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Ismaïla Balogoun

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# MEMOIRE DE DOCTORAT DE

L'ECOLE CENTRALE DE NANTES

Ecole Doctorale N°602  
*Sciences de l'Ingénierie et des Systèmes*  
Spécialité : Automatique  
Par

**Ismäïla BALOGOUN**

**Contributions à la théorie du contrôle des systèmes de dimension infinie soumis à des perturbations/incertitudes**

Projet de recherche doctoral présenté et soutenu à l'École Centrale de Nantes le 20 juillet 2023  
Unité de recherche : UMR 6004, Laboratoire des Sciences du Numérique de Nantes (LS2N)

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---

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# NOTATION

- Let  $c \in \mathbb{C}$ ,  $\Re_e(c)$  (resp.  $\Im_m(c)$ ) denote the real part (resp., the imaginary part) of  $c$ .
- Given  $n \in \mathbb{N}$ , the norm of the space  $\mathbb{R}^n$  is denoted by  $|\cdot|$ .
- A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be a class  $\mathcal{K}$  function if  $\alpha$  is nonnegative, increasing and vanishing at 0. It is said to be a class  $\mathcal{K}_\infty$  function if moreover it satisfies

$$\lim_{s \rightarrow +\infty} \alpha(s) = +\infty.$$

- A function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be a class  $\mathcal{KL}$  function if, for each nonnegative value  $s$ , the function  $r \rightarrow \beta(r, s)$  is a class  $\mathcal{K}$  function and, for every positive value  $r$ , the function  $s \rightarrow \beta(r, s)$  is strictly decreasing and satisfies

$$\lim_{s \rightarrow +\infty} \beta(r, s) = 0.$$

- The identity operator associated to a Hilbert space  $H$  is denoted by  $I_H$ .
- Given a Hilbert space  $H$  equipped with the scalar product  $\langle \cdot, \cdot \rangle_H$ .
- Given a positive value  $L$ , let us consider the function  $(t, x) \in \mathbb{R}_{\geq 0} \times [0, L] \mapsto z(t, x) \in \mathbb{R}$ . The term  $z_t$  or  $\partial_t z$  (respectively  $z_x$  or  $\partial_x z$ ) stands for the partial derivative of  $z$  with respect to  $t$  (respectively with respect to  $x$ ). Similarly,  $z_{xxx}$  or  $\partial_{xxx} z$  stands for the third derivative of  $z$  with respect to  $x$ . When the function  $z$  depends only on  $x$ ,  $z'$  (respectively  $z'''$ ) denotes the first (respectively the third) spatial derivative of  $z$ . When the function  $z$  depends only on  $t$ ,  $\dot{z} = \frac{d}{dt} z$  denotes the time derivative of  $z$ . We use the notation  $\dot{z}_d$  to indicate the Dini derivative of  $z$  at  $t = 0$ . We recall that  $\dot{z}_d = \limsup_{t \rightarrow 0} \frac{z(t) - z(0)}{t}$ .
- Given two vector spaces  $E$  and  $F$ ,  $\mathcal{L}(E, F)$  denotes the space of linear continuous applications from  $E$  into  $F$ . If  $E$  is a normed vector space, we denote by  $\|\cdot\|_E$  the norm on  $E$ . We denote by  $E'$  the dual space of  $E$ , that is, the space of all continuous linear functionals on  $E$  and we denote by  $\langle \cdot, \cdot \rangle_{E, E'}$  the dual product on  $E \times E'$ .
- Given a positive value  $T \in [0, \infty)$ , the space of continuous functions on  $[0, T]$  is denoted by  $C(0, T)$ . Given  $k \in \mathbb{N}^*$ , a function  $f$  is said to be of class  $C^k(0, T)$  if  $\frac{d^l}{dt^l} f(t)$ , where  $l \in \{1, \dots, k\}$ , belongs to the space  $C(0, T)$ .

- Given any subset of  $\mathbb{R}$  denoted by  $\Omega$  ( $\mathbb{R}_+$  or an interval, for instance),  $L^p(\Omega; \mathbb{R}^n)$  denotes the set of (Lebesgue) measurable functions  $f_1, \dots, f_n$  such that, for  $i \in \{1, \dots, n\}$ ,  $\int_\Omega |f_i(x)|^p dx < +\infty$  when  $p \neq +\infty$  and such that  $\sup \text{ess}_{x \in \Omega} |f_i(x)| < +\infty$  when  $p = +\infty$ . The associated norms are, for  $p \neq +\infty$ ,  $\|(f_1, \dots, f_n)\|_{L^p}^p := \int_\Omega |f_1(x)|^p dx + \dots + \int_\Omega |f_n(x)|^p dx$  and, for  $p = +\infty$ ,  $\|f_1, \dots, f_n\|_{L^\infty(\Omega; \mathbb{R}^n)} := \sup \text{ess}_{x \in [0, L]} |f_1(x)| + \dots + |f_n(x)|$ . When  $n = 1$ , we simplify the notation



and use  $L^p(\Omega)$ . For any  $p \in [1, \infty]$ , the Sobolev space  $\mathcal{W}^{1,p}(\Omega)$  is defined by the set  $\{f \in L^p(\Omega) \mid f' \in L^p(\Omega)\}$ . For  $m \geq 2$ , the Sobolev space  $\mathcal{W}^{m,p}(\Omega)$  is defined by the set  $\{f \in \mathcal{W}^{m-1,p}(\Omega) \mid f' \in \mathcal{W}^{m-1,p}(\Omega)\}$ . We also set  $\mathcal{H}^m(\Omega) = \mathcal{W}^{m,2}(\Omega)$ . We say that the function  $f \in L^p_{loc}(\Omega)$  (resp.  $f \in \mathcal{W}^{m,p}_{loc}(\Omega)$ ) if the restriction of  $f\chi_K \in L^p(\Omega)$  (resp.  $f\chi_K \in \mathcal{W}^{m,p}(\Omega)$ ) for every compact set  $K$  contained in  $\Omega$ , where  $\chi_K$  is the characteristic function of the set  $K$ .

- The function  $\text{sign}$  is defined by

$$\text{sign}_0(z) := \begin{cases} -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0, \\ 1 & \text{if } z > 0. \end{cases}$$

- The set-valued function  $\text{sign}$  is defined by

$$\text{sign}(z) := \begin{cases} -1 & \text{if } z < 0, \\ [-1, 1] & \text{if } z = 0, \\ 1 & \text{if } z > 0. \end{cases}$$

- For every  $x \in \mathbb{R}$ , we use  $[x]$  to denote the integer part of a number  $x$  and we use  $[x]^m$  to denote  $|x|^m \text{sign}(x)$  with  $m \geq 0$ .
- For  $x \in \mathbb{R}^n$  with  $n \in \mathbb{N}$ , we use  $x_i$  with  $i = 0, 1, \dots, n-1$  to indicate the coordinates of  $x$ .

# INTRODUCTION

---

In recent decades, considerable efforts have been devoted to the development of tools for the design of control laws ensuring the desired performances despite disturbances/uncertainties. The general problems discussed in this thesis are the design of stabilizing feedback laws and output regulation for infinite-dimensional systems subjected to disturbances/uncertainties.

Sliding mode control (see [116] for a good introduction to this topic) has been one of the most popular robust control techniques in recent decades. These types of control laws are known to be robust control and quite simple to implement. Sliding mode control design is split into two steps: firstly, a sliding variable (sliding surface) is selected such that, once this variable equals zero, global asymptotic stability is ensured; secondly, a discontinuous feedback-law is designed such that the trajectory reaches in a finite time the sliding surface, that is defined from the sliding variable. This thesis aims to contribute on the topic of sliding mode control in the context of infinite-dimensional systems. From this problem, a question arises.

## Question 1

How to propose a systematic methodology for the design of sliding variables for linear infinite-dimensional systems?

Various methods of sliding variable design are already known for linear finite-dimensional systems (see e.g. [116, Chapter 2],[39]). At the best of our knowledge, this question started in [98, 97, 96, 94] for infinite-dimensional systems. These articles use Lyapunov technique or regularization-based technique or state space decomposition synthesis to tackle the asymptotic stability of linear infinite-dimensional systems subjected to disturbances/uncertainties, with sliding mode control. The aim of this thesis is to provide complementary results for linear infinite-dimensional systems subjected to disturbances/uncertainties.

Furthermore, sliding mode control needs the full information of the sliding variable. To answer the previous question, we consider the sliding variable to be the scalar product of the state of the system and an eigenfunction of the adjoint operator of the closed-loop system without disturbance. Thus, sliding mode control needs the measurement of this scalar product.

However, in most cases, sensors can measure only a part of the state. Thus, in practical terms, it may not be possible to measure the previous sliding variable. Therefore, instead of assuming that we measure this sliding variable, we propose to assume that we measure some moments of the state of the system. Hence, a second question arises.

**Question 2**

Is it possible to maintain the asymptotic stability of linear infinite-dimensional systems subjected to disturbances by using the measurement of a few moments of the state of the system?

Finally, we aim to contribute to the output regulation problem. The output regulation problem consists in designing a feedback-law such that the output converges asymptotically towards a desired reference and such that disturbances are rejected. This thesis aims at contributing to this topic for a nonlinear Korteweg-de Vries equation by using integral action. To achieve this objective, we follow a Lyapunov approach. Now, a third and last question can be stated

**Question 3**

Is it possible to design a PI controller for Korteweg-de Vries equation using a Lyapunov method?

The manuscript is divided into three chapters. Chapter 1 tackles Question 1. Chapter 2 solves Question 2. Finally, Chapter 3 answers Question 3.

Here is the precise outline of the thesis.

**Outline**

- Chapter 2 deals with linear infinite-dimensional systems with an unbounded control operator. After a presentation of our design methodology on a finite-dimensional example, we present a sliding mode based control law, a super-twisting based control law for the infinite-dimensional system. The well-posedness of the closed-loop system and its asymptotic stability are established using techniques derived from the semigroup theory. Two illustrative examples are provided.
- Chapter 3 deals with the stabilization of a linear hyperbolic system subject to a boundary disturbance. Our feedback design relies on the active disturbance rejection control. The well-posedness of the closed-loop system and its asymptotic/exponential stability are established using techniques derived from the semigroup theory combined with Lyapunov techniques. A numerical example is given to illustrate the efficiency of our strategy.
- Chapter 4 deals with the output regulation of a nonlinear Korteweg-de Vries equation subject to a distributed disturbance. The control input and the regulated output are located at the boundary. To achieve this objective, we follow a Lyapunov approach. To this end inspired by a strictification methodology recently introduced in the finite-dimensional context [107], we construct an ISS-Lyapunov functional for the Korteweg-de Vries equation thanks to the use of an observer which is designed following the backstepping approach. Then, thanks to this Lyapunov functional, we apply the forwarding approach in order to solve the desired output regulation problem.

---

## List of Related Publications

### Journal articles

- I. Balogoun, S. Marx, D. Astolfi. ISS Lyapunov strictification via observer design and integral action control for a Korteweg-de Vries equation. *SIAM Journal on Control and Optimization*.  
Chapter 4 is adapted from this work.
- T. Liard, I. Balogoun, S. Marx, F. Plestan. Boundary sliding control of a system of linear hyperbolic equations: a Lyapunov approach. *Automatica*, 135, 2022.  
This article is a preliminary work containing some results of the Chapter 2 in the case of Banach space  $L^p$  with  $p \in [1, \infty]$ .
- I. Balogoun, S. Marx, T. Liard, F. Plestan. Super-twisting sliding mode control for the stabilization of a linear hyperbolic system. *IEEE Control Systems Letters*, vol. 7, pp. 1-6, 2023.  
This article a preliminary work containing some results of the Chapter 2 in the case of Hilbert space  $L^2$ .
- I. Balogoun, S. Marx, F. Plestan. Sliding mode control for a class of linear infinite dimensional systems. Submitted to *IEEE TAC*.  
Chapter 2 is adapted from this work.
- I. Balogoun, S. Marx, Y. Orlov, F. Plestan. Active disturbance rejection control for the stabilization of a linear hyperbolic system. To be submitted to *International Journal of Robust and Nonlinear Control*.  
Chapter 3 is adapted from this work.

### Conference papers

- I. Balogoun, S. Marx, Y. Orlov, F. Plestan. Active disturbance rejection control for a transport equation via a differentiator. *IFAC World Congress 2023*.  
This article is a preliminary work containing some results of Chapter 3.



# SLIDING MODE CONTROL FOR LINEAR INFINITE-DIMENSIONAL SYSTEMS

---

## 2.1 Introduction

This chapter is concerned with the stabilization of a class of linear infinite-dimensional systems with unbounded control operators and subject to a boundary disturbance (see e.g. [14, 37, 71] for a review on this class of system). To be more precise, we aim to design a sliding mode control (SMC) [44, 116, 128] for the stabilization of boundary or pointwise control for linear partial differential equations (PDEs). We further propose a super-twisting control (STC), where, in contrast with the standard SMC strategy, the control is continuous.

The boundary control of systems described by partial differential equations has received a lot of attention for decades. It is still an important research line today because its application in many important engineering systems is natural (see e.g., [11]). Such a problem has been studied in [110, 113, 31, 54] in the controllability context, in [117, 127, 32, 29, 143, 68, 38] in terms of stabilization, to cite just few papers dealing with this topic.

In this chapter, as mentioned earlier, we focus on the case where infinite-dimensional systems are subject to a disturbance. Therefore, we are not only interested in the stabilization, but also in the rejection of this disturbance. This might be interpreted as a regulation problem. In the case where the disturbance is constant, one can follow a proportional integral (PI) strategy, which is quite well-known in the linear finite-dimensional context, but which is still nowadays an active topic when dealing with PDEs (see e.g., [9, 76, 69, 122, 99]). For more complicated cases, i.e., when the disturbance is time-dependant, one may apply the internal-model approach [99, 41, 109], which consists, roughly speaking, in adding the dynamics of the disturbance in the loop of the controller, which requires therefore the knowledge of this dynamics. Our strategy, based on SMC controllers, is in contrast with the latter one, since only the bound of the disturbance is needed, at the price of assuming that the disturbance matches with the control (i.e., the control and the disturbance are located at the same place).

The SMC strategy has been proved to be efficient for robust control of nonlinear systems of ordinary differential equations (ODEs) [116, 128, 44, 137]. Such controllers allow to force, thanks to discontinuous terms, the trajectories of the system to reach in a finite-time a manifold, called the sliding surface, and to evolve on it, this manifold being defined from control objectives. Basically, the design of the control is split into two steps: firstly, a sliding variable is selected such that, once this variable equals zero, global asymptotic stability is ensured; secondly, a discontinuous feedback-law is designed such that the

trajectory reaches the sliding surface, that is defined thanks to the sliding variable. Once the surface is reached, the disturbance is rejected. The generalization of the SMC procedure to the PDEs case is not new. In [98, 97], a definition of equivalent control (which is the control applied to the system after reaching the sliding surface, to ensure that the trajectories stays on the surface thereafter) for systems governed by semilinear differential equations in Banach spaces has been proposed. One can refer also to [72, 73] where differential inclusions and viability theory are combined to design sliding mode controllers for semilinear differential equations in Banach spaces. We also mention the use of spectral reduction methods in [95] and the case where there are uncertain measurements in [144]. In the last decade, a backstepping strategy has been used to select a sliding variable [55, 102, 130, 119, 79]. Note also that the SMC feedback-law is discontinuous, which creates chattering phenomena when implementing the control numerically. Recall that chattering phenomena are characterized by high frequency oscillations. Therefore, in practical control cases, it is important to reduce this phenomena by providing continuous or smooth controller.

Based on second-order sliding mode techniques (see e.g, [116, Chapter 4]), the super-twisting algorithm has been developed for systems whose the sliding variable admits a relative degree (see [116, Definition 1.6]) equal to 1. The essential feature of the super-twisting control is to require only the measurement of the sliding variable to guarantee the convergence in finite-time to zero of the sliding variable and its derivative. Moreover, the super-twisting feedback-law is continuous with respect to the state, and this drastically attenuates the chattering phenomenon.

The aim of this chapter is to propose a strategy different from the ones that have been mentioned earlier in order to design "classical" sliding mode controls and super-twisting sliding mode controls for general linear infinite-dimensional systems. Indeed, the sliding variable is defined as the scalar product of the state and an eigenfunction of the adjoint operator of the closed-loop system without disturbance. This requires measurement of the scalar product of the state with some function. Such a sliding variable allows to directly use well-known results on the stabilization of abstract infinite-dimensional systems with unbounded control operators in the absence of disturbance [46, 127, 117] together with well-known results about the finite-time convergence of the sliding variable in the context of the finite dimension [91, 105, 128]. In comparison with [98, 97, 72, 73], the approach proposed in this chapter allows to define explicitly and systematically the sliding variable for a large class of linear infinite-dimensional systems.

This chapter is organized as follows. Section 2.2 provides an overview of our design strategy through a finite-dimensional example. Section 2.3 presents a class of linear infinite-dimensional system with an unbounded control operator, the sliding mode based control law, the super-twisting based control law and the main results of the chapter. Section 2.4 introduces an illustrative example. Finally, Section 2.5 collects some remarks

## 2.2 Basic idea of SMC

The purpose of this section is to provide an overview of the strategy that will be proposed for infinite-dimensional systems through a finite-dimensional example. It should be noted that the sliding surface design strategy proposed in this section is different from existing methods given in [116, Chapter 2].

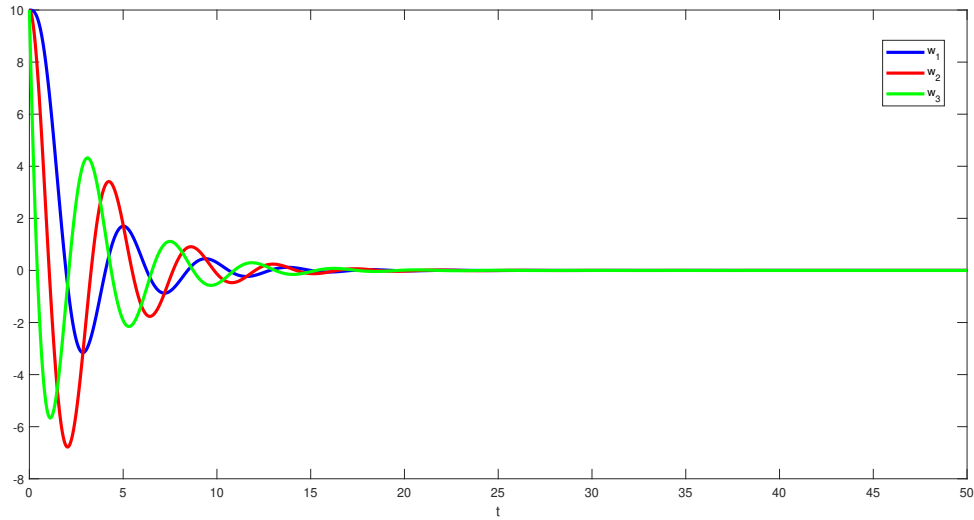


Figure 2.1: System (2.1) for  $d = 0$ :  $w_1$ ,  $w_2$ ,  $w_3$  versus time, with the feedback law (2.2).

