Dilation properties of Schur and Fourier multipliers

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Propriétés de dilatation des multiplicateurs de Schur et des Multiplicateurs de Fourier

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Introduction (français)

Introduction générale

Le théorème de dilatation d'Akcoglu [1, 2] affirme que pour tout $1 < p < +\infty$, pour tout espace mesuré (Σ, μ) et pour toute contraction positive $T\colon L^p(\Sigma) \to L^p(\Sigma)$, il existe une isométrie $J\colon L^p(\Sigma) \to L^p(\Sigma')$, il existe un isomorphisme isométrique $U\colon L^p(\Sigma') \to L^p(\Sigma')$ et il existe une contraction $Q\colon L^p(\Sigma') \to L^p(\Sigma)$ tels que pour tout $k \geqslant 0$ on a $T^k = QU^kJ$. Ce résultat est primordial dans de nombreux domaines des opérateurs sur L^p, par exemple il est utile pour le calcul fonctionnel, en analyse harmonique ou dans la théorie ergodique. Pour plus de détails, voir [8, 14, 31, 42, 44] ou [33, Section I].

Pendant les deux dernières décennies, la théorie des opérateurs et l'analyse harmonique dans les espaces L^p non -commutatifs ont été étudiés et beaucoup de résultats sur $L^p(\Sigma)$, ont été transférés dans la cadre non-commutatif. Cependant le théorème d'Akcoglu ne peut être étendu au cas non-commutatif. Plus précisément, si on prend une algèbre de von Neumann N équipée d'une trace normale semi-finie fidèle τ, on dit que (N, τ) est une algèbre de von Neumann traciale. Pour tout $1 < p < +\infty$, on note $L^p(N)$ l'espace L^p non-commutatif associé à (N, τ). On dit qu'une contraction $T\colon L^p(N) \to L^p(N)$ admet une p-dilatation isométrique s'il existe une autre algèbre de von Neumann traciale $(\mathcal{N}, \tau')$, il existe une isométrie $J\colon L^p(N) \to L^p(\mathcal{N})$, il existe un isomorphisme isométrique $U\colon L^p(\mathcal{N}) \to L^p(\mathcal{N})$ et une contraction $Q\colon L^p(\mathcal{N}) \to L^p(N)$ tels que pour tout $k \geqslant 0$ $T^k = QU^kJ$.

$$\begin{array}{ccc} L^p(\mathcal{N}) & \xrightarrow{U^k} & L^p(\mathcal{N}) \\ J \uparrow & & \downarrow Q \\ L^p(N) & \xrightarrow{T^k} & L^p(N) \end{array}$$

Il est démontré dans [34] que certaines contractions positives $T\colon L^p(N) \to L^p(N)$ n'admettent pas de p-dilatation isométrique. Après que ce résultat a été publié, il est devenu important d'exhiber des classes de contractions positives dans les espaces L^p non-commutatifs qui admettent une p-dilatation isométrique. Pour des résultats de ce type, on peut regarder par exemple dans [35, Chapters 8-10], [4, 5] ou [32]. En particulier, dans l'article [4], C. Arhancet montre que tout multiplicateur de Schur $B(l^2_{\mathbb{Z}}) \to B(l^2_{\mathbb{Z}})$ auto-adjoint, unitaire et positif est absolument dilatable, donc il admet une p-dilatation isométrique (voir ci-dessous). Il démontre aussi que, pour tout groupe discret G, tout multiplicateur de Fourier $VN(G) \to VN(G)$ auto-adjoint, unitaire et complètement positif est absolument dilatable.

La notion d'absolument dilatable est la suivante. Soit (N, τ) une algèbre de von Neumann traciale et soit une contraction $T\colon N \to N$, on dit que T est absolument dilatable (ou admet une dilatation absolue) s'il existe une autre algèbre de von Neumann

traciale $(\mathcal{N}, \tau')$, il existe un $*$-homomorphisme préfaiblement continu, préservant la trace et unitaire $J\colon N \to \mathcal{N}$ et un $*$-automorphisme préservant la trace $U\colon \mathcal{N} \to \mathcal{N}$ tel que pour tout $k \geqslant 0$ $T^k = \mathbb{E}U^k J$, où $\mathbb{E}\colon \mathcal{N} \to N$ est l'espérance conditionnelle associée à J.

$$\begin{array}{ccc} (\mathcal{N}, \tau') & \xrightarrow{U^k} & (\mathcal{N}, \tau') \\ J \uparrow & & \downarrow \mathbb{E} \\ (N, \tau) & \xrightarrow{T^k} & (N, \tau) \end{array} \tag{0.0.1}$$

Il s'avère que si $T\colon N \to N$ est absolument dilatable, alors, pour tout $1 < p < +\infty$, T s'étend en une contraction positive admettant une p-dilatation isométrique (voir la proposition 1.3.5). La notion de dilatation absolue apparaît implicitement dans [4, 5, 41, 28, 35]. Beaucoup de constructions de p-dilatations isométriques pour des classes d'opérateurs de L^p proviennent de dilatations absolues. De plus les dilatations absolues sont étroitement liées aux opérateurs factorisables [3, 28, 56].

Une autre propriété de dilatation est la notion de complètement p-dilatable. Elle est analogue à la notion de p-dilatation isométrique mais J et Q sont des contractions complètes et U est une isométrie complète inversible. De plus si $T\colon N \to N$ est une dilatation absolue, alors pour tout $1 < p < +\infty$, T s'étend en une contraction complètement p-dilatable sur $L^p(N)$ (voir la proposition 1.3.8).

L'objet de cette thèse est l'étude des propriétés de dilatation de différentes classes d'opérateurs. En particulier on regarde les multiplicateurs de Schur dans les parties 2,3,4 et les multiplicateurs de Fourier dans la partie 5. De plus, dans chacun cas on regarde un dilatation jointe pour une famille finie de tels opérateurs. L'article [54] donne des résultats dans ce cas.

Contenu de la thèse

La thèse commence avec toutes les définitions et propriétés nécessaires sur les multiplicateurs de Schur, les multiplicateurs de Fourier et les dilatations. D'abord, on se focalisera sur les multiplicateurs de Schur. On prouvera que tout multiplicateur de Schur mesurable, auto-adjoint, unitaire et positif est absolument dilatable. On caractérisera alors le fait qu'un multiplicateur de Schur est absolument dilatable. Ensuite on démontrera une caractérisation pour les multiplicateurs de Schur unitaux, positifs et complètement p-dilatable dans le cas de la dimension finie puis dans le cas discret. On finira par étudier les multiplicateurs de Fourier. On prouvera que tout multiplicateur de Fourier auto-adjoint, unitaire et complètement positif est absolument dilatable.

Première partie

L'objet de ce chapitre est de présenter les outils qui seront utilisés dans notre travail. On présentera d'abord les espaces L^p non commutatifs. Ainsi, on travaillera avec une algèbre de von Neumann équipée d'une trace normale semi-finie et fidèle. Les espaces L^p non-commutatifs sont une généralisation des espaces $L^p(\Sigma)$ classiques où $(\Sigma, \mathcal{F}(\Sigma), \mu)$ est un espace mesuré. On considérera aussi les algèbres de von Neumann à un poids. La construction des espaces L^p à partir d'une algèbre de von Neumann à poids est fondée sur le produit croisé. Les propriétés de ces espaces L^p à poids sont similaires au cas tracial.

Dans cette partie on présentera aussi les différents types de dilatation cités précédemment. Ensuite on donnera la définition des multiplicateurs de Schur et la définition des multiplicateurs de Fourier ainsi que leur propriétés de base. On note $B(L^2(\Sigma))$ l'espace des opérateurs bornés sur $L^2(\Sigma)$ associé à un espace mesuré σ-fini (Σ, μ). L'étude des multiplicateurs de Schur agissant sur $B(L^2(\Sigma))$ remonte au moins à Haagerup [26] et Spronk [59]. Le papier de Haagerup-Musat [28] donne une caractérisation des multiplicateurs de Schur admettant une dilatation absolue dans le cas de la dimension finie. On présentera trois types de multiplicateurs de Schur en fonction de le forme de l'espace L^2 considéré. La définition dans le cas mesurable est la suivante: soit (Σ, μ) un espace mesuré σ-fini, pour tout $f \in L^2(\Sigma^2)$, on note $S_f \colon L^2(\Sigma) \to L^2(\Sigma)$ l'opérateur borné défini par

$$[S_f(h)](s) = \int_\Sigma f(s,t)h(t)\,dt, \qquad h \in L^2(\Sigma).$$

L'opérateur $S_f \in S^2(L^2(\Sigma))$ est un opérateur de Hilbert-Schmidt. De plus, l'application $f \mapsto S_f$ est un unitaire de $L^2(\Sigma^2)$ dans $S^2(L^2(\Sigma))$.

Soit $\varphi \in L^\infty(\Sigma^2)$, on définit $M_\varphi \colon S^2(L^2(\Sigma)) \to S^2(L^2(\Sigma))$ par

$$M_\varphi(S_f) = S_{\varphi f}, \qquad f \in L^2(\Sigma^2).$$

Cet opérateur est appelé opérateur de multiplication de Schur. De plus, s'il existe $C > 0$ telle que

$$\|M_\varphi(S)\|_{B(L^2(\Sigma))} \leqslant C\|S\|_{B(L^2(\Sigma))}, \qquad S \in S^2(L^2(\Sigma)).$$

On dit que M_φ est un multiplicateur de Schur mesurable sur $B(L^2(\Sigma))$ (ou multiplicateur de Schur borné).

Une autre famille classique d'opérateurs est la famille des multiplicateurs de Fourier sur $VN(G)$ où G est un groupe localement compact. Ce sujet a beaucoup d'applications aux propriétés d'approximation d'algèbres d'opérateurs, en probabilités non-commutatives, en information quantique, aux intégrales singulières ou aux opérateurs de Calderon-Zygmund (voir [6, 24, 36, 37, 43, 47, 48]). On introduira les multiplicateurs de Fourier dans le cas non unimodulaire puis dans le cas unimodulaire.

Soit G un groupe localement compact, on note $\lambda \colon G \to B(L^2(G))$ la représentation régulière à gauche de G. On définit l'algèbre de von Neumann associée au groupe G par $VN(G) = \lambda(G)'' \subset B(L^2(G))$. On note $\omega_G : VN(G)_+ \longrightarrow [0; +\infty]$ le poids de Plancherel sur $VN(G)$, qui est une trace dans le cas unimodulaire. On dit qu'un opérateur préfaiblement continu $T : VN(G) \to VN(G)$ est un multiplicateur de Fourier, s'il existe une fonction continue bornée $u \colon G \to \mathbb{C}$ telle que pour tout $s \in G$, $T(\lambda(s)) = u(s)\lambda(s)$.

La dernière partie de ce chapitre portera sur les espaces de Fock. Le but est de cet espace est de construire une algèbre de von Neumann à partir d'un espace de Hilbert réel.

Seconde partie

On a mentionné plus tôt l'article de C.Arhancet [4]. Il a prouvé que tout multiplicateur de Schur auto-adjoint unitaire et positif de $B(\ell^2_{\mathbb{Z}})$ dans $B(\ell^2_{\mathbb{Z}})$ admet une dilatation absolue. Son article repose sur une construction d'E. Ricard introduite dans [56]. L'objectif de cette partie est de généraliser ce résultat aux multiplicateurs de Schur mesurables.

Théorème (A). *Soit Σ un espace mesuré σ-fini et soit $\varphi \in L^\infty(\Sigma^2)$, on suppose que M_φ est un multiplicateur de Schur auto-adjoint unitaire et positif sur $B(L^2(\Sigma))$. Alors M_φ est absolument dilatable.*

On regardera aussi la version multi-variables du théorème précédent. On obtient le théorème suivant:

Théorème (C). *Soit $\varphi_1, \ldots, \varphi_n \in L^\infty(\Sigma^2)$ et supposons que les M_{φ_i} sont des multiplicateurs de Schur auto-adjoints unitaires et positifs sur $B(L^2(\Sigma))$. Alors il existe une algèbre de von Neumann traciale (M, τ), un n-uple d'$*$-homomorphismes préservant la trace $(U_1, \ldots, U_n)$ qui commutent entre eux sur M et un homomorphisme injectif préservant la trace et préfaiblement continu $J : B(L^2(\Sigma)) \to M$ tels que pour tout $k_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant n$,*

$$M_{\varphi_1}^{k_1} \cdots M_{\varphi_n}^{k_n} = \mathbb{E} U_1^{k_1} \cdots U_n^{k_n} J$$

où $\mathbb{E} : M \to B(L^2(\Sigma))$ est l'espérance conditionnelle associée à J.

On a ainsi une condition suffisante pour un multiplicateur de Schur d'admettre une dilatation absolue. L'idée de la prochaine partie est de retirer l'hypothèse auto-adjoint du multiplicateur de Schur et de voir ce que l'on obtient.

Troisième partie

Le but de cette partie est d'établir une caractérisation pour les multiplicateurs de Schur de $B(L^2(\Sigma))$ dans $B(L^2(\Sigma))$ absolument dilatables. On introduit dans cette partie une nouvelle notion de dilatation. On dit qu'un opérateur $T : N \to N$ admet une dilatation absolue séparable si N a un prédual séparable et que T vérifie (0.0.1) avec un $\mathcal{N}$ possédant un prédual séparable aussi.

On regarde déjà le cas discret. Pour tout $n \geqslant 1$, on note M_n l'espace des matrices de taille $n \times n$ à coefficients complexes. Soit $\mathfrak{m} = \{m_{ij}\}_{i,j \geqslant 1}$ une famille de nombres complexes. On rappelle que $\mathfrak{m}$ est un multiplicateur de Schur borné s'il existe un (nécessairement unique) opérateur préfaiblement continu $M_\mathfrak{m} \colon B(\ell^2) \to B(\ell^2)$ tel que pour tout $n \geqslant 1$ et pour tout $[a_{ij}] \in M_n \subset B(\ell^2)$ $M_\mathfrak{m}\big([a_{ij}]\big) = [m_{ij} a_{ij}]$. On montrera que $M_\mathfrak{m}$ admet une dilatation absolue si et seulement si il existe une algèbre de von Neumann traciale normalisée (N, τ_N) avec un prédual séparable et il existe une suite d'unitaires de N $(d_k)_{k \geqslant 1}$, tels que

$$m_{ij} = \tau_N(d_i^* d_j), \qquad i, j \geqslant 1. \tag{0.0.2}$$

Dans ce cas, $M_\mathfrak{m}$ admet en fait une dilatation absolue séparable, voir corollaire 3.5.2.

Dans le cas général, l'espace des familles bornées, dont les coefficients sont dans une algèbre de von Neumann N qui a un prédual séparable, est remplacé par l'espace $L^\infty_\sigma(\Sigma; N)$ qui est l'ensemble des (classes de) fonctions essentiellement bornées et w^*-mesurables de Σ dans N. Ces espaces sont utilisés dans e.g [58, Theorem 1.22.13]. On a l'identification $L^\infty_\sigma(\Sigma; N) \simeq L^\infty(\Sigma) \overline{\otimes} N$, pour plus de détails voir la section 3.1.

A noter que dans le cas général, (0.0.2) est remplacée par (0.0.3).

Théorème. *On suppose que (Σ, μ) est séparable et soit $\varphi \in L^\infty(\Sigma^2)$, les assertions suivantes sont équivalentes:*

(i) *la fonction φ est un multiplicateur de Schur borné et $M_\varphi\colon B(L^2(\Sigma)) \to B(L^2(\Sigma))$ admet une dilatation absolue séparable;*

(ii) *il existe une algèbre de von Neumann traciale normalisée (N, τ_N) qui est à prédual séparable et un unitaire $\mathfrak{d} \in L^\infty_\sigma(\Sigma; N)$ tels que*

$$\varphi(s,t) = \tau_N\big(\mathfrak{d}(s)^*\mathfrak{d}(t)\big), \qquad \text{pour presque tout } (s,t) \in \Sigma^2. \tag{0.0.3}$$

On donnera une version du théorème précédent dans le cas des multiplicateurs de Schur qui admettent une dilatation absolue (non nécessairement séparable). Cette version est dans le langage des produits tensoriels.

Théorème. *Soit $\varphi \in L^\infty(\Sigma^2)$ et soit $e \in L^2(\Sigma)$ tel que $\|e\|_2 = 1$, les assertions suivantes sont équivalentes:*

(i) *la fonction φ est un multiplicateur de Schur et $M_\varphi\colon B(L^2(\Sigma)) \to B(L^2(\Sigma))$ est absolument dilatable;*

(ii) *il existe une algèbre de von Neumann traciale normalisée (N, τ_N) agissant sur un espace de Hilbert H, et un unitaire $D \in L^\infty(\Sigma)\overline{\otimes}B(H)$ tels que:*

$(ii)_1$ *le produit $(1_t \otimes D_s^*)(1_s \otimes D_t)$ est dans $L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}N$ et on a*

$$\varphi = \big[I_{L^\infty(\Sigma^2)}\overline{\otimes}\tau_N\big]\big((1_t \otimes D_s^*)(1_s \otimes D_t)\big);$$

$(ii)_2$ *la trace de $D^*(e \otimes e \otimes I_H)D$ dans $B(L^2(\Sigma))\overline{\otimes}N$ est égale 1.*

Quatrième partie

Dans la section 4.1, on considérera des propriétés de dilatation pour les multiplicateurs de Schur, unitaires et positifs, sur les matrices. On prend $T_M : M_n \to M_n$ un multiplicateur de Schur défini par $T_M([a_{ij}]_{1\leqslant i,j\leqslant n}) = [m_{ij}a_{ij}]_{1\leqslant i,j\leqslant n}$, pour toute matrice $A = [a_{ij}]_{1\leqslant i,j\leqslant n}$. On suppose que T_M est unitaire et positif. On prouvera que si T_M (vu comme agissant sur S^p_n) est complètement p-dilatable pour un certain $1 < p \neq 2 < +\infty$ alors il existe une algèbre de von Neumann traciale normalisée (N, τ) et il existe des unitaires $v_1, \ldots, v_n$ de N tels que pour tout $1 \leqslant i, j \leqslant n$,

$$m_{ij} = \tau(v_i^* v_j).$$

En combinant ceci avec les résultats de [28], on en déduit que tout multiplicateur de Schur T_M comme précédemment est complètement p-dilatable pour un certain $1 \leqslant p \neq 2 < +\infty$ si et seulement si T_M est absolument dilatable.

Dans la section 4.2, on étendra le théorème précédent aux multiplicateurs de Schur sur $B(l^2(I))$ où I est une famille infinie d'indices. On obtiendra le théorème suivant:

Théorème. *Soit T_M un multiplicateur de Schur unitaire, positif et borné sur $B(l^2(I))$, les assertions suivantes sont équivalentes:*

1. *il existe $1 < p \neq 2 < +\infty$ tel que T_M est complètement p-dilatable;*
2. *pour tout $1 \leqslant p < +\infty$, T_M est complètement p-dilatable;*

3. *T_M est absolument dilatable;*

4. *il existe une algèbre de von Neumann traciale normalisée (N, τ) et il existe une famille d'unitaires de N $(v_i)_{i\in I}$ tel que pour tout $i, j \in I$:*

$$m_{ij} = \tau(v_i^* v_j).$$

Dans la dernière section de ce chapitre, on considérera des familles finies de multiplicateurs de Schur qui sont simultanément absolument dilatable. On obtiendra que toute famille finie de multiplicateurs de Schur est simultanément absolument dilatable si et seulement si chaque multiplicateur de Schur est absolument dilatable.

Cinquième partie

L'objectif de cette partie est de prouver que tout multiplicateur de Fourier sur $VN(G)$ auto-adjoint, unitaire et complètement positif est absolument dilatable. On traitera deux cas. Le premier: on supposera que G est un groupe unimodulaire. Dans un second temps on retirera l'hypothèse unimodulaire. La démonstration est assez similaire dans les deux cas. Cependant dans le cas non unimodulaire, on perd la propriété traciale du poids de Plancherel. La notion de dilatation absolue doit ainsi être modifiée pour inclure les espaces L^p à poids. Le théorème principal de cette partie est le suivant:

Théorème. *Soit $T_u : VN(G) \to VN(G)$ un multiplicateur de Fourier auto-adjoint, unitaire et complètement positif, alors T_u est absolument dilatable.*

Comme dans la deuxième partie, on généralisera le théorème précédent pour un n-uple de multiplicateurs de Fourier auto-adjoints, unitaires et complètement positifs. On établira donc le résultat suivant.

Théorème (D). *Soit G un groupe localement compact et soient $T_{u_1}, \dots, T_{u_n}$ des multiplicateurs de Fourier sur $VN(G)$ auto-adjoints, unitaires et complètement positifs. Alors il existe une algèbre de von Neumann traciale (M, τ), un n-uplet de $*$-automorphismes sur M préservant la trace et commutant entre eux $(U_1, \dots, U_n)$ et un $*$-homomorphisme injectif, préservant la trace et préfaiblement continu $J : VN(G) \to M$ tels que pour tout $k_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant n$*

$$T_{u_1}^{k_1} \cdots T_{u_n}^{k_n} = \mathbb{E} U_1^{k_1} \cdots U_n^{k_n} J,$$

où $\mathbb{E} : M \to B(L^2(\Sigma))$ est l'espérance conditionnelle associée à J.

Cette thèse repose sur trois articles. La partie 2 et le début de la partie 5 viennent de l'article:

• C.Duquet, *Dilation properties of measurable Schur multipliers and Fourier multipliers*, Positivity. Volume 26, 2022, n.4, Paper No. 69.

La troisième partie est fondée sur l'article:

• C.Duquet and C.Le Merdy, *A characterization of absolutely dilatable Schur Multipliers*, https://arxiv.org/abs/2303.08436, 2023.

Enfin la quatrième partie fera l'objet d'un article en cours d'écriture nommé *Unital positive Schur multipliers on S_n^p with a completely isometric dilation.*

Introduction

General introduction

Akcoglu's dilation theorem [1, 2] asserts that for any $1 < p < \infty$, for any measure space (Σ, μ) and for any positive contraction $T\colon L^p(\Sigma) \to L^p(\Sigma)$, there exist another measure space (Σ', μ'), an isometry $J\colon L^p(\Sigma) \to L^p(\Sigma')$, an isometric isomorphism $U\colon L^p(\Sigma') \to L^p(\Sigma')$ and a contraction $Q\colon L^p(\Sigma') \to L^p(\Sigma)$ such that $T^k = QU^kJ$ for all integer $k \geqslant 0$. This result is of paramount importance in various topics concerning L^p-operators, such as functional calculus, harmonic analysis and ergodic theory. For specific results, see [8, 14, 31, 42, 44] as well as [33, Section I] and the references therein.

In the last two decades, operator theory and harmonic analysis on non-commutative L^p-spaces gained a lot of interest and many classical results concerning $L^p(\Sigma)$-spaces have been transferred to the non-commutative setting. However Akcoglu's theorem does not extend to non-commutative L^p-spaces. More precisely, let N be a von Neumann algebra equipped with a normal semi-finite faithful trace τ. In this situation, we say that (N, τ) is a tracial von Neumann algebra. Let $1 < p < \infty$ and let $L^p(N)$ be the associated non-commutative L^p-space. Let us say that a contraction $T\colon L^p(N) \to L^p(N)$ admits an isometric p-dilation if there exist another tracial von Neumann algebra $(\mathcal{N}, \tau')$, an isometry $J\colon L^p(N) \to L^p(\mathcal{N})$, an isometric isomorphism $U\colon L^p(\mathcal{N}) \to L^p(\mathcal{N})$ and a contraction $Q\colon L^p(\mathcal{N}) \to L^p(N)$ such that $T^k = QU^kJ$ for all integer $k \geqslant 0$.

$$\begin{array}{ccc} L^p(\mathcal{N}) & \xrightarrow{U^k} & L^p(\mathcal{N}) \\ J \uparrow & & \downarrow Q \\ L^p(N) & \xrightarrow{T^k} & L^p(N) \end{array}$$

Then it was shown in [34] that not all positive contractions $T\colon L^p(N) \to L^p(N)$ admit an isometric p-dilation. After this negative result was published, it became important to exhibit classes of positive contractions on non-commutative L^p-spaces which do admit an isometric p-dilation. For results of this type, see for example [35, Chapters 8-10], [4, 5] and [32]. More precisely, in the article [4], C. Arhancet proved that any self-adjoint unital positive Schur multiplier $B(l^2_{\mathbb{Z}}) \to B(l^2_{\mathbb{Z}})$ is absolutely dilatable and that whenever G is discrete group, any self-adjoint unital completely positive Fourier multiplier $VN(G) \to VN(G)$ is absolutely dilatable. Thus it admits an isometric p-dilation.

The notion of absolutely dilatable is the following. Consider (N, τ) as above and let $T\colon N \to N$ be a contraction. We say that T is absolutely dilatable (or admits an absolute dilation) if there exist another tracial von Neumann algebra $(\mathcal{N}, \tau')$, a w^*-continuous trace preserving unital $*$-homomorphim $J\colon N \to \mathcal{N}$ and a trace preserving $*$-automomorphim $U\colon \mathcal{N} \to \mathcal{N}$ such that $T^k = \mathbb{E}U^kJ$ for all $k \geqslant 0$, where $\mathbb{E}\colon \mathcal{N} \to N$ is

the conditional expectation associated with J.

$$\begin{array}{ccc} (\mathcal{N},\tau') & \xrightarrow{U^k} & (\mathcal{N},\tau') \\ J\uparrow & & \downarrow \mathbb{E} \\ (N,\tau) & \xrightarrow{T^k} & (N,\tau) \end{array} \tag{0.0.4}$$

It turns out that if $T\colon N \to N$ is absolutely dilatable, then for all $1 < p < \infty$, T extends to a positive contraction on $L^p(N)$ which admits an isometric p-dilation (see Proposition 1.3.5). Absolute dilations appear implicitly in [4, 5, 41, 28, 35] and many known constructions of isometric p-dilations for specific classes of $L^p(N)$-operators arise from absolute dilations. Also, absolute dilations are closely related to the so-called factorisable operators [3, 28, 56].

Another related dilation property is the completely p-dilatable property. This notion is analogue to the notion isometric p-dilation but J and Q are complete contractions and U is a invertible complete isometry. In addition, if $T : N \to N$ is absolutely dilatable, then for all $1 < p < +\infty$, T extends to a positive completely p-dilatable contraction on $L^p(N)$ (See Proposition 1.3.8).

The purpose of this thesis is to study dilation properties of different classes of operators. In particular we look at Schur multipliers in part 2,3,4 and Fourier multipliers in part 5. In addition, we establish a multi-variable version in each case. In Article [54], results about mutli-variables are proved.

Contents of the thesis

This thesis starts with all definitions and needed properties of Schur Multipliers, Fourier Multipliers and Dilations. Firstly we will focus on Schur multipliers. We prove that any self-adjoint unital positive measurable Schur multiplier admits a absolute dilation. Then we characterize the fact that a measurable Schur multiplier is absolutely dilatable. Next we prove a characterisation for unital, positive, completely p-dilatable Schur multipliers in the finite dimensional case and in the discrete case. In the last part, we study Fourier multiplier. We will prove that any self-adjoint, unital, completely positive Fourier multiplier is absolutely dilatable.

First part

This first chapter is devoted to all tools that we use in our work. Firstly we present the classical non-commutative L^p-Spaces. In this case, we work with a von Neumann algebra equipped with a normal semi-finite faithful trace. Non-commutative L^p-Spaces are a generalisation of the L^p-spaces $L^p(\Sigma)$ where $(\Sigma, \mathcal{F}(\Sigma), \mu)$ a measure space. We also consider von Neumann algebras equipped with a weight, and the non-commutative L^p-spaces associated to them. The construction of these L^p-spaces is based on the crossed product. Properties of these spaces are similar to the ones associated to traces.

In this part, we also present the different types of dilation previously mentioned. Next we give definitions of Schur Multipliers and Fourier Multipliers. We also recall basic properties. The study of bounded Schur multipliers acting on $B(L^2(\Sigma))$, the space of bounded operators on the L^2-space associated with a σ-finite measure space (Σ, μ) goes back at least to Haagerup [26] and Spronk [59]. Further the Haagerup-Musat paper [28]

provides a characterization of absolutely dilatable Schur multipliers in the finite dimensional setting. We present three types of Schur multipliers depending of the considered L^2 space. We recall the definition in measurable case. Let (Σ, μ) be a σ-finite measure space. For any $f \in L^2(\Sigma^2)$, we let $S_f \colon L^2(\Sigma) \to L^2(\Sigma)$ be the bounded operator defined by

$$[S_f(h)](s) = \int_\Sigma f(s,t)h(t)\,dt, \qquad h \in L^2(\Sigma).$$

We recall that S_f is a Hilbert-Schmidt operator, that is, $S_f \in S^2(L^2(\Sigma))$. Further the mapping $f \mapsto S_f$ is a unitary from $L^2(\Sigma^2)$ onto $S^2(L^2(\Sigma))$.

Let $\varphi \in L^\infty(\Sigma^2)$. We define $M_\varphi \colon S^2(L^2(\Sigma)) \to S^2(L^2(\Sigma))$ by setting

$$M_\varphi(S_f) = S_{\varphi f}, \qquad f \in L^2(\Sigma^2).$$

This operator is called a Schur multiplication operator. Furthermore there exists $C > 0$ such that

$$\|M_\varphi(S)\|_{B(L^2(\Sigma))} \leqslant C\|S\|_{B(L^2(\Sigma))}, \qquad S \in S^2(L^2(\Sigma)).$$

We say that M_φ is a measurable Schur multiplier on $B(L^2(\Sigma))$ (or a bounded Schur multiplier).

Another classical family of operators is the family of Fourier multipliers $VN(G) \to VN(G)$ where G is a locally compact group. This topic have a lot of applications to approximation properties of operator algebras, to noncommutative probability and quantum information and to singular integrals and Calderon-Zygmund operators. See [6, 24, 36, 37, 43, 47, 48]. We introduce Fourier multipliers in two cases: in the non unimodular case and in the unimodular case.

Let G be a locally compact group. We let $VN(G) = \lambda(G)'' \subset B(L^2(G))$ be the group von Neumann algebra of G where $\lambda \colon G \to B(L^2(G))$ is the left regular representation of G. We denote the Plancherel weight on $VN(G)$ by ω_G, which is a trace in the unimodular case. We say that a w^*-continuous operator $T : VN(G) \to VN(G)$ is a Fourier multiplier, if there exists a bounded continuous function $u : G \to \mathbb{C}$ such that for all $s \in G$, $T(\lambda(s)) = u(s)\lambda(s)$.

The last part of this chapter is devoted to Fock Spaces. The purpose of this space is to construct a von Neumann algebra from real Hilbert space.

Second part

We mentioned earlier the C.Arhancet article [4]. He proved that any self-adjoint unital positive Schur multiplier from $B(\ell^2_{\mathbb{Z}})$ into $B(\ell^2_{\mathbb{Z}})$ admits an absolute dilation. This article is base on a construction of E. Ricard in [56]. The purpose of the part is to generalize this result to measurable Schur multipliers.

Theorem (A). *Let Σ be a σ-finite measure space. Let $\varphi \in L^\infty(\Sigma^2)$ and assume that M_φ is a self-adjoint, unital, positive Schur multiplier on $B(L^2(\Sigma))$. Then M_φ is absolutely dilatable.*

We will look at the mutli-variable of the previous theorem. We obtain the following theorem.

Theorem (C). *Let $\varphi_1, \dots, \varphi_n \in L^\infty(\Sigma^2)$ and assume each M_{φ_i} is a self-adjoint, unital, positive Schur multiplier on $B(L^2(\Sigma))$. Then there exist a tracial von Neumann algebra (M, τ), a commuting n-tuple $(U_1, \dots, U_n)$ of trace preserving $*$-automorphisms on M and a w^*-continuous trace preserving one-to-one $*$-homomorphism $J : B(L^2(\Sigma)) \to M$ such that*

$$M_{\varphi_1}^{k_1} \cdots M_{\varphi_n}^{k_n} = \mathbb{E} U_1^{k_1} \cdots U_n^{k_n} J$$

for all $k_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant n$, where $\mathbb{E} : M \to B(L^2(\Sigma))$ is the conditional expectation associated with J.

Thus we have a sufficient condition for a Schur multiplier to admit a absolute dilation. The idea of next part is to remove the self-adjoint hypothesis and see what we can obtain.

Third part

The objective of this part is to establish a characterization of absolutely dilatable Schur multipliers $B(L^2(\Sigma)) \to B(L^2(\Sigma))$ in the general case. We introduce a new notion of dilation. We say that an operator $T : N \to N$ admits a separable absolute dilation if N has a separable predual and if T verifies (0.0.4) for some $\mathcal{N}$ with a separable predual.

Let us start with the discrete case. For any $n \geqslant 1$, let M_n denote the space of $n \times n$ matrices with complex entries. Let $\mathfrak{m} = \{m_{ij}\}_{i,j \geqslant 1}$ be a family of complex numbers. Recall that $\mathfrak{m}$ is a bounded Schur multiplier if there exists a (necessarily unique) w^*-continuous operator $M_{\mathfrak{m}} \colon B(\ell^2) \to B(\ell^2)$ such that $M_{\mathfrak{m}}\big([a_{ij}]\big) = [m_{ij} a_{ij}]$ for all $n \geqslant 1$ and all $[a_{ij}] \in M_n \subset B(\ell^2)$. We show that $M_{\mathfrak{m}}$ admits an absolute dilation if and only if there exist a normalized tracial von Neumann algebra (N, τ_N) with separable predual and a sequence $(d_k)_{k \geqslant 1}$ of unitaries of N such that

$$m_{ij} = \tau_N(d_i^* d_j), \qquad i, j \geqslant 1. \tag{0.0.5}$$

In this case, $M_{\mathfrak{m}}$ actually admits a separable absolute dilation, see Corollary 3.5.2.

In the general separable case, spaces of bounded families with entries in a von Neumann algebra N with a separable predual are replaced by spaces $L^\infty_\sigma(\Sigma; N)$ consisting of (classes of) w^*-measurable essentially bounded functions $\Sigma \to N$. These spaces are used e.g. in [58, Theorem 1.22.13] and we have an identification $L^\infty_\sigma(\Sigma; N) \simeq L^\infty(\Sigma) \overline{\otimes} N$, see Section 3.1 for details and information.

Note that is this general case, (0.0.6) replaces (0.0.5).

Theorem. *Assume that (Σ, μ) is separable and let $\varphi \in L^\infty(\Sigma^2)$. The following assertions are equivalent.*

(i) The function φ is a bounded Schur multiplier and $M_\varphi \colon B(L^2(\Sigma)) \to B(L^2(\Sigma))$ admits a separable absolute dilation.

(ii) There exist a normalized tracial von Neumann algebra (N, τ_N) with a separable predual and a unitary $\mathfrak{d} \in L^\infty_\sigma(\Sigma; N)$ such that

$$\varphi(s,t) = \tau_N\big(\mathfrak{d}(s)^* \mathfrak{d}(t)\big), \qquad \text{for a.e. } (s,t) \in \Sigma^2. \tag{0.0.6}$$

We give a version of the previous theorem for Schur multipliers which admit a (not necessarily separable) absolute dilation. This version is in the language of von Neumann tensor products.

Theorem. *Let $\varphi \in L^\infty(\Sigma^2)$ and let $e \in L^2(\Sigma)$ with $\|e\|_2 = 1$. The following assertions are equivalent.*

(i) The function φ is a bounded Schur multiplier and $M_\varphi \colon B(L^2(\Sigma)) \to B(L^2(\Sigma))$ is absolutely dilatable.

(ii) There exist a normalized tracial von Neumann algebra (N, τ_N) acting on some Hilbert space H, and a unitary $D \in L^\infty(\Sigma)\overline{\otimes}B(H)$ such that:

$(ii)_1$ The product $(1_t \otimes D_s^*)(1_s \otimes D_t)$ belongs to $L_s^\infty(\Sigma)\overline{\otimes}L_t^\infty(\Sigma)\overline{\otimes}N$ and we have

$$\varphi = \big[I_{L^\infty(\Sigma^2)}\overline{\otimes}\tau_N\big]\big((1_t \otimes D_s^*)(1_s \otimes D_t)\big).$$

$(ii)_2$ The trace of $D^*(e \otimes e \otimes I_H)D$ in $B(L^2(\Sigma))\overline{\otimes}N$ is equal to 1.

Fourth part

In Section 4.1, we consider dilation properties of unital positive Schur multipliers on matrices. Let $T_M : M_n \to M_n$ be the Schur multiplier defined by $T_M([a_{ij}]_{1\leqslant i,j\leqslant n}) = [m_{ij}a_{ij}]_{1\leqslant i,j\leqslant n}$, for some $A = [a_{ij}]_{1\leqslant i,j\leqslant n}$ in M_n. Assume that T_M is unital and positive. We prove that if T_M (regarded as acting on S_n^p) is completely p-dilatable for some $1 < p \neq 2 < +\infty$, then there exist a tracial normalised von Neumann algebra (N, τ) and unitaries $v_1, \ldots, v_n$ in N such that for all $1 \leqslant i, j \leqslant n$,

$$m_{ij} = \tau(v_i^* v_j).$$

By combining with results of [28], we deduce that any Schur multiplier T_M as before is completely p-dilatable for some $1 \leqslant p \neq 2 < +\infty$ if and only if T_M is absolutely dilatable.

In Section 4.2, we extend the above result to Schur multipliers on $B(l^2(I))$ for any infinite index set I. We obtain this theorem

Theorem. *Let T_M be a unital positive bounded Schur multiplier on $B(l^2(I))$. The following assertions are equivalent:*

1. *there exists $1 < p \neq 2 < +\infty$ such that T_M is completely p-dilatable;*
2. *for all $1 \leqslant p < +\infty$, T_M is completely p-dilatable;*
3. *T_M is absolutely dilatable;*
4. *there exist a tracial normalised von Neumann algebra (N, τ) and a family $(v_i)_{i\in I}$ of unitaries of N such that for all $i, j \in I$:*

$$m_{ij} = \tau(v_i^* v_j).$$

In the last section of this chapter, we consider simultaneous absolute dilations for finite families of Schur multipliers. We obtain that a finite family of Schur multipliers is simultaneously absolutely dilatable if and only if each of these Schur multipliers is absolutely dilatable.

Fifth part

The purpose of the part is to prove that any self-adjoint, unital and completely positive Fourier multiplier on $VN(G)$ is absolutely dilatable. We will treat two different cases. In the first one, we will suppose that G is unimodular locally group. In the second, we remove the unimodular hypothesis. The proof of both cases is quite the same. However, in the non unimodular case, we lose the tracial property of the Plancherel weight. We have to change the notion of absolute dilation to include weighted L^p-spaces. The main theorem is the following:

Theorem (B). *Let $T_u : VN(G) \to VN(G)$ be a self-adjoint, unital, completely positive Fourier multiplier. Then T_u is absolutely dilatable.*

As in the second part, we will generalize above Theorem for any n-tuple of either Fourier multipliers satisfying the assumptions of Theorems **B**. We will establish the following result.

Theorem (D). *Let G be a locally compact group and let $T_{u_1}, \dots, T_{u_n}$ be self-adjoint, unital, completely positive Fourier multipliers on $VN(G)$. Then there exist a tracial von Neumann algebra (M, τ), a commuting n-tuple $(U_1, \dots, U_n)$ of trace preserving $*$-automorphisms on M and a w^*-continuous trace preserving one-to-one $*$-homomorphism $J : VN(G) \to M$ such that*

$$T_{u_1}^{k_1} \cdots T_{u_n}^{k_n} = \mathbb{E} U_1^{k_1} \cdots U_n^{k_n} J$$

for all $k_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant n$, where $\mathbb{E} : M \to B(L^2(\Sigma))$ is the conditional expectation associated with J.

This Thesis is based on three articles. Part 2 and Beginning of Part 5 comes from Article:

• C.Duquet, *Dilation properties of measurable Schur multipliers and Fourier multipliers*, Positivity. Volume 26, 2022, n.4, Paper No. 69.

The third part is based on Article:

• C.Duquet and C.Le Merdy, *A characterization of absolutely dilatable Schur Multipliers*, https://arxiv.org/abs/2303.08436, 2023.

Finally the fourth part will be a future article named *Unital positive Schur multipliers on S_n^p with a completely isometric dilation.*

Chapter 1

Tools

1.1 Notation

For any Banach space X and any $(x, x^*) \in X \times X^*$, we let $\langle x^*, x\rangle$ denote the action of x^* on x. Whenever H is a Hilbert space, we let $\langle \cdot, \cdot \rangle_H$ denote the inner product on H, that we assume linear in the first variable and anti-linear in the second variable.

1.2 Non-commutative L^p-spaces

1.2.1 Tracial L^p-spaces

Let N be a von Neumann algebra. We let N_+ denote its positive part and unless otherwise stated, we let 1_N denote its unit. If N is equiped with a normal semi-finite faithful trace $\tau_N \colon N_+ \to [0, \infty]$, we say that (N, τ_N) is a tracial von Neumann algebra. We further say that (N, τ_N) is normalized if $\tau_N(1_N) = 1$. In this case, τ_N is finite and extends to a normal tracial state on N.

Given any tracial von Neumann algebra (N, τ_N) acting on some Hilbert space H. Let $L^0(N, \tau_N)$ denote the $*$-algebra of all possibly unbounded operators on H which are measurable respect to τ_N. Let x be a closed densely defined operator on H. We let $D(x) \subset H$ denote the domain of x. If x is selfadjoint, then we let $s(x) : H \to H$ denote the support of x, which is equal to the orthogonal projection whose range is $\overline{\mathrm{Im}(x)}$. The map τ_N extends naturally to a functional on the positive part of $L^0(N, \tau_N)$ into $[0, \infty]$. We still denote it by τ_N. Any $x \in L_0(N)$ admits a (unique) polar decomposition $x = u|x|$ with $u \in N$ and $|x| \in L_0(N)$. For $1 \leqslant p < +\infty$, the noncommutative L^p-space associated with (N, τ_N) is defined as

$$L^p(N) := \{x \in L_0(N) : \tau_N(|x|^p) < +\infty\}.$$

This is a Banach space for the norm $\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}$. In addition we recall that $L^p(N, \tau_N)_+ = L^p(N, \tau_N) \cap L_0(N, \tau_N)_+$ and for all elements x of $L^p(N, \tau_N)$, x^* belongs to $L^p(N, \tau_N)$. We further set $L^\infty(N) = N$. It turns out that $N \cap L^p(N, \tau_N)$ is dense in $L^p(N, \tau_N)$.

The trace τ_N is finite on $L^1(N, \tau_N) \cap N_+$ and extends to a functional on $L^1(N, \tau_N)$, still denoted by τ_N. Let $1 \leqslant p < \infty$ and let $p' = \frac{p}{p-1}$ be its conjugate index. For any

$x \in L^p(N,\tau_N)$ and any $y \in L^{p'}(N,\tau_N)$, the product xy belongs to $L^1(N,\tau_N)$ and we have a Hölder inequality $|\tau_N(xy)| \leqslant \|x\|_p \|y\|_{p'}$. We further have an isometric isomorphism

$$L^p(N,\tau_N)^* \simeq L^{p'}(N,\tau_N),$$

for the duality pairing given by $\langle y, x\rangle = \tau_N(xy)$ for all $x \in L^p(N,\tau_N)$ and $y \in L^{p'}(N,\tau_N)$. In particular,

$$N \simeq L^1(N,\tau_N)^*. \tag{1.2.1}$$

The space $L^2(N,\tau_N)$ is a Hilbert space, with inner product given by $\langle x, y\rangle_{L^2(N,\tau_N)} = \tau_N(xy^*)$ for all $x, y \in L^2(N,\tau_N)$.

Let H be a Hilbert space and let $B(H)$ denote the von Neumann algebra of all bounded operators on H. Let $\mathrm{tr}\colon B(H)_+ \to [0,\infty]$ denote the usual trace. Then for all $1 \leqslant p < \infty$, $L^p(B(H),\mathrm{tr})$ is equal to the Schatten p-class $S^p(H)$. We note that $S^2(H)$ coincides with the space of Hilbert-Schmidt operators on H. Let I be an index set. If $H = l^2(I)$ then we denote the Schatten p-class $S^p(l^2(I))$ by S^p_I for all $1 \leqslant p < \infty$.

The reader is referred to [62, Section V.2] for information on traces and to [52] and [30, Section 4] for details on non-commutative L^p-spaces.

Lemma 1.2.1. *Let (N,τ) be a tracial von Neumann algebra, let $x \in L_0(N)$ be a positive element and let $z \in s(x)Ns(x)$ such that $xzx = 0$. Then $z = 0$.*

Proof. Changing N into $s(x)Ns(x)$, we may assume that $s(x) = 1$. This means that $\ker(s) = \{0\}$. Let $z \in N$ such that $xzx = 0$. For any $\xi \in D(xzx)$, we have,

$$xzx(\xi) = 0,$$

hence $zx(\xi) = 0$. By density of $D(xzx)$ in $D(zx)$, this implies that $zx = 0$. We deduce that $xz^* = 0$. Then arguing as above, we obtain that $z^* = 0$, and hence $z = 0$. □

Remark 1.2.2. Let $(\Sigma,\mathcal{F},\mu)$ be a σ-finite measure space. The space $(L^\infty(\Sigma),\int_\Sigma)$ is a tracial von Neumann algebra. For any $1 \leqslant p \leqslant +\infty$, the non-commutative L^p-space $L^p(L^\infty(\Sigma))$ coincides with the usual $L^p(\Sigma)$.

Let (N,τ_N) and (N_1,τ_{N_1}) be two tracial von Neumann algebras and let $U\colon N \to N_1$ be a positive map, that is, $U(N_+) \subset (N_1)_+$. We say that U is trace preserving if $\tau_{N_1} \circ U = \tau_N$ on N_+. If U is a trace preserving $*$-homomorphism, then U is 1-1 (because τ_N is faithful) and the following is well-known.

Lemma 1.2.3. *Let $(N,\tau),(N_1,\tau')$ be two tracial von Neumann algebras and let $T : N \to N_1$ be a positive trace preserving contraction. Then for all $1 \leqslant p < \infty$, there exists a necessarily unique contraction $T_p : L^p(N,\tau) \to L^p(N_1,\tau')$ such that for all $x \in N \cap L^p(N,\tau)$, $T(x) = T_p(x)$.*

If further T is a one-to-one $*$-homomorphism, then T_p is an isometry. Moreover T_p is onto if T is onto.

We refer to [67, Theorem 2] for the last fact and to [39, Lemma 1.1] for complements.

