



Limite de diffusion fractionnaire et problème de persistance

Loïc Bethencourt

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SORBONNE UNIVERSITÉ
Ecole Doctorale de Sciences Mathématiques de Paris Centre
Laboratoire de Probabilités, Statistique et Modélisation

THÈSE DE DOCTORAT
Discipline : Mathématiques

présentée par
Loïc Béthencourt

**Limite de diffusion fractionnaire et problème de
persistance**

sous la direction de **Nicolas Fournier** et **Camille Tardif**

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Résumé :

Les travaux de cette thèse portent sur l'étude asymptotique de certaines fonctionnelles de processus de Markov. Plus précisément, on s'intéresse à deux types de problèmes : (i) établir des limites d'échelles et (ii) étudier asymptotiquement des probabilités de *persistance*, encore appelées probabilités de *survie*. Ce travail est divisé en cinq parties.

Dans un premier temps, nous établissons un théorème central limite α -stable pour des fonctionnelles additives de diffusions unidimensionnelles. Le cas des fluctuations gaussiennes est un problème classique et de nombreux résultats existent à ce sujet. Mais de manière surprenante, très peu de résultats concernent les limites d'échelles α -stable, pour $\alpha \in (0, 2)$. Ces travaux généralisent les méthodes et résultats de Fournier-Tardif [FT21] qui traitent d'un cas spécifique.

Dans un second travail en commun avec Quentin Berger et Camille Tardif, nous nous intéressons au problème de persistance pour des fonctionnelles additives de processus de Markov, i.e. nous caractérisons asymptotiquement la probabilité que cette fonctionnelle reste en dessous d'un certain niveau jusqu'au temps t . Divers résultats y sont établis. Lorsque le processus de Markov sous-jacent est récurrent positif, nous donnons une condition nécessaire et suffisante pour que la probabilité de persistance soit à variations régulières. Lorsque le processus est récurrent nul, il nous faut des hypothèses supplémentaires pour établir le comportement asymptotique. Ces hypothèses étant un peu abstraites, nous les simplifions ensuite pour une sous-classe de processus appelés *diffusions généralisées*. Ceci nous amène dans une dernière partie à établir l'asymptotique de la queue de probabilité du temps de retour en zéro de la fonctionnelle, ce qui nous permet de construire la fonctionnelle additive conditionnée à rester négative.

Dans un troisième temps, nous étudions la limite d'échelle d'un modèle cinétique de Fokker-Planck avec conditions de bord diffusives. Plus précisément, on considère une particule qui vit dans $\mathbb{R}_+$ dont la vitesse est une diffusion récurrente positive ayant une mesure invariante à queues lourdes lorsque la particule est strictement positive. Quand la particule touche la frontière $x = 0$, elle en ressort instantanément avec une vitesse strictement positive tirée aléatoirement selon une mesure de probabilité sur $(0, \infty)$. Lorsque la particule n'est pas réfléchie, il a été montré par Fournier-Tardif [FT21] que dans un certain régime, la limite d'échelle de la particule est un processus α -stable symétrique. Nous montrons que pour la particule réfléchie, la limite d'échelle est un processus α -stable réfléchi sur son infimum.

Dans un quatrième travail en commun avec Quentin Berger, nous revisitons le théorème de Sparre Andersen pour des variables aléatoires échangeables et invariantes par signe. Nous utilisons ensuite ce résultat pour obtenir des bornes sur des probabilités de persistance de certaines chaînes de Markov intégrées.

Enfin, dans une dernière partie, nous revisitons les résultats du Chapitre 1 concernant les limites d'échelles de fonctionnelles additives en utilisant les outils introduits dans les Chapitres 2 et 3. Nous obtenons des résultats de tension et nous identifions partiellement les lois limites. Nous étudions aussi les limites d'échelles lorsque la diffusion sous-jacente est récurrente nulle.

Abstract:

This work is devoted to the asymptotic study of some functionals of Markov processes. More precisely, we are interested in two types of problems: (i) the establishment of scaling limits and (ii) the asymptotic study of *persistence* probabilities, also known as *survival* probabilities. This work is divided into five parts.

In a first part, we establish an α -stable central limit theorem for additive functionals of one-dimensional diffusions. The case of Gaussian fluctuations is a classical problem, and many results exist on the subject. But surprisingly few results concern the case of α -stable fluctuations, for $\alpha \in (0, 2)$. This work generalizes methods and results from Fournier-Tardif [FT21], which deal with a specific case.

In a second joint work with Quentin Berger and Camille Tardif, we address the problem of persistence for additive functionals of Markov processes, i.e. we asymptotically characterize the probability for this functional to remain below a certain level up until time t . Various results are established. When the underlying Markov process is positive recurrent, we give a necessary and sufficient condition for the probability of persistence to be regularly varying. When the process is null recurrent, we need additional assumptions to establish the asymptotic behavior. As these assumptions are somewhat abstract, we then simplify them for a subclass of processes called *generalized diffusions*. This finally leads us to establish the asymptotics of the probability tail of the first return time to zero of the functional, enabling us to construct the additive functional conditioned to stay negative.

In a third part, we study the scaling limit of a kinetic Fokker-Planck model with diffusive boundary conditions. More precisely, we consider a particle living in $\mathbb{R}_+$, whose velocity is a positive recurrent diffusion with heavy-tailed invariant distribution when the particle lives in $(0, \infty)$. When it hits the boundary $x = 0$, the particle restarts with a strictly positive random velocity, according to some probability measure on $(0, \infty)$. When the particle is not reflected, it was shown by Fournier-Tardif that in a certain regime, the scaling limit of the particle is a symmetric α -stable process. We show that when the particle is reflected diffusively, the scaling limit is a stable process reflected on its infimum.

In a fourth joint work with Quentin Berger, we revisit the famous result of Sparre Andersen for exchangeable and sign-invariant random variables. We then apply this result to obtain bounds on the persistence probabilities of certain integrated Markov chains.

Finally, in a last part, we revisit the results of Chapter 2 concerning scaling limits of additive functionals using the tools introduced in Chapters 3 and 4. We obtain tightness results and we partly identify the limiting laws. We also deal with the case when the underlying diffusion is null recurrent.

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Introduction

Les travaux de cette thèse portent sur l'étude asymptotique de certaines fonctionnelles de processus de Markov. Plus précisément, on s'intéresse à deux types de problèmes : (i) établir des limites d'échelles et (ii) étudier asymptotiquement des probabilités de *persistance*, encore appelées probabilités de *survie*. Dans ce chapitre, nous introduisons et discutons les différents thèmes étudiés dans cette thèse. Dans un premier temps, nous rappelons le théorème central limite classique, ainsi que sa généralisation à des variables aléatoires de variance infinie. Dans un second temps, nous discutons de principes d'invariance et de leurs liens avec les différentes topologies de *Skorokhod*, ainsi que du problème de persistance. Le fil conducteur sera l'étude de la marche aléatoire unidimensionnelle à travers ces différents thèmes. Ensuite, nous discutons de modèles cinétiques, de leur limite d'échelle et des résultats existants. Enfin, nous résumons les travaux réalisés dans cette thèse.

1.1 Autour du théorème central limite

1.1.1 Quelques heuristiques

On retrace ici succinctement l'histoire du *théorème central limite* (que l'on notera parfois TCL), et on discute d'une généralisation de ce dernier à des variables aléatoires qui ont une variance infinie. Une partie de cette thèse a pour but d'établir des généralisations similaires à ce résultat, que l'on appelle *théorème central limite α -stable* (ou TCL α -stable).

Ce sont les travaux de Jacques Bernoulli qui sont à l'origine du TCL. C'est en effet le premier à avoir établi une loi des grands nombres, qu'il publie en 1713 [Ber13]. Considérons une suite de variables aléatoires $(X_n)_{n \in \mathbb{N}}$ indépendantes et identiquement distribuées (i.i.d.), définie sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$, et à valeurs dans $\mathbb{R}$. La loi forte des grands nombres nous assure que, si $\mathbb{E}[|X_1|] < \infty$, la convergence suivante a lieu presque sûrement (p.s.) :

$$(LFGN) \quad \frac{1}{n} \sum_{k=1}^n X_k \longrightarrow \mathbb{E}[X_1], \quad \text{lorsque } n \rightarrow \infty. \quad (1.1)$$

Par la suite, une question naturelle se pose quant à l'étude des fluctuations : comment se comporte la moyenne empirique autour de sa "vraie" valeur ? Peut-on quantifier cet écart ? C'est De Moivre [DM33] en 1733 qui est le premier à faire apparaître la loi normale comme la loi limite d'une somme de variables de Bernoulli recentrées. Toutefois ce sont Laplace et Gauss qui ont ensuite majoritairement contribué à la quantification des erreurs ; et Laplace publie finalement en 1812 [Lap20] le théorème de Laplace que l'on appelle maintenant théorème central limite.

Théorème 1.1 (Théorème central limite). *Soit $(X_n)_{n \in \mathbb{N}}$ une suite de variables aléatoires i.i.d. à valeurs dans $\mathbb{R}$ telle que $\mathbb{E}[X_1] = 0$ et $\sigma^2 := \mathbb{E}[X_1^2] < \infty$. Alors on a la convergence en loi suivante :*

$$\frac{1}{\sigma\sqrt{n}} \sum_{k=1}^n X_k \xrightarrow{\mathcal{L}} G, \quad \text{lorsque } n \rightarrow \infty,$$

où G est une variable gaussienne centrée et réduite.

Ainsi, ce théorème nous indique que la forme des fluctuations est universelle et ne dépend pas réellement de la loi, mais seulement de la variance. Essayons de comprendre un peu ce résultat. Tout d'abord, on sait par (1.1) que, sous les hypothèses du théorème, $\frac{1}{n}S_n = \frac{1}{n} \sum_{k=1}^n X_k$ converge p.s. vers 0 et il est assez facile de voir qu'il faut effectivement multiplier par $\sqrt{n}$ pour obtenir le bon ordre. En effet, un rapide calcul montre que $\mathbb{E}[(S_n/\sqrt{n})^2] = \sigma^2$ est constant. Supposons maintenant que $S_n/\sqrt{n}$ converge en loi vers une variable G , et faisons la manipulation suivante :

$$\frac{1}{\sqrt{2n}} S_{2n} = \frac{1}{\sqrt{2n}} \left(\sum_{k=1}^n X_k + \sum_{k=n+1}^{2n} X_k \right) = \frac{S_n}{\sqrt{2n}} + \frac{\tilde{S}_n}{\sqrt{2n}} \quad (1.2)$$

où $\tilde{S}_n = \sum_{k=n+1}^{2n} X_k$. Puisque les variables aléatoires $(X_n)_{n \in \mathbb{N}}$ sont i.i.d., $\tilde{S}_n$ est indépendante de S_n et a la même loi. Il s'ensuit donc par convergence que la variable G doit satisfaire l'égalité en loi suivante :

$$\sqrt{2}G \stackrel{\mathcal{L}}{=} G + \tilde{G} \quad (1.3)$$

où $\tilde{G}$ est copie indépendante de G . On dit que la loi de G est *stable* (pour l'addition) et il s'avère que la loi normale est l'unique loi qui satisfait (1.3).

On peut ensuite se demander ce qu'il se passe si on suppose que X_1 a un moment d'ordre 2 infini, i.e. que $\mathbb{E}[X_1^2] = \infty$. Peut-on toujours dire quelque chose des fluctuations ? La réponse à cette question est précisément fournie par le TCL α -stable. Nous constaterons en effet que la notion de stabilité joue encore un rôle important ici. Avant d'introduire les objets nécessaires pour étudier ce problème, nous donnons quelques heuristiques simples. On commence par essayer de comprendre le coefficient de renormalisation que nous devons choisir. Lorsque $\mathbb{E}[X_1^2]$ est fini, il est clair qu'il faut multiplier S_n/n par $\sqrt{n}$. On pose

$$\alpha = \sup\{\theta \geq 0, \text{ tel que } \mathbb{E}[|X_1|^\theta] < \infty\}.$$

On a donc $\alpha \leq 2$ et on va supposer pour simplifier que $\alpha \in (0, 1)$ et que les variables aléatoires X_n sont à valeurs dans $\mathbb{R}_+$. On va essayer de comprendre pourquoi, ici, la bonne renormalisation est à peu près $S_n/n^{1/\alpha}$. Soit $\theta \in (0, \alpha)$, on va dans un premier temps montrer que $S_n/n^{1/\theta}$ converge vers 0 en probabilité lorsque $n \rightarrow \infty$. Pour cela, on peut calculer sa transformée de Laplace :

$$\mathbb{E} \left[e^{-\lambda n^{-1/\theta} S_n} \right] = \mathbb{E} \left[e^{-\lambda n^{-1/\theta} X_1} \right]^n,$$

où $\lambda > 0$, et donc on a

$$\log \mathbb{E} \left[e^{-\lambda n^{-1/\theta} S_n} \right] \underset{n \rightarrow \infty}{\sim} n \mathbb{E} \left[e^{-\lambda n^{-1/\theta} X_1} - 1 \right] = -\lambda n \int_0^\infty e^{-\lambda u} \mathbb{P}(X_1 > n^{1/\theta} u) du. \quad (1.4)$$

Soit $\theta' \in (\theta, \alpha)$ tel que $\mathbb{E}[|X_1|^{\theta'}] < \infty$. On a par l'inégalité de Markov que $\mathbb{P}(X_1 > n^{1/\theta} u)$ est plus petite que $n^{-\theta'/\theta} u^{-\theta'} \mathbb{E}[|X_1|^{\theta'}]$. Il vient alors que

$$n \int_0^\infty e^{-\lambda u} \mathbb{P}(X_1 > n^{1/\theta} u) du \leq n^{1-\theta'/\theta} \mathbb{E}[|X_1|^{\theta'}] \int_0^\infty e^{-\lambda u} u^{-\theta'} du.$$

Puisque $\theta' < 1$, l'intégrale à droite est bien convergente, et puisque $\theta' > \theta$, les quantités ci-dessus convergent vers 0 quand $n \rightarrow \infty$. On a donc montré que la transformée de Laplace de $S_n/n^{1/\theta}$ converge vers 1, ce qui implique que $S_n/n^{1/\theta}$ converge vers 0 en probabilité. On peut aussi montrer que pour $\theta > \alpha$, $S_n/n^{1/\theta}$ converge vers $+\infty$ en probabilité. On note par ailleurs que pour $\theta \geq 1$, ceci est une conséquence directe de la loi forte des grands nombres. Ainsi, le coefficient de normalisation doit être de l'ordre de $n^{1/\alpha}$.

Supposons maintenant que $S_n/n^{1/\alpha}$ converge en loi vers une variable aléatoire S_α . Alors, de manière analogue à (1.2), on obtient que la variable S_α doit satisfaire l'égalité en loi suivante :

$$2^{1/\alpha} S_\alpha \stackrel{\mathcal{L}}{=} S_\alpha + \tilde{S}_\alpha$$

où $\tilde{S}_\alpha$ est une copie indépendante de S_α . La loi limite est donc encore une fois *stable*. Observons aussi que si on remplace $n^{1/\alpha}$ par $n^{1/\alpha} \log(n)$, la loi limite satisfait la même relation car pour tout $\lambda > 0$, $\log(\lambda x)/\log(x) \rightarrow 1$ lorsque $x \rightarrow \infty$. Nous allons maintenant énoncer le TCL α -stable, mais il nous faut auparavant introduire quelques objets.

1.1.2 Stabilité et variation régulière

On définit ici d'abord plus rigoureusement la notion de stabilité pour une loi de probabilité sur $\mathbb{R}$. Dans tout ce qui suit, on exclura automatiquement les lois triviales concentrées en un point (du type $\mu = \delta_x$ pour $x \in \mathbb{R}$).

Définition 1.1. Soit μ une mesure de probabilité sur $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ et soit $(X_n)_{n \in \mathbb{N}}$ une suite de variables aléatoires distribuées selon μ . On dit que μ est stable si pour tout $n \in \mathbb{N}$, il existe des constantes $c_n \in (0, \infty)$ et $\gamma_n \in \mathbb{R}$ telles que

$$S_n \stackrel{\mathcal{L}}{=} c_n X_1 + \gamma_n.$$

On dit que μ est strictement stable si pour tout $n \in \mathbb{N}$, on peut choisir $\gamma_n = 0$.

Cette définition peut sembler *a priori* très large, mais on peut montrer le résultat suivant, qu'on peut retrouver dans le livre de Feller [Fel91, Chapitre VI, Section 1, Théorèmes 1 et 2].

Théorème 1.2. Soit μ une loi de probabilité stable. Alors

- (i) Il existe $\alpha \in (0, 2]$ tel que pour tout $n \in \mathbb{N}$, la constante c_n soit égale à $n^{1/\alpha}$.
- (ii) Soit X une variable aléatoire de loi μ . Si $\alpha \neq 1$, il existe $b \in \mathbb{R}$ tel que la loi de $X + b$ soit strictement stable.
- (iii) Si $\alpha = 2$, μ est une loi normale.

On pourra donc dire d'une loi qu'elle est " α -stable" pour spécifier son indice α associé. On dira qu'une variable aléatoire X est α -stable (ou strictement α -stable), si sa loi est α -stable (ou strictement α -stable). Lorsque $\alpha \neq 1$, seules les lois strictement stables sont intéressantes, puisque une loi stable n'est autre qu'une loi strictement stable translatée. Le cas $\alpha = 1$ est le seul cas où l'on a des lois stables, qui sont non strictement stables, et non-triviales. C'est Paul Lévy [Lév54] qui a initié la théorie des variables aléatoires stables, et a identifié la fonction caractéristique de toutes les lois strictement stables. Le cas $\alpha = 1$ fut traité plus tard avec Khintchine. On a le résultat suivant que l'on peut trouver dans le livre de Sato [Sat13, Theorem 14.15 page 86 et Definition 14.16 page 87].

Théorème 1.3. Soit $\alpha \in (0, 2) \setminus \{1\}$ et soit X une variable aléatoire strictement α -stable. Alors il existe $\nu > 0$ et $\beta \in [-1, 1]$ tels que pour tout $\xi \in \mathbb{R}$,

$$\mathbb{E}[\exp(i\xi X)] = \exp\left(-\nu|\xi|^\alpha \left(1 - i\beta \tan\left(\frac{\alpha\pi}{2}\right) \operatorname{sgn}(\xi)\right)\right).$$

Si X est une variable aléatoire 1-stable, alors il existe $\nu > 0$, $\beta \in [-1, 1]$ et $\tau \in \mathbb{R}$ tels que

$$\mathbb{E}[\exp(i\xi X)] = \exp\left(-\nu|\xi| \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(\xi) \log|\xi|\right) + i\tau\xi\right).$$

On souligne aussi que la seule loi strictement 1-stable est la loi de Cauchy, correspondant au cas où $\beta = \tau = 0$. Le paramètre β est un paramètre d'asymétrie et si $\beta = 0$ alors la loi est symétrique.

Pour pouvoir énoncer le TCL α -stable, il nous faut aussi parler de variations régulières et des théorèmes taubériens de Karamata.

Définition 1.2. Soit $\ell : (0, \infty) \rightarrow (0, \infty)$ une fonction mesurable et soit $\rho \in \mathbb{R}$. On dit que ℓ est à variations régulières d'indice ρ si pour tout $\lambda > 0$, on a

$$\lim_{x \rightarrow \infty} \frac{\ell(\lambda x)}{\ell(x)} = \lambda^\rho.$$

On notera $\ell \in \mathcal{R}_\rho$ et si $\rho = 0$, on dira que ℓ est à variations lentes.

On voit immédiatement que si $L \in \mathcal{R}_\rho$ pour $\rho \neq 0$, alors la fonction $\ell : (0, \infty) \rightarrow (0, \infty)$ définie par $\ell(x) = L(x)/x^\rho$ est à variations lentes et donc toute fonction à variations régulières n'est autre qu'une fonction à variations lentes multipliée par une fonction puissance. Voici quelques exemples simples de fonctions à variations lentes : toute puissance du logarithme, les logarithmes itérés, toute fonction ayant une limite finie à l'infini. La théorie des fonctions à variations régulières est intimement liée au TCL α -stable. Nous énonçons le résultat suivant qui relie le comportement asymptotique d'une fonction à variation régulière monotone à celui de sa transformée de Laplace. Il combine à la fois le théorème taubérien et le théorème de densité monotone, qu'on peut trouver dans le livre de Bingham, Goldie et Teugels [BGT87, Theorem 1.7.1 page 37 et Theorem 1.7.2 page 39].

Théorème 1.4. Soit $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ une fonction monotone. On note $\mathcal{L}u$ sa transformée de Laplace, i.e. la fonction définie par $\mathcal{L}u(\lambda) = \int_0^\infty e^{-\lambda x} u(x) dx$ pour tout $\lambda > 0$. Soit $\rho > -1$, Γ la fonction Gamma usuelle et ℓ une fonction lente. Alors les deux assertions suivantes sont équivalentes.

- (i) $u(x) \sim (\rho + 1)x^\rho \ell(x)$ lorsque $x \rightarrow \infty$.
- (ii) $\lambda \mathcal{L}u(\lambda) \sim \Gamma(\rho + 2)\lambda^{-\rho} \ell(1/\lambda)$ lorsque $\lambda \rightarrow 0$.

1.1.3 Le TCL α -stable

On définit ici la notion de domaine d'attraction pour une loi de probabilité.

Définition 1.3. Soient μ et μ_∞ deux mesures de probabilités sur $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $(X_n)_{n \in \mathbb{N}}$ une suite de variables aléatoires i.i.d. de loi μ et X_∞ une variable aléatoire de loi μ_∞ . On dit que μ est dans le domaine d'attraction de μ_∞ s'il existe des suites $(a_n)_{n \in \mathbb{N}}$ et $(b_n)_{n \in \mathbb{N}}$ à valeurs dans $\mathbb{R}$ telles que

$$\frac{1}{a_n}(S_n - b_n) \xrightarrow{\mathcal{L}} X_\infty.$$

On dira que μ_∞ possède un domaine d'attraction s'il existe μ dans le domaine d'attraction de μ_∞ .

Il est évident par définition, et par le Théorème 1.2, que toute loi α -stable possède un domaine d'attraction. En effet, il suffit de remarquer que μ_∞ est dans son propre domaine d'attraction (avec une égalité en loi, $a_n = n^{1/\alpha}$ et $b_n = \gamma_n$). La réciproque est aussi vraie, bien qu'elle soit moins évidente, et on se réfère à Feller [Fel91, Chapitre XVII, Section 5]. Ainsi, une loi possède un domaine d'attraction si et seulement si elle est stable, et on peut enfin énoncer le TCL α -stable, qui caractérise le domaine d'attraction d'une loi stable.

Théorème 1.5 (TCL α -stable). *Soit μ une mesure de probabilité sur $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ et soit X une variable aléatoire de loi μ . On définit la fonction V sur $[0, \infty)$ par $V(x) = \mathbb{E}[X^2 \mathbf{1}_{\{|X| \leq x\}}]$.*

(i) μ est dans le domaine d'attraction d'une loi normale si et seulement si $V \in \mathcal{R}_0$.

(ii) μ est dans le domaine d'attraction d'une loi α -stable, pour $\alpha \in (0, 2)$, si et seulement si $x \mapsto \mathbb{P}(|X| > x)$ est à variations régulières d'indice $-\alpha$ et s'il existe $p, q \in [0, 1]$ tels que $p + q = 1$ et

$$\frac{\mathbb{P}(X > x)}{\mathbb{P}(|X| > x)} \xrightarrow{x \rightarrow \infty} p, \quad \frac{\mathbb{P}(X < -x)}{\mathbb{P}(|X| > x)} \xrightarrow{x \rightarrow \infty} q.$$

Nous allons donner une partie de la preuve de ce résultat dans un cas particulier. On se placera dans le cas où μ est supportée par $\mathbb{R}_+$ et $\alpha \in (0, 1)$. Ce cas est un peu plus simple car on peut utiliser les transformées de Laplace, et appliquer directement les théorèmes Taubériens. Nous montrerons que si $x \mapsto \mathbb{P}(X > x)$ est à variations régulières d'indice $-\alpha$, alors il existe une suite $(a_n)_{n \in \mathbb{N}}$ telle que S_n/a_n converge en loi vers une loi stable. Dans ce cas la variable α -stable asymptotique, que l'on note S_α , est nécessairement à valeurs positives et on peut montrer qu'il existe $c > 0$ tel que $\mathbb{E}[\exp(-\lambda S_\alpha)] = \exp(-c\lambda^\alpha)$ pour tout $\lambda > 0$.

Preuve. On suppose donc que $u(x) = \mathbb{P}(X > x)$ est à variations régulières d'indice $-\alpha$. Il existe donc une fonction à variations lentes ℓ telle que $u(x) \sim x^{-\alpha} \ell(x)$ lorsque $x \rightarrow \infty$. Comme u est une fonction monotone, et que $\alpha < 1$, on peut appliquer le Théorème 1.4 qui nous dit que

$$\lambda \mathcal{L}u(\lambda) \sim \frac{\Gamma(2 - \alpha)}{1 - \alpha} \lambda^\alpha \ell(1/\lambda) \quad \text{lorsque } \lambda \rightarrow 0,$$

où $\mathcal{L}u(\lambda)$ est la transformée de Laplace de u . Considérons maintenant une suite $(a_n)_{n \in \mathbb{N}}$ telle que $\lim_{n \rightarrow \infty} a_n = \infty$. Par un calcul similaire à (1.4), on voit que

$$\log \mathbb{E} \left[e^{-\lambda a_n^{-1} S_n} \right] \underset{n \rightarrow \infty}{\sim} -n \lambda a_n^{-1} \int_0^\infty e^{-\lambda a_n^{-1} u} \mathbb{P}(X_1 > u) du = -n (\lambda a_n^{-1}) \mathcal{L}u(\lambda a_n^{-1}). \quad (1.5)$$

Ainsi, on a

$$\log \mathbb{E} \left[e^{-\lambda a_n^{-1} S_n} \right] \underset{n \rightarrow \infty}{\sim} -\frac{\Gamma(2 - \alpha)}{1 - \alpha} \frac{n \ell(a_n)}{a_n^\alpha} \lambda^\alpha.$$

On voit alors que si on prend $(a_n)_{n \in \mathbb{N}}$ telle que $\lim_{n \rightarrow \infty} n a_n^{-\alpha} \ell(1/a_n) = 1$, la transformée de Laplace de S_n/a_n converge vers $\exp(-c\lambda^\alpha)$ où $c = \Gamma(2 - \alpha)/(1 - \alpha)$, ce qui implique bien que S_n/a_n est dans le domaine d'attraction d'une loi stable. Il est possible de choisir de tels a_n , en posant par exemple $a_n = \inf\{x > 0, nx^{-\alpha} \ell(x) \leq 1\}$. $\square$

A vrai dire, la fonction $1/u \in \mathcal{R}_\alpha$ est croissante et possède une inverse généralisée que l'on note $g(x) = \inf\{y > 0, 1/u(y) > x\}$. On voit donc que $a(n) = g(n)$ et on peut montrer (voir [BGT87, Chapitre 1, Théorème 1.5.12]) que $g \in \mathcal{R}_{1/\alpha}$ et donc il existe une fonction lente L telle que $a_n = n^{1/\alpha} L(n)$.

On va à présent faire quelques remarques sur ce théorème. Tout d'abord, bien qu'elle ne soit pas spécifiée, la suite $(a_n)_{n \in \mathbb{N}}$ doit être choisie d'une manière particulière (à constante

multiplicative près). Si $V \in \mathbf{R}_0$, alors μ est dans le domaine d'attraction d'une loi normale et $(a_n)_{n \in \mathbb{N}}$ doit être telle que

$$\lim_{n \rightarrow \infty} na_n^{-2} V(a_n) = 1.$$

Si μ a un moment d'ordre 2, alors $V(x)$ converge vers σ^2 lorsque $x \rightarrow \infty$ et alors $a_n \sim \sigma\sqrt{n}$. On retrouve bien le TCL classique. Dans tous les cas, $(a_n)_{n \in \mathbb{N}}$ est une suite à variations régulières d'indice $1/2$, dans le sens où il existe une fonction lente L telle que $a(n) = n^{1/2}L(n)$.

Plaçons-nous maintenant sous les hypothèses de l'item (ii), i.e. il existe $\alpha \in (0, 2)$ et $\ell \in \mathbf{R}_0$ tels que $\mathbb{P}(|X| > x) \sim x^{-\alpha}\ell(x)$; et il existe $p, q \in [0, 1]$ tels que $p + q = 1$ et

$$\mathbb{P}(X > x) \underset{x \rightarrow \infty}{\sim} px^{-\alpha}\ell(x), \quad \mathbb{P}(X < -x) \underset{x \rightarrow -\infty}{\sim} qx^{-\alpha}\ell(x).$$

Alors dans ce cas $(a_n)_{n \in \mathbb{N}}$ doit être telle que $\lim_{n \rightarrow \infty} na_n^{-\alpha}\ell(a_n) = 1$, et alors c'est une suite à variations régulières d'indice $1/\alpha$. Concernant les constantes de recentrage $(b_n)_{n \in \mathbb{N}}$, on peut montrer la chose suivante, voir par exemple [Fel91, Chapitre XVII, Section 5, Théorème 3].

- (i) Si $\alpha \in (0, 1)$, alors on peut toujours choisir $b_n = 0$ pour tout $n \in \mathbb{N}$. La loi limite est alors strictement stable.
- (ii) Si $\alpha \in (1, 2]$, alors on peut toujours choisir $b_n = n\mathbb{E}[X_1]$ pour tout $n \in \mathbb{N}$.
- (iii) Si $\alpha = 1$, alors on peut prendre $b_n = n\mathbb{E}[X_1 \mathbf{1}_{\{|X_1| \leq a_n\}}]$ pour tout $n \in \mathbb{N}$.

On note aussi que si μ est dans le domaine d'attraction d'une loi α -stable, pour $\alpha \in (0, 2)$, alors μ possède des moments d'ordre θ pour tout $\theta < \alpha$ (c'est une conséquence immédiate du fait que les queues de probabilités sont à variations régulières d'indice $-\alpha$). En particulier, si $\alpha \in (1, 2)$, on a $\mathbb{E}[|X_1|] < \infty$.

1.2 Principe d'invariance et topologies de Skorokhod

Le TCL nous donne des informations sur la marche aléatoire au temps n , à savoir que $S_n \approx \sqrt{n}G$, où G est une variable gaussienne. Mais il ne nous donne pas d'informations sur l'entièreté de la trajectoire de la marche jusqu'au temps n , et il serait effectivement intéressant de savoir à quoi ressemble une marche typique, en temps long. Pour cela, il faut regarder la trajectoire d'une marche aléatoire comme une variable aléatoire à valeurs dans un espace de fonctions.

On dénote par $C = C([0, 1], \mathbb{R})$ l'espace des fonctions continues sur l'intervalle $[0, 1]$ à valeurs dans $\mathbb{R}$. On munit cet espace de la distance uniforme d_U , i.e. pour tout $x, y \in C$,

$$d_U(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|.$$

On munit aussi cet espace de la tribu borélienne associée (i.e. la tribu engendrée par les ouverts), que l'on note $\mathcal{C}$. Soient $(\mu_n)_{n \in \mathbb{N}}$ et μ_∞ des mesures de probabilité sur l'espace $(C, \mathcal{C})$. On rappelle que la suite $(\mu_n)_{n \in \mathbb{N}}$ converge faiblement vers μ_∞ si pour toute fonction continue bornée f de C dans $\mathbb{R}$, on a $\lim_{n \rightarrow \infty} \mu_n(f) = \mu_\infty(f)$ où $\mu(f) = \int_C f d\mu$. Une suite de variables aléatoires à valeurs dans C converge en loi si leurs lois convergent faiblement. Pour étudier la trajectoire, transformons-la en une fonction continue. Par exemple, on peut interpoler linéairement les points, et définir

$$S_t = S_{[t]} + (t - [t])X_{[t]+1},$$

où $t \geq 0$, et $[t]$ est la partie entière de t . Dans le cas où $\mathbb{E}[X_1^2] < \infty$, il faut renormaliser la marche, en temps et en espace, de la manière suivante :

$$S_t^n = \frac{S_{nt}}{\sqrt{n}}.$$

Ainsi, $\mathbf{S}^n = (\mathbf{S}_t^n)_{t \in [0,1]}$ est une variable aléatoire à valeurs dans C . Supposons à présent que $\mathbf{S}^n$ converge en loi dans (C, d_U) vers une variable $\mathbf{S}^\infty$. Ceci implique en particulier que pour tout $t \in [0, 1]$, $\mathbf{S}_t^n$ converge en loi vers $\mathbf{S}_t^\infty$ dans $\mathbb{R}$. On déduit facilement du théorème central limite classique que pour tout $t \in [0, 1]$, $\mathbf{S}_t^\infty$ est une variable gaussienne de variance $\sigma^2 t$.

Considérons maintenant $0 \leq s < t$, on voit de manière informelle que $\mathbf{S}_t^n - \mathbf{S}_s^n \approx \tilde{\mathbf{S}}_{t-s}^n$ où $\tilde{\mathbf{S}}^n$ est une copie indépendante de $\mathbf{S}^n$. On en déduit donc que $\mathbf{S}_t^\infty - \mathbf{S}_s^\infty$ est indépendante de $\mathbf{S}_s^\infty$, et est égale en loi à $\mathbf{S}_{t-s}^\infty$. Il vient donc que $\mathbf{S}^\infty$ est un mouvement brownien.

Théorème 1.6 (Théorème de Donsker, 1951). *Si $\mathbb{E}[X_1] = 0$ et $\mathbb{E}[X_1^2] < \infty$, alors $\mathbf{S}^n$ converge en loi dans (C, d_U) vers un mouvement brownien.*

Si on cherche maintenant à comprendre comment se comporte la marche lorsque la loi μ de son pas est dans le domaine d'attraction d'une loi α -stable μ_α , il nous faut introduire une autre topologie. En effet, l'objet limite dans l'espace des fonctions correspondant à la loi α -stable est l'unique processus $(Z_t^\alpha)_{t \geq 0}$ qui a des accroissements indépendants et stationnaires (c'est un processus de Lévy), tel que Z_1^α a pour loi μ_α . Ce dernier n'est pas continu mais càdlàg (continu à droite, limite à gauche). Il nous faut donc une topologie sur l'espace des fonctions càdlàg.

On note $D = D([0, 1], \mathbb{R})$ l'espace des fonctions càdlàg sur $[0, 1]$ à valeurs dans $\mathbb{R}$. Cet espace est muni d'une distance, qui remonte aux travaux de Skorokhod [Sko56]. On note Λ l'ensemble des bijections strictement croissantes de $[0, 1]$ dans lui-même. La topologie $\mathbf{J}_1$ de Skorokhod est engendrée par la distance $d_{\mathbf{J}_1}$ définie pour tout $x, y \in D$ par

$$d_{\mathbf{J}_1}(x, y) = \inf_{\lambda \in \Lambda} [d_U(x, y \circ \lambda) \vee d_U(\lambda, e)],$$

où e désigne la fonction identité sur $[0, 1]$. On note alors $\mathcal{D}$ la tribu borélienne associée. L'idée de Skorokhod est de dire que deux fonctions sont proches, si elles sont uniformément proches à une petite perturbation temporelle près (*via* la bijection λ). C'est aussi Skorokhod [Sko57] qui démontre un principe d'invariance de Donsker pour les marches dans le domaine d'attraction d'une loi stable.

Tout d'abord, on souligne que l'espace des fonctions continues C est un espace fermé dans D munie de la topologie $\mathbf{J}_1$: si une suite de fonctions continues converge pour la distance $d_{\mathbf{J}_1}$, alors la limite est forcément continue. Ce fait nous indique donc que pour espérer avoir une convergence en loi d'une marche dans le domaine d'attraction d'une loi stable, l'objet considéré doit être càdlàg et non continu ; et ainsi, on ne peut interpoler linéairement la marche comme on l'a fait précédemment. On considère alors simplement $\mathbf{S}_t = S_{[t]}$ qui forme alors une trajectoire càdlàg et non-continue. Soit μ une mesure de probabilité dans le domaine d'attraction d'une loi α -stable avec $\alpha \in (0, 2)$, et $(a_n)_{n \in \mathbb{N}}$ et $(b_n)_{n \in \mathbb{N}}$ les coefficients associés au Théorème 1.5. La marche dont le pas a pour loi μ , renormalisée, est alors définie par

$$\mathbf{S}_t^n = \frac{1}{a_n} (\mathbf{S}_{nt} - b_n t).$$

On a alors le résultat suivant.

Théorème 1.7. *Soit μ une mesure de probabilité dans le domaine d'attraction d'une loi α -stable μ_α . Alors $\mathbf{S}^n$ converge en loi dans $(D, d_{\mathbf{J}_1})$ vers le processus α -stable associé à la loi μ_α .*

Les objets que nous étudierons dans cette thèse sont intrinsèquement continus, et donc la topologie $\mathbf{J}_1$ n'est pas adaptée dans les cas où ils convergeraient en loi vers des processus α -stable. Il nous faut donc une topologie plus faible qui permette aux fonctions continues de converger vers des fonctions discontinues.

La topologie que nous cherchons se trouve encore dans les travaux originels de Skorokhod [Sko56] (dans lesquels il introduit quatre topologies sur l'espace des fonctions càdlàg). Une façon de comparer deux fonctions est de comparer leurs graphes. On définit le graphe complété d'une fonction $x \in D$ par l'ensemble

$$\Gamma(x) = \{(t, z) \in [0, 1] \times \mathbb{R}, z \in [x(t-), x(t)]\}.$$

Une façon naturelle de mesurer l'écart entre deux graphes serait d'utiliser la distance de Hausdorff, mais la topologie engendrée par cette distance s'avère être trop faible pour nos besoins. On note tout de même que cette distance engendre la topologie $\mathbf{M}_2$ de Skorokhod, voir [Sko56] et le livre de Whitt [Whi02, Chapitre 12].

On peut aussi voir $\Gamma(x)$ comme une courbe continue sur l'espace $[0, 1] \times \mathbb{R}$ et alors on peut quantifier la distance entre deux graphes comme on le fait sur les espaces de courbes, avec les représentations paramétriques. Cette distance engendre la topologie $\mathbf{M}_1$ de Skorokhod. Une paramétrisation de $\Gamma(x)$ est une fonction continue (u, r) de $[0, 1]$ dans $\Gamma(x)$, surjective et dont chaque coordonnée est croissante. On note $\Pi(x)$ l'ensemble des représentations paramétriques de $\Gamma(x)$. La distance $\mathbf{M}_1$ est définie pour $x_1, x_2 \in D$ par

$$d_{\mathbf{M}_1}(x_1, x_2) = \inf_{(u_i, r_i) \in \Pi(x_i)} (d_U(u_1, u_2) \wedge d_U(r_1, r_2)).$$

Cette distance a su trouver de nombreuses applications en probabilités et on se réfère au livre de Whitt [Whi02]. Cette topologie n'est pas trop "faible", dans le sens où un grand nombre de fonctions "intéressantes" continues pour la topologie $\mathbf{J}_1$, le reste pour la topologie $\mathbf{M}_1$, voir encore [Whi02, Chapitre 13]. Bien que nous n'ayons pas trouvé de références, il semblerait que si on considère une loi dans le domaine d'attraction d'une loi α -stable, pour $\alpha \in (0, 2)$, alors la marche interpolée linéairement converge en loi vers un processus α -stable, dans l'espace D muni de la topologie $\mathbf{M}_1$.

1.3 Problèmes de persistance

L'estimation des probabilités de persistance, encore appelées probabilités de survie, a été longuement étudiée et une littérature importante existe sur ce sujet. Il s'agit d'estimer, au moins d'une manière asymptotique, la probabilité qu'un processus stochastique reste sous un certain niveau jusque au temps t . Soit $\mathbb{T} = \mathbb{R}_+$ ou $\mathbb{N}$, et soit $(X_t)_{t \in \mathbb{T}}$ un processus stochastique à valeurs dans $\mathbb{R}$ et partant de 0. Pour $x \in (0, \infty)$, on introduit

$$T_x = \inf \{t \in \mathbb{T}, X_t \geq x\}.$$

On cherche donc à estimer $\mathbb{P}(T_x > t)$ lorsque $t \rightarrow \infty$. L'exemple le plus simple que l'on puisse donner concerne encore la marche aléatoire, et la question la plus naturelle que l'on peut se poser est de calculer la probabilité qu'une marche reste positive jusqu'à un temps n . Le résultat suivant, dû à Sparre-Andersen [SA54], traite du cas de la marche aléatoire symétrique non-atomique.

Théorème 1.8. *Soit μ une mesure de probabilité sur $\mathbb{R}$ symétrique et non-atomique. La marche aléatoire $(S_n)_{n \in \mathbb{N}}$ dont le pas a pour loi μ vérifie*

$$\mathbb{P}(S_k > 0 \text{ pour tout } 1 \leq k \leq n) = \frac{1}{4^n} \binom{2n}{n}.$$

Cette quantité est aussi égale à $\mathbb{P}(S_k \geq 0 \text{ pour tout } 1 \leq k \leq n)$ puisque μ est non-atomique. Ce résultat est frappant car cette quantité ne dépend pas de la loi μ , mais seulement de la symétrie. On verra en effet par la suite que la symétrie joue un rôle important pour ce genre de problème. Par la formule de Stirling, on peut estimer asymptotiquement cette quantité et on voit facilement qu'elle est équivalente à $cn^{-1/2}$. Ce résultat est bien connu et le livre de Feller [Fel91, Chapitre XII] en offre une preuve classique. Nous allons en donner une preuve simple, que l'on peut trouver dans Dembo-Ding-Gao [DDG13, Proposition 1.3].

Preuve. Notons pour tout $n \geq 1$, $p_n(\mu) = \mathbb{P}(S_k > 0 \text{ for all } 1 \leq k \leq n)$. Nous allons montrer par récurrence que cette quantité ne dépend pas de μ . Pour $n = 1$, on a évidemment $p_1(\mu) = p_1 = \mathbb{P}(X_1 > 0) = 1/2$ par hypothèse.

Supposons maintenant que pour $n \geq 1$ et pour tout $k \in \{1, \dots, n\}$, $p_k(\mu) = p_k$ ne dépend pas de μ . On introduit $W = \min\{k \in \{1, \dots, n+1\}, S_k = \max_{1 \leq i \leq n+1} S_i\}$, et on constate que pour tout $i \in \{0, \dots, n+1\}$, on a

$$\begin{aligned} \{W = i\} &= \{X_i > 0, X_i + X_{i+1} > 0, \dots, X_i + \dots + X_1 > 0\} \\ &\cap \{X_{i+1} \leq 0, X_{i+1} + X_{i+2} \leq 0, \dots, X_{i+1} + \dots + X_{n+1} \leq 0\}. \end{aligned}$$

Par indépendance, symétrie, et puisque μ est non-atomique, on voit alors que pour tout $i \in \{1, \dots, n\}$, on a $\mathbb{P}(W = i) = p_i p_{n+1-i}$. On voit aussi que $\mathbb{P}(W = 0) = \mathbb{P}(W = n+1) = p_{n+1}(\mu)$. De plus on a

$$1 = \sum_{i=0}^{n+1} \mathbb{P}(W = i) = 2p_{n+1}(\mu) + \sum_{i=1}^n p_i p_{n+1-i},$$

et on conclut donc que $p_{n+1}(\mu) = p_{n+1}$ ne dépend pas de μ .

On détermine maintenant la valeur de p_n , et on pose $p_0 = 1$ par convention. L'analyse précédente montre que la suite $(p_n)_{n \in \mathbb{N}}$ satisfait la relation $1 = \sum_{k=0}^n p_k p_{n-k}$ pour tout $n \geq 0$. Ainsi la fonction génératrice de $(p_n)_{n \in \mathbb{N}}$ satisfait l'égalité suivante

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} = \left(\sum_{k=0}^{\infty} p_k x^k \right)^2$$

pour $x \in (-1, 1)$. Cette fonction génératrice est donc égale à $(1-x)^{-1/2}$, et on déduit donc que $p_n = \frac{1}{4^n} \binom{2n}{n}$. $\square$

Il est ensuite intéressant de généraliser ces résultats pour des marches aléatoires générales. Par exemple, on pourrait s'attendre à ce que le comportement asymptotique de p_n soit aussi le même pour des marches asymptotiquement symétrique, i.e. lorsque $\mathbb{P}(S_n > 0) \rightarrow 1/2$. Pour étudier le problème général, il faut développer d'autres outils, tels que la factorisation de Wiener–Hopf, que nous n'aborderons pas ici, mais que l'on peut trouver dans les livres de Feller [Fel91, Chapitre XII] et de Doney [Don07]. On a le résultat suivant.

Théorème 1.9. *Soit $\rho \in (0, 1)$, les deux assertions suivantes sont équivalentes.*

$$(i) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{P}(S_k > 0) = \rho.$$

$$(ii) \text{ Pour tout } x > 0, \text{ l'application } n \mapsto \mathbb{P}(T_x > n) \text{ appartient à } \mathcal{R}_{-\rho}.$$

De plus, si (i) ou (ii) est vraie pour $\rho \in (0, 1)$, alors il existe une fonction lente ℓ et des réels $c_x > 0$, tels que, pour tout $x > 0$,

$$\mathbb{P}(T_x > n) \sim c_x \ell(n) n^{-\rho} \quad \text{lorsque } n \rightarrow \infty,$$

La condition (i) est connue sous le nom de *condition de Spitzer*. Elle est équivalente à la condition (i') $\lim_{n \rightarrow \infty} \mathbb{P}(S_n > 0) = \rho$, qui est en apparence plus forte, voir [BD97]. Ainsi, si $(S_n)_{n \in \mathbb{N}}$ est centrée et dans le bassin d'attraction d'une loi normale, on peut appliquer le théorème précédent avec $\rho = 1/2$.

Les problèmes de persistance ne se limitent pas à la marche aléatoire et ont été étudiés pour de nombreux processus. On se réfère à l'article de survol [AS15], et aussi à l'introduction du Chapitre 3 de cette thèse pour plus d'exemples et de références.

1.4 Limite d'échelle d'équations cinétiques

Dans cette thèse, on cherchera aussi à décrire les fluctuations asymptotiques des solutions de certaines équations cinétiques. Ces équations décrivent l'évolution temporelle de la densité de particules présentes dans un certain système dynamique. Elles donnent une description *mésoscopique* du système, entre la vision *microscopique* donnée par le système Hamiltonien décrivant les interactions entre chaque particule, et la vision *macroscopique* donnée par des équations hydrodynamiques (telles que les équations de Navier-Stokes ou d'Euler).

L'objet d'étude est donc une densité $f(t, x, v)$ de particules évoluant dans un domaine D de $\mathbb{R}^d$ où $d \geq 1$. On supposera que cette dernière est absolument continue par rapport à la mesure de Lebesgue sur $\mathbb{R}_+ \times D \times \mathbb{R}^d$. L'équation étudiée est

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = \mathcal{L}^* f(t, x, v), \quad (t, x, v) \in \mathbb{R}_+^* \times D \times \mathbb{R}^d. \quad (1.6)$$

Le membre de gauche ici décrit le transport libre de la particule (à travers sa vitesse), tandis que le terme de droite décrit l'interaction de la particule avec un environnement. L'opérateur $\mathcal{L}^*$ ici n'agit que sur la variable de vitesse v . Cette équation est bien entendu complétée d'une condition initiale f_0 telle que pour tout $(x, v) \in D \times \mathbb{R}^d$, $f(0, x, v) = f_0(x, v)$.

Le modèle phare de la théorie cinétique est l'équation de Boltzmann, où l'opérateur $\mathcal{L}^*$ correspondant est l'opérateur collisionnel de Boltzmann qui décrit comment les particules collisionnent. Lorsque l'on veut décrire le comportement de particules dans un plasma, l'équation de Landau est plus appropriée. Elle peut être vue comme une approximation de l'équation de Boltzmann lorsqu'on ne considère que les collisions rasantes. Il existe de nombreux modèles cinétiques et on se réfère à Villani [Vil02] ou l'article de survol plus récent de Mouhot-Villani [MV15] pour plus de détails. Supposons pour simplifier un instant que $D = \mathbb{R}^d$. Dans cette introduction, on discutera principalement de deux modèles :

- (i) *L'équation cinétique de Fokker-Planck*. Ce modèle décrit l'évolution de particules dont la vitesse diffuse tout en étant soumise à une dérive. Elles sont particulièrement utiles en astrophysiques, voir Chandrasekhar [Cha43], et aussi en théorie des semi-conducteurs. L'opérateur $\mathcal{L}^*$ correspondant est de la forme

$$\mathcal{L}^* g(v) = \frac{1}{2} \Delta g(v) - \operatorname{div}[F \cdot g](v),$$

où $g \in C^2(\mathbb{R}^d, \mathbb{R}_+)$, et $F \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ correspond à la force qui s'exerce sur les particules. D'un point de vue probabiliste, la partie diffusive correspond à des chocs aléatoires modélisés par un mouvement Brownien $(B_t)_{t \geq 0}$. On obtient alors l'équation différentielle stochastique suivante :

$$X_t = x_0 + \int_0^t V_s ds, \quad V_t = v_0 + \int_0^t F(V_s) ds + B_t. \quad (1.7)$$

Notons $f(t, dx, dv)$ la loi de (X_t, V_t) , i.e. $f(t, dx, dv) = \mathbb{P}(X_t \in dx, V_t \in dv)$. Si on suppose que la mesure $f(t, dx, dv)dt$ admet une densité régulière par rapport à $dx dv dt$, il est alors facile de voir en appliquant la formule d'Itô, en passant à l'espérance, et en effectuant des intégrations par parties, que cette densité est solution de (1.6). On note que l'opérateur $\mathcal{L}^*$ est l'adjoint du générateur infinitésimal du processus de Markov $(V_t)_{t \geq 0}$. Nous étudierons dans cette thèse (1.6) à travers le process $(X_t, V_t)_{t \geq 0}$ pour une certaine classe de forces F .

- (ii) *L'équation de scattering.* Ce modèle décrit la dispersion de particules dans un nuage aléatoire d'obstacles. Elle peut être vue comme une équation de Boltzmann linéaire, où les vitesses des particules qui collisionnent avec la particule que l'on regarde sont supposées à l'équilibre. Cette hypothèse est cohérente d'un point de vue physique lorsque l'on considère par exemple un gaz raréfié. Cette équation est particulièrement utilisée en théorie du transfert radiatif [Pom05], en théorie des réacteurs nucléaires [WW58] et en théorie des semi-conducteurs [MRS12]. L'opérateur $\mathcal{L}^*$ associé est ici donné par

$$\mathcal{L}^*g(v) = \int_{\mathbb{R}^d} b(v, w) [g(w)\nu(v) - g(v)\nu(w)] dw,$$

où ν est la densité de probabilité correspondant à l'équilibre de l'environnement, et $b(v, w)$ est un noyau de collision. Derrière cet opérateur se cache aussi un processus de Markov qui décrit la vitesse d'une particule. Ce processus $(V_t)_{t \geq 0}$ est un processus de saut dont la dynamique est la suivante : sachant que $V_t = v$, la vitesse saute à taux $\int_{\mathbb{R}^d} b(v, w)\nu(w)dw$ et la nouvelle vitesse est choisie selon $b(v, w)\nu(w)dw$. Lorsque $b(v, w)$ ne dépend pas de v (par exemple $b = 1$), le taux est constant et la vitesse est renouvelée à chaque fois ; on a alors $V_t = \xi_{N_t}$ où $(\xi_n)_{n \in \mathbb{N}}$ est une suite de variables aléatoires i.i.d. de loi μ et $(N_t)_{t \geq 0}$ est un processus de Poisson indépendant. Dans cette thèse, nous n'étudierons pas cette équation mais nous comparerons souvent nos résultats à des résultats similaires obtenus sur cette équation.

L'objectif ici est d'étudier les fluctuations asymptotiques des solutions de ces équations cinétiques. Au regard de ce qui a été expliqué dans la première partie de cette introduction, on distinguera principalement deux types de phénomènes :

- (i) *Fluctuations gaussiennes.* C'est le cas correspondant au théorème centrale limite classique, où le processus de position correctement remis à échelle converge en loi vers un mouvement Brownien. Plus précisément, si $(X_t)_{t \geq 0}$ désigne le processus position sous-jacent à (1.6), on a

$$(\varepsilon^{1/2}X_{t/\varepsilon})_{t \geq 0} \xrightarrow{f.d.} (W_t)_{t \geq 0}, \quad \text{lorsque } \varepsilon \rightarrow 0,$$

où la notation *f.d.* indique la convergence au sens des distributions fini-dimensionnelles et $(W_t)_{t \geq 0}$ est un mouvement Brownien. Si on note $h(t, x) = \int_{v \in \mathbb{R}^d} f(t, x, v)dv$ la densité de X_t , alors $h_\varepsilon(t, x) = \varepsilon^{-1/2}h(t/\varepsilon, \varepsilon^{-1/2}x)$ représente la densité de $\varepsilon^{1/2}X_{t/\varepsilon}$ et h_ε converge faiblement vers la densité de W_t qui est la solution fondamentale de l'équation de la chaleur

$$\partial_t \rho(t, x) - \Delta \rho(t, x) = 0.$$

On parle alors de *limite de diffusion*.

- (ii) *Fluctuations α -stables.* C'est le cas correspondant au TCL α -stable. On s'attend alors à avoir pour un certain $\alpha \in (0, 2)$

$$(\varepsilon^{1/\alpha}X_{t/\varepsilon})_{t \geq 0} \xrightarrow{f.d.} (Z_t^\alpha)_{t \geq 0}, \quad \text{lorsque } \varepsilon \rightarrow 0,$$

où $(Z_t^\alpha)_{t \geq 0}$ est un processus α -stable. Ainsi si on note $h_\varepsilon(t, x) = \varepsilon^{-1/\alpha} h(t/\varepsilon, \varepsilon^{-1/\alpha} x)$ la densité de $\varepsilon^{1/\alpha} X_{t/\varepsilon}$, cette dernière converge faiblement vers la densité Z_t^α qui est la solution fondamentale de l'équation de la chaleur fractionnaire

$$\partial_t \rho(t, x) + (-\Delta)^{\alpha/2} \rho(t, x) = 0,$$

où $(-\Delta)^{\alpha/2}$ est le Laplacien fractionnaire d'indice $\alpha/2$, qui se trouve être le générateur infinitésimal du processus α -stable. On parlera alors de *limite de diffusion fractionnaire* ou *limite de diffusion anormale*.

Résultats sur l'équation cinétique de Fokker-Planck : nos travaux ne porteront que sur le cas $d = 1$ et on cherchera à établir des limites d'échelles pour le processus de position $(X_t)_{t \geq 0}$ défini par (1.7). Supposons pour simplifier que la force F est impaire. Alors le processus $(V_t)_{t \geq 0}$ est récurrent et possède une mesure invariante μ qui est forcément paire. Le théorème ergodique pour les processus de Markov récurrents nous indique donc que p.s.

$$\frac{1}{t} X_t = \frac{1}{t} \int_0^t V_s ds \longrightarrow \int_{\mathbb{R}} v \mu(dv) = 0 \quad \text{lorsque } t \rightarrow \infty.$$

On peut interpréter ce résultat comme une loi forte des grands nombres, et il est ensuite naturelle de s'intéresser aux fluctuations de $(X_t)_{t \geq 0}$. Si la force F est assez “rentrante”, dans le sens où elle ramène le processus de vitesse rapidement dans un compact, on s'attend à obtenir des fluctuations gaussiennes. Ce cas a été largement étudié et il existe des théorèmes généraux dans ce cadre, voir par exemple Jacod-Shyriaev [JS03, Chapitre VIII], Pardoux-Veretennikov [PV01], ou encore l'article de survol récent de Cattiaux-Chafaï-Guillin [CCG12].

Il existe une classe de forces intéressantes pour ce genre de problèmes. Supposons que la force est telle que $F(v) \approx -\text{sgn}(v)|v|^\gamma$ lorsque $v \rightarrow \pm\infty$. On peut montrer que pour $\gamma < -1$, le processus vitesse est récurrent nul (la mesure invariante n'est pas de masse finie) et que pour $\gamma > -1$, la mesure invariante est sous-exponentielle et le processus est récurrent positif. Lorsque $\gamma = -1$, la mesure invariante est à queues lourdes, i.e. il existe $\delta > 0$ tel que $\mu(v) \approx |v|^{-\delta}$ lorsque v est grand. Cette classe de force est critique, et on peut s'attendre à observer des phénomènes résultant du TCL α -stable (car la mesure invariante est à queues lourdes).

Les premiers à s'être intéressés à cette classe critique sont Nasreddine-Puel [NP15], Cattiaux-Nasreddine-Puel [CNP19] et Lebeau-Puel [LP19]. Ils étudient l'équation cinétique de Fokker-Planck en toute dimension ($d \geq 1$) pour la force

$$F(v) = -\frac{\beta}{2} \frac{v}{1 + \|v\|^2}, \tag{1.8}$$

et $\beta > d$. La densité de la mesure invariante correspondante est $\mu(v) = (1 + \|v\|^2)^{-\beta/2}$ et donc la condition $\beta > d$ assure que cette dernière est de masse finie. Le premier article de Nasreddine-Puel [NP15] traite le cas $\beta > 4 + d$ et il est montré par des techniques analytiques que les fluctuations sont gaussiennes et qu'il y a une limite de diffusion. L'article [CNP19] traite le cas $\beta = 4 + d$ et il y a encore des fluctuations gaussiennes mais avec un taux de convergence “anormal”. Plus précisément, il est montré que $(|\varepsilon / \log \varepsilon|^{1/2} X_{t/\varepsilon})_{t \geq 0}$ converge vers un mouvement Brownien au sens des distributions fini-dimensionnelles. Ce cas correspond au TCL classique critique, où la variance tronquée $V(x)$ (voir théorème 1.5) est une fonction lente qui diverge. Enfin, l'article de Lebeau-Puel [LP19] traite de la dimension $d = 1$ et du cas $\beta \in (1, 5) \setminus \{2, 3, 4\}$, et il y est montré (encore de manière analytique) qu'il y a une limite de diffusion fractionnaire, pour $\alpha = (\beta + 1)/3$.

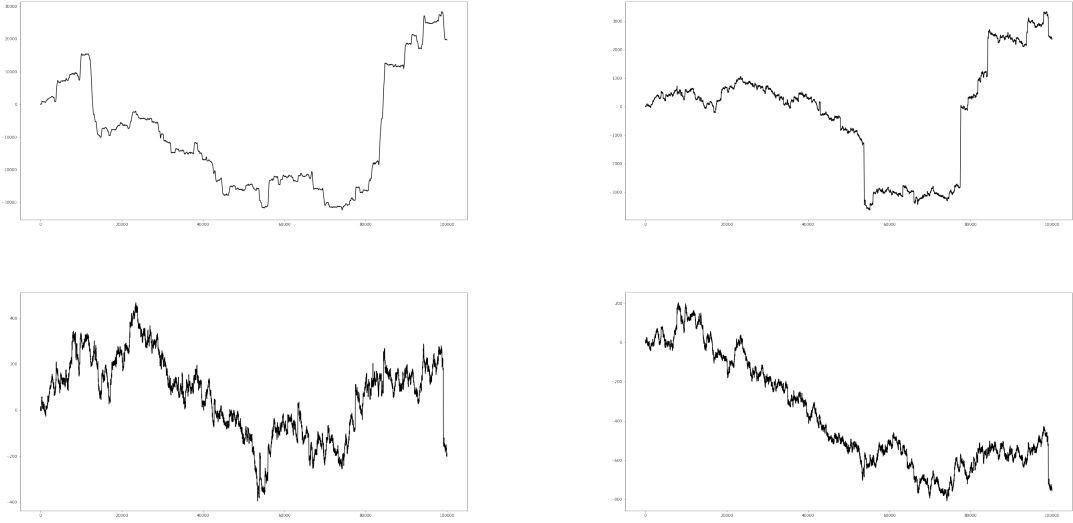


Figure 1.1: Représentation graphique de trajectoires $(X_t)_{t \geq 0}$ pour un même mouvement Brownien sous-jacent et différentes valeurs de β . Les deux premières simulations concernent les valeurs $\beta = 1.5$ et $\beta = 3$ tandis que les deux dernières correspondent aux valeurs $\beta = 4.5$ et $\beta = 6$. Quand $\beta = 6$, la trajectoire semble bien brownienne. Plus β diminue, plus des sauts apparaissent nettement.

Du côté probabiliste, l'article de Fournier-Tardif [FT21] traite ce problème pour une classe de forces légèrement plus générale du type $F(v) = -\frac{\beta}{2} \frac{\Theta'(v)}{\Theta(v)}$, où Θ est une fonction paire C^1 telle que $\Theta(v) \sim |v|^{-1}$ lorsque $v \rightarrow \pm\infty$. Tous les cas y sont traités, même le cas $\beta \in (0, 1]$ (dans ce cas la mesure invariante n'est plus de masse finie), dont nous discuterons plus tard. Les méthodes probabilistes introduites dans [FT21] sont à l'origine du Chapitre 2 de cette thèse, où nous les généralisons pour établir des TCL pour des fonctionnelles additives de diffusions unidimensionnelles. Dans [FT20], Fournier-Tardif montrent les résultats analogues pour le modèle multi-dimensionnel. Enfin, on peut résumer les résultats de [FT21, LP19, NP15, CNP19] en dimension 1 par le théorème suivant.

Théorème 1.10. *Soit $(X_t)_{t \geq 0}$ la solution de (1.7) avec la force F donnée par (1.8). Soit $(W_t)_{t \geq 0}$ un mouvement Brownien et $(Z_t^\alpha)_{t \geq 0}$ des processus α -stable symétriques. Il existe des constantes $(\sigma_\beta)_{\beta > 1}$ explicites telles que :*

(i) *Si $\beta > 5$, on a*

$$(\varepsilon^{1/2} X_{t/\varepsilon})_{t \geq 0} \xrightarrow{f.d.} (\sigma_\beta W_t)_{t \geq 0}, \quad \text{lorsque } \varepsilon \rightarrow 0.$$

(ii) *Si $\beta = 5$, on a*

$$(|\varepsilon / \log \varepsilon|^{1/2} X_{t/\varepsilon})_{t \geq 0} \xrightarrow{f.d.} (\sigma_\beta W_t)_{t \geq 0}, \quad \text{lorsque } \varepsilon \rightarrow 0.$$

(iii) *Si $\beta \in (1, 5)$, on a pour $\alpha = (\beta + 1)/3$,*

$$(\varepsilon^{1/\alpha} X_{t/\varepsilon})_{t \geq 0} \xrightarrow{f.d.} (\sigma_\beta Z_t^\alpha)_{t \geq 0}, \quad \text{lorsque } \varepsilon \rightarrow 0.$$

On peut observer cette transition de phase sur la Figure 1.1, où l'on trace différentes trajectoires de $(X_t)_{t \geq 0}$ pour un même mouvement Brownien sous-jacent et différentes valeurs de

β . On souligne aussi que l'étude de ce modèle est motivée par des phénomènes issus de la physique. En effet, des expériences ont montré que lorsque des atomes sont refroidis par un laser, ils diffusent de manière fractionnaire. Ces phénomènes ont été observés dans les articles de Castin, Dalibard et Cohen-Tannoudji [CDCT], Sagi, Brook, Almog et Davidson [SBAD12], et Marksteiner, Ellinger et Zoller [MEZ96]. Une étude théorique, réalisée par Barkai, Aghion and Kessler [BAK14]), modélise précisément le mouvement des atomes refroidis par (1.7) et F donnée par (1.8). Cette étude démontre avec un certain degré de rigueur les phénomènes observés.

Citons aussi l'article de Bouin-Mouhot [BM20] qui recouvre tous ces résultats pour $\beta > d$ et qui traite aussi d'autres équations cinétiques (dont l'équation de scattering). Les méthodes sont encore analytiques mais sont différentes de celles introduites dans [LP19, NP15, CNP19] et semblent unifier d'une certaine façon les différents modèles considérés. Enfin, on citera aussi les articles très récents de Dechicha-Puel [DP23a, DP23b] qui introduisent une nouvelle méthode analytique pour établir ces résultats. L'article [DP23b] traite de la dimension $d \geq 2$ et de la force $F(v) = -\frac{\beta}{2} \frac{v}{1+||v||^2}$ tandis que [DP23a] traite de la dimension 1 et de la même classe de force que dans Fournier-Tardif, avec une fonction Θ qui n'est pas forcément symétrique. Dans le chapitre 2, nous étendrons ces résultats à des forces asymétriques.

Résultats sur l'équation de scattering : si la mesure invariante μ du système décroît rapidement pour les grandes valeurs de v , on s'attend à obtenir des fluctuations gaussiennes et des limites de diffusion dans ce cadre ont été établies par exemple par Wigner [Wig61] et Bensoussan-Lions-Papanicolaou [BLP79]. Pour observer une limite de diffusion fractionnaire, il faut que la mesure invariante soit à queues lourdes, et les premiers à établir de tels résultats sont Mellet-Mischler-Mouhot [MMM11], Ben Abdallah-Mellet-Puel [BAMP11b] et Mellet [Mel10]. Ces trois articles traitent plus ou moins du même problème mais avec des méthodes analytiques différentes. Il y a aussi les résultats de Ben Abdallah-Mellet-Puel [BAMP11a] qui traitent du cas où la fréquence de collision est dégénérée.

Du côté probabiliste, l'article de Jara-Komorowski-Olla [JKO09] traite d'une équation de Boltzmann linéaire pour des phonons. Ils établissent d'abord un théorème central limite α -stable pour des fonctionnelles additives de certaines chaînes de Markov, puis appliquent leurs résultats à l'équation de scattering.

Conditions de bord pour des équations cinétiques : lorsque le domaine D n'est pas $\mathbb{R}^d$, il faut aussi compléter (1.6) d'une condition au bord. On notera ∂D la frontière de D . Pour tout $x \in \partial D$, on note n_x le vecteur unitaire normal rentrant à ∂D en x . On peut considérer différents types de conditions au bord.

- (i) *Conditions de bord spéculaires.* C'est la réflexion élastique de type billard. Lorsque la particule touche le bord avec une vitesse v , elle repart du même point avec une vitesse obtenue par réflexion miroir par rapport à la normale, comme on l'apprend en optique. La nouvelle vitesse est précisément égale à $v - 2(v \cdot n_x)n_x$ et la condition de bord s'écrit donc

$$f(t, x, v) = f(t, x, v - 2(v \cdot n_x)n_x), \quad t \geq 0, x \in \partial D, v \cdot n_x > 0.$$

Physiquement, cette condition traduit le fait que l'on considère le bord comme un mur élastique.

- (ii) *Conditions de bord diffusives.* Ces conditions traduisent le phénomène suivant : le bord est un mur qui est à l'équilibre thermodynamique et la particule considéré est typiquement plus petite que les particules qui constituent le mur. Ainsi lorsque la particule touche le mur, elle pénètre ce dernier et en ressort (presque) instantanément avec une vitesse à

l'équilibre du mur. Autrement dit, lorsqu'elle touche le bord au point $x \in \partial D$, elle repart avec une vitesse aléatoire distribuée selon une loi $M(x, v)dv$. La condition de bord s'écrit de la manière suivante :

$$(v \cdot n_x)f(t, x, v) = -M(x, v) \int_{\{w \cdot n_x < 0\}} (w \cdot n_x)f(t, x, w)dw, \quad t \geq 0, x \in \partial D, v \cdot n_x > 0.$$

Le noyau M doit être un noyau de probabilité.

- (iii) *Conditions de bord de Maxwell.* C'est Maxwell [Max79] lui-même qui a introduit les conditions de bord précédentes et il lui semblait qu'une combinaison convexe de ces deux conditions serait plus réaliste physiquement. La condition de bord de Maxwell s'écrit donc

$$(v \cdot n_x)f(t, x, v) = -\alpha(x)M(x, v) \int_{\{w \cdot n_x < 0\}} (w \cdot n_x)f(t, x, w)dw \\ + (1 - \alpha(x))(v \cdot n_x)f(t, x, v - 2(v \cdot n_x)n_x) \quad t \geq 0, x \in \partial D, v \cdot n_x > 0,$$

où α est une fonction de ∂D dans $[0, 1]$. Ainsi, lorsque la particule touche le bord au point $x \in \partial D$, elle est réfléchi de manière diffusive avec probabilité $\alpha(x)$ et de manière spéculaire avec probabilité $1 - \alpha(x)$.

- (iv) Il existe bien entendu d'autres conditions de bord que nous n'aborderons pas ici.

Concernant l'équation de scattering avec conditions de bord, on dénombre plusieurs travaux qui établissent des limites de diffusions fractionnaires. Tout d'abord il y a les travaux de Cesbron-Mellet-Puel [CMP20, CMP21] pour des conditions au bord diffusives. Dans leur premier article [CMP20], ils se placent dans le demi-plan, en toute dimension, et établissent une limite d'échelle α -stable. A vrai dire, leur résultat n'est pas tout à fait complet car ils ont des problèmes d'unicité concernant l'EDP limite. Ils obtiennent donc une convergence à sous-suite près. Ils règlent ce problème d'unicité dans [CMP21] où ils travaillent en dimension 1 et dans un intervalle borné. L'article de Cesbron [Ces20] traite aussi du demi-plan avec des conditions au bord de Maxwell et il se heurte aussi à des problèmes d'unicité.

Un phénomène intéressant ressort lorsqu'on considère des limites de diffusion fractionnaires pour des équations cinétiques dans des domaines : la condition de bord limite dépend de la condition de bord considérée. Ceci dénote par rapport au cas diffusif usuel, où l'on s'attend à obtenir l'équation de la chaleur avec conditions au bord de Neumann dans tous les cas. Le processus associé à cette équation est le mouvement Brownien réfléchi dans le domaine et il y a quelque part unicité dans la façon de réfléchir un mouvement Brownien, contrairement aux processus α -stables.

Du côté probabiliste, l'article de Komorowski-Olla-Ryzhik [KOR20] étudie l'équation de Boltzmann linéaire pour des phonons avec des conditions de bord spéculaires / transmissives / absorbantes en dimension 1. Lorsque le phonon atteint la barrière située en $x = 0$, il peut être transmis (il traverse la barrière), réfléchi de manière spéculaire, ou absorbé. A la limite, on trouve un processus stable dont les trajectoires sont transmises, réfléchies ou absorbées. L'article de Bogdan-Komorowski-Marino [BKM22] traite du même problème, avec une probabilité d'absorption qui tend vers 0.

Concernant l'équation cinétique de Fokker-Planck avec conditions de bord, il n'y a, à notre connaissance, pas de travaux sur ce sujet. Dans le Chapitre 4, nous établirons un premier résultat.

1.5 Résumé des travaux de thèse

1.5.1 Théorèmes limites α -stable pour des fonctionnelles additives de diffusions unidimensionnelles

Dans cette sous-section, nous résumons les travaux présentés dans le Chapitre 2, qui sont aussi contenus dans l'article [Bét21]. Dans ces travaux, on considère une diffusion unidimensionnelle $(X_t)_{t \geq 0}$ solution de l'équation différentielle stochastique

$$X_t = \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s, \quad (1.9)$$

où $(B_t)_{t \geq 0}$ est un mouvement Brownien. On suppose que les coefficients b et σ sont tels que l'EDS est bien posée et que le processus $(X_t)_{t \geq 0}$ est un processus récurrent positif. Ce dernier possède donc une mesure de probabilité invariante μ et le théorème ergodique pour les processus de Markov récurrents nous indique que pour tout $f \in L^1(\mu)$, on a p.s.

$$\frac{1}{t} \int_0^t f(X_s) ds \longrightarrow \mu(f) := \int_{\mathbb{R}} f d\mu \quad \text{lorsque } t \rightarrow \infty.$$

Comme expliqué précédemment, on peut interpréter cette convergence comme une loi forte des grands nombres, et on cherche ensuite naturellement à établir les fluctuations de

$$\frac{1}{t} \int_0^t [f(X_s) - \mu(f)] ds$$

lorsque $t \rightarrow \infty$. Le cas des fluctuations gaussiennes est un problème qui a été très largement étudié, mais de manière suprenante, celui des fluctuations α -stable était encore ouvert. Supposons que $\mu(f) = 0$ et notons $\zeta_t = \int_0^t f(X_s) ds$. Dans ce chapitre, nous donnons des conditions explicites sur la fonction f et les coefficients b et σ pour obtenir des limites d'échelles de $(\zeta_t)_{t \geq 0}$. Pour pouvoir énoncer le résultat principal, il nous faut introduire encore quelques objets. Tout d'abord, on définit la fonction d'échelle

$$\mathfrak{s}(x) = \int_0^x \exp \left(-2 \int_0^v b(u) \sigma^{-2}(u) du \right) dv$$

qui est supposée être une bijection croissante de $\mathbb{R}$ dans $\mathbb{R}$. On introduit aussi les fonctions suivantes, où $\mathfrak{s}^{-1}$ est l'inverse de $\mathfrak{s}$:

$$\psi = (\mathfrak{s}' \circ \mathfrak{s}^{-1}) \times (\sigma \circ \mathfrak{s}^{-1}) \quad \text{et} \quad \phi = (f \circ \mathfrak{s}^{-1}) / \psi^2.$$

On fait l'hypothèse suivante sur les fonctions f , b et σ .

Hypothèse 1.1. *La fonction $f : \mathbb{R} \rightarrow \mathbb{R}$ est localement bornée et mesurable. De plus, il existe $\alpha > 0$, des constantes $f_-, f_+ \in \mathbb{R}$ telles que $|f_+| + |f_-| > 0$ et une fonction lente $\ell \in \mathcal{R}_0$ tels que*

$$|x|^{2-1/\alpha} \ell(|x|) \phi(x) \longrightarrow f_{\pm} \quad \text{lorsque } x \rightarrow \pm\infty.$$

Si $f \in L^1(\mu)$, on impose $\mu(f) = 0$. Finalement, si $\alpha \geq 2$, on supposera f continue.

On introduit finalement la quantité $\rho = \int_1^\infty (\int_x^\infty \frac{dv}{v^{3/2} \ell(v)})^2 dx$ qui peut être infinie. Le théorème principal de ce chapitre est le suivant.

Théorème 1.11. *On suppose que l'Hypothèse 1.1 est vérifiée. Soient $(X_t)_{t \geq 0}$ une solution de (1.9), $(W_t)_{t \geq 0}$ un mouvement Brownien et $(Z_t^\alpha)_{t \geq 0}$ un processus α -stable tel que $\mathbb{E}[\exp(i\xi Z_t^\alpha)] = \exp(-t|\xi|^\alpha z_\alpha(\xi))$ où $\xi \in \mathbb{R}$ et $z_\alpha(\xi)$ est un complexe explicite. Alors il existe des constantes $(\sigma_\alpha)_{\alpha > 0}$ explicites telle que*

(i) Si $\alpha > 2$ ou $\alpha = 2$ et $\rho < \infty$, alors on a

$$\left(\varepsilon^{1/2}\zeta_{t/\varepsilon}\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha W_t)_{t \geq 0} \quad \text{lorsque } \varepsilon \rightarrow 0.$$

(ii) Si $\alpha = 2$ et $\rho = \infty$, on définit $\rho_\varepsilon = \int_1^{1/\varepsilon} \left(\int_x^\infty \frac{dv}{v^{3/2}\ell(v)}\right)^2 dx$ et on a

$$\left(|\varepsilon/\rho_\varepsilon|^{1/2}\zeta_{t/\varepsilon}\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_2 W_t)_{t \geq 0} \quad \text{lorsque } \varepsilon \rightarrow 0.$$

(iii) Si $\alpha \in (0, 2) \setminus \{1\}$, on a

$$\left(\varepsilon^{1/\alpha}\ell(1/\varepsilon)\zeta_{t/\varepsilon}\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha Z_t^\alpha)_{t \geq 0} \quad \text{lorsque } \varepsilon \rightarrow 0.$$

(iv) Si $\alpha = 1$ et si il existe $\lambda > 0$ tel que $|\mathfrak{s}|^\lambda \in L^1(\mu)$, alors il existe une famille déterministe de réels $(\xi_\varepsilon)_{\varepsilon > 0}$ telle que

$$\left(\varepsilon\ell(1/\varepsilon)\zeta_{t/\varepsilon} - \xi_\varepsilon t\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha Z_t^1)_{t \geq 0} \quad \text{lorsque } \varepsilon \rightarrow 0.$$

Pour ne pas rentrer dans trop de détails, nous ne donnons pas ici les constantes $z_\alpha(\xi)$ et σ_α mais ces dernières sont explicites et s'expriment avec les fonctions f , b , σ et les constantes f_+ , f_- . On souligne que les processus stables résultant ici ne sont pas nécessairement symétriques.

Lorsque $\alpha \geq 2$, le processus limite est un mouvement Brownien et on utilise les techniques classiques de martingales pour établir ce genre de résultats, voir par exemple [JS03, Chapitre VIII, Theorem 3.65]. On établit aussi dans ce chapitre un théorème général pour obtenir des fluctuations gaussiennes. Le cas $\alpha = 2$ et $\rho = \infty$ correspond au TCL classique critique et ce dernier ne semble pas avoir été traité de manière générale dans la littérature.

Lorsque $\alpha \in (0, 2)$, le taux de convergence est à variations régulières d'indices $1/\alpha$ et $(Z_t^\alpha)_{t \geq 0}$ est un processus stable non-nécessairement symétrique, dont on connaît explicitement la transformée de Fourier. On note que lorsque $\alpha = 1$, le processus limite n'est pas strictement stable, sauf dans le cas symétrique. Les méthodes utilisées pour établir ce résultat sont adaptées de l'article de Fournier-Tardif [FT21] qui traite le cas $f = \text{id}$ et $\sigma = 1$, $b = F$, comme expliqué précédemment.

La preuve dans le cas $\alpha \in (0, 2)$ repose sur la représentation de Feller des diffusions unidimensionnelles. Cette représentation nous dit qu'il existe un mouvement Brownien $(W_t)_{t \geq 0}$ tel que si on considère

$$A_t = \int_0^t \psi^{-2}(W_s) ds \quad \text{et} \quad \rho_t = \inf\{s > 0, A_s > t\},$$

on a p.s. pour tout $t \geq 0$, $X_t = \mathfrak{s}^{-1}(W_{\rho_t})$. Ainsi avec un changement de variable, on peut réexprimer ζ_t . L'Hypothèse 1.1 nous assure alors que la quantité $\int_0^{\rho_t} \phi(W_s) ds$ ressemble à peu près à $Z_t^\alpha = \int_0^{\tau_t} \text{sgn}_{f_+, f_-}(W_s) |W_s|^{1/\alpha-2} ds$ où $(\tau_t)_{t \geq 0}$ est l'inverse du temps local en 0 de $(W_t)_{t \geq 0}$ et $\text{sgn}_{f_+, f_-}(x) = f_+ \mathbf{1}_{\{x > 0\}} + f_- \mathbf{1}_{\{x < 0\}}$. Dans le cas symétrique où $f_- = -f_+$, ce processus est bien connu pour être un processus α -stable symétrique, voir Itô-McKean [IM74, page 226], Jeulin-Yor [JY81] et Biane-Yor [BY87].

Nous avons retrouvé *a posteriori* plusieurs traces de ces méthodes dans la littérature. Tout d'abord dans l'article de Bertoin-Werner [BW94], où ils redémontrent le théorème de Spitzer concernant les enroulements du mouvement Brownien plan avec cette méthode. On les retrouve aussi dans les articles de Yamada [Yam86], Hu-Shi-Yor [HSY99], Kasahara-Watanabe [KW06] et Fitzsimmons-Yano [FY08b].

1.5.2 Problèmes de persistance pour des fonctionnelles additive de processus de Markov

Nous résumons ici le contenu du Chapitre 3, lui même issu de l'article [BBT23] écrit en collaboration avec Quentin Berger et Camille Tardif. Dans ce chapitre, on traite du problème de persistance pour le processus non-Markovien

$$\zeta_t = \int_0^t f(X_s) ds,$$

où $(X_t)_{t \geq 0}$ est un processus de Markov fort càdlàg à valeurs dans $\mathbb{R}$ et f est une fonction borélienne. Dans ce qui suit $\mathbb{P}$ représente la loi de $(X_t)_{t \geq 0}$ partant de 0. On cherche à caractériser le comportement asymptotique de $\mathbb{P}(T_z > t)$, où $T_z = \inf\{t > 0, \zeta_t \geq z\}$ et $z > 0$. Ces travaux généralisent des résultats de Sinai [Sin92], Isozaki-Kotani [IK00] et Profeta [Pro21] où le processus $(X_t)_{t \geq 0}$ est un processus de Bessel biaisé (à valeurs dans $\mathbb{R}$) et f est une fonction homogène du type $f(x) = (\mathbf{1}_{\{x > 0\}} - c\mathbf{1}_{\{x < 0\}})|x|^\gamma$ avec $c, \gamma > 0$. Nous établissons des résultats similaires pour une large classe de processus de Markov et une large classe de fonction f . Concernant $(X_t)_{t \geq 0}$, nous faisons l'hypothèse suivante importante.

Hypothèse 1.2. *Le processus $(X_t)_{t \geq 0}$ n'est p.s. pas de signe constant. De plus, il ne peut pas changer de signe sans toucher 0, et 0 est un point régulier et récurrent pour $(X_t)_{t \geq 0}$.*

Le fait que 0 soit régulier pour $(X_t)_{t \geq 0}$ nous permet d'utiliser la théorie des excursions, qui est un outil important dans ce chapitre. Cette hypothèse est vérifiée par une très large classe de processus, telles que les diffusions régulières (processus de Markov fort continus) et ce qu'on appelle les diffusions généralisées. Elle n'est pas vérifiée pour la classe importante de processus qu'est la classe des processus de Lévy (sauf pour le mouvement Brownien et la différence de deux processus de Poisson indépendants). Concernant la fonction f , nous faisons l'hypothèse suivante.

Hypothèse 1.3. *La fonction f est mesurable et localement bornée, sauf éventuellement autour de 0. De plus f préserve le signe de x dans le sens où $xf(x) \geq 0$. On suppose aussi que $f(0) = 0$.*

Sous ces hypothèses, nous obtenons plusieurs résultats.

Résultats principaux I. Le premier résultat traite du cas où 0 est un point récurrent positif. Sous cette hypothèse, on obtient une condition nécessaire et suffisante pour que $t \mapsto \mathbb{P}(T_z > t)$ soit à variations régulières d'indice $-\rho$ où $\rho \in (0, 1)$. Cette condition est l'analogue de la condition de Spitzer.

Théorème 1.12. *Supposons que 0 soit un point récurrent positif et considérons $\rho \in (0, 1)$. Alors il y a équivalence entre les deux assertions suivantes.*

$$(i) \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}(\zeta_s \geq 0) ds = \rho.$$

$$(ii) \text{ Pour tout } z > 0, \text{ l'application } t \mapsto \mathbb{P}(T_z > t) \text{ est à variations régulières d'indice } -\rho.$$

De plus, si (i) ou (ii) est vraie pour $\rho \in (0, 1)$, alors il existe une fonction lente ℓ et une fonction croissante $\mathcal{V} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (relativement explicite) telles que pour tout $z > 0$

$$\mathbb{P}(T_z > t) \sim \mathcal{V}(z)\ell(t)t^{-\rho} \quad \text{lorsque } t \rightarrow \infty.$$

Lorsque 0 est un point récurrent nul, il nous faut être un peu plus précis et introduire d'autres objets. On note $(\tau_t)_{t \geq 0}$ l'inverse généralisée du temps local en 0 de $(X_t)_{t \geq 0}$. C'est un subordonateur (un processus de Lévy croissant). On introduit aussi le processus

$$Z_t = \zeta_{\tau_t} = \sum_{s < t} \int_{\tau_{s-}}^{\tau_s} f(X_u) du,$$

qui se trouve être aussi un processus de Lévy. On note aussi $g_t = \sup\{s < t, X_s = 0\}$ le dernier 0 avant t de $(X_u)_{u \geq 0}$ ainsi que $I_t = \int_{g_t}^t f(X_s) ds$.

Hypothèse 1.4. *Il existe $\beta \in (0, 1)$ et $\alpha \in (0, 2]$, et des fonctions $a(\cdot)$ et $b(\cdot)$ à variations régulières en 0 d'indices respectifs $1/\alpha$ et $1/\beta$ tels qu'on ait la convergence en loi suivante (pour la topologie de Skorokhod) :*

$$(b(h)\tau_{t/h}, a(h)Z_{t/h})_{t \geq 0} \xrightarrow{d} (\tau_t^0, Z_t^0)_{t \geq 0} \quad \text{lorsque } h \rightarrow 0,$$

où $(\tau_t^0, Z_t^0)_{t \geq 0}$ est un processus de Lévy. De plus, on suppose que $a(b^{-1}(q))I_e$ converge en loi lorsque $q \rightarrow 0$, où e est une variable exponentielle de paramètre $q > 0$ indépendante de $(X_t)_{t \geq 0}$, et b^{-1} est l'inverse asymptotique de b .

Sous cette hypothèse, on obtient le résultat suivant.

Théorème 1.13. *On suppose que l'Hypothèse 1.4 est vérifiée et que $\rho = \mathbb{P}(Z_t^0 \geq 0) \in (0, 1)$. Alors il existe une fonction lente ℓ et une fonction croissante $\mathcal{V} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ semi-explicite telles que pour tout $z > 0$*

$$\mathbb{P}(T_z > t) \sim \mathcal{V}(z)\ell(t)t^{-\beta\rho} \quad \text{lorsque } t \rightarrow \infty.$$

où β est donné par l'Hypothèse 1.4.

Résultats principaux II. On note que l'Hypothèse 1.4 porte sur les processus auxiliaires $(\tau_t, Z_t)_{t \geq 0}$ et I_t , et pas directement sur $(X_t)_{t \geq 0}$ et f . Elle peut donc *a priori* être difficile à vérifier en pratique.

Dans une deuxième partie, on s'attelle donc à simplifier cette hypothèse pour une certaine sous-classe de processus. Plus précisément, on suppose que $(X_t)_{t \geq 0}$ est une diffusion généralisée, i.e. un mouvement Brownien changé de temps et d'espace. Ces processus sont paramétrés par une fonction d'échelle $\mathfrak{s}$ et une mesure de vitesse $\mathfrak{m}$ (qui est en toute généralité une mesure de Radon sur $\mathbb{R}$), et on donne des conditions explicites sur $\mathfrak{s}$, $\mathfrak{m}$ et f pour que l'Hypothèse 1.4 soit vérifiée. Ces conditions sont intimement reliées aux conditions que l'on donne sur f , b et σ dans le Chapitre 2.

Résultats principaux III. Dans un troisième temps, on s'intéresse au temps de retour en 0 de $\zeta_t = z + \int_0^t f(X_s) ds$ partant d'une condition initiale $X_0 = x$. On se restreint ici aux diffusions régulières à valeurs dans un intervalle J qui contient 0 et on considère le processus $(\zeta_t, X_t)_{t \geq 0}$ en temps que processus de Markov fort. Pour $(z, x) \in \mathbb{R} \times J$, on note $\mathbf{P}_{(z,x)}$ la loi de $(\zeta_t, X_t)_{t \geq 0}$ partant de (z, x) . Nous établissons deux types de résultats.

- (i) Dans un premier temps, nous identifions une fonction $h : \mathbb{R} \times J \rightarrow \mathbb{R}_+$ telle que, sous les hypothèses des Théorèmes 1.12 et 1.13, on a pour tout $(z, x) \in \mathbb{R} \times J \setminus \{(0, 0)\}$,

$$\mathbf{P}_{(z,x)}(T_0 > t) \sim h(z, x)\ell(t)t^{-\beta\rho} \quad \text{lorsque } t \rightarrow \infty$$

où ℓ est une fonction lente (qui ne dépend pas de (z, x)) et $\beta \in (0, 1]$ et $\rho \in (0, 1)$ sont des paramètres donnés par les hypothèses.

- (ii) Dans un second temps, nous montrons que la fonction h est harmonique pour le processus tué $(\zeta_{t \wedge T_0}, X_{t \wedge T_0})_{t \geq 0}$. Ceci nous permet de construire, *via* une transformée de Doob, la fonctionnelle additive conditionnée à rester négative partant de $(z, x) \in \mathbb{R} \times J \setminus \{(0, 0)\}$.

Perspectives de travail et problèmes ouverts. Nous discutons maintenant de quelques perspectives de travaux futurs et de quelques problèmes ouverts. Tout d'abord, on souligne que les outils introduits dans ce chapitre permettent d'étudier de manière précise les fonctionnelles additives. Comme expliqué précédemment, ils permettent *in fine* de définir la fonctionnelle additive conditionnée à rester positive (ou négative), mais quelques questions sont ouvertes.

- (i) Il serait naturel de vouloir définir la fonctionnelle additive conditionnée à rester positive, partant de $(0, 0)$. Cela pourrait ensuite permettre de développer une théorie des excursions pour les fonctionnelles additives.
- (ii) On voudrait aussi établir des limites d'échelles pour les fonctionnelles additives conditionnées à rester positives. Par exemple, si la limite d'échelle de la fonctionnelle additive est un processus α -stable, $\alpha \in (0, 2]$, la limite d'échelle de la fonctionnelle additive conditionnée à rester positive devrait être le processus α -stable conditionné à rester positif.

Ensuite, on souhaiterait étudier le problème de persistance pour des processus de Markov de sauts, qui ne satisfont pas l'Hyptohèse 1.2. Par exemple, il serait naturel de regarder l'intégral d'un processus de Lévy. Dans ce sens, l'article de Profeta-Simon [PS15] établit l'exposant de persistance de l'intégrale d'un processus α -stable. Ce dernier est égal à $\rho/[1 + \alpha(1 - \rho)]$ où ρ est le paramètre de symétrie du processus α -stable. Il semblerait que les techniques développées dans ce chapitre ne puissent permettre, en l'état, de retrouver le résultat de Profeta-Simon. On pourrait naturellement conjecturer que pour les processus de Lévy et marches aléatoires dans le domaine d'attraction d'un processus α -stable, l'exposant de persistance de leur intégrale est aussi égal à $\rho/[1 + \alpha(1 - \rho)]$ où ρ est le paramètre de symétrie du processus limite.

1.5.3 Limite de diffusion fractionnaire pour une équation de Fokker-Planck cinétique avec conditions de bord diffusives

Nous résumons ici le contenu du chapitre 4, issu de l'article [Bét22]. Nous reprenons l'équation cinétique de Fokker-Planck étudiée par Fournier-Tardif [FT21] en dimension 1 et nous considérons ici le domaine $D = \mathbb{R}_+$ avec conditions de bord diffusives. Lorsque la position de la particule évolue dans $(0, \infty)$, sa vitesse est soumise à une force F et des chocs aléatoires modélisés par un mouvement Brownien. Lorsque la particule touche le bord $x = 0$, elle redémarre avec une vitesse aléatoire strictement positive.

Plus précisément, nous considérons une mesure de probabilité μ sur $(0, \infty)$, une suite de variables aléatoires i.i.d. $(M_n)_{n \in \mathbb{N}}$ de loi μ , et un mouvement Brownien $(B_t)_{t \geq 0}$ indépendant des variables $(M_n)_{n \in \mathbb{N}}$. L'objet d'étude ici est le processus $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ solution de l'équation différentielle stochastique

$$\begin{cases} \mathbf{X}_t = \int_0^t \mathbf{V}_s ds, \\ \mathbf{V}_t = v_0 + \int_0^t F(\mathbf{V}_s) ds + B_t + \sum_{n \in \mathbb{N}} (M_n - \mathbf{V}_{\tau_n-}) \mathbf{1}_{\{\tau_n \leq t\}}, \\ \tau_1 = \inf\{t > 0, \mathbf{X}_t = 0\} \quad \text{et} \quad \tau_{n+1} = \inf\{t > \tau_n, \mathbf{X}_t = 0\}, \end{cases}$$

où $v_0 > 0$. Concernant la force F , nous travaillons sous les mêmes hypothèses que [FT21], à savoir qu'il existe une fonction $\Theta : \mathbb{R} \rightarrow (0, \infty)$ qui est C^1 , paire et telle que $F(v) = -\frac{\beta}{2} \frac{\Theta'(v)}{\Theta(v)}$,

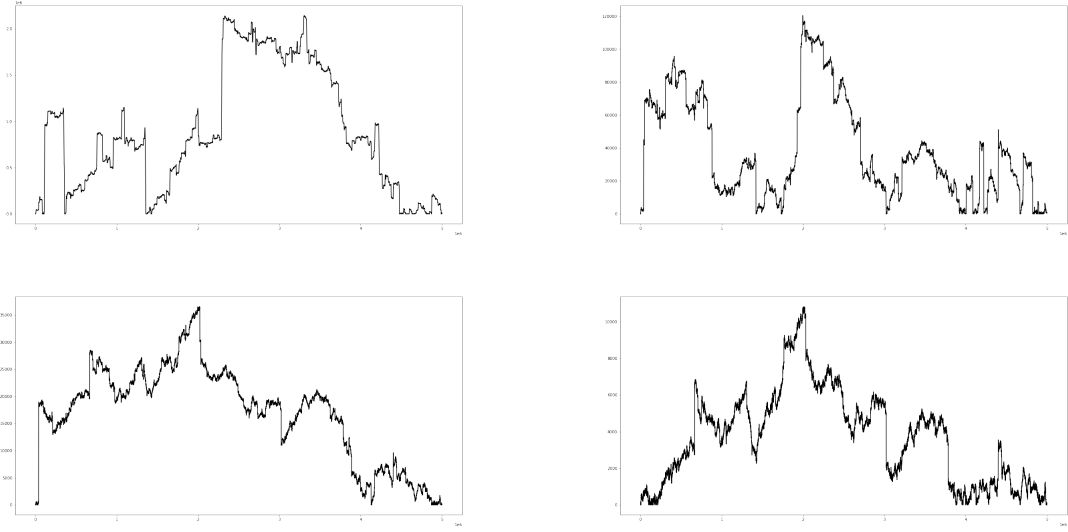


Figure 1.2: Représentation graphique de trajectoires $(\mathbf{X}_t)_{t \geq 0}$ pour un même mouvement Brownien sous-jacent, $\mu = \delta_1$ et différentes valeurs de β . Les deux premières simulations concernent les valeurs $\beta = 1.5$ et $\beta = 2.5$ tandis que les deux dernières correspondent aux valeurs $\beta = 3.5$ et $\beta = 4.5$.

où $\beta \in (1, 5)$. On suppose de plus que la force est bornée et lipschitzienne. On étudie dans ce chapitre la limite d'échelle de $(\mathbf{X}_t)_{t \geq 0}$ et on rappelle d'abord que pour $\beta \in (1, 5)$, Fournier et Tardif ont montré que le processus libre $(X_t, V_t)_{t \geq 0}$ solution de (1.7) est tel que

$$(\varepsilon^{1/\alpha} X_{t/\varepsilon})_{t \geq 0} \xrightarrow{f.d.} (Z_t^\alpha)_{t \geq 0}, \quad \text{lorsque } \varepsilon \rightarrow 0,$$

où $\alpha = (\beta+1)/3$ et $(Z_t^\alpha)_{t \geq 0}$ est un processus stable symétrique dont la fonction caractéristique est donnée par $\mathbb{E}[e^{i\xi Z_t^\alpha}] = \exp(-\sigma_\beta t |\xi|^\alpha)$ pour une certaine constante σ_β explicite. Nous établissons le théorème suivant.

Théorème 1.14. *On suppose qu'il existe $\eta > 0$ tel que la mesure μ admet un moment d'ordre $(\beta+1)/2 + \eta$. Soit $(Z_t^\alpha)_{t \geq 0}$ le processus α -stable symétrique tel que $\mathbb{E}[e^{i\xi Z_t^\alpha}] = \exp(-\sigma_\beta t |\xi|^\alpha)$ et soit $(R_t^\alpha)_{t \geq 0}$ le processus réfléchi sur son infimum défini par $R_t^\alpha = Z_t^\alpha - \inf_{s \in [0, t]} Z_s^\alpha$. On a*

$$(\varepsilon^{1/\alpha} \mathbf{X}_{t/\varepsilon})_{t \geq 0} \xrightarrow{f.d.} (R_t^\alpha)_{t \geq 0}, \quad \text{lorsque } \varepsilon \rightarrow 0.$$

De plus, si on suppose qu'il existe $\eta > 0$ tel μ admet un moment d'ordre $(\beta+1)(\beta+2)/6 + \eta$, alors il y a convergence en loi dans l'espace des fonctions càdlàg muni de la topologie $\mathbf{M}_1$.

Comme expliqué précédemment, la topologie usuelle de Skorokhod n'est pas adaptée ici car le processus $(\mathbf{X}_t)_{t \geq 0}$ est continu. Essayons d'expliquer informellement pourquoi on obtient un processus stable réfléchi sur son infimum à la limite. Tout d'abord, lorsque $\mathbf{X}_t > 0$, la dynamique du processus est régie par (1.7). Par conséquent, le théorème 1.10 nous indique que le processus limite doit se comporter comme un processus α -stable tant que ce dernier est strictement positif. Il suffit donc de comprendre comment le processus limite interagit avec la frontière. Lorsque $\mathbf{X}_t$ atteint le bord, il le fait avec une grande vitesse, ce qui correspond à un saut à la limite. Puis il est réfléchi, ralenti car sa nouvelle vitesse est distribuée selon μ et redémarre en oubliant son passé. Cette heuristique nous indique que le processus limite doit présenter le comportement suivant : lorsqu'il tente de sauter en dessous de la frontière, son saut est "coupé" et il redémarre

de 0. Le processus stable réfléchi sur son infimum possède exactement ce comportement : tant que $Z_t^\alpha > \inf_{s \in [0, t]} Z_s^\alpha$, R_t^α se comporte comme un processus α -stable translaté. Lorsque Z_t^α saute en dessous de son infimum, ce dernier est immédiatement “mis-à-jour” et R_t^α vaut donc 0.

Faisons à présent quelques remarques. Tout d’abord, le processus $(R_t^\alpha)_{t \geq 0}$ est toujours un processus de Markov. On souhaite ensuite souligner le fait que le processus limite dépend du mécanisme de réflexion initial. Considérons par exemple le même modèle cinétique avec des conditions de bord spéculaires : lorsque la particule touche 0, elle est réfléchiée avec une vitesse égale à l’opposée de la vitesse entrante. Il est alors facile de voir que la solution de l’équation correspondante est $(|X_t|, \text{sgn}(X_t)V_t)_{t \geq 0}$ où $(X_t, V_t)_{t \geq 0}$ est solution de l’équation (1.7). Par le Théorème 1.10, le processus limite correspondant est $(|Z_t^\alpha|)_{t \geq 0}$ et ce dernier a un comportement différent : lorsque il essaie de sauter en dessous de 0, le nouveau point de départ est obtenu par réflexion miroir.

Avec quelques ajustements, nous pourrions étendre ce résultat au cas diffusif $\beta \geq 5$. Le processus limite est alors un mouvement Brownien réfléchi sur son infimum, mais dans ce cas, il est bien connu que $(|B_t|)_{t \geq 0}$ est égal en loi à $(B_t - \inf_{s \in [0, t]} B_s)_{t \geq 0}$.

Dans ce chapitre, nous étudierons aussi le processus réfléchi sur une barrière inélastique. Informellement, on fait repartir le processus avec une vitesse nulle lorsqu’il touche la frontière, ce qui correspondrait au cas $\mu = \delta_0$ dans le précédent modèle. Bien entendu, il n’est pas très clair s’il est possible de définir un tel processus. Ce dernier serait solution de l’équation différentielle stochastique suivante :

$$\begin{cases} \mathbb{X}_t = \int_0^t \mathbb{V}_s ds, \\ \mathbb{V}_t = v_0 + \int_0^t F(\mathbb{V}_s) ds + B_t - \sum_{0 < s \leq t} \mathbb{V}_s - \mathbf{1}_{\{\mathbb{X}_s = 0\}}, \end{cases}$$

On s’appuiera sur des travaux de Bertoin [Ber07, Ber08] qui traite du cas $F = 0$ pour construire une solution à cette équation. Nous montrerons ensuite que $(\mathbb{X}_t)_{t \geq 0}$ a la même limite d’échelle que $(\mathbf{X}_t)_{t \geq 0}$. On souligne que nous nous inspirerons fortement de la construction de Bertoin dans la preuve du théorème principal.

Enfin, on souligne le fait que la preuve du résultat principal emprunte des résultats du Chapitre 3. En effet, il semble naturel dans ce genre de problème de vouloir estimer asymptotiquement la probabilité $\mathbb{P}(\tau_1 > t)$ quand $t \rightarrow \infty$, où on rappelle que $\tau_1 = \inf\{t > 0, \mathbf{X}_t = 0\}$. A vrai dire, les résultats issus du chapitre 3 ont en partie été motivés par ce problème.

Description informelle du résultat du point de vue des EDP. On introduit la mesure $f(dt, dx, dv)$ sur $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ définie par $f(dt, dx, dv) = \mathbb{P}(\mathbf{X}_t \in dx, \mathbf{V}_t \in dv)dt$. Alors cette mesure est une solution faible de l’équation de Fokker-Planck cinétique avec conditions de bord diffusives suivante :

$$\begin{cases} \partial_t f(t, x, v) + v \partial_x f(t, x, v) = \mathcal{L}^* f(t, x, v), & (t, x, v) \in (0, \infty)^2 \times \mathbb{R} \\ v f(t, 0, v) = -\mu(v) \int_{(-\infty, 0)} w f(t, 0, w) dw, & (t, v) \in (0, \infty)^2 \\ f(0, dx, dv) = \delta_{(0, v_0)}(dx, dv) \end{cases}$$

où on suppose ici pour simplifier que $\mu(dv) = \mu(v)dv$, et on rappelle que l’opérateur $\mathcal{L}^*$ est tel que

$$\mathcal{L}^* f(t, x, v) = \frac{1}{2} \partial_v^2 f(t, x, v) - \partial_v [F(v) f(t, x, v)].$$

On introduit maintenant $g(dt, dx) = \int_{v \in \mathbb{R}} f(dt, dx, dv) = \mathbb{P}(\mathbf{X}_t \in dx)dt$. Supposons pour simplifier que cette mesure admet une densité par rapport à $dt dx$ que l'on note $g(t, x)$. On note aussi $\rho(dt, dx) = \mathbb{P}(R_t^\alpha \in dx)dt$ qui admet une densité $\rho(t, x)$ par rapport à $dt dx$, voir Chaumont-Malecki [CMe20, Théorème 3]. Alors le théorème principal de ce chapitre implique que $g_\varepsilon(t, x) = \varepsilon^{-1/\alpha} g(t/\varepsilon, \varepsilon^{-1/\alpha} x)$ converge faiblement vers $\rho(t, x)$ lorsque ε tend vers 0.

Concernant le problème limite, nous obtenons le résultat suivant : ρ est une solution faible de l'EDP

$$\begin{cases} \partial_t \rho(t, x) = \mathcal{A}\rho(t, x) & \text{pour } (t, x) \in (0, \infty)^2, \\ \int_0^\infty \rho(t, x) dx = 1 & \text{pour } t \in (0, \infty), \\ \rho(0, \cdot) = \delta_0, \end{cases}$$

où l'opérateur $\mathcal{A}$ est défini pour toute fonction $\varphi \in C^2((0, \infty))$, et pour tout $x > 0$, par

$$\mathcal{A}\varphi(x) = \int_{\mathbb{R}} \frac{\varphi(x-z)\mathbf{1}_{\{x>z\}} - \varphi(x) + z\partial_x \varphi(x)\mathbf{1}_{\{|z|<x\}}}{|z|^{1+\alpha}} dz$$

Cet opérateur est intimement relié au générateur infinitésimal de $(R_t^\alpha)_{t \geq 0}$, que l'on note $\mathcal{L}^\alpha$. Plus précisément, on peut montrer que pour toute fonction $\varphi \in C^2(\mathbb{R}_+)$ et pour tout $x > 0$, on a $\mathcal{L}^\alpha \varphi(x) = \mathcal{A}\varphi(x) + \varphi(0)x^{-\alpha}/\alpha$. Dans le cas de ρ , il s'avère que pour tout $t > 0$, on a $\rho(t, x) \rightarrow \infty$ lorsque $x \rightarrow 0$ et donc $\mathcal{L}^\alpha \rho(t, x)$ n'a pas de sens ici.

L'équation limite de Cesbron-Mellet-Puel. Nous discutons aussi brièvement les résultats de Cesbron-Mellet-Puel [CMP20, CMP21] concernant l'équation de scattering avec conditions de bord diffusives. Nous nous placerons en dimension 1 sur $\mathbb{R}_+$ ici bien que leurs résultats sont établis dans le demi-plan en toute dimension dans [CMP20] et dans l'intervalle $[0, 1]$ dans [CMP21]. L'équation cinétique étudiée est la suivante :

$$\begin{cases} \partial_t f(t, x, v) + v\partial_x f(t, x, v) = \mathcal{L}^* f(t, x, v), & (t, x, v) \in (0, \infty)^2 \times \mathbb{R} \\ v f(t, 0, v) = -v\nu(v) \int_{(-\infty, 0)} w f(t, 0, w) dw, & (t, v) \in (0, \infty)^2 \end{cases}$$

où la densité ν correspond à la densité d'équilibre du système et où on rappelle que l'opérateur de scattering est tel que

$$\mathcal{L}^* f(t, x, v) = \int_{\mathbb{R}} b(v, w) [f(t, x, w)\nu(v) - f(t, x, v)\nu(w)] dw.$$

La densité d'équilibre ν est supposée être à queue lourde et on voit que la vitesse avec laquelle est redémarrée une particule lorsqu'elle touche le bord a donc pour loi $v\nu(v)dv$. Ils établissent une limite de diffusion fractionnaire pour $g(t, x) = \int_{v \in \mathbb{R}} f(t, x, v)dv$. La vitesse avec laquelle est redémarrée une particule lorsqu'elle touche le bord a donc pour loi $v\nu(v)dv$. Ils établissent une limite de diffusion fractionnaire pour $g(t, x) = \int_{v \in \mathbb{R}} f(t, x, v)dv$ et $\alpha > 1$. Correctement remise à échelle, cette fonction converge faiblement vers une fonction $\rho(t, x)$, qui est solution faible du problème au bord suivant :

$$\begin{cases} \partial_t \rho(t, x) - \mathcal{L}^\alpha \rho(t, x) = 0, & (t, x) \in (0, \infty)^2 \times \mathbb{R} \\ D^\alpha \rho(t, 0) = 0, & t \in (0, \infty) \end{cases}$$

où $\mathcal{L}^\alpha$ correspond à l'opérateur introduit précédemment, et D^α est défini pour tout φ de classe C^∞ et à support compact dans $\mathbb{R}_+$ par

$$D^\alpha \varphi(x) = c_\alpha \int_0^\infty \varphi'(y) |y - x|^{1-\alpha} dy,$$

pour une certaine constante c_α . Cette équation peut être vue comme une équation de la chaleur fractionnaire avec condition de Neumann, où la dérivée usuelle est ici remplacée par une dérivée fractionnaire.

Nous pensons que la solution $\rho(t, x)$ à cette équation décrit la densité d'un processus α -stable réfléchi dans $\mathbb{R}_+$, qui est différent du processus stable réfléchi sur son infimum du Théorème 1.14. Plus précisément, ce processus peut être construit à partir de ses excursions en dehors de 0, que nous décrivons maintenant.

On introduit $\mathcal{E}_+$, l'espace des fonctions càdlàg ε de $\mathbb{R}_+$ dans $\mathbb{R}_+$ telles que il existe $\ell(\varepsilon) \geq 0$ tel que pour tout $t \geq \ell(\varepsilon)$, $\varepsilon_t = 0$. Cet espace est muni de la topologie usuelle de Skorokhod et de la tribu Borelienne associée. On introduit, pour $\alpha \in (0, 2)$ et $\theta \in (0, 1)$, une mesure $\mathbf{n}_{\alpha, \theta}$ sur cet espace définie pour tout $f : \mathcal{E}_+ \rightarrow \mathbb{R}_+$ continue bornée par

$$\mathbf{n}_{\alpha, \theta}(f) = \int_0^\infty \frac{\mathbb{E}_x[f((0 \wedge Z_{t \wedge T_0}^\alpha)_{t \geq 0})]}{x^{1+\theta}} dx,$$

où $(Z_t^\alpha)_{t \geq 0}$ est un processus α -stable symétrique, $T_0 = \inf\{t \geq 0, Z_t^\alpha < 0\}$ et $\mathbb{P}_x$ est la loi de $(x + Z_t^\alpha)_{t \geq 0}$. Cette mesure est σ -finie et caractérise informellement la loi d'une excursion. Ainsi, une excursion démarre en x suivant la “loi” $x^{-1-\theta} dx$ et évolue ensuite comme un processus α -stable libre jusqu'à l'instant où ce dernier saute dans les négatifs et elle est alors “absorbée” par 0.

Concernant l'EDP limite de Cesbron-Mellet-Puel, nous pensons que le processus de Markov sous-jacent est le processus associé à la mesure d'excursions $\mathbf{n}_{\alpha, \alpha-1}$ (on rappelle que dans leur cas, $\alpha \in (1, 2)$). On peut d'ailleurs montrer que toute fonction g dans le domaine du générateur infinitésimal de ce processus de Markov, g satisfait forcément $D^\alpha g(0) = 0$, voir Itô [Itô15, Théorème 2.5.5].

Criticalité. Ceci nous amène à discuter de ce qui se passe pour le processus $(\mathbf{X}_t)_{t \geq 0}$ étudié dans le Théorème 1.14 lorsque la mesure μ est *critique*, i.e. lorsque $\mu((v, \infty)) \sim v^{-(\beta+1)/2}$. Dans ce cas, la condition de moment requise n'est plus satisfaite. Nous pensons dans ce cas obtenir une limite d'échelle similaire à celle obtenue dans [CMP20, CMP21]. Plus précisément, nous pensons que le processus limite correspond au processus associé à la mesure d'excursion $\mathbf{n}_{\alpha, \alpha/2}$. Ici, les vitesses de la particule lorsque elle ressort de la frontière ne sont plus négligeables, ce qui à la limite, donne naissance à un saut de “loi” $x^{-1-\alpha/2} dx$ à chaque fois que la particule touche le bord.

Travaux futurs. Les résultats de ce chapitre forment une base solide pour étudier un modèle un peu plus complet, où l'on considère des conditions de bord de Maxwell, toujours en dimension 1. Dans un second temps, on souhaiterait regarder le modèle réfléchi dans un domaine D convexe, en toute dimension $d \geq 2$. On s'appuiera bien entendu sur l'article de Fournier-Tardif [FT20] dans lequel ils établissent la limite d'échelle du processus “libre” (non-réfléchi) en toute dimension. Lorsque D est le demi-plan, il semblerait qu'il n'y ait pas trop de difficultés car on pourrait à peu près se ramener à la dimension 1. Si on considère un domaine convexe en toute généralité, il est possible que ce problème soit difficile, notamment si il faut estimer la queue de probabilité du temps de sortie du domaine, chose que nous avons dû faire en dimension 1.

1.5.4 Une application du théorème de Sparre Andersen pour des variables aléatoires échangeables et invariantes par signe

Nous résumons ici le contenu du Chapitre 5, lui-même issu de l'article [BB23] écrit en collaboration avec Quentin Berger. Nous revisitons le théorème de Sparre Andersen (voir Théorème

1.8) pour des variables aléatoires échangeables et invariantes par signe. Nous utilisons ensuite ce résultat pour obtenir des bornes sur des probabilités de persistance de certaines chaînes de Markov intégrées.

On fixe $n \geq 1$ et on considère un vecteur aléatoire $(\xi_1, \dots, \xi_n)$ à valeurs dans $\mathbb{R}^n$. On dit que la loi $\mathbf{P}$ de $(\xi_1, \dots, \xi_n)$ est :

- (E) échangeable si pour toute permutation $\sigma \in \mathfrak{S}_n$, le vecteur aléatoire $(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)})$ a la même loi que $(\xi_1, \dots, \xi_n)$.
- (S) invariant par signe si pour tout $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, $(\varepsilon_1 \xi_1, \dots, \varepsilon_n \xi_n)$ a la même loi que $(\xi_1, \dots, \xi_n)$.

Dans une première partie, nous étendons le Théorème 1.8 pour des lois $\mathbf{P}$ qui satisfont (E)-(S). Sparre-Andersen avait déjà observé ce fait, mais sa démonstration est un peu lacunaire, voir [SA54, Théorème 4]. Plus précisément, nous montrons le théorème suivant, où $g(n)$ est défini par $g(n) = \frac{1}{4^n} \binom{2n}{n}$.

Théorème 1.15. *On suppose que la loi $\mathbf{P}$ satisfait (E)-(S). Alors on a*

$$\mathbf{P}(S_k > 0 \text{ pour tout } 1 \leq k \leq n) \leq g(n) \leq \mathbf{P}(S_k \geq 0 \text{ pour tout } 1 \leq k \leq n).$$

De plus, si $\mathbf{P}$ n'a pas d'atomes, alors les trois quantités ci-dessus sont égales.

Pour démontrer ce théorème, on reprend la preuve exposée dans la Section 1.3, et on utilise les hypothèses (E)-(S) au lieu d'utiliser l'indépendance. On donne aussi des contre-exemples de lois qui ne satisfont pas (E)-(S) pour lesquelles le précédent théorème est faux.

Dans un second temps, on considère une chaîne de Markov $(X_n)_{n \in \mathbb{N}}$ de naissance et de mort telle que $X_0 = 0$ p.s., i.e. une chaîne à valeurs dans $\mathbb{Z}$ telle que pour tout $n \in \mathbb{N}$, on a $|X_{n+1} - X_n| \leq 1$ p.s. Ses probabilités de transitions sont données par

$$p_x = p(x, x+1), \quad q_x = p(x, x-1), \quad r_x = p(x, x) = 1 - p_x - q_x \quad \text{pour } x \geq 1.$$

On suppose de plus que la chaîne est symétrique au sens où $(-X_n)_{n \in \mathbb{N}}$ à la même loi que $(X_n)_{n \in \mathbb{N}}$, ce qui implique que $p(x, y) = p(-x, -y)$ pour tout $x, y \in \mathbb{Z}$. Concernant $x = 0$, on se donne $p_0 = p(0, 1) \in (0, \frac{1}{2}]$, $q_0 = p(0, -1) = p_0$ et $r_0 = p(0, 0) = 1 - 2p_0$. On se donne ensuite une fonction $f : \mathbb{Z} \rightarrow \mathbb{R}$ qui préserve le signe, i.e. telle que $f(x) > 0$ pour $x > 0$ et $f(x) < 0$ pour $x < 0$ (avec naturellement $f(0) = 0$). On définit enfin la chaîne f -intégrée $(\zeta_n)_{n \in \mathbb{N}}$ par $\zeta_0 = 0$ et pour tout $n \geq 1$,

$$\zeta_n = \sum_{k=1}^n f(X_k).$$

On cherche alors à estimer la probabilité que $(\zeta_k)_{k \in \mathbb{N}}$ reste positive jusqu'à un temps n donné. On cherche aussi à estimer cette probabilité lorsque l'on impose $X_n = 0$. Nous obtenons le théorème suivant où on rappelle que la fonction g est définie par $g(n) = \frac{1}{4^n} \binom{2n}{n}$.

