Inégalités systoliques optimales sur les surfaces

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Résumé

Une inégalité systolique sur une variété fermée M de dimension n est une inégalité de la forme

$$\mathrm{sys}^n(M,g) \leq C \cdot \mathrm{vol}(M,g)$$

valable pour toute métrique riemannienne g sur M, où $\mathrm{sys}(M,g)$ désigne la systole et $C = C(M)$ est une constante indépendante de g. Les inégalités systoliques les plus célèbres ont été démontrées sur le tore $\mathbb{T}^2$ par C. Loewner, sur le plan projectif réel $\mathbb{R}P^2$ par P. Pu et sur la bouteille de Klein $\mathbb{K}^2$ par C. Bavard. On établit dans cette thèse de nouvelles inégalités systoliques optimales sur des surfaces.

Dans un premier travail, on démontre l'existence d'une borne supérieure optimale pour la longueur de la plus courte géodésique fermée sur la sphère trouée avec trois ou quatre bouts munie d'une métrique riemannienne complète d'aire finie. Cette borne ne dépend pas de la courbure mais de l'aire de la sphère trouée. On décrit, dans les deux cas, les métriques extrémales. On établit ensuite des bornes pour les sphères trouées munies d'une métrique finslérienne réversible ou non-nécessairement réversible. Ces bornes sont exprimées en fonction de l'aire de Holmes-Thompson de la sphère trouée. On représente aussi une borne supérieure asymptotique sur la longueur de la plus courte géodésique pour des sphères avec un grand nombre de bouts.

Dans un deuxième travail, on démontre que le supremum local de la systole sur l'espace des surfaces d'Alexandrov à courbure au plus -1 est atteint par une surface hyperbolique. Ceci sans aucune hypothèse sur l'aire. On termine par une extension de ce résultat pour des variétés de dimension 3.

Mots-clés : Systole, espace métrique extrémale, sphère trouée, surface à courbure négative.

Optimal systolic inequalities on surfaces

Abstract

A systolic inequality on a closed manifold M of dimension n is an inequality of the form

$$\mathrm{sys}^n(M,g) \leq C \cdot \mathrm{vol}(M,g)$$

true for any Riemannian metric g on M, where $\mathrm{sys}(M,g)$ is the systole and $C = C(M)$ is a constant independent of g. The most famous systolic inequalities have been demonstrated on the torus $\mathbb{T}^2$ by C. Loewner, on the real projective plane $\mathbb{R}P^2$ by P. Pu and on the Klein bottle $\mathbb{K}^2$ by C. Bavard. We establish in this thesis new optimal systolic inequalities on surfaces.

In a first work, we prove the existence of an optimal upper bound on the length of the shortest closed geodesic on punctured sphere with three or four ends endowed with a complete Riemannian metric of finite area. This bound does not depend on the curvature but only on the area of the punctured sphere. In both cases, we describe the extremal metrics. Then, we establish upper bounds for the punctured spheres endowed with a reversible or not necessarily reversible Finsler metrics. These bounds are expressed in term of the Holmes-Thompson area of the punctured sphere. We also represent a roughly asymptotically optimal upper bound on the length of the shortest closed geodesic for spheres with a large number of ends.

In a second work, we show that the local supremum of the systole on the space of Alexandrov surfaces with curvature at most -1 is reached by a hyperbolic surface, without any area assumption. We end with an extension of this result for manifolds of dimension 3.

Keywords: Systole, extremal metric space, punctured sphere, surface with negative curvature.

Contents

Introduction

The present thesis is divided into two independent parts. In the first part, which corresponds to Chapters 1 and 2, we study geometric inequalities of surfaces related to an active research subject classically called the systolic geometry of surfaces. In the second part, which consists of Chapter 3, we study an analytical problem about the solutions of the generalized resolvent problem.

We introduce first the part one of the thesis and our related results. The systole of a non-simply connected manifold M with a Riemannian metric is the smallest length of a non-contractible loop on M. It is denoted by

$$\mathrm{sys}(M,g) = \inf\{\mathrm{length}(\gamma) \mid \gamma \text{ is a non-contractible loop of } M\}.$$

The term systole was coined by M. Berger in 1972, see [16]. If M is closed, this quantity is always attained by a closed non-homotopically trivial geodesic.

The existence of closed geodesics on a non-simply connected closed surface follows from a minimization process relying on Ascoli's theorem, while it follows from Birkhoff's minmax principle in the simply connected case. For surfaces with a complete Riemannian metric of finite area with one or two ends, the existence of closed geodesics was proved by V. Bangert [12], while this was proved by G. Thorbergsson [69] for surfaces of finite area with at least three ends.

A systolic inequality on a closed manifold M of dimension n is an inequality of the form

$$\mathrm{sys}^n(M,g) \leq C \cdot \mathrm{vol}(M,g)$$

which holds for any Riemannian metric g on M, where $C = C(M)$ is a constant independent of g . The first systolic inequality was established by C. Loewner on the two-torus in 1949 : for any Riemannian metric g on $\mathbb{T}^2$, we have

$$\mathrm{sys}^2(\mathbb{T}^2,g) \leq \frac{2}{\sqrt{3}}\ \mathrm{area}(\mathbb{T}^2,g). \tag{1}$$

Furthermore, the equality is sharp and is attained if and only if the torus is flat hexagonal. In 1952, his student P. Pu [56] established the following sharp systolic inequality on the real projective plane : for any Riemannian metric g on $\mathbb{R}P^2$, we have

$$\mathrm{sys}^2(\mathbb{R}P^2,g) \leq \frac{\pi}{2}\ \mathrm{area}(\mathbb{R}P^2,g).$$

The equality is attained precisely by round metrics. More than thirty years later, C. Bavard [13] proved a sharp systolic inequality on the Klein bottle : for any Riemannian metric g on the Klein bottle $\mathbb{K}$, we have

$$\mathrm{sys}^2(\mathbb{K},g) \leq \pi \cdot 2^{-\frac{3}{2}}\ \mathrm{area}(\mathbb{K},g).$$

In this case, the extremal metric is not smooth : it is spherical outside a singular curve.

The remaining closed surfaces, *i.e.*, closed surfaces of genus ≥ 2, also satisfy a systolic inequality but an optimal constant is still unknown for genus ≥ 2 (see for example [40] and [23]). Some sharp inequalities are known if we consider only metrics of constant curvature, of nonpositive curvature or in a given conformal class (see also [62] for a "local" inequality in genus 3). Higher dimensional systolic inequalities are known to exist for a large class of manifolds namely, esssential manifolds (including real projective spaces and aspherical manifolds) by a fundamental result of M. Gromov [36].

In Chapter 1, we present a joint project with S. Sabourau. We investigate a geometric problem in the same spirit as the systolic inequality by replacing the systole by the length of shortest closed geodesic denoted by

$$\mathrm{scg}(M,g) = \inf\{\mathrm{length}(\gamma) \mid \gamma \text{ is a closed geodesic of } M\}.$$

In this case, we do not need to eliminate the case of simply connected surfaces, *i.e.*, the sphere. In 1982, C. Croke [25] showed that for every Riemannian metric g on the two-sphere S^2, we have

$$\mathrm{scg}(S^2,g) \leq 31\sqrt{\mathrm{area}(S^2,g)}. \tag{2}$$

This bound was improved in [53], [60] and [58]. A classical conjecture (see [25] and [26]) asserts that the piecewise flat sphere with three conical singularities obtained by gluing two equilateral triangles along their sides (called the Calabi-Croke sphere) is the global maximum for the length of the shortest closed geodesic among Riemannian metrics with fixed area on the sphere. We note that a result of "local" maximality of the Calabi-Croke sphere was proved by F. Balacheff in [10] (see also [61]).

Inequality (2) holds for noncompact surfaces Σ with any complete Riemannian metric of finite area (see C. Croke [25]). The bound 31 in the inequality has recently been improved by Beach and Rotman in [18], where it was replaced by $4\sqrt{2}$ for surfaces with one puncture and by $2\sqrt{2}$ for surfaces with at least two punctures. The authors also conjectured that the optimal bound for a punctured sphere with at most three punctures is the same as the one for the sphere. That would imply that the optimal bound is equal to $2^{\frac{1}{2}} \cdot 3^{\frac{1}{4}}$. This conjecture is established in Chapter 1 where we prove an optimal inequality for spheres with three or four punctures, as well as non-optimal inequalities for spheres with k-punctures, where $k > 4$. We also prove other optimal and non-optimal inequalities of the same nature for punctured spheres with complete reversible and non reversible Finsler metrics of finite area.

The proofs of our optimal bounds do not rely on the conformal length method used in [45], [56] and [13]. Instead, we use ramified covers from the torus to the sphere. This technique was introduced by S. Sabourau in [59] in a similar context. Those ramified covers are useful because they allow us to apply Loewner's systolic inequality (1) after connecting the punctured two-spheres to the two-torus.

In the Riemannian case, we have the following theorem.

Theorem. Let $\Sigma = S^2 \setminus \{x_1, \dots, x_k\}$ be a k-punctured sphere with a complete Riemannian metric of finite area. Then the following holds.

1. If $k = 3$ then there exists a noncontractible figure-eight geodesic γ on Σ such that
$$\operatorname{length}(\gamma) < 2^{\frac{1}{2}} \cdot 3^{\frac{1}{4}} \sqrt{\operatorname{area}(\Sigma)}.$$
Furthermore, this inequality is optimal.

2. If $k \geq 4$ then there exists a noncontractible closed geodesic γ on Σ such that
$$\operatorname{length}(\gamma) < 2 \cdot 3^{-\frac{1}{4}} \sqrt{\operatorname{area}(\Sigma)}.$$
Furthermore, this inequality is optimal when $k = 4$.

The extremal metric on the three-punctured sphere is modelled on the Calabi-Croke sphere by attaching three cusps of arbitrarily small area around its singularities. This can be done by keeping the metric non-positively curved.

The extremal metric on the four-punctured sphere is modelled on the tetrahedral sphere by attaching four cusps of arbitrarily small area around its singularities. Here, the tetrahedral sphere is defined as the piecewise flat sphere with four conical singularities of angle π given by the regular tetrahedron. This can also be done by keeping the metric non-positively curved.

Then, we prove a Finsler version of the previous theorem, that is, for complete punctured spheres equipped with a reversible Finsler metric of finite Holmes-Thompson area. More precisely, we have

Theorem. Let $\Sigma = S^2 \setminus \{x_1, \dots, x_k\}$ be a k-punctured sphere with a complete reversible Finsler metric of finite area. Then the following holds.

1. If $k = 3$ then there exists a noncontractible figure-eight geodesic γ on Σ such that
$$\operatorname{length}(\gamma) < 2^{-\frac{1}{2}} \cdot 3^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\operatorname{area}(\Sigma)}.$$

2. If $k \geq 4$ then there exists a noncontractible closed geodesic γ on Σ such that

$$\text{length}(\gamma) < \pi^{\frac{1}{2}} \sqrt{\text{area}(\Sigma)}.$$

Furthermore, this inequality is optimal when $k \in \{4, \dots, 6\}$.

We note that the inequality on the three-punctured sphere is not necessarily optimal, *i.e.* it may exist a better constant less than $2^{-\frac{1}{2}} \cdot 3^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}}$. The extremal metric on the four-punctured sphere is modelled on the sphere obtained by gluing two copies of the square $[0,1] \times [0,1]$ endowed with the ℓ^1-metric along their boundary, with four cusps of arbitrarily small area attached around the four vertices of the squares. Also, we attach respectively one or two extra cusps of arbitrarily small area around the centers of the squares for the five-punctured and six-punctured spheres.

For punctured spheres equipped with non-necessarily reversible Finsler metrics, we prove the following non-optimal inequalities.

Theorem. Let $\Sigma = S^2 \setminus \{x_1, \dots, x_k\}$ be a k-punctured sphere with a complete non-necessarily reversible Finsler metric of finite area. Then the following holds.

1. If $k = 3$ then there exists a noncontractible figure-eight geodesic γ on Σ such that
$$\text{length}(\gamma) < 2^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\text{area}(\Sigma)}.$$

2. If $k \geq 4$ then there exists a non-contractible closed geodesic γ on Σ such that

$$\text{length}(\gamma) < 2 \cdot 3^{-\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\text{area}(\Sigma)}.$$

We conclude by proving the following result for spheres with a large number of punctures.

Theorem. Let $\Sigma = S^2 \setminus \{x_1, \dots, x_k\}$ be a k-punctured sphere with a complete Riemannian metric of finite area, where $k \geq 3$. Then there exists a noncontractible closed geodesic γ on Σ such that

$$\text{length}(\gamma) \leq 4\sqrt{2} \sqrt{\frac{\text{area}(\Sigma)}{k}}.$$

Furthermore, the upper bound is roughly asymptotically optimal in k.

In Chapter 2, we present a joint project with S. Sabourau. We study the maximum of the systole on surfaces of genus $g \geq 2$ with a Riemannian metric of curvature at most -1. With this curvature restriction condition, we prove that the extremal metrics are hyperbolic and we present some examples in dimension three.

An Alexandrov surface M is a surface endowed with a metric which is the limit of a sequence of Riemannian metrics with uniformly bounded absolute integral curvature. Such a surface is of curvature at most -1 if every small enough geodesic triangle in M has a comparison triangle in the hyperbolic plane (the sides are of same length) verifying the following condition: the distances between points in the boundary of the triangle in M are less or equal to the distances between the corresponding points in the hyperbolic triangle. For instance, our result holds true in the more general setting of Alexandrov curvature at most -1. A closed piecewise hyperbolic surface with conical singularities of total angle at least 2π is an Alexandrov surface of curvature at most -1. We refer the reader to Section 2.2 for further details about Alexandrov surfaces.

Alexandrov surfaces are interesting because of their compactness properties. Indeed, we know that the space of Alexandrov surfaces of curvature at most -1 homeomorphic to a non-simply connected closed surface with uniformly bounded systole is compact for the uniform distance topology; see [27]. Hence, the supremum of the systole over the space of Alexandrov surfaces with curvature at most -1 is attained by an Alexandrov surface.

If we only consider non-positively curved metrics of unit area, similar sharp inequalities have already been established for genus 2 surfaces (see [46]) and for the connected sum of three projective planes (see [47]). The extremal metrics are flat with conical singularities in both cases. Recently, this was generalized by M. Katz and S. Sabourau to all genus g surfaces (see [48]).

The main tool in the proofs given by Katz and Sabourau in [48] is the kite excision trick. The method consists of moving two conical singularities closer and closer until they merge together. We use the same technique in Chapter 2 to derive the following intermediate result.

Proposition. Let Σ be a non-simply connected closed surface. Let $\mathcal{U} \subseteq \mathcal{A}_\Sigma$ be an open set in the space $\mathcal{A}_\Sigma$ of Alexandrov surfaces of curvature at most -1 defining a local supremum of the systole. Then there exists a piecewise hyperbolic surface $M_0 \in \mathcal{U}$ with at most N_0 conical singularities such that

$$\mathrm{sys}(M_0) \geq \mathrm{sys}(M)$$

for every $M \in \mathcal{U}$, where N_0 is an integer depending only on the topology of Σ.

In order to eliminate the conical singularities and prove that the extremal metric is hyperbolic, we introduce the opposite technique, namely the kite insertion trick. The goal is to add as much area as possible to the piecewise hyperbolic surface with conical singularities with Alexandrov curvature at most -1 to obtain the desired hyperbolic surface. Hence, we have the following result.

Theorem. Let Σ be a nonsimply connected closed surface. Let $\mathcal{U} \subseteq \mathcal{A}_\Sigma$ be an open set in the space $\mathcal{A}_\Sigma$ of Alexandrov surfaces of curvature at most -1 defining a local supremum of the systole. Then there exists a hyperbolic surface $M \in \mathcal{U}$ such that

$$\operatorname{sys}(M) \geq \operatorname{sys}(M')$$

for every $M' \in \mathcal{U}$.

Since, see [14], the maximum of the systole among closed genus two hyperbolic surfaces is known and is attained by the Bolza surface, which is the smooth completion of the affine algebraic curve

$$y^2 = x^5 - x,$$

we obtain then the following corollary for closed surfaces of genus two.

Corollary. The maximal systole of a closed genus two surface M with a Riemannian metric of curvature K_M at most -1 is attained by the hyperbolic metric conformal to the Bolza surface and is equal to

$$\max_{K_M \leq -1} \operatorname{sys}(M) = 2 \operatorname{arccosh}(1 + \sqrt{2}).$$

Finally, we extend our work to higher dimension and obtain the following result for closed hyperbolic 3-manifolds. By definition, a *maximal hyperbolic surface* is a closed hyperbolic surface with maximal systole among all hyperbolic metrics of fixed genus.

Corollary. Let N be a closed hyperbolic 3-manifold admitting a totally geodesic immersion of a maximal hyperbolic surface M with $\operatorname{sys}(M) = \operatorname{sys}(N)$. Then the hyperbolic metric on N has maximal systole among all Riemannian metrics of (sectional) curvature at most -1.

A concrete example can be found in Example 2.13.3.

In Chapter 3, we present a work in the analysis of PDE done in collaboration with H. Al Baba and published in a conference paper [5]. It is unrelated to the first two chapters of this thesis.

We consider in a bounded cylindrical domain $\Omega \times (0,T)$ the following problem

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0,T) \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega \end{cases}$$

where the unknows $\boldsymbol{u}$ and π stand respectively for the velocity field and the pressure of a fluid occupying a domain Ω . The given data are the external force $\boldsymbol{f}$ and the initial velocity $\boldsymbol{u}_0$. In other terms, we consider the resolvent of the Stokes operator.

There exist several results for this problem with Dirichlet boundary conditions. For example, the reader can refer to the work of V. Solonnikov in [68] and its extended version by Y. Giga in [33]. The case where $\operatorname{div} \boldsymbol{u} \neq 0$ has important applications, especially in treating more general boundary value problems. It was studied by R. Farwig and H. Sohr [30]. The general problem is also studied with Robin boundary conditions by J. Saal [64], Y. Shibata and R. Shimada [67]. However, this type of boundary conditions is not always realistic since it does not necessarily reflect the behavior of the fluid on or near the boundary.

In 1824, H. Navier **[55]** suggested a type of boundary conditions based on a proportionality between the tangential components of the normal dynamic tensor and the velocity

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad 2\nu[\mathbb{D}\boldsymbol{u} \cdot \boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = 0 \qquad \text{on } \Gamma \times (0,T)$$

where ν is the viscosity, $\alpha \geqslant 0$ is the coefficient of friction and

$$\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$$

denotes the deformation tensor associated to the velocity field $\boldsymbol{u}$. The Navier boundary conditions defined above are often used to simulate the flows near rough walls as well as perforated walls. We also mention that such slip boundary conditions are used in the simulation of turbulent flows. Making use of the vorticity field $w = \mathbf{curl}\ \boldsymbol{u}$, and using classical identities, one can observe that in the case of a flat boundary and when $\alpha = 0$, the Navier boundary conditions may be replaced by

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \operatorname{curl} \boldsymbol{u} \times \boldsymbol{n} = 0 \qquad \text{on} \quad \Gamma \times (0,T).$$

We call them Navier-type boundary conditions.

Hence, one can study the resolvent of the Stokes operator with homogeneous Navier type boundary conditions by studying the following problem.

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0,T) \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on} \quad \Gamma \times (0,T) \end{cases} \tag{3}$$

In [52], T. Miyakawa shows that the Laplacian operator with homogeneous Navier-type boundary conditions generates a holomorphic semi-group on L^p-spaces when the domain Ω is of class C^∞. M. Mitrea and S. Monniaux [51] considered the problem in Lipschitz domains using differential forms on Lipschitz sub-domains of a smooth compact Riemannian manifold. In [2] and [3], H. Al Baba, C. Amrouche and M. Escobedo prove the existence of weak, strong and very weak solutions to this problem.

In this last part of the thesis, we study the Stokes operator with non-homegeneous Navier-type boundary conditions

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = \chi & \text{in} \quad \Omega \times (0,T) \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on} \quad \Gamma \times (0,T) \end{cases} \tag{4}$$

and we prove the existence of weak, strong and very weak solutions to the problem 4.

Introduction (version française)

La présente thèse est divisée en deux parties indépendantes. Dans la première partie, qui correspond aux chapitres 1 et 2, on étudie des inégalités géométriques des surfaces liées à un sujet de recherche actif classiquement appelé la géométrie systolique. Dans la deuxième partie, constituée du chapitre 3, on étudie un problème analytique concernant les solutions du problème généralisé de la resolvante.

On présente tout d'abord les résultats de la première partie de la thèse. La systole d'une variété non-simplement connexe M munie d'une métrique riemannienne est l'infimum des longueurs des lacets non-contractiles de M, définit par

$$\mathrm{sys}(M,g) = \inf\{\text{longueur}(\gamma) \mid \gamma \text{ est un lacet non-contractile de } M\}.$$

Le terme systole a été inventé par M. Berger en 1972, voir [16]. Si M est fermée, cette quantité est toujours atteinte par une géodésique fermée non-homotopiquement triviale.

L'existence de géodésiques fermées sur une surface fermée non-simplement connexe découle d'un processus de minimisation reposant sur le théorème d'Ascoli, alors qu'elle découle du principe de minmax de Birkhoff dans le cas simplement connexe. Pour les surfaces munies d'une métrique riemannienne complète d'aire finie à un ou deux bouts, l'existence des géodésiques fermées a été prouvée par V. Bangert [12], tandis que celle-ci a été prouvée par G. Thorbergsson [69] pour les surfaces d'aire finie à au moins trois bouts.

Une inégalité systolique sur une variété fermée M de dimension n est une inégalité de la forme

$$\mathrm{sys}^n(M,g) \leq C \cdot \mathrm{vol}(M,g)$$

valable pour toute métrique riemannienne g sur M, où $\mathrm{sys}(M,g)$ désigne la systole et $C = C(M)$ est une constante indépendante de g. La première inégalité systolique a été établie par C. Loewner sur le 2-tore en 1949 : pour toute métrique riemannienne g sur $\mathbb{T}^2$, on a

$$\mathrm{sys}^2(\mathbb{T}^2,g) \leq \frac{2}{\sqrt{3}}\ \mathrm{aire}(\mathbb{T}^2,g). \tag{5}$$

De plus, l'égalité est atteinte si et seulement si le tore est hexagonal plat. En 1952, son étudiant P. Pu [56] a établit l'inégalité systolique suivante sur le plan projectif réel : pour toute métrique riemannienne g sur $\mathbb{R}P^2$, on a

$$\mathrm{sys}^2(\mathbb{R}P^2,g) \leq \frac{\pi}{2}\ \mathrm{aire}(\mathbb{R}P^2,g).$$

L'égalité est atteinte précisement par les métriques rondes. Après plus que trente ans, C. Bavard [13] a démontré une inégalité systolique sur la bouteille de Klein :

pour toute métrique riemannienne g sur la bouteille de Klein $\mathbb{K}$, on a

$$\mathrm{sys}^2(\mathbb{K}, g) \leq \frac{\pi}{2\sqrt{2}} \,\mathrm{aire}(\mathbb{K}, g).$$

Dans ce cas, la métrique extrémale n'est pas lisse : elle est sphérique en dehors d'une courbe singulière.

Le reste des surfaces fermées, *i.e.*, les surfaces fermées de genre ≥ 2, satisfont également une inégalité systolique mais la constante optimale connue pour aucun genre ≥ 2 (voir par exemple [40] et [23]). Des inégalités optimales sont connues si on considère juste les métriques à courbure constante, à courbure négative ou appartenant à une classse conforme donnée (voir [62] pour une inégalité "locale" en genre 3). Les inégalités systoliques de dimension supérieure existent d'après un résultat fondamental de M. Gromov [36] pour une grande classe de variétés, celles des variétés essentielles (comprenant les espaces projectifs et les variétés asphériques) .

Dans le chapitre 1, on présente un travail en collaboration avec S. Sabourau. On étudie un problème géométrique dans le même esprit que l'inégalité systolique en remplaçant la systole par la longueur de la plus courte géodésique fermée notée

$$\mathrm{scg}(M, g) = \inf\{\text{longueur}(\gamma) \mid \gamma \text{ est une géodésique fermée de } M\}.$$

Dans ce cas, on n'élimine pas le cas des surfaces simplement connexe, *i.e.*, la sphère. En 1982, C. Croke [25] a démontré que pour toute métrique riemannienne sur g la 2-sphère S^2, on a

$$\mathrm{scg}(S^2, g) \leq 31\sqrt{\mathrm{aire}(S^2, g)}. \tag{6}$$

Cette borne a été améliorée dans [53], [60] et [58]. Une conjecture classique (voir [25] and [26]) affirme que la sphère plate par morceaux avec trois singularités coniques obtenue en collant deux triangles équilatéraux le long de leurs côtés (appelée sphère de Calabi-Croke) est le maximum global pour la longueur de la plus courte géodésique fermée parmi les métriques riemanniennes sur la sphère d'aire fixe. On note qu'un résultat de maximalité "locale" de la sphère de Calabi-Croke a été démontré par F. Balacheff dans [10] (voir aussi [61]).

L'inégalité (6) reste vraie pour les surfaces Σ non-compactes munies d'une métrique riemannienne complète d'aire finie (voir C. Croke [25]). La constante 31 dans l'inégalité a été améliorée par Beach et Rotman dans [18], où elle a été remplacée par $4\sqrt{2}$ pour les surfaces avec un bout et par $2\sqrt{2}$ pour les surfaces avec au moins deux bouts. Les auteurs ont également conjecturé que la borne optimale pour une sphère trouée avec au plus trois bouts est la même que celle de la sphère, c'est-à-dire que la borne optimale est égale à $2^{\frac{1}{2}} \cdot 3^{\frac{1}{4}}$. Cette conjecture est établie dans le chapitre 1 où on prouve une inégalité optimale pour les sphères à trois ou quatre bouts, ainsi que

des inégalités non-optimales pour les sphères avec k bouts, avec $k > 4$. On prouve également d'autres inégalités optimales et non-optimales de même nature pour des sphères trouées munies d'une métrique finslérienne réversible ou non-nécessairement réversible complète d'aire finie.