1.2.2 Weighted L^p-spaces

In the sequel, we will mostly follow [27]. We call weighted von Neumann algebra any pair (M, φ), where M is a von Neumann algebra acting on a Hilbert space H and φ is a normal semi-finite faithful weight (n.s.f in short) on M. We denote the modular automorphism group of the weight φ by $\sigma^\varphi := (\sigma_s^\varphi)_{s\in\mathbb{R}}$ (see [63, Section VIII.1]). We will work on Haagerup non-commutative L^p-spaces. We denote the crossed product $M \rtimes_{\sigma^\sigma} \mathbb{R}$ of M by $\mathbb{R}$ with respect to the modular automorphism group σ^φ by $\mathcal{R}$. $\mathcal{R}$ is the von Neumann subalgebra of $B(L^2(\mathbb{R}, H))$ generated by the operators $\pi_{\sigma^\varphi}(x)$, $x \in M$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by

$$(\pi_{\sigma^\varphi}(x)\xi)(s) := (\sigma_s^\varphi)^{-1}(x)(\xi(s)) \text{ and } (\lambda(s)(\xi))(t) := \xi(t-s),\ s,t \in \mathbb{R},\ \xi \in L^2(\mathbb{R}, H).$$

These operators satisfy the following commutation relation:

$$\lambda(s)\pi_{\sigma^\varphi}(x)\lambda(s)^* = \pi_{\sigma^\varphi}(\sigma_s^\varphi(x)),\ x \in M\ s \in \mathbb{R}.$$

For all $s \in \mathbb{R}$, we let $w(s)$ be the unitary operator on $L^2(\mathbb{R}, H)$ defined by:

$$(w(s)\xi)(t) := e^{-ist}\xi(t),\ \xi \in L^2(\mathbb{R}, H),\ t \in \mathbb{R}.$$

Following [63, Definition X.2.4], we denote by $\hat{\sigma} : \mathbb{R} \to Aut(\mathcal{R})$ the dual action defined by:

$$\hat{\sigma}_s(x) := w(s)xw(s)^*,\ x \in \mathcal{R}\ s \in \mathbb{R}.$$

By [63, Theorem X.2.3], $\pi_{\sigma^\varphi}(M)$ is exactly the fixed point algebra of $\mathcal{R}$ under $\hat{\sigma}$:

$$\pi_{\sigma^\varphi}(M) = \{x \in \mathcal{R}\ ;\ \hat{\sigma}_s(x) = x,\ \text{for all } s \in \mathbb{R}\}\,. \tag{1.2.2}$$

From now on, we identify M and $\pi_{\sigma^\varphi}(M) \subset \mathcal{R}$. Let ψ be a normal semifinite weight on M. We denote the dual weight on the crossed product $\mathcal{R}$ of ψ by $\hat{\psi}$ (see [63, definition X1.16]). By [25, Lemma 5.2 and Remark p.343], the crossed product $\mathcal{R}$ is semifinite and there exists a unique normal semi-finite faithful trace τ on $\mathcal{R}$ such that

$$(D\hat{\varphi} : D\tau)_s = \lambda(s),\ s \in \mathbb{R}, \tag{1.2.3}$$

where $(D\hat{\varphi} : D\tau)_s$ denotes the Radon-Nikodym cocycle of $\hat{\varphi}$ with respect to τ (see [63, Definition VIII.3.4]). In addition, by [25, Lemma 5.2], we have for all $s \in \mathbb{R}$,

$$\tau \circ \hat{\sigma}_s = e^{-s}\tau.$$

We denote the Radon-Nikodym derivative of the dual weight $\hat{\psi}$ respect to τ by h_ψ (see [65, Definition II.3]). Then we have the following relation,

$$\hat{\psi}(x) = \tau(h_\psi x),\ x \in \mathcal{R}_+.$$

By [65, Proposition II.4], the map $\mathfrak{i}\colon \psi \to h_\psi$ is a bijection from the set of all normal semifinite weights on M onto the set of all positive selfadjoint operators affiliated with $\mathcal{R}$ satisfying

$$\hat{\sigma}_s(h_\psi) = e^{-s}h_\psi,\ s \in \mathbb{R}.$$

We set $D_\varphi := h_\varphi$. By Equality (1.2.3), we have for all $s \in \mathbb{R}$,

$$\lambda(s) = D_\varphi^{is}.$$

We let $L_0(\mathcal{R}, \tau)$ be the $\star$-algebra of all unbounded operators on $L^2(\mathbb{R}, H)$ which are measurable respect to τ. Let $1 \leqslant p \leqslant +\infty$, we define the Haagerup L^p-space associated with (M, φ) by

$$L^p(M, \varphi) := \left\{ x \in L_0(\mathcal{R}, \tau) \ , \ \hat{\sigma}_s^\varphi(x) = e^{-\frac{s}{p}} x, \ \forall s \in \mathbb{R} \right\}.$$

By (1.2.2), M and $L^\infty(M, \varphi)$ coincide. The spaces $L^p(M, \varphi)$ are self-adjoint closed spaces of $L_0(\mathcal{R}, \tau)$. We also have $L^p(M, \varphi)_+ = L^p(M, \varphi) \cap L_0(\mathcal{R}, \tau)_+$. By [65, Theorem II.7], the restriction of $\mathfrak{i}$ on $(M_*)_+$ induces a one to one mapping $\mathfrak{j} \colon w \to h_w$ from $(M_*)_+$ onto $L^1(M, \varphi)_+$. In addition the mapping $\mathfrak{j}$ extends to a linear homeomorphism from M_* onto $L^1(M, \varphi)$. For any $x \in L^1(M, \varphi)$, we define the norm of x by $\|x\|_1 := \|\mathfrak{j}^{-1}(x)\|_{M_*}$. Moreover, we define $\mathrm{tr} : L^1(M, \varphi) \to \mathbb{C}$ by

$$\mathrm{tr}(h_w) = w(1) \ , \ w \in M_*.$$

This is a positive and contractive functional. Then for all $h \in L^1(M, \varphi)$, $\|h\|_1 = \mathrm{tr}(|h|)$. Let $1 \leqslant p < +\infty$, if $x = u|x|$ is the polar decomposition of $x \in L_0(\mathcal{R}, \tau)$, then by [65, Proposition II.12], we have

$$x \in L^p(M, \varphi) \iff u \in M \text{ and } |x| \in L^p(M, \varphi) \iff u \in M \text{ and } |x|^p \in L^1(M, \varphi).$$

We set for all $x \in L^p(M, \varphi)$

$$\|x\|_p = \||x|^p\|_1^{\frac{1}{p}}.$$

It defines a norm on $L^p(M, \varphi)$. The spaces $L^p(M, \varphi)$ are Banach spaces.

Let $1 \leqslant p, p' \leqslant +\infty$, with $\frac{1}{p} + \frac{1}{p'} = 1$, for all $x \in L^p(M, \varphi)$ and for all $y \in L^{p'}(M, \varphi)$, we have $xy, yx \in L^1(M, \varphi)$ and $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ (see [65, Proposition II.21]). Moreover, if $1 \leqslant p < +\infty$, the bilinear from $L^p(M, \varphi) \times L^{p'}(M, \varphi) \to \mathbb{C}$, $(x, y) \mapsto \mathrm{tr}(xy)$ defines a duality bracket which induces an isometric isomorphism

$$L^p(M, \varphi)^* \simeq L^{p'}(M, \varphi).$$

For the rest of this subsection, (M, φ) and (M_1, φ') are both weighted von Neumann algebra. We define these three classical spaces:

$$\begin{aligned} \mathcal{P}_\varphi^M &:= \{x \in M_+, \ \varphi(x) < +\infty\}; \\ \mathcal{N}_\varphi^M &:= \{x \in M, \ \varphi(x^*x) < +\infty\}; \\ \mathcal{M}_\varphi^M &:= \mathrm{span}\{y^*x : \ x, y \in \mathcal{N}_\varphi^M\}. \end{aligned}$$

We recall some properties of these three spaces in the following lemma.

Lemma 1.2.4.

1. *$\mathcal{N}_\varphi^M$ is a left ideal of M;*
2. *$\mathcal{M}_\varphi^M$ is a $*$-subalgebra of M such that $\mathcal{M}_\varphi^M \cap M_+ = \mathcal{P}_\varphi^M$ and every element of $\mathcal{M}_\varphi^M$ is a linear combination of four elements of $\mathcal{P}_\varphi^M$;*

3. $\mathcal{M}_\varphi^M \subset \mathcal{N}_\varphi^M \cap \left(\mathcal{N}_\varphi^M\right)^*$;

4. $\mathcal{M}_\varphi^M$ *is* w^**-dense in* M.

Proof. By [63, Definition VII.1.1 and Lemma VII.1.2], we have 1 and 2. By Definition of $\mathcal{M}_\varphi^M$ and 1, we deduce that $\mathcal{M}_\varphi^M \subset \mathcal{N}_\varphi^M \cap \left(\mathcal{N}_\varphi^M\right)^*$. The fourth one comes from the semi-finite property of φ. □

The weight φ can be extended to a functional on $\mathcal{M}_\varphi^M$. We still denote it by φ. We give two properties relating $\mathcal{M}_\varphi^M$ and $D_\varphi^{\frac{1}{2}}$ (cf e.g. [27, end of the page 2130 and Remark 5.6]).

Lemma 1.2.5.

1. $\forall x \in \mathcal{M}_\varphi^M \ \varphi(x) = tr(D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}})$;

2. *the space* $D_\varphi^{\frac{1}{2p}} \mathcal{M}_\varphi^M D_\varphi^{\frac{1}{2p}}$ *is dense in* $L^p(M,\varphi)$.

We say that a positive operator $T : (M,\varphi) \to (M_1,\varphi')$ between two weighted von Neumann algebras is weight preserving, if for all $x \in M_+ \cap \mathcal{M}_\varphi^M$, we have $\varphi'(T(x)) = \varphi(x)$. Spaces $\mathcal{N}_\varphi^M$ and $\mathcal{M}_\varphi^M$ are stable by weight preserving $*$-homomorphisms.

Lemma 1.2.6. *Let* (M,φ) *and* (M_1,φ') *be two weighted von Neumann algebras. Let* $J : (M,\varphi) \to (M_1,\varphi')$ *be a weight-preserving* $*$*-homomorphism. Then*

1. $m \in \mathcal{N}_\varphi^M \Rightarrow J(m) \in \mathcal{N}_{\varphi'}^{M_1}$;

2. $m \in \mathcal{M}_\varphi^M \Rightarrow J(m) \in \mathcal{M}_{\varphi'}^{M_1}$.

Proof. The proof is straightforward. □

We recall the definition of the centralizer of a weight (cf [63, Definition VIII.2.1].

$$\mathcal{M}_c^\varphi = \{x \in M : \sigma_s^\varphi(x) = x, \ \forall s \in \mathbb{R}\}.$$

This space is a von Neumann subalgebra of M and we have the following characterization (cf [63, Theorem VIII.2.6]).

Theorem 1.2.7. *Let* $a \in M$. *The following assertions are equivalent:*

1. $a \in \mathcal{M}_c^\varphi$;

2. (a) $a\mathcal{M}_\varphi^M \subset \mathcal{M}_\varphi^M$ *and* $\mathcal{M}_\varphi^M a \subset \mathcal{M}_\varphi^M$ *(*a *is a multiplier of* $\mathcal{M}_\varphi^M$*);*
   (b) $\varphi(az) = \varphi(za)$, $z \in \mathcal{M}_\varphi^M$.

Theorem 1.2.8. *Let* (M,φ) *and* (M_1,φ') *be two weighted von Neumann algebras. Let* $U : (M,\varphi) \to (M_1,\varphi')$ *be a weight preserving* $*$*-isomorphism. Then*

$$U(\mathcal{M}_c^\varphi) \subset \mathcal{M}_c^{\varphi'}$$

Proof. Let $x \in \mathcal{M}_c^{\varphi}$. By Theorem 1.2.7, we just have to prove that for all $y \in \mathcal{M}_{\varphi'}^{M_1}$, $U(x)y \in \mathcal{M}_{\varphi'}^{M_1}$, $yU(x) \in \mathcal{M}_{\varphi'}^{M_1}$ and $\varphi'(U(x)y) = \varphi'(yU(x))$. Firstly we observe that $U(x)y = U(xU^{-1}(y))$. By Lemma 1.2.6 and Theorem 1.2.7, we deduce $U(xU^{-1}(y)) \in \mathcal{M}_{\varphi'}^{M_1}$. Thus $yU(x) \in \mathcal{M}_{\varphi'}^{M_1}$. Likewise $U(x)y \in \mathcal{M}_{\varphi'}^{M_1}$. It remains to prove that $\varphi'(U(x)y) = \varphi'(yU(x))$. By weight preserving property of U, we obtain

$$\varphi'(U(x)y) = \varphi'(U(xU^{-1}(y))) = \varphi(xU^{-1}(y))$$

We recall that $U^{-1}(y) \in \mathcal{M}_{\varphi}^{M}$ and $x \in \mathcal{M}_c^{\varphi}$. By Theorem 1.2.7, the latter is equal to

$$\varphi(U^{-1}(y)x) = \varphi'(U(U^{-1}(y)x)) = \varphi'(yU(x)).$$

The result follows. □

We need one more property. We say that an element $x \in M$ is entire if the function $s \in \mathbb{R} \mapsto \sigma_s(x) \in M$ can be extended to a analytic function from $\mathbb{C}$ into M. We denote the set of all entire elements of M by M_a (see [63, Definition VIII.2.2]). This space is called the analytic part of M. Let

$$\begin{aligned}\mathcal{N}_{\varphi}^{\infty} &:= \{x \in M_a,\ \sigma_s(x) \in \mathcal{N}_{\varphi}^{M} \cap \left(\mathcal{N}_{\varphi}^{M}\right)^*, \text{ for all } s \in \mathbb{C}\};\\ \mathcal{M}_{\varphi}^{\infty} &:= \text{span}\{y^*x :\ x, y \in \mathcal{N}_{\varphi}^{\infty}\}.\end{aligned}$$

By [23, end of p.46] and Lemma 1.2.4, $\mathcal{N}_{\varphi}^{\infty}$ and $\mathcal{M}_{\varphi}^{\infty}$ are $*$-subalgebras of M such that $\mathcal{N}_{\varphi}^{\infty} \subset \mathcal{N}_{\varphi}^{M} \cap \left(\mathcal{N}_{\varphi}^{M}\right)^* \cap M_a$, $\mathcal{M}_{\varphi}^{\infty} \subset \mathcal{M}_{\varphi}^{M} \cap \mathcal{N}_{\varphi}^{\infty}$. Moreover there are both invariant under $\{\sigma_s :\ s \in \mathbb{C}\}$.

These two new spaces work like $\mathcal{N}_{\varphi}^{M}$ and $\mathcal{M}_{\varphi}^{M}$ in terms of density (cf [23, Theorem 2.4 and Proposition 2.11]).

Lemma 1.2.9.

1. *$\mathcal{N}_{\varphi}^{\infty}$ and $\mathcal{M}_{\varphi}^{\infty}$ are w^*-dense in M;*

2. *the space $D_{\varphi}^{\frac{1}{2p}} \mathcal{M}_{\varphi}^{\infty} D_{\varphi}^{\frac{1}{2p}}$ is dense in $L^p(M, \varphi)$ $(1 \leqslant p < +\infty)$.*

We state [23, Proposition 2.13] in our setting.

Lemma 1.2.10.

1. *$tr(xD_{\varphi}^{\frac{1}{2}} y D_{\varphi}^{\frac{1}{2}}) = \varphi(\sigma_{\frac{i}{2}}(y)x)$ for all $x \in \mathcal{N}_{\varphi}^{M}$ and $y \in \mathcal{M}_{\varphi}^{\infty}$;*

2. *$tr(xD_{\varphi}^{\frac{1}{2}} y D_{\varphi}^{\frac{1}{2}}) = \varphi(x\sigma_{-\frac{i}{2}}(y))$ for all $x \in \left(\mathcal{N}_{\varphi}^{M}\right)^*$ and $y \in \mathcal{M}_{\varphi}^{\infty}$.*

We give a version of Lemma 1.2.6 for $\mathcal{M}_{\varphi}^{\infty}$.

Lemma 1.2.11. *Let (M, φ) and (M_1, φ') be two weighted von Neumann algebras. Let $J : (M, \varphi) \to (M_1, \varphi')$ be a weight preserving $*$-homomorphism such that*

$$\sigma_s^{\varphi'}(J(m)) = J(\sigma_s^{\varphi}(m))\ s \in \mathbb{R} \text{ and } m \in M. \tag{1.2.4}$$

Then

1. $m \in \mathcal{N}_\varphi^\infty \Rightarrow J(m) \in \mathcal{N}_{\varphi'}^\infty$;

2. $m \in \mathcal{M}_\varphi^\infty \Rightarrow J(m) \in \mathcal{M}_{\varphi'}^\infty$.

Proof. Let $m \in \mathcal{N}_\varphi^\infty \subset M_a$. By (1.2.4) and linearity of J, we deduce that $J(m) \in (M_1)_a$. Let $s \in \mathbb{C}$, we have that

$$\begin{aligned}\varphi'(\sigma_s^{\varphi'}(J(m))^*\sigma_s^{\varphi'}(J(m))) &= \varphi'(J(\sigma_s^{\varphi}(m)^*\sigma_s^{\varphi}(m))) \\ &= \varphi(\sigma_s^{\varphi}(m)^*\sigma_s^{\varphi}(m)) < +\infty\end{aligned}$$

We obtain $\sigma_s^{\varphi'}(J(m)) \in \mathcal{N}_\varphi^{M_1}$. With the same computation, we have also $\sigma_s^{\varphi'}(J(m)) \in \left(\mathcal{N}_\varphi^{M_1}\right)^*$. Thus $J(m) \in \mathcal{N}_{\varphi'}^\infty$.

By linearity of J, we deduce the second point. □

Remark 1.2.12. 1. $L^p(M,\varphi) \cap L^q(M,\varphi) = \{0\}$ if and only if $p \neq q$.

2. $L^p(M,\varphi)$ is independent of φ up to an isometric isomorphism preserving the order and modular structure of $L^p(M,\varphi)$ (see [65, p.59] for details).

3. If φ is tracial, then we have $\sigma^\varphi = id$ and

$$\mathcal{R} = M \rtimes_\sigma \mathbb{R} = VN(\mathbb{R})\overline{\otimes}M = L^\infty(\mathbb{R})\overline{\otimes}M.$$

The last equality is obtained via the Fourier transform. The trace τ becomes $\int . e^t dt \otimes \varphi$. For any $1 \leqslant p < +\infty$, we denote $L^p(M)$ the L^p-space of Subsection 1.2.1. We have

$$L^p(M,\varphi) = \{t \mapsto e^{-\frac{t}{p}} y : y \in L^p(M)\} = e^{-\frac{\cdot}{p}} \otimes L^p(M),$$

with $\|e^{-\frac{t}{p}} \otimes y\|_{L^p(M,\varphi)} = \|y\|_{L^p(M)}$ (see [30, Remark 8.16 and Example 9.11]).

1.2.3 Complete properties

We recall definitions of a complete contraction, a complete isometry and a completely positive map.

For any $n \geqslant 1$ and any $1 \leqslant p \leqslant \infty$, we identify $L^p(M_n(N))$ with $S_n^p \otimes L^p(N)$ in the usual way. Let $T : L^p(M) \to L^p(N)$ be a linear map, where M and N are both tracial von Neumann algebras. For any $n \geqslant 1$, let $T_n : L^p(M_n(M)) \to L^p(M_n(N))$ be defined by

$$T_n([a_{ij}]) = [T(a_{ij})], \qquad [a_{ij}] \in L^p(M_n(M)).$$

We say that T is a complete contraction if for all $n \geqslant 1$, $\|T_n\|_{L^p(M_n(M)) \to L^p(M_n(N))} \leqslant 1$. If $T^* : L^{p'}(N) \to L^{p'}(M)$ denotes the adjoint of T and if T is a complete contraction, then T^* is a complete contraction. This follows from the fact that for any $n \geqslant 1$, $(T^*)_n = (T_n)^*$. In the same way, T is called a complete isometry if for all $n \geqslant 1$, T_n is an isometry. We further say that T is completely positive if for all $n \geqslant 1$, T_n is positive.

The previous definitions also work in the framework of weighted von Neumann algebras.

1.3 Dilation types

In the framework of tracial von Neumann algebras, the appropriate notion of dilation is the following.

Definition 1.3.1. *Let (N,τ) be a tracial von Neumann algebra and let $1 \leqslant p < \infty$. We say that an operator $T : L^p(N,\tau) \to L^p(N,\tau)$ admits an isometric p-dilation if there exist a tracial von Neumann algebra (N_1,τ'), two contractions $J : L^p(N,\tau) \to L^p(N_1,\tau')$, $Q : L^p(N_1,\tau') \to L^p(N,\tau)$ and an invertible isometry $U : L^p(N_1,\tau') \to L^p(N_1,\tau')$, such that for all $k \geqslant 0$, $T^k = QU^kJ$.*

$$\begin{array}{ccc} L^p(N_1,\tau') & \xrightarrow{U^k} & L^p(N_1,\tau') \\ J\uparrow & & \downarrow Q \\ L^p(N,\tau) & \xrightarrow{T^k} & L^p(N,\tau) \end{array}$$

1.3.1 Absolute dilations

Definition 1.3.2. *Let (N,τ_N) be a tracial von Neumann algebra. We say that an operator $T\colon N \to N$ is absolutely dilatable, or admits an absolute dilation, if there exist another tracial von Neumann algebra (N_1,τ_{N_1}), a w^*-continuous trace preserving unital $*$-homomorphism $J\colon N \to N_1$ and a trace preserving $*$-automorphism $U\colon N_1 \to N_1$ such that*

$$T^k = J_1^* U^k J, \qquad k \geqslant 0. \tag{1.3.1}$$

Here $J_1\colon L^1(N,\tau_N) \to L^1(N_1,\tau_{N_1})$ stands for the isometric extension of J and $J_1^*\colon N_1 \to N$ is defined using (1.2.1).

If N has a separable predual, we say that $T\colon N \to N$ admits a separable absolute dilation if the above property holds with some N_1 with a separable predual.

$$\begin{array}{ccc} (N_1,\tau_{N_1}) & \xrightarrow{U^k} & (N_1,\tau_{N_1}) \\ J\uparrow & & \downarrow J_1^* \\ (N,\tau_N) & \xrightarrow{T^k} & (N,\tau_N) \end{array}$$

Remark 1.3.3. Here are a few comments about the above definition.

(1) Let $J\colon N \to N_1$ be a trace preserving $*$-homomorphism. Then for all $x \in N$ and all $y \in N \cap L^1(N,\tau_N)$, we have

$$\langle x,y\rangle = \tau_N(xy) = \tau_{N_1}(J(xy)) = \tau_{N_1}(J(x)J_1(y)) = \langle J(x),J_1(y)\rangle.$$

This implies that $J_1^*J = I_N$, the identity operator on N. Hence the relation (1.3.1) for $k = 0$ is automatic. Thus (1.3.1) is significant for $k \geqslant 1$ only. See Remark 3.4.4 for more about this.

(2) If $J\colon N \to N_1$ is a w^*-continuous trace preserving unital $*$-homomorphism, then N can be regarded (through J) as a von Neumann subalgebra of N_1 whose trace τ_N is the restriction of τ_{N_1}. In this respect, $J_1^*\colon N_1 \to N$ is the natural conditional expectation onto N (see e.g. [62, Proposition V.2.36]). In particular, J_1^* is positive, trace preserving and unital. In the sequel, we call it the conditional expectation associated with J.

(3) If $T\colon N \to N$ is absolutely dilatable, then T is necessarily positive, trace preserving and unital. This follows from the factorization $T = J_1^*UJ$ and the previous paragraph.

(4) Assume that (N, τ_N) is normalized. According to [28, Theorem 4.4], an operator $T\colon N \to N$ is factorisable with respect to τ_N, in the sense of [3], if and only if it is absolutely dilatable and the tracial von Neumann algebra (N_1, τ_{N_1}) in Definition 1.3.2 can be chosen normalized. Thus, absolute dilatability should be regarded as a non-normalized version of factorisability.

(5) A von Neumann algebra N has a separable predual if and only if there exists a separable Hilbert space H such that $N \subset B(H)$ as a von Neumann subalgebra.

By Remark (2) 1.3.3, we may rewrite Definition 1.3.2, as follows.

Definition 1.3.4. *We say that an operator $T : (N, \tau) \to (N, \tau)$ is absolutely dilatable if there exist a tracial von Neumann algebra (N_1, τ'), a w^*-continuous trace preserving unital $*$-homomorphism $J : (N, \tau) \to (N_1, \tau')$ and a trace preserving $*$-automorphism $U : (N_1, \tau') \to (N_1, \tau')$ such that*

$$T^k = \mathbb{E}U^k J, \qquad k \geqslant 0, \tag{1.3.2}$$

where $\mathbb{E} : N_1 \to N$ is the the conditional expectation associated with J.

Proposition 1.3.5. *Let $T\colon N \to N$ be an absolutely dilatable operator and let $1 \leqslant p < \infty$. Then the restriction of T to $N \cap L^p(N)$ extends to a positive contraction $T_p\colon L^p(N, \tau_N) \to L^p(N, \tau_N)$ and the latter admits an isometric p-dilation.*

Proof. Let (N_1, τ_{N_1}), J and U as in Definition 1.3.2. Then $T_p = J_{p'}^* U_p J_p$ is a positive contraction which coincides with T on $N \cap L^p(N)$. Furthermore, we have $T_p^k = J_{p'}^* U_p^k J_p$ for all $k \geqslant 0$. Since J_p and U_p are isometries, we obtain that T_p admits an isometric p-dilation. □

1.3.2 Complete dilations

Now we define the completely p-dilatable property.

Definition 1.3.6. *Let (N, τ) be a tracial von Neumann algebra and let $1 \leqslant p < \infty$. We say that an operator $T : L^p(N, \tau) \to L^p(N, \tau)$ is completely p-dilatable if there exist a tracial von Neumann algebra (N_1, τ'), two complete contractions $J : L^p(N, \tau) \to L^p(N_1, \tau')$ and $Q : L^p(N_1, \tau') \to L^p(N, \tau)$, and an invertible complete isometry $U : L^p(N_1, \tau') \to L^p(N_1, \tau')$, such that for all $k \geqslant 0$, $T^k = QU^k J$.*

If $T : N \to N_1$ is a completely positive, trace preserving contraction and if we apply Lemma 1.2.3 to $T_n : M_n(N) \to M_n(N_1)$ for all $n \geqslant 1$, then we obtain the following analogue of Lemma 1.2.3 for completely positive maps.

Lemma 1.3.7. *Let $(N, \tau), (N_1, \tau')$ be two tracial von Neumann algebras and let $T : N \to N_1$ be a completely positive trace preserving contraction. Then for all $1 \leqslant p < \infty$, there exists a necessarily unique complete contraction $T_p : L^p(N, \tau) \to L^p(N_1, \tau')$ such that for all $x \in N \cap L^p(N, \tau)$, $T(x) = T_p(x)$.*

If further T is a one-to-one $\star$-homomorphism, then T_p is a complete isometry.

Assume that $J : (N, \tau) \to (N_1, \tau')$ is a unital one-to-one trace preserving and w^*-continuous $\star$-homomorphism. Let $J_1 : L^1(N, \tau) \to L^1(N_1, \tau')$ be induced by J, according to Lemma 1.3.7.

If T satisfies Definition 1.3.2, then applying Lemma 1.3.7 we obtain that for all $1 \leqslant p < \infty$, J (resp. $\mathbb{E}$) induces a complete contraction $L^p(N,\tau) \to L^p(N_1,\tau')$ (resp. $L^p(N_1,\tau') \to L^p(N,\tau)$) and that U induces an invertible complete isometry $L^p(N_1,\tau') \to L^p(N_1,\tau')$. Moreover (1.3.2) holds true on L^p-spaces. We therefore obtain the following strengthening of Proposition 1.3.5.

Proposition 1.3.8. *If $T : (N,\tau) \to (N,\tau)$ is absolutely dilatable, then for every $1 \leqslant p < \infty$, T induces a contraction $T_p : L^p(N,\tau) \to L^p(N,\tau)$ and T_p is completely p-dilatable.*

1.3.3 Absolute dilations on weighted von Neumann algebras

On weighted von Neumann algebras, the appropriate notion of dilation is the following.

Definition 1.3.9. *Let (M,φ) be a weighted von Neumann algebra and let $1 \leqslant p < \infty$. We say that an operator $T : L^p(M,\varphi) \to L^p(M,\varphi)$ admits an isometric p-dilation if there exist a weighted von Neumann algebra (M_1,φ'), two contractions $J : L^p(M,\varphi) \to L^p(M_1,\varphi')$, $Q : L^p(M_1,\varphi') \to L^p(M,\varphi)$ and an invertible isometry $U : L^p(M_1,\varphi') \to L^p(M_1,\varphi')$, such that for all $k \geqslant 0$, $T^k = QU^kJ$.*

Remark 1.3.10. We have two notions of isometric p-dilation but the framework is different. In the tracial case, both definitions are possible but we will keep the definition of Section 1.3.1.

For this subsection, (M,φ) and (M_1,φ') are both weighted von Neumann algebra.

We combine Remark 5.6 and Theorem 5.1 of [27] to obtain the following Theorem.

Theorem 1.3.11. *Let $T : (M,\varphi) \to (M_1,\varphi')$ be a weight-preserving completely positive operator then, for all $1 \leqslant p < +\infty$, the map $T_p : D_\varphi^{\frac{1}{2p}} x D_\varphi^{\frac{1}{2p}} \mapsto D_{\varphi'}^{\frac{1}{2p}} T(x) D_{\varphi'}^{\frac{1}{2p}}$ from $D_\varphi^{\frac{1}{2p}} \mathcal{M}_\varphi^M D_\varphi^{\frac{1}{2p}}$ into $D_{\varphi'}^{\frac{1}{2p}} \mathcal{M}_{\varphi'}^{M_1} D_{\varphi'}^{\frac{1}{2p}}$ extends into an operator from $L^p(M,\varphi)$ into $L^p(M_1,\varphi')$ (denote again T_p) such that $\|T_p\| \leqslant \|T\|_{M\to M_1}$.*

We give also a generalised version of [27, Proposition 5.4]. This generalization was communicated to me by Q. Xu.

Lemma 1.3.12. *Let $T : (M,\varphi) \to (M_1,\varphi')$ be a weight-preserving positive operator and T_1 be as in the previous theorem.*

1. *The map $T_1^* : (M_1,\varphi') \to (M,\varphi)$ is positive.*
2. *If T is unital, then T_1^* is weight preserving.*

As in the tracial case, we have the following result.

Lemma 1.3.13. *Let π be a weight preserving, $*$-isomorphism from M onto M_1; then for any $1 \leqslant p < +\infty$, the map $\pi_p : L^p(M,\varphi) \to L^p(M_1,\varphi')$ is isometric surjective.*

Proof. For any $x \in \mathcal{M}_{\varphi'}^M$, we have;

$$\begin{aligned}
\pi_p(\pi^{-1})_p(D_{\varphi'}^{\frac{1}{2p}} x D_{\varphi'}^{\frac{1}{2p}}) &= \pi_p(D_\varphi^{\frac{1}{2p}} \pi^{-1}(x) D_\varphi^{\frac{1}{2p}}) \\
&= D_{\varphi'}^{\frac{1}{2p}} \pi\pi^{-1}(x) D_{\varphi'}^{\frac{1}{2p}} \\
&= D_{\varphi'}^{\frac{1}{2p}} x D_{\varphi'}^{\frac{1}{2p}}
\end{aligned}$$

Likewise, we have $(\pi^{-1})_p\pi_p(D_\varphi^{\frac{1}{2p}}xD_\varphi^{\frac{1}{2p}}) = D_\varphi^{\frac{1}{2p}}xD_\varphi^{\frac{1}{2p}}$. By density and continuity, we obtain $(\pi^{-1})_p\pi_p = Id = \pi_p(\pi^{-1})_p$ and so $(\pi_p)^{-1} = (\pi^{-1})_p$. Since π_p and $(\pi^{-1})_p$ are contractive, this implies that π_p is an isometry. □

Definition 1.3.14. *Let $J : (M,\varphi) \to (M_1,\varphi')$ be a w^*-continuous, unital and weight preserving $*$-homomorphism. We say that $\mathbb{E}$ is a conditional expectation associated with J, if*

1. *$\mathbb{E} : M_1 \to M$ is a contraction;*
2. *$\mathbb{E} \circ J = id_M$;*
3. *$\varphi \circ \mathbb{E} = \varphi'$.*

We generalize part 2 of Remark 1.3.3.

Lemma 1.3.15. *Assume that $J : (M,\varphi) \to (M_1,\varphi')$ is a w^*-continuous, unital and weight preserving $*$-homomorphism and verifies for any $s \in \mathbb{R}$*

$$\sigma_s^{\varphi'} \circ J = J \circ \sigma_s^{\varphi}. \tag{1.3.3}$$

Then there exists an unique conditional expectation associated with J which is equal to J_1^*.

Proof. By [63, Theorem IX.4.2], we have the existence and the uniqueness. It remains to prove that J_1^* is a conditional expectation associated with J.

We will prove the following three properties,

1. J_1^* is a contraction;
2. for all $x \in M$, $J_1^*(J(x)) = x$;
3. $\varphi \circ J_1^* = \varphi'$.

J is a contraction. By Theorem 1.3.11, J_1^* is a contraction too. Now we will prove 2. Let $x \in \mathcal{M}_\varphi^M$ and $y \in \mathcal{M}_\varphi^\infty$. J is a weight preserving $*$-homomorphism, thus $J(x) \in \mathcal{M}_{\varphi'}^{M_1}$. By Lemma 1.2.11, we have also $J(y) \in \mathcal{M}_{\varphi'}^\infty$. In the following everything is well-defined and we have

$$\begin{aligned}\left\langle J_1^*(J(x)), D_\varphi^{\frac{1}{2}}yD_\varphi^{\frac{1}{2}}\right\rangle &= \left\langle J(x), J_1\left(D_\varphi^{\frac{1}{2}}yD_\varphi^{\frac{1}{2}}\right)\right\rangle \\ &= \mathrm{tr}_{\varphi'}\left(J(x)D_{\varphi'}^{\frac{1}{2}}J(y)D_{\varphi'}^{\frac{1}{2}}\right).\end{aligned}$$

By 2 of Lemma 1.2.10, the latter is equal to

$$\begin{aligned}\varphi'\left(J\left(x\right)\sigma_{-\frac{i}{2}}^{\varphi}(J(y))\right) &= \varphi'\left(J\left(x\sigma_{-\frac{i}{2}}^{\varphi}(y)\right)\right) \\ &= \varphi\left(x\sigma_{-\frac{i}{2}}^{\varphi}(y)\right)\end{aligned}$$

Again, by 2 of Lemma 1.2.10, we obtain

$$\left\langle J_1^*(J(x)), D_\varphi^{\frac{1}{2}}yD_\varphi^{\frac{1}{2}}\right\rangle = \mathrm{tr}_\varphi\left(xD_\varphi^{\frac{1}{2}}yD_\varphi^{\frac{1}{2}}\right) = \left\langle x, D_\varphi^{\frac{1}{2}}yD_\varphi^{\frac{1}{2}}\right\rangle.$$

By density of $D_\varphi^{\frac{1}{2}}\mathcal{M}_\varphi^\infty D_\varphi^{\frac{1}{2}}$ in $L^1(M)$, w^*-density of $\mathcal{M}_\varphi^M$ in M and w^*-continuity of J and J_1^*, we deduce for all $x \in M$, $J_1^*(J(x)) = x$. Last point comes from 2 of Lemma 1.3.12. □

Definition 1.3.16. *We say that an operator $T : (M, \varphi) \to (M, \varphi)$ is absolutely dilatable if there exist a weighted von Neumann algebra (M_1, φ'), a w*-continuous weight preserving unital $\star$-homomorphism $J : (M, \varphi) \to (M_1, \varphi')$ such that $\sigma_{\cdot}^{\varphi'} \circ J = J \circ \sigma_{\cdot}^{\varphi}$ and a weight preserving $\star$-isomorphism $U : (M_1, \varphi') \to (M_1, \varphi')$ such that*

$$T^k = J_1^* U^k J, \qquad k \geqslant 0, \tag{1.3.4}$$

Remark 1.3.17.

1. By Lemma 1.3.12, J_1^* is a positive weight preserving operator.
2. If T is absolutely dilatable, then T is weight preserving. Indeed with the above notation, (1.3.4) with $k = 1$ yields $T = J_1^* U J$ and J_1^* is weight preserving.
3. If φ is tracial and T absolutely dilatable (with weight), we can obtain a tracial von Neumann algebra for (M_1, φ'). Indeed we can replace (M_1, φ') by $(\mathcal{M}_c^{\varphi'}, \varphi')$ which is a tracial von Neumann algebra. Since $\sigma^{\varphi'} \circ J = J \circ \sigma^{\varphi} = J$, we deduce that $J(M) \subset \mathcal{M}_c^{\varphi'}$. Then we have J_1^* from $\mathcal{M}_c^{\varphi'}$ onto M. We apply Theorem 1.2.8 as U and U^{-1}, thus $U(\mathcal{M}_c^{\varphi'}) = \mathcal{M}_c^{\varphi'}$. We obtain a trace preserving $*$-isomorphism $U' = U_{|\mathcal{M}_c^{\varphi'}} : \mathcal{M}_c^{\varphi'} \to \mathcal{M}_c^{\varphi'}$. The relation (1.3.1) sill holds.

Proposition 1.3.18. *If $T : (M, \varphi) \to (M, \varphi)$ is absolutely dilatable, then for every $1 \leqslant p < \infty$, T induces a contraction $T_p : L^p(M, \varphi) \to L^p(M, \varphi)$ and T_p admits an isometric p-dilation.*

Proof. Let $p \in [1; +\infty[$, the maps T, U, J_1^* and J are positive and weight preserving. According to Theorem 1.3.11, there exist $T_p : L^p(M) \to L^p(M)$, $U_p : L^p(M_1) \to L^p(M_1)$, $(J_1^*)_p : L^p(M_1) \to L^p(M)$ and $J_p : L^p(M) \to L^p(M_1)$. In addition, $\|(J_1^*)_p\| \leqslant 1$ and $\|J_p\| \leqslant 1$. By Lemma 1.3.13, U_p is a surjective isometry. It remains to prove that for any $k \in \mathbb{N}$:

$$T_p^k = (J_1^*)_p U_p^k J_p \tag{1.3.5}$$

Let $x \in \mathcal{M}_\varphi^M$, we have:

$$\begin{aligned}
(J_1^*)_p U_p^k J_p \left(D_\varphi^{\frac{1}{2p}} x D_\varphi^{\frac{1}{2p}} \right) &= (J_1^*)_p U_p^k \left(D_{\varphi'}^{\frac{1}{2p}} J(x) D_{\varphi'}^{\frac{1}{2p}} \right) \\
&= (J_1^*)_p \left(D_{\varphi'}^{\frac{1}{2p}} U^k J(x) D_{\varphi'}^{\frac{1}{2p}} \right) \\
&= D_\varphi^{\frac{1}{2p}} (J_1^*) U^k J(x) D_\varphi^{\frac{1}{2p}} \\
&= \left(D_\varphi^{\frac{1}{2p}} T^k(x) D_\varphi^{\frac{1}{2p}} \right) \\
&= T_p^k \left(D_\varphi^{\frac{1}{2p}} x D_\varphi^{\frac{1}{2p}} \right).
\end{aligned}$$

By density of $D_\varphi^{\frac{1}{2p}} \mathcal{M}_\varphi^M D_\varphi^{\frac{1}{2p}}$ in $L^p(M)$, (1.3.5) follows. □

1.4 Schur multipliers

We will present Schur multipliers. There are really simple in the finite case and discrete case. However the measurable Schur multipliers are more involved. We will describe discrete Schur multipliers after the measurable Schur multipliers because some properties are needed for the discrete case.

1.4.1 Finite Schur multipliers

We denote by M_n the space of $n \times n$ matrices with complex coefficients. Let $M = (m_{ij})_{1\leqslant i,j\leqslant n}$ be an element of M_n. We denote by $T_M : M_n \to M_n$ the Schur multiplier associated with M, defined by

$$T_M([b_{ij}]_{1\leqslant i,j\leqslant n}) = [m_{ij}b_{ij}]_{1\leqslant i,j\leqslant n}, \qquad [b_{ij}]_{1\leqslant i,j\leqslant n} \in M_n.$$

We recall some links between T_M and M. In the next theorem, the first point is in [49, Theorem 3.7] and the second one is clear.

Theorem 1.4.1. *Let $M = (m_{ij})_{1\leqslant i,j\leqslant n} \in M_n$, then*

1. *T_M is positive if and only if T_M is completely positive if and only if M is positive semi-definite.*
2. *T_M is unital if and only if all diagonal entries of M are equal to 1.*

1.4.2 Measurable Schur multipliers

This subsection is devoted to preliminaries on measurable Schur multipliers and characterizations of some of their possible properties (positivity, unitality, etc.).

Let Σ be a σ-finite measure space with respect to a measure simply denoted by dt and let X be a Banach space. For any $1 \leqslant p \leqslant +\infty$, we let $L^p(\Sigma, X)$ denote the classical Bochner space of strongly measurable functions $f : \Sigma \to X$ (defined up to almost everywhere zero functions) such that the norm function $t \mapsto \|f(t)\|$ belongs to $L^p(\Sigma)$, equipped with $\|f\|_p = \|t \mapsto \|f(t)\|\|_{L^p(\Sigma)}$. (see [17, p. 49-50] for more explanations). We will use the fact that if p is finite, then $L^p(\Sigma) \otimes X$ is a dense subspace of $L^p(\Sigma, X)$. Furthermore if $Y = H$ is a Hilbert space, then the natural embedding of $L^2(\Sigma) \otimes H$ into $L^2(\Sigma; H)$ extends to a unitary identification

$$L^2(\Sigma; H) \simeq L^2(\Sigma) \overset{2}{\otimes} H, \tag{1.4.1}$$

where $\overset{2}{\otimes}$ stands for the Hilbertian tensor product.

Throughout we let $L^p_{\mathbb{R}}(\Sigma)$ denote the subset of real-valued functions in $L^p(\Sigma)$.

We denote by $X\hat{\otimes}Y$ the projective tensor product of any two Banach spaces X and Y (see [57]). We recall the isometric isomorphisms

$$L^1(\Sigma, Y) \simeq L^1(\Sigma)\hat{\otimes}Y \qquad \text{and} \qquad (L^1(\Sigma)\hat{\otimes}Y)^* \simeq B(L^1(\Sigma), Y^*), \tag{1.4.2}$$

see [57, example 2.19] and discussion p. 24 in the same book.

We recall that $S^2(L^2(\Sigma))$ is the space of all Hilbert-Schmidt operators on $L^2(\Sigma)$. We also recall that for any $f \in L^2(\Sigma^2)$, we may define a bounded operator

$$S_f : L^2(\Sigma^2) \longrightarrow L^2(\Sigma^2), \qquad S_f(h) = \int_\Sigma f(\cdot, t)h(t)dt, \tag{1.4.3}$$

and that S_f is a Hilbert-Schmidt operator, that is, $S_f \in S^2(L^2(\Sigma))$. Further the mapping $f \mapsto S_f$ yields a unitary identification

$$L^2(\Sigma^2) \simeq S^2(L^2(\Sigma)), \tag{1.4.4}$$

see e.g. [55, Theorem VI. 23].

For any $x, y \in L^2(\Sigma)$, we consider $x \otimes y$ as an element of $L^2(\Sigma^2)$ by writing $(x \otimes y)(s,t) = x(s)y(t)$ for $(s,t) \in \Sigma^2$. Then using (1.4.3), we have

$$S_{x\otimes y}(h) = \left(\int_\Sigma y(t)h(t)dt\right)x, \qquad h \in L^2(\Sigma).$$

Very often we will identify $x \otimes y$ and $S_{x\otimes y}$. Thus $L^2(\Sigma) \otimes L^2(\Sigma)$ is regarded as the space of finite rank operators on $L^2(\Sigma)$. This extends to an isometric isomorphism

$$L^2(\Sigma)\hat{\otimes}L^2(\Sigma) \simeq S^1(L^2(\Sigma)), \tag{1.4.5}$$

where $S^1(L^2(\Sigma))$ denotes the Banach space of trace class operators on $L^2(\Sigma)$.

Another useful result on the projective tensor product is the isometry $L^1(\Sigma_1)\hat{\otimes}L^1(\Sigma_2) \simeq L^1(\Sigma_1 \times \Sigma_2)$.

For any $\varphi \in L^\infty(\Sigma^2)$, we denote by $M_\varphi : S^2(L^2(\Sigma)) \to S^2(L^2(\Sigma))$ the operator defined by

$$M_\varphi(S_f) = S_{\varphi f}, \qquad f \in L^2(\Sigma^2). \tag{1.4.6}$$

We say that M_φ (or simply φ) is a (measurable) Schur multiplier on $B(L^2(\Sigma))$ (or a bounded Schur multiplier), if there exists $C > 0$ such that

$$\|M_\varphi(S)\|_{B(L^2(\Sigma))} \leqslant C\|S\|_{B(L^2(\Sigma))}, \qquad S \in S^2(L^2(\Sigma)). \tag{1.4.7}$$

It is plain that

$$\|M_\varphi\|_{B(S^2(L^2(\Sigma)))} = \|\varphi\|_{L^\infty(\Sigma^2)}. \tag{1.4.8}$$

In the sequel, we set $\check{\varphi}(s,t) = \varphi(t,s)$ for $(s,t) \in \Sigma^2$.

Lemma 1.4.2. *Let $\varphi \in L^\infty(\Sigma^2)$, the following assertions are equivalent:*

1. *φ is a Schur multiplier on $B(L^2(\Sigma))$;*
2. *M_φ extends to a w^*-continuous operator $M_\varphi^\infty : B(L^2(\Sigma)) \to B(L^2(\Sigma))$;*
3. *M_φ restricts to a bounded operator $M_\varphi^1 : S^1(L^2(\Sigma)) \to S^1(L^2(\Sigma))$;*
4. *M_φ extends to a bounded operator $M_\varphi^0 : K(L^2(\Sigma)) \to K(L^2(\Sigma))$, where $K(L^2(\Sigma))$ denotes the Banach space of compact operators on $L^2(\Sigma)$;*
5. *$\check{\varphi}$ verifies (3).*

In this case, $M_{\check{\varphi}}^1 = \big(M_\varphi^0\big)^*$, $\big(M_{\check{\varphi}}^1\big)^* = M_\varphi^\infty$ and $\|M_\varphi^1\| = \|M_{\check{\varphi}}^1\| = \|M_\varphi^\infty\| = \|M_\varphi^0\|$.

