1$) for all $t \in I$, and are given by

$$\hat{A}_j(t, x) = \begin{cases} \sum_{i=2}^n i A_i(t) [\bar{\phi}(t, x)^{i-1} - \phi^*(t)^{i-1} e_0(x)] , & j = 1 , \\ \sum_{i=j}^n \binom{i}{j} A_i(t) \bar{\phi}(t, x)^{i-j} , & j = 2, \dots, n . \end{cases}$$

Proof. The binomial formula for Hermite polynomials yields

$$:\phi(t, x)^i: = \sum_{j=0}^i \binom{i}{j} \bar{\phi}(t, x)^{i-j} : \phi_0(t, x)^j : .$$

Using the definition (5.1.2) of F and swapping the sums, we obtain

$$:F_0(t, x, \phi_0(t, x)): = \sum_{j=1}^n \left[\sum_{i=j}^n \binom{i}{j} A_i(t) \bar{\phi}(t, x)^{i-j} \right] : \phi_0(t, x)^j :$$

(note that the terms $j = 0$ cancel). This proves the claim for the terms with $j \geq 2$. For $j = 1$, we note that

$$a(t) = \partial_\phi F(t, \phi^*(t)) = \sum_{i=1}^n i A_i(t) \phi^*(t)^{i-1}.$$

Rearranging terms yields the claimed result. Proposition 5.4.2 shows that $\bar{\phi}(t, \cdot) \in H^1$, and that $\|\hat{A}_1(t, \cdot)\|_{H^1} = \mathcal{O}(\varepsilon)$. By [14, Théorème 7], powers of $\bar{\phi}$ belong to H^1 as well. $\square$

We rewrite (5.4.3) as

$$d\phi_0(t, x) = \frac{1}{\varepsilon} [\Delta\phi_0(t, x) + a(t)\phi_0(t, x) + :b(t, x, \phi_0(t, x)):] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x),$$

where $:b:$ denotes the sum over j in (5.4.4). Note that $:b:$ contains a linear term in ϕ_0 . However, it has a coefficient of order ε , since $\bar{\phi}$ and ϕ^* are at a distance of order ε . We now apply the Da Prato–Debussche trick, and consider the difference $\phi_1 = \phi_0 - \psi$. It satisfies the equation

$$d\phi_1(t, x) = \frac{1}{\varepsilon} [\Delta\phi_1(t, x) + a(t)\phi_1(t, x) + :b(t, x, \psi(t, x) + \phi_1(t, x)):] dt, \quad (5.4.5)$$

where

$$:b(t, x, \psi(t, x) + \phi_1(t, x)): = \sum_{j=1}^n \hat{A}_j(t, x) \sum_{\ell=0}^j \binom{j}{\ell} \phi_1(t, x)^{j-\ell} : \psi(t, x)^\ell :.$$

By Duhamel's principle, solution of (5.4.5) is given by

$$\phi_1(t, x) = \frac{1}{\varepsilon} \int_0^t e^{\alpha(t, t_1)/\varepsilon} e^{\frac{t-t_1}{\varepsilon} \Delta} :b(t_1, x, \psi(t_1, x) + \phi_1(t_1, x)): dt_1,$$

In what follows we give some technical results that will be needed several times in order to show that $\phi_1(t, \cdot)$ belongs to a certain Besov spaces with a regularity better than the regularity of the stochastic convolution. We recall first the following properties on products in Besov spaces which can be found, for instance, in [20, Proposition 2.1], in [20, Lemma 3.3].

Proposition 5.4.4 (Products in Besov spaces).

1. Let $p, r \geq 1$ and $\alpha + \beta > 0$, with $\alpha, \beta < \frac{2}{p}$. Then, if $\phi \in \mathcal{B}_{p,r}^\alpha$ and $\psi \in \mathcal{B}_{p,r}^\beta$, one has

$$\phi\psi \in \mathcal{B}_{p,r}^\gamma \quad \text{and} \quad \|\phi\psi\|_{\mathcal{B}_{p,r}^\gamma} \lesssim \|\phi\|_{\mathcal{B}_{p,r}^\alpha} \|\psi\|_{\mathcal{B}_{p,r}^\beta}, \quad (5.4.6)$$

where $\gamma = \alpha + \beta - \frac{2}{p}$.

2. Let $n \in \mathbb{N}$, $p, r \geq 1$ and $-\frac{2}{p(2n+1)} < \alpha < 0$. Set $s = \frac{2}{p} + 2\alpha$. Then, if $\phi \in \mathcal{B}_{p,r}^s$ and $\psi \in \mathcal{B}_{p,r}^\alpha$, one has

$$\phi^\ell \psi \in \mathcal{B}_{p,r}^{(2\ell+1)\alpha} \quad \text{and} \quad \|\phi^\ell \psi\|_{\mathcal{B}_{p,r}^{(2\ell+1)\alpha}} \lesssim \|\phi\|_{\mathcal{B}_{p,r}^s}^\ell \|\psi\|_{\mathcal{B}_{p,r}^\alpha} \quad (5.4.7)$$

for $\ell \in \{0, \dots, n-1\}$, with a constant depending on α, s, p, r and n .

Proposition 5.4.5 (Schauder estimate on the heat kernel). Let $g \in \mathcal{B}_{2,\infty}^\alpha$ for some $\alpha \in \mathbb{R}$, and let $e^{t\Delta}$ denote the heat kernel. Then there exists a constant M_0 depending on $\beta - \alpha$ such that

$$\|e^{t\Delta} g\|_{\mathcal{B}_{2,\infty}^\beta} \leq M_0 t^{-\frac{\beta-\alpha}{2}} \|g\|_{\mathcal{B}_{2,\infty}^\alpha}$$

holds for all $t \geq 0$ and all $\beta \leq \alpha + 2$.

Proof. Denoting by μ_k the eigenvalues of the Laplacian (cf. (5.1.3)) and by $(g_k)_{k \in \mathbb{Z}^2}$ the Fourier coefficients of g , there is a constant $c > 0$ such that

$$\|\delta_q(e^{t\Delta} g)\|_{L^2}^2 = \sum_{k \in \mathcal{A}_q} e^{-2\mu_k t} |g_k|^2 \leq e^{-c2^{2q}t} \|\delta_q g\|_{L^2}^2 \leq e^{-c2^{2q}t} 2^{-2q\alpha} \|g\|_{\mathcal{B}_{2,\infty}^\alpha}^2$$

for all q . Therefore,

$$\|e^{t\Delta} g\|_{\mathcal{B}_{2,\infty}^\beta} \leq \sup_{q \geq 0} 2^{q(\beta-\alpha)} e^{-\frac{1}{2}c2^{2q}t} \|g\|_{\mathcal{B}_{2,\infty}^\alpha}.$$

Now we observe that for any $\gamma \geq 0$,

$$2^{q(\beta-\alpha)} e^{-\frac{1}{2}c2^{2q}t} = 2^{q(\beta-\alpha-2\gamma)} t^{-\gamma} (2^{2q}t)^\gamma e^{-\frac{1}{2}c2^{2q}t} \leq M_0(2\gamma) 2^{q(\beta-\alpha-2\gamma)} t^{-\gamma}$$

by boundedness of the map $x \mapsto x^\gamma e^{-x}$. Choosing $\gamma = \frac{\beta-\alpha}{2}$ yields the result. $\square$

Corollary 5.4.6 (Schauder estimate on convolutions with the heat kernel). *Let $g(t) \in \mathcal{B}_{2,\infty}^\alpha$ for all $t \in [0, T]$, where $\alpha \in \mathbb{R}$. Let ϕ be the solution of*

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + a(t)\phi(t, x) + g(t, x)] dt, \quad (5.4.8)$$

starting from 0, where $a \in C^1([0, T], \mathbb{R}_-)$ is bounded away from 0 (cf. (5.3.1)). Then $\phi(t) \in \mathcal{B}_{2,\infty}^\beta$ for all $\beta < \alpha + 2$ and all $t \in [0, T]$, and there is a constant $M = M(\beta - \alpha)$ such that

$$\|\phi(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\beta} \leq M \varepsilon^{\frac{\beta-\alpha}{2}-1} \sup_{t_1 \in [0, T]} \|g(t_1, \cdot)\|_{\mathcal{B}_{2,\infty}^\alpha}$$

holds for all $\beta < \alpha + 2$ and all $t \in [0, T]$.

Proof. The solution of (5.4.8) can be written as

$$\phi(t, x) = \frac{1}{\varepsilon} \int_0^t e^{\alpha(t, t_1)/\varepsilon} (e^{\frac{t-t_1}{\varepsilon} \Delta} g)(t_1, x) dt_1,$$

where $\alpha(t, t_1) = \int_{t_1}^t a(t_2) dt_2$ is negative whenever $t \geq t_1$. Therefore

$$\begin{aligned} \|\phi(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\beta} &\leq \frac{1}{\varepsilon} \int_0^t \| (e^{\frac{t-t_1}{\varepsilon} \Delta} g)(t_1, \cdot) \|_{\mathcal{B}_{2,\infty}^\beta} dt_1 \\ &\leq \frac{1}{\varepsilon} M_0(\beta - \alpha) \int_0^t \left(\frac{t-t_1}{\varepsilon} \right)^{-\frac{\beta-\alpha}{2}} \|g(t_1, \cdot)\|_{\mathcal{B}_{2,\infty}^\alpha} dt_1 \\ &\leq \varepsilon^{\frac{\beta-\alpha}{2}-1} M_0(\beta - \alpha) \sup_{t_1 \in [0, T]} \|g(t_1, \cdot)\|_{\mathcal{B}_{2,\infty}^\alpha} \int_0^t (t-t_1)^{-\frac{\beta-\alpha}{2}} dt_1. \end{aligned}$$

The integral is bounded whenever $\beta < \alpha + 2$. $\square$

It follows from Proposition 5.4.4 that if $\phi_1 \in \mathcal{B}_{2,\infty}^\beta$ and $:\psi^\ell: \in \mathcal{B}_{2,\infty}^\alpha$ for $\alpha < 0$ and $\ell = 0, \dots, n-1$, then

$$:b(t, x, \psi + \phi_1): \in \mathcal{B}_{2,\infty}^{\bar{\alpha}} \quad \forall \bar{\alpha} < (2n-1)\alpha,$$

provided $\beta \geq 1 + 2\alpha$. By the Schauder estimate recalled in Proposition 5.4.5, the solution of (5.4.5) belongs to $\mathcal{B}_{2,\infty}^\gamma$ for $\gamma < 2 - (2n+1)|\alpha|$, which allows to close the fixed-point argument, in accordance with Theorem 5.2.1. By the embedding $\mathcal{B}_{2,\infty}^\gamma \hookrightarrow \mathcal{B}_{\infty,\infty}^{\gamma-1} = \mathcal{C}^{\gamma-1}$, we see that the solution of (5.4.5) is Hölder continuous, with exponent almost 1. In other words, the solution is almost Lipschitz continuous. The main result is then the following.