Les preuves de nos bornes optimales ne reposent pas sur la méthode de longueur conforme utilisée dans [45], [56] et [13]. Au lieu de cela, on utilise des revêtements ramifiés du tore sur la sphère. Cette technique a été introduite par S. Sabourau [59] dans un contexte similaire. Ces revêtements ramifiés sont utiles car ils permettent d'appliquer l'inégalité systolique de Loewner (5) après avoir relié les 2-sphères trouées aux 2-tores.

Dans le cas riemannien, on a le théorème suivant.

Théorème. Soit $\Sigma = S^2 \backslash \{x_1, \dots, x_k\}$ une sphère avec k bouts munie d'une métrique riemannienne complète d'aire finie.

1. Si $k = 3$ alors il existe une géodésique en huit γ non-contractile sur Σ telle que
$$\text{longueur}(\gamma) < 2^{\frac{1}{2}} \cdot 3^{\frac{1}{4}} \sqrt{\text{aire}(\Sigma)}.$$
De plus, cette inégalité est optimale.
2. Si $k \geq 4$ alors il existe une géodésique fermée γ non-contractile sur Σ telle que
$$\text{longueur}(\gamma) < 2 \cdot 3^{-\frac{1}{4}} \sqrt{\text{aire}(\Sigma)}.$$
De plus, cette inégalité est optimale lorsque $k = 4$.

La métrique extrémale sur la sphère à trois bouts est modelée par la sphère de Calabi-Croke en attachant trois cusps d'aire arbitrairement petite autour de ses singularités. Cela peut être fait en gardant la métrique à courbure négative.

La métrique extrémale sur la sphère à quatre bouts est modelée par la sphère tétraédrique en attachant quatre cusps d'aire arbitrairement petite autour de ses singularités. Ici, la sphère tétraédrique est définie comme la sphère plate par morceaux avec quatre singularités coniques d'angle π donnée par le tétraèdre régulier. Cela peut également être fait en gardant la métrique à courbure négative.

Ensuite, on démontre une version finslérienne du théorème précédent, c'est-à-dire pour les sphères trouées complètes équipées d'une métrique finslérienne réversible d'aire Holmes-Thompson finie. Plus précisément, nous avons

Théorème. Soit $\Sigma = S^2 \backslash \{x_1, \dots, x_k\}$ une sphère avec k bouts munie d'une métrique finslérienne réversible complète d'aire finie.

1. Si $k = 3$ alors il existe une géodésique en huit γ non-contractile sur Σ telle que
$$\text{longueur}(\gamma) < 2^{-\frac{1}{2}} \cdot 3^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\text{aire}(\Sigma)}.$$
2. Si $k \geq 4$ alors il existe une géodésique fermée γ non-contractile sur Σ telle que
$$\text{longueur}(\gamma) < \pi^{\frac{1}{2}} \sqrt{\text{aire}(\Sigma)}.$$
De plus, cette inégalité est optimale lorsque $k \in \{4, \ldots, 6\}$.

Notons que l'inégalité sur la sphère à trois bouts n'est pas nécessairement optimale. La métrique extrémale sur la sphère avec quatre bouts est modelée par la sphère obtenue en collant deux copies du carré $[0, 1] \times [0, 1]$ muni de la métrique ℓ^1 le long de leurs frontières, avec quatre cusps d'aire arbitrairement petite attachés autour des quatre sommets des carrés. De plus, on attache respectivement un ou deux cusps supplémentaires d'aire arbitrairement petite autour des centres des carrés pour les sphères avec cinq et six bouts.

Pour les sphères trouées munies d'une métrique finslérienne non-nécessairement réversible, on démontre les inégalités non-optimales suivantes.

Théorème. Soit $\Sigma = S^2 \backslash \{x_1, \ldots, x_k\}$ une sphère avec k bouts munie d'une métrique finslérienne non-nécessairement réversible complète d'aire finie.

1. Si $k = 3$ alors il existe une géodésique en huit γ non-contractile sur Σ telle que
$$\text{longueur}(\gamma) < 2^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\text{aire}(\Sigma)}.$$
2. Si $k \geq 4$ alors il existe une géodésique fermée γ non-contractile sur Σ telle que
$$\text{longueur}(\gamma) < 2 \cdot 3^{-\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\text{aire}(\Sigma)}.$$

On conclut en démontrant le résultat suivant pour des sphères avec un grand nombre de bouts.

Théorème. Soit $\Sigma = S^2 \backslash \{x_1, \ldots, x_k\}$ une sphère avec k bouts munie d'une métrique riemannienne complète d'aire finie, où $k \geq 3$. Alors il existe une géodésique fermée non contractile γ sur Σ telle que
$$\text{longueur}(\gamma) \leq 4\sqrt{2} \sqrt{\frac{\text{aire}(\Sigma)}{k}}.$$
De plus, la borne supérieure est à peu près asymptotiquement optimale en k.

Dans le chapitre 2, on présente un travail en collaboration avec S. Sabourau. On étudie le maximum de la systole sur les surfaces de genre $g \geq 2$ munies d'une métrique riemannienne de courbure au plus -1. Avec cette condition sur la courbure, on prouve que les métriques extrémales sont hyperboliques et on présente des exemples en dimension trois.

Une surface d'Alexandrov M est une surface munie d'une métrique qui est la limite d'une suite de métriques riemanniennes à courbure intégrale absolue uniformément bornée. Une telle surface est de courbure au plus -1 si tout triangle géodésique assez petit dans M possède un triangle de comparaison dans le plan hyperbolique (les côtés sont de même longueur) vérifiant la condition suivante: les distances entre les points du bord du triangle dans M sont inférieures ou égales aux distances entre les points correspondants dans le triangle hyperbolique. Par exemple, notre résultat est vrai dans le cadre plus général de la courbure d'Alexandrov au plus -1. Une surface fermée hyperbolique par morceaux avec des singularités coniques d'angle total au moins 2π est une surface d'Alexandrov à courbure au plus -1. On renvoie le lecteur à la section 2.2 pour plus de détails sur les surfaces d'Alexandrov.

Les surfaces d'Alexandrov sont intéressantes à cause de leurs propriétés de compacité. En effet, on sait que l'espace des surfaces d'Alexandrov de courbure au plus -1 homéomorphes à une surface fermée non simplement connexe de systole uniformément bornée est compact pour la topologie de la distance uniforme; voir [27]. Ainsi, le supremum de la systole sur l'espace des surfaces d'Alexandrov de courbure au plus -1 est atteint par une surface d'Alexandrov.

Si on ne considère que des métriques à courbure négative d'aire unitaire, des inégalités optimales similaires ont déjà été établies pour les surfaces de genre 2 (voir [46]) et pour la somme connexe de trois plans projectifs (voir [47]). Les métriques extrémales sont plates avec des singularités coniques dans les deux cas. Récemment, cela a été généralisé par M. Katz et S. Sabourau pour toutes les surfaces de genre g (voir [48]).

L'outil principal dans les preuves données par Katz et Sabourau dans [48] est l'astuce d'excision de cerfs-volants. La méthode consiste à rapprocher de plus en plus deux singularités coniques jusqu'à ce qu'elles se fusionnent. Nous utilisons la même technique dans le chapitre 2 pour obtenir le résultat intermédiaire suivant.

Proposition. Soit Σ une surface fermée non-simplement connexe. Soit $\mathcal{U} \subseteq \mathcal{A}_\Sigma$ un ouvert dans l'espace $\mathcal{A}_\Sigma$ des surfaces d'Alexandrov de courbure au plus -1 définissant un supremum local de la systole. Alors il existe une surface hyperbolique par morceaux $M_0 \in \mathcal{U}$ avec au plus N_0 singularités coniques telle que

$$\mathrm{sys}(M_0) \geq \mathrm{sys}(M)$$

pour toute surface $M \in \mathcal{U}$, où N_0 est un entier dépendant uniquement de la topologie de Σ.

Afin d'éliminer les singularités coniques et de prouver que la métrique extrémale est hyperbolique, nous introduisons la technique inverse, à savoir l'astuce d'insertion de cerfs-volants et non plus d'excision. Le but est d'ajouter autant d'aire que possible à la surface hyperbolique par morceaux avec singularités coniques de courbure d'Alexandrov au plus -1 pour obtenir la surface hyperbolique souhaitée. Ceci permet de montrer le résultat suivant.

Théorème. Soit Σ une surface fermée non-simplement connexe. Soit $\mathcal{U} \subseteq \mathcal{A}_\Sigma$ un ouvert dans l'espace $\mathcal{A}_\Sigma$ des surfaces d'Alexandrov de courbure au plus -1 définissant un supremum local de la systole. Alors il existe une surface hyperbolique $M \in \mathcal{U}$ telle que

$$\operatorname{sys}(M) \geq \operatorname{sys}(M')$$

pour toute surface $M' \in \mathcal{U}$.

Puisque le maximum de la systole parmi les surfaces hyperboliques fermées de genre deux est connu et est atteint par la surface de Bolza, qui est la surface de Riemann associée à la courbe algébrique affine

$$y^2 = x^5 - x,$$

on obtient alors le corollaire suivant pour les surfaces fermées de genre deux.

Corollaire. La systole maximale d'une surface fermée M de genre deux munie d'une métrique riemannienne de courbure K_M au plus -1 est atteinte par la métrique hyperbolique conforme à la surface de Bolza et est égale à

$$\max_{K_M \leq -1} \operatorname{sys}(M) = 2 \operatorname{arccosh}(1 + \sqrt{2}).$$

Enfin, on étend notre travail en dimension supérieure et on obtient le résultat suivant pour une variété hyperbolique fermée de dimension 3. Par définition, une *surface hyperbolique maximale* est une surface hyperbolique fermée avec une systole maximale parmi toutes les métriques hyperboliques d'un genre fixe.

Corollaire. Soit N une variété hyperbolique fermée de dimension 3 admettant une immersion totalement géodésique d'une surface hyperbolique maximale M avec $\operatorname{sys}(M) = \operatorname{sys}(N)$. Alors la métrique hyperbolique sur N a une systole maximale parmi toutes les métriques riemanniennes de courbure (sectionnelle) au plus -1.

Un exemple concret peut être trouvé dans Exemple 2.13.3.

Dans le chapitre 3, on présente un travail d'analyse d'EDP réalisé en collaboration avec H. Al Baba et publié dans un article de conférence [5]. Il n'est pas en rapport avec les deux premiers chapitres de cette thèse.

On considère le problème suivant dans un domaine cylindrique borné $\Omega \times (0,T)$

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{dans} \quad \Omega \times (0,T) \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{dans} \quad \Omega \end{cases}$$

où les inconnues $\boldsymbol{u}$ et π représentent respectivement le champ de vitesse et la pression d'un fluide occupant un domaine Ω . Les données fournies sont la force externe $\boldsymbol{f}$ et la vitesse initiale $\boldsymbol{u}_0$. En d'autres termes, on considère la résolvante de l'opérateur de Stokes.

Il existe plusieurs résultats pour ce problème avec des conditions aux limites de Dirichlet. Par exemple, le lecteur pourra se référer au travail de V. Solonnikov dans [68] et sa version étendue par Y. Giga dans [33]. Le cas où $\operatorname{div} \boldsymbol{u} \neq 0$ a des applications importantes, en particulier dans le traitement des problèmes avec des valeurs aux limites plus générales. Il a été étudié par R. Farwig et H. Sohr [30]. Le problème général est également étudié avec les conditions aux limites de Robin par J. Saal [64], Y. Shibata et R. Shimada [67]. Cependant, ce type de conditions aux limites n'est pas toujours réaliste car il ne reflète pas nécessairement le comportement du fluide sur ou près de la frontière.

En 1824, H. Navier **[55]** propose un type de conditions aux limites basé sur une proportionnalité entre les composantes tangentielles du tenseur dynamique normal et la vitesse

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad 2\nu[\mathbb{D}\boldsymbol{u} \cdot \boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = 0 \qquad \text{sur } \Gamma \times (0,T)$$

où ν est la viscosité, $\alpha \geqslant 0$ est le coefficient de frottement et

$$\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$$

désigne le tenseur de déformation associé au champ de vitesse $\boldsymbol{u}$. Les conditions aux limites de Navier définies ci-dessus sont souvent utilisées pour simuler l'écoulement à proximité des murs rugueux ainsi que des murs perforés. On mentionne également que de telles conditions aux limites sont utilisées dans la simulation des écoulements turbulents. En utilisant le champ de vorticité $w = \mathbf{curl}\ \boldsymbol{u}$, et quelques identités classiques, on peut observer que dans le cas d'une frontière plate et lorsque $\alpha = 0$, les conditions aux limites de Navier peuvent être remplacées par

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \operatorname{curl} \boldsymbol{u} \times \boldsymbol{n} = 0 \qquad \text{on} \quad \Gamma \times (0,T).$$

On les appelle conditions aux limites de type Navier.

Ainsi, on peut étudier la résolvante de l'opérateur de Stokes avec des conditions aux limites homogènes de type Navier en étudiant le problème suivant.

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{dans} & \Omega \times (0,T) \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{sur} & \Gamma \times (0,T) \end{cases} \tag{7}$$

Dans [52], T. Miyakawa montre que l'opérateur de Laplace avec des conditions aux limites homogènes de type Navier génère un semi-groupe holomorphe sur les L^p-espaces lorsque le domaine Ω est de classe C^∞. M. Mitrea et S. Monniaux [51] ont considéré le problème dans les domaines lipschtiziens en utilisant les formes différentielles sur les sous-domaines lipschitziens d'une variété riemannienne lisse et compacte. Dans [2] et [3], H. Al Baba, C. Amrouche et M. Escobedo prouvent l'existence de solutions faibles, fortes et très faibles à ce problème.

Dans cette dernière partie de la thèse, on étudie l'opérateur de Stokes avec des conditions aux limites de type Navier non homogènes

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = \chi & \text{dans} & \Omega \times (0,T) \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{sur} & \Gamma \times (0,T) \end{cases} \tag{8}$$

et on démontre l'existence de solutions faibles, fortes et très faibles au problème (8).

Chapter 1

Sharp bounds on the length of the shortest closed geodesic

We establish sharp universal upper bounds on the length of the shortest closed geodesic on a punctured sphere with three or four ends endowed with a complete Riemannian metric of finite area. These sharp curvature-free upper bounds are expressed in terms of the area of the punctured sphere. In both cases, we describe the extremal metrics, which are modeled on the Calabi-Croke sphere or the tetrahedral sphere. We also extend these optimal inequalities for reversible and non-necessarily reversible Finsler metrics. In this setting, we obtain optimal bounds for spheres with a larger number of punctures. Finally, we present a roughly asymptotically optimal upper bound on the length of the shortest closed geodesic for spheres/surfaces with a large number of punctures in terms of the area. This is joint work with S. Sabourau.

Contents

1.1 Introduction

This work deals with universal upper bounds on the length of the shortest closed geodesic on surfaces with a complete Riemannian metric of finite area. The existence of a closed geodesic on a closed surface with a Riemannian metric follows from a minimization process using Ascoli's theorem in the nonsimply connected case and from Birkhoff's minmax principle in the simply connected case. For noncompact surfaces with a complete Riemannian metric, closed geodesics may not exist (the Euclidean plane yields an obvious example). However, it was proved by Thorbergsson [69] for surfaces with at least three ends and by Bangert [12] for surfaces with one or two ends that every noncompact surface Σ with a complete Riemannian metric of finite area has a closed geodesic. This allows us to introduce

$$\operatorname{scg}(\Sigma) = \inf\{\operatorname{length}(\gamma) \mid \gamma \text{ is a closed geodesic of } \Sigma\}.$$

Note that in higher dimension the existence of a closed geodesic on a closed Riemannian manifold has been established by Fet and Lyusternik, but whether closed geodesics exist or not on any complete noncompact Riemannian n-manifold of finite volume with $n \geq 3$ is an open question; see [21, Question 2.3.1].

In this article, we are interested in finding good (if possible optimal) curvature-free upper bounds on $\operatorname{scg}(\Sigma)$ for every surface Σ with a complete Riemannian metric of finite area A. For every nonsimply connected closed surface, it was independently proved by Hebda [40] and Burago-Zalgaller [20] that $\operatorname{scg}(\Sigma) \leq \sqrt{2}\sqrt{\operatorname{area}(\Sigma)}$. In this case, optimal bounds are only known for the torus (Loewner 1949, unpublished, see [45]), the projective plane (Pu 1952, [56]) and the Klein bottle (Bavard 1986, [13]); see also Section 1.5 and Section 1.6 for a brief presentation of the extremal inequalities. For the sphere, it was proved by Croke [25] that $\operatorname{scg}(\Sigma) \leq 31\sqrt{\operatorname{area}(\Sigma)}$. This bound was subsequently improved in [53], [60] and [58], where the current best bound with 31 replaced with $4\sqrt{2}$ is due to Rotman [58]. It is conjectured that the global maximum for the length of the shortest closed geodesic among Riemannian metrics with fixed area on the sphere is attained by the Calabi-Croke sphere (see [25] and [26]), which would yield

$$\operatorname{scg}(S^2) \leq 2^{\frac{1}{2}}\, 3^{\frac{1}{4}}\, \sqrt{\operatorname{area}(S^2)}. \tag{1.1}$$

Recall that the Calabi-Croke sphere is defined as the piecewise flat sphere with three conical singularities of angle $\frac{2\pi}{3}$ obtained by gluing two copies of an equilateral triangle along their boundaries. Though this conjecture remains wide open, it was proved by Balacheff [10] that the Calabi-Croke sphere is a local maximum (see also [61] for an alternative proof extending to the Lipschitz distance topology). No conjecture is available for other surfaces, except in genus 3, where Calabi constructed nonpositively curved piecewise flat metrics with systolically extremal-like properties; see [23]

and [62] (and [63] for related systolic-like properties in genus 2). Under a nonpositive curvature assumption, extremal systolic inequalities have been established for the genus 2 surface and the connected sum of three projective planes; see [46] and [47]. In both cases, the extremal nonpositively curved metrics are piecewise flat with conical singularities. It was later proved that this structure is common to all extremal nonpositively curved surfaces; see [48]. Extremal systolic inequalities in a fixed conformal class have been investigated in relation with closed string field theory; see [38], [39], [54] for the most recent contributions and the references therein.

For noncompact surfaces Σ with a complete Riemannian metric of finite area, it was also shown by Croke [25] that $\mathrm{scg}(\Sigma) \leq 31\sqrt{\mathrm{area}(\Sigma)}$ (without any curvature assumption). This bound has recently been improved by Beach and Rotman [18], where the constant 31 is replaced with $4\sqrt{2}$ for surfaces with one puncture and with $2\sqrt{2}$ for surfaces with at least two punctures. The authors also conjectured that the optimal bound for a punctured sphere with at most three punctures is the same as that for the sphere; see (1.1).

In this article, we show that this conjecture is true for spheres with exactly three punctures and prove an optimal bound for spheres with four punctures. These are the only new optimal universal upper bounds on the length of the shortest closed geodesic obtained during the almost 35 years since Bavard's inequality on the Klein bottle [13]. We also improve the best known upper bounds for spheres with a higher number of punctures. More precisely, we have the following.

Theorem 1.1.1. *Let $\Sigma = S^2 \setminus \{x_1, \ldots, x_k\}$ be a k-punctured sphere with a complete Riemannian metric of finite area. Then the following holds.*

1. *If $k = 3$ then there exists a noncontractible figure-eight geodesic γ on Σ such that*
$$\mathrm{length}(\gamma) < 2^{\frac{1}{2}} \cdot 3^{\frac{1}{4}} \sqrt{\mathrm{area}(\Sigma)}. \tag{1.2}$$
Furthermore, this inequality is optimal.

2. *If $k \geq 4$ then there exists a noncontractible closed geodesic γ on Σ such that*
$$\mathrm{length}(\gamma) < 2 \cdot 3^{-\frac{1}{4}} \sqrt{\mathrm{area}(\Sigma)}. \tag{1.3}$$
Furthermore, this inequality is optimal when $k = 4$.

The extremal metric on the three-punctured sphere in (1.2) is modelled on the Calabi-Croke sphere by attaching three cusps of arbitrarily small area around its singularities. This can be done keeping the curvature nonpositively curved.

The extremal metric on the four-punctured sphere in (1.3) is modelled on the tetrahedral sphere by attaching four cusps of arbitrarily small area around its singularities. Here, the tetrahedral sphere is defined as the piecewise flat sphere with

four conical singularities of angle π given by the regular tetrahedron. This can also be done keeping the curvature nonpositively curved.

A version of this theorem holds true for complete Finsler punctured spheres of finite Holmes-Thompson area; see Section 1.2 for a brief account on reversible and non-necessarily reversible Finsler metrics and the Holmes-Thompson volume.

For reversible Finsler metrics, we have the following.

Theorem 1.1.2. *Let $\Sigma = S^2 \setminus \{x_1, \dots, x_k\}$ be a k-punctured sphere with a complete reversible Finsler metric of finite area. Then the following holds.*

1. *If $k = 3$ then there exists a noncontractible figure-eight geodesic γ on Σ such that*
$$\operatorname{length}(\gamma) < 2^{-\frac{1}{2}} \cdot 3^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\operatorname{area}(\Sigma)}. \tag{1.4}$$
2. *If $k \geq 4$ then there exists a noncontractible closed geodesic γ on Σ such that*
$$\operatorname{length}(\gamma) < \pi^{\frac{1}{2}} \sqrt{\operatorname{area}(\Sigma)}. \tag{1.5}$$
Furthermore, this inequality is optimal when $k \in \{4, \dots, 6\}$.

In contrast to (1.2), the inequality (1.4) on the three-punctured sphere is not necessarily optimal which means that it is possible to find a constant less than the one in the inequality (1.4). The extremal metric on the four-punctured sphere in (1.5) is modelled on the sphere S^2 obtained by gluing two copies of the square $[0,1] \times [0,1]$ endowed with the ℓ^1-metric along their boundary, with four cusps of arbitrarily small area attached around the four vertices of the squares. Note that the four vertices of the squares and the two centers of the squares are at distance $\frac{1}{2}\operatorname{scg}(S^2) = 1$ from each other. Similarly, the extremal metrics on the five- and six-punctured spheres in (1.5) are obtained by attaching one or two extra cusps of arbitrarily small area around the centers of the squares.

For non-necessarily reversible Finsler metrics, we have the following.

Theorem 1.1.3. *Let $\Sigma = S^2 \setminus \{x_1, \dots, x_k\}$ be a k-punctured sphere with a complete non-necessarily reversible Finsler metric of finite area. Then the following holds.*

1. *If $k = 3$ then there exists a noncontractible figure-eight geodesic γ on Σ such that*
$$\operatorname{length}(\gamma) < 2^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\operatorname{area}(\Sigma)}. \tag{1.6}$$
2. *If $k \geq 4$ then there exists a noncontractible closed geodesic γ on Σ such that*
$$\operatorname{length}(\gamma) < 2 \cdot 3^{-\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\operatorname{area}(\Sigma)}.$$

In contrast to Theorem 1.1.1 and Theorem 1.1.2, the inequalities in this theorem are not necessarily optimal. In Section 1.6, we present further optimal inequalities on punctured tori, punctured projective planes and punctured Klein bottle with a few ends, where the number of ends (interestingly) depends whether the metric is Riemannian, reversible Finsler or non-reversible Finsler. These inequalities immediately follow from the corresponding optimal bounds for the underlying closed surfaces.

The following table gives an approximation of the constant $c_{\#}$ (optimal in some cases) for the inequality

$$\operatorname{scg}(\Sigma) \leq c_{\#} \sqrt{\operatorname{area}(\Sigma)}$$

where Σ is a k-punctured sphere with a complete metric of finite area in the Riemannian, reversible Finsler and non-necessarily reversible Finsler cases when $k = 3$ or 4.

$c_{\#}$	Riemannian	reversible Finsler	non-reversible Finsler
k=3	$2^{\frac{1}{2}} \cdot 3^{\frac{1}{4}} \simeq 1.861...$	$2^{-\frac{1}{2}} \cdot 3^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \simeq 2.170...$	$2^{\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \simeq 2.506...$
k=4	$2 \cdot 3^{-\frac{1}{4}} \simeq 1.519...$	$\pi^{\frac{1}{2}} \simeq 1.772...$	$2 \cdot 3^{-\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \simeq 2.046...$

We conclude by proving a roughly asymptotically optimal upper bound on the length of the shortest noncontractible closed geodesic on spheres with a large number of punctures, *i.e.*, there exist a constant $c > 0$ and a sequence (g_{k_n}) of complete Riemannian metrics of finite area on the k_n-punctured sphere where k_n tends to infinity, such that

$$\frac{\operatorname{sys}(g_{k_n})}{\sqrt{\operatorname{area}(g_{k_n})/k_n}} \geq c > 0.$$

In Theorem 1.7.2, we present a more general statement for genus g surfaces with k punctures.

Theorem 1.1.4. *Let $\Sigma = S^2 \setminus \{x_1, \ldots, x_k\}$ be a k-punctured sphere with a complete Riemannian metric of finite area, where $k \geq 3$. Then there exists a noncontractible closed geodesic γ on Σ such that*

$$\operatorname{length}(\gamma) \leq 4\sqrt{2} \sqrt{\frac{\operatorname{area}(\Sigma)}{k}}.$$

Furthermore, the upper bound is roughly asymptotically optimal in k.