Let us consider the following system

$$\begin{cases} \dot{w}_1 = -w_1 + w_2, \\ \dot{w}_2 = -w_1 + w_3, \\ \dot{w}_3 = u + d, \end{cases} \quad (2.1)$$

where  $u$  is the control and  $d$  is an unknown disturbance. The goal is to design a feedback control  $u$  that drives the trajectories of (2.1) to the origin asymptotically, despite the presence of the unknown disturbance  $d$ . For example, when the system (2.1) is undisturbed i.e.,  $d = 0$ , the feedback-law

$$u = -w_1 - w_2 - w_3 \quad (2.2)$$

provides asymptotic stability of the origin of (2.1) (see Figure 2.1). If the unknown disturbance is bounded then the feedback-law (2.2) drives the states of (2.1) to a bounded domain (see Figure 2.2).

### 2.2.1 Sliding mode Control

Since the feedback-law (2.2) allows to stabilize asymptotically the origin of (2.1) with  $d = 0$ , we are going to add to the feedback-law a second term  $u_{SMC}$  that will compensate the effect of the disturbance and will make the closed-loop system asymptotically stable in the presence of the disturbance. Thus, if we select  $u$  as

$$u = -w_1 - w_2 - w_3 + u_{SMC} = \begin{pmatrix} -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + u_{SMC}, \quad (2.3)$$



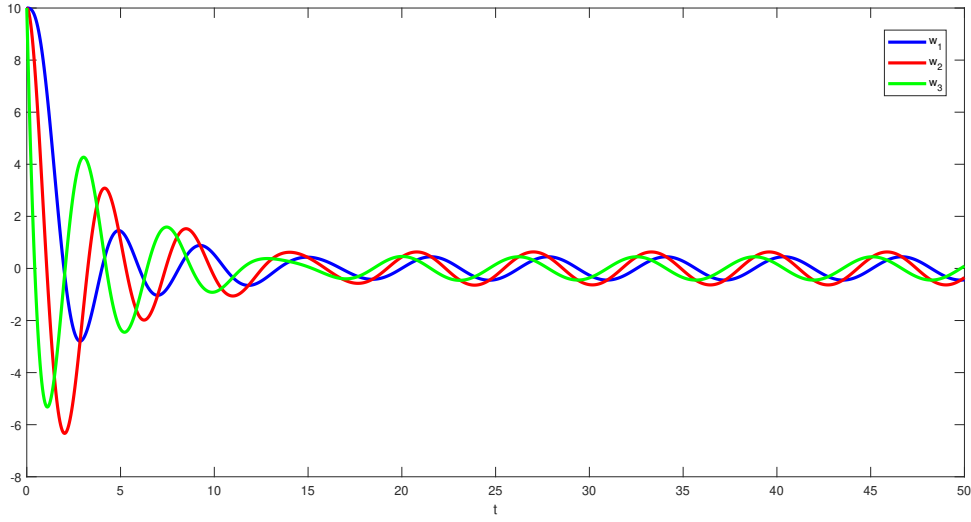


Figure 2.2: System (2.1) for  $d(t) = \sin(t)$ :  $w_1, w_2, w_3$  versus time, with the feedback law (2.2).

then the system (2.1) can be written as follows:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = A_L \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + B(u_{SMC} + d) \quad (2.4)$$

with

$$A_L = A + BL, \quad A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} -1 & -1 & -1 \end{pmatrix}. \quad (2.5)$$

Now, through a simple calculation, we can see that

$$\lambda = \frac{1}{3} \left( -2 - 5 \sqrt[3]{\frac{2}{9\sqrt{29} - 43}} + \sqrt[3]{\frac{1}{2}(9\sqrt{29} - 43)} \right) \approx -1.39265 \quad (2.6)$$

is an eigenvalue of  $A_L^\top$  and  $\varphi = \begin{pmatrix} 1 + \lambda + \lambda^2 \\ 1 + \lambda \\ 1 \end{pmatrix}$  is an eigenvector of  $A_L^\top$  associated to  $\lambda$ , where  $A_L^\top$  is the transpose matrix of  $A_L$ . Then, let us introduce the following surface

$$\Sigma := \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3 \mid \left\langle \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \varphi \right\rangle_{\mathbb{R}^3} = (1 + \lambda + \lambda^2)w_1 + (1 + \lambda)w_2 + w_3 = 0 \right\} \quad (2.7)$$

Since on the surface  $\Sigma$ ,  $w_3 = -aw_1 - bw_2$  with

$$\begin{cases} a = 1 + \lambda + \lambda^2, \\ b = 1 + \lambda, \end{cases} \quad (2.8)$$

then, on  $\Sigma$ , the system (2.4) is equivalent to

$$\begin{cases} \dot{w}_1 = -w_1 + w_2, \\ \dot{w}_2 = -(1+a)w_1 - bw_2, \\ w_3 = -aw_1 - bw_2. \end{cases} \quad (2.9)$$

With the following Lyapunov function

$$V(w_1, w_2) = \begin{pmatrix} w_1 & w_2 \end{pmatrix} P \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad (2.10)$$

we can prove that  $w_1$ ,  $w_2$  and  $w_3$  converge to zero asymptotically, where  $P$  is given by

$$P = \begin{pmatrix} \frac{-\gamma+\beta^2-\beta+1}{2(\beta-1)(\gamma+\beta)} & \frac{1-\gamma\beta}{2(\beta-1)(\gamma+\beta)} \\ \frac{1-\gamma\beta}{2(\beta-1)(\gamma+\beta)} & \frac{\gamma^2-\gamma-\beta+1}{2(\beta-1)(\gamma+\beta)} \end{pmatrix}, \quad \beta = -b, \quad \gamma = -(1+a). \quad (2.11)$$

Note that, the disturbance  $d$  does not appear in (2.9). Then, the idea of SMC is to drive in finite-time the trajectories of system (2.1) on the surface  $\Sigma$  and to maintain the motion on  $\Sigma$  by means of the control. To achieve this goal, let us introduce a new variable  $\sigma$  given by

$$\sigma = \left\langle \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \varphi \right\rangle_{\mathbb{R}^3}. \quad (2.12)$$

Thus, if we drive  $\sigma$  to zero in finite-time by means of the control, then asymptotic convergence of the state variable to zero, in the presence of the disturbance, can be achieved. Let us prove this. From (2.4), the  $\sigma$ -dynamics yields

$$\dot{\sigma} = \left\langle \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix}, \varphi \right\rangle_{\mathbb{R}^3} = \left\langle A_L \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + B(u_{SMC} + d), \varphi \right\rangle_{\mathbb{R}^3} = \left\langle A_L \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \varphi \right\rangle_{\mathbb{R}^3} + \langle B, \varphi \rangle_{\mathbb{R}^3} (u_{SMC} + d) \quad (2.13)$$

Since  $\varphi$  is an eigenvector of  $A_L^\top$  and  $\langle B, \varphi \rangle_{\mathbb{R}^3} = 1$ , then we obtain from (2.13)

$$\dot{\sigma} = \lambda\sigma + u_{SMC} + d. \quad (2.14)$$

Let us consider the following candidate Lyapunov function for the  $\sigma$ -dynamics

$$V(\sigma) = \frac{1}{2}\sigma^2. \quad (2.15)$$

Then, the derivative of  $V$  along of the trajectory of (2.14) yields

$$\dot{V}(\sigma) = \sigma \dot{\sigma} = \sigma (\lambda \sigma + u_{SMC} + d) = \lambda \sigma^2 + \sigma (u_{SMC} + d). \quad (2.16)$$

Since  $\lambda < 0$  then, we have

$$\dot{V}(\sigma) \leq \sigma u_{SMC} + |\sigma| |d|. \quad (2.17)$$

Therefore, if we assume that the disturbance  $d$  is bounded, i.e.,  $\|d\|_{L^\infty(\mathbb{R}_+)} \leq M$  with  $M$  a positive constant, then by selecting

$$u_{SMC} = -\rho \text{sign}(\sigma) \quad (2.18)$$

with  $\rho > M$ , we obtain

$$\dot{V}(\sigma) \leq -|\sigma|(\rho - M). \quad (2.19)$$

As a consequence,  $\sigma$  reaches zero in a finite-time  $t_r$  that is bounded by

$$t_r \leq \frac{|\sigma(0)|}{\rho - M}. \quad (2.20)$$

Finally, with the control (2.3)-(2.18), the trajectories of the system (2.1) reach the surface  $\Sigma$  in finite-time  $t_r$  and remain on it despite the presence of the bounded disturbance  $d$ . The control (2.3)-(2.18) is called sliding mode controller, the variable  $\sigma$  is called the sliding variable and the surface  $\Sigma$  is called the sliding surface.

The results of the simulation of the system (2.1) with the control (2.3)-(2.18), the control gain  $\rho = 1.2$  and the disturbance  $d(t) = \sin(t)$  are presented in Figures 2.3-2.5. Figure 2.3 shows the asymptotic convergence of the state variables to zero in the presence of the disturbance. In Figure 2.4, the control input  $u$  appears. Figure 2.5 shows that the sliding surface is reached in finite-time.

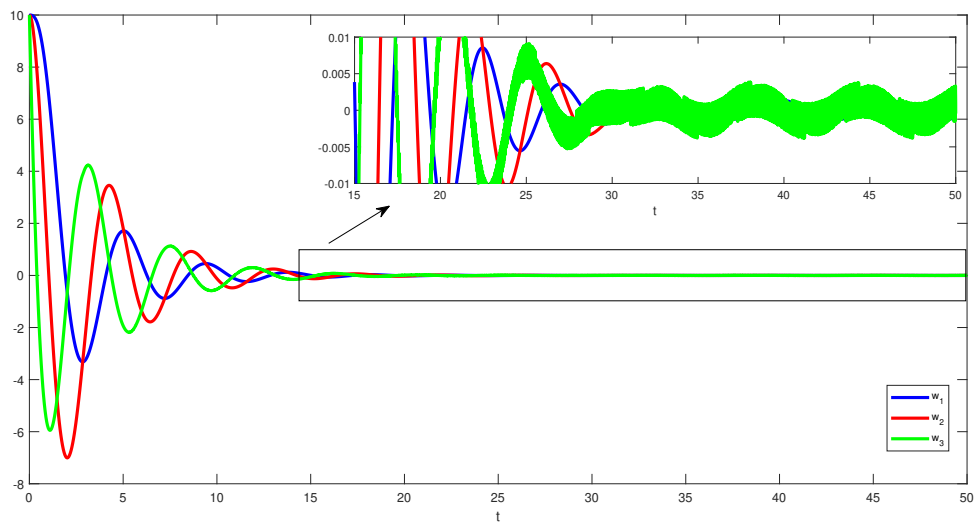


Figure 2.3: System (2.1) for  $d(t) = \sin(t)$ :  $w_1, w_2, w_3$  versus time, with the feedback law (2.3)-(2.18).

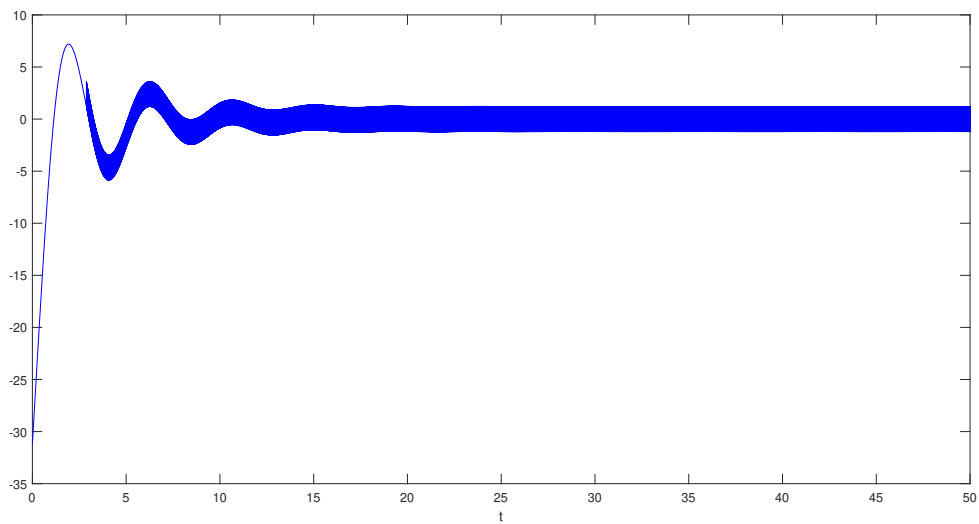


Figure 2.4: Control input (2.3)-(2.18) versus time.

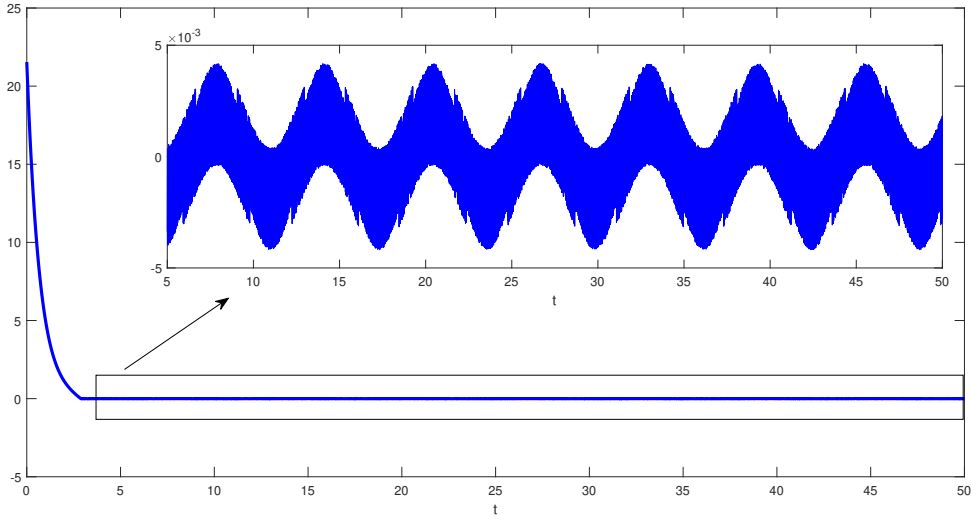


Figure 2.5: Sliding variable (2.12) versus time, with the feedback law (2.3)-(2.18).

As mentioned before, the SMC feedback-law is discontinuous, which creates chattering phenomenon when implementing the control numerically as we see in Figure 2.3 and Figure 2.4. We are going to propose in the next section a feedback-law which allows to attenuate the chattering phenomenon.

## 2.2.2 Super-twisting control

In the previous section, we saw that the control (2.3)-(2.18) is discontinuous, which created chattering phenomenon when implementing the control numerically. However, in an application, it is important to reduce this phenomenon by providing a continuous or a smooth controller, while keeping the robustness of the control with respect to the disturbance.

Let us start by assuming for instance that  $d = 0$  and consider the following continuous control

$$u = -w_1 - w_2 - w_3 - \lambda\sigma - \alpha|\sigma|^{\frac{1}{2}} \text{sign}(\sigma), \quad \alpha > 0. \quad (2.21)$$

Then, the sliding variable dynamics (2.13) becomes

$$\dot{\sigma} = -\alpha|\sigma|^{\frac{1}{2}} \text{sign}(\sigma). \quad (2.22)$$

Thus, the derivative of  $V$  defined in (2.15) along the trajectory of (2.22) yields

$$\dot{V}(\sigma) = -\alpha|\sigma|^{\frac{3}{2}} = -\alpha(\sqrt{2})^{\frac{3}{2}} V^{\frac{3}{4}}. \quad (2.23)$$

As a consequence,  $\sigma$  reaches zero in a finite-time  $t_r$  that is bounded by

$$t_r \leq 2 \frac{|\sigma(0)|^{\frac{1}{2}}}{\alpha}. \quad (2.24)$$

Now, let us consider the case where  $d \neq 0$ . Then, it is easy to see that the control (2.21) does not drive the sliding variable  $\sigma$  to zero. In order to compensate the effect of the disturbance, we add a term to the control in (2.21). Precisely, we consider the following feedback-law

$$\begin{cases} u = -w_1 - w_2 - w_3 - \lambda\sigma - \alpha|\sigma|^{\frac{1}{2}} \text{sign}(\sigma) + v, \\ \dot{v} = -\beta \text{sign}(\sigma) \end{cases} \quad (2.25)$$

with  $\beta > 0$ . Then, the dynamics of the sliding variable (2.13) becomes

$$\begin{cases} \dot{\sigma} = -\alpha|\sigma|^{\frac{1}{2}} \text{sign}(\sigma) + v + d \\ \dot{v} = -\beta \text{sign}(\sigma). \end{cases} \quad (2.26)$$

If we assume that  $d$  is globally Lipschitz over  $\mathbb{R}_+$  then, according to the following transformation

$$z = d + v, \quad (2.27)$$

the system (2.26) is rewritten as

$$\begin{cases} \dot{\sigma} = -\alpha|\sigma|^{\frac{1}{2}} \text{sign}(\sigma) + z \\ \dot{z} = -\beta \text{sign}(\sigma) + \dot{d}. \end{cases} \quad (2.28)$$

In addition, if we assume that the disturbance satisfy  $\|\dot{d}\|_{L^\infty(\mathbb{R}_+)} \leq C$  with  $C$  a positive constant, then by selecting

$$\beta > C \quad \text{and} \quad \alpha > \sqrt{\beta + C},$$

there exists a finite-time  $t_r > 0$  such that  $\sigma = 0$  and  $z = 0$  for any  $t > t_r$  according to [115, Theorem 1]. Thus, with the control (2.25), the trajectories of the system (2.1) reach the surface  $\Sigma$  in finite-time  $t_r$  and remain on it despite the presence of the disturbance  $d$ . The control defined in (2.25) is called super-twisting control. Note that, it is continuous since both  $-\alpha|\sigma|^{\frac{1}{2}} \text{sign}(\sigma)$  and  $v$  are continuous.