Theorem 5.4.7 (Concentration estimate for ϕ_1). *For any choice of $\gamma < 2$ and $\nu < 1 - \frac{\gamma}{2}$, there exist constants $C(T, \varepsilon)$, $M, \kappa, h_0, \varepsilon_0 > 0$ such that, whenever $\varepsilon < \varepsilon_0$ and $h < h_0 \varepsilon^\nu$, one has*

$$\begin{aligned} \mathbb{P} \left\{ \sup_{t \in [0, T]} \|\phi_1(t)\|_{\mathcal{B}_{2,\infty}^\gamma} > M \varepsilon^{-\nu} h (h + \varepsilon) \right\} &\leq C(T, \varepsilon) e^{-\kappa h^2 / \sigma^2}, \\ \mathbb{P} \left\{ \sup_{t \in [0, T]} \|\phi_1(t)\|_{\mathcal{C}^{\gamma-1}} > M \varepsilon^{-\nu} h (h + \varepsilon) \right\} &\leq C(T, \varepsilon) e^{-\kappa h^2 / \sigma^2}. \end{aligned}$$

This result shows in particular that sample paths of ϕ_1 are concentrated in a ball in $\mathcal{C}^{\gamma-1}$ -norm of size

$$\varepsilon^{-\nu} \sigma (\sigma + \varepsilon) \asymp \begin{cases} \varepsilon^{-\nu} \sigma^2 & \text{if } \sigma > \varepsilon, \\ \varepsilon^{1-\nu} \sigma & \text{if } \sigma \leq \varepsilon. \end{cases}$$

5.4.2.1 Proof of Theorem 5.4.7

Recall that $\phi_1(t, x)$ solves the equation

$$d\phi_1(t, x) = \frac{1}{\varepsilon} [\Delta \phi_1(t, x) + a(t) \phi_1(t, x) + :b(t, x, \psi(t, x) + \phi_1(t, x)):] dt,$$

where $\psi(t, x)$ is the stochastic convolution, and

$$:b(t, x, \psi(t, x) + \phi_1(t, x)): = \sum_{j=1}^n \hat{A}_j(t, x) \sum_{\ell=0}^j \binom{j}{\ell} \phi_1(t, x)^{j-\ell} : \psi(t, x)^\ell :. \quad (5.4.9)$$

Proof of Theorem 5.4.7. Assume that $\psi(t, \cdot) \in \mathcal{B}_{2,\infty}^\alpha$ and $\phi_1(t, \cdot) \in \mathcal{B}_{2,\infty}^\beta$ for all $t \in [0, T]$. The bound (5.4.7) on products in Besov spaces shows that

$$\|\phi_1(t, \cdot)^{j-\ell} : \psi(t, \cdot)^\ell : \|_{\mathcal{B}_{2,\infty}^{(2(j-\ell)+1)\alpha}} \leq M_1 \|\phi_1(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\beta}^{j-\ell} \| : \psi(t, \cdot)^\ell : \|_{\mathcal{B}_{2,\infty}^\alpha}$$

for a constant $M_1 = M_1(\beta, \alpha, n)$, provided $\beta \geq 1 + 2\alpha$.

We treat separately the term $j = 1$ in the sum (5.4.9) and the remaining terms. For $j = 1$, we use the fact that $\hat{A}_1(t, \cdot) \in \mathcal{B}_{2,\infty}^1$ has a norm of order ε and (5.4.6) to obtain that

$$\|\hat{A}_1(t, \cdot)(\phi_1(t, \cdot) + \psi(t, \cdot))\|_{\mathcal{B}_{2,\infty}^{\bar{\alpha}}} \leq M_2 \varepsilon \left(\|\phi_1(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\beta} + \|\psi(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\alpha} \right)$$

for any $\bar{\alpha} < \alpha$. For $j \geq 2$, we have a similar bound, but without the factor ε , since the $\hat{A}_j$ are of order 1 in H^1 . Now let $h, H \in (0, 1]$ be constants such that

$$\max_{1 \leq \ell \leq n} \| : \psi(t, \cdot)^\ell : \|_{\mathcal{B}_{2,\infty}^\alpha} \leq h, \quad \|\phi_1(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\beta} \leq H.$$

Summing over j , since $h, H \leq 1$ we get the existence of constants M_3, M_4 such that

$$\begin{aligned} \| : b(t, x, \psi + \phi_1) : \|_{\mathcal{B}_{2,\infty}^{\bar{\alpha}}} &\leq M_2 \varepsilon (H + h) + M_3 \left[H^2 + \sum_{j=2}^n \sum_{\ell=1}^j H^{j-\ell} h^\ell \right] \\ &\leq M_4 (H + h)(H + h + \varepsilon) \end{aligned}$$

for any $\bar{\alpha} < (2n - 1)\alpha$. We now fix a $\gamma < \bar{\alpha} + 2$ and introduce the stopping time

$$\tau = \inf \left\{ t \in [0, T] : \|\phi_1(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\gamma} > H \right\}.$$

Then we have

$$\begin{aligned} \mathbb{P}\{\tau < T\} &\leq \mathbb{P}\left\{ \exists \ell \in \{1, \dots, n\} : \sup_{t \leq T} \| : \psi(t, \cdot)^\ell : \|_{\mathcal{B}_{2,\infty}^\alpha} > h^\ell \right\} \\ &\quad + \mathbb{P}\left\{ \tau < T, \sup_{t \leq T} \| : \psi(t, \cdot)^\ell : \|_{\mathcal{B}_{2,\infty}^\alpha} \leq h^\ell \forall \ell \in \{1, \dots, n\} \right\}. \end{aligned} \quad (5.4.10)$$

The first term on the right-hand side can be bounded using Theorem 5.3.12. As for the second term, we use the fact that under the condition on the Wick powers of the stochastic convolution being small, the Schauder estimate given in Corollary 5.4.6 yields

$$\|\phi_1(T \wedge \tau, \cdot)\|_{\mathcal{B}_{2,\infty}^\gamma} \leq M \varepsilon^{-\nu} (H + h)(H + h + \varepsilon), \quad \nu = 1 - \frac{\gamma - \bar{\alpha}}{2}$$

for a constant M . Choosing first $H = 2M \varepsilon^{-\nu} h(h + \varepsilon)$, and then ε small enough and $h < h_0 \varepsilon^\nu$ for a sufficiently small h_0 , one can ensure that $(H + h)(H + h + \varepsilon) < 2h(h + \varepsilon)$, so that the second probability is actually equal to zero.

To conclude the proof, we first pick a $\gamma < 2$, and then $\bar{\alpha} \in (\gamma - 2, 0)$, and finally $\alpha \in (\frac{\bar{\alpha}}{2n-1}, 0)$. We also require that $\beta \leq \gamma$, which is possible by choosing $\beta = 1 + 2\alpha$ and asking that $\alpha \leq -\frac{1}{2}(1 - \gamma)$. This yields the claimed result, thanks to the embedding $\mathcal{B}_{2,\infty}^\gamma \hookrightarrow \mathcal{B}_{\infty,\infty}^{\gamma-1} = \mathcal{C}^{\gamma-1}$. $\square$

5.5 Concentration around pitchfork bifurcation

In this section, we comment on how the results of the last section can be extended to situations where the nonlinearity F fails to satisfy Assumption 5.4.1, that is, in the case of a bifurcation. In Chapter 3 and Chapter 4 which concerned SPDEs on the one-dimensional torus, we considered the case of an avoided transcritical bifurcation, where F is given locally by

$$F(t, \phi) = \delta + t^2 - \phi^2 + \mathcal{O}(|t| + |\phi|)^3$$

with $0 < \delta \ll 1$. In that case, there is a stable equilibrium branch $\phi_+^*(t) \simeq \sqrt{\delta + t^2}$ approaching an unstable branch $\phi_-^*(t) \simeq -\sqrt{\delta + t^2}$ at distance $2\sqrt{\delta}$. While the linearization $a(t) = \partial_\phi F(t, \phi^*(t))$ remains positive, its value becomes small in terms of δ near $t = 0$. As a result, while the system still behaves as in the stable case when $\sigma \ll (\delta \vee \varepsilon)^{3/4}$, a new behaviour emerges for $\sigma \gg (\delta \vee \varepsilon)^{3/4}$: it becomes likely for sample paths to cross the unstable equilibrium branch, and travel in a short time to a distant region of space.

Here we illustrate how these results can be transposed to singular SPDEs on the two-dimensional torus. However, for a change, we are going to take as an example the equation

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + a(t)\phi(t, x) - :\phi(t, x)^3:] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x), \quad (5.5.1)$$

which describes a pitchfork bifurcation when $a(t)$ changes from being negative to being positive at a time t^* . In the deterministic case $\sigma = 0$, there is a phenomenon known as bifurcation delay: solutions attracted by the stable equilibrium branch $\phi^*(t) = 0$ for $t < t^*$ remain close to 0 for a time of order 1 beyond the bifurcation time t^* , even though the equilibrium branch has become unstable. This is due to the solution becoming exponentially close to 0 during the stable phase, and a time of order 1 being required for the solution to reach again values of order 1.

In the one-dimensional SDE case, the effect of noise on such a system has been studied in [7]. The main result of that work is that sample path remain with high probability at a distance of order $\sigma\varepsilon^{-1/4}$ from zero up to a time $t^* + \mathcal{O}(\varepsilon^{1/2})$, but are unlikely to remain close to 0 after times of order $t^* + \mathcal{O}((\varepsilon \log(\sigma^{-1}))^{1/2})$. The effect of noise is thus to reduce the bifurcation delay from order 1 to order $(\varepsilon \log(\sigma^{-1}))^{1/2}$.

In order to analyse the SPDE (5.5.1), we start by carrying out the change of variables

$$\phi(t, x) = \psi_{\perp}(t, x) + \phi_1(t, x),$$

where the stochastic convolution $\psi_{\perp}$ solves the SPDE

$$d\psi_{\perp}(t, x) = \frac{1}{\varepsilon} [\Delta_{\perp}\psi_{\perp}(t, x) + a(t)\psi_{\perp}(t, x)] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_{\perp}(t, x)$$

with zero initial condition. Here the noise $dW_{\perp}$ acts only on non-zero Fourier modes, implying that the spatial average of $\psi_{\perp}(t, x)$ always remains equal to zero. We use the notation $\Delta_{\perp}$ to emphasize that the Laplacian only acts on non-zero Fourier modes, although it has the same effect as the usual Laplacian. The resulting equation for ϕ_1 reads

$$d\phi_1(t, x) = \frac{1}{\varepsilon} [\Delta\phi_1(t, x) + a(t)\phi_1(t, x) + :F(\psi_{\perp}(t, x), \phi_1(t, x)):] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_0(t, x),$$

where

$$:F(\psi_{\perp}, \phi_1): = -:\psi_{\perp}^3: - 3\phi_1:\psi_{\perp}^2: - 3\phi_1^2\psi_{\perp} - \phi_1^3.$$

The next step is to split ϕ_1 into its mean and oscillating spatial part, writing

$$\phi_1(t, x) = \phi_1^0(t)e_0(x) + \phi_1^{\perp}(t, x), \quad \phi_1^0(t) = \langle e_0, \phi_1(t, \cdot) \rangle.$$

This results in the coupled SDE–SPDE system

$$d\phi_1^0(t) = \frac{1}{\varepsilon} [a(t)\phi_1^0(t) - \phi_1^0(t)^3 + F_0(\psi_{\perp}, \phi_1^0, \phi_1^{\perp})] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_0(t), \quad (5.5.2)$$

$$d\phi_1^{\perp}(t, x) = \frac{1}{\varepsilon} [\Delta_{\perp}\phi_1^{\perp}(t, x) + a(t)\phi_1^{\perp}(t, x) + :F_{\perp}(\psi_{\perp}, \phi_1^0, \phi_1^{\perp}):] dt, \quad (5.5.3)$$

where F_0 and $:F_\perp:$ are nonlocal nonlinearities given by

$$\begin{aligned} F_0(\psi_\perp, \phi_1^0, \phi_1^\perp) &= \langle e_0, :F(\psi_\perp, \phi_1^0 e_0 + \phi_1^\perp): \rangle , \\ &= \delta_0(:F(\psi_\perp, \phi_1^0 e_0 + \phi_1^\perp):) , \\ F_\perp(\psi_\perp, \phi_1^0, \phi_1^\perp) &=: F(\psi_\perp, \phi_1^0 e_0 + \phi_1^\perp): - F_0(\psi_\perp, \phi_1^0, \phi_1^\perp) . \end{aligned}$$

We start by describing concentration properties of $\phi_1^\perp$. For that, given a parameter $H_0 > 0$, we introduce the stopping time

$$\tau_0(H_0) = \inf\{t \in [0, T]: |\phi_1^0(t)| > H_0\} .$$

Theorem 5.5.1 (Concentration estimate for $\phi_1^\perp$). *Assume there exists a constant $a_0 > 0$ such that $a(t) \leq (2\pi)^2 - a_0$ for all $t \in [0, T]$. Then for any choice of $\gamma < 2$ and $\nu < 1 - \frac{\gamma}{2}$, there exist constants $C(T, \varepsilon)$, $M, \kappa, h_0 > 0$ such that, whenever $h + H_0 \leq h_0 \varepsilon^{\nu/2}$, one has*

$$\mathbb{P}\left\{\sup_{t \in [0, T \wedge \tau_0(H_0)]} \|\phi_1^\perp(t)\|_{\mathcal{C}^{\gamma-1}} > M \varepsilon^{-\nu} (h + H_0)^3\right\} \leq C(T, \varepsilon) e^{-\kappa h^2 / \sigma^2} .$$