Proof. The flip mapping $\sum x_k \otimes y_k \mapsto \sum y_k \otimes x_k$ extends to an isometric automorphism $\rho : L^2(\Sigma)\hat{\otimes}L^2(\Sigma) \to L^2(\Sigma)\hat{\otimes}L^2(\Sigma)$. Using this notation and (1.4.5), it is easy to check that

$$M_{\check{\varphi}}(S_f) = \rho\big(M_\varphi(S_{\rho(f)})\big), \qquad f \in L^2(\Sigma) \otimes L^2(\Sigma),$$

provided that property 3 holds true. We easily deduce that $5 \Leftrightarrow 3$.

Let tr denote the usual trace on $B(L^2(\Sigma))$. We remark that $S_f^* = S_{\overline{\check{f}}}$, for any $f \in L^2(\Sigma^2)$. Hence the equality $\|S_f\|^2 = \|f\|^2$ reads $\mathrm{tr}(S_f S_{\overline{\check{f}}}) = \int f\overline{f}$.

By polarization, we obtain that for all $f, h \in L^2(\Sigma^2)$,

$$\mathrm{tr}\big(S_f S_g\big) = \int_{\Sigma^2} f(s,t)g(t,s)\,dtds. \tag{1.4.9}$$

Consequently,

$$\mathrm{tr}\left(S_{\varphi f} S_h\right) = \int_{\Sigma^2} \varphi(s,t)f(s,t)h(t,s)dsdt = \mathrm{tr}\left(S_f S_{\check{\varphi} h}\right), \qquad f, h \in L^2(\Sigma^2). \tag{1.4.10}$$

This yields $4 \Rightarrow 5 \Rightarrow 2$.

By definition, $M_\varphi(S^2(L^2(\Sigma))) \subset S^2(L^2(\Sigma)) \subset K(L^2(\Sigma))$, hence $1 \Rightarrow 4$ follows from the definition. This concludes the proof of the equivalence of the five properties. The rest of the statement follows from the above arguments. □

Schur multipliers as defined in this section go back at least to Haagerup [26] and Spronk [59].

The following description of bounded Schur multipliers has a rather long history. In the discrete case, Theorem 1.4.3 was stated by Paulsen in [49, Corolary 8.8] and by Pisier in [50, Theorem 5.1], who refers himself to some earlier work of Grothendieck. For the general case of this statement, we refer to Haagerup [26] and Spronk [59, Section 3.2].

In the next statement, $\langle \cdot\ ,\ \cdot \rangle_H$ stands for the inner product on some Hilbert space H.

Theorem 1.4.3. *Let $\varphi \in L^\infty(\Sigma^2)$ and let $C \geqslant 0$. The function φ is a bounded Schur multiplier and $\|M_\varphi^\infty \colon B(L^2(\Sigma)) \to B(L^2(\Sigma))\| \leqslant C$ if and only if there exist a Hilbert space H and two functions $\alpha, \beta \in L^\infty(\Sigma; H)$ such that $\|\alpha\|_\infty \|\beta\|_\infty \leqslant C$ and*

$$\varphi(s,t) = \langle \alpha(s)\,,\, \beta(t) \rangle_H \quad \text{a.e.-}(s,t) \in \Sigma^2. \tag{1.4.11}$$

In the rest of this subsection, we assume that $\varphi \in L^\infty(\Sigma^2)$ is a Schur multiplier on $B(L^2(\Sigma))$. The argument in the proof of Lemma 1.4.2 shows that the Hilbertian adjoint of $M_\varphi : S^2(L^2(\Sigma)) \to S^2(L^2(\Sigma))$ is equal to $M_{\overline{\varphi}}$. We can therefore characterize the self-adjointness of a Schur multiplier as follows (see [7, proposition 4.2]).

Proposition 1.4.4. *The operator $M_\varphi : S^2(L^2(\Sigma)) \to S^2(L^2(\Sigma))$ is self-adjoint if and only if φ is real-valued.*

As a complement to Theorem 1.4.3, we mention the following classical result on Schur multipliers. We refer to Subsection 1.2.3 for the notion of complete positively.

Theorem 1.4.5. *The following are equivalent:*

1. *M_φ is positive;*
2. *M_φ is completely positive;*
3. *there exist a Hilbert space H and a function $\alpha \in L^\infty(\Sigma, H)$ such that*

$$\varphi(s,t) = \langle \alpha(s), \alpha(t) \rangle_H \ \text{ a.e. on } \Sigma \times \Sigma. \tag{1.4.12}$$

This theorem can be proved in two steps. The first one consists in proving it in the finite dimensional case. This is done e.g. in [49, theorem 3.7]. The second step consists in approximating from the finite dimensional case. In [43, Theorem 1.7], the authors prove Theorem 1.4.3 using the finite dimensional case. Their argument shows as well that Theorem 1.4.5 can be deduced from the finite dimensional case. Our work in Subsection 2.2 will provide a direct proof of Theorem 1.4.5 (see the proof of Lemma 2.2.3, in particular (2.2.2)).

The following result is apparently new.

Theorem 1.4.6. *Let H and $\alpha, \beta \in L^\infty(\Sigma, H)$ such that (1.4.11) holds true. Then*

$$M_\varphi^\infty \text{ is unital } \iff \langle \alpha(t), \beta(t) \rangle_H = 1 \text{ a.e. on } \Sigma.$$

Proof. By linearity and density, we have:

$$M_\varphi^\infty(I) = I \iff \forall f, h, \in L^2(\Sigma),\ \left\langle M_\varphi^\infty(I), f \otimes h \right\rangle = \langle I, f \otimes h \rangle$$

Hence by Lemma 1.4.2,

$$\begin{aligned} M_\varphi^\infty(I) = I &\iff \forall f, h \in L^2(\Sigma),\ \left\langle I, M_{\check{\varphi}}^1(f \otimes h) \right\rangle = \langle I, f \otimes h \rangle \\ &\iff \forall f, h \in L^2(\Sigma),\ \mathrm{tr}\left(M_{\check{\varphi}}^1(f \otimes h)\right) = \int_\Sigma f(t)h(t)dt. \end{aligned}$$

We can suppose that H separable (and also that $\dim(H)$ is infinite), thanks to Pettis's measurability theorem (see [17, Theorem 2 p.42]). Thus there exists a Hilbertian basis $(e_n)_{n \geqslant 1}$ of H. We denote, for all n, $\alpha_n(s) = \langle \alpha(s), e_n \rangle_H$ and $\beta_n(t) = \langle e_n, \beta(t) \rangle_H$ for a.e. $s \in \Sigma$ and $t \in \Sigma$. We have

$$\langle \alpha(s), \beta(t) \rangle_H = \sum_n \alpha_n(s)\beta_n(t) \quad \text{a.e. on } \Sigma \times \Sigma.$$

Let $f, h \in L^2(\Sigma)$, we will prove that:

$$\sum_{n=1}^\infty \|\alpha_n h\|_2^2 < \infty \text{ and } \sum_{n=1}^\infty \|\beta_n f\|_2^2 < \infty. \tag{1.4.13}$$

First,

$$\begin{aligned} \sum_{n=1}^\infty \|\alpha_n h\|_2^2 = \sum_{n=1}^\infty \int_\Sigma |\alpha_n(s)|^2 |h(s)|^2 ds &= \int_\Sigma \sum_{n=1}^\infty |\alpha_n(s)|^2 |h(s)|^2 ds \\ = \int_\Sigma \|\alpha(s)\|_H^2 |h(s)|^2 ds &\leqslant \|\alpha\|_\infty^2 \|h\|_2^2 < \infty. \end{aligned}$$

Similarly, we have $\sum \|\beta_n f\|_2^2 < \infty$. By the Cauchy-Schwarz inequality, we deduce

$$\sum_{n=1}^\infty \int_\Sigma |\alpha_n(t)||h(t)||\beta_n(t)||f(t)|dt < \infty. \tag{1.4.14}$$

Moreover using (1.4.13) again, and (1.4.5), we may consider $\sum_{n=1}^{\infty} \beta_n f \otimes \alpha_n h$ in $S^1(L^2(\Sigma))$ and we have

$$\begin{aligned}
\operatorname{tr}\left(\sum_{n=1}^{\infty} \beta_n f \otimes \alpha_n h\right) &= \sum_{n=1}^{\infty} \operatorname{tr}\left(\beta_n f \otimes \alpha_n h\right) \\
&= \sum_{n=1}^{\infty} \int_{\Sigma} \alpha_n(t) h(t) \beta_n(t) f(t) dt \\
&= \int_{\Sigma} \sum_{n=1}^{\infty} \alpha_n(t) h(t) \beta_n(t) f(t) dt, \text{ by (1.4.14)}, \\
&= \int_{\Sigma} \langle \alpha(t), \beta(t) \rangle_H f(t) h(t) dt.
\end{aligned}$$

Let us now show that

$$M^1_{\breve{\varphi}}(S_{f\otimes h}) = \sum_{n=1}^{\infty} \beta_n f \otimes \alpha_n h.$$

Let $v_1, v_2 \in L^2(\Sigma)$. According to (1.4.13), we have

$$\sum_n \int_{\Sigma^2} |\alpha_n(t) h(t) v_1(t) \beta_n(s) f(s) v_2(s)| dt ds < +\infty. \tag{1.4.15}$$

Then we have (using the usual duality pairing $\langle \cdot, \cdot \rangle$ between $L^2(\Sigma)$ and itself)

$$\begin{aligned}
\langle M^1_{\breve{\varphi}}(f \otimes h) v_1, v_2 \rangle &= \langle S_{\breve{\varphi}(f\otimes h)} v_1, v_2 \rangle \\
&= \int_{\Sigma^2} \varphi(t,s) f(s) h(t) v_1(t) v_2(s) dt ds \\
&= \int_{\Sigma^2} \langle \alpha(t), \beta(s) \rangle_H f(s) h(t) v_1(t) v_2(s) dt ds \\
&= \int_{\Sigma^2} \sum_{n=1}^{\infty} \alpha_n(t) \beta_n(s) f(s) h(t) v_1(t) v_2(s) dt ds \\
&= \sum_{n=1}^{\infty} \int_{\Sigma^2} \alpha_n(t) \beta_n(s) f(s) h(t) v_1(t) v_2(s) dt ds, \text{ by (1.4.15)}, \\
&= \sum_{n=1}^{\infty} \langle \alpha_n h, v_1 \rangle \langle \beta_n f, v_2 \rangle \\
&= \left\langle \left(\sum_{n=1}^{\infty} \beta_n f \otimes \alpha_n h \right) v_1, v_2 \right\rangle
\end{aligned}$$

This proves the announced identity. We deduce that

$$\operatorname{tr}(M^1_{\breve{\varphi}}(S_{f\otimes h})) = \int_{\Sigma} f(t) h(t) \langle \alpha(t), \beta(t) \rangle_H \, dt.$$

Finally,

$$\begin{aligned}
M^{\infty}_{\varphi} \text{ is unital} &\iff \forall f, h \in L^2(\Sigma), \ \int_{\Sigma} f(t) h(t) \langle \alpha(t), \beta(t) \rangle_H \, dt = \int_{\Sigma} f(t) h(t) dt \\
&\iff \langle \alpha(t), \beta(t) \rangle_H = 1 \text{ a.e. on } \Sigma.
\end{aligned}$$

□

1.4.3 Discrete Schur multipliers

Let I be an index set and let $\mathcal{M}_I$ denote the space of $I \times I$ matrices with complex entries. We let $(e_i)_{i\in I}$ denote the canonical basis of $\ell^2(I)$. For any $\gamma \in B(\ell^2(I))$, for all $(i,j) \in I^2$ we let $\gamma_{i,j} := \langle \gamma(e_j), e_i\rangle$. Then we regard $B(\ell^2(I)) \subset \mathcal{M}_I$. Given any $M = (m_{ij})_{i,j\in I} \in \mathcal{M}_I$, the Schur multiplier on $B(\ell^2(I))$ associated with M is the unbounded operator T_M whose domain $D(T_M)$ is the space of all $A = (a_{ij})_{i,j\in I}$ in $B(\ell^2(I))$ such that $T_M(A) := (m_{ij}a_{ij})_{i,j\in I}$ belongs to $B(\ell^2(I))$. If $D(T_M) = B(\ell^2(I))$, then $T_M : B(\ell^2(I)) \to B(\ell^2(I))$ is a bounded Schur multiplier on $B(\ell^2(I))$. By Lemma, T_M restricts to a bounded operator $T_M : S^1_I \to S^1_I$. Then it is still true for any $1 \leqslant p < +\infty$ and we let $T_M \colon S^p_I \to S^p_I$.

For the rest of this subsection, we assume that T_M associated with $M \in \mathcal{M}_I$ is a bounded Schur multiplier on $B(\ell^2(I))$. We recall classical properties. By Lemma 1.4.2, T_M is w^*-continuous. Then T_M is positive (resp. completely positive) if and only if for all finite set $F \subset I$, $(T_M)_{|B(\ell^2(F))} : B(\ell^2(F)) \to B(\ell^2(F))$ is positive (resp. completely positive). Thus the following is a direct consequence of Theorem 1.4.1.

Theorem 1.4.7.

1. *T_M is positive if and only if T_M is completely positive if and only if for all finite set $F \subset I$, the matrix $(m_{ij})_{i,j\in F}$ is positive semi-definite.*

2. *T_M is unital if and only if for all $i \in I$, $m_{ii} = 1$.*

1.5 Fourier multipliers on $VN(G)$

This section provides some background on the group von Neumann algebra of a locally compact group and on Fourier multipliers.

1.5.1 General case (weighted von Neumann algebras)

Let G be a locally compact group and let Δ be its modular function. Let $\lambda \colon G \to B(L^2(G))$ be the left regular representation of G, that is, $\big[\lambda(s)f\big](t) = f(s^{-1}t)$ for any $f \in L^2(G)$ and $s, t \in G$. We let $VN(G) = \lambda(G)'' \subset B(L^2(G))$ be the group von Neumann algebra of G. We denote by

$$\omega_G : VN(G)_+ \longrightarrow [0; +\infty]$$

the Plancherel weight on $VN(G)$ (see [63, Definition VII.3.2]) and we denote by σ^G_s the modular automorphism group of ω_G.

We denote by $\lambda : L^1(G) \to VN(G) \subset B(L^2(G))$ the mapping defined by $\lambda(f) = \int_G f(s)\lambda(s)ds$. (The use of the same notation λ for these two maps should not create any confusion.) The above integral is defined in the strong sense. Indeed, for any $f_1 \in L^1(G)$ and $f_2 \in L^2(G)$, $[\lambda(f_1)](f_2) = f_1 * f_2$, where

$$(f_1 * f_2)(t) = \int_G f_1(s) f_2(s^{-1}t)ds, \qquad t \in G.$$

We recall that for any $f_1, f_2 \in L^2(G)$, the above formula defines a function $f_1 * f_2 \in C_0(G)$.

For any $f \in L^1(G)$, we define $f^* \in L^1(G)$ by

$$f^*(t) = \Delta_{t^{-1}}\overline{f(t^{-1})}, \qquad t \in G.$$

Lemma 1.5.1. *Let $f, h \in L^1(G) \cap L^2(G)$. We have:*

1. $\lambda(f), \lambda(h) \in \mathcal{N}_{\omega_G}^{VN(G)}$;
2. $\lambda(f)^*\lambda(h) = \lambda(f^* * h)$;
3. $\lambda(f)^*\lambda(h) \in \mathcal{M}_{\omega_G}^{VN(G)}$ *and*

$$\omega_G(\lambda(f)^*\lambda(h)) = \int_G \overline{f}(s)h(s)ds = (f^* * h)(e),$$

where e denotes the unit of G.

Proof. See [60, 18.17 (5)] . □

It is plain that the $*$-subalgebra $\lambda(L^1(G))$ is w^*-dense in $VN(G)$. Let $\mathcal{K}(G) \subset C_0(G)$ be the $*$-subalgebra of all continuous functions from G into $\mathbb{C}$ with compact support. Then $\lambda(\mathcal{K}(G))$ is w^*-dense in $VN(G)$. (See [22, 3.12] or [21]). In addition, $\lambda(\mathcal{K}(G))$ is a subset of $\mathcal{M}_{\omega_G}^{VN(G)}$.

We say that a w^*-continuous operator $T : VN(G) \to VN(G)$ is a Fourier multiplier, if there exists a bounded continuous function $u : G \to \mathbb{C}$ such that for all $s \in G$, $T(\lambda(s)) = u(s)\lambda(s)$. In this case, u is necessarily unique, we write $T = T_u$ and u is called the symbol of T_u. In addition, we have

$$T_u(\lambda(f)) = \lambda(uf), \qquad f \in L^1(G).$$

We say that an operator $T : VN(G) \to VN(G)$ is self-adjoint if

$$\omega_G(T(x)y^*) = \omega_G(xT(y)^*), \qquad x, y \in \mathcal{M}_{\omega_G}^{VN(G)}.$$

We recall that a function $u : G \to \mathbb{C}$ is called positive definite if, for all $n \geqslant 1$, for all $t_1, \ldots, t_n \in G$ and for all $z_1, \ldots, z_n \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{n} u(t_i^{-1}t_j)\overline{z_i}z_j \geqslant 0.$$

As in Section 1.4.2 on measurable Schur Multipliers, we relate the properties of a Fourier multiplier T_u to its symbol u.

Theorem 1.5.2. *Let $u : G \to \mathbb{C}$ be the symbol of a Fourier multiplier $T_u : VN(G) \to VN(G)$.*

1. *T_u is completely positive if and only if u is positive definite;*
2. *the map T_u is unital if and only if $u(e) = 1$;*
3. *the map T_u is self-adjoint if and only if u is real valued.*

The first point is [16, Proposition 4.2]. The other two ones are just computing. We recall that unlike for Schur Multiplier, there exist positive Fourier multipliers which are not completely positive, see [16, Corollary 4.8]. Similarly, a bounded Fourier multiplier is not necessarily completely bounded [9].

1.5.2 Unimodular case

If we add the unimodular hypothesis, things are easier. Since $\Delta = 1$, we have for any $f \in L^1(G)$, $f^* = \overline{\check{f}}$ where $\check{f} \in L^1(G)$ is defined by

$$\check{f}(t) = f(t^{-1}), \quad t \in G$$

The most important property is that the Plancherel weight is a trace. We do not need the weight theory in this case. The space $\mathcal{M}_{\omega_G}^{VN(G)}$ will be replaced by $L^1(VN(G))$. Then Lemma 1.5.1 becomes the following.

Lemma 1.5.3. *Let $f, h \in L^1(G) \cap L^2(G)$. We have:*

1. $\lambda(f)^*\lambda(h) = \lambda(f^* * h)$;
2. $\lambda(f)^*\lambda(h) \in L^1(VN(G))$ *and*

$$\omega_G(\lambda(f)^*\lambda(h)) = \int_G \overline{f}(s)h(s)ds = (f^* * h)(e),$$

where e denotes the unit of G.

Likewise, an operator $T : VN(G) \to VN(G)$ is self-adjoint if

$$\omega_G(T(x)y^*) = \omega_G(xT(y)^*), \qquad x, y \in VN(G) \cap L^1(VN(G)).$$

1.6 Antisymmetric Fock spaces and Fermions

We give some background on Fock spaces and Fermions. We refer to [10] and [11] for more details and information. Let H be a real Hilbert space and we let $H_{\mathbb{C}}$ denote its complexification. For any integer $n \geqslant 1$, we let $\Lambda_n(H_{\mathbb{C}})$ denote the n-fold antisymmetric space over $H_{\mathbb{C}}$, equipped with the inner product defined by

$$\langle h_1 \wedge \cdots \wedge h_n, k_1 \wedge \cdots \wedge k_n \rangle_{-1} = \det\left[\langle h_i, k_j \rangle_{H_{\mathbb{C}}}\right], \qquad h_i, k_j \in H_{\mathbb{C}}.$$

We also set $\Lambda_0(H_{\mathbb{C}}) = \mathbb{C}$. The antisymmetric Fock space over $H_{\mathbb{C}}$ is the Hilbertian direct sum

$$\mathcal{F}_{-1}(H) \;=\; \bigoplus_{n \geqslant 0} \Lambda_n(H_{\mathbb{C}}).$$

We let Ω be a fixed unit element of $\Lambda_0(H_{\mathbb{C}})$.

For all $e \in H$, we recall the creation operator $l(e) : \mathcal{F}_{-1}(H) \to \mathcal{F}_{-1}(H)$ satisfying $l(e)\Omega = e$ and $l(e)(h_1 \wedge \cdots \wedge h_n) = e \wedge h_1 \wedge \cdots \wedge h_n$ for all $h_1, \ldots, h_n \in H_{\mathbb{C}}$. These operators satisfy the following relation:

$$l(f)^*l(e) + l(e)l(f)^* = \langle f, e \rangle_H Id_{\mathcal{F}_{-1}(H)}, \qquad e, f \in H. \tag{1.6.1}$$

Let $\omega(e) : \mathcal{F}_{-1}(H) \to \mathcal{F}_{-1}(H)$ be the self-adjoint operator

$$\omega(e) = l(e) + l(e)^*.$$

It follows from (1.6.1) that

$$\omega(e)^2 = \|e\|_H^2 Id_{\mathcal{F}_{-1}(H)}, \qquad e \in H. \tag{1.6.2}$$

By definition,

$$\Gamma_{-1}(H) \subset B(\mathcal{F}_{-1}(H))$$

is the von Neumann algebra generated by $\{\omega(e);\, e \in H\}$. This is a finite von Neumann algebra equipped with a trace τ defined by $\tau(x) = \langle x\Omega, \Omega\rangle_{\mathcal{F}_{-1}(H)}$ for all $x \in \Gamma_{-1}(H)$. Clearly we have

$$\tau(\omega(e)\omega(f)) = \langle e, f\rangle_H, \qquad e, f \in H. \tag{1.6.3}$$

The space $\Gamma_{-1}(H)$ is called the Fermion algebra over H.

Let H and K be two real Hilbert spaces and let $T : H \to K$ be a contraction with complexification $T_{\mathbb{C}} : H_{\mathbb{C}} \to K_{\mathbb{C}}$. There exists a necessarily unique linear contraction

$$F_{-1}(T) : \mathcal{F}_{-1}(H) \longrightarrow \mathcal{F}_{-1}(K)$$

such that $F_{-1}(T)\Omega = \Omega$ and $F_{-1}(T)(h_1 \wedge \cdots \wedge h_n) = T_{\mathbb{C}}(h_1) \wedge \cdots \wedge T_{\mathbb{C}}(h_n)$ for all $h_1, \ldots, h_n \in H_{\mathbb{C}}$. Next, there exists a necessarily unique normal, unital, completely positive and trace preserving map

$$\Gamma_{-1}(T) : \Gamma_{-1}(H) \longrightarrow \Gamma_{-1}(K)$$

such that for every $x \in \Gamma_{-1}(H)$, we have:

$$(\Gamma_{-1}(T)(x))\Omega_K = F_{-1}(T)(x\Omega).$$

If further $T : H \to K$ is a isometry (resp. an onto isometry), then $\Gamma_{-1}(T)$ is a one-to-one $*$-homomorphism (resp. a one-to-one $*$-isomomorphism).

In particular we have

$$[\Gamma_{-1}(T)](\omega(e)) = \omega(T(e)), \qquad e \in H. \tag{1.6.4}$$

In the next lemma, we consider an integer $k \geqslant 1$ and we let $\mathcal{P}_2(2k)$ be the set of 2-partitions of the set $\{1, 2, \ldots, 2k\}$. Then for any $\nu \in \mathcal{P}_2(2k)$, we let $c(\nu)$ denote the number of crossings of ν. We refer to [20] for details. According to Corollary 2.1 in the latter paper, we have the following lemma, in which (2) is a straightforward consequence of (1).

Lemma 1.6.1. *Let $(f_i)_{i=1}^{2k}$ be a family of H.*

(1) We have

$$\tau(\omega(f_1)\cdots\omega(f_{2k})) = \sum_{\nu\in\mathcal{P}_2(2k)} (-1)^{c(\nu)} \prod_{(i,j)\in\nu} \langle f_i, f_j\rangle_H.$$

(2) If for all $1 \leqslant i < j \leqslant 2k$ such that $j \neq 2k - i + 1$, we have $\langle f_i, f_j\rangle_H = 0$, then

$$\tau(\omega(f_1)\cdots\omega(f_{2k})) = \langle f_1, f_{2k}\rangle_H \cdots \langle f_k, f_{k+1}\rangle_H$$

Chapter 2

Dilation properties of unital self-adjoint positive measurable Schur multipliers

All definitions of Schur multipliers and dilation properties are in Subsection 1.4.2 respectively in Subsection 1.3.1. The main result of this chapter is the following.

Theorem (A). *Let Σ be a σ-finite measure space. Let $\varphi \in L^\infty(\Sigma^2)$ and assume that M_φ is a self-adjoint, unital, positive Schur multiplier on $B(L^2(\Sigma))$. Then M_φ is absolutely dilatable.*

We will also give an multi-variable version.

Theorem (C). *Let $\varphi_1, \ldots, \varphi_n \in L^\infty(\Sigma^2)$ and assume each M_{φ_i} is a self-adjoint, unital, positive Schur multiplier on $B(L^2(\Sigma))$. Then there exist a tracial von Neumann algebra (M, τ), a commuting n-tuple $(U_1, \ldots, U_n)$ of trace preserving $*$-automorphisms on M and a w^*-continuous trace preserving one-to-one $*$-homomorphism $J : B(L^2(\Sigma)) \to M$ such that*

$$M_{\varphi_1}^{k_1} \cdots M_{\varphi_n}^{k_n} = \mathbb{E} U_1^{k_1} \cdots U_n^{k_n} J$$

for all $k_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant n$, where $\mathbb{E} : M \to B(L^2(\Sigma))$ is the conditional expectation associated with J.

2.1 Preliminaries and L^p_σ-spaces

Let N be a von Neumann algebra. We recall that the product is separately w^*-continuous on N. More precisely, for any $y \in N$ and $\eta \in N_*$, the map $z \mapsto \langle yz, \eta \rangle$ from N into $\mathbb{C}$ is continuous. We let $\eta y \in N_*$ such that $\langle yz, \eta \rangle = \langle z, \eta y \rangle$ for all $z \in N$. Likewise, the map $z \mapsto \langle zy, \eta \rangle$ from N to $\mathbb{C}$ is continuous and we let $y\eta \in N_*$ such that $\langle zy, \eta \rangle = \langle z, y\eta \rangle$ for all $z \in N$.

We now turn to a dual variant of the Bochner spaces $L^p(\Sigma, N)$. Bochner spaces are already introduced in Subsection 1.4.2. We mostly follow [29], to which we refer for more references and details. We say that a function $f : \Sigma \to N$ is w^*-measurable if, for all $\eta \in N_*$, the function $t \mapsto \langle f(t), \eta \rangle$ is measurable. Fix some $1 < q \leqslant \infty$. We recall that every order bounded subset of $L^q_{\mathbb{R}}(\Sigma)$ has a supremum in $L^q_{\mathbb{R}}(\Sigma)$, denoted by $L^q - \sup$. We define

$$\mathcal{L}^q_\sigma(\Sigma, N) := \left\{ \begin{array}{c} f : \Sigma \to N \ w^*\text{-measurable} \ ; \langle f, \eta \rangle \in L^q(\Sigma) \ \forall \eta \in N_* \\ \text{and } \{|\langle f, \eta \rangle| : \|\eta\| \leqslant 1\} \text{ is order bounded in } L^q_{\mathbb{R}}(\Sigma) \end{array} \right\}.$$

We define, for all $f \in \mathcal{L}^q_\sigma(\Sigma, N)$, the semi-norm

$$\|f\|_{\mathcal{L}^q} = \big\|L^q - \sup\{|\langle f, \eta\rangle| : \|\eta\| \leqslant 1\}\big\|_{L^q(\Sigma)}.$$

The kernel of this semi-norm is

$$N_\sigma := \{f \in \mathcal{L}^q_\sigma(\Sigma, N) :\ \forall \eta \in N_*, \langle f, \eta\rangle = 0 \text{ a.e.}\}.$$

We set $L^q_\sigma(\Sigma, N) = \mathcal{L}^q_\sigma(\Sigma, N)/N_\sigma$. This is a Banach space for the resulting norm, which we simply denote by $\|.\|_q$. We note that $L^q(\Sigma, N) \subset L^q_\sigma(\Sigma, N)$ isometrically.

Let $1 \leqslant p \leqslant \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. For any $h \in L^p(\Sigma, N_*)$ and $f \in L^q_\sigma(\Sigma, N)$, the function $t \mapsto \langle f(t), h(t)\rangle$ belongs to $L^1(\mathbb{R})$ and satisfies

$$\int_\Sigma |\langle f(t), h(t)\rangle| dt \leqslant \|h\|_p \|f\|_q. \tag{2.1.1}$$

This allows to define

$$\langle f, h\rangle = \int_\Sigma \langle f(t), h(t)\rangle dt, \qquad h \in L^p(\Sigma, N_*),\ f \in L^q_\sigma(\Sigma, N). \tag{2.1.2}$$

Theorem 2.1.1. *Let $1 \leqslant p < \infty$ and $1 < q \leqslant \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then the duality pairing (2.1.2) extends to an isometric isomorphism*

$$L^p(\Sigma, N_*)^* \simeq L^q_\sigma(\Sigma, N).$$

This theorem is due to Bukhvalov, see [12, Theorem 4.1], [13, Theorem 0.1]. We recall that $L^q(\Sigma) \otimes N$ is w^*-dense in $L^q_\sigma(\Sigma, N)$.

We have isometric identifications

$$L^\infty(\Sigma)\overline{\otimes}N \overset{(ii)}{\simeq} L^\infty_\sigma(\Sigma, N) \overset{(i)}{\simeq} B(L^1(\Sigma), N), \tag{2.1.3}$$

where $L^\infty(\Sigma)\overline{\otimes}N$ denotes the von Neumann tensor product of $L^\infty(\Sigma)$ and N. The identification (i) follows from (1.4.2) and Theorem 2.1.1. More explicitly, for any $F \in L^\infty_\sigma(\Sigma, N)$, the associated operator $T \in B(L^1(\Sigma), N)$ provided by (i) is given by

$$\forall h \in L^1(\Sigma),\ \forall \eta \in N_*,\ \langle T(h), \eta\rangle = \int_\Sigma \langle F(t), \eta\rangle h(t) dt. \tag{2.1.4}$$

The identification (ii) is proved in [58]. It is a combination of [58, Definition 1.22.10] and [58, Proposition 1.22.12].

Concerning $L^\infty(\Sigma)\overline{\otimes}N$, we mention that if N is equipped with a n.s.f. trace τ, then we equip $L^\infty(\Sigma)\overline{\otimes}N$ with the unique n.s.f. trace $\tilde{\tau} = \int \cdot \,\overline{\otimes}\tau$ such that $\tilde{\tau}(h \otimes x) = \tau(x)\int_\Sigma h(t)dt$ for all $h \in L^\infty(\Sigma)_+$ and $x \in N_+$. Then for all $1 \leqslant p < \infty$,

$$L^p(L^\infty(\Sigma)\overline{\otimes}N, \tilde{\tau}) \simeq L^p(\Sigma, L^p(N, \tau)). \tag{2.1.5}$$

Indeed let

$$\mathcal{E} = \operatorname{span}\big\{\chi_E : E \text{ measurable},\ |E| < \infty\big\}. \tag{2.1.6}$$

Then $\mathcal{E} \otimes (N \cap L^1(N))$ is both dense in $L^p(L^\infty\overline{\otimes}N, \tilde{\tau})$ and in $L^p(\Sigma, L^p(N))$, and the norms of $L^p(L^\infty\overline{\otimes}N, \tilde{\tau})$ and $L^p(\Sigma, L^p(N))$ coincide on this subspace.

Lemma 2.1.2. *Let $V : N_1 \to N_2$ be a w^*-continuous operator, where N_1 and N_2 are two von Neumann algebras. Then the map $V_\circ : L^\infty_\sigma(\Sigma, N_1) \to L^\infty_\sigma(\Sigma, N_2)$ given for all $F \in L^\infty_\sigma(\Sigma, N_1)$ by $V_\circ(F) = V \circ F$ is well-defined and w^*-continuous. Moreover, $\|V_\circ\| = \|V\|$.*

Proof. The operator V has a pre-adjoint $V_* : N_{2*} \to N_{1*}$. It is plain that $Id_{L_1(\Sigma)} \otimes V_*$ extends to a bounded map $Id_{L_1(\Sigma)} \overline{\otimes} V_* : L^1(\Sigma, N_{2*}) \to L^1(\Sigma, N_{1*})$, with norm $\|Id_{L^1(\Sigma)} \overline{\otimes} V_*\| = \|V_*\|$. Then using Theorem 2.1.1, $V_\circ$ coincides with $\big(Id_{L_1(\Sigma)} \overline{\otimes} V_*\big)^*$. □

The product of two N-valued measurable functions is measurable. However the product of two N-valued w^*-measurable functions is not necessarily w^*-measurable. We need the following to circumvent this difficulty.

Lemma 2.1.3. *Let $T_1 \in B(L^1(\Sigma); N)$ and $T_2 \in B(L^1(\Sigma); N)$. Then there exists a unique $P \in B(L^1(\Sigma^2), N)$ such that*

$$P(h_1 \otimes h_2) = T_1(h_1)T_2(h_2), \qquad h_1, h_2 \in L^1(\Sigma).$$

Proof. We define $Q : (h_1, h_2) \mapsto T_1(h_1)T_2(h_2)$ from $L^1(\Sigma) \times L^1(\Sigma)$ into N. This is a bounded bilinear map, with $\|Q\| \leqslant \|T_1\|\|T_2\|$. Hence there exists a bounded linear map $P : L^1(\Sigma)\hat{\otimes}L^1(\Sigma) \to N$ such that for all $h_1, h_2 \in L^1(\Sigma)$, $P(h_1 \otimes h_2) = T_1(h_1)T_2(h_2)$. Using $L^1(\Sigma)\hat{\otimes}L^1(\Sigma) \simeq L^1(\Sigma^2)$, this yields the result. □

Let $\varphi_1, \varphi_2 \in L^\infty_\sigma(\Sigma, N)$. Let $T_1, T_2 \in B(L^1(\Sigma); N)$ be the representatives of φ_1, and φ_2, respectively, through the identification (2.1.3), (i). We let

$$\varphi_1 \,\tilde{\times}\, \varphi_2 \in L^\infty_\sigma(\Sigma^2, N) \tag{2.1.7}$$

be the representative of the operator $P \in B(L^1(\Sigma^2), N)$ given by Lemma 2.1.3. Thus for all $h_1, h_2 \in L^1(\Sigma)$ and $\eta \in N_*$,

$$\langle T_1(h_1)T_2(h_2), \eta\rangle = \int_{\Sigma^2} \langle \varphi_1 \,\tilde{\times}\, \varphi_2(s,t), \eta\rangle h_1(s)h_2(t)dtds. \tag{2.1.8}$$

Remark 2.1.4. When $\varphi_1, \varphi_2 \in L^\infty(\Sigma, N)$, the product $\varphi_1 \,\tilde{\times}\, \varphi_2$ is simply given by $\varphi_1 \tilde{\times} \varphi_2(s,t) = \varphi_1(s)\varphi_2(t)$, since φ_1, φ_2 are both measurable (see Section 3.1 for more details on this case).

Lemma 2.1.5. *There exists a unique w^*-continuous contraction $\Gamma : L^2_\sigma(\Sigma^2, N) \to B(L^2(\Sigma))\overline{\otimes}N$ such that for all $\theta \in L^2(\Sigma^2)$ and for all $y \in N$, $\Gamma(\theta \otimes y) = S_\theta \otimes y$.*

Proof. Recall that $L^2_\sigma(\Sigma^2, N) = L^2(\Sigma^2, N_*)^*$, by Theorem 2.1.1. Let $|||.|||$ denote the norm on $S^1(L^2(\Sigma)) \otimes N_*$ induced by $(B(L^2(\Sigma))\overline{\otimes}N)_*$. Recall $\mathcal{E} \subset L^2(\Sigma)$ defined by (2.1.6). Then under the identification (1.4.5), $L^2(\Sigma) \otimes L^2(\Sigma) \otimes N_*$ is dense in $(B(L^2(\Sigma))\overline{\otimes}N)_*$, hence $\mathcal{E} \otimes \mathcal{E} \otimes N_*$ is dense in $(B(L^2(\Sigma))\overline{\otimes}N)_*$.

Let $L : \mathcal{E} \otimes \mathcal{E} \otimes N_* \to L^2(\Sigma^2, N_*)$ be the linear mapping such that for all $u, v \in \mathcal{E}$ and $\eta \in N_*$,

$$\big[L(u \otimes v \otimes \eta)\big](s,t) = u(s)v(t)\eta, \qquad (s,t) \in \Sigma^2.$$

Let $\Phi \in \mathcal{E} \otimes \mathcal{E} \otimes N_*$. It can be written as $\Phi = \sum_{i,j=1}^{M} \chi_{E_i} \otimes \chi_{F_j} \otimes \eta_{ij}$, where $M \geqslant 1$ is an integer, $E_1, \dots E_M$ are pairwise disjoint elements such that $0 < |E_i| < \infty$ for all i,

$F_1, \dots F_M$ are pairwise disjoint elements such that $0 < |F_j| < \infty$ for all j, and $\eta_{ij} \in N_*$. We define, for all $i, j = 1, \dots, M$, $z_{ij} = |E_i|^{\frac{1}{2}} |F_j|^{\frac{1}{2}} \eta_{ij} \in N_*$. By classical duality, there exists a family $(m_{ij})_{1 \leqslant i,j \leqslant M} \subset N$ such that

$$\left(\sum_{i,j=1}^{M} \|z_{ij}\|^2 \right)^{\frac{1}{2}} = \sum_{i,j=1}^{M} \langle z_{ij}, m_{ij} \rangle \qquad \text{and} \qquad \sum_{i,j=1}^{M} \|m_{ij}\|^2 \leqslant 1.$$

Thanks to the disjointness of the E_i and of the F_j, we have

$$\|L(\Phi)\|^2_{L^2(\Sigma^2, N_*)} = \sum_{ij} \|\eta_{ij}\|^2 |E_i||F_j| = \sum_{i,j} \|z_{ij}\|^2. \tag{2.1.9}$$

We now define

$$\Psi = \sum_{i,j=1}^{M} \frac{\chi_{E_i}}{|E_i|^{\frac{1}{2}}} \otimes \frac{\chi_{F_j}}{|F_j|^{\frac{1}{2}}} \otimes m_{ij} \in L^2(\Sigma) \otimes L^2(\Sigma) \otimes N \subset B(L^2(\Sigma)) \overline{\otimes} N.$$

We have

$$\langle \Phi, \Psi \rangle = \sum_{i,j} \langle z_{ij}, m_{ij} \rangle = \sum_{i,j} \|z_{ij}\|^2 = \|L(\Phi)\|_{L^2(\Sigma^2, N_*)},$$

by (2.1.9). Hence

$$\|L(\Phi)\|_{L^2(\Sigma^2, N_*)} \leqslant \|\Psi\|_{B(L^2(\Sigma)) \overline{\otimes} N} \|\Phi\|.$$

We will prove below that

$$\|\Psi\|_{B(L^2(\Sigma)) \overline{\otimes} N} \leqslant 1. \tag{2.1.10}$$

Taking this for granted, we obtain that L extends to a contraction (still denoted by)

$$L \colon \big(B(L^2(\Sigma)) \overline{\otimes} N \big)_* \longrightarrow L^2(\Sigma^2, N_*).$$

Its adjoint $\Gamma = L^*$ is a w^*-continuous contraction from $L^2_\sigma(\Sigma^2, N)$ into $B(L^2(\Sigma)) \overline{\otimes} N$ and it is easy to check that $\Gamma(\theta \otimes y) = S_\theta \otimes y$ for all $\theta \in L^2(\Sigma^2)$ and $y \in N$. This proves the existence result. The uniqueness comes from the w^*-continuity of Γ and the w^*-density of $L^2(\Sigma^2) \otimes N$.

It therefore remains to check (2.1.10). Let K be a Hilbert space such that $N \subset B(K)$ as a von Neumann algebra. Then

$$B(L^2(\Sigma)) \overline{\otimes} N \subset B(L^2(\Sigma)) \overline{\otimes} B(K) \simeq B(L^2(\Sigma, K)).$$

Let $\xi, \zeta \in L^2(\Sigma, K)$. Then

$$\langle \Psi(\xi), \zeta \rangle = \sum_{i,j=1}^{M} \frac{1}{|E_i|^{\frac{1}{2}} |F_j|^{\frac{1}{2}}} \int_{E_i \times F_j} \langle m_{ij}(\xi(t)), \zeta(s) \rangle dt ds.$$

Hence by Cauchy-Schwarz,

$$\begin{aligned}
|\langle \Psi(\xi), \zeta \rangle| &\leqslant \sum_{i,j=1}^{M} \frac{1}{|E_i|^{\frac{1}{2}} |F_j|^{\frac{1}{2}}} \|m_{ij}\| \Big(\int_{E_i} \|\xi(t)\| dt \Big) \Big(\int_{F_j} \|\zeta(s)\| ds \Big) \\
&\leqslant \left(\sum_{i,j=1}^{M} \|m_{ij}\|^2 \right)^{\frac{1}{2}} \left(\sum_{i,j=1}^{M} \frac{1}{|E_i||F_j|} \Big(\int_{E_i} \|\xi(t)\| dt \Big)^2 \Big(\int_{F_j} \|\zeta(s))\| ds \Big)^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i,j=1}^{M} \|m_{ij}\|^2 \right)^{\frac{1}{2}} \left(\sum_{i}^{M} \frac{1}{|E_i|} \Big(\int_{E_i} \|\xi(t)\| dt \Big)^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{M} \frac{1}{|F_j|} \Big(\int_{F_j} \|\zeta(s))\| ds \Big)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

By Cauchy-Schwarz again we have

$$\frac{1}{|E_i|}\Big(\int_{E_i}\|\xi(t)\|dt\Big)^2 \leqslant \int_{E_i}\|\xi(t)\|^2dt \qquad \text{and} \qquad \frac{1}{|F_j|}\Big(\int_{F_j}\|\zeta(s)\|dt\Big)^2 \leqslant \int_{F_j}\|\zeta(s)\|^2ds$$

for all $i,j=1,\ldots,M$. We derive

$$|\langle\Psi(\xi),\zeta\rangle| \leqslant \bigg(\int_\Sigma\|\xi(t)\|^2dt\bigg)^{\frac{1}{2}}\bigg(\int_\Sigma\|\zeta(s)\|^2ds\bigg)^{\frac{1}{2}},$$

which proves (2.1.10). □

2.2 Proof of Theorem (A)

The goal of the section is to prove Theorem (**A**), stated at the beginning of this chapter.

Throughout we let Σ be a σ-finite measure space, and we let $\varphi\in L^\infty(\Sigma^2)$ such that M_φ is a self-adjoint, unital, positive Schur multiplier on $B(L^2(\Sigma))$. We define a bilinear, symmetric map $V:L^1_{\mathbb{R}}(\Sigma)\times L^1_{\mathbb{R}}(\Sigma)\to\mathbb{R}$ by

$$V(f,h)=\int_{\Sigma^2}\varphi(s,t)h(t)f(s)dtds, \qquad (f,h)\in L^1_{\mathbb{R}}(\Sigma)\times L^1_{\mathbb{R}}(\Sigma).$$

We claim that V is positive. To prove it, fix some $f\in L^1_{\mathbb{R}}(\Sigma)\cap L^2_{\mathbb{R}}(\Sigma)$. For all $E\subset\Sigma$ such that $|E|<\infty$, we have $\chi_E\otimes\chi_E\geqslant 0$. Hence by the positivity of M_φ, we have

$$\langle[M_\varphi(\chi_E\otimes\chi_E)](f),f\rangle\geqslant 0.$$

Equivalently,

$$\int_{E\times E}\varphi(s,t)f(t)f(s)dtds\geqslant 0.$$

Passing to the supremum over E, we deduce

$$\int_{\Sigma\times\Sigma}\varphi(s,t)f(t)f(s)dtds\geqslant 0.$$

Since $L^1_{\mathbb{R}}(\Sigma)\cap L^2_{\mathbb{R}}(\Sigma)$ is dense in $L^1_{\mathbb{R}}(\Sigma)$, this proves the claim.

Let $K_V\subset L^1_{\mathbb{R}}(\Sigma)$ be the kernel of the seminorm $V(f,f)^{\frac{1}{2}}$. Equipped with the resulting norm, the quotient $L^1_{\mathbb{R}}(\Sigma)/K_V$ is a real pre-Hilbert space. We let $\mathbb{H}$ denote its completion. This is a real Hilbert space. Let N denote the tracial von Neumann algebra $(\Gamma_{-1}(\mathbb{H});\tau)$ (see Section 1.6).

In the sequel, wherever $h\in L^1_{\mathbb{R}}(\Sigma)$, we let $\dot{h}\in\mathbb{H}$ denote its class. We let $T:L^1(\Sigma)\to N$ be the unique linear map such that for all $h\in L^1_{\mathbb{R}}(\Sigma)$,

$$T(h)=\omega(\dot{h}).$$

We let $F\in L^\infty_\sigma(\Sigma,N)$ and $d_0\in L^\infty(\Sigma)\overline{\otimes}N$ be associated with $T\in B(L^1(\Sigma),N)$ through the identifications $B(L^1(\Sigma),N)\simeq L^\infty_\sigma(\Sigma,N)$ and $B(L^1(\Sigma),N)\simeq L^\infty(\Sigma)\overline{\otimes}N$ provided by (2.1.3).

Lemma 2.2.1. *We have* $\|T\| \leqslant 1$, $\|F\| \leqslant 1$ *and* $\|d_0\| \leqslant 1$.

Proof. We recall that M_φ is positive and unital. By the Russo-Dye Theorem, this implies that $\|M_\varphi^\infty\| = 1$. It follows from Lemma 1.4.2 that $\|M_\varphi^1\| = 1$. Hence by interpolation, we have $\|M_\varphi : S^2(L^2(\Sigma)) \to S^2(L^2(\Sigma))\| \leqslant 1$. Consequently, $\|\varphi\|_\infty \leqslant 1$ thanks to (1.4.8).

For all $f \in L^1_{\mathbb{R}}(\Sigma)$, we have

$$\|\dot{f}\|_{\mathbb{H}}^2 = \int_{\Sigma^2} \varphi(s,t) f(t) f(s) dt ds \leqslant \|\varphi\|_\infty \|f\|_1^2 \leqslant \|f\|_1^2.$$

So we have, thanks to (1.6.2),

$$\|\omega(\dot{f})\| = \|\dot{f}\|_{\mathbb{H}} \leqslant \|f\|_1. \tag{2.2.1}$$

Consider $h = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, where $E_1, \dots E_n$ are pairwise disjoint elements of finite measure and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Then we have $\|h\|_1 = \sum |\alpha_i||E_i|$. On the other hand,

$$T(h) = \sum \alpha_i T(\chi_{E_i}) = \sum \alpha_i w(\dot{\chi_{E_i}}).$$

Hence

$$\|T(h)\|_1 \leqslant \sum |\alpha_i| \, \|w(\dot{\chi_{E_i}})\| \leqslant \sum |\alpha_i| \|\chi_{E_i}\|_1 = \sum |\alpha_i||E_i| = \|h\|_1,$$

by (2.2.1). This shows that $\|T\| \leqslant 1$, and hence $\|F\| \leqslant 1$ and $\|d_0\| \leqslant 1$. □

The following lemma provides a link between F and φ.