The results of the simulation of the system (2.1) with the control (2.25), the control gains  $\beta = 1.2$ ,  $\alpha = 2$  and the disturbance  $d(t) = \sin(t)$  ( $C = 1$ ) are presented in Figures 2.6-2.9. Figure 2.6 shows the asymptotic convergence of the state variable to zero in the presence of the disturbance. In Figure 2.7, the continuous control input  $u$  appears. Figure 2.8 shows that the sliding surface is reached in finite-time. Figure 2.9 shows that, after a reaching time, the term  $v$  of the controller is almost equal to the disturbance  $-d$ . This shows that, once the sliding mode is reached, the perturbation can be reconstructed from the term  $v$  appearing in the feedback control law (2.25). If we compare Figures 2.3 and 2.4 with Figures 2.6 and 2.7, respectively, we remark that the chattering phenomenon is greatly attenuated when using the super-twisting control.

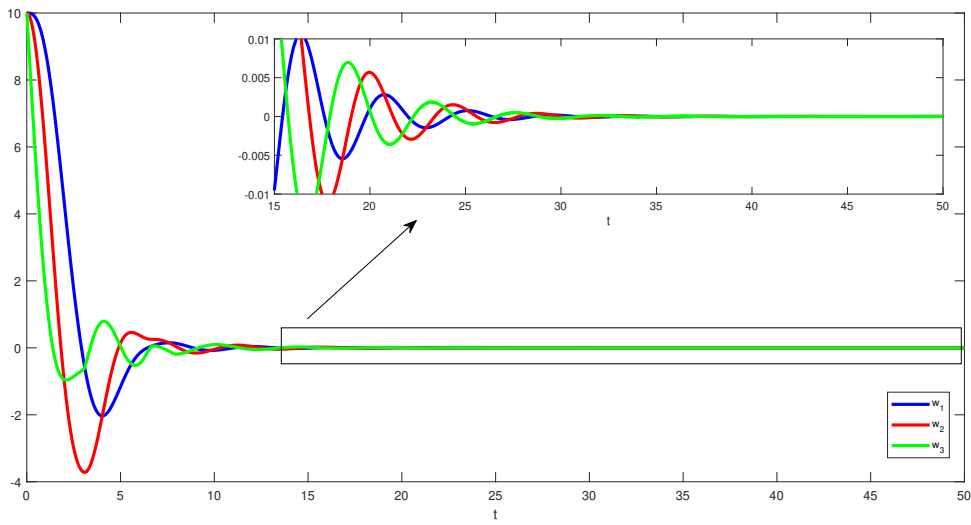


Figure 2.6: System (2.1) for  $d(t) = \sin(t)$ :  $w_1, w_2, w_3$  versus time, with the feedback law (2.25).

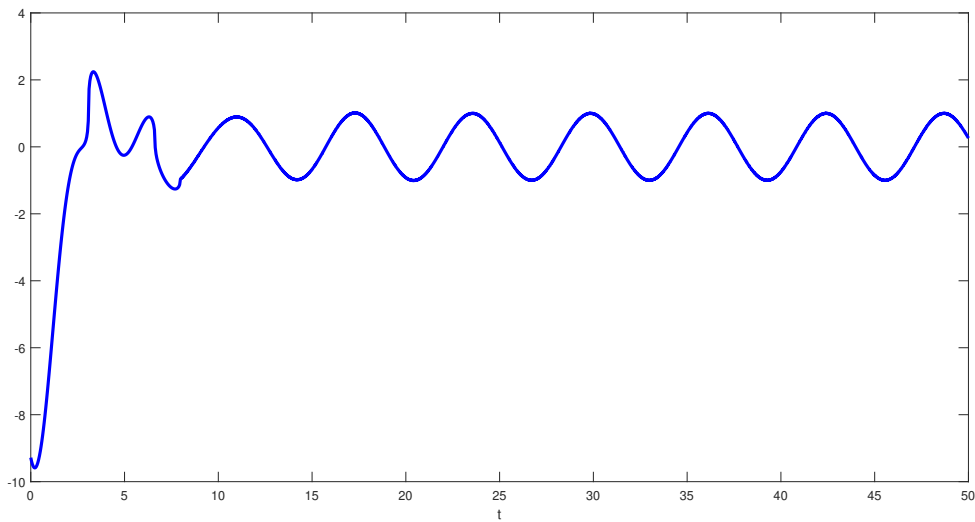


Figure 2.7: Control input (2.25) versus time.

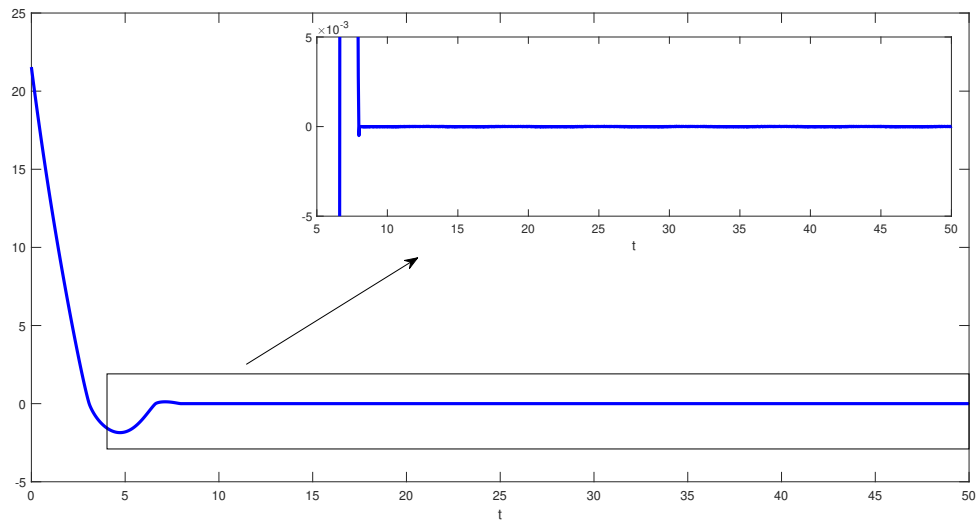


Figure 2.8: Sliding variable (2.12) versus time, with the feedback law (2.25).

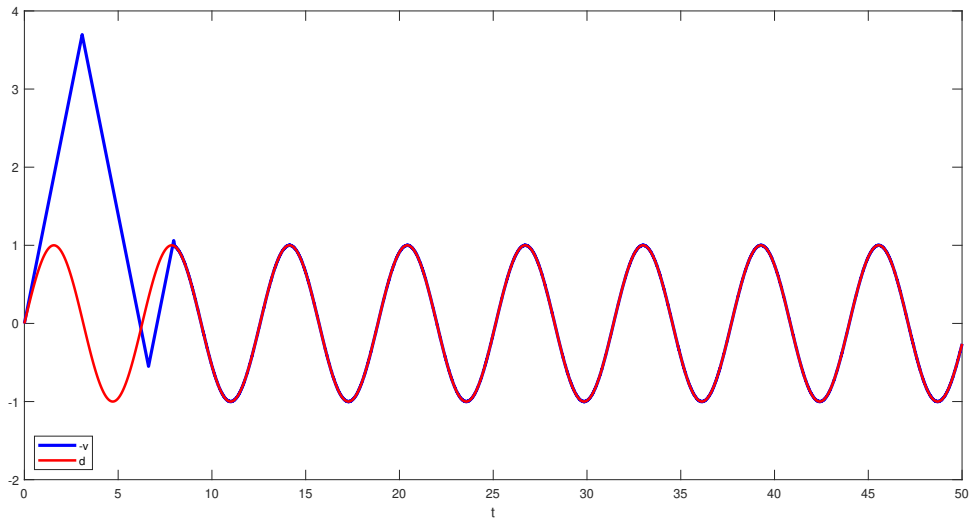


Figure 2.9: The term  $-v$  in (2.25) and the disturbance  $d(t) = \sin(t)$  versus time.

Note that the most important point of the strategy is the choice of the sliding variable. Once the suitable sliding variable is chosen, the remainder of the strategy can be easily applied. Note that sufficient conditions for this sliding variable design are as follows;



- the pair  $(A, B)$  is stabilizable i.e., there exists a matrix  $L$  such that the eigenvalues of  $A_L := A + BL$  belong to  $\mathbb{C}_-$ ;
- select an eigenvector  $\varphi$  of  $A_L^\top$  such that  $B^\top \varphi \neq 0$ .

Let us explain how one select such an eigenvector. For this, we distinguish the two possible cases.

**First case: the matrix  $L \neq 0$ .** In this case, there exists an eigenvalue  $\lambda$  of  $A_L^\top$  such that  $\ker(A_L^\top - \lambda I_d) \not\subset B^\perp$ , where  $B^\perp$  is the set of orthogonal vectors to  $B$  and  $I_d$  is the identity matrix. Indeed, let us assume that,

$$\text{for all eigenvalue } \lambda \text{ of } A_L^\top \text{ we have } \ker(A_L^\top - \lambda I_d) \subset B^\perp. \quad (2.29)$$

Then, for all eigenvalue  $\lambda$  of  $A_L^\top$  and for all  $\varphi \in \ker(A_L^\top - \lambda I_d)$ , we have

$$\begin{cases} A_L^\top \varphi = \lambda \varphi \\ B^\top \varphi = 0. \end{cases} \quad (2.30)$$

Since  $A_L^\top = A^\top + L^\top B^\top$ , then, from (2.30), we obtain for all eigenvalue  $\lambda$  of  $A_L^\top$  and for all  $\varphi \in \ker(A_L^\top - \lambda I_d)$

$$A_L^\top \varphi = \lambda \varphi \Leftrightarrow A^\top \varphi = \lambda \varphi. \quad (2.31)$$

Thus,  $A_L^\top$  and  $A^\top$  have the same spectrum, which is impossible. As a result, the assumption in (2.29) is not correct.

**Second case: the matrix  $L = 0$ .** In this case, the matrix  $A$  is Hurwitz. Then, either there is an eigenvector  $\varphi$  of  $A_L^\top$  such that  $B^\top \varphi \neq 0$ , or there is no such eigenvector. If such a vector does not exists and if the pair  $(A, B)$  is stabilizable, then we find ourselves in the previous case.

As a consequence, in both cases, one can always select an eigenvector  $\varphi$  such that  $B^\top \varphi \neq 0$  and, to apply our strategy, it suffices therefore to assume the stabilizability of  $(A, B)$ . This relaxes the restriction of the controllability of the pair  $(A, B)$  if we compare our sliding surface design for instance with the method based on eigenvalue placement algorithms [116, Chapter 2].

## 2.3 SMC for linear infinite-dimensional systems

All along this section,  $H$  is a Hilbert space on  $\mathbb{K}$  ( $\mathbb{K}$  is either  $\mathbb{R}$  or  $\mathbb{C}$ ) equipped with a scalar product  $\langle \cdot, \cdot \rangle_H$  and a norm  $\| \cdot \|_H$ .

### 2.3.1 Problem Statement

In this section we are interested in the stabilization (at the origin) problem for the system

$$\begin{cases} \frac{d}{dt}w = Aw + B(u + d), \\ w(0) = w_0, \end{cases} \quad (2.32)$$

where  $w(t) \in H$  is the state,  $u(t) \in \mathbb{K}$  is the control input and  $d(t) \in \mathbb{K}$  is an unknown disturbance. In system (2.32),  $A : D(A) \subseteq H \rightarrow H$  is a linear operator with  $D(A)$  densely defined in  $H$  and  $B \in \mathcal{L}(\mathbb{K}, D(A^*)')$ , with  $A^*$  the adjoint operator of  $A$ . Our objective is to provide a design method so that system (2.32) is globally stabilized despite the disturbance  $d$ . To do so, we will follow the sliding mode strategy.

This strategy can be applied thanks to the following set of assumptions.

**Assumption 1** *The following statements hold.*

- (i) *The operator  $A : D(A) \subseteq H \rightarrow H$  generates a strongly continuous semigroup, that is denoted by  $(\mathbb{T}(t))_{t \geq 0}$ .*
- (ii) *The operator  $B$  is admissible<sup>a</sup> for  $(\mathbb{T}(t))_{t \geq 0}$ .*
- (iii) *There exists an operator  $L : D(L) \rightarrow \mathbb{K}$  such that the operator*

$$\begin{cases} A_L = A + BL, \\ D(A_L) = \{w \in D(L); (A + BL)w \in H\}, \end{cases} \quad (2.33)$$

*is the infinitesimal generator of a strongly continuous semigroup  $(\mathbb{S}(t))_{t \geq 0}$  on  $H$  and the origin of the following system*

$$\begin{cases} \frac{d}{dt}w = (A + BL)w, \\ w(0) = w_0, \end{cases} \quad (2.34)$$

*is globally asymptotically stable.*

<sup>a</sup>See e.g [125, Section 4.2]

Items (i) and (ii) allow to state the well-posedness of system (2.32) in  $H$  with  $u \in L^2_{loc}([0, +\infty), \mathbb{K})$ . Finally, Item (iii) of Assumption 1 refers to a stabilizability property of system (2.32), needed to ensure that, without disturbance, the system can be stabilized.

The disturbance  $d$  is not supposed to be known entirely, but we assume the knowledge of its bound.

**Assumption 2** *The unknown disturbance  $d$  is supposed to be uniformly bounded measurable, i.e.,  $|d(t)| \leq K_d$  for some  $K_d > 0$  and for a.e.  $t \geq 0$ .*

**Remark 1** *Note that Item (iii) of Assumption 1 has been proven in [127, Theorem 2.1] in the case where*

the pair  $(A, B)$  is exactly controllable in time  $T$ .

Our goal is to find a state feedback control  $u$  which allows to reject the disturbance and to globally asymptotically stabilize the system (2.32) around 0. Precisely, we are looking for a sliding surface on which the system (2.32) has the same behavior than the system (2.34) in a finite-time. According to Item (iii) of Assumption 1, we know that 0 is globally asymptotically stable for the system (2.34). The next section will provide a definition of this sliding surface (and its related sliding variable), the associated sliding mode controllers and the associated super-twisting controllers.

### 2.3.2 Sliding surface

Let  $\varphi \in D(A_L^*) := \{\varphi \in H \mid \exists c > 0, \forall \phi \in D(A_L), |\langle \varphi, A_L(\phi) \rangle_H| \leq c \|\phi\|_H\}$  be an eigenfunction of the adjoint operator of  $A_L$  such that  $B^*\varphi \neq 0$  and  $\lambda$  the eigenvalue associated with  $\varphi$ . We introduce the following sliding surface  $\Sigma$

$$\Sigma := \{w \in H \mid \langle \varphi, w \rangle_H = 0\}.$$

Its related sliding variable  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{K}$  is defined by

$$\sigma(t) := \langle \varphi, w(t) \rangle_H \quad (2.35)$$

for any solution  $w$  of (2.32). This sliding variable represents the scalar product between the state and an eigenfunction of  $A_L^*$ .

This section is devoted to the design of a sliding mode controller and a super-twisting controller. In the following section, we begin with the design of the sliding mode control.

#### Sliding mode control

Since  $B^*\varphi \neq 0$  then, we consider the sliding mode controller  $u$  defined by, for a.e.  $t \geq 0$ ,

$$u(t) = Lw(t) - \frac{1}{B^*\varphi} \left( \lambda \sigma(t) + K \operatorname{sign}(\sigma(t)) \right), \quad (2.36)$$

where  $\sigma$  is given in (2.35),  $K$  is a positive constant that will be chosen later. Moreover, the set-valued function  $\operatorname{sign}$  is defined by

$$\operatorname{sign}(s) = \begin{cases} \frac{s}{|s|} & \text{if } s \neq 0, \\ \{z \in \mathbb{K} \mid |z|_{\mathbb{K}} \leq 1\} & \text{if } s = 0. \end{cases}$$

Note that, since  $B^* \in \mathcal{L}(D(A^*), \mathbb{K})$ , then  $B^*\varphi \in \mathbb{K}$ . Thus, we make the following assumption about the constant  $K$ .

**Assumption 3** The constant  $K$  is chosen such that  $\frac{K}{|B^*\varphi|} > K_d$ .

Then, the closed-loop system (2.32)-(2.36) can be written as

$$\begin{cases} \frac{d}{dt}w \in A_L w + B \left( d - \frac{1}{B^* \varphi} \left( \lambda \sigma(t) + K \operatorname{sign}(\sigma(t)) \right) \right), \\ w(0) = w_0. \end{cases} \quad (2.37)$$

The notion of solution for this system is defined in Definition 2.

Formally, the derivative of  $\sigma$  along the trajectory of (2.37) yields, for a.e.  $t \geq 0$

$$\begin{aligned} \dot{\sigma}(t) &= \langle \varphi, \frac{d}{dt}w(t) \rangle_H \\ &= \langle \varphi, A_L w(t) \rangle_H + B^* \varphi \left( d(t) - \frac{1}{B^* \varphi} \left( \lambda \sigma(t) + K \operatorname{sign}(\sigma(t)) \right) \right) \\ &= \langle A_L^* \varphi, w(t) \rangle_H + B^* \varphi \left( d(t) - \frac{1}{B^* \varphi} \left( \lambda \sigma(t) + K \operatorname{sign}(\sigma(t)) \right) \right) \\ &= \lambda \langle \varphi, w(t) \rangle_H + B^* \varphi \left( d(t) - \frac{1}{B^* \varphi} \left( \lambda \sigma(t) + K \operatorname{sign}(\sigma(t)) \right) \right) \\ \dot{\sigma}(t) &\in B^* \varphi \left( d(t) - \frac{K}{B^* \varphi} \operatorname{sign}(\sigma(t)) \right) \end{aligned} \quad (2.38)$$

with  $A_L^* \varphi = \lambda \varphi$ . Then, the following holds, for a.e.  $t \geq 0$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\sigma(t)|^2 &= \Re \left( \bar{\sigma}(t) \dot{\sigma}(t) \right) \\ &= \Re \left( \bar{\sigma}(t) B^* \varphi \left( d(t) - \frac{K}{B^* \varphi} \operatorname{sign}(\sigma(t)) \right) \right) \\ &\leq -(K - |B^* \varphi| K_d) |\sigma(t)|. \end{aligned} \quad (2.39)$$

as long as  $\sigma(t) \neq 0$ . Therefore, separating variables and integrating inequality (2.39) over the time interval  $0 \leq s \leq t$ , we obtain

$$|\sigma(t)| \leq |\sigma(0)| - (K - |B^* \varphi| K_d) t. \quad (2.40)$$

Thus, there exists a finite-time  $t_r > 0$ , for which we know a bound that will be given later on, such that  $\sigma(t) = 0$  for any  $t > t_r$ . This means that the system (2.37) reaches the sliding surface  $\Sigma$  in finite-time  $t_r$  and remains on it. Since  $\sigma(t) = 0$  for any  $t > t_r$ , then  $\dot{\sigma}(t) = 0$  for any  $t > t_r$ . Thus, from (2.38), we have  $0 \in d(t) - \frac{K}{B^* \varphi} \operatorname{sign}(\sigma(t))$  for any  $t > t_r$ .