Proof of Theorem 5.5.1. The proof is very similar to the proof of Theorem 5.4.7, so we only comment on the differences. Given $\alpha < 0$ and $H_\perp > 0$, we introduce stopping times

$$\begin{aligned} \tau_\psi(h) &= \inf\left\{t \in [0, T]: \max_{1 \leq \ell \leq 3} \|\psi_\perp(t, \cdot)^\ell\|_{\mathcal{B}_{2,\infty}^\alpha} > h\right\} , \\ \tau_\perp(H_\perp) &= \inf\left\{t \in [0, T]: \|\phi_1^\perp(t, \cdot)\|_{\mathcal{B}_{2,\infty}^\gamma} > H_\perp\right\} . \end{aligned}$$

For any $\bar{\alpha} < 5\alpha$, writing $\tau = \tau_\psi(h) \wedge \tau_\perp(H_\perp) \wedge \tau_0(H_0)$, one obtains the existence of a constant M such that, for any $t \leq \tau$, one has

$$\|:F_\perp(\psi_\perp(t, \cdot), \phi_1^0(t, \cdot), \phi_1^\perp(t, \cdot)): \|_{\mathcal{B}_{2,\infty}^{\bar{\alpha}}} \leq M(h + H_0 + H_\perp)^3 .$$

Using Duhamel's formula to write the solution of (5.5.3) in integral form, and the Schauder estimate in Corollary 5.4.6 (adapted to the eigenvalues of the new linear part), one obtains

$$\|\phi_1^\perp(t \wedge \tau)\|_{\mathcal{B}_{2,\infty}^\gamma} \leq M_1 \varepsilon^{-\nu} (h + H_0 + H_\perp)^3$$

for $\nu < 1 - \frac{1}{2}(\gamma - \bar{\alpha})$ and a constant $M_1(\nu) > 0$, provided $1 + 2\alpha \geq \gamma$. Then it suffices to decompose the probability as in (5.4.10). Choosing $H_\perp = 2M_1 \varepsilon^{-\nu} (h + H_0)^3$ and $h + H_0$ of order $\varepsilon^{\nu/2}$ yields the result. $\square$

Remark 5.5.2. Note in particular the weaker condition on $a(t)$: instead of having to stay negative, $a(t)$ may become positive, as long as it stays smaller than $(2\pi)^2$. This is because the eigenvalues of the Laplacian $\Delta_\perp$ are bounded above by $-(2\pi)^2$. One can easily extend the result to cases where $a(t)$ exceeds the value $(2\pi)^2$ by incorporating more Fourier modes in the variables ϕ_1^0 . $\diamond$

It is now relatively straightforward to extend the one-dimensional results from [7] to the SDE (5.5.2) governing the zeroth Fourier mode. The idea is that its solution is likely to remain close, on some time interval, to the solution of the linearised equation

$$d\phi^\circ(t) = \frac{1}{\varepsilon} a(t) \phi^\circ(t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_0(t) , \quad (5.5.4)$$

which is a Gaussian process, with variance

$$v^\circ(t) = v^\circ(0) + \frac{\sigma^2}{\varepsilon} \int_0^t e^{2\alpha(t,t_1)/\varepsilon} dt_1, \quad \alpha(t, t_1) = \int_{t_1}^t a(t, t_2) dt_2.$$

One can show (see [7, Lemma 4.2]) that for an initial variance $v^\circ(0)$ of order σ^2 , bounded away from zero, one has

$$v^\circ(t) \asymp \begin{cases} \frac{\sigma^2}{|t - t^*|} & \text{for } 0 \leq t \leq t^* - \sqrt{\varepsilon}, \\ \frac{\sigma^2}{\sqrt{\varepsilon}} & \text{for } -\sqrt{\varepsilon} \leq t - t^* \leq \sqrt{\varepsilon}, \\ \frac{\sigma^2}{\sqrt{\varepsilon}} e^{2\alpha(t, t^*)/\varepsilon} & \text{for } t \geq t^* + \sqrt{\varepsilon}. \end{cases}$$

Note that the variance increases slowly up to time $t^* + \sqrt{\varepsilon}$, and then increases exponentially fast. This suggests defining sets

$$\begin{aligned} \mathcal{B}_-(h_-) &= \left\{ (t, \phi_1^0) \in [0, t^* + \sqrt{\varepsilon}] \times \mathbb{R} : |\phi_1^0| \leq \frac{h_-}{\sigma} v^\circ(t) \right\}, \\ \mathcal{B}_+(h_+) &= \left\{ (t, \phi_1^0) \in [t^* + \sqrt{\varepsilon}, T] \times \mathbb{R} : |\phi_1^0| \leq \frac{h_+}{\sqrt{a(t)}} \right\}. \end{aligned}$$

The first set is a union of confidence intervals associated with the variance $v^\circ(t)$. The second set is motivated by the form of the exponential growth of the variance after the bifurcation. One then has the following generalisation of [7, Theorem 2.10] and [7, Proposition 4.7].

Theorem 5.5.3 (Behaviour of $\phi_1^0(t)$ near a pitchfork bifurcation). *There exist positive constants M, ε_0, h_0 such that, for any $\varepsilon < \varepsilon_0$ and $h_- \leq h_0 \varepsilon^{1/2}$, and any $t \leq t^* + \varepsilon^{1/2}$, one has*

$$\mathbb{P}\{\tau_{\mathcal{B}_-(h_-)} \leq t\} \leq C(t, \varepsilon) \exp\left\{-\frac{h_-^2}{2\sigma^2} \left[1 - \mathcal{O}(\sqrt{\varepsilon}) - \mathcal{O}\left(\frac{h_-^2}{\varepsilon}\right)\right]\right\}, \quad (5.5.5)$$

where $C(t, \varepsilon) = \mathcal{O}(\alpha(t)/\varepsilon^2)$. Furthermore, for $h_+ = \sigma \log(\sigma^{-1})^{1/2}$ and any $t \geq t^* + \varepsilon^{1/2}$, one has

$$\mathbb{P}\{\tau_{\mathcal{B}_+(h_+)} \geq t\} \leq \frac{h_+}{\sigma} \exp\left\{-\kappa \frac{\alpha(t, t^*)}{\varepsilon}\right\} + C(t, \varepsilon) e^{-\kappa \log(\sigma^{-1})/\sqrt{\varepsilon}} \quad (5.5.6)$$

for a constant $\kappa > 0$.

The bound (5.5.5) shows that when $\sigma \ll h_- \ll \sqrt{\varepsilon}$, sample paths are likely to stay in $\mathcal{B}_-(h_-)$ up to time $t^* + \sqrt{\varepsilon}$. At time $t^* + \sqrt{\varepsilon}$, typical fluctuations have a size of order $\sigma \varepsilon^{-1/4}$. Since $\alpha(t, t^*)$ grows like $(t - t^*)^2$, the bound (5.5.6) shows that sample paths are likely to leave a neighborhood of size σ of 0 at times of order $\sqrt{\varepsilon \log(\sigma^{-1})}$.

Proof of Theorem 5.5.3. The proof is essentially the same as the proof of [7, Theorem 2.10] and [7, Proposition 4.7], except that one has to account for the effect of the extra term $F_0(\psi_\perp, \phi_1^0, \phi_1^\perp)$ in the equation. The solution of (5.5.2) admits the integral representation

$$\phi_1^0(t) = \phi^\circ(t) + \frac{1}{\varepsilon} \int_0^t e^{\alpha(t, t_1)/\varepsilon} [-(\phi_1^0(t_1))^3 + F_0(\psi_\perp(t_1), \phi_1^0(t_1), \phi_1^\perp(t_1))] dt_1, \quad (5.5.7)$$

where ϕ° is the solution of the linear equation (5.5.4). Proposition 4.3 in [7] provides a similar estimate as (5.5.5) for ϕ° . Furthermore, up to time $\tau_{\mathcal{B}_-(h_-)} \wedge \tau_\psi(h) \wedge \tau_\perp(H_\perp)$, we have the bound

$$|-(\phi_1^0(t_1))^3 + F_0(\psi_\perp(t_1), \phi_1^0(t_1), \phi_1^\perp(t_1))| \leq M \left(\frac{h_-}{\varepsilon^{1/4}} + h + H_\perp \right)^3$$

for a constant M . The integral over t_1 in (5.5.7) yields an extra factor $1/\sqrt{\varepsilon}$. This allows to bound the supremum of $\phi_1^0(t)$ in terms of the supremum of $\phi^\circ(t)$ on the event

$$\Omega_{h, H_\perp} = \{ \tau_\psi(h) \wedge \tau_\perp(H_\perp) > \tau_{\mathcal{B}_-(h_-)} \}.$$

The probability of the complement $\Omega_{h, H_\perp}^c$ can be estimated by Theorems 5.3.12 and 5.5.1. Choosing $H_0 = h_- \varepsilon^{-1/4}$, $h = h_-$ and $H_\perp = \varepsilon^{-\nu}(h + H_0)^3$, one finds that $\mathbb{P}(\Omega_{h, H_\perp}^c)$ is negligible with respect to the probability of ϕ° leaving $\mathcal{B}_-(h_-[1 - \mathcal{O}(h_-^2/\varepsilon)])$, which proves (5.5.5).

To prove (5.5.6), we use the fact that for $t \leq \tau_{\mathcal{B}_+(h_+)} \wedge \tau_\psi(h) \wedge \tau_\perp(H_\perp)$, one has

$$\begin{aligned} |-(\phi_1^0(t_1))^3 + F_0(\psi_\perp(t_1), \phi_1^0(t_1), \phi_1^\perp(t_1))| &\leq M_1 \left(\frac{h_+}{a(t_1)^{1/2}} + h + H_\perp \right)^3 \\ &\leq M_2 \left(\frac{h_+}{\varepsilon^{1/4}} + h + H_\perp \right)^3 \end{aligned}$$

for constants M_1, M_2 . This motivates the choice $h = H_\perp = \varepsilon^{-1/4}h_+$. Proceeding as in the proof of [7, Proposition 4.7], one obtains

$$\mathbb{P}\{ \tau_{\mathcal{B}_+(h_+)} > t, \tau_\psi(h) \vee \tau_\perp(H_\perp) \geq t \} \leq \frac{h_+}{\sigma} \exp\left\{ -\kappa \frac{\alpha(t, t^*)}{\varepsilon} \right\}.$$

The second term on the right-hand side of (5.5.6) bounds $\mathbb{P}\{ \tau_\psi(h) \vee \tau_\perp(H_\perp) < t \}$. □

Some technical proofs for Section 5.3

A.1 Proof of Theorem 5.3.4 for $m = 2$

Fix $\gamma < 0$. By Definition 2.3.1, we have

$$\begin{aligned} \mathbb{P}\left\{\|\psi(\cdot)^2\|_{\mathcal{B}_{2,\infty}^\alpha} > h^2\right\} &= \mathbb{P}\left\{\sup_{q_0 \geq 0} 2^{-|\alpha|q_0} \|\delta_{q_0}(\psi(\cdot)^2)\|_{L^2} > h^2\right\} \\ &= \mathbb{P}\left\{\exists q_0 \geq 0 : \|\delta_{q_0}(\psi(\cdot)^2)\|_{L^2} > h^2 2^{|\alpha|q_0}\right\} \\ &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\|\delta_{q_0}(\psi(\cdot)^2)\|_{L^2} > h^2 2^{|\alpha|q_0}\right\}. \end{aligned}$$

The multinomial formula for Hermite polynomials given by Lemma 2.5.3 gives

$$\begin{aligned} \mathbb{P}\left\{\|\psi(\cdot)^2\|_{\mathcal{B}_{2,\infty}^\alpha} > h^2\right\} &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\left\|\delta_{q_0}\left(\sum_{|\mathbf{n}|=2} \frac{2!}{\mathbf{n}!} \prod_{q \geq 0} \delta_q \psi(\cdot)^{\mathbf{n}_q}\right)\right\|_{L^2} > h^2 2^{|\alpha|q_0}\right\} \\ &\leq \sum_{q_0 \geq 0} \mathbb{P}\left\{\sum_{|\mathbf{n}|=2} \frac{2!}{\mathbf{n}!} \left\|\delta_{q_0}\left(\prod_{q \geq 0} \delta_q \psi(\cdot)^{\mathbf{n}_q}\right)\right\|_{L^2} > h^2 2^{|\alpha|q_0}\right\}. \end{aligned}$$