Lemma 2.2.2. *For almost every* $(s,t) \in \Sigma^2$, $\tau(F \tilde{\times} F(s,t)) = \varphi(s,t)$.

Proof. We use the equation (2.1.8) with $\eta = \tau$. We obtain for all $f, h \in L^1_{\mathbb{R}}(\Sigma)$,

$$\begin{aligned}\int_{\Sigma^2} \tau(F \tilde{\times} F(s,t)) f(s) h(t) dt ds &= \tau(T(f)T(h)) \\ &= \tau(\omega(\dot{f})\omega(\dot{h})) \\ &= \int_{\Sigma^2} \varphi(s,t) f(s) h(t) dt ds,\end{aligned}$$

by (1.6.3). The expected equality follows at once. □

Lemma 2.2.3. *We have* $d_0^* = d_0$ *and* $d_0^2 = 1$.

Proof. We let $\tilde{\tau} = \int \cdot \overline{\otimes} \tau$ denote the usual trace on $L^\infty(\Sigma)\overline{\otimes}N$. For any $f \in L^1_{\mathbb{R}}(\Sigma)$ and $\eta \in N_* \simeq L^1(N)$, we have

$$\begin{aligned}\langle d_0^*, f \otimes \eta \rangle &= \tilde{\tau}(d_0^*(f \otimes \eta)) = \overline{\tilde{\tau}(d_0(f \otimes \eta^*))} \\ &= \overline{\tau(T(f)\eta^*)} = \overline{\tau(w(\dot{f})\eta^*)} \\ &= \tau(w(\dot{f})\eta) = \tau(T(f)\eta) = \langle d_0, f \otimes \eta \rangle.\end{aligned}$$

By linearity and density, this implies that $d_0^* = d_0$.

By assumption, (Σ, μ) is σ-finite. Thus there exists a positive function $v \in L^1(\Sigma)$ such that $\|v\|_1 = 1$. Changing $d\mu$ into $v d\mu$, we may therefore assume that μ is a probability measure. Since τ is normalized, the trace $\tilde{\tau}$ is normalized as well.

Let $j : N \to L^2(N)$ be the natural embedding and let $T_2 = j \circ T : L^1(\Sigma) \to L^2(N)$. Since $L^2(N)$ is a Hilbert space, it has the Radon-Nikodym property (see [17, corollary IV.1.4]). Hence there exists $\gamma \in L^\infty(\Sigma, L^2(N))$ such that for all $h \in L^1(\Sigma)$:

$$T_2(h) = \int_\Sigma h(t)\gamma(t)dt.$$

For any $f, h \in L^1_{\mathbb{R}}(\Sigma)$, we have

$$\begin{aligned}
\int_{\Sigma^2} \varphi(s,t)f(s)h(t)dtds &= \tau(\omega(\dot{f})\omega(\dot{h})), \quad \text{by (1.6.3)}, \\
&= \tau(\omega(\dot{f})\omega(\dot{h})^*) \\
&= \langle j(\omega(\dot{f})), j(\omega(\dot{h}))\rangle_{L^2(N)} \\
&= \langle T_2(f), T_2(h)\rangle_{L^2(N)} \\
&= \left\langle \int_\Sigma f(s)\gamma(s)ds, \int_\Sigma h(t)\gamma(t)dt \right\rangle_{L^2(N)} \\
&= \int_{\Sigma^2} \langle \gamma(s), \gamma(t)\rangle_{L^2(N)} f(s)h(t)dtds.
\end{aligned}$$

This implies that for almost every $(s,t) \in \Sigma^2$,

$$\langle \gamma(s), \gamma(t)\rangle_{L^2(N)} = \varphi(s,t). \tag{2.2.2}$$

We obtained a factorization of φ as in Theorem 1.4.3, with $\alpha = \beta = \gamma$. Now we can use Theorem 1.4.6 and we obtain that for almost every $t \in \Sigma$,

$$\langle \gamma(t), \gamma(t)\rangle_{L^2(N)} = 1$$

Since the measure on Σ is normalized, this yields

$$\int_\Sigma \|\gamma(t)\|^2_{L^2(N)} dt = 1.$$

Since $L^\infty(\Sigma)\overline{\otimes}N$ is normalized we have, using (2.1.5), a contractive inclusion

$$L^\infty(\Sigma)\overline{\otimes}N \subset L^2(\Sigma, L^2(N)).$$

By construction, $\gamma \in L^2(\Sigma, L^2(N))$ corresponds to d_0. Thus we have proved that

$$\tilde{\tau}(d_0^* d_0) = 1.$$

Since $\tilde{\tau}$ is normalized, this shows that $\tilde{\tau}(1 - d_0^* d_0) = 0$. However by Lemma 2.2.1, $\|d_0\| \leqslant 1$ hence $1 - d_0^* d_0 \geqslant 0$. Since $\tilde{\tau}$ is faithful, we deduce that $1 - d_0^* d_0 = 0$, that is, $d_0^2 = 1$. □

Remark 2.2.4. The identity (2.2.2) provides a new proof of Theorem 1.4.5.

We are now ready to introduce the maps U, J providing the absolute dilation of M_φ. We let N^∞ denote the infinite von Neumann tensor product $\overline{\otimes}_{\mathbb{Z}} N$ and we let τ^∞ denote the normal faithful finite trace on N^∞ (see [64]). We set

$$F^\infty = \cdots \otimes 1_N \otimes F \otimes 1_N \cdots \in L^\infty_\sigma(\Sigma, N^\infty),$$

where F is in 0 position. We denote by $T^\infty \in B(L^1(\Sigma), N^\infty)$ the element associated with F^∞ through the identification between $L^\infty_\sigma(\Sigma, N^\infty)$ and $B(L^1(\Sigma), N^\infty)$. We also let $d \in L^\infty(\Sigma)\overline{\otimes}N^\infty$ be associated with F^∞ through the identification between $L^\infty_\sigma(\Sigma, N^\infty)$ and $L^\infty(\Sigma)\overline{\otimes}N^\infty$. We know from above that

$$\|d\|_\infty \leqslant 1,\ d^* = d \text{ and } d^2 = 1.$$

In the sequel, we regard $L^\infty(\Sigma)$ as a von Neumann subalgebra of $B(L^2(\Sigma))$ by identifying any $\phi \in L^\infty(\Sigma)$ with the multiplication operator, $h \mapsto \phi h$ for $h \in L^2(\Sigma)$. Thus we have

$$L^\infty_\sigma(\Sigma, N^\infty) \simeq L^\infty(\Sigma)\overline{\otimes}N^\infty \subset B(L^2(\Sigma))\overline{\otimes}N^\infty.$$

We will see d as an element of $B(L^2(\Sigma))\overline{\otimes}N^\infty$.

We recall that tr denotes the trace on $B(L^2(\Sigma))$, we let $\mathcal{M} = B(L^2(\Sigma))\overline{\otimes}N^\infty$ and we let $\tau_{\mathcal{M}} = \mathrm{tr}\ \overline{\otimes}\tau^\infty$ be the natural semifinite normal faithful trace on $\mathcal{M}$. We define

$$J : B(L^2(\Sigma)) \longrightarrow B(L^2(\Sigma))\overline{\otimes}N^\infty, \qquad J(x) = x \otimes 1_{N^\infty}.$$

This is a w^*-continuous trace preserving one-to-one $*$-homomorphism. We let $\mathbb{E} : \mathcal{M} \to B(L^2(\Sigma))$ denote the conditional expectation associated with J.

We introduce the right shift $\mathcal{S} : N^\infty \to N^\infty$. This is a normal, trace preserving $*$-automorphism such that for all $(x_n)_{n\in\mathbb{Z}} \subset N$,

$$\mathcal{S}(\cdots \otimes x_0 \otimes x_1 \otimes x_2 \otimes \cdots) = \cdots \otimes x_{-1} \otimes x_0 \otimes x_1 \otimes \cdots .$$

We define

$$U : B(L^2(\Sigma))\overline{\otimes}N^\infty \longrightarrow B(L^2(\Sigma))\overline{\otimes}N^\infty$$

by

$$U(y) = d((Id \otimes \mathcal{S})(y))d, \qquad y \in B(L^2(\Sigma))\overline{\otimes}N^\infty.$$

Since d is a self-adoint symmetry, U is a normal, trace preserving $*$-automorphism.

We apply Lemma 2.1.5 with N^∞ and we let

$$\Gamma : L^2_\sigma(\Sigma^2, N^\infty) \longrightarrow B(L^2(\Sigma))\overline{\otimes}N^\infty$$

be the resulting w^*-continuous contraction.

Lemma 2.2.5. *The mapping $R : L^2_\sigma(\Sigma^2, N^\infty) \to L^2_\sigma(\Sigma, N^\infty)$ defined by $R(F) = \mathcal{S} \circ F$ is w^*- continuous. In addition, we have*

$$(Id \otimes \mathcal{S})\Gamma = \Gamma R. \tag{2.2.3}$$

Proof. We apply Lemma 2.1.2 with $V = \mathcal{S}$ to prove the first point. For all $\theta \in L^2(\Sigma)$, and $y \in N^\infty$, we have

$$(Id \otimes \mathcal{S})\Gamma(\theta \otimes y) = S_\theta \otimes \mathcal{S}(y) = \Gamma(\theta \otimes \mathcal{S}(y)) = \Gamma(R(\theta \otimes y)).$$

We deduce the equality $(Id \otimes \mathcal{S})\Gamma = \Gamma R$ on $L^2_\sigma(\Sigma^2, N^\infty)$, by linearity, w^*-continuity of the maps and w^*-density of $L^2(\Sigma^2) \otimes N^\infty$ in $L^2_\sigma(\Sigma^2, N^\infty)$. □

For any $k \in \mathbb{N}$ we set $N^k = \overbrace{N \overline{\otimes} \cdots \overline{\otimes} N}^{\text{k times}}$. Then we let $L_k : N^k \to N^\infty$ be the unique w^*-continuous $*$-homomorphism such that:

$$L_k(z_1 \otimes \cdots \otimes z_k) = \cdots \otimes 1_N \otimes z_1 \otimes \cdots \otimes z_k \otimes 1_N \cdots, \qquad z_1, \ldots, z_k \in N,$$

where z_j is at position j for all $1 \leqslant j \leqslant k$. For the rest of the chapter, we identify N^k with its image $L_k(N^k)$.

Lemma 2.2.6.

1. *Let $f \in L^\infty_\sigma(\Sigma, N)$, $k \in \mathbb{N}$, and let $A : L^\infty_\sigma(\Sigma, N^k) \to L^\infty_\sigma(\Sigma, N^\infty)$ be defined by*

$$A(g) = \cdots 1_N \otimes 1_N \otimes \quad f \quad \otimes \quad \underset{\uparrow}{g} \quad \otimes \quad \underset{\uparrow}{1_N} \quad \cdots$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \qquad k+1$$

   Then A is w^*-continuous.

2. *Let $\Psi \in L^\infty_\sigma(\Sigma, N^\infty)$ and let $B : L^\infty_\sigma(\Sigma, N^\infty) \to L^\infty_\sigma(\Sigma^2, N^\infty)$ be defined by*

$$B(\Phi) = \Phi \tilde{\times} \Psi.$$

   Then B is w^*-continuous.

Proof. In the definition of A, the 1_N play no role. So we may consider $A : L^\infty_\sigma(\Sigma, N^k) \to L^\infty_\sigma(\Sigma, N \overline{\otimes} N^k)$ such that $A(h) = f \otimes h$, instead of the map given in the statement. To prove that A is w^*-continuous, it suffices to show that for any bounded net (h_i) which converges to h in the w^*-topology of $L^\infty_\sigma(\Sigma, N^k)$ and for any $V \in (L^\infty_\sigma(\Sigma, N \overline{\otimes} N^k))_* = L^1(\Sigma, (N \overline{\otimes} N^k)_*)$, we have

$$\langle A(h_i), V \rangle \to \langle A(h), V \rangle. \tag{2.2.4}$$

We know that $L^1(\Sigma) \otimes N_* \otimes N_*^k$ is dense in $L^1(\Sigma, (N \overline{\otimes} N^k)_*$ (see the beginning of Section 1.4.2). Let $(h_i) \subset L^\infty_\sigma(\Sigma, N^k)$ be a bounded net which w^*-converges to h. Then for all $v \in L^1(\Sigma)$ and $\mu \in N_*^k$, we have:

$$\langle h_i, v \otimes \mu \rangle \to \langle h, v \otimes \mu \rangle \tag{2.2.5}$$

We note that for all $\eta \in N_*$, $\mu \in N_*^k$ and $l \in L^1(\Sigma)$:

$$\langle A(h_i), l \otimes \eta \otimes \mu \rangle = \int_\Sigma l(t) \langle f(t), \eta \rangle \langle h_i(t), \mu \rangle dt.$$

Using (2.2.5) with $v(t) = l(t)\langle f(t), \eta \rangle$, we derive

$$\langle A(h_i), l \otimes \eta \otimes \mu \rangle \to \langle A(h), l \otimes \eta \otimes \mu \rangle.$$

By linearity and density, and by the boundedness of (h_i), we obtain (2.2.4).

Now we prove the w^*- continuity of B. We let

$$\tilde{B} : B(L^1(\Sigma), N^\infty) \to B(L^1(\Sigma^2), N^\infty)$$

be the operator corresponding to B if we use the identifications $B(L^1(\Sigma); N^\infty) = L^\infty_\sigma(\Sigma, N^\infty)$ and $B(L^1(\Sigma^2); N^\infty) = L^\infty_\sigma(\Sigma^2, N^\infty)$. We will prove that $\tilde{B}$ is w^*-continuous. Let $S \in B(L^1(\Sigma); N^\infty)$ be corresponding to Ψ. Then for all $T \in B(L^1(\Sigma), N^\infty)$ and for all $f, h \in L^1(\Sigma)$, we have

$$\tilde{B}(T)(f \otimes h) = T(f)S(h)$$

We know that $L^1(\Sigma) \otimes L^1(\Sigma) \otimes N_*^\infty$ is dense in $L^1(\Sigma^2; N_*^\infty) = B(L^1(\Sigma^2); N^\infty)_*$. For all $f, h \in L^1(\Sigma)$ and $\eta \in N_*^\infty$, we have

$$\langle \tilde{B}(T), f \otimes h \otimes \eta \rangle = \langle T(f)S(h), \eta \rangle = \langle T(f), S(h)\eta \rangle = \langle T, f \otimes S(h)\eta \rangle. \tag{2.2.6}$$

Let $(T_i) \subset B(L^1(\Sigma); N^\infty)$ be a bounded net which converges to $T \in B(L^1(\Sigma); N^\infty)$ in the w^*-topology. We have:

$$\langle T_i, f \otimes S(h)\eta \rangle \to \langle T, f \otimes S(h)\eta \rangle$$

We therefore obtain, thanks to (2.2.6), that

$$\langle \tilde{B}(T_i), f \otimes h \otimes \eta \rangle \to \langle \tilde{B}(T), f \otimes h \otimes \eta \rangle.$$

By linearity, by density and by the boundedness of (T_i), we obtain that for all $V \in L^1(\Sigma, N_*^\infty)$:

$$\langle \tilde{B}(T_i), V \rangle \to \langle \tilde{B}(T), V \rangle.$$

This shows the w^*-continuity of $\tilde{B}$, and hence that of B. □

Remark 2.2.7. Let $\Psi \in L^\infty_\sigma(\Sigma, N^\infty)$, $\theta \in L^2(\Sigma^2)$ and $y \in N^\infty$. We deduce from the previous lemma the w^*-continuity of the map $\Phi \mapsto \theta(\Phi \tilde{\times} \Psi)y$ from $L^\infty_\sigma(\Sigma, N^\infty)$ into $L^2_\sigma(\Sigma^2, N^\infty)$.

The following lemma explains how to swap d and Γ.

Lemma 2.2.8. *For all $\theta \in L^2(\Sigma^2)$ and for all $y = \cdots \otimes y_{-1} \otimes 1_N \otimes y_1 \cdots \in N^\infty$, we have:*

$$d\Gamma(\theta \otimes y)d = \Gamma\left((s,t) \mapsto \theta(s,t)[\cdots \otimes y_{-1} \otimes F \tilde{\times} F(s,t) \otimes y_1 \otimes \cdots]\right) \tag{2.2.7}$$

Proof. Recall $\mathcal{E}$ given by (2.1.6). Instead of d, we first consider approximations by fnite sums

$$\phi = \sum_i \chi_{E_i} \otimes \cdots \otimes 1_N \otimes m_i \otimes 1_N \otimes \cdots \in \mathcal{E} \otimes N^\infty$$

and

$$\phi' = \sum_j \chi_{E'_j} \otimes \cdots \otimes 1_N \otimes m'_j \otimes 1_N \otimes \cdots \in \mathcal{E} \otimes N^\infty.$$

We have:

$$\begin{aligned}\phi\Gamma(\theta\otimes y)\phi' &= \phi(S_\theta\otimes y)\phi'\\ &= \sum_{i,j}\chi_{E_i}.S_\theta.\chi_{E'_j}\otimes\cdots\otimes y_{-1}\otimes m_i m'_j\otimes y_1\otimes\cdots\\ &= \sum_{i,j}\Gamma\Big((s,t)\mapsto\chi_{E_i}(s)\theta(s,t)\chi_{E'_j}(t)[\cdots\otimes y_{-1}\otimes m_i m'_j\otimes y_1\otimes\cdots]\Big)\\ &= \Gamma\left((s,t)\mapsto\theta(s,t)\Big[\cdots\otimes y_{-1}\otimes\Big(\sum_i\chi_{E_i}(s)m_i\Big)\Big(\sum_j\chi_{E'_j}(t)m'_j\Big)\otimes y_1\otimes\cdots\Big]\right)\\ &= \Gamma\left((s,t)\mapsto\theta(s,t)\phi(s)\phi'(t)y\right).\end{aligned}$$

Here ϕ and ϕ' are measurable, hence we have $\phi(s)\phi'(t)=\phi\,\tilde{\times}\,\phi'(s,t)$, by Remark 2.1.4. Hence the preceding equality gives:

$$\phi\Gamma(\theta\otimes y)\phi'=\Gamma\big((s,t)\mapsto\theta(s,t)\phi\,\tilde{\times}\,\phi'(s,t)y\big)$$

In this identity, the maps ϕ and ϕ' are regarded as elements of $B(L^2(\Sigma))\overline{\otimes}N^\infty$ on the left hand-side, and as elements of $L^\infty(\Sigma,N^\infty)$ on the right hand-side. Using the facts that $\mathcal{E}\otimes N^\infty$ is w^*-dense in $L^\infty_\sigma(\Sigma,N^\infty)$, that Γ is w^*- continuous, as well as Lemma 2.2.6 and Remark 2.2.7, we deduce that

$$\phi\Gamma(\theta\otimes y)d=\Gamma\left((s,t)\mapsto\theta(s,t)(\phi\tilde{\times}F^\infty)(s,t)y\right)$$

and then

$$d\Gamma(\theta\otimes y)d=\Gamma\left((s,t)\mapsto\theta(s,t)(F^\infty\,\tilde{\times}\,F^\infty)(s,t)y\right).$$

The last thing to observe is that $F^\infty\tilde{\times}F^\infty=\cdots\otimes 1_N\otimes F\tilde{\times}F\otimes 1_N\otimes\cdots$. The latter is true by the uniqueness of the construction. □

Remark 2.2.9. The reader not familiar with the approximation techniques used to establish Lemma 2.2.10 should first look at the finite-dimensional case to gain intuition on this proof. We provide this finite-dimensional proof for convenience. Thus, we now assume that $\Sigma=\{1,\dots,n\}$ for some integer $n\geqslant 1$, so that $L^\infty(\Sigma)=\ell^\infty_n$, $L^2(\Sigma)=\ell^2_n$, $B(L^2(\Sigma))=M_n$. Then $F\in\ell^n(N)$ is written as $F=(F_1,\dots,F_n)$, where each $F_k\in N$ is a unitary, and $\varphi=[\varphi(i,j)]_{1\leqslant i,j\leqslant n}$ is given by Lemma 2.2.2 by

$$\varphi(i,j)=\tau_N(F_i^*F_j),\qquad 1\leqslant i,j\leqslant n.$$

Write $u=(u_1,\dots,u_n)\in\ell^2_n$ and similarly for v,a,b. Then in the space $B(L^2(\Sigma))\overline{\otimes}N^\infty=M_n(N^\infty)$, we have $u\otimes v\otimes 1_\infty=[u_iv_j\cdot 1_\infty]$. Hence $\mathcal{S}(u\otimes v\otimes 1_\infty)=(u\otimes v\otimes 1_\infty)$, hence

$$UJ(S_{u\otimes v})=U(u\otimes v\otimes 1_\infty)=d(u\otimes v\otimes 1_\infty)d.$$

By Lemma 2.2.8, we obtain

$$d(u\otimes v\otimes 1_\infty)d=\big[u_iv_j(\cdots 1\otimes F_i^*F_j\otimes 1\cdots)\big],$$

where the $F_i^* F_j$ are in position 0. Next,

$$\mathcal{S}\big(U(u \otimes v \otimes 1_\infty)\big) = \big[u_i v_j(\cdots 1 \otimes 1 \otimes F_i^* F_j \otimes 1 \cdots)\big],$$

where the $F_i^* F_j$ are in position 1 and

$$U^2(u \otimes v \otimes 1_\infty) = \big[u_i v_j(\cdots 1 \otimes F_i^* F_j \otimes F_i^* F_j \otimes 1 \cdots)\big],$$

where the $F_i^* F_j$ appear in positions 0 and 1. By induction, we find that for all $k \geqslant 0$,

$$U^k(u \otimes v \otimes 1_\infty) = \big[u_i v_j(\cdots 1 \otimes F_i^* F_j \otimes F_i^* F_j \otimes \cdots \otimes F_i^* F_j \otimes 1 \cdots)\big],$$

where the $F_i^* F_j$ appear in positions $0, 1, \ldots k-1$, which is the requested identity (2.2.8) in this context.

Lemma 2.2.10. *For all $f, h \in L^2(\Sigma)$, and for all integer $k \geqslant 0$, we have:*

$$U^k J(S_{f\otimes h}) = \Gamma\Big((s,t) \mapsto f(s)h(t)[\cdots \otimes 1_N \otimes \underset{\substack{\uparrow \\ 0}}{F\tilde{\times}F(s,t)} \otimes \cdots \otimes \underset{\substack{\uparrow \\ (k-1)}}{F\tilde{\times}F(s,t)} \otimes 1_N \otimes \cdots]\Big). \tag{2.2.8}$$

Proof. We prove this lemma by induction. The result is true for $k = 0$ since

$$\begin{aligned} J(S_{f\otimes h}) &= S_{f\otimes h} \otimes \cdots \otimes 1_N \otimes \cdots \\ &= \Gamma((f \otimes h) \otimes 1_{N^\infty}) \\ &= \Gamma\big((s,t) \mapsto f(s)h(t)[\cdots \otimes 1_N \otimes \cdots]\big). \end{aligned}$$

We now suppose that the result holds true for some $k \geqslant 0$. By the w^*-density of $L^\infty(\Sigma^2) \otimes N$ into $L^\infty_\sigma(\Sigma^2, N)$, we may write

$$\begin{aligned} &(s,t) \mapsto f(s)h(t)[\cdots \otimes 1_N \otimes F\tilde{\times}F(s,t) \otimes \cdots \otimes F\tilde{\times}F(s,t) \otimes 1_N \otimes \cdots] \\ &= \overset{w^*}{\lim_i} \sum_{j \text{ finite}} \theta_j^i \otimes \cdots \otimes 1_N \otimes m_{i,j}^0 \otimes \cdots \otimes m_{i,j}^{k-1} \otimes 1_N \otimes \cdots, \end{aligned}$$

where $\theta_j^i \in L^\infty(\Sigma^2)$ and $m_{i,j}^l \in N$ for all i, j, l. We have:

$$\begin{aligned} &U^{k+1}(J(S_{f\otimes h})) = U(U^k J(S_{f\otimes h})) \\ &= U(\Gamma((s,t) \mapsto f(s)h(t)[\cdots \otimes 1_N \otimes F\tilde{\times}F(s,t) \otimes \cdots \otimes F\tilde{\times}F(s,t) \otimes 1_N \otimes \cdots]) \\ &= U\left(\Gamma\left(\overset{w^*}{\lim_i} \sum_{j \text{ finite}} \theta_j^i \otimes \cdots \otimes 1_N \otimes m_{i,j}^0 \otimes \cdots \otimes m_{i,j}^{k-1} \otimes 1_N \otimes \cdots\right)\right) \\ &= \overset{w^*}{\lim_i} \sum_{j \text{ finite}} U\left(\Gamma\left(\theta_j^i \otimes \cdots \otimes 1_N \otimes m_{i,j}^0 \otimes \cdots \otimes m_{i,j}^{k-1} \otimes 1_N \otimes \cdots\right)\right). \end{aligned}$$

By w*-continuity of Γ and U,

$$= \overset{w^*}{\lim_i} \sum_{j \text{ finite}} d\left(((Id \otimes \mathcal{S})(\Gamma(\theta_j^i \otimes \cdots \otimes 1_N \otimes m_{i,j}^0 \otimes \cdots \otimes m_{i,j}^{k-1} \otimes 1_N \otimes \cdots))\right) d$$
$$= \overset{w^*}{\lim_i} \sum_{j \text{ finite}} d(\Gamma(\theta_j^i \otimes \mathcal{S}(\cdots \otimes 1_N \otimes m_{i,j}^0 \otimes \cdots \otimes m_{i,j}^{k-1} \otimes 1_N \otimes \cdots)))d.$$

By Lemma 2.2.8, the latter is equal to

$$\overset{w^*}{\lim_i} \sum_j \Gamma\big((s,t) \mapsto \theta_j^i(s,t)[\cdots \otimes 1_N \otimes \underset{\uparrow 0}{F \tilde{\times} F(s,t)} \otimes m_{i,j}^0 \otimes \cdots \otimes \underset{\uparrow k}{m_{i,j}^{k-1}} \otimes 1_N \otimes \cdots]\big)$$

By Lemma 2.2.6, (1), and the w*-continuity of Γ, this is equal to

$$\Gamma\big((s,t) \mapsto [\cdots \otimes \underset{\uparrow 0}{F \tilde{\times} F(s,t)} \otimes \Big(\overset{w^*}{\lim_i} \textstyle\sum_{j \text{ finite}} \theta_j^i(s,t) m_{i,j}^0 \otimes \cdots \otimes m_{i,j}^{k-1}\Big) \otimes \underset{\uparrow k+1}{1_N} \cdots])$$

We let $\psi \in L^2_\sigma(\Sigma, N^\infty)$ be defined by

$$\psi(s,t) = \cdots \otimes F \tilde{\times} F(s,t) \otimes \left(\overset{w^*}{\lim_i} \sum_{j \text{ finite}} \theta_j^i(s,t) m_{i,j}^0 \otimes \cdots \otimes m_{i,j}^{k-1}\right) \otimes 1_N \cdots .$$

Then we have obtained that

$$U^{k+1}(J(S_{f \otimes h})) = \Gamma(\psi)$$

Recall the mapping $R : L^2(\Sigma^2, N^\infty) \to L^2(\Sigma^2, N^\infty)$ from Lemma 2.2.5. Then we have

$$\psi = [\cdots \otimes F \tilde{\times} F \otimes \left(\overset{w^*}{\lim_i} \sum_{j \text{ finite}} \theta_j^i m_{i,j}^0 \otimes \cdots \otimes m_{i,j}^{k-1}\right) \otimes 1_N \cdots].$$
$$= [\cdots \otimes \underset{\uparrow 0}{F \tilde{\times} F} \otimes 1_N \cdots] \overset{w^*}{\lim_i} \sum_{j \text{ finite}} \theta_j^i [\cdots \otimes \underset{\uparrow 0}{1_N} \otimes m_{i,j}^0 \otimes \cdots \otimes \underset{\uparrow k}{m_{i,j}^{k-1}} \otimes \cdots].$$
$$= [\cdots \otimes \underset{\uparrow 0}{F \tilde{\times} F} \otimes 1_N \cdots] \overset{w^*}{\lim_i} (R(\sum_{j \text{ finite}} \theta_j^i [\cdots \otimes \underset{\uparrow 0}{m_{i,j}^0} \otimes \cdots \otimes \underset{\uparrow (k-1)}{m_{i,j}^{k-1}} \otimes \cdots]))$$

By the w^*- continuity of R,

$$= [\cdots \otimes F\tilde{\times}F \otimes 1_N \cdots] R\left(\overset{w^*}{\lim_i} \sum_{j \text{ finite}} \theta_j^i [\cdots \otimes m_{i,j}^0 \otimes \cdots \otimes m_{i,j}^{k-1} \otimes \cdots]\right)$$

$$= [\cdots \otimes \underset{\substack{\uparrow \\ 0}}{F\tilde{\times}F} \otimes 1_N \cdots](f \otimes h)[\cdots \otimes \underset{\substack{\uparrow \\ 0}}{1_N} \otimes F\tilde{\times}F \otimes \cdots \otimes \underset{\substack{\uparrow \\ k}}{F\tilde{\times}F} \otimes \cdots]$$

$$= f \otimes h[\cdots \otimes 1_N \otimes \underset{\substack{\uparrow \\ 0}}{F\tilde{\times}F} \otimes \cdots \otimes \underset{\substack{\uparrow \\ k}}{F\tilde{\times}F} \otimes 1_N \otimes \cdots]).$$

We obtain the result for $k+1$. This proves the lemma. □

We now conclude the proof of Theorem (**A**) by proving

$$(M_\varphi^\infty)^k = \mathbb{E}U^k J, \qquad k \geqslant 0. \tag{2.2.9}$$

By linearity, density and duality, it suffices to prove that for all $f, h \in L^2(\Sigma)$ and for all $u, v \in L^2(\Sigma)$, we have

$$\left\langle \mathbb{E}U^k J(S_{f\otimes h}), u \otimes v \right\rangle = \left\langle M_\varphi^k(S_{f\otimes h}), u \otimes v \right\rangle. \tag{2.2.10}$$

Let $\gamma : \left(B(L^2(\Sigma))\overline{\otimes}N^\infty\right)_* \to L^2(\Sigma^2, N_*^\infty)$ such that $\gamma^* = \Gamma$. We write

$$\left\langle \mathbb{E}U^k J(S_{f\otimes h}), u \otimes v \right\rangle = \left\langle U^k J(S_{f\otimes h}), J_1(u \otimes v) \right\rangle.$$

By lemma 2.2.10, the latter is equal to

$$= \langle \Gamma((s,t) \mapsto f(s)h(t)[\cdots \otimes F\tilde{\times}F(s,t) \otimes \cdots \otimes F\tilde{\times}F(s,t) \otimes 1_N \otimes \cdots]), J_1(u \otimes v)\rangle$$
$$= \langle (s,t) \mapsto f(s)h(t)[\cdots \otimes F\tilde{\times}F(s,t) \otimes \cdots \otimes F\tilde{\times}F(s,t) \otimes 1_N \otimes \cdots], \gamma(J_1(u \otimes v))\rangle$$
$$= \left\langle (s,t) \mapsto f(s)h(t)[\cdots \otimes F\tilde{\times}F(s,t) \otimes \cdots \otimes F\tilde{\times}F(s,t) \otimes 1_N \otimes \cdots], \gamma(u \otimes v \otimes 1_{N_*^\infty})\right\rangle$$
$$= \left\langle (s,t) \mapsto f(s)h(t)[\cdots \otimes F\tilde{\times}F(s,t) \otimes \cdots \otimes F\tilde{\times}F(s,t) \otimes 1_N \otimes \cdots], (s,t) \mapsto u(s)v(t)1_{N_*^\infty})\right\rangle$$
$$= \int_{\Sigma^2} \left\langle f(s)h(t)[\cdots \otimes F\tilde{\times}F(s,t) \otimes \cdots \otimes F\tilde{\times}F(s,t) \otimes 1_N \otimes \cdots], u(s)v(t)1_{N_*^\infty})\right\rangle dsdt.$$

Consequently,

$$\begin{aligned}
\left\langle \mathbb{E}U^k J(S_{f\otimes h}), u \otimes v \right\rangle &= \int_{\Sigma^2} f(s)h(t)u(s)v(t)\langle F\tilde{\times}F(s,t), 1_{N_*}\rangle^k dsdt \\
&= \int_{\Sigma^2} f(s)h(t)u(s)v(t)\tau(F\tilde{\times}F(s,t))^k dsdt \\
&= \int_{\Sigma^2} f(s)h(t)u(s)v(t)\varphi(s,t)^k dsdt, \text{ by Lemma 2.2.2} \\
&= \left\langle M_\varphi^k(S_{f\otimes h}), u \otimes v \right\rangle
\end{aligned}$$

This shows (2.2.10) and concludes the proof.

By Proposition 1.3.5, we have the following explicit consequence of Theorem (**A**).

Theorem 2.2.11. *Let Σ be a σ-finite measure space. Let M_φ is a self-adjoint, unital, positive Schur multiplier on $B(L^2(\Sigma))$. Then for all $1 \leqslant p < \infty$, the map $(M_\varphi)_p : S^p(L^2(\Sigma)) \to S^p(L^2(\Sigma))$ is dilatable.*

2.3 Multi-variable Version

The purpose of this section is to prove Theorem (**C**); then we will state a consequence (Theorem 2.3.2).

Theorem (C). *Let $\varphi_1, \dots, \varphi_n \in L^\infty(\Sigma^2)$ and assume each M_{φ_i} is a self-adjoint, unital, positive Schur multiplier on $B(L^2(\Sigma))$. Then there exist a tracial von Neumann algebra (M, τ), a commuting n-tuple $(U_1, \dots, U_n)$ of trace preserving $*$-automorphisms on M and a w^*-continuous trace preserving one-to-one $*$-homomorphism $J : B(L^2(\Sigma)) \to M$ such that*

$$M_{\varphi_1}^{k_1} \cdots M_{\varphi_n}^{k_n} = \mathbb{E} U_1^{k_1} \cdots U_n^{k_n} J$$

for all $k_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant n$, where $\mathbb{E} : M \to B(L^2(\Sigma))$ is the conditional expectation associated with J.

The approach is similar to the one in [54], where the discrete case was considered. Since many arguments are similar to the ones in Section 2.2 we will not write all the details.

To make the presentation simpler, we only consider the case $n = 2$. The arguments for the general case are similar. So we consider $\varphi_1, \varphi_2 \in L^\infty(\Sigma^2)$ and we assume that $M_{\varphi_1}, M_{\varphi_2}$ are self-adjoint unital positive Schur multipliers on $B(L^2(\Sigma))$. We let $\mathbb{H}_1$, respectively $\mathbb{H}_2$, denote the real Hilbert space associated to φ_1, respectively φ_2, as at the beginning of Section 2.2. Then we consider the tracial von Neumann algebras $(N_1, \tau_1) := (\Gamma_{-1}(\mathbb{H}_1), \tau_1)$ and $(N_2, \tau_2) := (\Gamma_{-1}(\mathbb{H}_2), \tau_2)$, and we define

$$(N, \tau) := (N_1 \overline{\otimes} N_2, \tau_1 \overline{\otimes} \tau_2).$$

Next we let $T_1 \in B(L^1(\Sigma), N_1)$ and $T_2 \in B(L^1(\Sigma), N_2)$, be the unique linear maps such that

$$T_1(h) = \omega_1(\dot{h}) \quad \text{and} \quad T_2(h) = \omega_2(\dot{h}), \qquad h \in L^1_{\mathbb{R}}(\Sigma).$$

For $i = 1, 2$, we let $F_i \in L^\infty_\sigma(\Sigma, N_i)$ be associated to T_i. In addition we define $T_1^\otimes$ and $T_2^\otimes$ in $B(L^1(\Sigma), N)$ by

$$T_1^\otimes(h) = T_1(h) \otimes 1_{N_2} \quad \text{and} \quad T_2^\otimes(h) = 1_{N_1} \otimes T_2(h), \qquad h \in L^1(\Sigma).$$

As in Section 2.2, we consider the infinite tensor product N^∞, and we now let T_1^∞ and T_2^∞ in $B(L^1(\Sigma), N^\infty)$ be the natural extensions of $T_1^\otimes$ and $T_2^\otimes$, respectively. Finally we let F_1^∞ and F_2^∞ in $L^\infty_\sigma(\Sigma, N^\infty)$ be corresponding to T_1^∞ and T_2^∞, respectively, in the identification $B(L^1(\Sigma), N^\infty) \simeq L^\infty_\sigma(\Sigma, N^\infty)$. Likewise we let d_1 and d_2 in $L^\infty(\Sigma) \overline{\otimes} N^\infty$ be corresponding to T_1^∞ and T_2^∞, respectively. It follows from Section 2.2 that d_1, d_2 are self-adjoint symmetries. Moreover d_1 and d_2 commute, by construction.

We consider the shifts $\mathcal{S}_1, \mathcal{S}_2$ on N^∞ such that

$$\mathcal{S}_1(\cdots \otimes (x_0 \otimes y_0) \otimes (x_1 \otimes y_1) \otimes \cdots) := \cdots \otimes (x_{-1} \otimes y_0) \otimes (x_0 \otimes y_1) \otimes \cdots$$

and

$$\mathcal{S}_2(\cdots \otimes (x_0 \otimes y_0) \otimes (x_1 \otimes y_1) \otimes \cdots) := \cdots \otimes (x_0 \otimes y_{-1}) \otimes (x_1 \otimes y_0) \otimes \cdots$$

We set $M := B(L^2(\Sigma))\overline{\otimes}N^\infty$ and we let $J\colon B(L^2(\Sigma)) \to M$ be defined by $J(x) = x \otimes 1_{N^\infty}$. This is a trace preserving one-to-one $*$-homomorphism. We let $\mathbb{E} : M \to B(L^2(\Sigma))$ denote the conditional expectation associated with J.

For $i = 1, 2$, we define a trace preserving $*$-automorphism $U_i : M \to M$ by

$$U_i(y) = d_i((Id \otimes \mathcal{S}_i(y))d_i.$$

Let us now prove that U_1 and U_2 commute. We already have $d_1 d_2 = d_2 d_1$. We remark that $d_1 \in B(L^2(\Sigma))\overline{\otimes}N_1^\infty\overline{\otimes}1$ and $d_2 \in B(L^2(\Sigma))\overline{\otimes}1\overline{\otimes}N_2^\infty$. On the other hand, We can see $Id\otimes\mathcal{S}_1$ (respectively $Id\otimes\mathcal{S}_2$) as being the same as $Id\otimes\mathcal{S}\overline{\otimes}Id$ (respectively $Id\otimes Id\overline{\otimes}\mathcal{S}$). Further we have $Id \otimes \mathcal{S}_1(d_2) = d_2$ and $Id \otimes \mathcal{S}_2(d_1) = d_1$. Hence for any $x \in M$, we have

$$\begin{aligned}
U_1U_2(x) &= d_1(Id \otimes \mathcal{S}_1(d_2(Id \otimes \mathcal{S}_2(x))d_2)d_1 \\
&= d_1(Id \otimes \mathcal{S}_1(d_2)Id \otimes \mathcal{S}_1(Id \otimes \mathcal{S}_2(x))Id \otimes \mathcal{S}_1(d_2)d_1 \\
&= d_1d_2Id \otimes \mathcal{S}_1(Id \otimes \mathcal{S}_2(x))d_2d_1 \\
&= d_2d_1Id \otimes \mathcal{S}_2(Id \otimes \mathcal{S}_1(x))d_1d_2 \\
&\downarrow \text{ same computation} \\
&= U_2U_1(x).
\end{aligned}$$

This proves the commutation property.

We now apply Lemma 2.1.5 and we let

$$\Gamma : L^2_\sigma(\Sigma^2, N^\infty) \longrightarrow B(L^2(\Sigma))\overline{\otimes}N^\infty$$

be the resulting w^*-continuous contraction. Applying the argument in the proof of Lemma 2.2.8, we have:

Lemma 2.3.1.

1. *For all $\theta \in L^2(\Sigma^2)$ and for all $y = \cdots \otimes y_{-1} \otimes (1_{N_1} \otimes y_0^2) \otimes y_1 \cdots \in N^\infty$,*

$$d_1\Gamma(\theta \otimes y)d_1 = \Gamma\left((s,t) \mapsto \theta(s,t)[\cdots \otimes y_{-1} \otimes (F_1 \tilde{\times} F_1(s,t) \otimes y_0^2) \otimes y_1 \otimes \cdots]\right)$$

2. *For all $\theta \in L^2(\Sigma)$ and for all $y = \cdots \otimes y_{-1} \otimes (y_0^1 \otimes 1_{N_2}) \otimes y_1 \cdots \in N^\infty$:*

$$d_2\Gamma(\theta \otimes y)d_2 = \Gamma\left((s,t) \mapsto \theta(s,t)[\cdots \otimes y_{-1} \otimes (y_0^1 \otimes F_2 \tilde{\times} F_2(s,t)) \otimes y_1 \otimes \cdots]\right)$$

This is all that we need. Indeed, using this lemma and computations similar to the ones in the last part of Section 2.2, we obtain that for all integers $k, l \geqslant 0$ and for all $f, h \in L^2(\Sigma)$, we have

$$\mathbb{E}U_1^kU_2^lJ(S_{f\otimes h}) = M_{\varphi_1}^kM_{\varphi_2}^l(S_{f\otimes h}).$$

The full statement of Theorem (**C**) follows at once.

We finally mention that applying Lemma 1.2.3, we obtain the following consequence.

Theorem 2.3.2. *Let $\varphi_1, \ldots, \varphi_n \in L^\infty(\Sigma^2)$ and assume each M_{φ_i} is a self-adjoint, unital, positive Schur multiplier on $B(L^2(\Sigma))$. Then, for all $1 \leqslant p < \infty$, there exist a tracial von Neumann algebra (M, τ), a commuting n-tuple $(U_1, \ldots, U_n)$ of surjective isometries on $L^p(M)$, and two contractions $J : S^p(L^2(\Sigma)) \to L^p(M)$ and $Q : L^p(M) \to S^p(L^2(\Sigma))$ such that for all $k_1, \ldots, k_n$ in $\mathbb{N}_0$,*

$$M_{\varphi_1}^{k_1} \cdots M_{\varphi_n}^{k_n} = QU_1^{k_1} \cdots U_n^{k_n}J \qquad \text{on } S^p(L^2(\Sigma)).$$

Chapter 3

A characterization of absolutely dilatable measurable Schur multipliers

In the sequel, we say that (Σ, μ) is separable when the Hilbert space $L^2(\Sigma)$ is separable. In this chapter, we will prove the following theorem. After that, we will look at the non-separable case.

Theorem 3.0.1. *Assume that (Σ, μ) is separable and let $\varphi \in L^\infty(\Sigma^2)$. The following assertions are equivalent.*

(i) The function φ is a bounded Schur multiplier and $M_\varphi \colon B(L^2(\Sigma)) \to B(L^2(\Sigma))$ admits a separable absolute dilation.

(ii) There exist a normalized tracial von Neumann algebra (N, τ_N) with a separable predual and a unitary $\mathfrak{d} \in L^\infty_\sigma(\Sigma; N)$ such that

$$\varphi(s,t) = \tau_N\big(\mathfrak{d}(s)^*\mathfrak{d}(t)\big), \qquad \text{for a.e. } (s,t) \in \Sigma^2. \tag{3.0.1}$$

3.1 L^∞_σ-spaces and normal tensor products

Let (Σ, μ) be a σ-finite measure space. For any $1 \leqslant p \leqslant \infty$, we simply let $L^p(\Sigma)$ denote the associated L^p-space $L^p(\Sigma, \mu)$. In integration formulas, $d\mu(t)$ will be abbreviated to dt.

Let X be a separable Banach space. We say that a function $\mathfrak{d} \colon \Sigma \to X^*$ is w^*-measurable if for all $x \in X$, the scalar function $t \mapsto \langle \mathfrak{d}(t), x\rangle$ is measurable on Σ. The latter function is denoted by $\langle \mathfrak{d}, x\rangle$ for simplicity. If $\mathfrak{d} \colon \Sigma \to X^*$ is w^*-measurable, then the norm function $t \mapsto \|\mathfrak{d}(t)\|$ is measurable on Σ. Indeed if $(x_n)_{n\geqslant 1}$ is a dense sequence in the unit ball of X, then $\|\mathfrak{d}(\,\cdot\,)\| = \sup_n |\langle \mathfrak{d}, x_n\rangle|$. We let $L^\infty_\sigma(\Sigma; X^*)$ denote the space of all w^*-measurable functions $\mathfrak{d} \colon \Sigma \to X^*$, defined up to almost everywhere zero functions, such that $\|\mathfrak{d}(\,\cdot\,)\|$ is essentially bounded (in this case, we simply say that $\mathfrak{d}$ is essentially bounded). This is a Banach space for the norm

$$\|\mathfrak{d}\|_\infty := \big\|\|\mathfrak{d}(\,\cdot\,)\|\big\|_{L^\infty(\Sigma)}.$$

Let $F \in L^1(\Sigma; X)$ and let $\mathfrak{d} \in L^\infty_\sigma(\Sigma; X^*)$. Using the fact that $L^1(\Sigma) \otimes X$ is dense in $L^1(\Sigma; X)$, it is easy to see that the function $t \mapsto \langle \mathfrak{d}(t), F(t)\rangle$ is integrable and that

$$\langle \mathfrak{d}, F\rangle := \int_\Sigma \langle \mathfrak{d}(t), F(t)\rangle \, dt \tag{3.1.1}$$

satisfies $|\langle \mathfrak{d}, F\rangle| \leqslant \|F\|_1 \|\mathfrak{d}\|_\infty$. Consequently, (3.1.1) defines a duality pairing between $L^1(\Sigma; X)$ and $L^\infty_\sigma(\Sigma; X^*)$. It turns out that (3.1.1) extends to an isometric isomorphism

$$L^\infty_\sigma(\Sigma; X^*) \simeq L^1(\Sigma; X)^*. \tag{3.1.2}$$

This result is known as the Dunford-Pettis theorem [18, Theorem 2.1.6]. See [15, Theorem 11] for a simple proof.