The next section focuses on the design of the super-twisting control.

### Super-twisting control

In this section, we make the following assumption about the disturbance.

**Assumption 4** The disturbance  $d(\cdot)$  is globally Lipschitz over  $\mathbb{R}_+$  and there exists a known positive constant  $C$  such that, for a.e.  $t \in \mathbb{R}_+$ ,

$$|\dot{d}(t)| \leq C. \quad (2.41)$$

We assume that  $\mathbb{K} = \mathbb{R}$ . We do not treat the complex case, since we are not aware whether there exist super-twisting controllers for system whose state is in  $\mathbb{C}$ . We consider the super-twisting controller  $u$  defined by, for all  $t \geq 0$ ,

$$\begin{cases} u(t) = Lw(t) + \frac{1}{B^*\varphi} \left( -\lambda\sigma(t) - \alpha|\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) + v(t) \right), \\ \dot{v}(t) \in -\beta \text{sign}(\sigma(t)), \end{cases} \quad (2.42)$$

where  $\sigma$  is given in (2.35),  $\alpha$  and  $\beta$  are positive constants which will be chosen later.

Formally, the derivative of  $\sigma$  along the trajectory of (2.32)-(2.42) yields, for all  $t \geq 0$

$$\begin{aligned} \dot{\sigma}(t) &= \langle \varphi, \frac{d}{dt}w(t) \rangle_H \\ &= \langle \varphi, A_L w(t) \rangle_H + B^*\varphi d(t) - \lambda\sigma(t) + v(t) - \alpha|\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) \\ &= \lambda\sigma(t) - \lambda\sigma(t) + B^*\varphi d(t) + v(t) - \alpha|\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) \\ &= -\alpha|\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) + v(t) + B^*\varphi d(t). \end{aligned} \quad (2.43)$$

Then, according to the following transformation

$$z(t) = B^*\varphi d(t) + v(t), \quad (2.44)$$

we obtain

$$\begin{cases} \dot{\sigma}(t) = -\alpha|\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) + z(t), \\ \dot{z}(t) \in B^*\varphi \dot{d}(t) - \beta \text{sign}(\sigma(t)). \end{cases} \quad (2.45)$$

From [115, Theorem 1], all trajectories of (2.45) converge to zero in a finite time.

**Proposition 1** ([115, Theorem 1]) Assuming that

$$\beta > |B^*\varphi|C \quad \text{and} \quad \alpha > \sqrt{\beta + |B^*\varphi|C}, \quad (2.46)$$

there exists a finite-time  $t_r > 0$  such that  $\sigma(t) = 0$  and  $z(t) = 0$  for any  $t > t_r$ .

Then, the closed-loop system (2.32)-(2.42) can be written as

$$\begin{cases} \frac{d}{dt}w = A_L w + B \left( \frac{1}{B^*\varphi} \left( -\lambda\sigma(t) - \alpha|\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) + z(t) \right) \right), \\ \dot{z}(t) \in B^*\varphi \dot{d}(t) - \beta \text{sign}(\sigma(t)), \\ w(0) = w_0 \in H, z(0) = z_0 \in \mathbb{R}. \end{cases} \quad (2.47)$$

### Main results

The equations (2.38) and (2.45) are understood in the sense of Filippov [48, Chapter 2]. We recall the definition.

**Definition 1** A Filippov solution of (2.38) (resp., of (2.45)) is an absolutely continuous map that satisfies (2.38) (resp. (2.45)) for almost all  $t \geq 0$ .

The solutions of (2.37) are understood in the sense of the following definition.

**Definition 2** Let  $w_0 \in H$ . We say that the map  $w : [0, \infty) \rightarrow H$  is a mild solution of (2.37), if  $w \in C([0, \infty); H) \cap \mathcal{H}_{loc}^1([0, \infty); D(A^*)')$  such that, for all  $t \in [0, \infty)$ ,

$$w(t) = \mathbb{S}(t)w_0 + \int_0^t \mathbb{S}(t-s)Bh(s)ds, \quad (2.48)$$

where  $A^*$  is the adjoint operator of  $A$ ,  $(\mathbb{S}(t))_{t \geq 0}$  is the strongly continuous semigroup generated by the operator  $A_L$  and  $h : [0, \infty) \rightarrow \mathbb{K}$  is in  $L_{loc}^2([0, \infty); \mathbb{K})$  and satisfies, for a.e.  $t \geq 0$ ,

$$h(t) \in -\frac{1}{B^*\varphi} \left( \lambda\sigma(t) + K \operatorname{sign}(\sigma(t)) \right) + d(t) \quad (2.49)$$

with  $\sigma$  given in (2.35).

The following definition indicates how the solutions of (2.47) are understood.

**Definition 3** Let  $w_0 \in H$  and  $z_0 \in \mathbb{R}$ . We say that the map  $w : [0, \infty) \rightarrow H$  and  $z : [0, \infty) \rightarrow \mathbb{R}$  is a mild solution of (2.47), if  $w \in C([0, \infty); H) \cap \mathcal{H}_{loc}^1([0, \infty); D(A^*)')$  and  $z$  is absolutely continuous such that,

$$\text{for all } t \in [0, \infty), \quad w(t) = \mathbb{S}(t)w_0 + \int_0^t \mathbb{S}(t-s)B\omega(s)ds \quad (2.50)$$

and

$$\text{for a.e. } t \in [0, \infty), \quad \dot{z}(t) \in B^*\varphi\dot{d}(t) - \beta \operatorname{sign}(\sigma(t)), \quad (2.51)$$

where  $A^*$  is the adjoint operator of  $A$ ,  $(\mathbb{S}(t))_{t \geq 0}$  is the strongly continuous semigroup generated by the operator  $A_L$  and

$$\omega(t) = \frac{1}{B^*\varphi} \left( -\lambda\sigma(t) - \alpha|\sigma(t)|^{\frac{1}{2}} \operatorname{sign}(\sigma(t)) + z(t) \right) \quad (2.52)$$

with  $\sigma$  given in (2.35).

Note that Definition 2 and Definition 3 are based on the concept of mild solution<sup>1</sup>.

Before presenting the results of this chapter, we present the following definition of the equilibrium point of systems (2.37) and (2.47).

<sup>1</sup>See e.g [125, Definition 4.1.5]

**Definition 4** 1. We say that  $\tilde{w} \in H$  is an equilibrium point of system (2.37), if  $\tilde{w} \in D(A_L)$  and there exists  $\tilde{w}^* \in [-K_d, K_d] - \frac{K}{B^* \varphi} (\lambda \langle \varphi, \tilde{w} \rangle_H + \text{sign}(\langle \varphi, \tilde{w} \rangle_H))$  such that

$$A_L \tilde{w} + B \tilde{w}^* = 0. \quad (2.53)$$

2. We say that  $(\tilde{w}, \tilde{z}) \in H \times \mathbb{R}$  is an equilibrium point of system (2.47), if  $(\tilde{w}, \tilde{z}) \in D(A_L) \times \mathbb{R}$  such that

$$A_L \tilde{w} + B \left( \frac{1}{B^* \varphi} \left( -\lambda \langle \varphi, \tilde{w} \rangle_H + \tilde{z} - \alpha |\langle \varphi, \tilde{w} \rangle_H|^{\frac{1}{2}} \text{sign}(\langle \varphi, \tilde{w} \rangle_H) \right) \right) = 0$$

and

$$0 \in B^* \varphi [-C, C] - \beta \text{sign}(\langle \varphi, \tilde{w} \rangle_H). \quad (2.54)$$

**Remark 2** One can check that  $0 \in H$  (resp.,  $(0, 0) \in H \times \mathbb{R}$ ) is an equilibrium point of (2.37) (resp., (2.47)).

The main results of this chapter can be formulated as follows:

**Theorem 1 (Existence of solutions)** 1. Assume that Assumption 1, Assumption 2 and Assumption 3 are satisfied. Therefore, for any initial condition  $w_0 \in H$ , the closed-loop system (2.37) admits a mild solution.

2. Assume that Assumption 1, Assumption 4 and Equation (2.46) are satisfied. Therefore, for any initial condition  $w_0 \in H$  and  $z_0 \in \mathbb{R}$ , the closed-loop system (2.47) admits a mild solution.

The proof is given in the Appendix A. The next result of this chapter is stated as follows:

**Theorem 2 (Global asymptotic stability)** 1. Assume that Assumption 1, Assumption 2 and Assumption 3 are satisfied. For any initial condition  $w_0 \in H$ ,  $0 \in H$  is globally asymptotically stable for (2.37).

2. Assume that Assumption 1, Assumption 4 and Equation (2.46) are satisfied. For any initial condition  $(w_0, z_0) \in H \times \mathbb{R}$ ,  $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in H \times \mathbb{R}$  is globally asymptotically stable for (2.47).

The proof is given in the Appendix A.

## 2.4 Illustrative examples

### 2.4.1 Heat equation

Consider the following system

$$\begin{cases} \partial_t w(t, x) = \partial_{xx} w(t, x), & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ \partial_x w(t, 0) = c_0 w(t, 0), & t \in \mathbb{R}_+, \\ \partial_x w(t, 1) = u(t) + d(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), \end{cases} \quad (2.55)$$

where  $c_0$  is a positive constant,  $u(t) \in \mathbb{R}$  is the control input and  $d(t) \in \mathbb{R}$  is an unknown disturbance. This equation can be written in an abstract way as in (2.32) if one sets  $H = L^2(0, 1)$ ,

$$\begin{aligned} A : D(A) \subset L^2(0, 1) &\rightarrow L^2(0, 1), \\ w &\mapsto w'', \end{aligned} \quad (2.56)$$

where

$$D(A) := \{w \in \mathcal{H}^2(0, 1) \mid w'(0) = c_0 w(0); w'(1) = 0\}, \quad (2.57)$$

and the control operator  $B$  is the delta function in  $\mathcal{L}(\mathbb{R}, D(A)')$  defined as follows

$$\langle \varphi, Bu \rangle_{D(A), D(A)'} = \varphi(1)u \quad (2.58)$$

for all  $u \in \mathbb{R}$  and  $\varphi \in D(A)$ , where  $\langle \cdot, \cdot \rangle_{D(A), D(A)'}$  is a dual product. The adjoint operator of  $A$  is

$$\begin{aligned} A^* : D(A^*) \subset H &\rightarrow H, \\ w &\mapsto w'', \end{aligned} \quad (2.59)$$

with  $D(A^*) := \{w \in \mathcal{H}^2(0, 1) \mid w'(0) = c_0 w(0); w'(1) = 0\}$ . It can be checked that the operator  $A$  is self-adjoint in  $H$ . The adjoint of operator of  $B$  is

$$\begin{aligned} B^* : D(A^*) &\rightarrow \mathbb{R} \\ \varphi &\mapsto \varphi(1). \end{aligned} \quad (2.60)$$

According to [79, Lemma 2.1 and 2.2],  $A$  generates a strongly continuous semigroup  $(\mathbb{T}(t))_{t \geq 0}$  of contractions on  $H$  and the operator  $B$  is admissible for the semigroup  $(\mathbb{T}(t))_{t \geq 0}$ . Thus, the operators  $A$  and  $B$  satisfy the Items (i) and (ii) Assumption 1. Moreover, according to [79, Lemma 2.1], the origin of

$$\begin{cases} \partial_t w(t, x) = \partial_{xx} w(t, x), & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ \partial_x w(t, 0) = c_0 w(t, 0), & t \in \mathbb{R}_+, \\ \partial_x w(t, 1) = 0, & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), \end{cases} \quad (2.61)$$



is globally exponentially stable in  $H$ . As a consequence, Item (iii) of Assumption 1 holds for the operator  $L$  equal to the zero operator.

Since  $A$  is self-adjoint, then its spectrum is real. Therefore, a direct computation gives that the eigenpairs  $(\lambda, \varphi_\lambda)$  of  $A$  satisfies

$$\begin{cases} \varphi_\lambda(x) = \cos(\sqrt{-\lambda}x) + \frac{c_0}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda}x), \\ \sqrt{-\lambda} \tan(\sqrt{-\lambda}) = c_0. \end{cases} \quad (2.62)$$

The function  $x \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi; k \in \mathbb{Z}\} \mapsto \tan(x)$  is surjective. Thus, the equation  $\sqrt{-\lambda} \tan(\sqrt{-\lambda}) = c_0$  admits a solution. Note that  $\lambda$  is negative, since the origin of (2.61) is globally exponentially stable in  $H$ .

Let  $\varphi_\lambda \in D(A)$  the eigenfunction of the operator  $A$  associated to  $\lambda$ . The sliding variable and the feedback-law under consideration are as follows

$$\begin{aligned} \sigma(t) &= \int_0^1 w(t, x) \varphi_\lambda(x) dx \quad \text{and} \\ u(t) &= -\frac{1}{\varphi_\lambda(1)} (\lambda \sigma(t) + K \text{sign}(\sigma(t))). \end{aligned} \quad (2.63)$$

Thus, if we choose  $d$  and  $K$  as in Assumption 2 and 3, we can conclude that the origin of

$$\begin{cases} \partial_t w(t, x) = \partial_{xx} w(t, x), & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ \partial_x w(t, 0) = c_0 w(t, 0), & t \in \mathbb{R}_+, \\ \partial_x w(t, 1) \in -\frac{1}{\varphi_\lambda(1)} (\lambda \sigma(t) + K \text{sign}(\sigma(t))) + d(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), \end{cases} \quad (2.64)$$

is globally asymptotically stable in  $H$ . On the other hand, the super-twisting control under consideration is as follows

$$\begin{cases} u(t) = \frac{1}{\varphi_\lambda(1)} \left( -\lambda \sigma(t) - \alpha |\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) + v(t) \right), \\ \dot{v}(t) \in -\beta \text{sign}(\sigma(t)). \end{cases} \quad (2.65)$$

Therefore, if we choose  $d$  as in Assumption 4,  $\beta$  and  $\alpha$  as in (2.46) we can conclude that the origin of

$$\begin{cases} \partial_t w_t(t, x) = \partial_{xx} w(t, x), & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ \partial_x w(t, 0) = c_0 w(t, 0), & t \in \mathbb{R}_+, \\ \partial_x w(t, 1) = \frac{1}{\varphi_\lambda(1)} \left( -\lambda \sigma(t) - \alpha |\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) + v(t) \right) + d(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), \end{cases} \quad (2.66)$$

is globally asymptotically stable in  $H$ .

Using the finite difference method [77], we performed some numerical simulations. We choose  $\lambda = -2c_0 - \pi^2$  which is an approximated solution of  $\sqrt{-\lambda} \tan(\sqrt{-\lambda}) = c_0$ ,  $c_0 = 0.5$ ,  $K = 2.5$ ,  $w_0(x) = 10x^3$

and  $d(t) = 2 \sin(t)$ . The space and time steps are taken as 0.1 and 0.0001, respectively. In Figure 2.11 the control input  $u$  defined in (2.63) makes chattering phenomenon appearing once the sliding variable has converged (see Figure 2.12). In Figure 2.10, the stabilization of  $w$  of (2.64) is illustrated.

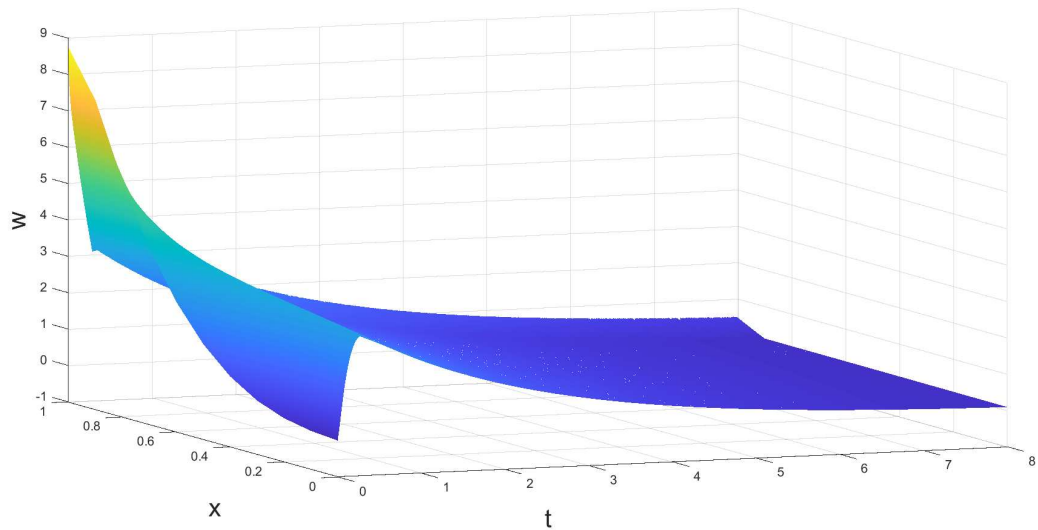


Figure 2.10: Sliding mode control (2.63): state of system (2.64) versus space and time.

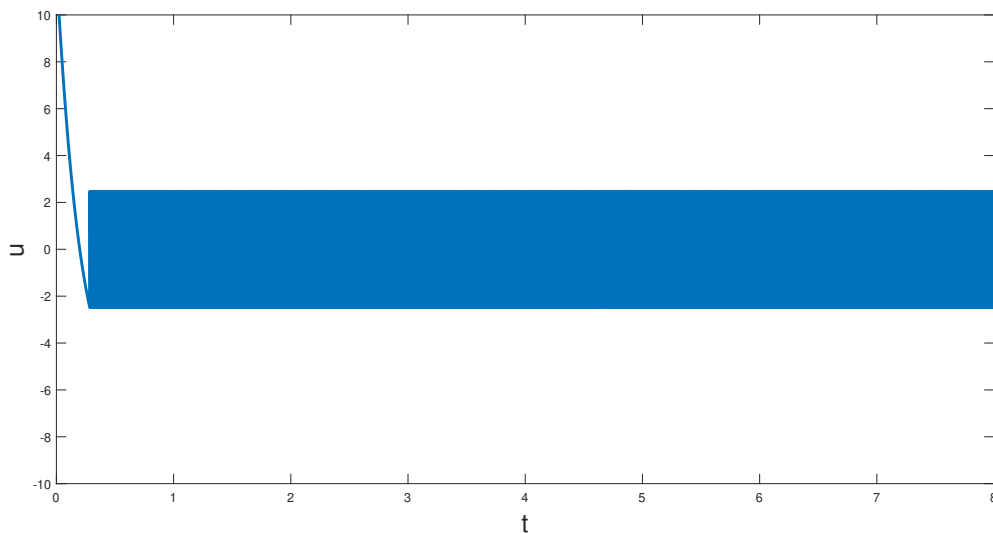


Figure 2.11: Sliding mode control (2.63): control input  $u$  versus time.