For every $\mathbf{n} = (\mathbf{n}_q)_{0 \leq q \leq \log_2(N)}$, we introduce the canonical basis $\mathbf{e}_q = (0, \dots, 0, 1, 0, \dots)$ where "1" is on the q th place. There are two ways of writing $|\mathbf{n}| = 2$, either $\mathbf{n} = 2\mathbf{e}_q$ or $\mathbf{n} = \mathbf{e}_{q_1} + \mathbf{e}_{q_2}$. Then,

$$\begin{aligned} \mathbb{P}\left\{\|\psi(\cdot)^2\|_{\mathcal{B}_{2,\infty}^\alpha} > h^2\right\} &\leq \sum_{q_0 \geq 0} \left[\mathbb{P}\left\{\sum_{q \geq 0} \frac{2!}{2!} \|\delta_{q_0}(\delta_q \psi(\cdot)^2)\|_{L^2} > \frac{1}{2} h^2 2^{|\alpha|q_0}\right\} \right. \\ &\quad \left. + \mathbb{P}\left\{\sum_{\substack{q_1, q_2 \\ q_1 < q_2}} \frac{2!}{1!1!} \|\delta_{q_0}(\delta_{q_1} \psi(\cdot) \delta_{q_2} \psi(\cdot))\|_{L^2} > \frac{1}{2} h^2 2^{|\alpha|q_0}\right\} \right] \\ &:= \sum_{q_0 \geq 0} [P_1(q_0) + P_2(q_0)] \end{aligned}$$

We establish a bound on every probability separately. First, we decompose $\delta_{q_0}(\delta_q \psi(x)^2)$ into its Fourier series

$$\delta_{q_0}(\delta_q \psi(x)^2) = \delta_{q_0}(\delta_q \psi(x)^2 - c_q) = \begin{cases} \sum_{k \in \mathcal{A}_{q_0}} \left(\sum_{\substack{k_1, k_2 \in \mathcal{A}_q \\ k_1 + k_2 = k}} \psi_{k_1} \psi_{k_2} \right) e_k(x) & \text{if } k > 0 \\ \left(\sum_{k_1 \in \mathcal{A}_q} \psi_{k_1} \psi_{-k_1} - c_q \right) e_0(x) & \text{if } k = 0. \end{cases}$$

We notice that c_q acts only on the zero-mode of Fourier and by convention $k = 0$ if and only if $q_0 = 0$ and for any $q \geq 0$, one has $|k| \leq |k_1| + |k_2| \leq 2^{q+1}$. Therefore

$$\delta_{q_0}(\cdot : \delta_q \psi(\cdot)^2) \neq 0 \text{ if and only if } q_0 \leq q + 1.$$

Let

$$Z_0(q) := \sum_{k_1 \in \mathcal{A}_q} \psi_{k_1} \psi_{-k_1} - c_q$$

and fix $\gamma \in \mathbb{R}$ such that $\hat{\gamma} = \frac{\gamma}{\sigma^2}$. By Markov's inequality, we get

$$\begin{aligned} \mathbb{P}\left\{\sum_{q \geq 0} \|Z_0(q)\|_{L^2} > \frac{1}{4} h^2\right\} &\leq e^{-\hat{\gamma} h^2/4} \mathbb{E}\left[e^{\hat{\gamma} \sum_{q \geq 0} \|Z_0(q)\|_{L^2}}\right] \\ &= e^{-\hat{\gamma} h^2/4} \prod_{q \geq 0} \mathbb{E}\left[e^{\hat{\gamma} |Z_0(q)|}\right], \end{aligned}$$

since the projections $\delta_q \psi$ are independent for different q . Expanding the exponential, we get

$$\mathbb{E}\left[e^{\hat{\gamma} |Z_0(q)|}\right] = \sum_{p \geq 0} \frac{\hat{\gamma}^p}{p!} \mathbb{E}\left[|Z_0(q)|^p\right]$$

Then, by the hypercontractivity argument (2.5.4)

$$\mathbb{E}\left[|Z_0(q)|^p\right] \leq (p-1)^p \mathbb{E}\left[|Z_0(q)|^2\right]^{p/2}.$$

In fact,

$$\begin{aligned} \mathbb{E}\left[|Z_0(q)|^2\right] &= \mathbb{E}\left[\left(\sum_{k_1 \in \mathcal{A}_q} \psi_{k_1} \psi_{-k_1}\right)^2\right] - 2c_q \mathbb{E}\left[\sum_{k_1 \in \mathcal{A}_q} \psi_{k_1} \psi_{-k_1}\right] + c_q^2 \\ &= \mathbb{E}\left[\sum_{k_1, k_2 \in \mathcal{A}_q} \psi_{k_1} \psi_{-k_1} \psi_{k_2} \psi_{-k_2}\right] - c_q^2 \\ &= \mathbb{E}\left[\sum_{k_1 \in \mathcal{A}_q} |\psi_{k_1}|^4\right] + \mathbb{E}\left[\sum_{k_1 \in \mathcal{A}_q} \psi_{k_1} \psi_{-k_1}\right] \mathbb{E}\left[\sum_{k_2 \in \mathcal{A}_q} \psi_{k_2} \psi_{-k_2}\right] - c_q^2 \\ &= \sum_{k_1 \in \mathcal{A}_q} \mathbb{E}\left[|\psi_{k_1}|^4\right], \end{aligned}$$

It can be bounded

$$\mathbb{E}\left[|Z_0(q)|^2\right] \leq \sigma^4 \sum_{k_1 \in \mathcal{A}_q} \frac{1}{\|k_1\|^4} \lesssim \sigma^4 \int_{2^{q-1}}^{2^q} \frac{1}{r^3} dr \lesssim \sigma^4 \frac{1}{2^{2q}}$$

Stirling's formula yields

$$\frac{p^p}{p!} \leq \frac{p^p}{p^{p+1/2} e^p} \leq e^p.$$

Therefore

$$\mathbb{E}\left[e^{\hat{\gamma} |Z_0(q)|}\right] \leq \sum_{p \geq 0} \frac{(\gamma e)^p \sigma^{2p}}{\sigma^{2p} 2^{pq}} \leq \frac{1}{1 - \gamma e 2^{-q}},$$

for any $\gamma < \frac{1}{e}2^q$. Then, we bound the product of expectations

$$\prod_{q \geq 0} \mathbb{E} \left[e^{\hat{\gamma} |Z_0(q)|} \right] \leq \exp \left\{ c\gamma \sum_{q \geq 0} \frac{1}{2^q} \right\} = \exp \{ c_0 \gamma \}$$

by summing a geometric series. We conclude that

$$\mathbb{P} \left\{ \sum_{q \geq 0} \|Z_0(q)\|_{L^2} > \frac{1}{4} h^2 \right\} \leq \exp \{ c_0 \gamma \} e^{-\gamma h^2 / (4\sigma^2)} \mathbf{1}_{\{q_0=0\}}. \quad (\text{A.1.1})$$

For any $k > 0$, let

$$Z_k(q) := \sum_{\substack{k_1, k_2 \in \mathcal{A}_q \\ k_1 + k_2 = k}} \psi_{k_1} \psi_{k_2},$$

and fix another $\hat{\gamma} = \frac{\gamma}{\sigma^2}$ where here we use the same symbol as before to not overload the notations. By Markov's inequality, we get

$$\begin{aligned} \mathbb{P} \left\{ \sum_{q \geq 0} \left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q) e_k(\cdot) \right\|_{L^2} > \frac{1}{4} h^2 2^{|\alpha|q_0} \right\} &\leq \exp \left\{ -\frac{\hat{\gamma}}{4} h^2 2^{|\alpha|q_0} \right\} \times \\ &\quad \prod_{q \geq 0} \mathbb{E} \left[\exp \left\{ \hat{\gamma} \left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q) e_k(\cdot) \right\|_{L^2} \right\} \right]. \end{aligned}$$

Expanding the exponential and applying the hypercontractivity inequality (2.5.4), we get

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ \hat{\gamma} \left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q) e_k(\cdot) \right\|_{L^2} \right\} \right] &= \sum_{p \geq 0} \frac{\gamma^p}{\sigma^{2p} p!} \mathbb{E} \left[\left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q) e_k(\cdot) \right\|_{L^2}^p \right] \\ &\leq \sum_{p \geq 0} \frac{\gamma^p}{\sigma^{2p} p!} p^p \mathbb{E} \left[\left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q) e_k(\cdot) \right\|_{L^2}^2 \right]^{p/2}. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E} \left[\left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q) e_k(\cdot) \right\|_{L^2}^2 \right] &= \sum_{k \in \mathcal{A}_{q_0}} \mathbb{E} \left[|Z_k(q)|^2 \right] \\ &= \sum_{k \in \mathcal{A}_{q_0}} \sum_{\substack{k_1, k_2, k'_1, k'_2 \in \mathcal{A}_q \\ k_1 + k_2 = k \\ k'_1 + k'_2 = -k}} \mathbb{E} \left[\psi_{k_1} \psi_{k_2} \psi_{k'_1} \psi_{k'_2} \right]. \end{aligned}$$

By Wick's theorem 2.5.1, one obtain 3 different pairings for 4 random variables

$$\mathbb{E} \left[\left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q) e_k(\cdot) \right\|_{L^2}^2 \right] = \sum_{k \in \mathcal{A}_{q_0}} \sum_{\substack{k_1, k_2, k'_1, k'_2 \in \mathcal{A}_q \\ k_1 + k_2 = k \\ k'_1 + k'_2 = -k}} \left(v_{k_1} v_{k'_1} \delta_{k_1, -k_2} \delta_{k'_1, -k'_2} + 2 v_{k_1} v_{k_2} \delta_{k_1, -k'_1} \delta_{k_2, -k'_2} \right).$$

Since $q_0 > 0$ due to $k > 0$, the first term in the bracket is zero ($k_1 = -k_2$ implies that $k = 0$). It remains

$$\mathbb{E} \left[\left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q) e_k(\cdot) \right\|_{L^2}^2 \right] = 2 \sum_{k \in \mathcal{A}_{q_0}} \sum_{\substack{k_1, k_2 \in \mathcal{A}_q \\ k_1 + k_2 = k}} \frac{\sigma^2}{1 + \|k_1\|^2} \frac{\sigma^2}{1 + \|k_2\|^2}.$$

A bound of this sum is given by

$$\sum_{\substack{k_1, k_2 \in \mathcal{A}_q \\ k_1 + k_2 = k}} \frac{1}{\|k_1\|^2} \frac{1}{\|k_2\|^2} \leq \sum_{k_1 \in \mathcal{A}_q} \frac{1}{\|k_1\|^2} \frac{1}{\|k - k_1\|^2}.$$

By the second triangle inequality, we get

$$\|k - k_1\| \geq \left| \|k\| - \|k_1\| \right| = \|k_1\| - \|k\| \gtrsim \|k_1\|,$$

the last inequality is due to the fact that $k \lesssim k_1$. Thus,

$$\begin{aligned} \mathbb{E} \left[\left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q) e_k(\cdot) \right\|_{L^2}^2 \right] &\lesssim \sigma^4 \sum_{k \in \mathcal{A}_{q_0}} \sum_{k_1 \in \mathcal{A}_q} \frac{1}{\|k_1\|^4} \lesssim \sigma^4 \sum_{k \in \mathcal{A}_{q_0}} \int_{2^{q-1}}^{2^q} \frac{1}{r^4} r \, dr \\ &\leq \sigma^4 \sum_{k \in \mathcal{A}_{q_0}} \frac{1}{2^{2q}} \leq \sigma^4 \frac{2^{2q_0}}{2^{2q}} \end{aligned}$$