Théorème 1.16. *On suppose que $(X_n)_{n \in \mathbb{N}}$ est une chaîne de naissance et de mort symétrique et récurrente, et que la fonction f est impaire et vérifie $xf(x) \geq 0$. Alors on a pour tout $n \geq 1$,*

$$p_0(1 - p_0)\mathbf{E}[g(L_n - 1)] \leq \mathbf{P}(\zeta_k > 0 \text{ pour tout } 1 \leq k \leq n) \leq \mathbf{E}[g(L_n)]$$

où $L_n = \sum_{k=1}^n \mathbf{1}_{\{X_k=0\}}$ est le temps local de la chaîne en 0 jusqu'au temps n . Concernant le pont, on a pour tout $n \geq 1$,

$$p_0\mathbf{E}[g(L_n - 1)\mathbf{1}_{\{X_n=0\}}] \leq \mathbf{P}(\zeta_k > 0 \text{ pour tout } 1 \leq k \leq n, X_n = 0) \leq \mathbf{E}[g(L_n)\mathbf{1}_{\{X_n=0\}}].$$

Dans un troisième temps, on se donne des hypothèses pour pouvoir estimer $(L_n)_{n \in \mathbb{N}}$ et obtenir des bornes plus explicites. Si la chaîne est récurrente positive, alors $L_n \sim cn$ lorsque $n \rightarrow \infty$ et on montre que

$$\mathbf{E}[g(L_n)] \sim \frac{c_1}{\sqrt{n}} \quad \text{et} \quad \mathbf{E}[g(L_n)\mathbf{1}_{\{X_n=0\}}] \sim \frac{c_2}{\sqrt{n}} \quad \text{lorsque } n \rightarrow \infty,$$

pour des constantes $c_1, c_2 > 0$. Lorsque la chaîne est récurrente nulle, il faut imposer des hypothèses plus précises. On note $\tau_0 = 0$, et on définit $\tau_{n+1} = \inf\{n \geq \tau_n + 1, X_n = 0\}$ de manière récursive. La suite $(\tau_n)_{n \in \mathbb{N}}$ forme un processus de renouvellement (par la propriété de Markov). On supposera que cette suite est dans le domaine d'attraction d'une loi α -stable complètement asymétrique, avec $\alpha \in (0, 1)$, ce qui nous permettra d'estimer plus précisément les quantités ci-dessus.

Lorsque $(X_n)_{n \in \mathbb{N}}$ est une marche aléatoire simple et symétrique, la probabilité que la marche intégrée reste positive jusqu'à l'instant n est de l'ordre de $n^{-1/4}$ et on retrouve le résultat bien connu établi par Sinai [Sin92]. Malheureusement, nos techniques ne nous permettent pas de donner des asymptotiques précises aux probabilités de persistance. Il est probable qu'en adaptant les méthodes utilisées dans le chapitre 3 au cas discret, on puisse établir ces asymptotiques.

1.5.5 Limites d'échelles pour des fonctionnelles additives

Dans un chapitre final, nous revisitons les résultats du Chapitre 2, en utilisant les méthodes et outils introduits dans les Chapitres 3 et 4. Nous étudions les limites d'échelles de fonctionnelles additives $\zeta_t = \int_0^t f(X_s)ds$ où $(X_t)_{t \geq 0}$ est une diffusion généralisée. Nous montrons que, sous certaines hypothèses, la fonctionnelle additive correctement remise à échelle est tendue dans l'espace des fonctions càdlàg muni de la topologie $\mathbf{M}_1$. Nous identifions aussi pour tout $t \geq 0$, la loi de tout processus limite (qui est un point d'accumulation) au temps t , ce qui ne suffit pas pour déterminer entièrement la loi du processus. Nous étudions aussi le cas où le processus $(X_t)_{t \geq 0}$ est récurrent nul.

Stable limit theorems for additive functionals of one-dimensional diffusion processes

Abstract

This chapter contains the results of [Bét21] which is accepted for publication at *Ann. Inst. H. Poincaré. Probab. Stat.* We consider a positive recurrent one-dimensional diffusion process with continuous coefficients and we establish stable central limit theorems for a certain type of additive functionals of this diffusion. In other words we find some explicit conditions on the additive functional so that its fluctuations behave like some α -stable process in large time for $\alpha \in (0, 2]$.

2.1 Introduction and main result

Consider a one-dimensional diffusion process with continuous coefficients b and σ , i.e. a continuous adapted process $(X_t)_{t \geq 0}$ satisfying the SDE

$$X_t = \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s, \quad (2.1)$$

where $(B_t)_{t \geq 0}$ is a Brownian motion. Without loss of generality, we assume $X_0 = 0$, even if it means changing b and σ . If σ does not vanish, then weak existence and uniqueness in law hold for (2.1), see Kallenberg [Kal02, Chapter 23 Theorem 23.1].

Assumption 2.1. *The functions $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are continuous and for all $x \in \mathbb{R}$, $\sigma(x) > 0$. Moreover b and σ are such that the process $(X_t)_{t \geq 0}$ is positive recurrent in the sense of Harris.*

We recall that a strong Markov process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}})$ valued in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is said to be recurrent in the sense of Harris if it has a σ -finite invariant measure μ such that for all $A \in \mathcal{B}(\mathbb{R})$

$$\mu(A) > 0 \quad \text{implies} \quad \limsup_{t \rightarrow \infty} \mathbf{1}_A(X_t) = 1, \quad \mathbb{P}_x - \text{almost surely for all } x \in \mathbb{R}.$$

The process is then said to be positive recurrent if $\mu(\mathbb{R}) < \infty$ and null recurrent otherwise. We introduce the scale function $\mathfrak{s}$ of $(X_t)_{t \geq 0}$ defined by

$$\mathfrak{s}(x) = \int_0^x \exp \left(-2 \int_0^v b(u) \sigma^{-2}(u) du \right) dv, \quad (2.2)$$

which is a C^2 , strictly increasing function solving $\frac{1}{2}s''\sigma^2 + s'b = 0$. We also introduce the speed measure density

$$m(x) = \sigma^{-2}(x) \exp \left(2 \int_0^x b(v) \sigma^{-2}(v) dv \right) = [\sigma^2(x) \mathfrak{s}'(x)]^{-1} \quad (2.3)$$

Remark 2.1. Assume b and σ are continuous and σ does not vanish. Then we have the equivalence between the two following propositions, see for instance Revuz-Yor [RY99, Chapter VII].

1. $(X_t)_{t \geq 0}$ is positive recurrent in the sense of Harris.
2. $\lim_{x \rightarrow \pm\infty} \mathfrak{s}(x) = \pm\infty$ and $\kappa^{-1} := \int_{\mathbb{R}} m(x) dx < \infty$.

Moreover, in this case, the measure $\mu(dx) = \kappa m(x) dx$ is the unique invariant probability measure for the process $(X_t)_{t \geq 0}$.

The ergodic theorem for Harris recurrent processes, see Azema-Duflo-Revuz [ADR69] or Revuz-Yor [RY99, Chapter X], tells us that for $f \in L^1(\mu)$, a.s.

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \mu(f), \quad (2.4)$$

where $\mu(f) = \int_{\mathbb{R}} f d\mu$. The convergence in (2.4) can be seen as a strong law of large numbers for the additive functional $\int_0^t f(X_s) ds$ of the process $(X_t)_{t \geq 0}$. Then it is very natural to study its fluctuations, i.e. to describe the asymptotic behavior of $\frac{1}{t} \int_0^t (f(X_s) - \mu(f)) ds$. In this paper, we give simple conditions on f for these asymptotic fluctuations to be α -stable for some $\alpha \in (0, 2]$. The conditions on f are entirely prescribed by the coefficients b and σ .

We recall that $\ell : [0, \infty) \mapsto (0, \infty)$ is said to be slowly varying if for every $\lambda > 0$, $\ell(\lambda x)/\ell(x) \rightarrow 1$ as x goes to infinity. The use of such functions is justified by the fact that domains of attraction of stable laws involve slowly varying functions.

Assumption 2.2. The function $f : \mathbb{R} \mapsto \mathbb{R}$ is locally bounded and Borel. Moreover, there exists $\alpha > 0$, $(f_+, f_-) \in \mathbb{R}^2$ and a continuous slowly varying function $\ell : [0, \infty) \mapsto (0, \infty)$ such that

$$[\sigma(x) \mathfrak{s}'(x)]^{-2} |\mathfrak{s}(x)|^{2-1/\alpha} \ell(|\mathfrak{s}(x)|) f(x) \xrightarrow[x \rightarrow \pm\infty]{} f_{\pm},$$

and $|f_+| + |f_-| > 0$. If $f \in L^1(\mu)$, we impose $\mu(f) = 0$. Finally, if $\alpha \geq 2$, we impose f to be continuous.

This assumption appears naturally in the computations. We refer to Section 2.4 for a collection of concrete applications and examples. In the critical stable regime $\alpha = 1$, we will add a mild assumption which ensures that the set $L^1(\mu)$ is big enough.

Assumption 2.3. There exists $\lambda > 0$ such that $|\mathfrak{s}|^\lambda \in L^1(\mu)$.

Under Assumption 2.2, we set $\rho = \int_1^\infty \left(\int_x^\infty \frac{dv}{v^{3/2} \ell(v)} \right)^2 dx$ which can be infinite, and we define the diffusive constant as follows:

- If $\alpha > 2$ or $\alpha = 2$ and $\rho < \infty$, $\sigma_\alpha^2 = 4\kappa \int_{\mathbb{R}} \mathfrak{s}'(x) \left(\int_x^\infty f(v) [\sigma^2(v) \mathfrak{s}'(v)]^{-1} dv \right)^2 dx$.
- If $\alpha = 2$ and $\rho = \infty$, $\sigma_2^2 = 4\kappa(f_+^2 + f_-^2)$.
- If $\alpha \in (0, 2)$, $\sigma_\alpha^\alpha = \frac{\kappa 2^{\alpha-2} \pi}{\alpha \sin(\alpha\pi/2)} \left(\frac{\alpha^\alpha}{\Gamma(\alpha)} \right)^2 (|f_+|^\alpha + |f_-|^\alpha)$.

We also introduce, for $\xi \in \mathbb{R}$, the complex numbers

- $z_\alpha(\xi) = 1 - i \frac{\text{sgn}(f_+)|f_+|^\alpha + \text{sgn}(f_-)|f_-|^\alpha}{|f_+|^\alpha + |f_-|^\alpha} \tan\left(\frac{\alpha\pi}{2}\right) \text{sgn}(\xi)$ if $\alpha \in (0, 2) \setminus \{1\}$.
- $z_1(\xi) = 1 + i \frac{f_+ + f_-}{|f_+| + |f_-|} \frac{2}{\pi} \text{sgn}(\xi) \left[\log\left(\frac{\pi|\xi|}{2(|f_+| + |f_-|)}\right) + 2\gamma + \log(2) + \frac{\log|f_+|f_+ + \log|f_-|f_-}{f_+ + f_-} \right]$, where γ is the Euler constant.

Finally, for a family of processes $((Y_t^\varepsilon)_{t \geq 0})_{\varepsilon > 0}$ valued in $\mathbb{R}$, we say that $(Y_t^\varepsilon)_{t \geq 0} \xrightarrow{f.d.} (Y_t^0)_{t \geq 0}$ if for all $n \geq 1$ and all $0 < t_1 < \dots < t_n$, the vector $(Y_{t_i}^\varepsilon)_{1 \leq i \leq n}$ converges in law to $(Y_{t_i}^0)_{1 \leq i \leq n}$ in $\mathbb{R}^n$. We are now ready to state our main theorem, which concerns positive recurrent diffusions.

Theorem 2.1. *Suppose Assumptions 2.1 and 2.2. Let $(X_t)_{t \geq 0}$ be a solution of (2.1), $(W_t)_{t \geq 0}$ a Brownian motion and $(S_t^{(\alpha)})_{t \geq 0}$ an α -stable process such that $\mathbb{E}[\exp(i\xi S_t^{(\alpha)})] = \exp(-t|\xi|^\alpha z_\alpha(\xi))$.*

(i) *If $\alpha > 2$, or $\alpha = 2$ and $\rho < \infty$,*

$$\left(\varepsilon^{1/2} \int_0^{t/\varepsilon} f(X_s) ds \right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha W_t)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0.$$

(ii) *If $\alpha = 2$ and $\rho = \infty$, we set $\rho_\varepsilon = \int_1^{1/\varepsilon} \left(\int_x^\infty \frac{dv}{v^{3/2}\ell(v)} \right)^2 dx$ and we have*

$$\left(|\varepsilon/\rho_\varepsilon|^{1/2} \int_0^{t/\varepsilon} f(X_s) ds \right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_2 W_t)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0.$$

(iii) *If $\alpha \in (0, 2) \setminus \{1\}$,*

$$\left(\varepsilon^{1/\alpha} \ell(1/\varepsilon) \int_0^{t/\varepsilon} f(X_s) ds \right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha S_t^{(\alpha)})_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0.$$

(iv) *If $\alpha = 1$ and Assumption 2.3 holds, there exists a deterministic family $(\xi_\varepsilon)_{\varepsilon > 0}$ of real numbers such that*

$$\left(\varepsilon \ell(1/\varepsilon) \int_0^{t/\varepsilon} f(X_s) ds - \xi_\varepsilon t \right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_1 S_t^{(1)})_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0.$$

Moreover $\xi_\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} \kappa(f_+ + f_-) \ell(1/\varepsilon) \zeta_\varepsilon$, where $\zeta_\varepsilon = -\int_{1/\varepsilon}^\infty \frac{dx}{x\ell(x)}$ if $f \in L^1(\mu)$ and $\zeta_\varepsilon = \int_1^{1/\varepsilon} \frac{dx}{x\ell(x)}$ otherwise.

For the reader more familiar with infinitesimal generators than with characteristic functions, let us mention the following remark.

Remark 2.2. *When $\alpha \in (0, 2)$, the process $(\sigma_\alpha S_t^{(\alpha)})_{t \geq 0}$ of Theorem 2.1 is a Lévy process with Lévy measure*

$$\nu(dx) = \left(c_+ x^{-1-\alpha} \mathbf{1}_{\{x>0\}} + c_- |x|^{-1-\alpha} \mathbf{1}_{\{x<0\}} \right) dx,$$

where

$$c_+ = \lambda_\alpha \left[\mathbf{1}_{\{f_+>0\}} |f_+|^\alpha + \mathbf{1}_{\{f_->0\}} |f_-|^\alpha \right], \quad c_- = \lambda_\alpha \left[\mathbf{1}_{\{f_+<0\}} |f_+|^\alpha + \mathbf{1}_{\{f_-<0\}} |f_-|^\alpha \right]$$

and $\lambda_\alpha = \frac{\kappa 2^{\alpha-2} \pi}{\alpha \sin(\alpha\pi/2)} \left(\frac{\alpha^\alpha}{\Gamma(\alpha)} \right)^2$. Its infinitesimal generator $\mathcal{L}^{(\alpha)}$ is such that for all $\phi \in C_c^\infty(\mathbb{R})$,

- $\mathcal{L}^{(\alpha)}\phi(x) = \int_{\mathbb{R} \setminus \{0\}} [\phi(x+z) - \phi(x)]\nu(dz)$ if $\alpha \in (0, 1)$,
- $\mathcal{L}^{(\alpha)}\phi(x) = \int_{\mathbb{R} \setminus \{0\}} [\phi(x+z) - \phi(x) - z\phi'(x)]\nu(dz)$ if $\alpha \in (1, 2)$,
- if $\alpha = 1$, $\mathcal{L}^{(\alpha)}\phi(x) = a\phi'(x) + \int_{\mathbb{R} \setminus \{0\}} [\phi(x+z) - \phi(x) - z\mathbf{1}_{\{|z| \leq 1\}}\phi'(x)]\nu(dz)$, where

$$a = -(f_+ + f_-) \left[2\gamma + \log(2) + \kappa \log(\kappa) + \frac{\log |f_+|f_+ + \log |f_-|f_-}{f_+ + f_-} + \kappa \frac{\pi}{2} A \right],$$

γ is the Euler constant and $A = \int_0^1 \frac{\sin(x)-x}{x^2} dx + \int_1^\infty \frac{\sin(x)}{x^2} dx$.

One can refer to Bertoin [Ber96, page 24] to understand how to go from the Lévy-Khintchine triplet to the generator and to Sato [Sat13, Chapter 3.14] to go from the Lévy-Khintchine triplet to the characteristic function.

Observe that in the Lévy regime $\alpha \in (0, 2)$, we have a convergence in finite dimensional distribution of a continuous process towards a discontinuous process (stable processes possess many jumps). Hence we cannot hope to obtain convergence in law as a process, for example for the usual Skorokhod distance. Observe also that if $f_- = 0$, the stable process has only positive jumps. Thus the limiting process is completely asymmetric although the process $(f(X_t))_{t \geq 0}$ may visit the whole space infinitely often.

When $f \in L^1(\mu)$ and $\mu(f) \neq 0$, we can use Theorem 2.1 with the function $f - \mu(f)$ provided it satisfies Assumption 2.2.

When $\alpha \geq 2$, the stable process obtained in the limit is a Brownian motion, which is the only 2-stable process. A standard strategy to show central limit theorems for the additive functional $\int_0^t f(X_s) ds$ is to solve the Poisson equation $\mathcal{L}g = f$, where $\mathcal{L}$ is the infinitesimal generator of $(X_t)_{t \geq 0}$ with domain $D_{\mathcal{L}}$. One can refer to Jacod-Shiryaev [JS03, Chapter VIII], Pardoux-Veretennikov [PV01] or Cattiaux, Chafaï and Guillin [CCG12]. Assume that f is a continuous function in $L^1(\mu)$ and define the function

$$g(x) = 2 \int_0^x \mathbf{s}'(v) \int_v^\infty f(u) [\sigma^2(u) \mathbf{s}'(u)]^{-1} du dv. \quad (2.5)$$

Then g is a C^2 function solving the Poisson equation $2bg' + \sigma^2 g'' = -2f$. Hence, applying the Itô formula with g , we express the additive functional as a martingale plus some remainder, and hope to obtain the result using a central limit theorem for martingales. This will be the strategy used when $\alpha \geq 2$, and we will actually prove the following more general result. Note that, for g to be well defined, it suffices that $f \in L^1(\mu)$ since $\mu(dx) = \kappa [\sigma^2(u) \mathbf{s}'(u)]^{-1} dx$, whence $\int_v^\infty f(u) [\sigma^2(u) \mathbf{s}'(u)]^{-1} du = \kappa^{-1} \mu(f \mathbf{1}_{(v, \infty)})$.

Theorem 2.2. *Suppose Assumption 2.1 and let $(X_t)_{t \geq 0}$ be the solution of (2.1). Let also f be a continuous function in $L^1(\mu)$ such that $\mu(f) = 0$ and g be defined by (2.5). If $g'\sigma \in L^2(\mu)$, then*

$$\left(\varepsilon^{1/2} \int_0^{t/\varepsilon} f(X_s) ds \right)_{t \geq 0} \xrightarrow{f.d.} (\gamma W_t)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0,$$

where $(W_t)_{t \geq 0}$ is a Brownian motion and $\gamma^2 = \int_{\mathbb{R}} [g'\sigma]^2 d\mu$. Moreover, if g is bounded, the finite dimensional convergence can be replaced by a convergence in law of continuous processes, with the topology of uniform convergence on compact time intervals.

Note that in the Lévy regime, we do not require f to be regular whereas we need f to be continuous in the diffusive regime, so we can apply the Itô formula with g .

References

The central limit theorem for Markov processes has a long history which goes back to Langevin [Lan08]. Roughly, he studied a one-dimensional particle, the velocity of which is subject to random shocks and a restoring force $F(v) = -v$, and showed that the position of the particle behaves like a Brownian motion, when t tends to infinity.

Probabilistic techniques to obtain such results can be found in Jacod-Shiryaev [JS03, Chapter VIII]: if there exists a solution to the Poisson equation having few properties, then we have a central limit theorem. This strategy is applied in Pardoux-Veretennikov [PV01] for very general multidimensional diffusion processes. However, [PV01] does not completely include the diffusive regime of Theorem 2.1: we have a little less assumptions but our result only holds in dimension one. It seems that the critical diffusive regime $\alpha = 2$ was not much studied. Finally, one can refer to Cattiaux, Chafaï and Guillin [CCG12] for a detailed state of the art of the techniques used to show such theorems.

Regarding the Lévy regime, we could not find many results for Markov processes and this does not seem to be well developed. Through the prism of ergodic theory, Gouëzel [Gou04] proved stable limit theorems for a certain class of maps. Applied to a positive recurrent Markov chain $(X_n)_{n \geq 0}$ with countable state space, its results tells us that if the function f is such that, up to a slowly varying function,

$$\mathbb{P}_i \left(\sum_{k=0}^{\tau_i-1} f(X_k) > x \right) \underset{x \rightarrow \infty}{\sim} c_+ |x|^{-\alpha} \quad \text{and} \quad \mathbb{P}_i \left(\sum_{k=0}^{\tau_i-1} f(X_k) < x \right) \underset{x \rightarrow -\infty}{\sim} c_- |x|^{-\alpha}, \quad (2.6)$$

where $\tau_i = \inf\{k > 0, X_k = i\}$, then we have

$$\frac{1}{n^\alpha} \sum_{k=0}^{n-1} f(X_k) \xrightarrow{d} S^\alpha,$$

where S^α is a stable random variable. This is more or less obvious in this case since we have a sum of i.i.d blocks in the domain of attraction of stable laws and we can thus apply classical stable central limit theorems.

Similar results with more tangible assumptions were proved by Jara, Komorowski and Olla [JKO09] for Markov chains with general state space. They assume the function f is such that, roughly, $\mu(\{|f| > x\}) \simeq |x|^{-\alpha}$, where μ is the invariant measure of the chain and a spectral gap condition, the latter one being more or less equivalent to return times having exponential moments. Since the return times are small, it is reasonable to think that we have $\mathbb{P}_i(\sum_{k=0}^{\tau_i-1} f(X_k) > x) \simeq \mu(\{f > x\})$, at least if we consider a countable state space. They actually give two proofs, one using a martingale approximation and a second one with a renewal method involving a coupling argument.

Mellet, Mischler and Mouhot [MMM11] showed that for a linearized Boltzmann equation with heavy-tailed invariant distribution, the rescaled distribution of the position converges to the solution of the fractional heat equation, i.e. the position process behaves like a stable process. Their work is closely related to [JKO09] although their proof is entirely analytic.

The method proposed in this article is rather powerful and gives a simple condition on the additive functional for stable limit theorems to occur. The assumption made on the function f does not seem to be a time-continuous equivalent of (2.6) and we do not assume anything on the return times, nor on $\mu(\{|f| > x\})$. We will see in Section 2.4 that, indeed, the index α is not always prescribed by $\mu(\{|f| > x\})$.

The strategy is the following: we classically express the process $(X_t)_{t \geq 0}$ as a Brownian motion $(W_t)_{t \geq 0}$ changed in time and in space. Then we write $\int_0^t f(X_s)ds = \int_0^{\tau_t} \phi(W_s)ds$ for a certain function ϕ , which roughly looks like $\int_0^{\tau_t} |W_s|^{1/\alpha-2}ds$ (up to asymmetry and principal values issues) in large time, where $(\tau_t)_{t \geq 0}$ is the generalized inverse of the local time at 0 of $(W_t)_{t \geq 0}$. This process is known to be a stable process, see Itô-McKean [IM74, page 226], Jeulin and Yor [JY81] and Biane and Yor [BY87]. The computations are thus tractable and the method is very robust.

This method was proposed by Fournier and Tardif [FT21], studying a kinetic model and the purpose of this paper is to extend this method to general one-dimensional diffusions and to more general additive functionals. The model they were studying was the following:

$$V_t = B_t - \frac{\beta}{2} \int_0^t \frac{V_s}{1+V_s^2} ds, \quad X_t = \int_0^t V_s ds, \quad (2.7)$$

where $(B_t)_{t \geq 0}$ is a Brownian motion and $\beta > 1$. The processes $(V_t)_{t \geq 0}$ and $(X_t)_{t \geq 0}$ are respectively the velocity and the position of a one-dimensional particle subject to the restoring force $F(v) = -\frac{\beta}{2} \frac{v}{1+v^2}$ and random shocks. Lebeau and Puel [LP19] showed that, when $\beta \in (1, 5) \setminus \{2, 3, 4\}$, the rescaled distribution of the position X_t converges to the solution of the fractional heat equation. Their work is analytical and relies on a deep spectral analysis leading to an impressive result. The diffusive regime ($\beta > 5$) and critical diffusive regime ($\beta = 5$) are treated by Nasreddine-Puel [NP15] and Cattiaux-Nasreddine-Puel [CNP19]. We stress that all these papers are P.D.E papers.

To summarize, if the restoring force field is strong enough ($\beta \geq 5$), the rescaled position process resembles a Brownian motion and we say that there is a *normal diffusion limit*. On the other hand, when the force is weak ($\beta \in (1, 5)$), the position resembles a stable process and there is an *anomalous diffusion limit* (or *fractional diffusion limit*).

Actually, physicists discovered that atoms diffuse anomalously when they are cooled by a laser. See for instance Castin, Dalibard and Cohen-Tannoudji [CDCT], Sagi, Brook, Almog and Davidson [SBAD12] and Marksteiner, Ellinger and Zoller [MEZ96]. A theoretical study (see Barkai, Aghion and Kessler [BAK14]) modeling the motion of atoms precisely by (2.7) proved with quite a high level of rigor the observed phenomenons.

In the Lévy regime, the index α is not prescribed by the function $f = \text{id}$ and the invariant measure μ . Indeed observe that we have $\alpha = (\beta + 1)/3$ whereas one can check, see Section 2.4, that

$$\mu(\{|f| > x\}) \simeq |x|^{1-\beta}.$$

In this situation, the return times are large and the dynamics are more complex.

Then, using probabilistic techniques, Fournier and Tardif [FT21] treated all cases of (2.7) for a larger class of symmetric forces. Naturally, we recover their result and enlarge it to asymmetrical forces, see Section 2.4. In a companion paper [FT20], they also prove the result in any dimension and the proof is much more involved.

We also found a paper of Bertoin and Werner [BW94] where they use similar techniques to redemonstrate *Spitzer's theorem*. It has been pointed out to us that Yamada [Yam86] had obtained limit theorems for additive functionals of Brownian motion, which is to be closely related to Lemma 2.8.

Plan of the paper

In Section 2.2, we recall some facts about slowly varying functions, local times and generalize some results on stable processes found in [IM74], [JY81] and [BY87]. Section 2.3 is dedicated to the proof of Theorem 2.1 and Theorem 2.2. Finally, we apply our result to several models in Section 2.4, illustrating some remarks made in the previous section.

Notations

Throughout the paper, we will use the function $\text{sgn}_{a,b}(x) = a\mathbf{1}_{\{x>0\}} + b\mathbf{1}_{\{x<0\}}$ where $(a, b) \in \mathbb{R}^2$. The function sgn will denote the usual sign function, i.e. $\text{sgn}(x) = \text{sgn}_{1,-1}(x)$.

2.2 Preliminaries

2.2.1 Slowly varying functions

In this section, we define slowly varying functions and give the properties of these functions that will be useful to us. One can refer to Bingham, Goldie and Teugels [BGT87] for more details.

Definition 2.1. *A measurable function $\ell : [0, \infty) \mapsto (0, \infty)$ is said to be slowly varying if for every $\lambda > 0$, $\ell(\lambda x)/\ell(x) \rightarrow 1$ as $x \rightarrow \infty$.*

Famous examples of slowly varying functions are powers of logarithm, iterated logarithms or functions having a strictly positive limit at infinity. It holds that for any $\theta > 0$, $x^\theta \ell(x) \rightarrow \infty$ and $x^{-\theta} \ell(x) \rightarrow 0$, as $x \rightarrow \infty$. We will need the following lemma.

Lemma 2.1. *Let ℓ be a continuous slowly varying function.*

(i) *For each $0 < a < b$, we have $\sup_{\lambda \in [a,b]} |\ell(\lambda x)/\ell(x) - 1| \xrightarrow{x \rightarrow \infty} 1$.*

(ii) *(Potter's bound). For any $\delta > 0$ and $C > 1$, there exists $x_0 > 0$, such that for all $x, y \geq x_0$,*

$$\frac{\ell(y)}{\ell(x)} \leq C \left(\left| \frac{y}{x} \right|^\delta \vee \left| \frac{x}{y} \right|^\delta \right).$$

(iii) *The functions $L : x \mapsto \int_1^x \frac{dv}{v\ell(v)}$ and $M : x \mapsto \int_1^x (\int_v^\infty \frac{du}{u^{3/2}\ell(u)})^2 dv$ are slowly varying, as well as $N : x \mapsto \int_x^\infty \frac{dv}{v\ell(v)}$ if $\int_1^\infty \frac{dv}{v\ell(v)} < \infty$.*

Proof. **Items (i) and (ii).** See [BGT87, pages 6 and 25].

Item (iii). Let us start with the function L . For $\lambda > 0$, we have

$$L(\lambda x) = \int_1^{\lambda x} \frac{dv}{v\ell(v)} = L(x) + \int_x^{\lambda x} \frac{dv}{v\ell(v)} = L(x) + \int_1^\lambda \frac{du}{u\ell(xu)}$$

where we used the substitution $v = xu$ in the last equality. By (i), $\sup_{u \in [1,\lambda]} |\ell(x)/\ell(xu) - 1| \rightarrow 0$ as $x \rightarrow \infty$, so that $\int_1^\lambda \frac{du}{u\ell(xu)} \sim \log(\lambda)/\ell(x)$ as x tends to infinity and it remains to show that $\ell(x)L(x) \rightarrow \infty$ at infinity. Let $A > 1$ and consider $x > A$: we have

$$\ell(x)L(x) = \ell(x) \int_1^x \frac{dv}{v\ell(v)} = \ell(x) \int_{1/A}^1 \frac{du}{u\ell(xu)} \geq \int_{1/A}^1 \frac{\ell(x)du}{u\ell(xu)} \xrightarrow{x \rightarrow \infty} \log(A),$$

by Point (i) again. Now let $A \rightarrow \infty$ and the result follows. Regarding the function N , we assume $\int_1^\infty \frac{dv}{v\ell(v)} < \infty$ and we write, for $\lambda > 0$,

$$N(\lambda x) = N(x) - \int_x^{\lambda x} \frac{dv}{v\ell(v)} = N(x) - \int_1^\lambda \frac{du}{u\ell(xu)}$$

and, as previously, we only need to show that $\ell(x)N(x) \rightarrow \infty$. We write

$$\ell(x)N(x) = \ell(x) \int_x^\infty \frac{dv}{v\ell(v)} = \int_1^\infty \frac{\ell(x)du}{u\ell(xu)} \geq \int_1^A \frac{\ell(x)du}{u\ell(xu)}$$

for any $A > 1$. But once again, $\int_1^A \frac{\ell(x)dv}{v\ell(xv)} \rightarrow \log(A)$ and the result holds letting A to infinity. Finally we show the result for M . We have, for $\lambda > 0$,

$$M(\lambda x) = M(x) + \int_x^{\lambda x} \left(\int_v^\infty \frac{du}{u^{3/2}\ell(u)} \right)^2 dv = M(x) + \int_1^\lambda \left(\int_y^\infty \frac{dz}{z^{3/2}\ell(xz)} \right)^2 dy$$

A little study shows that $\int_1^\lambda \left(\int_y^\infty \frac{dz}{z^{3/2}\ell(xz)} \right)^2 dy \sim \int_1^\lambda \left(\int_y^\infty \frac{dz}{z^{3/2}\ell(x)} \right)^2 dy = \frac{4\log(\lambda)}{\ell^2(x)}$. It remains to show that $\ell^2(x)M(x) \rightarrow \infty$. Let $A > 1$ and $x > A$, we have

$$\ell^2(x)M(x) = \int_{1/x}^1 \left(\int_y^\infty \frac{\ell(x)dz}{z^{3/2}\ell(xz)} \right)^2 dy \geq \int_{1/A}^1 \left(\int_y^\infty \frac{\ell(x)dz}{z^{3/2}\ell(xz)} \right)^2 dy \xrightarrow{x \rightarrow \infty} 4\log(A)$$

and the result follows by letting A to ∞ . $\square$

2.2.2 Brownian local times

Local times have been widely studied in Revuz-Yor [RY99] to which we refer for much more details. Let $(W_t)_{t \geq 0}$ be a Brownian motion. We introduce the local time of $(W_t)_{t \geq 0}$ at $x \in \mathbb{R}$, which is the process $(L_t^x)_{t \geq 0}$ defined by

$$L_t^x = |W_t - x| - |x| - \int_0^t \text{sgn}(W_s - x) dW_s.$$

The process $(L_t^x)_{t \geq 0}$ is continuous and non-decreasing and the random non-negative measure dL_t^x on $[0, \infty)$ is a.s. carried by the set $\{t \geq 0, W_t = x\}$, which is a.s. Lebesgue-null. We will heavily use the occupation times formula, see [RY99, Chapter 6 Corollary 1.6 page 224], which tells that a.s., for every $t \geq 0$ and for all Borel function $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$,

$$\int_0^t \varphi(W_s) ds = \int_{\mathbb{R}} \varphi(x) L_t^x dx.$$

We will also use the fact that the map $x \mapsto L_t^x$ is a.s. Hölder of order θ , for $\theta \in (0, 1/2)$, uniformly on compact time intervals, see [RY99, Corollary 1.8 page 226], i.e. that

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |x|^{-\theta} |L_t^x - L_t^0| < \infty \quad \text{a.s.} \quad (2.8)$$

We now introduce $\tau_t = \inf\{u \geq 0, L_u^0 > t\}$, the right-continuous generalized inverse of $(L_t^0)_{t \geq 0}$. Some properties of $(\tau_t)_{t \geq 0}$ will be needed. For all $t > 0$, $\mathbb{P}(\tau_{t-} < \tau_t) = 0$, which means that $(\tau_t)_{t \geq 0}$ has no deterministic jump time and a.s., for all $t \geq 0$, $L_{\tau_t}^0 = t$ since $(L_t^0)_{t \geq 0}$ is continuous. We will also use that a.s., for all $t \geq 0$, $W_{\tau_t} = 0$ and that a.s., $\tau_0 = 0$.

Finally, the processes $(\tau_t, W_t)_{t \geq 0}$ and $(L_{\tau_t}^x)_{x \in \mathbb{R}, t \geq 0}$ enjoy the following scaling property: for all $c > 0$

$$(\tau_t, W_t)_{t \geq 0} \stackrel{d}{=} (c^{-2}\tau_{ct}, c^{-1}W_{c^2t})_{t \geq 0} \quad \text{and} \quad (L_{\tau_t}^x)_{x \in \mathbb{R}, t \geq 0} \stackrel{d}{=} (c^{-1}L_{\tau_{ct}}^{xc})_{x \in \mathbb{R}, t \geq 0} \quad (2.9)$$

2.2.3 Stable processes

There is a huge litterature on stable processes and we will mainly refer to Sato [Sat13]. In this section we generalize some results on stable processes found in Itô-McKean [IM74, page 226], Jeulin-Yor [JY81] and Biane-Yor [BY87], where they worked in a symmetric or completely asymmetric framework. These kind of results were initiated by Lévy himself [L39], where he first proved that $(\tau_t)_{t \geq 0}$ is a $1/2$ -stable subordinator. Let us first recall a classic result on stable processes, see Sato [Sat13, Theorem 14.15 page 86 and Definition 14.16 page 87].

Theorem 2.3. *Let $\alpha \in (0, 2) \setminus \{1\}$ and let $(X_t)_{t \geq 0}$ be an α -strictly stable process, i.e. a Lévy process enjoying the scaling property $(X_t)_{t \geq 0} \stackrel{d}{=} (c^{-1/\alpha} X_{ct})_{t \geq 0}$ for all $c > 0$. Then there exist $\nu \geq 0$ and $\beta \in [-1, 1]$ such that for all $\xi \in \mathbb{R}$*

$$\mathbb{E}[\exp(i\xi X_t)] = \exp\left(-\nu t |\xi|^\alpha \left(1 - i\beta \tan\left(\frac{\alpha\pi}{2}\right) \operatorname{sgn}(\xi)\right)\right).$$

If $(X_t)_{t \geq 0}$ is a 1-stable process, i.e. a Lévy process such that for all $c > 0$, there exists $k_c \in \mathbb{R}$ with $(X_t)_{t \geq 0} \stackrel{d}{=} (c^{-1} X_{ct} + k_c t)_{t \geq 0}$, then there exist $\nu \geq 0$, $\beta \in [-1, 1]$ and $\tau \in \mathbb{R}$ such that for all $\xi \in \mathbb{R}$

$$\mathbb{E}[\exp(i\xi X_t)] = \exp\left(-\nu t |\xi| \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(\xi) \log |\xi|\right) + i\tau \xi\right).$$

In any case, if $(X_t)_{t \geq 0}$ has only positive jumps, i.e. if its Lévy measure is carried by $(0, \infty)$, then $\beta = 1$.

We now consider a Brownian motion $(W_t)_{t \geq 0}$, its local time $(L_t^0)_{t \geq 0}$ at 0, and the generalized inverse $(\tau_t)_{t \geq 0}$ of its local time. Our goal is to build all possible stable processes in terms of this Brownian motion and its inverse local time. This is important in order to adapt the method of Fournier-Tardif [FT21] who only treated the symmetric case. Let us recall the notation $\operatorname{sgn}_{a,b}(x) = a\mathbf{1}_{\{x>0\}} + b\mathbf{1}_{\{x<0\}}$. We have the following result.

Lemma 2.2. *Let $\alpha \in (0, 1)$, $(a, b) \in \mathbb{R}^2$ and define $K_t = \int_0^t \operatorname{sgn}_{a,b}(W_s) |W_s|^{1/\alpha-2} ds$. Then $(S_t)_{t \geq 0} = (K_{\tau_t})_{t \geq 0}$ is strictly α -stable and for all $\xi \in \mathbb{R}$ and all $t \geq 0$, we have*

$$\mathbb{E}[\exp(i\xi S_t)] = \exp\left(-c_{\alpha,a,b} t |\xi|^\alpha \left(1 - i\beta_{\alpha,a,b} \tan\left(\frac{\alpha\pi}{2}\right) \operatorname{sgn}(\xi)\right)\right),$$

where $c_{\alpha,a,b} = \frac{2^{\alpha-2}\pi}{\alpha \sin(\alpha\pi/2)} \left(\frac{\alpha^\alpha}{\Gamma(\alpha)}\right)^2 (|a|^\alpha + |b|^\alpha)$ and $\beta_{\alpha,a,b} = \frac{\operatorname{sgn}(a)|a|^\alpha + \operatorname{sgn}(b)|b|^\alpha}{|a|^\alpha + |b|^\alpha}$.

When $\alpha \in (0, 1)$, stable processes have finite variations and no drift part meaning they are pure jump processes and should be seen this way: $S_t = \sum_{s \in [0, t]} \int_{\tau_{s-}}^{\tau_s} \operatorname{sgn}_{a,b}(W_u) |W_u|^{1/\alpha-2} du$. The jumping times are the ones of $(\tau_t)_{t \geq 0}$ and the size of a jump is equal to the integral of the function $\operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2}$ over the corresponding excursion of the Brownian motion.

While $(K_t)_{t \geq 0}$ is well-defined when $\alpha \in (0, 1)$ since $\int_0^T |W_s|^{1/\alpha-2} ds < \infty$ a.s., this is not the case when $\alpha \in [1, 2)$ and we have to work a little more.

Lemma 2.3. *Let $\alpha \in (1, 2)$, $(a, b) \in \mathbb{R}^2$ and define $K_t = \int_{\mathbb{R}} \operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2} (L_t^x - L_t^0) dx$. Then $(S_t)_{t \geq 0} = (K_{\tau_t})_{t \geq 0}$ is strictly α -stable and for all $\xi \in \mathbb{R}$ and all $t \geq 0$, we have*

$$\mathbb{E}[\exp(i\xi S_t)] = \exp\left(-c_{\alpha,a,b} t |\xi|^\alpha \left(1 - i\beta_{\alpha,a,b} \tan\left(\frac{\alpha\pi}{2}\right) \operatorname{sgn}(\xi)\right)\right),$$

where $c_{\alpha,a,b} = \frac{2^{\alpha-2}\pi}{\alpha \sin(\alpha\pi/2)} \left(\frac{\alpha^\alpha}{\Gamma(\alpha)}\right)^2 (|a|^\alpha + |b|^\alpha)$ and $\beta_{\alpha,a,b} = \frac{\operatorname{sgn}(a)|a|^\alpha + \operatorname{sgn}(b)|b|^\alpha}{|a|^\alpha + |b|^\alpha}$.

The case $\alpha = 1$ is a little more tedious.

Lemma 2.4. *Let $(a, b) \in \mathbb{R}$ and define the process $K_t = \int_{\mathbb{R}} \operatorname{sgn}_{a,b}(x) |x|^{-1} (L_t^x - L_t^0 \mathbf{1}_{\{|x| \leq 1\}}) dx$. Then $(S_t)_{t \geq 0} = (K_{\tau_t})_{t \geq 0}$ is 1-stable. For all $\xi \in \mathbb{R}$ and all $t \geq 0$, we have*

$$\mathbb{E}[\exp(i\xi S_t)] = \exp\left(-c_{a,b}t|\xi| \left(1 + i\beta_{a,b} \frac{2}{\pi} \log(|\xi|) \operatorname{sgn}(\xi)\right) + it\tau_{a,b}\xi\right),$$

where $c_{a,b} = \frac{\pi}{2}(|a| + |b|)$, $\beta_{a,b} = \frac{a+b}{|a|+|b|}$, $\tau_{a,b} = -(a+b) \left[2\gamma + \log(2) + \frac{\log|a| + \log|b|}{a+b}\right]$ and γ is the Euler constant

To handle the proof, we need first to make the following general remark.

Remark 2.3. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function such that for all $T > 0$, $\int_0^T |\phi(W_s)| ds < \infty$ a.s. The process $Z_t = \int_0^t \phi(W_s) ds$ is a Lévy process with respect to the filtration $(\mathcal{F}_{\tau_t})_{t \geq 0}$. Indeed, for all $0 < s < t$, $Z_t - Z_s = \int_{\tau_s}^{\tau_t} \phi(W_u) du = \int_0^{\tau_t - \tau_s} \phi(W_{u+\tau_s}) du = \int_0^{\bar{\tau}_t - s} \phi(\bar{W}_u) du$, where $(\bar{W}_u)_{u \geq 0} = (W_{u+\tau_s})_{u \geq 0}$ is a Brownian motion independent of $\mathcal{F}_{\tau_s}$ thanks to the strong Markov property and since $\bar{W}_0 = W_{\tau_s} = 0$ a.s., and where $(\bar{\tau}_u)_{u \geq 0}$ is the generalized inverse of its local time. Hence the increments of $(Z_t)_{t \geq 0}$ are independent and stationary.*

Proof of Lemma 2.2. Let us first show that $(S_t)_{t \geq 0}$ is strictly α -stable. It is a Lévy process by Remark 2.3. For all $c > 0$, we have from (2.9)

$$\begin{aligned} S_t &\stackrel{d}{=} \int_0^{c^{-2}\tau_{ct}} \operatorname{sgn}_{a,b}(c^{-1}W_{c^2s}) |c^{-1}W_{c^2s}|^{1/\alpha-2} ds \\ &= c^{2-1/\alpha} \int_0^{c^{-2}\tau_{ct}} \operatorname{sgn}_{a,b}(W_{c^2s}) |W_{c^2s}|^{1/\alpha-2} ds \\ &= c^{-1/\alpha} \int_0^{\tau_{ct}} \operatorname{sgn}_{a,b}(W_u) |W_u|^{1/\alpha-2} du, \end{aligned}$$

which equals $c^{-1/\alpha} S_{ct}$. Hence $(S_t)_{t \geq 0}$ is strictly α -stable. We now focus on showing the explicit form of its characteristic function and introduce first the processes

$$C_t = \int_0^{\tau_t} \mathbf{1}_{\{W_s > 0\}} |W_s|^{1/\alpha-2} ds, \quad D_t = \int_0^{\tau_t} \mathbf{1}_{\{W_s < 0\}} |W_s|^{1/\alpha-2} ds$$

and $H_t = C_t - D_t$. We have $S_t = aC_t + bD_t$. It is clear that $(C_t)_{t \geq 0}$, $(D_t)_{t \geq 0}$ and $(H_t)_{t \geq 0}$ are also strictly stable processes because e.g. $(C_t)_{t \geq 0}$ is nothing but $(S_t)_{t \geq 0}$ when $a = 1$ and $b = 0$. It is also known from Biane-Yor [BY87] that for all $\xi \in \mathbb{R}$, for all $t \geq 0$ that

$$\mathbb{E}[\exp(i\xi H_t)] = \exp(-tc_\alpha |\xi|^\alpha) \quad \text{with} \quad c_\alpha = \frac{2^{\alpha-1}\pi}{\alpha \sin(\alpha\pi/2)} \left(\frac{\alpha^\alpha}{\Gamma(\alpha)}\right)^2. \quad (2.10)$$

Using the occupation times formula, we can write

$$C_t = \int_0^\infty |x|^{1/\alpha-2} L_{\tau_t}^x dx \quad \text{and} \quad D_t = \int_{-\infty}^0 |x|^{1/\alpha-2} L_{\tau_t}^x dx.$$

But it is well known, see [JY81, page 215], that for a fixed t , the processes $(L_{\tau_t}^x)_{x \geq 0}$ and $(L_{\tau_t}^x)_{x \leq 0}$ are independent and thus C_t and D_t are independent. It is also clear that they have the same law. Since $(C_t)_{t \geq 0}$ has only positive jumps, Theorem 2.3 tells us that its characteristic function can be written

$$\mathbb{E}[\exp(i\xi C_t)] = \exp\left(-c'_\alpha t |\xi|^\alpha \left(1 - i \tan\left(\frac{\alpha\pi}{2}\right) \operatorname{sgn}(\xi)\right)\right),$$

for some $c'_\alpha > 0$. Now since $\mathbb{E}[\exp(i\xi H_t)] = \mathbb{E}[\exp(i\xi C_t)] \mathbb{E}[\exp(-i\xi D_t)] = \exp(-2c'_\alpha t|\xi|^\alpha)$ it is clear from (2.10) that $c'_\alpha = c_\alpha/2$. Now having in mind that $S_t = aC_t + bD_t$, and C_t and D_t are i.i.d variables, we easily complete the proof. $\square$

Proof of Lemma 2.3. To show that $(S_t)_{t \geq 0}$ is a Lévy process, we first introduce, for $\eta > 0$,

$$\begin{aligned} K_t^\eta &= \int_0^t \operatorname{sgn}_{a,b}(W_s) |W_s|^{1/\alpha-2} \mathbf{1}_{\{|W_s| \geq \eta\}} ds - \left(\int_{\mathbb{R}} \operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2} \mathbf{1}_{\{|x| \geq \eta\}} dx \right) L_t^0 \\ &= \int_{\mathbb{R}} \operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2} \mathbf{1}_{\{|x| \geq \eta\}} (L_t^x - L_t^0) dx, \end{aligned}$$

where we used the occupation times formula. We show that $(K_t^\eta)_{t \geq 0}$ converges a.s., uniformly on compact time intervals to $(K_t)_{t \geq 0}$. For $\eta > 0$, we have

$$\begin{aligned} \sup_{t \in [0, T]} |K_t^\eta - K_t| &\leq \int_{\mathbb{R}} \left| \operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2} \mathbf{1}_{\{|x| \geq \eta\}} - \operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2} \right| \times \sup_{t \in [0, T]} |L_t^x - L_t^0| dx \\ &\leq \left(\sup_{(t, x) \in [0, T] \times \mathbb{R}} |x|^{-\theta} |L_t^x - L_t^0| \right) \int_{\mathbb{R}} \mathbf{1}_{\{|x| < \eta\}} |x|^{\theta+1/\alpha-2} dx, \end{aligned}$$

where we have fixed $\theta \in (0, 1/2)$ such that $1/\alpha - 2 + \theta > -1$, which is possible since $\alpha < 2$. We conclude using that $\sup_{(t, x) \in [0, T] \times \mathbb{R}} |x|^{-\theta} |L_t^x - L_t^0| < \infty$ a.s., see (2.8).

From Remark 2.3, and since a.s. for all $t \geq 0$, $L_{\tau_t}^0 = t$, we know that $(S_t^\eta)_{t \geq 0} = (K_{\tau_t}^\eta)_{t \geq 0}$ is a Lévy process for all $\eta > 0$. Hence $(S_t)_{t \geq 0}$ is a Lévy process: for all $t \geq 0$, a.s., $S_t = \lim_{\eta \rightarrow 0} S_t^\eta$ so that for all $t_1, \dots, t_n \geq 0$, a.s., $(S_{t_1}, \dots, S_{t_n}) = \lim_{\eta \rightarrow 0} (S_{t_1}^\eta, \dots, S_{t_n}^\eta)$ and thus $(S_t)_{t \geq 0}$ is the limit of $(S_t^\eta)_{t \geq 0}$ in the finite dimensional distribution sense, which is sufficient to conclude. We now show that it has the appropriate scaling property. By (2.9), we have, for any $c > 0$,

$$\begin{aligned} S_t &= \int_{\mathbb{R}} \operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2} (L_{\tau_t}^x - t) dx \\ &\stackrel{d}{=} \int_{\mathbb{R}} \operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2} (c^{-1} L_{\tau_{ct}}^{cx} - t) dx \\ &= c^{-1/\alpha} \int_{\mathbb{R}} \operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2} (L_{\tau_{ct}}^x - ct) dx, \end{aligned}$$

which equals $c^{-1/\alpha} S_{ct}$.

As in the previous proof, we consider the processes

$$C_t = \int_0^\infty |x|^{1/\alpha-2} (L_{\tau_t}^x - t) dx, \quad D_t = \int_{-\infty}^0 |x|^{1/\alpha-2} (L_{\tau_t}^x - t) dx$$

and $H_t = C_t - D_t$. For a fixed $t \geq 0$, C_t and D_t are independent and identically distributed random variables and it is clear that they only have positive jumps and thus their characteristic function is equal to $\exp(-c'_\alpha t|\xi|^\alpha (1 - i \tan(\frac{\alpha\pi}{2}) \operatorname{sgn}(\xi)))$ for some $c'_\alpha > 0$ by Theorem 2.3. But once again we know from Biane-Yor [BY87] that the characteristic function of H_t is equal to $\exp(-tc_\alpha |\xi|^\alpha)$ so once again it comes that $c'_\alpha = c_\alpha/2$ and we can conclude as in the previous proof. $\square$

Proof of Lemma 2.4. As previously, we can show that $(S_t)_{t \geq 0}$ is a Lévy process by approximation. Then it is enough to show that the characteristic function of S_t has the stated form. To this end, we approach $(S_t)_{t \geq 0}$ by the α -stable processes from Lemma 2.3, when $\alpha \in (1, 2)$. We

denote by $(K_t^\alpha)_{t \geq 0}$ and $(S_t^\alpha)_{t \geq 0}$ the processes from Lemma 2.3. We first show that for all $T > 0$, a.s.,

$$\sup_{t \in [0, T]} \left| K_t^\alpha + \left(\int_{|x| \geq 1} \operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2} dx \right) L_t^0 - K_t \right| \xrightarrow[\alpha \downarrow 1]{} 0. \quad (2.11)$$

Since $\int_{|x| \geq 1} \operatorname{sgn}_{a,b}(x) |x|^{1/\alpha-2} dx = \frac{(a+b)\alpha}{\alpha-1}$ and $1/\alpha - 2 < -1$ for $\alpha \in (1, 2)$, we can write

$$\begin{aligned} K_t^\alpha + \frac{(a+b)\alpha}{\alpha-1} L_t^0 - K_t &= \int_{|x| \leq 1} \operatorname{sgn}_{a,b}(x) \left(|x|^{1/\alpha-2} - |x|^{-1} \right) (L_t^x - L_t^0) dx \\ &\quad + \int_{|x| > 1} \operatorname{sgn}_{a,b}(x) \left(|x|^{1/\alpha-2} - |x|^{-1} \right) L_t^x dx. \end{aligned}$$

Let us introduce $M_t = \sup_{s \in [0, t]} |W_s|$. Then a.s., for any $x > M_t$, $L_t^x = 0$. Set $c = |a| \vee |b|$, for any $T > 0$, a.s.,

$$\begin{aligned} \sup_{t \in [0, T]} \left| K_t^\alpha + \frac{(a+b)\alpha}{\alpha-1} L_t^0 - K_t \right| &\leq c \int_{|x| \leq 1} \left(|x|^{1/\alpha-2} - |x|^{-1} \right) \sup_{t \in [0, T]} |L_t^x - L_t^0| dx \\ &\quad + c \mathbf{1}_{\{M_T \geq 1\}} \int_{1 \leq x \leq M_T} \left(|x|^{-1} - |x|^{1/\alpha-2} \right) L_T^x dx. \end{aligned}$$

Let us call $R_T^{1,\alpha}$ the first term on the right-hand-side of the above inequality and $R_T^{2,\alpha}$ the second. Then we have for any $T > 0$, a.s.,

$$R_T^{1,\alpha} \leq c \left(\sup_{(t,x) \in [0, T] \times [-1, 1]} |x|^{-1/3} |L_t^x - L_t^0| \right) \int_{|x| \leq 1} \left(|x|^{1/\alpha-2} - |x|^{-1} \right) |x|^{1/3} dx.$$

By (2.8) again, the quantity $\sup_{(t,x) \in [0, T] \times [-1, 1]} |x|^{-1/3} |L_t^x - L_t^0|$ is a.s. finite. For any $x \neq 0$, the integrand in the above quantity converges to 0 as α decreases to 1. We can use dominated convergence since for any $\alpha \in [1, 4/3)$, for any $x \in [-1, 1] \setminus \{0\}$, we have $(|x|^{1/\alpha-2} - |x|^{-1})|x|^{1/3} \leq (|x|^{-5/4} - |x|^{-1})|x|^{1/3}$, which is integrable on $(-1, 0)$ and $(0, 1)$. We conclude that $R_T^{1,\alpha}$ converges to 0 almost surely as $\alpha \downarrow 1$. Regarding $R_T^{2,\alpha}$, we have a.s., for any $T > 0$, a.s.

$$\begin{aligned} R_T^{2,\alpha} &\leq c \mathbf{1}_{\{M_T \geq 1\}} \sup_{(t,x) \in [0, T] \times \mathbb{R}} |L_t^x| \times \int_{1 \leq x \leq M_T} \left(|x|^{-1} - |x|^{1/\alpha-2} \right) dx \\ &\leq c \mathbf{1}_{\{M_T \geq 1\}} \sup_{(t,x) \in [0, T] \times \mathbb{R}} |L_t^x| \times \int_{1 \leq x \leq M_T} |x|^{-1} dx. \end{aligned}$$

Again, we conclude by dominated convergence that $R_T^{2,\alpha}$ converges to 0 almost surely as $\alpha \downarrow 1$. All in all, we showed that (2.11) holds.

As a consequence, for all $t \geq 0$ and all $\xi \in \mathbb{R}$,

$$\lim_{\alpha \downarrow 1} \mathbb{E} [\exp(i\xi S_t^\alpha)] e^{i\xi t \frac{(a+b)\alpha}{\alpha-1}} = \mathbb{E} [\exp(i\xi S_t)].$$

But we know from Lemma 2.3 that

$$\mathbb{E} [\exp(i\xi S_t^\alpha)] e^{i\xi t \frac{(a+b)\alpha}{\alpha-1}} = \exp \left(-c_{\alpha,a,b} t |\xi|^\alpha + it \xi \left(c_{\alpha,a,b} \beta_{\alpha,a,b} |\xi|^{\alpha-1} \tan \left(\frac{\alpha\pi}{2} \right) + \frac{(a+b)\alpha}{\alpha-1} \right) \right),$$

where $c_{\alpha,a,b} \beta_{\alpha,a,b} = \frac{2^{\alpha-2}\pi}{\alpha \sin(\alpha\pi/2)} \left(\frac{\alpha^\alpha}{\Gamma(\alpha)} \right)^2 (\operatorname{sgn}(a)|a|^\alpha + \operatorname{sgn}(b)|b|^\alpha) \xrightarrow[\alpha \downarrow 1]{} \frac{\pi}{2}(a+b)$. The function $h_{a,b}(\alpha) = c_{\alpha,a,b} \beta_{\alpha,a,b}$ defined for $\alpha > 0$ is C^1 in a neighborhood of 1 and a careful computation shows that

$$h'_{a,b}(1) = \frac{\pi}{2}(a+b) \left[1 + \log(2) + 2\gamma + \frac{\log|a| + \log|b|}{a+b} \right],$$

where γ is the Euler constant, which appears from the fact that $\Gamma'(1) = -\gamma$. Hence we have $c_{\alpha,a,b}\beta_{\alpha,a,b} \underset{\alpha \downarrow 1}{=} \frac{\pi}{2}(a+b) + h'_{a,b}(1)(\alpha-1) + o(\alpha-1)$. Now using that $|\xi|^{\alpha-1} \underset{\alpha \downarrow 1}{=} 1 + (\alpha-1)\log|\xi| + o(\alpha-1)$ and $\tan\left(\frac{\alpha\pi}{2}\right) \underset{\alpha \downarrow 1}{=} \frac{-2}{\pi(\alpha-1)} + o(1)$, it comes that

$$c_{\alpha,a,b}\beta_{\alpha,a,b}|\xi|^{\alpha-1}\tan\left(\frac{\alpha\pi}{2}\right) + \frac{(a+b)\alpha}{\alpha-1} = (a+b) - \frac{2}{\pi}h'_{a,b}(1) - (a+b)\log|\xi| + o(1),$$

and thus we have

$$\mathbb{E}[\exp(i\xi S_t^\alpha)] e^{i\xi t \frac{\alpha}{\alpha-1}} \underset{\alpha \downarrow 1}{\rightarrow} \exp\left(-c_{a,b}t|\xi|\left(1 + i\beta_{a,b}\frac{2}{\pi}\log(|\xi|)\text{sgn}(\xi)\right) + it\tau_{a,b}\xi\right),$$

where $c_{a,b} = \frac{\pi}{2}(|a| + |b|)$, $\beta_{a,b} = \frac{a+b}{|a|+|b|}$ and $\tau_{a,b} = -(a+b)\left[2\gamma + \log(2) + \frac{\log|a| + \log|b|}{a+b}\right]$. $\square$

2.2.4 Inverting time-changes

We remind some classical convergence results enabling to treat the convergence of the generalized inverses of time changes.

Lemma 2.5. *For all $n \geq 1$, let $(a_t^n)_{t \geq 0}$ from $[0, +\infty)$ to itself be a continuous, increasing and bijective function for all $n \geq 1$ and consider its inverse $(r_t^n)_{t \geq 0}$. Assume $(a_t^n)_{t \geq 0}$ simply converges, as $n \rightarrow \infty$, to a non-decreasing function $(a_t)_{t \geq 0}$ such that $\lim_{t \rightarrow +\infty} a_t = \infty$. Consider $r_t = \inf\{u \geq 0, a_u > t\}$ its right-continuous generalized inverse and $J = \{t \geq 0, r_{t-} < r_t\}$. For all $t \in [0, +\infty) \setminus J$, we have $\lim_{n \rightarrow +\infty} r_t^n = r_t$.*

This lemma is classical and we took the above statement in Fournier-Tardif [FT21, Lemma 8]. Actually, the proof of this lemma is not different from that of the pointwise convergence of inverse distribution functions, used to show *Skorokhod's representation theorem*, see for instance Billingsley [Bil95, Chapter 5, Theorem 25.6].

2.3 Main proofs

From now on, we will always suppose at least that Assumptions 2.1 and 2.2 holds. In Subsection 2.3.1, we characterize the integrability of f and $g'\sigma$. In Subsection 2.3.2, we represent the process $(X_{t/\varepsilon})_{t \geq 0}$ with a time-changed Brownian motion $(W_{\tau_t^\varepsilon})_{t \geq 0}$ which enables us to state that $\int_0^{t/\varepsilon} f(X_s)ds$ is equal in law to $H_{\tau_t^\varepsilon} = \int_0^{\tau_t^\varepsilon} \phi_\varepsilon(W_s)ds$.

In Subsection 2.3.3, we first check that A_t^ε , inverse of τ_t^ε , converges to the local time at 0 of $(W_t)_{t \geq 0}$ so that τ_t^ε converges towards τ_t . We also slightly prepare the proof of the diffusive and critical diffusive regime. Then, as $\varepsilon^{1/\alpha}\ell(1/\varepsilon)\phi_\varepsilon(x) \rightarrow \text{sgn}_{f_+, f_-}(x)|x|^{1/\alpha-2}$, we show that $H_{\tau_t^\varepsilon}$ converges to K_{τ_t} a.s. for fixed a fixed t , which is enough to conclude. We will work a little bit more in the critical Lévy regime.

In Subsection 2.3.4, we adopt the martingale strategy writing $\int_0^{t/\varepsilon} f(X_s)ds$ as a local martingale plus some remainder and we use classical central limit theorems for local martingales to conclude.

2.3.1 Integrability of f and $g'\sigma$

In this subsection, we characterize with the index α the integrability of f and $g'\sigma$ with respect to the measure μ . We first introduce the functions $\psi : \mathbb{R} \rightarrow (0, \infty)$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\psi = (\mathfrak{s}' \circ \mathfrak{s}^{-1}) \times (\sigma \circ \mathfrak{s}^{-1}) \quad \text{and} \quad \phi = (f \circ \mathfrak{s}^{-1})/\psi^2. \quad (2.12)$$

Assumption 2.2 precisely tells us that ϕ is a regular varying with index $1/\alpha - 2$. Indeed, since $\mathfrak{s}$ is an increasing bijection by Assumption 2.1 and Remark 2.1, we have

$$|x|^{2-1/\alpha} \ell(|x|) \phi(x) \xrightarrow{x \rightarrow \pm\infty} f_{\pm}. \quad (2.13)$$

We make the following proposition and let us insist on the fact that from now on, we work under Assumptions 2.1 and 2.2.

Proposition 2.1. *$f \in L^1(\mu)$ is equivalent to $\alpha > 1$ or $\alpha = 1$ and $\int_1^\infty \frac{dx}{x\ell(x)} < \infty$.*

Proof. If $\alpha > 1$, then $2 - 1/\alpha > 1$. Since ℓ is a slowly varying function, we can find $\delta > 0$ with $2 - 1/\alpha - \delta > 1$ such that $|x|^\delta \ell(|x|) \rightarrow \infty$ as $|x| \rightarrow \infty$. Thus we have by (2.13) that

$$|\phi(x)| \underset{x \rightarrow \pm\infty}{=} o(|x|^{\delta+1/\alpha-2})$$

and the function ϕ is integrable with respect to the Lebesgue measure on $\mathbb{R}$. But using the substitution $x = \mathfrak{s}(v)$, we have

$$\int_{\mathbb{R}} |\phi(x)| dx = \int_{\mathbb{R}} \frac{|f(v)|}{\sigma^2(v) \mathfrak{s}'(v)} dv = \frac{1}{\kappa} \int_{\mathbb{R}} |f| d\mu < \infty. \quad (2.14)$$

If $\alpha = 1$ and $\int_1^\infty \frac{dx}{x\ell(x)} < \infty$, then (2.13) tells us that $|x|\ell(|x|)\phi(x) \rightarrow f_{\pm}$, and thus ϕ is integrable with respect to the Lebesgue measure on $\mathbb{R}$ which tells us by (2.14) that f is integrable with respect to the Lebesgue measure.

We can make a similar reasoning to show that $\alpha < 1$ or $\alpha = 1$ and $\int_1^\infty \frac{dx}{x\ell(x)} = \infty$ implies that $f \notin L^1(\mu)$. $\square$

We now focus on the function g defined by (2.5) when $\alpha \geq 2$ and recall that

$$g(x) = 2 \int_0^x \mathfrak{s}'(v) \int_v^\infty f(u) [\sigma^2(u) \mathfrak{s}'(u)]^{-1} du dv = 2\kappa^{-1} \int_0^x \mathfrak{s}'(v) \mu(f \mathbf{1}_{(v,\infty)}) dv.$$

By Assumption 2.2, Proposition 2.1 and since $\alpha \geq 2$, we have $\mu(f) = 0$ and thus $\mu(f \mathbf{1}_{(v,\infty)}) = -\mu(f \mathbf{1}_{(-\infty,v)})$ so that

$$g(x) = -2 \int_0^x \mathfrak{s}'(v) \int_{-\infty}^v f(u) [\sigma^2(u) \mathfrak{s}'(u)]^{-1} du dv. \quad (2.15)$$

We now express the function $g \circ \mathfrak{s}^{-1}$, using the substitutions $v = \mathfrak{s}^{-1}(y)$ and $u = \mathfrak{s}^{-1}(z)$:

$$g(\mathfrak{s}^{-1}(x)) = 2 \int_0^x \int_y^\infty f(\mathfrak{s}^{-1}(z)) [\sigma(\mathfrak{s}^{-1}(z)) \mathfrak{s}'(\mathfrak{s}^{-1}(z))]^{-2} dz dy = 2 \int_0^x \int_y^\infty \phi(z) dz dy. \quad (2.16)$$

Using (2.15), we also get

$$g(\mathfrak{s}^{-1}(x)) = -2 \int_0^x \int_{-\infty}^y \phi(z) dz dy. \quad (2.17)$$

We are now ready to state the proposition characterizing the integrability of $g'\sigma$ with respect to μ .

Proposition 2.2. *If $\alpha > 2$, or $\alpha = 2$ and $\rho = \int_1^\infty \left(\int_x^\infty \frac{dv}{v^{3/2}\ell(v)} \right)^2 dx < \infty$, then $g'\sigma \in L^2(\mu)$.*

Proof. From (2.16) and (2.17), we get that

$$(g \circ \mathfrak{s}^{-1})'(x) = 2 \int_x^\infty \phi(v) dv = -2 \int_{-\infty}^x \phi(v) dv. \quad (2.18)$$

Assume $\alpha > 2$, then $2 - 1/\alpha > 3/2$. From Assumption 2.2 and since ℓ is a slowly varying function, we can find $\delta > 0$ such that $2 - 1/\alpha - \delta > 3/2$ and

$$|\phi(x)| \underset{x \rightarrow \pm\infty}{=} o\left(|x|^{\delta+1/\alpha-2}\right),$$

and thus

$$|(g \circ \mathfrak{s}^{-1})'(x)|^2 \underset{x \rightarrow \pm\infty}{=} o\left(|x|^{2(\delta+1/\alpha-1)}\right).$$

Since $2(\delta+1/\alpha-1) < -1$, $(g \circ \mathfrak{s}^{-1})' \in L^2(dx)$. Now using the fact that $(g \circ \mathfrak{s}^{-1})' = [\mathfrak{s}' \circ \mathfrak{s}^{-1}]^{-1} \times g' \circ \mathfrak{s}^{-1}$ and the substitution $x = \mathfrak{s}(v)$, we have

$$\int_{\mathbb{R}} \frac{|g'(\mathfrak{s}^{-1}(x))|^2}{[\mathfrak{s}'(\mathfrak{s}^{-1}(x))]^2} dx = \int_{\mathbb{R}} \frac{|g'(v)|^2}{\mathfrak{s}'(v)} dv = \frac{1}{\kappa} \int_{\mathbb{R}} |g'\sigma|^2 d\mu,$$

whence the result. Now assume $\alpha = 2$ and $\rho = \int_1^\infty \left(\int_x^\infty \frac{dv}{v^{3/2}\ell(v)} \right)^2 dx < \infty$. Assumption 2.2 tells us that $\phi(x) \underset{x \rightarrow \infty}{\sim} f_{\pm} \frac{1}{|x|^{3/2}\ell(|x|)}$. Integrating and using (2.18), we get

$$(g \circ \mathfrak{s}^{-1})'(x) \underset{x \rightarrow \infty}{\sim} 2f_+ \int_x^\infty \frac{dv}{v^{3/2}\ell(v)}.$$

Hence the fact that $\rho < \infty$ tells us precisely that $(g \circ \mathfrak{s}^{-1})' \in L^2(\mathbb{R}_+, dx)$. Using a similar computation involving (2.18), we find that $(g \circ \mathfrak{s}^{-1})' \in L^2(\mathbb{R}_-, dx)$. We conclude that $g'\sigma \in L^2(\mu)$ as when $\alpha > 2$. $\square$

2.3.2 Scale function and speed measure

In this subsection, we classically represent the solution $(X_t)_{t \geq 0}$ to (2.1) as a function of a time-changed Brownian motion. Roughly, the process $(\mathfrak{s}(X_t))_{t \geq 0}$ is a continuous local martingale and the Dubins-Schwarz theorem tells us $(\mathfrak{s}(X_t))_{t \geq 0}$ is a time-changed Brownian motion. The next lemma enables us to represent $X_{t/\varepsilon}$ and $\int_0^{t/\varepsilon} f(X_s) ds$ through the functions ψ , ϕ , defined by (2.12), and $\mathfrak{s}$, generalizing to our context the identity found in Fournier-Tardif [FT21, Lemma 6].

Lemma 2.6. *Let $\varepsilon > 0$ and $a_\varepsilon > 0$. We consider a Brownian motion $(W_t)_{t \geq 0}$ and we define*

$$A_t^\varepsilon = \varepsilon a_\varepsilon^{-2} \int_0^t \psi^{-2}(W_s/a_\varepsilon) ds$$

and its inverse $(\tau_t^\varepsilon)_{t \geq 0}$, which is a.s. a continuous and strictly increasing bijection from $[0, +\infty)$ in itself. Let us set

$$X_t^\varepsilon = \mathfrak{s}^{-1}\left(W_{\tau_t^\varepsilon}/a_\varepsilon\right) \quad \text{and} \quad F_t^\varepsilon = H_{\tau_t^\varepsilon}^\varepsilon \quad \text{where} \quad H_t^\varepsilon = a_\varepsilon^{-2} \int_0^t \phi(W_s/a_\varepsilon) ds.$$

For $(X_t)_{t \geq 0}$ the unique solution to (2.1), we have

$$\left(\int_0^{t/\varepsilon} f(X_s) ds, X_{t/\varepsilon} \right)_{t \geq 0} \stackrel{d}{=} (F_t^\varepsilon, X_t^\varepsilon)_{t \geq 0}.$$

We introduce the unusual degree of freedom a_ε that will allow us to make converge the time-change A_t^ε without normalization.

Proof. We recall that the function ψ and ϕ are defined by $\psi = (\mathfrak{s}' \circ \mathfrak{s}^{-1}) \times (\sigma \circ \mathfrak{s}^{-1})$ and $\phi = (f \circ \mathfrak{s}^{-1})/\psi^2$. Let us consider the function $\psi_\varepsilon(w) = \varepsilon^{-1/2} a_\varepsilon \psi(w/a_\varepsilon)$, so that $A_t^\varepsilon = \int_0^t \psi_\varepsilon^{-2}(W_s) ds$. We set $Y_t^\varepsilon = W_{\tau_t^\varepsilon}$ which classically solves, see [Kal02, page 452],

$$Y_t^\varepsilon = \int_0^t \psi_\varepsilon(Y_s^\varepsilon) dB_s^\varepsilon$$

where $(B_t^\varepsilon)_{t \geq 0}$ is a Brownian motion. Next, we consider the function $\varphi_\varepsilon(y) = \mathfrak{s}^{-1}(y/a_\varepsilon)$. We can apply Itô's formula to $X_t^\varepsilon = \varphi_\varepsilon(Y_t^\varepsilon)$ since $\mathfrak{s}^{-1}$ is C^2 and we get

$$X_t^\varepsilon = \int_0^t \varphi_\varepsilon'(Y_s^\varepsilon) \psi_\varepsilon(Y_s^\varepsilon) dB_s^\varepsilon + \frac{1}{2} \int_0^t \varphi_\varepsilon''(Y_s^\varepsilon) \psi_\varepsilon^2(Y_s^\varepsilon) ds.$$

The functions are such that:

$$\varphi_\varepsilon'(y) \psi_\varepsilon(y) = \frac{1}{a_\varepsilon \mathfrak{s}'(\mathfrak{s}^{-1}(y/a_\varepsilon))} \times \frac{a_\varepsilon \psi(y/a_\varepsilon)}{\varepsilon^{1/2}} = \frac{\sigma(\mathfrak{s}^{-1}(y/a_\varepsilon))}{\varepsilon^{1/2}} = \frac{\sigma(\varphi_\varepsilon(y))}{\varepsilon^{1/2}}$$

and,

$$\varphi_\varepsilon''(y) \psi_\varepsilon^2(y) = \frac{-1}{a_\varepsilon^2} \frac{\mathfrak{s}''(\mathfrak{s}^{-1}(y/a_\varepsilon))}{[\mathfrak{s}'(\mathfrak{s}^{-1}(y/a_\varepsilon))]^3} \times \frac{a_\varepsilon^2 \psi^2(y/a_\varepsilon)}{\varepsilon} = \frac{2b(\mathfrak{s}^{-1}(y/a_\varepsilon))}{\varepsilon} = \frac{2b(\varphi_\varepsilon(y))}{\varepsilon}.$$

Indeed remember that $\mathfrak{s}$ satisfies $\frac{1}{2} \mathfrak{s}'' \sigma^2 + \mathfrak{s}' b = 0$. Finally we have

$$X_t^\varepsilon = \varepsilon^{-1} \int_0^t b(X_s^\varepsilon) ds + \varepsilon^{-1/2} \int_0^t \sigma(X_s^\varepsilon) dB_s^\varepsilon$$

Now taking equation (2.1), we write

$$X_{t/\varepsilon} = \int_0^{t/\varepsilon} b(X_s) ds + \int_0^{t/\varepsilon} \sigma(X_s) dB_s = \varepsilon^{-1} \int_0^t b(X_{s/\varepsilon}) ds + \varepsilon^{-1/2} \int_0^t \sigma(X_{s/\varepsilon}) d\widehat{B}_s^\varepsilon,$$

where $(\widehat{B}_t^\varepsilon)_{t \geq 0} = (\varepsilon^{1/2} B_{t/\varepsilon})_{t \geq 0}$ is a Brownian motion. Thus, $\forall \varepsilon > 0$, the processes $(X_{t/\varepsilon})_{t \geq 0}$ and $(X_t^\varepsilon)_{t \geq 0}$ are solutions of the same SDE, for which we have uniqueness in law, driven by different Brownian motion, $(\widehat{B}_t^\varepsilon)_{t \geq 0}$ and $(B_t^\varepsilon)_{t \geq 0}$. Thus they are equal in law : $(X_{t/\varepsilon})_{t \geq 0} \stackrel{d}{=} (X_t^\varepsilon)_{t \geq 0}$. As $\int_0^{t/\varepsilon} f(X_s) ds = \varepsilon^{-1} \int_0^t f(X_{s/\varepsilon}) ds$, we have

$$\left(\int_0^{t/\varepsilon} f(X_s) ds, X_{t/\varepsilon} \right)_{t \geq 0} \stackrel{d}{=} \left(\varepsilon^{-1} \int_0^t f(X_{s/\varepsilon}) ds, X_t^\varepsilon \right)_{t \geq 0}.$$

Using the substitution $u = \tau_s^\varepsilon \Leftrightarrow s = A_u^\varepsilon$, we have $ds = \psi_\varepsilon^{-2}(W_u) du$, and

$$\varepsilon^{-1} \int_0^t f(X_{s/\varepsilon}) ds = \varepsilon^{-1} \int_0^t f \circ \varphi_\varepsilon(W_{\tau_s^\varepsilon}) ds = \varepsilon^{-1} \int_0^{\tau_t^\varepsilon} \frac{f \circ \varphi_\varepsilon(W_u)}{\psi_\varepsilon^2(W_u)} du = a_\varepsilon^{-2} \int_0^{\tau_t^\varepsilon} \phi(W_u/a_\varepsilon) du = F_t^\varepsilon$$

which ends the proof. $\square$

We end this section with the following proposition that we will use several times later on.

Proposition 2.3. *Let $\varepsilon > 0$ and $a_\varepsilon > 0$ such that $a_\varepsilon \rightarrow 0$ when $\varepsilon \rightarrow 0$. Let also $(W_t)_{t \geq 0}$ be a Brownian motion and $(L_t^0)_{t \geq 0}$ its local time at 0. For every $\varphi \in L^1(\mu)$, we have a.s., for all $T > 0$,*

$$\sup_{t \in [0, T]} \left| \kappa a_\varepsilon^{-1} \int_0^t \frac{\varphi \circ \mathfrak{s}^{-1}(W_s/a_\varepsilon)}{\psi^2(W_s/a_\varepsilon)} ds - \mu(\varphi) L_t^0 \right| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Proof. Denote $Y_t^\varepsilon = \kappa a_\varepsilon^{-1} \int_0^t \frac{\varphi \circ \mathfrak{s}^{-1}(W_s/a_\varepsilon)}{\psi^2(W_s/a_\varepsilon)} ds$. Using the occupation times formula, we get

$$Y_t^\varepsilon = \kappa a_\varepsilon^{-1} \int_{\mathbb{R}} \frac{\varphi \circ \mathfrak{s}^{-1}(x/a_\varepsilon)}{\psi^2(x/a_\varepsilon)} L_t^x dx = \kappa \int_{\mathbb{R}} \frac{\varphi(v) \mathfrak{s}'(v)}{\psi^2(\mathfrak{s}(v))} L_t^{a_\varepsilon \mathfrak{s}(v)} dv = \int_{\mathbb{R}} \varphi(v) L_t^{a_\varepsilon \mathfrak{s}(v)} \mu(dv),$$

where we used the substitution $x = a_\varepsilon \mathfrak{s}(v)$. We recall that $\mu(dx) = \kappa [\sigma^2(x) \mathfrak{s}'(x)]^{-1} dx$ and that $\psi = (\mathfrak{s}' \circ \mathfrak{s}^{-1}) \times (\sigma \circ \mathfrak{s}^{-1})$. Hence we have

$$\sup_{t \in [0, T]} |Y_t^\varepsilon - \mu(\varphi) L_t^0| \leq \int_{\mathbb{R}} |\varphi(v)| \sup_{t \in [0, T]} |L_t^{a_\varepsilon \mathfrak{s}(v)} - L_t^0| \mu(dv).$$

But for all $v \in \mathbb{R}$, $\sup_{t \in [0, T]} |L_t^{a_\varepsilon \mathfrak{s}(v)} - L_t^0| \xrightarrow{\varepsilon \rightarrow 0} 0$ a.s. (see [RY99] Corollary 1.8, page 226), and $\sup_{t \in [0, T]} |L_t^{a_\varepsilon \mathfrak{s}(v)} - L_t^0| \leq 2 \sup_{(t, x) \in [0, T] \times \mathbb{R}} L_t^x < \infty$ a.s.. We can conclude by dominated convergence since $\varphi \in L^1(\mu)$ by assumption. $\square$

2.3.3 The Lévy regime

In this section, which generalizes [FT21, Lemma 9], we first show two lemmas, that will enable us to conclude. The first lemma says that if in Lemma 2.6 we choose carefully a_ε , the time change A_t^ε converges to the local time of the Brownian motion at 0. It also recovers a weak version of the ergodic theorem and prepares some useful results for the diffusive case.

Lemma 2.7. *Let $(W_t)_{t \geq 0}$ be a Brownian motion and $(L_t^0)_{t \geq 0}$ its local time at 0. For all $\varepsilon > 0$, we consider the processes $(A_t^\varepsilon)_{t \geq 0}$, $(\tau_t^\varepsilon)_{t \geq 0}$ and $(X_t^\varepsilon)_{t \geq 0}$ from Lemma 2.6 with the choice $a_\varepsilon = \varepsilon/\kappa$.*

(i) *We have a.s., for all $T > 0$,*

$$\sup_{t \in [0, T]} |A_t^\varepsilon - L_t^0| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

(ii) *For all $t > 0$, a.s., $\tau_t^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \tau_t$, the generalized inverse of $(L_t^0)_{t \geq 0}$.*

(iii) *Suppose Assumption 2.3. For any slowly varying function γ , we have a.s., for all $T > 0$,*

$$\sup_{t \in [0, T]} \gamma(1/\varepsilon) |A_t^\varepsilon - L_t^0| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

(iv) *If $\alpha = 2$ and $\rho = \int_1^\infty \left(\int_x^\infty \frac{dv}{v^{3/2} \ell(v)} \right)^2 dx = \infty$, we have a.s., for all $T > 0$,*

$$\sup_{t \in [0, T]} |T_t^\varepsilon - \sigma_2^2 L_t^0| \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{where} \quad T_t^\varepsilon = \kappa^2 |\varepsilon \rho_\varepsilon|^{-1} \int_0^t \chi(W_s/a_\varepsilon) ds,$$

where $\chi = [(g \circ \mathfrak{s}^{-1})']^2$ with g defined by (2.5) and $\rho_\varepsilon = \int_1^{1/\varepsilon} \left(\int_x^\infty \frac{dv}{v^{3/2} \ell(v)} \right)^2 dx$.

(v) *For all $\varphi \in L^1(\mu)$, for all $t > 0$, we have a.s.*

$$\int_0^t \varphi(X_s^\varepsilon) ds \xrightarrow{\varepsilon \rightarrow 0} \mu(\varphi) t.$$

Proof. **Item (i).** The first point is just the application of Proposition 2.3 with $\varphi = 1$.

Item (ii). The second point follows immediately from the first and Lemma 2.5, since for all $t \geq 0$, as recalled in Subsection 2.2.2, $\mathbb{P}(\tau_{t-} < \tau_t) = 0$.

Item (iii). Using the same scheme as in the proof of Proposition 2.3, we have

$$\sup_{t \in [0, T]} \gamma(1/\varepsilon) |A_t^\varepsilon - L_t^0| = \sup_{t \in [0, T]} \left| \kappa \gamma(1/\varepsilon) \int_{\mathbb{R}} \frac{L_t^{\varepsilon y/\kappa} - L_t^0}{\psi^2(y)} dy \right| \leq \int_{\mathbb{R}} \gamma(1/\varepsilon) \sup_{t \in [0, T]} |L_t^{\varepsilon \mathfrak{s}(v)/\kappa} - L_t^0| \mu(dv).$$

Now we use (2.8) with some $\theta \in (0, \frac{1}{2} \wedge \lambda)$, where λ refers to the constant of Assumption 2.3, so that $|\mathfrak{s}|^\lambda \in L^1(\mu)$. We thus have $\gamma(1/\varepsilon) \sup_{t \in [0, T]} |L_t^{\varepsilon \mathfrak{s}(v)/\kappa} - L_t^0| \leq C |\mathfrak{s}(v)|^\theta \gamma(1/\varepsilon) \varepsilon^\theta$, for some random C . Since γ is slowly varying, we conclude that $\gamma(1/\varepsilon) \sup_{t \in [0, T]} |L_t^{\varepsilon \mathfrak{s}(v)/\kappa} - L_t^0| \rightarrow 0$ as $\varepsilon \rightarrow 0$, for each fixed v . Moreover

$$\sup_{\varepsilon \in (0, 1)} \gamma(1/\varepsilon) \sup_{t \in [0, T]} |L_t^{\varepsilon \mathfrak{s}(v)/\kappa} - L_t^0| \leq C \left(\sup_{\varepsilon \in (0, 1)} \gamma(1/\varepsilon) \varepsilon^\theta \right) |\mathfrak{s}(v)|^\theta$$

and we can conclude by dominated convergence since $|\mathfrak{s}|^\theta \in L^1(\mu)$.

Item (iv). *Step 0 :* We first estimate asymptotically the function χ . Let us recall from (2.18) that

$$(g \circ \mathfrak{s}^{-1})'(x) = 2 \int_x^\infty \phi(v) dv = -2 \int_{-\infty}^x \phi(v) dv.$$

Let us define the function h on $(0, \infty)$:

$$h(x) = \left(\int_x^\infty \frac{dv}{v^{3/2} \ell(v)} \right)^2.$$

By (2.13), we have $|x|^{3/2} \ell(|x|) \phi(x) \xrightarrow{x \rightarrow \pm\infty} f_\pm$ and thus

$$\left(\int_x^\infty \frac{dv}{v^{3/2} \ell(v)} \right)^{-1} \int_x^\infty \phi(v) dv \xrightarrow{x \rightarrow \infty} f_+$$

and

$$\left(\int_{|x|}^\infty \frac{dv}{v^{3/2} \ell(v)} \right)^{-1} \int_{-\infty}^x \phi(v) dv \xrightarrow{x \rightarrow -\infty} f_-.$$

All in all, recalling that $\chi = [(g \circ \mathfrak{s}^{-1})']^2$, we have $[h(|x|)]^{-1} \chi(x) \xrightarrow{x \rightarrow \pm\infty} 4f_\pm^2$ from which

$$\int_{-x}^x \chi(z) dz \underset{x \rightarrow \infty}{\sim} 4(f_+^2 + f_-^2) \int_1^x h(v) dv. \quad (2.19)$$

We used that $\rho = \int_1^\infty h(v) dv = \infty$ by assumption. Moreover there exists $A > 0$ such that for all x , we have $\chi(x) \leq Ah(|x|)$. Now we write for $\delta > 0$ fixed, $T_t^\varepsilon = D_t^{\varepsilon, \delta} + E_t^{\varepsilon, \delta}$ where

$$D_t^{\varepsilon, \delta} = \int_0^t \kappa^2 |\varepsilon \rho_\varepsilon|^{-1} \chi(\kappa W_s / \varepsilon) \mathbf{1}_{\{|W_s| > \delta\}} ds \quad \text{and} \quad E_t^{\varepsilon, \delta} = \int_0^t \kappa^2 |\varepsilon \rho_\varepsilon|^{-1} \chi(\kappa W_s / \varepsilon) \mathbf{1}_{\{|W_s| \leq \delta\}} ds.$$

Step 1 : We show that $\sup_{t \in [0, T]} |D_t^{\varepsilon, \delta}| \rightarrow 0$ as $\varepsilon \rightarrow 0$. We have $\chi(\kappa W_s/\varepsilon) \leq Ah(\kappa|W_s|/\varepsilon)$. Now notice that in this case, we have a.s.,

$$\begin{aligned} \mathbf{1}_{\{|W_s| > \delta\}} h(\kappa|W_s|/\varepsilon) &= \mathbf{1}_{\{|W_s| > \delta\}} \left(\int_{\kappa|W_s|/\varepsilon}^{\infty} \frac{du}{u^{3/2}\ell(u)} \right)^2 \\ &= \mathbf{1}_{\{|W_s| > \delta\}} \frac{\varepsilon}{\kappa|W_s|} \left(\int_1^{\infty} \frac{dv}{v^{3/2}\ell(\kappa|W_s|v/\varepsilon)} \right)^2 \\ &\leq \mathbf{1}_{\{|W_s| > \delta\}} \frac{\varepsilon}{\kappa\delta} \left(\int_1^{\infty} \frac{dv}{v^{3/2}\ell(\kappa|W_s|v/\varepsilon)} \right)^2. \end{aligned}$$

We know from Lemma 2.1-(i) (with $a = \delta$ and $b = \sup_{s \in [0, T]} |W_s|$) that the following uniform convergence holds a.s.

$$\sup_{s \in [0, T]} \mathbf{1}_{\{|W_s| > \delta\}} \left| \frac{\ell(v/\varepsilon)}{\ell(\kappa|W_s|v/\varepsilon)} - 1 \right| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Hence there exists a random $M > 0$ such that for any $s \in [0, T]$ and $\varepsilon \in (0, 1)$, $|W_s| > \delta$ implies that $1/\ell(\kappa|W_s|v/\varepsilon) \leq M/\ell(v/\varepsilon)$. Consequently,

$$\sup_{s \in [0, T]} \mathbf{1}_{\{|W_s| > \delta\}} |\chi(\kappa W_s/\varepsilon)| \leq \sup_{s \in [0, T]} \mathbf{1}_{\{|W_s| > \delta\}} Ah(\kappa|W_s|/\varepsilon) \leq \frac{\varepsilon AM^2}{\kappa\delta} \left(\int_1^{\infty} \frac{dv}{v^{3/2}\ell(v/\varepsilon)} \right)^2.$$

We know from Potter's bound (Lemma 2.1-(ii)) that there exists $x_0 > 1$ such that for all $x, y \geq x_0$,

$$\frac{\ell(y)}{\ell(x)} \leq 2 \left(\left| \frac{y}{x} \right|^{1/4} \vee \left| \frac{x}{y} \right|^{1/4} \right).$$

Hence, if $\varepsilon \in (0, 1/x_0)$, we have

$$\left(\int_1^{\infty} \frac{dv}{v^{3/2}\ell(v/\varepsilon)} \right)^2 \leq \frac{4}{\ell^2(1/\varepsilon)} \left(\int_1^{\infty} \frac{|v|^{1/4} \vee |v|^{-1/4} dv}{v^{3/2}} \right)^2.$$

All in all, there exists a random constant that we name M again, such that

$$\sup_{t \in [0, T]} |D_t^{\varepsilon, \delta}| \leq \frac{MT}{\delta \ell^2(1/\varepsilon) \rho_\varepsilon}.$$

It holds that $\ell^2(1/\varepsilon) \rho_\varepsilon \rightarrow \infty$, see the end of the proof of Lemma 2.1-(iii) and use that $\rho_\varepsilon = M(1/\varepsilon)$, and thus for each $\delta > 0$, $\sup_{t \in [0, T]} |D_t^{\varepsilon, \delta}| \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Step 2 : We concentrate on $E_t^{\varepsilon, \delta}$. Using the occupation times formula, we have

$$E_t^{\varepsilon, \delta} = \left(\int_{-\delta}^{\delta} \frac{\kappa^2 \chi(\kappa x/\varepsilon) dx}{|\varepsilon \rho_\varepsilon|} \right) L_t^0 + \int_{-\delta}^{\delta} \frac{\kappa^2 \chi(\kappa x/\varepsilon) (L_t^x - L_t^0) dx}{|\varepsilon \rho_\varepsilon|} = r_{\varepsilon, \delta} L_t^0 + R_t^{\varepsilon, \delta},$$

the last equality standing for a definition. We have, using a substitution and (2.19),

$$r_{\varepsilon, \delta} = \frac{\kappa}{\rho_\varepsilon} \int_{-\kappa\delta/\varepsilon}^{\kappa\delta/\varepsilon} \chi(z) dz \underset{\varepsilon \rightarrow 0}{\sim} \frac{4\kappa(f_+^2 + f_-^2)}{\rho_\varepsilon} \int_1^{\kappa\delta/\varepsilon} h(x) dx \underset{\varepsilon \rightarrow 0}{\sim} 4\kappa(f_+^2 + f_-^2) = \sigma_2^2,$$

where we used that $\rho_\varepsilon = \int_1^{1/\varepsilon} h(x) dx$ and that the function $M(x) = \int_1^x h(v) dv$ is slowly varying by Lemma 2.1-(iii).

Step 3 : We conclude. Recalling that $T_t^\varepsilon = r_{\varepsilon,\delta} L_t^0 + R_t^{\varepsilon,\delta} + D_t^{\varepsilon,\delta}$, we proved that a.s., for all $T > 0$ and all $\delta > 0$,

$$\limsup_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} |T_t^\varepsilon - \sigma_2^2 L_t^0| \leq \limsup_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} |R_t^{\varepsilon,\delta}|.$$

But

$$\sup_{t \in [0, T]} |R_t^{\varepsilon,\delta}| \leq r_{\varepsilon,\delta} \sup_{(t,x) \in [0, T] \times [-\delta, \delta]} |L_t^x - L_t^0|,$$

which implies

$$\limsup_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} |T_t^\varepsilon - \sigma_2^2 L_t^0| \leq \sigma_2^2 \sup_{(t,x) \in [0, T] \times [-\delta, \delta]} |L_t^x - L_t^0|.$$

Now, we let δ to 0 which completes the proof.

Item (v). Using the definition of X_t^ε , and making the substitution $u = \tau_s^\varepsilon \Leftrightarrow s = A_u^\varepsilon$, we have

$$\int_0^t \varphi(X_s^\varepsilon) ds = \varepsilon a_\varepsilon^{-2} \int_0^{\tau_t^\varepsilon} \frac{\varphi \circ \mathfrak{s}^{-1}(W_s/a_\varepsilon)}{\psi^2(W_s/a_\varepsilon)} ds = \kappa a_\varepsilon^{-1} \int_0^{\tau_t^\varepsilon} \frac{\varphi \circ \mathfrak{s}^{-1}(W_s/a_\varepsilon)}{\psi^2(W_s/a_\varepsilon)} ds.$$

Let $T = \sup_{\varepsilon \in (0,1)} \tau_t^\varepsilon$ which is a.s. finite from the second point. We have

$$\begin{aligned} \left| \int_0^t \varphi(X_s^\varepsilon) ds - \mu(\varphi)t \right| &\leq \sup_{u \in [0, T]} \left| \kappa a_\varepsilon^{-1} \int_0^u \frac{\varphi \circ \mathfrak{s}^{-1}(W_s/a_\varepsilon)}{\psi^2(W_s/a_\varepsilon)} ds - \mu(\varphi)L_u^0 \right| \\ &\quad + \mu(\varphi) \left| L_{\tau_t^\varepsilon}^0 - t \right|. \end{aligned}$$

The first term on the right hand side goes to 0 by Proposition 2.3 and the second goes to 0 since $\tau_t^\varepsilon \rightarrow \tau_t$ by Point 2, $L_{\tau_t}^0 = t$ a.s., and $(L_t^0)_{t \geq 0}$ is continuous. $\square$

Let us state two immediate consequences of the previous lemma.

Remark 2.4. Point (5) of the preceding Lemma recovers a weak version of the ergodic theorem for the process $(X_t)_{t \geq 0}$. Although it is straightforward since the process is Harris recurrent, this provides a self-contained proof of this weak version, which we only need. For all $\varphi \in L^1(\mu)$ and all $t \geq 0$, $\varepsilon \int_0^{t/\varepsilon} \varphi(X_s) ds \stackrel{d}{=} \int_0^t \varphi(X_s^\varepsilon) ds$, hence for all $t \geq 0$

$$\varepsilon \int_0^{t/\varepsilon} \varphi(X_s) ds \xrightarrow{\mathbb{P}} \mu(\varphi)t.$$

Remark 2.5. Point (4) prepares the critical diffusive regime. Using arguments similar to those in the proof of point (5), we easily see that if $\alpha = 2$ and $\rho = \infty$, then for every $t \geq 0$, a.s., $T_{\tau_t^\varepsilon}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \sigma_2^2 t$. Now one can also see that as $|\varepsilon/\rho_\varepsilon| \int_0^{t/\varepsilon} [g'(X_s)\sigma(X_s)]^2 ds \stackrel{d}{=} T_{\tau_t^\varepsilon}^\varepsilon$, we have

$$|\varepsilon/\rho_\varepsilon| \int_0^{t/\varepsilon} [g'(X_s)\sigma(X_s)]^2 ds \xrightarrow{\mathbb{P}} \sigma_2^2 t.$$

The next result is the last preliminary before proving the main result in the Lévy case.

Lemma 2.8. Let $(W_t)_{t \geq 0}$ be a Brownian motion, and $(L_t^0)_{t \geq 0}$ its local time in 0. Let also $(K_t)_{t \geq 0}$ be the process defined in section 2.2.3, with $(a, b) = (f_+, f_-)$. For all $\varepsilon > 0$, we consider the process $(H_t^\varepsilon)_{t \geq 0}$ from Lemma 2.6 with the choice $a_\varepsilon = \varepsilon/\kappa$.

(i) If $\alpha \neq 1$, then a.s., for all $T > 0$,

$$\sup_{t \in [0, T]} \left| \varepsilon^{1/\alpha} \ell(1/\varepsilon) H_t^\varepsilon - \kappa^{1/\alpha} K_t \right| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

(ii) If $\alpha = 1$, there exists $(\xi_\varepsilon)_{\varepsilon > 0}$ such that, a.s., for all $T > 0$,

$$\sup_{t \in [0, T]} \left| \varepsilon \ell(1/\varepsilon) H_t^\varepsilon - \xi_\varepsilon L_t^0 - \kappa K_t \right| \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where $\xi_\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} -\kappa(f_+ + f_-)\ell(1/\varepsilon) \int_{1/\varepsilon}^\infty \frac{dx}{x\ell(x)}$ if $f \in L^1(\mu)$ and $\xi_\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} \kappa(f_+ + f_-)\ell(1/\varepsilon) \int_1^{1/\varepsilon} \frac{dx}{x\ell(x)}$ otherwise.

Proof. Technical estimates. We first show some technical estimates that we will use in this proof.

(i) For all $x \in \mathbb{R}^*$, it holds (and this is the case for all $\alpha \in (0, 2)$) that

$$\kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \kappa^{1/\alpha} \text{sgn}_{a,b}(x) |x|^{1/\alpha-2}. \quad (2.20)$$

Indeed we have by (2.13), since ℓ is slowly varying and since $(a, b) = (f_+, f_-)$,

$$\forall x > 0, \quad \kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \kappa^{1/\alpha} a x^{1/\alpha-2},$$

$$\forall x < 0, \quad \kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \kappa^{1/\alpha} b |x|^{1/\alpha-2}.$$

(ii) For all $\delta > 0$, there exist $A > 0$ and $\varepsilon_0 > 0$ (and this is the case for all $\alpha \in (0, 2)$) such that for all $x \in \mathbb{R}$ and all $\varepsilon \in (0, \varepsilon_0)$,

$$\varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) \leq A(|x|^{1/\alpha-2} + 1)(|x|^\delta + |x|^{-\delta}) \quad (2.21)$$

Indeed, we start using (2.13) again and that ϕ and ℓ are locally bounded: there exists $A > 0$ such that for all $x \in \mathbb{R}$, $\ell(|x|)|\phi(x)| \leq A(|x|^{1/\alpha-2} + 1)$. Hence we have

$$\varepsilon^{1/\alpha-2} \ell(1/\varepsilon) |\phi(\kappa x/\varepsilon)| \leq A(|x|^{1/\alpha-2} + 1) \frac{\ell(1/\varepsilon)}{\ell(\kappa|x|/\varepsilon)},$$

the value of A being allowed to change. We know from Potter's bound (Lemma 2.1-(ii)) that there exists x_0 such that for all $x, y \geq x_0$,

$$\frac{\ell(y)}{\ell(x)} \leq 2 \left(\left| \frac{y}{x} \right|^\delta \vee \left| \frac{x}{y} \right|^\delta \right).$$

Thus, for $\varepsilon \in (0, 1/x_0)$ we have

$$\varepsilon^{1/\alpha-2} \ell(1/\varepsilon) |\phi(\kappa x/\varepsilon)| \mathbf{1}_{\{\kappa|x| \geq \varepsilon x_0\}} \leq A(|x|^{1/\alpha-2} + 1)(|x|^\delta \vee |x|^{-\delta}).$$

The function ℓ is strictly positive and bounded from below on every compact set and thus there exists $\ell_0 > 0$ such that for all $x \in [0, x_0]$, $\ell(x) \geq \ell_0$. Now if $\kappa|x| < \varepsilon x_0$, we have $\ell(\kappa|x|/\varepsilon) \geq \ell_0$ and $1 \leq (\kappa|x|/\varepsilon x_0)^{-\delta}$. Hence

$$\varepsilon^{1/\alpha-2} \ell(1/\varepsilon) |\phi(\kappa x/\varepsilon)| \mathbf{1}_{\{\kappa|x| < \varepsilon x_0\}} \leq A(|x|^{1/\alpha-2} + 1) |x|^{-\delta} \sup_{\varepsilon \in (0, 1)} \varepsilon^\delta \ell(1/\varepsilon).$$

We proved (2.21).

Item (i) when $\alpha \in (0, 1)$. Let us remind that $K_t = \int_0^t \text{sgn}_{a,b}(W_s) |W_s|^{1/\alpha-2} ds$. We have $\varepsilon^{1/\alpha} \ell(1/\varepsilon) H_t^\varepsilon = \kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \int_0^t \phi(\kappa W_s/\varepsilon) ds$ and

$$\sup_{t \in [0, T]} \left| \varepsilon^{1/\alpha} \ell(1/\varepsilon) H_t^\varepsilon - \kappa^{1/\alpha} K_t \right| \leq \int_0^T \left| \kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \phi(\kappa W_s/\varepsilon) - \kappa^{1/\alpha} \text{sgn}_{a,b}(W_s) |W_s|^{1/\alpha-2} \right| ds$$

which tends to 0 a.s. by dominated convergence. Indeed, since $\{t \in [0, T], W_t = 0\}$ is a.s. Lebesgue-null, we have by (2.20) that a.s., for a.e. $s \in [0, T]$,

$$|\kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \phi(\kappa W_s/\varepsilon) - \kappa^{1/\alpha} \text{sgn}_{a,b}(W_s) |W_s|^{1/\alpha-2}| \rightarrow 0$$

Next we can dominate $\varepsilon^{1/\alpha-2} \ell(1/\varepsilon) |\phi(\kappa W_s/\varepsilon)|$ using (2.21) by

$$A(|W_s|^{1/\alpha-2} + 1)(|W_s|^\delta + |W_s|^{-\delta})$$

which is a.s. integrable on $[0, T]$ if $\delta > 0$ is small enough so that $1/\alpha - 2 - \delta > -1$.

Item (i) when $\alpha \in (1, 2)$. We notice that $\mu(f) = \kappa \int_{\mathbb{R}} \phi(z) dz$, which equals 0 by Assumption 2.2. Indeed,

$$\int_{\mathbb{R}} \phi(z) dz = \int_{\mathbb{R}} \frac{f(\mathbf{s}^{-1}(z))}{\sigma^2(\mathbf{s}^{-1}(z)) \mathbf{s}'(\mathbf{s}^{-1}(z))^2} dz = \int_{\mathbb{R}} \frac{f(v)}{\sigma^2(v) \mathbf{s}'(v)} dv = \frac{\mu(f)}{\kappa}. \quad (2.22)$$

We can now use the occupation times formula to write

$$\begin{aligned} \varepsilon^{1/\alpha} \ell(1/\varepsilon) H_t^\varepsilon &= \kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \int_0^t \phi(\kappa W_s/\varepsilon) ds \\ &= \int_{\mathbb{R}} \kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) L_t^x dx \\ &= \int_{\mathbb{R}} \kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) (L_t^x - L_t^0) dx, \end{aligned}$$

since $\int_{\mathbb{R}} \phi(z) dz = 0$ as noted above. Thus, recalling that $K_t = \int_{\mathbb{R}} \text{sgn}_{a,b}(x) |x|^{1/\alpha-2} (L_t^x - L_t^0) dx$ in this case, we have

$$\begin{aligned} &\sup_{t \in [0, T]} \left| \varepsilon^{1/\alpha} \ell(1/\varepsilon) H_t^\varepsilon - \kappa^{1/\alpha} K_t \right| \\ &\leq \int_{\mathbb{R}} \left| \kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) - \kappa^{1/\alpha} \text{sgn}_{a,b}(x) |x|^{1/\alpha-2} \right| \sup_{t \in [0, T]} |L_t^x - L_t^0| dx. \end{aligned}$$

We conclude by dominated convergence. First, by (2.20), the integrand converges to 0 for each $x \in \mathbb{R}^*$.

We next use (2.21) to dominate $R_\varepsilon(x) = |\kappa^2 \varepsilon^{1/\alpha-2} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon)| \sup_{t \in [0, T]} |L_t^x - L_t^0|$ by

$$A(|x|^{1/\alpha-2} + 1)(|x|^\delta + |x|^{-\delta}) \sup_{t \in [0, T]} |L_t^x - L_t^0| \leq M(|x|^{1/\alpha-2} + 1)(|x|^\delta + |x|^{-\delta}) \min(1, |x|^\theta),$$

for some random constant $M > 0$ and $\theta \in (0, 1/2)$ fixed. We used (2.8) and the fact that $\sup_{(t,x) \in [0, T] \times \mathbb{R}} |L_t^x - L_t^0| < \infty$ a.s. This bound is integrable on $\mathbb{R}$ if we choose $\delta > 0$ and $\theta \in (0, 1/2)$ such that $1/\alpha - 2 + \delta < -1$ and $1/\alpha - 2 + \theta - \delta > -1$. This is possible because $-3/2 < 1/\alpha - 2 < -1$.

Item (ii): Convergence. Setting $\xi_\varepsilon = \kappa^2 \varepsilon^{-1} \ell(1/\varepsilon) \int_{-1}^1 \phi(\kappa x/\varepsilon) dx$ and using the occupation times formula, we have

$$\varepsilon \ell(1/\varepsilon) H_t^\varepsilon - \xi_\varepsilon L_t^0 = \int_{|x| \geq 1} \kappa^2 \varepsilon^{-1} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) L_t^x dx + \int_{|x| \leq 1} \kappa^2 \varepsilon^{-1} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) (L_t^x - L_t^0) dx$$

Then, remembering that $K_t = \int_{\mathbb{R}} \operatorname{sgn}_{a,b}(x) |x|^{-1} (L_t^x - L_t^0 \mathbf{1}_{\{|x| \leq 1\}}) dx$, we write

$$\begin{aligned} & \sup_{t \in [0, T]} |\varepsilon H_t^\varepsilon - \xi_\varepsilon L_t^0 - \kappa K_t| \\ & \leq \int_{|x| \leq 1} \left| \kappa^2 \varepsilon^{-1} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) - \kappa \operatorname{sgn}_{a,b}(x) |x|^{-1} \right| \sup_{t \in [0, T]} |L_t^x - L_t^0| dx \\ & \quad + \int_{|x| > 1} \left| \kappa^2 \varepsilon^{-1} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon) - \kappa \operatorname{sgn}_{a,b}(x) |x|^{-1} \right| L_T^x dx, \end{aligned}$$

Again, we conclude by dominated convergence. By (2.20), the integrand in both terms converges to 0 for each $x \in \mathbb{R}^*$.

We first use (2.21) to dominate $R_\varepsilon(x) = |\kappa^2 \varepsilon^{-1} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon)| \sup_{t \in [0, T]} |L_t^x - L_t^0|$, for all $|x| \leq 1$, by

$$M(|x|^{-1} + 1)(|x|^\delta + |x|^{-\delta}) \times |x|^\theta,$$

for some random constant $M > 0$ and $\theta \in (0, 1/2)$ fixed. We used (2.8) and the fact that $\sup_{(t,x) \in [0, T] \times \mathbb{R}} |L_t^x - L_t^0| < \infty$ a.s. This bound is integrable on $[-1, 1]$ if we choose $\delta > 0$ and $\theta \in (0, 1/2)$ such that $-1 + \theta - \delta > -1$, which is possible: $\theta = 1/4$, $\delta = 1/8$.

Now we dominate the quantity $P_\varepsilon(x) = |\varepsilon^{-1} \ell(1/\varepsilon) \phi(\kappa x/\varepsilon)| L_T^x$ for all $|x| > 1$. The second integral is actually supported by the compact set $\{|x| \in [1, S_T]\}$ where $S_T = \sup_{s \in [0, T]} |W_s|$ since $L_T^x = 0$ as soon as $|x| > S_T$. Using (2.21) with $\delta = 1/2$,

$$P_\varepsilon(x) \leq A(|x|^{-1} + 1)(|x|^{1/2} + |x|^{-1/2}) L_T^x,$$

which is a.s. bounded and thus integrable on $\{|x| \in [1, S_T]\}$.

Item (ii): Asymptotics of ξ_ε . First remember from Proposition 2.1 that $f \in L^1(\mu)$ if and only if $\int_1^\infty \frac{dx}{x\ell(x)} < \infty$.

Case 1: Let us first suppose $\int_1^\infty \frac{dx}{x\ell(x)} = \infty$ and show that $\xi_\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} \kappa(f_+ + f_-) \ell(1/\varepsilon) \int_1^{1/\varepsilon} \frac{dx}{x\ell(x)}$. For $\varepsilon \in (0, \kappa)$

$$\begin{aligned} \xi_\varepsilon &= \kappa \ell(1/\varepsilon) \int_{-\kappa/\varepsilon}^{\kappa/\varepsilon} \phi(x) dx \\ &= \kappa \ell(1/\varepsilon) \int_{-\kappa/\varepsilon}^{-1} \phi(x) dx + \kappa \ell(1/\varepsilon) \int_{-1}^1 \phi(x) dx + \kappa \ell(1/\varepsilon) \int_1^{\kappa/\varepsilon} \phi(z) dz. \end{aligned} \tag{2.23}$$

The middle term is $o(\ell(1/\varepsilon) \int_1^{1/\varepsilon} \frac{dx}{x\ell(x)})$ and will not contribute. Assume e.g. $f_+ \neq 0$, then $\phi(x) \underset{x \rightarrow \infty}{\sim} f_+ / [x\ell(x)]$ and it comes that

$$\kappa \int_1^{\kappa/\varepsilon} \phi(x) dx \underset{\varepsilon \rightarrow 0}{\sim} \kappa f_+ \int_1^{\kappa/\varepsilon} \frac{dx}{x\ell(x)} \underset{\varepsilon \rightarrow 0}{\sim} \kappa f_+ \int_1^{1/\varepsilon} \frac{dx}{x\ell(x)},$$

where we used the fact that $L(x) = \int_1^x \frac{dv}{v\ell(v)}$ is a slowly varying function by Lemma 2.1-(iii).

If $f_- \neq 0$, then $\kappa \int_{-\kappa/\varepsilon}^{-1} \phi(x) dx \underset{\varepsilon \rightarrow 0}{\sim} \kappa f_- \int_1^{1/\varepsilon} \frac{dx}{x\ell(x)}$ as previously. If next $f_- = 0$, we have $\phi(x) \underset{x \rightarrow -\infty}{=} o(|x\ell(|x|)|^{-1})$, then $\kappa \int_{-\kappa/\varepsilon}^{-1} \phi(x) dx \underset{\varepsilon \rightarrow 0}{=} o(\int_1^{1/\varepsilon} \frac{dx}{x\ell(x)})$ because $\int_1^\infty \frac{dx}{x\ell(x)} = \infty$.

Now, if $f_+ = 0$, then necessarily $f_- \neq 0$ and we can proceed as previously.

Case 2: $\int_1^\infty \frac{dx}{x\ell(x)} < \infty$. We must show $\xi_\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} -\kappa(f_+ + f_-)\ell(1/\varepsilon) \int_{1/\varepsilon}^\infty \frac{dx}{x\ell(x)}$. In this case, $f \in L^1(\mu)$ and we impose $\mu(f) = 0$ so that $\int_{\mathbb{R}} \phi(x)dx = 0$ by (2.22) and thus

$$\xi_\varepsilon = -\kappa\ell(1/\varepsilon) \int_{-\infty}^{-\kappa/\varepsilon} \phi(x)dx - \kappa\ell(1/\varepsilon) \int_{\kappa/\varepsilon}^\infty \phi(x)dx.$$

Let us treat the case $f_+ \neq 0$ and $f_- = 0$, the other cases being treated similarly. Since $f_+ \neq 0$, we can integrate the equivalence $\phi(x) \underset{x \rightarrow \infty}{\sim} f_+/[x\ell(x)]$ and get that $\int_{\kappa/\varepsilon}^\infty \phi(x)dx \underset{x \rightarrow \infty}{\sim} f_+ \int_{1/\varepsilon}^\infty \frac{dx}{x\ell(x)}$, where we used that $N(x) = \int_x^\infty \frac{dx}{x\ell(x)}$ is slowly varying by Lemma 2.1. Next $\int_{-\infty}^{-\kappa/\varepsilon} \phi(x)dx \underset{x \rightarrow -\infty}{=} o(\int_{1/\varepsilon}^\infty \frac{dx}{x\ell(x)})$ since $\phi(x) \underset{x \rightarrow -\infty}{=} o(|x\ell(|x|)|^{-1})$. $\square$

We are now ready to give the proof of Theorem 2.1-(iii)-(iv).

Proof of Theorem 2.1-(iii)-(iv). Consider a Brownian motion $(W_t)_{t \geq 0}$, its local time $(L_t^0)_{t \geq 0}$ at 0 and $\tau_t = \inf\{u \geq 0, L_u^0 > t\}$. Consider also the process $(K_t)_{t \geq 0}$ defined in Subsection 2.2.3 for $(a, b) = (f_+, f_-)$.

For each $\varepsilon > 0$, consider the processes $(A_t^\varepsilon)_{t \geq 0}$, $(\tau_t^\varepsilon)_{t \geq 0}$ and $(H_t^\varepsilon)_{t \geq 0}$ from Lemma 2.6 with the choice $a_\varepsilon = \varepsilon/\kappa$. We know from this Lemma that

$$\left(\int_0^{t/\varepsilon} f(X_s)ds \right)_{t \geq 0} \stackrel{d}{=} (H_{\tau_t^\varepsilon}^\varepsilon)_{t \geq 0} \quad (2.24)$$

And we know from Lemma 2.7-(ii) that $\tau_t^\varepsilon \rightarrow \tau_t$ a.s. for each t fixed. Thus $T = \sup_{\varepsilon \in (0,1)} \tau_t^\varepsilon$ is a.s. finite.

Item (iii). We aim at showing that $(\varepsilon^{1/\alpha}\ell(1/\varepsilon) \int_0^{t/\varepsilon} f(X_s)ds)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha S_t^{(\alpha)})_{t \geq 0}$. By (2.24) it is enough to show that a.s.,

$$\Delta_t(\varepsilon) = \left| \varepsilon^{1/\alpha}\ell(1/\varepsilon)H_{\tau_t^\varepsilon}^\varepsilon - \kappa^{1/\alpha}K_{\tau_t} \right| \xrightarrow{\varepsilon \rightarrow 0} 0,$$

for each fixed $t \geq 0$. Indeed, this would imply that a.s., for any $t_1, \dots, t_n \geq 0$, the vector $(\Delta_{t_1}(\varepsilon), \dots, \Delta_{t_n}(\varepsilon))$ converges to 0 which implies the convergence in finite dimensional distribution. Moreover, setting $S_t^{(\alpha)} = \sigma_\alpha^{-1}\kappa^{1/\alpha}K_{\tau_t}$, Lemmas 2.2 and 2.3 tell us that $(S_t^{(\alpha)})_{t \geq 0}$ is a stable process such that

$$\mathbb{E}[\exp(i\xi S_t^{(\alpha)})] = \exp\left(-c_{\alpha,a,b}t|\sigma_\alpha^{-1}\kappa^{1/\alpha}\xi|^\alpha \left(1 - i\beta_{\alpha,a,b} \tan\left(\frac{\alpha\pi}{2}\right) \text{sgn}(\xi)\right)\right) = \exp(-t|\xi|^\alpha z_\alpha(\xi)),$$

where $z_\alpha(\xi) = 1 - i\beta_{\alpha,a,b} \tan\left(\frac{\alpha\pi}{2}\right) \text{sgn}(\xi)$ as in the statement.

We have

$$\Delta_t(\varepsilon) \leq \sup_{s \in [0, T]} \left| \varepsilon^{1/\alpha}\ell(1/\varepsilon)H_s^\varepsilon - \kappa^{1/\alpha}K_s \right| + \kappa^{1/\alpha} |K_{\tau_t^\varepsilon} - K_{\tau_t}|.$$

The first term goes to 0 from Lemma 2.8-(i) and the second by continuity of $(K_t)_{t \geq 0}$.

Item (iv). We want to show that with $(\xi_\varepsilon)_{\varepsilon > 0}$ defined in Lemma 2.8-2,

$$\left(\varepsilon\ell(1/\varepsilon) \int_0^{t/\varepsilon} f(X_s)ds - \xi_\varepsilon t \right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_1 S_t^{(1)})_{t \geq 0}.$$

Once again, by (2.24), it is enough to show that

$$\Delta_t(\varepsilon) = \left| \varepsilon \ell(1/\varepsilon) H_{\tau_t^\varepsilon}^\varepsilon - \xi_\varepsilon t - \kappa K_{\tau_t} \right| \xrightarrow{\varepsilon \rightarrow 0} 0,$$

for each fixed $t \geq 0$. Indeed, setting $S_t^{(1)} = \sigma_1^{-1} \kappa K_{\tau_t}$, Lemma 2.4 tells us that $(S_t^{(1)})_{t \geq 0}$ is a stable process such that

$$\begin{aligned} \mathbb{E}[\exp(i\xi S_t^{(1)})] &= \exp\left(-c_{a,b}t|\sigma_1^{-1}\kappa\xi| \left(1 + i\beta_{a,b}\frac{2}{\pi} \log(\sigma_1^{-1}\kappa|\xi|)\operatorname{sgn}(\xi)\right) + it\tau_{a,b}\sigma_1^{-1}\kappa\xi\right) \\ &= \exp\left(-t|\xi| \left(1 + i\beta_{a,b}\frac{2}{\pi} \log(\sigma_1^{-1}\kappa|\xi|)\operatorname{sgn}(\xi) - i\tau_{a,b}\sigma_1^{-1}\kappa \operatorname{sgn}(\xi)\right)\right) \\ &= \exp(-t|\xi|z_1(\xi)), \end{aligned}$$

where $z_1(\xi) = 1 + i\beta_{a,b}\frac{2}{\pi} \log(\sigma_1^{-1}\kappa|\xi|)\operatorname{sgn}(\xi) - i\tau_{a,b}\sigma_1^{-1}\kappa \operatorname{sgn}(\xi)$ as in the statement.