It is clear that

$$L^\infty(\Sigma; X^*) \subset L^\infty_\sigma(\Sigma; X^*)$$

According to [17, Theorem IV.1.1], this inclusion is an equality if and only if X^* has the so-called Radon-Nikodym property. We will use the fact that Hilbert spaces have this property [17, Corollary IV.14]. Thus, for any separable Hilbert space $\mathcal{H}$, we have

$$L^\infty(\Sigma; \mathcal{H}) = L^\infty_\sigma(\Sigma; \mathcal{H}). \tag{3.1.3}$$

Note that more generally, all reflexive Banach spaces have the Radon-Nikodym property. See [17, Section VII.7] for more information.

Let $Z \subset X^*$ be any w^*-closed subspace. Then Z has a natural predual, equal to the quotient space $X/Z_\perp$, and the latter is separable. This allows to define $L^\infty_\sigma(\Sigma; Z)$. It is clear that

$$L^\infty_\sigma(\Sigma; Z) \subset L^\infty_\sigma(\Sigma; X^*)$$

isometrically. Furthermore, $L^1(\Sigma; Z_\perp) \subset L^1(\Sigma; X)$ isometrically and under the identification (3.1.2), we have

$$L^\infty_\sigma(\Sigma; Z) = L^1(\Sigma; Z_\perp)^\perp. \tag{3.1.4}$$

In the separating case, the above construction and the construction in Section 2.1 of $L_\sigma(\Sigma, X^*)$ give the same space. It follows from this lemma.

Lemma 3.1.1. *Let X be a separable Banach space. For any w^*-measurable function $\mathfrak{d} : \Sigma \to X^*$, We have,*

$$\|t \mapsto \|\mathfrak{d}(t)\|\|_\infty = \big\|L^\infty - \sup\{|\langle \mathfrak{d}, x\rangle| : x \in X, \|x\| \leqslant 1\}\big\|_{L^\infty(\Sigma)}$$

Proof. We let $c = \|t \mapsto \|\mathfrak{d}(t)\|\|_\infty$. We let $(x_n)_{n\geqslant 1}$ be a dense sequence in the unit ball of X. Then

$$\|t \mapsto \|\mathfrak{d}(t)\|\|_\infty = \|\sup_n |\langle \mathfrak{d}, x_n\rangle|\|_\infty \leqslant \big\|L^\infty - \sup\{|\langle \mathfrak{d}, x\rangle| : x \in X, \|x\| \leqslant 1\}\big\|_{L^\infty(\Sigma)}.$$

Now we will prove that

$$\big\|L^\infty - \sup\{|\langle \mathfrak{d}, x\rangle| : x \in X, \|x\| \leqslant 1\}\big\|_{L^\infty(\Sigma)} \leqslant c.$$

Let $x \in X$, such that $\|x\| \leqslant 1$ and let $f \in L^1(\Sigma)$. There exists a sub-sequence $(x'_n)_{n\geqslant 1}$ of $(x_n)_{n\geqslant 1}$, such that $x'_n \to x$. We have for all $n \in \mathbb{N}$,

$$\left|\int_\Sigma f(t)\langle \mathfrak{d}(t), x'_n\rangle dt\right| \leqslant c\|f\|_1. \tag{3.1.5}$$

Using (2.1.1), we obtain,

$$\left|\int_\Sigma f(t)\langle \mathfrak{d}(t), x\rangle dt\right| \leqslant \left|\int_\Sigma f(t)\langle \mathfrak{d}(t), x'_n\rangle dt\right|. \tag{3.1.6}$$

By 3.1.6 and 3.1.5, we have

$$\left|\int_\Sigma f(t)\langle \mathfrak{d}(t), x\rangle dt\right| \leqslant c\|f\|_1.$$

Consequently,

$$\|\langle \mathfrak{d}(t), x\rangle\|_\infty \leqslant c.$$

Finally, we obtain the other inequality. So the equality between $\|t \mapsto \|\mathfrak{d}(t)\|\|_\infty$ and $\left\|L^\infty - \sup\{|\langle \mathfrak{d}, x\rangle| : x \in X, \|x\| \leqslant 1\}\right\|_{L^\infty(\Sigma)}$ holds. □

Given any von Neumann algebras N_1, N_2, we let $N_1 \overline{\otimes} N_2$ denote their normal tensor product. We will assume that the reader is familiar with the normal tensor product and its basic properties, for which we refer either to [62, Section IV.5] (where it is called the W^*-tensor product) or to [40, Section 11.2], see also [58, Section 1.22]. We recall a few facts that can be found in the latter references.

First assume that $N_1 \subset B(H_1)$ and $N_2 \subset B(H_2)$ as von Neumann subalgebras, for some Hilbert spaces H_1, H_2. Then

$$N_1 \overline{\otimes} N_2 \subset B(H_1)\overline{\otimes}B(H_2) \simeq B(H_1 \overset{2}{\otimes} H_2).$$

More precisely, if we regard $N_1 \otimes N_2 \subset B(H_1 \overset{2}{\otimes} H_2)$ in the usual way, then $N_1 \overline{\otimes} N_2$ is the w^*-closure of $N_1 \otimes N_2$ in $B(H_1 \overset{2}{\otimes} H_2)$.

Second, consider $\eta_1 \in N_{1*}$ and $\eta_2 \in N_{2*}$. Let I_{N_1} (resp. I_{N_2}) denote the identity operator on N_1 (resp. N_2). Then the tensor map $\eta_1 \otimes I_{N_2} \colon N_1 \otimes N_2 \to N_2$ uniquely extends to a w^*-continuous operator $\eta_1 \overline{\otimes} I_{N_2} \colon N_1 \overline{\otimes} N_2 \to N_2$. Likewise, the tensor map $I_{N_1} \otimes \eta_2$ uniquely extends to a w^*-continuous operator $I_{N_1} \overline{\otimes} \eta_2 \colon N_1 \overline{\otimes} N_2 \to N_1$. Further $\eta_1 \otimes \eta_2$ uniquely extends to a w^*-continuous functional on $N_1 \overline{\otimes} N_2$. We still denote this extension by $\eta_1 \otimes \eta_2$. It is plain that

$$\eta_1 \otimes \eta_2 = \eta_1 \circ (I_{N_1} \overline{\otimes} \eta_2) = \eta_2 \circ (\eta_1 \overline{\otimes} I_{N_2}).$$

Third, the above definition of $\eta_1 \otimes \eta_2 \in (N_1 \overline{\otimes} N_2)_*$ yields a natural embedding

$$N_{1*} \otimes N_{2*} \subset (N_1 \overline{\otimes} N_2)_*. \tag{3.1.7}$$

Remark 3.1.2. The operators $\eta_1 \overline{\otimes} I_{N_2}$ and $I_{N_1} \overline{\otimes} \eta_2$ considered above are usually called the slice maps associated with η_1 and η_2, respectively.

Assume $N_1 \subset B(H_1)$ and $N_2 \subset B(H_2)$ as above and let $w \in B(H_1)\overline{\otimes}B(H_2)$. Then $w \in N_1\overline{\otimes}N_2$ if and only if

$$\big(I_{B(H_1)}\overline{\otimes}\eta_2\big)(w) \in N_1 \qquad \text{and} \qquad \big(\eta_1\overline{\otimes}I_{B(H_2)}\big)(w) \in N_2,$$

for all $\eta_1 \in B(H_1)_*$ and all $\eta_2 \in B(H_2)_*$. This is the so-called slice map property of von Neumann algebras, see [66]. In this chapter, we will use this property only when $N_2 = B(H_2)$, in which case the proof is simple (see e.g. [19, Theorem 2.4] and its proof).

We conclude this section with a natural connection between L^∞_σ-spaces and normal tensor products.

Lemma 3.1.3. *Let N be a von Neumann algebra with a separable predual.*

(1) For any $\mathfrak{c}, \mathfrak{d} \in L^\infty_\sigma(\Sigma; N)$, the functions $t \mapsto \mathfrak{c}(t)\mathfrak{d}(t)$ and $t \mapsto \mathfrak{d}(t)^*$ are w^*-measurable. Furthermore, $L^\infty_\sigma(\Sigma; N)$ is a von Neumann algebra for the pointwise product and the pointwise involution.

(2) The natural embedding of $L^\infty(\Sigma) \otimes N$ into $L^\infty_\sigma(\Sigma; N)$ extends to a von Neumann algebra identification

$$L^\infty_\sigma(\Sigma; N) \simeq L^\infty(\Sigma)\overline{\otimes}N. \tag{3.1.8}$$

Proof. This follows from [58, Theorem 1.22.13] and its proof. □

In the sequel, we write

$$\mathfrak{d} \sim D$$

if $\mathfrak{d} \in L^\infty_\sigma(\Sigma; N)$ and $D \in L^\infty(\Sigma)\overline{\otimes}N$ are corresponding to each other under the identification (3.1.8).

Remark 3.1.4. It is plain that for any $\mathfrak{d} \in L^\infty_\sigma(\Sigma; N)$ and $D \in L^\infty(\Sigma)\overline{\otimes}N$, we have $\mathfrak{d} \sim D$ if and only if

$$\int_\Sigma f(t)\langle \mathfrak{d}(t), \eta\rangle \, dt \; = [f \otimes \eta](D)$$

for all $f \in L^1(\Sigma)$ and all $\eta \in N_*$.

Lemma 3.1.5. *Assume that N has a separable predual, let $\mathfrak{d} \in L^\infty_\sigma(\Sigma; N)$ and let $D \in L^\infty(\Sigma)\overline{\otimes}N$ such that $\mathfrak{d} \sim D$. Then for all $x \in N$, we have*

$$x\mathfrak{d} \sim (1 \otimes x)D,$$

where $x\mathfrak{d} \in L^\infty_\sigma(\Sigma; N)$ is defined by $(x\mathfrak{d})(t) = x\mathfrak{d}(t)$, and $1 = 1_{L^\infty(\Sigma)}$.

Proof. This readily follows from Lemma 3.1.3, using the constant function $\mathfrak{c}(t) = x$. □

As a complement to Lemma 3.1.3, we note that the natural embedding of $L^\infty(\Sigma) \otimes L^\infty(\Sigma)$ into $L^\infty(\Sigma^2)$ extends to a von Neumann algebra identification

$$L^\infty(\Sigma^2) \simeq L^\infty(\Sigma)\overline{\otimes}L^\infty(\Sigma). \tag{3.1.9}$$

Indeed, the embedding of $L^1(\Sigma)\otimes L^1(\Sigma)$ into $L^1(\Sigma^2)$ extends to an isometric identification $L^1(\Sigma; L^1(\Sigma)) \simeq L^1(\Sigma^2)$, by Fubini's theorem. Hence (3.1.9) follows from [58, 1.22.10–1.22.12].

Remark 3.1.6. We finally recall the following classical fact, for which we refer to [62, Proposition V.2.14]. Let (N, τ_N) be any tracial von Neumann algebra. Let I be an index set and let $(e_i)_{i\in I}$ denote the standard Hilbertian basis of ℓ^2_I. Then $\mathrm{tr}\otimes\tau_N$ uniquely extends to a normal semi-finite faithful trace

$$\mathrm{tr}\overline{\otimes}\tau_N\colon (B(\ell^2_I)\overline{\otimes}N)_+ \longrightarrow [0,\infty],$$

which can be described as follows. Any $z \in B(\ell^2_I)\overline{\otimes}N$ may be naturally regarded as an infinite matrix $[z_{ij}]_{(i,j)\in I^2}$ with entries $z_{ij} \in N$. If z is positive, then each z_{ii} is positive and we have

$$\big[\mathrm{tr}\overline{\otimes}\tau_N\big](z) = \sum_{i\in I} \tau_N(z_{ii}).$$

3.2 Notation and Remark

Given any $u, v \in L^2(\Sigma)$, we regard $u \otimes v$ as an element of $L^2(\Sigma^2)$ using the convention $(u\otimes v)(s,t) = u(t)v(s)$, for $(s,t) \in \Sigma^2$. We warn the reader, it is not the same convention as the subsection 1.4.2. In the sequel, the operator $S_{u\otimes v}$ will be abusively denoted by $u \otimes v$. In other words, we use the notation $u \otimes v$ to also denote the element of $B(L^2(\Sigma))$ defined by

$$[u\otimes v](h) = \Big(\int_\Sigma u(t)h(t)\,dt\Big)v, \qquad h \in L^2(\Sigma).$$

Recall that we let tr denote the usual trace on $B(L^2(\Sigma))$. Let $\varphi \in L^\infty(\Sigma^2)$. Applying (1.4.9), we obtain that for all $u, v, a, b \in L^2(\Sigma)$, we have

$$\mathrm{tr}\Big(\big[M_\varphi(u\otimes v)\big]\big[a\otimes b\big]\Big) = \int_{\Sigma^2} \varphi(s,t)u(t)v(s)a(s)b(t)\,dtds. \tag{3.2.1}$$

In the sequel, we keep the notation

$$M_\varphi\colon B(L^2(\Sigma)) \longrightarrow B(L^2(\Sigma))$$

to denote M^∞_φ of theorem 1.4.2.

Recall that (Σ,μ) is called separable if $L^2(\Sigma)$ is separable. For any $1 \leqslant p < \infty$, this is equivalent to the separability of $L^p(\Sigma)$.

Remark 3.2.1. Assume that (Σ,μ) is separable, let $\varphi \in L^\infty(\Sigma^2)$ and assume that φ satisfies the assertion (ii) of Theorem 3.0.1. Since (N,τ_N) is normalized, we have a contractive embedding $\kappa\colon N \to L^2(N,\tau_N)$. Let $\alpha := \kappa\circ\mathfrak{d} \in L^\infty_\sigma(\Sigma; L^2(N,\tau_N))$. Then $\|\alpha\|_\infty \leqslant 1$. According to (3.1.3), α actually belongs to $L^\infty(\Sigma; L^2(N,\tau_N))$. Moreover for almost every $(s,t) \in \Sigma^2$, we have

$$\tau_N\big(\mathfrak{d}(s)^*\mathfrak{d}(t)\big) = \big\langle \alpha(t)\,\big|\,\alpha(s)\big\rangle_{L^2(N,\tau_N)}.$$

Hence φ satisfies (1.4.12), by (3.0.1). Thus, (3.0.1) should be regarded as a strengthening of the factorization (1.4.12). In this respect, Theorem 3.0.1 says that this strengthened factorization property characterizes Schur multipliers with a separable absolute dilation.

3.3 Preparatory results

Throughout we fix a σ-finite measure space (Σ, μ). For any $\theta \in L^\infty(\Sigma)$, let $R_\theta \in B(L^2(\Sigma))$ be the pointwise multiplication operator defined by $R_\theta(h) = \theta h$ for all $h \in L^2(\Sigma)$. Since $\theta \mapsto R_\theta$ is a w^*-continuous 1-1 $*$-homomorphism, we may identify $L^\infty(\Sigma)$ with $\{R_\theta : \theta \in L^\infty(\Sigma)\}$. Using this identification, we will henceforth regard $L^\infty(\Sigma)$ as a von Neumann subalgebra of $B(L^2(\Sigma))$, and simply write

$$L^\infty(\Sigma) \subset B(L^2(\Sigma)). \tag{3.3.1}$$

We recall that $L^\infty(\Sigma)$ is equal to its commutant. That is, an operator $T \in B(L^2(\Sigma))$ is of the form R_{θ_0} for some $\theta_0 \in L^\infty(\Sigma)$ if and only if $TR_\theta = R_\theta T$ for all $\theta \in L^\infty(\Sigma)$, that is, $T(\theta h) = \theta T(h)$ for all $\theta \in L^\infty(\Sigma)$ and all $h \in L^2(\Sigma)$.

For any measurable set $E \subset \Sigma$, we let χ_E denote the indicator function of E. Note that χ_E belongs to $L^2(\Sigma)$ if E has finite measure.

Lemma 3.3.1. *Let $T \in B(L^2(\Sigma))$. Then T belongs to $L^\infty(\Sigma)$ (in the above sense) if and only if $T(\chi_E) = \chi_E T(\chi_E)$ for all measurable $E \subset \Sigma$ with finite measure.*

Proof. The 'only if' part is obvious. To prove the 'if part', let $\mathscr{A}$ be the collection of all measurable sets $E \subset \Sigma$ with finite measure and let

$$\mathcal{E} = \mathrm{Span}\big\{\chi_E : E \in \mathscr{A}\big\}.$$

We assume that $T(\chi_E) = \chi_E T(\chi_E)$ for all $E \in \mathscr{A}$. This readily implies that $T(\chi_E) = \chi_{E'} T(\chi_E)$ for all $E, E' \in \mathscr{A}$ with $E \subset E'$.

Let $E, F \in \mathscr{A}$. Since $\chi_E \chi_F = \chi_{E\cap F}$ and $E \cap F \subset E$, it follows from above that we have $\chi_E T(\chi_E \chi_F) = T(\chi_E \chi_F)$. Let E^c be the complement of E. Since $\chi_{E^c}\chi_F = \chi_{E^c\cap F}$, we have $T(\chi_{E^c}\chi_F) = \chi_{E^c\cap F} T(\chi_{E^c}\chi_F)$, hence $\chi_E T(\chi_{E^c}\chi_F) = 0$. Writing a decomposition $T(\chi_F) = T(\chi_E\chi_F) + T(\chi_{E^c}\chi_F)$, we deduce that

$$T(\chi_E \chi_F) = \chi_E T(\chi_F). \tag{3.3.2}$$

By linearity, the identity (3.3.2) implies that $T(\theta h) = \theta T(h)$ for all $\theta, h \in \mathcal{E}$. Since $\mathcal{E}$ is dense in $L^2(\Sigma)$, we deduce that $T(\theta h) = \theta T(h)$ for all $h \in L^2(\Sigma)$ and all $\theta \in \mathcal{E}$. Further $\mathcal{E}$ is w^*-dense in $L^\infty(\Sigma)$. If $(\theta_\iota)_\iota$ is a net of $\mathcal{E}$ converging to some $\theta \in L^\infty(\Sigma)$ in the w^*-topology, then for all $h \in L^2(\Sigma)$, $\theta_\iota h \to \theta h$ weakly in $L^2(\Sigma)$. We deduce that $T(\theta_\iota h) \to T(\theta h)$ and $\theta_\iota T(h) \to \theta T(h)$ weakly in $L^2(\Sigma)$. Consequently, we have $T(\theta h) = \theta T(h)$ for all $h \in L^2(\Sigma)$ and all $\theta \in L^\infty(\Sigma)$. According to the discussion after (3.3.1), T is therefore a multiplication operator, that is, an element of $L^\infty(\Sigma)$. □

In the sequel we write $L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)$ instead of $L^\infty(\Sigma)\overline{\otimes}L^\infty(\Sigma)$ to indicate that the variable in the first factor is denoted by $s \in \Sigma$ and the variable in the second factor is denoted by $t \in \Sigma$. In accordance with this convention, we re-write (3.1.9) as

$$L^\infty(\Sigma^2) \simeq L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma). \tag{3.3.3}$$

Let N be a von Neumann algebra and let

$$Z = L^\infty(\Sigma^2)\overline{\otimes}N = L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}N.$$

We let 1_t (resp. 1_s) denote the constant function 1 in $L^\infty_t(\Sigma)$ (resp. $L^\infty_s(\Sigma)$). Consider D and C in $L^\infty(\Sigma)\overline{\otimes}N$. We let

$$1_s \otimes D_t \in Z \tag{3.3.4}$$

be obtained from D by identifying $L^\infty(\Sigma)\overline{\otimes}N$ with $1_s\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}N \subset Z$ in the natural way. Likewise, we let

$$1_t \otimes C_s \in Z \tag{3.3.5}$$

be obtained from C by identifying $L^\infty(\Sigma)\overline{\otimes}N$ with $L^\infty_s(\Sigma)\overline{\otimes}1_t\,\overline{\otimes}N \subset Z$.

Lemma 3.3.2. *Assume that N has a separable predual. Let C, D as above and let $\mathfrak{c}, \mathfrak{d} \in L^\infty_\sigma(\Sigma; N)$ such that $\mathfrak{c} \sim C$ and $\mathfrak{d} \sim D$. Then the function $\mathfrak{c}\tilde{\times}\mathfrak{d}\colon \Sigma^2 \to N$ defined by*

$$(\mathfrak{c}\tilde{\times}\mathfrak{d})(s,t) = \mathfrak{c}(s)\mathfrak{d}(t), \qquad (s,t) \in \Sigma^2, \tag{3.3.6}$$

is w^*-measurable, hence belongs to $L^\infty_\sigma(\Sigma^2; N)$. Moreover we have

$$\mathfrak{c}\tilde{\times}\mathfrak{d} \sim (1_t \otimes C_s)(1_s \otimes D_t). \tag{3.3.7}$$

Proof. Let $\eta \in N_*$. For any $\delta \in N$, let $\eta\delta \in N_*$ be defined by $\langle a, \eta\delta\rangle = \langle \delta a, \eta\rangle$ for all $a \in N$. This is well-defined since the product on N is separately w^*-continuous. Similarly, $\delta\eta \in N_*$ is defined by $\langle a, \delta\eta\rangle = \langle a\delta, \eta\rangle$.

Writing $\langle \delta, \mathfrak{d}(t)\eta\rangle = \langle \mathfrak{d}(t), \eta\delta\rangle$ for any $\delta \in N$, we see that the function $t \mapsto \mathfrak{d}(t)\eta$ is weakly measurable from Σ to N_*. Hence it is measurable, by the separability assumption and [17, Theorem II. 1.2]. The function $t \mapsto \mathfrak{d}(t)\eta$ is therefore an almost everywhere limit of a sequence $(w_n)_{n\geqslant 1}$ of $L^\infty_t(\Sigma) \otimes N_*$. Since $\mathfrak{c}$ is w^*-measurable, the function $(s,t) \mapsto \langle \mathfrak{c}(s), w_n(t)\rangle$ is measurable for any $n \geqslant 1$. We deduce that $(s,t) \mapsto \langle \mathfrak{c}(s), \mathfrak{d}(t)\eta\rangle = \langle \mathfrak{c}(s)\mathfrak{d}(t), \eta\rangle$ is measurable. This shows that $\mathfrak{c}\tilde{\times}\mathfrak{d}$ is w^*-measurable. It is clear that $\mathfrak{c}\tilde{\times}\mathfrak{d}$ is essentially bounded. Thus, we obtain that $\mathfrak{c}\tilde{\times}\mathfrak{d} \in L^\infty_\sigma(\Sigma^2; N)$.

Set $\Delta = (1_t \otimes C_s)(1_s \otimes D_t)$ in this proof. We introduce

$$m = \int_\Sigma f(s)\mathfrak{c}(s)\,ds \in N,$$

this integral being defined in the w^*-topology. Consider the slice map

$$\widetilde{f} := f\overline{\otimes}I_{L^\infty_t(\Sigma)\overline{\otimes}N}\colon Z \longrightarrow L^\infty(\Sigma)\overline{\otimes}N.$$

For any $u, v \in L^\infty(\Sigma)$ and any $a, b \in N$, we have $(u \otimes 1 \otimes a)(1 \otimes v \otimes b) = (u \otimes v \otimes ab)$ in Z, hence

$$\widetilde{f}\big((u \otimes 1 \otimes a)(1 \otimes v \otimes b)\big) = \Big(\int_\Sigma fu\Big)v \otimes ab = \Big(1 \otimes \Big(\int_\Sigma f(s)u(s)a\,ds\Big)\Big)(v \otimes b).$$

By linearity, this implies that if C and D belong to the algebraic tensor product $L^\infty(\Sigma) \otimes N$, we have

$$\widetilde{f}(\Delta) = (1 \otimes m)D. \tag{3.3.8}$$

If $(C_\iota)_\iota$ is a net of $L^\infty(\Sigma)\overline{\otimes}N$ converging to C in the w^*-topology of $L^\infty(\Sigma)\overline{\otimes}N$ and if $\mathfrak{c}_\iota \in L^\infty_\sigma(\Sigma; N)$ are such that $\mathfrak{c}_\iota \sim C_\iota$, then $\int_\Sigma f(s)\mathfrak{c}_\iota(s)\,ds$ converges to m in the w^*-topology of N. Since the product on von Neumann algebras is separately w^*-continuous

and $\widetilde{f}$ is w^*-continuous, we deduce that the identity (3.3.8) actually holds in the general case. Consequently,

$$[f\otimes g\otimes\eta](\Delta)=[g\otimes\eta]\big((1\otimes m)D\big).$$

Since $d\sim D$, Lemma 3.1.5 and the 'only if' part of Remark 3.1.4 ensure that

$$[g\otimes\eta]\big((1\otimes m)D\big)=\int_\Sigma g(t)\langle md(t),\eta\rangle\,dt.$$

Writing

$$\langle m\mathfrak{d}(t),\eta\rangle=\langle m,\mathfrak{d}(t)\eta\rangle=\int_\Sigma f(s)\langle\mathfrak{c}(s),\mathfrak{d}(t)\eta\rangle\,ds=\int_\Sigma f(s)\langle\mathfrak{c}(s)\mathfrak{d}(t),\eta\rangle\,ds,$$

and using the measurability of $\langle\mathfrak{c}\tilde{\times}\mathfrak{d},\eta\rangle$, we deduce that

$$[f\otimes g\otimes\eta](\Delta)=\int_{\Sigma^2}g(t)f(s)\big\langle\mathfrak{c}(s)\mathfrak{d}(t),\eta\big\rangle\,dsdt.$$

Applying the 'if' part of Remark 3.1.4, we deduce (3.3.7). □

Let H_1,H_2 be two Hilbert spaces and let $H_1\overset{2}{\oplus}H_2$ denote their Hilbertian direct sum. Let $B(H_1,H_2)$ denote the Banach space of all bounded operators from H_1 into H_2. Any $W\in B(H_1\overset{2}{\oplus}H_2)$ can be naturally written as a 2×2 matrix

$$W=\begin{pmatrix}W_{11}&W_{12}\\W_{21}&W_{22}\end{pmatrix}$$

with $W_{ij}\in B(H_j,H_i)$ for all $(i,j)\in\{1,2\}^2$. Thus, we may regard

$$B(H_1,H_2)\subset B(H_1\overset{2}{\oplus}H_2)\tag{3.3.9}$$

as a closed subspace. Note that $B(H_1,H_2)$ is actually a w^*-closed subspace of $B(H_1\overset{2}{\oplus}H_2)$, hence a dual space. If H_1,H_2 are separable, then the predual of $B(H_1,H_2)$ is separable as well.

For any von Neumann algebra N_0, we let $N_0\overline{\otimes}B(H_1,H_2)$ denote the w^*-closure of the algebraic tensor product $N_0\otimes B(H_1,H_2)$ into $N_0\overline{\otimes}B(H_1\overset{2}{\oplus}H_2)$. For any

$$D\in N_0\overline{\otimes}B(H_1,H_2)\qquad\text{and}\qquad C\in N_0\overline{\otimes}B(H_2,H_1),$$

the product of D and C in the von Neumann algebra $N_0\overline{\otimes}B(H_1\overset{2}{\oplus}H_2)$ yields

$$CD\in N_0\overline{\otimes}B(H_1).$$

Let K be another Hilbert space and let $H=H_1\overset{2}{\oplus}H_2$. We note that the identification $B(K\overset{2}{\otimes}H)\simeq B(K)\overline{\otimes}B(H)$ reduces to a w^*-homeomorphic and isometric identification

$$B(K\overset{2}{\otimes}H_1,K\overset{2}{\otimes}H_2)\simeq B(K)\overline{\otimes}B(H_1,H_2).\tag{3.3.10}$$

Part (2) of Lemma 3.1.3 extends to the case when N is replaced by $B(H_1, H_2)$. More precisely, if H_1, H_2 are separable, then the natural embedding of $L^\infty(\Sigma)\otimes B(H_1, H_2)$ into $L^\infty_\sigma(\Sigma; B(H_1, H_2))$ extends to a w^*-homeomorphic and isometric identification

$$L^\infty_\sigma(\Sigma; B(H_1, H_2)) \simeq L^\infty(\Sigma)\overline{\otimes}B(H_1, H_2). \tag{3.3.11}$$

To prove it, apply Lemma 3.1.3 with $N = B(H_1 \overset{2}{\oplus} H_2)$ and use the embedding (3.3.9).

For any $D \in L^\infty(\Sigma)\overline{\otimes}B(H_1, H_2)$, we may associate, similarly to what is done before Lemma 3.3.2,

$$1_s \otimes D_t \in L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}B(H_1, H_2)$$

by identifying $L^\infty(\Sigma)\overline{\otimes}B(H_1, H_2)$ with $1_s\,\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}B(H_1, H_2)$.

Similarly, for any $C \in L^\infty(\Sigma)\overline{\otimes}B(H_2, H_1)$, we may associate

$$1_t \otimes C_s \in L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}B(H_2, H_1).$$

by identifying $L^\infty(\Sigma)\overline{\otimes}B(H_2, H_1)$ with $L^\infty_s(\Sigma)\overline{\otimes}1_t\overline{\otimes}B(H_2, H_1)$.

Lemma 3.3.3. *Let D, C as above and recall from (3.3.1) that they may be regarded as elements of $B(L^2(\Sigma))\overline{\otimes}B(H_1, H_2)$ and $B(L^2(\Sigma))\overline{\otimes}B(H_2, H_1)$, respectively. Let $N_1 \subset B(H_1)$ be a von Neumann subalgebra. The following assertions are equivalent.*

(i) For all $u, v \in L^2(\Sigma)$, $C(u \otimes v \otimes I_{H_2})D$ belongs to $B(L^2(\Sigma))\overline{\otimes}N_1$.

(ii) The product $(1_t \otimes C_s)(1_s \otimes D_t)$ belongs to $L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}N_1$.

Proof. Let $u, v, a, b \in L^2(\Sigma)$. Consider $a \otimes b$ as an element of $S^1(L^2(\Sigma))$ and let $(a \otimes b)\overline{\otimes}I$ be the associated slice map $B(L^2(\Sigma))\overline{\otimes}B(H_1) \to B(H_1)$. Likewise, regard $av \otimes bu$ as an element of $L^1_s(\Sigma) \otimes L^1_t(\Sigma)$ and let $(av \otimes bu)\overline{\otimes}I$ be the associated slice map $L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}B(H_1) \to B(H_1)$. We claim that

$$\big[(a \otimes b)\overline{\otimes}I\big]\big(C(u \otimes v \otimes I_{H_2})D\big) = \big[(av \otimes bu)\overline{\otimes}I\big]\big((1_t \otimes C_s)(1_s \otimes D_t)\big). \tag{3.3.12}$$

To prove this, assume first that $D \in L^\infty(\Sigma) \otimes B(H_1, H_2)$ and $C \in L^\infty(\Sigma) \otimes B(H_2, H_1)$. Thus,

$$D = \sum_j f_j \otimes T_j \qquad \text{and} \qquad C = \sum_i g_i \otimes S_i$$

for some finite families $(f_j)_j$ in $L^\infty(\Sigma)$, $(T_j)_j$ in $B(H_1, H_2)$, $(g_i)_i$ in $L^\infty(\Sigma)$ and $(S_i)_i$ in $B(H_2, H_1)$.

Recall from (3.3.1) that each f_j (resp. g_i) is identified with the pointwise multiplication by f_j (resp. g_i). Then an elementary calculation yields $g_i(u \otimes v)f_j = (uf_j) \otimes (vg_i)$ in $B(L^2(\Sigma))$. Consequently,

$$C(u \otimes v \otimes I_{H_2})D = \sum_{i,j}(uf_j) \otimes (vg_i) \otimes S_iT_j.$$

Hence, the left handside of (3.3.12) is

$$\begin{aligned}\big[(a \otimes b)\overline{\otimes}I\big]\big(C(u \otimes v \otimes I_{H_2})D\big) &= \sum_{i,j} \mathrm{tr}\big((a \otimes b)[(uf_j) \otimes (vg_i)]\big)S_iT_j \\ &= \sum_{i,j}\Big(\int_\Sigma avg_i\Big)\Big(\int_\Sigma buf_j\Big)S_iT_j.\end{aligned}$$

To compute the right handside of (3.3.12), write

$$(1_t \otimes C_s)(1_s \otimes D_t) = \Big(\sum_i g_i \otimes 1 \otimes S_i\Big)\Big(\sum_j 1 \otimes f_j \otimes T_j\Big) = \sum_{i,j} g_i \otimes f_j \otimes S_i T_j.$$

This implies

$$\big[(av \otimes bu)\overline{\otimes} I\big]\big((1_t \otimes C_s)(1_s \otimes D_t)\big) = \sum_{i,j}\Big(\int_\Sigma avg_i\Big)\Big(\int_\Sigma buf_j\Big) S_i T_j,$$

and (3.3.12) follows. Now since the product on von Neumann algebras is separately w^*-continuous, the above special case implies that (3.3.12) holds true in the general case.

We now prove the equivalence. Assume (i) and let $h, k \in L^1(\Sigma)$. There exist $u, v, a, b \in L^2(\Sigma)$ such that $h = av$ and $k = bu$. Since $C(u \otimes v \otimes I_{H_2})D$ belongs to $B(L^2(\Sigma))\overline{\otimes}N_1$, it follows from (3.3.12) that $\big[(h \otimes k)\overline{\otimes} I\big]\big((1_t \otimes C_s)(1_s \otimes D_t)\big)$ belongs to N_1. Since $L^1(\Sigma)_s \otimes L^1_t(\Sigma)$ is dense in $(L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma))_*$, we deduce that $(1_t \otimes C_s)(1_s \otimes D_t) \in L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}N_1$ by applying the slice map property (see Remark 3.1.2).

Conversely, assume (ii) and fix $u, v \in L^2(\Sigma)$. Since $(1_t \otimes C_s)(1_s \otimes D_t) \in L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}N_1$, it follows from (3.3.12) that $\big[(a \otimes b)\overline{\otimes} I\big]\big(C(u \otimes v \otimes I_{H_2})D\big)$ belongs to N_1 for all $a, b \in L^2(\Sigma)$. Since $L^2(\Sigma) \otimes L^2(\Sigma)$ is dense in $B(L^2(\Sigma))_*$, we deduce that $C(u \otimes v \otimes I_{H_2})D \in B(L^2(\Sigma))\overline{\otimes}N_1$ by applying again the slice map property. □

3.4 Proof of Theorem 3.0.1

3.4.1 Proof of "$(i) \Rightarrow (ii)$"

We let $\varphi \in L^\infty(\Sigma^2)$ and we assume that φ satisfies the assertion (i) of Theorem 3.0.1.

Applying (1.3.1) with $k = 1$, we obtain a tracial von Neumann algebra $(\mathcal{M}, \tau_\mathcal{M})$ with a separable predual, a trace preserving unital w^*-continuous $*$-homomorphism $J\colon B(L^2(\Sigma)) \to \mathcal{M}$ and a trace preserving $*$-automorphism $U\colon \mathcal{M} \to \mathcal{M}$ such that

$$M_\varphi = J_1^* U J. \tag{3.4.1}$$

We fix a Hilbertian basis $(e_i)_{i\in I}$ of $L^2(\Sigma)$ and we define $E_{ij} = e_j \otimes e_i \in B(L^2(\Sigma))$ for all $(i, j) \in I^2$. Since J is a unital w^*-continuous $*$-homomorphism, $\{J(E_{ij}) : (i, j) \in I^2\}$ is a system of matrix units of $\mathcal{M}$ (in the sense of [62, Definition IV.1.7]). Fix $k_0 \in I$ and set $q = J(E_{k_0k_0})$. This is a projection of $\mathcal{M}$. According to [62, Proposition IV.1.8] and its proof,

$$m_{ij} := J(E_{k_0 i}) m J(E_{jk_0})$$

belongs to $q\mathcal{M}q$ for all $m \in \mathcal{M}$ and all $(i, j) \in I^2$, and the mapping

$$\rho\colon \mathcal{M} \longrightarrow B(L^2(\Sigma))\overline{\otimes}(q\mathcal{M}q), \qquad \rho(m) = \sum_{(i,j)\in I^2} E_{ij} \otimes m_{ij},$$

is a $*$-isomorphism. (Here the summation is taken in the w^*-topology of $\mathcal{M}$.) The restriction of $\tau_\mathcal{M}$ to $q\mathcal{M}q$ is semi-finite, let us call it τ_1. Equip $B(L^2(\Sigma))\overline{\otimes}(q\mathcal{M}q)$ with $\mathrm{tr}\overline{\otimes}\tau_1$, see Remark 3.1.6. Then ρ is trace preserving. To check this, observe that

$J(E_{ik_0})J(E_{k_0i}) = J(E_{ii})$ for all $i \in I$ and that $\sum_{i\in I} J(E_{ii}) = 1_{\mathcal{M}}$. Hence for any $m \in \mathcal{M}_+$, each m_{ii} belongs to $(q\mathcal{M}q)_+$ and we have

$$\begin{aligned}(\mathrm{tr}\overline{\otimes}\tau_1)\big(\rho(m)\big) &= \sum_{i\in I} \tau_{\mathcal{M}}\big(J(E_{k_0i})mJ(E_{ik_0})\big)\\ &= \sum_{i\in I} \tau_{\mathcal{M}}\big(J(E_{ii})m\big)\\ &= \tau_{\mathcal{M}}\Big(\Big(\sum_{i\in I} J(E_{ii})\Big)m\Big) = \tau_{\mathcal{M}}(m).\end{aligned}$$

It is plain that $\rho\big(J(E_{ij})\big) = E_{ij}\otimes q$ for all $(i,j) \in I^2$. Since J is w^*-continuous, this implies that $\rho\circ J(z) = z\otimes q$, for all $z \in B(L^2(\Sigma))$. Since both J and ρ are trace preserving, this implies that $\tau_1(q) = 1$.

With $N_1 = q\mathcal{M}q$, it follows from above that changing J into $\rho\circ J$ and changing U into $\rho\circ U\circ\rho^{-1}$, we may assume that

$$\mathcal{M} = B(L^2(\Sigma))\overline{\otimes}N_1,$$

for some normalized tracial von Neumann algebra (N_1,τ_1), and

$$J(z) = z\otimes 1_{N_1}, \qquad z \in B(L^2(\Sigma)). \tag{3.4.2}$$

Moreover N_1 has a separable predual.

We fix a Hilbert space H_1 such that $N_1 \subset B(H_1)$ as a von Neumann subalgebra. We may assume that H_1 is separable, see Remark 1.3.3, (5). By (1.4.1), we have $B(L^2(\Sigma))\overline{\otimes}B(H_1) \simeq B(L^2(\Sigma;H_1))$. Hence $\mathcal{M} \subset B(L^2(\Sigma;H_1))$ as a von Neumann subalgebra. We use again [62, Proposition IV.1.8] and its proof as follows. We note that since U is a unital w^*-continuous $*$-homomorphism, $\{U(E_{ij}\otimes 1_{N_1}) : (i,j) \in I^2\}$ is a system of matrix units of $\mathcal{M}$. Then we set

$$q_0 = U(E_{k_0k_0}\otimes 1_{N_1}), \quad H_2 = q_0\big(L^2(\Sigma;H_1)\big) \quad\text{and}\quad N_2 = q_0\mathcal{M}q_0,$$

and we equip the latter von Neumann algebra with the restriction of the trace of $\mathcal{M}$. We let τ_2 denote this trace on N_2. Then arguing as above, we find a trace preserving $*$-isomorphism

$$\pi\colon B(L^2(\Sigma))\overline{\otimes}N_1 \longrightarrow B(L^2(\Sigma))\overline{\otimes}N_2 \tag{3.4.3}$$

such that $\pi\circ U(z\otimes 1_{N_1}) = z\otimes 1_{N_2}$ for all $z \in B(L^2(\Sigma))$. We deduce that τ_2 is normalized. Moreover, $N_2 \subset B(H_2)$ as a von Neumann subalgebra, and H_2 is separable.

Regard $B(L^2(\Sigma))\overline{\otimes}N_1 \subset B(L^2(\Sigma;H_1))$ as above and $B(L^2(\Sigma))\overline{\otimes}N_2 \subset B(L^2(\Sigma;H_2))$ similarly. Then the argument in the proof of [62, Proposition IV.1.8] actually shows the existence of a unitary operator

$$D\colon L^2(\Sigma;H_1) \longrightarrow L^2(\Sigma;H_2)$$

such that

$$\pi(R) = DRD^*, \qquad R \in B(L^2(\Sigma))\overline{\otimes}N_1. \tag{3.4.4}$$

This implies that

$$U(z\otimes 1_{N_1}) = D^*(z\otimes 1_{N_2})D, \qquad z \in B(L^2(\Sigma)). \tag{3.4.5}$$

We shall now use properties of Schur multipliers. Note that $1_{N_1} = I_{H_1} \in L^1(N_1, \tau_1)$ and recall the embedding $S^1(L^2(\Sigma))\otimes L^1(N_1,\tau_1) \subset \mathcal{M}_*$ from (3.1.7). The relation (3.4.1) and the identity (3.4.2) imply that for all $u, v, a, b \in L^2(\Sigma)$, we have

$$\big\langle U(u\otimes v\otimes I_{H_1}), a\otimes b\otimes I_{H_1}\big\rangle_{\mathcal{M},\mathcal{M}_*} \;=\; \mathrm{tr}\Big(\big[M_\varphi(u\otimes v)\big]\big[a\otimes b\big]\Big). \tag{3.4.6}$$

Note the right handside can be expressed by (3.2.1).

According to (3.3.10), we regard D as an element of $B(L^2(\Sigma))\overline{\otimes}B(H_1,H_2)$.

Lemma 3.4.1. *Under the representation (3.3.1), we have*

$$D \in L^\infty(\Sigma)\overline{\otimes}B(H_1,H_2).$$

Proof. Let $E, F \subset \Sigma$ be measurable sets of finite measure. Consider

$$V_{E,F} = (\chi_F\otimes\chi_F\otimes I_{H_2})D(\chi_E\otimes\chi_E\otimes I_{H_1}) \in B(L^2(\Sigma))\overline{\otimes}B(H_1,H_2).$$

Since $(\chi_F\otimes\chi_F\otimes I_{H_2})$ is a self-adjoint projection, we have

$$V_{E,F}^*V_{E,F} = (\chi_E\otimes\chi_E\otimes I_{H_1})D^*(\chi_F\otimes\chi_F\otimes I_{H_2})D(\chi_E\otimes\chi_E\otimes I_{H_1}).$$

We regard this product as an element of $B(L^2(\Sigma))\overline{\otimes}B(H_1)$. According to (3.4.5),

$$D^*(\chi_F\otimes\chi_F\otimes I_{H_2})D = U(\chi_F\otimes\chi_F\otimes I_{H_1})$$

belongs to $B(L^2(\Sigma))\overline{\otimes}N_1$. Hence $V_{E,F}^*V_{E,F}$ actually belongs to $B(L^2(\Sigma))\overline{\otimes}N_1$. Moreover using (3.4.6) and (3.2.1), we have

$$\begin{aligned}
(\mathrm{tr}\overline{\otimes}\tau_1)\big(V_{E,F}^*V_{E,F}\big) &= (\mathrm{tr}\overline{\otimes}\tau_1)\Big(\big[U(\chi_F\otimes\chi_F\otimes I_{H_1})\big]\big[\chi_E\otimes\chi_E\otimes I_{H_1}\big]\Big)\\
&= \big\langle U(\chi_F\otimes\chi_F\otimes I_{H_1}), \chi_E\otimes\chi_E\otimes I_{H_1}\big\rangle_{\mathcal{M},\mathcal{M}_*}\\
&= \mathrm{tr}\Big(\big[M_\varphi(\chi_F\otimes\chi_F)\big]\big[\chi_E\otimes\chi_E\big]\Big)\\
&= \int_{\Sigma^2}\varphi(s,t)\chi_F(t)\chi_F(s)\chi_E(s)\chi_E(t)\,dtds.
\end{aligned}$$

Consequently, $(\mathrm{tr}\overline{\otimes}\tau_1)\big(V_{E,F}^*V_{E,F}\big) = 0$ if E and F are disjoint. Hence $V_{E,F} = 0$ if E and F are disjoint.

For any $\eta \in B(H_1,H_2)_*$, set $D_\eta = (I_{B(L^2)}\overline{\otimes}\eta)(D) \in B(L^2(\Sigma))$. For all $z \in B(L^2(\Sigma))$ and $w \in B(H_1,H_2)$, we have

$$\begin{aligned}
&(I_{B(L^2)}\otimes\eta)\big[(\chi_F\otimes\chi_F\otimes I_{H_2})(z\otimes w)(\chi_E\otimes\chi_E\otimes I_{H_1})\big]\\
&\qquad= (I_{B(L^2)}\otimes\eta)\big[(\chi_F\otimes\chi_F)z(\chi_E\otimes\chi_E)\otimes w\big]\\
&\qquad= \eta(w)(\chi_F\otimes\chi_F)z(\chi_E\otimes\chi_E)\\
&\qquad= (\chi_F\otimes\chi_F)\big[(I_{B(L^2)}\otimes\eta)(z\otimes w)\big](\chi_E\otimes\chi_E).
\end{aligned}$$

Approximating D by elements of the algebraic tensor product $B(L^2(\Sigma))\otimes B(H_1,H_2)$, we deduce that

$$(I_{B(L^2)}\overline{\otimes}\eta)(V_{E,F}) = (\chi_F\otimes\chi_F)D_\eta(\chi_E\otimes\chi_E).$$

Hence

$$(\chi_F \otimes \chi_F) D_\eta (\chi_E \otimes \chi_E) = 0$$

if E and F are disjoint. Since

$$(\chi_F \otimes \chi_F) D_\eta (\chi_E \otimes \chi_E) = \Big(\int_\Sigma D_\eta(\chi_E)\chi_F \Big) \chi_E \otimes \chi_F,$$

this implies that

$$\int_\Sigma D_\eta(\chi_E)\chi_F \ = 0$$

if E and F are disjoint. We deduce that $D_\eta(\chi_E)$ has support in E whenever $E \subset \Sigma$ is a measurable set of finite measure. Applying Lemma 3.3.1, we obtain that $D_\eta \in L^\infty(\Sigma)$ for all $\eta \in B(H_1, H_2)_*$. The result therefore follows from the slice map property (see Remark 3.1.2). □

Lemma 3.4.2. *The product* $(1_t \otimes D_s^*)(1_s \otimes D_t)$ *belongs to* $L_s^\infty(\Sigma)\overline{\otimes}L_t^\infty(\Sigma)\overline{\otimes}N_1$.