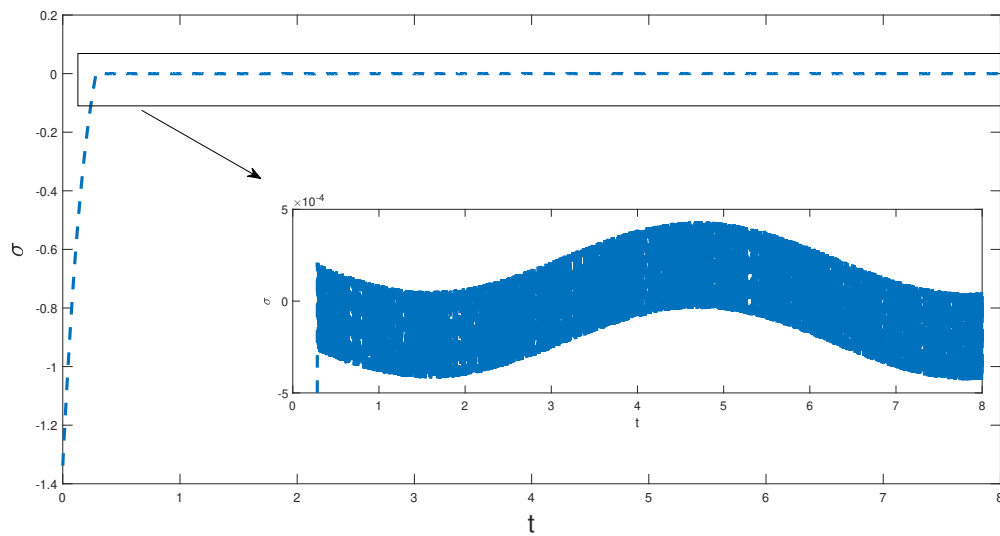


Figure 2.12: Sliding mode control (2.63): sliding variable  $\sigma$  versus time.

Figures 2.14, 2.15 and 2.13 are obtained with the same settings as the Figures 2.11, 2.12 and 2.10 with  $\beta = 2.5$  and  $\alpha = 2.2$  in the case of super-twisting control (2.65). It must be noted that, thanks to the use of super-twisting algorithm, the chattering on  $u$  has been removed (super-twisting is continuous) whereas the stabilization is kept.

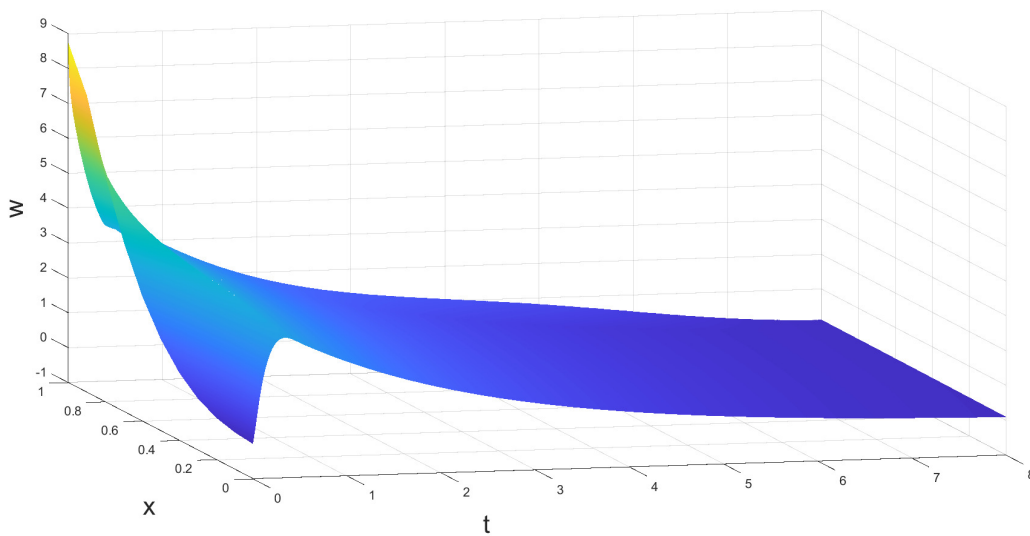
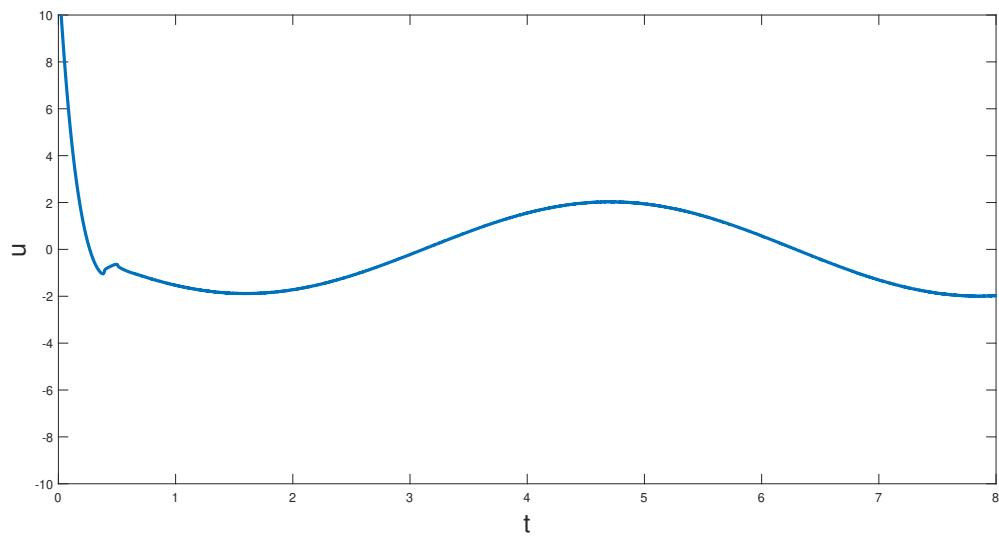
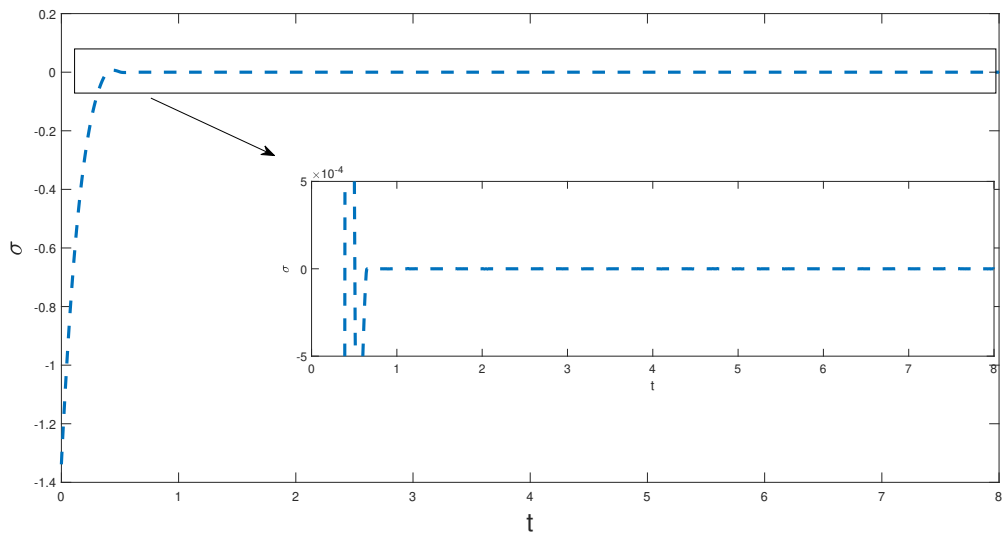


Figure 2.13: Super-twisting control (2.65): state of the system (2.66) versus space and time.

Figure 2.14: Super-twisting control (2.65): control input  $u$  versus time.Figure 2.15: Super-twisting control (2.65): sliding variable  $\sigma$  versus time.

## 2.4.2 Linear hyperbolic system

Consider the following system

$$\begin{cases} \partial_t w_1(t, x) + \lambda_1 \partial_x w_1(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ \partial_t w_2(t, x) - \lambda_2 \partial_x w_2(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ w_1(t, 0) = u(t) + d(t), \\ w_2(t, 1) = k_2 w_1(t, 1), \\ w_1(0, x) = w_1^0(x), w_2(0, x) = w_2^0(x), \end{cases} \quad (2.67)$$

where  $\lambda_1, \lambda_2$  are positive constants,  $k_2 \in \mathbb{R} \setminus \{0\}$ ,  $u(t) \in \mathbb{R}$  is the control input and  $d(t) \in \mathbb{R}$  is an unknown disturbance. This system has been studied in [78] and the results were obtained in  $L^p(0, 1)$  with  $p \in [1, \infty]$ . Here, we present the results in  $L^2(0, 1)$ .

The system (2.67) can be written in an abstract way as in (2.32) if one sets  $H = L^2(0, 1; \mathbb{R}^2)$ ,

$$\begin{aligned} A : D(A) \subset L^2(0, 1; \mathbb{R}^2) &\rightarrow L^2(0, 1; \mathbb{R}^2), \\ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &\mapsto \begin{pmatrix} -\lambda_1 w_1' \\ \lambda_2 w_2' \end{pmatrix}, \end{aligned} \quad (2.68)$$

where

$$D(A) := \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{H}^1(0, 1; \mathbb{R}^2) \mid w_1(0) = 0, w_2(1) = k_2 w_1(1) \right\}. \quad (2.69)$$

Since  $\lambda_1$  and  $\lambda_2$  are positive constants, define a scalar product on  $H$  as follows: for all  $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in H$ ,

$$\left\langle \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right\rangle := \int_0^1 \frac{1}{\lambda_1} w_1(x) \varphi_1(x) dx + \int_0^1 \frac{1}{\lambda_2} w_2(x) \varphi_2(x) dx. \quad (2.70)$$

Thus, the adjoint operator of  $A$  is

$$\begin{aligned} A^* : D(A^*) \subset L^2(0, 1; \mathbb{R}^2) &\rightarrow L^2(0, 1; \mathbb{R}^2), \\ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &\mapsto \begin{pmatrix} \lambda_1 w_1' \\ -\lambda_2 w_2' \end{pmatrix}, \end{aligned} \quad (2.71)$$

where

$$D(A^*) := \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{H}^1(0, 1; \mathbb{R}^2) \mid w_2(0) = 0, w_1(1) = k_2 w_2(1) \right\}. \quad (2.72)$$

Then, the control operator  $B \in \mathcal{L}(\mathbb{R}, D(A)')$  is defined as follow

$$\left\langle \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, Bu \right\rangle_{D(A), D(A)'} = \varphi_1(0)u \quad (2.73)$$

for all  $u \in \mathbb{R}$  and  $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in D(A)$ , where  $\langle \cdot, \cdot \rangle_{D(A), D(A)'}$  is a dual product. The adjoint of operator of  $B$  is

$$\begin{aligned} B^* : D(A^*) &\rightarrow \mathbb{R} \\ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &\mapsto \varphi_1(0). \end{aligned} \quad (2.74)$$

We can prove as in the proof of [11, Theorem A.1] that  $A$  generates a strongly continuous semigroup  $(\mathbb{T}(t))_{t \geq 0}$  of contractions on  $H$ . As in Appendix B, we can prove that the operator  $B$  is admissible for the semigroup  $(\mathbb{T}(t))_{t \geq 0}$ . Thus, the operators  $A$  and  $B$  satisfy the Items (i) and (ii) Assumption 1. Moreover, according [11, Theorem 2.4.], the origin of

$$\begin{cases} \partial_t w_1(t, x) + \lambda_1 \partial_x w_1(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ \partial_t w_2(t, x) - \lambda_2 \partial_x w_2(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ w_1(t, 0) = k_1 w_2(0), \\ w_2(t, 1) = k_2 w_1(t, 1), \\ w_1(0, x) = w_1^0(x), w_2(0, x) = w_2^0(x), \end{cases} \quad (2.75)$$

is globally exponentially stable in  $H$  with  $|k_1 k_2| < 1$ . As a consequence, Item (iii) of Assumption 1 holds with the operator  $L$  defined as

$$\begin{aligned} L : \mathcal{H}^1(0, 1; \mathbb{R}^2) \subset L^2(0, 1; \mathbb{R}) &\rightarrow \mathbb{R}, \\ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &\mapsto k_1 w_2(0), \end{aligned} \quad (2.76)$$

and the operator  $A_L$  define as

$$\begin{cases} A_L \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 w_1' \\ \lambda_2 w_2' \end{pmatrix}, \\ D(A_L) = \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{H}^1(0, 1; \mathbb{R}^2) \mid w_1(0) = k_1 w_2(0), w_2(1) = k_2 w_1(1) \right\}. \end{cases} \quad (2.77)$$

Thus, the adjoint operator of  $A_L$  is

$$\begin{aligned} A_L^* : D(A_L^*) \subset L^2(0, 1; \mathbb{R}^2) &\rightarrow L^2(0, 1; \mathbb{R}^2), \\ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &\mapsto \begin{pmatrix} \lambda_1 w_1' \\ -\lambda_2 w_2' \end{pmatrix}, \end{aligned} \quad (2.78)$$

with  $D(A_L^*) := \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{H}^1(0, 1; \mathbb{R}^2) \mid w_2(0) = k_1 w_1(0), w_1(1) = k_2 w_2(1) \right\}$ . Then, a direct computation

gives that the eigenpairs  $(\lambda, \varphi_{1\lambda}, \varphi_{2\lambda})$  of  $A_L^*$  satisfies

$$\begin{cases} \varphi_{1\lambda}(x) = c_0 e^{\frac{\lambda}{\lambda_1} x}, \\ \varphi_{2\lambda}(x) = k_1 c_0 e^{-\frac{\lambda}{\lambda_2} x}, \\ k_1 k_2 = e^{\lambda(\frac{1}{\lambda_1} + \frac{1}{\lambda_2})}, \\ \lambda < 0, c_0 \in \mathbb{R} \setminus \{0\}. \end{cases} \quad (2.79)$$

Thus, the sliding variable under consideration is as follows

$$\sigma(t) = \frac{1}{\lambda_1} \int_0^1 w_1(t, x) \varphi_{1\lambda}(x) dx + \frac{1}{\lambda_2} \int_0^1 w_2(t, x) \varphi_{2\lambda}(x) dx \quad (2.80)$$

and the sliding mode control under consideration is given by

$$u(t) = k_1 w_2(t, 0) - \frac{1}{\varphi_{1\lambda}(0)} (\lambda \sigma(t) + K \text{sign}(\sigma(t))). \quad (2.81)$$

Then, if we take  $d$  and  $K$  as in Assumptions 2 and 3, we can conclude that the origin of

$$\begin{cases} \partial_t w_1(t, x) + \lambda_1 \partial_x w_1(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ \partial_t w_2(t, x) - \lambda_2 \partial_x w_2(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ w_1(t, 0) = k_1 w_2(0) - \frac{1}{\varphi_{1\lambda}(0)} (\lambda \sigma(t) + K \text{sign}(\sigma(t))) + d(t), \\ w_2(t, 1) = k_2 w_1(t, 1), \\ w_1(0, x) = w_1^0(x), w_2(0, x) = w_2^0(x), \end{cases} \quad (2.82)$$

is globally asymptotically stable in  $H$ . On the other hand the super-twisting control under consideration is as follows

$$\begin{cases} u(t) = k_1 w_2(t, 0) + \frac{1}{\varphi_{1\lambda}(0)} \left( -\lambda \sigma(t) - \alpha |\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) + v(t) \right), \\ \dot{v}(t) \in -\beta \text{sign}(\sigma(t)). \end{cases} \quad (2.83)$$

Therefore, if we choose  $d$  as in Assumption 4,  $\beta$  and  $\alpha$  as in (2.46) we can conclude that the origin of

$$\begin{cases} \partial_t w_1(t, x) + \lambda_1 \partial_x w_1(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ \partial_t w_2(t, x) - \lambda_2 \partial_x w_2(t, x) = 0, & (t, x) \in \mathbb{R}_{\geq 0} \times (0, 1), \\ w_1(t, 0) = k_1 w_2(0) + \frac{1}{\varphi_{1\lambda}(0)} \left( -\lambda \sigma(t) - \alpha |\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) + v(t) \right) + d(t), \\ w_2(t, 1) = k_2 w_1(t, 1), \\ w_1(0, x) = w_1^0(x), w_2(0, x) = w_2^0(x), \end{cases} \quad (2.84)$$

is globally asymptotically stable in  $H$ .

Using the finite difference method [77], we performed some numerical simulations. We choose  $\lambda_1 =$

2,  $\lambda_2 = 1$ ,  $k_2 = 2$ ,  $c_0 = 1$ ,  $\lambda = -5$ ,  $d(t) = \sin(t)$ ,  $K = 1.2$  and

$$w_1^0(x) = -2\chi_{(0,0.5)}(x) + \chi_{(0.5,1)}(x),$$

$$w_2^0(x) = -3\chi_{(0,0.5)}(x) - \chi_{(0.5,1)}(x).$$

The space and time steps are taken as 0.005 and 0.0012, respectively. Figure 2.18 shows that the sliding surface is reached in finite-time. In Figure 2.19, the control input  $u$  defined in (2.81) appears. In Figures 2.16 and 2.17, the stabilization of the state of (2.82) is illustrated. Figures 2.18, 2.19 and 2.16 indicate that the chattering phenomenon occurs.