We write

$$\begin{aligned} \Delta_t(\varepsilon) &\leq \left| \varepsilon \ell(1/\varepsilon) H_{\tau_t^\varepsilon}^\varepsilon - \xi_\varepsilon L_{\tau_t^\varepsilon}^0 - \kappa K_{\tau_t^\varepsilon} \right| + \xi_\varepsilon \left| A_{\tau_t^\varepsilon}^\varepsilon - L_{\tau_t^\varepsilon}^0 \right| + \kappa \left| K_{\tau_t^\varepsilon} - K_{\tau_t} \right| \\ &\leq \sup_{t \in [0, T]} \left| \varepsilon \ell(1/\varepsilon) H_t^\varepsilon - \xi_\varepsilon L_t^0 - \kappa K_t \right| + \sup_{t \in [0, T]} \xi_\varepsilon \left| A_t^\varepsilon - L_t^0 \right| + \kappa \left| K_{\tau_t^\varepsilon} - K_{\tau_t} \right|. \end{aligned}$$

The first term goes to 0 from Lemma 2.8-(ii) and the last one by continuity of $(K_t)_{t \geq 0}$. Now remember that $\xi_\varepsilon = \kappa \ell(1/\varepsilon) \int_{-\kappa/\varepsilon}^{\kappa/\varepsilon} \phi(x) dx = \gamma(1/\varepsilon)$ where $\gamma(x) = \kappa \ell(x) \int_{-\kappa x}^{\kappa x} \phi(v) dv$. Using the notations of Lemma 2.1 and the asymptotic estimate of ξ_ε , see Lemma 2.8-(ii), we have

$$\gamma(x) \underset{x \rightarrow \infty}{\sim} -\kappa(f_+ + f_-)l(x)N(\kappa x) \quad \text{if } f \in L^1(\mu),$$

and

$$\gamma(x) \underset{x \rightarrow \infty}{\sim} \kappa(f_+ + f_-)l(x)L(\kappa x) \quad \text{if } f \notin L^1(\mu).$$

In any case, γ is equivalent to the product of two slowly varying functions, and thus, is a slowly varying function. Hence, Lemma 2.7-(iii) tells us that the second term goes to 0. $\square$

2.3.4 The diffusive regime and critical diffusive regime

This case is standard and we use the strategy explained in the introduction. It consists in solving the Poisson equation $\mathcal{L}g = f$, see Jacod-Shiryaev [JS03, Chapter VIII section 3f]. We recall the function g defined by

$$g(x) = 2 \int_0^x \mathbf{s}'(v) \int_v^\infty f(u) [\sigma^2(u) \mathbf{s}'(u)]^{-1} du dv,$$

which is a C^2 function solving $2bg' + \sigma^2 g'' = -2f$. Also since $(X_t)_{t \geq 0}$ is a regular positive recurrent diffusion by Assumption 2.1, we classically deduce that X_t tends in law to μ as t goes to infinity, see Kallenberg [Kal02, Theorem 23.15]. We first show Theorem 2.2.

Proof of Theorem 2.2. Using Itô formula with the function g , we have

$$\int_0^t f(X_s) ds = \int_0^t g'(X_s) \sigma(X_s) dB_s - g(X_t).$$

Thus $\varepsilon^{1/2} \int_0^{t/\varepsilon} f(X_s) ds = M_t^\varepsilon - Y_t^\varepsilon$ where $M_t^\varepsilon = \varepsilon^{1/2} \int_0^{t/\varepsilon} g'(X_s) \sigma(X_s) dB_s$ is a local martingale and $Y_t^\varepsilon = \varepsilon^{1/2} g(X_{t/\varepsilon})$. Since g is continuous, $g(X_{t/\varepsilon})$ converges in law as $\varepsilon \rightarrow 0$ and thus

Y_t^ε converges to 0 in probability, for each fixed t . Moreover if g is bounded, we have a.s. $\sup_{t \in [0, T]} Y_t^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$.

We now show that $(M_t^\varepsilon)_{t \geq 0}$ converges in law as a continuous process to $(\gamma W_t)_{t \geq 0}$. It is enough (see Jacod-Shiryaev [JS03, Theorem VII-3.11 page 473]) to show that for each $t \geq 0$, $\langle M^\varepsilon \rangle_t \xrightarrow{\mathbb{P}} \gamma^2 t$ as $\varepsilon \rightarrow 0$. But $\langle M^\varepsilon \rangle_t = \varepsilon \int_0^{t/\varepsilon} [g'(X_s)\sigma(X_s)]^2 ds$ and the result follows from Remark 2.4 since $g'\sigma \in L^2(\mu)$ by assumption.

All in all we have proved that $(\varepsilon^{1/2} \int_0^{t/\varepsilon} f(X_s) ds)_{t \geq 0}$ converges in finite dimensional distributions to $(\gamma W_t)_{t \geq 0}$ and as a continuous process if g is bounded. $\square$

Proof of Theorem 2.1-(i)-(ii). The first point is covered by Theorem 2.2 since $g'\sigma \in L^2(\mu)$ by Proposition 2.2. The diffusive constant $\sigma_\alpha^2 = \int_{\mathbb{R}} [g'\sigma]^2 d\mu$ is indeed the one specified in the introduction. Remember that $g'(x) = 2\mathfrak{s}'(x) \int_x^\infty f(v)[\sigma^2(v)\mathfrak{s}'(v)]^{-1} dv$ and $\mu(dx) = \kappa[\sigma^2(x)\mathfrak{s}'(x)]^{-1} dx$ so that in the end

$$\sigma_\alpha^2 = 4\kappa \int_{\mathbb{R}} \mathfrak{s}'(x) \left(\int_x^\infty f(v)[\sigma^2(v)\mathfrak{s}'(v)]^{-1} dv \right)^2 dx.$$

In the case $\alpha = 2$ and $\rho = \infty$, we can use exactly the same proof: we write again that $|\varepsilon/\rho_\varepsilon|^{1/2} \int_0^{t/\varepsilon} f(X_s) ds = M_t^\varepsilon - Y_t^\varepsilon$, where $M_t^\varepsilon = |\varepsilon/\rho_\varepsilon|^{1/2} \int_0^{t/\varepsilon} g'(X_s)\sigma(X_s) dB_s$ is a local martingale and $Y_t^\varepsilon = |\varepsilon/\rho_\varepsilon|^{1/2} g(X_{t/\varepsilon})$. First Y_t^ε tends to 0 in probability for each t fixed, as previously, and since $|\varepsilon/\rho_\varepsilon| \rightarrow 0$. Finally it only remains to show that for each $t \geq 0$,

$$\langle M^\varepsilon \rangle_t = |\varepsilon/\rho_\varepsilon| \int_0^{t/\varepsilon} [g'(X_s)\sigma(X_s)]^2 ds \xrightarrow{\mathbb{P}} \sigma_2^2,$$

which is the case by Remark 2.5. $\square$

Remark 2.6. We stress that it is plausible we can show the limit theorem in the Lévy regime, at least when $f_+ = -f_-$, using the martingale strategy, i.e. writing $\int_0^t f(X_s) ds = M_t - g(X_t)$ where $(M_t)_{t \geq 0}$ is a local martingale with bracket $\int_0^t [g'(X_s)\sigma(X_s)]^2 ds$. Using similar arguments as in Lemma 2.8, we can show that, correctly rescaled, this bracket behaves like a stable subordinator. Hence M_t might behave like a Brownian motion subordinate by a stable subordinator, which is known to be a stable process.

2.4 Applications

Here we give some examples of application. Each time, we try to explain intuitively why the resulting limiting stable process is not a Brownian motion. It can be either because (a) f is large near infinity even if the diffusion (2.1) has small return times to 0 (as studied by Jara-Komorowski-Olla [JKO09] in the case of Markov chains), or (b) the diffusion (2.1) has large return times to 0, meaning that its invariant distribution has a rather slow decay; (c) or for both reasons.

In case (a), and as we will see only in case (a), one easily determines the index α of the limiting stable process from the behavior of $\mu(\{f > x\})$ as $x \rightarrow \pm\infty$; namely, $\mu(\{f > x\}) \sim |x|^{-\alpha}$, up to a constant or a slowly varying function.

Diffusions with fast decay invariant measure : Consider the following stochastic differential equation

$$X_t = -\frac{\theta + 1}{2} \int_0^t \operatorname{sgn}(X_s) |X_s|^\theta ds + B_t, \quad (2.25)$$

where $\theta > 0$ and $(B_t)_{t \geq 0}$ is a Brownian motion. This model includes the Ornstein-Uhlenbeck process ($\theta = 1$). Following equations (2.2) and (2.3), the invariant measure and the scale function are given by

$$\mu(dx) = \kappa e^{-|x|^{\theta+1}} dx, \quad \mathfrak{s}(x) = \int_0^x e^{|v|^{\theta+1}} dv.$$

One can check that $\mathfrak{s}(x) \underset{x \rightarrow \pm\infty}{\sim} \operatorname{sgn}(x) e^{|x|^{\theta+1}} / [(\theta+1)|x|^\theta]$. Hence, as soon as f is a continuous function such that there exists $\alpha > 0$ and $(f_+, f_-) \in \mathbb{R}^2$ satisfying

$$|x|^{(1/\alpha-2)\theta} e^{-|x|^{\theta+1}/\alpha} f(x) \xrightarrow{x \rightarrow \pm\infty} f_\pm,$$

and $|f_+| + |f_-| > 0$, we can apply Theorem 2.1 with $\ell \equiv 1$, provided $\int_0^{t/\varepsilon} f(X_s) ds$ is replaced by $\int_0^{t/\varepsilon} [f(X_s) - \mu(f)] ds$ when $\alpha > 1$. We can also generalize using slowly varying functions.

Roughly, the return times of $(X_t)_{t \geq 0}$ have exponential moments, because its invariant distribution has an exponential decay. To get an α -stable process at the limit with some $\alpha \in (0, 2)$, the function f really has to be large near infinity. Namely, we need that there exists a slowly varying function L and $c_+, c_- > 0$ such that

$$L(x)|x|^\alpha \mu(\{f > x\}) \xrightarrow{x \rightarrow \infty} c_+ \quad \text{and} \quad L(x)|x|^\alpha \mu(\{f < x\}) \xrightarrow{x \rightarrow -\infty} c_+. \quad (2.26)$$

A toy kinetic model : Here we slightly generalize the results of [NP15], [CNP19], [LP19] and [FT21]. Consider a one-dimensional particle with position $X_t \in \mathbb{R}$ and velocity $V_t \in \mathbb{R}$ subject to random shocks and a force field $F(v) = \frac{\beta}{2} \frac{\Theta'(v)}{\Theta(v)}$ for some $\beta > 1$ and a function $\Theta : \mathbb{R} \mapsto (0, +\infty)$ of class C^1 . The Newton equations describing the motion of the particle are

$$V_t = B_t + \int_0^t F(V_s) ds, \quad X_t = \int_0^t V_s ds, \quad (2.27)$$

where $(B_t)_{t \geq 0}$ is a Brownian motion modeling the random shocks. The invariant measure and the scale function are given by

$$\mu(dx) = \kappa [\Theta(x)]^\beta dx, \quad \mathfrak{s}(x) = \int_0^x [\Theta(v)]^{-\beta} dv.$$

We make the following assumption on the function Θ :

Assumption 2.4. *There exists $(c_-, c_+) \in \mathbb{R}_+^2$ such that $|x|\Theta(x) \rightarrow c_\pm$ as $x \rightarrow \pm\infty$ and $c_+ + c_- > 0$. If $c_+ = 0$ (respectively $c_- = 0$), we moreover impose there exists $A > 0$ such that for all $x \geq A$, $\Theta'(x) < 0$ (respectively for all $x \leq -A$, $\Theta'(x) > 0$).*

Obviously, $\lim_{x \rightarrow \pm\infty} \mathfrak{s}(x) = \pm\infty$ and since $\beta > 1$, $\mu(\mathbb{R}) < \infty$. Now take $f = \operatorname{id}$, we want to find $\alpha > 0$ such that Assumption 2.2 holds. Let us show that, with $\alpha = (\beta + 1)/3$, we have

$$|\mathfrak{s}'(x)|^{-2} |\mathfrak{s}(x)|^{2-1/\alpha} x \xrightarrow{x \rightarrow \infty} (\beta + 1)^{1/\alpha-2} c_+^{\beta/\alpha}, \quad (2.28)$$

$$|\mathfrak{s}'(x)|^{-2} |\mathfrak{s}(x)|^{2-1/\alpha} x \xrightarrow{x \rightarrow -\infty} -(\beta + 1)^{1/\alpha-2} c_-^{\beta/\alpha}. \quad (2.29)$$

Observe that we have the following estimations

$$|x|^\beta |\mathfrak{s}'(x)|^{-1} \xrightarrow{x \rightarrow \pm\infty} c_\pm^\beta \quad \text{and} \quad |x|^{\beta+1} |\mathfrak{s}(x)|^{-1} \xrightarrow{x \rightarrow \pm\infty} (\beta + 1) c_\pm^\beta. \quad (2.30)$$

Hence if $c_+ > 0$ and $c_- > 0$, the result is immediate. We consider e.g. the case $c_+ = 0$. First we notice that $\mathfrak{s}''(x) = -\beta\Theta'(x)[\Theta(x)]^{-\beta-1}$ and thus Assumption 2.4 tells us that there exists $A > 0$, such that $\mathfrak{s}'$ is increasing on $[A, \infty)$. Hence for all $x \geq A$, we have

$$0 \leq \mathfrak{s}(x) \leq \mathfrak{s}(A) + \mathfrak{s}'(x)(x - A) \leq \mathfrak{s}(A) + \mathfrak{s}'(x)x,$$

and thus for all $x \geq A$, since $2 - 1/\alpha > 0$ (because $\beta > 1$),

$$|\mathfrak{s}'(x)|^{-2}|\mathfrak{s}(x)|^{2-1/\alpha}x \leq |\mathfrak{s}'(x)|^{-2}|\mathfrak{s}(A) + \mathfrak{s}'(x)x|^{2-1/\alpha}x \underset{x \rightarrow \infty}{\sim} |\mathfrak{s}'(x)|^{-1/\alpha}|x|^{3-1/\alpha}$$

But since $\alpha = (\beta + 1)/3$, we have $3 - 1/\alpha = \beta/\alpha$ and thus $|\mathfrak{s}'(x)|^{-1/\alpha}|x|^{3-1/\alpha} = |x^\beta[\mathfrak{s}'(x)]^{-1}|^{1/\alpha}$ which goes to 0 by (2.30).

We can apply Theorem 2.1: with $\alpha = (\beta + 1)/3$, $m = \mu(\text{id})$, $\ell \equiv 1$, $f_+ = (\beta + 1)^{1/\alpha-2}c_+^{\beta/\alpha}$ and $f_- = -(\beta + 1)^{1/\alpha-2}c_-^{\beta/\alpha}$,

- (i) If $\beta > 5$, $\left(\varepsilon^{1/2}(X_{t/\varepsilon} - mt/\varepsilon)\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha W_t)_{t \geq 0}$.
- (ii) If $\beta = 5$, $\left(|\varepsilon/\log \varepsilon|^{1/2}(X_{t/\varepsilon} - mt/\varepsilon)\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_2 W_t)_{t \geq 0}$.
- (iii) If $\beta \in (2, 5)$, $\left(\varepsilon^{1/\alpha}(X_{t/\varepsilon} - mt/\varepsilon)\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha S_t^{(\alpha)})_{t \geq 0}$.
- (iv) If $\beta = 2$, $\left(\varepsilon(X_{t/\varepsilon} - \xi_\varepsilon t/\varepsilon)\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha S_t^{(\alpha)})_{t \geq 0}$.
- (v) If $\beta \in (1, 2)$, $\left(\varepsilon^{1/\alpha}X_{t/\varepsilon}\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha S_t^{(\alpha)})_{t \geq 0}$.

Remark that α ranges from $2/3$ to infinity. The additional assumption that Θ is eventually monotonic when $c_+ = 0$ (or $c_- = 0$) is not optimal, the true assumption is (2.28) and (2.29).

We stress that this model is the starting point of this paper as it was explained in the references section. Considering the *symmetric* function $\Theta(v) = (1 + v^2)^{-1/2}$ and reasoning on the law of (X_t, V_t) , which is the solution of the associated kinetic Fokker-Planck equation, a series of P.D.E papers (Nasreddine-Puel [NP15] : $\beta > 5$, Cattiaux-Nasreddine-Puel [CNP19] : $\beta = 5$ and Lebeau-Puel [LP19] : $\beta \in (1, 5) \setminus \{2, 3, 4\}$) answered these questions, finding some symmetric stable processes at the limit. Then, using probabilistic techniques, Fournier and Tardif [FT21] treated all the cases (even when $\beta \in (0, 1]$), still in a symmetric context. We thus recover their results and extend them to asymmetrical forces.

When $\beta \in (0, 1)$, the process $(V_t)_{t \geq 0}$ is null recurrent and Fournier and Tardif [FT21] show the rescaled process $(\varepsilon^{1/2}V_{t/\varepsilon})_{t \geq 0}$ converges in finite dimensional distribution to a symmetrized Bessel process and $(\varepsilon^{3/2}X_{t/\varepsilon})_{t \geq 0}$ to an integrated symmetrized Bessel process (their proof actually holds for $\beta \in (-1, 1)$). Although we can extend their result to asymmetrical forces, this phenomenon seems specific to this equation and we were not able to extend it to general diffusions.

Remark that for this model, the return times do not have exponential moments, because the invariant distribution has a slow decay. The exponent α is not prescribed by the invariant measure. Indeed we have

$$|x|^{\beta-1}\mu(\{f > x\}) \xrightarrow{x \rightarrow \infty} \kappa c_+^\beta/(\beta - 1) \quad \text{and} \quad |x|^{\beta-1}\mu(\{f < x\}) \xrightarrow{x \rightarrow \infty} \kappa c_-^\beta/(\beta - 1)$$

and $\alpha = (\beta + 1)/3$, while one might expect $\alpha = \beta - 1$. Observe that if $\beta \in (1, 2)$, we have $(\beta + 1)/3 > \beta - 1$ and if $\beta > 2$, we have $(\beta + 1)/3 < \beta - 1$. The dynamics are a little bit more complicated and one must take into account the fact that the return times can be long.

SDE without drift : Consider the following SDE

$$X_t = \int_0^t (1 + |X_s|)^{\beta/2} dB_s, \quad (2.31)$$

with $\beta > 1$. We have $\mathfrak{s}(x) = x$ (indeed $(X_t)_{t \geq 0}$ is a martingale) and $\mu(dx) = \kappa(1 + |x|)^{-\beta} dx$. Set for instance $f(x) = x/(1 + |x|)^{1-\gamma}$ with $\gamma > \beta - 2$, then $\mu(f) = 0$ and one can see that, with $\alpha = 1/(\gamma + 2 - \beta) > 0$, $f_+ = 1$ and $f_- = -1$, we have

$$|x|^{2-1/\alpha-\beta} f(x) \xrightarrow{x \rightarrow \pm\infty} f_{\pm}.$$

Hence we can apply Theorem 2.1 with $\ell \equiv 1$.

- (i) If $\beta - 2 < \gamma < \beta - 3/2$, then $\left(\varepsilon^{1/2} \int_0^{t/\varepsilon} f(X_s) ds\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_{\alpha} W_t)_{t \geq 0}$.
- (ii) If $\gamma = \beta - 3/2$, then $\left(|\varepsilon / \log \varepsilon|^{1/2} \int_0^{t/\varepsilon} f(X_s) ds\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_2 W_t)_{t \geq 0}$.
- (iii) If $\beta - 3/2 < \gamma < \beta - 1$, then $\left(\varepsilon^{1/\alpha} \int_0^{t/\varepsilon} f(X_s) ds\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_{\alpha} S_t^{(\alpha)})_{t \geq 0}$.
- (iv) If $\gamma = \beta - 1$, then $\left(\varepsilon \int_0^{t/\varepsilon} f(X_s) ds\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_{\alpha} S_t^{(\alpha)})_{t \geq 0}$.
- (v) If $\gamma > \beta - 1$, then $\left(\varepsilon^{1/\alpha} \int_0^{t/\varepsilon} f(X_s) ds\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_{\alpha} S_t^{(\alpha)})_{t \geq 0}$.

Now if $\gamma \leq \beta - 2$, one can see that the function g , solution of the Poisson equation, is such that

$$g'(x) = \int_x^{\infty} f(v) \sigma^{-2}(v) dv \underset{x \rightarrow \infty}{\sim} x^{\gamma-\beta+1} \quad \text{and} \quad g'(x) = \int_{-\infty}^x f(v) \sigma^{-2}(v) dv \underset{x \rightarrow -\infty}{\sim} -|x|^{\gamma-\beta+1}.$$

Hence $g' \in L^2(dx)$ since $2(\gamma - \beta + 1) < -1$ and thus $g' \sigma \in L^2(\mu)$ (since $\mu(dx) = \sigma^{-2}(x) dx$). We can apply Theorem 2.2 which tells us that the diffusive regime holds for $\gamma \leq \beta - 2$.

Once again, the return times are slow and the index α is not entirely determined by the invariant measure. Indeed we have for $\gamma > 0$

$$\mu(\{f > x\}) \underset{x \rightarrow \infty}{\sim} C|x|^{(1-\beta)/\gamma} \quad \text{and} \quad \mu(\{f < x\}) \underset{x \rightarrow -\infty}{\sim} C|x|^{(1-\beta)/\gamma}$$

for some constant C .

Assume $\beta = 5/4$ and let f be a function going to 0 at $\pm\infty$ such that $\mu(f) \neq 0$. If $\alpha = 4/3$, we have

$$|x|^{2-1/\alpha-\beta} (f(x) - \mu(f)) \xrightarrow{x \rightarrow \pm\infty} -\mu(f),$$

and thus we can apply Theorem 2.1-(iii) which tells us that

$$\left(\varepsilon^{3/4} \int_0^{t/\varepsilon} [f(X_s) - \mu(f)] ds\right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_{4/3} S_t^{(4/3)})_{t \geq 0}.$$

Persistence problems for additive functionals of one-dimensional Markov processes

Abstract

This chapter contains the results of [BBT23] which has been written with Quentin Berger and Camille Tardif and is submitted for publication. We consider additive functionals $\zeta_t = \int_0^t f(X_s)ds$ of a càdlàg Markov process $(X_t)_{t \geq 0}$ on $\mathbb{R}$. Under some general conditions on the process $(X_t)_{t \geq 0}$ and on the function f , we show that the persistence probabilities verify $\mathbb{P}(\zeta_s < z \text{ for all } s \leq t) \sim \mathcal{V}(z)\varsigma(t)t^{-\theta}$ as $t \rightarrow \infty$, for some (explicit) $\mathcal{V}(\cdot)$, some slowly varying function $\varsigma(\cdot)$ and some $\theta \in (0, 1)$. This extends results in the literature, which mostly focused on the case of a self-similar process $(X_t)_{t \geq 0}$ (such as Brownian motion or skew-Bessel process) with a homogeneous functional f (namely a pure power, possibly asymmetric). In a nutshell, we are able to deal with processes which are only asymptotically self-similar and functionals which are only asymptotically homogeneous. Our results rely on an excursion decomposition of $(X_t)_{t \geq 0}$, together with a Wiener–Hopf decomposition of an auxiliary (bivariate) Lévy process, with a probabilistic point of view. This provides an interpretation for the asymptotic behavior of the persistence probabilities, and in particular for the exponent θ , which we write as $\theta = \rho\beta$, with β the scaling exponent of the local time of $(X_t)_{t \geq 0}$ at level 0 and ρ the (asymptotic) positivity parameter of the auxiliary Lévy process.

3.1 Introduction

The study of persistence (or survival) probabilities for stochastic processes is a widely investigated problem and an extensive literature exists on this matter. It consists in estimating the probability for a given stochastic process to remain below some level (or more generally below some barrier), at least in some asymptotic regime. For instance, for a general random walk $(S_n)_{n \geq 0}$ or a Lévy process $(Z_t)_{t \geq 0}$ in $\mathbb{R}$, fluctuation’s theory and the Wiener–Hopf factorization gives the following characterization, see e.g. [Don07]. Letting T_z , for $z \geq 0$, denote the first hitting of $(S_n)_{n \geq 0}$ (resp. of $(Z_t)_{t \geq 0}$) above level z , then $t \mapsto \mathbb{P}(T_z > t)$ is regularly varying with index $-\rho$, $\rho \in (0, 1)$, if and only if $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{P}(S_k > 0) = \rho$ (resp. if and only if $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}(Z_s > 0)ds = \rho$). The latter condition is known as *Spitzer’s condition*.

3.1.1 Persistence problems for additive functionals

In this paper, we consider a one-dimensional càdlàg strong Markov process $(X_t)_{t \geq 0}$ with values in $\mathbb{R}$, and we assume that 0 is recurrent for $(X_t)_{t \geq 0}$. We denote by $\mathbb{P}_x$ the law of the process starting from $X_0 = x$; we also write $\mathbb{P} = \mathbb{P}_0$ for simplicity. For a measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$,

we consider the following additive functional of $(X_t)_{t \geq 0}$:

$$\zeta_t = \int_0^t f(X_s) ds. \quad (3.1)$$

We are now interested in the asymptotic behavior of the persistence (or survival) probabilities for the (non-Markovian) process $(\zeta_t)_{t \geq 0}$, *i.e.* probabilities that the process ζ avoids a barrier during a long period of time.

For $z > 0$, we denote by $T_z = \inf\{t > 0, \zeta_t \geq z\}$ the first hitting time of $(\zeta_t)_{t \geq 0}$ at the level z . We aim at describing the asymptotic behavior of the probability $\mathbb{P}_x(T_z > t)$. More precisely we show that, under some natural conditions on f and $(X_t)_{t \geq 0}$ (presented below in Section 3.2.2), there exist a persistence exponent $\theta \in (0, 1)$, a slowly varying function $\varsigma(\cdot)$ and a constant $\mathcal{V}(z)$ such that

$$\mathbb{P}(T_z > t) \sim \mathcal{V}(z) \varsigma(t) t^{-\theta} \quad \text{as } t \rightarrow \infty. \quad (3.2)$$

3.1.2 Overview of the literature and main contribution

The study of persistence probabilities for additive functionals of random walks or Lévy processes processes has a long history; we refer to the survey [AS15] for a thorough review. Let us here present some quick (and updated) overview of the literature and outline what are the main novelties of our paper.

The discrete case: integrated random walks. A question that has attracted a lot of attention since the seminal work of Sinai [Sin92] is that of persistence probabilities for the integrated random walk. Let $(U_i)_{i \geq 1}$ be i.i.d. random variables and let $X_n := \sum_{i=1}^n U_i$. Then setting $\zeta_n = \sum_{k=1}^n X_k$, Sinai [Sin92] proved that if $(X_n)_{n \geq 0}$ is the simple random walk (*i.e.* U_i is uniform in $\{-1, 1\}$), then

$$\mathbb{P}(\zeta_k \geq 0 \text{ for all } 0 \leq k \leq n) \asymp n^{-1/4},$$

where $u_n \asymp v_n$ means that there are two constants $c, c' > 0$ such that $c'v_n \leq u_n \leq cv_n$.

In the case where $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] < +\infty$, this result has then been extended to include the case of more general random walks. Vysotsky [Vys10] gave the same $\asymp n^{-1/4}$ asymptotic for double-sided exponential and double-sided geometric walks (not necessarily symmetric). Dembo, Ding and Gao [DDG13] gave a general proof for the $\asymp n^{-1/4}$ asymptotic. Finally, Denisov and Wachtel [DW15] proved the precise asymptotic behavior, *i.e.* there is some constant c_0 such that

$$\mathbb{P}(\zeta_k \geq 0 \text{ for all } 0 \leq k \leq n) \sim c_0 n^{-1/4} \quad \text{as } n \rightarrow \infty.$$

The case where $\mathbb{E}[X_i] = 0$ and $(X_i)_{i \geq 1}$ is in the domain of attraction of an α -stable random variable with $\alpha \in (1, 2)$ is still mostly open. An example of a *one-sided* random variable X_i with pure power tail is given in [DDG13], for which one has $\mathbb{P}(\zeta_k \geq 0 \text{ for all } 0 \leq k \leq n) \asymp n^{-\theta}$ with $\theta = \frac{\alpha-1}{2\alpha}$. Let us also mention [Vys14], which gives the sharp asymptotic $\sim c_0 n^{-\theta}$ in the case of a *one-sided* random variables (more precisely, right-exponential or skip free), with the same exponent θ .

Let us stress that in the discrete case, the main focus in the literature has been so far on integrated random walks rather than some more general additive functionals of a more general Markov process. Nevertheless, let us mention a series of article by I. Grama and R. Lauvergnat and E. Le Page [GLLP18, GLLP20] where the authors study additive functional of discrete-time Markov chain under a spectral-gap assumption. They show that the probability of persistence behave like $n^{-1/2}$ as the simple random walk and they continue the analysis further by proving local limit theorem for the additive functional conditioned to be positive.

We do not pursue further in this article the case of an additive functional $\zeta_n = \sum_{k=1}^n f(X_k)$ of a discrete-time Markov chain $(X_n)_{n \geq 0}$, but we mention [BB23] which uses a generalization of Sparre Andersen's formula to treat general cases where X is symmetric (and skip free) and f is also symmetric; we also refer to Section 3.2.5 where continuous-time Bessel random walks are considered. We believe that the methods of the present paper are general and could be adapted to treat some more general (*i.e.* non-symmetric) cases.

Integrated Brownian motion and α -stable Lévy processes. Analogously to the discrete case, the question that first attracted a lot of attention was that of persistence probabilities for the integrated Brownian motion. Consider $(X_t)_{t \geq 0}$ a standard Brownian motion and let $\zeta_t := \int_0^t X_s ds$ and $T_z := \inf\{t, \zeta_t \geq z\}$. It is then proven in [Gol71, IW94] that one also has

$$\mathbb{P}(T_z > t) \sim c_0 z^{1/6} t^{-1/4} \quad \text{as} \quad \frac{t}{z^{2/3}} \rightarrow +\infty,$$

with an explicit constant $c_0 = \frac{3^{4/3} \Gamma(2/3)}{\pi^{2/3} \Gamma(3/4)}$. Let us also stress that there exists an explicit formula for the density of (T_z, B_{T_z}) , given by Lachal [Lac91].

The case where $(X_t)_{t \geq 0}$ is a strictly α -stable Lévy process has been treated more recently. First, Simon [Sim07] proved that if $\alpha \in (1, 2)$ and $(X_t)_{t \geq 0}$ is spectrally positive, then, as $t \rightarrow \infty$ we have $\mathbb{P}(T_z > t) = t^{-\theta+o(1)}$ with $\theta = \frac{\alpha-1}{2\alpha}$. (Note that this is the same exponent as in the case of random walks mentioned above.) In the case $\alpha \in (0, 2]$ and if $(X_t)_{t \geq 0}$ has a positivity parameter $\varrho := \mathbb{P}(X_t > 0)$, Profeta and Simon [PS15] proved that, as $t \rightarrow \infty$,

$$\mathbb{P}(T_z > t) = t^{-\theta+o(1)} \quad \text{with} \quad \theta = \frac{\varrho}{1 + \alpha(1 - \varrho)}.$$

These results do not include the case of general Lévy processes $(X_t)_{t \geq 0}$. A natural conjecture, which is still open, is that the above asymptotic behavior for the persistence probabilities remains valid if $(X_t)_{t \geq 0}$ is in the domain of attraction of an α -stable Lévy process with positivity parameter ϱ .

Let us also mention the work [AD13] which deals with fractionally integrated Lévy processes $\zeta_t = \int_0^t K(t-s) X_s ds$ for some function $K : [0, \infty) \rightarrow [0, \infty)$ that includes the specific example $K(s) = \frac{1}{\Gamma(\beta)} s^{\beta-1}$ (corresponding to β -integration if β is an integer). They show that if $(X_s)_{s \geq 0}$ admits exponential moments, then the persistence exponent θ_β is the same as if $(X_s)_{s \geq 0}$ were a Brownian motion; additionally the persistence exponent θ_β is non-increasing in β .

Homogeneous additive functional of (skew-)Bessel processes. In the above, we only accounted for the literature concerning integrated processes, *i.e.* additive functionals ζ_t with the identity function $f(x) = x$. The case of a more general function f has also been considered, starting with the work of Isozaki [Iso96].

First, in the case where $(X_t)_{t \geq 0}$ is a Brownian motion, Isozaki considered the function f is homogeneous (and symmetric), given by $f(x) = \text{sign}(x)|x|^\gamma$ for some $\gamma \geq 0$ and proved that $\mathbb{P}(T_z > t) \asymp t^{-1/4}$ as $t \rightarrow +\infty$. This work was inspired by the one of Sinai [Sin92] and exploits the underlying idea that the fluctuations of ζ_t are related to the one of $Z_t := \zeta_{\tau_t}$ where τ_t is the inverse local time at 0 of the Brownian motion. Observing that Z_t is a Lévy process for which fluctuation theory is well known, Kotani managed to solve the persistence problem for ζ_t by establishing a Wiener–Hopf factorization for the bi-dimensional Lévy process $(\tau_t, Z_t)_{t \geq 0}$. Later, Isozaki and Kotani [IK00] considered the case where f is homogeneous but possibly asymmetric, *i.e.* $f(x) = |x|^\gamma (c_+ \mathbf{1}_{\{x > 0\}} - c_- \mathbf{1}_{\{x < 0\}})$ for some $c_+, c_- > 0$ and $\gamma > -1$: they prove the precise asymptotic estimate $\mathbb{P}(T_z > t) \sim C_z t^{-\rho/2}$ as $t \rightarrow +\infty$, where ρ is some asymmetry

parameter that depends (explicitly) on γ and the ratio c_+/c_- . Note that the tools used in [IK00] are somewhat analytical and do not rely on the Wiener–Hopf factorization established by Isozaki in the previous article [Iso96].

Let us also mention the work of McGill [McG08] who considers the case of a Brownian motion, and a generalized function f taking $f(x)dx = \mathbf{1}_{\{x \geq 0\}}m_+(dx) - \mathbf{1}_{\{x < 0\}}m_-(dx)$, with m_+ and m_- being Radon measures respectively on $\mathbb{R}_+$ and $\mathbb{R}_-$. The question raised in [McG08] is to give the asymptotic behavior of $\mathbb{P}(T_z > t)$ as $z \downarrow 0$, when t is fixed, which is slightly different from our persistence problem. Using excursion theory of the Brownian motion, McGill proves, under some technical conditions on $m_{\pm}$, that $\mathbb{P}(T_z > t) \sim C_t \mathcal{V}(z)$ as $z \rightarrow 0$ where $\mathcal{V}$ is the renewal function of the ladder height process associated to $(Z_t)_{t \geq 0}$ (see (3.6) for a definition), generalizing results in [IK00].

More recently, Profeta [Pro21] treated the case where $(X_t)_{t \geq 0}$ is a *skew-Bessel process* with dimension $\delta \in [1, 2)$ and skewness parameter $\eta \in (-1, 1)$, *i.e.* roughly speaking a Bessel process of dimension $\delta \in [1, 2)$ which has some asymmetry η when it touches 0 (see Example 3.2 for a proper definition). Profeta also considers the case of a homogeneous but possibly asymmetric function f , namely $f(x) = |x|^\gamma(c_+\mathbf{1}_{\{x > 0\}} - c_-\mathbf{1}_{\{x < 0\}})$ for some $c_+, c_- > 0$ but his work is restricted to the case $\gamma > 0$. He proves that $\mathbb{P}(T_z > t) \sim C_z t^{-\theta}$ as $t \rightarrow \infty$, with some explicit expression for the constant C_z and the exponent θ (see section 3.2.5 below). Actually, [Pro21] goes further and provides some explicit expression for the law of different quantities related to this problem.

Let us also mention the work of Simon [Sim07], which deals with the additive functional of a strictly α -stable Lévy process $(X_t)_{t \geq 0}$ with $\alpha \in (1, 2]$ and a (symmetric) homogeneous functional $f(x) = \text{sign}(x)|x|^\gamma$ for some $\gamma > -\frac{1}{2}(1 + \alpha)$. Letting $\theta = \frac{\alpha-1}{2\alpha}$, the first result of [Sim07] is that one always have $\mathbb{P}(T_z > t) \leq C t^{-\theta}$; the second result is that if $(X_t)_{t \geq 0}$ is spectrally positive, then $\mathbb{P}(T_z > t) = t^{-\theta+o(1)}$.

Our main contribution. Before we briefly describe our contribution, let us make a few comments on the above-mentioned results.

First of all, all the results on persistence probabilities for additive functionals of processes are limited to: (i) self-similar Markov processes, *i.e.* Brownian motion, (skew-)Bessel processes and strictly stable Lévy processes; (ii) functions f that are homogeneous, *i.e.* also enjoy some scaling property. These two points are important in the proofs, since it immediately entails some scaling property for the additive functional $(\zeta_t)_{t \geq 0}$; which is then easily seen to be self-similar.

Second, the method of proof of Isozaki [Iso96] relies on an excursion decomposition of the process $(X_t)_{t \geq 0}$, together with a Wiener–Hopf decomposition for the auxiliary process $(\tau_t, Z_t)_{t \geq 0}$. Further works, in particular [IK00, McG08], did not completely rely on this decomposition to obtain the sharp asymptotic behavior (and in particular the constant C_z), but rather on a more analytical approach. For instance, Profeta’s approach in [Pro21] uses exact calculations to derive the densities of various quantities of interest; the exact formulas available when dealing with Bessel processes then becomes crucial.

Let us keep the following example in mind (from [Pro21]), that we will use as a common thread:

Example 3.1.

- (i) $(X_t)_{t \geq 0}$ is a skew-Bessel process of dimension $\delta \in (0, 2)$ and skewness parameter $\eta \in (-1, 1)$.
- (ii) f is homogeneous $f(x) = |x|^\gamma(c_+\mathbf{1}_{\{x > 0\}} - c_-\mathbf{1}_{\{x < 0\}})$ for some $\gamma \in \mathbb{R}$ and $c_+, c_- > 0$.

With that said, here are the main contribution of our paper

- We treat the case of a general Markov process $(X_t)_{t \geq 0}$ (with some minimal assumption); in particular, we only need asymptotic properties on $(X_t)_{t \geq 0}$. Note that we also treat the

case where $(X_t)_{t \geq 0}$ positive recurrent (e.g. an Ornstein-Uhlenbeck process), which seems to have been left outside of the literature so far.

- We treat the case of a general function f ; in particular, we only need asymptotic properties on f . In the case of Example 3.1, we also extend the range of parameters where (3.2) holds, compared to [IK00, Pro21], see Section 3.2.5.

Another contribution of our paper is that it somehow unifies all the above results, by using a probabilistic approach. We push the excursion decomposition employed in [Iso96, IK00] (our main assumption on $(X_t)_{t \geq 0}$ ensures that it possess an excursion decomposition) and we prove on a slightly more general Wiener–Hopf factorization for the bivariate process $(\tau_t, Z_t)_{t \geq 0}$. In particular, this provides a probabilistic interpretation of the exponent θ in (3.2), which we decompose into two parts: $\theta = \beta\rho$, with (i) $\beta \in (0, 1]$ which describes the scaling exponent of the local time of $(X_t)_{t \geq 0}$ at level 0 (we have $\beta = 1 - \delta/2$ in Example 3.1); (ii) $\rho \in (0, 1)$ which is an asymmetry parameter, namely the asymptotic positivity parameter of $(Z_t)_{t \geq 0}$ (explicit in Example 3.1, see (3.10)). Our approach also provides a natural interpretation of the constant $\mathcal{V}(z)$ in (3.2).

3.2 Main results

3.2.1 Main assumptions and notation of the article

Throughout this paper, we will consider a strong Markov process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}})$ with càdlàg paths and valued in $E \subset \mathbb{R}$. We assume that the filtration is right-continuous and complete and that the process is conservative, *i.e.* it has an infinite lifetime $\mathbb{P}_x$ -a.s. for every $x \in \mathbb{R}$. We also assume that $0 \in E$ and we define $(\zeta_t)_{t \geq 0}$ as in (3.1), with a function f that verifies the following assumption.

Assumption 3.1. *The function f is measurable and locally bounded, except possibly around 0. It is such that a.s. $|\zeta_t| < \infty$ for any $t \geq 0$. Moreover f preserves the sign of x , in the sense that $f(x) \geq 0$ if $x > 0$ and $f(x) \leq 0$ if $x < 0$. Finally, we assume that $f(0) = 0$.*

As far as the Markov process $(X_t)_{t \geq 0}$ is concerned, we assume that 0 is regular for itself, that is $\mathbb{P}_0(\eta_0 = 0) = 1$, where $\eta_0 = \inf\{t > 0, X_t = 0\}$. We also assume that 0 is recurrent for $(X_t)_{t \geq 0}$. We make the following important assumption on $(X_t)_{t \geq 0}$ (the most restricting one).

Assumption 3.2. *Under $\mathbb{P}_0$, the process $(X_t)_{t \geq 0}$ is a.s. not of constant sign. Additionally, it cannot change sign without touching 0.*

These assumptions allow us to introduce the local time $(L_t)_{t \geq 0}$ of the process $(X_t)_{t \geq 0}$ at level 0. Its right-continuous inverse $(\tau_t)_{t \geq 0}$ is a subordinator and we denote by Φ its Laplace exponent:

$$t\Phi(q) = -\log \mathbb{E}[e^{-q\tau_t}] \quad \text{for any } q \geq 0. \quad (3.3)$$

We then introduce the process $(Z_t)_{t \geq 0} = (\zeta_{\tau_t})_{t \geq 0}$ which we will refer to as the Lévy process associated to the additive functional $(\zeta_t)_{t \geq 0}$. Indeed, $(Z_t)_{t \geq 0}$ is a pure jump Lévy process with finite variations and should be understood this way:

$$Z_t := \zeta_{\tau_t} = \int_0^{\tau_t} f(X_s) ds = \sum_{s \leq t} \int_{\tau_{s-}}^{\tau_s} f(X_u) du. \quad (3.4)$$

We also introduce g_t , the last zero before t , and I_t , the contribution of the last (unfinished) excursion:

$$g_t := \sup\{s < t, X_s = 0\} \quad \text{and} \quad I_t = \zeta_t - \zeta_{g_t} = \int_{g_t}^t f(X_r) dr. \quad (3.5)$$

To state our theorems, we need to introduce the renewal function $\mathcal{V}(\cdot)$ of the usual *ladder height process* $(H_t)_{t \geq 0}$ associated with $(Z_t)_{t \geq 0}$, see Section 3.4.3 for a proper definition of $(H_t)_{t \geq 0}$:

$$\mathcal{V}(z) = \int_0^\infty \mathbb{P}(H_t \leq z) dt, \quad z \in \mathbb{R}_+. \quad (3.6)$$

3.2.2 Main results I: persistence probabilities

There are two main assumptions under which we are able to obtain the exact asymptotic behavior for the persistence probability. The first one corresponds to the case where 0 is positive recurrent for $(X_t)_{t \geq 0}$, in which case the last part of the integral, *i.e.* the term I_t (see (3.5)), becomes irrelevant. The second one is a bit more involved since the last term I_t plays a role; the assumption is discussed in more detail in Section 3.2.3 below.

Assumption 3.3. *The point 0 is positive recurrent for $(X_t)_{t \geq 0}$.*

Under this hypothesis, we have a necessary and sufficient condition so that (3.2) holds. This condition is the analog of *Spitzer's condition* for Lévy processes or random walks.

Theorem 3.1. *Suppose that Assumption 3.3 holds and let $\rho \in (0, 1)$. The two following assertions are equivalent:*

- (i) $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}(\zeta_s \geq 0) ds = \rho$
- (ii) *For any $z > 0$, the map $t \mapsto \mathbb{P}(T_z > t)$ is regularly varying at ∞ with index $-\rho$.*

Moreover, if (i) or (ii) holds for some $\rho \in (0, 1)$, then there exists a slowly varying function $\varsigma(\cdot)$ such that for any $z > 0$

$$\mathbb{P}(T_z > t) \sim \mathcal{V}(z)\varsigma(t)t^{-\rho} \quad \text{as } t \rightarrow \infty.$$

For instance, Theorem 3.1 applies to an Ornstein-Uhlenbeck process $(X_t)_{t \geq 0}$, see Section 3.2.5 below. Let us stress that the slowly varying function $\varsigma(\cdot)$ and the so-called *renewal function* $\mathcal{V}(\cdot)$ can be described explicitly in some cases, see Section 3.2.5. In particular, if the process $(Z_t)_{t \geq 0}$ is symmetric, for instance if $(X_t)_{t \geq 0}$ is symmetric and f is odd, then $\rho = \frac{1}{2}$ and the slowly varying function $\varsigma(\cdot)$ is constant (the expression in (3.32) is equal to 1).

Remark 3.1. *It is shown below (see Theorem 3.4) that for any $\rho \in (0, 1)$, condition (i) above is equivalent to $\frac{1}{t} \int_0^t \mathbb{P}(Z_s \geq 0) ds \rightarrow \rho$ as $t \rightarrow \infty$. Therefore, it is also equivalent to the stronger condition $\mathbb{P}(Z_t \geq 0) \rightarrow \rho$ as $t \rightarrow \infty$, see for instance [BD97]. It is not clear whether or not these conditions are equivalent to $\mathbb{P}(\zeta_t \geq 0) \rightarrow \rho$ as $t \rightarrow \infty$. Condition (i) of Theorem 3.1 is satisfied if, for instance, there is some central limit theorem for $(\zeta_t)_{t \geq 0}$; we refer to [Bét21, Thm 5 and 7] for such instances, in the case of one-dimensional diffusions.*

We now turn to the case where 0 is null recurrent: our assumption is the following.

Assumption 3.4. *The point 0 is null recurrent for $(X_t)_{t \geq 0}$. Moreover, there exist $\alpha \in (0, 2]$, $\beta \in (0, 1)$, some functions $a(\cdot)$ and $b(\cdot)$ that are regularly varying around 0 with respective indices $1/\alpha$ and $1/\beta$ such that the following convergence in distribution holds (for the Skorokhod topology):*

$$\left(\tau_t^h, Z_t^h \right)_{t \geq 0} := \left(b(h)\tau_{t/h}, a(h)Z_{t/h} \right)_{t \geq 0} \xrightarrow{(d)} \left(\tau_t^0, Z_t^0 \right)_{t \geq 0} \quad \text{as } h \rightarrow 0,$$

where $(\tau_t^0, Z_t^0)_{t \geq 0}$ is a Lévy process. We additionally assume that $a(b^{-1}(q))I_e$ converges in distribution as $q \rightarrow 0$, where $e = e(q)$ is an independent exponential random variable of parameter $q > 0$ and b^{-1} is an asymptotic inverse of b .

Let us stress that Assumption 3.4 is about the auxiliary process $(\tau_t, Z_t)_{t \geq 0}$ (and not directly about the process $(X_t)_{t \geq 0}$ and the function f), which may be difficult to verify. We present below in Section 3.2.3 some conditions on $(X_t)_{t \geq 0}$ and f for Assumption 3.4 to hold; these conditions are easier to verify in practice.

Remark 3.2. *The limiting process $(\tau_t^0, Z_t^0)_{t \geq 0}$ necessarily satisfies the following scaling property:*

$$(\tau_t^0, Z_t^0)_{t \geq 0} \stackrel{d}{=} (c^{-1/\beta} \tau_{ct}^0, c^{-1/\alpha} Z_{ct}^0)_{t \geq 0} \quad \text{for any } c > 0.$$

Also, the fact that $(b(h)\tau_{t/h})_{t \geq 0}$ converges in distribution to a β -stable subordinator $(\tau_t^0)_{t \geq 0}$ is actually equivalent to the fact that the Laplace exponent $\Phi(q)$ of $(\tau_t)_{t \geq 0}$ is regularly varying with exponent $\beta \in (0, 1)$ as $q \downarrow 0$. In that case, $b(\cdot)$ is an asymptotic inverse of $\Phi(\cdot)$ (up to a constant factor), see Section 3.6.

Theorem 3.2. *Suppose that Assumption 3.4 holds and that $(Z_t^0)_{t \geq 0}$ has positivity parameter $\rho = \mathbb{P}(Z_t^0 \geq 0) \in (0, 1)$. Then there exists a slowly varying function $\varsigma(\cdot)$ such that, for any $z > 0$,*

$$\mathbb{P}(T_z > t) \sim \mathcal{V}(z)\varsigma(t)t^{-\beta\rho} \quad \text{as } t \rightarrow \infty,$$

where $\beta \in (0, 1)$ is given by Assumption 3.4.

Let us observe that Example 3.1, where $(X_t)_{t \geq 0}$ is a skew-Bessel process and f is homogeneous, verify our Assumption 3.4. Indeed, if $\gamma > -\delta$, the bivariate Lévy process $(\tau_t, Z_t)_{t \geq 0}$ directly enjoys a (β, α) -scaling property (and similarly for I_t) with $\beta = 1 - \delta/2$ and $\alpha = (2 - \delta)/(2 + \gamma) \in (0, 1)$, so it trivially satisfies Assumption 3.4. Thus, Theorem 3.2 applies and the parameter ρ is also explicit; we refer to Section 3.2.5 for further details. The advantage of our result is that we are able to treat a general class of Markov processes $(X_t)_{t \geq 0}$ (for instance that are “asymptotically skew-Bessel processes”) and of function f (that are “asymptotically” homogeneous).

3.2.3 Main results II: application to one-dimensional generalized diffusions

In this section we apply our result to a large class of one-dimensional Markov processes. We recall the Itô-McKean [IMJ63, IMJ96] construction of generalized one-dimensional diffusions, based on a Brownian motion changed of scale and time. Our main goal is to provide conditions on the function f , the *scale function* $\mathfrak{s}$ and *speed measure* $\mathfrak{m}$ that ensure that Assumption 3.4 holds.

Let $\mathfrak{m} : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing right-continuous function such that $\mathfrak{m}(0) = 0$, and $\mathfrak{s} : \mathbb{R} \rightarrow \mathbb{R}$ a continuous increasing function. We assume that $\mathfrak{s}(\mathbb{R}) = \mathbb{R}$, $\mathfrak{s}(0) = 0$ and abusively, we also denote by $\mathfrak{m}$ the Radon measure associated to $\mathfrak{m}$, that is $\mathfrak{m}((a, b]) = \mathfrak{m}(b) - \mathfrak{m}(a)$ for all $a < b$. We introduce $\mathfrak{m}^\mathfrak{s}$ the image of $\mathfrak{m}$ by $\mathfrak{s}$, *i.e.* the Stieltjes measure associated to the non-decreasing function $\mathfrak{m} \circ \mathfrak{s}^{-1}$, where $\mathfrak{s}^{-1}$ is the inverse function of $\mathfrak{s}$. Then, we define $A_t^{\mathfrak{m}^\mathfrak{s}}$ the continuous additive functional of a Brownian motion $(B_t)_{t \geq 0}$ given by

$$A_t^{\mathfrak{m}^\mathfrak{s}} = \int_{\mathbb{R}} L_t^x \mathfrak{m}^\mathfrak{s}(dx),$$

where $(L_t^x)_{t \geq 0, x \in \mathbb{R}}$ denotes the usual family of local times of the Brownian motion, assumed to be continuous in the variables x and t . We let ρ_t the right-continuous inverse of $A_t^{\mathfrak{m}^\mathfrak{s}}$, and we set

$$X_t = \mathfrak{s}^{-1}(B_{\rho_t}).$$

Then it holds that $(X_t)_{t \geq 0}$ is a strong Markov process valued in $\text{supp}(\mathfrak{m})$, where $\text{supp}(\mathfrak{m})$ is the support of the measure $\mathfrak{m}$. We refer to Section 3.7 for more details. We will therefore assume

that $0 \in \text{supp}(\mathbf{m})$ and in this framework, 0 is a recurrent point for $(X_t)_{t \geq 0}$. When $\text{supp}(\mathbf{m})$ is some interval J , $(X_t)_{t \geq 0}$ is a one-dimensional diffusion living in J and its generator is formally given by

$$\mathcal{A} = \frac{d}{d\mathbf{m}} \frac{d}{d\mathfrak{s}}.$$

Example 3.2. A skew-Bessel process of dimension $\delta \in (0, 2)$ and skewness parameter $\eta \in (-1, 1)$, is the linear diffusion on $\mathbb{R}$ whose scale function $\mathfrak{s}$ and speed measure $\mathbf{m}$ are defined as

$$\mathfrak{s}(x) = \text{sgn}(x) \frac{1 - \text{sgn}(x)\eta}{2 - \delta} |x|^{2-\delta} \quad \text{and} \quad \mathbf{m}(dx) = \frac{1}{1 - \text{sgn}(x)\eta} \mathbf{1}_{\{x \neq 0\}} |x|^{\delta-1} dx.$$

Informally, this process can be constructed by concatenating independent excursions of the usual Bessel process, flipped to the negative half-line with probability $(1 - \eta)/2$.

Example 3.3. When $\mathbf{m}^\mathfrak{s}$ is a sum of Dirac masses, then $(X_t)_{t \geq 0}$ is a birth and death process, see [Sto63]. For instance, if $\mathfrak{s} = \text{id}$ and $\mathbf{m} = \sum_{n \in \mathbb{N}} \delta_n$, then $(X_t)_{t \geq 0}$ is a continuous-time simple random walk on $\mathbb{Z}$.

For a function f as in Assumption 3.1, we also set $d\mathbf{m}^f := f \circ \mathfrak{s}^{-1} d\mathbf{m}^\mathfrak{s}$. Note that $\mathbf{m}^f$ is a signed measure (recall that f preserves the sign). We suppose in addition that $f \circ \mathfrak{s}^{-1}$ is locally integrable with respect to $\mathbf{m}^\mathfrak{s}$ so that $\mathbf{m}^f$ is also a Radon measure. We will also denote by $\mathbf{m}^f$ the associated function, i.e. $\mathbf{m}^f(x) = \int_0^x f \circ \mathfrak{s}^{-1}(u) \mathbf{m}^\mathfrak{s}(du)$, which is non-decreasing on $\mathbb{R}_+$ and non-increasing on $\mathbb{R}_-$.

We now give practical conditions on the scale function $\mathfrak{s}$, the speed measure $\mathbf{m}$ and the function f so that Assumption 3.4 holds. We will consider three different assumptions.

Assumption 3.5. There exist $\beta \in (0, 1)$, a slowing variation function $\Lambda_\mathfrak{s}$ at $+\infty$, and two non-negative constants m_-, m_+ with $m_- + m_+ > 0$, such that

$$\begin{cases} \mathbf{m}^\mathfrak{s}(x) \sim m_+ \Lambda_\mathfrak{s}(x) x^{1/\beta-1} & \text{as } x \rightarrow +\infty, \\ \mathbf{m}^\mathfrak{s}(x) \sim -m_- \Lambda_\mathfrak{s}(|x|) |x|^{1/\beta-1} & \text{as } x \rightarrow -\infty. \end{cases}$$

Assumption 3.6. The function $\mathfrak{s}$ is $\mathcal{C}^1$, and there exist a constant $\mathbf{m}^f(\infty) \in (0, \infty)$ such that $\lim_{x \rightarrow \pm\infty} \mathbf{m}^f(x) = \mathbf{m}^f(\infty)$ and the function $\mathbf{m}^f(\infty) - \mathbf{m}^f$ belongs to $L^2(dx)$.

Assumption 3.7. There exist $\alpha \in (0, 2)$, a slowing variation function Λ_f at $+\infty$, and two non-negative constants f_-, f_+ with $f_- + f_+ > 0$, such that according to the value of α , we have

(i) If $\alpha \in (0, 1)$, then

$$\begin{cases} \mathbf{m}^f(x) \sim f_+ \Lambda_f(x) x^{1/\alpha-1} & \text{as } x \rightarrow +\infty, \\ \mathbf{m}^f(x) \sim f_- \Lambda_f(|x|) |x|^{1/\alpha-1} & \text{as } x \rightarrow -\infty. \end{cases} \quad (3.7)$$

(ii) If $\alpha = 1$, then the following limit exists $\lim_{h \downarrow 0} \frac{1}{\Lambda_f(1/h)} (\mathbf{m}^f(1/h) - \mathbf{m}^f(-1/h)) = \mathbf{c}$ and

$$\begin{cases} \lim_{h \rightarrow 0} \frac{1}{\Lambda_f(1/h)} (\mathbf{m}^f(x/h) - \mathbf{m}^f(1/h)) = f_+ \log x, & \forall x > 0, \\ \lim_{h \rightarrow 0} \frac{1}{\Lambda_f(1/h)} (\mathbf{m}^f(x/h) - \mathbf{m}^f(-1/h)) = f_- \log |x|, & \forall x < 0. \end{cases} \quad (3.8)$$

Note that if the limit $\mathbf{c}$ exists, this implies that $f_+ = f_-$; we will assume for simplicity that $f_+ = f_- = 1$.

(iii) If $\alpha \in (1, 2)$, then there is a constant $\mathbf{m}^f(\infty) \in (0, \infty)$ such that $\lim_{x \rightarrow \pm\infty} \mathbf{m}^f(x) = \mathbf{m}^f(\infty)$ and

$$\begin{cases} \mathbf{m}^f(\infty) - \mathbf{m}^f(x) \sim f_+ \Lambda_f(x) x^{1/\alpha-1} & \text{as } x \rightarrow +\infty, \\ \mathbf{m}^f(\infty) - \mathbf{m}^f(x) \sim f_- \Lambda_f(|x|) |x|^{1/\alpha-1} & \text{as } x \rightarrow -\infty. \end{cases} \quad (3.9)$$

Note that if Assumption 3.6 holds, then Assumption 3.7 can not hold and conversely. The main results of this section are the following.

Proposition 3.1 (Gaussian case). *Suppose that Assumptions 3.5 and 3.6 hold. Then, Assumption 3.4 is verified with $\beta \in (0, 1)$, $\alpha = 2$, and the following choice for $a(\cdot)$, $b(\cdot)$:*

$$a(h) = h^{1/2}, \quad b(h) = h^{1/\beta} / \Lambda_s(1/h).$$

As a consequence, Theorem 3.2 holds under Assumptions 3.5 and 3.6, with $\rho = \mathbb{P}(Z_t^0 \geq 0) = 1/2$.

Proposition 3.2 (α -stable case, $\alpha \in (0, 2)$). *Suppose that Assumptions 3.5 and 3.7 hold. Then, Assumption 3.4 is verified with $\alpha \in (0, 2)$, $\beta \in (0, 1)$ and the following choice for $a(\cdot)$, $b(\cdot)$:*

$$a(h) = h^{1/\alpha} / \Lambda_f(1/h), \quad b(h) = h^{1/\beta} / \Lambda_s(1/h).$$

As a consequence, Theorem 3.2 holds under Assumptions 3.5 and 3.7, with the following asymmetry parameter $\rho = \mathbb{P}(Z_t^0 \geq 0)$,

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan(\vartheta) \quad \text{where} \quad \vartheta = \begin{cases} \frac{f_+^\alpha - f_-^\alpha}{f_+^\alpha + f_-^\alpha} \tan(\pi\alpha/2) & \text{if } \alpha \neq 1, \\ \mathbf{c} & \text{if } \alpha = 1. \end{cases}$$

3.2.4 Main results III: starting with a non-zero velocity

In this section, we are interested in the hitting time of zero of the additive functional $z + \zeta_t$, with some initial velocity $X_0 = x$. A motivation to consider such a question is to construct the additive functional conditioned to stay negative; of course, the only reason we deal with a condition to remain negative (and not positive) is because we have treated above the asymptotics of $\mathbb{P}(T_z > t)$ for $z > 0$.

To avoid the introduction of lengthy notation, we only give an outline of our results, summarizing the content of Section 3.8: the precise statement of the results are presented there. We restrict ourselves to the case where $(X_t)_{t \geq 0}$ is a regular diffusion process, valued in some open interval J containing 0. We consider the process $(\zeta_t, X_t)_{t \geq 0}$ as a strong Markov process. For a pair $(z, x) \in \mathbb{R} \times J$, we denote by $\mathbf{P}_{(z,x)}$ the law of $(\zeta_t, X_t)_{t \geq 0}$ when started at (z, x) , i.e. the law of $(z + \int_0^t f(X_s) ds, X_t)_{t \geq 0}$ under $\mathbb{P}_x$. Note that the previous sections were dealing with the probability $\mathbb{P}(T_z > t) = \mathbf{P}_{(-z,0)}(T_0 > t)$ for $z > 0$. Roughly, we derive two kind of results:

(i) We identify some finite function $h : \mathbb{R} \times J \rightarrow \mathbb{R}_+$ such that, under Assumption 3.3 or 3.4, we have for any $(z, x) \in \mathbb{R} \times J \setminus \{(0, 0)\}$,

$$\mathbf{P}_{(z,x)}(T_0 > t) \sim h(z, x) \varsigma(t) t^{-\rho\beta} \quad \text{as } t \rightarrow \infty,$$

for some slowly varying function ς (which does not depend on (z, x)) and some parameters $\beta \in (0, 1]$, $\rho \in (0, 1)$ (given by the assumption). We refer to Theorem 3.6 for the precise statement.

(ii) Secondly, we show that the function h is harmonic for the killed process $(\zeta_{t \wedge T_0}, X_{t \wedge T_0})_{t \geq 0}$, see Corollary 3.3. This classically enables us to construct the additive functional conditioned to stay negative, through Doob's h -transform, see Proposition 3.12. This result generalizes the previous work [GJW99] on the integrated Brownian motion conditioned to be positive and have the same flavor of some results from Grama-Lauvergnat-Le Page [GLLP18] in a discrete setting. It is also related to Profeta's article [Pro15] where he investigated other penalizations for the integral of a Brownian motion.

3.2.5 A series of examples

In this section, we provide several examples of application to our main theorems. We start with examples where Assumptions 3.3 or 3.4 are easy to verify; we then turn to examples where the reformulation in terms of Assumptions 3.5, 3.6 or 3.7 are useful.

Ornstein-Uhlenbeck. Let $(X_t)_{t \geq 0}$ be an Ornstein-Uhlenbeck process and f be some odd function. Then 0 is positive recurrent for $(X_t)_{t \geq 0}$ and since the Ornstein-Uhlenbeck process started at 0 is symmetric (in the sense that the law of $(-X_t)_{t \geq 0}$ equals the law of $(X_t)_{t \geq 0}$) it is clear that $\mathbb{P}(\zeta_t \geq 0) = 1/2$ for any $t > 0$. Therefore Theorem 3.1 holds with $\rho = 1/2$ and the slowly varying function ς is constant (the term (3.32) is equal to 1 since Z_t is symmetric).

Skew-Bessel and homogeneous functional, back to Example 3.1. Let $(X_t)_{t \geq 0}$ be a skew-Bessel process of dimension $\delta \in (0, 2)$ and skewness parameter $\eta \in (-1, 1)$, as defined in Example 3.2 by its scale function $\mathfrak{s}$ and speed measure $\mathfrak{m}$. This process can be constructed by the following informal procedure: concatenate independent excursions of the (usual) Bessel process, flipped to the negative half-line with probability $(1 - \eta)/2$. Let c_+, c_- be positive constants and consider the function f defined as $f(x) = (c_+ \mathbf{1}_{\{x > 0\}} - c_- \mathbf{1}_{\{x < 0\}}) |x|^\gamma$, with $\gamma > -\delta$. The persistence probability of $(\zeta_t)_{t \geq 0}$ is studied in Profeta [Pro21] (for $\delta \in [1, 2)$ and $\gamma > 0$).

The condition $\gamma > -\delta$ is here to ensure that $|\zeta_t| < \infty$ a.s. for all $t > 0$. One can verify in this case that $(\tau_t)_{t \geq 0}$ is a β -stable subordinator where $\beta = 1 - \delta/2$. By the self-similarity of the skew-Bessel process, it holds that the law of $(\tau_t, X_t)_{t \geq 0}$ is equal to the law of $(c^{-1/\beta} \tau_{ct}, c^{-1/2\beta} X_{c^{1/\beta} t})_{t \geq 0}$ for any $c > 0$. This entails that the law of $(\tau_t, Z_t)_{t \geq 0}$ is equal to the law of $(c^{-1/\beta} \tau_{ct}, c^{-1/\alpha} Z_{ct})_{t \geq 0}$ for any $c > 0$, where $\alpha = (2 - \delta)/(\gamma + 2) \in (0, 1)$. It also entails that for any $t > 0$, the law of $t^{-\beta/\alpha} I_t$ is equal to the law of I_1 . These facts imply that Assumption 3.4 holds with $a(h) = h^{1/\alpha}$ and $b(h) = h^{1/\beta}$ (with an equality rather than a convergence in distribution); one can also verify Assumptions 3.5 and 3.7 directly with the expressions of the scale function $\mathfrak{s}$ and speed measure $\mathfrak{m}$ (which are pure powers, so the scaling properties are clear).

Therefore, Theorem 3.2 holds with $\beta = 1 - \delta/2$ and $\rho = \mathbb{P}(Z_t \geq 0)$. The positivity parameter ρ can be computed, see for instance Zolotarev [Zol86, §2.6]: we have

$$\rho = \frac{1}{2} + \frac{\arctan(\vartheta \tan(\pi\alpha/2))}{\pi\alpha} \quad \text{where} \quad \vartheta = \frac{1 + \eta - (1 - \eta)(\frac{c_-}{c_+})^\alpha}{1 + \eta + (1 - \eta)(\frac{c_-}{c_+})^\alpha}. \quad (3.10)$$

The computation of ϑ can be done as in [Bét21, Lem. 11]. Finally, since $(Z_t)_{t \geq 0}$ is a stable process, the renewal function $\mathcal{V}$ is such that $\mathcal{V}(z) = b_+ z^{\alpha\rho}$ for some constant $b_+ > 0$. It is also clear that, by self-similarity, the slowly varying function ς is constant. Hence, we fully recover and extend the results of Profeta [Pro21] to $\delta \in (0, 1)$ and $\gamma \in (-\delta, 0]$.

Kinetic Fokker-Planck. Let $(X_t)_{t \geq 0}$ be a solution of the following stochastic differential equation

$$X_t = x_0 - \frac{\mu}{2} \int_0^t \frac{X_s}{1 + X_s^2} ds + B_t$$

where $(B_t)_{t \geq 0}$ is a Brownian motion, $\mu > -1$ and $x_0 \in \mathbb{R}$. The scaling limit of the process $(\zeta_t)_{t \geq 0} = (\int_0^t X_s ds)_{t \geq 0}$, *i.e.* with the choice $f = \text{id}$, is studied in [FT21, LP19, NP15, CNP19]. The corresponding scale function and speed measure are given by

$$\mathfrak{s}(x) = \int_0^x (1 + v^2)^{\mu/2} dv \quad \text{and} \quad \mathfrak{m}(x) = \int_0^x (1 + v^2)^{-\mu/2} dv,$$

see for instance [FT21]. Then one can check that:

- (i) If $\theta \in (-1, 1)$, Assumption 3.5 is satisfied with $\beta = \frac{1}{2}(\mu + 1) \in (0, 1)$ and Assumption 3.7 is satisfied with $\alpha = \frac{1}{3}(\mu + 1) \in (0, \frac{2}{3})$. Since $\mathfrak{m}$, $\mathfrak{s}$ and f are odd functions, Theorem 3.2 holds with $\beta = \frac{1}{2}(\mu + 1)$, $\rho = \frac{1}{2}$ and a constant slowly varying function $\varsigma(\cdot)$.
- (ii) When $\mu > 1$, 0 is positive recurrent for $(X_t)_{t \geq 0}$ and since $x \mapsto x/(1 + x^2)$ is odd, the process $(\zeta_t, X_t)_{t \geq 0}$ is symmetric (when $X_0 = 0$) so that $\mathbb{P}(\zeta_t \geq 0) = 1/2$ for any $t > 0$. Therefore Theorem 3.1 holds with $\rho = 1/2$ and a constant slowly varying function $\varsigma(\cdot)$.

Note that our results would also be able to deal with $(\zeta_t)_{t \geq 0} = (\int_0^t f(X_s) ds)_{t \geq 0}$ for more general functions f .

Non-homogeneous functionals of Bessel processes. The previous examples are limited to the case where $\alpha \in (0, 1)$ in Assumption 3.4 (or Assumption 3.7). Let us give here an example where one has $\alpha \in [1, 2]$; we consider a simplified example for pedagogical purposes.

Consider a symmetric Bessel process $(X_t)_{t \geq 0}$ of dimension $\delta \in (0, 2)$, *i.e.* a diffusion with scale function $\mathfrak{s}(x) = \frac{\text{sign}(x)}{2-\delta} |x|^{2-\delta}$ and speed measure $\mathfrak{m}(dx) = \mathbf{1}_{\{x \neq 0\}} |x|^{\delta-1} dx$. Then, one can check that Assumption 3.5 holds with $\beta = 1 - \delta/2$. Now, let f be some odd function such that: $\int_0^1 f(u^{1/(2-\delta)}) u^{1/\beta-1} du < +\infty$, for instance if f is bounded, to ensure that $f \circ \mathfrak{s}^{-1}$ is locally integrable with respect to $\mathfrak{m}^\mathfrak{s}$ (so that $\zeta_t < +\infty$ for all $t > 0$); $f(x) \sim \text{sign}(x)|x|^\gamma$ as $x \rightarrow \infty$, for some $\gamma \in \mathbb{R}$. Then, we can check that

- (i) if $\gamma > -(1 + \delta)$ then Assumption 3.7 holds with $\alpha = (2 - \delta)(\gamma + 2) \in (0, 2)$ and $f_+ = f_-$ (by symmetry);
- (ii) if $\gamma < -(1 + \delta)$, then Assumption 3.6 holds.

In all cases, we have the asymptotic behavior $\mathbb{P}(T_z > t) \sim c_0 \mathcal{V}(z) t^{-\beta/2}$, since $\rho = \frac{1}{2}$ and $\varsigma(\cdot)$ is constant, by symmetry. Note that our Assumption 3.6 does not deal with the case $\gamma = -(1 + \delta)$, but the result should still hold in that case (one would need to deal with non-normal domain of attraction to the normal law, which would require further technicalities).

Continuous-time birth and death chains (and Bessel-like walks). Let $(\tilde{X}_n)_{n \geq 0}$ be a birth and death process on $\mathbb{Z}$ with transition probabilities given by

$$\mathbb{P}(\tilde{X}_{n+1} = i + 1 | \tilde{X}_{n+1} = i) = p_i \in (0, 1) \quad \text{and} \quad \mathbb{P}(\tilde{X}_{n+1} = i - 1 | \tilde{X}_{n+1} = i) = q_i = 1 - p_i,$$

for $i \in \mathbb{Z}$. We then define $X_t := \tilde{X}_{N_t}$, with $(N_t)_{t \geq 0}$ an independent Poisson process of unit intensity. Then $(X_t)_{t \geq 0}$ is a continuous-time birth and death chain, and can be described as a

generalized diffusion associated to a scale function $\mathfrak{s}$ and a speed measure $\mathfrak{m}$ as follows; we refer to Stone [Sto63] for more details. Let us define $\Delta_0 = 1$ and

$$\Delta_i = \prod_{k=1}^i \frac{q_k}{p_k}, \quad \forall i \geq 1 \quad \text{and} \quad \Delta_i = \prod_{k=i+1}^0 \frac{p_k}{q_k}, \quad \forall i \leq -1.$$

The scale function $\mathfrak{s} : \mathbb{R} \rightarrow \mathbb{R}$ is increasing piecewise linear and such that $\mathfrak{s}(i) = x_i$ for $i \in \mathbb{Z}$, with $(x_i)_{i \in \mathbb{Z}}$ defined iteratively by $x_0 = 0$ and $\Delta_i = x_{i+1} - x_i$, $\forall i \in \mathbb{Z}$. The speed measure $\mathfrak{m}$ is defined as

$$\mathfrak{m} := \sum_{i \in \mathbb{Z}} \left(\frac{1}{2\Delta_i} + \frac{1}{2\Delta_{i-1}} \right) \delta_i.$$

In a companion paper [BB23], the first two authors use an elementary approach to obtain two-sided bounds for the persistence of integrated symmetric (discrete-time) birth and death process with an odd function f ; they apply their results to symmetric Bessel-like random walk (see [Ale11] for a recent account). Let us now observe that we can apply our machinery to continuous-time Bessel-like random walks and obtain sharp asymptotics for the persistence probabilities $\mathbb{P}(\zeta_s \leq 0 \text{ for all } s \leq t)$.

We define a symmetric Bessel-like random walk as a birth and death process with transition probabilities

$$p_i := \frac{1}{2} \left(1 - \frac{\mu + \varepsilon_i}{2i} \right), \quad \text{for } i \geq 1, \quad p_i = q_{-i} \text{ for } i \leq -1, \quad p_0 = q_0 = \frac{1}{2}.$$

Here, μ is a real parameter and ε_i is such that $\lim_{i \rightarrow \infty} \varepsilon_i = 0$. Then, we have that the process is recurrent if $\mu > 1$ and null-recurrent if $\mu \in (-1, 1)$; the case $\mu = 1$ depends on $(\varepsilon_i)_{i \geq 0}$.

(i) In the case $\mu > 1$, by Theorem 3.1 we directly obtain the asymptotics

$$\mathbb{P}(T_z > t) \sim c_0 \mathcal{V}(z) t^{-1/2} \quad \text{as } t \rightarrow \infty. \quad (3.11)$$

(We have $\rho = \frac{1}{2}$ and $\varsigma(\cdot)$ constant thanks to the symmetry.)

(ii) In the case $\mu \in (-1, 1)$, we use the following asymptotics: there exists a constant C_0 and a slowly varying function $L(i) = \exp(-\sum_{k=1}^i \frac{\varepsilon_k}{k})$ such that $\Delta_i \sim C_0 i^\mu L(i)$ as $i \rightarrow +\infty$ (and symmetrically for $i \rightarrow -\infty$). From this asymptotics we obtain that $\mathfrak{s}(i) = x_i = \sum_{k=0}^{i-1} \Delta_k \sim C_0' i^{1+\mu} L(i)$ as $i \rightarrow +\infty$ and also that $\mathfrak{m}(x) \sim C_0'' x^{1-\mu} L(x)^{-1}$ as $x \rightarrow \infty$. One can therefore show that Assumption 3.5 is satisfied with $\beta = \frac{1}{2}(1+\mu) \in (0, 1)$ (and $m_+ = m_-$). If we consider the function $f(x) = x$, Assumption 3.7 is satisfied with $\alpha = \frac{1}{3}(1+\mu)$ and $\rho = \frac{1}{2}$ (by symmetry). Finally, Theorem 3.2 states that

$$\mathbb{P}(T_z > t) \sim \mathcal{V}(z) \varsigma(t) t^{-(1+\mu)/4} \quad \text{as } t \rightarrow \infty, \quad (3.12)$$

with ς some slowing variation function (depending on $(\varepsilon_i)_{i \geq 0}$).

Let us stress that this matches the results from [BB23] in the discrete case and additionally gives the sharp asymptotics of the persistence probabilities; obviously one could consider a function $f(x) = \text{sign}(x)|x|^\gamma$ with $\gamma \in \mathbb{R}$ without affecting the conclusion (the exponent α of Assumption 3.4 does not affect the persistence exponent θ in the symmetric case).

3.3 Ideas of the proof and further comments

3.3.1 Ideas of the proof: path decomposition of trajectories

Recall from (3.5) that $I_t = \zeta_t - \zeta_{g_t}$, we introduce

$$\xi_t = \sup_{[0, t]} \zeta_s \quad \text{and} \quad \Delta_t = I_t - (\xi_{g_t} - \zeta_{g_t}). \quad (3.13)$$

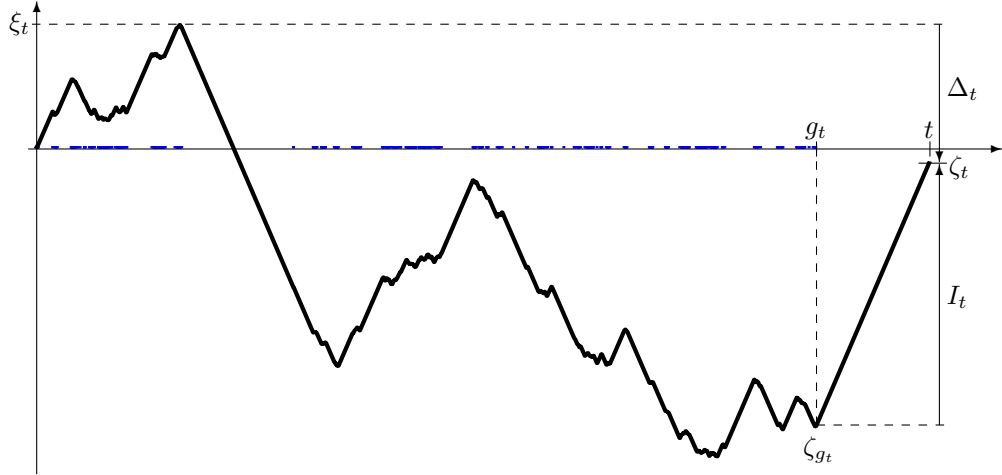


Figure 3.1: Graphical representation of the trajectory of a realization of $(\zeta_s)_{0 \leq s \leq t}$ and of its decomposition. The above is a simulation in the setting of Bessel-like random walk (see Section 3.2.5): $(X_t)_{t \geq 0}$ is a (symmetric) Bessel-like random walk with $\mu = 0.4$, *i.e.* $\beta = 0.7$, and with the function $f(x) = \text{sign}(x)$. The dots represent the returns to 0 of $(X_s)_{0 \leq s \leq t}$. In this particular realization, we have $I_t > 0$ and $\Delta_t < 0$.

We refer to Figure 4.1 for an illustration of $\zeta_t, \xi_t, \xi_{g_t}, I_t$ and Δ_t . Then, to study the probability $\mathbb{P}(T_z > t) = \mathbb{P}(\xi_t < z)$, the main idea is to decompose it into two parts: for any $z > 0$, we have

$$\mathbb{P}(\xi_t < z) = \mathbb{P}(\xi_{g_t} < z, \Delta_t \leq 0) + \mathbb{P}(\xi_{g_t} + \Delta_t < z, 0 < \Delta_t \leq z). \quad (3.14)$$

(The second term will turn out to be negligible.) As a first consequence of (3.14), we see that $\mathbb{P}(\xi_{g_t} < z, \Delta_t \leq 0) \leq \mathbb{P}(\xi_{g_t} < z) \leq \mathbb{P}(\xi_{g_t} < z, \Delta_t \leq z)$, from which one easily gets that

$$c_t \mathbb{P}(\xi_{g_t} < z) \leq \mathbb{P}(\xi_t < z) \leq \mathbb{P}(\xi_{g_t} < z).$$

with $c_t = \mathbb{P}(I_t \leq 0) \in (0, 1)$. We will show that we actually have some constant $c_1 \in (0, 1]$ such that

$$\mathbb{P}(\xi_t < z) \sim c_1 \mathbb{P}(\xi_{g_t} < z) \quad \text{as } t \rightarrow \infty. \quad (3.15)$$

We will then control the probability $\mathbb{P}(\xi_{g_t} < z)$ by using the fact that $\xi_{\tau_t} = \sup_{[0, t]} Z_s$ (see Remark 3.3 below).

A standard trick to gain independence. To handle the quantities in (3.14), we will use the following trick: letting $e = e(q)$ be an exponential random time e with parameter q independent of $(X_t)_{t \geq 0}$, we will look at the quantity $\mathbb{P}(\xi_e < z)$ instead of looking directly at $\mathbb{P}(\xi_t < z)$. This corresponds to taking the Laplace transform of $\mathbb{P}(\xi_t < z)$ and we loose no information by doing this: indeed, a combination of the Tauberian theorem and the monotone density theorem (see [BGT87, Thms. 1.7.1 and 1.7.2]) tells us that having an asymptotic of $\mathbb{P}(\xi_e < z)$ as $q \rightarrow 0$ is equivalent to having an asymptotic of $\mathbb{P}(\xi_t < z)$ as $t \rightarrow \infty$.

The first advantage of this trick is that it allows us to factorize functionals of trajectories before time g_e and functionals of trajectories between times g_e and e . The precise statement is presented in Proposition 3.3, which is inspired by [SVY07]. This enables us to operate a first reduction, treating I_e separately from ξ_{g_e}, ζ_{g_e} . More precisely, using Proposition 3.3 below, by independence, the first term in (3.14) (with t replaced by an exponential random variable e) can be rewritten as

$$\mathbb{P}(\xi_{g_e} < y, \Delta_e \leq 0) = \mathbb{P}(\xi_{g_e} < y, I_e \leq \xi_{g_e} - \zeta_{g_e}) = \mathbb{E}[F_{I_e}(\xi_{g_e} - \zeta_{g_e}) \mathbf{1}_{\{\xi_{g_e} < y\}}],$$

where F_{I_e} is the c.d.f. of I_e .

Wiener–Hopf factorization. A second key tool is a Wiener–Hopf factorization for the bivariate Lévy Process $(\tau_t, Z_t)_{t \geq 0}$, that among other things allows us to obtain the joint distribution of ξ_{g_e} and $\xi_{g_e} - \zeta_{g_e}$, see Corollary 3.1 below. In particular, it shows that ξ_{g_e} and $\xi_{g_e} - \zeta_{g_e}$ are independent, so (3.16) can further be decomposed as

$$\mathbb{P}(\xi_{g_e} < y, \Delta_e \leq 0) = \mathbb{E}[F_{I_e}(\xi_{g_e} - \zeta_{g_e})] \mathbb{P}(\xi_{g_e} < y) = \mathbb{P}(\Delta_e \leq 0) \mathbb{P}(\xi_{g_e} < y). \quad (3.16)$$

From this, we will be able to prove that $\mathbb{P}(\xi_e < y) \sim c_1 \mathbb{P}(\xi_{g_e} < t)$ as $q \downarrow 0$, *i.e.* (3.15), where the constant c_1 is $c_1 = \lim_{q \downarrow 0} \mathbb{P}(I_e \leq \xi_{g_e} - \zeta_{g_e}) \in (0, 1]$. Moreover, the Wiener–Hopf factorization will also help us obtain the asymptotic behavior of $\mathbb{P}(\xi_{g_e} < y)$ as $q \downarrow 0$.

Conclusion. With this picture in mind, we split our results into two categories, that correspond to Assumptions 3.3 and 3.4:

(i) If X is positive recurrent. Then, $\lim_{t \rightarrow \infty} \frac{1}{t} g_t = 1$ and I_t will typically be much smaller than $\xi_{g_t} - \zeta_{g_t}$ as $t \rightarrow \infty$, so $\lim_{q \downarrow 0} \mathbb{P}(I_e \leq \xi_{g_e} - \zeta_{g_e}) = 1$: we will have $c_1 = 1$ in (3.15), that is $\mathbb{P}(\xi_e < y) \sim \mathbb{P}(\xi_{g_e} < y)$ as $q \downarrow 0$. Loosely speaking, the part of the trajectory between time g_t and t will have no impact on the behavior of the persistence probability. Then, the behavior of $\mathbb{P}(\xi_{g_e} < y)$ is studied thanks to the Wiener–Hopf factorization, with the assumption that $(Z_t)_{t \geq 0}$ satisfies the so-called Spitzer’s condition.

(ii) If X is null recurrent then there are two cases, depending on whether $\alpha = 2$ or $\alpha \in (0, 2)$ in Assumption 3.4 (or corresponding to Assumptions 3.6 and 3.7). First, if $(Z_t)_{t \geq 0}$ is in the (normal) domain of attraction of a normal law, then also in that case I_t will typically be much smaller than $\xi_{g_t} - \zeta_{g_t}$ as $t \rightarrow \infty$ and we again have $\lim_{q \downarrow 0} \mathbb{P}(I_e \leq \xi_{g_e} - \zeta_{g_e}) = 1$, that is $c_0 = 1$ in (3.15). Second, if $(Z_t)_{t \geq 0}$ is in the domain of attraction of some α -stable law, $\alpha \in (0, 2)$, then under Assumption 3.4 we have that $\lim_{q \downarrow 0} \mathbb{P}(I_e \leq \xi_{g_e} - \zeta_{g_e}) = \mathbb{P}(I \leq W) =: c_1 \in (0, 1)$, where I, W are independent random variables, the respective limits in law of $a(b^{-1}(q))I_e$ and $a(b^{-1}(q))(\xi_{g_e} - \zeta_{g_e})$. Then, the behavior of $\mathbb{P}(\xi_{g_e} < y)$ is again studied thanks to the Wiener–Hopf factorization, with the assumption that $(\tau_t)_{t \geq 0}$ is in the domain of attraction of a stable subordinator.

3.3.2 Comparison with the literature

We now discuss the novelty of our results and techniques and compare them with the existing litterature.

Let us first start with the work of Isozaki [Iso96], which treats the case of integrated powers of the Brownian motion. Isozaki first uses the following Wiener–Hopf factorization of $(\tau_t, Z_t)_{t \geq 0}$. If $e = e(q)$ denotes an independent exponential random variable and $S_t = \sup_{[0, t]} Z_s$, then the product of the Laplace transforms of (τ_e, S_e) and $(\tau_e, Z_e - S_e)$ can be expressed in terms of the characteristic function of $(\tau_t, Z_t)_{t \geq 0}$. The rest of Isozaki’s method is somehow more analytic and involves inversion of Fourier transforms. Also, he exploits deeply the self-similarity of the Brownian motion. Of course, our work has been inspired by [Iso96], but regarding the Wiener–Hopf factorization, we go one step further. If we set $G_t = \sup\{s < t, Z_t = S_t\}$, then we are able to show that the law of (G_e, τ_{G_e}, S_e) and $(e - G_e, \tau_e - \tau_{G_e}, Z_e - S_e)$ are independent, infinitely divisible and can be expressed with the law of $(\tau_t, Z_t)_{t \geq 0}$. This enables us to the study the quantities of interest in the spirit of fluctuation’s theory for Lévy processes. We refer to Subsection 3.4.3 and Appendix 3.A for more details.

Let us now discuss the work of McGill [McG08], which seems to be the closest to our work. McGill considers general additive functionals of the Brownian, *i.e.* $\zeta_t = \int_{\mathbb{R}} L_t^x m(dx)$ where

$(L_t^x)_{t \geq 0, x \in \mathbb{R}}$ is the family of local times of the Brownian motion and $m(dx) = \mathbf{1}_{\{x \geq 0\}} m_+(dx) - \mathbf{1}_{\{x < 0\}} m_-(dx)$ is a signed measure. With the above notation, he characterizes the behavior of $\mathbb{P}(\xi_e < z)$ as $z \rightarrow 0$, for a fixed $q > 0$; whereas we study $\mathbb{P}(\xi_e < z)$ when $q \rightarrow 0$, for a fixed z . Let us quickly explain the content of [McG08]. First, if we fix $q > 0$ and set $v(z) = \mathbb{P}(\xi_e < z)$, he establishes the following identity:

$$v(z) = \int_0^z \mathcal{V}(dy) \int_{\mathbb{R}_-} \widehat{\mathcal{V}}(dx) k(z - y - x), \quad (3.17)$$

where $\widehat{\mathcal{V}}$ denotes the dual renewal function, $\mathcal{V}(dx)$ and $\widehat{\mathcal{V}}(dx)$ the Stieltjes measures associated with the non-decreasing function $\mathcal{V}$ and $\widehat{\mathcal{V}}$ and the function k depends on v and on q . This identity is derived using martingale techniques and tools from the excursion theory of $(Z_t)_{t \geq 0}$ below its supremum. With this key identity at hand, McGill proves under some technical conditions his main result: $v(z) \sim C(q)\mathcal{V}(z)$ as $z \rightarrow 0$, where $C(q)$ is some unknown constant. We emphasize that we can (more or less) recover (3.17) and his main results from our work. Our interpretation of (3.17) is the following: the random variable ξ_e can be decomposed as

$$\xi_e = \xi_{g_e} + \Delta_e \vee 0,$$

where ξ_{g_e} and $\Delta_e := I_e - (\xi_{g_e} - \zeta_{g_e})$ are independent, see (3.13). Our bivariate Wiener–Hopf factorization yields the following result (see Section 3.4.4): $\mathbb{P}(\xi_{g_e} \in dz) = C(q)\mathcal{V}_q(dz)$, where the constant $C(q)$ is known and $\mathcal{V}_q$ is a non-decreasing function, which is close to $\mathcal{V}$ in the sense that for any fixed $z \geq 0$, $\mathcal{V}_q(z)$ increases to $\mathcal{V}(z)$ as $q \rightarrow 0$. Then, we claim that we are able to show that $v(z) \sim C(q)\mathcal{V}_q(z)$ as $z \rightarrow 0$. To match the result of McGill, it remains to show that $\mathcal{V}_q(z) \sim \mathcal{V}(z)$ as $z \rightarrow 0$ and we believe this should hold, at least if $\mathcal{V}$ is regularly varying at 0: we can show that their Laplace transforms are equivalent at infinity.

We insist on the fact that all of the above approach more or less relies on the fact that the integrated process is a Brownian motion, whereas we are able to consider a very large class of Markov process and of functions f , leading to a wide range of possible behaviors.

3.3.3 Related problems and open questions

Let us now give an overview of questions that we have chosen not to develop in the present paper, that either fall in the scope of our method or represent important challenges.

Further examples in our framework. Let us give a couple of additional examples that we are able to treat with our method (but that we have chosen not to develop), since they are defined via their excursions (and satisfy Assumption 3.2).

Jumping-in diffusions. A jumping-in diffusion in $\mathbb{R}$ is a strong Markov process $(X_t)_{t \geq 0}$ which has continuous paths up until its first hitting time of 0. When the process touches 0, it immediately jumps back into $\mathbb{R} \setminus \{0\}$ and starts afresh. These processes are somehow again a generalization of diffusion processes and are typically constructed via excursion theory; we refer to the book of Itô [Itô15] for more details. Such processes obviously satisfy Assumption 3.2 and we can apply our results.

Stable processes reflected on their infimum. The following example is not related to a diffusion process and shows that our method are not only concerned with generalized diffusions. Let $(S_t^\alpha)_{t \geq 0}$ be an α -stable process with some $\alpha \in (0, 2)$ such that $(|S_t^\alpha|)_{t \geq 0}$ is not a subordinator. We consider $(R_t^\alpha)_{t \geq 0} = (S_t^\alpha - \inf_{[0, t]} S_s^\alpha)_{t \geq 0}$ the process reflected on its infimum. It is a positive strong Markov process and 0 is a regular and recurrent point for $(R_t^\alpha)_{t \geq 0}$. Informally, we construct the strong Markov process $(X_t)_{t \geq 0}$ by concatenating independent excursions of $(R_t^\alpha)_{t \geq 0}$

flipped to the negative half-line with probability $1/2$. Then it is also clear that $(X_t)_{t \geq 0}$ satisfies Assumption 3.2. It is also clear that this process is self-similar with index α , and as for (skew-)Bessel processes (see Section 3.2.5), it entails that Assumption 3.4 is satisfied if we consider f to be homogeneous, for instance $f(x) = \text{sgn}(x)|x|^\gamma$ for some $\gamma \in \mathbb{R}$ (with some restriction to ensure that $\zeta_t < +\infty$ a.s., e.g. $\gamma \geq 0$).

Asymptotics uniform in t, z . A natural question that we have chosen not to investigate further is the case when the barrier z is “far away”. In other words, we are interested in knowing for which regimes of t, z the asymptotics (3.2) remains valid. We would like to find some function $\varphi(\cdot)$ with $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$ so that the following statement holds:

$$\mathbb{P}(T_z > t) \sim \mathcal{V}(z)\zeta(t)t^{-\theta} \quad \text{uniformly for } t, z > 0 \text{ with } \varphi(t)/z \rightarrow \infty. \quad (3.18)$$

In particular, this would allow to consider both the case when z is fixed and $t \rightarrow \infty$ and the case when t fixed and $z \downarrow 0$.

We have not pursued this issue further to avoid lengthening further the paper, but we believe that our method should work to obtain such a result. Indeed, thanks to (3.15), we are reduced to estimating $\mathbb{P}(\xi_{g_e} < z)$ in terms of t, z , which is standard in fluctuation theory for Lévy processes, see [Kyp14]. Since we have identified that the correct scaling for ξ_{g_e} (and $\xi_{g_e} - \zeta_{g_e}$) is $a(b^{-1}(q))$ under Assumption 3.4, it is natural to expect that (3.18) holds with $\varphi(t) = a(b^{-1}(t))$.

In view of our proof, the main (and only?) step where an improvement is needed is in a control of the convergence $\mathcal{V}_q(z) \uparrow \mathcal{V}(z)$ as $q \downarrow 0$, where $\mathcal{V}_q$ is defined in (3.28), see Corollary 3.2.

Additive functional conditioned on being negative. In Section 3.8, we construct the additive functional (ζ_t, X_t) (under $\mathbb{P}_{(z,x)}$ with $(z,x) \neq 0$) conditioned to remain negative, see Proposition 3.12. But there are several questions that we have left open.

Starting from $(0,0)$. First of all, a natural question would be to construct the additive functional conditioned to stay negative, but with starting point $(0,0)$, that is starting from 0 with a zero speed. One should take the limit $(z,x) \rightarrow 0$ inside Proposition 3.12, but this brings several technical difficulties and requires further work.

Scaling limit of the conditioned process. Another natural question is that of obtaining the long-term behavior of the additive functional (conditioned to be negative or not) and in particular scaling limits. Our approach should yield all necessary tools to obtain such results, and let us give an outline of what one could expect:

(i) If $(X_t)_{t \geq 0}$ is positive recurrent, i.e. under Assumption 3.3, then one should have that the conditioned process converges to either a Brownian motion or an α -stable Lévy process conditioned to be negative (depending on whether the process $(Z_t)_{t \geq 0}$ is in the domain of attraction of an α -stable law with $\alpha = 2$ or $\alpha \in (0, 2)$).

(ii) If $(X_t)_{t \geq 0}$ is null-recurrent and Assumption 3.4 holds, then one should have that the conditioned process converges to the following non-Markovian processes:

- If $\alpha = 2$, a time-changed Brownian motion conditioned to stay negative, namely $(Z_{L_t^0}^0)_{t \geq 0}$, where Z^0 is a Brownian motion conditioned to stay negative and L_t^0 is the inverse of the β -stable subordinator τ^0 ; note that necessarily τ^0 and Z^0 are independent.
- If $\alpha \in (0, 2)$, a “skeleton” given by a time-changed α -stable Lévy process conditioned to stay negative, namely $Z_{L_t^0}^0$ as above (with here Z^0 an α -stable Lévy process), then “filled” with the integrals of excursions conditioned to bridge the skeleton (i.e. excursions of lengths $\tau_t^0 - \tau_{t-}^0$ with the integral $\int_0^\ell (\varepsilon_s)^\gamma ds$ conditioned to be equal to $Z_t^0 - Z_{t-}^0$).

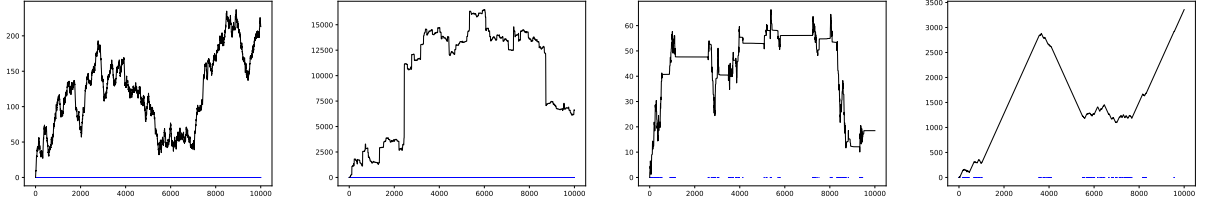


Figure 3.2: Graphical representation of the trajectory of a realization of $(\zeta_s)_{0 \leq s \leq t}$ conditioned to be positive. These are simulations in the setting of Bessel-like random walks: $(X_t)_{t \geq 0}$ is a (symmetric) Bessel-like random walk and $f(x) = \text{sign}(x)|x|^\gamma$ for some $\gamma \in \mathbb{R}$; the dots represent the returns to 0 of $(X_s)_{0 \leq s \leq t}$. The first two simulations are when $(X_t)_{t \geq 0}$ is positive recurrent and $(Z_t)_{t \geq 0}$ converges either to a Brownian motion (first) or an α -stable Lévy process (second). The last two simulations are when $(X_t)_{t \geq 0}$ is null recurrent and Assumption 3.4 holds either with $\alpha = 2$ (third, corresponding to Assumption 3.6) or with $\alpha \in (0, 2)$ (fourth, corresponding to Assumption 3.7).

We refer to Figure 3.2 for illustrative simulations. Let us also mention that one might also be interested in some local limit theorems for the conditioned process, in the spirit of [Car05] for random walk and [GLLP18] for additive functional of Markov chain under a spectral gap assumption.

Additive functionals of Markov processes with jumps. Finally we stress that our main assumption on the process $(X_t)_{t \geq 0}$, *i.e.* Assumption 3.2, is not satisfied for a large class of processes, including Lévy processes with jumps (except for the difference of two independent Poisson processes). When Assumption 3.2 is not satisfied, it is not clear how to attack the problem and the only result so far in this direction seems to focus only on (symmetric) *homogeneous* functionals of *strictly* α -stable Lévy processes, see [Sim07]. Generalizing this result, for instance to asymptotically α -stable processes, remains an important open problem.

3.3.4 Overview of the rest of the paper

The rest of the paper is organized as follows.

- In Section 3.4, we develop several key tools that are crucial in our proof. We decompose the paths of the process $(X_t)_{t \geq 0}$ around the last excursion and we establish a Wiener–Hopf factorization for the bi-dimensional Lévy process $(\tau_t, Z_t)_{t \geq 0}$, using the excursions of Z_t below its supremum, in the spirit of Greenwood–Pitman [GP80]. With these tools at hand we are able to compute the Laplace transform of $(\xi_{g_e}, \xi_{g_e} - \zeta_{g_e})$ which is the key identity of our paper, see Corollary 3.1, and seems to be new to the best of our knowledge.

- Using the tools of Section 3.4, we are able to show our persistence result in two different settings. In Section 3.5 we prove Theorem 3.1 under the assumption that 0 is positive recurrent; in Section 3.6 we prove Theorem 3.2 under Assumption 3.4 (corresponding to the case where 0 is null recurrent).

- In Section 3.7 we consider the case of generalized diffusions and we give tangible conditions which ensure that Assumption 3.4 holds.

- Finally in section 3.8 we treat the case where $(X_t, \zeta_t)_{t \geq 0}$ does not start at $(0, 0)$. To handle the proof we assume that $(X_t)_{t \geq 0}$ has continuous paths.

- We also collect some technical results in Appendix: in Appendix 3.A we prove our Wiener–Hopf factorization are related results (based on well-established techniques), in Appendix 3.B we

present results on convergences of Lévy processes and of their Laplace exponents that serve in Section 3.7, in Appendix 3.C we give technical results on generalized one-dimensional diffusions that are used in Section 3.8.

3.4 Path decomposition and Wiener–Hopf factorization

3.4.1 Preliminaries on excursion theory

Recall we assumed that 0 is regular for itself, that is $\mathbb{P}(\eta_0 = 0) = 1$ where $\eta_0 := \inf\{t > 0, X_t = 0\}$ (and $\mathbb{P} = \mathbb{P}_0$), and that 0 is a recurrent point. In this setting, we have a theory of excursions of $(X_t)_{t \geq 0}$ away from 0, see for instance Bertoin [Ber96, Chapter IV] or Gettoor [Get79, Section 7]. Setting $S = \inf\{t \geq 0, X_t \neq 0\}$, then Blumenthal 0-1 law gives that either $\mathbb{P}(S = 0) = 1$ or $\mathbb{P}(S = 0) = 0$: in the first case, 0 is called an *instantaneous* point; in the second case, 0 is called a *holding* point.

The process $(X_t)_{t \geq 0}$ possesses a local time $(L_t)_{t \geq 0}$ at the level 0, in the sense that there is a non-decreasing and continuous additive functional whose support is the closure of the zero set of $(X_t)_{t \geq 0}$. Then there exists some $m \geq 0$ such that a.s. for any $t \geq 0$

$$mL_t = \int_0^t \mathbf{1}_{\{X_s=0\}} ds.$$

The right-continuous inverse $(\tau_t)_{t \geq 0}$ of the local time is a (càdlàg) subordinator and we will denote by Φ its Laplace exponent, see (3.3). The subordinator $(\tau_t)_{t \geq 0}$ has the following representation: for any $t \geq 0$

$$\tau_t = mt + \sum_{s \leq t} \Delta \tau_s, \quad \text{with } \Delta \tau_s := \tau_s - \tau_{s-}.$$

Let us also recall that there are two cases:

- If 0 is an instantaneous point, then the Lévy measure of $(\tau_t)_{t \geq 0}$ has infinite mass.
- If 0 is a holding point, then the Lévy measure of $(\tau_t)_{t \geq 0}$ has finite mass and $m > 0$. In other words, $(\tau_t)_{t \geq 0}$ is a drifted compound Poisson process.

We denote by $\mathcal{D}$ the usual space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}$, and for $\varepsilon \in \mathcal{D}$, let us introduce the length of ε

$$\ell(\varepsilon) = \inf\{t > 0, \varepsilon_t = 0\},$$

and we will often write for simplicity ℓ instead of $\ell(\varepsilon)$.

Then the set of excursions $\mathcal{E}$ is the set of functions $\varepsilon \in \mathcal{D}$ such that: (i) $0 < \ell(\varepsilon) < \infty$; (ii) $\varepsilon_t = 0$ for every $t \geq \ell(\varepsilon)$; (iii) $\varepsilon_t \neq 0$ for every $0 < t < \ell(\varepsilon)$. This space is endowed with the usual Skorokhod's topology and the associated Borel σ -algebra.

We now introduce the excursion processes of $(X_t)_{t \geq 0}$, denoted by $(e_t)_{t \geq 0}$, which take values in $\mathcal{E}_0 = \mathcal{E} \cup \{\Upsilon\}$, where Υ is an isolated cemetery point. The excursion process is given by

$$e_t = \begin{cases} (X_{\tau_t-+s})_{s \in [0, \Delta \tau_t]} & \text{if } \Delta \tau_t > 0, \\ \Upsilon & \text{otherwise,} \end{cases}$$

A famous result, essentially due to Itô [Itô72], states that $(e_t)_{t \geq 0}$ is a Poisson point process and we denote by $\mathbf{n}$ its characteristic measure, which is defined for any measurable set Γ by

$$\mathbf{n}(\Gamma) = \frac{1}{t} \mathbb{E}[N_{t,\Gamma}] \quad \text{where} \quad N_{t,\Gamma} = \sum_{s \leq t} \mathbf{1}_\Gamma(e_s).$$

Here again, let us distinguish two cases:

- If 0 is an instantaneous point, then $\mathbf{n}$ has infinite mass
- If 0 is a holding point, then $\mathbf{n}$ has finite mass and is proportional to the law of e_1 under $\mathbb{P}$.

For a non-negative measurable function $F : \mathcal{E}_0 \rightarrow \mathbb{R}$, we also write $\mathbf{n}(F) = \int_{\mathcal{E}} F(\varepsilon) \mathbf{n}(d\varepsilon)$. Let us stress that, from the exponential formula for Poisson point processes, we can express the Laplace exponent Φ of $(\tau_t)_{t \geq 0}$ in terms of $\mathbf{m}$ and $\mathbf{n}$:

$$\Phi(q) = \mathbf{m}q + \mathbf{n}(1 - e^{-q\ell}). \quad (3.19)$$

Recall the definition (3.4) of the Lévy process $(Z_t)_{t \geq 0}$, and recall Assumption 3.1 on the function f . Let us stress that if 0 is an instantaneous point, then the Lévy measure ν has infinite mass, whereas if 0 is a holding point, then $(Z_t)_{t \geq 0}$ is a compound Poisson process.

Recalling the last expression in (3.4), the Lévy measure ν of $(Z_t)_{t \geq 0}$ can be expressed through the excursion measure $\mathbf{n}$. For $\varepsilon \in \mathcal{E}$, we define

$$\mathfrak{F}(\varepsilon) = \int_0^\ell f(\varepsilon_s) ds. \quad (3.20)$$

Then the exponential formula for Poisson point processes ensures that $\nu(dz) = \mathbf{n}(\mathfrak{F} \in dz)$. Then, Assumption 3.2 that the process $(X_t)_{t \geq 0}$ cannot change sign without touching 0 (and is not of constant sign) can be reformulated as follows:

Assumption 2’. *The measure $\mathbf{n}$ is supported by the set of excursions of constant sign. Moreover, $\mathbf{n}$ is neither supported by the set of positive excursions, nor by the set of negative excursions.*

This assumption, combined with Assumption 3.1 on the function f , leads to the following remark, which is crucial in our study.

Remark 3.3. *Under Assumptions 3.1 and 2’, $(|Z_t|)_{t \geq 0}$ is not a subordinator. Moreover we have almost surely $\sup_{[0, \tau_t]} \zeta_s = \sup_{[0, t]} Z_s$, for every $t \geq 0$. Indeed, for every $s \geq 0$, $(X_t)_{t \geq 0}$ is of constant sign on the time-interval $[\tau_{s-}, \tau_s]$ and consequently $t \mapsto \zeta_t$ is monotone on every such interval. As a consequence, the supremum is necessarily reached at the extremities, i.e. we have $\sup_{u \in [\tau_{s-}, \tau_s]} \zeta_u = \zeta_{\tau_{s-}} \vee \zeta_{\tau_s}$.*

3.4.2 Decomposing paths around the last excursion

We now show a path decomposition at an exponential random time. Recall the definition (3.5) of $g_t = \sup\{s \leq t, X_s = 0\}$, the last return to 0 of X before time t . The following result, inspired by Salminen, Vallois and Yor [SVY07, Thm 9] allows us to decouple the path of X at an exponential random time into two independent parts, before and after the last return to 0. An important application of this independence has already been presented in (3.16). Additionally, Proposition 3.3 provides useful formulas for computing functionals of the path before and after the last zero, respectively.

To make the statement more precise, let us introduce some notation: for $t > 0$ we let $\mathcal{D}_t$ denote the space of càdlàg functions from $[0, t)$ to $\mathbb{R}$, and for $t \geq 0$ we let $\tilde{\mathcal{D}}_t$ denote the space of càdlàg functions from $[0, t]$ to $\mathbb{R}$.

Proposition 3.3. *Let $e = e(q)$ be an exponential random variable of parameter q , independent of $(X_t)_{t \geq 0}$. Then the processes $(X_u)_{0 \leq u < g_e}$ and $(X_{g_e+v})_{0 \leq v \leq e - g_e}$ are independent. Moreover, for all non negative functionals $F_1 : \bigcup_{t > 0} \mathcal{D}_t \rightarrow \mathbb{R}_+$ and $F_2 : \bigcup_{t \geq 0} \tilde{\mathcal{D}}_t \rightarrow \mathbb{R}_+$, we have*

$$\mathbb{E}[F_1((X_u)_{0 \leq u < g_e})] = \Phi(q) \int_0^\infty \mathbb{E}[F_2((X_u)_{0 \leq u < \tau_t}) e^{-q\tau_t}] dt \quad (3.21)$$

and

$$\mathbb{E}[F_2((X_{g_e+v})_{0 \leq v \leq e-g_e})] = \frac{q}{\Phi(q)} \left(mF_2(\tilde{0}) + n \left(\int_0^\ell e^{-qu} F_2((\varepsilon_v)_{0 \leq v \leq u}) du \right) \right), \quad (3.22)$$

where $\tilde{0}$ denotes the null function of $\tilde{\mathcal{D}}_0$.

Let us stress that the strict inequality in considering $(X_u)_{0 \leq u < g_e}$ is crucial in the proof.

Proof. Let F_1 and F_2 be as in the statement. We have

$$\begin{aligned} \mathbb{E}[F_1((X_u)_{0 \leq u < g_e}) F_2((X_{g_e+v})_{0 \leq v \leq e-g_e})] \\ = \mathbb{E} \left[q \int_0^\infty e^{-qt} F_1((X_u)_{0 \leq u < g_t}) F_2((X_{g_t+v})_{0 \leq v \leq t-g_t}) dt \right]. \end{aligned}$$

We decompose the above integral in two parts, according to whether $X_t = 0$ or not. If g and d denotes the left and right endpoints of an excursion, then the excursion straddling time t is the only excursion such that $g \leq t \leq d$. Indexing the excursions by their left endpoint g , *i.e.* by $\mathcal{G} = \{\tau_{t-}; \Delta\tau_t > 0, t \in \mathbb{R}_+\}$, the previous display is then equal to

$$\begin{aligned} \mathbb{E} \left[q \int_0^\infty e^{-qt} F_1((X_u)_{0 \leq u < t}) F_2(\tilde{0}) \mathbf{1}_{\{X_t=0\}} dt \right] \\ + \mathbb{E} \left[q \int_0^\infty e^{-qt} \sum_{g \in \mathcal{G}} F_1((X_u)_{0 \leq u < g}) F_2((X_{g+v})_{0 \leq v \leq t-g}) \mathbf{1}_{\{g < t < d\}} dt \right]. \end{aligned}$$

Regarding the first term, it is equal to

$$mF_2(\tilde{0})q \mathbb{E} \left[\int_0^\infty e^{-qt} F_1((X_u)_{0 \leq u < t}) dL_t \right] = mF_2(\tilde{0})q \int_0^\infty \mathbb{E} [F_1((X_u)_{0 \leq u < \tau_t}) e^{-q\tau_t}] dt.$$

For the second term, it is equal to

$$\mathbb{E} \left[\sum_{g \in \mathcal{G}} F_1((X_u)_{0 \leq u < g}) e^{-qg} \int_0^\ell q e^{-qt} F_2((X_{g+v})_{0 \leq v \leq t}) dt \right],$$

where we simply used that $(\tau_t)_{t \geq 0}$ is the right-continuous inverse of L_t .

Using the compensation formula for Poisson point processes, we end up with

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty F_1((X_u)_{0 \leq u < \tau_t}) e^{-q\tau_t} n \left(\int_0^\ell e^{-qu} F_2((\varepsilon_v)_{0 \leq v \leq u}) du \right) dt \right] \\ = \int_0^\infty \mathbb{E} [F_1((X_u)_{0 \leq u < \tau_t}) e^{-q\tau_t}] dt \times n \left(\int_0^\ell e^{-qu} F_2((\varepsilon_v)_{0 \leq v \leq u}) du \right). \end{aligned}$$

To summarize, we have showed that $\mathbb{E}[F_1((X_u)_{0 \leq u < g_e}) F_2((X_{g_e+v})_{0 \leq v \leq e-g_e})]$ is equal to

$$\int_0^\infty \mathbb{E} [F_1((X_u)_{0 \leq u < \tau_t}) e^{-q\tau_t}] dt \times q \left(mF_2(\tilde{0}) + n \left(\int_0^\ell e^{-qu} F_2((\varepsilon_v)_{0 \leq v \leq u}) du \right) \right),$$

and the result follows since we have $\Phi(q) = (\int_0^\infty \mathbb{E}[e^{-q\tau_t}] dt)^{-1} = mq + n(1 - e^{-q\ell})$. $\square$

3.4.3 Wiener–Hopf factorization

In this section, we derive a Wiener–Hopf factorization for the bivariate Lévy process $(\tau_t, Z_t)_{t \geq 0}$. This factorization is similar to the one in Isozaki [Iso96] although the factorization therein is somehow incomplete and our method is a bit different: we follow the approach of Greenwood and Pitman [GP80] which can also be found in Bertoin [Ber96, Chapter VI]. We will only display in this section the results needed as it is not the main purpose of the paper: the proofs are postponed to Appendix 3.A. In fact, our results hold for any bivariate Lévy process $(\tau_t, Z_t)_{t \geq 0}$ for which $(\tau_t)_{t \geq 0}$ is a subordinator.

We define $S_t = \sup_{[0,t]} Z_t$ the running supremum of Z_t . Then the reflected process $(R_t)_{t \geq 0} = (S_t - Z_t)_{t \geq 0}$ is a strong Markov process (see [Ber96, Proposition VI.1]) and possesses a local time $(L_t^R)_{t \geq 0}$ at 0 and we denote by $(\sigma_t)_{t \geq 0}$ its right-continuous inverse, called the *ladder time* process. Next we define $\theta_t := \tau_{\sigma_t}$ and $H_t := S_{\sigma_t}$ the *ladder heights processes*. Then $(\sigma_t, \theta_t, H_t)_{t \geq 0} := (\sigma_t, \tau_{\sigma_t}, S_{\sigma_t})_{t \geq 0}$ is a trivariate subordinator possibly killed at some exponential random time, according to whether or not 0 is recurrent for $(R_t)_{t \geq 0}$ (see Lemma 3.8 in Appendix 3.A). We denote by κ its Laplace exponent: for any non-negative α, β, γ :

$$\kappa(\alpha, \beta, \gamma) = -\log \mathbb{E} \left[e^{-\alpha \sigma_1 - \beta \theta_1 - \gamma H_1} \mathbf{1}_{\{1 < L_\infty^R\}} \right]. \quad (3.23)$$

If 0 is transient for $(R_t)_{t \geq 0}$, then $\kappa(0, 0, 0) > 0$ and L_∞^R has an exponential distribution of parameter $\kappa(0, 0, 0)$ whereas if 0 is recurrent, then $\kappa(0, 0, 0) = 0$ and $L_\infty^R = \infty$ a.s. In any case, using the convention $e^{-\infty} = 0$, we have for any $\alpha, \beta, \gamma \geq 0$ such that $\alpha + \beta + \gamma > 0$,

$$\mathbb{E} \left[e^{-\alpha \sigma_t - \beta \theta_t - \gamma H_t} \right] = \exp(-\kappa(\alpha, \beta, \gamma)t). \quad (3.24)$$

Let us also set $G_t = \sup\{s < t, Z_s = S_s\}$ be the last time before t where $(Z_t)_{t \geq 0}$ attains its supremum, or equivalently the last return to 0 before t of $(R_t)_{t \geq 0}$.

We now state a Wiener–Hopf factorization for the process $(\tau_t, Z_t)_{t \geq 0}$, which will allow us below to obtain (among other things), the joint Laplace transform of ξ_{g_e} and $\xi_{g_e} - \zeta_{g_e}$, see Corollary 3.1 below. Let us stress that we need to separate two cases, according to whether 0 is regular or irregular for the reflected process $(R_t)_{t \geq 0}$.

Theorem 3.3 (Wiener–Hopf factorization). *Let $e = e(q)$ be an exponential random variable of parameter q , independent of $(\tau_t, Z_t)_{t \geq 0}$.*

(A) *If 0 is irregular for the reflected process $(R_t)_{t \geq 0}$, then we have the following:*

(i) *The triplets (G_e, τ_{G_e}, S_e) and $(e - G_e, \tau_e - \tau_{G_e}, Z_e - S_e)$ are independent.*

(ii) *The law of (G_e, τ_{G_e}, S_e) is infinitely divisible and its Lévy measure is*

$$\mu_+(dt, dr, dx) = \frac{e^{-qt}}{t} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt, \quad \text{for } t > 0, r \geq 0, x \geq 0.$$

(iii) *The law of $(e - G_e, \tau_e - \tau_{G_e}, Z_e - S_e)$ is infinitely divisible and its Lévy measure is*

$$\mu_-(dt, dr, dx) = \frac{e^{-qt}}{t} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt, \quad \text{for } t > 0, r \geq 0, x < 0.$$

(B) *If 0 is regular for the reflected process, then the same statements hold with τ_{G_e} replaced by τ_{G_e-} and $\tau_e - \tau_{G_e}$ replaced by $\tau_e - \tau_{G_e-}$.*

Let us give a corollary, which is key in our analysis, that expresses the Laplace transform of $(S_e, S_e - Z_e, \tau_e)$; let us stress that a more general formula is available, see (3.54) in Appendix 3.A.