Stirling formula and summing the geometric series yield

$$\mathbb{E} \left[\exp \left\{ \hat{\gamma} \left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q) e_k(\cdot) \right\|_{L^2} \right\} \right] \leq \frac{1}{1 - \gamma e^{2^{q_0-q}}},$$

for any $\gamma < e^{-1} 2^{q-q_0}$. Then, the product of expectations is bounded by

$$\begin{aligned} \prod_{q \geq q_0-1} \mathbb{E} \left[\exp \left\{ \hat{\gamma} \left\| \sum_{k \in \mathcal{A}_q} Z_k(q) e_k(\cdot) \right\|_{L^2} \right\} \right] &\leq \prod_{q \geq q_0-1} \frac{1}{1 - \gamma e^{2^{q_0-q}}} \\ &\leq \exp \left\{ c\gamma \sum_{q \geq q_0-1} \frac{1}{2^{q-q_0}} \right\} \end{aligned}$$

where

$$\sum_{q \geq q_0-1} \frac{1}{2^{q-q_0}} = \sum_{q' \geq 0} \frac{1}{2^{q'-1}} = 4.$$

We conclude that

$$\mathbb{P} \left\{ \sum_{q \geq 0} \left\| \sum_{k \in \mathcal{A}_q} Z_k(q) e_k(\cdot) \right\|_{L^2} > \frac{1}{4} h^2 2^{|\alpha|q_0} \right\} \leq \exp\{4c\gamma\} e^{-\gamma h^2 2^{|\alpha|q_0}/4\sigma^2}. \quad (\text{A.1.2})$$

Combining (A.1.1) and (A.1.2), we get the following bound

$$P_1(q_0) \leq \exp\{c_0\gamma\} e^{-\gamma h^2/(4\sigma^2)} \mathbf{1}_{\{q_0=0\}} + \exp\{4c\gamma\} e^{-\gamma h^2 2^{|\alpha|q_0}/4\sigma^2}. \quad (\text{A.1.3})$$

In second part, we develop a bound for $P_2(q_0)$, on the two annulus $\mathcal{A}_{q_1}$ and $\mathcal{A}_{q_2}$ such that $q_1 < q_2$. We express the projection into its Fourier series and we get

$$\delta_{q_0}(\delta_{q_1}\psi(\cdot)\delta_{q_2}\psi(\cdot)) = \sum_{k \in \mathcal{A}_{q_0}} \left(\sum_{\substack{k_1 \in \mathcal{A}_{q_1}, k_2 \in \mathcal{A}_{q_2} \\ k_1 + k_2 = k}} \psi_{k_1}\psi_{k_2} \right) e_k(\cdot) := \sum_{k \in \mathcal{A}_{q_0}} Z_k(q_1, q_2) e_k(\cdot).$$

We notice that for any $q_1, q_2 > 0$, one has $|k| \leq |k_1 + k_2| \leq 2^{q_1} + 2^{q_2} \leq 2^{\max\{q_1, q_2\}+1}$. Therefore

$$\delta_{q_0}(\delta_{q_1}\psi(\cdot)\delta_{q_2}\psi(\cdot)) \neq 0 \text{ if and only if } q_0 \leq \max\{q_1, q_2\} + 1 = q_2 + 1.$$

Following same arguments as before, we fix another $\gamma \in \mathbb{R}$ such that $\hat{\gamma} = \gamma/\sigma^2$. We use again the same symbol to not overload the notations. By Markov's inequality, we get

$$\mathbb{P}\left\{ \sum_{\substack{q_1, q_2 \\ q_1 < q_2}} \|\delta_{q_0}(\delta_{q_1}\psi(\cdot)\delta_{q_2}\psi(\cdot))\|_{L^2} > \frac{1}{2} h^2 2^{|\alpha|q_0} \right\} \leq \exp\left\{ -\frac{\hat{\gamma}}{2} h^2 2^{|\alpha|q_0} \right\} \times \\ \prod_{\substack{q_1, q_2 \\ 0 \leq q_1 < q_2}} \mathbb{E}\left[\exp\left\{ \hat{\gamma} \left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q_1, q_2) e_k(\cdot) \right\|_{L^2} \right\} \right],$$

By equivalence of moments (2.5.4), we get

$$\mathbb{E}\left[\exp\left\{ \hat{\gamma} \left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q_1, q_2) e_k(\cdot) \right\|_{L^2} \right\} \right] \leq \sum_{p \geq 0} \frac{\gamma^p}{\sigma^{2p} p!} (p-1)^p \mathbb{E}\left[\left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q_1, q_2) e_k(\cdot) \right\|_{L^2}^2 \right]^{p/2}.$$

Theorem 2.5.1 of Wick gives the following pairings

$$\begin{aligned} \mathbb{E}\left[\left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q_1, q_2) e_k(\cdot) \right\|_{L^2}^2 \right] &= \sum_{k \in \mathcal{A}_{q_0}} \mathbb{E}\left[|Z_k(q_1, q_2)|^2 \right] \\ &= \sum_{k \in \mathcal{A}_{q_0}} \sum_{\substack{k_1, k'_1 \in \mathcal{A}_{q_1} \\ k_2, k'_2 \in \mathcal{A}_{q_2} \\ k_1 + k_2 = k \\ k'_1 + k'_2 = -k}} \mathbb{E}\left[\psi_{k_1} \psi_{k_2} \psi_{k'_1} \psi_{k'_2} \right] \\ &= \sum_{k \in \mathcal{A}_{q_0}} \sum_{\substack{k_1, k'_1 \in \mathcal{A}_{q_1} \\ k_2, k'_2 \in \mathcal{A}_{q_2} \\ k_1 + k_2 = k \\ k'_1 + k'_2 = -k}} \left(2v_{k_1} v_{k'_1} \delta_{k_1, -k_2} \delta_{k'_1, -k'_2} + v_{k_1} v_{k_2} \delta_{k_1, -k'_1} \delta_{k_2, -k'_2} \right). \end{aligned}$$

Since $q_1 < q_2$ then $|k_1|, |k'_1| < |k_2|, |k'_2|$ therefore the first term in the bracket is zero.

$$\mathbb{E}\left[\left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q_1, q_2) e_k(\cdot) \right\|_{L^2}^2 \right] \leq \sigma^4 \sum_{k \in \mathcal{A}_{q_0}} \sum_{\substack{k_1 \in \mathcal{A}_{q_1}, k_2 \in \mathcal{A}_{q_2} \\ k_1 + k_2 = k}} \frac{1}{\|k_1\|^2} \frac{1}{\|k_2\|^2}$$

The second triangle inequality yields

$$\|k - k_2\| \geq \left| \|k\| - \|k_2\| \right| = \|k_2\| - \|k\| \gtrsim \|k_2\|.$$

Thus,

$$\begin{aligned} \sum_{\substack{k_1 \in \mathcal{A}_{q_1}, k_2 \in \mathcal{A}_{q_2} \\ k_1 + k_2 = k}} \frac{1}{\|k_1\|^2} \frac{1}{\|k_2\|^2} &= \sum_{k_2 \in \mathcal{A}_{q_2}} \frac{1}{\|k - k_2\|^2} \frac{1}{\|k_2\|^2} \\ &\lesssim \sum_{k_2 \in \mathcal{A}_{q_2}} \frac{1}{\|k_2\|^4} \\ &\lesssim \int_{2^{q_2-1}}^{2^{q_2}} \frac{1}{r^4} r \, dr \leq \frac{1}{2^{2q_2}}, \end{aligned}$$

and

$$\sum_{k \in \mathcal{A}_{q_0}} \sum_{\substack{k_1 \in \mathcal{A}_{q_1}, k_2 \in \mathcal{A}_{q_2} \\ k_1 + k_2 = k}} \frac{1}{\|k_1\|^2} \frac{1}{\|k_2\|^2} \leq \frac{2^{2q_0}}{2^{2q_2}}.$$

Therefore, by summing the geometric series we get

$$\mathbb{E} \left[\exp \left\{ \hat{\gamma} \left\| \sum_{k \in \mathcal{A}_{q_0}} Z_k(q_1, q_2) e_k(\cdot) \right\|_{L^2} \right\} \right] \leq \frac{1}{1 - \gamma e^{2^{q_0 - q_2}}},$$

for any $\gamma < e^{-1} 2^{q_2 - q_0}$. Then, the product over q_1 and q_2 is bounded

$$\begin{aligned} \prod_{\substack{q_1, q_2 \\ 0 \leq q_1 < q_2}} \mathbb{E} \left[e^{\hat{\gamma} \left\| \delta_{q_0}(\delta_{q_1} \psi(x) \delta_{q_2} \psi(x)) \right\|_{L^2}} \right] &\leq \prod_{\substack{q_1, q_2 \\ 0 \leq q_1 < q_2}} \frac{1}{1 - \gamma e^{2^{q_0 - q_2}}} \\ &\leq \exp \left\{ c \gamma e \sum_{\substack{q_1, q_2 \\ 0 \leq q_1 < q_2}} 2^{q_0 - q_2} \right\}. \end{aligned}$$

One can see the previous sum as geometric series

$$\begin{aligned} \sum_{\substack{q_1 < q_2 \\ q_2 \geq q_0 - 1}} \frac{1}{2^{q_2 - q_0}} &= \sum_{q_2 \geq q_0 - 1} \sum_{q_1 = 0}^{q_2 - 1} \frac{1}{2^{q_2 - q_0}} \leq \sum_{q_2 \geq q_0 - 1} \frac{q_2}{2^{q_2 - q_0}} \\ &\leq \sum_{q' \geq 0} \frac{q_0 + q' - 1}{2^{q' - 1}} \leq c(q_0 + 1). \end{aligned}$$

Finally,

$$P_2(q_0) \leq e^{c\gamma(q_0+1)} e^{-\gamma h^2 2^{|\alpha|q_0} / (2\sigma^2)}. \quad (\text{A.1.4})$$

Choosing $\gamma = 1/(2e)$ and combining (A.1.3) and (A.1.4), we obtain

$$\begin{aligned} \mathbb{P} \left\{ \left\| \psi(t, \cdot) \right\|_{\mathcal{B}_{2,\infty}^\alpha}^2 > h^2 \right\} &\leq \sum_{q_0 \geq 0} \left[C_1 e^{-\kappa_1 h^2 / \sigma^2} \mathbf{1}_{\{q_0=0\}} + C_2 e^{-\kappa_2 h^2 2^{|\alpha|q_0} / \sigma^2} \right. \\ &\quad \left. + e^{c'(q_0+1)} e^{-\kappa_3 h^2 2^{|\alpha|q_0} / \sigma^2} \right], \end{aligned}$$

for $C_1, C_2, \kappa_1, \kappa_3$ and κ_3 are positive constants. We approximate the sums by integrals. We have from inequality (5.3.9) that

$$\sum_{q_0 \geq 0} e^{-\kappa_2 h^2 2^{|\alpha|q_0}/\sigma^2} \leq C(\alpha) e^{-\kappa(\alpha) h^2/\sigma^2}.$$

We consider now $f(x) = e^{c'x} e^{-\beta 2^{2|\alpha|x}}$, where $\beta = \kappa_3 \frac{h^2}{\sigma^2}$. One can easily check that f is decreasing. Therefore we obtain

$$\sum_{q \geq 0} f(q) = f(0) + f(1) + \sum_{q=2}^{\infty} f(q) \leq e^{-\beta} + e^{c'} e^{-\beta 2^{2|\alpha|}} + \int_1^{\infty} f(x) dx.$$

Thus,

$$\int_1^{\infty} f(x) dx \leq \int_1^{\infty} e^{-2\beta|\alpha|x \ln(2) + c'x} dx = \frac{1}{2\beta|\alpha| \ln(2) - c'} e^{-2\beta|\alpha| \ln(2) + c'}.$$

We conclude that,

$$\sum_{q_0 \geq 0} e^{c'q_0} e^{-\kappa_3 2^{|\alpha|q_0} h^2/\sigma^2} \leq C(\alpha) e^{-\kappa(\alpha) h^2/\sigma^2},$$

yielding the claimed result.

A.2 Proof of Lemma 5.3.9

We divide the proof of the lemma into the following two parts.