Similar upper bounds hold true both in the reversible and non-necessarily reversible Finsler cases (albeit with a different multiplicative constant).

The proofs of our optimal bounds, namely Theorem 1.1.1, Theorem 1.1.2 and Theorem 1.1.3, do not rely on the conformal length method used in [45], [56] and [13]

to establish optimal systolic inequalities. Instead, we exploit ramified covers by the torus to the sphere (the first one was introduced in [59] and used in [9] and [61] in the same context) to connect the extremal properties of the punctured spheres with the extremal equilateral flat torus in Loewner's systolic inequality; see Theorem 1.5.3.

1.2 Finsler metrics and Holmes-Thompson volume

This section aims at introducing the notions of Finsler metrics and Holmes-Thompson volume.

Let us recall the definition of a Finsler metric.

Definition 1.2.1. A *Finsler metric* on a manifold M is a continuous function $F : TM \to [0, \infty)$ on the tangent bundle TM of M which is smooth outside the zero section of TM and whose restriction $F_x := F_{|T_xM}$ to each tangent space T_xM is a (possibly asymmetric) norm, that is,

1. Subadditivity: $F_x(u + v) \leq F_x(u) + F_x(v)$ for every $u, v \in T_xM$;
2. Homogeneity: $F_x(tu) = tF_x(u)$ for every $u \in T_xM$;
3. Positive definiteness: $F_x(u) > 0$ for every nonzero $u \in T_xM$.

A Finsler metric is *reversible* if $F_x(-u) = F_x(u)$ for every $x \in M$ and $u \in T_xM$.

The length of a piecewise smooth curve $\gamma : [0, 1] \to M$ is defined as

$$\text{length}(\gamma) = \int_0^1 F(\gamma'(t))\, dt$$

and the distance between two points x and y in M is the infimal length of a curve γ in M joining x to y.

We will consider the following notion of volume.

Definition 1.2.2. The *Holmes-Thompson volume* of an n-dimensional Finsler manifold M is defined as the symplectic volume of its unit co-ball bundle $B^*M \subseteq T^*M$ divided by the volume ϵ_n of the Euclidean unit ball in $\mathbb{R}^n$. That is,

$$\text{vol}(M) = \frac{1}{\epsilon_n} \int_{B^*M} \tfrac{1}{n!}\, \omega_M^n$$

where ω_M is the standard symplectic form on T^*M.

The Holmes-Thompson volume of a Finsler manifold is bounded from above by its Hausdorff measure, with equality if and only if the metric is Riemannian. This is a consequence of the Blaschke-Santaló inequality, see [29]. Note also that the Holmes-Thompson volume of a Riemannian manifold agrees with its Riemannian volume.

1.3 Degree-three ramified cover from the torus onto the Calabi-Croke sphere

Consider the piecewise flat sphere (S^2, g_0) with three conical singularities x_1, x_2, x_3 obtained by gluing two copies of a flat unit-side equilateral triangle along their boundaries. The sphere (S^2, g_0) is referred to as the Calabi-Croke sphere.

By the standard covering theory [28], there exists a degree-three cover $\pi_0 : \mathbb{T}^2 \to S^2$ ramified over the three vertices x_1, x_2, x_3 of S^2, and a deck transformation map $\rho_0 : \mathbb{T}^2 \to \mathbb{T}^2$ fixing only the ramification points of $\pi_0 : \mathbb{T}^2 \to S^2$ with $\rho_0^3 = \mathrm{id}_{\mathbb{T}^2}$ and $\pi_0 \circ \rho_0 = \pi_0$.

The ramified cover $\pi_0 : \mathbb{T}^2 \to S^2$ can also be constructed in a more geometric way as follows. First, cut the sphere along the two minimizing arcs of g_0 joining x_1 to x_2 and x_1 to x_3. This yields a parallelogram with all sides of unit length. Then, glue three copies of this parallelogram along the two sides between x_3 and the two copies of x_1 to form a hexagon; see Figure 1.1. By identifying the opposite sides of this parallelogram, we obtain an equilateral flat torus $\mathbb{T}^2$. The isometric rotation, defined on the hexagon, centered at x_3 and permuting the parallelograms, passes to the quotient and induces a map $\rho_0 : \mathbb{T}^2 \to \mathbb{T}^2$. This map gives rise to a degree-three ramified cover $\pi_0 : \mathbb{T}^2 \to S^2$.

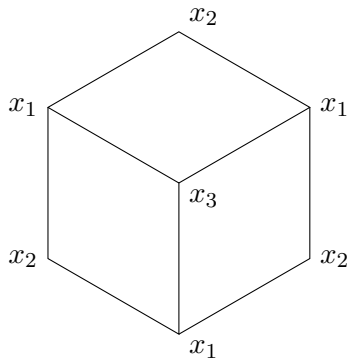


Figure 1.1: Degree-three ramified cover of the Calabi-Croke sphere

Thus, the Calabi-Croke sphere can be described as the quotient of an equilateral flat torus by the deck transformation map $\rho_0 : \mathbb{T}^2 \to \mathbb{T}^2$.

Given a Riemannian metric with conical singularities on S^2, we will endow $\mathbb{T}^2$ with the metric pulled back by $\pi_0 : \mathbb{T}^2 \to S^2$ and its universal cover $\mathbb{R}^2$ with the metric pulled back by the composite map

$$\mathbb{R}^2 \to \mathbb{T}^2 \to S^2.$$

Since the degree of the Riemannian ramified cover $\pi_0 : \mathbb{T}^2 \to S^2$ is equal to three, we have

$$\operatorname{area}(\mathbb{T}^2) = 3 \operatorname{area}(S^2). \tag{1.7}$$

Remark 1.3.1. The degree-three ramified cover $\pi_0 : \mathbb{T}^2 \to S^2$ was first used in this type of setting in [59] in relation with extremal properties of the Calabi-Croke sphere regarding the length of the shortest closed geodesic. It was later used in [10] to show that the Calabi-Croke sphere is a local extremum for the length of the shortest closed geodesic among metrics with fixed area. A different proof which does not require the uniformization theorem, but still makes use of the degree-three ramified cover $\pi_0 : \mathbb{T}^2 \to S^2$, can be found in [61].

1.4 Degree-two ramified cover from the torus onto the tetrahedral sphere

Consider the piecewise flat sphere (S^2, g_1) with four conical singularities x_1, x_2, x_3, x_4 given by the unit-side regular tetrahedron. The sphere (S^2, g_1) is referred to as the tetrahedral sphere.

By the standard covering theory, there exists a degree-two cover $\pi_1 : \mathbb{T}^2 \to S^2$ ramified over the four vertices x_1, x_2, x_3, x_4 of S^2, and a deck transformation map $\rho_1 : \mathbb{T}^2 \to \mathbb{T}^2$ fixing only the ramification points of $\pi_1 : \mathbb{T}^2 \to S^2$ with $\rho_1^2 = \mathrm{id}_{\mathbb{T}^2}$ and $\pi_1 \circ \rho_1 = \pi_1$.

The ramified cover $\pi_1 : \mathbb{T}^2 \to S^2$ can also be constructed in a more geometric way as follows. First, cut the sphere along the three minimizing arcs of g_1 joining x_1 to x_2, x_1 to x_3 and x_1 to x_4. This yields an equilateral triangle with side length two. Then, glue two copies of this triangle along the side passing through x_4 to form a parallelogram; see Figure 1.2. By identifying the opposite sides of this parallelogram, we obtain an equilateral flat torus $\mathbb{T}^2$. The symmetry, defined on the parallelogram, centered at x_4 and switching the two equilateral triangles, passes to the quotient and induces a map $\rho_1 : \mathbb{T}^2 \to \mathbb{T}^2$. This map gives rise to a degree-two ramified cover $\pi_1 : \mathbb{T}^2 \to S^2$.

Thus, the tetrahedral sphere (S^2, g_1) can be described as the quotient of an equilateral flat torus by the deck transformation map $\rho_1 : \mathbb{T}^2 \to \mathbb{T}^2$.

Given a Riemannian metric with conical singularities on S^2, we will endow $\mathbb{T}^2$ with the metric pulled back by $\pi_1 : \mathbb{T}^2 \to S^2$ and its universal cover $\mathbb{R}^2$ with the metric pulled back by the composite map

$$\mathbb{R}^2 \to \mathbb{T}^2 \to S^2.$$

Since the degree of the Riemannian ramified cover $\pi_1 : \mathbb{T}^2 \to S^2$ is equal to two, we have

$$\operatorname{area}(\mathbb{T}^2) = 2 \operatorname{area}(S^2). \tag{1.8}$$

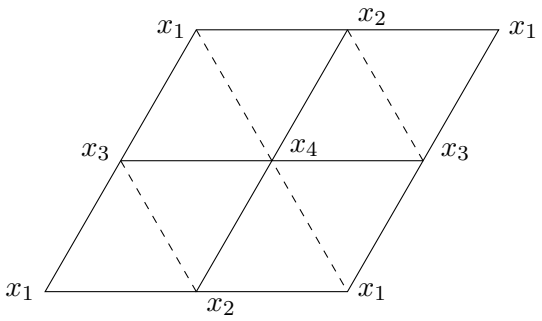


Figure 1.2: Degree-two ramified cover of the tetrahedral sphere

1.5 Proof of the main theorem

In this section, we recall some basic results in systolic geometry and prove the main theorem of this article, both in the Riemannian case and in the Finsler case.

Definition 1.5.1. Let M be a surface with a complete (Riemannian or Finsler) metric. The *systole* of M is defined as

$$\mathrm{sys}(M) = \inf\{\mathrm{length}(\gamma) \mid \gamma \text{ is a noncontractible loop of } M\}.$$

When M is closed, the systole is attained by the length of a noncontractible closed geodesic referred to as a *systolic loop* of M.

We will also need the following extension of the notion of systole.

Definition 1.5.2. Let M be a surface with k punctures and p marked points $x_1, \ldots, x_p$, endowed with a complete (Riemannian or Finsler) metric. A loop of M is *admissible* if it lies in $M' = M \setminus \{x_1, \ldots, x_p\}$ and is not homotopic in M' to a point, a multiple of a noncontractible simple loop of a cusp, or a multiple of a simple loop of M' surrounding a single marked point x_i. The *marked homotopy systole* of M is the infimal length of admissible loops of M. It is denoted by $\mathrm{sys}_*(M)$.

Let us recall Loewner's systolic inequality in the Riemannian case (unpublished), see [45], in the reversible Finsler case, see [61], and in the non-necessarily reversible case, see [1].

Theorem 1.5.3 ([45], [61], [1])**.** *Let $\mathbb{T}^2$ be a torus. Then the following statements hold true.*

1. *For every Riemannian metric on $\mathbb{T}^2$,*

$$\mathrm{sys}(\mathbb{T}^2) \leq 2^{\frac{1}{2}} \cdot 3^{-\frac{1}{4}} \sqrt{\mathrm{area}(\mathbb{T}^2)} \tag{1.9}$$

with equality if and only if $\mathbb{T}^2$ is an equilateral flat torus.

2. *For every reversible Finsler metric on* $\mathbb{T}^2$,

$$\mathrm{sys}(\mathbb{T}^2) \leq 2^{-\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\mathrm{area}(\mathbb{T}^2)} \tag{1.10}$$

with equality if $\mathbb{T}^2$ *is a square flat torus endowed with the* ℓ^1*- or* ℓ^∞*-metric.*

3. *For every non-necessarily reversible Finsler metric on* $\mathbb{T}^2$,

$$\mathrm{sys}(\mathbb{T}^2) \leq 2^{\frac{1}{2}} \cdot 3^{-\frac{1}{2}} \cdot \pi^{\frac{1}{2}} \sqrt{\mathrm{area}(\mathbb{T}^2)} \tag{1.11}$$

with equality if $\mathbb{T}^2$ *is homothetic to the quotient of* $\mathbb{R}^2$*, endowed with the non-symmetric norm whose unit disk is the triangle with vertices* $(1,0)$*,* $(0,1)$ *and* $(-1,-1)$*, by the lattice* $\mathbb{Z}^2$*.*

We can now proceed to the proof of the main theorem.

Proof of Theorem 1.1.1. Consider case (1). Let Σ be a 3-punctured sphere with a complete Riemannian metric of finite area. Take three cylindrical ends $\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3$ of Σ. For every $i = 1, \ldots, 3$, take a cylindrical end $C_i \subseteq \bar{C}_i$ with

$$d(C_i, \partial \bar{C}_i) > 2^{\frac{1}{2}} \cdot 3^{\frac{1}{4}} \sqrt{\mathrm{area}(\Sigma)}. \tag{1.12}$$

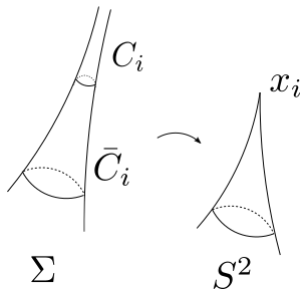


Figure 1.3: Collapsing C_i to a point x_i

Collapse every end C_i to a point x_i; see Figure 1.3. This gives rise to a sphere S^2 with a Riemannian metric with three singularities x_1, x_2, x_3. Note that $\mathrm{area}(S^2) < \mathrm{area}(\Sigma)$.

Consider the degree-three ramified cover $\pi_0 : \mathbb{T}^2 \to S^2$ with branched points x_1, x_2, x_3 described in Section 1.3. Denote by p_i the preimage of x_i under $\pi_0 : \mathbb{T}^2 \to S^2$ for $i = 1, \ldots, 3$. Endow $\mathbb{T}^2$ with the singular pullback Riemannian metric. The metric on $\mathbb{T}^2$ can be smoothed out in the neighborhood of its singularities, keeping the area and the systole fixed. By Loewner's inequality (1.9) and the relation (1.7), there exists a noncontractible closed geodesic γ on $\mathbb{T}^2$ with

$$\mathrm{length}(\gamma) \leq 2^{\frac{1}{2}} \cdot 3^{\frac{1}{4}} \sqrt{\mathrm{area}(S^2)}. \tag{1.13}$$

The loop γ does not pass through any singularity p_i. Otherwise, by the length upper bound (1.13) and the distance lower bound (1.12), it would lie in the topological disk of $\mathbb{T}^2$ given by the quotient $\bar{C}_i/C_i$. This would contradict the noncontractibility of γ in $\mathbb{T}^2$. To end the proof of the case (1) in Theorem 1.1.1, we need to prove the following Claim 1.5.4, similar to [61, Lemma 7.1].

Claim 1.5.4. *The systolic loop γ of $\mathbb{T}^2$ projects to a figure-eight geodesic of S^2. Furthermore, this figure-eight geodesic decomposes S^2 into three domains with exactly one vertex in each of them.*

Proof. Recall that γ does not pass through a ramification point of π. The projection of γ on S^2 forms a graph α with geodesic edges. Consider the shortest simple loop c_1 of α which seperates one vertex of S^2, say x_1 after renumbering, from the other two. This loop exists, otherwise the lift γ of α in $\mathbb{T}^2$ would be contractible. Consider also the shortest simple loop c_2 lying in the closure of $\alpha \setminus c_1$ which separates x_2 from x_3. This loop exists for the same reason as above. Switching the roles of x_2 and x_3 if necessary, we can assume the winding number of c_i around the vertex x_j in $S^2 \setminus \{x_3\}$ is equal to 1 if $i = j$ and 0 otherwise, where $i, j \in \{1, 2\}$.
Let c_3 be the shortest arc of α connecting c_1 to c_2. We can construct a loop c made of c_1, c_2 and two copies of c_3 with winding numbers 1 and -1 around x_1 and x_2 in $S^2 \setminus \{x_3\}$. Clearly,

$$\text{length}(c) \leq \text{length}(\alpha) = \text{length}(\gamma). \tag{1.14}$$

The loop c lifts to a noncontractible loop of $\mathbb{T}^2$. From (1.14), we conclude that c is the projection of a systolic loop of $\mathbb{T}^2$ like α. Thus, c is a geodesic loop of the same length as α. This implies that c is a figure-eight geodesic loop (with c_3 reduced to a point) which agrees with α. Hence the claim. □

In the case (2), the proof is similar. Start with a 4-punctured sphere Σ with a complete Riemannian metric of finite area. Take four cylindrical ends C_1, C_2, C_3, C_4 of Σ of small area located far away from the core of the surface. Collapse the cylindrical ends into points x_i. This gives rise to a sphere S^2 with a Riemannian metric with four singularities x_i. Consider the degree-two ramified cover $\pi_1 : \mathbb{T}^2 \to S^2$ with branched points x_i described in Section 1.4. By Loewner's inequality (1.9) and the relation (1.8), there exists a noncontractible closed geodesic γ on $\mathbb{T}^2$ with

$$\text{length}(\gamma) \leq 2 \cdot 3^{-\frac{1}{4}} \sqrt{\text{area}(S^2)}.$$

As previously, the systolic loop γ does not pass through a ramification point of $\pi_1 : \mathbb{T}^2 \to S^2$ and projects to a closed geodesic[1] of $S^2 \setminus \{x_1, x_2, x_3, x_4\} \subseteq \Sigma$. This

[1] Arguing as in [61, Lemma 7.1], one can show that the systolic loop γ of $\mathbb{T}^2$ projects either to a simple closed geodesic surrounding exactly two branched points of S^2 on each side, or to a figure-eight geodesic with exactly one or two branched points in each of the three domains of S^2 it bounds.

concludes the proof of the case (2) in Theorem 1.1.1.

Both inequalities are optimal since the extremal metric on $\mathbb{T}^2$ passes to the quotient by the transformation groups of the ramified covers π_0 and π_1. The extremal metrics on the punctured spheres thus-obtained are described in the introduction right after Theorem 1.1.1. □

Remark 1.5.5. In the Finsler case, we simply need to replace Loewner's inequality (1.9) with (1.10) for reversible Finsler metrics, and with (1.11) for non-necessarily reversible Finsler metrics. This leads to the Finsler version of the main theorem given by Theorem 1.1.2 and Theorem 1.1.3. The only minor novelty is when $k = 5$ or 6. In this case, we take k cylindrical ends C_i of Σ of small area located far away from the core of the surface, and collapse the cylindrical ends into points x_i. Consider the degree-two ramified cover $\pi_1 : \mathbb{T}^2 \to S^2$ branched *only* at four points $x_1, \ldots, x_4$ as previously. Apply the Finsler version of Loewner's inequality and observe that the systolic loops of $\mathbb{T}^2$ do not pass through the preimages $\pi_1^{-1}(x_i)$ of the singularities of S^2 and project to closed geodesics of $S^2 \setminus \{x_1, \ldots, x_k\} \subseteq \Sigma$ as required.

Remark 1.5.6. In contrast to the Riemannian case, the extremal (reversible or non-reversible) Finsler metric on $\mathbb{T}^2$ does not pass to the quotient under the deck transformation groups of $\pi_0 : \mathbb{T}^2 \to S^2$, which explains why the inequalities (1.4) and (1.6) may not be optimal. The same occurs for the extremal non-reversible Finsler metric on $\mathbb{T}^2$ with the deck transformation group of $\pi_1 : \mathbb{T}^2 \to S^2$. However, the extremal reversible Finsler metric on $\mathbb{T}^2$ does pass to the quotient under the deck transformation group of $\pi_1 : \mathbb{T}^2 \to S^2$. In this case, the inequality (1.5) is optimal and the approximating metrics are described in the introduction right after Theorem 1.1.2.

Remark 1.5.7. One may wonder if our technique can be applied to other ramified covers $\mathbb{T}^2 \to S^2$ in order to derive sharp upper bounds on the length of the shortest closed geodesics on other Riemannian punctured spheres Σ. At the heart of the matter is the property that the extremal equilateral flat metric on $\mathbb{T}^2$ should induce an extremal Riemannian metric on Σ but also on S^2 with marked points/branched points x_i corresponding to the ends of Σ. In particular, the marked homotopy systole of S^2 should be greater or equal to the systole of $\mathbb{T}^2$. This implies that the ramification points p_i of $\mathbb{T}^2$ must be at distance at least $\frac{1}{2}\operatorname{sys}(\mathbb{T}^2)$ from each other. Thus, the open disks $D(p_i, \frac{1}{4}\operatorname{sys}(\mathbb{T}^2))$ must be disjoint. Since the area of each of these flat disks is equal to $\frac{\pi}{16}\operatorname{sys}(\mathbb{T}^2)^2$, we deduce that the number of ramification points of $\mathbb{T}^2$ does not exceed

$$\frac{\operatorname{area}(\mathbb{T}^2)}{\frac{\pi}{16}\operatorname{sys}(\mathbb{T}^2)^2} = \frac{8}{\pi}\sqrt{3} = 4.4...$$

Therefore, the number of ramification points is at most 4. In conclusion, our method to find extremal Riemannian metrics based on Loewner's inequality on the torus cannot apply to punctured spheres with more than 4 ends.

1.6 Extremal metrics on noncompact surfaces

In this section, we present other examples of noncompact surfaces admitting sharp upper bounds on the length of their shortest closed geodesic.

Proposition 1.6.1. *Let M be a closed surface with a systolically extremal (Riemannian or Finsler) metric. Denote by $\Sigma = M \setminus \{x_1, \dots, x_k\}$ the surface M with k punctures. Then every complete (Riemannian or Finsler) metric on Σ satisfies*

$$\operatorname{sys}_*(\Sigma) \leq c(M) \sqrt{\operatorname{area}(\Sigma)} \tag{1.15}$$

where $c(M) = \frac{\operatorname{sys}(M)}{\sqrt{\operatorname{area}(M)}}$.

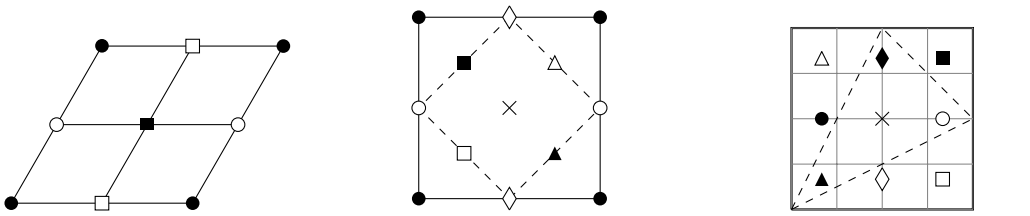
Proof. To prove this upper bound on $\operatorname{sys}_*(\Sigma)$, simply collapse small enough and far enough cylindrical ends C_i of Σ. The resulting surface M' (where the metric is smoothed out) is homeomophic to M and satisfies $\operatorname{sys}_*(\Sigma) \leq \operatorname{sys}(M')$ and $\operatorname{area}(M') \leq \operatorname{area}(\Sigma)$. Since the metric on M is systolically extremal, we clearly have $c(M') \leq c(M)$ and the desired result immediately follows. □

The inequality (1.15) is not optimal when k is large, see Theorem 1.7.2, but it is for small values of k. For this, one needs to find k points on M at distance at least $\frac{1}{2}\operatorname{sys}(M)$ from each other. Compare with Remark 1.5.7.

For instance, we can consider the extremal (Riemannian or Finsler) metrics on the torus as follows; see Theorem 1.5.3. The equilateral flat torus (see Theorem 1.5.3.(1)) admits 4 such points; see Figure 1.4a. Attaching cusps of arbitrarily small area around these 4 points, we construct an almost extremal Riemannian metric on the torus with k punctures, where $k \leq 4$. Similarly, the square flat torus with the ℓ^1-metric (see Theorem 1.5.3.(2)) admits 8 such points; see Figure 1.4b. As previously, we can construct an almost extremal reversible Finsler metric on the torus with k punctures, where $k \leq 8$. Finally, the square torus with the extremal non-reversible Finsler metric (see Theorem 1.5.3.(3)) admits 9 points whose distance, back and forth, between any pair of them is at least $\operatorname{sys}(\mathbb{T}^2)$; see Figure 1.4c. (Note that the asymmetric distance between two of these points might be less than $\frac{1}{2}\operatorname{sys}(\mathbb{T}^2)$ but the distance in the opposite direction makes up for it and their sum is at least $\operatorname{sys}(\mathbb{T}^2)$.) As previously, we can construct an almost extremal non-reversible Finsler metric on the torus with k punctures, where $k \leq 9$.

The same construction applies to the projective plane where the extremal metric is given by the canonical metric both in the Riemannian and Finsler cases; see [56] and [41]. More precisely, we can construct an almost extremal metric on the projective plane with k punctures, where $k \leq 3$.

This construction also applies to the Klein bottle where the extremal metric is known both in the Riemannian and reversible Finsler settings. Specifically, the extremal Riemannian Klein bottle is obtained by attaching along their boundary



(a) Riemannian (b) reversible Finsler (c) non-reversible Finsler

Figure 1.4: Separated points on the torus

two copies of the Mobius band defined as the quotient of the $\frac{\pi}{4}$-neighborhood of the equator on the standard sphere by the antipodal map; see [13]. While the extremal Finsler Klein bottle is the square flat Klein bottle with the ℓ^1-metric; see [24]. Thus, we can construct an almost extremal metric on the Klein bottle with k punctures, where $k \leq 4$ in the Riemannian case and $k \leq 8$ in the reversible Finsler case.

1.7 Surfaces with many punctures

In this section, we show a roughly asymptotically optimal upper bound on the length of the shortest closed geodesic on a surface with a large number of punctures.

We will need the following result, which can be found in [11, Lemma 6.5].

Lemma 1.7.1. *Let M be a closed surface with a Riemannian metric and k marked points $x_1, \ldots, x_k$, with $k \geq 3$. Fix $R \in (0, \frac{1}{4}\operatorname{sys}_*(M)]$. Then there exists a closed Riemannian surface $\bar{M}$ such that*

$$\operatorname{area}(\bar{M}) \leq \operatorname{area}(M) \tag{1.16}$$

$$\operatorname{sys}_*(\bar{M}) = \operatorname{sys}_*(M) \tag{1.17}$$

$$\operatorname{area} \bar{D}(R) \geq \tfrac{1}{2}R^2 \tag{1.18}$$

for every disk $\bar{D}(R)$ of radius R in $\bar{M}$.