Proposition 3.4. *Let $e = e(q)$ be an exponential random variable of parameter q , independent of $(\tau_t, Z_t)_{t \geq 0}$. There exists a constant $c > 0$ such that*

$$\kappa(\alpha, \beta, \gamma) = c \exp \left(\int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha t - \beta r - \gamma x}}{t} \mathbf{1}_{\{x \geq 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt \right).$$

Also, for any positive α, β, γ , we have

$$\mathbb{E} \left[e^{-\alpha S_e - \beta(Z_e - S_e) - \gamma \tau_e} \right] = \frac{\kappa(q, 0, 0)}{\kappa(q, \gamma, \alpha)} \frac{\bar{\kappa}(q, 0, 0)}{\bar{\kappa}(q, \gamma, \beta)}, \quad (3.25)$$

where we have defined

$$\bar{\kappa}(\alpha, \beta, \gamma) = \exp \left(\int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha t - \beta r + \gamma x}}{t} \mathbf{1}_{\{x < 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt \right).$$

Remark 3.4. *We can also introduce the same objects as above for the dual process $(\hat{Z}_t)_{t \geq 0} = (-Z_t)_{t \geq 0}$. If we set $(\hat{S}_t)_{t \geq 0} = (\sup_{[0, t]} \hat{Z}_t)_{t \geq 0}$, then the dual reflected process $(\hat{R}_t)_{t \geq 0} = (\hat{S}_t - \hat{Z}_t)_{t \geq 0}$ also possesses a local time at 0 denoted by $(L_t^{\hat{R}})_{t \geq 0}$, with right-continuous inverse $(\hat{\sigma}_t)_{t \geq 0}$. Finally, we set $(\hat{\sigma}_t, \hat{\theta}_t, \hat{H}_t)_{t \geq 0} = (\hat{\sigma}_t, \tau_{\hat{\sigma}_t}, \hat{S}_{\hat{\sigma}_t})_{t \geq 0}$, which is again a trivariate subordinator possibly killed at some exponential random time with Laplace exponent that we denote $\hat{\kappa}$, i.e. such that $\hat{\kappa}(\alpha, \beta, \gamma) = -\log \mathbb{E} \left[e^{-\alpha \sigma_1 - \beta \theta_1 - \gamma H_1} \mathbf{1}_{\{1 < L_\infty^{\hat{R}}\}} \right]$ for any non-negative α, β, γ . Then, we prove in Appendix 3.A that if $(Z_t)_{t \geq 0}$ is not a compound Poisson process, there exists a constant $\hat{c} > 0$ such that $\hat{\kappa} = \hat{c} \bar{\kappa}$.*

3.4.4 Consequences of the Wiener–Hopf factorization

From Proposition 3.4, using also Proposition 3.3, we can compute the Laplace transform of $(\xi_{g_e}, \xi_{g_e} - \zeta_{g_e})$ in terms of κ and $\bar{\kappa}$, where $e = e(q)$ is an exponential random variable of parameter $q > 0$ independent of the rest.

Corollary 3.1. *Let $e = e(q)$ be an exponential random variable of parameter q , independent of $(X_t)_{t \geq 0}$. Then for every $\lambda, \mu \geq 0$, we have*

$$\mathbb{E} \left[e^{-\lambda \xi_{g_e} - \mu(\xi_{g_e} - \zeta_{g_e})} \right] = \frac{\kappa(0, q, 0)}{\kappa(0, q, \lambda)} \frac{\bar{\kappa}(0, q, 0)}{\bar{\kappa}(0, q, \mu)}. \quad (3.26)$$

As a consequence, ξ_{g_e} and $\xi_{g_e} - \zeta_{g_e}$ are independent, the law of ξ_{g_e} , $\xi_{g_e} - \zeta_{g_e}$ and ζ_{g_e} are infinitely divisible.

Proof. Since $(\zeta_t)_{t \geq 0}$ and $(\xi_t)_{t \geq 0}$ are continuous, we have $\zeta_{g_e} = \zeta_{g_e-}$ and $\xi_{g_e} = \xi_{g_e-}$ and we can apply Proposition 3.3-(3.21):

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda \xi_{g_e} - \mu(\xi_{g_e} - \zeta_{g_e})} \right] &= \Phi(q) \int_0^\infty \mathbb{E} \left[e^{-\lambda \xi_{\tau_t} - \mu(\xi_{\tau_t} - \zeta_{\tau_t}) - q \tau_t} \right] dt \\ &= \Phi(q) \int_0^\infty \mathbb{E} \left[e^{-\lambda S_t - \mu(S_t - Z_t) - q \tau_t} \right] dt. \end{aligned}$$

In the last equality, we have used Remark 3.3 which tells us that $\xi_{\tau_t} = S_t$ a.s. We now introduce some additional Laplace variable $\tilde{q} > 0$ so that the above integral becomes a quantity evaluated at an exponential random variable $\tilde{e}$ of parameter $\tilde{q}$. We write:

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda \xi_{g_e} - \mu(\xi_{g_e} - \zeta_{g_e})} \right] &= \lim_{q \rightarrow 0} \Phi(q) \int_0^\infty \mathbb{E} \left[e^{-\lambda S_t - \mu(S_t - Z_t) - q \tau_t - \tilde{q} t} \right] dt \\ &= \lim_{q \rightarrow 0} \frac{\Phi(q)}{\tilde{q}} \mathbb{E} \left[e^{-\lambda S_{\tilde{e}} - \mu(S_{\tilde{e}} - Z_{\tilde{e}}) - q \tau_{\tilde{e}}} \right], \end{aligned}$$

where $\tilde{e} = \tilde{e}(\tilde{q})$ is an independent exponential random variable of parameter $\tilde{q}$. Therefore, using (3.25), we obtain

$$\mathbb{E} \left[e^{-\lambda \xi_{g_e} - \mu(\xi_{g_e} - \zeta_{g_e})} \right] = \lim_{\tilde{q} \rightarrow 0} \frac{\Phi(q)}{\tilde{q}} \frac{\kappa(\tilde{q}, 0, 0)}{\kappa(\tilde{q}, q, \lambda)} \frac{\bar{\kappa}(\tilde{q}, 0, 0)}{\bar{\kappa}(\tilde{q}, q, \mu)}. \quad (3.27)$$

Using Frullani's formula $\int_0^\infty \frac{e^{-u} - e^{-bu}}{u} du = \ln(b)$, we can write

$$\Phi(q) = \exp \left(\int_0^\infty \frac{e^{-t} - e^{-\Phi(q)t}}{t} dt \right), \quad \frac{1}{\tilde{q}} = \exp \left(- \int_0^\infty \frac{e^{-t} - e^{-\tilde{q}t}}{t} dt \right),$$

so that writing $e^{-\Phi(q)t} = \mathbb{E}[e^{-q\tau_t}]$, we obtain

$$\frac{\Phi(q)}{\tilde{q}} = \exp \left(\int_0^{+\infty} \int_0^{+\infty} \frac{e^{-\tilde{q}t} - e^{-qr}}{t} \mathbb{P}(\tau_t \in dr) dt \right) = \left(\frac{\Phi(q)}{\tilde{q}} \right)^+ \left(\frac{\Phi(q)}{\tilde{q}} \right)^-,$$

where we have set

$$\begin{aligned} \left(\frac{\Phi(q)}{\tilde{q}} \right)^+ &:= \exp \left(\int_0^{+\infty} \int_{[0, +\infty) \times \mathbb{R}} \frac{e^{-\tilde{q}t} - e^{-qr}}{t} \mathbf{1}_{\{x \geq 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt \right), \\ \left(\frac{\Phi(q)}{\tilde{q}} \right)^- &:= \exp \left(\int_0^{+\infty} \int_{[0, +\infty) \times \mathbb{R}} \frac{e^{-\tilde{q}t} - e^{-qr}}{t} \mathbf{1}_{\{x < 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt \right). \end{aligned}$$

Then, by Proposition 3.4, we observe that for all $\tilde{q} > 0$,

$$\begin{aligned} \left(\frac{\Phi(q)}{\tilde{q}} \right)^+ \kappa(\tilde{q}, 0, 0) &= c \exp \left(\int_0^{+\infty} \int_{[0, +\infty) \times \mathbb{R}} \frac{e^{-t} - e^{-qr}}{t} \mathbf{1}_{\{x \geq 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt \right) \\ &= \kappa(0, q, 0), \end{aligned}$$

and

$$\begin{aligned} \left(\frac{\Phi(q)}{\tilde{q}} \right)^- \bar{\kappa}(\tilde{q}, 0, 0) &= \exp \left(\int_0^{+\infty} \int_{[0, +\infty) \times \mathbb{R}} \frac{e^{-t} - e^{-qr}}{t} \mathbf{1}_{\{x < 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt \right) \\ &= \bar{\kappa}(0, q, 0). \end{aligned}$$

Finally equation (3.27) gives that

$$\mathbb{E} \left[e^{-\lambda \xi_{g_e} - \mu(\xi_{g_e} - \zeta_{g_e})} \right] = \lim_{\tilde{q} \rightarrow 0} \frac{\kappa(0, q, 0)}{\kappa(\tilde{q}, q, \lambda)} \frac{\bar{\kappa}(0, q, 0)}{\bar{\kappa}(\tilde{q}, q, \mu)} = \frac{\kappa(0, q, 0)}{\kappa(0, q, \lambda)} \frac{\bar{\kappa}(0, q, 0)}{\bar{\kappa}(0, q, \mu)},$$

which is the desired result.

Finally, we show that the law of ξ_{g_e} and $\xi_{g_e} - \zeta_{g_e}$ are infinitely divisible. Let us start with ξ_{g_e} . By Proposition 3.4 and the expression of $\kappa(\alpha, \beta, \gamma)$, we have for any $\lambda > 0$,

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda \xi_{g_e}} \right] &= \exp \left(\int_0^\infty \int_{[0, +\infty) \times \mathbb{R}} t^{-1} e^{-qr} (e^{-\lambda x} - 1) \mathbf{1}_{\{x \geq 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt \right) \\ &= \exp \left(\int_{(0, \infty)} (e^{-\lambda x} - 1) \mu(dx) \right), \end{aligned}$$

where we have set

$$\mu(dx) = \mathbf{1}_{\{x > 0\}} \int_{[0, \infty)^2} t^{-1} e^{-qr} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt.$$

It only remains to show that $\int_{(0,\infty)} 1 \wedge x \mu(dx) < \infty$. We have

$$\int_{(0,\infty)} 1 \wedge x \mu(dx) = \int_0^\infty t^{-1} \mathbb{E} \left[e^{-q\tau_t} (1 \wedge Z_t) \mathbf{1}_{\{Z_t > 0\}} \right] dt.$$

It can be shown, see for instance Lemma 3.11, that there exists a constant $C > 0$ such that for any $t \geq 0$, $\mathbb{E}[1 \wedge |Z_t|] \leq C\sqrt{t}$ which is enough to check that $\int_{(0,\infty)} 1 \wedge x \mu(dx) < \infty$. This shows that ξ_{g_e} is infinitely divisible. The same proof carries on for $\xi_{g_e} - \zeta_{g_e}$. Since these variables are independent, it comes that $\zeta_{g_e} = \xi_{g_e} + (\zeta_{g_e} - \xi_{g_e})$ is also infinitely divisible. $\square$

From this result, we can deduce a formula for $\mathbb{P}(\xi_{g_e} < z)$ with $z > 0$ by inverting Laplace transforms. For $q > 0$, define the function $\mathcal{V}_q$ on $\mathbb{R}_+$ as

$$\mathcal{V}_q(z) = \mathbb{E} \left[\int_0^\infty e^{-q\theta_t} \mathbf{1}_{\{H_t \leq z\}} dt \right], \quad (3.28)$$

where we recall that $(\theta_t, H_t)_{t \geq 0}$ are the ladder heights, defined in Section 3.4.3. Recalling from (3.6) that for any $z \geq 0$, $\mathcal{V}(z) = \int_0^\infty \mathbb{P}(H_t \leq z) dt$, we observe that $\mathcal{V}_q(z)$ increases to $\mathcal{V}(z)$ as $q \rightarrow 0$.

Corollary 3.2. *For any $z > 0$, $\mathbb{P}(\xi_{g_e} < z) = \kappa(0, q, 0) \mathcal{V}_q(z)$.*

Proof. By Corollary 3.1, we have for any $q > 0$ and any $\lambda > 0$

$$\lambda \int_0^\infty e^{-\lambda z} \mathbb{P}(\xi_{g_e} < z) dz = \mathbb{E} \left[e^{-\lambda \xi_{g_e}} \right] = \frac{\kappa(0, q, 0)}{\kappa(0, q, \lambda)}.$$

By the definition of $\mathcal{V}_q$, we also have

$$\lambda \int_0^\infty e^{-\lambda z} \mathcal{V}_q(z) dz = \mathbb{E} \left[\int_0^\infty e^{-q\theta_t - \lambda H_t} dt \right] = \int_0^\infty e^{-t\kappa(0, q, \lambda)} dt = \frac{1}{\kappa(0, q, \lambda)},$$

and the results holds by injectivity of the Laplace transform. $\square$

We finally end this section with the following proposition, which allows us to decompose $\mathbb{P}(\xi_e < z)$, our quantity of interest.

Proposition 3.5. *The random variables ξ_{g_e} , $\xi_{g_e} - \zeta_{g_e}$ and I_e are mutually independent. Moreover, we have for any $z > 0$,*

$$\mathbb{P}(\xi_e < z) = \mathbb{P}(\xi_{g_e} < z) \mathbb{P}(\Delta_e \leq 0) + \mathbb{P}(\xi_{g_e} + \Delta_e < z, \Delta_e \in (0, z)), \quad (3.29)$$

where $\Delta_e = I_e + \zeta_{g_e} - \xi_{g_e} = \zeta_e - \xi_{g_e}$.

Proof. By Proposition 3.3 and since $\xi_{g_e} = \xi_{g_e-}$ and $\zeta_{g_e} = \zeta_{g_e-}$, we get that I_e is independent of $(\xi_{g_e}, \xi_{g_e} - \zeta_{g_e})$. Then by Corollary 3.1, ξ_{g_e} and $\xi_{g_e} - \zeta_{g_e}$ are independent. This implies that the three random variables are mutually independent.

Now observe that thanks to Assumptions 3.1 and 3.2, $s \mapsto \zeta_s$ is monotone on the interval $[g_t, t]$ for every $t > 0$. This implies that $\sup_{s \in [g_t, t]} \zeta_s = \zeta_{g_t} \vee \zeta_t$ which in turn implies that $\xi_t = \xi_{g_t} \vee \zeta_t$. We get the identity

$$\xi_t = \xi_{g_t} \mathbf{1}_{\{\Delta_t \leq 0\}} + (\xi_{g_t} + \Delta_t) \mathbf{1}_{\{\Delta_t > 0\}} = \xi_{g_t} + \max(\Delta_t, 0). \quad (3.30)$$

This allows to derive the desired identity. $\square$

3.5 The positive recurrent case: proof of Theorem 3.1

In this section, we focus on the case $\mathbf{n}(\ell) < \infty$, which corresponds to the positive recurrent case, *i.e.* Assumption 3.3. In this case, we have that $\tau_t \approx t$ as $t \rightarrow \infty$ and thus we should expect ξ_t to behave as S_t , as we shall see. We recall that the Laplace exponent Φ is expressed as $\Phi(q) = mq + \mathbf{n}(1 - e^{-q\ell})$. Therefore, if $\mathbf{n}(\ell) < \infty$, we have $\Phi(q) \sim (m + \mathbf{n}(\ell))q$ as $q \rightarrow 0$ which shows that $\mathbb{E}[\tau_1] = m + \mathbf{n}(\ell) = m_1 < \infty$. Then by the strong law of large numbers for subordinators, it holds that a.s.

$$\lim_{t \rightarrow \infty} \frac{1}{t} \tau_t = m_1. \quad (3.31)$$

Our main goal is to prove Theorem 3.2 under Assumption 3.3. We procede in three steps: we deal with the contribution of the unfinished excursion (this will show that I_e can be neglected); we study the behavior of $\kappa(0, q, 0)$ as $q \downarrow 0$; we conclude the proof by combining the above with Corollary 3.2 and Proposition (3.5).

3.5.1 Estimate of the last excursion

Let us first estimate I_e as $q \downarrow 0$. It turns out that in the positive recurrent case, it converges in law.

Lemma 3.1. *If $\mathbf{n}(\ell) < \infty$, then I_e converges in law as $q \rightarrow 0$ to some random variable I_0 which is a.s. finite and such that $\mathbb{P}(I_0 \leq 0) > 0$.*

Proof. Let $F : \mathbb{R} \rightarrow \mathbb{R}_+$ be a bounded continuous function. Using Proposition 3.3, we have

$$\mathbb{E}[F(I_e)] = \frac{q}{\Phi(q)} \left(mF(0) + \mathbf{n} \left(\int_0^\ell e^{-qt} F \left(\int_0^t f(\varepsilon_r) dr \right) dt \right) \right).$$

Again, $\Phi(q) \sim m_1 q$ as $q \rightarrow 0$, and we get by the dominated convergence theorem that

$$\mathbb{E}[F(I_e)] \rightarrow \frac{mF(0) + \mathbf{n} \left(\int_0^\ell F \left(\int_0^t f(\varepsilon_r) dr \right) dt \right)}{m_1} \quad \text{as } q \rightarrow 0,$$

which shows the result. Since 0 is recurrent for $(X_t)_{t \geq 0}$, for $\mathbf{n}$ -almost every excursions ε , ε has a finite lifetime ℓ , which shows that I_0 is a.s. finite. By Assumption 2', $\mathbf{n}$ charges the set of negative excursions so that we necessarily have $\mathbb{P}(I_0 \leq 0) > 0$. $\square$

3.5.2 Asymptotic behavior of the Laplace exponent

Let us now study the asymptotic behavior of $\kappa(0, q, 0)$ as $q \downarrow 0$. Since by Corollary 3.2 we have that $\mathbb{P}(\xi_{g_e} < z) = \kappa(0, q, 0) \mathcal{V}_q(z)$ with $\mathcal{V}_q(z) \uparrow \mathcal{V}(z)$ as $q \downarrow 0$, the behavior of $\kappa(0, q, 0)$ characterizes the behavior of $\mathbb{P}(\xi_{g_e} < z)$.

Theorem 3.4. *Assume that $\mathbf{n}(\ell) < \infty$ and let e be an exponential variable with parameter q , independent of $(X_t)_{t \geq 0}$. Then the following assertions are equivalent for any $\rho \in (0, 1)$:*

- (i) $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}(Z_s \geq 0) ds = \rho$;
- (ii) $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}(\zeta_s \geq 0) ds = \rho$;
- (iii) *The function $q \mapsto \kappa(0, q, 0)$ is regularly varying as $q \downarrow 0$, with index ρ .*

Proof. Step 1: We show that (i) is equivalent to (iii). By Proposition 3.4, we have for any $q > 0$,

$$\begin{aligned} \log \kappa(0, q, 0) &= \log c + \int_0^\infty \frac{\mathbb{E}[(e^{-t} - e^{-q\tau_t})\mathbf{1}_{\{Z_t \geq 0\}}]}{t} dt \\ &= \log c + \rho \int_0^\infty \frac{e^{-t} - e^{-\Phi(q)t}}{t} dt + \int_0^\infty \frac{\mathbb{E}[(e^{-t} - e^{-q\tau_t})(\mathbf{1}_{\{Z_t \geq 0\}} - \rho)]}{t} dt \\ &= \log c + \rho \log \Phi(q) + \int_0^\infty \frac{\mathbb{E}[(e^{-t} - e^{-q\tau_t})(\mathbf{1}_{\{Z_t \geq 0\}} - \rho)]}{t} dt, \end{aligned}$$

where we used Frullani's identity in the third equality. Let us define the function ς for $q > 0$ as

$$\varsigma(q) = \exp \left(\int_0^\infty \frac{\mathbb{E}[(e^{-t} - e^{-q\tau_t})(\mathbf{1}_{\{Z_t \geq 0\}} - \rho)]}{t} dt \right). \quad (3.32)$$

Since $\Phi(q) \sim m_1 q$ as $q \rightarrow 0$, it is clear that $q \mapsto \kappa(0, q, 0)$ is regularly varying as $q \rightarrow 0$ with index ρ if and only if ς is slowly varying at 0. We have for $\lambda > 0$ and $q > 0$

$$\begin{aligned} \log \varsigma(\lambda q) - \log \varsigma(q) &= \int_0^\infty \frac{\mathbb{E}[(e^{-q\tau_t} - e^{-\lambda q\tau_t})(\mathbf{1}_{\{Z_t \geq 0\}} - \rho)]}{t} dt \\ &= \int_0^\infty \frac{\mathbb{E}[(e^{-q\tau_{t/q}} - e^{-\lambda q\tau_{t/q}})(\mathbf{1}_{\{Z_{t/q} \geq 0\}} - \rho)]}{t} dt = \mathbf{I} + \mathbf{II}, \end{aligned} \quad (3.33)$$

where we have set

$$\begin{aligned} \mathbf{I} &:= \int_0^\infty \frac{e^{-m_1 t} - e^{-m_1 \lambda t}}{t} (\mathbb{P}(Z_{t/q} \geq 0) - \rho) dt, \\ \mathbf{II} &:= \int_0^\infty \frac{1}{t} \mathbb{E}[(e^{-q\tau_{t/q}} - e^{-m_1 t} + e^{-m_1 \lambda t} - e^{-\lambda q\tau_{t/q}})(\mathbf{1}_{\{Z_{t/q} \geq 0\}} - \rho)] dt. \end{aligned}$$

We first show that the term $\mathbf{II}$ always converges to 0 under the assumption $\mathbf{n}(\ell) < \infty$. Let us set $F(t, q) = F_1(t, q) + F_2(t, q)$ where $F_1(t, q) = \mathbb{E}[(e^{-q\tau_{t/q}} - e^{-m_1 t})(\mathbf{1}_{\{Z_{t/q} \geq 0\}} - \rho)]$ and $F_2(t, q) = \mathbb{E}[(e^{-m_1 \lambda t} - e^{-\lambda q\tau_{t/q}})(\mathbf{1}_{\{Z_{t/q} \geq 0\}} - \rho)]$. By (3.31), we have that for any $t \geq 0$, $q\tau_{t/q} \rightarrow m_1 t$ a.s. as $q \rightarrow 0$. Therefore by the dominated convergence theorem, we have for any $t \geq 0$, $F(t, q) \rightarrow 0$ as $q \rightarrow 0$, and it only remains to dominate $F(t, q)$ to get that $\mathbf{II} = \int_0^\infty \frac{1}{t} F(t, q) dt$ goes to zero as $q \downarrow 0$. We only show that we can dominate F_1 ; the same domination can be done on F_2 . We write

$$\begin{aligned} |F_1(t, q)| &\leq 2\mathbb{E}[|e^{-q\tau_{t/q}} - e^{-m_1 t}|] \leq 2\mathbb{E}[|e^{-q\tau_{t/q}} - e^{-m_1 t}|^2]^{1/2} \\ &\leq 2 \left(e^{-2t\Phi(2q)/2q} - 2e^{-(m_1 + \Phi(q)/q)t} + e^{-2m_1 t} \right)^{1/2}. \end{aligned}$$

We set $b_+ = \sup_{q \in (0, 1)} \Phi(q)/q$ and $b_- = \inf_{q \in (0, 1)} \Phi(q)/q$ which are two finite and strictly positive real numbers. Then it is clear that for any $q \in (0, \frac{1}{2})$, for any $t > 0$,

$$\frac{1}{t} |F_1(t, q)| \leq \frac{2}{t} (e^{-2tb_-} - 2e^{-(m_1 + b_+)t} + e^{-2m_1 t})^{1/2}.$$

The term on the right-hand side is integrable: indeed, for $t \in [0, 1]$ it is bounded by a constant times $t^{-1/2}$, whereas for $t > 1$ it is bounded by $2(e^{-2tb_-} + e^{-2m_1 t})^{1/2}$. We can conclude by the dominated convergence theorem that $\mathbf{II} = \int_0^\infty \frac{1}{t} F(t, q) dt$ goes to zero as $q \downarrow 0$.

At this point, we know that (iii) holds if and only if for any $\lambda > 0$, the term $\mathbf{I}$ goes to 0 as $q \downarrow 0$. We now use well-known results about Lévy processes. The usual Laplace exponent of the ladder time process of $(Z_t)_{t \geq 0}$ is $q \mapsto \kappa(q, 0, 0)$ and it is well-known, see for instance [Ber96, Prop. VI.18 and its proof] that (i) holds if and only if $q \mapsto \kappa(q, 0, 0)$ is regularly varying at 0 with index ρ . Recalling Proposition 3.4, we have for any $q > 0$,

$$\begin{aligned} \log \kappa(q, 0, 0) &= \log c + \int_0^\infty \frac{e^{-t} - e^{-qt}}{t} \mathbb{P}(Z_t \geq 0) dt \\ &= \log c + \rho \log q + \int_0^\infty \frac{e^{-t} - e^{-qt}}{t} (\mathbb{P}(Z_t \geq 0) - \rho) dt. \end{aligned}$$

We therefore obtain that (i) holds if and only if, for any $\lambda > 0$

$$\log \kappa(\lambda q, 0, 0) - \log \kappa(q, 0, 0) - \rho \log \lambda = \int_0^\infty \frac{e^{-q\lambda t} - e^{-qt}}{t} (\mathbb{P}(Z_t \geq 0) - \rho) dt \longrightarrow 0$$

as $q \downarrow 0$. But this is clearly equivalent to having the term $\mathbf{I}$ going to 0 as $q \downarrow 0$, for any $\lambda > 0$. This concludes the proof that (i) holds if and only if (iii) holds.

Step 2: Next, we show that (i) holds if and only if we have (ii') $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}(\zeta_{g_s} \geq 0) ds = \rho$. Let $e = e(q)$ be an independent exponential random variable of parameter $q > 0$ and $e' = e(m_1 q)$ be an independent exponential random variable of parameter $m_1 q > 0$. We have

$$\mathbb{P}(\zeta_{g_e} \geq 0) = q \int_0^\infty e^{-qt} \mathbb{P}(\zeta_{g_t} \geq 0) dt \quad \text{and} \quad \mathbb{P}(Z_{e'} \geq 0) = m_1 q \int_0^\infty e^{-m_1 q t} \mathbb{P}(Z_t \geq 0) dt.$$

Then by the Tauberian theorem, see [BGT87, Thm. 1.7.1], we have that $\lim_{q \downarrow 0} \mathbb{P}(Z_{e'} \geq 0) = \rho$ if and only if (i) holds, and $\lim_{q \downarrow 0} \mathbb{P}(\zeta_{g_e} \geq 0) = \rho$ if and only if $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}(\zeta_{g_s} \geq 0) ds = \rho$. Therefore, to conclude, it only remains to show that $\lim_{q \rightarrow 0} |\mathbb{P}(\zeta_{g_e} \geq 0) - \mathbb{P}(Z_{e'} \geq 0)| = 0$.

Applying Proposition 3.3-(3.21) with the functional $F_1 = \mathbf{1}_{\{\zeta_{g_t} \geq 0\}}$, we get

$$\mathbb{P}(\zeta_{g_e} \geq 0) = \Phi(q) \int_0^\infty \mathbb{E} \left[e^{-q\tau_t} \mathbf{1}_{\{Z_t \geq 0\}} \right] dt.$$

Then we obtain that $|\mathbb{P}(\zeta_{g_e} \geq 0) - \mathbb{P}(Z_{e'} \geq 0)|$ is bounded by

$$\begin{aligned} & \left| \int_0^\infty \mathbb{E} \left[(\Phi(q) e^{-q\tau_t} - m_1 q e^{-m_1 q t}) \mathbf{1}_{\{Z_t \geq 0\}} \right] dt \right| \\ & \leq |\Phi(q) - m_1 q| \int_0^\infty \mathbb{E} \left[e^{-q\tau_t} \mathbf{1}_{\{Z_t \geq 0\}} \right] dt + m_1 \int_0^\infty \mathbb{E} \left[|q e^{-q\tau_t} - q e^{-m_1 q t}| \mathbf{1}_{\{Z_t \geq 0\}} \right] dt \\ & \leq |\Phi(q) - m_1 q| \frac{1}{\Phi(q)} + m_1 \int_0^\infty \mathbb{E} \left[|e^{-q\tau_t/q} - e^{-m_1 t}| \right] dt. \end{aligned}$$

By assumption, we have $\lim_{q \rightarrow 0} \Phi(q)/q = m_1$, so the first term goes to 0. By the law of large numbers (3.31) and dominated convergence, we have that for any $t \geq 0$, $\mathbb{E}[|e^{-q\tau_t/q} - e^{-m_1 t}|]$ converges to 0 as $q \downarrow 0$. We conclude again by dominated convergence that the second term goes to 0, since $\mathbb{E}[|e^{-q\tau_t/q} - e^{-m_1 t}|] \leq \mathbb{E}[e^{-q\tau_t/q}] + e^{-m_1 t} = e^{-t\Phi(q)/q} + e^{-m_1 t} \leq e^{-b-t} + e^{-m_1 t}$.

Step 3: Finally, we show that (ii) is equivalent to (ii') $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}(\zeta_{g_s} \geq 0) ds = \rho$. Again, by the Tauberian theorem, (ii) holds if and only if $\lim_{q \downarrow 0} \mathbb{P}(\zeta_e \geq 0) = \rho$ and (ii') holds if and only if $\lim_{q \rightarrow 0} \mathbb{P}(\zeta_{g_e} \geq 0) = \rho$ (with $e = e(q)$ an independent exponential random variable of parameter $q > 0$). Therefore it is enough to show that $\lim_{q \rightarrow 0} |\mathbb{P}(\zeta_{g_e} \geq 0) - \mathbb{P}(\zeta_e \geq 0)| = 0$ in the case $n(\ell) < \infty$. To prove this, we write

$$\mathbb{P}(\zeta_{g_e} \geq 0) - \mathbb{P}(\zeta_e \geq 0) = \mathbb{P}(\zeta_{g_e} \geq 0, \zeta_e < 0) - \mathbb{P}(\zeta_{g_e} < 0, \zeta_e \geq 0).$$

We will only show that $\lim_{q \rightarrow 0} \mathbb{P}(\zeta_{g_e} \geq 0, \zeta_e < 0) = 0$ as the proof for the other term is similar. By Lemma 3.1 above, the term $I_e = \zeta_e - \zeta_{g_e}$ converges in law as $q \rightarrow 0$ to some real random variable I_0 . We let $A > 0$ and we decompose the probability as

$$\mathbb{P}(\zeta_{g_e} \geq 0, \zeta_e < 0) = \mathbb{P}(\zeta_{g_e} \geq 0, \zeta_{g_e} + I_e < 0, I_e < -A) + \mathbb{P}(\zeta_{g_e} \geq 0, \zeta_{g_e} + I_e < 0, I_e \geq -A),$$

which yields the following inequality:

$$\mathbb{P}(\zeta_{g_e} \geq 0, \zeta_e < 0) \leq \mathbb{P}(I_e < -A) + \mathbb{P}(\zeta_{g_e} \in [0, A]).$$

By Proposition 3.3, we have

$$\mathbb{P}(\zeta_{g_e} \in [0, A]) = \Phi(q) \int_0^\infty \mathbb{E} \left[e^{-q\tau_t} \mathbf{1}_{\{Z_t \in [0, A]\}} \right] dt = \frac{\Phi(q)}{q} \int_0^\infty \mathbb{E} \left[e^{-q\tau_{t/q}} \mathbf{1}_{\{Z_{t/q} \in [0, A]\}} \right] dt.$$

For every $A > 0$, we have $\lim_{t \rightarrow \infty} \mathbb{P}(Z_t \in [0, A]) = 0$, see e.g. Sato [Sat13, Ch. 9 Lem. 48.3]. Since $\lim_{q \rightarrow 0} \Phi(q)/q = m_1$ and $\mathbb{E}[e^{-q\tau_{t/q}} \mathbf{1}_{\{Z_{t/q} \in [0, A]\}}] \leq \mathbb{E}[e^{-q\tau_{t/q}}] = e^{-t\Phi(q)/q} \leq e^{-b-t}$, it follows from the dominated convergence theorem that $\lim_{q \rightarrow 0} \mathbb{P}(\zeta_{g_e} \in [0, A]) = 0$. We deduce that

$$\limsup_{q \rightarrow 0} \mathbb{P}(\zeta_{g_e} \geq 0, \zeta_e < 0) \leq \limsup_{q \rightarrow 0} \mathbb{P}(I_e < -A) \leq \mathbb{P}(I_0 \leq A).$$

Since I_0 is a.s. finite, the result follows by letting $A \rightarrow \infty$. $\square$

3.5.3 Conclusion of the proof of Theorem 3.1 under Assumption 3.3

We are finally able to prove our main theorem in the positive recurrent case.

Proof. We assume that $\mathbf{n}(\ell) < \infty$, and will first show that, by setting $c_0 = \liminf_{q \rightarrow 0} \mathbb{P}(\Delta_e \leq 0)$, we have $c_0 \in (0, 1]$ and for any $z > 0$,

$$c_0 \mathcal{V}(z) \leq \liminf_{q \rightarrow 0} \frac{\mathbb{P}(\xi_e < z)}{\kappa(0, q, 0)} \leq \limsup_{q \rightarrow 0} \frac{\mathbb{P}(\xi_e < z)}{\kappa(0, q, 0)} \leq \mathcal{V}(z). \quad (3.34)$$

Then we will see that if $q \mapsto \mathbb{P}(\xi_e < z)$ is regularly varying at 0 with index $\rho \in (0, 1)$, or if $t^{-1} \int_0^t \mathbb{P}(\zeta_s \geq 0) ds \rightarrow \rho$ as $t \rightarrow \infty$, then necessarily $c_0 = 1$. Using (3.29), we deduce the following inequalities

$$\mathbb{P}(\xi_{g_e} < z) \mathbb{P}(\Delta_e \leq 0) \leq \mathbb{P}(\xi_e < z) \leq \mathbb{P}(\xi_{g_e} < z).$$

By Corollary 3.2, we have $\mathbb{P}(\xi_{g_e} < z) = \kappa(0, q, 0) \mathcal{V}_q(z)$ for any $z > 0$, and since $\mathcal{V}_q(z) \rightarrow \mathcal{V}(z)$, (3.34) holds if we can show that $\mathbb{P}(\Delta_e \leq 0)$ converges to some positive constant as $q \rightarrow 0$.

Remember that I_e and $\zeta_{g_e} - \xi_{g_e}$ are independent, that $\Delta_e = I_e + \zeta_{g_e} - \xi_{g_e}$, and that by Lemma 3.1, I_e converges in law as $q \rightarrow 0$. We will show that, either $\zeta_{g_e} - \xi_{g_e}$ converges to $-\infty$ as $q \rightarrow 0$ in probability, or $\zeta_{g_e} - \xi_{g_e}$ converges in law to some (finite) random variable as $q \rightarrow 0$. By Proposition 3.3, we have for any $\mu > 0$,

$$\mathbb{E} \left[e^{-\mu(\xi_{g_e} - \zeta_{g_e})} \right] = \Phi(q) \int_0^\infty \mathbb{E} \left[e^{-q\tau_t - \mu(S_t - Z_t)} \right] dt = \frac{\Phi(q)}{q} \int_0^\infty \mathbb{E} \left[e^{-q\tau_{t/q} - \mu(S_{t/q} - Z_{t/q})} \right] dt.$$

Recall that $\Phi(q)/q \rightarrow m_1$ as $q \rightarrow 0$ and that $q\tau_{t/q} \rightarrow m_1 t$ as $q \rightarrow 0$. Duality entails that for every $t \geq 0$, $S_t - Z_t$ is equal in law to $\hat{S}_t$ where $\hat{S}_t = \sup_{s \in [0, t]} \hat{Z}_s$ and $\hat{Z}_t = -Z_t$, see [Ber96, Ch. VI Prop. 3]. Since $(\hat{S}_t)_{t \geq 0}$ is increasing, $\hat{S}_t \rightarrow \hat{S}_\infty$ as $t \rightarrow \infty$. By the 0-1 law, $\mathbb{P}(\liminf_{t \rightarrow \infty} Z_t = \infty)$ is equal to 0 or 1, two cases are to be considered:

- If $\liminf_{t \rightarrow \infty} Z_t = \infty$ a.s., then $\hat{S}_\infty = \infty$ a.s. and thus, $S_t - Z_t$ converges in probability to ∞ as $t \rightarrow \infty$. Since $\mathbb{E}[e^{-q\tau_t/q - \mu(S_{t/q} - Z_{t/q})}] \leq \mathbb{E}[e^{-q\tau_t/q}] = e^{-t\Phi(q)/q}$, we can apply the dominated convergence theorem which shows that $\xi_{g_e} - \zeta_{g_e}$ converges in probability to ∞ as $q \rightarrow 0$. It comes that $\mathbb{P}(\Delta_e \leq 0) \rightarrow 1$ as $q \rightarrow 0$.
- If $\liminf_{t \rightarrow \infty} Z_t < \infty$ a.s., then $\hat{S}_\infty < \infty$ a.s. and thus $S_t - Z_t$ converges in law to $\hat{S}_\infty$ as $t \rightarrow \infty$. By Slutsky's lemma, for any $t \geq 0$, $(q\tau_t/q, S_{t/q} - Z_{t/q})$ converges in law to $(m_1 t, \hat{S}_\infty)$ and we can again apply the dominated convergence theorem, which shows that $\xi_{g_e} - \zeta_{g_e}$ converges in law to some non-negative finite random variable that we name $\xi_0 - \zeta_0$. Then we have $\mathbb{P}(\Delta_e \leq 0) \rightarrow \mathbb{P}(\zeta_0 - \xi_0 + I_0 \leq 0)$ where $\zeta_0 - \xi_0$ and I_0 are independent. By Lemma 3.1, $\mathbb{P}(I_0 \leq 0) > 0$ which shows that $\mathbb{P}(\zeta_0 - \xi_0 + I_0 \leq 0) > 0$.

We showed that in any case (3.34) holds. Let us now assume first that $t^{-1} \int_0^t \mathbb{P}(\zeta_s \geq 0) ds \rightarrow \rho$ as $t \rightarrow \infty$ for some $\rho \in (0, 1)$. Then by Theorem 3.4, it also holds that $t^{-1} \int_0^t \mathbb{P}(Z_s \geq 0) ds \rightarrow \rho$ as $t \rightarrow \infty$, which implies that $\liminf_{t \rightarrow \infty} Z_t = \infty$ a.s., see for instance [Ber96, Theorem VI.12], and therefore the constant c_0 in (3.34) is equal to 1. We then have $\mathbb{P}(\xi_e < z) \sim \kappa(0, q, 0)\mathcal{V}(z)$ which shows by Theorem 3.4 that $q \mapsto \mathbb{P}(\xi_e < z)$ is regularly varying with index ρ .

Let us now assume that $q \mapsto \mathbb{P}(\xi_e < z)$ is regularly varying at 0 with index $\rho \in (0, 1)$. Then by (3.34), for any $\delta > 0$, $q^{\rho+\delta}/\kappa(0, q, 0) \rightarrow 0$ as $q \rightarrow 0$. Now remember from Proposition 3.4 that

$$\kappa(0, q, 0) = c \exp \left(\int_0^\infty \frac{\mathbb{E}[(e^{-t} - e^{-q\tau_t}) \mathbf{1}_{\{Z_t \geq 0\}}] dt}{t} \right)$$

and

$$\bar{\kappa}(0, q, 0) = \exp \left(\int_0^\infty \frac{\mathbb{E}[(e^{-t} - e^{-q\tau_t}) \mathbf{1}_{\{Z_t < 0\}}] dt}{t} \right).$$

We see by Frullani's identity that $\kappa(0, q, 0)\bar{\kappa}(0, q, 0) = c\Phi(q)$ and since $\Phi(q) \sim m_1 q$ as $q \rightarrow 0$ it comes that $\bar{\kappa}(0, q, 0) \rightarrow 0$ as $q \rightarrow 0$. Recall that by Corollary 3.1, for any $\mu > 0$, we have $\mathbb{E}[e^{-\mu(\xi_{g_e} - \zeta_{g_e})}] = \bar{\kappa}(0, q, 0)/\bar{\kappa}(0, q, \mu)$, and since $\bar{\kappa}(0, q, \mu) \rightarrow \bar{\kappa}(0, 0, \mu) > 0$ as $q \rightarrow 0$, we see that $\mathbb{E}[e^{-\mu(\xi_{g_e} - \zeta_{g_e})}] \rightarrow 0$ as $q \rightarrow 0$ which shows that, in this case $\xi_{g_e} - \zeta_{g_e}$ converges in probability to ∞ . From the previous analysis, we see that we are necessarily in the case $\liminf_{t \rightarrow \infty} Z_t = \infty$ a.s. and therefore the constant c_0 from (3.34) is equal to 1 so that $\mathbb{P}(\xi_e < z) \sim \kappa(0, q, 0)\mathcal{V}(z)$ as $q \rightarrow 0$. This shows that $q \mapsto \kappa(0, q, 0)$ is also regularly varying at 0 with index ρ and thus, by Theorem 3.4, $t^{-1} \int_0^t \mathbb{P}(\zeta_s \geq 0) ds \rightarrow \rho$ as $t \rightarrow \infty$. $\square$

3.6 The null recurrent case: proof of Theorem 3.2

In this section, we assume that 0 is null recurrent, i.e. that $\mathbf{n}(\ell) = \infty$ and suppose that Assumption 3.4 holds: in a nutshell,

$$(b(h)\tau_{t/h}, a(h)Z_{t/h})_{t \geq 0} \longrightarrow (\tau_t^0, Z_t^0)_{t \geq 0} \quad \text{as } h \rightarrow 0,$$

with $(\tau_t^0, Z_t^0)_{t \geq 0}$ a bivariate Lévy process, $(\tau_t^0)_{t \geq 0}$ being a β -stable subordinator and $(Z_t^0)_{t \geq 0}$ an α -stable process. Assumption 3.4 also entails that $a(b^{-1}(h))I_{1/h}$ converges in law as $q \rightarrow 0$ to some (possibly degenerate) random variable I .

Remark 3.5. *Let us stress that since $(b(h)\tau_{t/h})_{t \geq 0}$ converges in law to $(\tau_t^0)_{t \geq 0}$ as $h \rightarrow 0$, the Laplace exponent $\Phi(q)$ of $(\tau_t)_{t \geq 0}$ is regularly varying at 0 with index β . More precisely, $\Phi(\cdot)$ is an asymptotic inverse (up to a constant) of $b(\cdot)$ near 0. Indeed, $(b(q)\tau_{t/q})$ converges in law to a β -stable subordinator, and therefore*

$$-\frac{1}{t} \log \mathbb{E} [e^{-b(q)\tau_{t/q}}] = \frac{\Phi(b(q))}{q} \xrightarrow{q \rightarrow 0} -\frac{1}{t} \log \mathbb{E} [e^{-\tau_t^0}].$$

This also implies, see [BGT87, Thm. 1.5.12], that $q/b(\Phi(q))$ converges to some positive constant $\bar{b}$ as $q \rightarrow 0$. For simplicity and without loss of generality, we assume in the following that $\bar{b} = 1$. Since $(b(\Phi(q))\tau_{t/\Phi(q)}, a(\Phi(q))Z_{t/\Phi(q)})_{t \geq 0}$ converges in law to $(\tau_t^0, Z_t^0)_{t \geq 0}$ as $q \rightarrow 0$, it comes that

$$(q\tau_{t/\Phi(q)}, a(\Phi(q))Z_{t/\Phi(q)})_{t \geq 0} \longrightarrow (\tau_t^0, Z_t^0)_{t \geq 0} \quad \text{as } q \rightarrow 0. \quad (3.35)$$

Below, we use some convergence results for Lévy processes, whose proofs are collected in Appendix 3.B.

3.6.1 Laplace exponent and convergence of scaled $\xi_{g_e} - \zeta_{g_e}$

We start with the following result.

Proposition 3.6. *Suppose that Assumption 3.4 holds. Then the function $q \mapsto \kappa(0, q, 0)$ is regularly varying as $q \rightarrow 0$ with index $\beta\rho$ with $\rho := \mathbb{P}(Z_t^0 \geq 0)$.*

Proof. Recalling Proposition 3.4, we start by writing that

$$\begin{aligned} \log \kappa(0, q, 0) &= \log c + \int_0^\infty \frac{\mathbb{E}[(e^{-t} - e^{-q\tau_t})\mathbf{1}_{\{Z_t \geq 0\}}]}{t} dt \\ &= \log c + \int_0^\infty \frac{e^{-t/\Phi(q)} - e^{-t}}{t} \mathbb{P}(Z_{t/\Phi(q)} \geq 0) dt + \int_0^\infty \frac{\mathbb{E}[(e^{-t} - e^{-q\tau_{t/\Phi(q)}})\mathbf{1}_{\{a(\Phi(q))Z_{t/\Phi(q)} \geq 0\}}]}{t} dt. \end{aligned}$$

Using the convergence (3.35) and Proposition 3.14 in Appendix 3.B, the last term converges as $q \rightarrow 0$. Then, with $\rho = \mathbb{P}(Z_t^0 \geq 0)$, the second term becomes, by Frullani's identity

$$\rho \log \Phi(q) + \int_0^\infty \frac{e^{-t} - e^{-\Phi(q)t}}{t} (\mathbb{P}(Z_t \geq 0) - \rho) dt.$$

As we argued in the proof of Theorem 3.4, the function

$$q \mapsto \exp \left(\int_0^\infty \frac{e^{-t} - e^{-\Phi(q)t}}{t} (\mathbb{P}(Z_t \geq 0) - \rho) dt \right)$$

is slowly varying as $q \rightarrow 0$. Since Φ is regularly varying with index β , it comes that $q \mapsto \kappa(0, q, 0)$ is regularly varying with index $\beta\rho$, which establishes the results. $\square$

Proposition 3.7. *Suppose that Assumption 3.4 holds and let e be an exponential variable with parameter $q > 0$, independent of $(X_t)_{t \geq 0}$. Then $a(\Phi(q))(\xi_{g_e} - \zeta_{g_e})$ converges in law as $q \rightarrow 0$: more precisely, we have that for any $\lambda > 0$*

$$\lim_{q \rightarrow 0} \mathbb{E} \left[e^{-\lambda a(\Phi(q))(\xi_{g_e} - \zeta_{g_e})} \right] = \frac{\bar{\kappa}^0(0, 1, 0)}{\bar{\kappa}^0(0, 1, \lambda)},$$

where $\bar{\kappa}^0$ is defined, analogously to $\bar{\kappa}$, as

$$\bar{\kappa}^0(\alpha, \beta, \gamma) = \exp \left(\int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha t - \beta r + \gamma x}}{t} \mathbf{1}_{\{x < 0\}} \mathbb{P}(\tau_t^0 \in dr, Z_t^0 \in dx) dt \right).$$

Proof. By Corollary 3.1, we have that the Laplace transform of $a(\Phi(q))(\xi_{g_e} - \zeta_{g_e})$ is

$$\mathbb{E} \left[e^{-\lambda a(\Phi(q))(\xi_{g_e} - \zeta_{g_e})} \right] = \frac{\bar{\kappa}(0, q, 0)}{\bar{\kappa}(0, q, \lambda a(\Phi(q)))}.$$

Now, by the definition of $\bar{\kappa}$ (in Proposition 3.4), we have

$$\log \bar{\kappa}(0, q, 0) - \log \bar{\kappa}(0, q, \lambda a(\Phi(q))) = \int_0^\infty \frac{\mathbb{E} \left[(e^{-q\tau_t + \lambda a(\Phi(q))Z_t} - e^{-q\tau_t}) \mathbf{1}_{\{Z_t < 0\}} \right]}{t} dt.$$

Making the change of variables $t = u/\Phi(q)$ and splitting the integral in two parts, we get that the above quantity is equal to

$$\begin{aligned} \int_0^\infty t^{-1} \mathbb{E} \left[(e^{-t} - e^{-q\tau_{t/\Phi(q)}}) \mathbf{1}_{\{Z_{t/\Phi(q)} < 0\}} \right] dt \\ - \int_0^\infty t^{-1} \mathbb{E} \left[(e^{-t} - e^{-q\tau_{t/\Phi(q)} + \lambda a(\Phi(q))Z_{t/\Phi(q)}}) \mathbf{1}_{\{Z_{t/\Phi(q)} < 0\}} \right] dt. \end{aligned}$$

By (3.35) and Proposition 3.14, $\log \bar{\kappa}(0, q, 0) - \log \bar{\kappa}(0, q, \lambda a(\Phi(q)))$ is a difference of two converging terms. More precisely, Proposition 3.14 entails that it converges to $\log \bar{\kappa}^0(0, 1, 0) - \log \bar{\kappa}^0(0, 1, \lambda)$, which completes the proof. $\square$

3.6.2 Conclusion of the proof of Theorem 3.2 under Assumption 3.4

Finally, we are able to prove our main theorem in the null recurrent case.

Proof. We start by recalling that, from (3.29), for any $z > 0$ we have

$$\mathbb{P}(\xi_e < z) = \mathbb{P}(\xi_{g_e} < z) \mathbb{P}(\Delta_e \leq 0) + \mathbb{P}(\xi_{g_e} + \Delta_e < z, \Delta_e \in (0, z)).$$

We also remind that by Proposition 3.5, the random variable I_e , ξ_{g_e} and $\xi_{g_e} - \zeta_{g_e}$ are mutually independent. Since $\Delta_e = I_e + \zeta_{g_e} - \xi_{g_e}$, it is independent from ξ_{g_e} and we get that

$$\mathbb{P}(\xi_{g_e} < z) \mathbb{P}(\Delta_e \leq 0) \leq \mathbb{P}(\xi_e < z) \leq \mathbb{P}(\xi_{g_e} < z) \mathbb{P}(\Delta_e < z).$$

By Corollary 3.2, we have $\mathbb{P}(\xi_{g_e} < z) = \kappa(0, q, 0) \mathcal{V}_q(z)$ and since $\kappa(0, q, 0)$ is regularly varying at 0 with index $\beta\rho$ and $\mathcal{V}_q(z)$ increases to $\mathcal{V}(z)$, it only remains to control $\mathbb{P}(\Delta_e \leq 0)$ and $\mathbb{P}(\Delta_e < z)$. By Assumption 3.4, and recalling that $\Phi(\cdot)$ is an asymptotic inverse of $b(\cdot)$, we have that $a(\Phi(h))I_{1/h}$ converges in law as $h \rightarrow 0$ to some random variable I . This easily implies that $a(\Phi(q))I_e$ converges in law as $q \downarrow 0$ and thanks to Proposition 3.7 so does $a(\Phi(q))(\zeta_{g_e} - \xi_{g_e})$. Since I_e and $\zeta_{g_e} - \xi_{g_e}$ are independent, it gives that $a(\Phi(q))\Delta_e$ converges in law as $q \rightarrow 0$ to some random variable, that we denote Δ_∞ . Finally, we end up with

$$\lim_{q \downarrow 0} \mathbb{P}(\Delta_e \leq 0) = \lim_{q \downarrow 0} \mathbb{P}(\Delta_e < z) = \mathbb{P}(\Delta_\infty \leq 0),$$

which completes the proof. $\square$

3.6.3 The case of Gaussian fluctuations

In this section, we treat the special case where Assumption 3.4 is satisfied with $\alpha = 2$, *i.e.* when the limiting process $(Z_t^0)_{t \geq 0}$ is a Brownian motion. This case is somehow simpler since $(\tau_t^0)_{t \geq 0}$ and $(Z_t^0)_{t \geq 0}$ are then necessarily independent (this can be seen directly from the Lévy-Khintchine formula), and the convergence of $(b(h)\tau_{t/h})_{t \geq 0}$ and $(a(h)Z_{t/h})_{t \geq 0}$ alone implies the convergence of the bi-dimensional process, see Lemma 3.12. We are going to prove that in that case, we can somehow weaken Assumption 3.4 (relaxing the assumption on the convergence of the last part $a(b^{-1}(h))I_{1/h}$).

Recall that for a generic excursion ε , we denote $\mathfrak{F} = \mathfrak{F}(\varepsilon) = \int_0^\ell f(\varepsilon_s) ds$, see (3.20).

Assumption 3.8. *There exists some $\beta \in (0, 1)$ such that Φ is regularly varying at 0 with index β , and $\mathfrak{n}(\mathfrak{F}) = 0$, $\mathfrak{n}(\mathfrak{F}^2) < \infty$.*

We then have the following result.

Theorem 3.5. *Suppose that Assumption 3.8 holds. Then there exists a slowly varying function ς such that for any $z > 0$,*

$$\mathbb{P}(T_z > t) = \mathcal{V}(z)\varsigma(t)t^{-\beta/2} \quad \text{as } t \rightarrow \infty,$$

where β is given by Assumption 3.8.

Proof. We simply check that Assumption 3.8 implies that Assumption 3.4 is satisfied. Let us denote by $\nu(dz) = \mathfrak{n}(\mathfrak{F} \in dz)$ the Lévy measure of $(Z_t)_{t \geq 0}$. Then Assumption 3.8 entails that $\int_{\mathbb{R} \setminus \{0\}} z^2 \nu(dz) < \infty$, which implies that $(h^{1/2}Z_{t/h})_{t \geq 0}$ converges in law towards a Brownian motion $(Z_t^0)_{t \geq 0}$ as $h \rightarrow 0$ (this can be easily checked via the characteristic function).

Moreover, if b is an asymptotic inverse of Φ at 0, we have that $(b(h)\tau_{t/h})_{t \geq 0}$ converges in law to a β -stable subordinator $(\tau_t^0)_{t \geq 0}$ as $h \rightarrow 0$. By Lemma 3.12, this shows that the bivariate process $(b(h)\tau_{t/h}, h^{1/2}Z_{t/h})_{t \geq 0}$ converges in law to $(\tau_t^0, Z_t^0)_{t \geq 0}$ as $h \rightarrow 0$.

Finally, we show that $\Phi(q)^{1/2}I_e$ converges to 0 in L^2 as $q \rightarrow 0$. By Proposition 3.3-(3.22), setting $\mathfrak{F}_u(\varepsilon) = \int_0^u f(\varepsilon_v)dv$ for any $u \leq \ell(\varepsilon)$, we have

$$\mathbb{E} [\Phi(q)I_e^2] = q \int_{\mathcal{E}} \int_0^{\ell(\varepsilon)} e^{-qu} \mathfrak{F}_u(\varepsilon)^2 du \mathfrak{n}(d\varepsilon) \leq \int_{\mathcal{E}} \mathfrak{F}(\varepsilon)^2 (1 - e^{-q\ell(\varepsilon)}) \mathfrak{n}(d\varepsilon).$$

Indeed Assumptions 3.1 and 2' entail that the map $u \mapsto \mathfrak{F}_u(\varepsilon)$ is monotonic on $[0, \ell(\varepsilon)]$ and thus $\mathfrak{F}_u(\varepsilon)^2 \leq \mathfrak{F}(\varepsilon)^2$. Since $\mathfrak{n}(\mathfrak{F}^2) < +\infty$, we can use dominated convergence to deduce that $\lim_{q \rightarrow 0} \mathbb{E}[\Phi(q)I_e^2] = 0$. We can then apply Theorem 3.2 with $\rho = \mathbb{P}(Z_t^0 \geq 0) = 1/2$. $\square$

3.7 Application to generalized diffusions

In this section we apply our result to a large class of one-dimensional Markov processes called *generalized diffusions*. These processes are defined as a time and space changed Brownian motion. Originally, it was noted by Itô and McKean [IMJ63, IMJ96] that regular diffusions, *i.e.* regular strong Markov processes with continuous paths can be represented as a time and space changed Brownian motion through a *scale function* $\mathfrak{s}$ and a *speed measure* $\mathfrak{m}$. In the mean time, Stone [Sto63] also observed that continuous-in-time birth and death processes could be represented this way. As we shall see, this construction leads to a general class of Markov processes. Our main goal is to provide conditions on the function f , the *scale function* $\mathfrak{s}$ and *speed measure* $\mathfrak{m}$ that ensure that Assumption 3.4 holds. Let us now recall the notation of Section 3.2.3.

Let $\mathfrak{m} : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing right-continuous function such that $\mathfrak{m}(0) = 0$, and $\mathfrak{s} : \mathbb{R} \rightarrow \mathbb{R}$ a continuous increasing function such that $\mathfrak{s}(0) = 0$ and $\mathfrak{s}(\mathbb{R}) = \mathbb{R}$. We assume moreover that $\mathfrak{m}$ is not constant and will also denote by $\mathfrak{m}$ the Radon measure associated to $\mathfrak{m}$, that is $\mathfrak{m}((a, b]) = \mathfrak{m}(b) - \mathfrak{m}(a)$ for all $a < b$. Recall that $\mathfrak{m}^\mathfrak{s}$ is the image of $\mathfrak{m}$ by $\mathfrak{s}$, *i.e.* the Stieltjes measure associated to the non-decreasing function $\mathfrak{m} \circ \mathfrak{s}^{-1}$. We consider a Brownian motion $(B_t)_{t \geq 0}$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with $(L_t^x)_{t \geq 0, x \in \mathbb{R}}$ the usual family of its local times and we introduce

$$A_t^{\mathfrak{m}^\mathfrak{s}} = \int_{\mathbb{R}} L_t^x \mathfrak{m}^\mathfrak{s}(dx).$$

The process $(A_t^{\mathfrak{m}^s})_{t \geq 0}$ is a non-decreasing continuous additive functional of the Brownian motion $(B_t)_{t \geq 0}$. For every $x \in \mathbb{R}$ we have $L_\infty^x = \infty$ a.s., and since the support of the measure $\mathfrak{m}^s$ is not empty we see by Fatou's lemma that $A_\infty^{\mathfrak{m}^s} = \infty$ a.s. Now we introduce $(\rho_t)_{t \geq 0}$ the right-continuous inverse of $(A_t^{\mathfrak{m}^s})_{t \geq 0}$ and we set

$$X_t = \mathfrak{s}^{-1}(B_{\rho_t}).$$

As the change of time through a continuous non-decreasing additive functional of a strong Markov process preserves the strong Markovianity, see Sharpe [Sha88, Ch. VIII Thm. 65.9], and since $\mathfrak{s}$ is bijective, it holds that $(X_t)_{t \geq 0}$ is a strong Markov process with respect to the filtration $(\mathcal{F}_{\rho_t})_{t \geq 0}$. Let us denote by $\text{supp}(\mathfrak{m}^s)$ the support of the measure $\mathfrak{m}^s$. It is rather classical, see again Sharpe [Sha88, Ch. VIII Thm. 65.9] or Revuz-Yor [RY99, Ch. X Prop. 2.17], that $(X_t)_{t \geq 0}$ is valued in $\mathfrak{s}^{-1}(\text{supp}(\mathfrak{m}^s)) = \text{supp}(\mathfrak{m})$. From now on, we will always assume that $0 \in \text{supp}(\mathfrak{m}^s)$ so that, since $\mathfrak{s}(0) = 0$, $(X_t)_{t \geq 0}$ spends time in 0. Since 0 is recurrent for the Brownian motion, it is also recurrent for $(X_t)_{t \geq 0}$. We have the following proposition, which shows that the process $(X_t)_{t \geq 0}$ satisfies the hypothesis of this article and that its local time at the level 0 can be expressed with the local time of the Brownian motion. The proof is postponed to Appendix 3.C.1.

Proposition 3.8. *The following assertions hold.*

- (i) *The family $(L_{\rho_t}^{\mathfrak{s}(x)})_{t \geq 0, x \in \mathbb{R}}$ defines a family of local times of $(X_t)_{t \geq 0}$ in the sense that a.s., for any non-negative Borel functions h , for any $t \geq 0$, the following occupation times formula holds:*

$$\int_0^t h(X_s) ds = \int_{\mathbb{R}} h(x) L_{\rho_t}^{\mathfrak{s}(x)} \mathfrak{m}(dx).$$

- (ii) *The point 0 is regular for $(X_t)_{t \geq 0}$.*

- (iii) *The process $(L_{\rho_t}^0)_{t \geq 0}$ is a proper local time for $(X_t)_{t \geq 0}$ at the level 0 in the sense that it is a continuous additive functional whose support almost surely coincides with the closure of the zero set of $(X_t)_{t \geq 0}$.*

- (iv) *Let $(\tau_t)_{t \geq 0}$ be the right-continuous inverse of $(L_{\rho_t}^0)_{t \geq 0}$ and $(\tau_t^B)_{t \geq 0}$ be the right continuous inverse of $(L_t^0)_{t \geq 0}$, then we have*

$$\tau_t = \int_{\mathbb{R}} L_{\tau_t^B}^x \mathfrak{m}^s(dx) = A_{\tau_t^B}^{\mathfrak{m}^s}.$$

Moreover, it also holds that for any $t \geq 0$, $\rho_{\tau_t} = \tau_t^B$.

In this framework, the Lévy process $Z_t = \int_0^{\tau_t} f(X_s) ds$ can be expressed, thanks to items (i) and (iv) of Proposition 3.8, as

$$Z_t = \int_{\mathbb{R}} L_{\tau_t^B}^x \mathfrak{m}^f(dx),$$

where we have set $\mathfrak{m}^f(dx) := f \circ \mathfrak{s}^{-1}(x) \mathfrak{m}^s(dx)$. Note that $\mathfrak{m}^f$ is a signed measure (recall that by Assumption 3.1, the function f preserves the sign). We suppose in addition that $f \circ \mathfrak{s}^{-1}$ is locally integrable with respect to $\mathfrak{m}^s$ so that $\mathfrak{m}^f$ is also a Radon measure. We will also denote by $\mathfrak{m}^f$ the associated function, *i.e.* $\mathfrak{m}^f(x) = \int_0^x f \circ \mathfrak{s}^{-1}(u) \mathfrak{m}^s(du)$, which is non-decreasing on $\mathbb{R}_+$ and non-increasing on $\mathbb{R}_-$. Then it holds that $(Z_t)_{t \geq 0}$ is a Lévy process with finite variations, with zero drift (since $f(0) = 0$). The aim of this section is to show Propositions 3.1 and 3.2

3.7.1 Excursions of $(X_t)_{t \geq 0}$ using those of $(B_t)_{t \geq 0}$

Let us first describe the excursions away from 0 of $(X_t)_{t \geq 0}$ in terms of the excursions of the Brownian motion. Let us denote by $\mathcal{D}$ the usual space of cadlag functions from $\mathbb{R}_+$ to $\mathbb{R}$, and for $\varepsilon \in \mathcal{D}$, let us introduce

$$\ell(\varepsilon) = \inf\{t > 0, \varepsilon_t = 0\}.$$

Then the set of excursions $\mathcal{E}$ is the set of functions $\varepsilon \in \mathcal{D}$ such that $0 < \ell(\varepsilon) < \infty$, for every $t \geq \ell(\varepsilon)$, $\varepsilon_t = 0$, and for every $0 < t < \ell(\varepsilon)$, $\varepsilon_t \neq 0$. This space is endowed with the usual Skorokhod's topology and the associated Borel σ -algebra.

We now introduce the excursion processes of $(B_t)_{t \geq 0}$ and $(X_t)_{t \geq 0}$, denoted by $(e_t^B)_{t \geq 0}$ and $(e_t^X)_{t \geq 0}$, which take values in $\mathcal{E} \cup \{\Upsilon\}$, where Υ is an isolated cemetery point, and are given by

$$e_t^B = \begin{cases} (B_{\tau_{t-}^B + s})_{s \in [0, \Delta\tau_t^B]} & \text{if } \Delta\tau_t^B > 0 \\ \Upsilon & \text{otherwise} \end{cases} \quad \text{and} \quad e_t^X = \begin{cases} (X_{\tau_{t-} + s})_{s \in [0, \Delta\tau_t]} & \text{if } \Delta\tau_t > 0 \\ \Upsilon & \text{otherwise} \end{cases}$$

A famous result, essentially due to Itô [Itô72], states that $(e_t^B)_{t \geq 0}$ and $(e_t^X)_{t \geq 0}$ are Poisson point processes and we will respectively denote by $\mathbf{n}^B$ and $\mathbf{n}$ their characteristic measure, which are defined by

$$\mathbf{n}^B(\Gamma) = \frac{1}{t} \mathbb{E} [N_{t, \Gamma}^B] \quad \text{and} \quad \mathbf{n}(\Gamma) = \frac{1}{t} \mathbb{E} [N_{t, \Gamma}^X], \quad (3.36)$$

for any measurable set Γ , where

$$N_{t, \Gamma}^B = \sum_{s \leq t} \mathbf{1}_\Gamma(e_s^B) \quad \text{and} \quad N_{t, \Gamma}^X = \sum_{s \leq t} \mathbf{1}_\Gamma(e_s^X).$$

Our first aim is to describe the measure $\mathbf{n}$ in terms of $\mathbf{n}^B$; we will see $\mathbf{n}$ as a push-forward of $\mathbf{n}^B$ by some application T . To this end, we first introduce the subset $\mathcal{C}$ of $\mathcal{E}$ of functions which are continuous. It is clear that every function in $\mathcal{C}$ has constant sign, and that $\mathbf{n}^B(\mathcal{E} \setminus \mathcal{C}) = 0$. We have the following lemma.

Lemma 3.2. *Under $\mathbf{n}^B$, almost every path possesses a family of local times $(\mathbf{L}_t^x)_{t \geq 0, x \in \mathbb{R}}$, in the sense that for any Borel function g and every $t \geq 0$, we have*

$$\int_0^t g(\varepsilon_s) ds = \int_{\mathbb{R}} g(x) \mathbf{L}_t^x dx.$$

The family $(\mathbf{L}_t^x)_{t \geq 0, x \in \mathbb{R}}$ is jointly continuous and for any $\gamma \in (0, 1/2)$ the map $x \mapsto \mathbf{L}_t^x$ is Hölder of order γ uniformly on compact time intervals.

Proof. Consider some $t \geq 0$ such that $\Delta\tau_t^B > 0$ and set for any $x \in \mathbb{R}$ and any $s \in [0, \Delta\tau_t^B]$, $\mathbf{L}_s^x = L_{\tau_{t-}^B + s}^x - L_{\tau_{t-}^B}^x$. Then $(\mathbf{L}_s^x)_{s \in [0, \Delta\tau_t^B], x \in \mathbb{R}}$ is the family of local time of e_t^B . Indeed, for any Borel function g and $s \in [0, \Delta\tau_t^B]$, we have,

$$\int_0^s g(e_t^B(u)) du = \int_0^s g(B_{\tau_{t-}^B + u}) du = \int_{\mathbb{R}} g(x) (L_{\tau_{t-}^B + s}^x - L_{\tau_{t-}^B}^x) dx.$$

Since the Brownian local times are jointly continuous and almost surely Hölder of order γ (for any $\gamma \in (0, 1/2)$) in the variable x uniformly on compact time intervals, so is $(\mathbf{L}_t^x)_{t \geq 0, x \in \mathbb{R}}$. Hence we showed, almost surely, for any t such $\Delta\tau_t^B > 0$, e_t^B has the property stated in the lemma: by (3.36), this shows that outside of a negligible set for $\mathbf{n}^B$, every path has the stated property. $\square$

We denote by $\mathcal{C}_+$ and $\mathcal{C}_-$ the subsets of $\mathcal{C}$ of positive and negative (continuous) excursions. We introduce the first point of increase and decrease of $\mathbf{m}^s$ around 0, *i.e.* the real numbers defined by

$$x_+ = \inf\{x > 0, \mathbf{m}^s(x) > \mathbf{m}^s(0)\} \quad \text{and} \quad x_- = \sup\{x < 0, \mathbf{m}^s(x) < \mathbf{m}^s(0-)\}.$$

(Note that for standard diffusions we have $x_+ = x_- = 0$, but one may have $x_- < 0 < x_+$, for instance for birth and death chains, see Section 3.2.5.) Under Assumption 2' they are finite, *i.e.* $\mathbf{m}^s$ eventually increases and decreases. For a path $\varepsilon \in \mathcal{C}$, we let $M(\varepsilon) = \sup\{|\varepsilon_t|, t \geq 0\}$ and we introduce the measurable set

$$\mathcal{C}_{x_+, x_-} = (\mathcal{C}_+ \cap \{M(\varepsilon) > x_+\}) \cup (\mathcal{C}_- \cap \{M(\varepsilon) > |x_-|\}).$$

For $\varepsilon \in \mathcal{C}_{x_+, x_-}$, we define the time-change

$$(A_t^s)_{0 \leq t \leq \ell} = \left(\int_{\mathbb{R}} \mathbf{L}_t^x \mathbf{m}^s(dx) \right)_{0 \leq t \leq \ell}. \quad (3.37)$$

Observe that $A_\ell^s > 0$ if $\varepsilon \in \mathcal{C}_{x_+, x_-}$ (whereas $A_\ell^s = 0$ if $\varepsilon \in \mathcal{C} \setminus \mathcal{C}_{x_+, x_-}$). We denote by $(\rho_t^s)_{0 \leq t \leq A_\ell^s}$ the right-continuous inverse of $(A_t^s)_{0 \leq t \leq \ell}$ and finally define the measurable application $T : \mathcal{C}_{x_+, x_-} \rightarrow \mathcal{E}$ such that,

$$T(\varepsilon)_t = \mathfrak{s}^{-1}(\varepsilon_{\rho_t^s}) \quad \text{if } t < A_\ell^s \quad \text{and} \quad T(\varepsilon)_t = 0 \quad \text{if } t \geq A_\ell^s.$$

A key tool in this section is the following result, which expresses the measure $\mathbf{n}$ as the pushforward measure of $\mathbf{n}^B$ by T .

Proposition 3.9. *For any measurable set Γ , we have $\mathbf{n}(\Gamma) = \mathbf{n}^B(T^{-1}(\Gamma))$.*

Let us note that, since $T^{-1}(\Gamma) \subset \mathcal{C}_{x_+, x_-}$, the measure $\mathbf{n}$ is a finite measure if and only if $x_+ > 0$ and $x_- < 0$, since in this case $\mathbf{n}^B(\mathcal{C}_{x_+, x_-}) < \infty$.

Proof. We first emphasize that thanks to Proposition 3.8-(iv), we have a.s., for any $t \geq 0$,

$$\Delta\tau_t = A_{\tau_t^B}^{\mathbf{m}^s} - A_{\tau_{t-}^B}^{\mathbf{m}^s} = \int_{\mathbb{R}} (L_{\tau_t^B}^x - L_{\tau_{t-}^B}^x) \mathbf{m}^s(dx).$$

Thus, it should be clear that

$$\Delta\tau_t > 0 \quad \text{if and only if} \quad \Delta\tau_t^B > 0 \quad \text{and} \quad e_t^B \in \mathcal{C}_{x_+, x_-}.$$

Let us now consider some $t \geq 0$ such that $\Delta\tau_t > 0$, *i.e.* some $t \geq 0$ such that $\Delta\tau_t^B > 0$ and $e_t^B \in \mathcal{C}_{x_+, x_-}$. We set for $x \in \mathbb{R}$ and $s \in [0, \Delta\tau_t^B]$, $\mathbf{L}_s^x = L_{\tau_{t-}^B + s}^x - L_{\tau_{t-}^B}^x$, the local time of e_t^B .

Then we introduce the time change $(A_s^t)_{s \in [0, \Delta\tau_t^B]}$, defined by

$$A_s^t = \int_{\mathbb{R}} \mathbf{L}_s^x \mathbf{m}^s(dx) = A_{\tau_{t-}^B + s}^{\mathbf{m}^s} - \tau_{t-}.$$

If $(\rho_s^t)_{s \in [0, \Delta\tau_t]}$ denotes the right-continuous inverse of $(A_s^t)_{s \in [0, \Delta\tau_t^B]}$, then for every $s \in [0, \Delta\tau_t]$, we get the following identity.

$$\rho_s^t = \inf\{u > 0, A_{\tau_{t-}^B + u}^{\mathbf{m}^s} > \tau_{t-} + s\} = \rho_{\tau_{t-} + s}^{\mathbf{m}^s} - \tau_{t-}^B.$$

Therefore, we conclude that, if t is such that $\Delta\tau_t > 0$, then we have

$$e_t^X = (\mathfrak{s}^{-1}(B_{\rho_{\tau_{t-} + s}}))_{s \in [0, \Delta\tau_t]} = (\mathfrak{s}^{-1}(B_{\tau_{t-}^B + \rho_s^t}))_{s \in [0, \Delta\tau_t]} = T(e_t^B).$$

Therefore, for any measurable set Γ , we have for any $t \geq 0$, a.s. $N_{t, \Gamma}^X = N_{t, T^{-1}(\Gamma)}^B$, which, by (3.36), shows the result. $\square$

We are now able to express some quantities of interest of $(\tau_t, Z_t)_{t \geq 0}$ using the Brownian excursion measure. We first define for any $\varepsilon \in \mathbf{C}_{x_+, x_-}$

$$(A_t^f)_{0 \leq t \leq \ell} = \left(\int_{\mathbb{R}} \mathbf{L}_t^x \mathbf{m}^f(dx) \right)_{0 \leq t \leq \ell}. \quad (3.38)$$

Lemma 3.3. *The following assertions hold.*

(i) *For any $\lambda > 0$ and any $\mu \in \mathbb{R}$,*

$$-\log \mathbb{E} \left[e^{-\lambda \tau_1 + i\mu Z_1} \right] = \mathbf{n}^B \left(1 - \exp \left(-\lambda A_\ell^{\mathfrak{s}} + i\mu A_\ell^f \right) \right).$$

(ii) *Let $e = e(q)$ be an independent exponential random variable of parameter $q > 0$. Then for any $\mu \in \mathbb{R}$,*

$$\mathbb{E} \left[e^{i\mu I_e} \right] = \frac{q}{\Phi(q)} \left(\mathbf{m} + \mathbf{n}^B \left(\int_0^\ell \exp(-q A_t^{\mathfrak{s}} + i\mu A_t^f) dA_t^{\mathfrak{s}} \right) \right),$$

where we recall that $\mathbf{m}$ is the drift coefficient of $(\tau_t)_{t \geq 0}$, see Section 3.4.1.

Proof. Let us start with the first item. By the exponential formula for Poisson point processes, it is straightforward that for any $\lambda > 0$ and any $\mu \in \mathbb{R}$,

$$\psi(\lambda, \mu) := -\log \mathbb{E} \left[e^{-\lambda \tau_1 + i\mu Z_1} \right] = \mathbf{n} \left(1 - \exp \left(-\lambda \ell + i\mu \int_0^\ell f(\varepsilon_s) ds \right) \right).$$

Applying Proposition 3.9, we get that

$$\psi(\lambda, \mu) = \mathbf{n}^B \left(\mathbf{1}_{\{\varepsilon \in \mathbf{C}_{x_+, x_-}\}} \left(1 - \exp \left(-\lambda A_\ell^{\mathfrak{s}} + i\mu \int_0^{A_\ell^{\mathfrak{s}}} f(\mathfrak{s}^{-1}(\varepsilon_{\rho_s^{\mathfrak{s}}})) ds \right) \right) \right)$$

For any $\varepsilon \in \mathbf{C}_{x_+, x_-}$, we have $\int_0^{A_\ell^{\mathfrak{s}}} f(\mathfrak{s}^{-1}(\varepsilon_{\rho_s^{\mathfrak{s}}})) ds = \int_0^\ell f(\mathfrak{s}^{-1}(\varepsilon_u)) dA_u^{\mathfrak{s}} = A_\ell^f$. Remark that if $\varepsilon \notin \mathbf{C}_{x_+, x_-}$, then $A_\ell^{\mathfrak{s}} = A_\ell^f = 0$ so that

$$\psi(\lambda, \mu) = \mathbf{n}^B \left(1 - \exp \left(-\lambda A_\ell^{\mathfrak{s}} + i\mu A_\ell^f \right) \right).$$

We now show the second item. By Proposition 3.3, we have for any $\mu \in \mathbb{R}$,

$$\mathbb{E} \left[e^{i\mu I_e} \right] = \frac{q}{\Phi(q)} \left(\mathbf{m} + \mathbf{n} \left(\int_0^\ell \exp \left(-qt + i\mu \int_0^t f(\varepsilon_s) ds \right) dt \right) \right) =: \frac{q}{\Phi(q)} (\mathbf{m} + G(\mu, q)).$$

Then, by Proposition 3.9 again, we get

$$G(\mu, q) = \mathbf{n}^B \left(\mathbf{1}_{\{\varepsilon \in \mathbf{C}_{x_+, x_-}\}} \int_0^{A_\ell^{\mathfrak{s}}} \exp \left(-qt + i\mu \int_0^t f(\mathfrak{s}^{-1}(\varepsilon_{\rho_s^{\mathfrak{s}}})) ds \right) dt \right).$$

Performing the change of variables $t = A_u^{\mathfrak{s}}$ and using the occupation time formula for the Brownian excursion, we get

$$G(\mu, q) = \mathbf{n}^B \left(\mathbf{1}_{\{\varepsilon \in \mathbf{C}_{x_+, x_-}\}} \int_0^\ell \exp(-q A_t^{\mathfrak{s}} + i\mu A_t^f) dA_t^{\mathfrak{s}} \right).$$

If $\varepsilon \notin \mathbf{C}_{x_+, x_-}$, then $A_t^{\mathfrak{s}} = 0$ for any $t \in [0, \ell]$ and we can remove $\mathbf{1}_{\{\varepsilon \in \mathbf{C}_{x_+, x_-}\}}$ from the above expression, which gives the desired statement. $\square$

3.7.2 Additive functionals for strings with regular variation

Assumptions 3.5 and 3.7 state that $\mathbf{m}^s$ and $\mathbf{m}^f$ are regularly varying and we will show that it implies that $(\tau_t, Z_t)_{t \geq 0}$ belongs to the domain of attraction of a bivariate stable process. To prove that, we remark that the jumps of the rescaled process $(\tau_t^h, Z_t^h)_{t \geq 0}$ (with $h \downarrow 0$) are also additive functionals of the Brownian motion $(B_t)_{t \geq 0}$ but involving rescaled measures $\mathbf{m}_h^s$ and $\mathbf{m}_h^f$. We need technical tools to go from the convergence of $(\mathbf{m}_h^s, \mathbf{m}_h^f)$ to the convergence of the law of those additive functionals. The goal of this section is to provide those technical tools which are mainly inspired by a work by Fitzsimmons and Yano [FY08a] and are based on classical result from regular variation theory [BGT87]. All the proofs of the following results are postponed to Appendix 3.C.2.

Let us now give some convergence results for sequences for additive functionals of Brownian excursions, in the spirit of [FY08a, Thm. 2.9]; we will apply them to $\mathbf{m}^s$ and $\mathbf{m}^f$ (more precisely to their restrictions to $\mathbb{R}_+$ and $\mathbb{R}_-$). We consider a *string*, *i.e.* a non-decreasing right-continuous function $\mathbf{m}$ on $\mathbb{R}_+$ such that $\mathbf{m}(0) = 0$, with regular variation in the sense that there are some $\alpha \in (0, 2)$, and some smooth, locally bounded slowly varying function $\Lambda : \mathbb{R}_+ \rightarrow (0, \infty)$, such that, according to the values of α , we have

- If $\alpha < 1$, then $\mathbf{m}(x) \sim \Lambda(x)x^{1/\alpha-1}$ as $x \rightarrow \infty$.
- If $\alpha = 1$, then for any $x > 0$, $(\mathbf{m}(x/h) - \mathbf{m}(1/h))/\Lambda(1/h) \rightarrow \log x$ as $h \rightarrow 0$.
- If $\alpha \in (1, 2)$, then $\lim_{x \rightarrow \infty} \mathbf{m}(x) = \mathbf{m}(\infty) < \infty$ and $\mathbf{m}(\infty) - \mathbf{m}(x) \sim \Lambda(x)x^{1/\alpha-1}$ as $x \rightarrow \infty$.

For such a string $\mathbf{m}$, we define the family of rescaled strings $\mathbf{m}_h, h > 0$ as follows:

- If $\alpha < 1$, then $\mathbf{m}_h(x) = h^{1/\alpha-1}\mathbf{m}(x/h)/\Lambda(1/h)$.
- If $\alpha = 1$, then $\mathbf{m}_h(x) = (\mathbf{m}(x/h) - \mathbf{m}(1/h))/\Lambda(1/h)$.
- If $\alpha \in (1, 2)$, then $\mathbf{m}_h(x) = h^{1/\alpha-1}(\mathbf{m}(x/h) - \mathbf{m}(\infty))/\Lambda(1/h)$.

Note that in all cases, we have $\mathbf{m}_h(dx) = \frac{h^{1/\alpha-1}}{\Lambda(1/h)}\mathbf{m}(d(x/h))$. Moreover, it is easily checked that

- If $\alpha \in (0, 1)$, then for any $x > 0$, $\mathbf{m}_h(x) \rightarrow x^{1/\alpha-1}$ as $h \rightarrow 0$.
- If $\alpha = 1$, then for any $x > 0$, $\mathbf{m}_h(x) \rightarrow \log x$ as $h \rightarrow 0$.
- If $\alpha \in (1, 2)$, then for any $x > 0$, $\mathbf{m}_h(x) \rightarrow -x^{1/\alpha-1}$ as $h \rightarrow 0$.

The following lemma shows that the regular variations of $\mathbf{m}$ implies the convergence for additive functionals with respect to the rescaled strings $\mathbf{m}_h$ (as $h \rightarrow 0$). This will reveal to be a key result in the proof of Proposition 3.2 and is close from results in [FY08a].

Lemma 3.4. *Let $\mathbf{m}$ be such a string with regular variation and $g : [0, \infty) \rightarrow \mathbb{R}$ some compactly supported continuous function such that $g(0) = 0$ and which is γ -Hölder at 0 for any $\gamma < 1/2$. Then*

$$\lim_{h \downarrow 0} \int_{\mathbb{R}_+} g(x) \mathbf{m}_h(dx) = c_\alpha \int_{\mathbb{R}_+} g(x) x^{1/\alpha-2} dx,$$

where $c_\alpha = |1/\alpha - 1|$ if $\alpha \neq 1$ and $c_\alpha = 1$ for $\alpha = 1$.

Remark 3.6. Thanks to Lemma 3.2, the previous lemma applies to the family of local times of the Brownian excursion, i.e. $x \mapsto \mathbf{L}_t^x$. For $t \in [0, \ell]$, we have $\mathbf{n}_+^B$ -a.e.,

$$\lim_{h \downarrow 0} \int_{\mathbb{R}_+} \mathbf{L}_t^x \mathbf{m}_h(dx) = c_\alpha \int_{\mathbb{R}_+} \mathbf{L}_t^x x^{1/\alpha-2} dx.$$

Note also that since $t \mapsto \int_{\mathbb{R}_+} \mathbf{L}_t^x \mathbf{m}_h(dx)$ is non-decreasing and since the limiting function $t \mapsto \int_{\mathbb{R}_+} \mathbf{L}_t^x x^{1/\alpha-2} dx$ is continuous, the convergence holds in uniform norm on compact set.

We will use the following technical lemma to go from the convergence of additive functionals for an excursion to the convergence in measure (it is a truncation and domination lemma).

Lemma 3.5. Define $\mathbf{A}_\delta := \{\varepsilon \in \mathcal{C}, \sup_{s \in [0, \ell]} |\varepsilon_s| \leq \delta\}$ then for $0 < \delta < 1$:

(i) If $\alpha \in (0, 1)$ then $\limsup_{h \downarrow 0} \mathbf{n}_+^B \left[\left(\int_{\mathbb{R}_+} \mathbf{L}_\ell^x \mathbf{m}_h(dx) \right) \mathbf{1}_{\mathbf{A}_\delta} \right] \leq 2\delta^{\frac{1}{\alpha}-1}$, which goes to 0 as $\delta \downarrow 0$.

(ii) If $\alpha \in [1, 2)$ then

$$\lim_{\delta \downarrow 0} \mathbf{n}_+^B \left[\sup_{h \in (0, 1)} \left(\int_{\mathbb{R}_+} \mathbf{L}_\ell^x \mathbf{m}_h(dx) \right)^2 \mathbf{1}_{\mathbf{A}_\delta} \right] = 0.$$

(iii) If $\alpha \in (1, 2)$ then

$$\mathbf{n}_+^B \left[\sup_{h \in (0, 1)} \left(\int_{\mathbb{R}_+} \mathbf{L}_\ell^x \mathbf{m}_h(dx) \right) \mathbf{1}_{\mathbf{A}_\delta^c} \right] < +\infty.$$

(iv) If $\alpha = 1$ then

$$\mathbf{n}_+^B \left[\sup_{h \in (0, 1)} \left(\int_0^1 \mathbf{L}_\ell^x \mathbf{m}_h(dx) \right) \mathbf{1}_{\mathbf{A}_\delta^c} \right] < +\infty.$$

Although these results are stated for functionals of positive Brownian excursions, they obviously hold for functionals of negative excursions as the measure $\mathbf{n}^B$ is invariant by the application $\varepsilon \mapsto -\varepsilon$. In the following, we apply these results to the strings $\mathbf{m}^s$ and $\mathbf{m}^f$. Indeed, since we assumed that $\mathbf{m}^s(0) = 0$ and $\mathbf{m}^f(0) = 0$, we can first apply the results to $\mathbf{m}^s$ and $\mathbf{m}^f$ restricted to $\mathbb{R}_+$, denoted $\mathbf{m}_+^s$ and $\mathbf{m}_+^f$. Then we can also apply them to $\mathbf{m}_-^s$ and $\mathbf{m}_-^f$ defined on $\mathbb{R}_-$ as $\mathbf{m}_-^s = \mathbf{m}^s - \mathbf{m}^s(0-)$, $\mathbf{m}_-^f = \mathbf{m}^f - \mathbf{m}^f(0-)$ on $(-\infty, 0)$ and $\mathbf{m}_-^s(0) = 0$, $\mathbf{m}_-^f(0) = 0$.

3.7.3 Proof of Proposition 3.2

We finally prove here Proposition 3.2. We let the reader recall Assumptions 3.5 and 3.7, which are assumed throughout this proof (they tell that $\mathbf{m}_\pm^s$ and $\mathbf{m}_\pm^f$ are *strings* with regular variation). Let us also introduce the following notation: for $a, b, x \in \mathbb{R}$, we set $\text{sgn}_{a,b}(x) = a\mathbf{1}_{\{x>0\}} + b\mathbf{1}_{\{x<0\}}$. The function sgn will denote the usual sign function, i.e. $\text{sgn}(x) = \text{sgn}_{1,-1}(x)$. Then, for $\varepsilon \in \mathcal{C}$, let us denote:

$$A_\ell^\beta(\varepsilon) = c_\beta \int_0^\ell \text{sgn}_{m_+, m_-}(\varepsilon_s) |\varepsilon_s|^{1/\beta-2} ds, \quad A_\ell^\alpha(\varepsilon) = c_\alpha \int_0^\ell \text{sgn}_{f_+, -f_-}(\varepsilon_s) |\varepsilon_s|^{1/\alpha-2} ds,$$

where c_α is as in Lemma 3.4. In the case $\alpha = 1$, $f_+ = f_- = 1$ so $A_\ell^1(\varepsilon) = \int_0^\ell \frac{ds}{\varepsilon_s}$.

We proceed in two steps: first, we prove that the rescaled Lévy process $(\tau_t^h, Z_t^h)_{t \geq 0}$ converges as $h \downarrow 0$; then we prove the convergence of the other term $a(b^{-1}(q))I_e$ as $q \downarrow 0$.

Step 1: Convergence of (τ_t^h, Z_t^h) . Let us set $b(h) = h^{1/\beta}/\Lambda_s(1/h)$ and $a(h) = h^{1/\alpha}/\Lambda_f(1/h)$. We show that $(\tau_t^h, Z_t^h)_{t \geq 0} := (b(h)\tau_{t/h}, a(h)Z_{t/h})_{t \geq 0}$ converges in distribution to $(\tau_t^0, Z_t^0)_{t \geq 0}$, where $(\tau_t^0, Z_t^0)_{t \geq 0}$ is a Lévy process, without drift (except in the case $\alpha = 1$ where the drift is some constant $\tilde{c}$) and without Brownian component, whose Lévy measure $\pi^0(dr, dz)$ is defined as

$$\pi^0(dr, dz) = \mathbf{n}^B \left(A_\ell^\beta(\varepsilon) \in dr, A_\ell^\alpha(\varepsilon) \in dz \right).$$

To this end, we will show that the following convergence holds for every $\lambda > 0$ and $\mu \in \mathbb{R}$:

$$\lim_{h \downarrow 0} \mathbb{E} \left[e^{-\lambda \tau_t^h + i\mu Z_t^h} \right] = \mathbb{E} \left[e^{-\lambda \tau_t^0 + i\mu Z_t^0} \right]. \quad (3.39)$$

Let us set for any $\lambda, h > 0$, and any $\mu \in \mathbb{R}$, $\psi_h(\lambda, \mu) = -\log \mathbb{E}[e^{-\lambda \tau_1^h + i\mu Z_1^h}]$. Then we get by Lemma 3.3-(i) that

$$\psi_h(\lambda, \mu) = \frac{1}{h} \mathbf{n}^B \left(1 - \exp \left(-\lambda b(h) \int_{\mathbb{R}} \mathbf{L}_\ell^x \mathbf{m}^s(dx) + i\mu a(h) \int_{\mathbb{R}} \mathbf{L}_\ell^x \mathbf{m}^f(dx) \right) \right).$$

We will now use the scaling property of the Brownian excursion measure: for every $h > 0$, we have $\mathbf{n}_B = h \mathbf{n}_B \circ \lambda_h^{-1}$, where $\lambda_h : \mathcal{E} \rightarrow \mathcal{E}$ is defined by $\lambda_h(\varepsilon) = (h^{-1}\varepsilon_{th^2})_{t \geq 0}$. Now if $(\mathbf{L}_t^x)_{x \in \mathbb{R}, t \in [0, \ell]}$ denotes the family of local times of ε , then $(h^{-1}\mathbf{L}_{th^2}^{hx})_{x \in \mathbb{R}, t \in [0, \ell/h^2]}$ is the family of local times of $\lambda_h(\varepsilon)$. Then we get

$$\psi_h(\lambda, \mu) = \mathbf{n}^B \left(1 - \exp \left(-\lambda \int_{\mathbb{R}} \mathbf{L}_\ell^x \mathbf{m}_h^s(dx) + i\mu \int_{\mathbb{R}} \mathbf{L}_\ell^x \mathbf{m}_h^f(dx) \right) \right),$$

where the measures $\mathbf{m}_h^s$ and $\mathbf{m}_h^f$ are defined as follows:

- For every $x \in \mathbb{R}$, $\mathbf{m}_h^s(x) = h^{1/\beta-1} \mathbf{m}^s(x/h)/\Lambda_s(1/h)$ (recall $\beta \in (0, 1)$).
- According to the values of α , we have:
 - (i) If $\alpha \in (0, 1)$, for every $x \in \mathbb{R}$, $\mathbf{m}_h^f(x) = h^{1/\alpha-1} \mathbf{m}^f(x/h)/\Lambda_f(1/h)$
 - (ii) If $\alpha = 1$, for every $x \geq 0$, $\mathbf{m}_h^f(x) = (\mathbf{m}^f(x/h) - \mathbf{m}^f(1/h))/\Lambda_f(1/h)$ and for every $x < 0$, $\mathbf{m}_h^f(x) = (\mathbf{m}^f(x/h) - \mathbf{m}^f(-1/h))/\Lambda_f(1/h)$.
 - (iii) If $\alpha \in (1, 2)$, then for every $x \in \mathbb{R}$, $\mathbf{m}_h^f(x) = h^{1/\alpha-1} (\mathbf{m}^f(x/h) - \mathbf{m}^f(\infty))/\Lambda_f(1/h)$.

Before going further, we introduce the following notation, analogous to (3.37)-(3.38)

$$A_\ell^{s,h}(\varepsilon) = \int_{\mathbb{R}} \mathbf{L}_\ell^x \mathbf{m}_h^s(dx), \quad A_\ell^{f,h}(\varepsilon) = \int_{\mathbb{R}} \mathbf{L}_\ell^x \mathbf{m}_h^f(dx), \quad (3.40)$$

so that $\psi_h(\lambda, \mu) = \mathbf{n}^B(\varphi(A_\ell^{s,h}, A_\ell^{f,h}))$ with $\varphi(x, y) = 1 - e^{-\lambda x + i\mu y}$.

Then, thanks to Assumptions 3.5 and 3.7 we can apply Lemma 3.4: we have for $\mathbf{n}^B$ -almost every excursion $\varepsilon \in \mathcal{C}$,

$$\lim_{h \downarrow 0} A_\ell^{s,h}(\varepsilon) = A_\ell^\beta(\varepsilon) \quad \text{and} \quad \lim_{h \downarrow 0} A_\ell^{f,h}(\varepsilon) = A_\ell^\alpha(\varepsilon). \quad (3.41)$$

It remains to prove that we can exchange the limits inside $\psi_h(\lambda, \mu)$. Recall that we set, for $\delta > 0$, $\mathbf{A}_\delta = \{\varepsilon \in \mathcal{C}, M(|\varepsilon|) \leq \delta\}$ and that $\mathbf{n}^B(\mathbf{A}_\delta^c) < +\infty$.

Case $\alpha \in (0, 1)$. We need to prove that $\lim_{h \downarrow 0} \psi_h(\lambda, \mu) = \psi_0(\lambda, \mu) := \mathbf{n}^B[\varphi(A_\ell^\beta, A_\ell^\alpha)]$, where ψ_0 is the characteristic exponent of the limit process $(\tau_t^0, Z_t^0)_{t \geq 0}$. We have

$$\begin{aligned} |\psi_h(\lambda, \mu) - \psi_0(\lambda, \mu)| &\leq \mathbf{n}^B \left[|\varphi(A_\ell^{s,h}, A_\ell^{f,h}) - \varphi(A_\ell^\beta, A_\ell^\alpha)| \mathbf{1}_{\mathbf{A}_\delta^c} \right] + \mathbf{n}^B \left[|\varphi(A_\ell^{s,h}, A_\ell^{f,h})| \mathbf{1}_{\mathbf{A}_\delta} \right] \\ &\quad + \mathbf{n}^B \left[|\varphi(A_\ell^\beta, A_\ell^\alpha)| \mathbf{1}_{\mathbf{A}_\delta} \right]. \end{aligned}$$

Since φ is bounded and $\mathbf{n}^B(\mathbf{A}_\delta) < +\infty$, by (3.41) and the dominated convergence theorem, it comes

$$\limsup_{h \rightarrow 0} \mathbf{n}^B \left[|\varphi(A_\ell^{s,h}, A_\ell^{f,h}) - \varphi(A_\ell^\beta, A_\ell^\alpha)| \mathbf{1}_{\mathbf{A}_\delta^c} \right] = 0.$$

Since there is some $C_{\lambda,\mu} > 0$ such that $|\varphi(x, y)| \leq C_{\lambda,\mu}(x + |y|)$ for all $x \geq 0, y \in \mathbb{R}$, from item (i) of Lemma 3.5 (decomposing for positive and negative excursions), we get that

$$\limsup_{h \rightarrow 0} \mathbf{n}^B \left[|\varphi(A_\ell^{s,h}, A_\ell^{f,h})| \mathbf{1}_{\mathbf{A}_\delta} \right] \longrightarrow 0 \quad \text{as } \delta \downarrow 0.$$

By Fatou's lemma, we also get that $\mathbf{n}^B[|\varphi(A_\ell^\beta, A_\ell^\alpha)| \mathbf{1}_{\mathbf{A}_\delta}]$ converges to 0 as $\delta \downarrow 0$. We now quickly explain the value of the constant $\rho = \mathbb{P}(Z_t^0 \geq 0)$ in Proposition 3.2. One can easily see that a representation of the limiting α -stable process $(Z_t^0)_{t \geq 0}$ is

$$Z_t^0 = \int_0^{\tau_t^B} \operatorname{sgn}_{f_+, -f_-}(B_s) |B_s|^{1/\alpha-2} ds$$

where we recall that $(B_t)_{t \geq 0}$ is a Brownian motion and $(\tau_t^B)_{t \geq 0}$ is its inverse local time at 0. The characteristic function of $(Z_t^0)_{t \geq 0}$ is computed in [Bét21, Lemma 11] and we can deduce the value of ρ using a formula due to Zolotarev [Zol86, §2.6].

Case $\alpha \in (1, 2)$. Recall that there exists a positive constant $\mathbf{m}^f(\infty) < \infty$ such that $\mathbf{m}^f(x) \rightarrow \mathbf{m}^f(\infty)$ as $x \rightarrow \pm\infty$. It follows that $\mathbf{m}^f(\mathbf{1}) = 0$ and by Lemma 3.6 below (whose proof is postponed to Appendix 3.C.1), we have that $\mathbf{n}(\int_0^\ell f(\varepsilon_s) ds) = 0$, which in turn implies $\mathbf{n}^B(A_\ell^f) = 0$. Obviously, we also have $\mathbf{n}^B(A_\ell^{f,h}) = 0$ for any $h > 0$.

Lemma 3.6. *Let g be a positive Borel function such that $\mathbf{m}_+^f(g) < \infty$ and $|\mathbf{m}_-^f(g)| < \infty$. Then we have $\mathbf{m}^f(g) = \mathbf{n}(\int_0^\ell ((g \circ \mathfrak{s}) \times f)(\varepsilon_s) ds)$.*

Let us introduce $\bar{\varphi}(x, y) = 1 - e^{-\lambda x + i\mu y} - i\mu\chi(y)$ where $\chi(y) = y$ on $[-1, 1]$ and is continuous with compact support and odd. Thus

$$\psi_h(\lambda, \mu) = \mathbf{n}^B \left[\bar{\varphi}(A_\ell^{s,h}, A_\ell^{f,h}) \right] + i\mu \mathbf{n}^B \left[\chi(A_\ell^{f,h}) \right].$$

Since $\bar{\varphi}$ is bounded and since there exists a constant $C_{\lambda,\mu} > 0$ such that $|\bar{\varphi}(x, y)| \leq C_{\lambda,\mu}(x + y^2)$ for any $x \geq 0, y \in \mathbb{R}$, we obtain by points (i)-(ii) of Lemma 3.5 and decomposing as previously on $\mathbf{A}_\delta$ and $\mathbf{A}_\delta^c$ that

$$\lim_{h \downarrow 0} \mathbf{n}^B \left[\bar{\varphi}(A_\ell^{s,h}, A_\ell^{f,h}) \right] = \mathbf{n}^B \left[\bar{\varphi}(A_\ell^\beta, A_\ell^\alpha) \right].$$

It remains to check that $\mathbf{n}^B[\chi(A_\ell^{f,h})]$ converges as $h \downarrow 0$. Recall that $\mathbf{n}^B(A_\ell^{f,h}) = 0$, so that $\mathbf{n}^B[\chi(A_\ell^{f,h})] = -\mathbf{n}^B[A_\ell^{f,h} - \chi(A_\ell^{f,h})]$. Note that there exists $C > 0$ such that $|x - \chi(x)| \leq Cx^2$, by item (ii) of Lemma 3.5, it comes that

$$\limsup_{h \downarrow 0} \left| \mathbf{n}^B \left[(A_\ell^{f,h} - \chi(A_\ell^{f,h})) \mathbf{1}_{\mathbf{A}_\delta} \right] \right| \longrightarrow 0 \quad \text{as } \delta \downarrow 0.$$

Moreover, for any $\delta > 0$, by item (iii) of Lemma 3.5 and since $|x - \chi(x)| \leq C|x|$, we get by the dominated convergence theorem that for any $\delta > 0$,

$$\lim_{h \downarrow 0} \mathbf{n}^B \left[(A_\ell^{f,h} - \chi(A_\ell^{f,h})) \mathbf{1}_{\mathbf{A}_\delta^c} \right] = \mathbf{n}^B \left[(A_\ell^\alpha - \chi(A_\ell^\alpha)) \mathbf{1}_{\mathbf{A}_\delta^c} \right].$$

We conclude that that

$$\lim_{h \downarrow 0} \mathbf{n}^B \left[A_\ell^{f,h} - \chi(A_\ell^{f,h}) \right] = \mathbf{n}^B \left[A_\ell^\alpha - \chi(A_\ell^\alpha) \right],$$

so that $\mathbf{n}^B[\chi(A_\ell^{f,h})]$ converges to $\mathbf{n}^B[\chi(A_\ell^\alpha)]$ as $h \downarrow 0$. Again, let us quickly explain the value of ρ . In the case $\alpha \in (1, 2)$, the limiting process has the following representation

$$Z_t^0 = \int_{\mathbb{R}} \operatorname{sgn}_{f_+, -f_-}(x) |x|^{1/\alpha-2} (L_{\tau_t^B}^x - t) dx$$

where we recall that $(L_t^x)_{t \geq 0, x \in \mathbb{R}}$ denotes the family of local times of $(B_t)_{t \geq 0}$. Again, the characteristic function of $(Z_t^0)_{t \geq 0}$ is computed in [Bét21, Lemma 12] which is sufficient to deduce the value of ρ .

Case $\alpha = 1$. Using the same notation as in the previous case and with the same reasoning, we also have that $\mathbf{n}^B[\bar{\varphi}(A_\ell^{s,h}, A_\ell^{f,h})]$ converges to $\mathbf{n}^B[\bar{\varphi}(A_\ell^\beta, A_\ell^\alpha)]$ when h goes to 0. It remains to prove that the extra assumption $\lim_{h \downarrow 0} \frac{1}{\Lambda_f(1/h)} (\mathbf{m}^f(1/h) - \mathbf{m}^f(-1/h)) = \mathbf{c}$ implies that $\mathbf{n}^B[\chi(A_\ell^{f,h})]$ converges. Note that the limit is not $\mathbf{n}^B[\chi(A_\ell^\alpha)]$ which is infinite when $\alpha = 1$. Writing that $\mathbf{m}^f(1/h) - \mathbf{m}^f(-1/h) = \int_{\mathbb{R}} \mathbf{1}_{\{x \in (-1/h, 1/h]\}} \mathbf{m}^f(dx)$ and using Lemma 3.6 with $g(x) = \mathbf{1}_{\{x \in (-1/h, 1/h]\}}$, we get

$$\mathbf{m}^f(1/h) - \mathbf{m}^f(-1/h) = \mathbf{n} \left(\int_0^\ell ((g \circ \mathfrak{s}) \times f)(\varepsilon_s) ds \right).$$

Then, using Proposition 3.9 and the occupation time formula for the Brownian excursion, we easily get that

$$\frac{1}{\Lambda_f(1/h)} (\mathbf{m}^f(1/h) - \mathbf{m}^f(-1/h)) = \mathbf{n}^B [\tilde{A}_\ell^{f,h}]$$

where $\tilde{A}_\ell^{f,h} := \int_{\mathbb{R}} \mathbf{L}_\ell^x \mathbf{1}_{\{x \in (-1, 1]\}} \mathbf{m}_h^f(dx)$. Note that as in (3.41), thanks to Lemma 3.4 (and Assumption 3.7) we have for $\mathbf{n}^B$ -almost every excursion $\varepsilon \in \mathcal{C}$

$$\lim_{h \downarrow 0} \tilde{A}_\ell^{f,h} = \tilde{A}_\ell^\alpha := c_\alpha \int_{(-1, 1]} \mathbf{L}_\ell^x \operatorname{sgn}_{f_+, f_-}(x) x^{1/\alpha-2} dx.$$

(Recall $f_+ = f_-$ in the case $\alpha = 1$). Remark that we have $\tilde{A}_\ell^{f,h} = A_\ell^{f,h}$ for every $\varepsilon \in \mathbf{A}_1$: we can then decompose

$$\begin{aligned} \mathbf{n}^B[\chi(A_\ell^{f,h})] &= \mathbf{n}^B[\chi(\tilde{A}_\ell^{f,h}) \mathbf{1}_{\mathbf{A}_1}] + \mathbf{n}^B[\chi(A_\ell^{f,h}) \mathbf{1}_{\mathbf{A}_1^c}] \\ &= \mathbf{n}^B \left[\left(\chi(\tilde{A}_\ell^{f,h}) - \tilde{A}_\ell^{f,h} \right) \mathbf{1}_{\mathbf{A}_1} \right] + \mathbf{n}^B[\tilde{A}_\ell^{f,h}] - \mathbf{n}^B[\tilde{A}_\ell^{f,h} \mathbf{1}_{\mathbf{A}_1^c}] + \mathbf{n}^B[\chi(A_\ell^{f,h}) \mathbf{1}_{\mathbf{A}_1^c}]. \end{aligned}$$

The second term converges (to $\mathbf{c}$) by assumption. It remains to check that the other three terms converge. Regarding the first term, using item (ii) in Lemma 3.5, and the fact that $|\chi(x) - x| \leq Cx^2$ for some constant $C > 0$, we get that $\mathbf{n}^B[\sup_{h \in (0, 1)} (\chi(\tilde{A}_\ell^{f,h}) - \tilde{A}_\ell^{f,h}) \mathbf{1}_{\mathbf{A}_1}] < +\infty$. By the dominated convergence theorem, it comes that

$$\lim_{h \downarrow 0} \mathbf{n}^B \left[\left(\chi(\tilde{A}_\ell^{f,h}) - \tilde{A}_\ell^{f,h} \right) \mathbf{1}_{\mathbf{A}_1} \right] = \mathbf{n}^B \left[\left(\chi(\tilde{A}_\ell^\alpha) - \tilde{A}_\ell^\alpha \right) \mathbf{1}_{\mathbf{A}_1} \right].$$

We obtain the convergence of the third term applying point (iv) in Lemma 3.5 (and dominated convergence). As a conclusion, we have

$$\lim_{h \downarrow 0} \mathbf{n}^B[\chi(A_\ell^{f,h})] = \mathbf{n}^B \left[\left(\chi(\tilde{A}_\ell^\alpha) - \tilde{A}_\ell^\alpha \right) \mathbf{1}_{\mathbf{A}_1} \right] - \mathbf{n}^B[\tilde{A}_\ell^\alpha \mathbf{1}_{\mathbf{A}_1^c}] + \mathbf{n}^B[\chi(A_\ell^\alpha) \mathbf{1}_{\mathbf{A}_1^c}] + \mathbf{c}.$$

Now we point out that, since χ is odd and $f_+ = f_-$, the first three terms in the above limit are null by symmetry. Hence the limiting process $(Z_t^0)_{t \geq 0}$ is a Cauchy process drifted by $\mathbf{c}$ (whence the value of ρ from the statement).

Step 2: Convergence of $a(b^{-1}(q))I_e$. Let us stress that since $b(h)\tau_{t/h}$ converges in distribution, b is an asymptotic inverse of Φ (see Remark 3.5); in particular, Φ is regularly varying at 0 with index $\beta \in (0, 1)$.

By Lemma 3.3-(ii), we have that for any $\mu \in \mathbb{R}$,

$$\mathbb{E} \left[e^{ia(b^{-1}(q))\mu I_e} \right] = \frac{q}{\Phi(q)} \left(m + n^B \left(\int_0^\ell \exp(-qA_t^s + ia(b^{-1}(q))\mu A_t^f) dA_t^s \right) \right) =: \frac{q}{\Phi(q)} (m + G(\mu, q)),$$

where we recall that m is the drift coefficient of $(\tau_t)_{t \geq 0}$ and A_t^s, A_t^f are defined in (3.37) and (3.38) respectively. Since Φ is regularly varying with index $\beta \in (0, 1)$ we have $\lim_{q \downarrow 0} m/\Phi(q) = 0$.

Therefore, it remains to show that $qG(\mu, q)/\Phi(q)$ converges as $q \downarrow 0$, or equivalently that $qG(\mu, q)/b^{-1}(q)$ converges. Using the scaling property of the Brownian excursion measure, *i.e.* $n_B = h n_B \circ \lambda_h^{-1}$, with $h = b^{-1}(q)$, we get

$$\frac{q}{b^{-1}(q)} G(\mu, q) = n^B \left(\int_0^\ell \exp(-A_t^{s,q} + i\mu A_t^{f,q}) dA_t^{s,q} \right), \quad (3.42)$$

where, similarly as in (3.37)-(3.38), we have set

$$A_t^{s,q} = \int_{\mathbb{R}} \mathbf{L}_t^x \mathbf{m}_{b^{-1}(q)}^s(dx), \quad A_t^{f,q} = \int_{\mathbb{R}} \mathbf{L}_t^x \mathbf{m}_{b^{-1}(q)}^f(dx),$$

and the measures $\mathbf{m}_h^s$ and $\mathbf{m}_h^f$ are the measures defined in Step 1, see above (3.37). By Lemma 3.4 (and Assumptions 3.5 and 3.7), we have that for n^B -almost every excursion $\varepsilon \in \mathcal{C}$, for any $t \in [0, \ell]$, $\lim_{q \downarrow 0} A_t^{s,q} = A_t^\beta$ and $\lim_{q \downarrow 0} A_t^{f,q} = A_t^\alpha$.

Let us show that for n^B -almost every excursion,

$$\lim_{q \downarrow 0} \int_0^\ell \exp(-A_t^{s,q} + i\mu A_t^{f,q}) dA_t^{s,q} = \int_0^\ell \exp(-A_t^\beta + i\mu A_t^\alpha) dA_t^\beta. \quad (3.43)$$

First, the finite measures $dA_t^{s,q}$ on $([0, \ell], \mathcal{B}([0, \ell]))$ converge weakly to the finite measure dA_t^β as $q \rightarrow 0$. Since $t \mapsto \exp(-cA_t^\beta + i\mu A_t^\alpha)$ is continuous and bounded, we deduce that

$$\lim_{q \downarrow 0} \int_0^\ell \exp(-A_t^\beta + i\mu A_t^\alpha) dA_t^{s,q} = \int_0^\ell \exp(-A_t^\beta + i\mu A_t^\alpha) dA_t^\beta.$$

Next, we bound

$$\begin{aligned} & \left| \int_0^\ell \left(\exp(-A_t^{s,q} + i\mu A_t^{f,q}) - \exp(-A_t^\beta + i\mu A_t^\alpha) \right) dA_t^{s,q} \right| \\ & \leq A_\ell^{s,q} \sup_{t \in [0, \ell]} \left| \exp(-A_t^{s,q} + i\mu A_t^{f,q}) - \exp(-A_t^\beta + i\mu A_t^\alpha) \right|. \end{aligned}$$

By Remark 3.6, the convergence of $A_t^{s,q}$ and $A_t^{f,q}$ is uniform on $[0, \ell]$, *i.e.* $\sup_{t \in [0, \ell]} |A_t^{s,q} - A_t^\beta| + |A_t^{f,q} - A_t^\alpha|$ vanishes as $q \downarrow 0$. Since $(x, y) \mapsto \exp(-x + iy)$ is Lipschitz, it is clear the above quantities converges to 0 as $q \downarrow 0$, which proves (3.43).

Finally, we bound the integrand in (3.42) to apply dominated convergence. We have

$$\begin{aligned} \left| \int_0^\ell \exp(-A_t^{s,q} + i\mu A_t^{f,q}) dA_t^{s,q} \right| & \leq \int_0^\ell \exp(-A_t^{s,q}) dA_t^{s,q} \\ & = 1 - \exp(-A_\ell^{s,q}) \leq 1 \wedge A_\ell^{s,q}. \end{aligned} \quad (3.44)$$

We conclude as before, by introducing some arbitrary small $\delta > 0$ and by splitting the integral (3.42) into two parts, on the sets $\mathbf{A}_\delta$ and $\mathbf{A}_\delta^c$. Using the bound (3.44), we obtain thanks to item (i) in Lemma 3.5 (recall $\beta \in (0, 1)$) that

$$\limsup_{q \rightarrow 0} \mathbf{n}^B \left[\left(\int_0^\ell \exp \left(-A_t^{s,q} + i\mu A_t^{f,q} \right) dA_t^{s,q} \right) \mathbf{1}_{\mathbf{A}_\delta} \right] \longrightarrow 0 \quad \text{as } \delta \downarrow 0.$$

Moreover, the bound (3.44) dominates uniformly in q the integrand and we conclude using the dominated convergence theorem (recall that $\mathbf{n}^B$ is finite on $\mathbf{A}_\delta^c$) that

$$\lim_{q \downarrow 0} \mathbf{n}^B \left[\left(\int_0^\ell \exp \left(-A_t^{s,q} + i\mu A_t^{f,q} \right) dA_t^{s,q} \right) \mathbf{1}_{\mathbf{A}_\delta^c} \right] = \mathbf{n}^B \left[\left(\int_0^\ell \exp \left(-A_t^\beta + i\mu A_t^\alpha \right) dA_t^\beta \right) \mathbf{1}_{\mathbf{A}_\delta^c} \right].$$

From this, we get that

$$\lim_{q \downarrow 0} \mathbf{n}^B \left[\int_0^\ell \exp \left(-A_t^{s,q} + i\mu A_t^{f,q} \right) dA_t^{s,q} \right] = \mathbf{n}^B \left[\int_0^\ell \exp \left(-A_t^\beta + i\mu A_t^\alpha \right) dA_t^\beta \right], \quad (3.45)$$

which completes the proof of Step 2. $\square$

3.7.4 Proof of Proposition 3.1

In this section, we prove Proposition 3.1. We let the reader recall Assumptions 3.5 and 3.6 (which corresponds to the case $\alpha = 2$). Our goal is to show that Assumption 3.8 holds, *i.e.* that Φ is regularly varying at 0 with index $\beta \in (0, 1)$, that $\mathbf{n}(\mathfrak{F}) = 0$ and that $\mathbf{n}(\mathfrak{F}^2) < \infty$. First, as in the proof of Proposition 3.2, we can show that, by setting $b(h) = h^{1/\beta}/\Lambda_s(1/h)$, the rescaled process $(b(h)\tau_{t/h})_{t \geq 0}$ converges as $h \rightarrow 0$ toward a β -stable subordinator. This entails that Φ is an asymptotic inverse of b (see Remark 3.5) and in particular that it is regularly varying with index β .