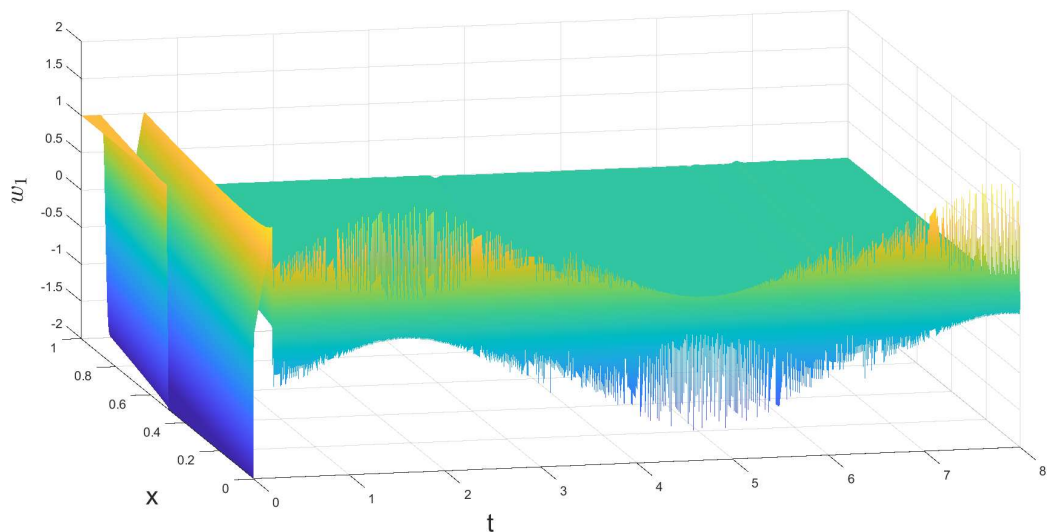


Figure 2.16: Sliding mode control (2.81): state variable  $w_1$  (2.82) versus space and time.



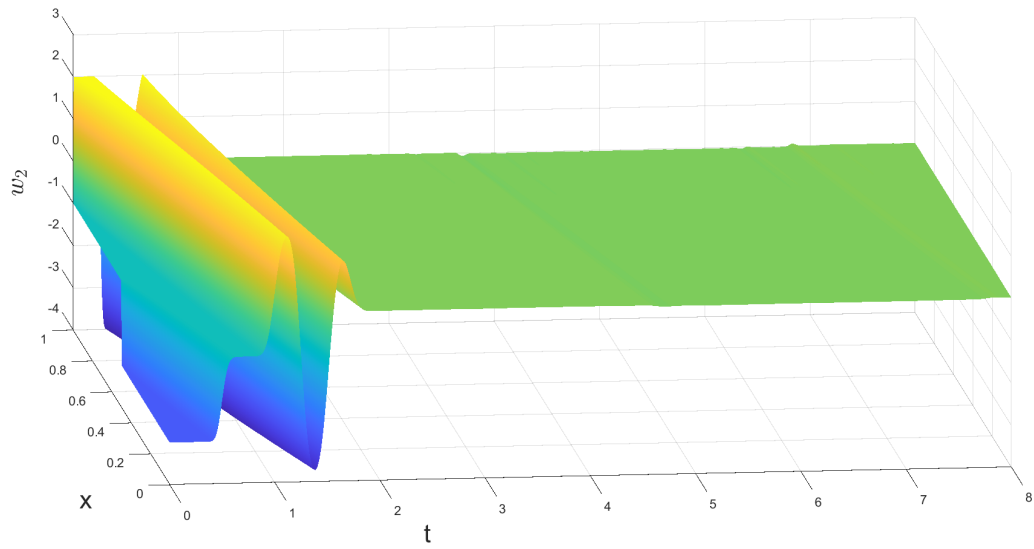


Figure 2.17: Sliding mode control (2.81): state variable  $w_2$  (2.82) versus space and time.

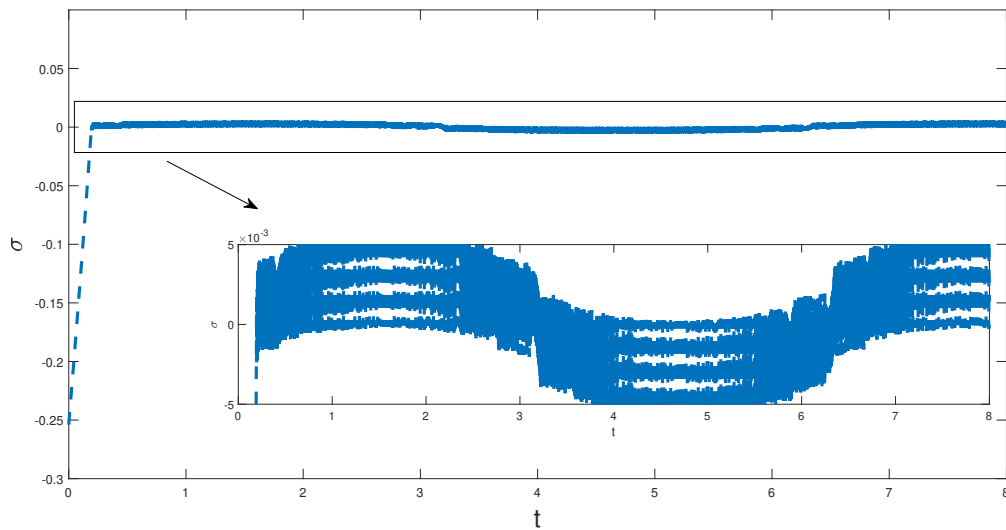


Figure 2.18: Sliding mode control (2.81): sliding variable  $\sigma$  (2.80) versus time.

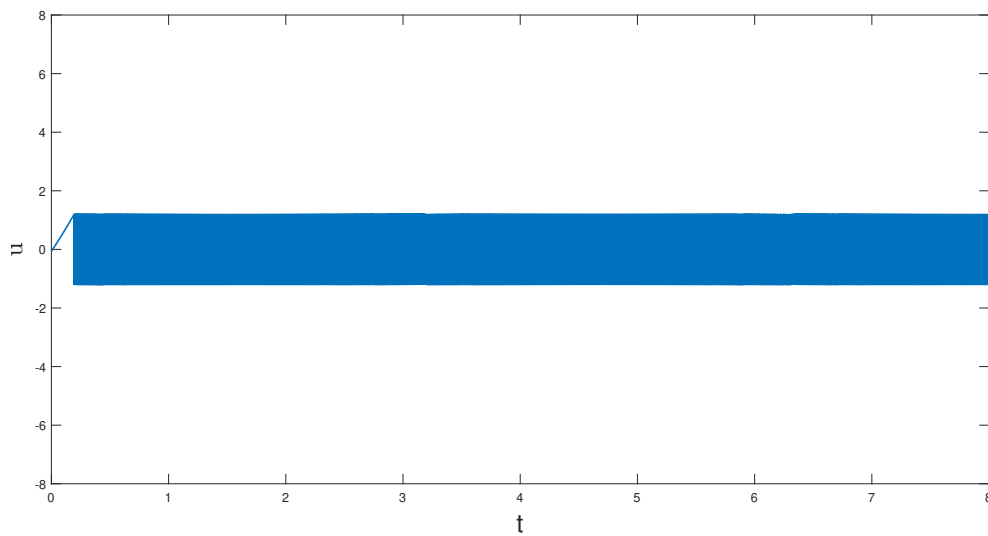


Figure 2.19: Sliding mode control (2.81): control input  $u$  versus time.

Figures 2.23, 2.22, 2.20 and 2.21 are obtained with the same settings as the Figures 2.18, 2.19, 2.16 and 2.17 with  $\beta = 1.5$  and  $\alpha = 2.1$  in the case of super-twisting control (2.83). It must be noted that, thanks to the use of super-twisting algorithm, the chattering phenomenon is greatly attenuated whereas the stabilization is kept.

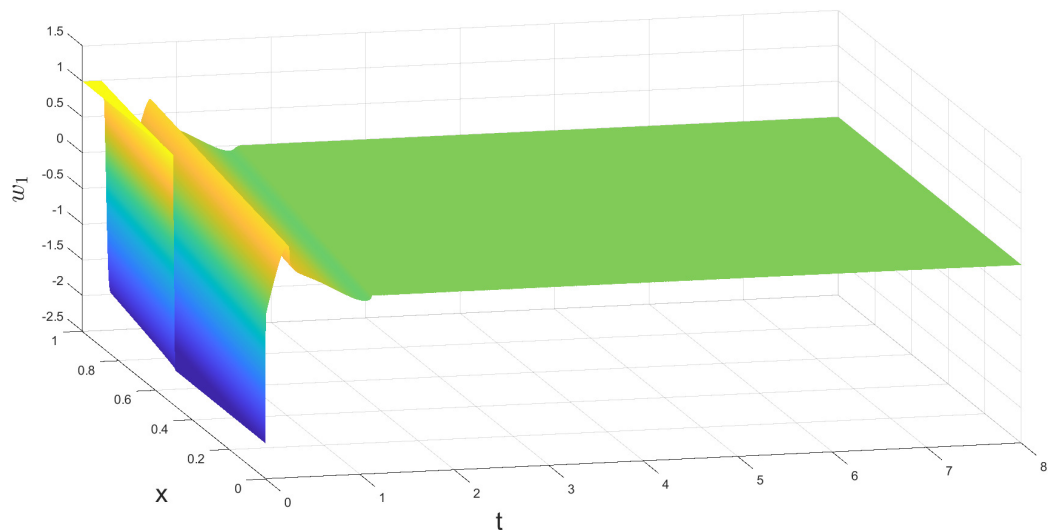


Figure 2.20: Super-twisting control (2.83): state variable  $w_1$  (2.84) versus space and time.

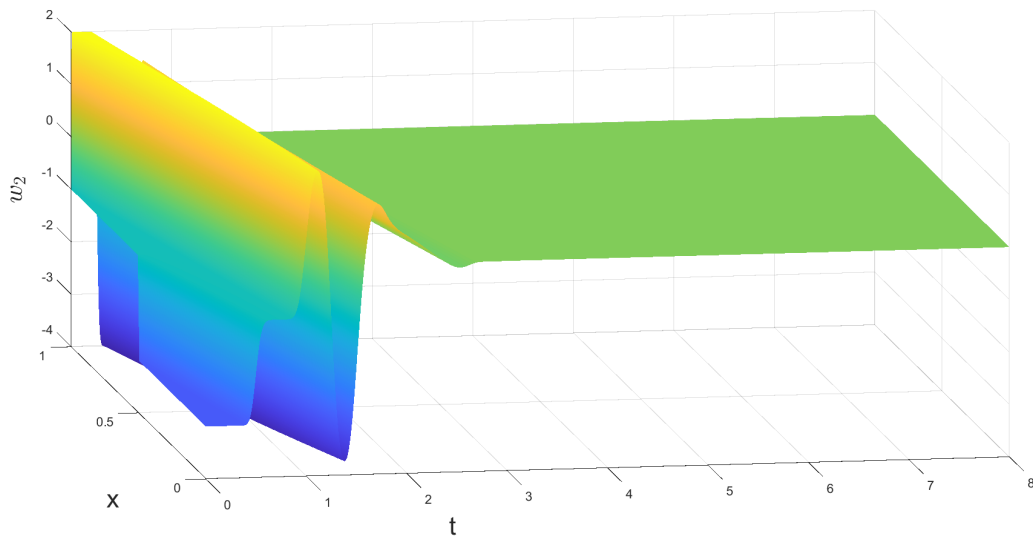


Figure 2.21: Super-twisting control (2.83): state variable  $w_2$  (2.84) versus space and time.

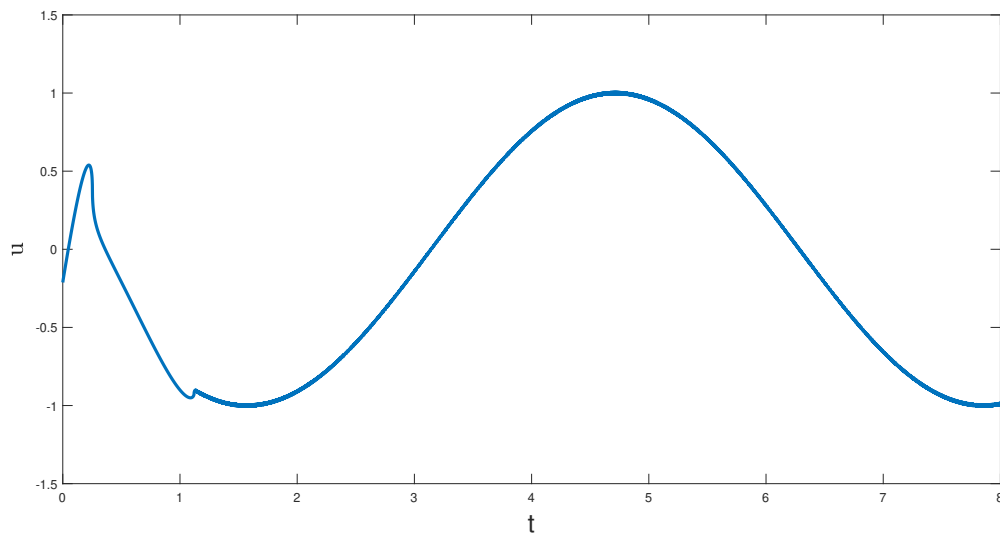


Figure 2.22: Super-twisting control (2.83): control input  $u$  versus time.

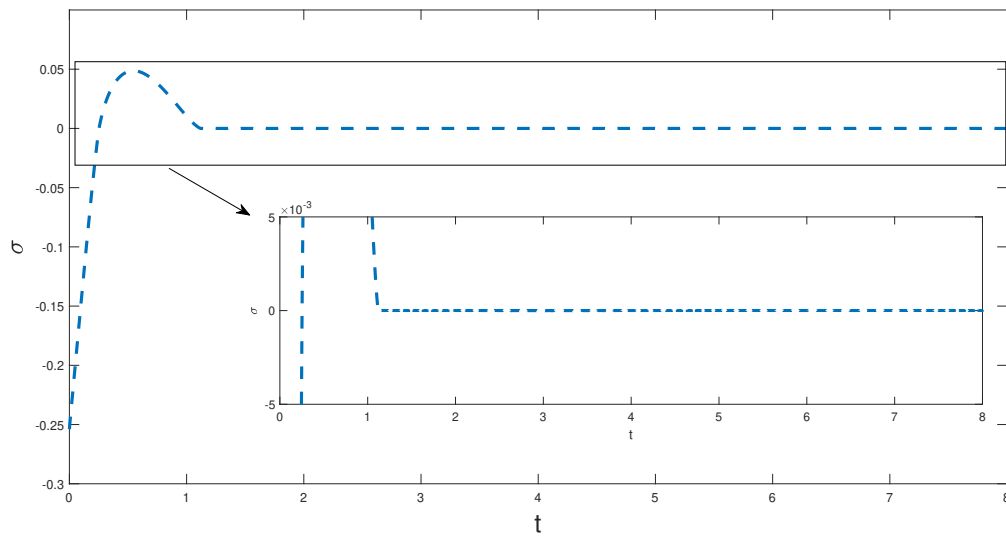


Figure 2.23: Super-twisting control (2.83): sliding variable  $\sigma$  (2.80) versus time.

## 2.5 Conclusion

In this chapter, we have proposed a design method based on sliding mode control for the stabilization of a class of linear infinite-dimensional systems with unbounded control operators and subject to a boundary disturbance. The sliding variable is defined as the scalar product of the state and an eigenfunction of the adjoint operator of the closed-loop system without disturbance. This requires measurement of the scalar product of the state with some function, which from a practical point of view is too restrictive. The existence of solutions of the closed-loop system has been proved as well as the disturbance rejection and the asymptotic stability of the closed-loop control system. We further have extended the super-twisting method for the same class of linear infinite-dimensional systems.



# ACTIVE DISTURBANCE REJECTION CONTROL FOR THE STABILIZATION OF A HYPERBOLIC SYSTEM

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## 3.1 Introduction

This chapter is concerned with the stabilization of a hyperbolic system with a boundary control and subject to a disturbance (see e.g. [11] for a review on this class of system). Normally, the sliding mode control proposed in the previous Chapter can solve this problem. These controls depend on the sliding variable, meaning that the latter is assumed to be measured. But, from a practical point of view, it may not be possible to measure the sliding variable proposed in the Chapter 2. Thus, in this chapter, we are going to follow another control design, namely the active disturbance rejection control strategy [47, 132].

Systems of transport equations have received much attention for many years due to the many physical phenomena that can be modelled by this way: e.g pressure drilling [70], water management systems [42], aeronomy [114] and cable vibration dynamics [129]. A good overview of the actual research lines concerning this topic is provided in [11] and [64].

Stabilization of this kind of systems where the disturbances appear is not a new topic. The reader can refer to [122, 28] in which PI controllers are proposed or [40] and [41] with controllers based on backstepping method and observer design. Note that, in these cases, the disturbances might not be located at the same position as the control. However, the dynamics of the disturbance is assumed to be known. The objective of the current chapter is to propose a control strategy for a larger class of uncertainties/disturbances as it was the case in Chapter 2 and in [78, 119, 10] with sliding mode control.

As the sliding mode control method, the active disturbance rejection control (ADRC) is a powerful method to deal with disturbances. It was initially proposed in [61] in the context of finite dimensional systems. The main idea of the ADRC is to build an observer to estimate the disturbance. Then, the disturbance is compensated in a feedback-law by its estimated value. Recently, this approach has been successfully applied to systems described by one-dimensional partial differential equations (PDEs) [79, 56, 140, 139, 19, 57, 59].

The contribution of this chapter is to apply the ADRC strategy to design a feedback-law which allows to reject the disturbance asymptotically and to ensure that the resulting closed-loop system is globally asymptotically stable. The proposed ADRC is based on an extended state observer [141, 134, 58] or a

robust observer [75, 51, 35, 34, 3] which allows to estimate the disturbance asymptotically. The extended state observer is a class of high-gain observer [67, 2, 8, 6] and provides estimation of the state and the disturbance. Moreover, the robust observer provides the estimation of the state despite the presence of the disturbance. To apply this strategy, we assume that we know some moments of the PDE's state, which is often possible in practice [4, 93]. In contrast with [56, 57], the dynamics of the output are not derived from the weak formulation of the PDE, which, in general, leads to a scalar ODE. We rather compute the dynamics of all the moments up to a certain degree  $n$ , which leads to a much more complicated system. We are, in general, closer to practical cases, since the output that is assumed to be known in [56, 57] does not correspond exactly to a moment.

This chapter is organized as follows. Section 3.2 provides an overview on the ADRC strategy through a finite-dimensional example. Section 3.3 presents a linear hyperbolic equation, the ADRC strategy and the main results of the chapter. Finally, Section 3.4 collects some remarks.