The following result implies Theorem 1.1.4 when $g = 0$.

Theorem 1.7.2. *Let Σ be a surface of genus g with k punctures, endowed with a complete Riemannian metric of finite area. Then*

$$\operatorname{sys}_*(\Sigma) \leq C \frac{\log(g+2)}{\sqrt{g+k+1}} \sqrt{\operatorname{area}(\Sigma)}$$

where C is an explicit universal constant.

Proof. Take k cylindrical ends $C_i \subseteq \Sigma$ far away from the core of Σ so that

$$d(C_i, C_j) > \mathrm{sys}_*(\Sigma) \tag{1.19}$$

for every $i \neq j$, and

$$\mathrm{length}(\alpha) > \mathrm{sys}_*(\Sigma) \tag{1.20}$$

for every arc α of Σ with endpoints in C_i inducing a nontrivial class in $\pi_1(\Sigma, C_i)$.

Collapse every end C_i to a point x_i. Denote by M the resulting closed surface with k marked points $x_1, \ldots, x_k$. The Riemannian metric on Σ induces a metric on M that can be smoothed out in the neighborhood of the singularities x_i, keeping the area and the marked homotopy systole fixed. Note that $\mathrm{area}(M) \leq \mathrm{area}(\Sigma)$.

Claim 1.7.3. *We have*

$$\mathrm{sys}_*(\Sigma) \leq \mathrm{sys}(M).$$

Proof. Let us show that $\mathrm{length}(\gamma) \geq \mathrm{sys}_*(\Sigma)$ for every noncontractible loop γ of M. By (1.19), we can assume that the loop γ passes through at most one singularity of M, otherwise we are done. We can further assume that the loop γ does not pass through any singularity x_i of M. Otherwise, it would admit an arc $\alpha \subseteq \Sigma$ with endpoints in C_i inducing a nontrivial class in $\pi_1(\Sigma, C_i)$ as a lift under the quotient map $\Sigma \to M$. By (1.20), we would be done. Thus, the loop γ of M also lies in Σ. Furthermore, the loop γ is noncontractible in Σ, even after collapsing the ends of Σ. It follows that γ is an admissible loop of Σ. Therefore, $\mathrm{length}(\gamma) \geq \mathrm{sys}_*(\Sigma)$. □

The roughly asymptotically optimal systolic inequality for closed surfaces of large genus [36] [37] (see also [9] and [49] for alternate proofs) applied to M, combined with the relations $\mathrm{sys}_*(\Sigma) \leq \mathrm{sys}(M)$ and $\mathrm{area}(M) \leq \mathrm{area}(\Sigma)$, shows that

$$\mathrm{sys}_*(\Sigma) \leq C' \frac{\log(g+2)}{\sqrt{g+1}} \sqrt{\mathrm{area}(\Sigma)} \tag{1.21}$$

for some explicit universal constant C'. This proves the theorem when $k = 0$.

Now, consider the closed surface $\bar{M}$ obtained by applying Lemma 1.7.1 to the closed surface M with its k marked points, with $R = \frac{1}{4}\,\mathrm{sys}_*(M)$. Observe that

$$d_{\bar{M}}(x_i, x_j) \geq \tfrac{1}{2}\,\mathrm{sys}_*(M).$$

Otherwise we could find a figure-eight curve on $\bar{M}$ of length less than $\mathrm{sys}_*(M)$, in the neighborhood of the segment $[x_i, x_j]$, surrounding both x_i and x_j. This would contradict the relation (1.17).

It follows that the open disks $\bar{D}(x_i, \frac{1}{4}\,\mathrm{sys}_*(M))$ of $\bar{M}$ are disjoint. Combined with (1.16), we derive

$$\mathrm{area}(\Sigma) \geq \mathrm{area}(\bar{M}) \geq \sum_{i=1}^{k} \mathrm{area}\,\bar{D}(x_i, \tfrac{1}{4}\,\mathrm{sys}_*(M)).$$

Now, by (1.18), we have

$$\operatorname{area} \bar{D}(x_i, \tfrac{1}{4}\operatorname{sys}_*(M)) \geq \tfrac{1}{32}\operatorname{sys}_*(M)^2.$$

Hence,

$$\operatorname{sys}_*(\Sigma) \leq \frac{4\sqrt{2}}{\sqrt{k}}\sqrt{\operatorname{area}(\Sigma)}. \tag{1.22}$$

Now, if $k \geq 10\,\dfrac{g+1}{\log(g+2)^2}$, then $k \geq \dfrac{1}{10}\,\dfrac{g+k+1}{\log(g+2)^2}$ and the desired upper bound on $\operatorname{sys}_*(\Sigma)$ follows from (1.22). Otherwise, if $k \leq 10\,\dfrac{g+1}{\log(g+2)^2}$, then $\sqrt{g+1} \geq \frac{1}{10}\sqrt{g+k+1}$ and the desired upper bound follows from (1.21). □

Remark 1.7.4. Theorem 1.7.2 extends to Finsler metrics. Indeed, given a non-necessarily reversible Finsler metric F on Σ, we can replace F with a reversible Finsler metric F' defined by $F'(v) = F(v) + F(-v)$. Then we replace F' with the continuous Riemannian metric g whose unit disk agrees with the inner Loewner ellipsoid associated to the unit tangent disk of F'. By construction, $\operatorname{sys}_*(\Sigma, F) \leq \operatorname{sys}_*(\Sigma, g)$ and $\operatorname{area}(\Sigma, g) \leq \lambda \operatorname{area}(\Sigma, F)$ for some explicit universal constant λ. (We refer to the proofs of Corollary 4.12 and Theorem 4.13 in [1] for the details.) Thus, we can apply Theorem 1.7.2 to the Riemannian metric g and immediately derive a similar upper bound on the length of the shortest closed geodesic of F in terms of the Holmes-Thompson area of F (with a different multiplicative constant).

Chapter 2

Sharp systolic bounds on negatively curved surfaces

We show that every local supremum of the systole over the space of Riemannian metrics of curvature at most -1 on a given nonsimply connected closed surface is attained by a hyperbolic metric. As an application, we also present a partial extension of this result to 3-manifolds. This is joint work with S. Sabourau.

Contents

2.1 Introduction

Sharp upper bounds on the length of the shortest closed geodesic have been established only in a few instances, namely on the torus (see [45]), the projective plane [56], the Klein bottle [13] and the three and four punctured spheres [42], for complete Riemannian metrics of unit area, without any curvature assumption. For metrics of unit area and nonpositive curvature, sharp upper bounds are known for the genus two surface [46] and the connected sum of three projective planes [47], also called Dyck's surface. In these last two cases, the extremal metrics are flat with conical singularities. This has recently been generalized to surfaces of higher genus; see [48]. Namely, the supremum of the systole over the space of nonpositively curved Riemannian metrics of unit area on a given closed surface of nonzero genus is attained by a piecewise flat metric. Systolic inequalities have also been studied for hyperbolic surfaces, where the area is fixed depending only on the topology of the surface; see [65] [15], [22] for some foundational works.

In this article, we replace the normalization of the metric by the area with an upper bound on the curvature. More precisely, we study the supremum of the systole over the space of Riemannian metrics with curvature at most -1 on a given closed surface. It follows from the Gauss-Bonnet formula that the area of such metrics is bounded and from general (nonsharp) systolic inequalities on surfaces (see [45]) that the systole is also bounded.

We establish the following result regarding the existence of systolically extremal metrics of curvature at most -1.

Theorem 2.1.1. *Let Σ be a nonsimply connected closed surface. Then the maximal systole of a Riemannian metric of curvature at most -1 on Σ is attained by a hyperbolic metric.*

In fact, we prove a more general version dealing with local maxima and singular metrics; see our main result, namely Theorem 2.1.4.

Since the maximum of the systole among closed genus two hyperbolic surfaces is known (see [43], [14], [65, Theorem 5.2]), we immediately derive the following result.

Corollary 2.1.2. *The maximal systole of a closed genus two surface M with a Riemannian metric of curvature K_M at most -1 is attained by the hyperbolic metric conformal to the Bolza surface and is equal to*

$$\max_{K_M \leq -1} \operatorname{sys}(M) = 2 \operatorname{arccosh}(1 + \sqrt{2}).$$

Recall that the Bolza surface is the unique Riemann surface of genus two with a group of holomorphic automorphisms of order 48 and is defined as the smooth completion of the affine algebraic curve

$$y^2 = x^5 - x.$$

In view of their geometric and compactness properties, it is natural to consider Alexandrov surfaces (*i.e.*, surfaces with an intrinsic metric for which there exists a natural notion of curvature defined as a Radon measure) instead of surfaces with a Riemannian metric; see Section 2.2. Indeed, given a nonsimply connected closed surface Σ, the space of Alexandrov surfaces of curvature at most -1 homeomorphic to Σ with systole uniformly bounded away from zero is compact for the uniform distance topology (and the Gromov-Hausdorff topology); see [27] for a general compactness result. By continuity of the systole over the space of Alexandrov surfaces endowed with the uniform distance topology, the supremum of the systole over the space $\mathcal{A}_\Sigma$ of Alexandrov surfaces of curvature at most -1 homeomorphic to Σ is attained by an Alexandrov surface in $\mathcal{A}_\Sigma$. This shows it is natural to work within the class of Alexandrov surfaces.

Actually, we obtain a version of Theorem 2.1.1 which holds for local suprema of the systole (and not only for its global supremum). Before stating our result, we need to introduce the following definition.

Definition 2.1.3. Consider one of the three natural topologies on the space $\mathcal{A}_\Sigma$ of Alexandrov surfaces of curvature at most -1 homeomorphic to Σ, namely the bilipschitz distance topology, the uniform distance topology and the Gromov-Hausdorff topology; see Section 2.2 for a more detailed presentation.

A *local supremum* of the systole on $\mathcal{A}_\Sigma$ is a real number $\mu > 0$ such that there exists an open set $\mathcal{U} \subseteq \mathcal{A}_\Sigma$ satisfying a strict inequality

$$\mu = \sup_{M \in \mathcal{U}} \operatorname{sys}(M) > \sup_{M \in \partial\mathcal{U}} \operatorname{sys}(M).$$

Note that this definition is not entirely local. Indeed, the strict inequality may hold for some open set $\mathcal{U}$, but fail for arbitrarily small ones.

An Alexandrov surface $M \in \mathcal{A}_\Sigma$ of curvature at most -1 is *locally extremal* for the systole if there exists an open set $\mathcal{U} \subseteq \mathcal{A}_\Sigma$ containing M such that

$$\operatorname{sys}(M) = \sup_{M' \in \mathcal{U}} \operatorname{sys}(M') > \sup_{M' \in \partial\mathcal{U}} \operatorname{sys}(M').$$

In this case, we say that the local supremum $\mu = \sup_{M \in \mathcal{U}} \operatorname{sys}(M)$ is attained by M.

The notions of local supremum and locally extremal surface depend *a priori* on the chosen topology on $\mathcal{A}_\Sigma$.

We can now state our main result.

Theorem 2.1.4. *Let Σ be a nonsimply connected closed surface. Let $\mathcal{U} \subseteq \mathcal{A}_\Sigma$ be an open set in the space $\mathcal{A}_\Sigma$ of Alexandrov surfaces of curvature at most -1 defining a local supremum of the systole (for any of the natural topologies on $\mathcal{A}_\Sigma$ considered in Definition 2.1.3). Then there exists a hyperbolic surface $M \in \mathcal{U}$ such that*

$$\operatorname{sys}(M) \geq \operatorname{sys}(M')$$

for every $M' \in \mathcal{U}$.

Since the systole has only finitely many local maxima on the moduli space of hyperbolic metric on Σ, see [65, Theorem 2.6], we immediately derive the following corollary.

Corollary 2.1.5. *Let Σ be a nonsimply connected closed surface. Then the systole has only finitely many local suprema on the space $\mathcal{A}_\Sigma$ of Alexandrov surfaces of curvature at most -1 (for any of the natural topologies considered in Definition 2.1.3).*

A natural problem would be to extend our main result to higher dimension (recall that in higher dimension, a closed hyperbolic n-manifold has a unique hyperbolic metric, up to isometry, by Mostow's rigidity theorem). More specifically, given a closed hyperbolic n-manifold M with $n \geq 3$, is the systole of a Riemannian metric of curvature at most -1 on M always bounded by the systole of the hyperbolic metric on M? In the last section of this article, we present a partial result in this direction and give specific examples of closed hyperbolic 3-manifolds where the answer is affirmative.

Corollary 2.1.6. *There exist (explicit) closed hyperbolic 3-manifolds M whose systole is maximal among all Riemannian metrics of (sectional) curvature at most -1 on M.*

The strategy of the proof of the main result decomposes into two parts. First, we show that every local supremum of the systole on $\mathcal{A}_\Sigma$ is attained by a piecewise hyperbolic surface M of Alexandrov curvature at most -1 with finitely many conical singularities. The argument closely follows [48] and relies on the kite excision trick. This trick has the effect of moving a pair of singularities lying in the same systolic domain closer and closer until they merge, keeping the same curvature upper bound, while not decreasing the systole and strictly decreasing the area. Second, we eliminate the conical singularities. For this purpose, observe that the area of an Alexandrov surface of curvature at most -1 satisfies a sharp upper bound given by the Gauss-Bonnet formula and is maximal exactly when the metric is hyperbolic. Thus, to prove that the locally extremal piecewise hyperbolic surface M is hyperbolic, we would like to add as much area as possible to it without changing the systole or the curvature upper bound. Though the argument is actually slightly different, this captures the motivation. Thus, in order to remove the conical singularities, we apply the kite insertion trick, introduced for the occasion, which has the opposite effect to the kite excision trick: it still does not decrease the systole but increases the area while keeping the same curvature upper bound.

Concretely, we start by taking a surface of Alexandrov curvature at most -1 in an open set $\mathcal{U} \subseteq \mathcal{A}_\Sigma$ defining a local supremum of the systole. In Section 2.3, we recall that every surface of $\mathcal{A}_\Sigma$ can be bilipschitz approximated by a piecewise hyperbolic

surface of Alexandrov curvature at most -1 with N conical singularities. In Section 2.4, we state a compactness result on the space of piecewise hyperbolic surfaces of Alexandrov curvature at most -1 with at most N conical singularities and a systole bounded away from zero. This compactness result allows us to take a piecewise hyperbolic surface M in $\mathcal{U}$ with at most N conical singularities of maximal systole. In Section 2.5, we introduce the kite excision trick. In Section 2.6, we decompose every negatively curved Alexandrov surface along its systolic loops into polygonal systolic domains and show that the kite excision trick does not decrease the systole for small kites in a systolic domain. In Section 2.7, we relate the number of edges and vertices in the systolic decomposition to the number of pairwise nonhomotopic (systolic) loops intersecting at most twice. This yields an *a priori* upper bound on the complexity of the systolic decomposition of a negatively curved Alexandrov surface in terms of its Euler characteristic. In Section 2.8, we exploit the kite excision trick as in [48] to show that every systolic domain and every edge of a systolic domain of the surface M contains at most one small conical singularity. This yields an *a priori* upper bound on the number of conical singularities of M. We deduce that the surface M is extremal not only among piecewise hyperbolic surfaces of $\mathcal{U}$ with at most N conical singularities, but among all Alexandrov surfaces of $\mathcal{U}$. Thus, every local supremum of the systole on $\mathcal{A}_\Sigma$ is attained by a piecewise hyperbolic surface with finitely many conical singularities. Then, we consider an extremal piecewise hyperbolic surface M in $\mathcal{A}_\Sigma$ with a minimal number of systolic loops and show that the surface M has no conical singularity. For this purpose, we introduce the kite insertion trick which consists of cutting open a piecewise hyperbolic surface of $\mathcal{A}_\Sigma$ along a geodesic segment passing through a conical singularity and pasting a kite along its boundary; see Section 2.9. When done properly, the kite insertion trick has the effect of never decreasing the length of a minimizing loop in a given homotopy class keeping the same curvature upper bound; see Section 2.10. Applying the kite insertion trick to a piecewise hyperbolic surface of $\mathcal{A}_\Sigma$ either increases the systole or keeps the systole fixed decreasing its number of systolic loops; see Section 2.11. Since the surface M has a minimal number of systolic loops, we deduce that it has no conical singularities, otherwise we could derive a contradiction by applying the kite insertion trick around one of its conical singularities; see Section 2.12. Therefore, the local supremum of the systole among all Alexandrov metrics of curvature at most -1 is attained by a hyperbolic metric.

2.2 Alexandrov surfaces

We start by recalling that the *systole* of a nonsimply connected closed surface M with a length metric (*e.g.*, a piecewise Riemannian metric), denoted by $\mathrm{sys}(M)$, is the least length of a noncontractible loop in M.

We also need to introduce some definitions related to the metrics we consider in

this article.

Definition 2.2.1. A closed surface M with a Riemannian metric with conical singularities is locally isometric to the complex plane endowed with the metric

$$ds^2 = e^{2u(z)}|z|^{2\beta}|dz|^2$$

where $\beta > -1$ and $u : \mathbb{C} \to \mathbb{R}$ is a continuous function, smooth everywhere except possibly at the origin. The *total angle* of the conical singularity corresponding to the origin of $\mathbb{C}$ is equal to $\theta = 2\pi(\beta + 1)$. See [70] for a detailed description.

Alexandrov surfaces (or surfaces with bounded integral curvature) are surfaces endowed with a singular metric. They include polyhedral surfaces and surfaces endowed with a Riemannian metric with conical singularities. Geometrically, they can be defined as follows; see [8], [57], [71] and [27] for a detailed account on the subject.

Definition 2.2.2. A closed surface M with an intrinsic metric compatible with the topology of M is an *Alexandrov surface* if for every point of M there exists a neighborhood U homeomorphic to an open disk such that the sum of the (upper) excesses of the geodesic triangles of any finite system of non-overlapping simple triangles in U is bounded from above by a constant depending only on U. Recall that the (upper) excess of a geodesic triangle T is the sum of the upper angles of T minus π and that a geodesic triangle is simple if it is homeomorphic to a disk and convex.

Alexandrov surfaces can also be characterized by metric approximation as follows. A closed surface M with an intrinsic metric compatible with the topology of M is an Alexandrov surface if and only if the metric is the uniform limit of a sequence of Riemannian metrics with uniformly bounded absolute integral curvature.

On an Alexandrov surface M, there is no need to distinguished upper angles and lower angles since the angle between two geodesics always exists and for a geodesic triangle T with angles a, b, c, the excess of T is defined as $\delta(T) := a + b + c - \pi$.

The *curvature measure* $\omega = \omega^+ - \omega^-$ of M is defined by two non-negative Radon measures ω^+ and ω^- as follows. For every open subset $U \subseteq M$, let

$$\omega^{\pm}(U) = \sup \sum_{i=1}^{n} \delta(T_i)^{\pm}$$

where the supremum is taken over all finite systems $(T_i)_{1 \leq i \leq n}$ of non-overlapping simple triangles contained in U. Note that $\delta^+ = \max\{\delta, 0\}$ and $\delta^- = -\min\{\delta, 0\}$. Then for every Borel set $A \subseteq M$, we set

$$\omega^{\pm}(A) = \inf_{U \supseteq A} \omega^{\pm}(U)$$

where the infimum is taken over all open subsets U containing A. It can be shown that the curvature measure has locally finite total variation on M, see [8, Chapter 5].

By definition, a *peak point* of a closed Alexandrov surface is a point where the curvature measure is atomic of mass 2π. Note that closed Riemannian surfaces (possibly with conical singularities) do not have any peak points.

Upper bounds on the curvature can be introduced as follows.

Definition 2.2.3. An Alexandrov surface M is of (Alexandrov) *curvature at most* -1 if every small enough geodesic triangle $\Delta \subseteq M$ has a comparison triangle $\bar{\Delta} \subseteq \mathbb{H}^2$ in the hyperbolic plane, with sides of the same length as the sides of Δ, such that the distances between points in $\partial\Delta$ are less or equal to the distances between corresponding points in $\partial\bar{\Delta}$.

For instance, a closed piecewise hyperbolic surface with conical singularities of total angle at least 2π is an Alexandrov surface of curvature at most -1.

Alexandrov surfaces satisfy a Gauss-Bonnet formula, namely

$$\omega(M) = 2\pi\chi(M).$$

For closed Riemannian surfaces M of curvature K with N conical singularities of total angles θ_i, this formula can be written

$$\int_M K\, dA - \sum_{i=1}^{N}(\theta_i - 2\pi) = 2\pi\chi(M) \tag{2.1}$$

where $\chi(M)$ is the Euler characteristic of M. When the surface M is of Alexandrov curvature at most -1 (*i.e.*, when $K \leq -1$ and $\theta_i \geq 2\pi$), we obtain

$$\text{area}(M) \leq 2\pi|\chi(M)|. \tag{2.2}$$

We can consider several topologies on the space of Alexandrov surfaces.

Definition 2.2.4. Let Σ be a closed surface. There are three natural topologies on the space $\mathcal{A}_\Sigma$ of Alexandrov surfaces of curvature at most -1 homeomorphic to Σ, namely the bilipschitz distance topology, the uniform distance topology and the Gromov-Hausdorff distance topology, which are induced by the following distances.

Let M and M' be two closed Alexandrov surfaces homeomorphic to Σ. Define

1. the *bilipschitz distance* as

$$d_{\text{Lip}}(M, M') = \inf_f \max\{\log \text{dil}(f), \log \text{dil}(f^{-1})\}$$

where the infimum is taken over all bilipschitz homeomorphisms $f : M \to M'$ and $\text{dil}(f)$ is the dilatation/Lipschitz constant of f.

2. the *uniform distance* as

$$d_{\mathrm{unif}}(M, M') = \sup_{x,y \in M} \inf_{\varphi} |d_{M'}(\varphi(x), \varphi(y)) - d_M(x, y)|$$

where the infimum is taken over all homeomorphisms $\varphi : M \to M'$.

3. the *Gromov-Hausdorff distance* as

$$d_{\mathrm{GH}}(M, M') = \inf_{f,\, f'} \sup_{x,x' \in \Sigma} \max\{d_X(f(x), f'(M')), d_X(f'(x'), f(M))\}$$

where the infimum is taken over all isometric embeddings $f : M \hookrightarrow X$ and $f' : M' \hookrightarrow X$ into a metric space $X = (X, d_X)$.

In [19], it is shown that the bilipschitz distance between two closed Alexandrov surfaces is always finite and so are the other two distances.

Remark 2.2.5. The bilipschitz distance topology is finer than the uniform distance topology, which is finer than the Gromov-Hausdorff distance topology. Note also that the compactness result on the space of Alexandrov surfaces of curvature at most -1 homeomorphic to Σ with systole bounded away from zero, mentioned in the introduction, see [27], holds for the uniform distance topology (and so for the Gromov-Hausdorff distance topology), but not for the bilipschitz distance topology. Indeed, we can construct a deformation of a piecewise hyperbolic surface with conical singularities and Alexandrov curvature at most -1 by merging two conical singularities adding their total angles (and keeping the systole bounded away from zero). This deformation is continuous for the uniform distance topology, but it is not continuous for the bilipschitz distance topology because of the merging of the singularities. As a consequence, we can derive that every local supremum of the systole on the space $\mathcal{A}_\Sigma$ endowed with the uniform distance topology or the Gromov-Hausdorff distance topology is attained by an Alexandrov metric, but this result does not necessarily hold when $\mathcal{A}_\Sigma$ is endowed with the bilipschitz distance topology, except for the global supremum since the topology of the space of metrics does not play any role in this case.

2.3 Lipschitz approximation

The following Lipschitz approximation result of Alexandrov surfaces by piecewise hyperbolic surfaces essentially follows from [19, Lemma 6].

Proposition 2.3.1. *Every closed Alexandrov surface M without peak point is bilipschitz close to a piecewise hyperbolic surface M' with conical singularities. In particular, the systoles of the two surfaces are close.*

In addition, if the initial Alexandrov surface M is of curvature at most -1 then the piecewise hyperbolic surface M' can be assumed to be of Alexandrov curvature at most -1.

Proof. By [19, Lemma 6], the Alexandrov surface M admits a partition $\mathcal{T}$ into arbitrarily small geodesic triangles where each triangle Δ of $\mathcal{T}$ is bilipschitz close to its Euclidean comparison triangle Δ_0 with the same side lengths. Now, if the size of the triangles of $\mathcal{T}$ is sufficiently small, the Euclidean comparison triangles Δ_0 are bilipschitz close to their hyperbolic comparison triangle Δ_{-1}. Replacing every triangle Δ of $\mathcal{T}$ with its hyperbolic comparison triangle Δ_{-1} gives rise to a piecewise hyperbolic surface M' with conical singularities. Putting together the bilipschitz maps between the triangles Δ and Δ_{-1} yields a bilipschitz map between the two surfaces M and M' with bilipschitz constant close to 1.

By the Alexandrov–Toponogov comparison theorem, the angles of every triangle Δ in the partition $\mathcal{T}$ are less or equal to the corresponding angles in the hyperbolic comparison triangle Δ_{-1}. Thus, the total angles of the conical singularities of the associated piecewise hyperbolic surface M' are at least 2π, otherwise there would exist a point of M corresponding to a conical singularity of M' of total angle less than 2π through which passes no geodesic. Therefore, the piecewise hyperbolic surface M' has Alexandrov curvature at most -1. □

2.4 Metric compactness

Let us prove the following compactness result for closed piecewise hyperbolic surfaces, which is the equivalent of [48, Proposition 3.7] for piecewise flat nonpositively curved surfaces.