Next, we apply Lemma 3.6 with $g = \mathbf{1}$. This tells us that $\mathbf{m}^f(\mathbf{1}) = \mathbf{n}(\int_0^\ell f(\varepsilon_s) ds) = \mathbf{n}(\mathfrak{F})$. Since $\mathbf{m}^f(\mathbf{1}) = \lim_{x \rightarrow \infty} (\mathbf{m}^f(x) - \mathbf{m}^f(-x)) = 0$, it comes that $\mathbf{n}(\mathfrak{F}) = 0$. It remains to show that $\mathbf{n}(\mathfrak{F}^2) < \infty$. Recalling that the Lévy measure of $(Z_t)_{t \geq 0}$ is $\mathbf{n}(\mathfrak{F} \in dx)$, this is equivalent to having $\mathbb{E}[Z_t^2] < \infty$ for any $t \geq 0$, see for instance Sato [Sat13, Thm. 25.3]. We will show that the condition $\mathbf{m}^f \in L^2(dx)$ implies the latter one (it is actually equivalent). To do so, let us first define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ as

$$g(x) = \int_0^x \mathfrak{s}'(v) \int_v^\infty f(u) \mathbf{m}(du) dv.$$

Making two changes of variables, it holds that

$$g(\mathfrak{s}^{-1}(x)) = \int_0^x h(v) dv, \quad \text{where} \quad h(v) = \int_v^\infty f \circ \mathfrak{s}^{-1}(u) \mathbf{m}^s(du) = \mathbf{m}^f(\infty) - \mathbf{m}^f(v).$$

Since $\mathbf{m}^f$ is non-decreasing on $\mathbb{R}_+$ and non-increasing on $\mathbb{R}_-$, $g \circ \mathfrak{s}^{-1}$ is a difference of two convex function and we can apply the Itô-Tanaka formula, see [RY99, Ch. VI Thm. 1.5], which tells us that

$$g(\mathfrak{s}^{-1}(B_t)) = \int_0^t h(B_s) dB_s - \frac{1}{2} \int_{\mathbb{R}} f(\mathfrak{s}^{-1}(x)) L_t^x \mathbf{m}^s(dx).$$

Substituting ρ_t in the preceding equation, and using the occupation time formula from item (i) of Proposition 3.8, we see that (with a change of variable $y = \mathfrak{s}^{-1}(x)$)

$$g(X_t) = \int_0^{\rho_t} h(B_s) dB_s - \frac{1}{2} \int_0^t f(X_s) ds.$$

We now substitute τ_t in the preceding equation, and using that $X_{\tau_t} = 0$ so $g(X_{\tau_t}) = 0$ and $\rho_{\tau_t} = \tau_t^B$ (see Proposition 3.8-(iv)), we get that

$$Z_t = \int_0^{\tau_t} f(X_s) ds = 2 \int_0^{\tau_t^B} h(B_s) dB_s.$$

Let us set for any fixed $t \geq 0$, $(M_u^t)_{u \geq 0} = (\int_0^{u \wedge \tau_t^B} h(B_s) dB_s)_{u \geq 0}$ which is a (centered) local martingale. Its quadratic variation is such that, for any $u \geq 0$,

$$\mathbb{E}[\langle M^t \rangle_u] = \mathbb{E}\left[\int_{\mathbb{R}} L_{u \wedge \tau_t^B}^x h(x)^2 dx\right] \leq \int_{\mathbb{R}} \mathbb{E}[L_{\tau_t^B}^x] [h(x)]^2 dx = t \int_{\mathbb{R}} h(x)^2 dx,$$

where in the last equality, we used that by Ray-Knight's theorem, $\mathbb{E}[L_{\tau_t^B}^x] = t$ for any $x \in \mathbb{R}$. Since $h \in L^2(dx)$, $(M_u^t)_{u \geq 0}$ is a martingale bounded in $L^2(\mathbb{P})$, *i.e.* $\sup_{u \geq 0} \mathbb{E}[(M_u^t)^2] < \infty$. Therefore, it converges in $L^2(\mathbb{P})$, which entails that

$$\mathbb{E}\left[\left(\int_0^{\tau_t^B} h(B_s) dB_s\right)^2\right] = \mathbb{E}\left[\int_0^{\tau_t^B} h(B_s)^2 ds\right] = t \int_{\mathbb{R}} h(x)^2 dx.$$

We have shown that for any $t \geq 0$, $\mathbb{E}[Z_t^2] = 4t \int_{\mathbb{R}} h(x)^2 dx < +\infty$, which completes the proof. $\square$

3.8 Hitting time of zero

Let us assume for simplicity that the process $(X_t)_{t \geq 0}$ is a regular diffusion valued in an open interval $J = (a, b)$ with $a \in [-\infty, 0)$ and $b \in (0, +\infty]$ and let us consider the process $(\zeta_t, X_t)_{t \geq 0}$ as a strong Markov process. For a pair $(z, x) \in \mathbb{R} \times J$, we will denote by $\mathbf{P}_{(z, x)}$ the law of $(\zeta_t, X_t)_{t \geq 0}$ when started at (z, x) , *i.e.* the law of $(z + \int_0^t f(X_s) ds, X_t)_{t \geq 0}$ under $\mathbb{P}_x$, with $\mathbb{P}_x$ the law of $(X_t)_{t \geq 0}$ started at x .

The aim of this section is to describe the asymptotic behavior of $\mathbf{P}_{(z, x)}(T_0 > t)$ as $t \rightarrow \infty$ when $(z, x) \neq (0, 0)$. Let us stress that in the previous section we were considering the probability $\mathbb{P}(T_z > t) = \mathbf{P}_{(-z, 0)}(T_0 > t)$ for $z > 0$. We will naturally place ourselves under Assumption 3.3 or 3.4. As a rather classical application, we will identify a harmonic function for the killed process: through Doob's h -transform, this enables us to construct the additive functional conditioned to stay negative.

3.8.1 Preliminaries

First, we recall that a regular diffusion is a continuous strong Markov process such that, if set $\eta_x = \inf\{t > 0, X_t = x\}$, then for any $(x, y) \in J^2$, $\mathbb{P}_x(\eta_y < \infty) > 0$. It is well-known that regular diffusions are space and time changed Brownian motion. First, there exists a continuous and increasing function $\mathfrak{s} : J \rightarrow \mathbb{R}$ such that $(\mathfrak{s}(X_t))_{t \geq 0}$ is a local martingale, see Kallenberg [Kal02, Ch. 23, Thm. 23.7]. We assume that $\mathfrak{s}$ is such that $\mathfrak{s}(J) = \mathbb{R}$ and we denote by $\mathfrak{s}^{-1}$ its inverse, defined on $\mathbb{R}$ and valued in J . Let $(W_t)_{t \geq 0}$ be a Brownian on some probability space and denote by $(L_t^x)_{t \geq 0, x \in \mathbb{R}}$ its family of local times. Then there exists an increasing function $\mathfrak{m}^{\mathfrak{s}} : \mathbb{R} \rightarrow \mathbb{R}$ such that, if we set

$$A_t = \int_{\mathbb{R}} L_t^x \mathfrak{m}^{\mathfrak{s}}(dx) \quad \text{and} \quad \rho_t = \inf\{s \geq 0, A_s > t\},$$

then for any $x \in J$, the probability measure $\mathbb{P}_x$ coincides with the law of $(\mathfrak{s}^{-1}(W_{\rho_t}))_{t \geq 0}$ with $(W_t)_{t \geq 0}$ started at $\mathfrak{s}(x)$. We refer to Kallenberg [Kal02, Ch. 23, Thm. 23.9] for details. We will first show the following lemma. Let us also define

$$J_+^* := J \cap (0, +\infty), \quad \text{and} \quad J_-^* = J \cap (-\infty, 0).$$

Lemma 3.7. Recall that we denote by $\mathfrak{F}(\varepsilon) = \int_0^\ell f(\varepsilon_s)ds$ for some excursion $\varepsilon \in \mathcal{E}$ of $(X_t)_{t \geq 0}$, see (3.20) and let us set $M = M(\varepsilon) = \sup_{s \geq 0} |\varepsilon_s|$. The two following assertions hold.

(i) For any $x \in J_+^*$, we have

$$\mathbf{n}(M > x, \mathfrak{F} < 0) = \frac{1}{2|\mathfrak{s}(x)|} \quad \text{and} \quad \mathbf{n}(M > x, \mathfrak{F} > 0) = \frac{1}{2|\mathfrak{s}(x)|}.$$

(ii) For any $x \in J_+^*$, respectively any $x \in J_-^*$, the law of $(X_{g_{\eta_x}+t})_{t \in [0, d_{\eta_x} - g_{\eta_x}]}$ under $\mathbb{P}$ is exactly $\mathbf{n}(\cdot \mid M > x, \mathfrak{F} > 0)$, respectively $\mathbf{n}(\cdot \mid M > x, \mathfrak{F} < 0)$.

Proof. Remember first that, since excursions have constant sign, $\mathfrak{F}(\varepsilon) > 0$ if and only if ε is a positive excursion. By Proposition 3.9 and since A_t is increasing it comes that for any $x \in J_+^*$, we have $\mathbf{n}(M > x, \mathfrak{F} > 0) = \mathbf{n}_B^+(M > \mathfrak{s}(x))$, respectively $\mathbf{n}(M > x, \mathfrak{F} < 0) = \mathbf{n}_B^-(M > \mathfrak{s}(x))$, where $\mathbf{n}_B^+$ is the Brownian excursion measure restricted to the set of positive excursions, respectively $\mathbf{n}_B^-$ is restricted to the set of negative excursions. The first result follows immediately since $\mathbf{n}_B^+(M > x) = \mathbf{n}_B^-(M > x) = \frac{1}{2x}$ for any $x > 0$.

Regarding the second point, for $x \in J_+^*$ let us denote $U_x^+ = \{\varepsilon \in \mathcal{E}, M(\varepsilon) > x, \mathfrak{F}(\varepsilon) > 0\}$. Recall that $(e_t)_{t \geq 0}$ denote the excursion process and set $T_{U_x^+} = \inf\{t > 0, e_t \in U_x^+\}$. Clearly, we have $e_{T_{U_x^+}} = (X_{g_{\eta_x}+t})_{t \in [0, d_{\eta_x} - g_{\eta_x}]}$ and since $\mathbf{n}(U_x^+) < \infty$ by the first point, the result follows from [RY99, Ch. XII Lem. 1.13] which tells us that for any measurable set Γ , we have

$$\mathbf{n}(\Gamma \cap U_x^+) = \mathbb{P}(e_{T_{U_x^+}} \in \Gamma) \mathbf{n}(U_x^+).$$

The same proof holds for $x \in J_-^*$. □

Let us consider the set of starting points $\mathbf{H} := [(-\infty, 0) \times J] \cup [\{0\} \times J_-^*]$. Note that for starting points $(z, x) \in \mathbf{H}$ we have $\mathbf{P}_{(z,x)}(T_0 > 0) = 1$ and the process $(\zeta_{t \wedge T_0})_{t \geq 0}$ is negative under $\mathbf{P}_{(z,x)}$. Recall that $\mathcal{V}$ denotes the renewal function: for any $x \geq 0$, $\mathcal{V}(x) = \int_0^\infty \mathbb{P}(H_t \leq x)dt$, see (3.6). Finally, we set $\eta_0 = \inf\{t > 0, X_t = 0\}$ the first hitting time of zero of $(X_t)_{t \geq 0}$, and we consider the positive function $h : \mathbf{H} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ defined for every $(z, x) \in \mathbf{H}$ by

$$h(z, x) = \mathbf{E}_{(z,x)} \left[\mathcal{V}(-\zeta_{\eta_0}) \mathbf{1}_{\{\zeta_{\eta_0} < 0\}} \right]. \quad (3.46)$$

Proposition 3.10. For any $(z, x) \in \mathbf{H}$ we have $h(z, x) < \infty$.

This result is not obvious at first sight. However since ζ_{η_0} is a piece of an integrated excursion, its law is directly linked with the Lévy measure of $(Z_t)_{t \geq 0}$, and so is the function $\mathcal{V}$. As we shall see, this link is given by the *équations amicales* of Vigon [Vig02a, Vig02b].

Proof. Let $(z, x) \in \mathbf{H}$ and remember that

$$h(z, x) = \mathbb{E}_x \left[\mathcal{V} \left(-z - \int_0^{\eta_0} f(X_s)ds \right) \mathbf{1}_{\{z + \int_0^{\eta_0} f(X_s)ds < 0\}} \right].$$

First case: $z < 0$ and $x \geq 0$. If $x = 0$, then $\eta_0 = 0$ a.s. and $h(z, x) = \mathcal{V}(-z)$. If $x > 0$, then $\int_0^{\eta_0} f(X_s)ds > 0$ a.s. and since $\mathcal{V}$ is non-decreasing, $h(z, x) \leq \mathcal{V}(-z) < +\infty$.

Second case: $z \leq 0$ and $x < 0$. Then $\int_0^{\eta_0} f(X_s)ds < 0$ a.s. By the strong Markov property, and since $(X_t)_{t \geq 0}$ is continuous, the law of $\int_0^{\eta_0} f(X_s)ds$ under $\mathbb{P}_x$ is equal to the law of $\int_{\eta_x}^{d_{\eta_x}} f(X_s)ds$ under $\mathbb{P}_0$. But since $\int_{\eta_x}^{d_{\eta_x}} f(X_s)ds \geq \int_{g_{\eta_x}}^{d_{\eta_x}} f(X_s)ds$ and since the law of $\int_{g_{\eta_x}}^{d_{\eta_x}} f(X_s)ds$ under $\mathbb{P}_0$ is

equal to the law of $\mathfrak{F}(\varepsilon)$ under $\mathbf{n}(\cdot \mid M > x, \mathfrak{F} < 0)$ by Lemma 3.7-(ii), we obtain the following bound:

$$h(z, x) \leq \frac{\mathbf{n}\left(\mathcal{V}(-z - \mathfrak{F})\mathbf{1}_{\{M > x\}}\mathbf{1}_{\{\mathfrak{F} < 0\}}\right)}{\mathbf{n}(M > x, \mathfrak{F} < 0)}.$$

Since $\mathbf{n}(M > x, \mathfrak{F} < 0) > 0$, it remains to show that $\mathbf{n}(\mathcal{V}(-z - \mathfrak{F})\mathbf{1}_{\{M > x\}}\mathbf{1}_{\{\mathfrak{F} < 0\}}) < \infty$. We split this quantity in two parts:

$$\begin{aligned} \mathbf{n}\left(\mathcal{V}(-z - \mathfrak{F})\mathbf{1}_{\{M > x\}}\mathbf{1}_{\{\mathfrak{F} < 0\}}\right) &= \mathbf{n}\left(\mathcal{V}(-z - \mathfrak{F})\mathbf{1}_{\{M > x\}}\mathbf{1}_{\{\mathfrak{F} \in [-1, 0)\}}\right) \\ &\quad + \mathbf{n}\left(\mathcal{V}(-z - \mathfrak{F})\mathbf{1}_{\{M > x\}}\mathbf{1}_{\{\mathfrak{F} < -1\}}\right). \end{aligned}$$

The first term on the right-hand-side is smaller than $\mathcal{V}(-z + 1)\mathbf{n}(M > x) < \infty$, since $\mathcal{V}$ is non-decreasing. Regarding the second term, we first recall that $\mathcal{V}$ is subadditive, see for instance [Ber96, Ch. III], *i.e.* $\mathcal{V}(x + y) \leq \mathcal{V}(x) + \mathcal{V}(y)$ for any $x, y \geq 0$: hence, we can write

$$\mathbf{n}\left(\mathcal{V}(-z - \mathfrak{F})\mathbf{1}_{\{M > x\}}\mathbf{1}_{\{\mathfrak{F} < -1\}}\right) \leq \mathcal{V}(-z + 1/2)\mathbf{n}(M > x) + \mathbf{n}\left(\mathcal{V}(-1/2 - \mathfrak{F})\mathbf{1}_{\{\mathfrak{F} < -1\}}\right).$$

We finally show that $\mathbf{n}(\mathcal{V}(-1/2 - \mathfrak{F})\mathbf{1}_{\{\mathfrak{F} < -1\}}) < \infty$. Recall now that the Lévy measure ν of $(Z_t)_{t \geq 0}$ is $\nu(du) = \mathbf{n}(\mathfrak{F} \in du)$, so that

$$\begin{aligned} \mathbf{n}\left(\mathcal{V}(-1/2 - \mathfrak{F})\mathbf{1}_{\{\mathfrak{F} < -1\}}\right) &= \int_{(-\infty, -1)} \mathcal{V}(-1/2 - u)\nu(du) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \mathbf{1}_{\{u < -1\}} \mathbf{1}_{\{y \leq -1/2 - u\}} \mathcal{V}(dy)\nu(du), \end{aligned}$$

where $\mathcal{V}(dy)$ stands for the Stieltjes measure associated to the non-increasing function $\mathcal{V}$. We then have the following bound

$$\mathbf{n}\left(\mathcal{V}(-1/2 - \mathfrak{F})\mathbf{1}_{\{\mathfrak{F} < -1\}}\right) \leq \int_{\mathbb{R}} \int_{\mathbb{R}_+} \mathbf{1}_{\{y+u \leq -1/2\}} \nu(du)\mathcal{V}(dy) = (\nu * \mathcal{V})((-\infty, -1/2]),$$

where $\nu * \mathcal{V}$ is the convolution of the measures ν and $\mathcal{V}$. The *équations amicales* of Vigon, see for instance [Vig02b, Vig02a], state that the measure $\nu * \mathcal{V}$ coincides on $(-\infty, 0)$ with the Lévy measure of the dual ladder height process $(-\tilde{H}_t)_{t \geq 0}$. Therefore $(\nu * \mathcal{V})((-\infty, -1/2]) < \infty$, which completes the proof. $\square$

3.8.2 Hitting probabilities

We now show the following technical result, which describes the asymptotic behavior of the Laplace transform $\mathbf{P}_{(z,x)}(T_0 > t)$. Recall that $e = e(q)$ is an independent exponential random variable of parameter q .

Proposition 3.11. *Suppose that Assumption 3.3 or 3.4 holds. Then there exists a constant $c_0 > 0$ such that for any $(z, x) \in \mathbb{H}$, we have*

$$\mathbf{P}_{(z,x)}(T_0 > e) \sim c_0 h(z, x) \kappa(0, q, 0) \quad \text{as } q \downarrow 0.$$

Moreover, there exists a constant $K > 0$ such that for any $(z, x) \in \mathbb{H}$, for any $q \in (0, 1)$,

$$\mathbf{P}_{(z,x)}(T_0 > e) \leq K \kappa(0, q, 0)(|\mathfrak{s}(x)| + h(z, x)).$$

This proposition, combined with the Tauberian theorem, the monotone density theorem, Theorem 3.4 and Proposition 3.6 leads to the following theorem.

Theorem 3.6. *Suppose that Assumption 3.3 or 3.4 holds. Then there exists a slowly varying function ς such that for any $(z, x) \in \mathbf{H}$,*

$$\mathbf{P}_{(z,x)}(T_0 > t) \sim h(z, x)\varsigma(t)t^{-\theta} \quad \text{as } t \rightarrow \infty,$$

where $\theta = \rho \in (0, 1)$ if Assumption 3.3 holds, with ρ given by Theorem 3.4 (or Theorem 3.1), or $\theta = \beta\rho$ if Assumption 3.4 holds, with $\rho = \mathbb{P}(Z_t^0 \geq 0)$ as given by Proposition 3.6 (or Theorem 3.2).

Proof of Proposition 3.11. Step 1. We first focus on the quantity $\mathbb{P}_x(\eta_0 > e)$ where $x \in J$. We will first show that there exists a constant $C > 0$ such that for any $q \in (0, 1)$ and any $x \in J$,

$$\mathbb{P}_x(\eta_0 > e) \leq C\kappa(0, q, 0)|\mathfrak{s}(x)|. \quad (3.47)$$

Let $x \in J_+^*$ and remember that $\eta_x = \inf\{t > 0, X_t = x\}$; recall also that by the strong Markov property the law of η_0 under $\mathbb{P}_x$ is equal to the law of $d_{\eta_x} - \eta_x$ under $\mathbb{P}_0$. Therefore we get

$$\mathbb{P}_x(\eta_0 > e) \leq \mathbb{P}_0(d_{\eta_x} - g_{\eta_x} > e) = q \int_0^\infty e^{-qt} \mathbb{P}_0(d_{\eta_x} - g_{\eta_x} > t) dt.$$

By Lemma 3.7-(ii), the law of $d_{\eta_x} - g_{\eta_x}$ under $\mathbb{P}_0$ is the law of ℓ under $\mathbf{n}(\cdot \mid M > x, \mathfrak{F} > 0)$ and by Lemma 3.7-(i) we have $\mathbf{n}(M > x, \mathfrak{F} > 0) = 1/(2|\mathfrak{s}(x)|)$. Hence we obtain

$$\mathbb{P}_x(\eta_0 > e) \leq 2|\mathfrak{s}(x)| \mathbf{n}(1 - e^{-q\ell}). \quad (3.48)$$

This bound also holds for $x \in J_-^*$, again by Lemma 3.7, and also for $x = 0$ since $\mathbb{P}_x(\eta > e) = 0$. Remember now that $\mathbf{n}(1 - e^{-q\ell}) = \Phi(q) - mq$, see (3.19), and that under Assumption 3.3 or 3.4, Φ is regularly varying at 0 with index $\beta \in (0, 1]$ (with $\beta = 1$ under Assumption 3.3 and $\beta \in (0, 1)$ under Assumption 3.4). By Theorem 3.4 and Proposition 3.6, $q \mapsto \kappa(0, q, 0)$ is regularly varying at 0 with some index $\theta \in (0, \beta)$. It is therefore clear that $\lim_{q \downarrow 0} \mathbf{n}(1 - e^{-q\ell})/\kappa(0, q, 0) = 0$, which shows that (3.47) holds and that for any $x \in J$,

$$\frac{\mathbb{P}_x(\eta_0 > e)}{\kappa(0, q, 0)} \longrightarrow 0 \quad \text{as } q \downarrow 0. \quad (3.49)$$

This completes the first step of the proof.

Step 2. We now focus on the quantity $\mathbf{P}_{(z,x)}(T_0 - \eta_0 > e)$ for $(z, x) \in \mathbf{H}$. We first remark that $\mathbf{P}_{(z,x)}$ -almost surely, $T_0 > \eta_0$ if and only if $\zeta_{\eta_0} < 0$. First, if $x < 0$, then $\mathbf{P}_{(z,x)}$ -almost surely, $(\zeta_t)_{t \geq 0}$ is decreasing on $[0, \eta_0]$ so that $\mathbf{P}_{(z,x)}(\zeta_{\eta_0} < 0) = \mathbf{P}_{(z,x)}(T_0 > \eta_0) = 1$. Next, if $x \geq 0$, then $\mathbf{P}_{(z,x)}$ -almost surely, $(\zeta_t)_{t \geq 0}$ is non-increasing on $[0, \eta_0]$ and thus, we see that if $\zeta_{\eta_0} \geq 0$, then $T_0 \leq \eta_0$ whereas if $\zeta_{\eta_0} < 0$, then $T_0 > \eta_0$.

We define the processes $(\bar{\zeta}_t, \bar{X}_t)_{t \geq 0} = (\zeta_{t+\eta_0} - \zeta_{\eta_0}, X_{t+\eta_0})_{t \geq 0}$ and $\bar{\xi}_t = \sup_{s \in [0, t]} \bar{\zeta}_s$. By the strong Markov property again, $(\bar{\zeta}_t, \bar{X}_t)_{t \geq 0}$ and ζ_{η_0} are independent under $\mathbf{P}_{(z,x)}$. Moreover the law of $(\bar{\zeta}_t, \bar{X}_t)_{t \geq 0}$ under $\mathbf{P}_{(z,x)}$ is equal to the of $(\zeta_t, X_t)_{t \geq 0}$ under $\mathbf{P}_{(0,0)} = \mathbb{P}$. We see that $\mathbf{P}_{(z,x)}$ -almost surely, $T_0 - \eta_0 > e$ if and only if $\xi_e < -\zeta_{\eta_0}$ and $\zeta_{\eta_0} < 0$. By independence, we get

$$\mathbf{P}_{(z,x)}(T_0 - \eta_0 > e) = \int_{(-\infty, 0)} \mathbb{P}(\xi_e < -u) \mathbf{P}_{(z,x)}(\zeta_{\eta_0} \in du). \quad (3.50)$$

Let us first show that for any $q \in (0, 1)$, for any $(z, x) \in \mathbf{H}$, we have

$$\mathbf{P}_{(z,x)}(T_0 - \eta_0 > e) \leq h(z, x) \kappa(0, q, 0). \quad (3.51)$$

Recall from (3.30) that $\xi_e = \xi_{g_e} + \max(\Delta_e, 0)$ and that ξ_{g_e} and Δ_e are independent. Since moreover $\mathbb{P}(\xi_{g_e} < -u) = \kappa(0, q, 0)\mathcal{V}_q(-u) \leq \kappa(0, q, 0)\mathcal{V}(-u)$ we get the following bound:

$$\mathbb{P}(\xi_e < -u) \leq \mathbb{P}(\xi_{g_e} < -u)\mathbb{P}(\Delta_e < -u) \leq \kappa(0, q, 0)\mathcal{V}(-u),$$

which, combined with (3.50) and the definition (3.46) of $h(z, x)$, leads to (3.51).

Let us now show that there exists a constant $c_0 > 0$ such that for any $(z, x) \in \mathbf{H}$,

$$\mathbf{P}_{(z,x)}(T_0 - \eta_0 > e) \sim c_0 h(z, x) \kappa(0, q, 0) \quad \text{as } q \downarrow 0. \quad (3.52)$$

The proofs of Theorems 3.1 and 3.2 show that $\mathbb{P}(\xi_e < -u) \sim c_0 \mathcal{V}(-u) \kappa(0, q, 0)$ as $q \downarrow 0$, where $c_0 = \lim_{q \rightarrow 0} \mathbb{P}(\Delta_e \leq 0)$, see Sections 3.5.3 and 3.6.2. Since $\mathbb{P}(\xi_e < -u) \leq \kappa(0, q, 0)\mathcal{V}(-u)$, it is clear that we can apply the dominated convergence theorem in (3.50), using Proposition 3.10: we deduce that (3.52) holds. This completes the second step.

Step 3. Conclusion. First, by (3.52), we have that for any $(z, x) \in \mathbf{H}$,

$$c_0 h(z, x) = \liminf_{q \rightarrow 0} \frac{\mathbf{P}_{(z,x)}(T_0 - \eta_0 > e)}{\kappa(0, q, 0)} \leq \liminf_{q \rightarrow 0} \frac{\mathbf{P}_{(z,x)}(T_0 > e)}{\kappa(0, q, 0)}.$$

For the other bound, let $\delta \in (0, 1)$ and write

$$\begin{aligned} \mathbf{P}_{(z,x)}(T_0 > e) &= \mathbf{P}_{(z,x)}(T_0 > e, T_0 - \eta_0 > (1 - \delta)e) + \mathbf{P}_{(z,x)}(T_0 > e, T_0 - \eta_0 \leq \delta e) \\ &\leq \mathbf{P}_{(z,x)}(T_0 - \eta_0 > (1 - \delta)e) + \mathbb{P}_x(\eta_0 > \delta e). \end{aligned}$$

Thanks to (3.49), the second term divided by $\kappa(0, q, 0)$ vanishes as $q \rightarrow 0$, using also that δe is an exponential random variable of parameter q/δ and that $\kappa(0, q, 0)$ is regularly varying at 0 with exponent θ . For the first term, using that $(1 - \delta)e$ is an exponential random variable of parameter $q/(1 - \delta)$, we get from (3.52) and the regular variation of $\kappa(0, q, 0)$ that

$$\mathbf{P}_{(z,x)}(T_0 - \eta_0 > \delta e) \sim c_0 (1 - \delta)^{-\theta} h(z, x) \kappa(0, q, 0) \quad \text{as } q \downarrow 0.$$

We have therefore obtained that

$$\limsup_{q \rightarrow 0} \frac{\mathbf{P}_{(z,x)}(T_0 > e)}{\kappa(0, q, 0)} \leq c_0 (1 - \delta)^{-\theta} h(z, x),$$

which concludes the proof of the first part of the proposition by letting $\delta \downarrow 0$.

For the second part of the proposition, take $\delta = 1/2$ above. Then, it comes from (3.47) that there exists a constant $C > 0$ such that for any $q \in (0, 1)$ and any $(z, x) \in \mathbf{H}$, we have $\mathbb{P}_x(\eta_0 > e/2) \leq C |\mathfrak{s}(x)| \kappa(0, q, 0)$. Also, by (3.51), we see that for $q \in (0, 1)$ and any $(z, x) \in \mathbf{H}$, we have

$$\mathbf{P}_{(z,x)}(T_0 - \eta_0 > e/2) \leq h(z, x) \kappa(0, 2q, 0) \leq h(z, x) \kappa(0, q, 0) \times \sup_{q \in (0, 1)} \frac{\kappa(0, 2q, 0)}{\kappa(0, q, 0)}.$$

This concludes the proof. $\square$

3.8.3 Process conditioned to stay negative

Using Proposition 3.11, we are able to show that the function h defined in (3.46) is harmonic. As a consequence, we are able to define the additive functional conditioned to stay negative.

Corollary 3.3. *Suppose that Assumption 3.3 or 3.4 holds and assume that $\mathbb{E}_x[|\mathfrak{s}(X_t)|] < \infty$ for any $x \in J$ and $t \geq 0$. Then the function h is harmonic for the killed process $(\zeta_{t \wedge T_0}, X_{t \wedge T_0})_{t \geq 0}$, i.e. for any $(z, x) \in \mathbf{H}$ and any $t \geq 0$, we have*

$$\mathbf{E}_{(z,x)} \left[h(\zeta_t, X_t) \mathbf{1}_{\{T_0 > t\}} \right] = h(z, x).$$

We emphasize that the assumption $\mathbb{E}_x[|\mathfrak{s}(X_t)|] < \infty$ for any $x \in J$ and any $t \geq 0$ is satisfied by a large class of processes (as for example Brownian motion or Ornstein-Uhlenbeck processes). However it is possible to find a recurrent process for which it is not satisfied. This assumption is technical: we believe that it is an artifact of the proof and that might be possible to obtain the result without it.

Proof. Step 1. We first show that for any $(z, x) \in \mathbf{H}$, we have

$$\mathbf{E}_{(z,x)} [h(\zeta_t, X_t) \mathbf{1}_{\{T_0 > t\}}] \leq h(z, x) < \infty. \quad (3.53)$$

Let $t \geq 0$ and note that as long as $T_0 > t$, $(\zeta_t, X_t) \in \mathbf{H}$. Then, using Proposition 3.11, we have

$$\begin{aligned} \mathbf{E}_{(z,x)} [h(\zeta_t, X_t) \mathbf{1}_{\{T_0 > t\}}] &= \mathbf{E}_{(z,x)} \left[\liminf_{q \rightarrow 0} \frac{\mathbf{P}_{(\zeta_t, X_t)}(T_0 > e)}{c_0 \kappa(0, q, 0)} \mathbf{1}_{\{T_0 > t\}} \right] \\ &\leq \liminf_{q \rightarrow 0} \mathbf{E}_{(z,x)} \left[\frac{\mathbf{P}_{(\zeta_t, X_t)}(T_0 > e)}{c_0 \kappa(0, q, 0)} \mathbf{1}_{\{T_0 > t\}} \right] = \liminf_{q \rightarrow 0} \frac{\mathbf{P}_{(z,x)}(T_0 > e + t)}{c_0 \kappa(0, q, 0)}, \end{aligned}$$

where we have applied Fatou's lemma and then the Markov property. Since $\mathbf{P}_{(z,x)}(T_0 > e + t) \leq \mathbf{P}_{(z,x)}(T_0 > e)$, we obtain (3.53) from Proposition 3.11.

Step 2. We now use the dominated convergence theorem. Indeed, by Proposition 3.11 we have

$$\mathbf{E}_{(z,x)} [h(\zeta_t, X_t) \mathbf{1}_{\{T_0 > t\}}] = \mathbf{E}_{(z,x)} \left[\lim_{q \rightarrow 0} \frac{\mathbf{P}_{(\zeta_t, X_t)}(T_0 > e)}{c_0 \kappa(0, q, 0)} \mathbf{1}_{\{T_0 > t\}} \right].$$

Additionally, there exists a constant $K > 0$ such that for any $t \geq 0$ and any $q \in (0, 1)$,

$$\frac{\mathbf{P}_{(\zeta_t, X_t)}(T_0 > e)}{c_0 \kappa(0, q, 0)} \mathbf{1}_{\{T_0 > t\}} \leq K (|\mathfrak{s}(X_t)| + h(\zeta_t, X_t)) \mathbf{1}_{\{T_0 > t\}}.$$

By assumption $\mathbb{E}_x[|\mathfrak{s}(X_t)|] < \infty$ and by the first step, $\mathbf{E}_{(z,x)} [h(\zeta_t, X_t) \mathbf{1}_{\{T_0 > t\}}] < \infty$ so that we have by dominated convergence the strong Markov property

$$\mathbf{E}_{(z,x)} [h(\zeta_t, X_t) \mathbf{1}_{\{T_0 > t\}}] = \lim_{q \rightarrow 0} \frac{\mathbf{P}_{(z,x)}(T_0 > e + t)}{c_0 \kappa(0, q, 0)} = h(z, x),$$

provided that we can show that $\mathbf{P}_{(z,x)}(T_0 > e + t) \sim \mathbf{P}_{(z,x)}(T_0 > e) \sim c_0 h(z, x) \kappa(0, q, 0)$ as $q \downarrow 0$, for any $t \geq 0$ and $(z, x) \in \mathbf{H}$.

To obtain such an estimate, note that for any $(z, x) \in \mathbf{H}$ and any $t \geq 0$ we have

$$\mathbf{P}_{(z,x)}(T_0 > e + t) = q \int_0^\infty e^{-qs} \mathbf{P}_{(z,x)}(T_0 > s + t) ds = \mathbf{E}_{(z,x)} \left[(1 - e^{-q(T_0 - t)}) \mathbf{1}_{\{T_0 > t\}} \right].$$

Since $\mathbf{P}_{(z,x)}(T_0 > e) = \mathbf{E}_{(z,x)} [1 - e^{-qT_0}]$, we get that

$$\begin{aligned} \mathbf{P}_{(z,x)}(T_0 > e) - \mathbf{P}_{(z,x)}(T_0 > e + t) &= \mathbf{E}_{(z,x)} \left[(1 - e^{-qT_0}) \mathbf{1}_{\{T_0 \leq t\}} \right] + \mathbf{E}_{(z,x)} \left[e^{-qT_0} (e^{qt} - 1) \mathbf{1}_{\{T_0 > t\}} \right] \\ &\leq 2(1 - e^{-qt}) \leq 2qt, \end{aligned}$$

where we have simply bounded $1 - e^{-qT_0} \leq 1 - e^{-qt}$ in the first expectation and $e^{-qT_0} \leq e^{-qt}$ in the second one. At this point, using Proposition 3.11 together with the fact that $\kappa(0, q, 0)$ is regularly varying with index $\theta < 1$ as $q \downarrow 0$, we get that $\mathbf{P}_{(z,x)}(T_0 > e + t) \sim c_0 h(z, x) \kappa(0, q, 0)$, as required. This completes the proof. $\square$

Let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by $(X_t)_{t \geq 0}$. For $(z, x) \in \mathbf{H}$, we introduce a new probability measure $\mathbf{Q}_{(z,x)}$ in the following way: for every $t \geq 0$, for every $A \in \mathcal{F}_t$,

$$\mathbf{Q}_{(z,x)}(A) := \frac{1}{h(z, x)} \mathbf{E}_{(z,x)} \left[h(\zeta_t, X_t) \mathbf{1}_{A \cap \{T_0 > t\}} \right].$$

Corollary 3.3 ensures that this measure is indeed a probability measure. In fact, it corresponds to the law of $(\zeta_t, X_t)_{t \geq 0}$ where $(\zeta_t)_{t \geq 0}$ is conditioned to remain negative. Using similar arguments as in the previous proof, we could easily show the following result, which justifies the terminology.

Proposition 3.12. *Suppose that Assumption 3.3 or 3.4 holds and assume that $\mathbb{E}_x[|\mathfrak{s}(X_t)|] < \infty$ for any $x \in J$ and $t \geq 0$. For any $(z, x) \in \mathbf{H}$, for any $t \geq 0$ and any $A \in \mathcal{F}_t$, we have*

$$\mathbf{Q}_{(z,x)}(A) = \lim_{q \rightarrow 0} \mathbf{P}_{(z,x)}(A \mid T_0 > q).$$

Appendices to Chapter 3

3.A Wiener–Hopf factorization

In this section, we derive the Wiener-Hopf factorization from Subsection 3.4.3, *i.e.* Theorem 3.3. We consider, in all generality, a bivariate Lévy process $(\tau_t, Z_t)_{t \geq 0}$ with respect to some filtration $(\mathcal{F}_t)_{t \geq 0}$, where $(\tau_t)_{t \geq 0}$ is a subordinator. This section is very close to [Ber96, Ch. VI] and follows the same approach. However, our result requires some new arguments and we chose to establish properly Theorem 3.3.

3.A.1 Preliminaries

Let us recall briefly the notation introduced in Section 3.4.3. Let $S_t = \sup_{[0,t]} Z_t$ the running supremum of Z_t and consider the reflected process $(R_t)_{t \geq 0} = (S_t - Z_t)_{t \geq 0}$, which is a strong Markov process (see [Ber96, Prop. VI.1]) and possesses a local time $(L_t^R)_{t \geq 0}$ at 0. We denote by $(\sigma_t)_{t \geq 0}$ its right-continuous inverse, and we define $(\sigma_t, \theta_t, H_t)_{t \geq 0} := (\sigma_t, \tau_{\sigma_t}, S_{\sigma_t})_{t \geq 0}$.

Lemma 3.8. *The two following assertions hold.*

- (i) *If 0 is recurrent for the reflected process, then $(\sigma_t, \theta_t, H_t)_{t \geq 0}$ is a trivariate subordinator.*
- (ii) *If 0 is transient for the reflected process, then there exists some $\mathbf{q} > 0$ such that L_∞^R has an exponential distribution with parameter $\mathbf{q}$. Moreover, the process $(\sigma_t, \theta_t, H_t)_{0 \leq t < L_\infty^R}$ is a trivariate subordinator killed at rate $\mathbf{q}$.*

Proof. The proof is essentially the same as in Kyprianou [Kyp14, Ch. VI Thm 6.9] and we will show the two items at once. Let $t \geq 0$, we first place ourselves on the event $\{t < L_\infty^R\} = \{\sigma_t < \infty\}$ so that σ_t is a finite stopping time.

The strong Markov property tells us that the process $(\tilde{\tau}_s, \tilde{Z}_s)_{s \geq 0} = (\tau_{\sigma_t+s} - \tau_{\sigma_t}, Z_{\sigma_t+s} - Z_{\sigma_t})_{s \geq 0}$ is a Lévy process independent of $\mathcal{F}_{\sigma_t}$. Then it is clear that the corresponding local time $(\tilde{L}_s)_{s \geq 0}$ is such that for any $s \geq 0$, $\tilde{L}_s = L_{\sigma_t+s}^R - t$ so that its right-continuous inverse $(\tilde{\sigma}_s)_{s \geq 0}$ is $\tilde{\sigma}_s = \sigma_{t+s} - \sigma_t$. Moreover, since $S_{\sigma_t} = Z_{\sigma_t}$, it comes that for any $s \geq 0$

$$\tilde{S}_s = \sup_{u \in [0,s]} \tilde{Z}_u = \sup_{u \in [0,s]} (Z_{\sigma_t+u} - Z_{\sigma_t}) = S_{\sigma_t+s} - S_{\sigma_t}.$$

This shows that, on the event $\{t < L_\infty^R\}$, the shifted process

$$(\tilde{\sigma}_s, \tilde{\theta}_s, \tilde{H}_s)_{s \geq 0} = (\tilde{\sigma}_s, \tilde{\tau}_{\tilde{\sigma}_s}, \tilde{S}_{\tilde{\sigma}_s})_{s \geq 0} = (\sigma_{t+s} - \sigma_t, \theta_{t+s} - \theta_t, H_{t+s} - H_t)_{s \geq 0}$$

is independent of $\mathcal{F}_{\sigma_t}$ and has the same law as $(\sigma_s, \theta_s, H_s)_{s \geq 0}$. Finally, we see that for any $s, t \geq 0$ and any $\alpha, \beta, \gamma \geq 0$,

$$\begin{aligned} & \mathbb{E} \left[e^{-\alpha\sigma_{t+s} - \beta\theta_{t+s} - \gamma H_{t+s}} \mathbf{1}_{\{t+s < L_\infty^R\}} \right] \\ &= \mathbb{E} \left[e^{-\alpha\sigma_t - \beta\theta_t - \gamma H_t} \mathbf{1}_{\{t < L_\infty^R\}} \mathbb{E} \left[e^{-\alpha\tilde{\sigma}_s - \beta\tilde{\theta}_s - \gamma\tilde{H}_s} \mathbf{1}_{\{s < \tilde{L}_\infty\}} \mid \mathcal{F}_{\sigma_t} \right] \right] \\ &= \mathbb{E} \left[e^{-\alpha\sigma_t - \beta\theta_t - \gamma H_t} \mathbf{1}_{\{t < L_\infty^R\}} \right] \mathbb{E} \left[e^{-\alpha\sigma_s - \beta\theta_s - \gamma H_s} \mathbf{1}_{\{s < L_\infty^R\}} \right]. \end{aligned}$$

We classically deduce from this, and the right-continuity of $(\sigma_t, \theta_t, H_t)_{s \geq 0}$, that for any $t \geq 0$ and any $\alpha, \beta, \gamma \geq 0$, we have

$$\mathbb{E} \left[e^{-\alpha\sigma_t - \beta\theta_t - \gamma H_t} \mathbf{1}_{\{t < L_\infty^R\}} \right] = e^{-\kappa(\alpha, \beta, \gamma)t},$$

where

$$\kappa(\alpha, \beta, \gamma) = -\log \mathbb{E} \left[e^{-\alpha\sigma_1 - \beta\theta_1 - \gamma H_1} \mathbf{1}_{\{1 < L_\infty^R\}} \right] \geq 0.$$

We also see that for any $t \geq 0$, $\mathbb{P}(L_\infty^R > t) = e^{-\kappa(0,0,0)t}$ so that L_∞^R is exponentially distributed with parameter $\kappa(0,0,0) \geq 0$ (if $\kappa(0,0,0) = 0$, then $L_\infty^R = \infty$ a.s.). It is a well-known fact that 0 is recurrent for the reflected process if and only if $L_\infty^R = \infty$ a.s., which completes the proof. $\square$

Using the convention $e^{-\infty} = 0$, we have for any $t \geq 0$ and any $\alpha, \beta, \gamma \geq 0$ such that $\alpha + \beta + \gamma > 0$,

$$\mathbb{E} \left[e^{-\alpha\sigma_t - \beta\theta_t - \gamma H_t} \right] = e^{-\kappa(\alpha, \beta, \gamma)t},$$

From now on, we will assume for simplicity that 0 is recurrent for the reflected process but the proof carries through if it is not the case, with minor adaptations. Recall now that we have defined $G_t = \sup\{s < t, Z_s = S_s\}$ the last return to 0 before t of $(R_t)_{t \geq 0}$. Let us state the following lemma, of which we omit the proof since it is no different from the proof of [Ber96, Ch. VI Lem. 6].

Lemma 3.9. *Let $e = e(q)$ be an exponential random variable of parameter q , independent of $(\tau_t, Z_t)_{t \geq 0}$.*

- (i) *If 0 is irregular for the reflected process $(R_t)_{t \geq 0}$, then the processes $(\tau_t, Z_t)_{t \in [0, G_e]}$ and $(\tau_{G_e+t} - \tau_{G_e}, Z_{t+G_e} - Z_{G_e})_{t \in [0, e-G_e]}$ are independent.*
- (ii) *If 0 is regular for the reflected process $(R_t)_{t \geq 0}$, then the processes $(\tau_t, Z_t)_{t \in [0, G_e]}$ and $(\tau_{G_e+t} - \tau_{G_e-}, Z_{t+G_e} - Z_{G_e-})_{t \in [0, e-G_e]}$ are independent.*

Recalling Remark 3.4, we also introduce the same objects for the dual process $(\hat{Z}_t)_{t \geq 0} = (-Z_t)_{t \geq 0}$. If we set $(\hat{S}_t)_{t \geq 0} = (\sup_{[0,t]} \hat{Z}_t)_{t \geq 0}$, then the dual reflected process $(\hat{R}_t)_{t \geq 0} = (\hat{S}_t - \hat{Z}_t)_{t \geq 0}$ also possesses a local time at 0 denoted by $(L_t^{\hat{R}})_{t \geq 0}$, with inverse $(\hat{\sigma}_t)_{t \geq 0}$. Lemma 3.8 also holds and so the process $(\hat{\sigma}_t, \hat{\theta}_t, \hat{H}_t)_{t \geq 0} = (\hat{\sigma}_t, \tau_{\hat{\sigma}_t}, \hat{S}_{\hat{\sigma}_t})_{t \geq 0}$ is a trivariate subordinator with Laplace exponent that we denote by $\hat{\kappa}$.

3.A.2 Laplace transform of (G, τ_G, S)

An crucial step in the proof of Theorem 3.3 is the following result, where we recall that κ is the Laplace exponent of $(\sigma_t, \theta_t, H_t)_{t \geq 0}$, defined in (3.23).

Proposition 3.13. *Let $e = e(q)$ be an exponential random variable of parameter q , independent of $(\tau_t, Z_t)_{t \geq 0}$.*

(A) *If 0 is irregular for the reflected process $(R_t)_{t \geq 0}$, then*

$$\mathbb{E} \left[e^{-\alpha G_e - \beta \tau_{G_e} - \gamma S_e} \right] = \frac{\kappa(q, 0, 0)}{\kappa(\alpha + q, \beta, \gamma)}.$$

(B) *If 0 is regular for the reflected process, then the same result holds with τ_{G_e} replaced by τ_{G_e-} .*

We split the proof into two parts: we first treat the case where 0 is irregular for the reflected process and then we treat the case where 0 is regular (which is more involved and needs some preliminary estimates).

Remark 3.7. *We mention the following classification, see for instance Bertoin [Ber97]: if $(Z_t)_{t \geq 0}$ has infinite variations, then 0 is regular for $(R_t)_{t \geq 0}$. If it has finite variation, then denote by d its drift coefficient. If $d > 0$ then 0 is regular for $(R_t)_{t \geq 0}$ whereas if $d < 0$, it is irregular. In the remaining case $d = 0$, let us set $Z_t^+ = \sum_{s \leq t} \Delta Z_s \mathbf{1}_{\{\Delta Z_s > 0\}}$ and $Z_t^- = -\sum_{s \leq t} \Delta Z_s \mathbf{1}_{\{\Delta Z_s < 0\}}$ so that $Z_t = Z_t^+ - Z_t^-$. Then 0 is irregular for $(R_t)_{t \geq 0}$ if and only if $\lim_{t \rightarrow 0} Z_t^+ / Z_t^- = 0$ a.s.*

Case (A): assume 0 is irregular for the reflected process

Proof of Proposition 3.13 in case (A). We assume here that 0 is irregular for the reflected process. Therefore, the zero set of $(R_t)_{t \geq 0}$ is discrete (without accumulation points) and we can define for any $n \geq 0$, $T_{n+1} = \inf\{t > T_n, R_t = 0\}$ with $T_0 = 0$. Then the sequence $(T_n)_{n \geq 0}$ is an increasing random walk. Moreover, for any $n \geq 0$ and any $t \in [T_n, T_{n+1})$, $G_t = T_n$ and $S_t = S_{T_n}$.

Note that the ladder time process $(\sigma_t)_{t \geq 0}$ is a compound Poisson process and its Lévy measure is proportional to the law of T_1 , where $T_1 = \inf\{t > 0, R_t = 0\}$. This forces the trivariate subordinator $(\sigma_t, \theta_t, H_t)_{t \geq 0}$ to be a trivariate compound Poisson process with Lévy measure proportional to the law of $(T_1, \tau_{T_1}, S_{T_1})$. For any non-negative α, β, γ , we have

$$\begin{aligned} \mathbb{E} \left[e^{-\alpha G_e - \beta \tau_{G_e} - \gamma S_e} \right] &= \mathbb{E} \left[q \int_0^\infty e^{-qt - \alpha G_t - \beta \tau_{G_t} - \gamma S_t} dt \right] \\ &= \sum_{n \geq 0} \mathbb{E} \left[e^{-(\alpha+q)T_n - \beta \tau_{T_n} - \gamma S_{T_n}} \int_0^{T_{n+1}-T_n} q e^{-qt} dt \right]. \end{aligned}$$

By the strong Markov property, $(\tau_{t+T_n} - \tau_{T_n}, R_{t+T_n})_{t \in [0, T_{n+1}-T_n]}$ is independent of $\mathcal{F}_{T_n}$ and is equal in law to $(\tau_t, R_t)_{t \in [0, T_1]}$. Therefore, we get

$$\mathbb{E} \left[e^{-\alpha G_e - \beta \tau_{G_e} - \gamma S_e} \right] = \mathbb{E} \left[1 - e^{-qT_1} \right] \sum_{n \geq 0} \mathbb{E} \left[e^{-(\alpha+q)T_n - \beta \tau_{T_n} - \gamma S_{T_n}} \right].$$

By the strong Markov property, the sequence $(T_n, \tau_{T_n}, S_{T_n})_{n \geq 0}$ is a random walk and thus, we get

$$\mathbb{E} \left[e^{-\alpha G_e - \beta \tau_{G_e} - \gamma S_e} \right] = \frac{\mathbb{E} \left[1 - e^{-qT_1} \right]}{\mathbb{E} \left[1 - e^{-(\alpha+q)T_1 - \beta \tau_{T_1} - \gamma S_{T_1}} \right]} = \frac{\kappa(q, 0, 0)}{\kappa(\alpha + q, \beta, \gamma)},$$

where we recall that κ is the Laplace exponent of $(\sigma_t, \theta_t, H_t)_{t \geq 0}$. □

Case (B): assume 0 is regular for the reflected process. Before proving Proposition 3.13 in the regular case, we first recall and show a few results when 0 is regular.

Lemma 3.10. *If 0 is regular for the reflected process $(R_t)_{t \geq 0}$, then a.s. we have $S_e = S_{G_e-} = Z_{G_e-}$. Moreover, if $(Z_t)_{t \geq 0}$ is not a compound Poisson process and 0 is also regular for the dual reflected process $(\tilde{R}_t)_{t \geq 0}$, we have $Z_{G_e-} = Z_{G_e}$ and $\tau_{G_e-} = \tau_{G_e}$.*

Proof. It is proved properly in [Ber96, Ch. VI Thm. 5-(i)] that if 0 is regular, then $S_e = S_{G_e-} = Z_{G_e-}$, so we refer the reader to it. It is also proved there that $(Z_t)_{t \geq 0}$ cannot make a positive jump at time G_e , i.e. that $Z_{G_e-} \geq Z_{G_e}$.

We now turn to the second part of the lemma: we assume now that $(Z_t)_{t \geq 0}$ is not a compound Poisson process. Let us define the process $(\tilde{Z}_t)_{t \geq 0} = (Z_{(e-t)-} - Z_e)_{t \in [0, e]}$, which, by duality, is equal in law to $(-Z_t)_{t \in [0, e]}$. Since $(Z_t)_{t \geq 0}$ is not a compound Poisson process, [Ber96, Proposition 4 Chapter VI] implies that the supremum is reached at a unique time. This ensures that, in the obvious notations, we have $\tilde{G}_e = e - G_e$ and $\tilde{S}_e = S_e - Z_e$. Now if 0 is regular for the dual reflected process, we have $\tilde{Z}_{\tilde{G}_e-} \geq \tilde{Z}_{\tilde{G}_e}$, i.e. $Z_{G_e} \geq Z_{G_e-}$ which shows that $Z_{G_e-} = Z_{G_e}$.

Finally, we show that if G_e is a point of continuity of $(Z_t)_{t \geq 0}$, then it is also a point of continuity of $(\tau_t)_{t \geq 0}$. We denote by Π the Lévy measure of $(\tau_t, Z_t)_{t \geq 0}$, and $m \geq 0$ the drift coefficient of $(\tau_t)_{t \geq 0}$. Now remark that the Lévy measure Π_τ of $(\tau_t)_{t \geq 0}$ can be decomposed into two parts:

$$\Pi_\tau(dr) = \int_{z \in \mathbb{R}} \Pi(dr, dz) = \int_{z \in \mathbb{R}^*} \Pi(dr, dz) + \int_{z=0} \Pi(dr, dz).$$

In other words, we can decompose the subordinator $(\tau_t)_{t \geq 0}$ as

$$\tau_t = mt + \sum_{s < t} \Delta \tau_s \mathbf{1}_{\{|\Delta Z_s| > 0\}} + \sum_{s < t} \Delta \tau_s \mathbf{1}_{\{|\Delta Z_s| = 0\}} = mt + \tau_t^* + \tau_t^0,$$

which is then a sum of a drift and two independent independent subordinator (since they do not jump at the same time). We see first that if t is a jumping time of $(\tau_s^*)_{s \geq 0}$, then it is also a jumping time of $(Z_s)_{s \geq 0}$ so that any point of continuity of $(Z_s)_{s \geq 0}$ is a point of continuity of $(\tau_s^*)_{s \geq 0}$. Next, we see that $(\tau_t^0)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ are independent since they do not jump at the same time and since G_e is a functional of $(Z_t)_{t \geq 0}$, it is also independent from $(\tau_t^0)_{t \geq 0}$ and therefore it can not be a point of discontinuity of $(\tau_t^0)_{t \geq 0}$. This completes the proof. $\square$

When 0 is regular for $(R_t)_{t \geq 0}$, the ladder time process $(\sigma_t)_{t \geq 0}$ is a strictly increasing subordinator. For the local time $(L_t^R)_{t \geq 0}$ of the reflected process at the level 0, we have that there exists some $m_R \geq 0$ such that a.s., for any $t \geq 0$,

$$m_R L_t^R = \int_0^t \mathbf{1}_{\{R_s = 0\}} ds \quad \text{and} \quad \sigma_t = m_R t + \sum_{s \leq t} \Delta \sigma_s.$$

Then, it is well-known that the excursion process $(e_t^R)_{t \geq 0}$ defined by

$$e_t^R = \begin{cases} (R_{\sigma_{t-}+s})_{s \in [0, \Delta \sigma_t]} & \text{if } \Delta \sigma_t > 0, \\ \Upsilon & \text{otherwise,} \end{cases}$$

is a Poisson point process valued in the excursion space $\mathcal{E}_0$ (with the notation introduced in Section 3.4.1). We will denote by $\mathbf{n}_R$ its characteristic measure. Finally, since $(\sigma_t)_{t \geq 0}$ is strictly increasing if 0 is regular, we have almost surely for every $t \geq 0$, $\theta_{t-} = \tau_{(\sigma_{t-})-}$ and $H_{t-} = S_{(\sigma_{t-})-}$.

Proof of Proposition 3.13 in case (B). The idea of the proof is similar to that in case (A), although it is a little bit more tedious. Using that $S_e = S_{G_e-}$, and summing over all excursions of $(R_t)_{t \geq 0}$, indexed by their left end-point g and their right end-point d (using the same decomposition and similar notation as in the proof of Proposition 3.3), we have for any non-negative α, β, γ

$$\begin{aligned} \mathbb{E} \left[e^{-\alpha G_e - \beta \tau_{G_e-} - \gamma S_e} \right] &= \mathbb{E} \left[\int_0^\infty q e^{-qt - \alpha G_t - \beta \tau_{G_t-} - \gamma S_{G_t-}} dt \right] \\ &= \mathbb{E} \left[\int_0^\infty q e^{-(\alpha+q)t - \beta \tau_t - \gamma S_t} \mathbf{1}_{\{R_t=0\}} dt \right] + \mathbb{E} \left[\sum_{g \in \mathcal{G}_R} e^{-(\alpha+q)g - \beta \tau_g - \gamma S_g} \left(1 - e^{-q(d-g)} \right) \right]. \end{aligned}$$

Since for every $t \geq 0$, almost surely t is not a discontinuity of $(\tau_t)_{t \geq 0}$ and $(S_t)_{t \geq 0}$, the first term on the right-hand-side is equal to

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty q e^{-(\alpha+q)t - \beta \tau_t - \gamma S_t} \mathbf{1}_{\{R_t=0\}} dt \right] &= q m_R \mathbb{E} \left[\int_0^\infty e^{-(\alpha+q)t - \beta \tau_t - \gamma S_t} dL_t^R \right] \\ &= q m_R \int_0^\infty \mathbb{E} \left[e^{-(\alpha+q)\sigma_t - \beta \theta_t - \gamma H_t} \right] dt = \frac{q m_R}{\kappa(\alpha + q, \beta, \gamma)}, \end{aligned}$$

where we have also used that $(\sigma_t)_{t \geq 0}$ is the right-continuous inverse of $(L_t^R)_{t \geq 0}$ for the second identity. For the second term, we use the Master formula for Poisson point processes which tells us that

$$\mathbb{E} \left[\sum_{g \in \mathcal{G}_R} e^{-(\alpha+q)g - \beta \tau_g - \gamma S_g} \left(1 - e^{-q(d-g)} \right) \right] = n_R (1 - e^{-q\ell}) \mathbb{E} \left[\int_0^\infty e^{-(\alpha+q)\sigma_t - \beta \tau_{\sigma_t-} - \gamma S_{\sigma_t-}} dt \right].$$

Then, using that almost surely for every $t \geq 0$, $\theta_{t-} = \tau_{(\sigma_{t-})-}$ and $H_{t-} = S_{(\sigma_{t-})-}$, and that for every $t \geq 0$, almost surely, t is not a discontinuity of $(\sigma_t, \theta_t, H_t)_{t \geq 0}$, this is equal to

$$n_R (1 - e^{-q\ell}) \int_0^\infty \mathbb{E} \left[e^{-(\alpha+q)\sigma_t - \beta \theta_t - \gamma H_t} \right] dt = \frac{n_R (1 - e^{-q\ell})}{\kappa(\alpha + q, \beta, \gamma)}.$$

Since $\kappa(q, 0, 0)$ is the Laplace exponent of $(\sigma_t)_{t \geq 0}$, it follows from the exponential formula for Poisson point processes that $\kappa(q, 0, 0) = q m_R + n_R (1 - e^{-q\ell})$, which gives the desired result. $\square$

3.A.3 Proof of Theorem 3.3 and Proposition 3.4

We are finally ready to give the proof of Theorem 3.3. Afterwards, we deduce Proposition 3.4 from it.

Proof of Theorem 3.3. The independence of the two triplets follows from Lemma 3.9 and that $S_e = Z_{G_e-}$ in the regular case and $S_e = Z_{G_e}$ in the irregular case (recall Lemma 3.10). We now turn to proving items (ii)-(iii): we first deal with the case where $(Z_t)_{t \geq 0}$ is not a compound Poisson process, and then we treat the case of a compound Poisson process by approximation. The main idea is to prove that the law of (G_e, τ_{G_e}, S_e) and $(e - G_e, \tau_e - \tau_{G_e}, S_e - Z_e)$ (resp. of (G_e, τ_{G_e-}, S_e) and $(e - G_e, \tau_e - \tau_{G_e-}, S_e - Z_e)$ in the irregular case) are infinitely divisible; we then deduce their Lévy measure.

Step 1: We show that the law of (G_e, τ_{G_e}, S_e) , resp. of (G_e, τ_{G_e-}, S_e) , is infinitely divisible if 0 is irregular, resp. regular, for the reflected process $(R_t)_{t \geq 0}$. We only prove it in the irregular case as the proofs are identical.

For any positive q, α, β, γ , the Lévy-Khintchine formula tells us that

$$\kappa(\alpha + q, \beta, \gamma) - \kappa(q, 0, 0) = \int_{[0, \infty)^3} (1 - e^{-\alpha t - \beta r - \gamma x}) e^{-qt} \Pi(dt, dr, dx),$$

where Π is the Lévy measure of $(\sigma_t, \theta_t, H_t)_{t \geq 0}$. By Proposition 3.13, we have

$$\mathbb{E} \left[e^{-\alpha G_e - \beta \tau_{G_e} - \gamma S_e} \right] = \frac{\kappa(q, 0, 0)}{\kappa(q, 0, 0) + (\kappa(\alpha + q, \beta, \gamma) - \kappa(q, 0, 0))},$$

and we therefore see that the law of (G_e, τ_{G_e}, S_e) is equal to the law of a pure jump Lévy process with Lévy measure $e^{-qt} \Pi(dt, dr, dx)$, evaluated at an independent exponential time of parameter $\kappa(q, 0, 0)$. Therefore, it is infinitely divisible and it has no drift part and no Brownian part, so it is characterized by its Lévy measure.

Step 2: We assume that $(Z_t)_{t \geq 0}$ is not a compound Poisson process, and we show that the law of $(e - G_e, \tau_e - \tau_{G_e}, S_e - Z_e)$, resp. of $(e - G_e, \tau_e - \tau_{G_{e-}}, S_e - Z_e)$, is infinitely divisible if 0 is irregular, resp. regular for the reflected process $(R_t)_{t \geq 0}$. Note that the corresponding distributions are again characterized by their Lévy measure.

Let us define the process $(\tilde{\tau}_t, \tilde{Z}_t)_{t \geq 0} = (\tau_e - \tau_{(e-t)-}, Z_{(e-t)-} - Z_e)_{t \in [0, e]}$, which, by duality, is equal in law to $(\tau_t, -Z_t)_{t \in [0, e]}$. Since $(Z_t)_{t \geq 0}$ is not a compound Poisson process its supremum is reached at a unique time so we have $\tilde{G}_e = e - G_e$ and $\tilde{S}_e = S_e - Z_e$, with the obvious notation. Moreover, we have $\tilde{\tau}_{\tilde{G}_e} = \tau_e - \tau_{G_{e-}}$ and $\tilde{\tau}_{\tilde{G}_{e-}} = \tau_e - \tau_{G_e}$.

If 0 is irregular for $(R_t)_{t \geq 0}$, then it is necessarily regular for $(\hat{R}_t)_{t \geq 0}$, see Remark 3.7 and we can apply Step 1 with $(\tilde{G}_e, \tilde{\tau}_{\tilde{G}_{e-}}, \tilde{S}_e)$. If 0 is regular for $(R_t)_{t \geq 0}$ and irregular for $(\hat{R}_t)_{t \geq 0}$, then we can apply Step 1 with $(\tilde{G}_e, \tilde{\tau}_{\tilde{G}_e}, \tilde{S}_e)$. Finally, if 0 is regular for both sides, then $\tau_{G_e} = \tau_{G_{e-}}$ by Lemma 3.10 and we can again apply the first step.

Step 3: We are now able to conclude the proof of items (ii) and (iii) when $(Z_t)_{t \geq 0}$ is not a compound Poisson process. Again, we only show it in the irregular case. It is well-known that (e, τ_e, Z_e) is infinitely divisible, see [Ber96, Ch. VI Lem. 7], and that its Lévy measure is given by

$$\mu(dt, dr, dx) = t^{-1} e^{-qt} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt.$$

Let us denote the Lévy measures of (G_e, τ_{G_e}, S_e) and $(e - G_e, \tau_e - \tau_{G_e}, Z_e - S_e)$ by μ_+ and μ_- and recall that by Steps 1 and 2 that they characterize their distributions. Now, observe that

$$(e, \tau_e, Z_e) = (G_e, \tau_{G_e}, S_e) + (e - G_e, \tau_e - \tau_{G_e}, Z_e - S_e),$$

with the triplets being independent. It follows that $\mu = \mu_+ + \mu_-$. Since $(Z_t)_{t \geq 0}$ is not a compound Poisson process, we have $\mathbb{P}(Z_t = 0) = 0$ for every $t \geq 0$, and since μ_+ is supported on $[0, +\infty)^3$ and μ_- on $[0, +\infty)^2 \times (-\infty, 0]$, we get

$$\begin{aligned} \mu_+(dt, dr, dx) &= t^{-1} e^{-qt} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt, & (t > 0, r > 0, x > 0) \\ \mu_-(dt, dr, dx) &= t^{-1} e^{-qt} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt, & (t > 0, r > 0, x < 0), \end{aligned}$$

which finishes this step.

Step 4: We now extend the proof of items (ii) and (iii) when $(Z_t)_{t \geq 0}$ is a compound Poisson process. Note that in this case, 0 is regular for both sides. We proceed by approximation and consider $(Z_t^\varepsilon)_{t \geq 0} = (Z_t + \varepsilon t)_{t \geq 0}$ which is a Lévy process but not a compound Poisson process. It is easy to see that 0 is regular for $(R_t^\varepsilon)_{t \geq 0}$ but is irregular for $(\hat{R}_t^\varepsilon)_{t \geq 0}$, with the obvious notation.

Let us now show that a.s. $G_e^\varepsilon \rightarrow G_e$, $S_e^\varepsilon \rightarrow S_e$ and $\tau_{G_e^\varepsilon} \rightarrow \tau_{G_{e-}}$ as $\varepsilon \rightarrow 0$. This is obvious on the event $\{G_e = e\}$, since we have $G_e^\varepsilon = G_e$ and $S_e^\varepsilon = S_e + \varepsilon e$. Now, on the event $\{G_e < e\}$,

since $(Z_t)_{t \geq 0}$ is a compound Poisson process, we have $Z_t < Z_{G_e-} = S_e$ for any $t \in [G_e, e]$. Let $D_e = \sup_{t \in [G_e, e]} Z_t < S_e$ and set $\varepsilon_0 = (S_e - D_e)/e$: we see that for any $t \in [G_e, e]$ and any $\varepsilon \in (0, \varepsilon_0)$,

$$Z_t^\varepsilon = Z_t + \varepsilon t < S_e < Z_{G_e-}^\varepsilon.$$

Therefore, for any $\varepsilon \in (0, \varepsilon_0)$, we have $G_e^\varepsilon = G_e$, which implies again that a.s. $G_e^\varepsilon \rightarrow G_e$, $S_e^\varepsilon \rightarrow S_e$ and $\tau_{G_e^\varepsilon-} \rightarrow \tau_{G_e-}$ as $\varepsilon \rightarrow 0$.

Finally, as $\varepsilon \rightarrow 0$, the measures

$$\mathbf{1}_{\{x > 0\}} \mathbb{P}(\tau_t \in dr, Z_t^\varepsilon \in dx) \quad \text{and} \quad \mathbf{1}_{\{x < 0\}} \mathbb{P}(\tau_t \in dr, Z_t^\varepsilon \in dx)$$

respectively converge to

$$\mathbf{1}_{\{x \geq 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx) \quad \text{and} \quad \mathbf{1}_{\{x < 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx).$$

Then the result follows by approximation. $\square$

Proof of Proposition 3.4. Paraphrasing Corollary VI.10 p 165 in [Ber96]: for $\alpha' > 1$, the formula

$$\kappa(\alpha', \beta, \gamma) = c \exp \left(\int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha' t - \beta r - \gamma x}}{t} \mathbf{1}_{\{x \geq 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt \right)$$

follows from Theorem 3.3 and Proposition 3.13 applied with $q = 1$ (and $\alpha' = \alpha + q$). The identity is extended by analyticity.

Recall also the definition

$$\bar{\kappa}(\alpha, \beta, \gamma) = \exp \left(\int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha t - \beta r + \gamma x}}{t} \mathbf{1}_{\{x < 0\}} \mathbb{P}(\tau_t \in dr, Z_t \in dx) dt \right).$$

Then, if 0 is irregular for $(R_t)_{t \geq 0}$, we have for any positive $\alpha, \beta, \gamma, \hat{\alpha}, \hat{\beta}, \hat{\gamma}$, we have

$$\mathbb{E} \left[e^{-\alpha G_e - \beta \tau_{G_e} - \gamma S_e - \hat{\alpha}(e - G_e) - \hat{\beta}(\tau_e - \tau_{G_e}) - \hat{\gamma}(Z_e - S_e)} \right] = \frac{\kappa(q, 0, 0)}{\kappa(\alpha + q, \beta, \gamma)} \frac{\bar{\kappa}(q, 0, 0)}{\bar{\kappa}(\hat{\alpha} + q, \hat{\beta}, \hat{\gamma})}. \quad (3.54)$$

If 0 is regular for $(R_t)_{t \geq 0}$, then the same identity holds with τ_{G_e} replaced by τ_{G_e-} and $\tau_e - \tau_{G_e}$ replaced by $\tau_e - \tau_{G_e-}$. This formula yields (3.25).

Moreover, the duality entails that, if $(Z_t)_{t \geq 0}$ is not a compound Poisson process, there exists a constant $\hat{c} > 0$ such that $\hat{\kappa}(\alpha, \beta, \gamma) = \hat{c} \bar{\kappa}(\alpha, \beta, \gamma)$. $\square$

3.B Convergence of Lévy processes

In this section, we give some results on converging sequences of Lévy processes.

3.B.1 Convergence of processes and convergence of Laplace exponents

We consider a family of Lévy processes $\{(\tau_t^h, Z_t^h)_{t \geq 0}, h > 0\}$ with $(\tau_t^h)_{t \geq 0}$ a subordinator (for every $h > 0$), which converges in law for the Skorokhod topology as $h \rightarrow 0$, to some Lévy process $(\tau_t^0, Z_t^0)_{t \geq 0}$, with $(\tau_t^0)_{t \geq 0}$ also a subordinator and $(Z_t^0)_{t \geq 0}$ not a compound Poisson process. Note that, according to Jacod-Shiryaev [JS03, Thm. 2.9 p 396], $(\tau_t^h, Z_t^h)_{t \geq 0}$ converges in law for the Skorokhod topology if and only if (τ_1^h, Z_1^h) converges in law to (τ_1^0, Z_1^0) as $h \rightarrow 0$. For non-negative α, β, γ such that $\alpha + \beta + \gamma > 0$, we introduce the quantities

$$\phi_h(\alpha, \beta, \gamma) = \int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha t - \beta r - \gamma x}}{t} \mathbf{1}_{\{x \geq 0\}} \mathbb{P}(\tau_t^h \in dr, Z_t^h \in dx) dt$$

and

$$\bar{\phi}_h(\alpha, \beta, \gamma) = \int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha t - \beta r + \gamma x}}{t} \mathbf{1}_{\{x < 0\}} \mathbb{P}(\tau_t^h \in dr, Z_t^h \in dx) dt.$$

We will denote by $\phi_0(\alpha, \beta, \gamma)$ and $\bar{\phi}_0(\alpha, \beta, \gamma)$ the corresponding quantities for $(\tau_t^0, Z_t^0)_{t \geq 0}$. As we should expect, we have the following result.

Proposition 3.14. *Let $(\tau_t^h, Z_t^h)_{t \geq 0}$ and $(\tau_t^0, Z_t^0)_{t \geq 0}$ be as above. Then we have for any non-negative α, β, γ such that $\alpha + \beta > 0$,*

$$\lim_{h \rightarrow 0} \phi_h(\alpha, \beta, \gamma) = \phi_0(\alpha, \beta, \gamma) \quad \text{and} \quad \lim_{h \rightarrow 0} \bar{\phi}_h(\alpha, \beta, \gamma) = \bar{\phi}_0(\alpha, \beta, \gamma).$$

Proof. We will only show the convergence of ϕ_h as the proof for $\bar{\phi}_h$ is similar. We only stress that since $\mathbb{P}(Z_t^0 = 0) = 0$, by dominated convergence we have

$$\lim_{h \rightarrow 0} \mathbb{E} \left[(e^{-t} - e^{-\alpha t - \beta \tau_t^h - \gamma Z_t^h}) \mathbf{1}_{\{Z_t^h \geq 0\}} \right] = \mathbb{E} \left[(e^{-t} - e^{-\alpha t - \beta \tau_t^0 - \gamma Z_t^0}) \mathbf{1}_{\{Z_t^0 \geq 0\}} \right].$$

Now since

$$\phi_h(\alpha, \beta, \gamma) = \int_0^\infty \frac{1}{t} \mathbb{E} \left[(e^{-t} - e^{-\alpha t - \beta \tau_t^h - \gamma Z_t^h}) \mathbf{1}_{\{Z_t^h \geq 0\}} \right] dt,$$

it only remains to dominate the integrand. We first dominate for $t \in (0, 1)$. We have

$$\begin{aligned} \left| \mathbb{E} \left[(e^{-t} - e^{-\alpha t - \beta \tau_t^h - \gamma Z_t^h}) \mathbf{1}_{\{Z_t^h \geq 0\}} \right] \right| &\leq 1 - e^{-t} + \mathbb{E} \left[(1 - e^{-\alpha t - \beta \tau_t^h - \gamma Z_t^h}) \mathbf{1}_{\{Z_t^h \geq 0\}} \right] \\ &\leq 1 - e^{-t} + \mathbb{E} \left[(\beta \tau_t^h + \gamma |Z_t^h|) \wedge 1 \right]. \end{aligned}$$

Then by Lemma 3.11 below (applied to the sequence of bivariate Lévy process (τ_t^h, Z_t^h) valued in $\mathbb{R}^2$), there exists a constant $C_{\beta, \gamma} > 0$ such that for any $h \in (0, 1)$ and for any $t \in (0, 1)$, $\mathbb{E}[(\beta \tau_t^h + \gamma |Z_t^h|) \wedge 1] \leq C_{\beta, \gamma} \sqrt{t}$. Thus we get for any $h \in (0, 1)$ and any $t \in (0, 1)$,

$$t^{-1} \left| \mathbb{E} \left[(e^{-t} - e^{-\alpha t - \beta \tau_t^h - \gamma Z_t^h}) \mathbf{1}_{\{Z_t^h \geq 0\}} \right] \right| \leq 1 + \frac{C_{\beta, \gamma}}{\sqrt{t}},$$

which is integrable on $(0, 1)$. Next, we dominate on $[1, \infty)$. We have

$$\left| \mathbb{E} \left[(e^{-t} - e^{-\alpha t - \beta \tau_t^h - \gamma Z_t^h}) \mathbf{1}_{\{Z_t^h \geq 0\}} \right] \right| \leq e^{-t} + \mathbb{E} \left[e^{-\alpha t - \beta \tau_t^h - \gamma Z_t^h} \mathbf{1}_{\{Z_t^h \geq 0\}} \right].$$

If $\alpha > 0$, then we can dominate by $e^{-t} + e^{-\alpha t}$ which, divided by t , is integrable on $[1, \infty)$. If $\alpha = 0$, then $\beta > 0$ and we have

$$\left| \mathbb{E} \left[(e^{-t} - e^{-\alpha t - \beta \tau_t^h - \gamma Z_t^h}) \mathbf{1}_{\{Z_t^h \geq 0\}} \right] \right| \leq e^{-t} + e^{-\Phi_h(\beta)t},$$

where Φ_h is the Laplace exponent of $(\tau_t^h)_{t \geq 0}$. Since the latter converges in law to $(\tau_t)_{t \geq 0}$, we get that $\lim_{h \rightarrow 0} \Phi_h(\beta) = \Phi_0(\beta) > 0$ as $h \rightarrow 0$ where Φ_0 is the Laplace exponent of $(\tau_t^0)_{t \geq 0}$. Therefore, there exists a constant $c_\beta > 0$ such that for any $h \in (0, 1)$, $\Phi_h(\beta) \geq c_\beta$. Finally, we see that for any $h \in (0, 1)$, for any $t \geq 1$, we have

$$\left| \mathbb{E} \left[(e^{-t} - e^{-\alpha t - \beta \tau_t^h - \gamma Z_t^h}) \mathbf{1}_{\{Z_t^h \geq 0\}} \right] \right| \leq e^{-t} + e^{-c_\beta t},$$

which, divided by t , is integrable on $[1, \infty)$. This completes the proof. $\square$

Proposition 3.15. *Let $(\tau_t^h, Z_t^h)_{t \geq 0}$ and $(\tau_t^0, Z_t^0)_{t \geq 0}$ be as above and assume that 0 is recurrent for the reflected limiting process $(R_t^0)_{t \geq 0}$. Then we also have for any $\gamma > 0$, $\phi_h(0, 0, \gamma) \rightarrow \phi_0(0, 0, \gamma)$ as $h \rightarrow 0$.*

Here, we cannot use the dominated convergence theorem as it is not clear how to dominate $\mathbb{E}[e^{-\gamma Z_t^h} \mathbf{1}_{\{Z_t^h \geq 0\}}]$. We will instead use an argument of tightness combined with the continuity of the Laplace transform.

Proof. For every $h > 0$, let $(\sigma_t^h, \theta_t^h, H_t^h)_{t \geq 0}$ be the trivariate subordinator associated with $(\tau_t^h, Z_t^h)_{t \geq 0}$, as in Appendix 3.A. Since the local time of the reflected process $(R_t^h)_{t \geq 0}$ is defined up to a constant, we can normalize it so that for any non-negative α, β, γ , we have

$$\kappa_h(\alpha, \beta, \gamma) = \exp(\phi_h(\alpha, \beta, \gamma)),$$

where κ_h is the Laplace exponent of $(\sigma_t^h, \theta_t^h, H_t^h)_{t \geq 0}$. In other words, we can choose the constant c_h in Proposition 3.4 to be equal to 1. Similarly, we consider $(\sigma_t^0, \theta_t^0, H_t^0)_{t \geq 0}$ the trivariate subordinator associated with $(\tau_t^0, Z_t^0)_{t \geq 0}$, whose Laplace exponent κ_0 is such that for any non-negative α, β, γ , we have $\kappa_0(\alpha, \beta, \gamma) = \exp(\phi_0(\alpha, \beta, \gamma))$. Then by Proposition 3.14, $\kappa_h(\alpha, \beta, \gamma)$ converges to $\kappa_0(\alpha, \beta, \gamma)$ for any non-negative α, β, γ such that $\alpha + \beta > 0$. In particular, we have for any $\gamma > 0$,

$$\lim_{h \rightarrow 0} \mathbb{E} \left[e^{-\gamma(\sigma_1^h + H_1^h)} \right] = \lim_{h \rightarrow 0} e^{-\kappa_h(\gamma, 0, \gamma)} = e^{-\kappa_0(\gamma, 0, \gamma)} = \mathbb{E} \left[e^{-\gamma(\sigma_1^0 + H_1^0)} \right],$$

which implies that $\sigma_1^h + H_1^h$ converges in law to $\sigma_1^0 + H_1^0$ as $h \rightarrow 0$. Since 0 is recurrent for $(R_t^0)_{t \geq 0}$, the random variables σ_1^0 and H_1^0 are a.s. finite. This implies that the family of random variables $(\sigma_1^h + H_1^h)_{h \in (0,1)}$ is tight in $\mathbb{R}_+$, which in turn implies that the family $((\sigma_1^h, H_1^h))_{h \in (0,1)}$ is tight in $\mathbb{R}_+^2$. Let $((\sigma_1^{h_k}, H_1^{h_k}))_{k \in \mathbb{N}}$ be a subsequence which converges in law towards some finite random variables $(\bar{\sigma}, \bar{H})$. Using Proposition 3.14 again, we see that for any $\alpha > 0$ and any $\gamma \geq 0$, we have

$$\mathbb{E} \left[e^{-\alpha \bar{\sigma} - \gamma \bar{H}} \right] = \mathbb{E} \left[e^{-\alpha \sigma_1^0 - \gamma H_1^0} \right].$$

Since $\bar{H}$ and H_1^0 are finite, the above equality also holds for $\alpha = 0$ and $\gamma > 0$ by the monotone convergence theorem, which shows that the law of $(\bar{\sigma}, \bar{H})$ is uniquely determined and therefore (σ_1^h, H_1^h) converges in law as $h \rightarrow 0$ to (σ_1^0, H_1^0) . Hence H_1^h converges in law to H_1^0 , which completes the proof, since this implies that $\kappa_h(0, 0, \gamma)$ converges to $\kappa_0(0, 0, \gamma)$ for any $\gamma > 0$. $\square$

3.B.2 Technical lemmas

Lemma 3.11. *Let $(Z_t^n)_{t \geq 0}$, $n \geq 1$ be a sequence of Lévy processes on $\mathbb{R}^d$ which converges in law to a Lévy process $(Z_t)_{t \geq 0}$. Then there exists a constant $C > 0$ such that for any $n \geq 1$, for any $t \geq 0$, we have*

$$\mathbb{E}[|Z_t^n| \wedge 1] \leq C\sqrt{t}.$$

Proof. We classically decompose the sequence of Lévy processes as a sum of a Brownian motion, a compound Poisson process, a pure jump martingale and a drift: $Z_t^n = c_n B_t^n + C_t^n + M_t^n + b_n t$, where $b_n \in \mathbb{R}^d$, c_n is a $d \times d$ matrix. Let us denote by ν_n the Lévy measure of X^n , then the Lévy measure of C^n is $\mathbf{1}_{\{|x| \geq 1\}} \nu_n(dx)$ and M^n has Lévy measure $\mathbf{1}_{\{|x| < 1\}} \nu_n(dx)$. For any $n \geq 1$, for any $t \geq 0$, we have

$$\mathbb{E}[|Z_t^n|^2 \wedge 1] \leq 4 \left(\mathbb{E}[|c_n B_t^n|^2] + \mathbb{E}[|M_t^n|^2] + \mathbb{E}[|C_t^n|^2 \wedge 1] + t^2 |b_n|^2 \wedge 1 \right).$$

By the maximum inequality for compensated sums, we have

$$\mathbb{E} \left[|M_t^n|^2 \right] \leq \mathbb{E} \left[\sup_{[0,t]} |M_s^n|^2 \right] \leq t \int_{\{|x| < 1\} \setminus \{0\}} |x|^2 \nu_n(dx).$$

We introduce the truncation function $\chi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ which is a bounded continuous function such that $\chi(x) = x$ if $|x| \leq 1$. Setting $\tilde{c}_n := c_n^* c_n + \int \chi(x) \otimes \chi(x) \nu_n(dx)$, which is a $d \times d$ symmetric nonnegative matrix, we get by the above inequalities $\mathbb{E}[|c_n B_t^n|^2] + \mathbb{E}[|M_t^n|^2] \leq \text{Tr}(\tilde{c}_n)t$.

Let us now introduce the smooth function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^+$ such that $\varphi(x) = |x|^2$ if $|x| \leq 1$ and $\varphi(x) = 2$ if $|x| \geq 2$ and $1 \leq \varphi(x) \leq 2$ if $|x| \in [1, 2]$. By Itô formula, we have

$$\mathbb{E}[|C_t^n|^2 \wedge 1] \leq \mathbb{E}[\varphi(C_t^n)] = \int_0^t \int_{|x|>1} \mathbb{E}[\varphi(C_s^n + x) - \varphi(C_s^n)] \nu_n(dx) \leq 2t \int_{|x|>1} \nu_n(dx).$$

By Theorem 2.9 p 396 in [JS03], since $(Z_t^n)_{t \geq 0}$ converges in law to $(Z_t)_{t \geq 0}$:

$$b_n \rightarrow b, \quad \tilde{c}_n \rightarrow \tilde{c} \quad \text{and} \quad \int g(x) \nu_n(dx) \rightarrow \int g(x) \nu(dx),$$

for all continuous bounded function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ such that g is equal to 0 in a neighborhood of 0. Thus, setting $C_1 = \sup_n \text{Tr}(\tilde{c}_n)$, $C_2 = 2 \sup_n \int_{|x|>1} \nu_n(dx)$, $C_3 = (\sup_n |b_n|^2)^{1/2}$, $C_4 = C_3 \vee 1$ and $C = 4(C_1 + C_2 + C_4)$, which are finite quantities, we get for any $n \geq 1$, for any $t \geq 0$,

$$\mathbb{E}[|Z_t^n|^2 \wedge 1] \leq Ct.$$

Indeed, we have $(t^2 |b_n|^2) \wedge 1 \leq (t^2 C_3^2) \wedge 1 \leq C_4 t$. The result follows from Cauchy-Schwartz inequality. $\square$

Lemma 3.12. *Let $(\tau_t^n, Z_t^n)_{t \geq 0}$ be a sequence of Lévy processes taking values in $\mathbb{R}_+ \times \mathbb{R}$. Suppose that the sequence of subordinators $(\tau_t^n)_{t \geq 0}$ converges in law to $(\tau_t)_{t \geq 0}$ and $(Z_t^n)_{t \geq 0}$ converges to a Brownian motion $(Z_t)_{t \geq 0}$. Then $(\tau_t^n, Z_t^n)_{t \geq 0}$ converges in law to $(\tau_t, Z_t)_{t \geq 0}$ where $(\tau_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ are independent.*

Proof. Denote by $\pi_n(dr, dx)$ the Lévy measure of $(\tau_t^n, Z_t^n)_{t \geq 0}$ and define by $\chi(r, x) = (1 \wedge r, -1 \vee (x \wedge 1))$ a truncation function on $\mathbb{R}_+ \times \mathbb{R}$. It is continuous, bounded and equals (r, x) on a neighborhood of 0 in $\mathbb{R}_+ \times \mathbb{R}$. Then for all $\alpha > 0$, $\beta \in \mathbb{R}$,

$$\mathbb{E}[e^{-\alpha \tau_t^n + i\beta Z_t^n}] = \exp(t\psi_n(\alpha, \beta))$$

where

$$\psi_n(\alpha, \beta) = -\alpha b_n + i\beta \hat{b}_n - \frac{\sigma_n^2}{2} \beta^2 + \int_{\mathbb{R}_+ \times \mathbb{R}} e^{-\alpha r + i\beta x} - 1 - (-\alpha, i\beta) \cdot \chi(r, x) \pi_n(dr, dx).$$

We denoted by $(b_n, \hat{b}_n) \in \mathbb{R}_+ \times \mathbb{R}$ the drift coefficient (which depends on the choice of the truncation function) and σ_n^2 the Brownian coefficient of $(Z_t^n)_{t \geq 0}$. Then making $\alpha = 0$ and $\beta = 0$ in the expression we deduce that the characteristics of $(\tau_t^n)_{t \geq 0}$ and $(Z_t^n)_{t \geq 0}$ are respectively (b_n, ν_n) and $(\hat{b}_n, \sigma_n^2, \mu_n)$, where ν_n and μ_n are defined as

$$\nu_n(dr) := \int_{\mathbb{R}} \pi_n(dr, dx), \quad \mu_n(dx) := \int_{\mathbb{R}_+} \pi_n(dr, dx).$$

By Theorem VII.2.9 p 396 [JS03], the convergence in law of $(\tau_t^n)_{t \geq 0}$ and $(Z_t^n)_{t \geq 0}$ implies that $b_n, \hat{b}_n, \int_{\mathbb{R}} 1 \wedge r^2 \nu_n(dr)$ and $\sigma_n^2 + \int_{\mathbb{R}} 1 \wedge x^2 \mu_n(dx)$ converges and that for all $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\hat{g} : \mathbb{R} \rightarrow \mathbb{R}$ continuous bounded which are 0 around 0

$$\int_{\mathbb{R}_+} g(r) \nu_n(dr) \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}_+} g(r) \nu(dr), \quad \text{and} \quad \int_{\mathbb{R}_+} \hat{g}(x) \mu_n(dx) \xrightarrow{n \rightarrow +\infty} 0.$$

Here we denote by $\nu(dr)$ the Lévy measure of the limit process $(\tau_t)_{t \geq 0}$. It follows that for all $\delta > 0$, $\int_{\mathbb{R}} \mathbf{1}_{|x| \geq \delta} \mu_n(dx)$ goes to 0 and which implies that

$$\int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}_{|x| \geq \delta} \pi_n(dr, dx) \xrightarrow{n \rightarrow +\infty} 0. \quad (3.55)$$

By Theorem VII.2.9 p 396 [JS03] again, the convergence of $(\tau_t^n, Z_t^n)_{t \geq 0}$ will follow if we can show that (i)

$$\int_{\mathbb{R}_+ \times \mathbb{R}} (1 \wedge r)(-1 \vee (x \wedge 1)) \pi_n(dr, dx)$$

converges as $n \rightarrow \infty$, and that (ii) for all continuous function $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ which is constant outside a compact set and is 0 around 0, $\int g(r, x) \pi_n(dr, dx)$ converges.

Regarding item (i), we set $f(r, x) := (1 \wedge r)(-1 \vee (x \wedge 1))$ and fix $\delta > 0$. By (3.55), and since f is bounded, $\int_{|x| > \delta} |f(r, x)| \pi_n(dr, dx)$ vanishes as $n \rightarrow \infty$. It follows that

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}_+ \times \mathbb{R}} f(r, x) \pi_n(dr, dx) \right| \leq \limsup_{n \rightarrow \infty} \int_{|x| \leq \delta} |f(r, x)| \pi_n(dr, dx) \leq \delta \limsup_{n \rightarrow \infty} \int_{\mathbb{R}} 1 \wedge r \nu_n(dr).$$

Now, remark that $b_n - \int_{\mathbb{R}} 1 \wedge r \nu_n(dr)$ is the drift coefficient of the subordinator $(\tau_t^n)_{t \geq 0}$ and therefore, it must be positive. Since b_n converges, it follows that

$$0 \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}} 1 \wedge r \nu_n(dr) \leq \limsup_{n \rightarrow \infty} b_n < \infty.$$

Making δ to 0 shows that $\int_{\mathbb{R}_+ \times \mathbb{R}} f(r, x) \pi_n(dr, dx)$ converges to 0.

For item (ii), we consider a function $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ which is constant outside a compact set and is 0 around 0, say on $K := \{r \leq 1, |x| \leq 1\}$. It is uniformly continuous and for all $\varepsilon > 0$ there is some $1 > \delta > 0$ such that $|x| \leq \delta$ implies $|g(r, x) - g(r, 0)| \leq \varepsilon$. Using (3.55) again, and using that $\int_{\mathbb{R}_+} g(r, 0) \nu_n(dr) = \int_{\mathbb{R}_+ \times \mathbb{R}} g(r, 0) \mu_n(dr, dx)$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}_+ \times \mathbb{R}} g(r, x) \pi_n(dr, dx) - \int_{\mathbb{R}_+} g(r, 0) \nu_n(dr) \right| &\leq \limsup_{n \rightarrow \infty} \int_{|x| \leq \delta} |g(r, x) - g(r, 0)| \pi_n(dr, dx) \\ &= \limsup_{n \rightarrow \infty} \int_{K^c \cap \{|x| \leq \delta\}} |g(r, x) - g(r, 0)| \pi_n(dr, dx) \leq \varepsilon \limsup_{n \rightarrow \infty} \nu_n(\mathbb{R}_+ \setminus [0, 1]). \end{aligned}$$

Since $\limsup_{n \rightarrow \infty} \nu_n(\mathbb{R}_+ \setminus [0, 1]) < +\infty$ and since $\int_{\mathbb{R}_+} g(r, 0) \nu_n(dr)$ converges to $\int g(r, 0) \nu(dr)$, we conclude that

$$\int g(r, x) \pi_n(dr, dx) \xrightarrow{n \rightarrow +\infty} \int g(r, 0) \nu(dr).$$

To summarize we get that $(\tau_t^n, Z_t^n)_{t \geq 0}$ converges in law and the limiting characteristic Laplace-Fourier exponent is

$$\psi(\alpha, \beta) = -\alpha b + i\beta \hat{b} - \frac{\sigma^2}{2} \beta^2 + \int_{\mathbb{R}_+} (e^{-\alpha r} - 1 + \alpha 1 \wedge r) \nu(dr),$$

where $b := \lim_n b_n$, $\hat{b} := \lim_n \hat{b}_n$ and $\sigma^2 := \lim_n \sigma_n^2 + \int 1 \wedge x^2 \mu_n(dx)$. This is the characteristic exponent of $(\tau_t, Z_t)_{t \geq 0}$ where τ_t and Z_t are independent. $\square$

3.C Technical results on generalized one-dimensional diffusions

In this appendix, we prove some technical results from Section 3.7. We do not recall here the notation here, so we refer the reader to the introduction of Section 3.7.

3.C.1 Local times and the occupation time formula

The goal of this section is to prove Proposition 3.8 and Lemma 3.6. We do not recall the statements and refer the reader to Section 3.7.

Proof of Proposition 3.8. Item (i). Let $t \geq 0$ and h be a Borel function. Using a change of variables, we have

$$\int_0^t h(X_s) ds = \int_0^t h(\mathfrak{s}^{-1}(B_{\rho_s})) ds = \int_0^{\rho_t} h(\mathfrak{s}^{-1}(B_s)) dA_s^{\mathfrak{m}^s}.$$

By Corollary 2.13 in [RY99, Ch. X], it holds that for any $u \geq 0$,

$$\int_0^u h(\mathfrak{s}^{-1}(B_s)) dA_s^{\mathfrak{m}^s} = \int_{\mathbb{R}} h(\mathfrak{s}^{-1}(x)) L_u^x \mathfrak{m}^s(dx),$$

and the first item follows by substituting ρ_t in the preceding equation and performing a the change of variables $y = \mathfrak{s}^{-1}(x)$.

Item (ii). Let us now show that 0 is regular for $(X_t)_{t \geq 0}$. More precisely, we will show that the time $t = 0$ is an accumulation point of $\mathcal{Z}_X = \{t \geq 0, X_t = 0\} = \{t \geq 0, B_{\rho_t} = 0\}$. First we remark that for every $(\mathcal{F}_t)_{t \geq 0}$ -stopping time T such that $B_T = 0$, we have $A_{T+\varepsilon}^{\mathfrak{m}^s} - A_T^{\mathfrak{m}^s} > 0$ for every $\varepsilon > 0$. Indeed, the family $(L_{T+\varepsilon}^x - L_T^x)_{t \geq 0, x \in \mathbb{R}}$ is the family of local times of the Brownian motion $(B_{T+t})_{t \geq 0}$. Therefore, $L_{T+\varepsilon}^0 - L_T^0 > 0$ for every $\varepsilon > 0$ and by the continuity of the Brownian local times in the space variable, $L_{T+\varepsilon}^x - L_T^x > 0$ in a neighbourhood of 0. Finally, since $0 \in \text{supp}(\mathfrak{m}^s)$, we get

$$A_{T+\varepsilon}^{\mathfrak{m}^s} - A_T^{\mathfrak{m}^s} = \int_{\mathbb{R}} (L_{T+\varepsilon}^x - L_T^x) \mathfrak{m}^s(dx) > 0$$

for every $\varepsilon > 0$. Hence, for such a stopping time T , we have $\rho_{A_T^{\mathfrak{m}^s}} = \inf\{s > 0, A_s^{\mathfrak{m}^s} > A_T^{\mathfrak{m}^s}\} = T$. Let us define for any $t \geq 0$, $d_t = \inf\{s > t, B_s = 0\}$ the first zero of $(B_u)_{u \geq 0}$ after time t , which is an $(\mathcal{F}_t)_{t \geq 0}$ -stopping time. Consider a decreasing sequence $(t_n)_{n \in \mathbb{N}}$ which converges to 0 as $n \rightarrow \infty$. Then for any $n \in \mathbb{N}$, d_{t_n} is a stopping time such that $B_{d_{t_n}} = 0$ and therefore $\rho_{A_{d_{t_n}}^{\mathfrak{m}^s}} = d_{t_n}$. Let us set $(u_n)_{n \in \mathbb{N}} = (A_{d_{t_n}}^{\mathfrak{m}^s})_{n \in \mathbb{N}}$, then it is clear that it is a non-increasing sequence of points in $\mathcal{Z}_X$. Since $d_{t_n} \rightarrow 0$ as $n \rightarrow \infty$ and since $(A_t^{\mathfrak{m}^s})_{t \geq 0}$ is continuous, it comes that $u_n \rightarrow 0$ as $n \rightarrow \infty$ which shows that 0 is an accumulation point of $\mathcal{Z}_X$.

Item (iii). We show item (ii) and start by showing that $(L_{\rho_t}^0)_{t \geq 0}$ is continuous. Let Σ_L be the complement of the open set $\bigcup_{s \geq 0}]\tau_{s-}^B, \tau_s^B[$ where $(\tau_t^B)_{t \geq 0}$ is the right-continuous inverse of $(L_t^0)_{t \geq 0}$. The set Σ_L a.s. coincides with the set $\mathcal{Z}_B = \{t \geq 0, B_t = 0\}$. Let Σ_A be the complement of $\bigcup_{s \geq 0}]\rho_{s-}, \rho_s[$. It is shown in [RY99, Ch. X Proposition 2.17] that Σ_A a.s. coincides with the set $\Gamma_A = \{t \geq 0, B_t \in \text{supp}(\mathfrak{m}^s)\}$ and with the support of the measure $dA_t^{\mathfrak{m}^s}$. Since $0 \in \text{supp}(\mathfrak{m}^s)$, it comes that $\Gamma_L \subset \Gamma_A$ a.s. and therefore, we have

$$\bigcup_{s \geq 0}]\rho_{s-}, \rho_s[\subset \bigcup_{s \geq 0}]\tau_{s-}^B, \tau_s^B[.$$

Since $\bigcup_{s \geq 0}]\tau_{s-}^B, \tau_s^B[$ also coincides with the flat sections of $(L_t^0)_{t \geq 0}$ we get that for any $t \geq 0$ such that $\rho_{t-} < \rho_t$, then $L_{\rho_{t-}}^0 = L_{\rho_t}^0$ which shows that $(L_{\rho_t}^0)_{t \geq 0}$ is continuous. The fact that $(L_t^0)_{t \geq 0}$ remains an additive functional after time-change is rather classical. It follows from the fact that $(\rho_t)_{t \geq 0}$ is itself the inverse of a continuous additive functional, and $(L_t^0)_{t \geq 0}$ is a *strong* additive functional. We refer to Revuz-Yor [RY99, Ch. X, Propositions 1.2 and 1.3] for more details.

Let us show that the support of the measure $dL_{\rho_t}^0 = \mu(dt)$ is included in the closure of $\mathcal{Z}_X$. We introduce the measure $\nu(dt)$ which is the pushforward measure of μ by $(\rho_t)_{t \geq 0}$. It is the Stieltjes measure associated to the non-decreasing process $(L_{\rho_{A_t^{\mathbf{m}^s}}})_{t \geq 0}$. Let $t \geq 0$ be fixed, then if $A_{t+\varepsilon}^{\mathbf{m}^s} > A_t^{\mathbf{m}^s}$ for every $\varepsilon > 0$, we have $L_{\rho_{A_t^{\mathbf{m}^s}}}^0 = L_t^0$. On the other hand, if t is such that there exists $\varepsilon > 0$ such that $A_t^{\mathbf{m}^s} = A_{t+\varepsilon}^{\mathbf{m}^s}$, then $A_t^{\mathbf{m}^s}$ is a jumping time of $(\rho_s)_{s \geq 0}$ and since $(L_s^0)_{s \geq 0}$ is constant on $[\rho_{u-}, \rho_u]$ for every u such that $\Delta \rho_u > 0$, we get $L_t^0 = L_{\rho_{A_t^{\mathbf{m}^s}}}^0$. In the end, we see that ν is the Stieltjes measure associated to (L_t^0) and, applying the change of variables, we get

$$\int_0^t \mathbf{1}_{\{X_s \neq 0\}} dL_{\rho_s}^0 = \int_{[0,t]} \mathbf{1}_{\{B_{\rho_s} \neq 0\}} \mu(ds) = \int_{\rho([0,t])} \mathbf{1}_{\{B_s \neq 0\}} \nu(ds) \leq \int_0^{\rho_t} \mathbf{1}_{\{B_s \neq 0\}} dL_s^0 = 0.$$

This shows that the support of the measure μ is included in the closure of $\mathcal{Z}_X$.

We now show the converse and we will assume that 0 is an instantaneous point for $(X_t)_{t \geq 0}$, as the proof is easier when 0 is a holding point. In this case, the closure of $\mathcal{Z}_x$ is a perfect set with empty interior. For any $t \geq 0$, we denote by $d_t^X = \inf\{s > t, X_s = 0\}$ the first zero of $(X_s)_{s \geq 0}$ after t . Then for any $t \geq 0$, $B_{\rho_{d_t^X}} = 0$ and since $\rho_{d_t^X}$ is an $(\mathcal{F}_t)$ -stopping time, $L_{\rho_{d_t^X} + \varepsilon}^0 > L_{\rho_{d_t^X}}^0$ for every $\varepsilon > 0$. Since $(A_t^{\mathbf{m}^s})_{t \geq 0}$ is continuous, $(\rho_t)_{t \geq 0}$ is increasing and therefore $L_{\rho_{d_t^X} + \varepsilon}^0 > L_{\rho_{d_t^X}}^0$ for any $\varepsilon > 0$. This shows that almost surely, for any $t \in \mathbb{Q}_+$, d_t^X belongs to the support of μ . Consider now some t in the closure of $\mathcal{Z}_x$ and some $\varepsilon > 0$. Then since $\overline{\mathcal{Z}_X}$ has empty interior, there exists $r \in \mathbb{Q} \cap [t - \varepsilon, t)$ such that $r \notin \overline{\mathcal{Z}_X}$ and $d_r^X \leq t$. Hence t is a limit of points belonging to the support of μ , which is a closed set, and therefore t belongs to this set.

Item (iv). We now consider $(\tau_t)_{t \geq 0}$ the right-continuous inverse of $(L_{\rho_t}^0)_{t \geq 0}$. Let us show that for any $t \geq 0$, $\tau_t = A_{\tau_t}^{\mathbf{m}^s}$. The continuous non-decreasing process $(A_t^{\mathbf{m}^s})_{t \geq 0}$ is also the right-continuous inverse of $(\rho_t)_{t \geq 0}$ so that we have for any $t > 0$,

$$A_{\tau_t}^{\mathbf{m}^s} = \inf\{s > 0, \rho_s > \tau_t^B\} = \inf\{s > 0, L_{\rho_s}^0 > t\} = \tau_t.$$

For any $t \geq 0$, τ_t^B is a stopping time such that $B_{\tau_t^B} = 0$ and thus, by the argument from the second step, we have $\rho_{A_{\tau_t^B}^{\mathbf{m}^s}} = \rho_{\tau_t} = \tau_t^B$. $\square$

Proof of Lemma 3.6. We introduce the Lévy process $Z_t^g = \int_0^{\tau_t} ((g \circ \mathbf{s}) \times f)(X_s) ds$. It is clear that, by assumption and by the occupation time formula from item (i) of Proposition 3.8, this process is well defined. Let us denote by $\nu_g(dz) = \mathbf{n}(\int_0^\ell ((g \circ \mathbf{s}) \times f)(\varepsilon_s) ds \in dz)$ the Lévy measure of $(Z_t)_{t \geq 0}$. On one hand, we have

$$\mathbb{E}[Z_t] = t \int_{\mathbb{R}} z \nu_g(dz) = t \mathbf{n} \left(\int_0^\ell ((g \circ \mathbf{s}) \times f)(\varepsilon_s) ds \right).$$

On the other hand, it holds by items (i) and (iv) of Proposition 3.8 that

$$\mathbb{E}[Z_t] = \mathbb{E} \left[\int_{\mathbb{R}} L_{\rho_{\tau_t}}^x g(x) \mathbf{m}^f(dx) \right] = \int_{\mathbb{R}} \mathbb{E}[L_{\tau_t^B}^x] g(x) \mathbf{m}^f(dx).$$

By Ray-Knight's theorem, see [RY99, Chp. XI, Thm. 2.3], for any fixed $t \geq 0$, the processes $(L_{\tau_t^B}^x)_{x \geq 0}$ and $(L_{\tau_t^B}^x)_{x \leq 0}$ are two independent squared Bessel processes of dimensions 0 starting at t and therefore $\mathbb{E}[L_{\tau_t^B}^x] = t$ for any $x \in \mathbb{R}$, which completes the proof. $\square$

3.C.2 Convergence of strings under the Brownian excursion measure

The goal of this section is to prove the technical Lemmas 3.4 and 3.5 about strings with regular variation. Again, we do not recall the notation nor the statements of the results; we refer the reader to Section 3.7.2. In particular $\mathbf{m}$ is a string of regular variation with regular variation of index $\alpha \in (0, 2)$ and $\mathbf{m}_h$ denote its rescaled version.

Let us start with a technical lemma in the case $\alpha \in [1, 2)$, which is an almost direct application of Potter's bound [BGT87, Thm. 1.5.6].

Lemma 3.13. *Let $\mathbf{m}$ be a string with regular variation of index $\alpha \in [1, 2)$. Then for any $\eta > 0$, there exists a constant $C > 0$, such that for any $x > 0$ and any $h \in (0, 1)$,*

$$|\mathbf{m}_h(x)| \leq Cx^{1/\alpha-1-\eta} \vee x^{1/\alpha-1+\eta}. \quad (3.56)$$

Proof of Lemma 3.13. When $\alpha = 1$, this is a direct application of [BGT87, Thm. 3.8.6]. When $\alpha \in (1, 2)$, we have that there is a constant C_α such that

$$\mathbf{m}(\infty) - \mathbf{m}(y) \leq C_\alpha \Lambda(y) y^{1/\alpha-1} \quad \text{for any } y > 0,$$

using also that $1/\alpha - 1 < 0$ so the upper bound diverges as $x \downarrow 0$. It then simply remains to show that $\frac{\Lambda(x/h)}{\Lambda(1/h)} \leq Cx^{-\eta} \vee x^\eta$ for any $x > 0$, which is exactly the content of Potter's bound [BGT87, Thm. 1.5.6]. $\square$

Proof of Lemma 3.4. When $\alpha \in (0, 1)$ there is nothing to prove since $x^{1/\alpha-1} = c_\alpha \int_0^x y^{1/\alpha-2} dy$ is the cumulant to the Radon measure $x^{1/\alpha-2} dx$ on $\mathbb{R}_+$ and $x \mapsto g(x)$ is continuous with compact support. When $\alpha \in [1, 2)$, then for any $\delta \in (0, 1)$, then we have

$$\lim_{\delta \downarrow 0} \int_\delta^\infty \mathbf{L}_t^x \mathbf{m}_h(dx) = c_\alpha \int_\delta^\infty \mathbf{L}_t^x x^{1/\alpha-2} dx.$$

Again, this is a direct consequence of the fact that the measure $\mathbf{m}_h$ restricted to (δ, ∞) converges weakly to the measure $x^{1/\alpha-2} dx$ restricted to (δ, ∞) , and that $x \mapsto \mathbf{L}_t^x$ is continuous with compact support.

Now consider some $\gamma \in (0, 1/2)$ such that $1/\alpha - 1 + \gamma > 0$, which is possible since $\alpha < 2$. Since $x \mapsto g(x)$ is Hölder of order γ , and $g(0) = 0$, there exists a constant $D > 0$ such for any $x \in (0, \delta)$, $g(x) \leq Dx^\gamma$. Therefore we have

$$c_\alpha \int_0^\delta g(x) x^{1/\alpha-2} dx \leq D' \delta^{1/\alpha-1+\gamma},$$

for some constant $D' > 0$. Now pick some $\eta \in (0, 1/\alpha - 1 + \gamma)$, so by (3.56) there exists $C > 0$ such that $|\mathbf{m}_h(x)| \leq Cx^{1/\alpha-1-\eta}$ for any $x \in (0, \delta]$. Therefore we have

$$\int_0^\delta g(x) \mathbf{m}_h(dx) \leq D \int_0^\delta x^\gamma \mathbf{m}_h(dx) \leq \delta^\gamma |\mathbf{m}_h(\delta)| + \gamma \int_0^\delta x^{\gamma-1} |\mathbf{m}_h(x)| dx \leq C \delta^{1/\alpha-1+\gamma-\eta},$$

for some constant $C > 0$, the second inequality being obtained by integration by parts. Therefore, we have just proved that for all $\delta \in (0, 1)$, we have

$$\limsup_{h \rightarrow 0} \left| \int_{\mathbb{R}_+} g(x) \mathbf{m}_h(dx) - c_\alpha \int_{\mathbb{R}_+} g(x) x^{1/\alpha-2} dx \right| \leq D' \delta^{1/\alpha-1+\gamma} + C \delta^{1/\alpha-1+\gamma-\eta}.$$

Letting $\delta \downarrow 0$ proves the result. $\square$

Proof of Lemma 3.5. Before we start with the proof of the four items, let us recall Williams's decomposition of the excursion measure, see [RY99, Chp. XII, Thm. 4.5]. Take $(U_t)_{t \geq 0}$ and $(\tilde{U}_t)_{t \geq 0}$ two independent 3-dimensional Bessel processes starting from 0, defined on some probability space. For any $c > 0$, let T_c and $\tilde{T}_c$ be their respective first hitting of c . We define the process $(Q_t^c)_{t \geq 0}$ by

$$Q_t^c = \begin{cases} U_t & \text{if } 0 \leq t \leq T_c \\ c - \tilde{U}_{t-T_c} & \text{if } T_c \leq t \leq T_c + \tilde{T}_c \\ 0 & \text{if } t \geq T_c + \tilde{T}_c. \end{cases}$$

Then, the law of a positive excursion conditioned to $M(\varepsilon) = c$ is equal to the law of $(Q_t^c)_{t \geq 0}$, and for any measurable set Γ , we have

$$\mathbf{n}_+^B(\Gamma) = \frac{1}{2} \int_0^\infty \mathbb{P}((Q_t^y)_{t \geq 0} \in \Gamma) y^{-2} dy. \quad (3.57)$$

Then, denoting by $L_t^x(X)$ the local time in x at time t of a process $(X_s)_{s \geq 0}$, we have

$$L_{T_y + \tilde{T}_y}^x(Q^y) = L_{T_y}^x(U) + L_{\tilde{T}_y}^x(\tilde{U}) \leq L_\infty^x(U) + L_\infty^x(\tilde{U}) =: R_x. \quad (3.58)$$

Recalling the variant of the second Ray-Knight's theorem, see [Yor92, RK2.a)], $(L_\infty^x(U))_{x \geq 0}$ and $(L_\infty^x(\tilde{U}))_{x \geq 0}$ are two independent 2-dimensional squared Bessel processes started at 0 and so $(R_u)_{u \geq 0}$ is a square Bessel process of dimension 4 started at 0.

Item (i). Consider the first point and take $\alpha \in (0, 1)$, then combining (3.57) and (3.58) we have

$$\begin{aligned} \mathbf{n}_+^B \left[\left(\int_{\mathbb{R}_+} \mathbf{L}_\ell^x \mathbf{m}_h(dx) \right) \mathbf{1}_{\mathbf{A}_\delta} \right] &= \frac{1}{2} \int_0^\delta \mathbb{E} \left[\int_0^y L_{T_y + \tilde{T}_y}^u(Q^y) \mathbf{m}_h(du) \right] y^{-2} dy \\ &\leq \frac{1}{2} \int_0^\delta \mathbb{E} \left[\int_0^y R_u \mathbf{m}_h(du) \right] y^{-2} dy, \end{aligned}$$

Since $\mathbb{E}[R_u] = 4u$, this is bounded by

$$2 \int_0^\delta \left[\int_0^y u \mathbf{m}_h(du) \right] y^{-2} dy = 2 \int_0^\delta u \int_u^{+\infty} \frac{dy}{y^2} \mathbf{m}_h(du) = 2 \int_0^\delta \mathbf{m}_h(du) = 2\mathbf{m}_h(\delta).$$

The first point of the Lemma follows since $\lim_{h \rightarrow 0} \mathbf{m}_h(\delta) = \delta^{1/\alpha-1}$ and $\alpha \in (0, 1)$.

Item (ii). Take $\alpha \in [1, 2)$ and consider $\varphi = x^2$ so that

$$\mathbf{n}_+^B \left[\sup_{h \in (0, 1]} \left(\int_{\mathbb{R}_+} \mathbf{L}_\ell^x \mathbf{m}_h(dx) \right)^2 \mathbf{1}_{\mathbf{A}_\delta} \right] \leq \frac{1}{2} \int_0^\delta \mathbb{E} \left[\sup_{h \in (0, 1]} \left(\int_0^y R_u \mathbf{m}_h(du) \right)^2 \right] y^{-2} dy.$$

Fix $\eta > 0$ (how small depends on α). Since $(R_u)_{u \geq 0}$ is a sum of independent squares of Brownian processes, by Kolmogorov's regularity criterion (see [RY99, Thm. 2.1]), we have that $R_u \leq C_\eta u^{1-\eta}$ for all $u \in [0, 1]$, where $C_\eta > 0$ is a square integrable positive random variable. So for $\delta \in (0, 1]$,

$$\mathbf{n}_+^B \left[\sup_{h \in (0, 1]} \left(\int_{\mathbb{R}_+} \mathbf{L}_\ell^x \mathbf{m}_h(dx) \right)^2 \mathbf{1}_{\mathbf{A}_\delta} \right] \leq \frac{1}{2} \mathbb{E}[C_\eta^2] \int_0^\delta \sup_{h \in (0, 1]} \left(\int_0^y u^{1-\eta} \mathbf{m}_h(du) \right)^2 y^{-2} dy. \quad (3.59)$$

Moreover, since $\int_0^y u^{1-\eta} \mathbf{m}_h(du) = y^{1-\eta} \mathbf{m}_h(y) - (1-\eta) \int_0^y u^{-\eta} \mathbf{m}_h(u) du$, we have

$$\left(\int_0^y u^{1-\eta} \mathbf{m}_h(du) \right)^2 \leq 4 \left(y^{2-2\eta} \mathbf{m}_h(y)^2 + \left(\int_0^y u^{-\eta} |\mathbf{m}_h(u)| du \right)^2 \right).$$

Take $0 < 2\eta < 1/\alpha - 1/2$ (which is possible since $\alpha < 2$), by Lemma 3.13 we get that $|\mathbf{m}_h(u)| \leq c_\eta u^{1/\alpha-1-\eta}$ uniformly in $h \in (0, 1)$ and $u \in (0, 1]$, for some constant c_η . Since $1/\alpha - 1 - 2\eta > -1$ there is a constant $\widehat{c}_\eta > 0$ such that

$$\sup_{h \in (0, 1]} \left(\int_0^y u^{1-\eta} \mathbf{m}_h(du) \right)^2 \leq \widehat{c}_\eta y^{2/\alpha-4\eta}, \quad \forall y \in [0, 1].$$

Hence for $\eta > 0$ small enough, the left-hand side of (3.59) is bounded by a constant (that depends on η) times $\delta^{2/\alpha-1-4\eta}$, which goes to 0 as $\delta \downarrow 0$.

Item (iii). We proceed in the same vein. Similarly as above, we have

$$\mathbf{n}_+^B \left[\sup_{h \in (0, 1]} \left(\int_0^{+\infty} \mathbf{L}_\ell^x \mathbf{m}_h(dx) \right) \mathbf{1}_{\mathbf{A}_\delta^c} \right] \leq \frac{1}{2} \int_\delta^\infty \mathbb{E} \left[\sup_{h \in (0, 1]} \left(\int_0^y R_u \mathbf{m}_h(du) \right) \right] y^{-2} dy.$$

Fix $\eta > 0$. By time-inversion, $(\widetilde{R}_u)_{u \geq 0} := (u^2 R_{1/u})_{u \geq 0}$ has the same law at $(R_u)_{u \geq 0}$, hence Kolmogorov's regularity theorem applied both to $(\widetilde{R}_u)_{u \leq 1}$ and $(R_u)_{u \leq 1}$ gives that

$$R_u \leq C_\eta (u^{1-\eta} \vee u^{1+\eta}), \quad \forall u \geq 0,$$

with C_η some integrable random variable. Thus we have,

$$\mathbf{n}_+^B \left[\sup_{h \in (0, 1]} \left(\int_0^{+\infty} \mathbf{L}_\ell^x \mathbf{m}_h(dx) \right) \mathbf{1}_{\mathbf{A}_\delta^c} \right] \leq \frac{1}{2} \mathbb{E}[C_\eta] \int_\delta^{+\infty} \sup_{h \in (0, 1]} \left(\int_0^y u^{1-\eta} \vee u^{1+\eta} \mathbf{m}_h(du) \right) y^{-2} dy$$

Then we write for all $y \geq \delta$,

$$\int_0^y u^{1-\eta} \vee u^{1+\eta} \mathbf{m}_h(du) = \int_0^{y \wedge 1} u^{1-\eta} \mathbf{m}_h(du) + \int_1^{y \vee 1} u^{1+\eta} \mathbf{m}_h(du).$$

If η is small enough so that $2\eta < 1 - 1/\alpha$ and $2\eta < 1/\alpha$, which is possible since $\alpha \in (1, 2)$, it follows from Lemma 3.13 that $|\mathbf{m}_h(y)| \leq C y^{1/\alpha-1-\eta} \vee y^{1/\alpha-1+\eta}$. Then, recalling that $\mathbf{m}_h(0) = 0$, we see that

$$\int_0^{y \wedge 1} u^{1-\eta} \mathbf{m}_h(du) \leq (y \wedge 1)^{1-\eta} \mathbf{m}_h(y \wedge 1) \leq C(y \wedge 1)^{1/\alpha-2\eta} \leq C.$$

Similarly we have

$$\int_1^{y \vee 1} u^{1+\eta} \mathbf{m}_h(du) \leq (y \vee 1)^{1+\eta} (\mathbf{m}_h(y \vee 1) - \mathbf{m}_h(1)) \leq C(y \vee 1)^{1/\alpha+2\eta}.$$

In the end, we have for any $y \geq \delta$ the bound $\int_0^y u^{1-\eta} \vee u^{1+\eta} \mathbf{m}_h(du) \leq C + C(y \vee 1)^{1/\alpha+2\eta}$ which is integrable with respect to the measure $y^{-2} dy$ on $[\delta, \infty)$ since $2 - 1/\alpha - 2\eta > 1$. This concludes the proof item (iii).