Lemma A.2.1. *For any $q_0 \geq 0$, $t \in I_l$ and $\mathbf{n}$, one has*

$$\mathbb{E}[X_{\mathbf{n}}^2] = \mathbf{n}! \sum_{\substack{k_1^{(q)}, k_2^{(q)}, \dots, k_{n_q}^{(q)} \in \mathcal{A}_q \forall q \geq 0 \\ \sum_{q \geq 0} \sum_{i=1}^{n_q} k_i^{(q)} \in \mathcal{A}_{q_0}}} \prod_{q \geq 0} \prod_{i=1}^{n_q} \hat{\psi}_{k_i^{(q)}}(t). \quad (\text{A.2.1})$$

Proof. Let $\varphi(t, \cdot) = \prod_{q \geq 0} : \delta_q \hat{\psi}(t, \cdot)^{n_q} : .$ The L^2 -norm of its projection on $\mathcal{A}_{q_0}$ is given by

$$\|\delta_{q_0} \varphi(t, \cdot)\|_{L^2}^2 = \sum_{k \in \mathcal{A}_{q_0}} |(P_k \varphi)(t, \cdot)|^2,$$

where $(P_k \varphi)(t, x)$ is the projection of φ on the k th Fourier basis vector $e_k(x)$, given by

$$(P_k \varphi)(t, x) = \int_{\mathbb{T}^2} e_{-k}(x_1) \varphi(t, x_1) dx_1 e_k(x).$$

Therefore,

$$\mathbb{E}[\|\delta_{q_0} \varphi(t, \cdot)\|_{L^2}^2] = \sum_{k \in \mathcal{A}_{q_0}} \mathbb{E}[|(P_k \varphi)(t, \cdot)|^2].$$

For a fixed $k \in \mathcal{A}_{q_0}$, we have

$$\begin{aligned} \mathbb{E}[|(P_k \varphi)(t, x)|^2] &= \mathbb{E}\left[\int_{\mathbb{T}^2} \int_{\mathbb{T}^2} e_{-k}(x_1) \varphi(t, x_1) e_k(x_2) \bar{\varphi}(t, x_2) dx_1 dx_2 e_k(x) e_{-k}(x)\right] \\ &= \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} e_{-k}(x_1 - x_2) \mathbb{E}[\varphi(t, x_1) \bar{\varphi}(t, x_2)] dx_1 dx_2, \end{aligned}$$

where

$$\begin{aligned}\mathbb{E}[\varphi(t, x_1)\bar{\varphi}(t, x_2)] &= \mathbb{E}\left[\prod_{q \geq 0} : \delta_q \hat{\psi}(t, x_1)^{\mathbf{n}_q} : : \overline{\delta_q \hat{\psi}(t, x_2)^{\mathbf{n}_q}} : \right] \\ &= \prod_{q \geq 0} \mathbb{E}\left[: \delta_q \hat{\psi}(t, x_1)^{\mathbf{n}_q} : : \overline{\delta_q \hat{\psi}(t, x_2)^{\mathbf{n}_q}} : \right],\end{aligned}$$

since the projections δ_q and $\delta_{q'}$ are independent for $q \neq q'$. By Lemma 2.5.1, we get

$$\mathbb{E}\left[: \delta_q \hat{\psi}(t, x_1)^{\mathbf{n}_q} : : \overline{\delta_q \hat{\psi}(t, x_2)^{\mathbf{n}_q}} : \right] = \mathbf{n}_q! \mathbb{E}[\delta_q \hat{\psi}(t, x_1) \overline{\delta_q \hat{\psi}(t, x_2)}]^{\mathbf{n}_q},$$

where

$$\begin{aligned}\mathbb{E}[\delta_q \hat{\psi}(t, x_1) \overline{\delta_q \hat{\psi}(t, x_2)}] &= \sum_{k_1, k_2 \in \mathcal{A}_q} \mathbb{E}[\hat{\psi}_{k_1}(t) \overline{\hat{\psi}_{k_2}(t)}] e_{k_1}(x_1) e_{-k_2}(x_2) \\ &= \sum_{k_1, k_2 \in \mathcal{A}_q} \hat{v}_{k_1}(t) \delta_{k_1, k_2} e_{k_1}(x_1) e_{-k_2}(x_2) \\ &= \sum_{k_1 \in \mathcal{A}_q} \hat{v}_{k_1}(t) e_{k_1}(x_1 - x_2).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[\varphi(t, x_1)\bar{\varphi}(t, x_2)] &= \prod_{q \geq 0} \mathbf{n}_q! \left(\sum_{k_1 \in \mathcal{A}_q} \hat{v}_{k_1}(t) e_{k_1}(x_1 - x_2) \right)^{\mathbf{n}_q} \\ &= \left(\prod_{q \geq 0} \mathbf{n}_q! \right) \prod_{q \geq 0} \left(\sum_{k_1, \dots, k_{\mathbf{n}_q} \in \mathcal{A}_q} \hat{v}_{k_1}(t) \cdots \hat{v}_{k_{\mathbf{n}_q}}(t) e_{k_1 + \dots + k_{\mathbf{n}_q}}(x_1 - x_2) \right).\end{aligned}$$

Integrating over x_1 and x_2 , we get

$$\begin{aligned}\mathbb{E}\left[|(P_k \varphi)(t, x)|^2\right] &= \mathbf{n}! \sum_{k_1^{(q)}, \dots, k_{\mathbf{n}_q}^{(q)} \in \mathcal{A}_q, \forall q} \prod_{q \geq 0} \prod_{i=1}^{\mathbf{n}_q} \hat{v}_{k_i^{(q)}}(t) \\ &\quad \times \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} e_{-k}(x_1 - x_2) e_{\sum_{q \geq 0} k_1^{(q)} + \dots + k_{\mathbf{n}_q}^{(q)}}(x_1 - x_2) dx_1 dx_2 \\ &= \mathbf{n}! \sum_{k_1^{(q)}, \dots, k_{\mathbf{n}_q}^{(q)} \in \mathcal{A}_q, \forall q} \prod_{q \geq 0} \prod_{i=1}^{\mathbf{n}_q} \hat{v}_{k_i^{(q)}}(t) \mathbf{1}_{\{\sum_{q \geq 0} \sum_{i=1}^{\mathbf{n}_q} k_i^{(q)} = k\}} \\ &= \mathbf{n}! \sum_{\substack{k_1^{(q)}, \dots, k_{\mathbf{n}_q}^{(q)} \in \mathcal{A}_q, \forall q \\ \sum_{q \geq 0} \sum_{i=1}^{\mathbf{n}_q} k_i^{(q)} = k}} \prod_{q \geq 0} \prod_{i=1}^{\mathbf{n}_q} \hat{v}_{k_i^{(q)}}(t).\end{aligned}$$

Summing over $k_0 \in \mathcal{A}_{q_0}$ yields the claimed result. $\square$

Lemma A.2.2. *There exists a numerical constant C_0 such that*

$$\mathbb{E}[X_{\mathbf{n}}^2] \leq C_0^m \mathbf{n}! \sigma^{2m} \frac{2^{2q_0}}{2^{2q_{[\mathbf{n}]}}} \leq C_0^m m! \sigma^{2m} \frac{2^{2q_0}}{2^{2q_{[\mathbf{n}]}}}.$$

Proof. We have to evaluate the sum given by (A.2.1). Recall that $q_1 < q_2 < \dots < q_{[n]}$ denote the indices of the $[n]$ nonzero entries of $\mathbf{n}$, and that there is a numerical constant c_0 such that

$$\hat{v}_k(t) \leq \frac{c_0 \sigma^2}{1 + \|k\|^2}$$

for all $t \in I_l$. For a fixed $k_0 \in \mathcal{A}_{q_0}$, we get the bound

$$S_{\mathbf{n}, k_0} := \sum_{\substack{k_1^{(q)}, \dots, k_{\mathbf{n}_q}^{(q)} \in \mathcal{A}_q, \forall q \\ \sum_{q \geq 0} \sum_{i=1}^{\mathbf{n}_q} k_i^{(q)} = k_0}} \prod_{q \geq 0} \prod_{i=1}^{\mathbf{n}_q} \hat{v}_{k_i^{(q)}}(t) \leq \sum_{\substack{k_1^{(q)}, \dots, k_{\mathbf{n}_q}^{(q)} \in \mathcal{A}_q, \forall q \\ \sum_{q \geq 0} \sum_{i=1}^{\mathbf{n}_q} k_i^{(q)} = k_0}} \prod_{q \geq 0} \prod_{i=1}^{\mathbf{n}_q} \frac{c_0 \sigma^2}{1 + \|k_i^{(q)}\|^2}.$$

Note that

$$\prod_{q \geq 0} \prod_{i=1}^{\mathbf{n}_q} (c_0 \sigma^2) = \prod_{q \geq 0} (c_0 \sigma^2)^{\mathbf{n}_q} = (c_0 \sigma^2)^{\sum_{q \geq 0} \mathbf{n}_q} = (c_0 \sigma^2)^m,$$

and that we can write

$$k_1^{(q_1)} = k_0 - \sum_{j=2}^{[n]} k_1^{(q_j)} - \sum_{j=1}^{[n]} \sum_{i=2}^{\mathbf{n}_q} k_i^{(q_j)}. \quad (\text{A.2.2})$$

Since $\|k_i^{(q_1)}\| < \|k_i^{(q_2)}\| < \dots < \|k_i^{(q_{[n]})}\|$ and $\|k_0\| \leq \|k_i^{(q_{[n]})}\|$, by the second triangle inequality, we get

$$\begin{aligned} \left\| k_0 - \sum_{j=2}^{[n]} k_1^{(q_j)} - \sum_{j=1}^{[n]} \sum_{i=2}^{\mathbf{n}_q} k_i^{(q_j)} \right\| &\geq \left\| k_0 \right\| - \sum_{j=2}^{[n]} \|k_1^{(q_j)}\| - \sum_{j=1}^{[n]} \sum_{i=2}^{\mathbf{n}_q} \|k_i^{(q_j)}\| \\ &\geq \sum_{j=2}^{[n]} \|k_1^{(q_j)}\| + \sum_{j=1}^{[n]} \sum_{i=2}^{\mathbf{n}_q} \|k_i^{(q_j)}\| - \|k_0\| \\ &\geq c \|k_1^{(q_{[n]})}\| \end{aligned} \quad (\text{A.2.3})$$

for a numerical constant $c > 0$. Replacing $k_1^{(q_1)}$ by (A.2.2) and bounding its norm by (A.2.3), we obtain

$$S_{\mathbf{n}, k_0} \lesssim (c_0 \sigma^2)^m \prod_{j=1}^{[n]} \left(\sum_{k_1^{(q_j)}, \dots, k_{\mathbf{n}_{q_j}}^{(q_j)} \in \mathcal{A}_{q_j}} \frac{1}{\|k_i^{(q_j)}\|^2} \right)^{\mathbf{n}_{q_j}} \sum_{k_1^{(q_{[n]})} \in \mathcal{A}_{q_{[n]}}} \frac{1}{\|k_1^{(q_{[n]})}\|^4}.$$

For a fixed q_j , we view these sums as Riemann sums, and integrating using polar coordinates yields

$$\sum_{k_1^{(q_j)}, \dots, k_{\mathbf{n}_{q_j}}^{(q_j)} \in \mathcal{A}_{q_j}} \frac{1}{\|k_i^{(q_j)}\|^2} \lesssim \int_{2^{q_j-1}}^{2^{q_j}} \frac{r}{r^2} dr \leq \log(2^{q_j}) - \log(2^{q_j-1}) = \log(2),$$

and

$$\sum_{k_1^{(q_{[n]})} \in \mathcal{A}_{q_{[n]}}} \frac{1}{\|k_1^{(q_{[n]})}\|^4} \lesssim \int_{2^{q_{[n]}-1}}^{2^{q_{[n]}}} \frac{r}{r^4} dr \lesssim \frac{1}{2^{2q_{[n]}}}.$$

We conclude that

$$S_{\mathbf{n},k_0} \lesssim (c_0 \sigma^2)^m \prod_{j=1}^{[\mathbf{n}]} (c_1 \log(2))^{n_{q_j}} \frac{1}{2^{2q_{[\mathbf{n}]}}} = C_0^m \sigma^{2m} \frac{1}{2^{2q_{[\mathbf{n}]}}}$$

for some numerical constant c_1, C_0 . The result follows again by summing over $k \in \mathcal{A}_{q_0}$. $\square$

A.3 Proof of Lemma 5.3.11

We decompose the sum (5.3.13) as

$$\sum_{|\mathbf{n}|=m} \frac{1}{2^{2(q_{[\mathbf{n}]}-q_0)/m}} = \sum_{[\mathbf{n}]=1}^m S_m([\mathbf{n}], 0), \quad (\text{A.3.1})$$

where for $a \in \{1, \dots, m\}$ and $b \in \mathbb{N}_0$, we define

$$S_m(a, b) = \sum_{\substack{\mathbf{n}: |\mathbf{n}|=m \\ [\mathbf{n}]=a, q_{[\mathbf{n}]} + \mathbf{n}_{q_{[\mathbf{n}]}} \geq q_0}} \frac{q_{[\mathbf{n}]}^b}{2^{2(q_{[\mathbf{n}]}-q_0)/m}}.$$