Proof. For all $u, v \in L^2(\Sigma)$, we have $D^*(u \otimes v \otimes I_{H_2})D = \pi^{-1}(u \otimes v \otimes I_{H_2})$, by (3.4.4), and the latter belongs to $B(L^2(\Sigma))\overline{\otimes}N_1$. The result therefore follows from Lemma 3.3.3. □

Lemma 3.4.3. *In the space* $L^\infty(\Sigma^2)$*, we have*

$$\varphi = \big[I_{L^\infty(\Sigma^2)} \overline{\otimes} \tau_1 \big] \big((1_t \otimes D_s^*)(1_s \otimes D_t) \big). \tag{3.4.7}$$

Proof. Let $\widetilde{\varphi} \in L^\infty(\Sigma^2)$ be equal to the right handside of (3.4.7). Let $a, b, u, v \in L^2(\Sigma)$. By (3.2.1), (3.4.6), (3.4.5) and (3.3.12), we have

$$\begin{aligned}
\int_{\Sigma^2} \varphi(s,t)u(t)v(s)a(s)b(t)\,dtds &= \big\langle U(u \otimes v \otimes I_{H_1}), a \otimes b \otimes I_{H_1} \big\rangle \\
&= \big\langle D^*(u \otimes v \otimes 1_{N_2})D, a \otimes b \otimes 1_{N_1} \big\rangle \\
&= \tau_1 \Big(\big[a \otimes b \overline{\otimes} I \big] \big(D^*(u \otimes v \otimes 1_{N_2})D \big) \Big) \\
&= \tau_1 \Big(\big[av \otimes bu \overline{\otimes} I \big] \big((1_t \otimes D_s^*)(1_s \otimes D_t) \big) \Big) \\
&= \big[av \otimes bu \otimes \tau_1 \big] \big((1_t \otimes D_s^*)(1_s \otimes D_t) \big) \\
&= \int_{\Sigma^2} \widetilde{\varphi}(s,t) a(s) v(s) b(t) u(t)\,dtds.
\end{aligned}$$

Consequently,

$$\int_{\Sigma^2} \varphi(s,t) f(s) g(t)\,dtds = \int_{\Sigma^2} \widetilde{\varphi}(s,t) f(s) g(t)\,dtds$$

for all $f, g \in L^1(\Sigma)$. The result follows. □

We can now prove the assertion (ii) of Theorem 3.0.1. Recall that H_1 and H_2 are separable. Let $\mathfrak{d} \in L_\sigma^\infty(\Sigma; B(H_1, H_2))$ be associated with D in the identification (3.3.11). By Lemmas 3.3.2 and 3.4.2, we have $\mathfrak{d}^* \tilde{\times} \mathfrak{d} \sim (1_t \otimes D_s^*)(1_s \otimes D_t)$ and

$$\mathfrak{d}^* \tilde{\times} \mathfrak{d} \in L_\sigma^\infty(\Sigma; N_1). \tag{3.4.8}$$

Further for any $f, g \in L^1(\Sigma)$, it follows from (3.4.7) and Remark 3.1.4 that

$$\begin{aligned}\int_{\Sigma^2} f(s)g(t)\varphi(s,t)\,dtds &= [f\otimes g\otimes\tau_1]\big((1_t\otimes D_s^*)(1_s\otimes D_t)\big)\\ &= \int_{\Sigma^2} f(s)g(t)\tau_1\big(\mathfrak{d}(s)^*\mathfrak{d}(t)\big)\,dtds.\end{aligned}$$

This implies that

$$\varphi(s,t) = \tau_1\big(\mathfrak{d}(s)^*\mathfrak{d}(t)\big), \qquad \text{for a.e. } (s,t)\in\Sigma^2. \tag{3.4.9}$$

.

Recall that as an operator from $L^2(\Sigma; H_1)$ to $L^2(\Sigma; H_2)$, D is a unitary. Hence we both have $D^*D = 1_{L^\infty(\Sigma)}\otimes I_{H_1}$ and $DD^* = 1_{L^\infty(\Sigma)}\otimes I_{H_2}$. By Lemma 3.1.3, this implies that $\mathfrak{d}(s)^*\mathfrak{d}(s) = 1_{H_1}$ and $\mathfrak{d}(s)\mathfrak{d}(s)^* = I_{H_2}$ for almost every $s\in\Sigma$. Changing the values of $\mathfrak{d}$ on a null subset of Σ, we can henceforth assume that for all $s\in\Sigma$, $\mathfrak{d}(s)\colon H_1\to H_2$ is a unitary.

Consider the pre-annihilator $N_{1\perp}\subset B(H_1)_*$. It follows from (3.4.8) that for all $f, g\in L^1(\Sigma)$ and for all $\eta\in N_{1\perp}$, we have

$$\int_{\Sigma^2} f(s)g(t)\langle\mathfrak{d}(s)^*\mathfrak{d}(t),\eta\rangle\,dtds = 0.$$

Since (Σ,μ) is separable, $L^1(\Sigma)$ is separable. Also, $B(H_1)_*$ is separable, hence $N_{1\perp}$ is separable. Let $(g_k)_{k\geqslant 1}$ and $(\eta_n)_{n\geqslant 1}$ be dense sequences of $L^1(\Sigma)$ and $N_{1\perp}$, respectively. It follows from above that for all $n, m\geqslant 1$, $\int_\Sigma g_k(t)\langle\mathfrak{d}(s)^*\mathfrak{d}(t),\eta_n\rangle\,dt = 0$ for almost every $s\in\Sigma$. Consequently, there exists $s_0\in\Sigma$ such that

$$\int_\Sigma g_k(t)\langle\mathfrak{d}(s_0)^*\mathfrak{d}(t),\eta_n\rangle\,dt = 0 \tag{3.4.10}$$

for all $n, m\geqslant 1$.

Define $\widetilde{\mathfrak{d}}\colon\Sigma\to B(H_1)$ by $\widetilde{\mathfrak{d}}(t) = \mathfrak{d}(s_0)^*\mathfrak{d}(t)$. Clearly $\widetilde{\mathfrak{d}}$ is w^*-measurable and essentially bounded. Since $\mathfrak{d}(s_0)$ is a unitary, we have $\widetilde{\mathfrak{d}}(t)^*\widetilde{\mathfrak{d}}(t) = \mathfrak{d}(t)^*\mathfrak{d}(t) = I_{H_1}$ for almost every $t\in\Sigma$. Hence as an element of $L^\infty_\sigma(\Sigma, B(H_1))$, $\widetilde{\mathfrak{d}}$ is a unitary. According to (3.4.10), $\widetilde{\mathfrak{d}}$ belongs to the annihilator of

$$\mathrm{Span}\{g_k\otimes\eta_n\,:\,k,n\geqslant 1\}\subset L^1(\Sigma; B(H_1)_*).$$

Since $\{g_k\otimes\eta_n\,:\,k,n\geqslant 1\}$ is dense in $L^1(\Sigma; N_{1\perp})$, this implies that $\widetilde{\mathfrak{d}}$ belongs to $L^1(\Sigma; N_{1\perp})^\perp$. By (3.1.4), this means that $\widetilde{\mathfrak{d}}\in L^\infty_\sigma(\Sigma, N_1)$.

Finally, using again the fact that $\mathfrak{d}(s_0)$ is a unitary, we deduce from (3.4.9) that $\varphi(s,t) = \tau_1\big(\widetilde{\mathfrak{d}}(s)^*\widetilde{\mathfrak{d}}(t)\big)$ for almost every $(s,t)\in\Sigma^2$. Hence the assertion (ii) of Theorem 3.0.1 is satisfied, with $N = N_1$ and $\widetilde{\mathfrak{d}}$ instead of $\mathfrak{d}$.

3.4.2 Proof of "$(ii)\Rightarrow(i)$"

Assume that $\varphi\in L^\infty(\Sigma^2)$ satisfies the assertion (ii) of Theorem 3.0.1, for some normalized tracial von Neumann algebra (N,τ_N) with a separable predual and some unitary $\mathfrak{d}\in L^\infty_\sigma(\Sigma; N)$.

Since $\tau_N\colon N \to \mathbb{C}$ is a faithful state, we may define

$$N^\infty = \overline{\bigotimes_{\mathbb{Z}}} N,$$

the infinite tensor product of (N, τ_N) over the index set $\mathbb{Z}$, as considered in [64, Definition XIV.1.6]. The associated state $\tau_\infty := \overline{\otimes}_{\mathbb{Z}}\, \tau_N$ on N^∞ is a normal faithful tracial state. Thus, (N^∞, τ_∞) is a normalized tracial von Neumann algebra. We let

$$\mathcal{M} = B(L^2(\Sigma))\overline{\otimes}N^\infty,$$

equipped with its natural trace (see Remark 3.1.6). The von Neumann algebra N^∞ has a separable predual, hence $\mathcal{M}$ has a separable predual.

Let 1_∞ denote the unit of N^∞ and let $J\colon B(L^2(\Sigma)) \to \mathcal{M}$ be the w^*-continuous trace preserving unital $*$-homomorphism defined by

$$J(z) = z \otimes 1_\infty, \qquad z \in B(L^2(\Sigma)).$$

We are now going to construct a trace preserving $*$-automorphism $U\colon \mathcal{M} \to \mathcal{M}$ such that $T = M_\varphi$ satisfies (1.3.1).

Let 1 denote the unit of N. For any integer $m \geqslant 1$ and for any finite family $(x_k)_{k=-m}^m$ in N, set

$$\bigotimes_k x_k = \cdots 1 \otimes 1 \otimes x_{-m} \otimes \cdots \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_m \otimes 1 \otimes 1 \cdots,$$

where each x_k is in position k. Let $\sigma\colon N^\infty \to N^\infty$ be the $*$-automorphism such that

$$\sigma\big(\bigotimes_k x_k\big) = \bigotimes_k x_{k-1}$$

for all finite families $(x_k)_k$ in N. It is plain that σ is trace preserving. We set

$$S = I_{B(L^2(\Sigma))}\overline{\otimes}\sigma\colon \mathcal{M} \longrightarrow \mathcal{M}.$$

Consider $N_+ = \overline{\otimes}_{k\geqslant 1}\, N$ and $N_- = \overline{\otimes}_{k\leqslant -1}\, N$, so that we can write

$$N^\infty = N_-\overline{\otimes}N\overline{\otimes}N_+.$$

This allows us to write

$$\mathcal{M} = N_-\overline{\otimes}\big(B(L^2(\Sigma))\overline{\otimes}N\big)\overline{\otimes}N_+. \tag{3.4.11}$$

Let $D \in L^\infty(\Sigma)\overline{\otimes}N$ such that $\mathfrak{d} \sim D$ and recall that $L^\infty(\Sigma)\overline{\otimes}N \subset B(L^2(\Sigma))\overline{\otimes}N$, by (3.3.1). Thus we may define

$$\gamma\colon B(L^2(\Sigma))\overline{\otimes}N \longrightarrow B(L^2(\Sigma))\overline{\otimes}N, \qquad \gamma(y) = D^*yD.$$

Since D is a unitary, γ is a trace preserving $*$-automorphism. Owing to the decomposition (3.4.11), we can define

$$\Gamma = I_{N_-}\overline{\otimes}\gamma\overline{\otimes}I_{N_+}\colon \mathcal{M} \longrightarrow \mathcal{M}.$$

Now we set

$$U = \Gamma \circ S.$$

By construction, $U\colon \mathcal{M} \to \mathcal{M}$ is a trace preserving $*$-automorphism.

Let $u, v, a, b \in L^2(\Sigma)$. On the one hand, by a simple induction argument, it follows from (3.2.1) that for all $k \geqslant 0$, we have

$$\mathrm{tr}\Big(\big[M_\varphi^k(u \otimes v)\big]\big[a \otimes b\big]\Big) = \int_{\Sigma^2} \varphi(s,t)^k u(t)v(s)a(s)b(t)\, dtds.$$

On the other hand, using (3.0.1), it follows from the arguments in the proof of Lemma 2.2.10 and in the proof of Theorem (**A**) of Chapter 2 coming right after it, that for all $k \geqslant 0$, we have

$$\big\langle U^k(u \otimes v \otimes 1_\infty), a \otimes b \otimes 1_\infty\big\rangle_{\mathcal{M},\mathcal{M}_*} = \int_{\Sigma^2} \varphi(s,t)^k u(t)v(s)a(s)b(t)\, dtds. \tag{3.4.12}$$

According to the definition of J, these identities imply that

$$\big\langle J_1^* U^k J(u \otimes v), a \otimes b\big\rangle = \big\langle M_\varphi^k(u \otimes v), a \otimes b\big\rangle.$$

Since the linear span of all $u \otimes v$ as above is w^*-dense in $B(L^2(\Sigma))$, the linear span of all $a \otimes b$ as above is dense in $S^1(L^2(\Sigma))$ and J, U and M_φ are w^*-continuous, this implies that $T = M_\varphi$ satisfies (1.3.1).

3.4.3 Two remarks

Remark 3.4.4. In the proof of the implication "$(i) \Rightarrow (ii)$" in Theorem 3.0.1, we use the identity (1.3.1) only with $k = 1$. Consequently, the assertions (i) and (ii) in Theorem 3.0.1 are also equivalent to:

(i)' *There exist a tracial von Neumann algebra $(\mathcal{M}, \tau_\mathcal{M})$ with a separable predual, a w^*-continuous trace preserving unital $*$-homomorphim $J\colon B(L^2(\Sigma)) \to \mathcal{M}$ and a trace preserving $*$-automomorphim $U\colon \mathcal{M} \to \mathcal{M}$ such that $M_\varphi = J_1^* U J$.*

Remark 3.4.5. Let $\varphi \in L^\infty(\Sigma^2)$. It follows from the proof of Theorem (**A**) of Chapter 2 that if (Σ, μ) is separable, then any positive unital self-adjoint $M_\varphi\colon B(L^2(\Sigma)) \to B(L^2(\Sigma))$ admits a separable absolute dilation.

3.5 The non separable case

We recall some issues of the non separable case, see Section 2.1 for more details. Let (Σ, μ) be a σ-finite measure space. If X is a non separable Banach space, it is still possible to define a space $L^\infty_\sigma(\Sigma; X^*)$ of classes of w^*-measurable functions $\Sigma \to X^*$ in such a way that $L^1(\Sigma; X)^* \simeq L^\infty_\sigma(\Sigma; X^*)$ isometrically, see [12, 13]. However these spaces are delicate to define and to use, because if $\mathfrak{d}\colon \Sigma \to X^*$ is a w^*-measurable function, then the norm function $t \mapsto \|\mathfrak{d}(t)\|$ may not be measurable. Further given a von Neumann algebra N and w^*-measurable functions $\mathfrak{c}, \mathfrak{d}\colon \Sigma \to N$, the function $\mathfrak{c}\tilde{\times}\mathfrak{d}\colon \Sigma^2 \to N$ defined by (3.3.6) is likely to be non measurable. For these reasons, it seems to be out of reach to extend Theorem 3.0.1 verbatim to the non separable case.

In this general setting, we will content ourselves with a version of Theorem 3.0.1 stated in the language of von Neumann tensor products. In the next statement, we use some notations introduced in Section 3.3, see in particular (3.3.4) and (3.3.5). The most important implication here is "$(\mathrm{i}) \Rightarrow (\mathrm{ii})_1$", which extends the implication "$(\mathrm{i}) \Rightarrow (\mathrm{ii})$" of Theorem 3.0.1 to the non separable case.

Theorem 3.5.1. *Let $\varphi \in L^\infty(\Sigma^2)$ and let $e \in L^2(\Sigma)$ with $\|e\|_2 = 1$. The following assertions are equivalent.*

(i) The function φ is a bounded Schur multiplier and $M_\varphi \colon B(L^2(\Sigma)) \to B(L^2(\Sigma))$ is absolutely dilatable.

(ii) There exist a normalized tracial von Neumann algebra (N, τ_N) acting on some Hilbert space H, and a unitary $D \in L^\infty(\Sigma)\overline{\otimes}B(H)$ such that:

$(ii)_1$ The product $(1_t \otimes D_s^*)(1_s \otimes D_t)$ belongs to $L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}N$ and under the identification (3.3.3), we have

$$\varphi = \big[I_{L^\infty(\Sigma^2)}\overline{\otimes}\tau_N\big]\big((1_t \otimes D_s^*)(1_s \otimes D_t)\big). \tag{3.5.1}$$

$(ii)_2$ The trace of $D^*(e \otimes e \otimes I_H)D$ in $B(L^2(\Sigma))\overline{\otimes}N$ is equal to 1.

Proof. $\underline{(i) \Rightarrow (ii)}$: Assume (i) and apply the construction devised in Subsection 3.4.1. Note that until Lemma 3.4.3, this construction does not use separability. Recall from Lemma 3.4.1 that we are working with a unitary $D \in L^\infty(\Sigma)\overline{\otimes}B(H_1, H_2)$. Let K be a (large enough) Hilbert space such that $H_1 \overset{2}{\otimes} K$ is isomorphic to $H_2 \overset{2}{\otimes} K$ and let $W \colon H_1 \overset{2}{\otimes} K \to H_2 \overset{2}{\otimes} K$ be a unitary. We define

$$\widetilde{D} = [I_{L^\infty(\Sigma)}\overline{\otimes}W^*](D \otimes I_K) \in L^\infty(\Sigma)\overline{\otimes}B(H_1 \overset{2}{\otimes} K),$$

where $D \otimes I_K$ is regarded as an element of $L^\infty(\Sigma)\overline{\otimes}B(H_1 \overset{2}{\otimes} K, H_2 \overset{2}{\otimes} K)$. Clearly $\widetilde{D}$ is a unitary. We will now check that the assertion (ii) of Theorem 3.5.1 is satisfied with the Hilbert space $H = H_1 \overset{2}{\otimes} K$, the von Neumann algebra

$$N = N_1 \otimes I_K \subset B(H),$$

the tracial state τ_N on N defined by $\tau_N(x \otimes I_K) = \tau_1(x)$ for all $x \in N_1$ and the unitary $\widetilde{D}$ in place of D.

First we have $(1_s \otimes \widetilde{D}_s^*)(1_s \otimes \widetilde{D}_t) = (1_s \otimes D_s^*)(1_s \otimes D_t) \otimes I_K$, because $WW^* = I_{H_2} \otimes I_K$. Hence by Lemma 3.4.2, the product $(1_s \otimes \widetilde{D}_s^*)(1_s \otimes \widetilde{D}_t)$ belongs to $L^\infty_s(\Sigma)\overline{\otimes}L^\infty_t(\Sigma)\overline{\otimes}N$. Moreover we have

$$\big[I_{L^\infty(\Sigma^2)}\overline{\otimes}\tau_N\big]\big((1_t \otimes \widetilde{D}_s^*)(1_s \otimes \widetilde{D}_t)\big) = \big[I_{L^\infty(\Sigma^2)}\overline{\otimes}\tau_{N_1}\big]\big((1_t \otimes D_s^*)(1_s \otimes D_t)\big).$$

Applying Lemma 3.4.3, we deduce that the assertion $(ii)_1$ is satisfied.

Next we note, using again $WW^* = I_{H_2} \otimes I_K$, that

$$\widetilde{D}^*(e \otimes e \otimes I_H)\widetilde{D} = D^*(e \otimes e \otimes I_{H_2})D \otimes I_K = \pi^{-1}(e \otimes e \otimes 1_{N_2}) \otimes I_K,$$

where π comes from (3.4.3). Since π is trace preserving, the trace of $\widetilde{D}^*(e \otimes e \otimes I_H)\widetilde{D}$ in $B(L^2(\Sigma))\overline{\otimes}N$ is equal to the trace of $e \otimes e \otimes 1_{N_2}$ in $B(L^2(\Sigma))\overline{\otimes}N_2$. The latter is equal to 1 (recall that τ_2 is normalized). Hence the assertion $(ii)_2$ is satisfied.

$\underline{(ii) \Rightarrow (i)}$: Assume (ii) and consider the $*$-automorphism

$$\pi \colon B(L^2(\Sigma))\overline{\otimes}B(H) \longrightarrow B(L^2(\Sigma))\overline{\otimes}B(H), \qquad \pi(y) = DyD^*.$$

Let $B = \pi\big(B(L^2(\Sigma))\overline{\otimes}N\big)$. This a von Neumann subalgebra of $B(L^2(\Sigma))\overline{\otimes}B(H)$. We equip it with the trace $\tau_B = (\mathrm{tr}\overline{\otimes}\tau_N)\circ\pi^{-1}$, so that the restriction $B(L^2(\Sigma))\overline{\otimes}N \to B$ of π is trace preserving by nature.

For any $u, v \in L^2(\Sigma)$, $D^*(u\otimes v\otimes I_H)D$ belongs to $B(L^2(\Sigma))\overline{\otimes}N$, by the assumption $(\mathrm{ii})_1$ and Lemma 3.3.3. (Note that this is the reason why it makes sense to consider $D^*(e\otimes e\otimes I_H)D$ as an element of $B(L^2(\Sigma))\overline{\otimes}N$ in $(\mathrm{ii})_2$.) Since D is a unitary, we have $u\otimes v\otimes I_H = \pi(D^*(u\otimes v\otimes I_H)D)$, hence $u\otimes v\otimes I_H$ belongs to B. Since the linear span of all the $u\otimes v$ as above is w^*-dense in $B(L^2(\Sigma))$, we deduce that

$$B(L^2(\Sigma))\otimes I_H \subset B \subset B(L^2(\Sigma))\overline{\otimes}B(H).$$

Let $\{E_{ij} : (i,j)\in I^2\}$ be a system of matrix units in $B(L^2(\Sigma))$. Then it follows from above that $\{E_{ij}\otimes I_H : (i,j)\in I^2\}$ is a system of matrix units in B. Using again [62, Proposition IV.1.8] and its proof, we obtain the existence of a von Neumann subalgebra $\mathcal{N}\subset B(H)$ such that $B = B(L^2(\Sigma))\overline{\otimes}\mathcal{N}$. According to the assumption $(\mathrm{ii})_2$ and the definition of τ_B, we have

$$\tau_B(e\otimes e\otimes 1_{\mathcal{N}}) = (\mathrm{tr}\overline{\otimes}\tau_N)\big(D^*(e\otimes e\otimes I_H)D\big) = 1.$$

This implies that $\tau_{\mathcal{N}}\colon m\mapsto \tau_B(e\otimes e\otimes m)$ is a well-defined tracial state on $\mathcal{N}$. A classical uniqueness argument shows that $\tau_B = \mathrm{tr}\overline{\otimes}\tau_{\mathcal{N}}$.

We can now adapt the proof given in Subsection 3.4.2, as follows. We set

$$\mathcal{N}_- = \bigoverline{\bigotimes_{k\leqslant -1}}\mathcal{N} \qquad \text{and} \qquad N_+ = \bigoverline{\bigotimes_{k\geqslant 1}}N.$$

More precisely, $\mathcal{N}_-$ is the infinite tensor product of $(\mathcal{N},\tau_{\mathcal{N}})$ over the index set $\{-1,-2,\ldots\}$ whereas N_+ is the infinite tensor product of (N,τ_N) over the index set $\{1,2,\ldots\}$. Then we define

$$\mathcal{M} = B(L^2(\Sigma))\overline{\otimes}\mathcal{N}_-\overline{\otimes}\mathcal{N}\overline{\otimes}N\overline{\otimes}N_+,$$

that we equip with its natural trace. We define $J\colon B(L^2(\Sigma))\to\mathcal{M}$ by $J(z) = z\otimes 1$, where 1 is the unit of $\mathcal{N}_-\overline{\otimes}\mathcal{N}\overline{\otimes}N\overline{\otimes}N_+$. We set

$$\mathcal{M}_1 = \mathcal{N}_-\overline{\otimes}\big(B(L^2(\Sigma))\overline{\otimes}\mathcal{N}\big)\overline{\otimes}N\overline{\otimes}N_+ \qquad \text{and} \qquad \mathcal{M}_2 = \mathcal{N}_-\overline{\otimes}\mathcal{N}\overline{\otimes}\big(B(L^2(\Sigma))\overline{\otimes}N\big)\overline{\otimes}N_+,$$

and we let $\kappa_1\colon\mathcal{M}\to\mathcal{M}_1$ and $\kappa_2\colon\mathcal{M}\to\mathcal{M}_2$ be the canonical $*$-automorphisms. We let $\sigma_-\colon\mathcal{N}_-\to\mathcal{N}_-\overline{\otimes}\mathcal{N}$ be the $*$-isomorphism such that

$$\sigma_-\Big(\bigotimes_{k\leqslant -1}x_k\Big) = \Big(\bigotimes_{k\leqslant -1}x_{k-1}\Big)\otimes x_{-1},$$

for all finite families $(x_k)_{k\leqslant -1}$ in $\mathcal{N}$. Likewise, we let $\sigma_+\colon N\overline{\otimes}N_+\to N_+$ be the $*$-isomorphism such that

$$\sigma_+\Big(x_0\otimes\Big(\bigotimes_{k\geqslant 1}x_k\Big)\Big) = \bigotimes_{k\geqslant 1}x_{k-1},$$

for all finite families $(x_k)_{k\geqslant 0}$ in N. These two $*$-isomorphisms are trace preserving.

It follows from the construction of $\mathcal{N}$ that we may define a $*$-isomorphism

$$\gamma\colon B(L^2(\Sigma))\overline{\otimes}\mathcal{N}\longrightarrow B(L^2(\Sigma))\overline{\otimes}N, \qquad \gamma(y) = D^*yD.$$

This is the inverse of the restriction of π to $B(L^2(\Sigma))\overline{\otimes}N$, hence it is trace preserving.

Finally, we define $U\colon \mathcal{M} \to \mathcal{M}$ by

$$U = \kappa_2^{-1} \circ \big(\sigma_- \overline{\otimes} \gamma \overline{\otimes} \sigma_+\big) \circ \kappa_1.$$

Then U is a trace preserving $*$-automorphism. Next arguing again as in Chapter 2, we obtain that (3.4.12) holds true for all $u, v, a, b \in L^2(\Sigma)$. As in Subsection 3.4.2, we deduce that $T = M_\varphi$ satisfies (1.3.1). □

We now focus on the discrete case, that is $\Sigma = \mathbb{N}^* = \{1, 2, \ldots\}$. An element of $L^\infty(\Sigma^2)$ is just a bounded family $\mathfrak{m} = \{m_{ij}\}_{i,j\geqslant 1}$ of complex numbers. Then $\mathfrak{m}$ is a bounded Schur multiplier if and only if there exists a constant $C \geqslant 0$ such that for all $n \geqslant 1$ and all $[a_{ij}]_{1\leqslant i,j\leqslant n} \in M_n$,

$$\big\|[m_{ij}a_{ij}]\big\|_{M_n} \leqslant C\big\|[a_{ij}]\big\|_{M_n}.$$

Moreover for any von Neumann algebra N, we may identify $L^\infty(\Sigma)\overline{\otimes}N$ with $\ell^\infty(N)$, the space of all bounded sequences with entries in N. Likewise, $L^\infty(\Sigma)\overline{\otimes}L^\infty(\Sigma)\overline{\otimes}N \simeq \ell^\infty_{\mathbb{N}^*\times\mathbb{N}^*}(N)$. If $D = (d_k)_{k\geqslant 1} \in \ell^\infty(N)$, then D is a unitary if and only if d_k is a unitary for all $k \geqslant 1$. Moreover, the product $(1_j \otimes D_i^*)(1_i \otimes D_j)$ coincides with the doubly indexed sequence $(d_i^* d_j)_{i,j\geqslant 1}$.

Corollary 3.5.2. *For any bounded family $\mathfrak{m} = \{m_{ij}\}_{i,j\geqslant 1}$ of complex numbers, the following assertions are equivalent.*

(i) The family $\mathfrak{m}$ is a bounded Schur multiplier and $T_\mathfrak{m}\colon B(\ell^2) \to B(\ell^2)$ is absolutely dilatable.

(ii) There exist a normalized tracial von Neumann algebra (N, τ_N) and a sequence $(d_k)_{k\geqslant 1}$ of unitaries of N such that

$$m_{ij} = \tau_N\big(d_i^* d_j\big), \qquad i, j \geqslant 1.$$

(iii) The family $\mathfrak{m}$ is a bounded Schur multiplier and $T_\mathfrak{m}\colon B(\ell^2) \to B(\ell^2)$ admits a separable absolute dilation.

(iv) There exist a normalized tracial von Neumann algebra (N, τ_N) with a separable predual and a sequence $(d_k)_{k\geqslant 1}$ of unitaries of N such that

$$m_{ij} = \tau_N\big(d_i^* d_j\big), \qquad i, j \geqslant 1.$$

Proof. The implication "(iv) ⇒ (iii)" follows from Theorem 3.0.1 and "(iii) ⇒ (i)" is obvious. So we only need to prove (i) ⇒ (ii) ⇒ (iv).

Assume (i) and apply Theorem 3.5.1 to $\mathfrak{m}$. We find a normalized tracial von Neumann algebra (N, τ_N) acting on some H and a sequence $(d_k)_{k\geqslant 1}$ of unitaries of $B(H)$ such that $d_i^* d_j \in N$ and $m_{ij} = \tau_N\big(d_i^* d_j\big)$ for all $i, j \geqslant 1$. Set $d'_k = d_1^* d_k$, $k \geqslant 1$. The d'_k are unitaries of N and $d_i'^* d'_j = d_i^* d_j$ for all $i, j \geqslant 1$. Hence $m_{ij} = \tau_N\big(d_i'^* d'_j\big)$, for all $i, j \geqslant 1$. This proves (ii).

Assume (ii) and let $\widetilde{N} \subset N$ be the von Neumann subalgebra generated by $(d_k)_{k\geqslant 1}$. Then $\widetilde{N}$ admits a sequence $(a_n)_{n\geqslant 1}$ such that $V := \mathrm{Span}\{a_n : n \geqslant 1\}$ is w^*-dense in $\widetilde{N}$. Let $\tau_{\widetilde{N}}$ be the restriction of τ_N to $\widetilde{N}$ and recall that we have a contractive embedding $\widetilde{N} \subset L^1(\widetilde{N}, \tau_{\widetilde{N}})$. If $b \in \widetilde{N}$ is such that $\tau_{\widetilde{N}}(a_n b) = 0$ for all $n \geqslant 1$, then as an element

of $L^1(\widetilde{N}, \tau_{\widetilde{N}})$, b belongs to $V_\perp$. Hence $b = 0$. In turn, this shows that V is dense in $L^1(\widetilde{N}, \tau_{\widetilde{N}})$. Thus, $\widetilde{N}$ has a separable predual. Consequently, (iv) holds true, with $\widetilde{N}$ in place of N. □

Remark 3.5.3. In the finite dimensional case, we find similarly that the Schur multiplier operator $T_{\mathfrak{m}} \colon M_n \to M_n$ associated with a family $\mathfrak{m} = \{m_{ij}\}_{1 \leqslant i,j \leqslant n}$ admits a separable absolute dilation if and only if it admits an absolute dilation, if and only if there exist a normalized tracial von Neumann algebra (N, τ_N) and unitaries $d_1, \ldots, d_n$ in N such that $m_{ij} = \tau_N\big(d_i^* d_j\big)$ for all $1 \leqslant i, j \leqslant n$. The latter equivalence result is implicit in [28]. Indeed it follows from Proposition 2.8 and Theorem 4.4 in this reference.

Taking this equivalence into account, [28, Example 3.2] exhibits a positive unital Schur multiplier $M_4 \to M_4$ which is not absolutely dilatable.

We also mention that there are plenty of absolutely dilatable Schur multiplier operators $T_{\mathfrak{m}} \colon M_n \to M_n$ such that $\mathfrak{m}$ is not real valued. For example, it follows from above that for any complex number ω with $|\omega| = 1$, the operator $T_{\mathfrak{m}} \colon M_2 \to M_2$ associated with $\mathfrak{m} = \begin{bmatrix} 1 & \omega \\ \overline{\omega} & 1 \end{bmatrix}$ is absolutely dilatable.

Chapter 4

Characterisation of discrete unital positive Schur multipliers with a completely isometric dilation

All backgrounds are introduced in Section Tools or during the chapter. The two main theorems of this chapter are 4.1.9 for the finite dimensional case and Theorem 4.2.1 for the discrete case. We also give a characterisation for the multivariable case.

4.1 Finite Schur multipliers

In the following, we denote by M_n the space of $n \times n$ matrices with complex coefficients. Let tr_n be the usual trace on M_n. Unless otherwise stated we will consider M_n as equipped with the normalized trace $\tau_n := \frac{1}{n}\mathrm{tr}_n$.

Remark 4.1.1. The two traces τ_n and tr_n are needed in the following. We warn the reader to be careful. It can be confusing.

We recall the following definition from [62, Definition 4.1.7].

Definition 4.1.2. *Let I be an index set. A family $\{w_{ij} : i, j \in I\}$ of elements in a von Neumann algebra N is called a set of matrix units if*

1. *For all $i, j \in I$, $w_{ij}^* = w_{ji}$;*
2. *For all $i, j, k, l \in I$, $w_{ij}w_{kl} = \delta_{jk}w_{il}$;*
3. *$\sum_{i\in I} w_{ii} = 1$ in the strong topology.*

We give an alternative version of [62, Theorem 4.1.8]. In the sequel we let e_{ij}, $1 \leqslant i, j \leqslant n$, denote the standard matrix units of M_n.

Theorem 4.1.3. *Let (N, τ) be a tracial von Neumann algebra. Suppose that $\{w_{ij} : 1 \leqslant i, j \leqslant n\}$ is a set of matrix units in N. Let $e = w_{11}$ and let $N_1 = eNe$ be equipped with the restriction τ_1 of τ to N_1. Then the following mapping is a trace preserving $\star$-isomorphism:*

$$\rho : N \to (M_n, tr_n) \otimes (N_1, \tau_1),\ x \mapsto \sum_{1\leqslant i,j\leqslant n} e_{ij} \otimes w_{1i}xw_{j1}.$$

Proof. See [46, Lemma 2.2] and its proof. □

We refer to [28, lemma 2.1] for the following lemma.

Lemma 4.1.4. *Let P be a von Neumann algebra, let $n \geqslant 1$ be an integer and let $\{f_{ij} : 1 \leqslant i, j \leqslant n\}$ and $\{g_{ij} : 1 \leqslant i, j \leqslant n\}$ be two sets of matrix units in P. Then there exists an unitary $u \in P$ such that:*

$$uf_{ij}u^* = g_{ij}, \qquad 1 \leqslant i, j \leqslant n.$$

We recall a classical duality result and we give a proof for the sake of completeness.

Lemma 4.1.5. *Let $1 < p, q < +\infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and let (N, τ) be a tracial von Neumann algebra. For all $x \in L^p(N)\backslash\{0\}$, there exists a unique $y \in L^q(N)$ such that $\|y\|_q = 1$ and $\tau(xy) = \|x\|_p$. (This element is called the norming functional of x.)*

Assume that $\|x\|_p = 1$ and write the polar decomposition of x as $x = u|x|$. Then the norming functional y of x satisfies

$$y = |x|^{p-1}u^* \quad \textit{and} \quad |y| = u|x|^{p-1}u^*. \tag{4.1.1}$$

Moreover, its polar decomposition is $y = u^*|y|$.

Proof. The first part of the statement means that $L^p(N)$ is smooth. This well-known fact follows e.g. from [53, Corollary 5.2].

For the second part, assume that $\|x\|_p = 1$ and set $y = |x|^{p-1}u^*$. Then we have $y^* = u|x|^{p-1}$ and $yx = |x|^p$. This implies that $\tau(yx) = \|x\|_p^p = 1$ and $\|y\|_q^p = \tau(|x|^p) = 1$. Hence y is the norming functional of x.

We will now prove that $|y| = u|x|^{p-1}u^*$ and $y = u^*|y|$. We have,

$$(u|x|^{p-1}u^*)^2 = u|x|^{p-1}u^*u|x|^{p-1}u^* = u|x|^{2(p-1)}u^* = y^*y.$$

So we obtain $|y| = u|x|^{p-1}u^*$. We deduce that

$$u^*|y| = u^*u|x|^{p-1}u^* = |x|^{p-1}u^* = y.$$

For the polar decomposition, it remains to show that $\mathrm{Im}(|y|)^\perp \subset \ker u^*$. We have $\mathrm{Im}(|y|)^\perp = \ker(|y|) = \ker(y)$. Let h be an element of $\mathrm{Im}(|y|)^\perp$, then $|x|^{p-1}u^*(h) = y(h) = 0$, and so $u^*(h) \in \ker(|x|^{p-1}) = \ker(|x|)$. Moreover $u^*(h) \in \mathrm{Im}(u^*) \subset \ker(|x|)^\perp$. Consequently, $u^*(h) = 0$, and hence h belongs to $\ker(u^*)$. □

The next definition is from [28, Definition 1.3].

Definition 4.1.6. *Let (N, τ) be a tracial von Neumann algebra and assume that τ is normalised. We say that $T : (N, \tau) \to (N, \tau)$ is factorisable if there exist another tracial normalised von Neumann algebra $(\mathcal{N}, \tau')$ and two unital trace-preserving $\star$-homomorphisms $\widetilde{J}, J : (N, \tau) \to (\mathcal{N}, \tau')$ such that $T = \widetilde{J}_1^* J$.*

By Theorem 1.4.1, a Schur multiplier T_M is positive and unital if and only if M is a positive semi-definite complex matrix having all diagonal entries equal to 1.

The following two theorems are [28, Proposition 2.8] and [28, Theorem 4.4], respectively.

Theorem 4.1.7. *Let $M = (m_{ij})_{1\leqslant i,j\leqslant n}$ be a positive semi-definite complex matrix having all diagonal entries equal to 1. Then the Schur multiplier T_M associated with M is factorisable if and only if there exist a tracial normalised von Neumann algebra (N,τ) and an n-tuple $(v_1,\dots,v_n)$ of unitaries of N such that for all $1 \leqslant i,j \leqslant n$:*

$$m_{ij} = \tau(v_i^* v_j).$$

Theorem 4.1.8. *Let (N,τ) be a normalized tracial von Neumann algebra. Then $T : N \to N$ is factorisable if and only if T is absolutely dilatable.*

The main result of this section is the equivalence $1 \Leftrightarrow 5$ of the next theorem. The other equivalences in this theorem are already known.

Theorem 4.1.9. *Let $M = (m_{ij})_{1\leqslant i,j\leqslant n}$ be a positive semi-definite complex matrix having all diagonal entries equal to 1. The following assertions are equivalent:*

1. *there exists $1 < p \neq 2 < +\infty$ such that $T_M : S_n^p \to S_n^p$ is completely p-dilatable;*
2. *for all $1 \leqslant p < +\infty$, $T_M : S_n^p \to S_n^p$ is completely p-dilatable;*
3. *T_M is absolutely dilatable;*
4. *T_M is factorisable;*
5. *there exist a tracial normalised von Neumann algebra (N,τ_N) and an n-tuple $(v_1,\dots,v_n)$ of unitaries of N such that for all $1 \leqslant i,j \leqslant n$:*

$$m_{ij} = \tau(v_i^* v_j).$$

Proof. Using Theorem 4.1.8 and Theorem 4.1.7, we obtain the equivalences $3 \Leftrightarrow 4 \Leftrightarrow 5$. The implications $3 \Rightarrow 2 \Rightarrow 1$ follow from Proposition 1.3.8. It therefore remains to prove that 1 implies 5.

Let $1 < p \neq 2 < +\infty$ and suppose that $T_M : S_n^p \to S_n^p$ is completely p-dilatable. Let $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. There exist a tracial von Neumann algebra (N,τ_N), two complete contractions $J : S_n^p \to L^p(N)$ and $Q : L^p(N) \to S_n^p$ and an invertible complete isometry $U : L^p(N) \to L^p(N)$ such that for all $k \in \mathbb{N}$:

$$T_M^k = QU^kJ. \tag{4.1.2}$$

We set $V = UJ : S_n^p \to L^p(N)$ and $J' := Q^* : S_n^q \to L^q(N)$. Applying (4.1.2) with $k = 1$, we obtain that

$$T_M = (J')^* V. \tag{4.1.3}$$

Next applying (4.1.2) with $k = 0$, we obtain that $QJ = Id$. Since Q, J are complete contractions, this implies that J is a complete isometry. Likewise, $J^*Q^* = Id$, hence Q^* is a complete isometry. Consequently V and J' are both complete isometries.

Firstly, we will show that we can suppose that J' and V are positive. Since $p \neq 2$, Yeadon's Theorem describing L^p-isometries applies to J' and V. More precisely, by [38, Proposition 3.2], there exist partial isometries $W_1, W_2 \in N$, positive operators B_1, B_2

affiliated with N and $\star$-homomorphisms $J_1 : M_n \to N$ and $J_2 : M_n \to N$ such that for all $x \in M_n$, and $i = 1, 2$,

$$\begin{aligned}
&B_i \text{ commute with } J_i(x),\\
&\tau_n(x) = \tau_N(B_1^p J_1(x)) = \tau_N(B_2^q J_2(x)),\\
&V(x) = W_1 B_1 J_1(x) \text{ and } J'(x) = W_2 B_2 J_2(x),\\
&W_i^* W_i = s(B_i) = J_i(1).
\end{aligned}$$

Note that since the domain space of V has a finite trace, we actually have that $B_1 = |V(1)|$ belongs to $L^p(N)$ and the polar decomposition of $V(1)$ is $W_1 B_1$. Likewise $B_2 = |J'(1)|$ belongs to $L^q(N)$ and the polar decomposition of $J'(1)$ is $W_2 B_2$.

It follows from (4.1.3) that for all $x \in S_n^p$ and $y \in S_n^q$, we have

$$\tau_n(T_M(x)y) = \tau_N(V(x)J'(y)). \tag{4.1.4}$$

By Theorem 1.4.1, the operator T_M is unital. Hence if we take $x = y = 1$ in the last formula, we obtain

$$1 = \tau_n(1) = \langle W_1 B_1, W_2 B_2 \rangle.$$

In addition, we have $\|W_1 B_1\|_p = \|V(1)\|_p = 1$ and $\|W_2 B_2\|_q = \|J'(1)\|_q = 1$. According to Lemma 4.1.5, we deduce that $W_2 B_2 = B_1^{p-1} W_1^*$ and $B_2 = W_1 B_1^{p-1} W_1^*$. Moreover by the uniqueness of the polar decomposition of $J'(1)$, which is both $W_2 B_2$ and $W_1^* B_2$, we obtain that $W_1 = W_2^*$. We define $\tilde{V} : S_n^p \to L^p(N)$ and $\tilde{J'} : S_n^q \to L^q(N)$ by setting

$$\tilde{V}(x) := B_1 J_1(x) \quad \text{and} \quad \tilde{J'}(y) := B_1^{p-1} W_1^* J_2(y) W_1.$$

for all $x \in S_n^p$ and $y \in S_n^q$. Note that we both have $\tilde{V}(.) = W_1^* V(.)$ and $V(.) = W_1 \tilde{V}(.)$. This implies that $\|\tilde{V}(x)\|_p \leqslant \|V(x)\|_p$ and $\|V(x)\|_p \leqslant \|\tilde{V}(x)\|_p$ for all $x \in S_n^p$. Then we have $\|V(x)\|_p = \|\tilde{V}(x)\|_p$ for all $x \in S_n^p$. Hence $\tilde{V}$ is an isometry. We may show as well that $\tilde{V}$ is a complete isometry. Since J_1 and B_1 commute, the operator $\tilde{V}$ is positive.

Let us prove that $\tilde{J'}$ is also a positive complete isometry. We first note that B_1^{p-1} and $W_1^* J_2(y) W_1$ commute for all $y \in M_n$. Indeed, given any $y \in M_n$, we have

$$\begin{aligned}
B_1^{p-1} W_1^* J_2(y) W_1 &= W_2 B_2 J_2(y) W_1 = W_2 J_2(y) B_2 W_1 = W_2 J_2(y) W_2^* W_2 B_2 W_1\\
&= W_2 J_2(y) W_2^* B_1^{p-1} W_1^* W_1 = W_2 J_2(y) W_2^* B_1^{p-1} = W_1^* J_2(y) W_1 B_1^{p-1},
\end{aligned}$$

which proves the commutation property. Since B_1^{p-1} and $J_2(y)$ are positive, if y is positive, the latter implies that $\tilde{J'}$ is positive.

We now prove that $W_1^* J_2(.) W_1$ is multiplicative. Let $x, y \in M_n$. Since $W_1^* = W_2$, we have

$$W_1^* J_2(xy) W_1 = W_1^* J_2(x) J_2(y) W_1 = W_1^* J_2(x) W_2^* W_2 J_2(y) W_1 = W_1^* J_2(x) W_1 W_1^* J_2(y) W_1,$$

which proves the result. Consequently, $W_1^* J_2(.) W_1 : M_n \to N$ is a $\star$-homomorphism.

In addition we have, for all $y \in M_n$,

$$\begin{aligned}
\tau_n(y) &= \tau_N\left(B_2^q J_2(y)\right)\\
&= \tau_N\left(\left(W_1 (B_1^{p-1}) W_1^*\right)^q J_2(y)\right)\\
&= \tau_N\left(W_1 (B_1^{p-1)})^q W_1^* J_2(y)\right)\\
&= \tau_N\left((B_1^{p-1})^q W_1^* J_2(y) W_1\right).
\end{aligned}$$

We also have

$$W_1^* J_2(1) W_1 = W_1^* W_2^* W_2 W_1 = W_1^* W_1 W_1^* W_1 = s(B_1)s(B_1) = s(B_1).$$

By the characterisation of positive complete isometries (see [38, Theorem 3.1] and [38, Proposition 3.2]), we deduce that the operator $\tilde{J'}$ is an positive complete isometry.

We now check that $T_M = \tilde{J'}^* \tilde{V}$. Let $x, y \in M_n$,

$$\begin{aligned}\tau_n(\tilde{J'}^* \tilde{V}(x)y) &= \tau_N(\tilde{V}(x)\tilde{J'}(y)) = \tau_N(B_1 J_1(x) B_1^{p-1} W_1^* J_2(y) W_1) \\ &= \tau_N(W_1 B_1 J_1(x) W_2 B_2 J_2(y)) = \tau_N(V(x)J'(y)) \\ &= \tau_n(T_M(x)y),\end{aligned}$$

by (4.1.4).

It follows from above that we may now suppose that J' and V are positive. Once again we use the Yeadon decomposition for the complete isometries of J' and V. In the positive case, we can remove the partial isometry in the Yeadon decomposition. We therefore obtain that there exist two positive operators $B_1 \in L^p(N)$ and $B_2 \in L^q(N)$ as well as two $\star$-homomorphisms $J_1 : M_n \to N$ and $J_2 : M_n \to N$ such that for all $x \in M_n$, and $i = 1, 2$,

$$B_i \text{ commutes with } J_i(x), \tag{4.1.5}$$

$$\tau_n(x) = \tau_N(B_1^p J_1(x)) = \tau_N(B_2^q J_2(x)), \tag{4.1.6}$$

$$V(x) = B_1 J_1(x) \text{ and } J'(x) = B_2 J_2(x), \tag{4.1.7}$$

$$s(B_i) = J_i(1). \tag{4.1.8}$$

Argunig as above, we obtain that $1 = \tau_N(B_1 B_2)$ from which we deduce that

$$B_2 = B_1^{p-1}.$$

We set $B := B_1^p = B_2^q$, this is an element of $L^1(N)$. Thanks to (4.1.5), B commutes with $J_1(x)$ and $J_2(x)$ for all $x \in M_n$.