Item (iv). This point is similar. We write similarly as above

$$\begin{aligned} \mathbf{n}_+^B \left[\sup_{h \in (0, 1]} \left(\int_0^1 \mathbf{L}_\ell^x \mathbf{m}_h(dx) \right) \mathbf{1}_{\mathbf{A}_\delta^c} \right] &= \frac{1}{2} \int_\delta^\infty \mathbb{E} \left[\sup_{h \in (0, 1]} \int_0^{1 \wedge y} R_u \mathbf{m}_h(du) \right] y^{-2} dy \\ &\leq \frac{1}{2} \mathbb{E}[C_\eta] \int_\delta^{+\infty} \sup_{h \in (0, 1]} \left(\int_0^{y \wedge 1} u^{1-\eta} \mathbf{m}_h(du) \right) y^{-2} dy, \end{aligned}$$

which is finite as in the case of the previous item. $\square$

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Fractional diffusion limit for a kinetic Fokker-Planck equation with diffusive boundary conditions in the half-line

Abstract

This chapter contains the results of [Bét22] which is submitted for publication. We consider a particle with position $(X_t)_{t \geq 0}$ living in $\mathbb{R}_+$, whose velocity $(V_t)_{t \geq 0}$ is a positive recurrent diffusion with heavy-tailed invariant distribution when the particle lives in $(0, \infty)$. When it hits the boundary $x = 0$, the particle restarts with a random strictly positive velocity. We show that the properly rescaled position process converges weakly to a stable process reflected on its infimum. From a P.D.E. point of view, the time-marginals of $(X_t, V_t)_{t \geq 0}$ solve a kinetic Fokker-Planck equation on $(0, \infty) \times \mathbb{R}_+ \times \mathbb{R}$ with diffusive boundary conditions. Properly rescaled, the space-marginal converges to the solution of some fractional heat equation on $(0, \infty) \times \mathbb{R}_+$.

4.1 Introduction

In the last two decades, many mathematical works showed how to derive *anomalous diffusion limit* results, also called *fractional diffusion limits*, from different kinetic equations with heavy-tailed equilibria. In short, these types of results state that the properly rescaled density of the position of a particle subject to some kinetic equation, is asymptotically non-gaussian. A case of particular interest is when the scaling limit of the position of the particle is a stable process, of which the time-marginals satisfies the fractional heat equation.

Mellet, Mischler and Mouhot [MMM11] showed a fractional diffusion limit result for a linearized Boltzmann equation with heavy-tailed invariant distribution using Fourier transform arguments, a result which was improved by Mellet [Mel10] using a moments method. The proofs rely entirely on analytic tools. A similar result was shown in Jara, Komorowski and Olla [JKO09], although it is derived from an α -stable central limit theorem for additive functional of Markov chains, i.e. from a probabilistic result.

Anomalous diffusion limits also occur for kinetic equations with degenerate collision frequency, see Ben Abdallah, Mellet and Puel [BAMP11a], and in transport of particles in plasma, see Cesbron, Mellet and Trivisa [CMT12]. In [BM20], Bouin and Mouhot propose a unified approach to derive fractional diffusion limits from several linear collisional kinetic equations.

In [LP19], Lebeau and Puel showed a fractional diffusion limit result for a one-dimensional Fokker-Planck equation with heavy-tailed equilibria. From a probabilistic point of view, the corresponding toy model is a one-dimensional particle whose velocity is subject to a restoring force F and random shocks modeled by a Brownian motion $(B_t)_{t \geq 0}$, which leads to the following stochastic differential equation:

$$V_t = v_0 + \int_0^t F(V_s) ds + B_t, \quad X_t = x_0 + \int_0^t V_s ds, \quad (4.1)$$

where $(x_0, v_0) \in \mathbb{R}^2$, $(V_t)_{t \geq 0}$ and $(X_t)_{t \geq 0}$ are the velocity and position processes of the particle. The force at stake is $F(v) = -\frac{\beta}{2} \frac{v}{1+v^2}$, where $\beta \in (1, 5) \setminus \{2, 3, 4\}$ leading to an invariant measure which behaves as $(1 + |v|)^{-\beta}$ as $|v| \rightarrow \infty$. Their result states that in this case, the properly rescaled position process resembles a stable process in large time. When $\beta > 5$, Nasreddine and Puel [NP15] established that $(\varepsilon^{1/2} X_{t/\varepsilon})_{t \geq 0}$ resembles a Brownian motion as $\varepsilon \rightarrow 0$, corresponding to a classical diffusion limit type theorem. Then Cattiaux, Nasreddine and Puel [CNP19] later showed that in the critical case $\beta = 5$, the same result holds up to a logarithmic correction term.

This phenomenon was actually observed by physicists who discovered experimentally that atoms cooled by a laser diffuse anomalously, see for instance Castin, Dalibard and Cohen-Tannoudji [CDCT], Sagi, Brook, Almog and Davidson [SBAD12] and Marksteiner, Ellinger and Zoller [MEZ96]. A theoretical study (see Barkai, Aghion and Kessler [BAK14]) modeling the motion of atoms precisely by (4.1) proved with quite a high level of rigor the observed phenomena.

Then, using probabilistic techniques, Fournier and Tardif [FT21] treated all cases of (4.1) (i.e. $\beta > 0$) for a slightly larger class of symmetric forces. When $\beta \geq 5$, the limiting distribution is Gaussian whereas when $\beta \in (1, 5)$, they show that the following convergence in finite dimensional distributions holds, for any initial condition $v_0 \in \mathbb{R}$:

$$\left(\varepsilon^{1/\alpha} X_{t/\varepsilon} \right)_{t \geq 0} \xrightarrow{f.d.} (\sigma_\alpha Z_t^\alpha)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0, \quad (4.2)$$

where $(Z_t^\alpha)_{t \geq 0}$ is a symmetric α -stable process with $\alpha = (\beta + 1)/3$, and σ_α is some positive diffusive constant. Naturally, they recover the result of [NP15, CNP19, LP19] and even go beyond, treating the case $\beta \in (0, 1)$ which was new. In this regime, the velocity is null recurrent, and the rescaled process was shown to converge to a symmetric Bessel process of dimension $\delta \in (0, 2)$. Then the position process naturally converges to an integrated symmetric Bessel process, which is no longer Markov. Their proof heavily relies on Feller's representation of diffusion processes through their scale functions and speed measures, enabling them to treat all cases at once, even the critical cases $\beta = 1, 2, 5$. This method was generalized in [Bét21] and (4.2) was shown to be a special case of an α -stable central limit theorem for additive functional of one-dimensional Markov processes. In a companion paper, Fournier and Tardif [FT20] also showed that these results hold in any dimension and the proofs are much more involved than in dimension 1.

As it is very natural in kinetic theory to consider gas particles interacting with a surface in thermodynamical equilibrium, corresponding to the case of diffusive boundary conditions, we propose in this article to study a version of the process $(X_t)_{t \geq 0}$ living in $\mathbb{R}_+$ and reflected diffusively through its velocity when the particles hits 0. In other words, we consider the case of a particle governed by (4.1), which interacts with a wall located at $x = 0$. When the particle hits the boundary, it reemerges from the wall with a random velocity distributed according to some probability measure.

The aim of this paper is to study the scaling limit of such a particle. More precisely, we will show that in the Lévy regime ($\beta \in (1, 5)$), the rescaled position process converges in law to a stable process reflected on its infimum. This result should be related to the recent articles of Cesbron, Mellet and Puel [CMP20, CMP21] and Cesbron [Ces20], which extend the results obtained in [BAMP11a, MMM11, Mel10]. They deal with the kinetic scattering equation describing particles living in the half-space or in bounded domains, with specular and / or diffusive boundary conditions. They obtain as a scaling limit a fractional heat equation with some boundary conditions depending on the original boundary conditions. This differs from the normal diffusive case where one would always obtain the classical heat equation with Neumann boundary conditions. Here, the kinetic equation at stake is different, but we obtain a similar limiting equation as in [CMP20, CMP21, Ces20] with however some different behaviour at the boundary. We refer to the PDE section below for more details. In [CMP20, Ces20], the limiting PDE is clearly identified, but they have a uniqueness issue due to the weakness of their solutions. This problem was solved in dimension 1 in [CMP21]. We emphasize that we have no such issue and that the limiting process is quite explicit. We also point out that their result only holds for $\alpha \in (1, 2)$.

We should also mention the works of Komorowski, Olla, Ryzhik [KOR20] and Bogdan, Komorowski, Marino [BKM22], which are, to our knowledge, the only probabilistic results of fractional diffusion limits with boundary interactions. They both study a scattering equation (linear Boltzmann), as in [CMP20, CMP21, Ces20], but with a mixed reflective / transmissive / absorbing boundary conditions. We refer to the comments section below, where these papers are further discussed.

Let us now introduce more formally the model studied. We will denote by $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of positive integers. Let $v_0 > 0$, $(B_t)_{t \geq 0}$ be a Brownian motion and $(M_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d. random variables whose law μ is supported in $(0, \infty)$. Everything is assumed to be independent. The object at stake in this article is the strong Markov process $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ valued in $E = ((0, \infty) \times \mathbb{R}) \cup (\{0\} \times (0, \infty))$ and defined by the following stochastic differential equation

$$\begin{cases} \mathbf{X}_t = x_0 + \int_0^t \mathbf{V}_s ds, \\ \mathbf{V}_t = v_0 + \int_0^t F(\mathbf{V}_s) ds + B_t + \sum_{n \in \mathbb{N}} (M_n - \mathbf{V}_{\tau_n-}) \mathbf{1}_{\{\tau_n \leq t\}}, \\ \tau_1 = \inf\{t > 0, \mathbf{X}_t = 0\} \quad \text{and} \quad \tau_{n+1} = \inf\{t > \tau_n, \mathbf{X}_t = 0\}, \end{cases} \quad (4.3)$$

where F fulfills Assumption 4.1 below and $(x_0, v_0) \in E$. This equation is well-posed and we refer to Subsection 4.2.1 for more details. It describes the motion of a particle evolving in $[0, \infty)$ and being reflected when it hits 0. More precisely, the velocity $(\mathbf{V}_t)_{t \geq 0}$ and position $(\mathbf{X}_t)_{t \geq 0}$ processes are governed by (4.1) when $\mathbf{X}_t > 0$, and the particle is reflected through the velocity when it hits the boundary, i.e. when $\mathbf{X}_t = 0$. Note that $t \mapsto \mathbf{V}_t$ is a.s. càdlàg and that the jumps only occur when the particle hits the boundary, i.e. when $t = \tau_n$ for some $n \in \mathbb{N}$. In this case, the value of the velocity after the jump is

$$\mathbf{V}_{\tau_n} = \mathbf{V}_{\tau_n-} + \Delta \mathbf{V}_{\tau_n} = M_n$$

As for every $n \in \mathbb{N}$, $M_n > 0$ a.s., the particle reaches $(0, \infty)$ instantaneously after hitting the boundary and thus spend a strictly positive amount of time in $(0, \infty)$. Hence the zeros of $(\mathbf{X}_t)_{t \geq 0}$ are countable and the successive hitting times $(\tau_n)_{n \in \mathbb{N}}$ are well defined. Finally, we point out that since a solution $(X_t, V_t)_{t \geq 0}$ of (4.1) reaches $\{0\} \times \mathbb{R}$ with a necessarily non-positive velocity, the process $(\mathbf{X}_{t-}, \mathbf{V}_{t-})_{t \geq 0}$ is valued in $((0, \infty) \times \mathbb{R}) \cup (\{0\} \times (-\infty, 0])$.

Let us mention some related works, dealing with Langevin-type models reflected in the half-space with diffusive-reflective boundary conditions. In [Jac12, Jac13], Jacob studies the classical Langevin process, namely the process $(B_t, \int_0^t B_s ds)_{t \geq 0}$, reflected at a partially elastic boundary. In [JP19], Jabir and Profeta study a stable Langevin process with diffusive-reflective boundary conditions. Roughly, their work treats the well-posedness of the equations and whether or not, the obtained process is conservative, observing some phase transitions. The question of the existence of a scaling limit does not make sense since the processes are intrinsically self-similar (at least in the purely reflective case).

In the rest of the article, the process $(X_t, V_t)_{t \geq 0}$ will always be defined by (4.1), and will be referred to as the "free process". On the other hand, $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ will always refer to the process defined by (4.3) and will be called the "reflected process" or "reflected process with diffusive boundary conditions". Let us now be a little more precise on the assumptions we will suppose. Regarding the force field F , we assume the following.

Assumption 4.1. *The Lipschitz and bounded force $F : \mathbb{R} \rightarrow \mathbb{R}$ is such that $F = \frac{\beta}{2} \frac{\Theta'}{\Theta}$ where $\beta \in (1, 5)$ and $\Theta : \mathbb{R} \rightarrow (0, \infty)$ is a C^1 even function satisfying $\lim_{v \rightarrow \pm\infty} |v| \Theta(v) = 1$.*

This assumption is a little bit stronger than the one in [FT21], and thus (4.2) holds for the free process. The typical force we have in mind is $F(v) = -\frac{\beta}{2} \frac{v}{1+v^2}$ studied in [LP19, NP15, CNP19], corresponding to the function $\Theta(v) = (1 + v^2)^{-1/2}$. The value of the diffusive constant σ_α from (4.2) is given by

$$\sigma_\alpha = \frac{3^{1-2\alpha} 2^{\alpha-1} \pi c_\beta}{\Gamma^2(\alpha) \sin(\pi\alpha/2)}, \quad \text{where} \quad c_\beta = \left(\int_{\mathbb{R}} \Theta^\beta(v) dv \right)^{-1} \quad (4.4)$$

Regarding the probability measure μ governing the reflection, we will have two different assumptions.

Assumption 4.2.

- (i) *There exists $\eta > 0$ such that μ has a moment of order $(\beta + 1)/2 + \eta$.*
- (ii) *There exists $\eta > 0$ such that μ has a moment of order $(\beta + 1)(\beta + 2)/6 + \eta$.*

Since $\beta > 1$, Assumption 4.2-(ii) is obviously stronger than Assumption 4.2-(i). These assumptions are satisfied by many probability measures of interest such as the Gaussian density, every sub-exponential distributions, and many heavy-tailed distributions.

During the whole paper, Assumption 4.1 is always in force and Assumption 4.2 will be mentioned when necessary.

For a family of processes $((Y_t^\varepsilon)_{t \geq 0})_{\varepsilon > 0}$, we say that $(Y_t^\varepsilon)_{t \geq 0} \xrightarrow{f.d.} (Y_t^0)_{t \geq 0}$ as $\varepsilon \rightarrow 0$ if for all $n \geq 1$, for all $t_1, \dots, t_n \geq 0$, the vector $(Y_{t_i}^\varepsilon)_{1 \leq i \leq n}$ converges in law to $(Y_{t_i}^0)_{1 \leq i \leq n}$ in $\mathbb{R}^n$. Most of the convergence results obtained are actually stronger than convergence in finite dimensional distributions, i.e. we obtain convergence in law of processes in the space of càdlàg functions. As the usual Skorokhod topology is not suited for convergence of continuous processes to a discontinuous process, we will instead use a weaker topology, namely the $\mathbf{M}_1$ -topology, and we refer to Section 4.2.2 for more details. The following theorem is the main result of this paper.

Theorem 4.1. *Grant Assumptions 4.1 and 4.2-(i), and let $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ be a solution of (4.3) starting at $(0, v_0)$ with $v_0 > 0$. Let $(Z_t^\alpha)_{t \geq 0}$ be a symmetric stable process with $\alpha = (\beta + 1)/3$ and such that $\mathbb{E}[e^{i\xi Z_t^\alpha}] = \exp(-t\sigma_\alpha |\xi|^\alpha)$ where σ_α is defined in (4.4). Let $(R_t^\alpha)_{t \geq 0}$ be the stable process reflected on its infimum, i.e. $R_t^\alpha = Z_t^\alpha - \inf_{s \in [0, t]} Z_s^\alpha$. Then we have*

$$(\varepsilon^{1/\alpha} \mathbf{X}_{t/\varepsilon})_{t \geq 0} \xrightarrow{f.d.} (R_t^\alpha)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0.$$

Moreover, if we grant Assumption 4.2-(ii), this convergence in law holds in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology.

It is likely that this theorem could be extended for any initial condition $(x_0, v_0) \in E$ with some small adjustments but for the sake of simplicity we will only consider the case $x_0 = 0$ and $v_0 > 0$.

Let us try to explain informally why the limiting process should be the stable process reflected on its infimum. Remember that when $\mathbf{X}_t > 0$, the process is governed by (4.1), and therefore, we should expect the limit process to behave like a stable process in $(0, \infty)$. Only the behavior at the boundary is to be identified. When $\mathbf{X}_t$ reaches the boundary with a very high speed, corresponding to a jump in the limit, it is suddenly reflected and slowed down, as the new velocity is distributed according to μ ; and the process somehow restarts afresh. As a consequence, we should expect the following behavior for the limiting process: when it tries to jump across the boundary, the jump is "cut" and the process restarts from 0. This is exactly the behavior of $(R_t^\alpha)_{t \geq 0}$: when $R_t^\alpha > 0$, it behaves as Z_t^α and when Z_t^α jumps below its past infimum, the latter one is immediately "updated", corresponding to R_t^α trying to cross the boundary and being set to 0.

We emphasize that $(R_t^\alpha)_{t \geq 0}$ is a Markov process, see [Ber96, Chapter 6, Proposition 1]. Let us now point out an interesting phenomena: the limiting process really depends on the way we reflect the initial process. For instance, let us consider (4.3) with a specular boundary condition, i.e. when the process hits the boundary, it is reflected with the same incoming velocity, which is flipped. Then it is easy to see that, since the force field is symmetric, $(|X_t|, \text{sgn}(X_t)V_t)_{t \geq 0}$ is a solution of the corresponding reflected equation, where $(X_t, V_t)_{t \geq 0}$ is a solution of (4.1). Then by (4.2), the limiting process is $(|Z_t^\alpha|)_{t \geq 0}$ whose behavior at the boundary is different from $(R_t^\alpha)_{t \geq 0}$: when it tries to cross the boundary, the process is moved back in $\mathbb{R}_+$ by a mirror reflection. Note that $(|Z_t^\alpha|)_{t \geq 0}$ is a Markov process only in the symmetric case, so it is not clear what happens in the asymmetric case.

Unlike the Brownian motion, there is no unique way to reflect a stable process. Indeed, by a famous result of Paul Lévy, it is well known that $(|B_t|)_{t \geq 0} \stackrel{d}{=} (B_t - \inf_{s \in [0, t]} B_s)_{t \geq 0}$. This is to be related with the fact that unlike the Laplacian, the fractional Laplacian is a non-local operator. We believe that, with a little bit of work, we could extend Theorem 4.1 to the diffusive regime, i.e. when $\beta \geq 5$, and the rescaled process would converge to a reflected Brownian motion.

On our way to establish Theorem 4.1, we will encounter a singular equation which describes the motion of a particle reflected at a completely inelastic boundary, which is very close to the equation studied by Bertoin in [Ber08]. We will study a solution of the following stochastic differential equation.

$$\begin{cases} \mathbb{X}_t = \int_0^t \mathbb{V}_s ds, \\ \mathbb{V}_t = v_0 + \int_0^t F(\mathbb{V}_s) ds + B_t - \sum_{0 < s \leq t} \mathbb{V}_{s-} \mathbf{1}_{\{\mathbb{X}_s = 0\}}, \end{cases} \quad (4.5)$$

where $v_0 > 0$. Let $a > 0$ and consider a solution $(\mathbf{X}_t^a, \mathbf{V}_t^a)_{t \geq 0}$ of (4.3) starting at $(0, v_0)$ for the particular choice $\mu = \delta_a$. Then, informally, $(\mathbf{X}_t^a, \mathbf{V}_t^a)_{t \geq 0}$ should tend to $(\mathbb{X}_t, \mathbb{V}_t)_{t \geq 0}$ as we let $a \rightarrow 0$. Since for every $a > 0$, the rescaled process $(\varepsilon^{1/\alpha} \mathbf{X}_{t/\varepsilon}^a)_{t \geq 0}$ converges in law to $(R_t^\alpha)_{t \geq 0}$, it should be expected that $(\mathbb{X}_t)_{t \geq 0}$ has the same scaling limit. We will see that it is indeed the case, see Theorem 4.6 below. While it is clear that we can construct a solution to (4.3), it is non-trivial that (4.5) possesses a solution and let us quickly explain why.

Consider a solution $(X_t, V_t)_{t \geq 0}$ of (4.1) starting at $(0, 0)$, then one can easily see that 0 is an accumulation point of the instants at which $X_t = 0$. Now if we consider a solution $(\mathbb{X}_t, \mathbb{V}_t)_{t \geq 0}$ of (4.5), its energy is fully absorbed when the particle hits the boundary i.e. when $\mathbb{X}_t = 0$, the value of the velocity is $\mathbb{V}_t = \mathbb{V}_{t-} - \mathbb{V}_{t-} = 0$. Hence it is not clear at all whether the particle will ever reemerge in $(0, \infty)$ or will remain stuck at 0. As we will see, it turns out that (4.5) admits a conservative solution, which never gets stuck at 0.

In [Ber07, Ber08], Bertoin studies equation (4.5) in the special case $F = 0$ and he establishes the existence and uniqueness (in some sense). We will not be interested in studying the uniqueness of (4.5) in our case but we will use the construction developed in [Ber07] and [Ber08].

Comments and comparison with the litterature

At this point, we should discuss the works of Komorowski, Olla, Ryzhik [KOR20] and Bogdan, Komorowski, Marino [BKM22]. In [KOR20], they study a scattering equation with a reflective / transmissive / absorbing boundary conditions. When the particle hits the boundary $x = 0$, it is either reflected (by flipping the incoming velocity), either unchanged (transmitted), or killed. Their model is such that the rescaled particle always satisfies $\mathbf{X}_0^\varepsilon = x$ for some $x > 0$, and they find the following limiting process. Consider a symmetric α -stable process $(Z_t^\alpha)_{t \geq 0}$ started at x with $\alpha \in (1, 2)$, as well as its successive crossing times $(\sigma_n)_{n \in \mathbb{N}}$ at the level 0. Consider also i.i.d. random variables $(\xi_n)_{n \in \mathbb{N}}$ such that $\mathbb{P}(\xi_1 = 1) = p_+$, $\mathbb{P}(\xi_1 = -1) = p_-$ and $\mathbb{P}(\xi_1 = 0) = p_0$, with $p_+, p_-, p_0 > 0$ such that $p_+ + p_- + p_0 = 1$. The reflected stable process $(R_t^\alpha)_{t \geq 0}$ is then defined as follows: $R_t^\alpha = Z_t^\alpha$ on $[0, \sigma_1)$, and for any $n \in \mathbb{N}$ and any $t \in [\sigma_n, \sigma_{n+1})$,

$$R_t^\alpha = \left(\prod_{k=1}^n \xi_k \right) Z_t^\alpha.$$

In other words, each time $(Z_t^\alpha)_{t \geq 0}$ crosses the boundary, the trajectory of $(R_t^\alpha)_{t \geq 0}$ is transmitted with probability p_+ , reflected with probability p_- or absorbed with probability p_0 . As it is well-known, a symmetric stable process with index $\alpha > 1$ eventually touches 0, but crosses the boundary infinitely many times before doing so, see for instance [Ber96, Chapter VIII, Proposition 8]. Therefore $\sigma_\infty = \lim_{n \rightarrow \infty} \sigma_n$ is a.s. finite but since $p_0 > 0$, the reflected process is a.s. absorbed before σ_∞ and $(R_t^\alpha)_{t \geq 0}$ is naturally set to 0 after σ_∞ .

In [BKM22], they study the very same model, but the probability p_0^ε for the kinetic process to be absorbed when it hits the boundary vanishes as $\varepsilon \rightarrow 0$ and behaves as $1/|\log \varepsilon|$. The limiting process obtained is the same as above with $p_0 = 0$ for $t < \sigma_\infty$, and the process is absorbed at 0 at $t = \sigma_\infty$.

Observe that in both cases, the limiting process is killed before (or precisely when) hitting the boundary. In the present paper, we thus have a substantial additional difficulty which is to characterize the limiting process when starting from 0. Indeed, a symmetric stable process started from 0 touches 0 infinitely many times immediately after (when $\alpha > 1$), so that the behavior of the limiting process started from 0 is not trivially defined. We cannot avoid this difficulty as the kinetic process is restarted with some small velocity.

We emphasize that these kinds of results are very recent and were mostly treated from an analytic point of view, see [CMP20, CMP21, Ces20]. With the papers [KOR20, BKM22], our work seems to be the only probabilistic study of fractional diffusion limit with boundary conditions. Our proof borrows different tools from stochastic analysis such as excursion theory, Wiener-Hopf factorization and a bit of enlargement of filtrations.

Regarding Assumption 4.2-(i), we believe that it is near optimality. As we will see, it appears naturally in the proofs at several places. Moreover, we believe the limiting process should differ

at criticality, i.e. when $v \mapsto \mu((v, \infty))$ is regularly varying with index $-(\beta + 1)/2$ as $v \rightarrow \infty$. Roughly, the restarting velocities are no longer negligible and, the limiting process directly enters the domain after hitting 0 by a jump with "law" $x^{-\alpha}dx$. Observe that this is the reason why they [CMP20, CMP21] have to assume $\alpha > 1$. We think that, at criticality, we might obtain the same limiting distribution as in [CMP20, CMP21].

Assumption 4.2-(ii) is technical and we believe the convergence in the $\mathbf{M}_1$ -topology should also hold under Assumption 4.2-(i).

The present results might be extended, through the same line of proof, to integrated powers of the velocity, i.e. to $\mathbf{X}_t = \int_0^t \text{sgn}(\mathbf{V}_s) |\mathbf{V}_s|^\gamma ds$ and some $\gamma > 0$. At least, we already know from [Bét21, Theorem 5] that a α -stable central limit theorems holds for the free process.

Finally, it would be interesting to study what happens in higher dimensions (using the results of [FT20]), as well as a more complete model, with both diffusive and specular boundary conditions: when the particle hits the boundary, it is reflected diffusively with probability $p \in (0, 1)$ and specularly with probability $1 - p$.

Informal PDE description of the result

Let us expose our result with a kinetic theory point of view, making a bridge with the P.D.E. papers [NP15, CNP19, LP19, CMP20, CMP21, Ces20]. Let us denote by f_t the law of the process $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ solution to (4.3) starting at $(0, v_0)$ with $v_0 > 0$, i.e. $f_t(dx, dv) = \mathbb{P}(\mathbf{X}_t \in dx, \mathbf{V}_t \in dv)$ which is a probability measure on $\mathbb{R}_+ \times \mathbb{R}$. Then, see Proposition 4.8 and Remark 4.1, $(f_t)_{t \geq 0}$ is a weak solution of the kinetic Fokker-Planck equation with diffusive boundary conditions

$$\begin{cases} \partial_t f_t + v \partial_x f_t = \frac{1}{2} \partial_v^2 f_t - \partial_v [F(v) f_t] & \text{for } (t, x, v) \in (0, \infty)^2 \times \mathbb{R} \\ v f_t(0, v) = -\mu(v) \int_{(-\infty, 0)} w f_t(0, w) dw & \text{for } (t, v) \in (0, \infty)^2 \\ f_0 = \delta_{(0, v_0)} \end{cases}$$

where we assume for simplicity that $\mu(dv) = \mu(v)dv$.

We now set $\rho_t(dx) = \mathbb{P}(R_t^\alpha \in dx)$ where $(R_t^\alpha)_{t \geq 0}$ is the limiting process defined in Theorem 3.2. Then, see Proposition 4.9 and Remark 4.2, $(\rho_t)_{t \geq 0}$ is a weak solution of

$$\begin{cases} \partial_t \rho_t(x) = \frac{\sigma_\alpha}{2} \int_{\mathbb{R}} \frac{\rho_t(x-z) \mathbf{1}_{\{x > z\}} - \rho_t(x) + z \partial_x \rho_t(x) \mathbf{1}_{\{|z| < x\}}}{|z|^{1+\alpha}} dz & \text{for } (t, x) \in (0, \infty)^2, \\ \int_0^\infty \rho_t(x) dx = 1 & \text{for } t \in (0, \infty), \\ \rho_0(\cdot) = \delta_0. \end{cases}$$

We believe that the above equation might be well-posed, just as for the heat equation $\partial_t \rho_t(x) = \partial_{xx} \rho_t(x)$ on $(0, \infty)^2$ where the Neumann boundary condition $\partial_x \rho_t(0) = 0$ can classically be replaced by the constraint $\int_0^\infty \rho_t(x) dx = 1$. The following statement immediately follows from Theorem 3.2

Corollary 4.1. *Grant Assumptions 4.1 and 4.2-(i). Let $g_t(dx) = \int_{v \in \mathbb{R}} f_t(dx, dv) = \mathbb{P}(\mathbf{X}_t \in dx)$ and set with an abuse of notation $g_t^\varepsilon(x) = \varepsilon^{-1/\alpha} g_{t/\varepsilon}(\varepsilon^{-1/\alpha} x)$, that is g_t^ε is the pushforward measure of $g_{t/\varepsilon}$ by the map $x \mapsto \varepsilon^{1/\alpha} x$. It holds that for each $t \geq 0$, g_t^ε converges weakly (in the sense of measures) to ρ_t as $\varepsilon \rightarrow 0$.*

It is classical, see e.g. Bertoin [Ber96, Chapter VI, Proposition 3], that for any fixed $t \geq 0$, R_t^α has the same law as $\sup_{s \in [0, t]} Z_s^\alpha$. Hence, the results of Doney-Savov [DS10, Theorem 1] tell us that ρ_t has a continuous density $\rho_t(x)$ satisfying the following asymptotics:

$$\rho_t(x) \sim A t x^{-1-\alpha} \quad \text{as } x \rightarrow \infty, \quad \text{and} \quad \rho_t(x) \sim B t^{-1/2} x^{\alpha/2-1} \quad \text{as } x \rightarrow 0.,$$

for some constants $A, B > 0$ and for all $t > 0$.

Regarding the limiting fractional diffusion equation, we stress that it is not the same as the one from [CMP20, CMP21, Ces20]. Both corresponding Markov processes possess the same infinitesimal generator, with however a different domain: they both behave like a stable process when strictly positive but are not reflected in the same manner when hitting 0. We do not find the same limiting P.D.E. inside the domain: they have an additional term expressing that particles can jump from the boundary to the interior of the domain, whereas in our case the process $(R_t^\alpha)_{t \geq 0}$ leaves 0 continuously. More precisely, their limiting P.D.E. can be written as

$$\begin{aligned} \partial_t \rho_t(x) &= \int_{\mathbb{R}} \frac{\rho_t((x-z)_+) - \rho_t(x) + z \partial_x \rho_t(x) \mathbf{1}_{\{|z| < x\}}}{|z|^{1+\alpha}} dz \\ &= \int_{\mathbb{R}} \frac{\rho_t(x-z) \mathbf{1}_{\{x > z\}} - \rho_t(x) + z \partial_x \rho_t(x) \mathbf{1}_{\{|z| < x\}}}{|z|^{1+\alpha}} dz + \rho_t(0) \frac{x^\alpha}{\alpha}, \end{aligned}$$

together with some boundary condition ensuring the mass conservation.

Plan of the paper and sketch of the proof

Once the limiting process is identified, it is very natural to try to establish the scaling limit of $(X_t - \inf_{s \in [0, t]} X_s)_{t \geq 0}$, where $(X_t)_{t \geq 0}$ is defined by (4.1). It should be clear that we need more than the convergence in finite dimensional distributions, and we need at least the convergence of past supremum and infimum. This is why we will use the convergence in the $\mathbf{M}_1$ -topology.

In Section 4.2, we first explain why (4.3) is well-posed. Then we recall and define rigorously the notion of convergence in the $\mathbf{M}_1$ -topology. This section ends with Theorem 4.4, which states that the convergence (4.2) actually holds in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology, strengthening the result of [FT21]. The proof is postponed to Section 4.5.

In Section 4.3, we study the particle reflected at a completely inelastic boundary, i.e. the solution of (4.5). We will see that $(X_t - \inf_{s \in [0, t]} X_s)_{t \geq 0}$ plays a central role in the construction of a solution to (4.5). This construction is essentially the same as in [Ber07, Ber08]. Then we prove, see Theorem 4.6, that $(X_t - \inf_{s \in [0, t]} X_s)_{t \geq 0}$ and $(\mathbb{X}_t)_{t \geq 0}$ have the same scaling limit, which is $(R_t^\alpha)_{t \geq 0}$. The proof relies on Theorem 4.4, the continuous mapping theorem and Skorokhod's representation theorem.

In Section 4.4, we finally establish the scaling limit of $(\mathbf{X}_t)_{t \geq 0}$. The proof consists in using the scaling limit of $(X_t - \inf_{s \in [0, t]} X_s)_{t \geq 0}$ and to compare this process with $(\mathbf{X}_t)_{t \geq 0}$. More precisely, we will show two comparison results. First we will see that $\mathbf{X}_t \geq X_t - \inf_{s \in [0, t]} X_s$. Then, inspired by the work of Bertoin [Ber07, Ber08] and its construction of a solution to (4.5), we will show how we can construct a solution to (4.3) from the free process $(X_t, V_t)_{t \geq 0}$. From this construction, we will remark that, up to a time-change $(A'_t)_{t \geq 0}$, we have $\mathbf{X}_{A'_t} \leq X_t - \inf_{s \in [0, t]} X_s$. Then the proof is almost complete if we can show that $A'_t \sim t$ as $t \rightarrow \infty$ and we will see that it is indeed the case. Subsections 4.4.3 and 4.4.6 are dedicated to the proof of this result. We believe that these are the most technical parts of the paper. We stress that Assumption 4.2 is only used in Subsection 4.4.6.

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4.2 Preliminaries

4.2.1 Well-posedness

In this subsection, we explain quickly why (4.3) possesses a unique solution. To do so, we will regard (4.1) as an ordinary differential equation with a continuous random source. The force F being Lipschitz continuous, for any $(x_0, v_0) \in \mathbb{R}^2$, for every $\omega \in \Omega$, there exists a unique global solution $(X_t(\omega), V_t(\omega))_{t \geq 0}$ to

$$V_t(\omega) = v_0 + \int_0^t F(V_s(\omega)) ds + B_t(\omega), \quad X_t(\omega) = x_0 + \int_0^t V_s(\omega) ds. \quad (4.6)$$

One can easily construct a solution to (4.3) by "gluing" together solutions of (4.1) until their first hitting times of $\{0\} \times \mathbb{R}$. First consider the solution $(X_t^1, V_t^1)_{t \geq 0}$ of (4.1) starting at $(x_0, v_0) \in E$, define $\tau_1 = \inf\{t > 0, X_t^1 = 0\}$ and set $(\mathbf{X}_t, \mathbf{V}_t)_{t \in [0, \tau_1)} = (X_t^1, V_t^1)_{t \in [0, \tau_1)}$. Then consider the solution $(X_t^2, V_t^2)_{t \geq 0}$ of (4.1) starting at $(0, M_1)$ with B_t replaced by $B_{t+\tau_1} - B_{\tau_1}$, define $\tau_2 - \tau_1 = \inf\{t > 0, X_t^2 = 0\}$ and set $(\mathbf{X}_t, \mathbf{V}_t)_{t \in [\tau_1, \tau_2)} = (X_t^2, V_t^2)_{t \in [0, \tau_2 - \tau_1)}$. Iterating this operation indefinitely, we obtain a process $(\mathbf{X}_t, \mathbf{V}_t)_{t \in [0, \tau_\infty)}$ defined on $[0, \tau_\infty)$ where $\tau_\infty = \lim_{n \rightarrow \infty} \tau_n$, which solves (4.3) on $[0, \tau_\infty)$.

Consider now two solutions $(\mathbf{X}_t^1, \mathbf{V}_t^1)_{t \in [0, \tau_\infty^1)}$ and $(\mathbf{X}_t^2, \mathbf{V}_t^2)_{t \in [0, \tau_\infty^2)}$ of (4.3) with the same initial condition, together with their respective sequence of hitting times $(\tau_n^1)_{n \in \mathbb{N}}$ and $(\tau_n^2)_{n \in \mathbb{N}}$. For every $\omega \in \Omega$, $(X_t^1(\omega), V_t^1(\omega))_{t \geq 0}$ and $(X_t^2(\omega), V_t^2(\omega))_{t \geq 0}$ are two solutions of (4.6) on the time interval $[0, \tau_1^1(\omega) \wedge \tau_1^2(\omega))$. Hence they are equal on this interval and $\tau_1^1(\omega) = \tau_1^2(\omega)$, and we can extend this reasoning to deduce that $\tau_\infty^1 = \tau_\infty^2 = \tau_\infty$ and that $(\mathbf{X}_t^1, \mathbf{V}_t^1)_{t \in [0, \tau_\infty)}$ and $(\mathbf{X}_t^2, \mathbf{V}_t^2)_{t \in [0, \tau_\infty)}$ are equal. Therefore, uniqueness holds for (4.3) for each $\omega \in \Omega$. Note that so far, we did not need the use of filtrations.

For a solution $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ of (4.3), we set $(\mathcal{F}_t)_{t \geq 0}$ for the usual completion of the filtration generated by the process. Then $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ is a strong Markov process in the filtration $(\mathcal{F}_t)_{t \geq 0}$. Since $(\tau_n)_{n \in \mathbb{N}}$ is the sequence of successive hitting times of $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ in $\{0\} \times \mathbb{R}$, it is a sequence of $(\mathcal{F}_t)_{t \geq 0}$ -stopping times and we deduce from the strong Markov property that the sequence $(\tau_{n+1} - \tau_n)_{n \in \mathbb{N}}$ is i.i.d. and therefore $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$ almost surely. In other words, any solution of (4.3) is global.

Finally, we stress that uniqueness in law holds for (4.3) as pathwise uniqueness for S.D.E. implies uniqueness in law. To summarize, (4.3) is well-posed, i.e. there exists a unique and global solution to (4.3) for any initial condition $(x_0, v_0) \in E$.

4.2.2 $\mathbf{M}_1$ -topology and the scaling limit of the free process

The main result of [FT21], see (4.2), is a convergence in the finite dimensional distributions sense and we cannot hope to obtain a convergence in law as a process in the usual Skorokhod distance, namely the $\mathbf{J}_1$ -topology. This is due to the fact that the space of continuous functions is closed in the space of càdlàg functions endowed with the $\mathbf{J}_1$ -topology. But the process may converge in a weaker topology and we will show that the process actually converges in the $\mathbf{M}_1$ -topology, first introduced in the seminal work of Skorokhod [Sko56]. In this subsection, we recall the

definition and a few properties of this topology. All of the results stated here may be found in Skorokhod [Sko56] or in the book of Whitt [Whi02, Chapter 12].

For any $T > 0$, we denote by $\mathcal{D}_T = \mathcal{D}([0, T], \mathbb{R})$ the usual sets of càdlàg functions on $[0, T]$ valued in $\mathbb{R}$. For a function $x \in \mathcal{D}_T$, we define the completed graph $\Gamma_{T,x}$ of x as follows:

$$\Gamma_{T,x} = \{(t, z) \in [0, T] \times \mathbb{R}, z \in [x(t-), x(t)]\}.$$

The $\mathbf{M}_1$ -topology on $\mathcal{D}_T$ is metrizable through parametric representations of the complete graphs. A parametric representation of x is a continuous non-decreasing function (u, r) mapping $[0, 1]$ onto $\Gamma_{T,x}$. Let us denote by $\Pi_{T,x}$ the set of parametric representations of x . Then the $\mathbf{M}_1$ -distance on $\mathcal{D}_T$ is defined for $x_1, x_2 \in \mathcal{D}_T$ as

$$d_{\mathbf{M}_1, T}(x_1, x_2) = \inf_{(u_i, r_i) \in \Pi_{T, x_i}} (||u_1 - u_2|| \vee ||r_1 - r_2||),$$

where $||\cdot||$ is the uniform distance. The metric space $(\mathcal{D}_T, d_{\mathbf{M}_1, T})$ is separable and topologically complete.

Let us now denote by $\mathcal{D} = \mathcal{D}(\mathbb{R}_+, \mathbb{R})$ the set of càdlàg functions on $\mathbb{R}_+$. We introduce for any $t \geq 0$, the usual restriction map $\mathbf{r}_t$ from $\mathcal{D}$ to $\mathcal{D}_t$. Then the $\mathbf{M}_1$ -distance on $\mathcal{D}$ is defined for $x, y \in \mathcal{D}$ as

$$d_{\mathbf{M}_1}(x, y) = \int_0^\infty e^{-t} (d_{\mathbf{M}_1, t}(\mathbf{r}_t(x), \mathbf{r}_t(y)) \wedge 1) dt.$$

Again, the metric space $(\mathcal{D}, d_{\mathbf{M}_1})$ is separable and topologically complete. We now briefly recall some characterization of converging sequences in $(\mathcal{D}, d_{\mathbf{M}_1})$. To this end, we first introduce for $x \in \mathcal{D}$ and $\delta, T > 0$, the following oscillation function

$$w(x, T, \delta) = \sup_{t \in [0, T]} \sup_{t_{\delta-} \leq t_1 < t_2 < t_3 \leq t_{\delta+}} d(x(t_2), [x(t_1), x(t_3)]),$$

where $t_{\delta-} = 0 \vee (t - \delta)$, $t_{\delta+} = T \wedge (t + \delta)$ and $d(x(t_2), [x(t_1), x(t_3)])$ is the distance of $x(t_2)$ to the segment $[x(t_1), x(t_3)]$, i.e.

$$d(x(t_2), [x(t_1), x(t_3)]) = \begin{cases} 0 & \text{if } x(t_2) \in [x(t_1), x(t_3)], \\ |x(t_2) - x(t_1)| \wedge |x(t_2) - x(t_3)| & \text{otherwise.} \end{cases}$$

We finally introduce, for $x \in \mathcal{D}$, the set $\text{Disc}(x) = \{t \geq 0, \Delta x(t) \neq 0\}$ of the discontinuities of x . We have the following characterization, which can be found in Whitt [Whi02, Theorem 12.5.1 and Theorem 12.9.3].

Theorem 4.2. *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{D}$ and let x in $\mathcal{D}$. Then the following assertions are equivalent.*

(i) $d_{\mathbf{M}_1}(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

(ii) $x_n(t)$ converges to $x(t)$ in a dense subset of $\mathbb{R}_+$ containing 0, and for every $T \notin \text{Disc}(x)$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} w(x_n, T, \delta) = 0.$$

In any case, we will write $x_n \xrightarrow{\mathbf{M}_1} x$ if x_n converges to x in $(\mathcal{D}, d_{\mathbf{M}_1})$.

We are now ready to characterize convergence in law for sequences of random variables valued in $(\mathcal{D}, d_{\mathbf{M}_1})$. We have the following result, see for instance [Sko56, Theorem 3.2.1 and Theorem 3.2.2].

Theorem 4.3. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables valued in $\mathcal{D}$ and let X be a random variable valued in $\mathcal{D}$ such that for any $t \geq 0$, $\mathbb{P}(\Delta X(t) \neq 0) = 0$. Then the following assertions are equivalent.*

- (i) $(X_n)_{n \in \mathbb{N}}$ converges in law to X as $n \rightarrow \infty$ in $\mathcal{D}$ endowed with the $\mathbf{M}_1$ -topology.
- (ii) The finite dimensional distributions of X_n converge to those of X in some dense subset of $\mathbb{R}_+$ containing 0, and for any $T > 0$ and any $\eta > 0$ we have

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(w(X_n, T, \delta) > \eta) = 0.$$

The following theorem shows that the convergence proved in [FT21] can be enhanced, in a stronger convergence.

Theorem 4.4. *Grant Assumption 4.1 and let $(X_t, V_t)_{t \geq 0}$ be a solution of (4.1) starting at $(0, v_0)$ with $v_0 \in \mathbb{R}$. Let $(Z_t^\alpha)_{t \geq 0}$ be a symmetric stable process with $\alpha = (\beta + 1)/3$ and such that $\mathbb{E}[e^{i\xi Z_t^\alpha}] = \exp(-t\sigma_\alpha|\xi|^\alpha)$, where σ_α is defined by (4.4). Then we have*

$$(\varepsilon^{1/\alpha} X_{t/\varepsilon})_{t \geq 0} \longrightarrow (Z_t^\alpha)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0$$

in law for the $\mathbf{M}_1$ -topology.

The proof is postponed to Section 4.5.

4.3 The particle reflected at an inelastic boundary and its scaling limit

In this section, we construct a weak solution of (4.5) following Bertoin [Ber07, Ber08]. Although the author uses a Brownian motion instead of the process $(V_t)_{t \geq 0}$, some of the trajectorial properties shown in [Ber07, Ber08] still hold in our case.

Consider a weak solution $(X_t, V_t)_{t \geq 0}$ of (4.1) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ supporting some Brownian motion $(B_t)_{t \geq 0}$, starting at $(0, v_0)$ where $v_0 \geq 0$. We introduce

$$\mathcal{X}_t = X_t - \inf_{s \in [0, t]} X_s. \tag{4.7}$$

Before introducing the solution of (4.5), we will first study a little bit the process $(\mathcal{X}_t)_{t \geq 0}$. It is a continuous process, which has the same motion as X_t as long as $X_t > \inf_{s \in [0, t]} X_s$. When X_t reaches its past infimum, it is necessarily with a non-positive velocity. When this happens with a strictly negative velocity, say at time t_0 , the infimum process decreases until the velocity hits zero, i.e. on the interval $[t_0, d_{t_0}]$ where $d_{t_0} = \inf\{s \geq t_0, V_s = 0\}$. On this time interval, $X_t = \inf_{s \in [0, t]} X_s$ and thus $\mathcal{X}_t = 0$. It is interesting and useful to study

$$\mathcal{I}_{\mathcal{X}} = \{t \geq 0, \mathcal{X}_t = 0\}.$$

As explained in [Ber07, page 2025], the following random set appears naturally in the study of $\mathcal{I}_{\mathcal{X}}$:

$$\mathcal{H} = \{t \geq 0, V_t < 0, \mathcal{X}_t = 0 \text{ and } \exists \varepsilon > 0, \forall s \in [t - \varepsilon, t], \mathcal{X}_s > 0\}.$$

Each element of $\mathcal{H}$ is the first time at which X_t reaches its infimum, after an "excursion" above its infimum. Each point of $\mathcal{H}$ is necessarily isolated and thus $\mathcal{H}$ is countable. We are now able to state the following lemma, which is identical to [Ber07, Lemma 2], and which characterizes the set $\mathcal{I}_{\mathcal{X}}$.

Lemma 4.1. *The decomposition of the interior of $\mathcal{I}_{\mathcal{X}}$ as a union of disjoint intervals is given by*

$$\mathring{\mathcal{I}}_{\mathcal{X}} = \bigcup_{s \in \mathcal{H}}]s, d_s[,$$

where $d_s = \inf\{u > s, V_u = 0\}$. Moreover, the boundary $\partial\mathcal{I}_{\mathcal{X}} = \mathcal{I}_{\mathcal{X}} \setminus \mathring{\mathcal{I}}_{\mathcal{X}}$ has zero Lebesgue measure.

Proof. The idea is to use the result from [Ber07] and the Girsanov theorem. Indeed, the same result is proved in [Ber07] when $F = 0$ and $v_0 = 0$.

Step 1: We first show that the results holds when $v_0 = 0$. We first define the following local martingales

$$L_t = - \int_0^t F(V_s) dB_s \quad \text{and} \quad \mathcal{E}(L)_t = \exp\left(L_t - \frac{1}{2} \langle L \rangle_t\right)$$

For any $T > 0$, $\mathbb{E}[e^{\frac{1}{2} \langle L \rangle_T}] \leq e^{\frac{1}{2} T \|F^2\|_{\infty}} < \infty$, by Assumption 4.1, and therefore, by the Novikov criterion and the Girsanov theorem, the measure $\mathbb{Q}_T = \mathcal{E}(L)_T \cdot \mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F}_T)$ and the process

$$V_t = \int_0^t F(V_s) ds + B_t = B_t - \langle B, L \rangle_t$$

is an $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion starting at 0 under $\mathbb{Q}_T$. We deduce from Lemma 2 in [Ber07] that $\mathbb{Q}_T$ -a.s.,

$$\mathring{\mathcal{I}}_{\mathcal{X}} \cap [0, T] = \left(\bigcup_{s \in \mathcal{H}}]s, d_s[\right) \cap [0, T]$$

and $\partial\mathcal{I}_{\mathcal{X}} \cap [0, T]$ has zero Lebesgue measure. Since this holds for every $T > 0$, and since $\mathbb{Q}_T \sim \mathbb{P}$, this establishes our result for the initial condition $(0, 0)$.

Step 2: We show the results when $v_0 > 0$. Let us define $\tau = \inf\{t > 0, X_t = 0\}$. Since $v_0 > 0$, it is clear that $\tau > 0$ a.s. and that $\tau \in \mathcal{H}$ and

$$\mathcal{I}_{\mathcal{X}} \cap [0, d_{\tau}] = \{0\} \cup [\tau, d_{\tau}].$$

Now the process $(\bar{X}_t, \bar{V}_t)_{t \geq 0} = (X_{t+d_{\tau}} - X_{d_{\tau}}, V_{t+d_{\tau}})_{t \geq 0}$ is a solution of (4.1) starting at $(0, 0)$. Moreover, since $X_{d_{\tau}} = \inf_{s \in [0, d_{\tau}]} X_s$, we clearly have on the event $\{d_{\tau} < t\}$,

$$\inf_{s \in [0, t]} X_s = X_{d_{\tau}} + \inf_{s \in [0, t-d_{\tau}]} \bar{X}_s,$$

which yields

$$\mathcal{I}_{\mathcal{X}} = \{0\} \cup [\tau, d_{\tau}] \cup \bar{\mathcal{I}}, \quad \text{where} \quad \bar{\mathcal{I}} = \left\{ t \geq d_{\tau}, \bar{X}_{t-d_{\tau}} = \inf_{s \in [0, t-d_{\tau}]} \bar{X}_s \right\} = \mathcal{I}_{\mathcal{X}} \cap [d_{\tau}, \infty).$$

By the first step,

$$\bar{\mathcal{I}} = \bigcup_{s \in \mathcal{H} \cap [d_{\tau}, \infty)}]s, d_s[,$$

whence the result. $\square$

We are now ready to complete the construction of the solution of (4.5). We introduce the following time-change, as well as its right-continuous inverse

$$A_t = \int_0^t \mathbf{1}_{\{X_s > 0\}} ds \quad \text{and} \quad T_t = \inf\{s > 0, A_s > t\}.$$

We claim that by using the same arguments as in [Ber07, page 2024-2025], or by using the Girsanov theorem as in the proof of Lemma 4.1, it holds that

$$\mathcal{X}_t = \int_0^t V_s \mathbf{1}_{\{\mathcal{X}_s > 0\}} ds = \int_0^t V_s dA_s. \quad (4.8)$$

Therefore, by the change of variables theorem for Stieltjes integrals, we get

$$\mathbb{X}_t := \mathcal{X}_{T_t} = \int_0^{T_t} V_s dA_s = \int_0^t \mathbb{V}_s ds, \quad \text{where} \quad \mathbb{V}_t := V_{T_t}. \quad (4.9)$$

We finally set $\mathcal{G}_t = \mathcal{F}_{T_t}$ and state the main theorem of this section.

Theorem 4.5. *There exists a $(\mathcal{G}_t)_{t \geq 0}$ -Brownian motion such that $(\mathbb{X}_t, \mathbb{V}_t)_{t \geq 0}$ is a solution of (4.5) on $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t \geq 0}, \mathbb{P})$.*

Proof. We have already seen that $\mathbb{X}_t = \int_0^t \mathbb{V}_s ds$. Concerning the velocity, we paraphrase the proof of Proposition 1 in [Ber08] and start by decomposing the process $(\mathbb{V}_t)_{t \geq 0}$ as follows

$$\mathbb{V}_t = v_0 + \int_0^{T_t} \mathbf{1}_{\{\mathcal{X}_s = 0\}} dV_s + \int_0^{T_t} \mathbf{1}_{\{\mathcal{X}_s > 0\}} dV_s =: v_0 + C_t + D_t.$$

Let us first deal with D_t . Using (4.1), we get

$$D_t = \int_0^{T_t} \mathbf{1}_{\{\mathcal{X}_s > 0\}} F(V_s) ds + \int_0^{T_t} \mathbf{1}_{\{\mathcal{X}_s > 0\}} dB_s.$$

The last term is a $(\mathcal{G}_t)_{t \geq 0}$ -local martingale whose quadratic variation at time t equals $A_{T_t} = t$ by the very definition of $(T_t)_{t \geq 0}$ and thus, it is a Brownian motion that we will denote $(\mathbb{B}_t)_{t \geq 0}$. Regarding the second term, we use again the change of variables for Stieltjes integrals, to deduce that $\int_0^{T_t} \mathbf{1}_{\{\mathcal{X}_s > 0\}} F(V_s) ds = \int_0^{T_t} F(V_s) dA_s = \int_0^t F(\mathbb{V}_s) ds$. We have proved that $D_t = \int_0^t F(\mathbb{V}_s) ds + \mathbb{B}_t$. We now deal with C_t . First we note that the semimartingale

$$\int_0^t \mathbf{1}_{\{s \in \partial \mathcal{I}_\mathcal{X}\}} dV_s = \int_0^t \mathbf{1}_{\{s \in \partial \mathcal{I}_\mathcal{X}\}} F(V_s) ds + \int_0^t \mathbf{1}_{\{s \in \partial \mathcal{I}_\mathcal{X}\}} dB_s$$

is a.s. null. Indeed since $\partial \mathcal{I}_\mathcal{X}$ has zero Lebesgue measure by Lemma 4.1, the first term is obviously equal to zero. By the same argument, the second term is a local martingale whose quadratic variation is equal to zero, and therefore it is null. Then we can write as in [Ber07]

$$C_t = \int_0^{T_t} \mathbf{1}_{\{s \in \mathcal{I}_\mathcal{X}\}} dV_s = \int_0^{T_t} \mathbf{1}_{\{s \in \mathring{\mathcal{I}}_\mathcal{X}\}} dV_s = \sum_{u \in \mathcal{H}, u \leq T_t} (V_{d_u} - V_u) = - \sum_{u \in \mathcal{H}, u \leq T_t} V_u.$$

In the third equality, we used Lemma 4.1 and the fact that, by definition, $T_t \notin \mathring{\mathcal{I}}_\mathcal{X}$ and in the fourth that $V_{d_u} = 0$ for every $u \in \mathcal{H}$. To every point $u \in \mathcal{H}$ corresponds a unique jumping time s of $(T_t)_{t \geq 0}$ at which $\mathcal{X}_t$ hits the boundary, i.e. $u = T_{s-}$ and $\mathcal{X}_{T_{s-}} = \mathcal{X}_{T_s} = 0$. Indeed, the flat sections of $(A_t)_{t \geq 0}$ are precisely $\mathring{\mathcal{I}}_\mathcal{X}$, and therefore, for every $u \in \mathcal{H}$, we have $A_u = A_{d_u}$ and thus if we set $s = A_u$, then $T_{s-} = u$ and $T_s = d_u$. Hence we have

$$C_t = - \sum_{0 < s \leq t} V_{T_{s-}} \mathbf{1}_{\{\mathcal{X}_{T_s} = 0\}} = - \sum_{0 < s \leq t} \mathbb{V}_s \mathbf{1}_{\{\mathbb{X}_s = 0\}},$$

which completes the proof. $\square$

We are now ready to study the scaling limits of $(\mathcal{X}_t)_{t \geq 0}$ and $(\mathbb{X}_t)_{t \geq 0}$.

Theorem 4.6. *Let $(\mathcal{X}_t)_{t \geq 0}$ and $(\mathbb{X}_t)_{t \geq 0}$ be defined as in (4.7) and (4.9). Let also $(Z_t^\alpha)_{t \geq 0}$ be a symmetric stable process with $\alpha = (\beta + 1)/3$ and such that $\mathbb{E}[e^{i\xi Z_t^\alpha}] = \exp(-t\sigma_\alpha|\xi|^\alpha)$ where σ_α is defined by (4.4). Let $R_t^\alpha = Z_t^\alpha - \inf_{s \in [0, t]} Z_s^\alpha$. Then we have*

$$(\varepsilon^{1/\alpha} \mathcal{X}_{t/\varepsilon})_{t \geq 0} \longrightarrow (R_t^\alpha)_{t \geq 0} \quad \text{and} \quad (\varepsilon^{1/\alpha} \mathbb{X}_{t/\varepsilon})_{t \geq 0} \longrightarrow (R_t^\alpha)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0$$

in law for the $\mathbf{M}_1$ -topology.

Proof. The convergence of $(\varepsilon^{1/\alpha} \mathcal{X}_{t/\varepsilon})_{t \geq 0}$ is straightforward by the continuous mapping theorem and Theorem 4.4. Indeed the reflection map from $\mathcal{D}$ into itself, which maps a function x to the function y , defined for every $t \geq 0$ by $y(t) = x(t) - 0 \wedge \inf_{s \in [0, t]} x(s)$, is continuous with respect to the $\mathbf{M}_1$ -topology, see Whitt [Whi02, Chapter 13, Theorem 13.5.1].

We now study $(\mathbb{X}_t)_{t \geq 0} = (\mathcal{X}_t)_{t \geq 0}$. By Skorokhod's representation theorem, there exist a family of processes $(\mathcal{X}_t^\varepsilon)_{t \geq 0}$ indexed by $\varepsilon > 0$, and a reflected symmetric stable process $(R_t^\alpha)_{t \geq 0}$, both defined on the probability space $([0, 1], \mathcal{B}([0, 1]), \lambda)$, where λ denotes the Lebesgue measure on $[0, 1]$, such that for every $\varepsilon > 0$, $(\mathcal{X}_t^\varepsilon)_{t \geq 0} \stackrel{d}{=} (\mathcal{X}_{t/\varepsilon})_{t \geq 0}$ and such that,

$$\lambda - \text{a.s.}, \quad d_{\mathbf{M}_1}((\varepsilon^{1/\alpha} \mathcal{X}_t^\varepsilon)_{t \geq 0}, (R_t^\alpha)_{t \geq 0}) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Let us denote by J be the set of discontinuities of $(R_t^\alpha)_{t \geq 0}$. Then, by [Whi02, Chapter 12, Lemma 12.5.1], we get that

$$\lambda - \text{a.s.}, \quad \text{for every } t \notin J, \quad \varepsilon^{1/\alpha} \mathcal{X}_t^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} R_t^\alpha.$$

We introduce the time-change process $A_t^\varepsilon = \int_0^t \mathbf{1}_{\{\varepsilon^{1/\alpha} \mathcal{X}_s^\varepsilon > 0\}} ds$ for every $\varepsilon > 0$ and every $t \geq 0$. Then we get by the Fatou lemma that λ -a.s., for every $t \geq 0$,

$$\int_0^t \liminf_{\varepsilon \rightarrow 0} \mathbf{1}_{\{\varepsilon^{1/\alpha} \mathcal{X}_s^\varepsilon > 0\}} ds \leq \liminf_{\varepsilon \rightarrow 0} A_t^\varepsilon \leq \limsup_{\varepsilon \rightarrow 0} A_t^\varepsilon \leq t.$$

Since J is countable, we have λ -a.s., for a.e. $s \in [0, t]$, $\mathbf{1}_{\{R_s^\alpha > 0\}} \leq \liminf_{\varepsilon \rightarrow 0} \mathbf{1}_{\{\varepsilon^{1/\alpha} \mathcal{X}_s^\varepsilon > 0\}}$. Since the zero set of the reflected stable process is a.s. Lebesgue-null, we conclude that λ -a.s., for every $t \geq 0$, $A_t^\varepsilon \rightarrow t$ as $\varepsilon \rightarrow 0$.

Let us denote by $(T_t^\varepsilon)_{t \geq 0}$ the right-continuous inverse of $(A_t^\varepsilon)_{t \geq 0}$. As an immediate consequence, we have λ -a.s., for every $t \geq 0$, $T_t^\varepsilon \rightarrow t$ as $\varepsilon \rightarrow 0$. Since the $\mathbf{M}_1$ -topology on the space of non-increasing functions reduces to pointwise convergence on a dense subset including 0, see [Whi02, Corollary 12.5.1], we have $(T_t^\varepsilon)_{t \geq 0} \longrightarrow (\text{id}_t)_{t \geq 0}$ as $\varepsilon \rightarrow 0$ in law in the $\mathbf{M}_1$ -topology, where $\text{id}_t = t$ is the identity process.

By a simple substitution, we see that $(\varepsilon A_{t/\varepsilon})_{t \geq 0} \stackrel{d}{=} (A_t^\varepsilon)_{t \geq 0}$, from which we deduce that $(\varepsilon T_{t/\varepsilon})_{t \geq 0} \stackrel{d}{=} (T_t^\varepsilon)_{t \geq 0}$ and that $(\varepsilon T_{t/\varepsilon})_{t \geq 0} \longrightarrow (\text{id}_t)_{t \geq 0}$ as $\varepsilon \rightarrow 0$ in law in the $\mathbf{M}_1$ -topology. Hence, by a generalization of Slutsky theorem, see for instance [Bil99, Section 3, Theorem 3.9], we get that $(\varepsilon^{1/\alpha} \mathcal{X}_{t/\varepsilon}, \varepsilon T_{t/\varepsilon})_{t \geq 0}$ converges in law to $(R_t^\alpha, \text{id}_t)_{t \geq 0}$ in $\mathcal{D} \times \mathcal{D} = \mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)$ endowed with the $\mathbf{M}_1$ -topology.

We are now able to conclude. Let us denote by $\mathcal{D}_\uparrow$ the set of càdlàg and non-decreasing functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, and $\mathcal{C}_{\uparrow\uparrow}$ the set of continuous and strictly increasing functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then the composition map, from $\mathcal{D} \times \mathcal{D}_\uparrow$ to $\mathcal{D}$, which maps (x, y) to $x \circ y$, is continuous on $\mathcal{D} \times \mathcal{C}_{\uparrow\uparrow}$, see [Whi02, Chapter 13, Theorem 13.2.3]. Hence $(\varepsilon^{1/\alpha} \mathbb{X}_{t/\varepsilon})_{t \geq 0} = (\varepsilon^{1/\alpha} \mathcal{X}_{T_{t/\varepsilon}})_{t \geq 0}$ converges to $(R_t^\alpha)_{t \geq 0}$ by the continuous mapping theorem. $\square$

4.4 The reflected process with diffusive boundary condition

In this section, we finally study the process $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$, solution of (4.3), and its scaling limit. To establish our result, we will rely on the convergence of $(\mathcal{X}_t)_{t \geq 0}$ from Theorem 4.6. First, we will show that $\mathbf{X}_t \geq \mathcal{X}_t = X_t - \inf_{s \in [0, t]} X_s$ where $(\mathbf{X}_t)_{t \geq 0}$ and $(X_t)_{t \geq 0}$ are the solutions to (4.3) and (4.1) with the same Brownian motion. Then, inspired by the work of Bertoin [Ber07], we will give another construction of $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ and we will prove that, up to a time-change, $\mathbf{X}_t \leq \mathcal{X}_t$. Finally, we will conclude since the time-change at stake is asymptotically equivalent to t . We believe that the proof of the limit of the time-change is the most technical part of the paper and this will be the subject of Subsections 4.4.3 and 4.4.6.

4.4.1 A comparison result

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space supporting a Brownian motion $(B_t)_{t \geq 0}$ and a sequence of i.i.d μ -distributed random variables $(M_n)_{n \in \mathbb{N}}$, independent of each other. Consider the solution $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ of (4.3), starting at $(0, v_0)$ where $v_0 > 0$, as well as its sequence of hitting times $(\tau_n)_{n \in \mathbb{N}}$. We also consider on the same probability space the solution $(X_t, V_t)_{t \geq 0}$ of (4.1) starting at $(0, v_0)$, with the same driving Brownian motion $(B_t)_{t \geq 0}$. Let also $(\mathcal{X}_t)_{t \geq 0}$ be defined as in (4.7), i.e. $\mathcal{X}_t = X_t - \inf_{s \in [0, t]} X_s$. We have the following proposition.

Proposition 4.1. *Almost surely, for any $t \geq 0$, $\mathbf{X}_t \geq \mathcal{X}_t$.*

Proof. Step 1: We first prove that a.s. for any $t \geq 0$, $V_t \leq \mathbf{V}_t$ and to do so, we use the classical comparison theorem for O.D.E's. We prove recursively that for any $n \in \mathbb{N}$, a.s. for any $t \in [0, \tau_n)$, $V_t \leq \mathbf{V}_t$. This is true for $n = 1$, since the processes are both solutions of the same well-posed O.D.E. on $[0, \tau_1)$, with the same starting point. Hence they are equal on this interval.

Now let us assume that for some $n \in \mathbb{N}$, a.s. for any $t \in [0, \tau_n)$, $V_t \leq \mathbf{V}_t$. Then a.s. $V_{\tau_n} = V_{\tau_n-} \leq \mathbf{V}_{\tau_n-} \leq 0$. We also see that $(\mathbf{V}_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ are two solutions of the same O.D.E. on the interval $[\tau_n, \tau_{n+1})$. Indeed, a.s. for any $t \in [\tau_n, \tau_{n+1})$, we have

$$\mathbf{V}_t = M_n + \int_{\tau_n}^t F(\mathbf{V}_s) ds + B_t - B_{\tau_n} \quad \text{and} \quad V_t = V_{\tau_n} + \int_{\tau_n}^t F(V_s) ds + B_t - B_{\tau_n}.$$

Since $M_n > 0$ with probability one, we have a.s. $M_n \geq V_{\tau_n}$ and thus, by the comparison theorem, we deduce that a.s., for any $t \in [\tau_n, \tau_{n+1})$, $V_t \leq \mathbf{V}_t$. This achieves the first step.

Step 2: We conclude. Almost surely, for any $0 \leq s \leq t$, we have

$$X_t - X_s = \int_s^t V_s ds \leq \int_s^t \mathbf{V}_s ds = \mathbf{X}_t - \mathbf{X}_s \leq \mathbf{X}_t,$$

the last inequality holding since $\mathbf{X}_s \geq 0$ a.s. This implies that a.s., $\inf_{s \in [0, t]} X_s \geq X_t - \mathbf{X}_t$, i.e. $\mathbf{X}_t \geq \mathcal{X}_t$, for any $t \geq 0$. $\square$

4.4.2 A second construction

In this subsection, we give another construction of $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$, which is inspired by the construction given by Bertoin [Ber07] of the reflected Langevin process at an inelastic boundary.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a Brownian motion $(B_t)_{t \geq 0}$ and a sequence of i.i.d μ -distributed random variables $(M_n)_{n \in \mathbb{N}}$, independent from the Brownian motion. We set

$(\mathcal{F}_t)_{t \geq 0}$ to be the filtration generated by $(B_t)_{t \geq 0}$ after the usual completions, and we introduce the filtration $(\mathcal{G}_t)_{t \geq 0}$ defined for every $t \geq 0$ as

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{(M_n)_{n \in \mathbb{N}}\}).$$

It is clear that $(B_t)_{t \geq 0}$ remains a Brownian motion in the filtration $(\mathcal{G}_t)_{t \geq 0}$. Next, we consider the strong solution $(X_t, V_t)_{t \geq 0}$ of (4.1) starting at $(0, v_0)$, which remains a strong Markov process in this filtration. We set $\sigma_0 = 0$, then we set $\tau_1 = \inf\{t > 0, X_t = 0\}$ and we define recursively the sequence of random times

$$\sigma_n = \inf\{t > \tau_n, V_t = M_n\} \quad \text{and} \quad \tau_{n+1} = \inf\{t > \sigma_n, X_t = X_{\sigma_n}\},$$

where $n \in \mathbb{N}$. We have the following lemma.

Lemma 4.2. *The random times $(\sigma_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ are $(\mathcal{G}_t)_{t \geq 0}$ -stopping times which are almost surely finite. Moreover, the sequence $(\tau_n - \sigma_{n-1}, \sigma_n - \sigma_{n-1})_{n \geq 2}$ forms a sequence of identically distributed random variables and the subsequences $(\tau_{2n} - \sigma_{2n-1}, \sigma_{2n} - \sigma_{2n-1})_{n \geq 1}$ and $(\tau_{2n+1} - \sigma_{2n}, \sigma_{2n+1} - \sigma_{2n})_{n \geq 1}$ form sequences of i.i.d. random variables.*

Proof. The process $(V_t)_{t \geq 0}$ being a recurrent diffusion, see for instance [FT21], we have almost surely, $\liminf_{t \rightarrow \infty} V_t = -\infty$ and $\limsup_{t \rightarrow \infty} V_t = \infty$. Moreover, by Theorem 4.4, we also have $\liminf_{t \rightarrow \infty} X_t = -\infty$ and $\limsup_{t \rightarrow \infty} X_t = \infty$ a.s. Hence, the random times previously defined are almost surely finite. Moreover, it is clear by a simple induction that the random times $(\sigma_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ are $(\mathcal{G}_t)_{t \geq 0}$ -stopping times. Indeed, if σ_n is a $(\mathcal{G}_t)_{t \geq 0}$ -stopping time for some $n \geq 0$, then $\tau_{n+1} = \inf\{t > \sigma_n, X_t = X_{\sigma_n}\}$ is obviously a $(\mathcal{G}_t)_{t \geq 0}$ -stopping time and since σ_{n+1} is the first hitting of zero after τ_{n+1} of the $(\mathcal{G}_t)_{t \geq 0}$ -adapted process $(V_t - M_{n+1})_{t \geq 0}$, it is also a stopping time for $(\mathcal{G}_t)_{t \geq 0}$.

Then, for any $n \geq 2$, applying the strong Markov property of $(X_t, V_t)_{t \geq 0}$ at time σ_{n-1} , we see that for any $n \geq 2$, $(\tau_n - \sigma_{n-1}, \sigma_n - \sigma_{n-1})$ has the same law as $(\tau_2 - \sigma_1, \sigma_2 - \sigma_1)$. The fact that the subsequences form sequences of i.i.d. random variables follows again from the strong Markov property and the fact that for any $n \geq 2$, $(\tau_n - \sigma_{n-1}, \sigma_n - \sigma_{n-1})$ only depends on $(B_t - B_{\sigma_{n-1}})_{t \in [\sigma_{n-1}, \sigma_n]}$ and (M_{n-1}, M_n) $\square$

We are finally ready to start the construction of the reflected process. We define the $(\mathcal{G}_t)_{t \geq 0}$ -adapted processes

$$\mathfrak{X}_t = \sum_{n \in \mathbb{N}} (X_t - X_{\sigma_{n-1}}) \mathbf{1}_{\{\sigma_{n-1} \leq t < \tau_n\}} \quad \text{and} \quad \mathfrak{V}_t = \mathbf{1}_{\{\mathfrak{X}_t > 0\}} V_t. \quad (4.10)$$

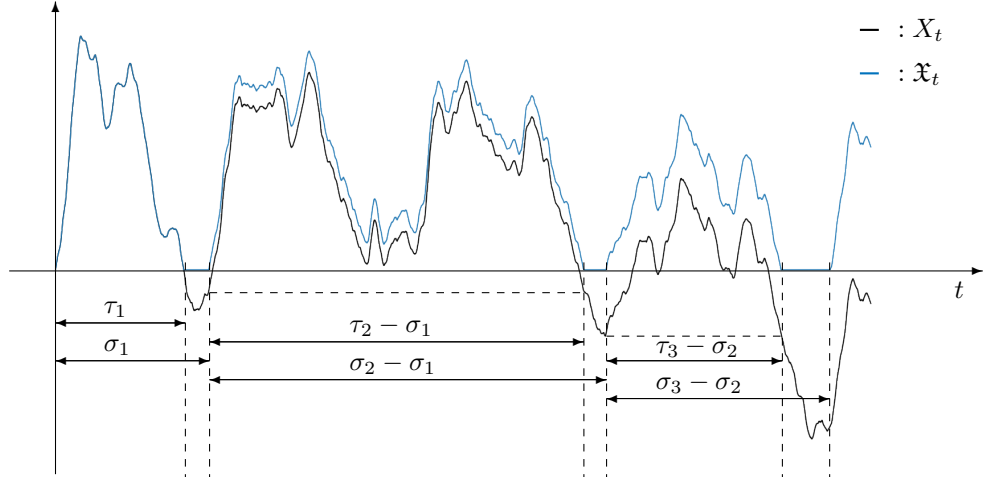
We refer to Figure 4.1 for a visual representation. For every $n \in \mathbb{N}$, the process $(\mathfrak{X}_t)_{t \geq 0}$ has the same trajectory as $(X_t)_{t \geq 0}$ on $[\sigma_{n-1}, \tau_n]$ shifted by $X_{\sigma_{n-1}}$ and is null on $[\tau_n, \sigma_n]$. Now by the very definition of σ_n and τ_n , X_t is above $X_{\sigma_{n-1}}$ on $[\sigma_{n-1}, \tau_n]$, and it should be clear that a.s., for every $t \geq 0$, $\mathfrak{X}_t \geq 0$ and that

$$\mathcal{I}_{\mathfrak{X}} = \{t \geq 0, \mathfrak{X}_t = 0\} = \{0\} \cup \left(\bigcup_{n \in \mathbb{N}} [\tau_n, \sigma_n] \right). \quad (4.11)$$

As in (4.9), we also have

$$\mathfrak{X}_t = \int_0^t \mathfrak{V}_s ds \quad (4.12)$$

Indeed, this can be easily checked using that for every $s \geq 0$, $\mathbf{1}_{\{\mathfrak{X}_s > 0\}} = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{\sigma_{n-1} < s < \tau_n\}}$ and that $X_{\tau_n} = X_{\sigma_{n-1}}$. As in the previous subsection, we can compare $\mathfrak{X}_t$ to $\mathcal{X}_t = X_t - \inf_{s \in [0, t]} X_s$.


 Figure 4.1: Graphical representation of a realization of $(X_t)_{t \geq 0}$ and $(\mathfrak{X}_t)_{t \geq 0}$.

Proposition 4.2. *Almost surely, for any $t \geq 0$, $\mathfrak{X}_t \leq X_t$.*

Proof. The result follows almost immediately from the definition of $(\mathfrak{X}_t)_{t \geq 0}$. Indeed, almost surely, for any $n \in \mathbb{N}$ and any $t \geq 0$,

$$X_{\sigma_{n-1}} \mathbf{1}_{\{\sigma_{n-1} \leq t < \tau_n\}} \geq \inf_{s \in [0, t]} X_s \times \mathbf{1}_{\{\sigma_{n-1} \leq t < \tau_n\}},$$

and therefore we have almost surely, for any $t \geq 0$,

$$\mathfrak{X}_t = \sum_{n \in \mathbb{N}} (X_t - X_{\sigma_{n-1}}) \mathbf{1}_{\{\sigma_{n-1} \leq t < \tau_n\}} \leq X_t \times \sum_{n \in \mathbb{N}} \mathbf{1}_{\{\sigma_{n-1} \leq t < \tau_n\}} \leq X_t,$$

which achieves the proof. $\square$

We are now ready to complete the construction of the solution of (4.3), as we did in Section 4.3. We introduce the $(\mathcal{G}_t)_{t \geq 0}$ -adapted time-change $A'_t = \int_0^t \mathbf{1}_{\{\mathfrak{X}_s > 0\}} ds$ as well as its right-continuous inverse T'_t , and we define the processes

$$\mathbf{X}_t = \mathfrak{X}_{T'_t} = \int_0^{T'_t} V_s dA'_s = \int_0^t V_s ds \quad \text{where} \quad \mathbf{V}_t = V_{T'_t}.$$

We finally define the filtration $(\mathcal{F}_t)_{t \geq 0} = (\mathcal{G}_{T'_t})_{t \geq 0}$. We have the following theorem.

Theorem 4.7. *There exists an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion such that the process $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ is a solution of (4.3) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.*

Proof. The proof is very similar to the proof of Theorem 4.5. We start by decomposing the process $(\mathbf{V}_t)_{t \geq 0}$ as follows

$$\mathbf{V}_t = v_0 + \int_0^{T'_t} \mathbf{1}_{\{\mathfrak{X}_s = 0\}} dV_s + \int_0^{T'_t} \mathbf{1}_{\{\mathfrak{X}_s > 0\}} dV_s =: v_0 + C_t + D_t.$$

As in the proof of Theorem 4.5, we easily see, using (4.1), the definition of $(T'_t)_{t \geq 0}$ and the change of variables theorem for Stieltjes integrals, that

$$D_t = \int_0^t F(\mathbf{V}_s) ds + \mathbf{B}_t,$$

where $(\mathbf{B}_t)_{t \geq 0} = (\int_0^{T'_t} \mathbf{1}_{\{x_s > 0\}} dB_s)_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion. We now deal with C_t and we write

$$C_t = \int_0^{T'_t} \mathbf{1}_{\{s \in \mathcal{I}_x\}} dV_s = \sum_{n \in \mathbb{N}, \tau_n \leq T'_t} (V_{\sigma_n} - V_{\tau_n}) = \sum_{n \in \mathbb{N}, \tau_n \leq T'_t} (M_n - V_{\tau_n}).$$

In the second inequality, we used (4.11) and that, by definition, $T'_t \notin \hat{\mathcal{I}}_x$. Let us now define for all $n \in \mathbb{N}$, $\tau_n = \sum_{k=1}^n (\tau_k - \sigma_{k-1})$. Then by definition of A'_t , we have for all $n \in \mathbb{N}$, $A'_{\tau_n} = \tau_n$, which leads to $\tau_n = T'_{\tau_n-}$. Indeed the flat section of $(A'_t)_{t \geq 0}$ consists in $\cup_{n \in \mathbb{N}} [\tau_n, \sigma_n]$ and thus the jumping times of $(T'_t)_{t \geq 0}$ are precisely the times τ_n . Then we get

$$C_t = \sum_{n \in \mathbb{N}} (M_n - V_{\tau_n-}) \mathbf{1}_{\{\tau_n \leq t\}}.$$

Finally it is clear that $\tau_1 = \inf\{t > 0, \mathbf{X}_t = 0\}$ and that $\tau_{n+1} = \inf\{t > \tau_n, \mathbf{X}_t = 0\}$. $\square$

4.4.3 Convergence of the time-change

The goal of this subsection is to see that the time change $(A'_t)_{t \geq 0}$ is asymptotically equivalent to t , i.e. the size of $\sigma_n - \tau_n$ is small compared to the size of $\tau_n - \sigma_{n-1}$. Recall we are concerned with process $(\mathfrak{X}_t)_{t \geq 0}$ defined in (4.10) and $A'_t = \int_0^t \mathbf{1}_{\{x_s > 0\}} ds$. The main result of this subsection is the following result.

Proposition 4.3. *Under Assumption 4.2-(i), we have $t^{-1}A'_t \xrightarrow{\mathbb{P}} 1$ as $t \rightarrow \infty$.*

We recall from Lemma 4.2 that $(\tau_n - \sigma_{n-1}, \sigma_n - \sigma_{n-1})_{n \geq 2}$ is a sequence of identically distributed random variables and each element is equal in law to the random variable (τ, σ) defined as follows: let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space supporting an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion $(B_t)_{t \geq 0}$ and two independent μ -distributed random variables V_0 and M , also independent of the Brownian motion. Consider the process $(X_t, V_t)_{t \geq 0}$ solution of (4.1) starting at $(0, V_0)$. Then τ and σ are defined as

$$\tau = \inf\{t > 0, X_t = 0\} \quad \text{and} \quad \sigma = \inf\{t \geq \tau, V_t = M\}.$$

For $t \geq 0$, we define $N_t = \sup\{n \geq 0, \sigma_n \leq t\}$, for which $\sigma_{N_t} \leq t < \sigma_{N_t+1}$. Then the time-change $(A'_t)_{t \geq 0}$ satisfies, see (4.10),

$$A'_t = \sum_{k=1}^{N_t} (\tau_k - \sigma_{k-1}) + \tau_{N_t+1} \wedge t - \sigma_{N_t}. \quad (4.13)$$

Roughly, the reason why Proposition 4.3 is true is that $\sigma - \tau$ is actually small compared to τ . More precisely, we will show that τ and σ have exactly the same probability tail.

First, since $(X_t)_{t \geq 0}$ resembles a symmetric stable process as t is large, we should expect $\mathbb{P}(\tau > t)$ to behave like the probability for a symmetric stable process started at $\eta > 0$, to stay positive up to time t , which is well-known to behave like $t^{-1/2}$ as $t \rightarrow \infty$.

Then, since $(V_t)_{t \geq 0}$ is positive recurrent, we should expect $\mathbb{P}(\sigma - \tau > t)$ to have a lighter tail than τ . Indeed, X_t reaches 0 at time τ with some random negative velocity V_τ , and $\sigma - \tau$ is the amount of time it takes for $(V_t)_{t \geq 0}$ to reach M , which should not be too big, thanks to the positive recurrence of the velocity and the assumption on μ .

However, we did not manage to employ this strategy, as the law of V_τ is unknown and it is not clear at all how to get an exact asymptotic of $\mathbb{P}(\tau > t)$ by approaching $(X_t)_{t \geq 0}$ by its scaling limit. We will rather use tools introduced in Berger, B  thencourt and Tardif [BBT23].

We state the following crucial lemma which describes the tails of τ and σ . Its proof is rather technical and a bit independent of the rest, so it is postponed to Subsection 4.4.6.

Lemma 4.3. *Grant Assumptions 4.2-(i). Then there exists a constant $C > 0$ such that*

$$\mathbb{P}(\tau > t) \sim \mathbb{P}(\sigma > t) \sim Ct^{-1/2} \quad \text{as } t \rightarrow \infty.$$

With these results at hand, we are able to prove Proposition 4.3.

Proof of Proposition 4.3. We seek to show that $(1 - t^{-1}A'_t)$ converges to 0 in probability as $t \rightarrow \infty$. From (4.13), we have

$$\begin{aligned} t - A'_t &= \sum_{k=1}^{N_t} (\sigma_{k-1} - \tau_k) + (t - \tau_{N_t+1}) \mathbf{1}_{\{\tau_{N_t+1} \leq t\}} + \sigma_{N_t} \\ &= \sum_{k=1}^{N_t} (\sigma_{k-1} - \tau_k) + (t - \tau_{N_t+1}) \mathbf{1}_{\{\tau_{N_t+1} \leq t\}} + \sum_{k=1}^{N_t} (\sigma_k - \sigma_{k-1}) \\ &= \sum_{k=1}^{N_t} (\sigma_k - \tau_k) + (t - \tau_{N_t+1}) \mathbf{1}_{\{\tau_{N_t+1} \leq t\}}. \end{aligned}$$

Since by definition, $\sigma_{N_t+1} \geq t$, we have the following bound:

$$0 \leq 1 - \frac{A'_t}{t} \leq \frac{1}{t} \sum_{k=1}^{N_t+1} (\sigma_k - \tau_k) = \frac{\sigma_1 - \tau_1}{t} + \frac{1}{t} \sum_{k=2}^{N_t+1} (\sigma_k - \tau_k).$$

Obviously, the first term on the right-hand side almost surely vanishes as $t \rightarrow \infty$. We now use Lemma 4.3 to study the asymptotic behavior of $\sum_{k=2}^{N_t+1} (\sigma_k - \tau_k)$ as $t \rightarrow \infty$. We divide the rest of the proof in two steps.

Step 1: We first show that

$$\lim_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}(N_t \geq At^{1/2}) = 0. \quad (4.14)$$

To show this, we introduce $\tilde{N}_t = \min\{n \geq 1, \sum_{k=1}^n (\sigma_{2k} - \sigma_{2k-1}) \leq t\}$. Since for any $n \geq 1$, we have $\sigma_{2n} \geq \sum_{k=1}^n (\sigma_{2k} - \sigma_{2k-1})$, it should be clear that for any $t > 0$, $N_t \leq \tilde{N}_t/2$. Next, by Lemma 4.2, $(\sigma_{2n} - \sigma_{2n-1})_{n \in \mathbb{N}}$ is a sequence of i.i.d. random variables whose common law is that of σ . Thanks to Lemma 4.3, we can apply the classical α -stable central limit theorem, and it is clear that

$$\frac{1}{n^2} \sum_{k=1}^n (\sigma_{2k} - \sigma_{2k-1}) \xrightarrow{\mathcal{L}} \mathcal{S}^{1/2} \quad \text{as } n \rightarrow \infty,$$

where $\mathcal{S}^{1/2}$ is a positive $1/2$ -stable random variable, see [Fel71, Chapter XII.6 Theorem 2]. Then it is immediate that $\tilde{N}_t/t^{1/2}$ converges in law to some random variable N_∞ . Finally, we get that

$$\limsup_{t \rightarrow \infty} \mathbb{P}(N_t \geq At^{1/2}) \leq \limsup_{t \rightarrow \infty} \mathbb{P}(\tilde{N}_t \geq 2At^{1/2}) = \mathbb{P}(N_\infty \geq 2A).$$

Letting $A \rightarrow \infty$ shows that (4.14) holds.

Step 2: We show that $t^{-1} \sum_{k=1}^{N_t+1} (\sigma_k - \tau_k)$ converges to 0 as $t \rightarrow \infty$ in probability. First, for any $n \geq 1$, we can write

$$\sum_{k=1}^n (\sigma_k - \tau_k) = \sum_{k \leq n, k \text{ even}} (\sigma_k - \tau_k) + \sum_{k \leq n, k \text{ odd}} (\sigma_k - \tau_k), \quad (4.15)$$

which is then by Lemma 4.2 the sum of two sums of i.i.d. random variables distributed as $\sigma - \tau$. Moreover, by Lemma 4.3 and Lemma 4.12 in Appendix 4.A with $X = \tau$ and $Y = \sigma - \tau$, the tail of $\sigma - \tau$ is lighter than the tail of σ :

$$\lim_{t \rightarrow \infty} t^{1/2} \mathbb{P}(\sigma - \tau > t) = 0.$$

This entails, see Proposition 4.7 with $\alpha = 1/2$, that the two terms in the right-hand-side of (4.15) divided by n^2 , converges to 0 as $n \rightarrow \infty$ in probability. At this point, we conclude that

$$\frac{1}{n^2} \sum_{k=2}^n (\sigma_k - \tau_k) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty. \quad (4.16)$$

Finally, for any $\eta > 0$ and any $A > 0$, we have

$$\mathbb{P}\left(t^{-1} \sum_{k=1}^{N_t+1} (\sigma_k - \tau_k) > \eta\right) \leq \mathbb{P}\left(t^{-1} \sum_{k=1}^{\lfloor At^{1/2} \rfloor} (\sigma_k - \tau_k) > \eta\right) + \mathbb{P}(N_t + 1 \geq At^{1/2}).$$

Making $t \rightarrow \infty$ using (4.16), then $A \rightarrow \infty$ using (4.14), we complete the step. $\square$

4.4.4 Scaling limit of $(\mathfrak{X}_t)_{t \geq 0}$

In this subsection, we show that under Assumption 4.2-(ii), the scaling limit of $(\mathfrak{X}_t)_{t \geq 0}$ is the stable process reflected on its infimum. The following proposition will help us showing the second part of Theorem 4.1.

Proposition 4.4. *Grant Assumptions 4.1 and 4.2-(ii), and let $(\mathfrak{X}_t, \mathfrak{V}_t)_{t \geq 0}$ be defined by (4.10) with $v_0 > 0$. Then we have, in law for the $\mathbf{M}_1$ -topology,*

$$(\varepsilon^{1/\alpha} \mathfrak{X}_{t/\varepsilon})_{t \geq 0} \longrightarrow (R_t^\alpha)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0,$$

where $R_t^\alpha = Z_t^\alpha - \inf_{s \in [0, t]} Z_s^\alpha$ and $(Z_t^\alpha)_{t \geq 0}$ is the stable process from Theorem 4.1.

Proof. We consider, for $v_0 > 0$, the processes $(\mathcal{X}_t, \mathcal{V}_t)_{t \geq 0}$ and $(\mathfrak{X}_t, \mathfrak{V}_t)_{t \geq 0}$ defined in Sections 4.3 and 4.4, starting at $(0, v_0)$ and both constructed from the process $(X_t, V_t)_{t \geq 0}$ solution of (4.1) also starting at $(0, v_0)$. We recall that almost surely, for all $t \geq 0$,

$$\mathcal{X}_t = \int_0^t V_s \mathbf{1}_{\{\mathcal{X}_s > 0\}} ds \quad \text{and} \quad \mathfrak{X}_t = \int_0^t V_s \mathbf{1}_{\{\mathfrak{X}_s > 0\}} ds,$$

see (4.8), (4.10) and (4.12). We show that, under Assumption 4.2-(ii), we have for any $T > 0$,

$$\Delta_{T, \varepsilon} := \sup_{t \in [0, T]} \varepsilon^{1/\alpha} (\mathcal{X}_{t/\varepsilon} - \mathfrak{X}_{t/\varepsilon}) \xrightarrow{\mathbb{P}} 0 \quad \text{as } \varepsilon \rightarrow 0,$$

which will prove the result thanks to Theorem 4.6 and [Bil99, Section 3, Theorem 3.1]. Since a.s. for any $t \geq 0$, we have $\mathcal{X}_t \geq \mathfrak{X}_t$ by Proposition 4.2, we can write

$$0 \leq \mathcal{X}_t - \mathfrak{X}_t = \int_0^t V_s \mathbf{1}_{\{\mathcal{X}_s > 0\} \cap \{\mathfrak{X}_s = 0\}} ds \leq \int_0^t (0 \vee V_s) \mathbf{1}_{\{\mathfrak{X}_s = 0\}} ds \leq \sum_{k=1}^{N_t+1} M_k (\sigma_k - \rho_k),$$

where for any $k \geq 1$, $\rho_k = \inf\{t \geq \tau_k, V_t = 0\} \leq \sigma_k$ and $N_t = \sup\{n \geq 0, \sigma_n \leq t\}$, as in the previous subsection. Indeed, for any $t \geq 0$, we have

$$\mathcal{I}_{\mathfrak{X}} \cap [0, t] = \bigcup_{k=1}^{N_t+1} [\tau_k, \sigma_k],$$

and for any $k \geq 1$, the velocity V_s is non-positive on $[\tau_k, \rho_k]$ and is smaller than M_k on $[\rho_k, \sigma_k]$. The sequence $(M_n(\sigma_n - \rho_n))_{n \in \mathbb{N}}$ is a sequence of i.i.d random variables and we claim that there exists $\delta' \in (0, 1 - \alpha/2)$ such that $\mathbb{E}[(M_1(\sigma_1 - \rho_1))^{\alpha/2 + \delta'}] < \infty$, see Lemma 4.4-(ii) below. This

implies by the Markov inequality that $t^{\alpha/2}\mathbb{P}(M_1(\sigma_1 - \rho_1) > t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore, by Proposition 4.7, we have

$$\frac{1}{n^{2/\alpha}} \sum_{k=1}^n M_k(\sigma_k - \rho_k) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty. \quad (4.17)$$

But since we have

$$\Delta_{T,\varepsilon} \leq \varepsilon^{1/\alpha} \sum_{k=1}^{N_{T/\varepsilon}+1} M_k(\sigma_k - \rho_k),$$

it comes that for any $\eta > 0$ and any $A > 0$,

$$\mathbb{P}(\Delta_{T,\varepsilon} > \eta) \leq \mathbb{P}\left(\varepsilon^{1/\alpha} \sum_{k=1}^{\lfloor A\varepsilon^{-1/2} \rfloor} M_k(\sigma_k - \rho_k) > \eta\right) + \mathbb{P}(N_{T/\varepsilon} + 1 \geq A\varepsilon^{-1/2}).$$

Letting $\varepsilon \rightarrow 0$ using (4.17), and letting $A \rightarrow \infty$ using (4.14), we conclude that $\mathbb{P}(\Delta_{T,\varepsilon} > \eta) \rightarrow 0$ as $\varepsilon \rightarrow 0$. $\square$

4.4.5 Proof of the main result

Proof of Theorem 4.1. Step 1: We start by showing the convergence in the finite dimensional sense. Let $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ and $(X_t, V_t)_{t \geq 0}$ be solutions of (4.3) and (4.1) starting at $(0, v_0)$. Let also $(\mathcal{X}_t)_{t \geq 0}$ be defined as in (4.7). Let $n \geq 1$, $t_1, \dots, t_n > 0$ and $x_1, \dots, x_n \geq 0$. By Proposition 4.1, we have

$$\mathbb{P}\left(\varepsilon^{1/\alpha} \mathbf{X}_{t_1/\varepsilon} \geq x_1, \dots, \varepsilon^{1/\alpha} \mathbf{X}_{t_n/\varepsilon} \geq x_n\right) \geq \mathbb{P}\left(\varepsilon^{1/\alpha} \mathcal{X}_{t_1/\varepsilon} \geq x_1, \dots, \varepsilon^{1/\alpha} \mathcal{X}_{t_n/\varepsilon} \geq x_n\right),$$

from which we deduce, by Theorem 4.6, that

$$\liminf_{\varepsilon \rightarrow 0} \mathbb{P}\left(\varepsilon^{1/\alpha} \mathbf{X}_{t_1/\varepsilon} \geq x_1, \dots, \varepsilon^{1/\alpha} \mathbf{X}_{t_n/\varepsilon} \geq x_n\right) \geq \mathbb{P}\left(R_{t_1}^\alpha \geq x_1, \dots, R_{t_n}^\alpha \geq x_n\right).$$

Let us now consider the process $(\mathfrak{X}_t, \mathfrak{V}_t)_{t \geq 0}$ starting at $(0, v_0)$, recall (4.10), built from the process $(X_t, V_t)_{t \geq 0}$, as well as the time change $(A'_t)_{t \geq 0} = (\int_0^t \mathbf{1}_{\{\mathfrak{X}_s > 0\}} ds)_{t \geq 0}$ and its right-continuous inverse $(T'_t)_{t \geq 0}$. Then by Theorem 4.7, the process $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0} = (\mathfrak{X}_{T'_t}, \mathfrak{V}_{T'_t})_{t \geq 0}$ is a solution of (4.3). By Proposition 4.3, we have

$$(\varepsilon T'_{t_1/\varepsilon}, \dots, \varepsilon T'_{t_n/\varepsilon}) \xrightarrow{\mathbb{P}} (t_1, \dots, t_n) \quad \text{as } \varepsilon \rightarrow 0. \quad (4.18)$$

Let $\delta > 0$ such that for every $k \in \{1, \dots, n\}$, $\delta < t_k$. We introduce the events

$$A_\varepsilon = \bigcap_{k=1}^n \left\{ \varepsilon^{1/\alpha} \mathbf{X}_{t_k/\varepsilon} \geq x_k \right\}, \quad B_{\varepsilon, \delta} = \bigcap_{k=1}^n \left\{ |\varepsilon T'_{t_k/\varepsilon} - t_k| \leq \delta \right\},$$

and

$$C_{\varepsilon, \delta} = \bigcap_{k=1}^n \left\{ \sup_{s \in [t_k - \delta, t_k + \delta]} \varepsilon^{1/\alpha} \mathfrak{X}_{s/\varepsilon} \geq x_k \right\} \quad \text{and} \quad D_{\varepsilon, \delta} = \bigcap_{k=1}^n \left\{ \sup_{s \in [t_k - \delta, t_k + \delta]} \varepsilon^{1/\alpha} \mathcal{X}_{s/\varepsilon} \geq x_k \right\}.$$

By (4.18), it is clear that $\mathbb{P}(B_{\varepsilon, \delta}^c) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We easily see that $A_\varepsilon \cap B_{\varepsilon, \delta} \subset C_{\varepsilon, \delta}$, and $C_{\varepsilon, \delta} \subset D_{\varepsilon, \delta}$ by Proposition 4.2. Therefore $\mathbb{P}(A_\varepsilon \cap B_{\varepsilon, \delta}) \leq \mathbb{P}(C_{\varepsilon, \delta}) \leq \mathbb{P}(D_{\varepsilon, \delta})$. As a consequence, we have $\mathbb{P}(A_\varepsilon) \leq \mathbb{P}(D_{\varepsilon, \delta}) + \mathbb{P}(B_{\varepsilon, \delta}^c)$, whence

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{P}(A_\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(D_{\varepsilon, \delta}).$$

We claim that the following convergence holds

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(D_{\varepsilon, \delta}) = \mathbb{P}(D_\delta) \quad \text{where} \quad D_\delta = \bigcap_{k=1}^n \left\{ \sup_{s \in [t_k - \delta, t_k + \delta]} R_s^\alpha \geq x_k \right\}. \quad (4.19)$$

Since a.s., $t_1, \dots, t_n$ are not jumping times of $(R_t^\alpha)_{t \geq 0}$, $\mathbb{P}(D_\delta) \rightarrow \mathbb{P}(R_{t_1}^\alpha \geq x_1, \dots, R_{t_n}^\alpha \geq x_n)$ as $\delta \rightarrow 0$. Hence (4.19) would imply

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\varepsilon^{1/\alpha} \mathbf{X}_{t_1/\varepsilon} \geq x_1, \dots, \varepsilon^{1/\alpha} \mathbf{X}_{t_n/\varepsilon} \geq x_n) \leq \mathbb{P}(R_{t_1}^\alpha \geq x_1, \dots, R_{t_n}^\alpha \geq x_n),$$

which would achieve the first step. We now show that (4.19) holds.

By Theorem 4.6 and Skorokhod's representation theorem, there exist a family of processes $(\mathcal{X}_t^\varepsilon)_{t \geq 0}$ indexed by $\varepsilon > 0$, and a reflected symmetric stable process $(R_t^\alpha)_{t \geq 0}$, both defined on the probability space $([0, 1], \mathcal{B}([0, 1]), \lambda)$, where λ denotes the Lebesgue measure on $[0, 1]$, such that for every $\varepsilon > 0$, $(\mathcal{X}_t^\varepsilon)_{t \geq 0} \stackrel{d}{=} (\mathcal{X}_{t/\varepsilon})_{t \geq 0}$ and such that,

$$\lambda - \text{a.s.}, \quad d_{\mathbf{M}_1}((\varepsilon^{1/\alpha} \mathcal{X}_t^\varepsilon)_{t \geq 0}, (R_t^\alpha)_{t \geq 0}) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

We now use the fact the $\mathbf{M}_2$ -topology, originally introduced by Skorokhod in his seminal paper [Sko56], is weaker than the $\mathbf{M}_1$ -topology. The convergence in $\mathcal{D}$ endowed with $\mathbf{M}_2$ can be characterized, see [Sko56, page 267], as follows: a sequence $(x_n)_{n \in \mathbb{N}}$ converges to x in $\mathcal{D}$ endowed with the $\mathbf{M}_2$ if and only if

$$\inf_{u \in [s, t]} x_n(u) \xrightarrow{n \rightarrow \infty} \inf_{u \in [s, t]} x(u) \quad \text{and} \quad \sup_{u \in [s, t]} x_n(u) \xrightarrow{n \rightarrow \infty} \sup_{u \in [s, t]} x(u)$$

for any $0 \leq s < t$ points of continuity of x . Therefore, since λ -a.s., for any $k \in \{1, \dots, n\}$, $t_k - \delta$ and $t_k + \delta$ are points of continuity of $(R_t^\alpha)_{t \geq 0}$, we deduce that λ -a.s., for any $k \in \{1, \dots, n\}$, the following convergence holds

$$\sup_{s \in [t_k - \delta, t_k + \delta]} \varepsilon^{1/\alpha} \mathcal{X}_s^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \sup_{s \in [t_k - \delta, t_k + \delta]} R_s^\alpha, \quad \text{as } \varepsilon \rightarrow 0,$$

which implies (4.19).

Step 2: We now grant Assumption 4.2-(ii) and we seek to show that, under this assumption, $(\varepsilon^{1/\alpha} \mathbf{X}_{t/\varepsilon})_{t \geq 0}$ is tight for the $\mathbf{M}_1$ -topology. Here we use the representation $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0} = (\mathfrak{X}_{T'_t}, \mathfrak{V}_{T'_t})_{t \geq 0}$, see Step 1. Our goal is to show that the conditions of Theorem 4.3-(ii) are satisfied. By the first step, it suffices to show that for any $\eta > 0$, for any $T > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(w(\varepsilon^{1/\alpha} \mathbf{X}_{t/\varepsilon}, T, \delta) > \eta) = 0, \quad (4.20)$$

By Proposition 4.4, the process $(\varepsilon^{1/\alpha} \mathfrak{X}_{t/\varepsilon})_{t \geq 0}$ is tight and by Proposition 4.3 and Dini's theorem, we have the following convergence in probability:

$$\sup_{t \in [0, T]} |\varepsilon T'_{t/\varepsilon} - t| \xrightarrow{\mathbb{P}} 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Let $T > 0$ and $\delta \in (0, 1)$, we will first place ourselves on the event $A_{T, \delta}^\varepsilon = \{\sup_{t \in [0, T]} |\varepsilon T'_{t/\varepsilon} - t| < \delta\}$. On this event, $T'_{t/\varepsilon} \in ((t - \delta)/\varepsilon, (t + \delta)/\varepsilon)$ for any $t \in [0, T]$. Let $t \in [0, T]$ and recall

that $t_{\delta-} = 0 \vee (t - \delta)$ and $t_{\delta+} = T \wedge (t + \delta)$. We set $t_{2\delta-} = 0 \vee (t - 2\delta)$ and (abusively) $t_{2\delta+} = (T + 1) \wedge (t + 2\delta)$. Then on the event $A_{T,\delta}^\varepsilon$, we have

$$\sup_{t_{\delta-} \leq t_1 < t_2 < t_3 \leq t_{\delta+}} d(\mathfrak{X}_{T'_{t_2/\varepsilon}}, [\mathfrak{X}_{T'_{t_1/\varepsilon}}, \mathfrak{X}_{T'_{t_3/\varepsilon}}]) \leq \sup_{t_{2\delta-} \leq t_1 < t_2 < t_3 \leq t_{2\delta+}} d(\mathfrak{X}_{t_2/\varepsilon}, [\mathfrak{X}_{t_1/\varepsilon}, \mathfrak{X}_{t_3/\varepsilon}]),$$

from which we deduce that

$$w(\varepsilon^{1/\alpha} \mathfrak{X}_{T'_{t/\varepsilon}}, T, \delta) \leq w(\varepsilon^{1/\alpha} \mathfrak{X}_{t/\varepsilon}, T + 1, \delta)$$

on the event $A_{T,\delta}^\varepsilon$. As a consequence, we have for any $\eta, \delta > 0$ and for any $T > 0$

$$\mathbb{P}\left(w(\varepsilon^{1/\alpha} \mathfrak{X}_{t/\varepsilon}, T, \delta) > \eta\right) \leq \mathbb{P}\left(w(\varepsilon^{1/\alpha} \mathfrak{X}_{t/\varepsilon}, T + 1, \delta) > \eta\right) + \mathbb{P}\left(\sup_{t \in [0, T]} |\varepsilon T'_{t/\varepsilon} - t| \geq \delta\right).$$

Therefore, since $(\varepsilon^{1/\alpha} \mathfrak{X}_{t/\varepsilon})_{t \geq 0}$ is tight, we have for any $\eta > 0$, for any $T > 0$

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}\left(w(\varepsilon^{1/\alpha} \mathfrak{X}_{t/\varepsilon}, T, \delta) > \eta\right) = 0,$$

which completes the proof. $\square$

4.4.6 Some persistence problems

The aim of this subsection is to prove Lemma 4.3. We recall that the random variable (τ, σ) are defined as follows: let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space supporting an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion $(B_t)_{t \geq 0}$ and two independent μ -distributed random variables V_0 and M , also independent of the Brownian motion. Consider the process $(X_t, V_t)_{t \geq 0}$ solution of (4.1) starting at $(0, V_0)$. During the whole subsection, $\alpha = (\beta + 1)/3$. Then τ and σ are defined as

$$\tau = \inf\{t > 0, X_t = 0\} \quad \text{and} \quad \sigma = \inf\{t \geq \tau, V_t = M\}.$$

We first introduce the random times

$$T_0 = \inf\{t \geq 0, V_t = 0\} \quad \text{and} \quad \rho = \inf\{t \geq \tau, V_t = 0\}.$$

It should be clear that a.s., $T_0 \leq \tau \leq \rho \leq \sigma$, see Figure 4.2. Indeed since V_t is strictly positive on $(0, T_0)$, so is X_t and $T_0 \leq \tau$. Moreover, since V_τ is non-positive, $M > 0$ and $(V_t)_{t \geq 0}$ is continuous, $\rho \leq \sigma$. Lemma 4.3 heavily relies on the two following lemmas.

Lemma 4.4.

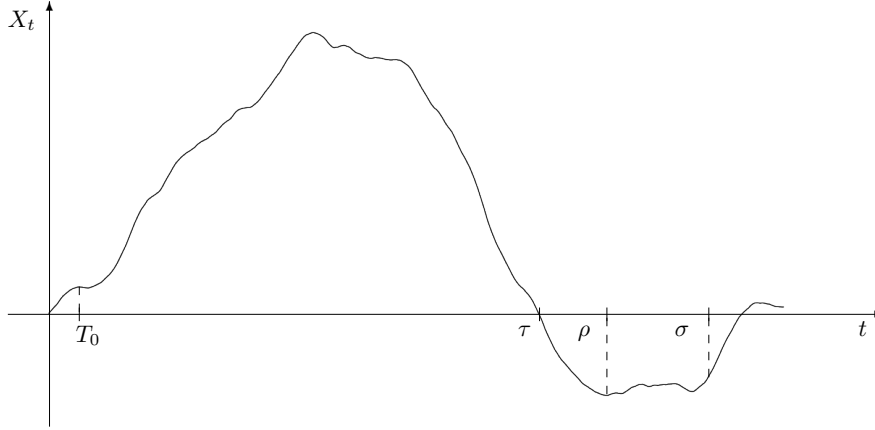
- (i) Under Assumption 4.2-(i), we can find some $\delta \in (0, 1/2)$ such that $\mathbb{E}[T_0^{1/2+\delta}] < \infty$ and $\mathbb{E}[(\sigma - \rho)^{1/2+\delta}] < \infty$. As a consequence, we have

$$\lim_{t \rightarrow \infty} t^{1/2} \mathbb{P}(T_0 > t) = \lim_{t \rightarrow \infty} t^{1/2} \mathbb{P}(\sigma - \rho > t) = 0.$$

- (ii) Under Assumption 4.2-(ii), there exists $\delta' \in (0, 1/\alpha - 2)$ such that $\mathbb{E}[(M(\sigma - \rho))^{\alpha/2+\delta'}] < \infty$.

Lemma 4.5. Grant Assumptions 4.2-(i). Then there exists a constant $C > 0$ such that

$$\mathbb{P}(\tau - T_0 > t) \sim \mathbb{P}(\rho - T_0 > t) \sim Ct^{-1/2} \quad \text{as } t \rightarrow \infty.$$


 Figure 4.2: Graphical representation of the random times T_0 , τ , ρ and σ .

Proof of Lemma 4.3. Using Lemma 4.12 in Appendix 4.A with $X = \tau - T_0$ and $Y = T_0$ and Lemmas 4.4-(i) and 4.5, it comes

$$\mathbb{P}(\tau > t) \sim Ct^{-1/2} \quad \text{as } t \rightarrow \infty.$$

By Lemma 4.12 again with $X = \rho - T_0$ and $Y = \sigma - \rho + T_0$, we get

$$\mathbb{P}(\sigma > t) \sim Ct^{-1/2} \quad \text{as } t \rightarrow \infty,$$

which completes the proof. $\square$

We now seek to show Lemma 4.4. To do so, we will use Feller's representation of regular diffusions i.e. we represent the velocity process through its scale function $\mathfrak{s}$ and its speed measure m :

$$\mathfrak{s}(v) = \int_0^v \Theta^{-\beta}(u) du \quad \text{and} \quad m(v) = \Theta^\beta(v).$$

Remember from Assumption 4.1 that $\Theta : \mathbb{R} \rightarrow (0, \infty)$ is a C^1 even function such that $F = \frac{\beta}{2} \frac{\Theta'}{\Theta}$ and satisfying $\lim_{v \rightarrow \pm\infty} |v| \Theta(v) = 1$. The function $\mathfrak{s}$ is an increasing bijection from $\mathbb{R}$ to $\mathbb{R}$ and we denote by $\mathfrak{s}^{-1}$ its inverse. We also define the function $\psi = \mathfrak{s}' \circ \mathfrak{s}^{-1}$. Now consider another Brownian motion $(W_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ (or an enlargement of the space) and for $v \in \mathbb{R}$, we set

$$W_t^v = \mathfrak{s}(v) + W_t, \quad A_t^v = \int_0^t [\psi(W_s^v)]^{-2} ds \quad \text{as well as} \quad \rho_t^v = \inf\{s > 0, A_s^v > t\}.$$

Then the process defined by $V_t^v = \mathfrak{s}^{-1}(W_{\rho_t^v}^v)$ and $X_t^v = \int_0^t V_s^v ds$ is a solution of (4.1) starting at $(v, 0)$. This result is standard and we refer to Kallenberg [Kal02, Chapter 23, Theorem 23.1 and its proof] for more details. As a consequence, using the substitution $u = \rho_s^v$, we can write that almost surely, for any $t \geq 0$,

$$X_t^v = \int_0^t \mathfrak{s}^{-1}(W_{\rho_s^v}^v) ds = \int_0^{\rho_t^v} \phi(W_s^v) ds, \quad (4.21)$$

where for any $v \in \mathbb{R}$, $\phi(v) = \mathfrak{s}^{-1}(v)/\psi^2(v)$. We set, for any $v \in \mathbb{R}$, $T_0^v = \inf\{t \geq 0, V_t^v = 0\}$, the first hitting time of $(V_t^v)_{t \geq 0}$ at the level 0. We also note that since $\mathfrak{s}(v) \rightarrow \pm\infty$ as $v \rightarrow \pm\infty$ and

$\int_{\mathbb{R}} m(v)dv < \infty$ because $\beta > 1$, the process $(V_t^v)_{t \geq 0}$ is a positive recurrent diffusion. Note that Assumption 4.1 yields the following asymptotics:

$$m(v) \sim v^{-\beta}, \quad \mathfrak{s}(v) \sim (\beta + 1)^{-1}v^{\beta+1}, \quad \phi(v) \sim (\beta + 1)^{\frac{1-2\beta}{\beta+1}}v^{\frac{1-2\beta}{\beta+1}} \quad \text{as } v \rightarrow \infty. \quad (4.22)$$

Finally, we stress that by the strong Markov property, since V_0 is μ -distributed, for any non negative functional G of continuous functions, we have

$$\mathbb{E}[G((V_t)_{t \geq 0})] = \int_0^\infty \mathbb{E}[G((V_t^v)_{t \geq 0})] \mu(dv). \quad (4.23)$$

Proof of Lemma 4.4. Step 1: We first deal with T_0 . Let $\delta \in (0, 1/2)$, then by (4.23) and the Hölder inequality, we have

$$\mathbb{E}[T_0^{1/2+\delta}] = \int_0^\infty \mathbb{E}[(T_0^v)^{1/2+\delta}] \mu(dv) \leq \int_0^\infty \mathbb{E}[T_0^v]^{1/2+\delta} \mu(dv).$$

We will show that the quantity on the right-hand-side is finite if δ is small enough and to do so, we need to understand the behavior of $\mathbb{E}[T_0^v]$ as v tends to infinity. We use Kac's moment formula (see for instance Löcherbach [Löc13, Corollary 3.5]) which, applied to our case, tells us that

$$\mathbb{E}[T_0^v] = \mathfrak{s}(v) \int_v^\infty m(u)du + \int_0^v \mathfrak{s}(u)m(u)du \leq \mathfrak{s}(v) \int_{\mathbb{R}} m(u)du + \int_0^v \mathfrak{s}(u)m(u)du. \quad (4.24)$$

By (4.22), we deduce that $\int_0^v \mathfrak{s}(u)m(u)du \sim (2(\beta+1))^{-1}v^2$ as $v \rightarrow \infty$. Since $\mathfrak{s}(v) \sim (\beta+1)^{-1}v^{\beta+1}$ and $\beta+1 > 2$, the dominant term on the right-hand-side in (4.24) is the first one and we deduce that there exists some positive constant K such that for any $v \geq 0$, $\mathbb{E}[T_0^v] \leq K(1+v^{\beta+1})$. Hence we have

$$\mathbb{E}[T_0^{1/2+\delta}] \leq \int_0^\infty \mathbb{E}[T_0^v]^{1/2+\delta} \mu(dv) \leq K^{1/2+\delta} \int_0^\infty (1+v^{\beta+1})^{1/2+\delta} \mu(dv),$$

which is finite by Assumption 4.2-(i) if $\delta = \eta/(\beta+1)$.

Step 2: Note that the random time $\sigma - \rho$ only depends on the speed process. More precisely, since ρ is a stopping time and $V_\rho = 0$, the process $(V_{t+\rho})_{t \geq 0}$ is a solution of $dY_t = F(Y_t)dt + dB_t$ starting at 0, and thus, $\sigma - \rho$ is equal in law to $T_M^0 = \inf\{t \geq 0, V_t^0 = M\}$. Since M is independent of $(V_t)_{t \geq 0}$, we can write, as in the first step, that

$$\mathbb{E}[(\sigma - \rho)^{1/2+\delta}] = \int_0^\infty \mathbb{E}[(T_v^0)^{1/2+\delta}] \mu(dv) \leq \int_0^\infty \mathbb{E}[T_v^0]^{1/2+\delta} \mu(dv),$$

where $T_v^0 = \inf\{t \geq 0, V_t^0 = v\}$. We use again Kac's moment formula, see [Löc13, Corollary 3.5]:

$$\mathbb{E}[T_v^0] = \mathfrak{s}(v) \int_{\mathbb{R}} m(u)du - \mathfrak{s}(v) \int_v^\infty m(u)du - \int_0^v \mathfrak{s}(u)m(u)du \leq \mathfrak{s}(v) \int_{\mathbb{R}} m(u)du, \quad (4.25)$$

and we can conclude as in the previous step that $\mathbb{E}[T_v^0] \leq K(1+v^{\beta+1})$, whence $\mathbb{E}[(\sigma - \rho)^{1/2+\delta}]$ is finite with the same value of δ .

Step 3: We deal with the second item and we grant Assumption 4.2-(ii). Let $\delta' \in (0, 1 - \alpha/2)$. Since M is independent of $(V_t)_{t \geq 0}$ and μ is the law of M , we get

$$\mathbb{E}[(M(\sigma - \rho))^{\alpha/2+\delta'}] = \int_0^\infty v^{\alpha/2+\delta'} \mathbb{E}[(T_v^0)^{\alpha/2+\delta'}] \mu(dv) \leq \int_0^\infty v^{\alpha/2+\delta'} \mathbb{E}[T_v^0]^{\alpha/2+\delta'} \mu(dv).$$

Then, since $\mathbb{E}[T_v^0] \leq K(1 + v^{\beta+1})$ as in Step 2,

$$\mathbb{E}[(M(\sigma - \rho))^{\alpha/2+\delta'}] \leq K^{\alpha/2+\delta'} \int_0^\infty v^{\alpha/2+\delta'} (1 + v^{\beta+1})^{\alpha/2+\delta'} \mu(dv)$$

which is finite by Assumption 4.2-(ii) if $\delta' = \eta/(\beta + 2)$, recall that $\alpha = (\beta + 1)/3$. $\square$

We now seek to show Lemma 4.5, and we will need to estimate the moments of X_{T_0} .