Proposition 2.4.1. *Fix a nonnegative integer N_0 and a positive real number s_0. Let Σ be a nonsimply connected closed surface. Then the space of piecewise hyperbolic surfaces $M \simeq \Sigma$ of Alexandrov curvature at most -1 with at most N_0 conical singularities and systole at least s_0 is compact.*

Proof. By assumption, the systole of M is at least s_0 and, by the Gauss-Bonnet formula (2.2), the area of M is at most $2\pi|\chi(\Sigma)|$. Thus, the systolic area

$$\sigma(M) = \frac{\operatorname{area}(M)}{\operatorname{sys}(M)^2}$$

of M is uniformly bounded. By [36, §5] (see [48, Proposition 3.6] for a detailed proof), this implies that the space of conformal classes of Riemannian metrics (possibly with conical singularities) on Σ with uniformly bounded systolic area is a compact set $\mathcal{K}$ in the conformal moduli space $\mathcal{M}_\Sigma$.

By [20, Theorem 5.3.1], every nonsimply connected closed surface satisfies the systolic inequality

$$\operatorname{area}(M) \geq \tfrac{1}{2} \operatorname{sys}(M)^2.$$

Since the Alexandrov curvature of M is at most -1 and its systole is at least s_0, we derive from the Gauss-Bonnet relation (2.1) that

$$\tfrac{1}{2}s_0^2 \leq \tfrac{1}{2}\operatorname{sys}(M)^2 \leq \operatorname{area}(M) \leq 2\pi|\chi(\Sigma)| - \sum_{i=1}^{N_0}(\theta_i - 2\pi).$$

Conversely, by [70, Theorem A], every conformal class of Σ carries a unique piecewise hyperbolic conformal metric with at most N_0 prescribed conical singularities p_i of given total angles θ_i, provided that the following relation

$$\sum_{i=1}^{N_0}(\theta_i - 2\pi) \leq 2\pi|\chi(\Sigma)| - \tfrac{1}{2}s_0^2 \tag{2.3}$$

is satisfied. Furthermore, the dependence on the parameters is continuous.

As the Alexandrov curvature of M is nonpositive, the total angles θ_i of the conical singularities are at least 2π and so lie in the interval $[2\pi, 2\pi(1-\chi(\Sigma))]$. Since the relation (2.3) and the conditions on the Alexandrov curvature and the number of conical singularities of M are closed, the N_0-tuple $(\theta_1, ..., \theta_{N_0})$ ranges through a compact set $L \subseteq \mathbb{R}^{N_0}$. Thus, the space of piecewise hyperbolic surfaces $M \simeq \Sigma$ of Alexandrov curvature at most -1 with at most N_0 conical singularities and systole at least s_0 is homeomorphic to a compact subset of $\mathcal{K}\times M^{N_0}\times L \subseteq \mathcal{M}_\Sigma \times M^{N_0} \times \mathbb{R}^{N_0}$. □

2.5 Kite excision trick

In this section, we present the kite excision trick for piecewise hyperbolic surfaces. Such a tool was recently introduced in [48] for nonpositively curved piecewise flat surfaces. We also establish two basic results by adapting the arguments developed in [48] to the piecewise hyperbolic case.

Consider a closed piecewise hyperbolic surface M of Alexandrov curvature at most -1.

Definition 2.5.1. A conical singularity $p \in M$ is said to be *large* if the total angle at p is at least 3π, and *small* otherwise. Observe that since M has nonpositive Alexandrov curvature, the total angles of its conical singularities are greater than 2π.

Let us introduce the definition of an (exact) kite.

Definition 2.5.2 (Kite)**.** Let $p, q \in M$ be two conical singularities connected by a geodesic arc $[p, q]$ with no conical singularity lying in the interior (p, q). Let $r \in M$ (not on $[p, q]$) be a point such that the triangle pqr is hyperbolic with acute angle at p and q. Consider the reflection pqr' of the hyperbolic triangle pqr with respect to $[p, q]$. Define the *kite* $K = prqr'$ as the union of the two symmetric hyperbolic

triangles; see Figure 2.1. The two opposite vertices p and q of K are referred to as the *main vertices* of the kite. The *width* w of K is the length of the diagonal $[r, r'] \subseteq K$. Note that all the angles of K are less than π.

Assume that p is a small singularity. The kite K is *exact* at p if its angles at the main vertices p and q are related to the angle excesses of the conical singularities p and q as follows:

$$\begin{aligned} \measuredangle rpr' &= \theta_p - 2\pi < \pi \\ \measuredangle rqr' &\leq \min\{\theta_q - 2\pi, \pi\} \end{aligned}$$

where θ_p and θ_q are the total angles at p and q.

Notice that the angle of the exact Kite K at p is *exactly* the angle excess of the conical singularity p, hence the name.

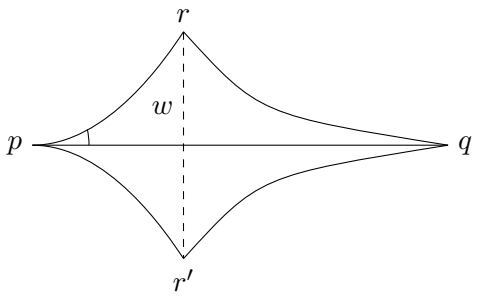


Figure 2.1: The hyperbolic kite K of width w.

We can now present the kite excision process.

Definition 2.5.3 (Excised surface M_w)**.** Let $K_w \subseteq M$ be a kite of width w. We excise the kite $K_w \subseteq M$ and introduce identifications on the boundary of $M \setminus K_w$ by setting $[p, r] \sim [p, r']$ and $[q, r] \sim [q, r']$. The result is a closed piecewise hyperbolic surface

$$M_w = (M \setminus K_w)/\sim$$

with conical singularities homeomorphic to M.

The quotient map

$$\pi_w : M \to M_w \tag{2.4}$$

is obtained by collapsing each segment of K_w parallel to the diagonal $[r, r']$ to a point. The map π_w is a homotopy equivalence.

The excised surface M_w of a closed piecewise hyperbolic surface M of Alexandrov curvature at most -1 satisfies the two following propositions; see [48, Propositions 6.7 and 6.8] for similar versions in the case of nonpositively curved piecewise flat surfaces.

Proposition 2.5.4. *If K_w is an exact kite (at one of its main vertices) then the excised surface M_w is piecewise hyperbolic of Alexandrov curvature at most -1 with at most as many conical singularities as M.*

Proof. Clearly, the excised surface M_w is a closed piecewise hyperbolic surface. The kite excision creates a new conical singularity at the point $r = r' \in M_w$ of total angle $4\pi - \measuredangle prq - \measuredangle qr'p > 2\pi$. Similarly, the total angles at the points p and q in the excised surface M_w are $\theta_p - \measuredangle r'pr = 2\pi$ and $\theta_q - \measuredangle rqr' \geq 2\pi$. Thus, the point p of total angle 2π is no longer a singularity in the new surface M_w. The other conical singularities of M remain unchanged in M_w. Therefore, the excised surface has at most as many conical singularities as M and their total angles are at least 2π, which implies that the Alexandrov curvature of M_w is at most -1. □

We also have the following convergence result.

Proposition 2.5.5. *Consider an exact kite K_w at p with main diagonal $[p, q]$ and width w. Then the excised surface M_w converges to M for the Lipschitz distance (and the uniform/Gromov-Hausdorff distances) as w tends to zero.*

Proof. Fix a kite $K_E = pr_0qr_0'$ exact at p. Consider a point p_* close to p such that p_* is on a geodesic extension p_*q of $[p, q]$ so that the rotation angle of p_*q at p is equal to $\frac{\theta_p}{2} \geq \pi$ on either side of the segment p_*q. Since the kite K_E is exact at p, we have $p \in [p_*, r_0]$. Fix a circular arc $\widehat{p_*r_0} \subseteq M \setminus K_E$ bounding a hyperbolic region $\mathcal{H}$ together with the segment $[p_*, r_0]$ containing p. Consider a smaller kite $K_w = prqr' \subseteq K_E$ of width w, exact at p, where $r \in (p, r_0)$. Let $p_r \in [p_*, p]$ with $|pp_r| = |pr|$. There exists a $(1 + \epsilon)$-bilipschitz homeomorphism

$$h_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$$

which fixes the circular arc $\widehat{p_*r_0}$ pointwise and linearly maps $[r_0, r]$, $[r, p]$, and $[p, p_*]$ to $[r_0, p]$, $[p, p_r]$, and $[p_r, p_*]$, respectively, where ϵ tends to 0 as r approches p. We combine the map $h_{\mathcal{H}}$ with $(1 + \epsilon)$-bilipschitz map from rr_0q to pr_0q fixing $[r_0, q]$, and perfom a symmetric construction on the other half of the kite. This produces a bilipschitz map $\phi_w : M_w \to M$ which agrees with the identity map on the complement of $\mathcal{H} \cup \mathcal{H}' \cup K_E \setminus K_w$ in $M_w = (M \setminus K_w)/\sim$, where $\mathcal{H}'$ is the symmetric of $\mathcal{H}$ and the bilipschitz constant of ϕ_w tends to 1 as w goes to zero. □

2.6 Systole comparison

In this section, we introduce the systolic decomposition of a surface and show that the kite excision trick does not decrease the systole for small kites chosen within a systolic domain.

Consider a closed piecewise hyperbolic surface M of Alexandrov curvature at most -1. In the following proposition, we will prove that every pair of intersecting systolic loops of M meet exactly at one or two points, or along a segment; see [48, Proposition 4.6] for similar proof.

Proposition 2.6.1. *Let M be a closed Alexandrov surface of curvature at most -1 of genus $g \geq 2$ with finitely many conical singularities. Then each pair of intersecting systolic loops of M meet exactly at one or two points, or along an arc.*

Proof. Let α and β be two systolic loops that meet in a single connected component, then they meet either in one point or along an arc.
Suppose now that their intersection $\alpha \cap \beta$ has at least two connected components. Then there exist two subarcs $\beta_1 \subseteq \beta$ and $\beta_2 \subseteq \beta$ in the complement of $\alpha \cap \beta$ with disjoint interior meeting α only at their endpoints. The endpoints of the arc β_i decompose α into two arcs denoted α_i' and α_i''. Observe that none of the four loops $\alpha_1' \cup \beta_1$, $\alpha_1'' \cup \beta_1$, $\alpha_2' \cup \beta_2$ and $\alpha_2'' \cup \beta_2$ is contractible, otherwise two distinct geodesic arcs with the same endpoints would be homotopic, which is impossible on a nonpositively curved surface. Thus, each of these four loops is of length at least $\mathrm{sys}(M)$. The sum of their lengths is at least $4\ \mathrm{sys}(M)$ and at most twice the total length of α and β:

$$4\ \mathrm{sys}(M) \leq |\alpha_1'| + |\alpha_1''| + |\alpha_2'| + |\alpha_2''| + 2|\beta_1| + 2|\beta_2| \leq 2|\alpha| + 2|\beta|. \tag{2.5}$$

Hence both inequalities in (2.5) are equalities and the same holds for the four inequalities involved in the sum. It follows that each of the arcs α_i', α_i'' and β_i is of length $\frac{1}{2}\mathrm{sys}(M)$. Therefore, the loops α and β have exactly two intersection points, which are antipodal along each loop. □

Definition 2.6.2. The *systolic decomposition* of M is the collection of open domains defined as the connected components of the complementary set in M of the systolic loops of M. The systolic decomposition of M induces a geodesic polygonal structure described as follows.

The *vertices* of this geodesic polygonal structure are of two types:

1. the intersection points between pairs of systolic loops when they meet at one or two points.

2. if systolic loops meet along a segment $I \subseteq M$ then the endpoints of I are also taken to be vertices.

The *edges* of this geodesic polygonal structure are the geodesic arcs given by the connected components of ∂D minus the vertices of ∂D, where D is a domain of the systolic decomposition of M.

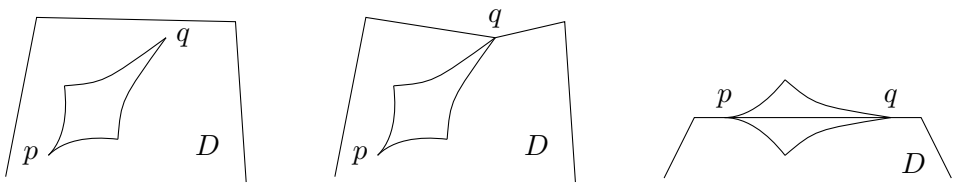
We will consider three positions for a kite within a systolic domain.

Definition 2.6.3. Consider a domain $D \subseteq M$ in the systolic decomposition of M. Let p and q be two conical singularities in the closure $\overline{D}$ of D, joined by a geodesic arc $[p,q] \subseteq \overline{D}$ such that no conical singularity of M lies in the interior (p,q) of the segment. Assume also that p is a small singularity. Choose an exact kite $K_w \subseteq M$ at p of width w with main diagonal $[p,q]$ according to the following cases (see Figure 2.2):

(E_1) if $[p,q] \subseteq D$, take K_w exact at p of sufficiently small width so that it lies in the open domain D;

(E_2) if $[p,q) \subseteq D$ with $q \in \partial D$, and the angle of D at q is greater than π, take K_w exact at p of sufficiently small width so that $K_w \setminus q$ lies in the open domain D;

(E_3) if $[p,q]$ is contained in (the interior of) an edge ∂D, take K_w exact at p.



Case (E_1) Case (E_2) Case (E_3)

Figure 2.2: Three exact kite configurations.

Let us establish the following comparison result between the systoles of a surface and its excised surface. A similar result was obtained in [48] where further cases were also considered. Here, we streamline the argument, correcting a false claim[1], to what is really required in our case.

Theorem 2.6.4. *Let M be a closed piecewise hyperbolic surface of Alexandrov curvature at most -1. Consider a kite $K_w \subseteq M$ with main diagonal $[p,q]$, exact at p, of width w satisfying one of the three cases (E_1), (E_2) or (E_3). If the width w of K_w is sufficiently small, then*

$$\operatorname{sys}(M_w) \geq \operatorname{sys}(M).$$

We will prove this result by analyzing the three cases (E_1), (E_2) and (E_3) separately. Observe that since the interior of K_w does not intersect any systolic loop of M in the cases (E_1) and (E_2), we immediately derive the reverse inequality $\operatorname{sys}(M_w) \leq \operatorname{sys}(M)$ in these two cases.

[1] It is incorrectly stated that $\pi_w : M_w \to M$ is nonexpanding at the end of the introduction of [48, §7]. As a consequence, we only have the inequality $\operatorname{sys}(M_w) \geq \operatorname{sys}(M)$ in [48, Proposition 7.3], which is enough to conclude.

Proposition 2.6.5 (Cases (E_1) and (E_2))**.** *Consider a kite K_w with main diagonal $[p,q]$, exact at p, of width w satisfying (E_1) or (E_2). If the width w of K_w is sufficiently small, then*

$$\mathrm{sys}(M_w) = \mathrm{sys}(M).$$

Proof. Consider a sequence of surfaces M_w with $\mathrm{sys}(M_w)$ going to its infimum limit as w goes to zero. Let γ_w be a systolic loop of M_w. Since the surfaces M_w converge to M for the Lipschitz distance as w goes to zero, see Proposition 2.5.5, we can extract from the systolic loop γ_w of M_w a sequence converging to a noncontractible loop γ in M by the Arzela-Ascoli theorem. We can assume that γ is a systolic loop of M, otherwise we derive that

$$\mathrm{sys}(M_w) = |\gamma_w| \geq |\gamma| - \varepsilon > \mathrm{sys}(M)$$

for some $\varepsilon > 0$ and every sufficiently small w, contradicting a previous inequality. By taking w small enough, we can further assume that γ_w and γ lie in the same nontrivial free homotopy class C in $M_w \simeq M$.

Suppose that γ is disjoint from K_w, which occurs in the case (E_1) and in the case (E_2) if γ does not pass through the singularity $q \in \partial D$. Then the projected loop $\pi_w(\gamma)$ remains a closed geodesic in M_w and $|\pi_w(\gamma)| = |\gamma|$ for w small enough, where $\pi_w : M \to M_w$ is the quotient map (2.4). Thus, the geodesic loops γ_w and $\pi_w(\gamma)$ of M_w agree (up to reparametrization) since they represent the same free homotopy class C and the surface M_w is negatively curved. That is, $\pi_w(\gamma) = \gamma_w$. Therefore,

$$\mathrm{sys}(M_w) = |\gamma_w| = |\pi_w(\gamma)| = |\gamma| = \mathrm{sys}(M).$$

Suppose that γ passes through K_w. Then the loop γ intersects K_w only at $q \in \partial D$ (corresponding to the case (E_2)) and $|\pi_w(\gamma)| = |\gamma|$. Since the angle of D at q is greater than π, the rotation angle R_q of γ at q from the side of D is also greater than π. Since the angle $\measuredangle r'qr$ at q of the kite K_w tends to zero as w goes to zero, we can assume that $\measuredangle r'qr < R_q - \pi$. This ensures that the rotation angle at q of the projected loop $\pi_w(\gamma)$ in M_w is still greater than π. By the local characterisation of geodesics, the projected loop $\pi_w(\gamma)$ is still a closed geodesic of M_w. As previously the geodesic loops γ_w and $\pi_w(\gamma)$ of M_w agree, that is, $\pi_w(\gamma) = \gamma_w$. Therefore, $\mathrm{sys}(M_w) = \mathrm{sys}(M)$ in this case too. □

Proposition 2.6.6 (Case (E_3))**.** *Consider a kite K_w with main diagonal $[p,q]$, exact at p, of width w satisfying (E_3). If the width w of K_w is sufficiently small, then*

$$\mathrm{sys}(M_w) \geq \mathrm{sys}(M).$$

Proof. Consider a sequence of systolic loops γ_w of M_w converging to a systolic loop γ of M and lying in the same free homotopy class C as at the beginning of the proof of Proposition 2.6.5. If γ is disjoint from $[p,q]$ (and then from K_w if w sufficiently

small), we argue as in the first case of the proof of Proposition 2.6.5 and conclude that $\mathrm{sys}(M_w) = \mathrm{sys}(M)$. Thus, we can assume that γ meets $[p, q]$. By the condition (E_3), the systolic loop γ contains $[p, q]$. Denote by $I_w = \pi_w(K_w)$ the segment of M_w given by the image of the kite K_w. Arguing as in the proof of [48, Lemma 7.4], we derive that $c_w = \gamma_w \setminus I_w$ is made of a single open geodesic arc of M_w. Denote by $\sigma_w \subseteq M$ the closure of the inverse image $\pi_w^{-1}(c_w)$ of c_w in M. Note that σ_w is a geodesic arc of M with endpoints in ∂K_w, which possibly closes up.

Suppose one of the endpoints of σ_w is one of the main vertices of the kite K_w, say p. Let y be the other endpoint of σ_w. The segment $[p, y] \subseteq K_w$ projects to the path of $I_w \subseteq M_w$ connecting $\pi_w(p)$ and $\pi_w(y)$. Then the loop $\bar{\gamma}_w = \sigma_w \cup [p, y] \subseteq M$ in the nontrivial free homotopy class C satisfies $|\bar{\gamma}_w| \leq |\gamma_w|$. Thus, $\mathrm{sys}(M) \leq \mathrm{sys}(M_w)$ as required. Therefore, we can assume that the endpoints of σ_w are disjoint from p and q.

Suppose one of the endpoints of the segment σ_w is a point other than r and r'. In such case, the minimizing loop $\gamma_w \subseteq M_w$ meets the interval I_w transversely at a regular (*i.e.*, non-singular) point of M_w. It follows that the endpoints of σ_w project to the same point on the closed geodesic $\gamma \subseteq M$. Hence the nearest-point projection of σ_w to γ closes up. By the assumption of nonpositive curvature, the projection map is distance-nonincreasing. Therefore, $\mathrm{sys}(M_w) = |\gamma_w| \geq |\sigma_w| \geq |\gamma| = \mathrm{sys}(M)$ in this case, as well.

Thus, we can assume that the endpoints of σ_w are the points $r, r' \in M$. In this case also the nearest-point projection of σ_w to the loop $\gamma \subseteq M$ closes up. Hence, $\mathrm{sys}(M_w) = |\gamma_w| \geq |\sigma_w| \geq |\gamma| = \mathrm{sys}(M)$, proving the proposition. □

2.7 Topological bounds on systolic decompositions

In this section, we establish topological bounds on the elements of the systolic decomposition of a closed piecewise hyperbolic surface.

Let us first bound the number of large conical singularities. Recall that a conical singularity of a closed piecewise hyperbolic surface is large if its total angle is at least 3π.

Proposition 2.7.1. *Let M be a closed piecewise hyperbolic surface of Alexandrov curvature at most -1. Then there are less than $2|\chi(M)|$ large conical singularities on M.*

Proof. Since M has nonpositive Alexandrov curvature, the total angles θ_i of its conical singularities are greater than 2π. Now, the result immediately follows from the Gauss–Bonnet formula (2.1) using the fact that the surface M is negatively curved with small and large conical singularities satisfying $\theta_i - 2\pi > 0$ and $\theta_i - 2\pi \geq \pi$. □

The following result follows from a bound on the number of pairwise nonhomotopic simple loops intersecting each other at most once/twice; see [48, Proposition 5.7].

Proposition 2.7.2. *Let M be a closed piecewise hyperbolic surface of Alexandrov curvature at most -1. Then the number of domains, edges and vertices in the systolic decomposition of M is at most N, where N is an integer depending only on the topology of M.*

Proof. Systolic loops are simple closed geodesics and, since the surface is negatively curved, are unique in their free homotopy classes. Furthermore, every pair of intersecting systolic loops of M meet exactly at one or two points, or along an arc; see [48, Proposition 4.5]. Thus, the number of systolic loops of M is bounded by the number of pairwise nonhomotopic simple loops on M intersecting at most twice. By [44], this number of pairwise nonhomotopic simple loops is finite and bounded by a number Q depending only on the topology of M. (The best estimate of Q can be found in [35], but we will not need a quantitative expression of Q.)

Now, the corners of any domain D of the systolic decomposition of M correspond to the intersection of two systolic loops, giving rise to at most eight corners. Since there are at most Q systolic loops, this yields at most $N = 8\binom{Q}{2}$ corners and so at most as many domains, edges and vertices in the systolic decomposition of M. □

2.8 Exploiting the kite excision trick

The goal of this section is to prove the following result on the existence of piecewise hyperbolic surfaces with conical singularities locally maximal for the systole by exploiting the kite excision trick. This is the analogue of the main theorem of [48] in our context.

Theorem 2.8.1. *Let Σ be a nonsimply connected closed surface. Let $\mathcal{U} \subseteq \mathcal{A}_\Sigma$ be an open set in the space $\mathcal{A}_\Sigma$ of Alexandrov surfaces of curvature at most -1 defining a local supremum of the systole; see Definition 2.1.3. Then there exists a piecewise hyperbolic surface $M_0 \in \mathcal{U}$ with at most N_0 conical singularities such that*

$$\operatorname{sys}(M_0) \geq \operatorname{sys}(M)$$

for every $M \in \mathcal{U}$, where N_0 is an integer depending only on the topology of Σ.

Proof. By definition of $\mathcal{U}$, for every $\varepsilon > 0$ small enough, there exists an Alexandrov surface $M \in \mathcal{U}$ of curvature at most -1, homeomorphic to Σ, such that

$$\operatorname{sys}(M) > \max_{M' \in \mathcal{U}} \operatorname{sys}(M') - \varepsilon > \max_{M' \in \partial\mathcal{U}} \operatorname{sys}(M'). \tag{2.6}$$

By metric approximation, see Proposition 2.3.1, we can assume that the surface M is piecewise hyperbolic with conical singularities (of Alexandrov curvature at most -1). Denote by N the number of conical singularities of M. By compactness (see Proposition 2.4.1) and the strict inequalities (2.6), there exists a surface $M_1 \in \mathcal{U}$ with maximal systole among all piecewise hyperbolic surfaces in $\mathcal{U}$ with at most N conical singularities. Take M_1 with minimal area. Note that $\mathrm{sys}(M_1) \geq \mathrm{sys}(M)$.

We will use the kite excision trick in the two following lemmas to obtain an *a priori* upper bound on the number of (small) conical singularities of M_1.

Lemma 2.8.2. *Every domain D of the systolic decomposition of M_1 contains at most one small conical singularity.*

Proof. We argue by contradiction. Suppose p and q' are two conical singularities in the open domain D with p small. Let $[p, q']$ be a length-minimizing arc in the closure of D joining the two points. We consider the following two cases.

1. If $[p, q']$ lies in the open domain D, we denote by q the first conical singularity along $(p, q']$ from p.

2. Otherwise, the arc $[p, q']$ meets ∂D, and the first point of intersection of $[p, q']$ with ∂D from p is a point, denoted q, at which D is strictly concave. We can further assume that no conical singularity of M lies in (p, q), otherwise we refer to the case (1).

In the second case, the angle of D at q is greater than π, which shows that the point q is a conical singularity.