We will estimate this sum by induction on a , for arbitrary $b \in \mathbb{N}_0$. For $a = 1$, the only possible $\mathbf{n}$ are those with one component, say q , equal to m , and all other components equal to 0. Therefore,

$$S_m(1, b) \leq \sum_{q \geq (q_0 - m) \vee 0} \frac{q^b}{2^{2(q-q_0)/m}}$$

(since one must have $q \geq 0$). The sum can be computed via the inequality

$$\sum_{q=0}^{\infty} (q+1)^b z^q \leq \frac{b!}{(1-z)^{b+1}},$$

valid for any $z \in [0, 1)$ and $b \in \mathbb{N}$, which follows directly from the definitions of the polylogarithm function and Eulerian numbers. Setting $z = \frac{1}{2^{2/m}}$, we have

$$\begin{aligned} S_m(1, b) &\leq \sum_{q \geq (q_0 - m) \vee 0} q^b z^{q-q_0} = z^{-m} \sum_{p \geq 0 \vee (m-q_0)} (q_0 - m + p)^b z^p \\ &\leq z^{-m} \sum_{\ell=0}^b \binom{b}{\ell} q_0^{b-\ell} \sum_{p \geq 0} p^\ell z^p \\ &\leq z^{-m} \frac{b!}{1-z} \sum_{\ell=0}^b \binom{b}{\ell} \frac{q_0^{b-\ell}}{(1-z)^\ell} \\ &\leq 4c_1 b! (q_0 + c_1)^b, \end{aligned}$$

where $(1-z)^{-1} \leq c_1$. Assume now that $a \geq 1$ and $[\mathbf{n}] = a + 1$. We decompose $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$, with $[\mathbf{n}_1] = a$ and $[\mathbf{n}_2] = 1$, and $|\mathbf{n}_1| = m_1, |\mathbf{n}_2| = m_2$ with $m_1 + m_2 = m$. We may assume that the largest

nonzero component of $\mathbf{n}$ appears in $\mathbf{n}_1$, so that $q_{[\mathbf{n}_1]} = q_{[\mathbf{n}]}$ and $q_{[\mathbf{n}_2]} = q < q_{[\mathbf{n}]}$. It follows that

$$\begin{aligned} S_m(a+1, b) &= \sum_{m_1+m_2=m} \sum_{q < q_{[\mathbf{n}]}} \sum_{\substack{|\mathbf{n}_1|=m_1 \\ [\mathbf{n}_1]=a \\ q_{[\mathbf{n}_1]}+\mathbf{n}_{q_{[\mathbf{n}_1]}} \geq q_0}} \frac{q_{[\mathbf{n}_1]}^b}{2^{2(q_{[\mathbf{n}_1]}-q_0)/m}} \\ &\leq \sum_{m_1=1}^{m-1} S_{m_1}(a, b+1), \end{aligned}$$

where we have bounded the sum over q by $q_{[\mathbf{n}]} = q_{[\mathbf{n}_1]}$. It is then straightforward to show by induction that

$$S_m(a, b) \leq c_0(q_0 + c_1)^{a+b-1}(a+b-1)!$$

for all a, b . In particular,

$$S_m(a, 0) \leq c_0(q_0 + c_1)^{a-1}(a-1)!.$$

Replacing this in (A.3.1) yields the result.

A.4 Proof of Lemma 5.3.19

We decompose the sum (5.3.27) as

$$K_m(q_0) = \sum_{[\mathbf{n}]=1}^m S_m([\mathbf{n}], 0), \quad (\text{A.4.1})$$

where for $a \in \{1, \dots, m\}$ and $b \in \mathbb{N}_0$, we define

$$S_m(a, b) = \sum_{\substack{\mathbf{n}: |\mathbf{n}|=m \\ [\mathbf{n}]=a, q_{[\mathbf{n}]}+\mathbf{n}_{q_{[\mathbf{n}]}} \geq q_0}} \frac{m!}{\mathbf{n}!} \frac{q_{[\mathbf{n}]}^b}{2^{(q_{[\mathbf{n}]}-q_0)/2}}.$$

We will estimate this sum by induction on a , for arbitrary $b \in \mathbb{N}_0$. For $a = 1$, the only possible $\mathbf{n}$ are those with one component, say q , equal to m , and all other components equal to 0. Therefore,

$$S_m(1, b) \leq \sum_{q \geq (q_0-m) \vee 0} \frac{q^b}{2^{(q-q_0)/2}}$$

(since one must have $q \geq 0$). The sum can be computed via the inequality

$$\sum_{q=0}^{\infty} (q+1)^b z^q \leq \frac{b!}{(1-z)^{b+1}},$$

valid for any $z \in [0, 1)$ and $b \in \mathbb{N}$, which follows directly from the definitions of the polylogarithm function and Eulerian numbers. Setting $1/\sqrt{2} = z$, we have

$$\begin{aligned}
 S_m(1, b) &\leq \sum_{q \geq (q_0 - m) \vee 0} q^b z^{q - q_0} = z^{-m} \sum_{p \geq 0 \vee (m - q_0)} (q_0 - m + p)^b z^p \\
 &\leq z^{-m} \sum_{\ell=0}^b \binom{b}{\ell} q_0^{b-\ell} \sum_{p \geq 0} p^\ell z^p \\
 &\leq z^{-m} \frac{b!}{1-z} \sum_{\ell=0}^b \binom{b}{\ell} \frac{q_0^{b-\ell}}{(1-z)^\ell} \\
 &= 2^{m/2} b! c_1 (q_0 + c_1)^b,
 \end{aligned}$$

where $c_1 = (1 - z)^{-1} = 2 + \sqrt{2}$.

Assume now that $a \geq 1$ and $[\mathbf{n}] = a + 1$. We decompose $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$, with $[\mathbf{n}_1] = a$ and $[\mathbf{n}_2] = 1$, and $|\mathbf{n}_1| = m_1$, $|\mathbf{n}_2| = m_2$ with $m_1 + m_2 = m$. We may assume that the largest nonzero component of $\mathbf{n}$ appears in $\mathbf{n}_1$, so that $q_{[\mathbf{n}_1]} = q_{[\mathbf{n}]}$ and $q_{[\mathbf{n}_2]} = q < q_{[\mathbf{n}]}$. It follows that

$$\begin{aligned}
 S_m(a + 1, b) &= \sum_{m_1 + m_2 = m} \sum_{q < q_{[\mathbf{n}]}} \sum_{\substack{|\mathbf{n}_1| = m_1 \\ [\mathbf{n}_1] = a \\ q_{[\mathbf{n}_1]} + \mathbf{n}_{q_{[\mathbf{n}_1]}} \geq q_0}} \frac{m!}{\mathbf{n}_1! m_2!} \frac{q_{[\mathbf{n}_1]}^b}{2^{(q_{[\mathbf{n}_1]} - q_0)/2}} \\
 &\leq \sum_{m_1=1}^{m-1} \binom{m}{m_1} S_{m_1}(a, b + 1),
 \end{aligned}$$

where we have bounded the sum over q by $q_{[\mathbf{n}]} = q_{[\mathbf{n}_1]}$. It is then straightforward to show by induction that

$$S_m(a, b) \leq c_1 (\sqrt{2} + a - 1)^m (q_0 + c_1)^{a+b-1} (a + b - 1)!$$

for all a, b . In particular,

$$S_m(a, 0) \leq c_1 (\sqrt{2} + a - 1)^m (q_0 + c_1)^{a-1} (a - 1)!.$$

Replacing this in (A.4.1) yields the result, with $c_2 = \sqrt{2} - 1$. □

A.5 Proof of Proposition 5.3.20

Using the relation (2.5.1) between Wick polynomials and monomials, we get

$$\begin{aligned}
 \prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \psi(t, \cdot); c_q) &= \prod_{q \geq 0} \left(\sum_{\mathbf{l}_q=0}^{[\mathbf{n}_q/2]} a_{\mathbf{n}_q \mathbf{l}_q} c_q^{\mathbf{l}_q} (\delta_q \psi(t, \cdot))^{\mathbf{n}_q - 2\mathbf{l}_q} \right) \\
 &= \sum_{\mathbf{l}: \mathbf{l} \leq [\mathbf{n}/2]} a_{\mathbf{n} \mathbf{l}} \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} c_q^{\mathbf{l}_q} (\delta_q \psi(t, \cdot))^{\mathbf{n}_q - 2\mathbf{l}_q},
 \end{aligned}$$

where

$$a_{\mathbf{n}l} = \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} a_{\mathbf{n}_q l_q} = \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \frac{(-1)^{l_q} \mathbf{n}_q!}{2^{l_q} l_q! (\mathbf{n}_q - 2l_q)!} = \frac{(-1)^{|\mathbf{l}|} \mathbf{n}!}{2^{|\mathbf{l}|} \mathbf{l}! (\mathbf{n} - 2\mathbf{l})!}.$$

Recall that

$$\delta_q \psi(t, x) = \sum_{k \in \mathcal{A}_q} \psi_k(t) e_k(x),$$

which implies

$$\begin{aligned} \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} (\delta_q \psi(t, x))^{\mathbf{n}_q - 2l_q} &= \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \left(\sum_{k_1, \dots, k_{\mathbf{n}_q - 2l_q} \in \mathcal{A}_q} \psi_{k_1}(t) \dots \psi_{k_{\mathbf{n}_q - 2l_q}}(t) e_{k_1 + \dots + k_{\mathbf{n}_q - 2l_q}}(x) \right) \\ &= \sum_{k_1^{(q)}, \dots, k_{\mathbf{n}_q - 2l_q}^{(q)} \in \mathcal{A}_q \forall q} \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \prod_{i=1}^{\mathbf{n}_q - 2l_q} \psi_{k_i^{(q)}}(t) e_{\sum_{\mathbf{n}_q > 0} \sum_{i=1}^{\mathbf{n}_q - 2l_q} k_i^{(q)}}(x), \end{aligned}$$

whose projection on the k_0 -th Fourier basis vector is given by

$$P_{k_0} \left(\prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} (\delta_q \psi(t, x))^{\mathbf{n}_q - 2l_q} \right) = \sum_{\mathcal{B}(k_0)} \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \prod_{i=1}^{\mathbf{n}_q - 2l_q} \psi_{k_i^{(q)}}(t) e_{k_0}(x),$$

where the sum runs over all tuples $(k_1^{(q)}, \dots, k_{\mathbf{n}_q - 2l_q}^{(q)})_{q \geq 0}$ in the set

$$\mathcal{B}(k_0) = \left\{ k_1^{(q)}, \dots, k_{\mathbf{n}_q - 2l_q}^{(q)} \in \mathcal{A}_q \forall q : \sum_{q \geq 0} \sum_{i=1}^{\mathbf{n}_q - 2l_q} k_i^{(q)} = k_0 \right\}.$$

Similar relations hold with $\hat{\psi}(t, x)$. We now note that

$$\|\delta_{q_0}(\varphi(t, \cdot) - \hat{\varphi}(t, \cdot))\|_{L^2}^2 = \sum_{k_0 \in \mathcal{A}_{q_0}} |\langle e_{k_0}, P_{k_0} \varphi(t, \cdot) - P_{k_0} \hat{\varphi}(t, \cdot) \rangle|^2,$$

where

$$\begin{aligned} &\langle e_{k_0}, P_{k_0} \varphi(t, \cdot) - P_{k_0} \hat{\varphi}(t, \cdot) \rangle \\ &= \sum_{1 \leq l \leq \lfloor \mathbf{n}/2 \rfloor} a_{\mathbf{n}l} \sum_{\mathcal{B}(k_0)} \left[\prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} c_q^{l_q} \prod_{i=1}^{\mathbf{n}_q - 2l_q} \psi_{k_i^{(q)}}(t) - \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \hat{c}_q^{l_q} \prod_{i=1}^{\mathbf{n}_q - 2l_q} \hat{\psi}_{k_i^{(q)}}(t) \right]. \quad (\text{A.5.1}) \end{aligned}$$