Lemma 4.6. *Grant Assumption 4.2-(i), then we have $\mathbb{E}[X_{T_0}^{\alpha/2}] < \infty$.*

Proof. For any $v > 0$, we set $\theta_0^v = \inf\{t \geq 0, W_t^v = 0\}$. Since $\mathfrak{s}(v) \geq 0$ if and only if $v \geq 0$, and $V_t^v = \mathfrak{s}^{-1}(W_{\rho_t^v}^v)$ with $(\rho_t^v)_{t \geq 0}$ the inverse of $(A_t^v)_{t \geq 0}$, it should be clear that $\mathbb{P}(T_0^v = A_{\theta_0^v}^v) = 1$. Hence, following (4.21), we have for any $v > 0$, almost surely

$$X_{T_0^v}^v = \int_0^{\theta_0^v} \phi(W_s^v) ds.$$

Step 1: We first introduce the non-negative random variable

$$Z_{v,\alpha} = \int_0^{\theta_0^v} (W_s^v)^{1/\alpha-2} ds.$$

By the scaling of the Brownian motion, the law of $(W_t^v, 0 \leq t \leq \theta_0^v)$ is the same as the law of $(\mathfrak{s}(v)W_{t/[\mathfrak{s}(v)]^2}^{\mathfrak{s}^{-1}(1)}, 0 \leq t \leq [\mathfrak{s}(v)]^2\theta_0^{\mathfrak{s}^{-1}(1)})$, and as a consequence,

$$\text{the law of } Z_{v,\alpha} \text{ is the same as the law of } [\mathfrak{s}(v)]^{1/\alpha} Z_{\mathfrak{s}^{-1}(1),\alpha}.$$

In fact, the law of $Z_{\mathfrak{s}^{-1}(1),\alpha}$ is explicit and it holds that $Z_{\mathfrak{s}^{-1}(1),\alpha}$ has the same law as α^2/Γ_α where Γ_α is a random variable whose law is the Gamma distribution of parameter $(\alpha, 1)$, see for instance Letemplier-Simon [LS19, page 93]. In particular, we have $\mathbb{P}(Z_{\mathfrak{s}^{-1}(1),\alpha} \geq x) \sim cx^{-\alpha}$ as $x \rightarrow \infty$ for some $c > 0$. Therefore, it holds that $\mathbb{E}[Z_{\mathfrak{s}^{-1}(1),\alpha}^{\alpha/2}] < \infty$.

Step 2: We conclude. Since $\alpha = (\beta + 1)/3$, we have $1/\alpha - 2 = (1 - 2\beta)/(\beta + 1)$ and by (4.22), there exists a constant $c > 0$ such that $\phi(v) \sim cv^{1/\alpha-2}$ as $v \rightarrow \infty$. Remark that $1/\alpha - 2 < 0$ as $\beta > 1$, and therefore there exists a constant $C > 0$ such that for any $v \geq 0$, $\phi(v) \leq Cv^{1/\alpha-2}$. As a consequence, we have for any $v > 0$, almost surely $0 < X_{T_0^v}^v < CZ_{v,\alpha}$. Putting the pieces together, by (4.23) and the previous step, we can write

$$\mathbb{E}[X_{T_0}^{\alpha/2}] = \int_0^\infty \mathbb{E}[(X_{T_0^v}^v)^{\alpha/2}] \mu(dv) \leq C^{\alpha/2} \int_0^\infty \mathbb{E}[(Z_{v,\alpha})^{\alpha/2}] \mu(dv)$$

whence

$$\mathbb{E}[X_{T_0}^{\alpha/2}] \leq C^{\alpha/2} \mathbb{E}[Z_{\mathfrak{s}^{-1}(1),\alpha}^{\alpha/2}] \int_0^\infty [\mathfrak{s}(v)]^{1/2} \mu(dv).$$

We can conclude since there exists a constant $K > 0$ such that for any $v > 0$, $\mathfrak{s}(v) \leq K(1 + v)^{\beta+1}$, see the proof of Lemma 4.4, and $\int_0^\infty (1 + v)^{(\beta+1)/2} \mu(dv) < \infty$ by Assumption 4.2-(i). $\square$

We now introduce the process $(\bar{V}_t, \bar{X}_t)_{t \geq 0} = (V_{t+T_0}, X_{t+T_0} - X_{T_0})_{t \geq 0}$ which is a solution of (4.1) starting at $(0, 0)$ and is independent of X_{T_0} . We emphasize that, since the restoring force F is odd by assumption, the processes $(\bar{V}_t)_{t \geq 0}$ and $(\bar{X}_t)_{t \geq 0}$ are symmetric. We also stress that the stopping times $\tau - T_0 = \inf\{t > 0, \bar{X}_t = -X_{T_0}\}$ and $\rho - T_0 = \inf\{t \geq \tau - T_0, \bar{V}_t = 0\}$ only

depend on $(\bar{V}_t)_{t \geq 0}$, $(\bar{X}_t)_{t \geq 0}$ and X_{T_0} . Let us introduce the supremum and infimum of $(\bar{X}_t)_{t \geq 0}$: $\xi_t = \sup_{s \in [0, t]} \bar{X}_s$ and $\Lambda_t = \inf_{s \in [0, t]} \bar{X}_s$. Let us also define for $t \geq 0$

$$g_t = \sup\{s \leq t, \bar{V}_s = 0\} \quad \text{and} \quad d_t = \inf\{s \geq t, \bar{V}_s = 0\}.$$

Then we have the following inclusions of events:

$$\{\Lambda_{d_t} > -X_{T_0}\} \subset \{\tau - T_0 > t\} \subset \{\rho - T_0 > t\} \subset \{\Lambda_{g_t} > -X_{T_0}\}.$$

The first two inclusions are straightforward since $\{\tau - T_0 > t\} = \{\Lambda_t > -X_{T_0}\}$, $d_t \geq t$ and $\tau \leq \rho$. Remember that $\rho - T_0$ is the first zero of $\bar{V}_t$ after $\tau - T_0$ and thus it should be clear that $\{\Lambda_{g_t} \leq -X_{T_0}\} = \{\tau - T_0 \leq g_t\} \subset \{\rho - T_0 \leq g_t\}$, which establishes the third inclusion since $g_t \leq t$. Since $(\bar{X}_t)_{t \geq 0}$ is symmetric, we then have

$$\mathbb{P}(\xi_{d_t} < X_{T_0}) \leq \mathbb{P}(\tau - T_0 > t) \leq \mathbb{P}(\rho - T_0 > t) \leq \mathbb{P}(\xi_{g_t} < X_{T_0}). \quad (4.26)$$

We will show the following lemma which, combined with (4.26), immediately implies Lemma 4.5.

Lemma 4.7. *Grant Assumptions 4.2-(i). Then there exists a constant $C > 0$ such that*

$$\mathbb{P}(\xi_{g_t} < X_{T_0}) \underset{t \rightarrow \infty}{\sim} \mathbb{P}(\xi_{d_t} < X_{T_0}) \underset{t \rightarrow \infty}{\sim} Ct^{-1/2}.$$

To prove this result, we will heavily use some results developed in [BBT23], which rely on works about Itô's excursion theory and the links with Lévy processes, in particular by Bertoin [Ber96] and Vallois, Salminen and Yor [SVY07]. The proof is rather long and we will segment it (again) into smaller pieces, see Lemmas 4.10 and 4.11 below.

We use the following standard trick: let $e = e(q)$ be an exponential random variable of parameter $q > 0$, independent of everything else, then we will look at the quantities $\mathbb{P}(\xi_{g_e} < X_{T_0})$ and $\mathbb{P}(\xi_{d_e} < X_{T_0})$ instead of looking at $\mathbb{P}(\xi_{g_t} < X_{T_0})$ and $\mathbb{P}(\xi_{d_t} < X_{T_0})$. We have nothing to loose doing this since a combination of the Tauberian theorem and the monotone density theorem (see Theorem 4.8 below) tells us that having an asymptotic of $\mathbb{P}(\xi_{g_t} < X_{T_0})$ (respectively $\mathbb{P}(\xi_{d_t} < X_{T_0})$) as $t \rightarrow \infty$ is equivalent to having an asymptotic of $\mathbb{P}(\xi_{g_e} < X_{T_0})$ (respectively $\mathbb{P}(\xi_{d_e} < X_{T_0})$) as $q \rightarrow 0$, and we will first study $\mathbb{P}(\xi_{g_e} < x)$ and $\mathbb{P}(\xi_{d_e} < x)$ for a fixed $x > 0$.

The first reason we do this is that it brings independence between the quantities we are interested in. The second reason is that by doing this, we actually have explicit formulas for some quantities of interest, for instance for $\mathbb{P}(\xi_{g_e} < x)$ and, as we will see, there is a strong link with some Lévy process associated to $(\bar{X}_t, \bar{V}_t)_{t \geq 0}$ and fluctuation's theory for Lévy processes, see for instance [Ber96, Chapter VI].

The velocity process $(\bar{V}_t)_{t \geq 0}$ possesses a local time at 0 and we will denote by $(\gamma_t)_{t \geq 0}$ its right-continuous inverse. The latter is a subordinator and we will denote by Φ its Laplace exponent, i.e. $\mathbb{E}[e^{-q\gamma_t}] = \exp(-t\Phi(q))$. The process $(\bar{V}_t)_{t \geq 0}$ being positive recurrent, we have $\mathbb{E}[\gamma_1] < \infty$, see [Ber99, Chapter 2, page 22], and we choose to normalize the local time so that $\mathbb{E}[\gamma_1] = 1$, whence $\Phi(q) \sim q$ as $q \rightarrow 0$. The strong law of large number for subordinators entails that a.s., $t^{-1}\gamma_t \rightarrow 1$ as $t \rightarrow \infty$. We will first prove the following which tells us that we only need to study $\mathbb{P}(\xi_{g_e} < x)$.

Lemma 4.8. *There exists a function $f : (0, \infty) \rightarrow [0, 1]$ such that $f(q) \rightarrow 1$ as $q \rightarrow 0$ and such that for any $q, x > 0$,*

$$\mathbb{P}(\xi_{g_e} < x)f(q) \leq \mathbb{P}(\xi_{d_e} < x) \leq \mathbb{P}(\xi_{g_e} < x). \quad (4.27)$$

Proof. Let us define for any $t > 0$ the processes

$$I_t = \int_{g_t}^{d_t} \bar{V}_s ds = \bar{X}_{d_t} - \bar{X}_{g_t} \quad \text{and} \quad \Delta_t = I_t + \bar{X}_{g_t} - \xi_{g_t} = \bar{X}_{d_t} - \xi_{g_t}.$$

Then we can express ξ_{d_t} in terms of ξ_{g_t} and Δ_t : remarking that $(\bar{X}_t)_{t \geq 0}$ is monotonic on every excursion of $(\bar{V}_t)_{t \geq 0}$, we get that $\sup_{s \in [g_t, d_t]} \bar{X}_s = \bar{X}_{g_t} \vee \bar{X}_{d_t}$. Therefore, if $\Delta_t \leq 0$, then $\bar{X}_{d_t} \leq \xi_{g_t}$ and $\xi_{d_t} = \xi_{g_t}$. On the other hand, if $\Delta_t > 0$, then $\xi_{d_t} = \bar{X}_{d_t} = \xi_{g_t} + \Delta_t$. All in all, we have

$$\xi_{d_t} = \xi_{g_t} \mathbf{1}_{\{\Delta_t \leq 0\}} + (\xi_{g_t} + \Delta_t) \mathbf{1}_{\{\Delta_t > 0\}}. \quad (4.28)$$

We can factorize functionals of the trajectories of $(\bar{V}_t)_{t \geq 0}$ before time g_e and functionals of the trajectories between g_e and d_e . More precisely, it is shown in [SVY07, Theorem 9] that the processes $(\bar{V}_u)_{0 \leq u \leq g_e}$ and $(\bar{V}_{u+g_e})_{0 \leq u \leq d_e - g_e}$ are independent. Therefore, I_e is independent of $(\xi_{g_e}, \bar{X}_{g_e} - \xi_{g_e})$. Moreover, it is shown in [BBT23, Corollary 4.6] that ξ_{g_e} and $\bar{X}_{g_e} - \xi_{g_e}$ are i.i.d., see also Lemma 4.9 below. Hence the random variables I_e , ξ_{g_e} and $\bar{X}_{g_e} - \xi_{g_e}$ are mutually independent and thus ξ_{g_e} is independent of $\Delta_e = I_e + \bar{X}_{g_e} - \xi_{g_e}$. We set $f(q) = \mathbb{P}(\Delta_e \leq 0)$ and we deduce from (4.28) that (4.27) holds for any $x > 0$.

Let us now show that $f(q) \rightarrow 1$ as $q \rightarrow 0$ and let us denote by n the excursion measure of $(\bar{V}_t)_{t \geq 0}$ away from zero. Let $\mathcal{E}$ the set of excursions, i.e. the set of continuous functions $\varepsilon = (\varepsilon_t)_{t \geq 0}$ such that $\varepsilon_0 = 0$ and such that there exists $\ell(\varepsilon) > 0$ for which $\varepsilon_s \neq 0$ for every $s \in (0, \ell(\varepsilon))$ and $\varepsilon_s = 0$ for every $s \geq \ell(\varepsilon)$. Then by Theorem 9 in [SVY07], we have for any measurable bounded function $G : \mathbb{R} \rightarrow \mathbb{R}$,

$$\mathbb{E}[G(I_e)] = \frac{1}{\Phi(q)} \int_{\mathcal{E}} G\left(\int_0^{\ell(\varepsilon)} \varepsilon_s ds\right) (1 - e^{-q\ell(\varepsilon)}) n(d\varepsilon).$$

But $\Phi(q) \sim q$ as $q \rightarrow 0$ and $\int_{\mathcal{E}} \ell(\varepsilon) n(d\varepsilon) = 1 < \infty$ since $\mathbb{E}[\gamma_1] = 1$ (by a direct application of the Master formula in the context of excursion theory), we get by dominated convergence that

$$\mathbb{E}[G(I_e)] \xrightarrow{q \rightarrow 0} \int_{\mathcal{E}} G\left(\int_0^{\ell(\varepsilon)} \varepsilon_s ds\right) \ell(\varepsilon) n(d\varepsilon),$$

and thus I_e converges in law as $q \rightarrow 0$.

Remember that $\xi_{g_e} - \bar{X}_{g_e}$ is equal in law to ξ_{g_e} . Moreover $\xi_{g_e} \rightarrow \infty$ in probability as $q \rightarrow 0$ because g_e tends in probability to infinity as $q \rightarrow 0$ and because $\xi_t = \sup_{s \in [0, t]} \bar{X}_s$ tends to infinity in probability (Theorem 4.4 clearly implies that $t^{-1/\alpha} \xi_t$ converges in law as $t \rightarrow \infty$). Since I_e converges in law and $\xi_{g_e} - \bar{X}_{g_e}$ converges in probability to ∞ as $q \rightarrow 0$, we have that $\Delta_e = I_e - (\xi_{g_e} - \bar{X}_{g_e})$ converges to $-\infty$ in probability, so that $\mathbb{P}(\Delta_e \leq 0) = f(q)$ converges to 1 as $q \rightarrow 0$. $\square$

To handle the quantity $\mathbb{P}(\xi_{g_e} < x)$, we will rely on a Wiener-Hopf factorization of the bivariate Lévy process $(\gamma_t, Z_t)_{t \geq 0} = (\gamma_t, \bar{X}_{\gamma_t})_{t \geq 0}$, which is developped in [BBT23, Appendix A]. Let us denote by $(S_t)_{t \geq 0}$ the supremum process of $(Z_t)_{t \geq 0}$, i.e. $S_t = \sup_{s \in [0, t]} Z_s$ and let $(R_t)_{t \geq 0} = (S_t - Z_t)_{t \geq 0}$ be the reflected process, which is a strong Markov process that also possesses a local time at 0, see [Ber96, Chapter VI] and we denote by $(\sigma_t)_{t \geq 0}$ its right-continuous inverse. The process $(\sigma_t, \theta_t, H_t)_{t \geq 0} = (\sigma_t, \gamma_{\sigma_t}, S_{\sigma_t})_{t \geq 0}$ is a trivariate subordinator, see [BBT23, Lemma A.1 in Appendix A]. Its Laplace exponent is denoted by κ , i.e. for any $\alpha, \beta, \delta \geq 0$,

$$\mathbb{E}\left[e^{-\alpha\sigma_t - \beta\theta_t - \delta H_t}\right] = \exp(-t\kappa(\alpha, \beta, \delta)).$$

Finally we introduce the renewal function $\mathcal{V}$ defined on $[0, \infty)$ by

$$\mathcal{V}(x) = \int_0^\infty \mathbb{P}(H_t \leq x) dt,$$

which is non-decreasing and right-continuous. We state the following lemma, which is borrowed from [BBT23, Proposition 4.4 and Corollary 4.6], which we will use several times.

Lemma 4.9. *There exists a constant $k > 0$ such that for every $\alpha, \beta, \delta \geq 0$, we have the following Fristedt formula*

$$\kappa(\alpha, \beta, \delta) = k \exp \left(\int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha t - \beta r - \delta x}}{t} \mathbf{1}_{\{x \geq 0\}} \mathbb{P}(\gamma_t \in dr, Z_t \in dx) dt \right).$$

Recall that $e = e(q)$ is an exponential random variable of parameter $q > 0$ independent of everything else. Then for every $\lambda, \mu \geq 0$, we have

$$\mathbb{E} \left[e^{-\lambda \xi_{g_e} - \mu (\xi_{g_e} - \bar{X}_{g_e})} \right] = \frac{\kappa(0, q, 0)}{\kappa(0, q, \lambda)} \frac{\kappa(0, q, 0)}{\kappa(0, q, \mu)}.$$

In particular, ξ_{g_e} and $\xi_{g_e} - \bar{X}_{g_e}$ are independent and have the same law.

Note that, when applying [BBT23, Corollary 4.6], we used the fact that $(Z_t)_{t \geq 0}$ is symmetric so that if $(\hat{Z}_t)_{t \geq 0} = (-Z_t)_{t \geq 0}$ is the dual process, then the corresponding Laplace exponent $\hat{\kappa} = \kappa$. Let us finally introduce two last tools that are key to the proof of our result and that we will use several times. Let us first remind Frullani's identity which holds for every $b \in (0, 1)$, see for instance [Ber96, page 73]:

$$\log b = \int_0^\infty \frac{e^{-x} - e^{-bx}}{x} dx. \quad (4.29)$$

Let us also remind the following classical theorem, which can be obtained by a careful application of the classical Karamata's Tauberian theorem and the monotone density theorem, see for instance [BGT87, Theorem 1.7.1 page 37 and Theorem 1.7.2 page 39].

Theorem 4.8. *Let $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a monotone function. Let us denote by $\mathcal{L}u$ its Laplace transform i.e. for any $\lambda > 0$, $\mathcal{L}u(\lambda) = \int_0^\infty e^{-\lambda x} u(x) dx$. Let $\rho > -1$ and Γ be the usual Gamma function, then the two following assertions are equivalent.*

$$(i) \quad u(x) \sim (\rho + 1)x^\rho \text{ as } x \rightarrow \infty.$$

$$(ii) \quad \lambda \mathcal{L}u(\lambda) \sim \Gamma(\rho + 2)\lambda^{-\rho} \text{ as } \lambda \rightarrow 0.$$

We have the following lemma.

Lemma 4.10. *Recall that $e = e(q)$ is an exponential random variable of parameter $q > 0$ independent of everything else. There exists $k > 0$ such that for any $x > 0$, we have*

$$\mathbb{P}(\xi_{g_e} < x) \underset{q \rightarrow 0}{\sim} \mathbb{P}(\xi_{d_e} < x) \underset{q \rightarrow 0}{\sim} kq^{1/2} \mathcal{V}(x). \quad (4.30)$$

Moreover, there exists a constant $M > 0$ such that for any $x > 0$, for any $q \in (0, 1)$,

$$\mathbb{P}(\xi_{d_e} < x) \leq \mathbb{P}(\xi_{g_e} < x) \leq Mq^{1/2} \mathcal{V}(x). \quad (4.31)$$

Proof. By Lemma 4.9, for any $q > 0$, we have

$$\lambda \int_0^\infty e^{-\lambda x} \mathbb{P}(\xi_{g_e} < x) dx = \mathbb{E} \left[e^{-\lambda \xi_{g_e}} \right] = \frac{\kappa(0, q, 0)}{\kappa(0, q, \lambda)}.$$

We introduce for any $q > 0$ the function $\mathcal{V}_q$ defined on $[0, \infty)$ by

$$\mathcal{V}_q(x) = \mathbb{E} \left[\int_0^\infty e^{-q\theta_t} \mathbf{1}_{\{H_t \leq x\}} dt \right],$$

which is such that

$$\lambda \int_0^\infty e^{-\lambda x} \mathcal{V}_q(x) dx = \mathbb{E} \left[\int_0^\infty e^{-q\theta_t - \lambda H_t} dt \right] = \int_0^\infty e^{-t\kappa(0, q, \lambda)} dt = \frac{1}{\kappa(0, q, \lambda)}.$$

Hence, by injectivity of the Laplace transform, we get $\mathbb{P}(\xi_{g_e} < x) = \kappa(0, q, 0) \mathcal{V}_q(x)$ for any $x, q > 0$. For any $x > 0$, $\mathcal{V}_q(x)$ increases to $\mathcal{V}(x)$ as $q \rightarrow 0$. Hence, by Lemma 4.8, to show (4.30) and (4.31), it is enough to show that $\kappa(0, q, 0) \sim kq^{1/2}$ as $q \rightarrow 0$ for some $k > 0$.

By the Fristedt formula from Lemma 4.9 and by symmetry of $(\bar{V}_t)_{t \geq 0}$, and thus of $(Z_t)_{t \geq 0}$ (observe $(Z_t)_{t \geq 0}$ and $(-Z_t)_{t \geq 0}$ share the same $(\gamma_t)_{t \geq 0}$), we also have

$$\kappa(0, q, 0) = k \exp \left(\int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-qr}}{t} \mathbf{1}_{\{x \leq 0\}} \mathbb{P}(\gamma_t \in dr, Z_t \in dx) dt \right).$$

Then we can write

$$\begin{aligned} \log[\kappa(0, q, 0)]^2 &= 2 \log k + \int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-qr}}{t} \mathbb{P}(\gamma_t \in dr, Z_t \in dx) dt \\ &= 2 \log k + \int_0^\infty \int_0^\infty \frac{e^{-t} - e^{-qr}}{t} \mathbb{P}(\gamma_t \in dr) dt \\ &= 2 \log k + \int_0^\infty \frac{e^{-t} - e^{-\Phi(q)t}}{t} dt \\ &= 2 \log k + \log \Phi(q) \end{aligned}$$

by Frullani's formula (4.29). Therefore $\kappa(0, q, 0) = k(\Phi(q))^{1/2} \sim kq^{1/2}$ as $q \rightarrow 0$. $\square$

We need to show one last result, after which we will be able to prove Lemma 4.7, which will close this subsection.

Lemma 4.11. *There exists a constant $v_\alpha > 0$ such that $\mathcal{V}(x) \sim v_\alpha x^{\alpha/2}$ as $x \rightarrow \infty$.*

Proof. Let us first remark that we have for any $\lambda > 0$,

$$\lambda \int_0^\infty e^{-\lambda x} \mathcal{V}(x) dy = \mathbb{E} \left[\int_0^\infty e^{-\lambda H_t} dt \right] = \int_0^\infty e^{-t\kappa(0, 0, \lambda)} dt = \frac{1}{\kappa(0, 0, \lambda)}.$$

Then, by Theorem 4.8, since $\mathcal{V}$ is non-decreasing and $\alpha/2 > -1$, it is enough to show that there exists a constant $c_\alpha > 0$ such that $\kappa(0, 0, \lambda) \sim c_\alpha \lambda^{\alpha/2}$ as $\lambda \rightarrow 0$.

To do so, we use the convergence in law of the rescaled Lévy process $(\varepsilon^{1/\alpha} Z_{t/\varepsilon})_{t \geq 0}$ to the symmetric stable process $(Z_t^\alpha)_{t \geq 0}$, see Proposition 4.6 below. Moreover, since $t^{-1} \gamma_t \rightarrow 1$ as $t \rightarrow \infty$, we get

$$(\varepsilon \gamma_{t/\varepsilon}, \varepsilon^{1/\alpha} Z_{t/\varepsilon})_{t \geq 0} \longrightarrow (t, Z_t^\alpha)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0, \quad (4.32)$$

in law for the usual $\mathbf{J}_1$ Skorokhod topology. Indeed, since $(\gamma_t, Z_t)_{t \geq 0}$ is Lévy, only the convergence in law of $(t^{-1}\gamma_t, t^{-1/\alpha}Z_t)$ to $(1, Z_1^\alpha)$ as $t \rightarrow \infty$ is required, see Jacod-Shiryaev [JS03, Chapter VII, Corollary 3.6], and the convergence follows from Slutsky's lemma.

Let us set, for every $\alpha, \beta, \lambda \geq 0$,

$$\kappa^\varepsilon(\alpha, \beta, \lambda) = k \exp \left(\int_0^\infty \int_{[0, \infty) \times \mathbb{R}} \frac{e^{-t} - e^{-\alpha t - \beta r - \lambda x}}{t} \mathbf{1}_{\{x \geq 0\}} \mathbb{P}(\varepsilon \gamma_{t/\varepsilon} \in dr, \varepsilon^{1/\alpha} Z_{t/\varepsilon} \in dx) dt \right),$$

where $k > 0$ is the constant from Lemma 4.9. We have

$$\begin{aligned} \log \kappa^\varepsilon(0, 0, \lambda) &= \log k + \int_0^\infty t^{-1} \mathbb{E} \left[(e^{-t} - e^{-\lambda \varepsilon^{1/\alpha} Z_{t/\varepsilon}}) \mathbf{1}_{\{Z_{t/\varepsilon} \geq 0\}} \right] dt \\ &= \log k + \int_0^\infty t^{-1} \mathbb{E} \left[(e^{-t} - e^{-\lambda \varepsilon^{1/\alpha} Z_t}) \mathbf{1}_{\{Z_t \geq 0\}} \right] dt \\ &\quad + \int_0^\infty t^{-1} (e^{-\varepsilon t} - e^{-t}) \mathbb{P}(Z_t \geq 0) dt \\ &= \log \kappa(0, 0, \lambda \varepsilon^{1/\alpha}) - \frac{1}{2} \log \varepsilon. \end{aligned}$$

In the third equality, we used Frullani's identity and the fact that $\mathbb{P}(Z_t \geq 0) = 1/2$. Therefore, we have $\kappa^\varepsilon(0, 0, \lambda) = \varepsilon^{-1/2} \kappa(0, 0, \lambda \varepsilon^{1/\alpha})$. Then it is shown in [BBT23, Proposition B.2 in Appendix B] that the convergence (4.32) entails that for any $\alpha, \beta, \lambda \geq 0$,

$$\kappa^\varepsilon(\alpha, \beta, \lambda) \xrightarrow{\varepsilon \rightarrow 0} k \exp \left(\int_0^\infty \int_0^\infty \frac{e^{-t} - e^{-(\alpha+\beta)t - \lambda x}}{t} \mathbb{P}(Z_t^\alpha \in dx) dt \right) =: \bar{\kappa}(\alpha, \beta, \lambda).$$

Now for $\delta > 0$, choosing $\lambda = 1$ and $\varepsilon = \delta^\alpha$, we see that $\kappa(0, 0, \delta) = \delta^{\alpha/2} \kappa^{\delta^\alpha}(0, 0, 1)$, which is equivalent to $\delta^{\alpha/2} \bar{\kappa}(0, 0, 1)$, as desired. $\square$

Proof of Lemma 4.7. Let $e = e(q)$ be an independent exponential random variable of parameter $q > 0$. Since $t \mapsto \mathbb{P}(\xi_{g_t} < X_{T_0})$ and $t \mapsto \mathbb{P}(\xi_{d_t} < X_{T_0})$ are non-decreasing, since $-1/2 > -1$, and since

$$\mathbb{P}(\xi_{g_e} < X_{T_0}) = q \int_0^\infty e^{-qt} \mathbb{P}(\xi_{g_t} < X_{T_0}) dt \quad \text{and} \quad \mathbb{P}(\xi_{d_e} < X_{T_0}) = q \int_0^\infty e^{-qt} \mathbb{P}(\xi_{d_t} < X_{T_0}) dt,$$

by Theorem 4.8, the result is equivalent to $\mathbb{P}(\xi_{g_e} < X_{T_0}) \sim \mathbb{P}(\xi_{d_e} < X_{T_0}) \sim \bar{C} q^{1/2}$ as $q \rightarrow 0$ for some $\bar{C} > 0$. Since $(\bar{X}_t)_{t \geq 0}$ is independent of X_{T_0} , we have

$$\mathbb{P}(\xi_{d_e} < X_{T_0}) = \int_0^\infty \mathbb{P}(\xi_{d_e} < x) \mathbb{P}(X_{T_0} \in dx).$$

By (4.30), we know that for any $x > 0$, $q^{-1/2} \mathbb{P}(\xi_{d_e} < x) \rightarrow k \mathcal{V}(x)$ as $q \rightarrow 0$, and by (4.31), we have $q^{-1/2} \mathbb{P}(\xi_{d_e} < x) \leq M \mathcal{V}(x)$. Finally, by Lemmas 4.6 and 4.11, $\mathbb{E}[\mathcal{V}(X_{T_0})] < \infty$ and thus we can apply the dominated convergence theorem, which tells us that

$$\lim_{q \rightarrow 0} q^{-1/2} \mathbb{P}(\xi_{d_e} < X_{T_0}) = k \mathbb{E}[\mathcal{V}(X_{T_0})].$$

The same proof holds for $\mathbb{P}(\xi_{g_e} < X_{T_0})$. $\square$

4.5 $\mathbf{M}_1$ -convergence of the free process

In this section, we give the proof of Theorem 4.4. Let $(X_t, V_t)_{t \geq 0}$ be the solution to (4.1) starting at $(0, v_0)$ where $v_0 \in \mathbb{R}$. Let $(Z_t^\alpha)_{t \geq 0}$ be the symmetric stable process with $\alpha = (\beta + 1)/3$ as in the statement. As the convergence in the finite dimensional distribution sense was already proved in [FT21], we only need to show the tightness for the $\mathbf{M}_1$ -topology. By Theorem 4.3, we need that for any $T > 0$, for any $\eta > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(w(\varepsilon^{1/\alpha} X_{t/\varepsilon}, T, \delta) > \eta) = 0, \quad (4.33)$$

The idea of the proof is as follows: we first show that the convergence holds for $(X_t, V_t)_{t \geq 0}$ starting at $(0, 0)$. To do so, we will show that some Lévy process $(Z_t)_{t \geq 0}$ associated to $(X_t)_{t \geq 0}$ converges in the $\mathbf{J}_1$ -topology to a symmetric stable process (which is immediately implied by the finite dimensional distribution convergence). Therefore, $(Z_t)_{t \geq 0}$ converges also in the $\mathbf{M}_1$ -topology and (4.33) is then satisfied by $(Z_t)_{t \geq 0}$, which will imply that (4.33) is also satisfied by $(X_t)_{t \geq 0}$. We will then extend the convergence to processes starting at $(0, v_0)$.

4.5.1 Some preliminary results

Before jumping in to the proof, we recall some results that were used / proved in [FT21]. The first result can be found in Jeulin-Yor [JY81] and Biane-Yor [BY87], and represents symmetric stable processes using one Brownian motion.

Theorem 4.9 (Biane-Yor). *Let $(W_t)_{t \geq 0}$ be a Brownian motion and $(\tau_t)_{t \geq 0}$ the inverse of its local time at 0. Let $\alpha \in (0, 2)$ and consider, for $\eta > 0$, the process*

$$K_t^\eta = \int_0^t \operatorname{sgn}(W_s) |W_s|^{1/\alpha-2} \mathbf{1}_{\{|W_s| > \eta\}} ds.$$

Then the process $(K_t^\eta)_{t \geq 0}$ converges a.s. toward a process $(K_t)_{t \geq 0}$ as $\eta \rightarrow 0$, uniformly on compact time intervals. Moreover the process $(S_t^\alpha)_{t \geq 0} = (K_{\tau_t})_{t \geq 0}$ is a symmetric stable process and for all $t \geq 0$ and all $\xi \in \mathbb{R}$, $\mathbb{E}[\exp(i\xi S_t^\alpha)] = \exp(-\kappa_\alpha t |\xi|^\alpha)$, where $\kappa_\alpha = \frac{2^\alpha \pi \alpha^{2\alpha}}{2\alpha \Gamma^2(\alpha) \sin(\pi\alpha/2)}$.

We now summarize the intermediate results that can be found in [FT21, Lemmas 6 and 9], enabling the authors to prove their main result, which is stated in (4.2). We mention that the main tool used in the proofs is the theory of scale function and speed measure.

Theorem 4.10 (Fournier-Tardif). *Let $(X_t, V_t)_{t \geq 0}$ be a solution of (4.1), with $\beta \in (1, 5)$ and starting at $(0, 0)$. There exists a Brownian motion $(W_t)_{t \geq 0}$ such that for any $\varepsilon > 0$, there exist a continuous process $(H_t^\varepsilon)_{t \geq 0}$ and a continuous, increasing and bijective time-change $(A_t^\varepsilon)_{t \geq 0}$ with inverse $(\rho_t^\varepsilon)_{t \geq 0}$, adapted to the filtration generated by $(W_t)_{t \geq 0}$, and having the following properties:*

(i) *For any $\varepsilon > 0$, $(X_{t/\varepsilon})_{t \geq 0} \stackrel{d}{=} (H_{\rho_t^\varepsilon}^\varepsilon)_{t \geq 0}$.*

(ii) *For every $t \geq 0$, a.s., $\rho_t^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \tau_t$, where $(\tau_t)_{t \geq 0}$ is the inverse of the local time at 0 of $(W_t)_{t \geq 0}$.*

(iii) *Almost surely, for every $t \geq 0$, $\sup_{[0, T]} |\varepsilon^{1/\alpha} H_t^\varepsilon - \theta K_t| \xrightarrow{\varepsilon \rightarrow 0} 0$, where $\alpha = (\beta + 1)/3$, where $\theta = (\beta + 1)^{1/\alpha-2} c_\beta^{1/\alpha}$ and where $(K_t)_{t \geq 0}$ is the process from Theorem 4.9.*

Recall that c_β is given in (4.4). We now slightly improve their result, showing the convergence of past infimum and supremum.

Proposition 4.5. *Grant Assumption 4.1 and let $(X_t, V_t)_{t \geq 0}$ be a solution of (4.1) with $\beta \in (1, 5)$ and starting at $(0, 0)$. Let $(Z_t^\alpha)_{t \geq 0}$ be a symmetric stable process with $\alpha = (\beta + 1)/3$ and such that $\mathbb{E}[e^{i\xi Z_t^\alpha}] = \exp(-t\sigma_\alpha|\xi|^\alpha)$. Then we have for every $0 \leq s < t$,*

$$\inf_{u \in [s, t]} \varepsilon^{1/\alpha} X_{u/\varepsilon} \xrightarrow{d} \inf_{u \in [s, t]} Z_u^\alpha \quad \text{and} \quad \sup_{u \in [s, t]} \varepsilon^{1/\alpha} X_{u/\varepsilon} \xrightarrow{d} \sup_{u \in [s, t]} Z_u^\alpha \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. By item (i) of Theorem 4.10, it is enough to show that for any $0 \leq s < t$, a.s., we have

$$\inf_{u \in [s, t]} \varepsilon^{1/\alpha} H_{\rho_u^\varepsilon}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \inf_{u \in [s, t]} \theta K_{\tau_u} \quad \text{and} \quad \sup_{u \in [s, t]} \varepsilon^{1/\alpha} H_{\rho_u^\varepsilon}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \sup_{u \in [s, t]} \theta K_{\tau_u}.$$

We will only show the result for the infimum as the proof for the supremum is identical.

Step 1 : We first show that a.s., for any $s < t$, $\inf_{u \in [s, t]} K_{\tau_u} = \inf_{u \in [\tau_s, \tau_t]} K_u$, which is not straightforward since $t \mapsto \tau_t$ is discontinuous. We will first treat the case $\alpha \in (0, 1)$. Observe that in this case, $\int_0^t |W_s|^{1/\alpha-2} ds < \infty$ since $1/\alpha - 2 > -1$ and thus we have

$$K_{\tau_t} = \int_0^{\tau_t} \text{sgn}(W_s) |W_s|^{1/\alpha-2} ds = \sum_{r \leq t} \int_{\tau_{r-}}^{\tau_r} \text{sgn}(W_s) |W_s|^{1/\alpha-2} ds,$$

which has finite variations and no drift part. For every $u \geq 0$, $(W_t)_{t \geq 0}$ is of constant sign on the time-interval $[\tau_{u-}, \tau_u]$ and consequently $t \mapsto K_t$ is monotone on every such interval. Hence, the infimum is necessarily reached at the extremities i.e. $\inf_{r \in [\tau_{u-}, \tau_u]} K_r = \min\{K_{\tau_{u-}}, K_{\tau_u}\}$.

If $\alpha \in [1, 2)$, we approximate $(K_t)_{t \geq 0}$ by the processes $(K_t^\eta)_{t \geq 0}$, from Theorem 4.9. Similarly, we have for every $t \geq 0$, $K_{\tau_t}^\eta = \sum_{s \leq t} \int_{\tau_{s-}}^{\tau_s} \text{sgn}(W_u) |W_u|^{1/\alpha-2} \mathbf{1}_{\{|W_u| > \eta\}} du$, and thus, by the previous reasoning, we have $\inf_{u \in [s, t]} K_{\tau_u}^\eta = \inf_{u \in [\tau_s, \tau_t]} K_u^\eta$. We can write

$$\begin{aligned} \left| \inf_{u \in [s, t]} K_{\tau_u} - \inf_{u \in [\tau_s, \tau_t]} K_u \right| &\leq \left| \inf_{u \in [s, t]} K_{\tau_u} - \inf_{u \in [s, t]} K_{\tau_u}^\eta \right| + \left| \inf_{u \in [\tau_s, \tau_t]} K_u^\eta - \inf_{u \in [\tau_s, \tau_t]} K_u \right| \\ &\leq \sup_{u \in [s, t]} |K_{\tau_u}^\eta - K_{\tau_u}| + \sup_{u \in [\tau_s, \tau_t]} |K_u^\eta - K_u| \\ &\leq 2 \sup_{u \in [0, \tau_t]} |K_u^\eta - K_u|. \end{aligned}$$

By Theorem 4.9, $(K_t^\eta)_{t \geq 0}$ converges a.s. to $(K_t)_{t \geq 0}$ uniformly on compact time intervals, and thus the last term vanishes as $\eta \rightarrow 0$.

Step 2 : For every $\varepsilon > 0$, $t \mapsto \rho_t^\varepsilon$ and $t \mapsto H_t^\varepsilon$ are almost surely continuous. Therefore, a.s. for every $s < t$, $\inf_{u \in [s, t]} \varepsilon^{1/\alpha} H_{\rho_u^\varepsilon}^\varepsilon = \inf_{u \in [\rho_s^\varepsilon, \rho_t^\varepsilon]} \varepsilon^{1/\alpha} H_u^\varepsilon$. Hence we can write, by Step 1,

$$\begin{aligned} \left| \inf_{u \in [s, t]} \varepsilon^{1/\alpha} H_{\rho_u^\varepsilon}^\varepsilon - \inf_{u \in [s, t]} \theta K_{\tau_u} \right| &\leq \left| \inf_{u \in [\rho_s^\varepsilon, \rho_t^\varepsilon]} \varepsilon^{1/\alpha} H_u^\varepsilon - \inf_{u \in [\rho_s^\varepsilon, \rho_t^\varepsilon]} \theta K_u \right| + \left| \inf_{u \in [\rho_s^\varepsilon, \rho_t^\varepsilon]} \theta K_u - \inf_{u \in [\tau_s, \tau_t]} \theta K_u \right| \\ &\leq \sup_{u \in [0, T]} \left| \varepsilon^{1/\alpha} H_u^\varepsilon - \theta K_u \right| + \left| \inf_{u \in [\rho_s^\varepsilon, \rho_t^\varepsilon]} \theta K_u - \inf_{u \in [\tau_s, \tau_t]} \theta K_u \right|, \end{aligned}$$

where $T = \sup_{\varepsilon \in (0, 1)} \rho_t^\varepsilon$ is a.s. finite by item (ii) of Theorem 4.10. Almost surely, the first term on the right-hand-side goes to 0 thanks to Theorem 4.10-(iii). By item (iii), we have for any $0 \leq s < t$, almost surely, $\rho_s^\varepsilon \rightarrow \tau_s$ and $\rho_t^\varepsilon \rightarrow \tau_t$ as $\varepsilon \rightarrow 0$. Since $(K_t)_{t \geq 0}$ is continuous, the second term vanishes as $\varepsilon \rightarrow 0$, which completes the proof. $\square$

4.5.2 Convergence of the associated Lévy process

In this subsection, we consider a solution $(X_t, V_t)_{t \geq 0}$ of (4.1) starting at $(0, 0)$. The velocity process possesses a local time at 0 that we will denote by $(L_t)_{t \geq 0}$ and we will also denote by $(\gamma_t)_{t \geq 0}$ its right-continuous inverse. The latter is a subordinator. The process $(V_t)_{t \geq 0}$ is positive recurrent which implies that $\mathbb{E}[\gamma_1] < \infty$ and we choose to normalize the local time so that $\mathbb{E}[\gamma_1] = 1$. The strong law of large number for subordinators entails that a.s. $t^{-1}\gamma_t \rightarrow 1$ as $t \rightarrow \infty$. This also implies the same result for $(L_t)_{t \geq 0}$, and by Dini theorem, we get that a.s., for any $t \geq 0$,

$$\sup_{s \in [0, t]} |\varepsilon L_{s/\varepsilon} - s| \xrightarrow{\varepsilon \rightarrow 0} 0 \quad (4.34)$$

Next we define $(Z_t)_{t \geq 0} = (X_{\gamma_t})_{t \geq 0}$ which is a pure jump Levy process with finite variations and should be seen the following way:

$$Z_t = \int_0^{\gamma_t} V_s ds = \sum_{s \leq t} \int_{\gamma_{s-}}^{\gamma_s} V_u du.$$

We establish an α -stable central limit theorem for the Levy process $(Z_t)_{t \geq 0}$, which seems more or less clear in the light of (4.2) and the strong law of large number for $(\gamma_t)_{t \geq 0}$.

Proposition 4.6. *Let $(Z_t^\alpha)_{t \geq 0}$ be the stable process of Proposition 4.5. Then we have*

$$(\varepsilon^{1/\alpha} Z_{t/\varepsilon})_{t \geq 0} \longrightarrow (Z_t^\alpha)_{t \geq 0} \quad \text{as } \varepsilon \rightarrow 0$$

in law in the $\mathbf{J}_1$ -topology

Proof. As $(Z_t)_{t \geq 0}$ is a Lévy process, it is enough to show that $t^{-1/\alpha} Z_t$ converges in law to Z_1^α , see for instance Jacod-Shiryaev [JS03, Chapter VII, Corollary 3.6]. Let $z \in \mathbb{R}$ and $\delta > 0$. On the one hand, we have

$$\mathbb{P}(t^{-1/\alpha} Z_t \geq z) \leq \mathbb{P}(t^{-1/\alpha} Z_t \geq z, |\gamma_t - t| \leq \delta t) + \mathbb{P}(|\gamma_t - t| > \delta t),$$

and on the other hand, we have

$$\mathbb{P}(t^{-1/\alpha} Z_t \geq z) \geq \mathbb{P}(|\gamma_t - t| \leq \delta t) - \mathbb{P}(t^{-1/\alpha} Z_t < z, |\gamma_t - t| \leq \delta t).$$

It follows from the strong law of large number that $\mathbb{P}(|\gamma_t - t| > \delta t)$ converges to 0 as $t \rightarrow \infty$. Now on the event $\{|\gamma_t - t| \leq \delta t\}$, we have $\gamma_t \in [(1 - \delta)t, (1 + \delta)t]$, and thus, reminding that $Z_t = X_{\gamma_t}$, we have

$$\mathbb{P}(t^{-1/\alpha} Z_t \geq z, |\gamma_t - t| \leq \delta t) \leq \mathbb{P}\left(\sup_{s \in [1-\delta, 1+\delta]} t^{-1/\alpha} X_{st} \geq z\right),$$

and

$$\mathbb{P}(t^{-1/\alpha} Z_t < z, |\gamma_t - t| \leq \delta t) \leq \mathbb{P}\left(\inf_{s \in [1-\delta, 1+\delta]} t^{-1/\alpha} X_{st} < z\right).$$

The two quantities on the right-hand-side of the above equations converge by Proposition 4.5 to $\mathbb{P}(\sup_{s \in [1-\delta, 1+\delta]} Z_s^\alpha \geq z)$ and $\mathbb{P}(\inf_{s \in [1-\delta, 1+\delta]} Z_s^\alpha < z)$ as t tends to infinity. Putting the pieces together, we have

$$\limsup_{t \rightarrow \infty} \mathbb{P}(t^{-1/\alpha} Z_t \geq z) \leq \mathbb{P}\left(\sup_{s \in [1-\delta, 1+\delta]} Z_s^\alpha \geq z\right).$$

and

$$\liminf_{t \rightarrow \infty} \mathbb{P}(t^{-1/\alpha} Z_t \geq z) \geq \mathbb{P}\left(\inf_{s \in [1-\delta, 1+\delta]} Z_s^\alpha \geq z\right),$$

Since almost surely, 1 is not a jumping time of $(Z_t^\alpha)_{t \geq 0}$, it should be clear that by letting $\delta \rightarrow 0$, we can conclude that $\lim_{t \rightarrow \infty} \mathbb{P}(t^{-1/\alpha} Z_t \geq z) = \mathbb{P}(Z_1^\alpha \geq z)$. $\square$

4.5.3 Proof of Theorem 4.4

Proof of Theorem 4.4. Step 1: As explained above, we start by showing that Theorem 4.4 holds for a solution $(X_t, V_t)_{t \geq 0}$ starting at $(0, 0)$. We need to show that (4.33) holds. Let $(Z_t)_{t \geq 0} = (X_{\gamma_t})_{t \geq 0}$ as in the previous subsection, where $(\gamma_t)_{t \geq 0}$ is the inverse of the local time $(L_t)_{t \geq 0}$ at 0 of $(V_t)_{t \geq 0}$. Since the M_1 -topology is weaker than the J_1 -topology, by Proposition 4.6 and Theorem 4.3, we get, for any $T > 0$, for any $\eta > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(w(\varepsilon^{1/\alpha} Z_{t/\varepsilon}, T, \delta) > \eta) = 0.$$

Now we show that for any $T > 0$, for any $\eta > 0$ and any $\delta \in (0, 1)$,

$$\mathbb{P}(w(\varepsilon^{1/\alpha} X_{t/\varepsilon}, T, \delta) > \eta) \leq \mathbb{P}(w(\varepsilon^{1/\alpha} Z_{t/\varepsilon}, T + 1, 2\delta) > \eta) + \mathbb{P}\left(\sup_{t \in [0, T]} |\varepsilon L_{t/\varepsilon} - t| \geq \delta\right) \quad (4.35)$$

This will achieve the first step by (4.34). We first introduce for $t \geq 0$,

$$g_t = \sup\{s \leq t, V_s = 0\} \quad \text{and} \quad d_t = \inf\{s \geq t, V_s = 0\}.$$

Note that g_t and d_t can be expressed in terms of the local time and its inverse, i.e. $g_t = \gamma_{L_t-}$ and $d_t = \gamma_{L_t}$. We also introduce the random function $\nu(t)$ such that $\nu(t) = 1$ if the excursion straddling the time t is positive and $\nu(t) = -1$ if it is negative, i.e. $\nu(t) = \mathbf{1}_{\{V_t > 0\}} - \mathbf{1}_{\{V_t < 0\}}$.

Let $T > 0$ and $\delta \in (0, 1)$, we first place ourselves on the event $A_{T, \delta} = \{\sup_{t \in [0, T]} |\varepsilon L_{t/\varepsilon} - t| < \delta\}$. Let $t \in [0, T]$ and $t_{\delta-} \leq t_1 < t_2 < t_3 \leq t_{\delta+}$, where $t_{\delta-} = 0 \vee (t - \delta)$ and $t_{\delta+} = T \wedge (t + \delta)$. We also introduce $t_{2\delta-} = 0 \vee (t - 2\delta)$ and $t_{2\delta+} = (T + 1) \wedge (t + 2\delta)$. We emphasize that, since we are on the event $A_{T, \delta}$, $d_{t/\varepsilon} = \gamma_{L_{t/\varepsilon}} \leq \gamma_{(t+\delta)/\varepsilon}$ for every $t \in [0, T]$. We first bound the distance $d(\varepsilon^{1/\alpha} X_{t_2/\varepsilon}, [\varepsilon^{1/\alpha} X_{t_1/\varepsilon}, \varepsilon^{1/\alpha} X_{t_3/\varepsilon}]) = \varepsilon^{1/\alpha} d(X_{t_2/\varepsilon}, [X_{t_1/\varepsilon}, X_{t_3/\varepsilon}])$. Without loss of generality, we will assume that $X_{t_1/\varepsilon} \leq X_{t_3/\varepsilon}$.

- First case: $X_{t_1/\varepsilon} \leq X_{t_2/\varepsilon} \leq X_{t_3/\varepsilon}$. Then we have

$$d(X_{t_2/\varepsilon}, [X_{t_1/\varepsilon}, X_{t_3/\varepsilon}]) = 0 \leq \sup_{t_{2\delta-} \leq t_1 < t_2 < t_3 \leq t_{2\delta+}} d(Z_{t_2/\varepsilon}, [Z_{t_1/\varepsilon}, Z_{t_3/\varepsilon}]).$$

- Second case: $X_{t_2/\varepsilon} < X_{t_1/\varepsilon} \leq X_{t_3/\varepsilon}$. In this case, $d(X_{t_2/\varepsilon}, [X_{t_1/\varepsilon}, X_{t_3/\varepsilon}]) = X_{t_1/\varepsilon} - X_{t_2/\varepsilon}$. Let us note that, since $(X_t)_{t \geq 0}$ is monotonic on every excursion of $(V_t)_{t \geq 0}$, t_1/ε and t_3/ε can not belong to the same excursion, i.e. $d_{t_1/\varepsilon} \leq g_{t_3/\varepsilon}$. We define, for $i \in \{1, 2, 3\}$ and $\varepsilon > 0$, the positive real numbers $u_{i, \varepsilon}$ defined as follows

1. $u_{2, \varepsilon} = g_{t_2/\varepsilon}$ if $\nu(t_2/\varepsilon) = 1$, $u_{2, \varepsilon} = d_{t_2/\varepsilon}$ if $\nu(t_2/\varepsilon) = -1$ and $u_{2, \varepsilon} = t_2/\varepsilon$ if $\nu(t_2/\varepsilon) = 0$. Since $(X_t)_{t \geq 0}$ is monotonic on every excursion of $(V_t)_{t \geq 0}$, we have $X_{t_2/\varepsilon} \geq X_{u_{2, \varepsilon}}$.
2. For $i \in \{1, 3\}$, $u_{i, \varepsilon} = d_{t_i/\varepsilon}$ if $\nu(t_i/\varepsilon) = 1$, $u_{i, \varepsilon} = g_{t_i/\varepsilon}$ if $\nu(t_i/\varepsilon) = -1$ and $u_{i, \varepsilon} = t_i/\varepsilon$ if $\nu(t_i/\varepsilon) = 0$. We have $X_{t_i/\varepsilon} \leq X_{u_{i, \varepsilon}}$.

Therefore, we have $X_{u_{2, \varepsilon}} < X_{u_{1, \varepsilon}}$ and $X_{u_{2, \varepsilon}} < X_{u_{3, \varepsilon}}$ so that necessarily, $u_{1, \varepsilon} < u_{2, \varepsilon} < u_{3, \varepsilon}$. Now we stress that, if $r \geq 0$ is such that $V_r = 0$ (i.e. such that $\nu(r) = 0$), then since the zero set of $(V_t)_{t \geq 0}$ has no isolated points, either $r = d_r$ or $r = g_r$. In any case, we always have for every $i \in \{1, 2, 3\}$, $u_{i, \varepsilon} = \gamma_{L_{t_i/\varepsilon}}$ or $u_{i, \varepsilon} = \gamma_{L_{t_i/\varepsilon}-}$. Let $\theta > 0$ and remember that on the event $A_{T, \delta}$, we have $L_{t_i/\varepsilon} \in ((t_i - \delta)/\varepsilon, (t_i + \delta)/\varepsilon)$ for every $i \in \{1, 2, 3\}$. Using the fact that $(\gamma_t)_{t \geq 0}$ is increasing and that $(X_t)_{t \geq 0}$ is continuous, we can always find $s_1 < s_2 < s_3 \in [t_{2\delta-}, t_{2\delta+}]$ such that

$$X_{t_2/\varepsilon} \geq X_{\gamma_{s_2/\varepsilon}} - \frac{\theta}{2}, \quad X_{t_1/\varepsilon} \leq X_{\gamma_{s_1/\varepsilon}} + \frac{\theta}{2} \quad \text{and} \quad X_{t_3/\varepsilon} \leq X_{\gamma_{s_3/\varepsilon}} + \frac{\theta}{2},$$

which leads to

$$d(X_{t_2/\varepsilon}, [X_{t_1/\varepsilon}, X_{t_3/\varepsilon}]) \leq (Z_{s_1/\varepsilon} - Z_{s_2/\varepsilon}) \wedge (Z_{s_3/\varepsilon} - Z_{s_2/\varepsilon}) + \theta = d(Z_{s_2/\varepsilon}, [Z_{s_1/\varepsilon}, Z_{s_3/\varepsilon}]) + \theta.$$

Since this holds for every $\theta > 0$, we deduce the following bound

$$d(X_{t_2/\varepsilon}, [X_{t_1/\varepsilon}, X_{t_3/\varepsilon}]) \leq \sup_{t_{2\delta-} \leq t_1 < t_2 < t_3 \leq t_{2\delta+}} d(Z_{t_2/\varepsilon}, [Z_{t_1/\varepsilon}, Z_{t_3/\varepsilon}]).$$

- Third case: $X_{t_1/\varepsilon} \leq X_{t_3/\varepsilon} < X_{t_2/\varepsilon}$. We can adapt the previous case to deduce that

$$d(X_{t_2/\varepsilon}, [X_{t_1/\varepsilon}, X_{t_3/\varepsilon}]) \leq \sup_{t_{2\delta-} \leq t_1 < t_2 < t_3 \leq t_{2\delta+}} d(Z_{t_2/\varepsilon}, [Z_{t_1/\varepsilon}, Z_{t_3/\varepsilon}]).$$

To summarize, we proved that, on the event $A_{T,\delta}$, the following bound holds

$$w(\varepsilon^{1/\alpha} X_{t/\varepsilon}, T, \delta) \leq w(\varepsilon^{1/\alpha} Z_{t/\varepsilon}, T + 1, 2\delta).$$

This implies (4.35). We proved Theorem 4.4 in the case $v_0 = 0$.

Step 2: We consider the solution $(X_t, V_t)_{t \geq 0}$ of (4.1) starting at $(0, v_0)$ where $v_0 \in \mathbb{R}$. We show that there exists a constant $C > 0$ such that for any $T > 0$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} |V_t| \right] \leq C(1 + T^{1/2}). \quad (4.36)$$

To this aim, we start by studying $\sup_{t \in [0, T]} |V_t|^{\beta+1}$ and we introduce, recall Assumption 4.1, the even function ℓ defined as

$$\ell(v) = 2 \int_0^v \Theta^{-\beta}(u) \int_0^u \Theta^\beta(w) dw du,$$

which solves the Poisson equation $\ell' F + \frac{1}{2} \ell'' = 1$. Then by the Itô formula, we have

$$\ell(V_t) = \ell(v_0) + t + \int_0^t \ell'(V_s) dB_s.$$

Moreover, remember that $|v| \Theta(v) \rightarrow 1$ as $v \rightarrow \pm\infty$ and that $\beta > 1$. As a consequence, there exist positive constants $c, c' > 0$ such that

$$\ell(v) \sim c|v|^{\beta+1} \quad \text{and} \quad \ell'(v) \sim c' \operatorname{sgn}(v)|v|^\beta \quad \text{as } v \rightarrow \pm\infty.$$

Therefore, there exist positive constants $M, M' > 0$ such that for all $v \in \mathbb{R}$,

$$|v|^{\beta+1} \leq M(1 + \ell(v)) \quad \text{and} \quad [\ell'(v)]^2 \leq M'(1 + v^{2\beta}). \quad (4.37)$$

Using (4.37) and Doob's inequality, we get

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |V_t|^{\beta+1} \right] &\leq M \left[1 + \ell(v_0) + T + 4\mathbb{E} \left[\left(\int_0^T [\ell'(V_s)]^2 ds \right)^{1/2} \right] \right] \\ &\leq M \left[1 + \ell(v_0) + T + 4(M'T)^{1/2} \right] + 4M(M'T)^{1/2} \mathbb{E} \left[\sup_{t \in [0, T]} |V_t|^\beta \right]. \end{aligned} \quad (4.38)$$

Finally, using Young's inequality with $p = \beta + 1$ and $q = (\beta + 1)/\beta$, we find some $C > 0$ such that

$$\begin{aligned} 4M(M'T)^{1/2}\mathbb{E}\left[\sup_{t\in[0,T]}|V_t|^\beta\right] &\leq CT^{(\beta+1)/2} + \frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]}|V_t|^\beta\right]^{(\beta+1)/\beta} \\ &\leq CT^{(\beta+1)/2} + \frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]}|V_t|^{\beta+1}\right], \end{aligned}$$

which, inserted in (4.38), implies that there exists a constant $K > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|V_t|^{\beta+1}\right] \leq K(1 + T^{(\beta+1)/2}).$$

We deduce (4.36) by Hölder's inequality again.

Step 3: We finally show the result for any solution $(X_t, V_t)_{t\geq 0}$ starting at $(0, v_0)$, where $v_0 \in \mathbb{R}$, and we extend the technique used in [FT21, page 21]. If we set $T_0 = \inf\{t \geq 0, V_t = 0\}$, then the process

$$(\bar{X}_t, \bar{V}_t)_{t\geq 0} = (X_{t+T_0} - X_{T_0}, V_{t+T_0})_{t\geq 0}$$

is a solution of (4.1) starting at $(0, 0)$. Therefore, by the first step, the process $(\varepsilon^{1/\alpha} \bar{X}_{t/\varepsilon})_{t\geq 0}$ converges to $(Z_t^\alpha)_{t\geq 0}$ in the space $\mathcal{D}$ endowed with the $\mathbf{M}_1$ -topology. Then, by a version of the Slutsky lemma, see for instance [Bil99, Section 3, Theorem 3.1], it is enough to show that for any $T > 0$,

$$\sup_{t\in[0,T]} \varepsilon^{1/\alpha} \left| X_{t/\varepsilon} - \bar{X}_{t/\varepsilon} \right| \xrightarrow{\mathbb{P}} 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (4.39)$$

Indeed, the $\mathbf{M}_1$ -topology is weaker than the topology induced by the uniform convergence on compact time-intervals. We distinguish two cases. First,

$$\mathbf{1}_{\{T_0 \geq t/\varepsilon\}} \left| X_{t/\varepsilon} - \bar{X}_{t/\varepsilon} \right| \leq \mathbf{1}_{\{T_0 \geq t/\varepsilon\}} \left| X_{t/\varepsilon} \right| + \mathbf{1}_{\{T_0 \geq t/\varepsilon\}} \left| X_{t/\varepsilon+T_0} - X_{T_0} \right| \leq \int_0^{2T_0} |V_s| ds = D_1.$$

Second,

$$\mathbf{1}_{\{T_0 < t/\varepsilon\}} \left| X_{t/\varepsilon} - \bar{X}_{t/\varepsilon} \right| \leq |X_{T_0}| + \mathbf{1}_{\{T_0 < t/\varepsilon\}} \left| X_{t/\varepsilon+T_0} - X_{t/\varepsilon} \right| \leq D_1 + \mathbf{1}_{\{T_0 < t/\varepsilon\}} \int_{t/\varepsilon}^{t/\varepsilon+T_0} |V_s| ds.$$

Hence, if we set $D_{t,\varepsilon}^2 = \mathbf{1}_{\{T_0 < t/\varepsilon\}} \int_{t/\varepsilon}^{t/\varepsilon+T_0} |V_s| ds$, we get

$$\sup_{t\in[0,T]} \varepsilon^{1/\alpha} \left| X_{t/\varepsilon} - \bar{X}_{t/\varepsilon} \right| \leq \varepsilon^{1/\alpha} D_1 + \sup_{t\in[0,T]} \varepsilon^{1/\alpha} D_{t,\varepsilon}^2.$$

The first term converges almost surely to 0 as $\varepsilon \rightarrow 0$. Regarding the second one, we have

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,T]} \varepsilon^{1/\alpha} D_{t,\varepsilon}^2 \middle| \mathcal{F}_{T_0}\right] &\leq \varepsilon^{1/\alpha} T_0 \mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{1}_{\{T_0 < t/\varepsilon\}} \sup_{s\in[t/\varepsilon, t/\varepsilon+T_0]} |V_s| \middle| \mathcal{F}_{T_0}\right] \\ &\leq \varepsilon^{1/\alpha} T_0 \mathbf{1}_{\{T_0 < T/\varepsilon\}} \mathbb{E}\left[\sup_{t\in[0, 2T/\varepsilon]} |V_s|\right] \\ &\leq \varepsilon^{1/\alpha} T_0 C(1 + (2T)^{1/2} \varepsilon^{-1/2}) \end{aligned}$$

by Step 2. This last quantity almost surely goes to 0 as $\alpha \in (0, 2)$. This implies the convergence in probability of $\sup_{t\in[0,T]} \varepsilon^{1/\alpha} D_{t,\varepsilon}^2$ to 0, so that (4.39) holds. $\square$

Appendices to Chapter 4

4.A Some useful results

Lemma 4.12. *Let X and Y be two positive random variables and let $C > 0$. Assume that we have $\mathbb{P}(X > t) \sim Ct^{-1/2}$ as $t \rightarrow \infty$. Then the following assertions are equivalent.*

- (i) $\lim_{t \rightarrow \infty} t^{1/2} \mathbb{P}(Y > t) = 0$.
- (ii) $\mathbb{P}(X + Y > t) \sim Ct^{-1/2}$ as $t \rightarrow \infty$.

Proof. Let us first show that (i) implies (ii). Let $\delta \in (0, 1)$ and write

$$\begin{aligned} \mathbb{P}(X + Y > t) &= \mathbb{P}(X + Y > t, X > \delta t) + \mathbb{P}(X + Y > t, X \leq \delta t) \\ &\leq \mathbb{P}(X > \delta t) + \mathbb{P}(Y > (1 - \delta)t). \end{aligned}$$

We deduce that

$$C = \liminf_{t \rightarrow \infty} t^{1/2} \mathbb{P}(X > t) \leq \liminf_{t \rightarrow \infty} t^{1/2} \mathbb{P}(X + Y > t) \leq \limsup_{t \rightarrow \infty} t^{1/2} \mathbb{P}(X + Y > t) \leq C\delta^{-1/2}.$$

Letting $\delta \rightarrow 1$ completes the first step.

We now show that (ii) implies (i). We first remark that

$$\{Y > 3t\} \subset \bigcup_{n \in \mathbb{N}} \{X + Y > (n + 1)t\} \cap \{X \leq nt\}.$$

Indeed, if $Y > 3t$, we set $n + 2 = \lfloor \frac{X+Y}{t} \rfloor \geq 3$, and get $\frac{X}{t} \leq \frac{X+Y}{t} - 3 \leq n$ and $\frac{X+Y}{t} > n + 1$. Now let $N \in \mathbb{N}$, then we have $(\cup_{n > N} \{X + Y > (n + 1)t\} \cap \{X \leq nt\}) \subset \{X + Y > (N + 2)t\}$ and therefore

$$\mathbb{P}(Y > 3t) \leq \mathbb{P}(X + Y > (N + 2)t) + \sum_{n=1}^N \mathbb{P}(X + Y > (n + 1)t, X \leq nt).$$

For any $n \geq 1$, we have

$$\begin{aligned} \mathbb{P}(X + Y > (n + 1)t, X \leq nt) &= \mathbb{P}(X + Y > (n + 1)t) - \mathbb{P}(X + Y > (n + 1)t, X > nt) \\ &\leq \mathbb{P}(X + Y > (n + 1)t) - \mathbb{P}(X > (n + 1)t), \end{aligned}$$

and thus $\lim_{t \rightarrow \infty} t^{1/2} \mathbb{P}(X + Y > (n + 1)t, X \leq nt) = 0$, from which we deduce that for any $N \in \mathbb{N}$,

$$\limsup_{t \rightarrow \infty} t^{1/2} \mathbb{P}(Y > 3t) \leq C(N + 2)^{-1/2}.$$

Letting $N \rightarrow \infty$ completes the proof. □

Proposition 4.7. *Let $(X_n)_{n \in \mathbb{N}}$ be i.i.d positive random variables and let $\alpha \in (0, 1)$ such that $\lim_{t \rightarrow \infty} t^\alpha \mathbb{P}(X_1 > t) = 0$. If $S_n = \sum_{k=1}^n X_k$, then $n^{-1/\alpha} S_n$ converges to 0 in probability as $n \rightarrow \infty$.*

Proof. We will show that the Laplace transform of $n^{-1/\alpha} S_n$ converges to 1 as $n \rightarrow \infty$. We have for any $\lambda > 0$,

$$\mathbb{E} \left[e^{-\lambda n^{-1/\alpha} S_n} \right] = \mathbb{E} \left[e^{-\lambda n^{-1/\alpha} X_1} \right]^n,$$

and thus

$$\log \mathbb{E} \left[e^{-\lambda n^{-1/\alpha} S_n} \right] \underset{n \rightarrow \infty}{\sim} n \mathbb{E} \left[e^{-\lambda n^{-1/\alpha} X_1} - 1 \right] = \frac{-\lambda}{n^{1/\alpha-1}} \int_0^\infty e^{-\lambda u/n^{1/\alpha}} \mathbb{P}(X_1 > u) du.$$

Then we use that for $u \geq n^{1/\alpha}$, $\mathbb{P}(X_1 > u) \leq \mathbb{P}(X_1 > n^{1/\alpha})$ and we get

$$\frac{\lambda}{n^{1/\alpha-1}} \int_0^\infty e^{-\lambda u/n^{1/\alpha}} \mathbb{P}(X_1 > u) du \leq \frac{\lambda}{n^{1/\alpha-1}} \int_0^{n^{1/\alpha}} \mathbb{P}(X_1 > u) du + e^{-\lambda} n \mathbb{P}(X_1 > n^{1/\alpha}).$$

The second term on the right-hand-side converges to 0 by assumption. Regarding the first term, since $\mathbb{P}(X_1 > u) = o(u^{-\alpha})$ as $u \rightarrow \infty$ and since $\lim_{x \rightarrow \infty} \int_1^x u^{-\alpha} du = \infty$, we have $\int_0^{n^{1/\alpha}} \mathbb{P}(X_1 > u) du = o(\int_1^{n^{1/\alpha}} u^{-\alpha} du) = o(n^{1/\alpha-1})$ as $n \rightarrow \infty$, which completes the proof. $\square$

4.B On the PDE result

In this subsection, we formalize the P.D.E. result briefly exposed in the introduction. We first explain how the law of $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ is linked with a kinetic Fokker-Planck equation with diffusive boundary conditions. For every $\varphi \in C_b^2(\mathbb{R})$, we define $\mathcal{L}\varphi = F\varphi' + \frac{1}{2}\varphi''$. Then $\mathcal{L}$ is the infinitesimal generator of the (free) speed process $(V_t)_{t \geq 0}$. We also denote by $\mathcal{L}^*$ its adjoint operator which is such that $\mathcal{L}^*\varphi = \frac{1}{2}\varphi'' - [F\varphi]'$.

Proposition 4.8. *Let $(\mathbf{X}_t, \mathbf{V}_t)_{t \geq 0}$ be a solution to (4.3) starting at $(x_0, v_0) \in ((0, \infty) \times \mathbb{R}) \cup (\{0\} \times (0, \infty))$. Let us denote by $f(dt, dx, dv) = \mathbb{P}(\mathbf{X}_t \in dx, \mathbf{V}_t \in dv)dt$ which is a measure on $\mathbb{R}_+^2 \times \mathbb{R}$. There exist two measures $\nu_- \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}_-)$ and $\nu_+ \in \mathcal{M}(\mathbb{R}_+^2)$ such that for every $\varphi \in C_c^\infty(\mathbb{R}_+^2 \times \mathbb{R})$, we have*

$$\begin{aligned} \varphi(0, x_0, v_0) + \int_{\mathbb{R}_+^2 \times \mathbb{R}} [\partial_t \varphi + v \partial_x \varphi + \mathcal{L}\varphi] f(ds, dx, dv) \\ + \int_{\mathbb{R}_+^2} \varphi(s, 0, v) \nu_+(ds, dv) - \int_{\mathbb{R}_+ \times \mathbb{R}_-} \varphi(s, 0, v) \nu_-(ds, dv) = 0. \end{aligned} \quad (4.40)$$

Moreover the measures ν_- and ν_+ satisfy $\nu_+(dt, dv) = \mu(dv) \int_{w \in \mathbb{R}_-} \nu_-(dt, dw)$.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}_+^2 \times \mathbb{R})$, then by Itô's formula, and passing to the expectation, we get for every $T > 0$ that

$$\begin{aligned} \mathbb{E} [\varphi(T, \mathbf{X}_T, \mathbf{V}_T)] &= \varphi(0, x_0, v_0) + \int_0^T \mathbb{E} [(\partial_t \varphi + v \partial_x \varphi + \mathcal{L}\varphi)(s, \mathbf{X}_s, \mathbf{V}_s)] ds \\ &\quad + \sum_{n \in \mathbb{N}} \mathbb{E} \left[(\varphi(\tau_n, 0, M_n) - \varphi(\tau_n, 0, \mathbf{V}_{\tau_n-})) \mathbf{1}_{\{\tau_n \leq t\}} \right]. \end{aligned}$$

The local martingale part $\int_0^T \partial_v \varphi(s, \mathbf{X}_s, \mathbf{V}_s) dB_s$ is indeed a true martingale since $\partial_v \varphi$ is bounded. Let us now define the measures $\nu_- \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}_-)$ and $\nu_+ \in \mathcal{M}(\mathbb{R}_+^2)$ by

$$\nu_-(dt, dv) = \sum_{n \in \mathbb{N}} \mathbb{P}(\tau_n \in dt, \mathbf{V}_{\tau_n-} \in dv) \quad \text{and} \quad \nu_+(dt, dv) = \sum_{n \in \mathbb{N}} \mathbb{P}(\tau_n \in dt, M_n \in dv).$$

Notice that for any $T > 0$, we have $\nu_-([0, T] \times \mathbb{R}_-) = \nu_+([0, T] \times \mathbb{R}_+) = \mathbb{E}[\sum_{n \in \mathbb{N}} \mathbf{1}_{\{\tau_n \leq T\}}]$, which is finite since $(\tau_{n+1} - \tau_n)_{n \geq 1}$ is an i.i.d. sequence of positive random variables. This also justifies the above exchange between $\mathbb{E}$ and $\sum$. Therefore we have for any $T > 0$

$$\begin{aligned} \mathbb{E}[\varphi(T, \mathbf{X}_T, \mathbf{V}_T)] &= \varphi(0, x_0, v_0) + \int_{\mathbb{R}_+^2 \times \mathbb{R}} [\partial_t \varphi + v \partial_x \varphi + \mathcal{L} \varphi] \mathbf{1}_{\{s \leq T\}} f(ds, dx, dv) \\ &\quad + \int_{\mathbb{R}_+^2} \varphi(s, 0, v) \mathbf{1}_{\{s \leq T\}} \nu_+(ds, dv) - \int_{\mathbb{R}_+ \times \mathbb{R}_-} \varphi(s, 0, v) \mathbf{1}_{\{s \leq T\}} \nu_-(ds, dv). \end{aligned}$$

Choosing T large enough so that the support of φ is included in $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$, we get the desired identity.

The relation between the two measures comes from the fact that for every $n \in \mathbb{N}$, τ_n and M_n are independent and that M_n is μ -distributed. Indeed it is clear that, since $(M_n)_{n \in \mathbb{N}}$ is i.i.d. and also independent from the driving Brownian motion, M_n is independent from $(\mathbf{X}_t, \mathbf{V}_t)_{0 \leq t < \tau_n}$ for every $n \in \mathbb{N}$. Hence we have

$$\nu_+(dt, dv) = \mu(dv) \sum_{n \in \mathbb{N}} \mathbb{P}(\tau_n \in dt) = \mu(dv) \int_{w \in \mathbb{R}_-} \nu_-(dt, dw),$$

which achieves the proof. $\square$

Remark 4.1. Assume for simplicity that $\mu(dv) = \mu(v)dv$. The preceding proposition shows that f is a weak solution of

$$\begin{cases} \partial_t f + v \partial_x f = \mathcal{L}^* f & \text{for } (t, x, v) \in (0, \infty)^2 \times \mathbb{R} \\ v f(t, 0, v) = -\mu(v) \int_{(-\infty, 0)} w f(t, 0, w) dw & \text{for } (t, v) \in (0, \infty)^2 \\ f(0, \cdot, \cdot) = \delta_{(0, v_0)} \end{cases}$$

Informally, it automatically holds that $\nu_+(ds, dv) = v f(s, 0, v) \mathbf{1}_{\{v > 0\}} ds dv$ and $\nu_-(ds, dv) = -v f(s, 0, v) \mathbf{1}_{\{v < 0\}} ds dv$.

For similar notions of weak solutions associated to closely related equations, we refer to Jabir-Profeta [JP19, Theorem 4.2.1] and Bernou-Fournier [BF22, Definition 4].

Proof. We assume that $f(dt, dx, dv) = f(t, x, v) dx dv dt$ with f smooth enough. Let φ be a function belonging to $C_c^\infty(\mathbb{R}_+^2 \times \mathbb{R})$. We perform some integrations by parts. We have

$$\int_{\mathbb{R}_+^2 \times \mathbb{R}} f \partial_t \varphi = -\varphi(0, x_0, v_0) - \int_{\mathbb{R}_+^2 \times \mathbb{R}} \varphi \partial_t f \quad \text{and} \quad \int_{\mathbb{R}_+^2 \times \mathbb{R}} f \mathcal{L} \varphi = \int_{\mathbb{R}_+^2 \times \mathbb{R}} \varphi \mathcal{L}^* f.$$

Regarding the integration by part in x , we have

$$\int_{\mathbb{R}_+^2 \times \mathbb{R}} v f \partial_x \varphi = - \int_{\mathbb{R}_+^2 \times \mathbb{R}} \varphi v \partial_x f - \int_{\mathbb{R}_+} \int_{\mathbb{R}} v \varphi(s, 0, v) f(s, 0, v) dv ds.$$

Inserting the previous identities in (4.40), it comes that

$$\begin{aligned} \int_{\mathbb{R}_+^2 \times \mathbb{R}} [\partial_t f + v \partial_x f - \mathcal{L}^* f] \varphi &+ \int_{\mathbb{R}_+} \int_{\mathbb{R}} v \varphi(s, 0, v) f(s, 0, v) dv ds \\ &= \int_{\mathbb{R}_+^2} \varphi(s, 0, v) \nu_+(ds, dv) - \int_{\mathbb{R}_+ \times \mathbb{R}_-} \varphi(s, 0, v) \nu_-(ds, dv). \end{aligned}$$

Since this holds for every $\varphi \in C_c^\infty(\mathbb{R}_+ \times (0, \infty) \times \mathbb{R})$, we first conclude that $\partial_t f + v \partial_x f = \mathcal{L}^* f$ for $(t, x, v) \in (0, \infty)^2 \times \mathbb{R}$. Then in a second time, we see that

$$vf(s, 0, v)dsdv = \nu_+(ds, dv) - \nu_-(ds, dv),$$

i.e. $\nu_+(ds, dv) = vf(s, 0, v)\mathbf{1}_{\{v>0\}}dsdv$ and $\nu_-(ds, dv) = -vf(s, 0, v)\mathbf{1}_{\{v<0\}}dsdv$. Then, since $\nu_+(ds, dv) = \mu(dv) \int_{w \in \mathbb{R}_-} \nu_-(ds, dw)$, we conclude that $vf(t, 0, v) = -\mu(v) \int_{(-\infty, 0)} wf(t, 0, w)dw$ for $(t, v) \in (0, \infty)^2$. $\square$

We finally study the limiting fractional diffusion equation. We have the following result.

Proposition 4.9. *Let $(R_t^\alpha)_{t \geq 0} = (Z_t^\alpha - \inf_{s \in [0, t]} Z_s^\alpha)_{t \geq 0}$ where $(Z_t^\alpha)_{t \geq 0}$ is the stable process from Theorem 4.1. Let us denote by $\rho_t(dx) = \mathbb{P}(R_t^\alpha \in dx)$. The following assertions hold.*

(i) *If $\alpha \in (0, 1)$, then for every $\varphi \in C_c^\infty(\mathbb{R}_+)$, we have*

$$\int_{\mathbb{R}_+} \varphi(x) \rho_t(dx) = \varphi(0) + \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx) ds,$$

where

$$\mathcal{L}^\alpha \varphi(x) = \frac{\sigma_\alpha}{2} \int_{\mathbb{R}} \frac{\varphi((x+z)_+) - \varphi(x)}{|z|^{1+\alpha}} dz = \frac{\sigma_\alpha}{2\alpha} \int_0^\infty \frac{\varphi'(y)(y-x)}{|y-x|^{\alpha+1}} dy.$$

(ii) *If $\alpha \in [1, 2)$, then for every $\varphi \in C_c^\infty(\mathbb{R}_+)$ such that $\varphi'(0) = 0$, we have*

$$\int_{\mathbb{R}_+} \varphi(x) \rho_t(dx) = \varphi(0) + \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx) ds,$$

where

$$\mathcal{L}^\alpha \varphi(x) = \frac{\sigma_\alpha}{2} \text{P.V.} \int_{\mathbb{R}} \frac{\varphi((x+z)_+) - \varphi(x)}{|z|^{1+\alpha}} dz = \frac{\sigma_\alpha}{2\alpha} \text{P.V.} \int_0^\infty \frac{\varphi'(y)(y-x)}{|y-x|^{\alpha+1}} dy,$$

where P.V. stands for principal values.

Proof. We will denote by $\nu(dz) = \frac{\sigma_\alpha}{2} |z|^{-\alpha-1} dz$ the Lévy measure of the symmetric stable process $(Z_t^\alpha)_{t \geq 0}$ and we set $I_t^\alpha = \inf_{s \in [0, t]} Z_s^\alpha$.

Item (i): Let us denote by $\Pi(dt, dz)$ the random Poisson measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $dt \otimes \nu(dz)$, associated with $(Z_t^\alpha)_{t \geq 0}$. Since $\alpha \in (0, 1)$, $(Z_t^\alpha)_{t \geq 0}$ has finite variations and we have $Z_t^\alpha = \int_0^t \int_{\mathbb{R}} z \Pi(ds, dz)$. The process $(R_t^\alpha)_{t \geq 0}$ is also a pure jump Markov process and we check that $\Delta R_t^\alpha = (R_{t-}^\alpha + \Delta Z_t^\alpha)_+ - R_{t-}^\alpha$. If first $\Delta Z_t^\alpha > -R_{t-}^\alpha = -Z_{t-}^\alpha + I_{t-}^\alpha$, then $Z_t^\alpha > I_{t-}^\alpha$ so that $I_t^\alpha = I_{t-}^\alpha$ and $\Delta R_t^\alpha = \Delta Z_t^\alpha$. If next $\Delta Z_t^\alpha \leq -R_{t-}^\alpha$, then $Z_t^\alpha \leq I_{t-}^\alpha$ so that $I_t^\alpha = Z_t^\alpha$ and thus $R_t^\alpha = 0$ i.e. $\Delta R_t^\alpha = -R_{t-}^\alpha$. Therefore, we have $R_t^\alpha = \int_0^t \int_{\mathbb{R}} [(R_{s-}^\alpha + z)_+ - R_{s-}^\alpha] \Pi(ds, dz)$. By Itô's formula, we get that for any $\varphi \in C_c^\infty(\mathbb{R}_+)$,

$$\mathbb{E}[\varphi(R_t^\alpha)] = \varphi(0) + \int_0^t \int_{\mathbb{R}} \mathbb{E}[\varphi((R_s^\alpha + z)_+) - \varphi(R_s^\alpha)] \nu(dx) ds,$$

which exactly means $\int_{\mathbb{R}_+} \varphi(x) \rho_t(dx) = \varphi(0) + \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx) ds$, where the operator $\mathcal{L}^\alpha$ is defined as $\mathcal{L}^\alpha \varphi(x) = \frac{\sigma_\alpha}{2} \int_{\mathbb{R}} [\varphi((x+z)_+) - \varphi(x)] |z|^{-1-\alpha} dz$. It only remains to prove the second identity for $\mathcal{L}^\alpha$. We assume that $x > 0$, the proof for $x = 0$ being similar. We have

$$\frac{2}{\sigma_\alpha} \mathcal{L}^\alpha \varphi(x) = \frac{\varphi(0) - \varphi(x)}{\alpha} x^{-\alpha} + \int_{-x}^0 \frac{\varphi(x+z) - \varphi(x)}{|z|^{1+\alpha}} dz + \int_0^\infty \frac{\varphi(x+z) - \varphi(x)}{|z|^{1+\alpha}} dz$$

Then, performing carefully two integration by parts, we see that

$$\int_{-x}^0 \frac{\varphi(x+z) - \varphi(x)}{|z|^{1+\alpha}} dz = -\frac{\varphi(0) - \varphi(x)}{\alpha} x^{-\alpha} + \frac{1}{\alpha} \int_{-x}^0 \frac{\varphi'(x+z)z}{|z|^{1+\alpha}} dz$$

and

$$\int_0^\infty \frac{\varphi(x+z) - \varphi(x)}{|z|^{1+\alpha}} dz = \frac{1}{\alpha} \int_0^\infty \frac{\varphi'(x+z)z}{|z|^{1+\alpha}} dz.$$

Putting the pieces together, we get

$$\frac{2}{\sigma_\alpha} \mathcal{L}^\alpha \varphi(x) = \frac{1}{\alpha} \int_{-x}^\infty \frac{\varphi'(x+z)z}{|z|^{1+\alpha}} dz = \frac{1}{\alpha} \int_0^\infty \frac{\varphi'(y)(y-x)}{|y-x|^{1+\alpha}} dy.$$

Item (ii): When $\alpha \in [1, 2)$, the second expression of $\mathcal{L}^\alpha$ is obtained as in the previous case. We need to remove the small jumps and work with the pure jump Lévy process $(Z_t^{\alpha, \delta})_{t \geq 0}$ with Lévy measure $\nu_\delta(dz) = \mathbf{1}_{\{|z| > \delta\}} \nu(dz)$. Since ν is symmetric, $Z_t^{\alpha, \delta} \rightarrow Z_t^\alpha$ in law as $\delta \rightarrow 0$ for every $t \geq 0$. This implies, see [JS03, Chapter VII, Corollary 3.6], that $(Z_t^{\alpha, \delta})_{t \geq 0}$ converges in law to $(Z_t^\alpha)_{t \geq 0}$ in the space of càdlàg functions endowed with the $\mathbf{J}_1$ -topology as $\delta \rightarrow 0$. But since the reflection map is continuous with respect to this topology, see [Whi02, Chapter 13, Theorem 13.5.1], the continuous mapping theorem implies that $(R_t^{\alpha, \delta})_{t \geq 0} = (Z_t^{\alpha, \delta} - \inf_{s \in [0, t]} Z_s^{\alpha, \delta})_{t \geq 0}$ converges weakly to $(R_t^\alpha)_{t \geq 0}$ as $\delta \rightarrow 0$. In particular, if we set $\rho_t^\delta(dx) = \mathbb{P}(R_t^{\alpha, \delta} \in dx)$, then the probability measure ρ_t^δ converges weakly to ρ_t as $\delta \rightarrow 0$.

But since $(R_t^{\alpha, \delta})_{t \geq 0}$ has finite variations, we can use the very same argument as in the first step to see that for every function $\varphi \in C_c^\infty(\mathbb{R}_+)$, we have

$$\int_{\mathbb{R}_+} \varphi(x) \rho_t^\delta(dx) = \varphi(0) + \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^{\alpha, \delta} \varphi(x) \rho_s^\delta(dx) ds,$$

where $\mathcal{L}^{\alpha, \delta}$ is such that for every $x \geq 0$

$$\mathcal{L}^{\alpha, \delta} \varphi(x) = \frac{\sigma_\alpha}{2} \int_{|z| > \delta} \frac{\varphi((x+z)_+) - \varphi(x)}{|z|^{1+\alpha}} dz.$$

First, it is clear that $\int_{\mathbb{R}_+} \varphi(x) \rho_t^\delta(dx)$ converges to $\int_{\mathbb{R}_+} \varphi(x) \rho_t(dx)$ as $\delta \rightarrow 0$. We will now conclude by showing that, if $\varphi'(0) = 0$,

$$\int_0^t \int_{\mathbb{R}_+} \mathcal{L}^{\alpha, \delta} \varphi(x) \rho_s^\delta(dx) ds \longrightarrow \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx) ds \quad \text{as } \delta \rightarrow 0. \quad (4.41)$$

Step 1: To do so, we show in Step 2 that there exists a positive constant C_φ such that

$$\|\mathcal{L}^\alpha \varphi - \mathcal{L}^{\alpha, \delta} \varphi\|_\infty \leq C_\varphi \delta^{2-\alpha}. \quad (4.42)$$

This shows that $\mathcal{L}^\alpha \varphi$ is a continuous and bounded function. Moreover, since

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx) ds - \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^{\alpha, \delta} \varphi(x) \rho_s^\delta(dx) ds \right| \\ & \leq \int_0^t \int_{\mathbb{R}_+} |\mathcal{L}^\alpha \varphi(x) - \mathcal{L}^{\alpha, \delta} \varphi(x)| \rho_s^\delta(dx) ds \\ & \quad + \left| \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s^\delta(dx) ds - \int_0^t \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx) ds \right|, \end{aligned}$$

it is clear that (4.42) implies (4.41). Indeed the first term clearly goes to 0 as $\delta \rightarrow 0$ and the second term converges since for every $s \in [0, t]$, $\int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s^\delta(dx) \rightarrow \int_{\mathbb{R}_+} \mathcal{L}^\alpha \varphi(x) \rho_s(dx)$ as $\mathcal{L}^\alpha \varphi$ is a continuous and bounded function and we can thus apply the dominated convergence theorem.

Step 2: We show that (4.42) holds. Let us first explicit a bit more $\mathcal{L}^\alpha \varphi(x)$. Let us remark that since $\varphi'(0) = 0$, there is no need for principal values for $\mathcal{L}^\alpha \varphi(0) = \int_0^\infty (\varphi(z) - \varphi(0))|z|^{-1-\alpha} dz$. If $x > 0$, we have for any $\varepsilon < x$, $\int_{-\varepsilon}^\varepsilon z|z|^{-1-\alpha} dz + \int_\varepsilon^x z|z|^{-1-\alpha} dz = 0$ and thus we can write

$$\mathcal{L}^\alpha \varphi(x) = \int_{\mathbb{R}} \frac{\varphi((x+z)_+) - \varphi(x) - z\varphi'(x)\mathbf{1}_{\{|z|<x\}}}{|z|^{1+\alpha}} dz. \quad (4.43)$$

For the very same reason, the same identity holds for $\mathcal{L}^{\alpha,\delta} \varphi(x)$ and therefore, for any $x > 0$

$$\begin{aligned} \mathcal{L}^\alpha \varphi(x) - \mathcal{L}^{\alpha,\delta} \varphi(x) &= \int_{|z|<\delta} \frac{\varphi((x+z)_+) - \varphi(x) - z\varphi'(x)\mathbf{1}_{\{|z|<x\}}}{|z|^{1+\alpha}} dz \\ &= \frac{\varphi(0) - \varphi(x)}{\alpha} (x^{-\alpha} - \delta^{-\alpha}) \mathbf{1}_{\{\delta>x\}} \\ &\quad + \int_{-(\delta \wedge x)}^\delta \frac{\varphi(x+z) - \varphi(x) - z\varphi'(x)\mathbf{1}_{\{|z|<x\}}}{|z|^{1+\alpha}} dz. \end{aligned}$$

Since $\varphi \in C_c^\infty(\mathbb{R}_+)$ and $\varphi'(0) = 0$, it is clear that if we set $D_\varphi = \|\varphi''\|_\infty$ we have for any $x \geq 0$ and any $z \geq -x$, $|\varphi(x+z) - \varphi(x) - z\varphi'(x)| \leq D_\varphi z^2/2$. Since $\varphi'(0) = 0$, we also have for any $x \geq 0$, $|\varphi'(x)| \leq D_\varphi |x|$, from which we get that $|\varphi(x+z) - \varphi(x)| \leq D_\varphi |x||z| + D_\varphi z^2$. In any case, we see that for any $x \geq 0$ and any $z \geq -x$

$$|\varphi(x+z) - \varphi(x) - z\varphi'(x)\mathbf{1}_{\{|z|<x\}}| \leq 2D_\varphi z^2,$$

It comes that

$$\left| \int_{-(\delta \wedge x)}^\delta \frac{\varphi(x+z) - \varphi(x) - z\varphi'(x)\mathbf{1}_{\{|z|<x\}}}{|z|^{1+\alpha}} dz \right| \leq \frac{4D_\varphi}{2-\alpha} \delta^{2-\alpha}.$$

We also get

$$\left| \frac{\varphi(0) - \varphi(x)}{\alpha} (x^{-\alpha} - \delta^{-\alpha}) \mathbf{1}_{\{\delta>x\}} \right| \leq \frac{D_\varphi}{\alpha} x^{2-\alpha} \mathbf{1}_{\{\delta>x\}} \leq \frac{D_\varphi}{\alpha} \delta^{2-\alpha}.$$

All in all, we showed that for any $x > 0$, $|\mathcal{L}^\alpha \varphi(x) - \mathcal{L}^{\alpha,\delta} \varphi(x)| \leq C_\varphi \delta^{2-\alpha}$. When $x = 0$, we have

$$|\mathcal{L}^\alpha \varphi(0) - \mathcal{L}^{\alpha,\delta} \varphi(0)| = \left| \int_0^\delta \frac{\varphi(z) - \varphi(0)}{|z|^{1+\alpha}} dz \right| \leq \frac{D_\varphi}{2-\alpha} \delta^{2-\alpha} \leq C_\varphi \delta^{2-\alpha}.$$

This shows that (4.42) holds. □

Remark 4.2. The above proposition shows that ρ is a weak solution of

$$\begin{cases} \partial_t \rho_t(x) = \frac{\sigma_\alpha}{2} \int_{\mathbb{R}} \frac{\rho_t(x-z)\mathbf{1}_{\{x>z\}} - \rho_t(x) + z\partial_x \rho_t(x)\mathbf{1}_{\{|z|<x\}}}{|z|^{1+\alpha}} dz & \text{for } (t, x) \in (0, \infty)^2, \\ \int_0^\infty \rho_t(x) dx = 1 & \text{for } t \in (0, \infty), \\ \rho_0(\cdot) = \delta_0. \end{cases}$$

The term $z\partial_x \rho_t(x)\mathbf{1}_{\{|z|<x\}}$ is useless when $\alpha \in (0, 1)$.

Proof. As usual, we do as if ρ_t was sufficiently regular so that all the computations below hold true. By the way, it might actually be the case, see for instance Chaumont-Malecki [CMe20], but this is not our purpose. Consider a function $\varphi \in C_c^\infty((0, \infty))$. Then, by Proposition 4.9, we have that

$$\frac{d}{dt} \int_{\mathbb{R}_+} \varphi(x) \rho_t(x) dx = \int_{\mathbb{R}_+} \rho_t(x) \mathcal{L}^\alpha \varphi(x) dx = \frac{\sigma_\alpha}{2} [I_t(\varphi) - J_t(\varphi)],$$

where, recalling (4.43),

$$I_t(\varphi) = \int_{\mathbb{R}_+} \int_{-x}^{\infty} \rho_t(x) \frac{\varphi(x+z) - \varphi(x) - z\varphi'(x) \mathbf{1}_{\{|z|<x\}}}{|z|^{1+\alpha}} dz dx$$

and, since $\varphi(0) = 0$,

$$J_t(\varphi) = \int_{\mathbb{R}_+} \frac{\varphi(x) \rho_t(x)}{\alpha x^\alpha}.$$

We first focus on $I_t(\varphi)$. Exchanging integrals, we get

$$I_t(\varphi) = \int_{\mathbb{R}} |z|^{-1-\alpha} \int_{(-z) \vee 0}^{\infty} \rho_t(x) [\varphi(x+z) - \varphi(x) - z\varphi'(x) \mathbf{1}_{\{|z|<x\}}] dx dz.$$

We next write, for any $z \in \mathbb{R}$,

$$\begin{aligned} & \int_{(-z) \vee 0}^{\infty} \rho_t(x) [\varphi(x+z) - \varphi(x) - z\varphi'(x) \mathbf{1}_{\{|z|<x\}}] dx \\ &= \int_{z \vee 0}^{\infty} \rho_t(x-z) \varphi(x) dx - \int_{(-z) \vee 0}^{\infty} \rho_t(x) \varphi(x) dx - z \int_{(-z) \vee 0}^{\infty} \rho_t(x) \varphi'(x) \mathbf{1}_{\{|z|<x\}} dx. \end{aligned}$$

Regarding the third term, we have $\int_{(-z) \vee 0}^{\infty} \rho_t(x) \varphi'(x) \mathbf{1}_{\{|z|<x\}} dx = \int_{|z|}^{\infty} \rho_t(x) \varphi'(x) dx$, so that, performing an integration by part, we have

$$z \int_{(-z) \vee 0}^{\infty} \rho_t(x) \varphi'(x) \mathbf{1}_{\{|z|<x\}} dx = z \rho_t(|z|) \varphi(|z|) - z \int_{(-z) \vee 0}^{\infty} \varphi(x) \partial_x \rho_t(x) \mathbf{1}_{\{|z|<x\}} dx.$$

All in all, it holds that

$$\begin{aligned} I_t(\varphi) &= \int_{\mathbb{R}_+} \varphi(x) \int_{\mathbb{R}} \frac{\rho_t(x-z) \mathbf{1}_{\{x>z\}} - \rho_t(x) \mathbf{1}_{\{x>-z\}} + z \partial_x \rho_t(x) \mathbf{1}_{\{|z|<x\}}}{|z|^{1+\alpha}} dz dx \\ &\quad + \int_{\mathbb{R}} \frac{z}{|z|^{1+\alpha}} \rho_t(|z|) \varphi(|z|) dz. \end{aligned}$$

The last term is equal to zero since the integrand is an odd function. Finally, we write

$$J_t(\varphi) = \int_{\mathbb{R}_+} \varphi(x) \int_{-\infty}^{-x} \frac{\rho_t(x)}{|z|^{1+\alpha}} dz.$$

Recombinig all the terms, we get that

$$\int_{\mathbb{R}_+} \rho_t(x) \mathcal{L}^\alpha \varphi(x) dx = \int_{\mathbb{R}_+} \varphi(x) \mathcal{A} \rho_t(x) dx,$$

where $\mathcal{A} \rho_t$ is defined for every $x > 0$ as

$$\mathcal{A} \rho_t(x) = \frac{\sigma_\alpha}{2} \int_{\mathbb{R}} \frac{\rho_t(x-z) \mathbf{1}_{\{x>z\}} - \rho_t(x) + z \partial_x \rho_t(x) \mathbf{1}_{\{|z|<x\}}}{|z|^{1+\alpha}} dz.$$

Therefore, we conclude that for any $\varphi \in C_c^\infty((0, \infty))$, we have

$$\int_{\mathbb{R}_+} \varphi(x) [\partial_t \rho_t(x) - \mathcal{A} \rho_t(x)] dx = 0,$$

which is enough to deduce that for any $t > 0$ and any $x > 0$, $\partial_t \rho_t(x) = \mathcal{A} \rho_t(x)$. $\square$

An application of Sparre Andersen's fluctuation theorem for exchangeable and sign-invariant random variables

Abstract

This chapter contains the results of [BB23] which has been written with Quentin Berger and is submitted for publication. We revisit the celebrated Sparre Andersen's fluctuation result on persistence (or survival) probabilities $\mathbf{P}(S_k > 0 \ \forall 0 \leq k \leq n)$ for symmetric random walks $(S_n)_{n \geq 0}$. We give a short proof of this result when considering sums of random variables that are only assumed *exchangeable* and *sign-invariant*. We then apply this result to the study of persistence probabilities of (symmetric) additive functionals of Markov chains, which can be seen as a natural generalization of integrated random walks.

5.1 Introduction

We discuss in this paper several results on persistence problems for symmetric random walks, in particular the well known (but nonetheless striking) fluctuation theorem due to Sparre Andersen [SA54]. To introduce the result, let $\xi_1, \dots, \xi_n$ be real random variables and let us denote $S_0 = 0$, and $S_k = \sum_{i=1}^k \xi_i$ for $1 \leq k \leq n$. Then, one is interested in estimating the persistence probabilities

$$\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n) \quad \text{or} \quad \mathbf{P}(S_k \geq 0 \text{ for all } 1 \leq k \leq n).$$

Sparre Andersen's result [SA54] states that if the $(\xi_i)_{1 \leq i \leq n}$ are i.i.d., symmetric and have no atom, then these probabilities do not depend on the law of ξ .

Theorem A. *If the $(\xi_i)_{1 \leq i \leq n}$ are i.i.d. and if the distribution of ξ_i is symmetric and has no atom, then we have*

$$\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n) = g(n) = \mathbf{P}(S_k \geq 0 \text{ for all } 1 \leq k \leq n),$$

where

$$g(n) := \frac{(2n-1)!!}{(2n)!!} = \frac{1}{4^n} \binom{2n}{n} = \prod_{k=1}^n \left(1 - \frac{1}{2k}\right).$$

In fact, [SA54] proves a number of results for exchangeable random variables, but the independence plays an important role in obtaining fluctuation's results, such as arcsine laws or

persistence probabilities. Chapter XII in Feller's book [Fel71] provides a streamlined approach to Sparre Andersen's result, using a duality argument together with a (purely combinatorial) cyclic lemma, see [Fel71, XII.6]: these two ingredients only require the law of $(X_1, \dots, X_n)$ to be exchangeable. Independence is required in the last step of the proof, in order to obtain that the (weak) ladder epochs defined iteratively by $T_0 = 0$ and $T_k = \min\{n > T_{k-1}, S_n \geq \max_{0 \leq j < n} \{S_j\}\}$ form a renewal sequence. The following is then deduced, see [Fel71, XII.7, Thm. 1]:

Theorem B. *If the $(\xi_i)_{i \geq 1}$ are i.i.d., then for any $s \in [0, 1)$ we have*

$$1 - \mathbf{E}[s^{T_1}] = \exp\left(-\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}(S_n \geq 0)\right).$$

Theorem B can be seen as a particular case of a Wiener–Hopf factorization, also known as Spitzer–Baxter formula, which gives the joint Laplace transform/characteristic function of the first ladder epoch and ladder height, see [Fel71, XII.9]. Note that if the law of X_i is symmetric and has no atom, one has $\mathbf{P}(S_n \geq 0) = \frac{1}{2}$ for all n , so the generating function of T_1 is given by $\mathbf{E}[s^{T_1}] = 1 - \sqrt{1-s}$.

Let us mention that another proof of Theorem A is presented in [DDG13, Prop. 1.3]: the proof is remarkably simple and elegant and relies on a decomposition with respect to the first time that $(S_k)_{k \geq 0}$ hits its minimum $\min_{1 \leq j \leq n} \{S_j\}$, using the independence of the $(\xi_i)_{1 \leq i \leq n}$. The goal of our paper is to present a version of Theorem A valid for exchangeable and sign-invariant random variables $(\xi_i)_{1 \leq i \leq n}$, going beyond the independent and symmetric setting. Sparre Andersen was already aware of this result, see [SA54, Thm. 4] (its proof is however a bit laconic), but we give here a short and self-contained proof (with no combinatorial lemma), taking inspiration from [DDG13]. We then present some application to persistence probabilities for symmetric additive functionals of birth-death chains, giving in particular a simple proof of Sinai's result [Sin92] on the integrated simple random walk.

5.2 The case of exchangeable and sign-invariant random variables

Below, we consider examples in which there is no independence: one is however still able to obtain the same conclusion as in Theorem A, using some weaker but natural assumption. Let us introduce some terminology: the law $\mathbf{P}$ of $(\xi_1, \dots, \xi_n)$ is said to be:

- (E) exchangeable if for any permutation $\sigma \in \mathfrak{S}_n$, $(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)})$ has the same distribution as $(\xi_1, \dots, \xi_n)$;
- (S) sign-invariant if for any $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, $(\varepsilon_1 \xi_1, \dots, \varepsilon_n \xi_n)$ has the same distribution as $(\xi_1, \dots, \xi_n)$.

One can think about these two assumptions as an invariance of $\mathbf{P}$ under the action of the groups: of permutations $\mathfrak{S}_n$ for condition (E); of sign changes $\{-1, 1\}^n$ for condition (S).

The following result is actually already mentioned in [SA54, Thm. 4]: the present paper is meant as a way to put more emphasis on this result; we then provide some applications.

Theorem 5.1. *Assume that $(\xi_1, \dots, \xi_n)$ satisfies conditions (E)-(S). Then, recalling that we have defined $g(n) = \frac{1}{4^n} \binom{2n}{n}$, we have*

$$\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n) \leq g(n) \leq \mathbf{P}(S_k \geq 0 \text{ for all } 1 \leq k \leq n).$$

Notably, if $\mathbf{P}$ has no atom, $\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n) = g(n) = \mathbf{P}(S_k \geq 0 \text{ for all } 1 \leq k \leq n)$.

Remark 5.1. *Let us note that one may also obtain a lower bound on $\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n)$ by imposing that $S_1 > 0, S_2 - S_1 \geq 0, \dots, S_n - S_1 \geq 0$: since conditionally on X_1 the random vector $(X_2, \dots, X_n)$ is still exchangeable and sign-invariant, we get from Theorem 5.1*

$$\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n) \geq \mathbf{P}(X_1 > 0)g(n-1), \quad (5.1)$$

so we have both an upper and a lower bound on $\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n)$ in terms of $g(\cdot)$. Let us also stress that we have $\frac{1}{\sqrt{\pi(n+1/2)}} \leq g(n) \leq \frac{1}{\sqrt{\pi n}}$ for all $n \geq 1$, see e.g. [BDG⁺23, Eq. (3.1)]. As far as $\mathbf{P}(S_k \geq 0 \text{ for all } 1 \leq k \leq n)$ is concerned, Theorem 5.1 provides a lower bound, but there is no general upper bound: [BDG⁺23, Conj. 6] conjectures that this probability is maximal when $(S_k)_{k \geq 0}$ is the simple symmetric random walk, i.e. that $\mathbf{P}(S_k \geq 0 \text{ for all } 1 \leq k \leq n) \leq g(\lfloor n/2 \rfloor)$.

One can easily see that both conditions (E)-(S) are necessary to obtain the statement of Theorem 5.1, see for instance Section 5.4.2 below.

Remark 5.2 (About sign-invariance). *The sign-invariance of $(\xi_1, \dots, \xi_n)$ can be seen as a strong form of symmetry for the law of $(\xi_1, \dots, \xi_n)$. Let us mention the article [Ber65] which derives some properties of sign-invariant sequences: in particular, [Ber65, Lem. 1.2] tells that sign-invariant $(\xi_1, \dots, \xi_n)$ are independent conditionally on $(|\xi_1|, \dots, |\xi_n|)$. In other words, any exchangeable and sign-invariant law can be obtained by taking a random vector $(Z_1, \dots, Z_n)$ (that can be constrained to have $0 \leq Z_1 \leq \dots \leq Z_n$) and by shuffling and changing the signs of its coordinates randomly, that is setting $\xi_i = \varepsilon_i Z_{\sigma(i)}$ for $1 \leq i \leq n$, where the $(\varepsilon_i)_{1 \leq i \leq n}$ are i.i.d. signs (i.e. such that $\mathbf{P}(\varepsilon_i = \pm 1) = \frac{1}{2}$) and σ is a random uniform permutation of $\{1, \dots, n\}$ (i.e. such that $\mathbf{P}(\sigma = \nu) = \frac{1}{n!}$ for all $\nu \in \mathfrak{S}_n$).*

Remark 5.3 (Finite vs. infinite sequences). *The law of an infinite sequence $(\xi_i)_{i \geq 1}$ of random variables is called exchangeable, resp. sign-invariant, if for any $n \geq 1$ the law of $(\xi_1, \dots, \xi_n)$ is exchangeable, resp. sign-invariant. By de Finetti's theorem, one knows that the law of $\xi = (\xi_i)_{i \geq 1}$ is exchangeable if and only if it is conditionally i.i.d.: in other words, there exists a random probability distribution μ such that $\mathbf{P}(\xi \in \cdot \mid \mu) = \mu^{\otimes \infty}$, where $\mu^{\otimes \infty}$ denotes the law of an infinite sequence of i.i.d. random variables with law μ . In [Ber62, Thm. 1], it is shown that an exchangeable sequence $(\xi_i)_{i \geq 1}$ is sign-invariant if and only if μ is almost surely symmetric. From this, it is easy to derive Theorem 5.1 for an exchangeable and sign-invariant sequence $(\xi_i)_{i \geq 1}$, simply by working conditionally on the realization of μ . Hence, the truly remarkable fact about Theorem 5.1 is that it holds for a finite exchangeable and sign-invariant $(\xi_1, \dots, \xi_n)$.*

5.3 Persistence probabilities of f -integrated birth-death chains

Let us consider $(X_n)_{n \geq 0}$ a birth and death Markov chain, starting from $X_0 = 0$: it is a Markov chain on $\mathbb{Z}$ such that $|X_n - X_{n-1}| \leq 1$, with transition probabilities

$$p_x = p(x, x+1), \quad q_x = p(x, x-1), \quad r_x = p(x, x) = 1 - p_x - q_x \quad \text{for } x \geq 1.$$

We assume that $(X_n)_{n \geq 0}$ is symmetric, in the sense that $(-X_n)_{n \geq 0}$ has the same distribution as $(X_n)_{n \geq 0}$; in other words, the transition probabilities verify $p(x, y) = p(-x, -y)$ for any $x, y \in \mathbb{Z}$. For $x = 0$, we also set $p_0 = p(0, 1) \in (0, \frac{1}{2}]$, $q_0 = p(0, -1) = p_0$ and $r_0 = p(0, 0) = 1 - 2p_0$.

Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be any anti-symmetric function which preserves the sign of x , i.e. such that $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$ (and naturally $f(0) = 0$). Then we define the f -integrated Markov chain, or *additive functional*, as follows: $\zeta_0 = 0$ and, for $n \geq 1$,

$$\zeta_n := \sum_{i=1}^n f(X_i). \quad (5.2)$$

We are now interested in the persistence (or survival) probabilities

$$\mathbf{P}(\zeta_k \geq 0 \text{ for all } 1 \leq k \leq n) \quad \text{or} \quad \mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n).$$

A classical, well-studied example is when $(X_n)_{n \geq 0}$ is the simple symmetric random walk and f is the identity: then $(\zeta_n)_{n \geq 0}$ is the integrated random walk and the persistence probabilities are known to be of order $n^{-1/4}$, see [Sin92].

As another motivation, let us point out that the persistence problem for integrated random walk bridges appeared in the context of a polymer pinning model [CD09] (see also [AS15]) and more recently in the study of graphic sequences in [BDG⁺23], *i.e.* on the number $G(n)$ of integer sequences $n-1 \geq d_1 \geq \dots \geq d_n \geq 0$ that are the degree sequence of a graph. We are therefore also interested in the behavior of

$$\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n, X_n = 0) \quad \text{or} \quad \mathbf{P}(\zeta_k \geq 0 \text{ for all } 1 \leq k \leq n, X_n = 0).$$

(Let us assume that $(X_n)_{n \geq 0}$ is aperiodic for the simplicity of exposition.)

Theorem 5.2. *Let $(X_n)_{n \geq 0}$ be a symmetric recurrent birth-death chain and let f be an odd function such that $xf(x) \geq 0$; recall that $p_0 := \mathbf{P}(X_1 > 0)$. Then, recalling that $g(n) := \frac{1}{4^n} \binom{2n}{n}$ and setting by convention $g(n) = 0$ for $n \leq 0$, we have for all $n \geq 1$*

$$p_0(1 - p_0)\mathbf{E}[g(L_n - 1)] \leq \mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \leq \mathbf{E}[g(L_n)]$$

where $L_n := \sum_{k=1}^n \mathbf{1}_{\{X_k=0\}}$ is the local time of the chain at 0 up to time n . Regarding the bridge, we have for every $n \geq 1$

$$p_0\mathbf{E}[g(L_n - 1)\mathbf{1}_{\{X_n=0\}}] \leq \mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n, X_n = 0) \leq \mathbf{E}[g(L_n)\mathbf{1}_{\{X_n=0\}}].$$

Let us mention that our approach is based on an excursion decomposition of the process $(X_n)_{n \geq 0}$. Our results would still be valid for Markov chains that “cannot jump above 0”, *i.e.* such that $\mathbf{P}(X_n \leq 0 \mid X_{n-1} = x) = \mathbf{P}(X_n = 0 \mid X_{n-1} = x)$ for any $x \in \mathbb{N}$. We have chosen to stick with the birth-death setting since it is the most natural example of such chains (and already contains a wide class of behaviors).

Since we have the asymptotic behavior $g(m) \sim (\pi m)^{-1/2}$ as $m \rightarrow \infty$ (we actually have explicit bounds $(\pi(m+1/2))^{-1/2} \leq g(m) \leq (\pi m)^{-1/2}$ for $m \geq 1$, see Remark 5.1), we can give sufficient conditions, often verified in practice, to obtain the asymptotic behavior of $\mathbf{E}[g(L_n)]$, $\mathbf{E}[g(L_n)\mathbf{1}_{\{X_n=0\}}]$ as $n \rightarrow \infty$. (The same asymptotics hold with $g(L_n - 1)$ in place of $g(L_n)$.)

Proposition 5.1. *Assume that $(X_n)_{n \geq 0}$ is aperiodic and let $\tau_1 := \inf\{n \geq 1, X_n = 0\}$.*

(i) *If $(X_n)_{n \geq 0}$ is positive recurrent, then we have, as $n \rightarrow \infty$*

$$\mathbf{E}[g(L_n)] \sim \frac{\sqrt{\mathbf{E}[\tau_1]}}{\sqrt{\pi n}} \quad \text{and} \quad \mathbf{E}[g(L_n)\mathbf{1}_{\{X_n=0\}}] \sim \frac{1}{\sqrt{\pi n \mathbf{E}[\tau_1]}}. \quad (5.3)$$

(ii) *If $\mathbf{P}(\tau_1 > n) \sim \ell(n)n^{-\alpha}$ for some $\alpha \in [0, 1]$ and some slowly varying function $\ell(\cdot)$, then we have, as $n \rightarrow \infty$*

$$\mathbf{E}[g(L_n)] \sim \frac{c_\alpha}{\sqrt{\pi b_n}} \quad \text{with} \quad b_n := \begin{cases} n/\mu(n) \\ n^\alpha/\ell(n) \\ 1/\ell(n) \end{cases} \quad c_\alpha := \begin{cases} 1 \\ \Gamma(1-\alpha)^{1/2} \mathbf{E}[(\mathcal{Z}_\alpha)^{\alpha/2}] \\ \sqrt{\pi} \end{cases} \quad \begin{array}{l} \text{if } \alpha = 1, \\ \text{if } \alpha \in (0, 1), \\ \text{if } \alpha = 0, \end{array} \quad (5.4)$$

where $\mu(n) := \mathbf{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq n\}}]$ and with Z_α a one-sided α -stable random variable of Laplace transform e^{-t^α} . Moreover, if $\alpha \in (2/3, 1)$, we have the asymptotic

$$\mathbf{E}[g(L_n) \mathbf{1}_{\{X_n=0\}}] \sim c'_\alpha n^{\frac{\alpha}{2}-1} \ell(n)^{-1/2}, \quad \text{with } c'_\alpha = \alpha \Gamma(1-\alpha)^{-1/2} \mathbf{E}[(Z_\alpha)^{-\alpha/2}]. \quad (5.5)$$

In order to have the same asymptotic (5.5) for $\alpha \in (0, 2/3]$, a sufficient condition is that there exists a constant $C > 0$ such that $\mathbf{P}(\tau_1 = n) \leq C \ell(n) n^{-(1+\alpha)}$ for all $n \geq 1$.

Example 5.1. We give below a class of example where Proposition 5.1 can be applied, but let us comment on the case of the simple symmetric random walk $(X_n)_{n \geq 0}$. In that case, Proposition 5.1 is verified (up to periodicity issues). We then get that the persistence probabilities for the integrated random walk verify

$$(1 + o(1)) \frac{1}{4} c_0 n^{-1/4} \leq \mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq 2n) \leq (1 + o(1)) c_0 n^{-1/4}, \quad (5.6)$$

with c_0 an explicit constant. This recovers a result by Sinai [Sin92] (in the case $f(x) = x$). For the integrated simple random walk bridge, we also have that

$$(1 + o(1)) \frac{1}{4} c_1 n^{-3/4} \leq \mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq 2n, X_{2n} = 0) \leq (1 + o(1)) c_1 n^{-3/4}, \quad (5.7)$$

recovering a result by Vysotsky [Vys14].

Remark 5.4. In the case $\alpha \in (0, 1)$ the condition $\mathbf{P}(\tau_1 > n) \sim \ell(n) n^{-\alpha}$ is exactly equivalent to τ_1 being in the domain of attraction of an α -stable law. Our sufficient condition $\alpha > 2/3$ to get (5.5) may seem mysterious at first, but it is reminiscent of the Garsia–Lamperti [GL62] condition $\alpha > 1/2$ to obtain a Strong Renewal Theorem (SRT), i.e. the sharp asymptotic behavior of $\mathbf{P}(X_n = 0) = \mathbf{P}(n \in \tau)$. In the case $\alpha \leq 2/3$, our condition on the local tail $\mathbf{P}(\tau_1 = n)$ is reminiscent of Doney’s condition in [Don97, Eq. (1.9)] to obtain a SRT. Let us also mention [CD19] where a necessary and sufficient condition for a SRT is found: with some effort, it should translate in a necessary and sufficient condition for (5.5) to hold. Note that for technical simplicity we only deal with the case $\alpha \in (0, 1)$ in (5.5), but an analogous result should hold in the cases $\alpha = 1$ and $\alpha = 0$.

Remark 5.5. We stress here that the asymptotic bounds that we obtain combining Theorem 5.2 and Proposition 5.1 do not depend on the (anti-symmetric) function f in the definition (5.2) of ζ_n . In particular, the bounds (5.6) and (5.7) obtained in the case of the simple random walk (and for Bessel-like random walks, see (5.9)-(5.10) below) are valid for any odd function f .

A class of examples: Bessel-like random walks

Let us give more precise result in the case of symmetric Bessel-like random walk, see [Ale11, HJR53, Lam62]. This is a class of birth-death chains that includes the simple symmetric random walk. They have the following transition probabilities: for $x \geq 1$

$$p_x := p(x, x+1) = \frac{1}{2} \left(1 - \frac{\delta + \varepsilon_x}{2x} \right), \quad p(x, x-1) = 1 - p_x, \quad (5.8)$$

where $\delta \in \mathbb{R}$ and ε_x is such that $\lim_{x \rightarrow \infty} \varepsilon_x = 0$; we take $p(-x, -x-1) = p(x, x+1)$ and $p(0, 1) = p(0, -1) = \frac{1}{2}$, for symmetry reasons. We also assume uniform ellipticity, i.e. there is some $\eta > 0$ such that $p_x \in [\eta, 1-\eta]$ for all $x \in \mathbb{Z}$. The parameter δ is called the *drift parameter* and we have the following behavior: the walk $(X_n)_{n \geq 0}$ is transient if $\delta < -1$, recurrent if $\delta > -1$, positive recurrent if $\delta > 1$; in the cases $\delta = -1$ and $\delta = 1$, the behavior depends on the function ε_x .