In either case, we apply the kite excision trick with a kite K_w of main diagonal $[p, q]$, exact at p, of width w small enough to satisfy (E_1) in the first case (when q lies in D) and (E_2) in the second case (when q lies in ∂D); see Definition 2.6.3. We also choose w small enough to ensure that the resulting piecewise hyperbolic surface $M_w = (M_1 \setminus K_w)/\!\sim$ lies in $\mathcal{U}$; see Proposition 2.5.5. By Proposition 2.5.4, the surface M_w has at most N conical singularities as M_1. By Proposition 2.6.5, the systole of M_w is equal to the systole of M_1. (Note that $\mathrm{sys}(M_w) \geq \mathrm{sys}(M)$ is enough.) Since the area of M_w is less than the area of M_1, this contradicts the assumption on the area of M_1. □

Lemma 2.8.3. *The interior of every edge $\mathcal{E}$ of a domain D of the systolic decomposition of M_1 contains at most one small conical singularity.*

Proof. We argue by contradiction. Let p be a small conical singularity in the interior of the edge $\mathcal{E}$. Let q be a conical singularity in the interior of $\mathcal{E}$ adjacent to p. Apply the kite excision trick with a kite K_w of main diagonal $[p, q]$, exact at p, of width w small enough to satisfy (E_3) (see Definition 2.6.3) and to ensure that the resulting piecewise hyperbolic surface M_w lies in $\mathcal{U}$; see Proposition 2.5.5.

We obtain a contradiction by arguing as in the proof of Lemma 2.8.2 by applying Proposition 2.6.6. □

Let us continue with the proof of Theorem 2.8.1. By Proposition 2.7.2, the number of domains, edges and vertices in the systolic decomposition of M_1 is at most N_Σ. (Without loss of generality, we can assume that $N_\Sigma \geq 2|\chi(\Sigma)|$.) Combined with Lemma 2.8.2 and Lemma 2.8.3, this shows that the piecewise hyperbolic surface M_1 has at most $3N_\Sigma$ small conical singularities. By Proposition 2.7.1, the surface M_1 has at most $2|\chi(M_1)| \leq N_\Sigma$ large conical singularities. Thus, we derive an *a priori* bound on the number of conical singularities of M_1. Namely, the piecewise hyperbolic surface M_1 has at most $N_0 = 4N_\Sigma$ conical singularities.

By compactness (see Proposition 2.3.1), and since the systole of M_1 is greater than $\max_{M' \in \partial\mathcal{U}} \operatorname{sys}(M')$, there exists a surface $M_0 \in \mathcal{U}$ with maximal systole among all piecewise hyperbolic surfaces in $\mathcal{U}$ with at most N_0 conical singularities. Note that the surface M_0 does not depend- on ε, M or M_1 (only on the topology of Σ). By construction, the surface M_0 satisfies

$$\operatorname{sys}(M_0) \geq \operatorname{sys}(M_1) \geq \operatorname{sys}(M) \geq \max_{M' \in \mathcal{U}} \operatorname{sys}(M') - \varepsilon$$

for every $\varepsilon > 0$ small enough. Hence, $\operatorname{sys}(M_0) = \max_{M' \in \mathcal{U}} \operatorname{sys}(M')$. □

Remark 2.8.4. It is not necessary to present an explicit estimate of N_0 since we will eventually show that the surface M_0 can be chosen hyperbolic (with $N_0 = 0$); see Theorem 2.12.1.

2.9 Kite insertion trick

In this section, we describe the kite insertion trick, which has the opposite effect to the kite excision trick and plays a key role in the proof of the main theorem.

Consider a piecewise hyperbolic surface $M \in \mathcal{A}_\Sigma$ of Alexandrov curvature at most -1 with conical singularities homeomorphic to a closed surface Σ.

Definition 2.9.1. Let $m \in M$ be a conical singularity of M of total angle $\theta_m > 2\pi$. Let (p, q) be a geodesic arc passing through m with angle π on one side of m and $\theta_m - \pi$ on the other side of m, such that m is the only conical singularity of M lying on $[p, q]$.

Fix $\alpha > 0$ with $\alpha < \frac{1}{2}(\theta_m - 2\pi)$. Let $q_\alpha \in M$ with $|mq_\alpha| = |mq|$ such that $\measuredangle pmq_\alpha = \pi + \alpha \in (\pi, \frac{\theta_m}{2})$; see Figure 2.3. Denote by M'_α the surface with boundary obtained by cutting open the surface M along $[p, q_\alpha] = [p, m] \cup [m, q_\alpha]$. The boundary $\partial M'_\alpha$ of M'_α is the geodesic quadrilateral $pm_1q_\alpha m_2$. Let $K_\alpha = \bar{p}\bar{m}_1\bar{q}_\alpha\bar{m}_2$ be a kite in the hyperbolic plane with $|\bar{p}\bar{m}_i| = |pm_i|$ and $|\bar{q}_\alpha\bar{m}_i| = |q_\alpha m_i|$ forming an angle $\pi - \alpha$ at $\bar{m}_1$ and $\bar{m}_2$, *i.e.*, $\measuredangle \bar{p}\,\bar{m}_i\bar{q}_\alpha = \pi - \alpha$. Attach K_α along the boundary of M'_α

by identifying $[\bar{p}, \bar{m}_i]$ to $[p, m_i]$ and $[\bar{q}_\alpha, \bar{m}_i]$ to $[q_\alpha, m_i]$. The resulting surface M_α is a piecewise hyperbolic surface with conical singularities homeomorphic to M; see Figure 2.3. The conical singularities of M different from m are also conical singularities of M_α (with the same total angles). In addition, there exist three conical singularities in M_α at p, q_α and m_2 of total angles $\theta_p^\alpha = 2\pi + \measuredangle \bar{m}_1 \bar{p}\, \bar{m}_2$, $\theta_{q_\alpha}^\alpha = 2\pi + \measuredangle \bar{m}_1 \bar{q}_\alpha \bar{m}_2$ and $\theta_{m_2}^\alpha = \theta_m - 2\alpha > 2\pi$. Note that there is no conical singularity at m_1. Therefore, the total angles of the conical singularities of M_α are greater than 2π. This shows that the piecewise hyperbolic surface M_α is of Alexandrov curvature at most -1, that is, $M_\alpha \in \mathcal{A}_\Sigma$.

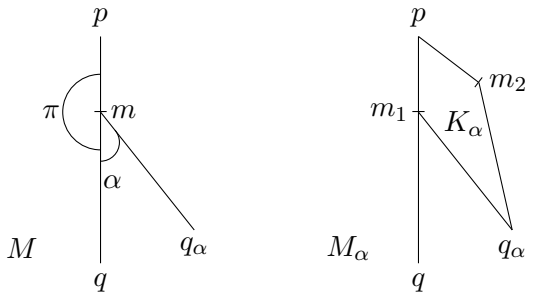


Figure 2.3: Inserting the kite K_α along $[p, q_\alpha]$.

Define $\pi_\alpha : M_\alpha \to M$ as the map collapsing every segment of the kite K_α parallel to $[\bar{m}_1, \bar{m}_2]$ to a point. This map is $(1 + \varepsilon(\alpha))$-Lipschitz with $\varepsilon(\alpha)$ tending to zero as α goes to zero, and is a homotopy equivalence.

We have the following convergence result.

Proposition 2.9.2. *The surface M_α converges to M for the Lipschitz distance (and the uniform/Gromov-Hausdorff distances) as α goes to zero.*

Proof. Fix $m' \in M$ close to m on the side of (p, q) with an angle π at m (say, $\measuredangle pmm' = \measuredangle qmm' = \frac{\pi}{2}$); see Figure 2.4. Consider the hyperbolic quadrilateral $pmq_\alpha m' \subseteq M$ and the hyperbolic quadrilateral $pm_2q_\alpha m' \subseteq M_\alpha$ obtained by attaching the kite K_α to the first quadrilateral $pmq_\alpha m'$ identifying $[p, m]$ with $[p, m_1]$ and $[q_\alpha, m]$ with $[q_\alpha, m_1]$. There exists a $(1 + \varepsilon(\alpha))$-bilipschitz homeomorphism

$$pmq_\alpha m' \to pm_2q_\alpha m'$$

between the two hyperbolic quadrilaterals of M and M_α, fixing p, q_α and m', and sending isometrically each side of $pmq_\alpha m' \subseteq M$ to their corresponding side in $pm_2q_\alpha m' \subseteq M_\alpha$, where $\varepsilon(\alpha)$ tends to zero as α goes to zero; see Figure 2.4.

Combining this map with the identity map

$$M \setminus pmq_\alpha m' \to M_\alpha \setminus pm_2q_\alpha m'$$

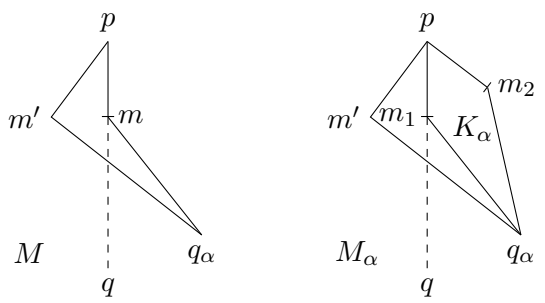


Figure 2.4: Lipschitz deformation of M_α.

off the quadrilaterals, we obtain a $(1+\varepsilon(\alpha))$-bilipschitz homeomorphism $M \to M_\alpha$ between M and M_α. □

2.10 Deforming systolic homotopy classes

In this section, we study the effect of the kite insertion trick on the length of a systolic loop in a given free homotopy class.

Let us introduce the following definition.

Definition 2.10.1. Let M be a negatively curved closed Alexandrov surface. Given a free homotopy class C of M, define $L_M(C)$ as the minimal length of a loop representing C. This minimal length is attained by the length of the unique closed geodesic representing C in M.

Let M be a closed piecewise hyperbolic surface of Alexandrov curvature at most -1 with at least one conical singularity m of total angle $\theta_m > 2\pi$. Fix a geodesic arc (p,q) of M passing through m, with angle π at m on one side of (p,q) and angle $\theta_m - \pi$ on the other side, such that m is the only conical singularity of M lying in $[p,q]$, as in Definition 2.9.1. By choosing the geodesic arc (p,q) carefully, we can further assume that at least one systolic loop of M transversely intersects (p,q) and that all the systolic loops of M meeting $[p,q]$ intersect the segment $[p,q]$ at a unique point $x \in (p,m]$.

Consider the surface M_α obtained from M by inserting a kite K_α along a segment $[p,q_\alpha]$ close to $[p,q]$ as in Definition 2.9.1. Recall that $\measuredangle pmq = \pi$ and $\measuredangle pmq_\alpha = \pi + \alpha \in (\pi, \frac{\theta_m}{2})$.

The free homotopy class of a systolic loop of M satisfies one of the two following propositions, namely Proposition 2.10.2 and Proposition 2.10.3.

Proposition 2.10.2. *Let C be the free homotopy class of a systolic loop γ of M which does not transversely intersect $[p, q]$. Then,*

$$L_{M_\alpha}(C) = L_M(C) = \mathrm{sys}(M)$$

for every $\alpha > 0$ small enough.

Proof. If γ does not intersect $[p, q]$, the loop γ does not intersect $[p, q_\alpha]$ either for every $\alpha > 0$ small enough. Thus, the loop γ is still geodesic in M_α after inserting the kite K_α along $[p, q_\alpha]$ and its length remains unchanged. Since the surface M_α has nonpositive curvature, it follows that

$$L_{M_\alpha}(C) = |\gamma|_{M_\alpha} = |\gamma|_M = L_M(C)$$

as desired.

If γ meets $[p, q]$, the intersection occurs at a conical singularity, namely at m, since the intersection is non-transverse by assumption. Furthermore, the geodesic curve γ lies on the side of $[p, q]$ where the angle at m is greater than π. In this case, the angle of γ at m on the side where the geodesic arc (p, q) lies is at least $\pi + 2\nu$, where ν is the minimal angle between (p, q) and γ. Similarly, the angle of γ at m on the side opposite to (p, q) is at least π. Moreover, the geodesic loop γ does not transversely intersect $[p, q_\alpha]$ for $\alpha < \nu$. Thus, after inserting a kite K_α along $[p, q_\alpha]$ with $\alpha < \nu$, the loop $\gamma \subseteq M_\alpha$ passes through m_2. Its angle in M_α at m_2 is at least $\pi + 2\nu - \alpha \geq \pi$ on the side of K_α and remains unchanged $\geq \pi$ on the side opposite to K_α. Thus, the loop γ is still a geodesic in M_α and its length remains unchanged. As previously, it follows that

$$L_{M_\alpha}(C) = |\gamma|_{M_\alpha} = |\gamma|_M = L_M(C).$$

□

The alternative case is covered by the following proposition.

Proposition 2.10.3. *Let C be the free homotopy class of a systolic loop γ of M which transversely intersects $[p, q]$. Then,*

$$L_{M_\alpha}(C) > L_M(C) = \mathrm{sys}(M)$$

for every $\alpha > 0$ small enough.

Proof. For every $\alpha > 0$ small enough, the geodesic loop γ transversely intersects $[p, q_\alpha]$. Let η_α be the geodesic loop of M_α representing the same free homotopy class C as γ under the homotopy equivalence $\pi_\alpha : M_\alpha \to M$ of Definition 2.9.1; see Figure 2.5.

Claim 2.10.4. *The image $\bar{\eta}_\alpha$ of η_α under the map $\pi_\alpha : M_\alpha \to M$ uniformly converges to γ in M, up to reparametrization.*

Furthermore, the loops η_α and $\bar{\eta}_\alpha$ decompose as follows.

- *The geodesic loop η_α of M_α decomposes as a geodesic arc η'_α lying in $M'_\alpha \subseteq M_\alpha$ with endpoints $x_1, x_2 \in \partial M'_\alpha$ (i.e., $x_i \in [p, m_i] \cup [m_i, q_\alpha]$) and a segment $[x_1, x_2]$ lying in K_α. That is,*
$$\eta_\alpha = \eta'_\alpha \cup [x_1, x_2].$$
- *The loop $\bar{\eta}_\alpha$ of M decomposes into a geodesic arc with endpoints $\bar{x}_1 = \pi_\alpha(x_1)$ and $\bar{x}_2 = \pi_\alpha(x_2)$ which identifies with η'_α, except at its endpoints, and a segment $[\bar{x}_1, \bar{x}_2]$ lying in $[p, q_\alpha] = [p, m] \cup [m, q_\alpha]$. That is,*
$$\bar{\eta}_\alpha = \eta'_\alpha \cup [\bar{x}_1, \bar{x}_2].$$

See Figure 2.5.

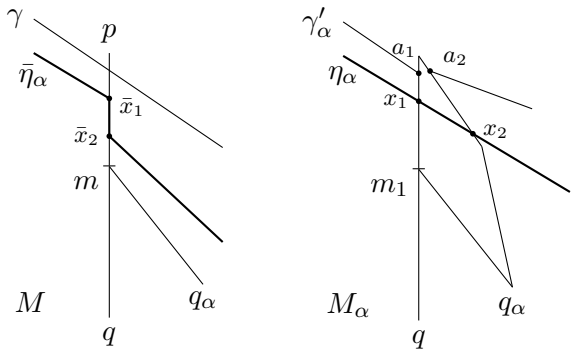


Figure 2.5: Decompositions of η_α and $\bar{\eta}_\alpha$ in M_α and M.

Proof. Since the image $\bar{\eta}_\alpha$ of η_α under the $(1 + \varepsilon(\alpha))$-Lipschitz map $\pi_\alpha : M_\alpha \to M$ lies in the same homotopy class as the geodesic loop γ, it follows that $|\gamma| \leq |\bar{\eta}_\alpha| \leq (1 + \varepsilon(\alpha))\,|\eta_\alpha|$. On the other hand, denote by γ'_α the geodesic arc of M'_α with endpoints $a_1, a_2 \in \partial M'_\alpha$ obtained from γ by cutting open the surface M along $[p, q_\alpha]$; see Figure 2.5. Since the loop $\gamma'_\alpha \cup [a_1, a_2]$ is homotopic to the geodesic loop η_α of M_α, where $[a_1, a_2]$ is the segment of K_α connecting a_1 and a_2, we derive that $|\eta_\alpha| \leq |\gamma| + \varepsilon_1(\alpha)$, where $\varepsilon_1(\alpha) = |a_1 a_2|_{K_\alpha}$ tends to zero as α goes to zero. Hence,

$$|\gamma| \leq |\bar{\eta}_\alpha| \leq (1 + \varepsilon_2(\alpha))\,|\gamma|$$

where $\varepsilon_2(\alpha)$ tends to zero as α goes to zero. Since the loop γ is uniquely minimizing in its homotopy class, the loop $\bar{\eta}_\alpha$ uniformly converges to γ in M, up to reparametrization.

Denote by $\eta'_\alpha = \eta_\alpha \cap M'_\alpha$ the part of η_α lying outside K_α in M_α. This part is made of geodesic subarcs of M'_α with endpoints lying in $\partial M'_\alpha$. Since the loop $\bar{\eta}_\alpha$ converges to γ and agrees with η_α on M'_α, one of these geodesic subarcs, denoted

by η''_α, converges to γ away from $[p, q]$. Since K_α is convex in the nonpositively curved surface M_α, the other geodesic subarcs of η'_α, which lie in the intersection of M'_α with a small neighborhood of K_α, in fact lie in $\partial M'_\alpha$ and so are trivial. Therefore, $\eta'_\alpha = \eta''_\alpha$ is a geodesic arc of $M'_\alpha \subseteq M_\alpha$ with endpoints x_1 and x_2 lying in M'_α. The other part of η_α with endpoints x_1 and x_2 lies in K_α and agrees with the segment $[x_1, x_2]$ of K_α; see Figure 2.5. Hence the decomposition $\eta_\alpha = \eta'_\alpha \cup [x_1, x_2]$.

The restriction of the map $\pi_\alpha : M_\alpha \to M$ to $M_\alpha \setminus K_\alpha$ is the identity map. Thus, the image of η'_α under the map π_α identifies with η'_α (except at its endpoints). Collapsing K_α to $[p, q_\alpha]$ under the map π_α takes the segment $[x_1, x_2]$ to the segment of $[p, q_\alpha]$ with endpoints $\bar{x}_1$ and $\bar{x}_2$; see Figure 2.5. Hence the decomposition $\bar{\eta}_\alpha = \eta'_\alpha \cup [\bar{x}_1, \bar{x}_2]$. □

For every $\alpha > 0$ small enough, the endpoints x_1 and x_2 of the geodesic arc η'_α satisfy the following alternative:

($A1$) either they lie in $(p, m_1]$ and $(p, m_2]$, or in $(q_\alpha, m_1]$ and $(q_\alpha, m_2]$;

($A2$) or they lie in (p, m_1) and (q_α, m_2), or in (q_α, m_1) and (p, m_2).

We write $\alpha \in \Lambda'$ in the first case and $\alpha \in \Lambda''$ in the second case.

The first case is covered by the following lemma.

Lemma 2.10.5. *For every $\alpha \in \Lambda'$ small enough, we have*

$$|\gamma| < |\eta_\alpha|.$$

Proof. Suppose that $x_1 \in (p, m_1]$ and $x_2 \in (p, m_2]$. The alternative case, where $x_1 \in (q_\alpha, m_1]$ and $x_2 \in (q_\alpha, m_2]$, is treated similarly. Recall that $\bar{\eta}_\alpha$ is the image of η_α in M under the map $\pi_\alpha : M_\alpha \to M$. Since γ is length-minimizing in its homotopy class and the loop $\bar{\eta}_\alpha$ is freely homotopic to γ in M, we obtain $|\gamma| \leq |\bar{\eta}_\alpha|$.

Now, by Claim 2.10.4, we have $\eta_\alpha = \eta'_\alpha \cup [x_1, x_2]$ and $\bar{\eta}_\alpha = \eta'_\alpha \cup [\bar{x}_1, \bar{x}_2]$. Thus, to prove that $|\bar{\eta}_\alpha| < |\eta_\alpha|$, it is enough to show that $|\bar{x}_1\bar{x}_2| < |x_1x_2|$. Assume that $|px_1| \leq |px_2|$. The other case is treated similarly. Let $y_1 \in (p, m_1)$ such that $|py_1| = |px_2|$; see Figure 2.6. Consider the three points x_1, y_1 and x_2 in the hyperbolic kite K_α. Note that $|\bar{x}_1\bar{x}_2| = |x_1y_1|$. The disk D_p centered at p of radius $|px_2|$ contains the disk D_{x_1} centered at x_1 of radius $|x_1y_1|$ since $y_1 \in \partial D_p$ and the surface is nonpositively curved; see Figure 2.6. Since x_2 does not lie in D_{x_1}, the circle ∂D_{x_1} intersects (x_1, x_2) at a point x'_2. Thus, $|x_1y_1| = |x_1x'_2| < |x_1x_2|$. Hence, $|\bar{x}_1\bar{x}_2| < |x_1x_2|$.

As a result, we derive that

$$|\gamma| \leq |\bar{\eta}_\alpha| < |\eta_\alpha|.$$

□

The second case is covered by the following lemma.

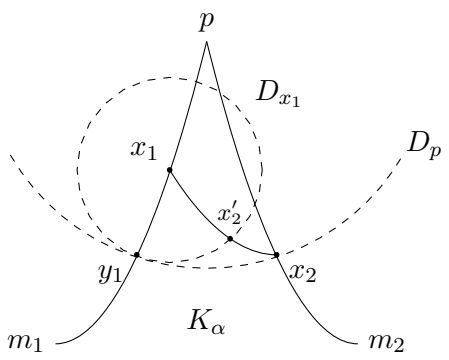


Figure 2.6: Collapsing K_α decreases distances between x_1 and x_2 when they lie on adjacent sides.

Lemma 2.10.6. *For every $\alpha \in \Lambda''$ small enough, we have*

$$|\gamma| < |\eta_\alpha|.$$

Proof. Recall that γ'_α is the geodesic arc of M'_α obtained from γ by cutting open the surface M along $[p, q_\alpha]$. By Claim 2.10.4, the endpoints x_1 and x_2 of the geodesic arc η'_α converge to the intersection point of γ with $[p, q]$. Since x_1 and x_2 lie in two opposite sides of K_α, the intersection point is given by $[p, m] \cap [q, m] = \{m\}$. Thus, the endpoints of γ'_α agree with m_1 and m_2; see Figure 2.7.

Suppose that $x_1 \in (p, m_1)$ and $x_2 \in (q_\alpha, m_2)$. The other case is treated similarly. Observe that the angle ν_1^α between γ'_α and $[m_1, p)$ at m_1 is less than $\frac{\pi}{2} + \varepsilon$ for every $\alpha \in \Lambda''$ small enough, where $\varepsilon > 0$ is fixed. (The angle ν_1^α does not depend on α, but we keep the index α by symmetry with the angle ν_2^α between γ'_α and $[m_2, q_\alpha]$ at m_2 which does depend on α.) Otherwise, there would exist a sequence of α's going to zero with $\nu_1^\alpha \geq \frac{\pi}{2} + \varepsilon$. Since the loop $\bar{\eta}_\alpha = \eta'_\alpha \cup [\bar{x}_1, \bar{x}_2]$ uniformly converges to γ in M, up to reparametrization, see Claim 2.10.4, the angle of the geodesic arc η'_α with $[x_1, p)$ at x_1 would be uniformly bounded away from below by $\frac{\pi}{2}$ for these $\alpha > 0$ small enough. (Note that the topology considered in Claim 2.10.4 is not the C^1-topology and therefore it does not lead to the convergence of angles for general arcs. However, in our situation, the arcs are geodesic, which is enough for our purpose.) In this case, the geodesic η_α entering the kite K_α at x_1 would leave it through (p, m_2) for these $\alpha > 0$ small enough. Hence a contradiction since the point x_2 where η_α leaves K_α lies in (q_α, m_2). Similarly, the angle ν_2^α between γ'_α and $[m_2, q_\alpha)$ at m_2 is less than $\frac{\pi}{2} + \varepsilon$ for every $\alpha \in \Lambda''$ small enough.

In order to deal with the extra ε, we slightly extend the kite $K_\alpha = p\, m_1 q_\alpha m_2$ of M_α into a new kite $K_\alpha^+ = p^+ m_1 q_\alpha^+ m_2$ by taking p^+ and q_α^+ close to p and q_α on geodesic extensions of $[m_2, p]$ and $[m_1, q_\alpha]$ (where the closeness depends on how small ε is) so that the unit tangent vectors u_1 and u_2 orthogonal to γ'_α at m_1 and m_2

point inward K_α^+ and the geodesic arcs c_1 and c_2 of K_α^+ they generate exit K_α^+ through (p^+, m_2) and (q_α^+, m_1) at t_1 and t_2 ; see Figure 2.7. We can choose p^+ and q_α^+ such that $\measuredangle p\, m_1 p^+ = \varepsilon$ and $\measuredangle q_\alpha m_2 q_\alpha^+ = \varepsilon$ with $\varepsilon > 0$ small enough. Since $t_1 \in (m_2, p^+)$ and $t_2 \in (q_\alpha^+, m_1)$, the segments $[m_1, t_1]$ and $[m_2, t_2]$ of K_α^+ lying in c_1 and c_2 do not intersect each other.

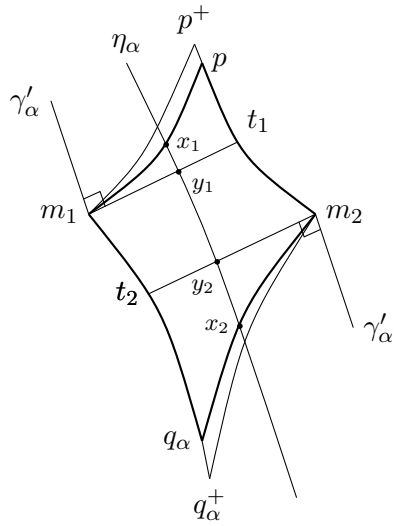


Figure 2.7: Positions of γ'_α and η_α around the kite K_α.