Consider the projection $v := J_1(1)$. Then we have

$$v := J_1(1) = s(B_1) = s(B) = s(B_2) = J_2(1).$$

We define $\mathcal{M} := vNv$, and we equip this von Neumann algebra with the trace $\tau_{\mathcal{M}} = \tau_{N|\mathcal{M}}$. The $\star$-homomorphisms J_1, J_2 are valued in $\mathcal{M}$. Henceforth we consider $J_1 : M_n \to \mathcal{M}$ and $J_2 : M_n \to \mathcal{M}$, they are now unital $\star$-homomorphisms. We set

$$f_{ij} = J_1(e_{ij}) \quad \text{and} \quad g_{ij} = J_1(e_{ij}), \qquad 1 \leqslant i, j \leqslant n.$$

Then $\{f_{ij} : 1 \leqslant i, j \leqslant n\}$ and $\{g_{ij} : 1 \leqslant i, j \leqslant n\}$ are two sets of matrix units. By Lemma 4.1.4, there exists a unitary $u_1 \in \mathcal{M}$ such that

$$u_1 f_{ij} u_1^* = g_{ij}, \qquad 1 \leqslant i, j \leqslant n. \tag{4.1.9}$$

We set $w_i := g_{i1} = J_2(e_{i1})$ for all $1 \leqslant i \leqslant n$. We set $\mathcal{N} = w_1 \mathcal{M} w_1$ and we equip it with $\tau_{\mathcal{N}} := \tau_{\mathcal{M}|\mathcal{N}}$. We define

$$\rho : \mathcal{M} \to (M_n, \mathrm{tr}_n) \otimes (\mathcal{N}, \tau_{\mathcal{N}}), \;\; x \mapsto \sum_{i,j=1}^{n} e_{ij} \otimes w_i^* x w_j.$$

According to Theorem 4.1.3, ρ is a trace preserving $\star$-isomorphism. By Lemma 1.3.7 we extend ρ to an isometry from $L^p(\mathcal{M})$ into $L^p(M_n \otimes \mathcal{N}) = S_n^p \otimes L^p(\mathcal{N})$, that we still denote by ρ.

We note that for all $x \in M_n$,

$$\rho(J_2(x)) = x \otimes 1_{\mathcal{N}}.$$

Indeed, for all $x = \sum_{i,j=1}^n \alpha_{ij} e_{ij} \in M_n$ we have;

$$\begin{aligned}
\rho(J_2(x)) &= \sum_{i,j=1}^n \alpha_{ij} \rho(J_2(e_{ij})) \\
&= \sum_{i,j=1}^n \alpha_{ij} \rho(g_{ij}) \\
&= \sum_{i,j=1}^n \alpha_{ij} \sum_{k,l=1}^n e_{kl} \otimes w_k^* g_{ij} w_l \\
&= \sum_{i,j=1}^n \alpha_{ij} \sum_{k,l=1}^n e_{kl} \otimes g_{1k} g_{ij} g_{l1} \\
&= \sum_{i,j=1}^n \alpha_{ij} e_{ij} \otimes g_{11} \\
&= x \otimes 1_{\mathcal{N}}.
\end{aligned}$$

Since ρ is an unital $\star$-homomorphism, $u := \rho(u_1) \in M_n \otimes \mathcal{N}$ is also a unitary. Thanks to property (4.1.9), we have $\rho(J_1(x)) = \rho(u_1^* J_2(x) u_1)$, for all $x \in M_n$. So we obtain that for all $x \in M_n$,

$$\rho(J_1(x)) = u^*(x \otimes 1_{\mathcal{N}})u.$$

Let us use the commutation of B with J_2. For all $x \in M_n$, we have

$$\rho(B)(x \otimes 1_{\mathcal{N}}) = \rho(BJ_2(x)) = \rho(J_2(x)B) = (x \otimes 1_{\mathcal{N}})\rho(B).$$

This implies that $\rho(B) = 1 \otimes b_{11}$, for some $b_{11} \in L^p(\mathcal{N})$. Since $\tau_M(B) = 1$ and ρ is trace preserving, we have

$$\tau_{\mathcal{N}}(b_{11}) = \frac{1}{n}.$$

Let $z \in M_n \otimes \mathcal{N}$ such that $\rho(B)^{\frac{1}{2}} z \rho(B)^{\frac{1}{2}} = 0$. Since ρ is a bijective homomorphism, we have $B^{\frac{1}{2}} \rho^{-1}(z) B^{\frac{1}{2}} = 0$. By Lemma 1.2.1, we obtain that $\rho^{-1}(z) = 0$. Finally, we have $z = 0$. We summarize this by writing that for all $z \in M_n \otimes \mathcal{N}$,

$$\left\{\rho(B)^{\frac{1}{2}} z \rho(B)^{\frac{1}{2}} = 0\right\} \Rightarrow z = 0. \tag{4.1.10}$$

We write $u = \sum_{i,j=1}^n e_{ij} \otimes u_{ij}$, with $u_{ij} \in \mathcal{N}$. Let $x \in S_n^p$ and $y \in S_n^q$. Owing to the commutation of B and $J_2(x)$, we have:

$$\langle B_2 J_2(y), B_1 J_1(x) \rangle = \tau_N(B_1^{p-1} J_2(y) B_1 J_1(x)) = \tau_N(B J_2(y) J_1(x)).$$

By (4.1.4), this implies that

$$\tau_n(y T_M(x)) = \tau_N(B J_2(y) J_1(x)). \tag{4.1.11}$$

We take $y = e_{kk}$ and $x = e_{ll}$ with $1 \leqslant k \neq l \leqslant n$. We deduce from above that

$$0 = m_{ll}\tau_n(e_{kk}e_{ll}) = \tau_n(e_{kk}T_M(e_{ll})) = \tau_N(BJ_2(e_{kk})J_1(e_{ll}))$$

Since $BJ_2(e_{kk})J_1(e_{ll}) \in \mathcal{M}$ and $J_2(e_{kk})^2 = J_2(e_{kk})$, this implies

$$\begin{aligned}
0 &= \tau_{\mathcal{M}}(BJ_2(e_{kk})J_1(e_{ll})) \\
&= \tau_{\mathcal{M}}(BJ_2(e_{kk})J_2(e_{kk})J_1(e_{ll})) \\
&= \tau_{\mathcal{M}}(J_2(e_{kk})BJ_2(e_{kk})J_1(e_{ll})) \\
&= \tau_{\mathcal{M}}(BJ_2(e_{kk})J_1(e_{ll})J_2(e_{kk})) \\
&= \tau_{M_n\otimes\mathcal{N}}(\rho(BJ_2(e_{kk})J_1(e_{ll})J_2(e_{kk}))) \\
&= \tau_{M_n\otimes\mathcal{N}}(\rho(B)\rho(J_2(e_{kk}))\rho(J_1(e_{ll}))\rho(J_2(e_{kk}))).
\end{aligned}$$

Replacing $\rho(J_2(e_{kk}))$ by $e_{kk}\otimes 1_{\mathcal{N}}$, $\rho(J_1(e_{ll}))$ by $u^*(e_{ll}\otimes 1_{\mathcal{N}})u$ and u^* by $\sum_{i,j=1}^n e_{ji}\otimes u_{ij}^*$, respectively, we deduce that

$$\begin{aligned}
0 &= \tau_{M_n\otimes\mathcal{N}}\left(\rho(B)(e_{kk}\otimes 1_{\mathcal{N}})\sum_{i,j=1}^n (e_{ji}\otimes u_{ij}^*)(e_{ll}\otimes 1_{\mathcal{N}})u(e_{kk}\otimes 1_{\mathcal{N}})\right) \\
&= \sum_{i,j=1}^n \tau_{M_n\otimes\mathcal{N}}\left(\rho(B)(e_{kk}e_{ji}e_{ll}\otimes u_{ij}^*)u(e_{kk}\otimes 1_{\mathcal{N}})\right) \\
&= \tau_{M_n\otimes\mathcal{N}}\left(\rho(B)(e_{kl}\otimes u_{lk}^*)u(e_{kk}\otimes 1_{\mathcal{N}})\right).
\end{aligned}$$

Next, replacing u by $\sum_{i,j=1}^n e_{ij}\otimes u_{ij}$, we deduce that

$$\begin{aligned}
0 &= \tau_{M_n\otimes\mathcal{N}}\left(\rho(B)(e_{kl}\otimes u_{lk}^*)\sum_{i,j=1}^n (e_{ij}\otimes u_{ij})(e_{kk}\otimes 1_{\mathcal{N}})\right) \\
&= \sum_{i,j=1}^n \tau_{M_n\otimes\mathcal{N}}\left(\rho(B)(e_{kl}e_{ij}e_{kk}\otimes u_{lk}^*u_{ij})\right) \\
&= \tau_{M_n\otimes\mathcal{N}}\left(\rho(B)(e_{kk}\otimes u_{lk}^*u_{lk}^*)\right) \\
&= \tau_{M_n\otimes\mathcal{N}}\left(\rho(B)(e_{kk}\otimes u_{lk}^*)(e_{kk}\otimes u_{lk})\right) \\
&= \tau_{M_n\otimes\mathcal{N}}\left(\rho(B)^{\frac{1}{2}}(e_{kk}\otimes u_{lk}^*)(e_{kk}\otimes u_{lk})\rho(B)^{\frac{1}{2}}\right).
\end{aligned}$$

Since $\rho(B)^{\frac{1}{2}}(e_{kk}\otimes u_{lk}^*)(e_{kk}\otimes u_{lk})\rho(B)^{\frac{1}{2}}$ is positive and the trace $\tau_{M_n\otimes\mathcal{N}}$ is faithful, we deduce that $\rho(B)^{\frac{1}{2}}(e_{kk}\otimes u_{lk}^*)(e_{kk}\otimes u_{lk})\rho(B)^{\frac{1}{2}} = 0$ for all $k \neq l$. According to (4.1.10), this implies that $(e_{kk}\otimes u_{lk}^*)(e_{kk}\otimes u_{lk}) = 0$, hence $e_{kk}\otimes u_{lk} = 0$. Finally for all $1 \leqslant k \neq l \leqslant n$, we have $u_{lk} = 0$. Therefore, the element u has the form $\sum_{i=1}^n e_{ii}\otimes u_{ii}$. We set $u_i := u_{ii}$, for $1 \leqslant i \leqslant n$. Then each u_i is a unitary.

We will use the commutation property of B and J_1. For all $x \in M_n$, we have

$$\rho(B)u^*(x\otimes 1_{\mathcal{N}})u = \rho(BJ_1(x)) = \rho(J_1(x)B) = u^*(x\otimes 1_{\mathcal{N}})u\rho(B).$$

Consider any $1 \leqslant k, l \leqslant n$. We have

$$\begin{aligned}\rho(B)u^*(e_{kl}\otimes 1_{\mathcal{N}})u &= \rho(B)\left(\sum_{i=1}^n e_{ii}\otimes u_i^*\right)(e_{kl}\otimes 1_{\mathcal{N}})\left(\sum_{j=1}^n e_{jj}\otimes u_j\right)\\ &= \sum_{i,j=1}^n \rho(B)e_{ii}e_{kl}e_{jj}\otimes u_i^*u_j = \rho(B)e_{kl}\otimes u_k^*u_l\\ &= e_{kl}\otimes b_{11}u_k^*u_l.\end{aligned}$$

In the same way,

$$u^*(x\otimes 1_{\mathcal{N}})u\rho(B) = e_{kl}\otimes u_k^*u_l b_{11}.$$

We therefore deduce from above that

$$u_k^*u_l b_{11} = b_{11}u_k^*u_l, \qquad 1 \leqslant k, l \leqslant n.$$

Thus for all $1 \leqslant k, l \leqslant n$, we have $u_l b_{11}u_l^* = u_k b_{11}u_k^*$. Hence $u_k b_{11}u_k^* = u_1 b_{11}u_1^*$ for all $1 \leqslant k \leqslant n$.

We will now relate m_{ij}, b_{11} and u_k. For any $1 \leqslant k, l \leqslant n$, we have

$$m_{kl} = m_{kl}n\tau_n(e_{lk}e_{kl}) = n\tau_n(e_{lk}T_M(e_{kl})).$$

By (4.1.11), this implies

$$\begin{aligned}m_{kl} &= n\tau_N(BJ_2(e_{lk})J_1(e_{kl})) = n\tau_{\mathcal{M}}(BJ_2(e_{lk})J_1(e_{kl}))\\ &= n\tau_{M_n\otimes\mathcal{N}}(\rho(B)\rho(J_2(e_{lk}))\rho(J_1(e_{kl})))\\ &= n\tau_{M_n\otimes\mathcal{N}}(\rho(B)(e_{lk}\otimes 1_{\mathcal{N}})u^*(e_{kl}\otimes 1_{\mathcal{N}})u)\\ &= n\tau_{M_n\otimes\mathcal{N}}\left(\rho(B)(e_{lk}\otimes 1_{\mathcal{N}})\left(\sum_{i=1}^n(e_{ii}\otimes u_i^*)\right)(e_{kl}\otimes 1_{\mathcal{N}})\left(\sum_{j=1}^n(e_{jj}\otimes u_j)\right)\right)\\ &= n\tau_{M_n\otimes\mathcal{N}}(\rho(B)(e_{lk}\otimes 1_{\mathcal{N}})(e_{kl}\otimes u_k^*u_l))\\ &= n\tau_{M_n\otimes\mathcal{N}}(\rho(B)(e_{ll}\otimes u_k^*u_l))\\ &= n\tau_{M_n\otimes\mathcal{N}}(e_{ll}\otimes b_{11}u_k^*u_l))\\ &= n\tau_{\mathcal{N}}(b_{11}u_k^*u_l)).\end{aligned}$$

We set $b = nu_1b_{11}u_1^*$ and $v_i = u_iu_1^*$, for all $1 \leqslant i \leqslant n$. By construction, $b \in L^1(\mathcal{N})$ and b is positive. Moreover $\tau_{\mathcal{N}}(b) = 1$. Furthermore, $v_1, \ldots, v_n$ are unitaries. For any $1 \leqslant i \leqslant n$, we have

$$v_i^*bv_i = nu_1u_i^*u_1b_{11}u_1^*u_iu_1^* = nu_1u_i^*u_ib_{11}u_i^*u_iu_1^* = nu_1b_{11}u_1^* = b.$$

Thus, b commutes with all the v_i. In addition, we obtain that for all $y \in \mathcal{N}$,

$$b^{\frac{1}{2}}yb^{\frac{1}{2}} = 0 \Rightarrow nu_1b_{11}^{\frac{1}{2}}u_1^*ynu_1b_{11}^{\frac{1}{2}}u_1^* = 0 \Rightarrow b_{11}u_1^*yu_1b_{11} = 0 \Rightarrow u_1^*yu_1 = 0 \Rightarrow y = 0,$$

by (4.1.10). Finally for any $1 \leqslant k, l \leqslant n$, we have,

$$\tau_{\mathcal{N}}(bv_k^*v_l) = n\tau_{\mathcal{N}}(u_1b_{11}u_1^*u_1u_k^*u_lu_1^*) = n\tau_{\mathcal{N}}(b_{11}u_k^*u_l) = m_{kl}.$$

Let us summarize the situation. We have obtained a tracial von Neumann algebra $(\mathcal{N}, \tau')$, a positive $b \in L^1(\mathcal{N})$ with $\tau'(b) = 1$, and unitaries $v_1, \ldots, v_n \in \mathcal{N}$ such that

$$\forall\, 1 \leqslant i, j \leqslant n,\ bv_i = v_i b,\ m_{ij} = \tau(bv_i^* v_j) \quad \text{and} \quad \forall y \in \mathcal{N},\ b^{\frac{1}{2}} y b^{\frac{1}{2}} = 0 \Rightarrow y = 0.$$

Let $\tilde{N}$ be the von Neumann algebra generated by $v_1, \ldots, v_n$ and let $\tau_{\tilde{N}} : \tilde{N} \to \mathbb{C}$ be defined by

$$\tau_{\tilde{N}}(x) = \tau'(bx).$$

Then $\tau_{\tilde{N}}$ is a normal state. Since b commutes with all the v_i, it commutes with all the elements of $\tilde{N}$. Hence for all $x, y \in \tilde{N}$, we have

$$\tau_{\tilde{N}}(xy) = \tau'(bxy) = \tau'(xby) = \tau'(byx) = \tau_{\tilde{N}}(yx).$$

That is, $\tau_{\tilde{N}}$ is a trace. Let $x \in \tilde{N}_+$, such that $\tau_{\tilde{N}}(x) = 0$. Then we have,

$$0 = \tau_{\tilde{N}}(x) = \tau'(bx) = \tau'(b^{\frac{1}{2}} x b^{\frac{1}{2}}).$$

Since τ' is faithful, we obtain that $b^{\frac{1}{2}} x b^{\frac{1}{2}} = 0$. By the above property of b, we deduce that $x = 0$. Hence $\tau_{\tilde{N}}$ is faithful. Thus $(\tilde{N}, \tau_{\tilde{N}})$ is a tracial normalized von Neumann algebra. In addition $(v_1, \ldots, v_n)$ are unitaries of $\tilde{N}$ and verify for all $1 \leqslant i, j \leqslant n$,

$$m_{ij} = \tau(bv_i^* v_j) = \tau_{\tilde{N}}(v_i^* v_j).$$

This shows property 5 in Theorem 4.1.9. □

Remark 4.1.10. Consider $T_M : S_n^p \to S_n^p$ as in Theorem 4.1.9 and for any $1 \leqslant p < \infty$, let us say that T_M is completely positively p-dilatable if there exist a tracial von Neumann algebra (N, τ), two completely positive and completely contrative maps $J : S_n^p \to L^p(N)$ and $Q : L^p(N) \to S_n^p$, and an invertible completely positive isometry $U : L^p(N) \to L^p(N)$ such that $T_M^k = QU^k J$ for all $k \geqslant 0$. Then the five conditions of Theorem 4.1.9 are also equivalent to:

1'. there exists $1 < p < \infty$ such that T_M is completely positively p-dilatable.

Note that the case $p = 2$ is admissible in this assertion.

To prove this, it suffices to observe that if T_M is completely positively p-dilatable for some $1 \leqslant p < \infty$, then (4.1.3) holds true for some completely positive isometries $V : S_n^p \to L^p(N)$ and $J' : S_n^q \to L^q(N)$ (here, $\frac{1}{p} + \frac{1}{q} = 1$). Furthemore, V and J' admit a Yeadon type factorization as in (4.1.5)-(4.1.6)-(4.1.7)-(4.1.8), see [45, Remark 5.2 and Theorem 4.2]. Therefore, the proof of Theorem 4.1.9 shows as well that $1' \Rightarrow 5$. Moreover the proof of Proppsition 1.3.8 shows that $5 \Rightarrow 1'$.

According to [34], there exists a completely positive contraction $u : S_n^p \to S_n^p$ which is not completely p-dilatable. The proofs of this result given in [34] do not provide any information on n. However we note that [28, Example 3.2] provides a unital completely positive Schur multiplier $T_M : M_4 \to M_4$ which is not factorisable. Applying Theorem 4.1.9, we deduce that for all $1 < p \neq 2 < +\infty$, $T_M : S_4^p \to S_4^p$ is not completely p-dilatable.

4.2 Discrete Schur multipliers

Let I be an index set and let $\mathcal{M}_I$ denote the space of the $I \times I$ matrices with complex entries. We assume that T_M associated with $M \in \mathcal{M}_I$ is a bounded Schur multiplier on $B(l^2(I))$.

We will generalize Theorem 4.1.9 to the present setting of discrete Schur multipliers. In the following, we simply say T_M is completely p-dilatable if $T_M : S_I^p \to S_I^p$ is completely p-dilatable.

Theorem 4.2.1. *Let T_M be a unital positive Schur multiplier. The following assertions are equivalent:*

1. *there exists $1 < p \neq 2 < +\infty$ such that T_M is completely p-dilatable;*
2. *for all $1 \leqslant p < +\infty$, T_M is completely p-dilatable;*
3. *T_M is absolutely dilatable;*
4. *there exist a tracial normalised von Neumann algebra (N, τ) and a family $(v_i)_{i\in I}$ of unitaries of N such that for all $i, j \in I$:*

$$m_{ij} = \tau(v_i^* v_j).$$

Proof. The implications $3 \Rightarrow 2 \Rightarrow 1$ are clear. The implication $4 \Rightarrow 3$ is proved implicitly in [4, Proof of Theorem 4.2]. It remains to show $1 \Rightarrow 4$.

Let $1 < p \neq 2 < +\infty$ and suppose that $T_M : S_I^p \to S_I^p$ is a completely p-dilatable. There exist a tracial von Neumann algebra $(\mathcal{M}, \tau_{\mathcal{M}})$, two complete contractions $J : S_I^p \to L^p(\mathcal{M})$ and $Q : L^p(\mathcal{M}) \to S_I^p$, as well as an inverible complete isometry U on $L^p(\mathcal{M})$, such that for all $k \geqslant 0$, $T^k = QU^kJ$. Let $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and set $J' := Q^* : S_I^q \to L^q(\mathcal{M})$. By the equality $T^k = QU^kJ$ for $k = 0$, we see (as in the proof of Theorem 4.1.9) that J and J' are two complete isometries.

For any finite subset $F \subset I$, we let $T_{M,F} : S_F^p \to S_F^p$ denote the restriction of T_M, that is, $T_{M,F}((a_{ij})_{i,j\in F}) = (m_{ij}a_{ij})_{i,j\in F}$. We consider $J_F = J_{|S_F^p}$ and $J'_F = J'_{|S_F^q}$. It is clear that on S_F^p, we have $T_{M,F}^k = (J'_F)^* U^k J_F$ for all $k \geqslant 0$. Hence $T_{M,F}$ is completely p-dilatable.

In the finite dimensional case, we have Theorem 4.1.9 at our disposal. We use it and we obtain that there exist a von Neumann algebra $\mathcal{M}_F$ equipped with a normal faithful normalized trace $\tau_{\mathcal{M}_F}$ and unitaries $(d_{i,F})_{i\in F}$ of $\mathcal{M}_F$ such that for all $i, j \in F$,

$$m_{ij} = \tau_{\mathcal{M}_F}(d_{i,F}^* d_{j,F}).$$

We recall the following ultraproduct construction, see [51, Section 11.5] for details. Let $\mathcal{F}$ be a non trivial ultrafilter on the index set $\{F \subset I \text{ finite}\}$. Let B be the C*-algebra defined by

$$B = \left\{ x = (x_F) \in \prod_{F\subset I \text{ finite}} \mathcal{M}_F \mid \sup_{F\subset I \text{ finite}} \|x_F\|_{\mathcal{M}_F} < +\infty \right\}.$$

Let $f_{\mathcal{F}} \in B^*$ be the tracial state defined for all $x = (x_F) \in B$ by

$$f_{\mathcal{F}}(x) = \lim_{\mathcal{F}} \tau_{\mathcal{M}_F}(x_F).$$

Let $H_{\mathcal{F}}$ be the Hilbert space associated with the tracial state $f_{\mathcal{F}}$ in the GNS construction. Let $L : B \to B(H_{\mathcal{F}})$ be the $\star$-homomorphism induced by left multiplication. Consider the ideal $\mathcal{I}_{\mathcal{F}}$ of B defined by

$$\mathcal{I}_{\mathcal{F}} := \{x \in B \mid f_{\mathcal{F}}(x^*x) = 0\}.$$

Since $f_{\mathcal{F}}$ is a trace, we have $\mathcal{I}_{\mathcal{F}} = \ker(L)$. Let $B/\mathcal{I}_{\mathcal{F}}$ denote the resulting quotient C*-algebra and let $q : B \to B/\mathcal{I}_{\mathcal{F}}$ be the quotient map. Then we have a one-to-one $\star$-homomorphism $L_{\mathcal{F}} : B/\mathcal{I}_{\mathcal{F}} \to B(H_{\mathcal{F}})$ and a faithful normalized trace $\tau_{\mathcal{F}} : B/\mathcal{I}_{\mathcal{F}} \to \mathbb{C}$ such that

$$L_{\mathcal{F}} \circ q = L \quad \text{and} \quad \tau_{\mathcal{F}} \circ q = f_{\mathcal{F}}.$$

A remarkable result is that

$$M_{\mathcal{F}} := L_{\mathcal{F}}(B/\mathcal{I}_{\mathcal{F}}) \subset B(H_{\mathcal{F}})$$

is a von Neumann algebra and $\rho_{\mathcal{F}} := \tau_{\mathcal{F}} \circ L_{\mathcal{F}}^{-1} : M_{\mathcal{F}} \to \mathbb{C}$ is normal. Thus, $(M_{\mathcal{F}}, \rho_{\mathcal{F}})$ is a tracial normalised von Neumann algebra.

For all $i \in I$, we define $d_i := L_{\mathcal{F}}(q((d_{i,F})_{F \subset I \text{ finite}})) = L((d_{i,F})_{F \subset I \text{ finite}})$. The d_i are well-defined because all $d_{i,F}$ are unitaries, hence $\|d_{i,F}\| = 1$. Since L is unital, d_i is a unitary for all $i \in I$. It remains to prove the formula $m_{ij} = \rho_{\mathcal{F}}(d_i^* d_j)$. Given any $i, j \in I$, we have

$$d_i^* d_j = L_{\mathcal{F}}\left(q\left((d_{i,F}^* d_{j,F})_F\right)\right),$$

hence

$$\begin{aligned}
\rho_{\mathcal{F}}(d_i^* d_j) &= \tau_{\mathcal{F}}\left(q\left((d_{i,F}^* d_{j,F})_F\right)\right) \\
&= f_{\mathcal{F}}\left((d_{i,F}^* d_{j,F})_F\right) \\
&= \lim_{\mathcal{F}} \tau_{\mathcal{M}_F}\left(d_{i,F}^* d_{j,F}\right) \\
&= m_{ij}.
\end{aligned}$$

□

The equivalence $3 \Leftrightarrow 4$ in Theorem 4.2.1 provides a new proof of Equivalence $(i) \Leftrightarrow (ii)$ of Corollary 3.5.2.

Remark 4.2.2. Following [56, page 4367], consider the normal faithful state on $B(l^2)$ with density equal to the diagonal operator $D = \sum_{i \geqslant 1} \lambda_i e_i \otimes e_i$, where $(e_i)_{i \geqslant 1}$ is the canonical basis of l^2, $\lambda_i > 0$ for all $i \geqslant 1$ and $\sum_{i \geqslant 1} \lambda_i = 1$.

We say that T_M is (D, D)-factorisable if T_M is factorisable in the sense of [28, Definition 1.3]. According to [28, Theorem 4.4], the equivalent conditions of Theorem 4.2.1 in the case $I = \mathbb{N}$ are also equivalent to "T_M is (D, D)-factorisable".

Remark 4.2.3. As in Remark 4.1.10, we can observe that the equivalent conditions of Theorem 4.2.1 are also equivalent to:

1' there exists $1 \leqslant p < \infty$ such that T_M is completely positively p-dilatable.

4.3 Multivariable case

In this last section, we introduce the notion of simultaneous absolute dilation for a commuting finite family of operators and consider the special case of Schur multipliers. Note that all bounded Schur multipliers are commuting.

Definition 4.3.1.

(a) Let (N,τ) be a tracial von Neumann algebra. We say that a commuting family $(T_1,\dots,T_n)$ of operators $T_l:(N,\tau)\to(N,\tau)$ is simultaneously absolutely dilatable if there exist a tracial von Neumann algebra $(\mathcal{N},\tau')$, a commuting n-tuple $(U_1,\dots,U_n)$ of trace preserving $\star$-automorphisms on $\mathcal{N}$ and a unital one-to-one trace preserving and w^*-continuous $\star$-homomorphism $J:N\to\mathcal{N}$ such that

$$T_1^{k_1}\cdots T_n^{k_n}=\mathbb{E}U_1^{k_1}\cdots U_n^{k_n}J$$

for all $k_l\geqslant 0$, $1\leqslant l\leqslant n$, where $\mathbb{E}:\mathcal{N}\to N$ is the conditional expectation associated with J.

(b) Let $1\leqslant p<+\infty$ and let (N,τ) be a tracial von Neumann algebra. We say that a commuting family $(T_1,\dots,T_n)$ of operators $T_l:L^p(N)\to L^p(N)$ is simultaneously completely p-dilatable, if there exist a tracial von Neumann algebra $(\mathcal{N},\tau)$, a commuting n-tuple $(U_1,\dots,U_n)$ of invertible complete isometries on $L^p(\mathcal{N})$ and two complete contractions $J:L^p(N)\to L^p(\mathcal{N})$ and $Q:L^p(\mathcal{N})\to L^p(N)$ such that

$$T_1^{k_1}\cdots T_n^{k_n}=QU_1^{k_1}\cdots U_n^{k_n}J\qquad\text{on } L^p(N)$$

for all $k_l\geqslant 0$, $1\leqslant l\leqslant n$.

As in the case of one operator (see Proposition 1.3.8), we see that a simultaneously absolutely dilatable family is simultaneously completely p-dilatable.

Theorem 4.3.2. *Let I be an index set and for any $1\leqslant l\leqslant n$, let $M^l=(m^l_{ij})_{i,j\in I}\in\mathcal{M}_I$ such that $T_{M_l}\colon B(l^2(I))\to B(l^2(I))$ is a unital positive bounded Schur multiplier. The following assertions are equivalent:*

1. *the family $(T_{M^l})_{l=1}^n$ is simultaneously absolutely dilatable;*
2. *each T_{M^l} is absolutely dilatable.*

Proof. Implication $1\Rightarrow 2$ is clear. Conversely, we suppose that each T_{M^l} is absolutely dilatable and we use Theorem 4.2.1. Thus for all $1\leqslant l\leqslant n$, there exist a tracial von Neumann algebra (N^l,τ^l), unitaries of N (u^l_j) such that for all $i,j\in I$:

$$m^l_{ij}=\tau^l\left(\left(u^l_i\right)^*u^l_j\right).$$

We denote the infinite von Neumann tensor product $\overline{\otimes}_{\mathbb{Z}}N$ by N^∞ and we let τ^∞ denote the normal faithful finite trace on N^∞ (see [64]). We denote the unit of N^∞ by 1_∞.

We let $(N,\tau_N)=(N^1\overline{\otimes}\cdots\overline{\otimes}N^n,\tau^1\overline{\otimes}\cdots\overline{\otimes}\tau^n)$ and $\mathcal{N}=B(l^2(I))\overline{\otimes}N^\infty$. We consider the standard n.s.f trace $\tau_{\mathcal{N}}:=\mathrm{tr}_I\overline{\otimes}\tau_N^\infty$ on $\mathcal{N}$. We set for all $j\in I$, the map $e_{jj}:l^2(I)\to$

$l^2(I)$ such that for all $x = (x_i)_{i\in I} \in l^2(I)$, $e_{jj}(x) = (\dots, 0, x_j, 0, \dots)$ where x_j is in the j position.

We let for all $1 \leqslant l \leqslant n$,

$$u^l = \sum_{j\in I} e_{jj} \otimes \cdots 1_N \otimes 1_N \otimes \underbrace{1_{N^1} \otimes \cdots \otimes 1_{N^{l-1}} \otimes u_j^l \otimes 1_{N^{l+1}} \otimes \cdots \otimes 1_{N^n}}_{0} \otimes 1_N \cdots$$

where the summation is taken in w^*-topology of $\mathcal{N}$.

The element u^l is unitary because all u_j^l are unitaries. We introduce the right shift $S : N^\infty \to N^\infty$. This is a normal, trace preserving $\star$-automorphism such that for all $(x_n)_{n\in\mathbb{Z}} \subset N$,

$$S(\cdots \otimes x_0 \otimes x_1 \otimes x_2 \otimes \cdots) = \cdots \otimes x_{-1} \otimes x_0 \otimes x_1 \otimes \cdots .$$

we let the following mappings:

$$J : B(l^2(I)) \to \mathcal{N} \quad x \mapsto x \otimes 1_\infty$$
$$U_l : \mathcal{N} \to \mathcal{N} \quad y \mapsto (u^l)^*((Id \otimes S)(y))u^l.$$

We remark that $\mathbb{E} = id \otimes \tau_N^\infty$ is the normal faithful canonical conditional expectation associated with J and it preserves the trace. In addition J is a $\star$-homomorphism which is trace preserving and the U_l are $\star$-isomorphisms which are trace preserving. All U_l are commuting to each other. With the same computation in [54, Proof of Theorem 1.9], we obtain $T_{M_1}^{k_1} \cdots T_{M_n}^{k_n} = \mathbb{E}U_1^{k_1} \cdots U_n^{k_n} J$, for all $k_l \in \mathbb{N}_0$, $1 \leqslant l \leqslant n$. □

Chapter 5

Absolute dilations of Fourier multipliers

In this chapter, we will prove that any self-adjoint, positive, unital Fourier multipliers admits an absolute dilation. We will treat the unimodular and the non-unimodular cases separately.

5.1 Unimodular case

The aim of this section is to prove the following theorem. All background on $VN(G)$ was introduced in Section 1.5. We will use the notation introduced in there.

Theorem (B). *Let $T_u : VN(G) \to VN(G)$ be a self-adjoint, unital, completely positive Fourier multiplier. Then T_u is absolutely dilatable.*

We adapt the construction of [4] to our general setting. We assume that $T_u : VN(G) \to VN(G)$ is a completely positive, self-adjoint unital Fourier multiplier associated with a bounded continuous function $u : G \to \mathbb{C}$. Let $\Theta : \ell^1_{\mathbb{R}}(G) \times \ell^1_{\mathbb{R}}(G) \to \mathbb{R}$ be the bilinear symmetric map defined by

$$\Theta : (f, h) \mapsto \sum_{G\times G} u(t^{-1}s)f(t)h(s).$$

This is well-defined, because u is bounded, and Θ is positive by Theorem 1.5.2, (1).

Let $K_\Theta \subset \ell^1_{\mathbb{R}}(G)$ be the kernel of the seminorm $\Theta(f, f)^{\frac{1}{2}}$ and let H_u be the completion of $\ell^1_{\mathbb{R}}(G)/K_\Theta$ for the norm induced by $\Theta(f, f)^{\frac{1}{2}}$. We denote by $\mathcal{H}$ the real Hilbert space $H_u \overset{2}{\otimes} \ell^2_{\mathbb{R}}(\mathbb{Z})$, to which we associate the tracial von Neumann algebra $(\Gamma_{-1}(\mathcal{H}), \tau)$ (See Section 1.6 for all details). Furthermore we let $H := \mathcal{F}_1(\mathcal{H})$ and we recall that by construction, $\Gamma_{-1}(\mathcal{H}) \subset B(H)$.

In the sequel, we let $\dot{f} \in H_u$ denote the class of any $f \in \ell^1_{\mathbb{R}}(G)$.

Lemma 5.1.1. *For any $t \in G$, the map $\theta_t : \dot{f} \mapsto \dot{\overbrace{s \mapsto f(t^{-1}s)}}$ is an isometry from H_u onto H_u.*

Proof. This follows from the following equality. For all $f, h \in \ell_1(G)$, we have:

$$\sum_{G\times G} u(s'^{-1}s)f(ts)h(ts') = \sum_{G\times G} u(s'^{-1}s)f(s)h(s').$$

□

For all $t \in G$, the map $\theta_t \otimes Id_{\ell^2_{\mathbb{R}}(\mathbb{Z})}$ extends to an onto isometry on $\mathcal{H}$. We denote this application by $\theta_t \overset{2}{\otimes} Id_{\ell^2_{\mathbb{R}}(\mathbb{Z})}$. We obtain for all $t \in G$, the map $\Gamma_{-1}(\theta_t \overset{2}{\otimes} Id_{\ell^2_{\mathbb{R}}(\mathbb{Z})})$ is a trace preserving $*$-automorphism.

We define a homomorphism

$$\alpha : G \to Aut(\Gamma_{-1}(\mathcal{H})), \qquad \alpha(t) = \Gamma_{-1}(\theta_t \overset{2}{\otimes} Id_{\ell^2_{\mathbb{R}}(\mathbb{Z})}).$$

For any $t \in G$ we let $\delta_t \in \ell^1_{\mathbb{R}}(G)$ be defined by $\delta_t(t) = 1$ and $\delta_t(s) = 0$ if $s \neq t$. According to (1.6.4), we have

$$\alpha(t)\big(\omega(\dot{\delta_s} \otimes z)\big) = \omega(\dot{\delta_{ts}} \otimes z), \qquad t, s \in G,\ z \in l^2_{\mathbb{R}}(\mathbb{Z}). \tag{5.1.1}$$

We consider below continuity of α in the sense of [63, Definition X.1.1] and [63, Proposition X.1.2].

Lemma 5.1.2. *The map $\alpha : G \to Aut(\Gamma_{-1}(\mathcal{H}))$ is point-w*-continuous, i.e for all $m \in \Gamma_{-1}(\mathcal{H})$ and for all $\eta \in \Gamma_{-1}(\mathcal{H})_*$, the map $t \mapsto \langle \alpha(t)(m), \eta\rangle_{\Gamma_{-1}(\mathcal{H}),\Gamma_{-1}(\mathcal{H})_*}$ is continuous.*

Proof. Firstly we prove the continuity of the map

$$t \mapsto \langle \alpha(t)(\omega(\dot{\delta_{s_1}} \otimes h_1) \cdots \omega(\dot{\delta_{s_n}} \otimes h_n)), \omega(\dot{\delta_{v_1}} \otimes k_1) \cdots \omega(\dot{\delta_{v_n}} \otimes k_n)\rangle,$$

for arbitrary $s_1, \ldots, s_n, v_1, \ldots, v_n \in G$ and $h_1, \ldots, h_n, k_1, \ldots, k_n \in l^2_{\mathbb{Z}}$.

For all $t \in G$ we have, by (5.1.1) and Lemma 1.6.1,

$$\begin{aligned}
&\langle \alpha(t)(\omega(\dot{\delta_{s_1}} \otimes h_1) \cdots \omega(\dot{\delta_{s_n}} \otimes h_n)), \omega(\dot{\delta_{v_1}} \otimes k_1) \cdots \omega(\dot{\delta_{v_n}} \otimes k_n)\rangle \\
&\qquad = \langle \alpha(t)(\omega(\dot{\delta_{s_1}} \otimes h_1)) \cdots \alpha(t)(\omega(\dot{\delta_{s_n}} \otimes h_n)), \omega(\dot{\delta_{v_1}} \otimes k_1) \cdots \omega(\dot{\delta_{v_n}} \otimes k_n)\rangle \\
&\qquad = \langle (\omega(\dot{\delta_{ts_1}} \otimes h_1)) \cdots (\omega(\dot{\delta_{ts_n}} \otimes h_n)), \omega(\dot{\delta_{v_1}} \otimes k_1) \cdots \omega(\dot{\delta_{v_n}} \otimes k_n)\rangle \\
&\qquad = \tau\big(\omega(\dot{\delta_{ts_1}} \otimes h_1) \cdots \omega(\dot{\delta_{ts_n}} \otimes h_n)\omega(\dot{\delta_{v_1}} \otimes k_1) \cdots \omega(\dot{\delta_{v_n}} \otimes k_n)\big) \\
&\qquad = \sum_{\nu \in \mathcal{P}_2(2n)} (-1)^{c(\nu)} \prod_{i,j \in \nu} \langle \dot{\delta_{t_i}}, \dot{\delta_{t_j}}\rangle_{H_u} \langle l_i, l_j\rangle_{l^2_{\mathbb{Z}}},
\end{aligned}$$

where $t_i = \begin{cases} ts_i & \text{if } \ i \leqslant n \\ v_{i-n} & \text{if } \ i > n \end{cases}$ and $l_i = \begin{cases} h_i & \text{if } \ i \leqslant n \\ k_{i-n} & \text{if } \ i > n \end{cases}$.

We have $\langle \dot{\delta_{ts_i}}, \dot{\delta_{v_j}}\rangle_{H_u} = u((ts_i)^{-1}v_j)$, and u is continuous. Hence we obtain the expected continuity.

Next we note that by the density of $\mathrm{span}\{\omega(.)\omega(.)\cdots\omega(.)\}$ in $L^2(\Gamma_{-1}(\mathcal{H}))$, this implies continuity of the map $t \mapsto \langle \alpha(t)a, b\rangle$ for all $a, b \in L^2(\Gamma_{-1}(\mathcal{H}))$. Finally let $m \in \Gamma_{-1}(\mathcal{H})$ and let $\eta \in \Gamma_{-1}(\mathcal{H})_* = L^1(\Gamma_{-1}(\mathcal{H}))$. We may write $\eta = ab$, with $a, b \in L^2(\Gamma_{-1}(\mathcal{H}))$. Then

$$\langle \alpha(t)m, \eta\rangle = \langle \alpha(t)m, ab\rangle = \langle \alpha(t)ma, b\rangle.$$

The continuity of $t \mapsto \langle \alpha(t)m, \eta\rangle$ follows. □

We recall some background about the crossed product $\Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G$ associated with α. Define, for all $m \in \Gamma_{-1}(\mathcal{H})$,

$$\pi_\alpha(m) : \xi \in L^2(G,H) \mapsto \alpha(.)^{-1}(m)(\xi)(.) \in L^2(G,H).$$

Then π_α is a 1-1 $*$-homomorphism from $\Gamma_{-1}(\mathcal{H})$ into $B(L^2(G,H))$. We note in passing that for every $m \in \Gamma_{-1}(\mathcal{H})$ the map $s \mapsto \alpha(s^{-1})(m)$ is a element of the space $L^\infty_\sigma(G, B(H))$. By definition,

$$\Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G := VN\big\{\{\pi_\alpha(m);\, m \in \Gamma_{-1}(\mathcal{H})\}, \{x \overline{\otimes} I_H :\, x \in VN(G)\}\big\} \subset B(L^2(G,H)).$$

We define

$$J : VN(G) \longrightarrow \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G, \qquad J(x) = x \overline{\otimes} I_H. \tag{5.1.2}$$

This w^*-continuous one-to-one $*$-homomorphism is the one to be used in Theorem (**B**).

The following is a classical fact.

Lemma 5.1.3. *For any $x \in \Gamma_{-1}(\mathcal{H})$ and $t \in G$,*

$$J(\lambda(t))\pi_\alpha(x) = \pi_\alpha(\alpha(t)(x))J(\lambda(t)).$$

This following will be applied a few times in this chapter.

Theorem 5.1.4. *Let φ and ψ two n.s.f. traces on a von Neumann algebra $\mathcal{M}$. Assume that there exists a w^*-dense $*$-subalgebra B of $\mathcal{M}$ such that $B \subset L^2(\mathcal{M}, \varphi)$ and for all $y \in B$,*

$$\psi(y^*y) = \varphi(y^*y).$$

Then $\varphi = \psi$.

Proof. It is a special case of the Pedersen-Takesaki theorem, see [60, §6.2]. □

We let τ_M denote the dual weight of τ on $M = \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G$, for which we refer to [63, p. 61, p. 248-249]. The action α considered in the present chapter is trace preserving hence the n.s.f. weight τ_M is actually a trace. Indeed it follows from in [63, Theorem X.1.17] that the modular group of τ_M acts trivially on $\pi(M)$ and on $J(VN(G))$.

We now recall a construction which is a slight variant of the one given in [63, Section X]. Let $\mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$ be the vector space of all compactly supported σ-strongly*-continuous $\Gamma_{-1}(\mathcal{H})$-valued functions on G. For all $F, F' \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$, we define $F * F' \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$ and $F^\sharp \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$ by

$$F * F'(t) = \int_G F(s^{-1})\alpha^{-1}(s)(F'(st))ds \quad \text{and} \quad F^\sharp(t) = \alpha(t)(F(t^{-1})^*).$$

Then for any $F \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$, we define $T^\alpha_F \in \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G$ by

$$\forall \eta \in (\Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G)_*,\ \langle T^\alpha_F, \eta\rangle = \int_G \langle \pi_\alpha(F(t))J(\lambda(t)), \eta\rangle dt.$$

We let $\mathcal{B} \subset \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G$ denote the set of all T^α_F, for $F \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$.

Following [63, Lemma X.1.8] or [60, 19.8], we have the following two results.

Lemma 5.1.5. *For all $F, F_1, F_2 \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$,*

$$(T_F^\alpha)^* = T_{F^\sharp}^\alpha \text{ and } T_{F_1}^\alpha T_{F_2}^\alpha = T_{F_1 * F_2}^\alpha$$

Lemma 5.1.6.

1. *The set $\mathcal{B}$ is a $*$-subalgebra and it is w^*- dense in $\Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G$;*
2. *For all $F \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$, we have $T_{F^\sharp * F}^\alpha \geqslant 0$.*
3. *Let e be the unit of G. If $F \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$ is such that $T_F^\alpha \geqslant 0$, then:*

$$\tau_M(T_F^\alpha) = \tau(F(e)).$$

We now prove a property necessary to apply Definition 1.3.4.

Lemma 5.1.7. *The map J defined by (5.1.2) is trace preserving, that is,*

$$\tau_M \circ J = \omega_G.$$

Proof. Firstly we remark that for all $f \in \mathcal{K}(G)$, $T_{f^* * f \otimes 1}^\alpha = J(\lambda(f^* * f))$. Using Lemma 1.5.3, it follows that

$$\tau_M(J(\lambda(f)^*\lambda(f))) = \tau_M(J(\lambda(f^* * f))) = \tau_M(T_{f^* * f \otimes 1}^\alpha)).$$

According to Lemma 5.1.6, we have

$$\tau_M(T_{f^* * f \otimes 1}^\alpha) = \tau(f^* * f \otimes 1(e)) = \tau(1) f^* * f(e) = \omega_G(\lambda(f)^*\lambda(f)).$$

Consequently,

$$\tau_M(J(\lambda(f)^*\lambda(f))) = \omega_G(\lambda(f)^*\lambda(f)).$$

The result follows by applying Theorem 5.1.4 to ω_G and the n.s.f. trace $\tau_M \circ J$. □

We now establish a link between the trace τ_M on $M = \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G$, the Plancherel trace ω_G and the trace τ on $\Gamma_{-1}(\mathcal{H})$.

Proposition 5.1.8. *For any $x \in \Gamma_{-1}(\mathcal{H})$ and $m \in VN(G) \cap L^1(VN(G))$, $\pi_\alpha(x)J(m)$ belongs to $M \cap L^1(M)$ and we have*

$$\tau_M(\pi_\alpha(x)J(m)) = \omega_G(m)\tau(x).$$

We first establish the above result in a special case.