## 3.2 Basic idea of ADRC

The purpose of this section is to provide a brief overview of the concept of ADRC for the reader who is discovering it for the first time. It should be noted that the techniques presented in this section are not new. For example, they can be found in [47].

Let us consider the following system

$$\begin{cases} \dot{w}_1 = w_2, \\ \dot{w}_2 = w_3, \\ \dot{w}_3 = u + d, \\ y = w_1 \end{cases} \quad (3.1)$$

where  $u$  is the control,  $y$  is the output and  $d$  is an unknown disturbance. The goal is to design a feedback control  $u$  that drives the trajectories of (3.1) to the origin asymptotically, despite the presence of the unknown disturbance  $d$ . For example, when the system (3.1) is undisturbed i.e disturbance  $d = 0$ , the feedback-law

$$u = -6w_1 - 11w_2 - 6w_3 \quad (3.2)$$

allows to asymptotically stabilize the origin of (3.1) (see Figure 3.1).

Note that for any initial state  $(w_1(0) \ w_2(0) \ w_3(0)) \in \mathbb{R}^3$  and for any  $u, d \in L^\infty(\mathbb{R}_+)$ ,  $y(t) = 0$  for all  $t \geq 0$  implies that  $w_1(t) = w_2(t) = w_3(t) = 0$  for all  $t \geq 0$ . This means that the system (3.1) is strongly<sup>1</sup> observable. Thus, the main idea of the ADRC is to build an observer to estimate the disturbance  $d$  in order to cancel the effect of the disturbance thanks to its estimate. We have at least two possibilities for the estimation of the disturbance by an observer. The first way is to design a robust observer [116, Chapter 7] for the system (3.1) and deduce the estimation of  $d$  from the observation errors. The second way is to consider the disturbance  $d$  as an extended state variable of the system (3.1) and design an extended state observer [134] for the system (3.1). In addition, in the first case [116, Chapter 7], we need

<sup>1</sup>see e.g [63, Definition 1.2]

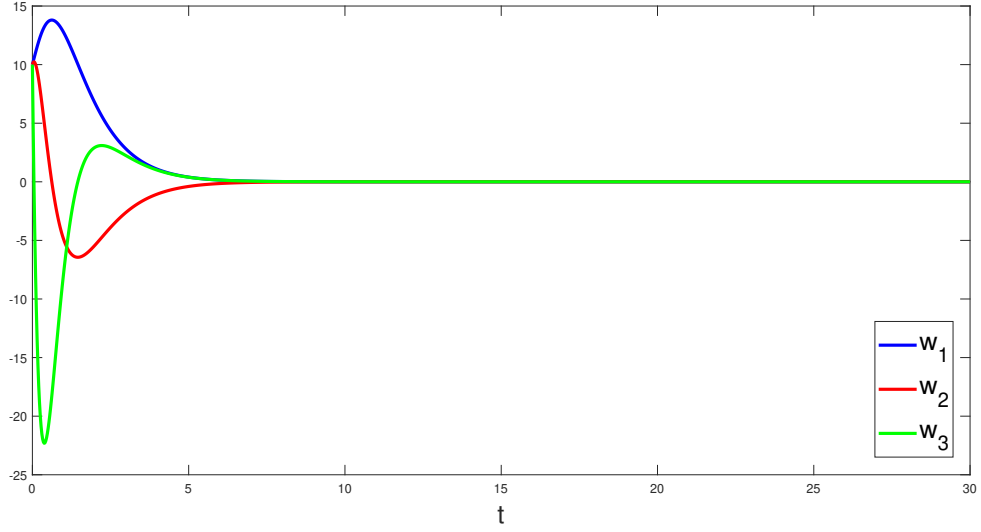


Figure 3.1:  $w_1$ ,  $w_2$  and  $w_3$  versus time, when  $d = 0$  and  $u$  defined by (3.2)

to assume that the disturbance  $d$  is bounded whereas in the second case [134], we need to assume that the derivative of the disturbance  $d$  is bounded, which means that the disturbance  $d$  is Lipschitz (in the weakest case).

Let us assume that the disturbance is bounded, i.e.  $|d(t)| \leq M$  with  $M$  a positive constant. Consider the following system

$$\begin{cases} \dot{\hat{w}}_1 = \hat{w}_2 - k_1 M^{\frac{1}{3}} [\hat{w}_1 - w_1]^{\frac{2}{3}}, \\ \dot{\hat{w}}_2 = \hat{w}_3 - k_2 M^{\frac{2}{3}} [\hat{w}_1 - w_1]^{\frac{1}{3}}, \\ \dot{\hat{w}}_3 \in u - k_3 M \operatorname{sign}(\hat{w}_1 - w_1), \end{cases} \quad (3.3)$$

where  $k_1$ ,  $k_2$  and  $k_3$  are positive constants. Notice that the system (3.3) is known as the Levant's differentiator [74]. Now, set  $e_i = \hat{w}_i - w_i$ ,  $i = 1, 2, 3$  the estimation errors. Then, we have

$$\begin{cases} \dot{e}_1 = e_2 - k_1 M^{\frac{1}{3}} [e_1]^{\frac{2}{3}}, \\ \dot{e}_2 = e_3 - k_2 M^{\frac{2}{3}} [e_1]^{\frac{1}{3}}, \\ \dot{e}_3 \in -k_3 M \operatorname{sign}(e_1) - d. \end{cases} \quad (3.4)$$

If we select

$$\begin{cases} k_1 = 3.34 k_3^{\frac{1}{3}}, \\ k_2 = 5.3 k_3^{\frac{2}{3}}, \\ k_3 > 0, \end{cases} \quad (3.5)$$

then according to [34, Theorem 1], the origin of (3.4) is finite-time stable. Thus, there exists  $t_r > 0$  such that for all  $t \geq t_r$ ,  $e_1(t) = e_2(t) = e_3(t) = 0$ . Therefore, according to the last line (3.4), we have for all



$t \geq t_r$

$$0 \in -k_3 M \operatorname{sign}(e_1(t)) - d(t). \quad (3.6)$$

Since  $\operatorname{sign}(e_1(t)) = [-1, 1]$  for all  $t \geq t_r$ , then for all  $t \geq t_r$  the set  $-k_3 M \operatorname{sign}(e_1(t))$  is closed, bounded, and upper semi-continuous. Thus, according to [49, Page 78] there exists a measurable function  $\hat{d}(t) \in -k_3 M \operatorname{sign}(e_1(t))$  such that

$$\forall t \geq t_r, d(t) = \hat{d}(t). \quad (3.7)$$

Therefore, for all  $t \geq t_r$ ,  $\hat{d}(t)$  is an estimate of  $d(t)$ . Finally, to stabilize system (3.1), we simply cancel the disturbance by using the feedback control:

$$u = -6w_1 - 11w_2 - 6w_3 - \hat{d} \quad (3.8)$$

The closed-loop of (3.1) under the feedback control (3.8) is given by

$$\begin{cases} \dot{w}_1 = w_2, \\ \dot{w}_2 = w_3, \\ \dot{w}_3 = -6w_1 - 11w_2 - 6w_3 - \hat{d} + d, \\ \hat{d} \in -k_3 M \operatorname{sign}(\hat{w}_1 - w_1) \\ \dot{\hat{w}}_1 = \hat{w}_2 - k_1 M^{\frac{1}{3}} \lceil \hat{w}_1 - w_1 \rceil^{\frac{2}{3}}, \\ \dot{\hat{w}}_2 = \hat{w}_3 - k_2 M^{\frac{2}{3}} \lceil \hat{w}_1 - w_1 \rceil^{\frac{1}{3}}, \\ \dot{\hat{w}}_3 = -6w_1 - 11w_2 - 6w_3, \end{cases} \quad (3.9)$$

Using the error variable  $e_i$ ,  $i = 1, 2, 3$ , we can rewrite the system (3.9) as:

$$\begin{cases} \dot{w}_1 = w_2, \\ \dot{w}_2 = w_3, \\ \dot{w}_3 = -6w_1 - 11w_2 - 6w_3 - \dot{e}_3, \\ \dot{e}_1 = e_2 - k_1 M^{\frac{1}{3}} \lceil e_1 \rceil^{\frac{2}{3}}, \\ \dot{e}_2 = e_3 - k_2 M^{\frac{2}{3}} \lceil e_1 \rceil^{\frac{1}{3}}, \\ \dot{e}_3 = \hat{d} - d \\ \hat{d} \in -k_3 M \operatorname{sign}(e_1). \end{cases} \quad (3.10)$$

Since for all  $t \geq t_r$ ,  $e_1(t) = e_2(t) = e_3(t) = 0$ , it immediately follows that  $(w_1(t), w_2(t), w_3(t)) \rightarrow 0$  as  $t \rightarrow \infty$ . The control in (3.8) is called active disturbance rejection control.

The results of the simulation of the system (3.1) with the control (3.8), the control gain  $\tau = 0.05$ ,  $k_3 = 1.1$ ,  $k_1$  and  $k_2$  being derived from (3.5) and the disturbance  $d(t) = \sin(t) + \frac{1}{2} \cos(t)$  are presented in Figures 3.2-3.3. The following low-pass filter

$$\tau \dot{\hat{d}} + \hat{d} = -k_3 M \operatorname{sign}_0(e_1) \quad (3.11)$$

is used to obtain  $\hat{d}$  in simulations. Figure 3.2 shows the asymptotic convergence of the state variables to zero in the presence of the disturbance. Figure 3.3 shows that  $\hat{d}$  estimates well  $d$ . The estimation of  $d$  is efficient once the observer has converged.

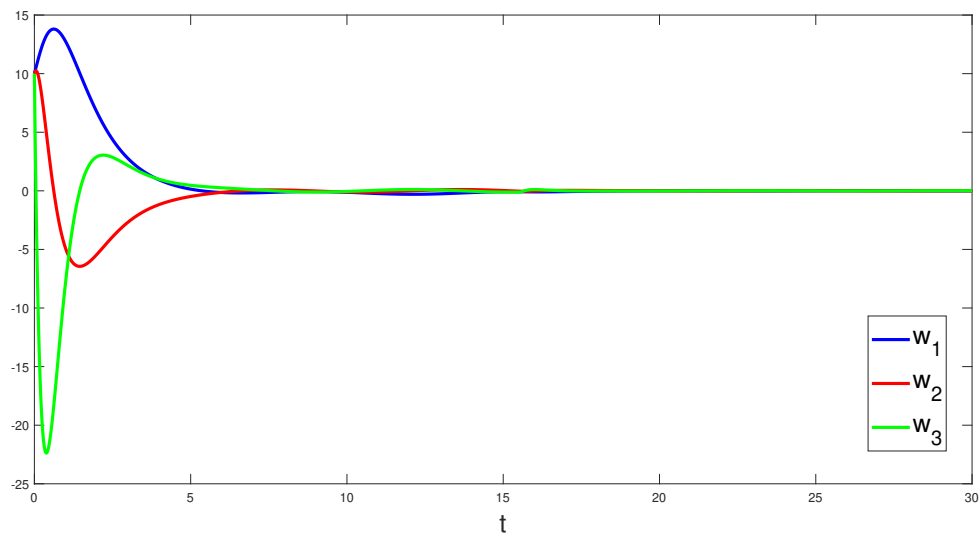


Figure 3.2:  $w_1$ ,  $w_2$  and  $w_3$  versus time, when  $d = \sin(t) + \frac{1}{2} \cos(t)$  and  $u$  defined by (3.8).

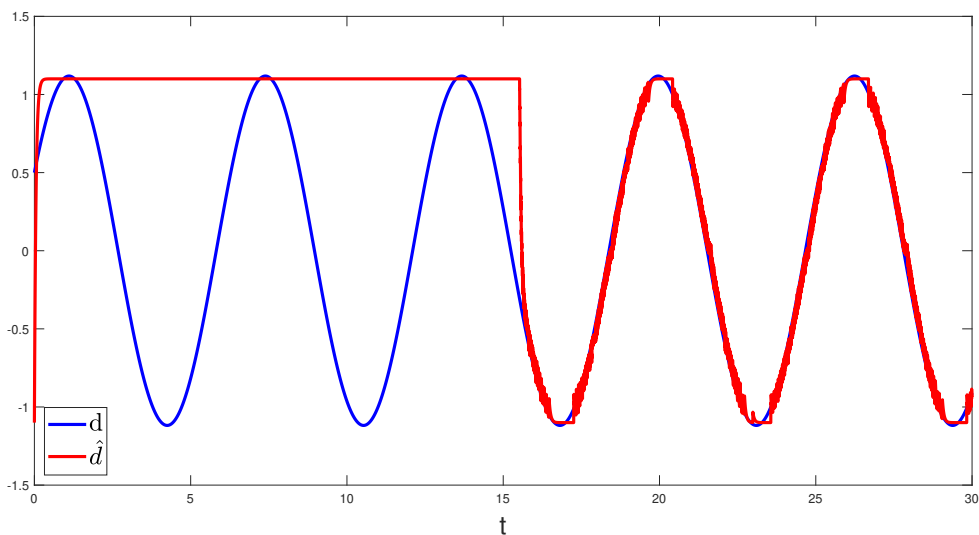


Figure 3.3:  $\hat{d}$  and the disturbance  $d$  versus time.

### 3.3 ADRC for hyperbolic systems

In Chapter 2, we proposed sliding mode controls to address the asymptotic stability of linear systems of infinite dimension in the presence of external disturbances. These controls depend mainly on the sliding variable, meaning that the latter is assumed to be measured. This means that in our case we measure the scalar product between the state of the system (2.32) and a function in  $D(A_L^*)$  ( $D(A_L^*)$  is defined in (2.78)). To be pragmatic, let us go back to Example 2.4.2 where the sliding variable is defined by

$$\sigma(t) = \frac{c_0}{\lambda_1} \int_0^1 w_1(t, x) e^{\frac{\lambda}{\lambda_1} x} dx + \frac{k_1 c_0}{\lambda_2} \int_0^1 w_2(t, x) e^{-\frac{\lambda}{\lambda_2} x} dx. \quad (3.12)$$

In this case, this means that we measure  $\sigma$  which is the sum of two scalar products in  $L^2(0, 1)$ . Thus, we measure these scalar products. But, from a practical point of view, how can we measure these scalar product without the knowledge of the full state  $w_1$  and  $w_2$  information? Do there exist sensors able to measure such outputs? Unfortunately, we cannot answer these questions with any certainty. Therefore, in the sequel, we will propose an ADRC in which what is measured can be achieved in practice [4, 93].

#### 3.3.1 Transport equation

Consider now the following linear transport equation

$$\begin{cases} \partial_t w(t, x) + \lambda \partial_x w(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times (0, L) \\ w(t, 0) = u(t) + d(t), \\ w(0, x) = w_0(x), \end{cases} \quad (3.13)$$

where  $L > 0$ ,  $\lambda > 0$ ,  $u(t) \in \mathbb{R}$  denotes the control and  $d(t) \in \mathbb{R}$  is an unknown disturbance. When the system (3.13) is undisturbed ( $d = 0$ ), it is well known that the feedback-law

$$u(t) := aw(t, L),$$

allows to stabilize the system if  $|a| < 1$ . If we assume that there exists a known positive constant  $C$  such that, for a.e  $t \in \mathbb{R}_+$ ,

$$|d(t)| \leq K_d \quad (3.14)$$

then, from Chapter 2, the following feedback-law

$$\begin{cases} u(t) := aw(t, L) - \lambda \sigma(t) - K \text{sign}(\sigma(t)) \\ \sigma(t) = \frac{1}{\lambda} \int_0^1 w(t, x) e^{\frac{\mu}{\lambda} x} dx \\ \mu < 0, |a| < 1, \\ K > K_d, \end{cases} \quad (3.15)$$

allows to stabilize the system (3.13) despite the presence of the unknown disturbance. This means that it is assume that  $w(t, L)$  and  $\sigma(t)$  are measured for all  $t > 0$ .

Let  $n \in \mathbb{N}$  fixed. Instead of assuming that  $\sigma(t)$  is measured for all  $t > 0$ , assume that the  $n$ -th moment defined by

$$\int_0^L x^n w(t, x) dx$$

is measured and denote by  $\eta_i$  the  $i$ -th moment of  $w$ , where  $i = 0, \dots, n$  i.e.

$$\eta_i(t) := \int_0^L x^i w(t, x) dx. \quad (3.16)$$

But,  $\eta_n$  does not stand for a sliding variable. Indeed, formally, the derivative of  $\eta_n$  along the trajectory of (3.13) yields, for all  $t \geq 0$

$$\dot{\eta}_n(t) = -\lambda L^n w(t, L) + n\eta_{n-1}(t).$$

This means that the relative degree is not equal to 1 anymore. Thus,  $\eta_n$  does not stand for a sliding variable for standard SMC strategy. Moreover, if we perform formal integrations by parts and using boundary conditions of (3.13), we obtain that the moments satisfy the following chain of integrators

$$\begin{cases} \dot{\eta}_n(t) &= -\lambda L^n w(t, L) + n\lambda\eta_{n-1}(t), \\ \dot{\eta}_{n-1}(t) &= -\lambda L^{n-1} w(t, L) + (n-1)\lambda\eta_{n-2}(t), \\ &\vdots \\ \dot{\eta}_1(t) &= -\lambda L w(t, L) + \lambda\eta_0(t), \\ \dot{\eta}_0(t) &= -\lambda w(t, L) + \lambda(u(t) + d(t)). \end{cases} \quad (3.17)$$

Thus, if  $w(\cdot, L) \in C^n(\mathbb{R}_+)$  and the measured output  $\eta_n$  is a class of  $C^{n+1}(\mathbb{R}_+)$ , then the relative degree of the system (3.13) is equal to  $n + 1$  according to (3.17). In this case, we can design a  $(n + 1)$ -order sliding mode control based on  $\eta_n$ . However,  $w(\cdot, L) \in C^n(\mathbb{R}_+)$  is an extremely restrictive condition because this would mean, for example, that we have to restrict ourselves to the initial value  $w_0$  that are in  $C^\infty(0, L)$ . Thus, for a general framework,  $\eta_n$  should not be chosen as a sliding variable. Instead of following the sliding mode strategy, we are going to estimate the disturbance  $d$  in order to cancel the effect of disturbance in the feedback-law by its estimate.