As previously, the geodesic η_α enters/leaves the new kite K_α^+ through the sides (p^+, m_1) and (q_α^+, m_2). Therefore, the geodesic loop η_α intersects $[m_1, t_1]$ and $[m_2, t_2]$ at two points $y_1, y_2 \in K_\alpha^+$. Let $\widehat{y_1 y_2}$ be the subarc of η_α joining y_1 to y_2 and going through $M_\alpha \setminus K_\alpha^+$. By the convergence result of Claim 2.10.4, we deduce that the arcs γ'_α and $\widehat{y_1 y_2}$ are homotopic in M_α through arcs with endpoints lying in $[m_1, t_1]$ and $[m_2, t_2]$; see Figure 2.8. Since the geodesic arc γ'_α is orthogonal to the segments $[m_1, t_1]$ and $[m_2, t_2]$, and the surface M_α has nonpositive curvature, it follows that

$$|\gamma| = |\gamma'_\alpha| \leq |\widehat{y_1 y_2}| < |\eta_\alpha|.$$

□

Let us conclude the proof of Proposition 2.10.3. Whether the endpoints of the geodesic arc η'_α satisfy the alternative $(A1)$ or $(A2)$, we deduce from Lemma 2.10.5 and Lemma 2.10.6 that

$$\mathrm{sys}(M) = |\gamma| < |\eta_\alpha| = L_{M_\alpha}(C)$$

for every $\alpha > 0$ small enough. □

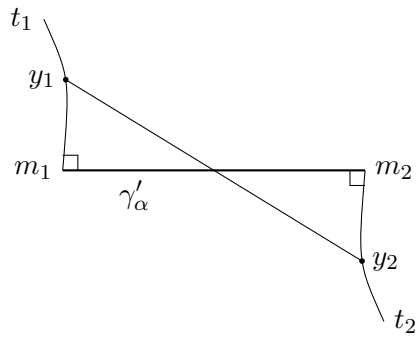


Figure 2.8: Comparing γ'_α and $\widehat{y_1y_2}$.

2.11 Systole comparison and the number of systolic loops

In this section, we examine the effect of the kite insertion trick on the systole and on the number of systolic loops.

Definition 2.11.1. Let M be a negatively curved closed Alexandrov surface. Define $\sharp_s(M)$ as the number of systolic loops of M.

We have the following result about the deformation of piecewise hyperbolic surfaces with conical singularities.

Theorem 2.11.2. *Let M be a closed piecewise hyperbolic surface of Alexandrov curvature at most -1 with at least one conical singularity m. Then the surface M can be deformed into a piecewise hyperbolic surface M_α with conical singularities of Alexandrov curvature at most -1 by inserting a kite K_α around m such that for every $\alpha > 0$ small enough, one of the following conditions is satisfied:*

(i) $\mathrm{sys}(M_\alpha) > \mathrm{sys}(M)$*;*

(ii) $\mathrm{sys}(M_\alpha) = \mathrm{sys}(M)$ *and* $\sharp_s(M_\alpha) < \sharp_s(M)$*.*

Proof. Choose a geodesic arc (p, q) of M passing through m as in Section 2.10 and consider the surface M_α obtained from M by inserting a kite K_α along a segment $[p, q_\alpha]$ close to $[p, q]$ as in Definition 2.9.1; see the beginning of Section 2.10. Let C be a free homotopy class of M. Three cases may occur:

1. If C is not represented by a systolic loop of M, then

$$L_{M_\alpha}(C) > \mathrm{sys}(M) + \varepsilon$$

for every $\alpha > 0$ small enough, where $\varepsilon > 0$ does not depend on C; see Proposition 2.9.2.

2. If C is represented by a systolic loop of M which does not transversely intersect $[p,q]$, then
$$L_{M_\alpha}(C) = L_M(C) = \mathrm{sys}(M)$$
for every $\alpha > 0$ small enough; see Proposition 2.10.2.

3. If C is represented by a systolic loop of M which transversely intersects $[p,q]$, then
$$L_{M_\alpha}(C) > \mathrm{sys}(M)$$
for every $\alpha > 0$ small enough; see Proposition 2.10.3.

It follows that $\mathrm{sys}(M_\alpha) \geq \mathrm{sys}(M)$. Furthermore, we derive the following.

Suppose that all the systolic loops of M meeting $[p,q]$ transversely intersect $[p,q]$. Then $\mathrm{sys}(M_\alpha) > \mathrm{sys}(M)$ for every $\alpha > 0$ small enough, since the case (2) does not occur (using also the fact that there are finitely many systolic loops in M). This yields the case (i) of Theorem 2.11.2.

Suppose now that a systolic loop of M does not transversely intersect $[p,q]$. Then $\mathrm{sys}(M_\alpha) = \mathrm{sys}(M)$ for every $\alpha > 0$ small enough from the case (2). Take a systolic loop of M transversely intersecting $[p,q]$. (This systolic loop exists by our choice of $[p,q]$.) By (3), the free homotopy class of this systolic loop is not represented by a systolic loop of M_α. Since every free homotopy class not represented by a systolic loop of M is not represented by a systolic loop of M_α either, we conclude that $\sharp_s(M_\alpha) < \sharp_s(M)$. This yields the case (ii) of Theorem 2.11.2. □

2.12 Extremality of hyperbolic surfaces

Let us show the main theorem of this article, namely, every local supremum of the systole among all Alexandrov metrics of curvature at most -1 on a closed surface is attained by a hyperbolic metric.

Theorem 2.12.1. *Let Σ be a nonsimply connected closed surface. Let $\mathcal{U} \subseteq \mathcal{A}_\Sigma$ be an open set in the space $\mathcal{A}_\Sigma$ of Alexandrov surfaces of curvature at most -1 defining a local supremum of the systole; see Definition 2.1.3. Then there exists a hyperbolic surface $M \in \mathcal{U}$ such that*
$$\mathrm{sys}(M) \geq \mathrm{sys}(M')$$
for every $M' \in \mathcal{U}$.

Proof. By Theorem 2.8.1, a local supremum of the systole on $\mathcal{U}$ is attained by a piecewise hyperbolic surface $M \in \mathcal{U}$ with conical singularities. Among these, take M with a minimal number of systolic loops $\sharp_s(M)$.

Assume that the piecewise hyperbolic surface M has a conical singularity m. By Theorem 2.11.2, we can deform M into a piecewise hyperbolic surface with conical singularities $M_\alpha \in \mathcal{A}_\Sigma$ of Alexandrov curvature at most -1 satisfying one of the following assertions:

(i) $\mathrm{sys}(M_\alpha) > \mathrm{sys}(M)$;

(ii) $\mathrm{sys}(M_\alpha) = \mathrm{sys}(M)$ and $\sharp_s(M_\alpha) < \sharp_s(M)$.

Furthermore, the surface M_α lies in $\mathcal{U}$ for every $\alpha > 0$ small enough; see Proposition 2.9.2. Now, since the systole of M is maximal among all Alexandrov surfaces of $\mathcal{U}$, only the assertion (ii) may occur. In this case, we obtain a contradiction since the surface M has a minimal number of systolic loops among all piecewise hyperbolic surfaces of $\mathcal{U}$ with maximal systole. As a result, the piecewise hyperbolic surface M has no conical singularity and is therefore hyperbolic. Thus, every local supremum of the systole among all Alexandrov metrics of curvature at most -1 on a closed surface is attained by a hyperbolic metric. □

2.13 Application to hyperbolic 3-manifolds

In this section, we obtain a partial extension of our sharp systolic bounds on surfaces to higher dimension.

Definition 2.13.1. A *maximal hyperbolic surface* is a closed hyperbolic surface with maximal systole among all hyperbolic metrics of fixed genus.

Theorem 2.12.1 immediately yields the following result about the systole of hyperbolic 3-manifolds.

Corollary 2.13.2. *Let N be a closed hyperbolic 3-manifold admitting a totally geodesic immersion of a maximal hyperbolic surface M with $\mathrm{sys}(M) = sys(N)$. Then the hyperbolic metric on N has maximal systole among all Riemannian metrics of (sectional) curvature at most -1.*

Proof. Let N' be the manifold N endowed with a Riemannian metric of curvature at most -1. Since the immersion $\Sigma \simeq M \to N'$ is π_1-injective, there exists a minimal immersion $\Sigma \to N'$ into N'; see [66]. Denote by M' the surface Σ with the metric induced by the minimal immersion $\Sigma \to N'$ into N'. By the Gauss equation, see [32, Theorem 5.5], the curvature of a minimal immersion is bounded from above by the curvature of the ambient space. Thus, the surface M' has curvature at most -1. Since the immersion $M' \to N'$ is π_1-injective and the hyperbolic surface M is maximal, we deduce from Theorem 2.12.1 that

$$\mathrm{sys}(N') \leq \mathrm{sys}(M') \leq \mathrm{sys}(M) = \mathrm{sys}(N).$$

□

Concrete examples of closed hyperbolic 3-manifolds satisfying the assumption of Corollary 2.13.2 can be obtained as follows.

Example 2.13.3. Consider a hyperbolic truncated tetrahedron T, whose four boundary hexagons are right-angled, such that the dihedral angles along the edges between a boundary triangle and a boundary hexagon are $\frac{\pi}{2}$ and the dihedral angles along the main edges $e_1, \dots, e_6$ between two boundary hexagons are $\frac{\pi}{4}, \frac{\pi}{8}, \dots, \frac{\pi}{8}, \frac{\pi}{2}$, where e_1 and e_6 are opposite edges; see Figure 2.9. The existence of such a hyperbolic truncated tetrahedron follows from [31, Proposition 2.1] since the sum of the dihedral angles along the main edges coming from every boundary triangle is less than π. Note that the angles of each boundary triangle are given by the corresponding dihedral angles (and their sum is less than π).

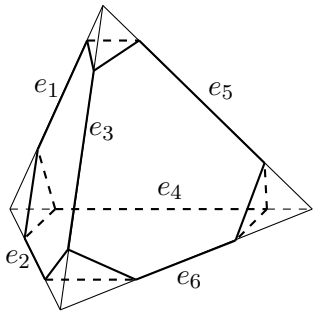


Figure 2.9: The hyperbolic truncated tetrahedron T.

Denote by $H(e_i, e_j, e_k)$ the boundary hexagon of T containing the main edges e_i, e_j, e_k. Take eight copies $T_1, \dots, T_8$ of T and glue them in cyclic order along their main edge e_1. More precisely, identify the boundary hexagon $H_i(e_1, e_2, e_3)$ of T_i with the boundary hexagon $H_{i+1}(e_1, e_5, e_4)$ of T_{i+1}, where the index i is taken modulo 8. The resulting space is a compact hyperbolic polyhedron P with two opposite faces $\mathcal{O}_T$ and $\mathcal{O}_B$ (T for top and B for bottom) centered at the endpoints of e_1 and isometric to the regular hyperbolic octagon with angles $\frac{\pi}{4}$. The other faces of P are either hyperbolic hexagons or hyperbolic quadrilaterals. Identify the boundary hexagons adjacent to the opposite sides of the faces of the octagons $\mathcal{O}_T$ and $\mathcal{O}_B$. By our choice of the dihedral angles, we obtain a compact singular hyperbolic 3-manifold Q with boundary and eight singular geodesic lines of dihedral angles π given by the main edges opposite to the axis e_1. Furthermore, the two connected boundary components of Q containing the endpoints of e_1 are isometric to the Bolza surface. These connected boundary components can be attached to form a new compact singular hyperbolic 3-manifold Q_+ with a single boundary component and eight singular geodesic lines of dihedral angles π, which contains a totally geodesic embedding of the Bolza surface. Attach two copies of Q_+ along their boundary component. The resulting space N' is a closed hyperbolic 3-manifold with eight singular closed geodesics of dihedral angles π disjoint from the Bolza surfaces. To eliminate the singularities of N', take a degree-two ramified cover N'' of N' with branching locus the eight singular closed geodesics of N'. By construction, the cover N'' is a closed hyperbolic

3-manifold containing a totally geodesic embedding of the Bolza surface. However, the systole of M'' is less than the systole of the Bolza surface. For the systole to attain the value of the systole of the Bolza surface, we can consider finite covers using the notion of subgroup separability. By [50], the surface subgroup H of $G = \pi_1(N'')$ corresponding to a totally geodesic Bolza surface M of N'' is separable. That is, for every $x \in G \setminus H$, there exists a finite-index subgroup $K \leqslant G$ containing H but not x. (More generally, we can use the fact that the fundamental group of every hyperbolic 3-manifold is subgroup separable; see [4, Corollary 4.2.3].) This implies that there exists a finite cover N of N'' (relative to H) containing the Bolza surface M with $\mathrm{sys}(N'') = \mathrm{sys}(M)$. Since the Bolza surface is maximal, we observe that the closed hyperbolic 3-manifold N satisfies the assumption of Corollary 2.13.2.

Chapter 3

Generalized resolvent of the Stokes Problem with Navier-type boundary conditions.

We study in this paper the generalized resolvent of the Stokes problem with Navier-type boundary conditions. Joint work with H. Al Baba published in [5].

Contents

3.1 Introduction

This paper is devoted to the existence and uniqueness of weak, strong and very weak solutions to the problem

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = \chi & \text{in} & \Omega \times (0,T) \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on} & \Gamma \times (0,T) \end{cases} \tag{3.1}$$

where we study the generalized resolvent of the Stokes operator with nonstandard Navier-type boundary conditions. Up to now, most research concerns the homogeneous boundary conditions and the case χ=0. Although the case $\chi \neq 0$ has many important applications, specially in treating more general boundary value problems and using cut-off procedure.

There exist several references on (3.1) when $\chi = 0$ in Ω. This question was already studied by Solonnikov in [68] for the homogeneous Dirichlet boundary condition (*i.e.*, $\boldsymbol{u} = \mathbf{0}$ on Γ). In this work, the author considered the resolvent problem when $|\arg \lambda| \leq \delta + \pi/2$ where $\delta \geq 0$ is small. Later on, the resolvent of the Stokes operator with Dirichlet boundary condition in bounded domains has been studied by Giga in [33] using the theory of pseudo-differential operators. The results in [33] extends those in [68] in two directions. First, he considers larger set of values of λ. More precisely, λ lies in the sector $|\arg \lambda| \leq \pi - \varepsilon$, for any $\varepsilon > 0$. Second, the resolvent of the Stokes operator is obtained explicitly and this enables him to describe the domains of fractional powers of the Stokes operator with Dirichlet boundary condition.

In exterior domains, Giga and Sohr [34] approximate the resolvent of the Stokes operator with Dirichlet boundary condition with the resolvent of the Stokes operator in the entire space.

Farwig and Sohr [30] investigate Problem (3.1) when $\operatorname{div} \boldsymbol{u} \neq 0$ in Ω and $\boldsymbol{u} = \mathbf{0}$ on Γ. Their results include bounded and unbounded domains. For the whole and the half space, the proof relies on multiplier technique. The problem is also investigated for bended half spaces and for cones by using perturbation criterion and referring to the half space problem.

Problem (3.1) is also studied with Robin boundary conditions by Saal [68], Shibata and Shimada [67]. In [68], Saal proves that the Stokes operator with homogeneous Robin boundary conditions is sectorial and admits an $\boldsymbol{H}^\infty$-calculus on L^p-spaces. Shibata and Shimada proved in [67] a generalized resolvent estimate for the Stokes equations with non-homogeneous Robin boundary conditions and divergence condition in $\boldsymbol{L}^p$-framework in a bounded or exterior domain by extending the argument of Farwig and Shor [30].

Concerning the Navier-type boundary conditions, Miyakawa [52] shows that the Laplacian operator with homogeneous Navier-type boundary conditions generates a holomorphic semi-group on L^p-spaces when the domain Ω is of class C^∞. Mitrea and Monniaux [51] consider the resolvent of the Stokes operator with homogeneous

Navier-type boundary conditions in Lipschitz domains using differential forms on Lipschitz sub-domains of a smooth compact Riemannian manifold. In [2] and [3], Al Baba et al. consider Problem (3.1) when $\chi = 0$ in Ω and $g = 0$, $\boldsymbol{h} = \mathbf{0}$ on Γ, and prove the existence of weak, strong and very weak solutions to this problem.

This paper is organized as follows. In Section 3.2, we give the functional framework and some preliminary results at the basis of our proofs. In Section 3.3, we prove our main results on the existence of weak, strong and very weak solutions to Problem (3.1).

3.2 Preliminaries

In this section we review some basic notations, definitions and functional framework which are essential in our work.

In what follows, unless stated otherwise, Ω will be considered as an open bounded domain of $\mathbb{R}^3$ of class $C^{2,1}$. The unit normal vector to the boundary, denoted by $\boldsymbol{n}$, is defined everywhere because $\boldsymbol{n}$ is $C^{1,1}$. A generic point in Ω is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$. The domain Ω is not necessarily simply-connected and the boundary Γ is not necessarily connected.

Let us introduce some functional spaces.
Let $\boldsymbol{L}^p(\Omega)$ denote the usual vector valued $\boldsymbol{L}^p$-space over Ω. Let us define the spaces:

$$\begin{aligned}\boldsymbol{H}^p(\mathbf{curl}, \Omega) =& \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},\\ \boldsymbol{H}^p(\mathrm{div}, \Omega) =& \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},\\ \boldsymbol{X}^p(\Omega) =& \boldsymbol{H}^p(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^p(\mathrm{div}, \Omega),\end{aligned}$$

equipped with their graph norms. Thanks to [6] and [7], we know that $D(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$, $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ and $\boldsymbol{X}^p(\Omega)$. We also define the subspaces:

$$\begin{aligned}\boldsymbol{H}^p_0(\mathbf{curl}, \Omega) &= \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}, \Omega);\quad \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\},\\ \boldsymbol{H}^p_0(\mathrm{div}, \Omega) &= \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div}, \Omega);\quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\},\\ \boldsymbol{X}^p_N(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\quad \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\},\\ \boldsymbol{X}^p_T(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}.\end{aligned}$$

We recall that for every function $\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}, \Omega)$ (respectively, $\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div}, \Omega)$), the tangential trace $\boldsymbol{v} \times \boldsymbol{n}$ (respectively, the normal trace $\boldsymbol{v} \cdot \boldsymbol{n}$) exists and belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)$ (respectively, to $W^{-1/p,p}(\Gamma)$). Thanks to [6], we know that $D(\Omega)$ is dense in $\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)$ and in $\boldsymbol{H}^p_0(\mathrm{div}, \Omega)$. Finally, we denote by $[\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)]'$ and $[\boldsymbol{H}^p_0(\mathrm{div}, \Omega)]'$ the dual spaces of $\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}^p_0(\mathrm{div}, \Omega)$ respectively.

Next, we review some known results which are essential in our work. First, we recall that the vector-valued Laplace operator of a vector field $\boldsymbol{v}= (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\ \boldsymbol{v} = \mathbf{grad}\ (\mathrm{div}\ \boldsymbol{v})\ \text{-}\ \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{v}.$$

We have the following lemmas; see [6].

Lemma 3.2.1. *The spaces $\boldsymbol{X}^p_N(\Omega)$ and $\boldsymbol{X}^p_T(\Omega)$ are continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$.*

In order to consider the case of nonhomogeneous boundary conditions, we introduce the following spaces:

$$\boldsymbol{X}^{1,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\boldsymbol{v} \in L^p(\Omega),\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} \in W^{1-1/p,p}(\Gamma)\},$$

$$\boldsymbol{Y}^{1,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\boldsymbol{v} \in L^p(\Omega), \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \text{ and } \boldsymbol{v}\times\boldsymbol{n} \in W^{1-1/p,p}(\Gamma)\}.$$

Lemma 3.2.2. *The spaces $\boldsymbol{X}^{1,p}(\Omega)$ and $\boldsymbol{Y}^{1,p}(\Omega)$ are continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$.*

Consider as well the spaces:

$$\boldsymbol{X}^{2,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\boldsymbol{v} \in W^{1,p}(\Omega), \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} \in W^{2-1/p,p}(\Gamma)\},$$

$$\boldsymbol{Y}^{2,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\boldsymbol{v} \in W^{1,p}(\Omega), \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \text{ and } \boldsymbol{v}\times\boldsymbol{n} \in W^{2-1/p,p}(\Gamma)\}.$$

We have the following theorem, see [7].

Theorem 3.2.3. *Assume that Ω is of class $C^{2,1}$. Then the spaces $\boldsymbol{X}^{2,p}(\Omega)$ and $\boldsymbol{Y}^{2,p}(\Omega)$ are continuously embedded in $\boldsymbol{W}^{2,p}(\Omega)$.*

Consider now the space

$$\boldsymbol{E}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta\boldsymbol{v} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\},$$

which is a Banach space for the norm $\|\boldsymbol{v}\|_{\boldsymbol{E}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\Delta\boldsymbol{v}\|_{[\boldsymbol{H}_o^{p'}(\mathrm{div},\Omega)]'}$. Thanks to [7, Lemma 4.1], we know that $D(\overline{\Omega})$ is dense in $\boldsymbol{E}^p(\Omega)$. Moreover, (see [7, Corollary 4.2]), the linear mapping $\gamma : \boldsymbol{v} \longmapsto \mathbf{curl}\boldsymbol{v} \times \boldsymbol{n}$ defined on $D(\overline{\Omega})$ can be extended to a linear and continuous mapping $\gamma : \boldsymbol{E}^p(\Omega) \longmapsto \boldsymbol{W}^{-1/p,p}(\Omega)$. Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{X}_\tau^{p'}(\Omega)$ such that div $\boldsymbol{\varphi}$= 0 in Ω,

$$-\langle\Delta\boldsymbol{v},\boldsymbol{\varphi}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} = \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\overline{\boldsymbol{\varphi}}\,\mathrm{d}x - \langle\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{\varphi}\rangle_\Gamma,$$

where $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{1/p,p'}(\Gamma)}$.

Next, we introduce the following space

$$\boldsymbol{T}^p(\Omega) = \{\boldsymbol{\phi} \in \boldsymbol{H}_0^p(\mathrm{div},\Omega);\ \mathrm{div}\boldsymbol{\phi} \in W_0^{1,p}(\Omega)\}.$$

The space $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{T}^p(\Omega)$ and for all $\chi \in W^{-1,p}(\Omega)$ and $\boldsymbol{\phi} \in \boldsymbol{T}^{p'}(\Omega)$, we have:

$$\langle\boldsymbol{\nabla}\chi,\boldsymbol{\phi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} = -\langle\chi,\mathrm{div}\boldsymbol{\phi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)}. \tag{3.2}$$

A distribution $\boldsymbol{f}$ belongs to $(\boldsymbol{T}^p(\Omega))'$ if and only if there exist $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $f_0 \in W^{-1,p'}(\Omega)$, such that $\boldsymbol{f} = \boldsymbol{\psi} + \boldsymbol{\nabla}f_0$. Moreover, we have the estimate

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|f_0\|_{W^{-1,p'}(\Omega)} \le C\|\boldsymbol{f}\|_{(\boldsymbol{T}^p(\Omega))'}.$$

We will need also the following space

$$\boldsymbol{H}_p(\Delta;\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \Delta\boldsymbol{v}\in(\boldsymbol{T}^{p'}(\Omega))'\},$$

which is a Banach space for the norm $\|\boldsymbol{v}\|_{\boldsymbol{H}_p(\Delta;\Omega)}=\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\Delta\boldsymbol{v}\|_{(\boldsymbol{T}^{p'}(\Omega))'}$. The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}_p(\Delta;\Omega)$ and the mapping γ: $\boldsymbol{v}\longmapsto\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}$ defined on $D(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping $\gamma:\boldsymbol{H}_p(\Delta;\Omega)\longmapsto\boldsymbol{W}^{-1-1/p,p}(\Omega)$. Moreover, we have the Green formula: for any $\boldsymbol{v}\in\boldsymbol{H}_p(\Delta;\Omega)$ and $\boldsymbol{\phi}\in\boldsymbol{Y}_\tau^{p'}(\Omega)$,

$$\langle\Delta\boldsymbol{v},\boldsymbol{\phi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}=\int_\Omega\boldsymbol{v}\cdot\Delta\overline{\boldsymbol{\phi}}\,\mathrm{d}x+\langle\mathbf{curl}\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{\phi}\rangle_\Gamma,\tag{3.3}$$

where $\langle\cdot,\cdot\rangle_\Gamma=\langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)\times\boldsymbol{W}^{1+1/p,p'}(\Gamma)}$ and

$$\boldsymbol{Y}_\tau^p(\Omega)=\{\boldsymbol{\phi}\in\boldsymbol{W}^{2,p}(\Omega);\ \boldsymbol{\phi}\cdot n=0,\ \mathrm{div}\boldsymbol{\phi}=0,\ \mathbf{curl}\boldsymbol{\phi}\times n=0\text{ on }\Gamma\}.$$

3.3 Generalized resolvent problem

In this section, we consider the generalized resolvent problem (3.1) and we prove the existence and uniqueness of weak, strong and very weak solution to this problem.

3.3.1 Weak solution

Consider the problem

$$\begin{cases}\lambda\boldsymbol{u}-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}, & \mathrm{div}\,\boldsymbol{u}=0 & \text{in} & \Omega\times(0,T)\\ \boldsymbol{u}\cdot\boldsymbol{n}=0, & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n} & \text{on} & \Gamma\times(0,T)\end{cases}\tag{3.4}$$

We start by the existence and uniqueness of weak solutions to (3.4).

Theorem 3.3.1. *Let $\varepsilon\in]0,\pi[$ be fixed and $\lambda\in\Sigma_\varepsilon$. Let $p\geq 2$, $\boldsymbol{f}\in(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'$ and $\boldsymbol{h}\times\boldsymbol{n}\in\boldsymbol{W}^{-1/p,p}(\Gamma)$. Then the problem (3.4) has a unique solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ satisfying the following estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C(\Omega,p)\left(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\right).\tag{3.5}$$

Proof. **Step 1 : Existence and uniqueness.** We can easily verify that problem (3.4) is equivalent to the variational problem: Find $\boldsymbol{u}\in\boldsymbol{V}_\tau^p(\Omega)$ such that for all $\boldsymbol{v}\in\boldsymbol{V}_\tau^{p\prime}(\Omega)$

$$\lambda\int_\Omega\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x+\int_\Omega\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x=\langle\boldsymbol{f},\boldsymbol{v}\rangle_\Omega+\langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_\Gamma,\tag{3.6}$$

where $\langle\cdot,\cdot\rangle_\Omega=\langle\cdot,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}$ and $\langle\cdot,\cdot\rangle_\Gamma=\langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{-1/p,p'}(\Gamma)}$.