More precisely, letting $\lambda_x = \prod_{k=1}^x \frac{p_x}{1-p_x}$, the random walk $(X_n)_{n \geq 1}$ is recurrent if and only if $\sum_{x=1}^{\infty} \lambda_x = +\infty$. Moreover, with the notation (5.8), there is a constant $K_0 > 0$ such that

$$\lambda_x = \prod_{k=1}^x \frac{1-p_k}{p_k} \sim K_0 x^\delta L(x)^{-1} \quad \text{as } x \rightarrow \infty, \quad \text{where } L(x) := \exp\left(\sum_{k=1}^x \frac{\varepsilon_k}{k}\right).$$

Note that L is a slowly varying function. Then, in [Ale11, Thm. 2.1], the sharp tail of the distribution of τ_1 is derived: Let $\delta \geq -1$ and set $\alpha := \frac{1+\delta}{2}$, then as $n \rightarrow \infty$

$$\begin{aligned} \text{if } \delta > -1 \ (\alpha > 0), \quad & \mathbf{P}(\tau_1 > n) \sim \frac{2^{1-\alpha}}{K_0 \Gamma(\alpha)} n^{-\alpha} L(\sqrt{n}), \\ \text{if } \delta = -1 \ (\alpha = 0), \quad & \mathbf{P}(\tau_1 > n) \sim \frac{1}{K_0} \nu(n)^{-1} \quad \text{where } \nu(n) := \sum_{x \leq n, x \text{ even}} \frac{1}{xL(\sqrt{x})}. \end{aligned}$$

Also, if $\delta = -1$, $(X_n)_{n \geq 0}$ is recurrent if and only if $\lim_{n \rightarrow \infty} \nu(n) = \infty$.

For $\alpha > 0$, [Ale11, Thm. 2.1] also gives that

$$\mathbf{P}(\tau_1 = n) \sim \frac{2^{2-\alpha}}{K_0 \Gamma(\alpha)} n^{-(1+\alpha)} L(\sqrt{n}) \quad \text{as } n \rightarrow \infty, \ n \text{ even.}$$

One can then simply apply Proposition 5.1 (up to periodicity issues) with $\ell(n) = cL(\sqrt{n})$ or $\ell(n) = c/\nu(n)$ to obtain that, in the null-recurrent case:

$$(1 + o(1)) \frac{1}{4} c_\alpha (b_n)^{-1/2} \leq \mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq 2n) \leq (1 + o(1)) c_\alpha (b_n)^{-1/2}, \quad (5.9)$$

with b_n given in (5.4). Also, for $\alpha \in (0, 1)$,

$$c' n^{\frac{\alpha}{2}-1} L(\sqrt{n})^{-1/2} \leq \mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq 2n, X_{2n} = 0) \leq c'' n^{\frac{\alpha}{2}-1} L(\sqrt{n})^{-1/2}. \quad (5.10)$$

Further comments and comparison with the literature

Let us stress that there are several directions in which one could extend our results. First, one could consider more general underlying random walks or Markov chains, for instance random walks with non-necessarily symmetric increments or Markov chains that can “jump over 0”. One could also consider more general functions f , not necessarily symmetric. Another room for improvement is to obtain sharp asymptotics for the persistence probabilities, *i.e.* for instance finding the correct constant $\hat{c}_\alpha$ such that $\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \sim \hat{c}_\alpha b_n^{-1/2}$ in (5.9).

A big part of the literature has considered the case of integrated random walks, that is considering the Markov chain $X_n = \sum_{k=1}^n \xi_k$ with $(\xi_k)_{k \geq 0}$ i.i.d. random variables and $\zeta_n = \sum_{i=1}^n X_i$ (*i.e.* taking $f(x) = x$), starting with the work of Sinai [Sin92]. Under the condition that the ξ_k 's are centered with a finite second moment, the persistence probability $\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n)$ has been proven to be of order $n^{-1/4}$ in [DDG13], and the sharp asymptotic $\sim c_1 n^{-1/4}$ in [DW15]. The case where ξ_k does not have a finite second moment remains mostly open, except in some specific (one-sided) cases, see e.g. [DDG13, Vys14]. The persistence probabilities of integrated random walk bridges has been studied for instance in [AS15, BDG⁺23, CD09, Vys14]: the order $n^{-3/4}$ is found in [Vys14, Prop. 1] for centered random walks with finite variance and the asymptotic $\sim cn^{-3/4}$ is given in [BDG⁺23, Prop. 1.2] in the case of the simple random walk. Our result can be seen as an extension to persistence problems for additive functionals of Markov chains $(X_n)_{n \geq 0}$; with the major restriction of symmetry and of the fact that the underlying Markov chain cannot jump above 0 (but with the advantage of having an elementary proof).

Another line of works considered persistence problems for additive functionals of *continuous-time* Markov processes $(X_t)_{t \geq 0}$. Let us mention [Gol71] that considered the case of an integrated Brownian motion, and [IK00, Pro21] who considered the f -integral of a Brownian motion or a

skew-Bessel process respectively, for the (possibly asymmetric) functional $f(x) = |x|^\gamma (c_+ \mathbf{1}_{\{x>0\}} - c_- \mathbf{1}_{\{x<0\}})$ for some $\gamma > -1$. More recently, in [BBT23], we have pushed further the existing techniques (based on a Wiener–Hopf decomposition of a bi-variate Lévy process associated to the problem): we obtained the sharp asymptotics for persistence probabilities for a wide class of Markov processes (including one-dimensional generalized diffusions, see [IMJ96]) and of functions f . In particular, [BBT23, Example 6] shows that the result applies to *continuous-time* birth-death processes, giving the existence of the constant $\hat{c}_\alpha$ mentioned above. The present article can be seen as a elementary approach to obtaining a sub-optimal result.

5.4 Exchangeable and symmetric sequences: proof of Theorem 5.1

Let us introduce some notation. For $n \in \mathbb{N}$ and for a fixed $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we define an exchangeable and symmetric vector $\xi = (\xi_1, \dots, \xi_n)$ with law denoted $\mathbf{P}^{(x)}$ by permuting the coordinates of $(x_1, \dots, x_n)$ by a random uniform permutation and by changing the signs of the coordinates uniformly at random. More precisely, let $(\varepsilon_i)_{1 \leq i \leq n}$ be i.i.d. random variables with law $\mathbf{P}(\varepsilon_1 = 1) = \mathbf{P}(\varepsilon_1 = -1) = \frac{1}{2}$ and let σ be a random permutation of $\{1, \dots, n\}$ with uniform distribution $\mathbf{P}(\sigma = \nu) = \frac{1}{n!}$ for all $\nu \in \mathfrak{S}_n$, independent of $(\varepsilon_i)_{1 \leq i \leq n}$: we then define

$$(\xi_1, \dots, \xi_n) := (\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)}). \quad (5.11)$$

Note that we can restrict to the case where $x_i \in \mathbb{R}_+$ for all $1 \leq i \leq n$.

We then construct the random walk $S_k = \sum_{i=1}^k \xi_i$ for any $0 \leq k \leq n$, and we are interested in the persistence probabilities

$$p_n(x) := \mathbf{P}^{(x)}(S_k > 0 \text{ for all } 1 \leq k \leq n) \quad \text{and} \quad \bar{p}_n(x) := \mathbf{P}^{(x)}(S_k \geq 0 \text{ for all } 1 \leq k \leq n).$$

We set by convention these probabilities equal to 1 for $n = 0$. The following proposition is the essence of Theorem 5.1.

Proposition 5.2. *If $x = (x_1, \dots, x_n)$ is such that*

$$\sum_{i \in I} x_i \neq \sum_{j \in J} x_j \quad \text{for all } I, J \subset \{1, \dots, n\} \text{ with } I \neq J \quad (\text{H})$$

(by convention $\sum_{i \in \emptyset} x_i = 0$), then we have that $p_n(x) = \bar{p}_n(x) = g(n) := \frac{1}{4^n} \binom{2n}{n}$; in particular it does not depend on x . In general, we have $p_n(x) \leq g(n) \leq \bar{p}_n(x)$.

Theorem 5.1 follows directly from Proposition 5.2, noting that the law $\mathbf{P}$ of $(\xi_1, \dots, \xi_n)$, conditionally on $|\xi| = (|\xi_1|, \dots, |\xi_n|)$, is $\mathbf{P}^{(|\xi|)}$. Let us also mention that a combinatorial proof of Proposition 5.2 is given in Sections 2.3-2.4 of [Bur07] (see also [BDG⁺23, Lem. 3.7]), in the spirit of that of [SA54]; our proof is quite simpler and does not use any combinatorial lemma.

5.4.1 Proof of Proposition 5.2

Our proof is greatly inspired by that of [DDG13, Prop. 1.3]. We start with the first statement: we are going to prove by recurrence on n that for any $x = (x_1, \dots, x_n)$ that verifies the assumption (H), the quantities $p_n(x) = \bar{p}_n(x)$ do not depend on x . The statement is trivial for $n = 1$ since we have $\mathbf{P}^{(x)}(\xi_1 > 0) = \mathbf{P}^{(x)}(\xi_1 \geq 0) = \frac{1}{2}$ if $x \neq 0$, so we directly proceed to the induction step.

Let us fix $n \geq 2$ and some $x = (x_1, \dots, x_n)$ that verifies assumption (H). We now apply a path decomposition used in [DDG13]. Let $W = \min\{k, S_k = \max_{1 \leq i \leq n} S_i\}$: then, for any $\ell \in \{0, \dots, n\}$, we have

$$\begin{aligned} \{W = \ell\} &= \{\xi_\ell > 0, \xi_\ell + \xi_{\ell-1} > 0, \dots, \xi_\ell + \dots + \xi_1 > 0\} \\ &\cap \{\xi_{\ell+1} \leq 0, \xi_{\ell+1} + \xi_{\ell+2} \leq 0, \dots, \xi_{\ell+1} + \dots + \xi_n \leq 0\}. \end{aligned}$$

Now, we have no independence at hand, but we can further decompose over permutations of $(x_1, \dots, x_n)$ with a fixed image $I = \sigma(\{1, \dots, \ell\})$: we get

$$\begin{aligned} \mathbf{P}^{(x)}(W = \ell) &= \sum_{I \subset \{1, \dots, n\}, |I|=\ell} \sum_{\nu_1: \{1, \dots, \ell\} \hookrightarrow I} \sum_{\nu_2: \{\ell+1, \dots, n\} \hookrightarrow I^c} \\ &\frac{1}{n!} \mathbf{P} \left(\sum_{i=0}^j \varepsilon_{\ell-i} x_{\nu_1(\ell-i)} > 0 \text{ for all } 0 \leq j \leq \ell-1, \sum_{i=1}^j \varepsilon_{\ell+i} x_{\nu_2(\ell+i)} \leq 0 \text{ for all } 1 \leq j \leq n-\ell \right), \end{aligned}$$

where $\nu: A \hookrightarrow B$ means that ν is a bijection from A to B . By independence and symmetry of the $(\varepsilon_i)_{1 \leq i \leq n}$, then recombining the sums over the permutations and using the exchangeability, we get that $\mathbf{P}^{(x)}(W = \ell)$ is equal to

$$\sum_{I \subset \{1, \dots, n\}, |I|=\ell} \frac{\ell!(n-\ell)!}{n!} \mathbf{P}^{(x_I)}(S_k > 0 \text{ for all } 1 \leq k \leq \ell) \mathbf{P}^{(x_{I^c})}(S_k \geq 0 \text{ for all } 1 \leq k \leq n-\ell),$$

where we have defined $x_J = (x_i)_{i \in J}$ for any $J \subset \{1, \dots, n\}$. Since both x_I and x_{I^c} verify the assumption (H), we can apply the induction hypothesis: we have that $p_\ell(x_I), p_{n-\ell}(x_{I^c})$ does not depend on x_I, x_{I^c} , for $1 \leq \ell \leq n-1$. Hence, denoting $p_\ell := p_\ell(x_I)$, $p_{n-\ell} := p_{n-\ell}(x_{I^c})$, we end up with $\mathbf{P}^{(x)}(W = \ell) = p_\ell p_{n-\ell}$, for any $\ell \in \{1, \dots, n-1\}$.

Since we have $\mathbf{P}^{(x)}(W = n) = \mathbf{P}^{(x)}(\xi_n > 0, \xi_n + \xi_{n-1} > 0, \dots, \xi_1 > 0) = p_n(x)$ by exchangeability and $\mathbf{P}^{(x)}(W = 0) = \mathbf{P}^{(x)}(S_k \leq 0 \text{ for all } 1 \leq k \leq n) = \bar{p}_n(x)$ by symmetry, we get

$$1 = \sum_{\ell=0}^n \mathbf{P}^{(x)}(W = \ell) = p_n(x) + \sum_{\ell=1}^{n-1} p_\ell p_{n-\ell} + \bar{p}_n(x).$$

Using that $\sum_{i \in I} x_i \neq \sum_{j \in J} x_j$ for all $I, J \subset \{1, \dots, n\}$ with $I \neq J$, we obtain that $p_n(x) = \bar{p}_n(x)$: the above identity shows that $p_n(x) = \bar{p}_n(x) =: p_n$ does not depend on x .

We can now determine the value of p_n , as done in [DDG13]. From the above, $(p_n)_{n \geq 0}$ satisfies the recursive relation $1 = \sum_{\ell=0}^n p_\ell p_{n-\ell}$ for every $n \geq 0$ (recall that $p_0 = 1$). Constructing the generating function, we get that for any $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = \left(\sum_{\ell=0}^{\infty} p_\ell x^\ell \right)^2.$$

Therefore, the generating function of $(p_n)_{n \geq 0}$ is equal to $(1-x)^{-1/2}$, from which one deduces that $p_n = \frac{1}{4^n} \binom{2n}{n} = \frac{(2n-1)!!}{2n!!}$ for all $n \geq 0$.

For the general bounds, for any fixed x , for any $\delta > 0$ fixed, one can choose $y = y(x, \delta)$ such that $x + y$ verifies the assumption (H) and $\sum_{i=1}^n |y_i| \leq \delta$ (take e.g. a typical realization of i.i.d. random variables uniform in $[0, \delta/n]$). Then, we clearly have that

$$\begin{aligned} \mathbf{P}^{(x+y)}(S_k \geq 0 \text{ for all } 1 \leq k \leq n) &\geq \mathbf{P}^{(x)}(S_k \geq \delta \text{ for all } 1 \leq k \leq n), \\ &\leq \mathbf{P}^{(x)}(S_k \geq -\delta \text{ for all } 1 \leq k \leq n). \end{aligned}$$

Since $x + y$ verifies assumption (H), we get that the probability on the left-hand side does not depend on x, y (and is equal to $\frac{(2n-1)!!}{2n!!}$), so

$$\mathbf{P}^{(x)}(S_k \geq \delta \text{ for all } 1 \leq k \leq n) \leq \frac{(2n-1)!!}{2n!!} \leq \mathbf{P}^{(x)}(S_k \geq -\delta \text{ for all } 1 \leq k \leq n).$$

Since δ is arbitrary, letting $\delta \downarrow 0$ concludes the proof. $\square$

5.4.2 On the necessity of conditions (E)-(S)

We stress that in the construction (5.11) of the exchangeable and symmetric vector $(\xi_1, \dots, \xi_n)$, the two assumptions are essential:

(a) *The signs $(\varepsilon_i)_{1 \leq i \leq n}$ need to be independent.* As a counter-example, take $(\varepsilon_i)_{1 \leq i \leq n}$ uniform on $\{\omega \in \{-1, 1\}^n, \sum_{i=1}^n \omega_i \in \{2-n, n-2\}\}$, so that ξ is still symmetric. Let $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ with $x_1 \geq x_2 \geq \dots \geq x_n$ (this is no restriction by definition of ξ). Then to have $S_k > 0$ for all $1 \leq k \leq n$, we need to have $\varepsilon_1 = +1$ (which happens with probability $1/2n$) and then, since all other signs are $\varepsilon_i = -1$, we need to place x_1 in the first position (which happens with probability $1/n$): we get

$$p_n(x) = \mathbf{P}^{(x)}(S_k > 0 \text{ for all } 1 \leq k \leq n) = \begin{cases} \frac{1}{2n^2} & \text{if } x_1 > \sum_{i=2}^n x_i, \\ 0 & \text{if } x_1 \leq \sum_{i=2}^n x_i. \end{cases}$$

Therefore we obtain that $p_n(x)$ depends on x .

(b) *The signs $(\varepsilon_i)_{1 \leq i \leq n}$ need to be independent from the permutation σ .* As a counter-example, take $n = 3$ and $x = (x_1, x_2, x_3) \in (\mathbb{R}_+)^3$ with $x_1 > x_2 > x_3 > 0$: then the probability $\mathbf{P}^{(x)}(S_1 > 0, S_2 > 0, S_3 > 0)$ is equal to (one needs to have $\varepsilon_1 = +1$)

$$\begin{aligned} & \mathbf{P}(\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1) + \mathbf{P}(\varepsilon_1 = \varepsilon_2 = 1, \varepsilon_3 = -1, x_{\sigma(3)} < x_{\sigma(1)} + x_{\sigma(2)}) \\ & + \mathbf{P}(\varepsilon_1 = \varepsilon_3 = 1, \varepsilon_2 = -1, x_{\sigma(2)} < x_{\sigma(1)}) + \mathbf{P}(\varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = -1, x_{\sigma(2)} + x_{\sigma(3)} < x_{\sigma(1)}). \end{aligned}$$

Now, if the joint distribution of (ε, σ) is such that $\varepsilon = (\varepsilon_i)_{1 \leq i \leq 3}$ is uniform on $\{-1, 1\}^3$ and $\mathbf{P}(\sigma(3) = 3 \mid \varepsilon_1 = \varepsilon_2 = +1, \varepsilon_3 = -1) = 1$, $\mathbf{P}(\sigma(2) = 3 \mid \varepsilon_1 = \varepsilon_2 = +1, \varepsilon_3 = -1) = 1$ and $\mathbf{P}(\sigma(1) = 1 \mid \varepsilon_1 = +1, \varepsilon_2 = \varepsilon_3 = -1) = 1$, since $x_1 > x_2 > x_3 > 0$, we get that

$$\mathbf{P}^{(x)}(S_1 > 0, S_2 > 0, S_3 > 0) = \frac{1}{4} + \frac{1}{8} \mathbf{1}_{\{x_1 > x_2 + x_3\}},$$

which depends on x . It simply remains to see that the above conditions on the joint distribution of (ε, σ) can be satisfied, which can be checked by hand.

5.5 Integrated birth and death chains

5.5.1 Persistence for integrated birth-death chains: Proof of Theorem 5.2

Let us define iteratively $\tau_0 = 0$ and, for $k \geq 1$, $\tau_k = \min\{n > \tau_{k-1}, X_n = 0\}$. Then, for $k \geq 1$, we define the random variable

$$\xi_k = \sum_{i=\tau_{k-1}+1}^{\tau_k} f(X_i),$$

i.e. the contribution of the k -th excursion of $(X_n)_{n \geq 0}$ to the f -integrated Markov chain. Note that by the Markov property, the $(\xi_k)_{k \geq 1}$ are i.i.d. We can therefore write $\zeta_n := \sum_{i=1}^n f(X_i)$ as

$$\zeta_n = \sum_{k=1}^{L_n} \xi_k + W_n, \quad W_n = \sum_{i=\tau_{L_n}+1}^n f(X_i)$$

where we recall that $L_n = \sum_{i=1}^n \mathbf{1}_{\{X_i=0\}}$ is the local time at 0. Since there is no change of sign during an excursion (recall that $(X_n)_{n \geq 0}$ is a birth-death chain), we have that $(\zeta_n)_{n \geq 0}$ is monotonous on each interval $(\tau_{k-1}, \tau_k]$. We therefore get the following bounds.

Removing the positivity condition on the last segment $(\tau_{L_n}, n]$, we get

$$\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \leq \mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 1 \leq \ell \leq L_n\right). \quad (5.12)$$

On the other hand, imposing that the excursion straddling over n is non-negative (with probability $1 - p_0$) so that $W_n \geq 0$, and using that the sign of W_n is independent from the past by symmetry, we get

$$\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \geq (1 - p_0) \mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 1 \leq \ell \leq L_n, L_n \geq 1\right). \quad (5.13)$$

The difficulty now to study $\mathbf{P}(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 1 \leq \ell \leq L_n)$ is that the number of terms is random and that ξ_i and L_n are not independent. However, for any $0 \leq m \leq j \leq n$, conditionally on $\{L_n = m, \tau_m = j\}$, the random variables $(\xi_i)_{1 \leq i \leq m}$, though not independent, are easily seen to be exchangeable and sign-invariant (thanks to the Markov property and the fact that the chain $(X_n)_{n \geq 0}$ is symmetric). We can therefore apply Theorem 5.1 and Remark 5.1 to obtain that

$$p_0 g(m-1) \leq \mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 0 \leq \ell \leq m \mid L_n = m, \tau_m = j\right) \leq g(m).$$

We therefore get that, conditioning on L_n, τ_{L_n} ,

$$p_0 g(L_n - 1) \leq \mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 0 \leq \ell \leq L_n \mid L_n, \tau_{L_n}\right) \leq g(L_n),$$

which gives the desired bounds thanks to (5.12)-(5.13), by taking the expectation.

As far as bridges are concerned, we have the identity

$$\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n, X_n = 0) = \mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 1 \leq \ell \leq L_n, X_n = 0\right), \quad (5.14)$$

so similarly as above we end up with the following conditional expectation

$$p_0 g(L_n - 1) \mathbf{1}_{\{X_n=0\}} \leq \mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 0 \leq \ell \leq L_n, X_n = 0 \mid L_n, \tau_{L_n}\right) \leq g(L_n) \mathbf{1}_{\{X_n=0\}},$$

which gives the desired bound, again by taking the expectation. $\square$

Remark 5.6. We also have, with the same line of proof and using Theorem 5.1

$$\begin{aligned} \mathbf{P}(\zeta_k \geq 0 \text{ for all } 1 \leq k \leq n) &\geq (1 - p_0) \mathbf{E}[g(L_n)], \\ \mathbf{P}(\zeta_k \geq 0 \text{ for all } 1 \leq k \leq n, X_n = 0) &\geq (1 - p_0) \mathbf{E}[g(L_n) \mathbf{1}_{\{X_n=0\}}]. \end{aligned}$$

5.5.2 Proof of Proposition 5.1

First of all, let us notice that since $(\pi(m+1/2))^{-1/2} \leq g(m) \leq (\pi m)^{-1/2}$ for all $m \geq 1$, we only have to obtain the asymptotics of

$$\mathbf{E}[(L_n)^{-1/2} \mathbf{1}_{\{L_n \geq 1\}}] \quad \text{and} \quad \mathbf{E}[(L_n)^{-1/2} \mathbf{1}_{\{X_n=0\}} \mathbf{1}_{\{L_n \geq 1\}}]. \quad (5.15)$$

The same asymptotics with $(L_n)^{-1/2}$ replaced with $(L_n + 1/2)^{-1/2}$ follow analogously.

Step 1. Convergence of $b_n^{-1}L_n$. Let us show that under the assumptions of Proposition 5.1, we have that $(b_n^{-1}L_n)_{n \geq 0}$ converges in distribution to a random variable $\mathcal{X}$. This is obvious in the positive recurrent case with $b_n = n$, thanks to the ergodic theorem, with $\mathcal{X} = \mathbf{E}[\tau_1]^{-1}$. In the null-recurrent case, we use the fact that $\mathbf{P}(b_n^{-1}L_n < t) = \mathbf{P}(\tau_{\lceil tb_n \rceil} > n)$ to deduce the convergence in law of $(b_n^{-1}L_n)_{n \geq 0}$ from that of $(\tau_k)_{k \geq 1}$. Let $(a_n)_{n \geq 1}$ be a sequence such that $\mathbf{P}(\tau_1 > a_n) \sim n^{-1}$ as $n \rightarrow \infty$. Then we consider the three cases $\alpha = 1$, $\alpha \in (0, 1)$ and $\alpha = 0$ separately.

Case $\alpha = 1$. We then have that $a_n^{-1}(\tau_n - n\mu(a_n))$ converges in distribution to a 1 stable law, see [Fel71, IX.8] (see Eq. (8.15) for the centering): in particular, $\frac{\tau_n}{n\mu(a_n)}$ converges in probability to 1 (recall that $\mu(n) := \mathbf{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq n\}}]$, which is slowly varying). Setting b_n such that $b_n\mu(a_{b_n}) \sim n$, then we have that $\frac{\tau_{tb_n}}{n}$ converges to t in probability, so $b_n^{-1}L_n$ converges in probability to 1. Then, we simply have to notice that if b_n is given by the above relation then we $\mu(a_{b_n}) \sim \mu(n)$, see e.g. [Ber19b, Lem. 4.3], so $b_n \sim n/\mu(n)$ as defined in (5.4).

Case $\alpha \in (0, 1)$. We then have that $\frac{1}{a_n}\tau_n$ converges in distribution to $\kappa_\alpha \mathcal{Z}_\alpha$, with $\kappa_\alpha = \Gamma(1-\alpha)^{1/\alpha}$ and $\mathcal{Z}_\alpha$ a one-sided α -stable random variable $\mathcal{Z}$ with Laplace transform e^{-t^α} , see [Fel71, XIII.6]. Then setting $b_n := n^\alpha/\ell(n)$ as in (5.4), we get that $a_{b_n} \sim n$ as $n \rightarrow \infty$; indeed, $\mathbf{P}(\tau_1 > n) \sim b_n^{-1} \sim \mathbf{P}(\tau_1 > a_{b_n})$ by definition of a_n . Using that a_n is regularly varying with exponent $1/\alpha$, we get that $a_{tb_n} \sim t^{1/\alpha}a_{b_n} \sim t^{1/\alpha}n$: we therefore get that $\lim_{n \rightarrow \infty} \mathbf{P}(\tau_{tb_n} > n) = \mathbf{P}(\kappa_\alpha \mathcal{Z}_\alpha > t^{-1/\alpha})$. This shows that $b_n^{-1}L_n$ converges in distribution to $(\kappa_\alpha \mathcal{Z}_\alpha)^{-\alpha}$.

Case $\alpha = 0$. We then have that $n\ell(\tau_n)$ converges in distribution to an $\text{Exp}(1)$ variable, see [Dar52, Thm. 8]. Let $b_n = 1/\ell(n)$ as in (5.4) and assume (without loss of generality) that ℓ is strictly decreasing. Then, for any $t > 0$,

$$\mathbf{P}(b_n^{-1}L_n < t) = \mathbf{P}(\tau_{tb_n} > n) = \mathbf{P}(tb_n\ell(\tau_{tb_n}) < tb_n\ell(n)) \xrightarrow[n \rightarrow \infty]{} 1 - e^{-t},$$

which shows that $b_n^{-1}L_n$ converges in distribution to an $\text{Exp}(1)$ variable.

Step 2a. Convergence of the expectation $\mathbf{E}[(L_n)^{-1/2} \mathbf{1}_{\{L_n \geq 1\}}]$. We have that

$$\begin{aligned} \mathbf{E}[(L_n)^{-1/2} \mathbf{1}_{\{L_n \geq 1\}}] &= \int_0^1 \mathbf{P}((L_n)^{-1/2} > t, L_n \geq 1) dt \\ &= b_n^{-1/2} \int_{b_n^{-1}}^\infty \mathbf{P}(b_n^{-1} \leq b_n^{-1}L_n < u) \frac{1}{2} u^{-3/2} du. \end{aligned} \quad (5.16)$$

Since by Step 1 we have that $\mathbf{P}(b_n^{-1} \leq b_n^{-1}L_n \leq u)$ converges to $\mathbf{P}(\mathcal{X} \leq u)$ as $n \rightarrow \infty$, we simply need some uniform bound on $\mathbf{P}(b_n^{-1}L_n \leq u)$ in order to be able to apply dominated convergence; note that we only need a bound for $u \leq 1$, otherwise we simply bound the probability by 1.

(i) *Positive recurrent case.* In that case $b_n = n$, so we easily get that uniformly in $n \geq 1$

$$\mathbf{P}(b_n^{-1}L_n \leq u) = \mathbf{P}(\tau_{un} > n) \leq n^{-1} \mathbf{E}[\tau_{un}] = u \mathbf{E}[\tau_1],$$

by Markov's inequality. Since $u \times u^{-3/2}$ is integrable on $(0, 1]$, we can apply dominated convergence to get that

$$\lim_{n \rightarrow \infty} \int_{b_n^{-1}}^\infty \mathbf{P}(b_n^{-1} \leq b_n^{-1}L_n < u) \frac{1}{2} u^{-3/2} du = \int_0^\infty \mathbf{P}(\mathcal{X} \leq u) \frac{1}{2} u^{-3/2} du = \mathbf{E}[\mathcal{X}^{-1/2}] = \mathbf{E}[\tau_1]^{1/2},$$

where we have used that $\mathcal{X} = \mathbf{E}[\tau_1]^{-1}$ in the positive recurrent case. Combined with (5.16), this gives the desired asymptotics.

(ii) *Null-recurrent case.* In the null-recurrent case, we use that

$$\mathbf{P}(b_n^{-1}L_n \leq u) = \mathbf{P}(\tau_{ub_n} > n) \leq C ub_n \mathbf{P}(\tau_1 > n),$$

for some constant $C > 0$. The last bound is standard and falls in the “big-jump” phenomenon (note that the event $\tau_{ub_n} > n$ is an upper large deviation). If $\alpha = 1$, this is given by [Nag79, Thm. 1.2]; note also that in this case $\lim_{n \rightarrow \infty} b_n \mathbf{P}(\tau_1 > n) = 0$, thanks to [BGT87, Prop. 1.5.9.a.]. If $\alpha \in (0, 1)$, this is contained in [Nag79, Thm. 1.1]; note that by definition of b_n we have that $b_n \mathbf{P}(\tau_1 > n)$ remains bounded in this case. In the case $\alpha = 0$, this is due to [NV08] (see also [AB16]); and $b_n \mathbf{P}(\tau_1 > n)$ also remains bounded in this case, by definition of b_n . This shows that in all cases we have $\mathbf{P}(b_n^{-1}L_n \leq u) \leq Cu$, so as above we can apply dominated convergence to get that

$$\lim_{n \rightarrow \infty} \int_{b_n^{-1}}^{\infty} \mathbf{P}(b_n^{-1}L_n < u) \frac{1}{2} u^{-3/2} du = \int_0^{\infty} \mathbf{P}(\mathcal{X} \leq u) \frac{1}{2} u^{-3/2} du = \mathbf{E}[\mathcal{X}^{-1/2}],$$

with $\mathcal{X} = 1$ if $\alpha = 1$, $\mathcal{X} = (\kappa_\alpha \mathcal{Z})^{-\alpha}$ if $\alpha \in (0, 1)$ and $\mathcal{X} \sim \text{Exp}(1)$ if $\alpha = 0$. Combined with (5.16), this gives the desired asymptotics.

Step 2b. Convergence of the expectation $\mathbf{E}[(L_n)^{-1/2} \mathbf{1}_{\{X_n=0\}} \mathbf{1}_{\{L_n \geq 1\}}]$. In the positive recurrent case, for any $\varepsilon > 0$, let us set $A_{\varepsilon,n} = \{n^{-1}L_n \in (1 - \varepsilon, 1 + \varepsilon) \mathbf{E}[\tau_1]^{-1}\}$. With the same argument as above, we have that

$$\lim_{n \rightarrow \infty} \sqrt{n} \mathbf{E}[(L_n)^{-1/2} \mathbf{1}_{\{X_n=0\}} \mathbf{1}_{\{L_n \geq 1\}} \mathbf{1}_{A_{\varepsilon,n}^c}] = 0.$$

Indeed, we first observe that, as in (5.16), we have

$$\sqrt{n} \mathbf{E}[(L_n)^{-1/2} \mathbf{1}_{\{L_n \geq 1\}} \mathbf{1}_{A_{\varepsilon,n}^c}] = \int_{1/n}^{\infty} \mathbf{P}(n^{-1} \leq n^{-1}L_n < u, A_{\varepsilon,n}^c) \frac{1}{2} u^{-3/2} du.$$

Since for every $\varepsilon > 0$, $\mathbf{P}(A_{\varepsilon,n}^c)$ vanishes as $n \rightarrow \infty$, the above limit holds using the same dominated convergence argument as in the previous step. Therefore, we only need to consider the remaining term $\mathbf{E}[(L_n)^{-1/2} \mathbf{1}_{\{X_n=0\}} \mathbf{1}_{A_{\varepsilon,n}}]$: by definition of $A_{\varepsilon,n}$, we have that

$$(1 + \varepsilon)^{-1/2} \mathbf{P}(X_n = 0, A_{\varepsilon,n}) \leq \sqrt{n} \mathbf{E}[(L_n)^{-1/2} \mathbf{1}_{\{X_n=0\}} \mathbf{1}_{A_{\varepsilon,n}}] \leq (1 - \varepsilon)^{-1/2} \mathbf{P}(X_n = 0).$$

Now, we have $\lim_{n \rightarrow \infty} \mathbf{P}(X_n = 0) = 1/\mathbf{E}[\tau_1]$ by the convergence to the stationary distribution, and we also have $\lim_{n \rightarrow \infty} \mathbf{P}(X_n = 0, A_{\varepsilon,n}) = 1/\mathbf{E}[\tau_1]$ since $\lim_{n \rightarrow \infty} \mathbf{P}(X_n = 0, A_{\varepsilon,n}^c) = 0$. This concludes the proof of (5.3).

For the null-recurrent case, we focus on the case $\alpha \in (0, 1)$. We write

$$\mathbf{E}[(L_n)^{-1/2} \mathbf{1}_{\{X_n=0\}} \mathbf{1}_{\{L_n \geq 1\}}] = \sum_{k=1}^{\infty} k^{-1/2} \mathbf{P}(L_n = k, X_n = 0) = \sum_{k=1}^{\infty} k^{-1/2} \mathbf{P}(\tau_k = n). \quad (5.17)$$

Our proof is quite standard and follows the line of [Don97, §3] (with some adaptation). We fix $\varepsilon > 0$ and we decompose the sum into three parts:

$$P_1 := \sum_{1 \leq k < \varepsilon b_n} k^{-1/2} \mathbf{P}(\tau_k = n), \quad P_2 := \sum_{k = \varepsilon b_n}^{\varepsilon^{-1} b_n} k^{-1/2} \mathbf{P}(\tau_k = n), \quad P_3 := \sum_{k > \varepsilon^{-1} b_n} k^{-1/2} \mathbf{P}(\tau_k = n).$$

The main contribution comes from the term P_2 . Gnedenko's local limit theorem [GK54] gives that $\delta_k := \sup_{x \in \mathbb{Z}} |a_k \mathbf{P}(\tau_k = x) - g_\alpha(\frac{x}{a_k})|$ goes to 0 as $k \rightarrow \infty$, where g_α is the density of the limiting α -stable distribution $\kappa_\alpha \mathcal{Z}_\alpha$. We therefore have that

$$\left| P_2 - \sum_{k=\varepsilon b_n}^{\varepsilon^{-1} b_n} \frac{k^{-1/2}}{a_k} g_\alpha\left(\frac{n}{a_k}\right) \right| \leq \bar{\delta}_{\varepsilon, n} \sum_{k=\varepsilon b_n}^{\varepsilon^{-1} b_n} k^{-1/2} \frac{1}{a_k},$$

with $\bar{\delta}_{\varepsilon, n} = \sup_{k \geq \varepsilon b_n} \delta_k$ going to 0 as $n \rightarrow \infty$. Since $k^{-1/2}/a_k$ is regularly varying, we have that $k^{-1/2}/a_k \leq C_\varepsilon b_n^{-1/2}/a_{b_n}$ uniformly in $k \in [\varepsilon b_n, \varepsilon^{-1} b_n]$, so using also that $a_{b_n} \sim n$, we get

$$\left| P_2 - \sum_{k=\varepsilon b_n}^{\varepsilon^{-1} b_n} \frac{k^{-1/2}}{a_k} g_\alpha\left(\frac{n}{a_k}\right) \right| \leq \bar{\delta}_{\varepsilon, n} \varepsilon^{-1} b_n \times C'_\varepsilon b_n^{-1/2}/n = o(b_n^{1/2}/n).$$

We now focus on the remaining sum. Since a_k is regularly varying with exponent $1/\alpha$ we get that $a_k = (1 + o(1))(k/b_n)^{1/\alpha} a_{b_n}$ as $n \rightarrow \infty$ with the $o(1)$ uniform in $k \in [\varepsilon b_n, \varepsilon^{-1} b_n]$ (it may depend on ε). Using also that $a_{b_n} \sim n$ we get that $\frac{n}{a_k} = (1 + o(1))(k/b_n)^{-1/\alpha}$: thanks to the uniform continuity of g_α on $[\varepsilon, \varepsilon^{-1}]$, we get that

$$\sum_{k=\varepsilon b_n}^{\varepsilon^{-1} b_n} \frac{k^{-1/2}}{a_k} g_\alpha\left(\frac{n}{a_k}\right) = (1 + o(1)) \sum_{k=\varepsilon b_n}^{\varepsilon^{-1} b_n} k^{-1/2} \frac{1}{n} \left(\frac{k}{b_n}\right)^{-\frac{1}{\alpha}} g_\alpha\left(\left(\frac{k}{b_n}\right)^{-1/\alpha}\right).$$

Therefore, a Riemann sum approximation for the last identity yields that

$$\lim_{n \rightarrow \infty} n b_n^{-1/2} P_2 = \int_\varepsilon^{\varepsilon^{-1}} t^{-\frac{1}{2} - \frac{1}{\alpha}} g_\alpha(t^{-1/\alpha}) dt.$$

One can also control P_3 thanks to the local limit theorem: there is a constant C such that $\mathbf{P}(\tau_k = n) \leq C/a_k$ for all $k \geq 1$, so that using the same Riemann sum approximation as above,

$$P_3 \leq C \sum_{k > \varepsilon^{-1} b_n} \frac{k^{-1/2}}{a_k} \leq C' \frac{b_n^{1/2}}{n} \int_{\varepsilon^{-1}}^\infty t^{-\frac{1}{2} - \frac{1}{\alpha}} dt = C'_\alpha \varepsilon^{\frac{1}{\alpha} - \frac{1}{2}} \frac{b_n^{1/2}}{n},$$

where we have used that $\frac{1}{2} + \frac{1}{\alpha} > 1$ in the last identity (since $\alpha \in (0, 1)$). This shows that $\limsup_{n \rightarrow \infty} n b_n^{-1/2} P_3$ goes to 0 as $\varepsilon \downarrow 0$.

It remains to deal with P_1 : we need to show that, as for P_3 ,

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} n b_n^{-1/2} P_1 = 0. \quad (5.18)$$

With this at hand, letting $n \rightarrow \infty$ then $\varepsilon \downarrow 0$, we would then get that

$$\lim_{n \rightarrow \infty} n b_n^{-1/2} \sum_{k=1}^\infty k^{-1/2} \mathbf{P}(\tau_k = n) = \int_0^\infty t^{-\frac{1}{2} - \frac{1}{\alpha}} g_\alpha(t^{-1/\alpha}) dt = \alpha \mathbf{E}[(\kappa_\alpha \mathcal{Z}_\alpha)^{-\alpha/2}],$$

the last identity following from a simple change of variable. Since $b_n = n^\alpha/\ell(n)$, we have shown (5.5).

Now, it only remains to prove (5.18), and this is where the condition $\alpha > 2/3$ will appear. We need to use local large deviations to bound $\mathbf{P}(\tau_k = n)$. First, a general bound is given in [Ber19a, Thm. 2.3] (see also [CD19, Thm. 6.1]): wthere is a constant $C > 0$ such that $\mathbf{P}(\tau_k = n) \leq \frac{C}{a_k} k \ell(n) n^{-\alpha}$ for all $n \geq k$. This gives that, for $\alpha > 2/3$,

$$P_1 = \sum_{1 \leq k < \varepsilon b_n} k^{-1/2} \mathbf{P}(\tau_k = n) \leq C n^{-\alpha} \ell(n) \sum_{k=1}^{\varepsilon b_n} \frac{k^{1/2}}{a_k} \leq C' b_n^{-1} \times \frac{(\varepsilon b_n)^{3/2}}{a_{\varepsilon b_n}}.$$

The fact that $\alpha > 2/3$ is crucial in the last inequality, where we have used that $k^{1/2}/a_k$ is regularly varying with exponent $\frac{1}{2} - \frac{1}{\alpha} > -1$. Now, since $a_{\varepsilon b_n} \sim \varepsilon^{1/\alpha} a_{b_n} \sim \varepsilon^{1/\alpha} n$, we get that $P_3 \leq C'' \varepsilon^{\frac{3}{2} - \frac{1}{\alpha}} n^{-1} b_n^{1/2}$ for all n . This shows that $\limsup_{n \rightarrow \infty} n b_n^{-1.2} P_1 \leq C'' \varepsilon^{\frac{3}{2} - \frac{1}{\alpha}}$, so that (5.18) holds (again, using that $\frac{3}{2} - \frac{1}{\alpha} > 0$ for $\alpha > 2/3$).

In the case where $\alpha \in (0, 2/3]$, one needs an extra assumption to obtain a better local large deviations estimate. From [Don97, Thm. 2] (or [Ber19a, Thm. 2.4]), the condition $\mathbf{P}(\tau_1 = n) \leq C n^{-(1+\alpha)} \ell(n)$ ensures that there is a constant $C' > 0$ such that $\mathbf{P}(\tau_k = n) \leq C' k \ell(n) n^{-(1+\alpha)}$. Hence, in this case, one ends up with

$$P_1 = \sum_{1 \leq k < \varepsilon b_n} k^{-1/2} \mathbf{P}(\tau_k = n) \leq C' n^{-(1+\alpha)} \ell(n) \sum_{k=1}^{\varepsilon b_n} k^{1/2} \leq C'' n^{-1} b_n^{-1} \times (\varepsilon b_n)^{3/2} = C'' \varepsilon^{3/2} n^{-1} b_n^{1/2}.$$

This shows that $\limsup_{n \rightarrow \infty} n b_n^{-1.2} P_1 \leq C'' \varepsilon^{3/2}$ so that (5.18) holds. This concludes the proof of (5.5).

A miscellany

Abstract

In this chapter, we study the scaling limits of additive functionals of generalized diffusion processes. We show that the correctly rescaled functional is tight in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology and we identify the law at time t of any limiting process, which yields a weak result since we do not entirely determine the limiting law. We also deal with the case where the Markov process is null recurrent. The proof heavily relies on the tools introduced in Chapters 3 and 4. Most part of the work is actually done in Chapter 3.

6.1 Introduction and main results

This final chapter is a miscellany of the first three chapters in this thesis. The main goal here is to recover some results on the scaling limiting of additive functionals, using techniques and results borrowed from Chapters 3 and 4. More precisely we will study scaling limits of additive functionals of generalized diffusion processes and we will show tightness in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology and will partly identify the limiting law.

We will consider, as in Section 3.7 of Chapter 3 a generalized diffusion on $\mathbb{R}$. Let $\mathbf{m} : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing right-continuous function such that $\mathbf{m}(0) = 0$, and $\mathfrak{s} : \mathbb{R} \rightarrow \mathbb{R}$ a continuous increasing function such that $\mathfrak{s}(0) = 0$ and $\mathfrak{s}(\mathbb{R}) = \mathbb{R}$. We assume moreover that $\mathbf{m}$ is not constant and will also denote by $\mathbf{m}$ the Radon measure associated to $\mathbf{m} : \mathbf{m}((a, b]) = \mathbf{m}(b) - \mathbf{m}(a)$ for all $a < b$. We introduce $\mathbf{m}_{\mathfrak{s}}$ the image of $\mathbf{m}$ by $\mathfrak{s}$, *i.e.* the Stieltjes measure associated to the non-decreasing function $\mathbf{m} \circ \mathfrak{s}^{-1}$, where $\mathfrak{s}^{-1}$ is the inverse function of $\mathfrak{s}$. Consider on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a Brownian motion $(B_t)_{t \geq 0}$, we introduce

$$A_t^{\mathbf{m}_{\mathfrak{s}}} = \int_{\mathbb{R}} L_t^x \mathbf{m}_{\mathfrak{s}}(dx),$$

where $(L_t^x)_{t \geq 0, x \in \mathbb{R}}$ denotes the usual family of local times of the Brownian motion, assumed to be continuous in the variables x and t . The process $(A_t^{\mathbf{m}_{\mathfrak{s}}})_{t \geq 0}$ is a non-decreasing continuous additive functional of the Brownian motion $(B_t)_{t \geq 0}$. Now we introduce $(\rho_t)_{t \geq 0}$ the right-continuous inverse of $(A_t^{\mathbf{m}_{\mathfrak{s}}})_{t \geq 0}$, and set

$$X_t = \mathfrak{s}^{-1}(B_{\rho_t}).$$

As the change of time through a continuous non-decreasing additive functional of a strong Markov process preserves the strong Markovianity, see Sharpe [Sha88, Ch. VIII Thm. 65.9], and since $\mathfrak{s}$ is bijective, it holds that $(X_t)_{t \geq 0}$ is a strong Markov process (with respect to the filtration $(\mathcal{F}_{\rho_t})_{t \geq 0}$). Let us denote by $S_{\mathbf{m}_{\mathfrak{s}}}$ the support of the measure $\mathbf{m}_{\mathfrak{s}}$. It is rather classical,

see again Sharpe [Sha88, Ch. VIII Thm. 65.9] or Revuz-Yor [RY99, Ch. X Prop. 2.17], that $(X_t)_{t \geq 0}$ is valued in $\mathfrak{s}^{-1}(S_{\mathfrak{m}_s})$. From now on, we will always assume that $0 \in S_{\mathfrak{m}_s}$ so that, since $\mathfrak{s}(0) = 0$, X_t spends time in 0. Since 0 is recurrent for the Brownian motion, it is also recurrent for $(X_t)_{t \geq 0}$. In this chapter, we will always assume that the process $(X_t)_{t \geq 0}$ starts from 0 and will denote by $\mathbb{P}$ the law of $(X_t)_{t \geq 0}$ starting from this point. We will study the scaling limit of the following (non-Markovian) process

$$\zeta_t = \int_0^t f(X_s) ds,$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel function satisfying the following assumption.

Assumption 6.1. *The function f is measurable and locally integrable with respect to $\mathfrak{m}$. It is such that a.s. for any $t \geq 0$, $|\zeta_t| < \infty$. Moreover f preserves the sign of x , in the sense that $f(x) \geq 0$ if $x > 0$ and $f(x) \leq 0$ if $x < 0$. Finally, we assume that $f(0) = 0$.*

This assumption will always be in force in this chapter. It ensures that the following auxiliary function $\mathfrak{m}_f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$\mathfrak{m}_f(x) = \int_0^x f(\mathfrak{s}^{-1}(u)) \mathfrak{m}_s(du)$$

is well defined. Since f is non-negative on $\mathbb{R}_+$ and non-positive on $\mathbb{R}_-$, the function $\mathfrak{m}_f$ is non-decreasing on $\mathbb{R}_+$ and non-increasing on $\mathbb{R}_-$. We will sometimes also denote by $\mathfrak{m}_f$ the signed and Radon measure associated to the function $\mathfrak{m}_f$. In order to study the scaling limit of $(\zeta_t)_{t \geq 0}$, we need a set of assumptions on $\mathfrak{m}_s$ and $\mathfrak{m}_f$. Let us start with $\mathfrak{m}_s$.

Assumption 6.2. *The function $\mathfrak{m}_s$ is bounded.*

This assumption is equivalent to the fact that $(X_t)_{t \geq 0}$ is positive recurrent. In the null recurrent case (i.e. when $\mathfrak{m}_s$ is not bounded), we need to specify its asymptotic behavior.

Assumption 6.3. *There exist $\beta \in (0, 1)$, a slowing variation function Λ_s at $+\infty$, and two non-negative constants m_-, m_+ with $m_- + m_+ > 0$, such that*

$$\begin{cases} \mathfrak{m}_s(x) \sim m_+ \Lambda_s(x) x^{1/\beta-1} & \text{as } x \rightarrow +\infty, \\ \mathfrak{m}_s(x) \sim -m_- \Lambda_s(|x|) |x|^{1/\beta-1} & \text{as } x \rightarrow -\infty. \end{cases}$$

Regarding the function $\mathfrak{m}_f$, we will consider the following two assumptions.

Assumption 6.4. *The function $\mathfrak{s}$ is a C^1 function, there exists a constant $\mathfrak{m}_f(\infty) \in (0, \infty)$ such that $\lim_{x \rightarrow \pm\infty} \mathfrak{m}_f(x) = \mathfrak{m}_f(\infty)$ and the function $\mathfrak{m}_f(\infty) - \mathfrak{m}_f$ belongs to $L^2(dx)$.*

Assumption 6.5. *There exist $\alpha \in (0, 2)$, a slowing variation function Λ_f at $+\infty$, and two non-negative constants f_-, f_+ with $f_- + f_+ > 0$, such that according to the value of α , we have*

(i) *If $\alpha \in (0, 1)$, then*

$$\begin{cases} \mathfrak{m}_f(x) \sim f_+ \Lambda_f(x) x^{1/\alpha-1} & \text{as } x \rightarrow +\infty, \\ \mathfrak{m}_f(x) \sim f_- \Lambda_f(|x|) |x|^{1/\alpha-1} & \text{as } x \rightarrow -\infty. \end{cases} \quad (6.1)$$

(ii) *If $\alpha = 1$, then the following limit exists $\lim_{h \downarrow 0} \frac{1}{\Lambda_f(1/h)} (\mathfrak{m}_f(1/h) - \mathfrak{m}_f(-1/h)) = \mathbf{c}$ and*

$$\begin{cases} \lim_{h \rightarrow 0} \frac{1}{\Lambda_f(1/h)} (\mathfrak{m}_f(x/h) - \mathfrak{m}_f(1/h)) = f_+ \log x, & \forall x > 0, \\ \lim_{h \rightarrow 0} \frac{1}{\Lambda_f(1/h)} (\mathfrak{m}_f(x/h) - \mathfrak{m}_f(-1/h)) = f_- \log |x|, & \forall x < 0. \end{cases} \quad (6.2)$$

Note that if the limit $\mathbf{c}$ exists, this implies that $f_+ = f_-$; we will assume for simplicity that $f_+ = f_- = 1$.

(iii) If $\alpha \in (1, 2)$, then there is a constant $\mathbf{m}_f(\infty) \in (0, \infty)$ such that $\lim_{x \rightarrow \pm\infty} \mathbf{m}_f(x) = \mathbf{m}_f(\infty)$ and

$$\begin{cases} \mathbf{m}_f(\infty) - \mathbf{m}_f(x) \sim f_+ \Lambda_f(x) x^{1/\alpha-1} & \text{as } x \rightarrow +\infty, \\ \mathbf{m}_f(\infty) - \mathbf{m}_f(x) \sim f_- \Lambda_f(|x|) |x|^{1/\alpha-1} & \text{as } x \rightarrow -\infty. \end{cases} \quad (6.3)$$

Note that if Assumption 6.4 holds, then Assumption 6.5 can not hold and conversely.

6.1.1 The limiting processes

We first introduce the limiting processes that will appear in the main result. Remember from Chapter 3 that the process $(X_t)_{t \geq 0}$ possesses a local time at 0 and we will denote by $(\tau_t)_{t \geq 0}$ its right continuous inverse, which is a subordinator. We also consider the space of continuous excursions $\mathcal{C}$ endowed with the usual topology, and will denote by $\mathbf{n}^B$ the Brownian excursion measure. Let m_-, m_+, f_- and f_+ be some non-negative constants (given by Assumptions 6.3 and 6.5), $\beta \in (0, 1)$, $\alpha \in (0, 2)$ and let $\varepsilon \in \mathcal{C}$. We introduce the following functionals of excursions:

$$A_\ell^\beta(\varepsilon) = c_\beta \int_0^\ell \text{sgn}_{m_+, m_-}(\varepsilon_s) |\varepsilon_s|^{1/\beta-2} ds, \quad A_\ell^\alpha(\varepsilon) = c_\alpha \int_0^\ell \text{sgn}_{f_+, -f_-}(\varepsilon_s) |\varepsilon_s|^{1/\alpha-2} ds, \quad (6.4)$$

where $\text{sgn}_{a,b}(x) = a \mathbf{1}_{\{x > 0\}} + b \mathbf{1}_{\{x < 0\}}$ and $c_m = |1/m - 1|$ if $m \in (0, 2) \setminus \{1\}$ and $c_m = 1$ if $m = 1$. For $\mathbf{n}^B$ -almost every excursions, these functionals are finite. Then, we define the two-dimensional Lévy measure $\pi_{\alpha, \beta}$ as

$$\pi_{\alpha, \beta}(dr, dz) = \mathbf{n}^B \left(A_\ell^\beta(\varepsilon) \in dr, A_\ell^\alpha(\varepsilon) \in dz \right). \quad (6.5)$$

as well as its marginal $\pi_\beta(dr) = \mathbf{n}^B(A_\ell^\beta(\varepsilon) \in dr)$ and $\pi_\alpha(dz) = \mathbf{n}^B(A_\ell^\alpha(\varepsilon) \in dz)$. We also define the following constants $\sigma^2 = 4 \int_{\mathbb{R}} [\mathbf{m}_f(\infty) - \mathbf{m}_f(x)]^2 dx$ and $m_1 = \mathbb{E}[\tau_1]$ which may be infinite.

The positive recurrent case. In this regime, we work under Assumption 6.2 and in this case the constant $m_1 = \mathbb{E}[\tau_1]$ is finite.

- For $\alpha \in (0, 2) \setminus \{1\}$, we define the process $(S_t^\alpha)_{t \geq 0} = (Z_{t/m_1}^\alpha)_{t \geq 0}$ where $(Z_t^\alpha)_{t \geq 0}$ is a stable process with Lévy measure π_α .
- For $\alpha = 1$, we define $(S_t^1)_{t \geq 0} = (Z_{t/m_1}^1)_{t \geq 0}$, where $(Z_t^1)_{t \geq 0}$ is a drifted Cauchy process with Lévy measure π_1 (and $f_+ = f_- = 1$), and drift $\mathbf{c}$, where $\mathbf{c}$ is the constant from Assumption 6.5-(ii).
- Finally, if $\sigma < \infty$, we define $(S_t^2)_{t \geq 0} = (\sigma W_{t/m_1})_{t \geq 0}$ where $(W_t)_{t \geq 0}$ is a Brownian motion.

The null recurrent case. In this regime, we work under Assumption 6.3. We set the parameter $\delta = 2 - 2\beta \in (0, 2)$ and define the skew-Bessel process $(R_t)_{t \geq 0}$ of parameter (δ, m_+, m_-) as the diffusion with scale function and speed measure defined as

$$\mathfrak{s}^R(x) = \text{sgn}(x) |x|^{2-\delta} \quad \text{and} \quad \mathbf{m}^R(dx) = \delta \text{sgn}_{m_+, m_-}(x) |x|^{\delta-1} dx.$$

Note that we have $\mathfrak{s}_s^R(x) = \text{sgn}_{m_+, m_-}(x) |x|^{\delta/(2-\delta)} = \text{sgn}_{m_+, m_-}(x) |x|^{1/\beta-1}$. Let $(\gamma_t^x)_{x \in \mathbb{R}, t \geq 0}$ be the family of local times of $(R_t)_{t \geq 0}$ and let $(\tau_t^R)_{t \geq 0}$ be the right-continuous inverse of $(\gamma_t^0)_{t \geq 0}$, which is exactly a β -stable subordinator with Lévy measure π_β . We also define $r_\pm = f_\pm/m_\pm$ and $\kappa = 2(\beta - \alpha)/\alpha$.

- For $\alpha \in (0, 1)$, we define the process $(U_t^\alpha)_{t \geq 0}$ as

$$U_t^\alpha = \frac{c_\alpha}{c_\beta} \int_0^t \text{sgn}_{r_+, -r_-}(R_s) |R_s|^\kappa ds,$$

- For $\alpha = 1$, we define $(U_t^1)_{t \geq 0}$ as

$$U_t^1 = \frac{\delta c_\alpha}{c_\beta} \int_{\mathbb{R}} \operatorname{sgn}(x) |x|^{\kappa+\delta-1} (\gamma_t^x - \gamma_t^0) dx + \mathbf{c} \gamma_t^0,$$

where $\mathbf{c}$ is the constant from Assumption 6.5-(ii).

- For $\alpha \in (1, 2)$, we define $(U_t^\alpha)_{t \geq 0}$ as

$$U_t^\alpha = \frac{\delta c_\alpha}{c_\beta} \int_{\mathbb{R}} \operatorname{sgn}_{f_+, -f_-}(x) |x|^{\kappa+\delta-1} (\gamma_t^x - \gamma_t^0) dx.$$

- Finally, if $\sigma < \infty$, we define $(U_t^2)_{t \geq 0} = (\sigma W_{\gamma_t^0})_{t \geq 0}$ where $(W_t)_{t \geq 0}$ is a Brownian motion independent of $(R_t)_{t \geq 0}$.

For $\alpha \in [1, 2)$, the process $(U_t^\alpha)_{t \geq 0}$ is indeed well defined since for every $t \geq 0$, a.s. the map $x \mapsto \gamma_t^x$ is θ -Hölder for any $\theta < \beta$. We have the following proposition.

Proposition 6.1. *If $\alpha \in (0, 2) \setminus \{1\}$, then $(\tau_t^R, U_{\tau_t^R}^\alpha)_{t \geq 0}$ is a Lévy process with Lévy measure $\pi_{\alpha, \beta}$ (without drift part and Brownian component). If $\alpha = 1$, then $(\tau_t^R, U_{\tau_t^R}^1)_{t \geq 0}$ is a Lévy process with Lévy measure $\pi_{1, \beta}$, drift part $(0, \mathbf{c})$ and no Brownian component.*

We omit the proof of this result as it is an easy consequence of Feller's representation of $(R_t)_{t \geq 0}$ and excursion theory, see also Proposition 3.9. Roughly, we already saw that $\mathbf{m}_s^R(x) = \operatorname{sgn}_{m_+, m_-}(x) |x|^{1/\beta-1}$ and one can check that the associated auxiliary function $\mathbf{m}_g$ (where $g(x) = \frac{c_\alpha}{c_\beta} \operatorname{sgn}_{r_+, -r_-}(x) |x|^\kappa$) is such that $\mathbf{m}_g(x) = \operatorname{sgn}_{f_+, -f_-}(x) |x|^{1/\alpha-1}$.

6.1.2 The main result

Under the different combinations of assumptions, we will establish the scaling limit of $(\zeta_t)_{t \geq 0}$. More precisely, we will show that there exists $c(\varepsilon)$ which vanishes as $\varepsilon \rightarrow 0$, and which is such that the process $(c(\varepsilon)\zeta_{t/\varepsilon})_{t \geq 0}$ converges in law as $\varepsilon \rightarrow 0$ in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology (introduced in Chapter 4). According, to the assumptions in force, we will use the following notations, where we recall that β and Λ_s (respectively α and Λ_f) are given by Assumption 6.3 (respectively Assumption 6.5).

- Under Assumptions 6.2 and 6.4, we set $b(\varepsilon) = \varepsilon$ and $a(\varepsilon) = \varepsilon^{1/2}$.
- Under Assumptions 6.2 and 6.5, we set $b(\varepsilon) = \varepsilon$ and $a(\varepsilon) = \varepsilon^{1/\alpha}/\Lambda_f(1/\varepsilon)$.
- Under Assumptions 6.3 and 6.4, we set $b(\varepsilon) = \varepsilon^{1/\beta}/\Lambda_s(1/\varepsilon)$ and $a(\varepsilon) = \varepsilon^{1/2}$.
- Under Assumptions 6.3 and 6.5, we set $b(\varepsilon) = \varepsilon^{1/\beta}/\Lambda_s(1/\varepsilon)$ and $a(\varepsilon) = \varepsilon^{1/\alpha}/\Lambda_f(1/\varepsilon)$.

We denote by b^{-1} an asymptotic inverse at 0 of b and we set $c(\varepsilon) = a(b^{-1}(\varepsilon))$. Our main result is the following.

Theorem 6.1. *Suppose that Assumption 6.1 holds and that one of the above combination of assumption holds. Then the family of processes $((c(\varepsilon)\zeta_{t/\varepsilon})_{t \geq 0})_{\varepsilon \in (0, 1)}$ is tight in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology. Moreover, for any sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ which decreases to zero and which is such that $(c(\varepsilon_n)\zeta_{t/\varepsilon_n})_{t \geq 0}$ converges in law to some limiting process $(\zeta_t^\infty)_{t \geq 0}$ we have:*

- (i) *If Assumptions 6.2 and 6.4 are in force, then for any $t \geq 0$, ζ_t^∞ is equal in law to S_t^2 .*

- (ii) If Assumptions 6.2 and 6.5 are in force, then for any $t \geq 0$, ζ_t^∞ is equal in law to S_t^α .
- (iii) If Assumptions 6.3 and 6.4 are in force, then for any $t \geq 0$, ζ_t^∞ is equal in law to U_t^2 .
- (iv) If Assumptions 6.3 and 6.5 are in force, then for any $t \geq 0$, ζ_t^∞ is equal in law to U_t^α .

We now make some important remarks on the main theorem. First, this theorem is obviously incomplete, but we stress that with a little bit of work, we could show the convergence in the finite dimensional distribution sense using for instance the techniques of Chapter 2 ; but we wanted to keep this chapter short. The idea here is to recover results from the tools introduced in Chapters 3 and 4. We stress that when $(X_t)_{t \geq 0}$ is a diffusion process, as in Chapter 2, this result completes the result of Chapter 2 since we have show therein the convergence of the finite-dimensional marginals.

Let us assume for a moment that the theorem is complete, i.e. we have convergence in law in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology. It is well-known that if a sequence of continuous functions converges to a continuous function in the $\mathbf{M}_1$ -topology, it also converges in the uniform topology (on compact-time intervals at least), see for instance the book of Whitt [Whi02, Chapter 3, Section 3.3]. In other words, the convergence in law in item (i), (iii) and (iv) of Theorem 6.1 would also hold in the space of continuous functions endowed with the uniform topology on compact-time intervals, since the limiting processes are continuous.

We also stress that in item (ii) of Theorem 6.1, we might also be able to remove from Assumption 6.1 the hypothesis

(S) The function f preserves the sign, i.e. we always have $xf(x) \geq 0$ and $f(0) = 0$.

by the less restricting assumption

(AS) There exists $A > 0$ such that for every $x \geq A$, $f(x) \geq 0$ and for every $x \leq -A$, we have $f(x) \leq 0$.

Indeed, suppose that the function f satisfies (AS) and that Assumptions 6.2 and 6.5 are in force. Then we can always find a function $\tilde{f}$ which preserves the sign and satisfies Assumption 6.1 and such that for every $|x| \geq A$, $\tilde{f}(x) = f(x)$. Letting $\tilde{\zeta}_t = \int_0^t \tilde{f}(X_s) ds$, we can apply the main theorem which tells us that $(c(\varepsilon)\tilde{\zeta}_{t/\varepsilon})_{t \geq 0}$ converges to a stable process as $\varepsilon \rightarrow 0$ in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology, where $c(\varepsilon)$ is specified above. Then we can study the remainder $\zeta_t - \tilde{\zeta}_t = \int_0^t [f - \tilde{f}](X_s) ds$ using the usual martingale technique: one can find a solution h to the Poisson equation $\mathcal{L}h = f - \tilde{f}$ which is compactly supported, see for instance the expression of the solution in Chapter 2. We deduce that $(\varepsilon^{1/2}(\zeta_{t/\varepsilon} - \tilde{\zeta}_{t/\varepsilon}))_{t \geq 0}$ converges in law as $\varepsilon \rightarrow 0$ towards a Brownian motion in the space of continuous functions endowed with the uniform topology on compact-time intervals. From this, we easily deduce that for every $T \geq 0$ and every $\eta > 0$,

$$\mathbb{P}\left(\sup_{s \in [0, T]} c(\varepsilon)|\zeta_{t/\varepsilon} - \tilde{\zeta}_{t/\varepsilon}| \geq \eta\right) \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

since we recall that in this case, $c(\varepsilon)$ is regularly varying of index $1/\alpha$ as $\varepsilon \rightarrow 0$ for some $\alpha \in (0, 2)$. This is enough to finally conclude that $(c(\varepsilon)\zeta_{t/\varepsilon})_{t \geq 0}$ converges to a stable process as $\varepsilon \rightarrow 0$ in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology.

Let us make some comments on the null recurrent case, i.e. on items (iii) and (iv) of Theorem 6.1. First, in this case, the limiting processes are not Markovian, which is a rather classical fact in the null recurrent regime. Regarding item (iii), the result is more or less known, see for instance the book of Höpfner and Löcherbach [HL03, Theorem 3.1 and Example 3.10].

Regarding item (iv), it was shown in Fournier-Tardif [FT21], that in some specific situation, some rescaled kinetic process converges to the integral of a symmetrized Bessel process which is recovered here by item (iv) for some $\alpha < 1$ and $\kappa = 1$. The limiting process seems new here when $\alpha \geq 1$.

6.2 Preliminaries

6.2.1 $\mathbf{M}_1$ -topology

In this subsection, we recall the definition and a few properties of the $\mathbf{M}_1$ -topology. For any $T > 0$, we denote by $\mathcal{D}_T = \mathcal{D}([0, T], \mathbb{R})$ the usual sets of càdlàg functions on $[0, T]$ valued in $\mathbb{R}$. For a function $x \in \mathcal{D}_T$, we define the completed graph $\Gamma_{T,x}$ of x as follows:

$$\Gamma_{T,x} = \{(t, z) \in [0, T] \times \mathbb{R}, z \in [x(t-), x(t)]\}.$$

The $\mathbf{M}_1$ -topology on $\mathcal{D}_T$ is metrizable through parametric representations of the complete graphs. A parametric representation of x is a continuous non-decreasing function (u, r) mapping $[0, 1]$ onto $\Gamma_{T,x}$. Let us denote by $\Pi_{T,x}$ the set of parametric representations of x . Then the $\mathbf{M}_1$ -distance on $\mathcal{D}_T$ is defined for $x_1, x_2 \in \mathcal{D}_T$ as

$$d_{\mathbf{M}_1,T}(x_1, x_2) = \inf_{(u_i, r_i) \in \Pi_{T,x_i}} (\|u_1 - u_2\| \vee \|r_1 - r_2\|),$$

where $\|\cdot\|$ is the uniform distance. The metric space $(\mathcal{D}_T, d_{\mathbf{M}_1,T})$ is separable and topologically complete.

Let us now denote by $\mathcal{D} = \mathcal{D}(\mathbb{R}_+, \mathbb{R})$ the set of càdlàg functions on $\mathbb{R}_+$. We introduce for any $t \geq 0$, the usual restriction map r_t from $\mathcal{D}$ to $\mathcal{D}_t$. Then the $\mathbf{M}_1$ -distance on $\mathcal{D}$ is defined for $x, y \in \mathcal{D}$ as

$$d_{\mathbf{M}_1}(x, y) = \int_0^\infty e^{-t} (d_{\mathbf{M}_1,t}(r_t(x), r_t(y)) \wedge 1) dt.$$

Again, the metric space $(\mathcal{D}, d_{\mathbf{M}_1})$ is separable and topologically complete. We now briefly recall some characterization of converging sequences in $(\mathcal{D}, d_{\mathbf{M}_1})$. To this end, we first introduce for $x \in \mathcal{D}$ and $\delta, T > 0$, the following oscillation function

$$w(x, T, \delta) = \sup_{t \in [0, T]} \sup_{t_{\delta-} \leq t_1 < t_2 < t_3 \leq t_{\delta+}} d(x(t_2), [x(t_1), x(t_3)]),$$

where $t_{\delta-} = 0 \vee (t - \delta)$, $t_{\delta+} = T \wedge (t + \delta)$ and $d(x(t_2), [x(t_1), x(t_3)])$ is the distance of $x(t_2)$ to the segment $[x(t_1), x(t_3)]$, i.e.

$$d(x(t_2), [x(t_1), x(t_3)]) = \begin{cases} 0 & \text{if } x(t_1) \in [x(t_2), x(t_3)], \\ |x(t_2) - x(t_1)| \wedge |x(t_2) - x(t_3)| & \text{otherwise.} \end{cases}$$

The following theorem characterizes tightness in the $\mathbf{M}_1$ -topology for sequences of random variables valued in $\mathcal{D}$. This result can be found in Whitt [Whi02, Chapter 12, Theorem 12.12.13].

Theorem 6.2. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables valued in $\mathcal{D}$. This sequence is tight in $(\mathcal{D}, d_{\mathbf{M}_1})$ if and only if the following two conditions hold:*

- (i) *There exists an increasing sequence $(T_p)_{p \in \mathbb{N}}$ such that $T_p \rightarrow \infty$ as $p \rightarrow \infty$ and for any $p \in \mathbb{N}$ and any $\eta > 0$,*

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(w(X_n, T_p, \delta) > \eta) = 0.$$

- (ii) *For any $T > 0$,*

$$\lim_{A \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [0, T]} |X_n(t)| > A\right) = 0.$$

6.2.2 Reminders from Chapter 3

In this section, we recall some results that were obtained in Chapter 3. First, the point 0 is regular for itself and is a recurrent point and in this setting, we have a theory of excursions of $(X_t)_{t \geq 0}$ away from 0. The process possesses a local time $(L_t)_{t \geq 0}$ at 0 and we denote by $(\tau_t)_{t \geq 0}$ its right continuous inverse, which is a subordinator. Its Laplace exponent is denoted by Φ .

Recall that $\mathcal{D}$ denote the usual space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}$, and for $\varepsilon \in \mathcal{D}$, let us introduce the length of ε

$$\ell(\varepsilon) = \inf\{t > 0, \varepsilon_t = 0\},$$

and we will often write for simplicity ℓ instead of $\ell(\varepsilon)$. Then the set of excursions $\mathcal{E}$ is the set of functions $\varepsilon \in \mathcal{D}$ such that: (i) $0 < \ell(\varepsilon) < \infty$; (ii) $\varepsilon_t = 0$ for every $t \geq \ell(\varepsilon)$; (iii) $\varepsilon_t \neq 0$ for every $0 < t < \ell(\varepsilon)$. This space is endowed with the usual Skorokhod's topology and the associated Borel σ -algebra. We will denote by $\mathbf{n}$ the excursion measure of $(X_t)_{t \geq 0}$ away from 0, which is a σ -finite measure on $\mathcal{E}$, see Chapter 3 for the precise definition.

Let us introduce some notations: for $t \geq 0$ we let $\mathcal{D}_t$ denote the space of càdlàg functions from $[0, t]$ to $\mathbb{R}$, and for $t > 0$ we let $\tilde{\mathcal{D}}_t$ denote the space of càdlàg functions from $[0, t)$ to $\mathbb{R}$. For $t > 0$, we also denote by $g_t = \sup\{s < t, X_s = 0\}$ the last zero of $(X_u)_{u \geq 0}$ before t . We recall the following proposition which holds for any generalized diffusion $(X_t)_{t \geq 0}$.

Proposition 6.2. *Let $e = e(q)$ be an exponential random variable of parameter q , independent of $(X_t)_{t \geq 0}$. Then the processes $(X_u)_{0 \leq u \leq g_e}$ and $(X_{g_e+v})_{0 \leq v \leq e-g_e}$ are independent. Moreover, for all non negative functionals $F_1 : \bigcup_{t>0} \tilde{\mathcal{D}}_t \rightarrow \mathbb{R}_+$ and $F_2 : \bigcup_{t \geq 0} \mathcal{D}_t \rightarrow \mathbb{R}_+$, we have*

$$\mathbb{E}[F_1((X_u)_{0 \leq u < g_e})] = \Phi(q) \int_0^\infty \mathbb{E}[F_1((X_u)_{0 \leq u < \tau_t})e^{-q\tau_t}] dt \quad (6.6)$$

and

$$\mathbb{E}[F_2((X_{g_e+v})_{0 \leq v \leq e-g_e})] = \frac{q}{\Phi(q)} \left(mF_2(\tilde{0}) + \mathbf{n} \left(\int_0^\ell e^{-qu} F_2((\varepsilon_v)_{0 \leq v \leq u}) du \right) \right), \quad (6.7)$$

where $\tilde{0}$ denotes the null function of $\tilde{\mathcal{D}}_0$.

We will also consider the Lévy process $(Z_t)_{t \geq 0}$ defined as

$$Z_t = \int_0^{\tau_t} f(X_s) ds = \zeta_{\tau_t}.$$

We also recall that $\mathbf{n}^B$ denotes the Brownian excursion measure, which only charges the subset of continuous excursions denoted by $\mathcal{C}$. Also recall the definition of the Lévy measures $\pi_{\alpha, \beta}$, π_α and π_β from (6.5) and the constants $\sigma^2 = 4 \int_{\mathbb{R}} [\mathbf{m}_f(\infty) - \mathbf{m}_f(x)]^2 dx$ and $m_1 = \mathbb{E}[\tau_1]$ which may be infinite. The following theorem was shown in Chapter 3.

Theorem 6.3. *Suppose that Assumption 6.1 holds. Then the following assertions hold.*

- (i) *Suppose that Assumptions 6.2 and 6.4 hold and let $(W_t)_{t \geq 0}$ be a Brownian motion. Then σ and m_1 are finite and we have*

$$(h\tau_{t/h}, h^{1/2}Z_{t/h})_{t \geq 0} \longrightarrow (m_1 t, \sigma W_t)_{t \geq 0} \quad \text{as } h \rightarrow 0$$

in law for the $\mathbf{J}_1$ -topology.

- (ii) *Suppose that Assumptions 6.2 and 6.5 hold and let $(Z_t^\alpha)_{t \geq 0}$ be a stable process with Lévy measure π_α . Then m_1 is finite and if we set $a(h) = h^{1/\alpha}/\Lambda_f(1/h)$, we have*

$$(h\tau_{t/h}, a(h)Z_{t/h})_{t \geq 0} \longrightarrow (m_1 t, Z_t^\alpha)_{t \geq 0} \quad \text{as } h \rightarrow 0$$

in law for the $\mathbf{J}_1$ -topology.

- (iii) Suppose that Assumptions 6.3 and 6.4 hold and let $(W_t)_{t \geq 0}$ be a Brownian motion and (τ_t^β) a β -stable subordinator with Lévy measure π_β , independent from the Brownian motion. Then σ is finite and if we set $b(h) = h^{1/\beta}/\Lambda_\beta(1/h)$, we have

$$\left(b(h)\tau_{t/h}, h^{1/2}Z_{t/h}\right)_{t \geq 0} \longrightarrow \left(\tau_t^\beta, \sigma W_t\right)_{t \geq 0} \quad \text{as } h \rightarrow 0$$

in law for the $\mathbf{J}_1$ -topology.

- (iv) Suppose that Assumptions 6.3 and 6.5 hold. If $\alpha \in (0, 2) \setminus \{1\}$, we let $(\tau_t^\beta, Z_t^\alpha)_{t \geq 0}$ be a Lévy process with Lévy measure $\pi_{\alpha, \beta}$ (and no drift and no Brownian part). If $\alpha = 1$, we let $(\tau_t^\beta, Z_t^1)_{t \geq 0}$ be a Lévy process with Lévy measure $\pi_{1, \beta}$ and drift $(0, \mathbf{c})$. Then, if we set $b(h) = h^{1/\beta}/\Lambda_\beta(1/h)$ and $a(h) = h^{1/\alpha}/\Lambda_\alpha(1/h)$, we have

$$\left(b(h)\tau_{t/h}, a(h)Z_{t/h}\right)_{t \geq 0} \longrightarrow \left(\tau_t^\beta, Z_t^\alpha\right)_{t \geq 0} \quad \text{as } h \rightarrow 0$$

in law for the $\mathbf{J}_1$ -topology.

Remark 6.1. Only items (iii) and (iv) were properly proved in Chapter 3. We justify here why items (i) and (ii) also hold.

- First, one can check that in the proof of Proposition 3.1 from Chapter 3, only Assumption 6.4 is used to show that $(h^{1/2}Z_{t/h})$ converges in law to a Brownian motion. Actually, the proof is quite simple, we only need to show that $(Z_t)_{t \geq 0}$ has a moment of order 2 (and that it is centered). It is shown therein that, if Assumption 6.4 holds, then $\mathbb{E}[Z_t^2] = t\sigma^2$ for any $t \geq 0$. This also justifies the value of σ .
- Second, one can carefully read the proof of Proposition 3.2 from Chapter 3 and see that, if Assumption 6.5 holds, then $(a(h)Z_{t/h})_{t \geq 0}$ converges in law to $(S_t^\alpha)_{t \geq 0}$, where $a(h)$ and $(S_t^\alpha)_{t \geq 0}$ are as in the above theorem.
- Third, if Assumption 6.2 holds, then 0 is a positive recurrent point for $(X_t)_{t \geq 0}$ which implies that $m_1 = \mathbb{E}[\tau_1] < \infty$. By the strong law of large numbers, this also implies that $(h\tau_{t/h})_{t \geq 0}$ converges in law to $(m_1 t)_{t \geq 0}$ as $h \rightarrow 0$. Then we can recover items (i) and (ii) from Slutsky's lemma and the two above points.

6.3 Tightness

In this section, we will show that the family of processes $((c(\varepsilon)\zeta_{t/\varepsilon})_{t \geq 0})_{\varepsilon \in (0,1)}$ is tight in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology. We will always assume that one of the following combination of assumptions hold: Assumptions 6.2 and 6.4 / Assumptions 6.2 and 6.5 / Assumptions 6.3 and 6.4 / Assumptions 6.3 and 6.5. Recall the definitions of $a(\varepsilon)$, $b(\varepsilon)$ and $c(\varepsilon)$ from Subsection 6.1.2. Then by Theorem 6.3, the process $(c(\varepsilon)Z_{t/b^{-1}(\varepsilon)})_{t \geq 0}$ converges in law as $\varepsilon \rightarrow 0$ in the space of càdlàg function endowed with the $\mathbf{J}_1$ -topology. Since the $\mathbf{M}_1$ -topology is weaker than the $\mathbf{J}_1$ -topology, it also converges in law in $(\mathcal{D}, d_{\mathbf{M}_1})$ and therefore the family of processes $((c(\varepsilon)Z_{t/b^{-1}(\varepsilon)})_{t \geq 0})_{\varepsilon \in (0,1)}$ is tight in both topology.

By Theorem 6.3 again, the process $(b(\varepsilon)\tau_{t/\varepsilon})_{t \geq 0}$ converges in law as $\varepsilon \rightarrow 0$ (in the $\mathbf{J}_1$ -topology). Then, if we denote by $(L_t)_{t \geq 0}$ the local time of $(X_t)_{t \geq 0}$ at the level 0, which is the right-continuous inverse of $(\tau_t)_{t \geq 0}$, it is rather classical that the rescaled process $(b^{-1}(\varepsilon)L_{t/\varepsilon})_{t \geq 0}$ also converges in law as $\varepsilon \rightarrow 0$ in the uniform topology to a limiting process which is continuous. Indeed, under Assumption 6.2, $(b(\varepsilon)\tau_{t/\varepsilon})_{t \geq 0}$ converges in law to $(m_1 t)_{t \geq 0}$ and $(b^{-1}(\varepsilon)L_{t/\varepsilon})_{t \geq 0}$ will then converges to $(t/m_1)_{t \geq 0}$. Under Assumption 6.3, $(b(\varepsilon)\tau_{t/\varepsilon})_{t \geq 0}$ converges in law to a

β -stable subordinator $(\tau_t^\beta)_{t \geq 0}$ and $(b^{-1}(\varepsilon)L_{t/\varepsilon})_{t \geq 0}$ will then converges to the right-continuous inverse of $(\tau_t^\beta)_{t \geq 0}$, which is continuous (it is the local time of a Bessel process). In any case, the family $((b^{-1}(\varepsilon)L_{t/\varepsilon})_{t \geq 0})_{\varepsilon \in (0,1)}$ is tight under the uniform topology and therefore, it holds that for any $T > 0$ and any $\theta > 0$, we have

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P} \left(\sup_{s, t \in [0, T], |s-t| < \delta} b^{-1}(\varepsilon) |L_{s/\varepsilon} - L_{t/\varepsilon}| > \theta \right) = 0. \quad (6.8)$$

With these facts in mind, we will show the following theorem.

Theorem 6.4. *The family of processes $((c(\varepsilon)\zeta_{t/\varepsilon})_{t \geq 0})_{\varepsilon \in (0,1)}$ is tight in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology.*

The proof is very similar to the proof given in Chapter 4. We recall that for $t \geq 0$, g_t and d_t respectively denotes the last zero before t and the first zero after t of $(X_t)_{t \geq 0}$. It holds that $g_t = \tau_{L_t-}$ and $d_t = \tau_{L_t}$. Moreover, if $t \geq 0$ is such that $X_t = 0$, then either $t = g_t$ or $t = d_t$. Indeed, if 0 is regular and holding for $(X_t)_{t \geq 0}$, then this is straightforward, whereas if 0 is regular and instantaneous, then the closure of the zero set of $(X_t)_{t \geq 0}$ has no isolated points. We finally introduce the random function $\nu : \mathbb{R}_+ \rightarrow \{-1, 0, 1\}$ defined as $\nu(t) = \mathbf{1}_{\{X_t > 0\}} - \mathbf{1}_{\{X_t < 0\}}$.

Proof. Step 1: We first show that for any $T > 0$, for any $\eta > 0$, we have

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P} \left(w((c(\varepsilon)\zeta_{t/\varepsilon})_{t \geq 0}, T, \delta) > \eta \right) = 0, \quad (6.9)$$

where we recall that $w(x, T, \delta)$ has been defined in Subsection 6.2.1 for a function $x \in \mathcal{D}$ and $T, \delta > 0$ as

$$w(x, T, \delta) = \sup_{t \in [0, T]} \sup_{t_{\delta-} \leq t_1 < t_2 < t_3 \leq t_{\delta+}} d(x(t_2), [x(t_1), x(t_3)]),$$

where $t_{\delta-} = 0 \vee (t - \delta)$, $t_{\delta+} = T \wedge (t + \delta)$ and $d(x(t_2), [x(t_1), x(t_3)])$ is the distance of $x(t_2)$ to the segment $[x(t_1), x(t_3)]$. For any $T, \varepsilon > 0$ and any $\theta, A > 0$, we introduce the event

$$A_{T, \delta}^\varepsilon = \left\{ \sup_{s, t \in [0, T], |s-t| < \delta} b^{-1}(\varepsilon) |L_{s/\varepsilon} - L_{t/\varepsilon}| < \theta \right\} \cap \left\{ b^{-1}(\varepsilon)L_{T/\varepsilon} < M \right\}.$$

We will first assume that we are on the event $A_{T, \delta}^\varepsilon$ for some $T, \theta, M > 0$ and $\varepsilon \in (0, 1)$ fixed (we omit the dependence in the variables θ and M). Let $t \in [0, T]$ and $t_{\delta-} \leq t_1 < t_2 < t_3 \leq t_{\delta+}$, where $t_{\delta-}$ and $t_{\delta+}$ are defined as above. We will bound the distance $d(\zeta_{t_2/\varepsilon}, [\zeta_{t_1/\varepsilon}, \zeta_{t_3/\varepsilon}])$ in order to show that

$$w((c(\varepsilon)\zeta_{t/\varepsilon})_{t \geq 0}, T, \delta) \mathbf{1}_{A_{T, \delta}^\varepsilon} \leq w((c(\varepsilon)Z_{t/b^{-1}(\varepsilon)})_{t \geq 0}, M, \theta) \mathbf{1}_{A_{T, \delta}^\varepsilon}. \quad (6.10)$$

Without loss of generality, we may assume that $\zeta_{t_1/\varepsilon} \leq \zeta_{t_3/\varepsilon}$.

- First case: $\zeta_{t_1/\varepsilon} \leq \zeta_{t_2/\varepsilon} \leq \zeta_{t_3/\varepsilon}$. Then we have

$$d(\zeta_{t_2/\varepsilon}, [\zeta_{t_1/\varepsilon}, \zeta_{t_3/\varepsilon}]) = 0 \leq w((c(\varepsilon)Z_{t/b^{-1}(\varepsilon)})_{t \geq 0}, M, \theta).$$

- Second case: $\zeta_{t_2/\varepsilon} < \zeta_{t_1/\varepsilon} \leq \zeta_{t_3/\varepsilon}$. In this case, $d(\zeta_{t_2/\varepsilon}, [\zeta_{t_1/\varepsilon}, \zeta_{t_3/\varepsilon}]) = \zeta_{t_1/\varepsilon} - \zeta_{t_2/\varepsilon}$. Let us note that, since $(\zeta_t)_{t \geq 0}$ is monotonic on every excursion of $(X_t)_{t \geq 0}$, t_1/ε and t_3/ε can not belong to the same excursion, i.e. $d_{t_1/\varepsilon} \leq g_{t_3/\varepsilon}$. We define, for $i \in \{1, 2, 3\}$ and $\varepsilon > 0$, the positive real numbers $u_{i, \varepsilon}$ defined as follows

1. $u_{2, \varepsilon} = g_{t_2/\varepsilon}$ if $\nu(t_2/\varepsilon) = 1$, $u_{2, \varepsilon} = d_{t_2/\varepsilon}$ if $\nu(t_2/\varepsilon) = -1$ and $u_{2, \varepsilon} = t_2/\varepsilon$ if $\nu(t_2/\varepsilon) = 0$. Since $(\zeta_t)_{t \geq 0}$ is monotonic on every excursion of $(X_t)_{t \geq 0}$, we have $\zeta_{t_2/\varepsilon} \geq \zeta_{u_{2, \varepsilon}}$.

2. For $i \in \{1, 3\}$, $u_{i,\varepsilon} = d_{t_i/\varepsilon}$ if $\nu(t_i/\varepsilon) = 1$, $u_{i,\varepsilon} = g_{t_i/\varepsilon}$ if $\nu(t_i/\varepsilon) = -1$ and $u_{i,\varepsilon} = t_i/\varepsilon$ if $\nu(t_i/\varepsilon) = 0$. We have $\zeta_{t_i/\varepsilon} \leq \zeta_{u_{i,\varepsilon}}$.

Therefore, we have $\zeta_{u_{2,\varepsilon}} < \zeta_{u_{1,\varepsilon}}$ and $\zeta_{u_{2,\varepsilon}} < \zeta_{u_{3,\varepsilon}}$ so that necessarily, $u_{1,\varepsilon} < u_{2,\varepsilon} < u_{3,\varepsilon}$. Observe that in any case, we have for every $i \in \{1, 2, 3\}$, $u_{i,\varepsilon} = \tau_{L_{t_i/\varepsilon}}$ or $u_{i,\varepsilon} = \tau_{L_{t_i/\varepsilon}-}$. Let $v > 0$ and remember that on the event A_T^ε , we have $|L_{t_i/\varepsilon} - L_{t_j/\varepsilon}| < \theta/b^{-1}(\varepsilon)$ for every $i, j \in \{1, 2, 3\}$ and $L_{t_i/\varepsilon} \leq M/b^{-1}(\varepsilon)$ for every $i \in \{1, 2, 3\}$. Using the fact that $(\tau_t)_{t \geq 0}$ is increasing and that $(\zeta_t)_{t \geq 0}$ is continuous, we can always find $s \in [0, M]$ and $s_{\theta-} \leq s_1 < s_2 < s_3 \leq s_{\theta+}$ where $s_{\theta-} = 0 \wedge (s - \theta)$ and $s_{\theta+} = M \vee (s + \theta)$, such that

$$\zeta_{t_2/\varepsilon} \geq \zeta_{\tau_{s_2/b^{-1}(\varepsilon)}} - \frac{v}{2}, \quad \zeta_{t_1/\varepsilon} \leq \zeta_{\tau_{s_1/b^{-1}(\varepsilon)}} + \frac{v}{2} \quad \text{and} \quad \zeta_{t_3/\varepsilon} \leq \zeta_{\tau_{s_3/b^{-1}(\varepsilon)}} + \frac{v}{2},$$

which leads to

$$\begin{aligned} d(\zeta_{t_2/\varepsilon}, [\zeta_{t_1/\varepsilon}, \zeta_{t_3/\varepsilon}]) &\leq (Z_{s_1/b^{-1}(\varepsilon)} - Z_{s_2/b^{-1}(\varepsilon)}) \wedge (Z_{s_3/b^{-1}(\varepsilon)} - Z_{s_2/b^{-1}(\varepsilon)}) + v \\ &= d(Z_{s_2/b^{-1}(\varepsilon)}, [Z_{s_1/b^{-1}(\varepsilon)}, Z_{s_3/b^{-1}(\varepsilon)}]) + v. \end{aligned}$$

As a consequence, we have

$$d(\zeta_{t_2/\varepsilon}, [\zeta_{t_1/\varepsilon}, \zeta_{t_3/\varepsilon}]) \leq \sup_{t \in [0, M]} \sup_{t_{\theta-} \leq t_1 < t_2 < t_3 \leq t_{\theta+}} d(Z_{t_2/b^{-1}(\varepsilon)}, [Z_{t_1/b^{-1}(\varepsilon)}, Z_{t_3/b^{-1}(\varepsilon)}]) + v.$$

Since this holds for every $v > 0$, we conclude that, on the event $A_{T,\delta}^\varepsilon$, it holds that

$$d(\zeta_{t_2/\varepsilon}, [\zeta_{t_1/\varepsilon}, \zeta_{t_3/\varepsilon}]) \leq w((c(\varepsilon)Z_{t/b^{-1}(\varepsilon)})_{t \geq 0}, M, \theta)$$

- Third case: $\zeta_{t_1/\varepsilon} \leq \zeta_{t_3/\varepsilon} < \zeta_{t_2/\varepsilon}$. We can adapt the previous case to deduce that on the event $A_{T,\delta}^\varepsilon$, the above inequality also holds.

This shows that (6.10) holds. Next, we introduce the events

$$B_{T,\delta}^\varepsilon = \left\{ \sup_{s,t \in [0,T], |s-t| < \delta} b^{-1}(\varepsilon) |L_{s/\varepsilon} - L_{t/\varepsilon}| \geq \theta \right\} \quad \text{and} \quad C_T^\varepsilon = \{b^{-1}(\varepsilon)L_{T/\varepsilon} \geq M\}$$

so that $(A_{T,\delta}^\varepsilon)^c = B_{T,\delta}^\varepsilon \cup C_T^\varepsilon$. Then, by (6.10), we have

$$\mathbb{P}\left(w((c(\varepsilon)\zeta_{t/\varepsilon})_{t \geq 0}, T, \delta) > \eta\right) \leq \mathbb{P}\left(w((c(\varepsilon)Z_{t/b^{-1}(\varepsilon)})_{t \geq 0}, M, \theta) > \eta\right) + \mathbb{P}(B_{T,\delta}^\varepsilon) + \mathbb{P}(C_T^\varepsilon).$$

Remember now that the process $(b^{-1}(\varepsilon)L_{t/\varepsilon})_{t \geq 0}$ converges in law as $\varepsilon \rightarrow 0$ and denote by $(\bar{L}_t)_{t \geq 0}$ the limiting process. By (6.10) and the Portmanteau theorem, we get

$$\begin{aligned} \limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}\left(w((c(\varepsilon)\zeta_{t/\varepsilon})_{t \geq 0}, T, \delta) > \eta\right) &\leq \limsup_{\varepsilon \rightarrow 0} \mathbb{P}\left(w((c(\varepsilon)Z_{t/b^{-1}(\varepsilon)})_{t \geq 0}, M, \theta) > \eta\right) \\ &\quad + \mathbb{P}(\bar{L}_T \geq M). \end{aligned}$$

Since the family of processes $((c(\varepsilon)Z_{t/b^{-1}(\varepsilon)})_{t \geq 0})$ is tight in $\mathcal{D}$ endowed with the $\mathbf{M}_1$ -topology, and since $\bar{L}_T$ is a.s. finite, we can let $\theta \rightarrow 0$ and $M \rightarrow \infty$ to deduce that (6.9) holds.

Step 2: We now show that for any $T > 0$, we have

$$\lim_{A \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}\left(\sup_{t \in [0, T]} |c(\varepsilon)\zeta_{t/\varepsilon}| \geq A\right) = 0. \quad (6.11)$$

Since the process $(\zeta_t)_{t \geq 0}$ is monotonic on every excursion of $(X_t)_{t \geq 0}$, it holds that for any $t \geq 0$, we have $\sup_{s \in [0, \tau_t]} |\zeta_s| = \sup_{s \in [0, t]} |Z_s|$. Consider some $A, M, T > 0$, and remember that $d_T = \tau_{L_T} \geq T$. We have the following bound:

$$\begin{aligned} \mathbb{P}\left(\sup_{t \in [0, T]} |c(\varepsilon)\zeta_{t/\varepsilon}| \geq A\right) &\leq \mathbb{P}\left(\sup_{t \in [0, L_T/\varepsilon]} |c(\varepsilon)Z_t| \geq A\right) \\ &\leq \mathbb{P}\left(\sup_{t \in [0, M]} |c(\varepsilon)Z_{t/b^{-1}(\varepsilon)}| \geq A\right) + \mathbb{P}\left(b^{-1}(\varepsilon)L_{T/\varepsilon} \geq M\right). \end{aligned}$$

Using that $(c(\varepsilon)Z_{t/b^{-1}(\varepsilon)})_{t \geq 0}$ and $(b^{-1}(\varepsilon)L_{t/\varepsilon})_{t \geq 0}$ converge in law as $\varepsilon \rightarrow 0$, and using the same notations as in the previous step, we get that

$$\limsup_{A \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}\left(\sup_{t \in [0, T]} |c(\varepsilon)\zeta_{t/\varepsilon}| \geq A\right) \leq \mathbb{P}(\bar{L}_T \geq M).$$

Letting $M \rightarrow \infty$ completes the proof. $\square$

6.4 Proof of Theorem 6.1

Before starting the proof, let us recall some notations and facts from Chapter 3. First remember that Φ denotes the Laplace exponent of $(\tau_t)_{t \geq 0}$, i.e. for any $q > 0$, $\mathbb{E}[e^{-q\tau_t}] = e^{-t\Phi(q)}$. Also recall the definition of $I_t = \zeta_t - \zeta_{g_t} = \int_{g_t}^t f(X_s)ds$. Let $e = e(q)$ be an independent exponential random variable of parameter $q > 0$. In Chapter 3, we showed that

- There exists a constant $r > 0$ such that $\Phi(q) \sim rb^{-1}(q)$ as $q \rightarrow 0$.
- Under Assumption 6.2, I_e converges in law as $q \rightarrow 0$ to some non-degenerate random variable, see Lemma 3.1.
- Under Assumption 6.4, $\Phi(q)^{1/2}I_e$ converges to 0 in probability as $q \rightarrow 0$, see the proof of Theorem 3.5.
- Under Assumptions 6.3 and 6.5, $c(q)I_e$ converges in law as $q \rightarrow 0$, see the proof of Proposition 3.2.

We can now move on to the proof.

Proof of Theorem 6.1. By Theorem 6.4, the family of processes $((c(\varepsilon)\zeta_{t/\varepsilon})_{t \geq 0})_{\varepsilon \in (0,1)}$ is tight in the space of càdlàg functions endowed with the $\mathbf{M}_1$ -topology. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a decreasing sequence which converges to 0 as $n \rightarrow \infty$ such that the sequence of processes $(c(\varepsilon_n)\zeta_{t/\varepsilon_n})_{t \geq 0}$ converges in law in the space $\mathcal{D}$ endowed with the $\mathbf{M}_1$ -topology towards a limiting process $(\zeta_t^\infty)_{t \geq 0}$. We will show that for any $t \geq 0$, the law of ζ_t^∞ is equal to the limiting law announced in Theorem 6.1.

First we consider for $n \in \mathbb{N}$ and $\lambda > 0$, an exponential random variable e_n of parameter ε_n , independent of $(X_t)_{t \geq 0}$. Then for any $\xi \in \mathbb{R}$, we have

$$\mathbb{E}\left[e^{i\xi c(\varepsilon_n)\zeta_{e_n}}\right] = \lambda \int_0^\infty e^{-\lambda t} \mathbb{E}\left[e^{i\xi c(\varepsilon_n)\zeta_{t/\varepsilon_n}}\right] dt \longrightarrow \lambda \int_0^\infty e^{-\lambda t} \mathbb{E}\left[e^{i\xi \zeta_t^\infty}\right] dt \quad \text{as } n \rightarrow \infty.$$

By Proposition 6.2, $\zeta_{g_{e_n}}$ and I_{e_n} are independent and for any $\xi \in \mathbb{R}$, we also have

$$\begin{aligned} \mathbb{E}\left[e^{i\xi c(\varepsilon_n)\zeta_{e_n}}\right] &= \mathbb{E}\left[e^{i\xi c(\varepsilon_n)I_{e_n}}\right] \times \mathbb{E}\left[e^{i\xi c(\varepsilon_n)\zeta_{g_{e_n}}}\right] \\ &= \mathbb{E}\left[e^{i\xi c(\varepsilon_n)I_{e_n}}\right] \times \Phi(\lambda \varepsilon_n) \int_0^\infty \mathbb{E}\left[e^{-\lambda \varepsilon_n \tau_t + i\xi c(\varepsilon_n)Z_t}\right] dt. \end{aligned} \quad (6.12)$$

We will now split the proof according to the four cases of Theorem 6.1.

Case (i): we work under Assumptions 6.2 and 6.4. Then in this case, we recall that $b(\varepsilon) = \varepsilon$ and $a(\varepsilon) = \varepsilon^{1/2}$. By Theorem 6.3, the process $(h\tau_{t/h}, h^{1/2}Z_{t/h})$ converges in law to $(m_1 t, \sigma W_t)_{t \geq 0}$ as $h \rightarrow 0$ in the space of càdlàg functions endowed with the $\mathbf{J}_1$ -topology, where $(W_t)_{t \geq 0}$ is a Brownian motion. Moreover, since in this case I_{e_n} converges in law as $n \rightarrow \infty$ and since $c(\varepsilon_n) \rightarrow 0$ as $n \rightarrow \infty$, it comes that $c(\varepsilon_n)I_{e_n}$ converges to 0 in probability as $n \rightarrow \infty$. By a small change of variables in (6.12), we see that

$$\mathbb{E}\left[e^{i\xi c(\varepsilon_n)\zeta_{e_n}}\right] = \mathbb{E}\left[e^{i\xi c(\varepsilon_n)I_{e_n}}\right] \times \frac{\Phi(\lambda\varepsilon_n)}{\varepsilon_n} \int_0^\infty \mathbb{E}\left[e^{-\lambda\varepsilon_n\tau_{t/\varepsilon_n} + i\xi c(\varepsilon_n)Z_{t/\varepsilon_n}}\right] dt.$$

Then, since $\Phi(q) \sim m_1 q$ as $q \rightarrow 0$, we get that $\Phi(\lambda\varepsilon_n)/\varepsilon_n \rightarrow \lambda m_1$ as $n \rightarrow \infty$. Moreover the constant $k := \inf_{q \in (0,1)} \Phi(q)/q$ is strictly positive and therefore for any $t \geq 0$

$$\left| \mathbb{E}\left[e^{-\lambda\varepsilon_n\tau_{t/\varepsilon_n} + i\xi c(\varepsilon_n)Z_{t/\varepsilon_n}}\right] \right| \leq \mathbb{E}\left[e^{-\lambda\varepsilon_n\tau_{t/\varepsilon_n}}\right] = e^{-t\Phi(\lambda\varepsilon_n)/\varepsilon_n} \leq e^{-\lambda k t}.$$

Consequently, we can use the dominated convergence theorem to deduce that

$$\mathbb{E}\left[e^{i\xi c(\varepsilon_n)\zeta_{e_n}}\right] \longrightarrow \lambda m_1 \int_0^\infty e^{-\lambda m_1 t} \mathbb{E}\left[e^{i\xi \sigma W_t}\right] dt = \lambda \int_0^\infty e^{-\lambda t} \mathbb{E}\left[e^{i\xi Z_t^2}\right] dt,$$

where we recall that $(Z_t^2)_{t \geq 0} = (\sigma W_{t/m_1})_{t \geq 0}$. Finally, we see that for any $\lambda > 0$,

$$\int_0^\infty e^{-\lambda t} \mathbb{E}\left[e^{i\xi \zeta_t^\infty}\right] dt = \int_0^\infty e^{-\lambda t} \mathbb{E}\left[e^{i\xi Z_t^2}\right] dt.$$

By injectivity of the Laplace transform, we get that for any $t \geq 0$, $\mathbb{E}[e^{i\xi \zeta_t^\infty}] = \mathbb{E}[e^{i\xi Z_t^2}]$, which shows the result.

Case (ii): we work under Assumptions 6.2 and 6.5. In this case, $b(\varepsilon) = \varepsilon$, $a(\varepsilon) = \varepsilon^{1/\alpha}/\Lambda_f(1/\varepsilon)$ and the process $(h\tau_{t/h}, a(h)Z_{t/h})$ converges in law to $(m_1 t, S_t^\alpha)_{t \geq 0}$ as $h \rightarrow 0$. We can apply the same line of proof as in the previous case.

Case (iii): we work under Assumptions 6.3 and 6.4. In this case, $b(\varepsilon) = \varepsilon^{1/\beta}/\Lambda_s(1/\varepsilon)$ and $a(\varepsilon) = \varepsilon^{1/2}$. By Theorem 6.3, $(b(h)\tau_{t/h}, a(h)Z_{t/h})_{t \geq 0}$ converges in law to $(\tau_t^\beta, \sigma W_t)_{t \geq 0}$ where $(\tau_t^\beta)_{t \geq 0}$ is a β -stable subordinator with Lévy measure π_β independent of the Brownian motion $(W_t)_{t \geq 0}$. Therefore, we have

$$-\log \mathbb{E}\left[e^{-b(q)\tau_{1/q}}\right] = \frac{\Phi(b(q))}{q} \longrightarrow \Phi^\beta(1) \quad \text{as } q \rightarrow 0,$$

where Φ^β is the Laplace exponent of $(\tau_t^\beta)_{t \geq 0}$. Since b^{-1} is an asymptotic inverse of b , we get that $\Phi(q) \sim \Phi^\beta(1)b^{-1}(q)$ as $q \rightarrow 0$. Since $c(q) = (b^{-1}(q))^{1/2}$ and since $\Phi(\varepsilon_n)^{1/2}I_{e_n}$ converges to 0 in probability, we see that $c(\varepsilon_n)I_{e_n}$ also converges to 0 in probability. Let us now make the change of variables $t = u/b^{-1}(\varepsilon_n)$ in (6.12), we get that

$$\mathbb{E}\left[e^{i\xi c(\varepsilon_n)\zeta_{e_n}}\right] = \mathbb{E}\left[e^{i\xi c(\varepsilon_n)I_{e_n}}\right] \times \frac{\Phi(\lambda\varepsilon_n)}{b^{-1}(\varepsilon_n)} \int_0^\infty \mathbb{E}\left[e^{-\lambda\varepsilon_n\tau_{t/b^{-1}(\varepsilon_n)} + i\xi c(\varepsilon_n)Z_{t/b^{-1}(\varepsilon_n)}}\right] dt.$$

Since Φ is regularly varying at 0 with index β , it is clear that $\Phi(\lambda\varepsilon_n)/b^{-1}(\varepsilon_n) \rightarrow \Phi^\beta(1)\lambda^\beta$. As in the previous cases, we can dominate the integrand in the above integral to deduce that

$$\mathbb{E}\left[e^{i\xi c(\varepsilon_n)\zeta_{e_n}}\right] \longrightarrow \Phi^\beta(1)\lambda^\beta \int_0^\infty \mathbb{E}\left[e^{-\lambda\tau_t^\beta + i\xi\sigma W_t}\right] dt \quad \text{as } n \rightarrow \infty.$$

Since $(\tau_t^\beta)_{t \geq 0}$ and $(W_t)_{t \geq 0}$ are independent, the above limit is equal to

$$\Phi^\beta(1)\lambda^\beta \int_0^\infty e^{-(\Phi^\beta(1)\lambda^\beta + \xi^2\sigma^2)t} dt = \frac{\Phi^\beta(1)\lambda^\beta}{\Phi^\beta(1)\lambda^\beta + \xi^2\sigma^2}. \quad (6.13)$$

Now we consider the process $(\gamma_t^0)_{t \geq 0}$ defined as the right-continuous inverse of $(\tau_t^\beta)_{t \geq 0}$. As it was announced in Subsection 6.1.1, this process can be seen as the local time at the level 0 of a skew Bessel process $(R_t)_{t \geq 0}$ of parameter (δ, m_+, m_-) where $\delta = 2 - 2\beta \in (0, 2)$. The Bessel process is taken to be independent from the Brownian motion $(W_t)_{t \geq 0}$. We will show that the limiting quantity that we found is exactly equal to the Laplace transform of the function $t \mapsto \mathbb{E}[\exp(i\xi\sigma W_{\gamma_t^0})]$, which will end the proof for this case. By independence, we get that

$$\lambda \int_0^\infty e^{-\lambda t} \mathbb{E}[\exp(i\xi\sigma W_{\gamma_t^0})] dt = \lambda \int_0^\infty e^{-\lambda t} \mathbb{E}[e^{-\xi^2\sigma^2\gamma_t^0}] dt = \mathbb{E}[e^{-\xi^2\sigma^2\gamma_{e(\lambda)}^0}]$$

where $e(\lambda)$ is an independent exponential random variable of parameter $\lambda > 0$. To compute this quantity, we can apply Proposition 6.2 with the Markov process $(R_t)_{t \geq 0}$ and the functional γ_t^0 . Since the zero set of $(R_t)_{t \geq 0}$ is almost surely Lebesgue-null and since for any $t \geq 0$, $\gamma_t^0 = \gamma_{g_t^R}^0$ where $g_t^R = \sup\{s < t, R_s = 0\}$, we have

$$\mathbb{E}[e^{-\xi^2\sigma^2\gamma_{e(\lambda)}^0}] = \Phi^\beta(\lambda) \int_0^\infty \mathbb{E}[\exp(-\lambda\tau_t^\beta - \xi^2\sigma^2\gamma_{\tau_t^\beta}^0)] dt.$$

But since $\Phi^\beta(\lambda) = \Phi^\beta(1)\lambda^\beta$ and almost surely for any $t \geq 0$, $\gamma_{\tau_t^\beta}^0 = t$, an easy computation shows that the above quantity is equal to the quantity from (6.13) and we deduce that for any $t \geq 0$ and any $\xi \in \mathbb{R}$, $\mathbb{E}[e^{i\xi\zeta_t^\infty}] = \mathbb{E}[e^{i\xi U_t^2}]$.

Case (iv): we work under Assumptions 6.3 and 6.5. In this case, $b(\varepsilon) = \varepsilon^{1/\beta}/\Lambda_\beta(1/\varepsilon)$ and $a(\varepsilon) = \varepsilon^{1/\alpha}/\Lambda_\alpha(1/\varepsilon)$ and the following convergence in law holds in the space of càdlàg functions endowed with the $\mathbf{J}_1$ -topology:

$$(b(h)\tau_{t/h}, a(h)Z_{t/h})_{t \geq 0} \longrightarrow (\tau_t^\beta, Z_t^\alpha)_{t \geq 0} \quad \text{as } h \rightarrow 0,$$

where $(\tau_t^\beta, Z_t^\alpha)_{t \geq 0}$ is a Lévy process with Lévy measure $\pi_{\alpha, \beta}$ and a drift equal to $(0, \mathbf{c})$ in the case $\alpha = 1$. Moreover, it was shown in Chapter 3, see (3.45) and Step 2 in the proof of Proposition 3.2, that we have

$$\mathbb{E}[e^{i\xi c(\varepsilon_n)I_{\varepsilon_n}}] \longrightarrow \frac{\lambda}{\Phi^\beta(\lambda)} \mathbf{n}^B \left(\int_0^\ell \exp(-\lambda A_t^\beta + i\xi A_t^\alpha) dA_t^\beta \right) =: \frac{G(\lambda, \xi)}{\Phi^\beta(\lambda)} \quad \text{as } n \rightarrow \infty,$$

where Φ^β denotes the Laplace exponent of $(\tau_t^\beta)_{t \geq 0}$ and for any $\varepsilon \in \mathcal{C}$ and any $t \in [0, \ell]$,

$$A_t^\beta = c_\beta \int_0^t \text{sgn}_{m_+, m_-}(\varepsilon_s) |\varepsilon_s|^{1/\beta-2} ds \quad \text{and} \quad A_t^\alpha = c_\alpha \int_0^t \text{sgn}_{f_+, -f_-}(\varepsilon_s) |\varepsilon_s|^{1/\alpha-2} ds.$$

As in the previous case, we see that

$$\mathbb{E}[e^{i\xi c(\varepsilon_n)\zeta_{\varepsilon_n}}] \longrightarrow G(\lambda, \xi) \int_0^\infty \mathbb{E}[e^{-\lambda\tau_t^\beta + i\xi Z_t^\alpha}] dt \quad \text{as } n \rightarrow \infty.$$

We now consider the process $(U_t^\alpha)_{t \geq 0}$ defined in Subsection 6.1.1. We recall that if we consider a skew-Bessel process $(R_t)_{t \geq 0}$ of parameter (δ, m_+, m_-) where $\delta = 2 - 2\beta \in (0, 2)$, as well as its family of local times $(\gamma_t^x)_{t \geq 0, x \in \mathbb{R}}$ and the constants $r_\pm = f_\pm/m_\pm$ and $\kappa = 2(\beta - \alpha)/\alpha$, the process $(U_t^\alpha)_{t \geq 0}$ is defined as

- If $\alpha \in (0, 1)$, then for any $t \geq 0$, $U_t^\alpha = \frac{c_\alpha}{c_\beta} \int_0^t \operatorname{sgn}_{r_+, -r_-}(R_s) |R_s|^\kappa ds$.
- If $\alpha = 1$, then for any $t \geq 0$, $U_t^1 = \frac{\delta c_\alpha}{c_\beta} \int_{\mathbb{R}} \operatorname{sgn}(x) |x|^{\kappa+\delta-1} (\gamma_t^x - \gamma_t^0) dx + \mathbf{c} \gamma_t^0$, where $\mathbf{c}$ is the constant from Assumption 6.5-(ii).
- If $\alpha \in (1, 2)$, then for any $t \geq 0$, $U_t^\alpha = \frac{\delta c_\alpha}{c_\beta} \int_{\mathbb{R}} \operatorname{sgn}_{f_+, -f_-}(x) |x|^{\kappa+\delta-1} (\gamma_t^x - \gamma_t^0) dx$.

We also consider an exponential random variable $e(\lambda)$ of parameter λ independent of the process $(R_t)_{t \geq 0}$. We will show that $\mathbb{E}[e^{i\xi U_{e(\lambda)}^\alpha}]$ is equal to $G(\lambda, \xi) \int_0^\infty \mathbb{E}[e^{-\lambda \tau_t^\beta + i\xi Z_t^\alpha}] dt$. As in the previous case, g_t^R denotes the last zero of $(R_s)_{s \geq 0}$ before time $t > 0$. We first note that for any $t > 0$, $U_t^\alpha - U_{g_t^R}^\alpha$ is a functional of $(R_s)_{s \in [g_t^R, t]}$. More precisely, we always have for any $t > 0$,

$$U_t^\alpha - U_{g_t^R}^\alpha = \frac{c_\alpha}{c_\beta} \int_{g_t^R}^t \operatorname{sgn}_{r_+, -r_-}(R_s) |R_s|^\kappa ds. \quad (6.14)$$

This is straightforward when $\alpha \in (0, 1)$. When $\alpha \geq 1$, we use the fact that $\gamma_t^0 = \gamma_{g_t^R}^0$ so that we have

$$\begin{aligned} U_t^\alpha - U_{g_t^R}^\alpha &= \frac{\delta c_\alpha}{c_\beta} \int_{\mathbb{R}} \operatorname{sgn}_{f_+, -f_-}(x) |x|^{\kappa+\delta-1} (\gamma_t^x - \gamma_{g_t^R}^x) dx \\ &= \frac{c_\alpha}{c_\beta} \int_{\mathbb{R}} \operatorname{sgn}_{r_+, -r_-}(x) |x|^\kappa (\gamma_t^x - \gamma_{g_t^R}^x) \mathbf{m}^R(dx), \end{aligned}$$

where we recall that the Radon measure $\mathbf{m}^R$ is the speed measure of $(R_s)_{s \geq 0}$, which is defined by $\mathbf{m}^R(dx) = \delta \operatorname{sgn}_{m_+, m_-}(x) |x|^{\delta-1} dx$. Then (6.14) is a consequence of the occupation time formula, see for instance Proposition 3.8. Since $(U_s^\alpha)_{s \geq 0}$ is continuous, we can apply Proposition 6.2 to see that

$$\mathbb{E} \left[\exp \left(i\xi U_{e(\lambda)}^\alpha \right) \right] = \mathbb{E} \left[\exp \left(i\xi (U_{e(\lambda)}^\alpha - U_{g_{e(\lambda)}^R}^\alpha) \right) \right] \mathbb{E} \left[\exp \left(i\xi U_{g_{e(\lambda)}^R}^\alpha \right) \right].$$

By Proposition 6.2, we also have

$$\mathbb{E} \left[\exp \left(i\xi U_{g_{e(\lambda)}^R}^\alpha \right) \right] = \Phi^R(\lambda) \int_0^\infty \mathbb{E} \left[e^{-\lambda \tau_t^R + i\xi U_{\tau_t^R}^\alpha} \right] dt,$$

where $(\tau_t^R)_{t \geq 0}$ is the right-continuous inverse of $(\gamma_t^0)_{t \geq 0}$, which is a subordinator; and Φ^R is its Laplace exponent. By Proposition 6.1, the process $(\tau_t^R, U_{\tau_t^R}^\alpha)_{t \geq 0}$ is equal in law to $(\tau_t^\beta, Z_t^\alpha)_{t \geq 0}$ so that the above quantity is equal to $\Phi^R(\lambda) \int_0^\infty \mathbb{E}[e^{-\lambda \tau_t^\beta + i\xi Z_t^\alpha}] dt$. Finally, by (6.14) and Proposition 6.2 again, we have

$$\mathbb{E} \left[\exp \left(i\xi (U_{e(\lambda)}^\alpha - U_{g_{e(\lambda)}^R}^\alpha) \right) \right] = \frac{\lambda}{\Phi^R(\lambda)} \mathbf{n}^R \left(\int_0^\ell e^{-\lambda u + i\xi \mathfrak{F}_u(\varepsilon)} du \right),$$

where $\mathbf{n}^R$ is the excursion measure of $(R_t)_{t \geq 0}$ and the functional $\mathfrak{F}_u(\varepsilon)$ is defined for any excursion $\varepsilon \in \mathcal{E}$ and any $u \leq \ell$ as

$$\mathfrak{F}_u(\varepsilon) = \frac{c_\alpha}{c_\beta} \int_0^u \operatorname{sgn}_{r_+, -r_-}(\varepsilon_s) |\varepsilon_s|^\kappa ds.$$

Finally, one can check that by using the representation of the measure $\mathbf{n}^R$ in terms of the Brownian excursion measure $\mathbf{n}^B$ (see Proposition 3.9 and Lemma 3.3) and making some changes of variables, we have exactly

$$\mathbf{n}^R \left(\int_0^\ell e^{-\lambda u + i\xi \mathfrak{F}_u(\varepsilon)} du \right) = \mathbf{n}^B \left(\int_0^\ell \exp \left(-\lambda A_t^\beta + i\xi A_t^\alpha \right) dA_t^\beta \right).$$

Finally, we conclude that

$$\mathbb{E}\left[\exp\left(i\xi U_{e(\lambda)}^\alpha\right)\right] = G(\lambda, \xi) \int_0^\infty \mathbb{E}\left[e^{-\lambda\tau_t^\beta + i\xi Z_t^\alpha}\right] dt,$$

which ends the proof. □

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