Lemma 5.1.9. *Let $x \in \Gamma_{-1}(\mathcal{H})$ and $f \in \mathcal{K}(G)$. We have:*

$$\tau_M(\pi_\alpha(x^*x)J(\lambda(f)^*\lambda(f))) = \tau(x^*x)\omega_G(\lambda(f)^*\lambda(f)).$$

Proof. We have by the tracial property of τ_M:

$$\begin{aligned}\tau_M(\pi_\alpha(x^*x)J(\lambda(f)\lambda(f)^*)) &= \tau_M(\pi_\alpha(x)^*\pi_\alpha(x)J(\lambda(f))J(\lambda(f))^*) \\ &= \tau_M(\ \underbrace{\pi_\alpha(x)J(\lambda(f))(\pi_\alpha(x)J(\lambda(f)))^*}_{\geqslant 0.}\ )\end{aligned}$$

We observe that $T^\alpha_{f\otimes x} = \pi_\alpha(x)J(\lambda(f))$ and $(f\otimes x)*(f\otimes x)^\sharp = f*f^*(.)x\alpha(.)(x^*)$. Then we have, using Lemmas 1.5.3, 5.1.5 and 5.1.6,

$$\begin{aligned}\tau_M(\pi_\alpha(x^*x)J(\lambda(f)\lambda(f)^*)) &= \tau_M(T^\alpha_{f\otimes x}(T^\alpha_{f\otimes x})^*) \\ &= \tau_M(T^\alpha_{(f\otimes x)*(f\otimes x)^\sharp}) \\ &= \tau_M(T^\alpha_{f*f^*(.)x\alpha(.)(x^*)}) \\ &= \tau(f*f^*(e)\otimes x\alpha(e)(x^*)) \\ &= f*f^*(e)\tau(xx^*) \\ &= \omega_G(\lambda(f)\lambda(f)^*)\tau(x^*x).\end{aligned}$$

We obtain the result by changing f into f^*. □

Proof of Proposition 5.1.8. For any $m\in VN(G)\cap L^1(VN(G))$, $J(m)\in M\cap L^1(M)$, by Lemma 5.1.7. Hence $\pi_\alpha(x)J(m)$ belongs to $M\cap L^1(M)$ for any $x\in\Gamma_{-1}(\mathcal{H})$.

We fix $x\in\Gamma_{-1}(\mathcal{H})$ and we may assume that $x\neq 0$. We define a n.s.f. trace Φ_x on $VN(G)$ by

$$\Phi_x(m) = \tau_M\left(\frac{\pi_\alpha(x^*x)}{\tau(x^*x)}J(m)\right), \qquad m\in VN(G)_+.$$

The faithful property comes from the faithful property of τ_M and the computation in the beginning of Proof of Lemma 5.1.9. By Lemma 5.1.9, Φ_x coincides with ω_G on $\{\lambda(f)^*\lambda(f) : f\in\mathcal{K}(G)\}$. Applying Theorem 5.1.4, we deduce that $\Phi_x=\omega_G$. Thus we have

$$\tau_M(\pi_\alpha(x^*x)J(m)) = \tau(x^*x)\omega_G(m), \qquad x\in\Gamma_{-1}(\mathcal{H}),\ m\in VN(G)_+.$$

By polarization, the result follows at once. □

As in Section 2.2 on Schur multipliers, J induces $J_1 : L^1(VN(G))\to L^1(\Gamma_{-1}(\mathcal{H})\rtimes_\alpha G)$ and we denote by $\mathbb{E} = J_1^* : \Gamma_{-1}(\mathcal{H})\rtimes_\alpha G\to VN(G)$ the conditional expectation associated with J.

Lemma 5.1.10. *For any $x\in\Gamma_{-1}(\mathcal{H})$ and $t\in G$, we have:*

$$\mathbb{E}(\pi_\alpha(x)J(\lambda(t))) = \tau(x)\lambda(t).$$

Proof. Let $\eta\in VN(G)\cap L^1(VN(G))$. Using Proposition 5.1.8, we have:

$$\begin{aligned}\langle\mathbb{E}(\pi_\alpha(x)J(\lambda(t))),\eta\rangle_{VN(G),L^1(VN(G))} &= \langle\pi_\alpha(x)J(\lambda(t)), J_1(\eta)\rangle_{M,L^1(M)} \\ &= \langle\pi_\alpha(x)\lambda(t)\overline{\otimes}I_H, \eta\overline{\otimes}I_H\rangle_{M,L^1(M)} \\ &= \tau_M(\pi_\alpha(x)(\lambda(t)\eta\overline{\otimes}I_H)) \\ &= \tau_M(\pi_\alpha(x)J(\lambda(t)\eta)) \\ &= \omega_G(\lambda(t)\eta)\tau(x).\end{aligned}$$

Since $VN(G)\cap L^1(VN(G))$ is dense in $L^1(VN(G))$, this yields the result. □

Let $\rho \in B(l^2_{\mathbb{Z}})$ be the shift operator, $\rho\big((a_n)_n\big) = \big((a_{n+1})_n\big)$. This is a unitary hence $\rho' := \Gamma_{-1}(Id \overset{2}{\otimes} \rho)$ is a normal completely positive, trace preserving automorphism. Further we have

$$\rho'(\omega(x)) = \omega(Id \overset{2}{\otimes} \rho(x)), \qquad x \in \Gamma_{-1}(\mathcal{H}), \tag{5.1.3}$$

by (1.6.4). It follows that for any $t \in G$, we have:

$$\alpha(t)\rho' = \rho'\alpha(t).$$

Theorem 5.1.11. *There exists a (necessarily unique) automorphism $\tilde{\rho} : \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G \to \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G$ such that:*

1. *$\tilde{\rho}$ is trace preserving;*
2. *for all $x \in \Gamma_{-1}(\mathcal{H})$, $\tilde{\rho}(\pi_\alpha(x)) = \pi_\alpha(\rho'(x))$;*
3. *for all $t \in G$, $\tilde{\rho}(J(\lambda(t))) = J(\lambda(t))$.*

Proof. We apply [61, Theorem 4.4.4 p. 149]. It provides all the expected properties except the fact that $\tilde{\rho}$ is trace preserving. Let us check this.

Let $F \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$. We observe that for all $t \in G$:

$$\begin{aligned}\rho'(F^\sharp * F(t)) &= \rho'\left(\int_G \alpha^{-1}(s)(F(s)^*)\alpha^{-1}(s)(F(ts))ds\right)\\ &= \int_G \alpha^{-1}(s)((\rho'(F(s)))^*)\alpha^{-1}(s)(\rho'(F(ts))ds.\end{aligned}$$

The map ρ' is a automorphism and α and ρ' commute, hence we have:

$$\rho'(F^\sharp * F(t)) = (\rho' \circ F)^\sharp * (\rho' \circ F)(t)$$

We have for all $\eta \in (\Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G)_*$:

$$\begin{aligned}\big\langle \tilde{\rho}((T^\alpha_F)^* T^\alpha_F), \eta\big\rangle &= \int_G \langle \tilde{\rho}(\pi_\alpha(F^\sharp * F)(t))\tilde{\rho}(J(\lambda(t)), \eta\rangle dt\\ &= \int_G \langle \pi_\alpha(\rho'(F^\sharp * F(t))J(\lambda(t)), \eta\rangle dt\\ &= \int_G \big\langle \pi_\alpha((\rho' \circ F)^\sharp * (\rho' \circ F)(t))J(\lambda(t)), \eta\big\rangle\, dt\\ &= \langle (T^\alpha_{\rho' \circ F})^* T^\alpha_{\rho' \circ F}, \eta\rangle.\end{aligned}$$

Hence $\tilde{\rho}((T^\alpha_F)^* T^\alpha_F) = (T^\alpha_{\rho' \circ F})^* T^\alpha_{\rho' \circ F}$. Computing the trace, we have:

$$\begin{aligned}\tau_M(\tilde{\rho}((T^\alpha_F)^* T^\alpha_F)) &= \tau_M((T^\alpha_{\rho' \circ F})^* T^\alpha_{\rho' \circ F})\\ &= \tau((\rho' \circ F)^*(\rho' \circ F)(e))\\ &= \tau(\rho'(F^\sharp * F(e)))\\ &= \tau(F^\sharp * F)(e))\\ &= \tau_M((T^\alpha_F)^* T^\alpha_F).\end{aligned}$$

Using the w^*-density of $\mathcal{B}$ and Theorem 5.1.4, we obtain the equality $\tau_M \circ \tilde{\rho} = \tau_M$. □

In the sequel we let $(\varepsilon_k)_{k\in\mathbb{Z}}$ denote the standard basis of $l^2_{\mathbb{Z}}$.

Lemma 5.1.12. *Let $d = \pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0)) \in \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G$, where e is the unit of G. Then $d^* = d$ and $d^2 = 1$.*

Proof. It is clear that $d^* = d$. Next observe that since T_u is unital, we have $u(e) = 1$ hence

$$\|\dot{\delta}_e \otimes \varepsilon_0\|_{\mathcal{H}} = \|\dot{\delta}_e\|_{H_u} = u(e)^{\frac{1}{2}} = 1.$$

Consequently, $d^2 = 1$ by (1.6.2). □

We define

$$U : \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G \longrightarrow \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G, \qquad U(x) = d^*\tilde{\rho}(x)d.$$

It follows from above that this is trace preserving automorphism.

Lemma 5.1.13. *For all $k \in \mathbb{N}_0$, for all $t \in G$, we have:*

$$U^k(J(\lambda(t))) =$$
$$\pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0)\omega(\dot{\delta}_e \otimes \varepsilon_1)\cdots\omega(\dot{\delta}_e \otimes \varepsilon_{k-1})\omega(\dot{\delta}_t \otimes \varepsilon_{k-1})\cdots\omega(\dot{\delta}_t \otimes \varepsilon_1)\omega(\dot{\delta}_t \otimes \varepsilon_0))J(\lambda(t)).$$

Proof. We prove the lemma by induction. The case $k = 0$ is easy. Let $k \in \mathbb{N}_0$ and assume that the result true for k. We remark that for all $t \in G$ and $x \in \Gamma_{-1}(\mathcal{H})$:

$$U(\pi_\alpha(x)J(\lambda(t))) = \pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0))\pi_\alpha(\rho'(x))J(\lambda(t))\pi_\alpha(\omega((\dot{\delta}_e \otimes \varepsilon_0))).$$

Hence for any $t \in G$ we have:

$$\begin{aligned}
&U^{k+1}(J(\lambda(t))) \\
&= U(\pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0)\omega(\dot{\delta}_e \otimes \varepsilon_1)\cdots\omega(\dot{\delta}_e \otimes \varepsilon_{k-1})\omega(\dot{\delta}_t \otimes \varepsilon_{k-1})\cdots\omega(\dot{\delta}_t \otimes \varepsilon_1)\omega(\dot{\delta}_t \otimes \varepsilon_0))J(\lambda(t)) \\
&= \pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0))\pi_\alpha(\rho'(\omega(\dot{\delta}_e \otimes \varepsilon_0)\cdots\omega(\dot{\delta}_t \otimes \varepsilon_0)))J(\lambda(t))\pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0)).
\end{aligned}$$

By (5.1.3), this is equal to

$$\pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0)\omega(\dot{\delta}_e \otimes \varepsilon_1)\cdots\omega(\dot{\delta}_e \otimes \varepsilon_k)\omega(\dot{\delta}_t \otimes \varepsilon_k)\cdots\omega(\dot{\delta}_t \otimes \varepsilon_1)))J(\lambda(t))\pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0)).$$

By lemma 5.1.3, the latter is equal to

$$\begin{aligned}
&\pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0)\omega(\dot{\delta}_e \otimes \varepsilon_1)\cdots\omega(\dot{\delta}_e \otimes \varepsilon_k)\omega(\dot{\delta}_t \otimes \varepsilon_k)\cdots\omega(\dot{\delta}_t \otimes \varepsilon_1)))\pi_\alpha(\omega(\dot{\delta}_t \otimes \varepsilon_0))J(\lambda(t)) \\
&= \pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0)\omega(\dot{\delta}_e \otimes \varepsilon_1)\cdots\omega(\dot{\delta}_e \otimes \varepsilon_k)\omega(\dot{\delta}_t \otimes \varepsilon_k)\cdots\omega(\dot{\delta}_t \otimes \varepsilon_1)\omega(\dot{\delta}_t \otimes \varepsilon_0))J(\lambda(t)).
\end{aligned}$$

This proves the lemma. □

We conclude the proof of Theorem (**B**) by showing:

$$\mathbb{E}U^k J = T_u^k, \qquad k \geqslant 0.$$

It suffices to prove that for all $t \in G$, $\mathbb{E}U^k J(\lambda(t)) = T^k(\lambda(t))$. To check this, fix $t \in G$ and $k \in N_0$ and for all $i \in [|1; 2k|]$, consider $f_i = \begin{cases} \dot{\delta}_e \otimes \varepsilon_{i-1} & \text{If} \quad 1 \leqslant i \leqslant k \\ \dot{\delta}_t \otimes \varepsilon_{2k-i} & \text{If} \quad k+1 \leqslant i \leqslant 2k \end{cases}$.

Then $\langle f_i, f_{2k-i+1}\rangle_{-1} = u(t)$ and for all $1 \leqslant i < j \leqslant 2k$ such that $j \neq 2k-i+1$, $\langle f_i, f_j\rangle_{\mathcal{H}} = 0$.

We have according to Lemma 5.1.13 :

$$\mathbb{E}U^k J(\lambda(t)) = \mathbb{E}(\pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0)\omega(\dot{\delta}_e \otimes \varepsilon_1)\cdots\omega(\dot{\delta}_t \otimes \varepsilon_1)\omega(\dot{\delta}_t \otimes \varepsilon_0))J(\lambda(t))).$$

Using Lemma 5.1.10, this is equal to

$$\begin{aligned}&\tau(\omega(\dot{\delta}_e \otimes \varepsilon_0)\omega(\dot{\delta}_e \otimes \varepsilon_1)\cdots\omega(\dot{\delta}_e \otimes \varepsilon_{k-1})\omega(\dot{\delta}_t \otimes \varepsilon_{k-1})\cdots\omega(\dot{\delta}_t \otimes \varepsilon_1)\omega(\dot{\delta}_t \otimes \varepsilon_0))\lambda(t)\\ &= \tau(\omega(f_1)\cdots\omega(f_{2k}))\lambda(t) = \langle f_1, f_{2k}\rangle_{\mathcal{H}}\cdots\langle f_k, f_{k+1}\rangle_{\mathcal{H}}\lambda(t).\end{aligned}$$

We derive, using lemma 1.6.1, that

$$\mathbb{E}U^k J(\lambda(t)) = u(t)^k\lambda(t) = T_u^k(\lambda(t)),$$

which concludes the proof.

By Proposition 1.3.5, we have the following explicit consequence of Theorem (**B**).

Theorem 5.1.14. *Let G be a unimodular locally compact group. Let $T : VN(G) \to VN(G)$ is a self-adjoint, unital, completely positive Fourier multiplier. Then for all $1 \leqslant p < \infty$, the map $T_p : L^p(VN(G)) \to L^p(VN(G))$ is dilatable.*

5.2 Non-unimodular case (weighted von Neumann algebras)

In this section, we will prove, the following theorem.

Theorem (B'). *Let G be a locally compact group, let $T_u : VN(G) \to VN(G)$ be a self-adjoint, unital, completely positive Fourier multiplier. Then T_u is absolutely dilatable.*

We follow the proof of theorem (**B**). The construction does not use unimodular hypothesis until Theorem 5.1.3. In general case, Lemma 5.1.4 becomes the following.

Theorem 5.2.1. *Let φ and ψ two n.s.f. weights on a von Neumann algebra M. Assume that there exists a w^*-dense $*$-subalgebra B of M such that $B \subset \mathcal{M}_\varphi$, σ^φ-invariant and for all $y \in B$, $s \in \mathbb{R}$,*

$$\psi(y^*y) = \varphi(y^*y) \text{ and } \psi \circ \sigma_s^\varphi = \psi.$$

Then $\varphi = \psi$.

Proof. See §6.2 in [60]. □

We denote the dual weight of τ on $M = \Gamma_{-1}(\mathcal{H}) \underset{\alpha}{\rtimes} G$ by τ_M, (cf [63, pages 248-249 and page 61]) and we denote the modular automorphism group of the dual weight τ_M by $\sigma_s^{\tau_M}$. The construction of T_F^α is the same in general setting. The difference is the definition of $F^\sharp$. It becomes for all $F \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$.

$$F^\sharp \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H})) \text{ and } F^\sharp(t) = \Delta_{t^{-1}}\alpha(t)(F(t^{-1})^*)\ t \in G.$$

The two lemmas 5.1.5 and 5.1.6 still hold true with the above definition.

Lemma 5.2.2. *The map $J : VN(G) \to \Gamma_{-1}(\mathcal{H})$ defined by $J(x) = x\overline{\otimes}I_H$ is weight preserving and for any $s \in \mathbb{R}$,*

$$\sigma_s^{\tau_M} \circ J = J \circ \sigma_s^{\omega_G}$$

We have for all $x \in \Gamma_{-1}(\mathcal{H})$ and for all $s \in \mathbb{R}$, $\sigma_s^{\tau_M}(\pi_\alpha(x)) = \pi_\alpha(x)$. Hence

$$\pi_\alpha(x) \in \{x \in \Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G;\ \sigma_s^{\tau_M}(x) = x,\ \forall\ s \in \mathbb{R}\} \text{ the centralizer of } \tau_M.$$

Remark 5.2.3.

1. By Theorem 1.2.7, for all $x \in \Gamma_{-1}(\mathcal{H})$, we have for all $z \in \mathcal{M}_{\tau_M}^M$, $z\pi_\alpha(x) \in \mathcal{M}_{\tau_M}^M$, $\pi_\alpha(x)z \in \mathcal{M}_{\tau_M}^M$ and $\tau_M(\pi_\alpha(x)z) = \tau_M(z\pi_\alpha(x))$.

2. By Lemma 1.2.11, for any $m \in \mathcal{M}_{\omega_G}^{VN(G)}$, $J(m)$ belongs to $\in \mathcal{M}_{\tau_M}^M$. Then $\pi_\alpha(x)J(m)$ belongs to $\mathcal{M}_{\tau_M}^M$ for any $x \in \Gamma_{-1}(\mathcal{H})$.

The following is an analogue of Proposition 5.1.8.

Proposition 5.2.4. *For any $x \in \Gamma_{-1}(\mathcal{H})$ and $m \in \mathcal{M}_{\omega_G}^{VN(G)}$, we have*

$$\tau_M(\pi_\alpha(x)J(m)) = \omega_G(m)\tau(x).$$

We prove the following result before the proof of the above proposition.

Lemma 5.2.5. *Let $x \in \Gamma_{-1}(\mathcal{H})$ and $f \in \mathcal{K}(G)$. We have:*

$$\tau_M(\pi_\alpha(x^*x)J(\lambda(f)^*\lambda(f)) = \tau(x^*x)\omega_G(\lambda(f)^*\lambda(f)).$$

Proof. By Remark 5.2.3 we have,

$$\begin{aligned}
\tau_M(\pi_\alpha(x^*x)J(\lambda(f)\lambda(f)^*)) &= \tau_M(\pi_\alpha(x)^*\pi_\alpha(x)J(\lambda(f))J(\lambda(f))^*) \\
&= \tau_M(\pi_\alpha(x)J(\lambda(f))J(\lambda(f))^*\pi_\alpha(x)^*) \\
&= \tau_M(\ \underbrace{\pi_\alpha(x)J(\lambda(f))(\pi_\alpha(x)J(\lambda(f)))^*}_{\geqslant 0.}\ )
\end{aligned}$$

We recall that $T_{f\otimes x}^\alpha = \pi_\alpha(x)J(\lambda(f))$ and $(f\otimes x) * (f\otimes x)^\sharp = f * f^*(.)x\alpha(.)(x^*)$. Then we obtain, using Lemmas 1.5.1, 5.1.5 and 5.1.6,

$$\begin{aligned}
\tau_M(\pi_\alpha(x^*x)J(\lambda(f)\lambda(f)^*)) &= \tau_M(T_{f\otimes x}^\alpha(T_{f\otimes x}^\alpha)^*) \\
&= \tau_M(T_{(f\otimes x)*(f\otimes x)^\sharp}^\alpha) \\
&= \tau_M(T_{f*f^*(.)x\alpha(.)(x^*)}^\alpha) \\
&= \tau(f * f^*(e)x\alpha(e)(x^*)) \\
&= f * f^*(e)\tau(xx^*) \\
&= \omega_G(\lambda(f)\lambda(f)^*)\tau(x^*x).
\end{aligned}$$

The result follows by changing f into f^*. □

Proof of Proposition 5.2.4. We fix $x \in \Gamma_{-1}(\mathcal{H})$ and we may assume that $x \neq 0$. We define a normal weight Φ_x on $VN(G)$ by

$$\Phi_x(m) = \tau_M\left(\frac{\pi_\alpha(x^*x)}{\tau(x^*x)}J(m)\right), \qquad m \in \mathcal{M}_{\omega_G}^{VN(G)}.$$

The map Φ_x is well defined by Remark 5.2.3. By faithfulness of τ_M and the computation in the beginning of Proof of Lemma 5.2.5, the weight Φ_x is faithful. By Lemma 5.2.5, Φ_x coincides with ω_G on $\{\lambda(f)^*\lambda(f) : f \in \mathcal{K}(G)\}$. We observe that $\lambda(\mathcal{K}(G)) \subset \mathcal{M}_{\omega_G}^{VN(G)}$ is σ-invariant and w^*-dense in $VN(G)$. Then Φ_x is semi-finite. Let $m \in \mathcal{M}_{\omega_G}^{VN(G)}$, we have for all $s \in \mathbb{R}$,

$$\begin{aligned}
\Phi_x(\sigma_s^{\omega_G}(m)) &= \tau_M\left(\frac{\pi_\alpha(x^*x)}{\tau(x^*x)}J(\sigma_s^{\omega_G}(m))\right)\\
&= \tau_M\left(\frac{\pi_\alpha(x^*x)}{\tau(x^*x)}\sigma_s^{\tau_M}(J(m))\right)\\
&= \tau_M\left(\sigma_s^{\tau_M}\left(\frac{\pi_\alpha(x^*x)}{\tau(x^*x)}J(m)\right)\right)\\
&= \tau_M\left(\frac{\pi_\alpha(x^*x)}{\tau(x^*x)}J(m)\right)\\
&= \Phi_x(m)
\end{aligned}$$

Applying Theorem 5.2.1, we deduce that $\Phi_x = \omega_G$. Then we obtain

$$\tau_M(\pi_\alpha(x^*x)J(m)) = \tau(x^*x)\omega_G(m), \qquad x \in \Gamma_{-1}(\mathcal{H}),\ m \in \mathcal{M}_{\omega_G}^{VN(G)}.$$

By polarization, the result follows. □

We see $VN(G)$ as a subset of $\Gamma_{-1}(\mathcal{H}) \rtimes_\alpha G$ and we have $\sigma_s^{\tau_M}(VN(G)) \subset VN(G)$ according to lemma 5.2.2.

Proposition 5.2.6. *For any $x \in \Gamma_{-1}(\mathcal{H})$, $m \in \mathcal{M}_{\omega_G}^{VN(G)}$ and $F \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$, we have:*

1. $J_1^*(\pi_\alpha(x)J(m)) = \tau(x)m$;
2. $J_1^*(T_F^\alpha) = \int_G \tau(F(s))\lambda(s)ds = \lambda(\tau \circ F)$;

Proof. Let $\eta \in \mathcal{M}_{\omega_G}^{\infty}$ and let $m \in \mathcal{M}_{\omega_G}^{VN(G)}$. By Lemma 1.2.11 and Remark 5.2.3, we obtain $J(\eta) \in \mathcal{M}_{\tau_M}^{\infty}$ and $\pi_\alpha(x)J(m) \in \mathcal{M}_{\tau_M}^{M}$. We have

$$\begin{aligned}
\left\langle J_1^*(\pi_\alpha(x)J(m)), D_{\omega_G}^{\frac{1}{2}}\eta D_{\omega_G}^{\frac{1}{2}}\right\rangle &= \left\langle \pi_\alpha(x)J(m), J_1\left(D_{\omega_G}^{\frac{1}{2}}\eta D_{\omega_G}^{\frac{1}{2}}\right)\right\rangle\\
&= \left\langle \pi_\alpha(x)J(m), D_{\tau_M}^{\frac{1}{2}}J(\eta)D_{\tau_M}^{\frac{1}{2}}\right\rangle\\
&= \mathrm{tr}_{\tau_M}\left(\pi_\alpha(x)J(m)D_{\tau_M}^{\frac{1}{2}}J(\eta)D_{\tau_M}^{\frac{1}{2}}\right).
\end{aligned}$$

By 2 of Lemma 1.2.10, we have

$$\begin{aligned}\left\langle J_1^*(\pi_\alpha(x)J(m)), D_{\omega_G}^{\frac{1}{2}}\eta D_{\omega_G}^{\frac{1}{2}}\right\rangle &= \tau_M\left(\pi_\alpha(x)J(m)\sigma^{\tau_M}_{-\frac{i}{2}}(J(\eta))\right)\\ &= \tau_M\left(\pi_\alpha(x)J(m\sigma^{\omega_G}_{-\frac{i}{2}}(\eta))\right).\end{aligned}$$

By Proposition 5.2.4, the latter is equal to,

$$\begin{aligned}\tau(x)\omega_G\left(m\sigma^{\omega_G}_{-\frac{i}{2}}(\eta)\right) &= \tau(x)\mathrm{tr}_{\omega_G}\left(mD_{\omega_G}^{\frac{1}{2}}\eta D_{\omega_G}^{\frac{1}{2}}\right)\\ &= \langle\tau(x)m, D_{\omega_G}^{\frac{1}{2}}\eta D_{\omega_G}^{\frac{1}{2}}\rangle.\end{aligned}$$

We used again 2 of Lemma 1.2.10 with ω_G. By density of $D_{\omega_G}^{\frac{1}{2}}\mathcal{M}_{\omega_G}^{\infty}D_{\omega_G}^{\frac{1}{2}}$ in $L^1(VN(G),\omega_G)$ and duality, we obtain the first point.

For the second one, we remark that for any $x\in\Gamma_{-1}(\mathcal{H})$ and $f\in\mathcal{K}(G)$ we have $\lambda(f)\in\mathcal{M}_{\omega_G}^{VN(G)}$. By the first point, we deduce that

$$J_1^*(T^\alpha_{f\otimes x}) = J_1^*(\pi_\alpha(x)J(\lambda(f))) = \tau(x)\lambda(f) = \int_G \tau((f\otimes x)(s))\lambda(s)ds.$$

By linearity we have the previous equality for any finite combination:

$$J_1^*\left(T^\alpha_{\sum_{i=1}^N f_i\otimes x_i}\right) = J_1^*\left(\sum_{i=1}^N T^\alpha_{f_i\otimes x_i}\right) = \int_G \tau\left(\sum_{i=1}^N (f_i\otimes x_i)(s)\right)\lambda(s)ds$$

Finally by w^*- density of $\mathcal{K}\otimes\Gamma_{-1}(\mathcal{H})$ in $\mathcal{K}(G,\Gamma_{-1}(\mathcal{H}))$ and w^*-continuity J_1^* and τ, we obtain the equality $J_1^*(T_F^\alpha) = \int_G\tau(F(s))\lambda(s)ds = \lambda(\tau\circ F)$.

□

We let $\rho' := \Gamma_{-1}(Id\overset{2}{\otimes}\rho)$ the same normal completely positive, trace preserving automorphism than in the previous section. Then ρ' also verifies (5.1.3) and commutes with α.

Theorem 5.2.7. *There exists a (necessarily unique) automorphism $\tilde{\rho}:\Gamma_{-1}(\mathcal{H})\underset{\alpha}{\rtimes}G\to\Gamma_{-1}(\mathcal{H})\underset{\alpha}{\rtimes}G$ such that:*

1. *$\tilde{\rho}$ is weight preserving;*
2. *for all $x\in\Gamma_{-1}(\mathcal{H})$, $\tilde{\rho}(\pi_\alpha(x)) = \pi_\alpha(\rho'(x))$;*
3. *for all $t\in G$, $\tilde{\rho}(J(\lambda(t))) = J(\lambda(t))$.*

Proof. Theorem 4.4.4 p.149 in [61] gives all the properties except the fact that $\tilde{\rho}$ is weight preserving. We will apply Theorem 5.2.1. We just have to prove that $\sigma_s^{\tau_M}(\mathcal{B})\subset\mathcal{B}$ and

$\tau_M \circ \tilde{\rho} \circ \sigma_s^{\tau_M} = \tau_M \circ \tilde{\rho}$. The rest is done in the proof of Theorem 5.1.11. For any $F \in \mathcal{K}(G, \Gamma_{-1}(\mathcal{H}))$, by [63, Theorem X.1.17], we have

$$\begin{aligned}
\sigma_t^{\tau_M}(T_F^\alpha) &= \sigma_t^{\tau_M}\left(\int_G \pi_\alpha(F(s))J(\lambda(s))ds\right) \\
&= \int_G \sigma_t^{\tau_M}(\pi_\alpha(F(s)))\sigma_t^{\tau_M}(J(\lambda(s)))ds \\
&= \int_G \pi_\alpha(F(s))\Delta_s^{it}(J(\lambda(s)))ds \\
&= \int_G \pi_\alpha(\Delta_s^{it}F(s))(J(\lambda(s)))ds \\
&= T^\alpha_{\Delta^{it}_\cdot F}
\end{aligned}$$

Hence $\sigma_s^{\tau_M}(\mathcal{B}) \subset \mathcal{B}$.
In addition,

$$\begin{aligned}
\tau_M\tilde{\rho}\sigma_t^{\tau_M}(T_F^\alpha) &= \tau_M\tilde{\rho}(T^\alpha_{\Delta^{it}_\cdot F}) \\
&= \tau_M(T^\alpha_{\rho'\circ\Delta^{it}_\cdot F}) \\
&= \tau_M(T^\alpha_{\Delta^{it}_\cdot \rho'\circ F})) \\
&= \tau_M\sigma_t^{\tau_M}(T^\alpha_{\rho'\circ F})) \\
&= \tau_M(T^\alpha_{\rho'\circ F})) \\
&= \tau_M\tilde{\rho}(T_F^\alpha))
\end{aligned}$$

By density, we have $\tau_M\tilde{\rho}\sigma_s^{\tau_M} = \tau_M\tilde{\rho}$. Using the w^*-density of $\mathcal{B}$ and Theorem 5.2.1, we obtain the equality $\tau_M \circ \tilde{\rho} = \tau_M$. □

During the proof, we show that $\sigma_s^{\tau_M}\tilde{\rho} = \tilde{\rho}\sigma_s^{\tau_M}$. We define $U : \Gamma_{-1}(\mathcal{H}) \underset{\alpha}{\rtimes} G \longrightarrow \Gamma_{-1}(\mathcal{H}) \underset{\alpha}{\rtimes} G$ as in Section 5.1. U have all the expected properties. We give an analogue of Lemma 5.1.13.

Lemma 5.2.8. *For all $k \in \mathbb{N}_0$, for all $f \in \mathcal{K}(G)$, we have:*

$$\begin{aligned}
&U^k(J(\lambda(f))) = \\
&\int_G \pi_\alpha(\omega(\dot{\delta}_e \otimes \varepsilon_0)\omega(\dot{\delta}_e \otimes \varepsilon_1)\dots\omega(\dot{\delta}_e \otimes \varepsilon_{k-1})\omega(\dot{\delta}_s \otimes \varepsilon_{k-1})\dots\omega(\dot{\delta}_s \otimes \varepsilon_1)\omega(\dot{\delta}_s \otimes \varepsilon_0))f(s)\otimes 1)J(\lambda(s))ds. \\
&= T^\alpha_{\omega(\dot{\delta}_e\otimes\varepsilon_0)\omega(\dot{\delta}_e\otimes\varepsilon_1)\dots\omega(\dot{\delta}_e\otimes\varepsilon_{k-1})\omega(\dot{\delta}_\cdot\otimes\varepsilon_{k-1})\dots\omega(\dot{\delta}_\cdot\otimes\varepsilon_1)\omega(\dot{\delta}_\cdot\otimes\varepsilon_0))f(\cdot)\otimes 1}
\end{aligned}$$

It remains to show that:

$$(J_1)^*U^kJ = T_u^k, \qquad k \geqslant 0.$$

The proof is quietly the same as at the end of Section 5.1.

5.3 Multi-variable Version

This section is devoted to multi-variable version of Theorem (**B**).

Theorem (D). *Let G be a unimodular locally compact group and let $T_{u_1}, \ldots, T_{u_n}$ be self-adjoint, unital, completely positive Fourier multipliers on $VN(G)$. Then there exist a tracial von Neumann algebra (M, τ), a commuting n-tuple $(U_1, \ldots, U_n)$ of trace preserving $*$-automorphisms on M and a w^*-continuous trace preserving one-to-one $*$-homomorphism $J : VN(G) \to M$ such that*

$$T_{u_1}^{k_1} \cdots T_{u_n}^{k_n} = (J_1)^* U_1^{k_1} \cdots U_n^{k_n} J$$

for all $k_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant n$, where $(J_1)^* : M \to B(L^2(\Sigma))$ is the conditional expectation associated with J.

Proof. As in the case of Theorem (**C**) (from the Chapter 2) we only treat the case $n = 2$. So we consider two self-adjoint, unital, completely positive Fourier multipliers T_{u_1}, T_{u_2} on $VN(G)$. As we did at the beginning of Section 5.1, we create real Hilbert spaces H_{u_1}, H_{u_2}, $\mathcal{H}_1 = H_{u_1} \overset{2}{\otimes} \ell^2_{\mathbb{R}}(\mathbb{Z})$, $\mathcal{H}_2 = H_{u_2} \overset{2}{\otimes} \ell^2_{\mathbb{R}}(\mathbb{Z})$, as well as the tracial von Neumann algebras $(\Gamma_{-1}(\mathcal{H}_1), \tau_1)$ and $(\Gamma_{-1}(\mathcal{H}_2), \tau_2)$. Futher we consider the resulting tracial von Neumann algebra

$$\Gamma_{-1}(\mathcal{H}_1) \overline{\otimes} \Gamma_{-1}(\mathcal{H}_2) \subset B(H).$$

Next for any $t \in G$, we consider, for $i = 1, 2$, the onto isometry θ_t^i on H_{u_i} given by

$$\theta_t^i(\dot{f}) : \overbrace{s \mapsto f(t^{-1}s)}^{\cdot},$$

and the trace preserving $*$-automorphism

$$\Gamma_{-1}(\theta_g^i \overset{2}{\otimes} Id_{\ell^2_{\mathbb{R}}(\mathbb{Z})}) : \Gamma_{-1}(\mathcal{H}_i) \longrightarrow \Gamma_{-1}(\mathcal{H}_i).$$

Then we define

$$\alpha : G \longrightarrow Aut(\Gamma_{-1}(\mathcal{H}_1) \overline{\otimes} \Gamma_{-1}(\mathcal{H}_2))$$

by

$$\alpha(t) = \Gamma_{-1}(\theta_t^1 \overset{2}{\otimes} Id_{\ell^2_{\mathbb{R}}(\mathbb{Z})}) \overline{\otimes} \Gamma_{-1}(\theta_t^2 \overset{2}{\otimes} Id_{\ell^2_{\mathbb{R}}(\mathbb{Z})}), \qquad t \in G.$$

This is a homomorphism, and this homomorphism is point-w^*-continuous (in the sense of lemma 5.1.2), by an argument similar to the one used in Section 5.1. This gives rise to the crossed product

$$M = \Gamma_{-1}(\mathcal{H}_1) \overline{\otimes} \Gamma_{-1}(\mathcal{H}_2) \rtimes_\alpha G.$$

As in Section 5.1, we see that the dual weight on this crossed product is a trace. We let

$$\pi_\alpha : \Gamma_{-1}(\mathcal{H}_1) \overline{\otimes} \Gamma_{-1}(\mathcal{H}_2) \longrightarrow M \qquad \text{and} \qquad J : VN(G) \longrightarrow M$$

be the canonical embeddings. The latter is a w^*-continuous trace preserving one-to-one $*$-homomorphism. We let $\mathbb{E} : M \to B(L^2(\Sigma))$ denote the conditional expectation associated with J.

Let ρ be the shift operator on $l^2_{\mathbb{Z}}$, as in Section 5.1. Then we may define completely positive, unital, trace preserving $*$-automorphisms :

$$\rho_1' := \Gamma_{-1}(Id_{H_{u_1}} \otimes \rho) \overline{\otimes} Id \qquad \text{and} \qquad \rho_2' := Id \overline{\otimes} \Gamma_{-1}(Id_{H_{u_2}} \otimes \rho).$$

It is plain that they commute. In addition, for all $t \in G$ and $i = 1, 2$, we have $\alpha(t)\rho_i' = \rho_i'\alpha(t)$. Now thanks to [61, Theorem 4.4.4 p. 149], we may obtain $\tilde{\rho}_1$ and $\tilde{\rho}_2$ as in Theorem 5.1.11. Then we have, for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$,

$$\rho_1'(\omega_1(x) \otimes y) = \omega_1(Id \overset{2}{\otimes} \rho(x)) \otimes y \quad \text{and} \quad \rho_1'(\omega_1(x) \otimes y) = x \otimes \omega_2(Id \overset{2}{\otimes} \rho(y)).$$

We now introduce the two self-adjoint symmetries

$$d_1 = \pi_\alpha(\omega_1(\dot{\delta}_e \otimes \epsilon_0) \otimes 1) \qquad \text{and} \qquad d_2 = \pi_\alpha(1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0)).$$

Then for all $t \in G$, we have

$$\alpha(t)(\omega_1(\dot{\delta}_e \otimes \epsilon_0) \otimes 1) = \omega_1(\dot{\delta}_t \otimes \epsilon_0) \otimes 1 \quad \text{and} \quad \alpha(t)(1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0)) = 1 \otimes \omega_2(\dot{\delta}_t \otimes \epsilon_0).$$

Finally we define the trace preserving $*$-automorphisms U_1, U_2 on $\Gamma_{-1}(\mathcal{H}_1)\overline{\otimes}\Gamma_{-1}(\mathcal{H}_2) \rtimes_\alpha G$ by

$$U_i : x \mapsto d_i^* \tilde{\rho}_i(x) d_i.$$

We claim that U_1 and U_2 commute.

To check this, we first remark that:

$$\begin{aligned} d_1 d_2 &= \pi_\alpha(\omega_1(\dot{\delta}_e \otimes \epsilon_0) \otimes 1)\pi_\alpha(1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0)) \\ &= \pi_\alpha((\omega_1(\dot{\delta}_e \otimes \epsilon_0) \otimes 1)(1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0))) \\ &= \pi_\alpha((1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0))(\omega_1(\dot{\delta}_e \otimes \epsilon_0) \otimes 1)) \\ &= d_2 d_1. \end{aligned}$$

Second, for all $x \in \Gamma_{-1}(\mathcal{H}_1)\overline{\otimes}\Gamma_{-1}(\mathcal{H}_2)$, we have

$$\tilde{\rho}_1\tilde{\rho}_2(\pi_\alpha(x)) = \pi_\alpha(\rho_1'\rho_2'(x)) = \pi_\alpha(\rho_2'\rho_1'(x)) = \tilde{\rho}_2\tilde{\rho}_1(\pi_\alpha(x)),$$

and for all $t \in G$, we have

$$\tilde{\rho}_1\tilde{\rho}_2(J(\lambda(t))) = J(\lambda(t)) = \tilde{\rho}_2\tilde{\rho}_1(J(\lambda(t))).$$

We deduce that $\tilde{\rho}_1$ and $\tilde{\rho}_2$ commuting. For all $x \in \Gamma_{-1}(\mathcal{H}_1)\overline{\otimes}\Gamma_{-1}(\mathcal{H}_2) \rtimes_\alpha G$, we obtain:

$$\begin{aligned} U_1(U_2(x)) &= U_1(d_2\tilde{\rho}_2(x)d_2) \\ &= d_1\tilde{\rho}_1(d_2\tilde{\rho}_2(x)d_2)d_1 \\ &= d_1\tilde{\rho}_1(d_2)\tilde{\rho}_1(\tilde{\rho}_2(x))\tilde{\rho}_1(d_2)d_1 \\ &= d_1\tilde{\rho}_1(\pi_\alpha(1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0)))\tilde{\rho}_2(\tilde{\rho}_1(x))\tilde{\rho}_1(\pi_\alpha(1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0)))d_1 \\ &= d_1\pi_\alpha(\rho_1'(1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0)))\tilde{\rho}_2(\tilde{\rho}_1(x))\pi_\alpha(\rho'(1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0)))d_1 \\ &= d_1\pi_\alpha(1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0))\tilde{\rho}_2(\tilde{\rho}_1(x))\pi_\alpha(1 \otimes \omega_2(\dot{\delta}_e \otimes \epsilon_0))d_1 \\ &= d_1 d_2\tilde{\rho}_2(\tilde{\rho}_1(x))d_2 d_1 \\ &= d_2 d_1\tilde{\rho}_2(\tilde{\rho}_1(x))d_1 d_2 \\ &\vdots \text{ same computation} \\ &= U_2(U_1(x)) \end{aligned}$$

With this commutation property in hands, the proof of Theorem (**D**) is obtained by arguments similar to the ones in Section 5.1. □

As the end of Section 5.1, we mention that applying Lemma 1.2.3, we obtain the following consequence of Theorem (**D**).

Theorem 5.3.1. *Let G be a unimodular locally compact group and let $T_{u_1}, \ldots, T_{u_n}$ be self-adjoint, unital, completely positive Fourier multipliers on $VN(G)$. Then for all $1 \leqslant p < \infty$, there exist a tracial von Neumann algebra (M, τ), a commuting n-tuple $(U_1, \ldots, U_n)$ of surjective isometries on $L^p(M)$, and two contractions $J : L^p(VN(G)) \to L^p(M)$ and $Q : L^p(M) \to L^p(VN(G))$ such that for all $k_1, \ldots, k_n \in \mathbb{N}_0$,*

$$T_{u_1}^{k_1} \cdots T_{u_n}^{k_n} = QU_1^{k_1} \cdots U_n^{k_n} J \qquad \text{on } L^p(VN(G)).$$

With the same arguments, we can obtain Theorem (**D**) in the non-unimodular case.

Theorem (D'). *Let G be a locally compact group and let $T_{u_1}, \ldots, T_{u_n}$ be self-adjoint, unital, completely positive Fourier multipliers on $VN(G)$. Then there exist a weighted von Neumann algebra (M, τ), a commuting n-tuple $(U_1, \ldots, U_n)$ of weight preserving $*$-automorphisms on M and a w^*-continuous weight preserving one-to-one $*$-homomorphism $J : VN(G) \to M$ such that*

$$T_{u_1}^{k_1} \cdots T_{u_n}^{k_n} = (J_1)^* U_1^{k_1} \cdots U_n^{k_n} J$$

for all $k_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant n$, where $(J_1)^* : M \to B(L^2(\Sigma))$ is the conditional expectation associated with J.

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Titre : Propriétés de Dilatation des Multiplicateurs de Schur et des Multiplicateurs de Fourier.

Mots clés : Dilatations, Multiplicateurs de Schur, Multiplicateurs de Fourier, Espaces L^p non-commutatifs.

Résumé : Soit N une algèbre de von Neumann équipée d'une trace normale semi-finie fidèle (n.s.f) et soit $T\colon N \to N$ une contraction, on dit que T est absolument dilatable s'il existe une autre algèbre de von Neumann équipée d'une trace n.s.f, un $*$-homomorphisme w^*-continu préservant la trace et unitaire $J\colon N \to \mathcal{N}$ et un $*$-automorphisme préservant la trace $U\colon \mathcal{N} \to \mathcal{N}$ tel que pour tout $k \in \mathbb{N}$, $T^k = \mathbb{E}U^kJ$ où $\mathbb{E}\colon \mathcal{N} \to N$ est l'espérance conditionnelle associée à J. Soit (Σ, μ) un espace mesuré σ-fini, nous prouvons que tout multiplicateur de Schur unitaire, positif, auto-adjoint et borné sur $B(L^2(\Sigma))$ est absolument dilatable. De même, nous obtenons que tout multiplicateur de Fourier sur $VN(G)$, unitaire, complètement positif, auto-adjoint est aussi absolument dilatable, où G est un groupe localement compact. De plus, nous établissons une version multi-variable de ces résultats. Ensuite, nous caractérisons les fonctions $\varphi \in L^\infty(\Sigma^2)$ induisant un multiplicateur de Schur borné $M_\varphi\colon B(L^2(\Sigma)) \to B(L^2(\Sigma))$ qui est absolument dilatable. Nous montrons aussi une nouvelle caractérisation pour les multiplicateurs de Schur $S_n^p \to S_n^p$ positifs et unitaires qui admettent une dilatation en une isométrie complète inversible sur un espace L^p non-commutatif avec $1 < p \neq 2 < +\infty$. Ensuite, nous traitons le cas de la dimension infinie.

Title : Dilation Properties of Schur and Fourier Multipliers.

Keywords : Dilation, Schur multipliers, Fourier multipliers, Non commutative L^p-spaces.

Abstract : Let N be a von Neumann algebra equipped with a normal semi-finite faithful trace (nsf trace in short) and let $T\colon N \to N$ be a contraction. We say that T is absolutely dilatable if there exist another von Neumann algebra $\mathcal{N}$ equipped with a nsf trace, a w^*-continuous trace preserving unital $*$-homomorphim $J\colon N \to \mathcal{N}$ and a trace preserving $*$-automomorphim $U\colon \mathcal{N} \to \mathcal{N}$ such that $T^k = \mathbb{E}U^kJ$ for all integer $k \geqslant 0$, where $\mathbb{E}\colon \mathcal{N} \to N$ is the conditional expectation associated with J. Given a σ-finite measure space (Σ, μ), we prove that any self-adjoint, unital, positive measurable bounded Schur multiplier on $B(L^2(\Sigma))$ is absolutely dilatable. We obtain a similar result for self-adjoint, unital, completely positive Fourier multiplier on $VN(G)$, when G is a locally compact group. Furthermore, we establish multi-variable versions of these results. In addition we characterize bounded Schur multipliers $\varphi \in L^\infty(\Sigma^2)$ such that the Schur multiplication operator $M_\varphi\colon B(L^2(\Sigma)) \to B(L^2(\Sigma))$ is absolutely dilatable. Moreover we prove new characterizations of unital positive Schur multipliers $S_n^p \to S_n^p$ which can be dilated into an invertible complete isometry acting on a non-commutative L^p-space where $1 < p \neq 2 < \infty$ and S_n^p be the Schatten von Neumann class over $n \times n$ matrices. Then we investigate the infinite dimensional case.

AMS Classification : 47A20, 43A22, 47L65, 47B65, 46L06, 46E40

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