The proof is done in two steps:

i) Case $2 \le p \le 6$**.** The case $p = 2$ can be directly obtained using Lax-Milgram theorem. Suppose that $2 < p \le 6$, then Problem (3.4) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. We write (3.4) in the form:

$$\left\{ \begin{array}{rl} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} - \lambda \boldsymbol{u} = \boldsymbol{F}, & div\, \boldsymbol{u} = 0 \qquad \text{in} \quad \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \quad \text{on} \quad \Gamma \end{array} \right. . \tag{3.7}$$

As $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$, we have $\boldsymbol{F} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}; \Omega))'$ and

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_\tau^{p'}(\Omega), \quad \langle \boldsymbol{F}, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v} \rangle_\Gamma = 0. \tag{3.8}$$

Theorem 4.4 of [7] implies that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)$.

Let $\boldsymbol{v} \in \boldsymbol{K}_\tau^{p'}(\Omega)$. Using the variational formulation, we have

$$\langle \boldsymbol{F}, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v} \rangle_\Gamma = 0.$$

Then our solution $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$.

ii) Case $p \ge 6$**.** Observe that $(\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))' \hookrightarrow (\boldsymbol{H}_0^{6/5}(\mathrm{div}, \Omega))'$ and $\boldsymbol{W}^{-1/p,p}(\Gamma) \hookrightarrow \boldsymbol{W}^{-1/6,6}(\Gamma)$. Then Problem (3.7) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,6}(\Omega) \times L^6(\Omega)/\mathbb{R}$. Thanks to the embedding $\boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$, we deduce that $\boldsymbol{F} = \boldsymbol{f} - \lambda \boldsymbol{u} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$. Moreover, $\boldsymbol{F}$ satisfies the compatibility condition (3.8). We conclude that $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$.

Step 2: Estimate. Let $B \in \mathcal{L}(\boldsymbol{V}_\tau^p(\Omega), (\boldsymbol{V}_\tau^{p'}(\Omega))')$ be the operator defined by

$$\forall\, \boldsymbol{u} \in \boldsymbol{V}_\tau^p(\Omega), \forall\, \boldsymbol{v} \in \boldsymbol{V}_\tau^{p'}(\Omega), \quad \langle B\boldsymbol{u}, \boldsymbol{v} \rangle_{(\boldsymbol{V}_\tau^{p'}(\Omega))' \times \boldsymbol{V}_\tau^p(\Omega)} = \lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x.$$

For all $p \ge 2$, the operator B is an isomorphism from $\boldsymbol{V}_\tau^p(\Omega)$ into $(\boldsymbol{V}_\tau^{p'}(\Omega))'$ and $\|\boldsymbol{u}\|_{\boldsymbol{X}_\tau^p} \approx \|B\boldsymbol{u}\|_{(\boldsymbol{V}_\tau^{p'}(\Omega))'}$ for all $\boldsymbol{u} \in \boldsymbol{V}_\tau^p(\Omega)$. Moreover, using the continuous embedding $\boldsymbol{X}_\tau^p(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$, we have for every $\boldsymbol{u} \in \boldsymbol{V}_\tau^p(\Omega)$ solution of problem (3.6),

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C(\Omega, p) \|\boldsymbol{u}\|_{\boldsymbol{X}_\tau^p(\Omega)} \le C(\Omega, p) \|B\boldsymbol{u}\|_{(\boldsymbol{V}_\tau^{p'}(\Omega))'}$$

and

$$\begin{aligned} \|B\boldsymbol{u}\|_{(\boldsymbol{V}_\tau^{p'}(\Omega))'} = \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_\tau^{p'}(\Omega) \\ \boldsymbol{v} \neq 0}} \frac{|\langle B\boldsymbol{u}, \boldsymbol{v} \rangle|}{\|\boldsymbol{v}\|_{\boldsymbol{X}_\tau^{p'}(\Omega)}} &= \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_\tau^{p'}(\Omega) \\ \boldsymbol{v} \neq 0}} \frac{|\langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v} \rangle_\Gamma|}{\|\boldsymbol{v}\|_{\boldsymbol{X}_\tau^{p'}(\Omega)}} \\ &\le C(\Omega, p) \left(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} \right), \end{aligned}$$

which is the estimate (3.5). □

Theorem 3.3.2. *Let $\lambda \in \Sigma_\varepsilon$. Let $p \geq 2$. Let $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$, $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$, $g \in W^{1-1/p,p}(\Gamma)$ and $\chi \in L^p(\Omega)$ verifying the following compatibility condition*

$$\int_\Omega \chi \, \mathrm{d}x = \int_\Gamma g \, \mathrm{d}\sigma. \tag{3.9}$$

Then Problem (3.1) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ satisfying the following estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C(\Omega, p, \lambda)(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}). \tag{3.10}$$

Proof. **i) Existence and uniqueness.** Consider the following Neumann problem

$$\Delta \boldsymbol{\theta} = \chi \quad \text{in} \quad \Omega \qquad \text{and} \qquad \frac{\partial \theta}{\partial n} = g \quad \text{on} \quad \Gamma. \tag{3.11}$$

Since $g \in W^{1-1/p,p}(\Gamma)$ and $\chi \in L^p(\Omega)$ verifying the compatibility condition (3.9), this problem has a unique solution $\theta \in W^{2,p}(\Omega)/\mathbb{R}$ such that

$$\|\theta\|_{W^{2,p}(\Omega)/\mathbb{R}} \leq C \left(\|g\|_{W^{1-1/p,p}(\Gamma)} + \|\chi\|_{L^p(\Omega)} \right). \tag{3.12}$$

Set $\boldsymbol{F} = \boldsymbol{f} - \lambda \nabla \theta + \nabla \chi$ and observe that $\boldsymbol{F} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$. Using Theorem 3.3.1, we deduce that the problem

$$\begin{cases} \lambda \boldsymbol{z} - \Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{F}, \quad \mathrm{div}\, \boldsymbol{z} = 0 & \text{in} \quad \Omega \\ \boldsymbol{z} \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on} \quad \Gamma \end{cases} \tag{3.13}$$

has a unique solution $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ satisfying the following estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega, p) \left(\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} \right). \tag{3.14}$$

Set $\boldsymbol{u} = \boldsymbol{z} + \nabla \theta$. Then $(\boldsymbol{u}, \pi)$ solves (3.1).

ii) Estimate. Observe that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega, p)(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + |\lambda| \|\nabla \theta\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\nabla \chi\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}) + \|\nabla \theta\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

Using estimate (3.12), one gets

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega, p, \lambda)(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}). \tag{3.15}$$

Moreover, $\|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C(\Omega,p)\,\|\nabla\pi\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} = \|\boldsymbol{f}-\lambda\boldsymbol{u}+\Delta\boldsymbol{u}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}$. Thus,

$$\|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C(\Omega,p,\lambda)(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\chi\|_{L^p(\Omega)}+\|g\|_{W^{1-1/p,p}(\Gamma)}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}). \tag{3.16}$$

Combining (3.15) together with (3.16), we obtain the estimate (3.10).

□

Theorem 3.3.3. *Let $1<p<2$, $\boldsymbol{f}\in(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'$, $\boldsymbol{h}\times\boldsymbol{n}\in\boldsymbol{W}^{-1/p,p}(\Gamma)$, $g\in W^{1-1/p,p}(\Gamma)$ and $\chi\in L^p(\Omega)$ verifying the following compatibility condition* (3.9)*. Then Problem* (3.1) *has a unique solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$.*

Proof. **Step 1:** We suppose that $g=0$. The problem

$$\begin{cases} \lambda\boldsymbol{u}-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}, & \mathrm{div}\,\boldsymbol{u}=\chi, & \text{in} & \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n}=0, & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n}, & \text{on} & \Gamma \end{cases} \tag{3.17}$$

has the following equivalent variational formulation: Find $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ satisfying $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on Γ such that for every $\boldsymbol{w}\in\boldsymbol{W}^{1,p'}$ satisfying $\boldsymbol{w}\cdot\boldsymbol{n}=0$ and $\mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{n}=0$ on Γ, we have

$$\lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{w}}\,\mathrm{d}x+\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{w}}\,\mathrm{d}x-\int_\Omega \pi\cdot\mathrm{div}\,\overline{\boldsymbol{w}}\,\mathrm{d}x=\langle\boldsymbol{f},\boldsymbol{w}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{w}\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{-1/p,p'}(\Gamma)}-\int_\Omega\chi\cdot\mathrm{div}\,\overline{\boldsymbol{w}}\,\mathrm{d}x.$$

According to theorem 3.3.2, for any $(\boldsymbol{F},\varphi)$ in $(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'\times L_0^{p'}(\Omega)$, there exists a unique solution $(\boldsymbol{w},\eta)\in\boldsymbol{W}^{1,p'}(\Omega)\times L^{p'}(\Omega)/\mathbb{R}$ solution to

$$\begin{cases} \lambda\boldsymbol{w}-\Delta\boldsymbol{w}+\nabla\eta=\boldsymbol{F}, & \mathrm{div}\,\boldsymbol{w}=\varphi, & \text{in} & \Omega \\ \boldsymbol{w}\cdot\boldsymbol{n}=0, & \mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{n}=0, & \text{on} & \Gamma \end{cases} \tag{3.18}$$

and satisfying

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p'}(\Omega)}+\|\eta\|_{L^{p'}(\Omega)/\mathbb{R}}\leq C(\Omega,p',\lambda)(\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'}+\|\varphi\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

Let $T:(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'\times L_0^{p'}(\Omega)\to\mathbb{C}$ be a linear form defined by

$$T(\boldsymbol{F},\varphi)=\langle\boldsymbol{f},\boldsymbol{w}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}+\langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{w}\rangle_\Gamma-\int_\Omega\chi\cdot\overline{\eta}\,\mathrm{d}x.$$

Observe that

$$|T(\boldsymbol{F},\varphi)|\leq\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}\|\boldsymbol{w}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\|\boldsymbol{w}\|_{\boldsymbol{W}^{1/p,p'}(\Gamma)}+\|\varphi\|_{L^{p'}(\Omega)}.$$

Then T is continuous on $(\boldsymbol{H}_0^p(\mathrm{div},\Omega))' \times \boldsymbol{L}^{p'}(\Omega)$ and we deduce that there exists a unique $(\boldsymbol{u},\pi) \in \boldsymbol{H}_0^p(\mathrm{div},\Omega) \times L^p(\Omega)/\mathbb{R}$ such that

$$T(\boldsymbol{F},\varphi) = \langle \boldsymbol{u}, \boldsymbol{F}\rangle_{\boldsymbol{H}_0^p(\mathrm{div},\Omega)\times(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'} - \int_\Omega \pi \cdot \overline{\varphi}\,\mathrm{d}x.$$

As a result,

$$\begin{aligned}&\lambda \int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{w}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{w}}\,\mathrm{d}x - \int_\Omega \pi\cdot\mathrm{div}\,\overline{\boldsymbol{w}}\,\mathrm{d}x\\ &= \langle \boldsymbol{f},\boldsymbol{w}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \langle \boldsymbol{h}\times\boldsymbol{n},\boldsymbol{w}\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{-1/p,p'}(\Gamma)} - \int_\Omega \chi\cdot\mathrm{div}\,\overline{\boldsymbol{w}}\,\mathrm{d}x.\end{aligned}$$

To finish, we shall prove that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. To this end, we write our problem in the form (3.7), where $\boldsymbol{F} = \boldsymbol{f} - \lambda\boldsymbol{u}$ belongs to $(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'$ and satisfies (3.8). Then, using [7, Remark 4.6], our solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$.
Step 2 : $g \neq 0$. Let $\theta \in W^{2,p}(\Omega)/\mathbb{R}$ be the unique solution of the Neumann problem (3.11) with $\chi \in L^p(\Omega)$ and $g \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ satisfying (3.9). Let $\boldsymbol{F} = \boldsymbol{f} + \nabla\chi - \lambda\nabla\theta \in (\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'$. Then there exists $(\boldsymbol{z},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ solution of (3.13). Set $\boldsymbol{u}{=}\boldsymbol{z}+\nabla\theta$. We can easily verify that $(\boldsymbol{u},\pi)$ solves (3.1). □

3.3.2 Strong solution

Theorem 3.3.4. *Let $1 < p < \infty$. Let $\boldsymbol{f} \in L^p(\Omega)$ and $\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. Then the problem* (3.4) *has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)/\mathbb{R}$ satisfying the following estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \le C(\lambda,p,\Omega)(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}). \tag{3.19}$$

Proof. We know that the problem (3.4) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$.
Moreover, the map π satisfies

$$\mathrm{div}(\nabla\pi - \boldsymbol{f}) = 0 \text{ in }\Omega, \qquad (\nabla\pi - \boldsymbol{f})\cdot\boldsymbol{n} = -\mathrm{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}) \text{ on }\Gamma.$$

Since $\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, we deduce that $\pi \in W^{1,p}(\Omega)$.

Set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$. Notice that $\boldsymbol{z}$ verifies the following problem:

$$\begin{cases} \lambda\boldsymbol{z} - \Delta\boldsymbol{z} = \mathbf{curl}\boldsymbol{f}, \quad \mathrm{div}\,\boldsymbol{z} = 0, & \text{in} \quad \Omega \\ \quad \boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}, & \text{on} \quad \Gamma \end{cases} \tag{3.20}$$

where $\mathbf{curl}\,\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega))'$ and $\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. Then, $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$ and satisfies

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C(\Omega)(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}).$$

Thus, $\boldsymbol{u} \in L^p(\Omega)$, div $\boldsymbol{u} = 0 \in W^{1,p}(\Omega)$, $\mathbf{curl}\ \boldsymbol{u} = \boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0 \in W^{1-1/p,p}(\Gamma)$. Then, $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\lambda, p, \Omega)(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}).$$

Finally, proceeding as in Step 2 of the proof of Theorem 3.3.2, we obtain that the solution $(\boldsymbol{u}, \pi)$ satisfies the estimation (3.19), which ends the proof. □

Corollary 3.3.5. *Let $1 < p < \infty$. Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, $g \in W^{2-1/p,p}(\Gamma)$ and $\chi \in W^{1,p}(\Omega)$ verifying the following compatibility condition* (3.9). *Then Problem* (3.1) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ satisfying*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\Omega, p, \lambda)(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{1,p}(\Omega)} + \|g\|_{W^{2-1/p,p}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}). \quad (3.21)$$

Proof. Let $\theta \in W^{2,p}(\Omega)$ be the unique solution of the Neumann problem (3.11). Set $\boldsymbol{F} = \boldsymbol{f} - \lambda\nabla\theta + \nabla\chi$ and observe that $\boldsymbol{F} \in \boldsymbol{L}^p(\Omega)$. Thanks to Theorem 3.3.4, the problem (3.13) has a unique solution $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ satisfying

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\Omega, p, \lambda)(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}).$$

By setting $\boldsymbol{u} = \boldsymbol{z} + \nabla\theta$, we can easily verify that $(\boldsymbol{u}, \pi)$ solves (3.1) and verifies (3.21). □

3.3.3 Very weak solution

In this section, we prove the existence of very week solution to Problem (3.1).

Theorem 3.3.6. *Let $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$, $\chi \in L^p(\Omega)$, $g \in W^{-1/p,p}(\Gamma)$ and $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{-1-1/p,p}(\Gamma)$ verifying the compatibility condition* (3.9). *Then Problem* (3.1) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, the following estimate holds*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C(\Omega, p, \lambda)(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}). \quad (3.22)$$

Proof. **Step 1.** Problem (3.1) is equivalent to the variational formulation: find $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ such that for any $\boldsymbol{\phi} \in \boldsymbol{Y}^{p'}_\tau(\Omega)$ and for any $q \in W^{1,p'}(\Omega)$,

$$\lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{\phi}}\, \mathrm{d}x - \int_\Omega \boldsymbol{u} \cdot \Delta\overline{\boldsymbol{\phi}}\, \mathrm{d}x - \langle \pi, \mathrm{div}\boldsymbol{\phi} \rangle_{W^{-1,p}(\Omega) \times W^{1,p'}_0(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{\phi} \rangle_\Gamma \quad (3.23)$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla\overline{q}\, \mathrm{d}x = -\int_\Omega \chi\overline{q}\, \mathrm{d}x + \langle g, q \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}, \quad (3.24)$$

where $\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}$ and $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)\times\boldsymbol{W}^{1+1/p,p'}(\Gamma)}$.
Indeed, using the Green formula (3.3), we can verify that every $(\boldsymbol{u},\pi)\in\boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)$ solution to (3.1) solves (3.23)-(3.24). Conversely, let $(\boldsymbol{u},\pi)\in\boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)$ be a solution to (3.23)-(3.24). Clearly, $-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}$ and $\operatorname{div}\boldsymbol{u}=\chi$ in Ω. Consequently, $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ and since $\nabla\pi\in(\boldsymbol{T}^{p'}(\Omega))'$, we have $\Delta\boldsymbol{u}=-\boldsymbol{f}+\lambda\boldsymbol{u}+\nabla\pi\in(\boldsymbol{T}^{p'}(\Omega))'$. Then $\boldsymbol{u}\in\boldsymbol{H}_p(\Delta,\Omega)$. Using (3.2) and (3.3), we obtain that for any $\boldsymbol{\phi}\in\boldsymbol{Y}^{p'}_\tau(\Omega)$:

$$\lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{\phi}}\,\mathrm{d}x-\int_\Omega\boldsymbol{u}\cdot\Delta\overline{\boldsymbol{\phi}}\,\mathrm{d}x-\langle\mathbf{curl}\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{\phi}\rangle_\Gamma-\langle\pi,\operatorname{div}\boldsymbol{\phi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)}=\langle\boldsymbol{f},\boldsymbol{\phi}\rangle_\Omega.$$

Thus, $\langle\mathbf{curl}\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{\phi}\rangle_\Gamma=\langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{\phi}\rangle_\Gamma$. Let $\boldsymbol{\mu}\in\boldsymbol{W}^{1+1/p,p'}(\Gamma)$, there exists a function $\boldsymbol{\phi}\in\boldsymbol{W}^{2,p}(\Omega)$ satisfying

$$\boldsymbol{\phi}_\tau=\boldsymbol{\mu}_\tau\quad\text{and}\quad\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{n}}=-\boldsymbol{n}\operatorname{div}_\Gamma\boldsymbol{\mu}_\tau+\sum_{j=1}^{2}\Big(\frac{\partial\boldsymbol{\mu_\tau}}{\partial s_j}\times\boldsymbol{T}_j\Big)\times\boldsymbol{n}\ \text{ on }\ \Gamma.$$

It is clear that $\boldsymbol{\phi}\in\boldsymbol{Y}^{p'}_\tau(\Omega)$ and

$$\langle\mathbf{curl}\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{\mu}\rangle_\Gamma-\langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{\mu}\rangle_\Gamma=\langle\mathbf{curl}\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{\phi}_\tau\rangle_\Gamma-\langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{\phi}_\tau\rangle_\Gamma=0.$$

Thus, $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n}$ on Γ. Next using that $\operatorname{div}\boldsymbol{u}=\chi$ in Ω, we deduce that for any $q\in W^{1,p'}(\Omega)$, we have

$$\langle\boldsymbol{u}\cdot\boldsymbol{n},q\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}=\langle g,q\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}.$$

Consequently, $\boldsymbol{u}\cdot\boldsymbol{n}=g\in W^{-1/p,p}(\Gamma)$.

Step 2. Let us now solve Problem (3.23)-(3.24). We suppose that

$$g=0\quad\text{on}\quad\Gamma\qquad\text{and}\qquad\int_\Omega\chi\,\mathrm{d}x=0.$$

Thanks to Theorem 3.3.5, for any pair $(\boldsymbol{F},\xi)\in\boldsymbol{L}^{p'}(\Omega)\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$, there exists a unique $(\boldsymbol{\phi},q)\in\boldsymbol{W}^{2,p'}(\Omega)\times W^{1,p'}(\Omega)/\mathbb{R}$ satisfying:

$$\left\{\begin{array}{rl}\lambda\boldsymbol{\phi}-\Delta\boldsymbol{\phi}+\nabla q=\boldsymbol{F},\quad\operatorname{div}\boldsymbol{\phi}=\xi, & \text{in}\quad\Omega,\\ \boldsymbol{\phi}\cdot\boldsymbol{n}=0,\quad\mathbf{curl}\,\boldsymbol{\phi}\times\boldsymbol{n}=0, & \text{on}\quad\Gamma,\end{array}\right.\tag{3.25}$$

with the estimate

$$\|\boldsymbol{\phi}\|_{\boldsymbol{W}^{2,p'}(\Omega)}+\|q\|_{\boldsymbol{W}^{1,p'}(\Omega)/\mathbb{R}}\le C(\lambda,\Omega,p')(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}+\|\xi\|_{W^{1,p'}(\Omega)}).$$

Let T be a linear form defined from $\boldsymbol{L}^{p'}(\Omega)\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$ onto $\mathbb{C}$ by

$$T:(\boldsymbol{F},\xi)\longmapsto\langle\boldsymbol{f},\boldsymbol{\phi}\rangle_\Omega+\langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{\phi}\rangle_\Gamma-\int_\Omega\chi q\,\mathrm{d}x.$$

An easy computation shows that

$$|T(\boldsymbol{F},\xi)| \leq C(\Omega,p',\lambda)(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}+\|\chi\|_{L^p(\Omega)})(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}+\|\xi\|_{W^{1,p'}(\Omega)}).$$

This means that T defines an element of the dual space of $\boldsymbol{L}^{p'}(\Omega)\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$ and according to the Riesz's representation theorem, there exists a unique $(\boldsymbol{u},\pi)\in \boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ such that

$$T(\boldsymbol{F},\xi) = \langle \boldsymbol{u},\boldsymbol{F}\rangle_{\boldsymbol{T}^{p'}(\Omega)\times(\boldsymbol{T}^{p'}(\Omega))'} - \int_\Omega \pi\xi \,\mathrm{d}x.$$

Thus, $(\boldsymbol{u},\pi)$ is a solution to (3.23)-(3.24) and satisfies (3.22).
Step 3. Suppose that $g\neq 0$ and the compatibility condition (3.9) holds. The Neumann problem (3.11) has a unique solution $\theta\in W^{1,p}(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\|\chi\|_{L^p(\Omega)}+\|g\|_{W^{-1/p,p}(\Gamma)}).$$

Set $\boldsymbol{F}=\boldsymbol{f}-\lambda\nabla\theta+\nabla\chi$. Then $\boldsymbol{F}\in(\boldsymbol{T}^{p'}(\Omega))'$ and Problem (3.13) has a unique solution $(\boldsymbol{z},\pi)\in\boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ satisfying the following estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)}+\|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C(\lambda,\Omega,p)\left(\|\boldsymbol{F}\|_{(\boldsymbol{T}^{p'}(\Omega))'}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}\right). \quad (3.26)$$

Thus, $(\boldsymbol{u},\pi)$ with $\boldsymbol{u}=\boldsymbol{z}+\nabla\theta$ solves (3.1) and satisfies (3.22). □

Remark 3.3.7. i) Consider the problem (3.1) with $\chi\in W^{1,p}(\Omega)$ such that $\int_\Omega \chi\,\mathrm{d}\,x = 0$, $g=0$ and $\boldsymbol{h}=\boldsymbol{0}$ on Γ. As in [34], we can prove that the solution $(\boldsymbol{u},\pi)$ satisfies the following estimate

$$|\lambda|\,\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}+\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \leq C\,(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}+\|\nabla\chi\|_{\boldsymbol{L}^p(\Omega)}+|\lambda|\,\|\chi\|_{W^{-1,p}(\Omega)}). \quad (3.27)$$

Indeed, let $\theta\in W^{2,p}(\Omega)/\mathbb{R}$ solution to $\Delta\,\theta=\chi$ in Ω, $\frac{\partial\theta}{\partial\boldsymbol{n}}=0$ on Γ and satisfying $\|\theta\|_{W^{2,p}(\Omega)}\leq C\,\|\chi\|_{W^{1,p}(\Omega)}$. Set $\boldsymbol{F}=\boldsymbol{f}-\lambda\nabla\theta+\nabla\chi$. Then, $\boldsymbol{F}\in\boldsymbol{L}^p(\Omega))$ and the problem

$$\begin{cases} \lambda\boldsymbol{z}-\Delta\boldsymbol{z}+\nabla\pi=\boldsymbol{F}, \quad \operatorname{div}\boldsymbol{z}=0 & \text{in} \quad \Omega \\ \boldsymbol{z}\cdot\boldsymbol{n}=0, \quad \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n}=\boldsymbol{0} & \text{on} \quad \Gamma \end{cases}$$

has a unique solution $(\boldsymbol{z},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ satisfying the following estimate

$$|\lambda|\,\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega))}+\|\nabla\chi\|_{\boldsymbol{L}^p(\Omega))}+|\lambda|\|\nabla\theta\|_{\boldsymbol{L}^p(\Omega))}\right)$$

Set $\boldsymbol{u}=\boldsymbol{z}+\nabla\theta$. Then $(\boldsymbol{u},\pi)$ is a solution to (3.1) and satisfies (3.27).
ii) Notice that when $\chi=0$, we recover the resolvent estimate established in [2] and [3].

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