Observe that

$$\begin{aligned} \prod_{i=1}^{\mathbf{n}_q - 2l_q} \psi_{k_i^{(q)}}(t) &= \exp \left\{ -\frac{1}{\varepsilon} \sum_{i=1}^{\mathbf{n}_q - 2l_q} \alpha_{k_i^{(q)}}(u_{l+1}, t) \right\} \prod_{i=1}^{\mathbf{n}_q - 2l_q} \hat{\psi}_{k_i^{(q)}}(t), \\ c_q^{l_q} &\leq \exp \left\{ -\frac{2}{\varepsilon} l_q \alpha_{k(q)}(u_{l+1}, t) \right\} \hat{c}_q^{l_q} \end{aligned}$$

for some $k(q) \in \mathcal{A}_q$. The definition of the partition implies that $|u_{l+1} - t|$ has order $2^{-2q_{[n]}}\gamma_0\varepsilon$, and therefore there is a numerical constant c_0 such that

$$-\frac{1}{\varepsilon}\alpha_k(u_{l+1}, t) \leq c_0 2^{-2(q_{[n]}-q)}\gamma_0$$

holds for all $k \in \mathcal{A}_q$. Therefore,

$$-\frac{1}{\varepsilon} \sum_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q} \alpha_{k_i^{(q)}}(u_{l+1}, t) - \frac{2}{\varepsilon} \mathbf{l}_q \alpha_{k(q)}(u_{l+1}, t) \leq c_0 \gamma_0 |\mathbf{n}_q| 2^{-2(q_{[n]}-q)}.$$

Replacing this in (A.5.1) yields

$$\begin{aligned} & |\langle e_{k_0}, P_{k_0} \varphi(t, \cdot) - P_{k_0} \hat{\varphi}(t, \cdot) \rangle| \\ & \leq \sum_{\mathbf{l}: \mathbf{l} \leq \lfloor \mathbf{n}/2 \rfloor} |a_{\mathbf{n}\mathbf{l}}| \sum_{\mathcal{B}(k_0)} \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \left(e^{c_0 \gamma_0 |\mathbf{n}_q| 2^{-2(q_{[n]}-q)}} - 1 \right) \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \left(\hat{c}_q(t)^{\mathbf{l}_q} \prod_{i=1}^{\mathbf{n}_q-2\mathbf{l}_q} |\hat{\psi}_{k_i^{(q)}}(t)| \right). \quad (\text{A.5.2}) \end{aligned}$$

Since the exponent $c_0 \gamma_0 |\mathbf{n}_q| 2^{-2(q_{[n]}-q)}$ is bounded, we can write, for a numerical constant c_1 ,

$$\begin{aligned} \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \left(e^{c_0 \gamma_0 |\mathbf{n}_q| 2^{-2(q_{[n]}-q)}} - 1 \right) & \leq \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \left(c_1 \gamma_0 |\mathbf{n}_q| 2^{-2(q_{[n]}-q)} \right) \\ & \leq (c_1 \gamma_0)^{[\mathbf{n}]} \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \mathbf{n}_q, \end{aligned}$$

since the product of powers of 2 is bounded by 1 (in fact, it can even be bounded by $2^{-2([\mathbf{n}]-1)}$, but this just decreases the constant c_1). Since

$$\|\delta_q \hat{\psi}(t, \cdot)\|_{L^2}^2 = \sum_{k \in \mathcal{A}_q} |\hat{\psi}_k(t)|^2,$$

we have the rough bound

$$|\hat{\psi}_k(t)|^2 \leq \|\delta_q \hat{\psi}(t, \cdot)\|_{L^2}^2 \quad \forall k \in \mathcal{A}_q.$$

Plugging the last bounds into (A.5.2), we get

$$\begin{aligned} & |\langle e_{k_0}, P_{k_0} \varphi(t, \cdot) - P_{k_0} \hat{\varphi}(t, \cdot) \rangle| \\ & \leq (c_1 \gamma_0)^{[\mathbf{n}]} \left(\prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \mathbf{n}_q \right) \sum_{\mathbf{l}: \mathbf{l} \leq \lfloor \mathbf{n}/2 \rfloor} |a_{\mathbf{n}\mathbf{l}}| \hat{c}(t)^{|\mathbf{l}|} \prod_{\substack{q \geq 0 \\ \mathbf{n}_q > 0}} \left(\|\delta_q \hat{\psi}(t, \cdot)\|_{L^2}^{\mathbf{n}_q-2\mathbf{l}_q} \right) \# \mathcal{B}(k_0). \end{aligned}$$

Finally, by counting the number of choices of the $k_i^{(q)}$, we obtain

$$\# \mathcal{B}(k_0) \leq 2^{2q_{[n]}([\mathbf{n}]-2|\mathbf{l}|)}.$$

This yields the claimed result, noticing that this bound is independent of k_0 , so summing over all $k_0 \in \mathcal{A}_{q_0}$ only yields an extra factor 2^{2q_0} in the L^2 -norm squared. $\square$

A.6 Proof of Lemma 5.3.23

We decompose the sum as

$$\bar{K}_{m,b}(q_0) = \sum_{a=1}^m S_{m,m}(a,b),$$

where for $a \in \{1, \dots, m\}$ and $b \in \mathbb{N}_0$, we define

$$S_{m,m_0}(a,b) = \sum_{\substack{\mathbf{n}: |\mathbf{n}|=m \\ [\mathbf{n}]=a, q_{[\mathbf{n}]} + \mathbf{n}_{q_{[\mathbf{n}]}} \geq q_0}} q_{[\mathbf{n}]}^b 2^{(m_0+3)q_{[\mathbf{n}]}} \exp\left\{-\beta(m_0, q_0) 2^{(q_{[\mathbf{n}]} - q_0)/m_0}\right\}.$$

We will proceed similarly to the proof of Lemma 5.3.23, and estimate this sum by induction on a , for arbitrary $b \in \mathbb{N}_0$. For $a = 1$, the only possible $\mathbf{n}$ are those with one component, say q , equal to m , and all other components equal to 0. Then $q_{[\mathbf{n}]} = q$, and writing $x_+ = x \vee 0$ we get

$$\begin{aligned} S_{m,m_0}(1,b) &\leq \sum_{q \geq (q_0 - m)_+} q^b 2^{(m_0+3)q} \exp\left\{-\beta(m_0, q_0) 2^{(q - q_0)/m_0}\right\} \\ &= \sum_{p \geq (m - q_0)_+} (q_0 - m + p)^b 2^{(m_0+3)(p + q_0 - m)} \exp\left\{-\beta(m_0, q_0) 2^{(p - m)/m_0}\right\} \\ &= 2^{(m_0+3)(q_0 - m)} \sum_{p \geq (m - q_0)_+} (q_0 - m + p)^b 2^{(m_0+3)p} \exp\left\{-\beta(m_0, q_0) 2^{(p - m)/m_0}\right\}. \end{aligned}$$

One checks that for $\beta(m_0, q_0)$ larger than a numerical constant of order 1, the general term of this sum is decreasing in p . Estimating the sum by an integral, we get

$$S_{m,m_0}(1,b) \leq c_1 (q_0 - m)^b 2^{(m_0+3)(q_0 - m)_+} \exp\left\{-2^{-m/m_0} \beta(m_0, q_0)\right\}$$

for a numerical constant c_1 . Assume now that $a \geq 1$ and $[\mathbf{n}] = a + 1$. We decompose $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$, with $[\mathbf{n}_1] = a$ and $[\mathbf{n}_2] = 1$, and $|\mathbf{n}_1| = m_1$, $|\mathbf{n}_2| = m_2$ with $m_1 + m_2 = m$. We may assume that the largest nonzero component of $\mathbf{n}$ appears in $\mathbf{n}_1$, so that $q_{[\mathbf{n}_1]} = q_{[\mathbf{n}]}$ and $q_{[\mathbf{n}_2]} = q < q_{[\mathbf{n}]}$. It follows that

$$\begin{aligned} S_{m,m_0}(a+1,b) &= \sum_{m_1+m_2=m} \sum_{q < q_{[\mathbf{n}]}} \sum_{\substack{|\mathbf{n}_1|=m_1 \\ [\mathbf{n}_1]=a \\ q_{[\mathbf{n}_1]} + \mathbf{n}_{q_{[\mathbf{n}_1]}} \geq q_0}} q_{[\mathbf{n}]}^b 2^{(m_0+3)q_{[\mathbf{n}]}} \exp\left\{-\beta(m_0, q_0) 2^{(q_{[\mathbf{n}]} - q_0)/m_0}\right\} \\ &\leq \sum_{m_1=1}^{m-1} S_{m_1,m_0}(a,b+1). \end{aligned}$$

where we have bounded the sum over q by $q_{[\mathbf{n}]} = q_{[\mathbf{n}_1]}$. It is then straightforward to show by induction that

$$S_{m,m_0}(a,b) \leq c_1 m^{a-1} q_0^{a+b-1} 2^{(m_0+3)q_0} \exp\left\{-\beta(m_0, q_0)\right\}$$

for all a, b . Summing over a and setting $m = m_0$ yields the result.

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**Métastabilité dans des EDP stochastiques lentement dépendantes du temps
non singulières et singulières**

Résumé :

On considère des équations aux dérivées partielles stochastiques (EDPS) lentement dépendantes du temps soumises à un bruit blanc espace-temps. Ces EDPS ne sont pas toujours bien posées. On montre que sur le tore de dimension une, ce problème n'apparaît pas. On s'intéresse à des EDPS soumises aussi à un forçage périodique en temps. Ce dernier s'annule en trois branches d'équilibres dont deux stables et une instable, qui s'approchent l'une de l'autre à un certain temps. On décrit l'effet de la résonance stochastique sur le système. On montre l'existence d'une intensité de bruit critique dépendante de la période du forçage et de la distance minimale entre les branches d'équilibres. Pour une intensité de bruit inférieure à l'intensité seuil, la probabilité que les solutions de l'EDPS passent d'un équilibre stable à l'autre est exponentiellement petite, tandis que ces transitions ont lieu avec une probabilité exponentiellement proche de 1 pour des intensités de bruit plus grandes. Les estimées de concentration des solutions sont données dans des normes de Sobolev. D'un autre côté, sur le tore de dimension deux, les EDPS ne sont pas bien définies et une renormalisation au sens de Wick est nécessaire pour définir une solution de l'équation. On donne des estimées de concentration des solutions dans des normes de Besov et Hölder et on montre qu'elles sont concentrées avec grande probabilité près de la branche d'équilibre stable. Ensuite, on discute le cas où le système s'approche d'une bifurcation fourche où un phénomène intéressant a lieu : le retard à la bifurcation. Les résultats obtenus sont une généralisation à une dimension infinie de résultats obtenus pour des EDS en dimension finie.

Mots clés : EDP stochastiques, résonance stochastique, estimées de concentration, renormalisation, bifurcations, systèmes lents-rapides

Metastability in slowly time-dependent non-singular or singular stochastic PDEs

Abstract :

We consider slowly time-dependent stochastic partial differential equations (SPDEs) driven by space-time white noise. These SPDEs are not always well-posed. On the one-dimensional torus, we show that this problem does not occur. We are interested in SPDEs subjected also to a time-periodic driving force which vanishes on three equilibrium branches, two of which come close to each other at particular times. We show the effect of stochastic resonance on the system: the dynamic changes when the noise intensity crosses a critical value. The probability that solutions of the SPDE make transitions between stable equilibria is exponentially small below the threshold, while for a larger noise intensity transitions happen with high probability. Concentration estimates of solutions are given in Sobolev norms. On the other hand, on the two-dimensional torus, the SPDEs are ill-defined and a renormalisation in the Wick sense is needed because space-time white noise is more singular in dimension two than in dimension one. We show that sample paths stay near stable equilibrium branches with high probability. This distance is measured in Besov and Hölder spaces. We discuss a case involving a pitchfork bifurcation characterized by a delay of the transition from the unstable to the stable state. The results on the one and two-dimensional torus generalise to an infinite-dimensional setting those obtained for finite-dimensional SDEs.

Keywords : Stochastic PDEs, stochastic resonance, sample-path estimates, renormalisation, bifurcations, slow-fast systems