### Estimation of disturbance

We assume that equation (3.14) holds. To estimate the disturbance, we design an observer for (3.17) using a Levant's differentiator [74] of the form

$$\begin{cases} \dot{\hat{\eta}}_n(t) &= -\lambda L^n w(t, L) + n\lambda\hat{\eta}_{n-1}(t) - k_n K_d^{\frac{1}{n+1}} [\hat{\eta}_n - \eta_n]^{\frac{n}{n+1}}, \\ \dot{\hat{\eta}}_{n-1}(t) &= -\lambda L^{n-1} w(t, L) + (n-1)\lambda\hat{\eta}_{n-2}(t) - k_{n-1} K_d^{\frac{2}{n+1}} [\hat{\eta}_n - \eta_n]^{\frac{n-1}{n+1}}, \\ &\vdots \\ \dot{\hat{\eta}}_1(t) &= -\lambda L w(t, L) + \lambda\hat{\eta}_0(t) - k_1 K_d^{\frac{n}{n+1}} [\hat{\eta}_n - \eta_n]^{\frac{1}{n+1}}, \\ \dot{\hat{\eta}}_0(t) &\in -\lambda w(t, L) + \lambda u(t) - k_0 K_d [\hat{\eta}_n - \eta_n]^0 \end{cases} \quad (3.18)$$

where  $k_i$  for  $i = 0, 1, \dots, n$ , are the output injection gains to be selected to ensure the convergence of the observer. We define the estimation error as  $e_i := \hat{\eta}_i - \eta_i$ . The error dynamics is given by

$$\begin{cases} \dot{e}_n(t) &= n\lambda e_{n-1}(t) - k_n K_d^{\frac{1}{n+1}} [e_n]^{\frac{n}{n+1}}, \\ \dot{e}_{n-1}(t) &= (n-1)\lambda e_{n-2}(t) - k_{n-1} K_d^{\frac{2}{n+1}} [e_n]^{\frac{n-1}{n+1}}, \\ &\vdots \\ \dot{e}_1(t) &= \lambda e_0(t) - k_1 K_d^{\frac{n}{n+1}} [e_n]^{\frac{1}{n+1}}, \\ \dot{e}_0(t) &\in -\lambda d(t) - k_0 K_d [e_n]^0 \end{cases} \quad (3.19)$$

that gives

$$\begin{cases} \dot{e}_i(t) &= i\lambda e_{i-1}(t) - k_i K_d^{\frac{n+1-i}{n+1}} [e_n]^{\frac{i}{n+1}}, \quad i = 1, \dots, n \\ \dot{e}_0(t) &\in -\lambda K_d [-1, 1] - k_0 K_d [e_n]^0. \end{cases} \quad (3.20)$$

Using the following transformation

$$\begin{cases} \hat{e}_i = \frac{(i+1)\lambda e_i}{k_{i+1} K_d}, \quad i = 1, \dots, n \\ \hat{e}_0 = \frac{e_0}{k_1 K_d}, \\ k_{n+1} = 1 \end{cases} \quad (3.21)$$

we obtain

$$\begin{cases} \dot{\hat{e}}_i(t) &= -\tilde{k}_i \left( \left[ \frac{w_n}{\lambda(n+1)} \right]^{\frac{i}{n+1}} - \hat{e}_{i-1}(t) \right), \quad i = 1, \dots, n \\ \dot{\hat{e}}_0(t) &\in -\tilde{k}_0 \left( \left[ \frac{w_n}{\lambda(n+1)} \right]^0 + \frac{\lambda}{k_0} [-1, 1] \right) \end{cases} \quad (3.22)$$

where

$$\tilde{k}_0 = \frac{k_0}{k_1} \quad \text{and} \quad \tilde{k}_i = \frac{\lambda(j+1)k_i}{k_{i+1}} \quad i = 1, \dots, n.$$

The solution of system (3.22) is understood in the sense of Filippov ([48]) and the existence of the solution will be proved later. From [34, Theorem 1 and Proposition 1], all trajectories of (3.22) converge to zero in finite-time. More precisely, we have the following Proposition.

**Proposition 2** [34, Theorem 1 and Proposition 1] *There exists a valid set of differentiator parameters  $k_i$ ,  $i = 1, \dots, n$  and a finite-time  $t_r > 0$  such that  $\hat{e}_i(t) = 0$ ,  $i = 1, \dots, n$  for any  $t > t_r$ .*

Then, according to (3.21),  $\hat{e}_i(t) = 0$ ,  $i = 1, \dots, n$  for all  $t \geq t_r$  implies that  $e_i(t) = 0$ ,  $i = 1, \dots, n$  for all  $t \geq t_r$ . Thus, according to the last line (3.19), we have for all  $t \geq t_r$

$$0 \in -k_0 K_d \text{sign}(e_n(t)) - \lambda d(t). \quad (3.23)$$

Since  $\text{sign}(e_n(t)) = [-1, 1]$  for all  $t \geq t_r$ , then for all  $t \geq t_r$  the set  $-\frac{k_0 K_d}{\lambda} \text{sign}(e_n(t))$  is closed, bounded, and upper semi-continuous. Thus, according to [49, Page 78] there exists a measurable function  $\tilde{d}(t) \in -\frac{k_0 K_d}{\lambda} \text{sign}(e_n(t))$  such that

$$\forall t \geq t_r, d(t) = \tilde{d}(t). \quad (3.24)$$

Therefore, for all  $t \geq t_r$ ,  $\tilde{d}(t)$  is an estimate of  $d(t)$ .

Then, to stabilize the system (3.13), consider now the following feedback controller:

$$u(t) = aw(t, L) - \tilde{d}(t), \quad |a| < 1. \quad (3.25)$$

Note that the first term in (3.25) is a usual control that makes the closed-loop system (3.13) exponentially stable without the disturbance. The second term is used to compensate the effect of the disturbance.

The closed-loop system (3.13)-(3.25) is finally given by

$$\begin{cases} \partial_t w(t, x) + \lambda \partial_x w(t, x) = 0, \\ w(t, 0) = aw(t, L) - \tilde{d}(t) + d(t), \\ \dot{\hat{\eta}}_i(t) = -\lambda L^i w(t, L) + i\lambda \hat{\eta}_{i-1}(t) - k_i K_d^{\frac{n+1-i}{n+1}} [\hat{\eta}_n - \eta_n]^{\frac{i}{n+1}}, \quad i = 1, \dots, n \\ \hat{\eta}_0(t) = \lambda w(t, L)(a - 1) \\ \dot{\tilde{d}} \in -\frac{k_0 K_d}{\lambda} \text{sign}(\hat{\eta}_n(t) - \eta_n(t)) \\ w(0, x) = w_0(x), \\ \hat{\eta}(0) = \hat{\eta}^0 \in \mathbb{R}^{n+1}. \end{cases} \quad (3.26)$$

**Remark 3** According to Proposition 2,  $e_i(t) = 0$ ,  $i = 1, \dots, n$  for all  $t > t_r$ . Thus,  $\dot{e}_i(t) = 0$ ,  $i = 1, \dots, n$  for all  $t > t_r$ . Then, the mild solution<sup>2</sup>  $w$  of (3.26) is a mild solution to

$$\begin{cases} \partial_t w(t, x) + \lambda \partial_x w(t, x) = 0, \\ w(t, 0) = aw(t, L) \end{cases} \quad (3.27)$$

for all  $t > t_r$ .

The main result of this section can be formulated as follows:

**Theorem 3 (Existence of solutions)** Assume that (3.14) holds. Then, for all  $T > 0$  and for all  $(w_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$ , the closed-loop system (3.26) admits a mild solution  $(y, \hat{\eta}) \in C(0, T; L^2(0, L) \times \mathbb{R}^{n+1})$ .

**Theorem 4 (Global asymptotic stability)** There exists a  $\mathcal{KL}$ -function  $\alpha$  such that, the following inequality

$$\|w(t, \cdot)\|_{L^2(0, L)} + |e(t)|_{\mathbb{R}^{n+1}} \leq \alpha(\|w_0\|_{L^2(0, L)} + |e^0|_{\mathbb{R}^{n+1}}, t) \quad (3.28)$$

is satisfied for any  $(w_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$ , for any  $t \geq 0$  and for all solution  $(y, \hat{\eta})$  of (3.26), where  $e = \hat{\eta} - \eta$ ,  $e^0 = \hat{\eta}^0 - \eta^0$  and  $\eta_i^0 = \eta_i(0)$ .

The proofs are given in the Appendix B.

<sup>2</sup>See e.g [125, Definition 4.1.5]

**Numerical illustration** Consider the system (3.13) with the control defined by (3.25)-(3.19). The parameters of the system (3.13) are  $L = 3$ ,  $\lambda = 2$ . We choose the disturbance as

$$d(t) = \sin(t) + 0.5 \cos(t) - 1. \quad (3.29)$$

The parameters of the control and estimator are  $n = 1$ ,  $k_0 = 2$ ,  $k_1 = \sqrt{k_0}$ ,  $a = 0.47$ ,  $\tau = 0.05$ ,  $K_d = 2.1$ . The time-space step variation  $(\Delta t, \Delta x) = (0.0015, 0.0150)$  satisfies the CFL condition  $\frac{\lambda \Delta t}{\Delta x} < 1$ . The following low-pass filter

$$\tau \dot{\tilde{d}} + \tilde{d} = -\frac{k_0 K_d}{\lambda} \text{sign}_0(e_n) \quad (3.30)$$

is used to obtain  $\tilde{d}$  in simulations. In Figure 3.4, the robust stabilization of  $w$  is illustrated. Figure 3.5 shows that  $\hat{d}$  estimates well  $d$ .

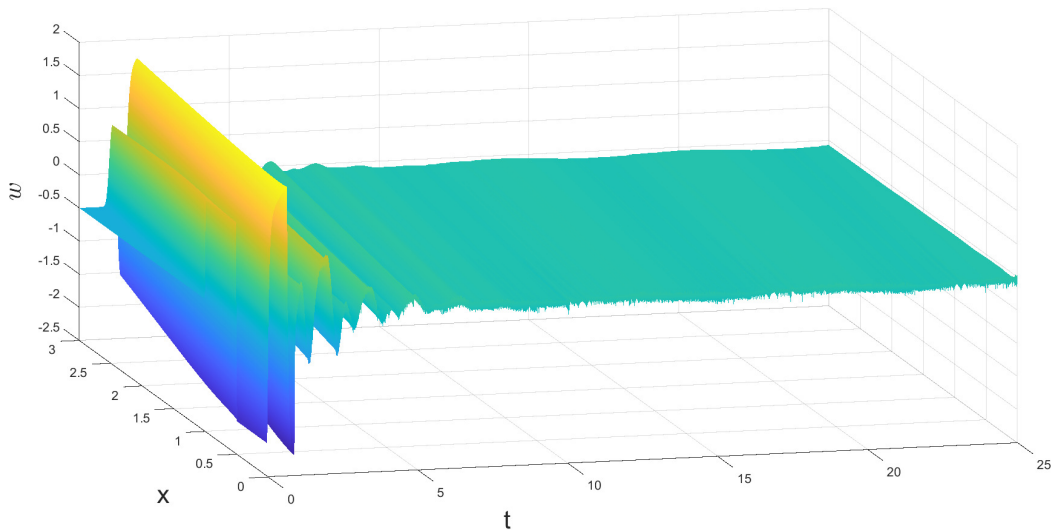
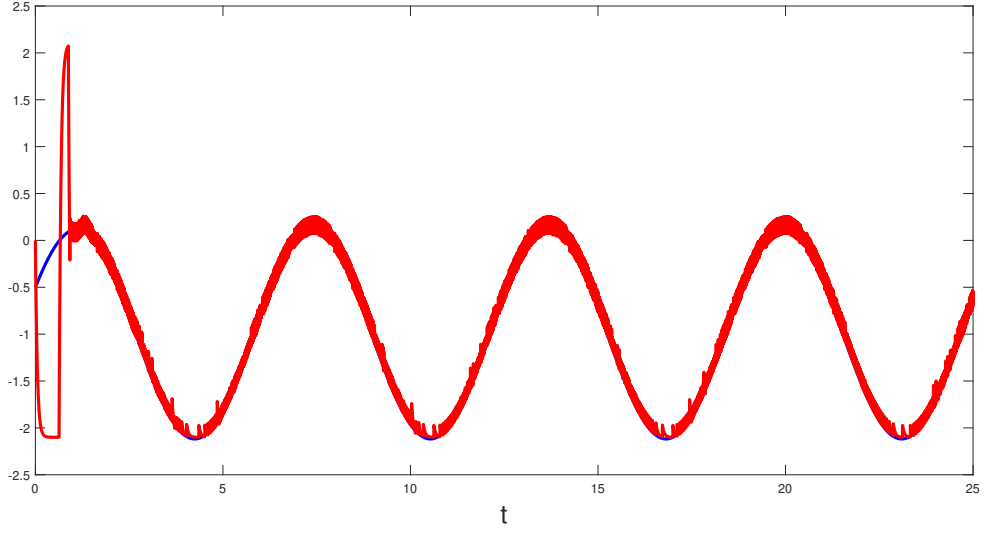


Figure 3.4: State variable  $w$  of (3.26) versus time and space.


 Figure 3.5:  $\tilde{d}$  (—) and the disturbance  $d$  (—) versus time.

### 3.3.2 System of transport equations

In the previous section, we saw how the ADRC works for a transport equation. In this section, let us consider two transport equations coupled at the boundary and inside the domain.

Consider the following system of transport equations with  $\lambda_1 > 0$ ,  $\lambda_2 > 0$ ,  $c_1, c_2 \in \mathbb{R}$  and  $a \in \mathbb{R}$

$$\begin{cases} \partial_t w_1(t, x) + \lambda_1 \partial_x w_1(t, x) = c_1 w_2(t, x), & (t, x) \in \mathbb{R}_+ \times (0, L) \\ \partial_t w_2(t, x) - \lambda_2 \partial_x w_2(t, x) = c_2 w_1(t, x), & (t, x) \in \mathbb{R}_+ \times (0, L) \\ w_1(t, 0) = u(t) + d(t), \\ w_2(t, L) = a w_1(t, L), \end{cases} \quad (3.31)$$

where  $u(t) \in \mathbb{R}$  denotes the control input and  $d(t) \in \mathbb{R}$  an unknown disturbance. When the system (3.31) is undisturbed ( $d = 0$ ), it is nowadays well-known that the control law

$$u(t) := b w_2(t, 0)$$

allows to stabilize the system (3.31) if there exist positive parameters  $p_1 > 0$ ,  $p_2 > 0$  and  $\mu > 0$  such that for all  $x \in [0, L]$

$$\mathcal{P}(x) := \begin{pmatrix} \mu \lambda_1 p_1 e^{-\mu x} & -c_1 p_1 e^{-\mu x} - c_2 p_2 e^{\mu x} \\ -c_1 p_1 e^{-\mu x} - c_2 p_2 e^{\mu x} & \mu \lambda_2 p_2 e^{\mu x} \end{pmatrix} \quad (3.32)$$

is positive definite and

$$b^2 \leq \frac{p_2 \lambda_2}{p_1 \lambda_1}, \quad a^2 \leq \frac{p_1 \lambda_1}{p_2 \lambda_2} e^{-2\mu L}, \quad (3.33)$$



see [11, Corollary 5.5.]. However, notice that if

$$\begin{cases} a = 1, \\ c_1 = c_2 = c < 0, \\ L \geq -\frac{\pi}{c} \end{cases} \quad (3.34)$$

then according to [11, Proposition 5.12], there is no  $b \in \mathbb{R}$  such that  $u(t) = bw_2(t, 0)$  allows to stabilize the system (3.31). We will therefore assume all along this section that the system (3.31) can be stabilized with a static output feedback.

As in the previous section, the goal is to propose an ADRC  $u$  allowing the rejection of the disturbance  $d$  and the global asymptotic stabilization of the system around the point  $(0, 0)$ .

### Canonical form

The vector space  $\mathcal{I} := L^2((0, L); \mathbb{R}^2)$  is equipped with the scalar product

$$\left( \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right)_\mu := p_1 \int_0^L \varphi_1 \psi_1 e^{-\mu x} dx + p_2 \int_0^L \varphi_2 \psi_2 e^{\mu x} dx \quad (3.35)$$

for every  $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in L^2$  and for every  $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in L^2$ , where  $p_1, p_2, \mu > 0$  are defined in (3.32) and (3.33). With this scalar product,  $\mathcal{I}$  is an Hilbert space. Let  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{I} \rightarrow \mathcal{I}$  be the linear operator defined by

$$\begin{cases} \mathcal{A} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 \varphi_1' + c_1 \varphi_2 \\ \lambda_2 \varphi_2' + c_2 \varphi_1 \end{pmatrix}, \\ D(\mathcal{A}) = \left\{ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathcal{H}^1(0, L, \mathbb{R}^2) \mid \varphi_1(0) = 0, \varphi_2(L) = a\varphi_1(L) \right\} \end{cases} \quad (3.36)$$

where  $\lambda_1, \lambda_2, c_1, c_2$  and  $a$  are given in system (3.31). We can prove as in the proof of [11, Theorem A.1] that  $\mathcal{A}$  generates a  $C_0$ -semigroup  $(\mathcal{T}(t))_{t \geq 0}$  on  $\mathcal{I}$ . Also, consider the operator  $\mathcal{B}$  defined as  $p_1 \lambda_1 \left\langle \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \mathcal{B}v \right\rangle_{D(\mathcal{A}^*), D(\mathcal{A}^*)'} = p_1 \lambda_1 \varphi_1(0)v$  for all  $v \in \mathbb{R}$  and  $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in D(\mathcal{A}^*)$  where  $\mathcal{A}^*$  is the adjoint operator of  $\mathcal{A}$  and  $\langle \cdot, \cdot \rangle_{D(\mathcal{A}^*), D(\mathcal{A}^*)'}$  is the dual product. We can prove as in Appendix B that the operator  $\mathcal{B}$  is admissible for the  $C_0$ -semigroup  $(\mathcal{T}(t))_{t \geq 0}$ . So, for any initial value  $\begin{pmatrix} w_1(0, \cdot) \\ w_2(0, \cdot) \end{pmatrix} \in \mathcal{I}$  and for any  $u \in L^2_{\text{loc}}(0, \infty)$ , there exists a unique mild solution  $(w_1, w_2) \in C([0, \infty); \mathcal{I}) \cap \mathcal{H}^1_{\text{loc}}((0, \infty); D(\mathcal{A}^*)')$  according to [125, Proposition 4.2.5]. Thus, for all  $t \geq 0$  we have

$$\begin{pmatrix} w_1(t, \cdot) \\ w_2(t, \cdot) \end{pmatrix} - \begin{pmatrix} w_1(0, \cdot) \\ w_2(0, \cdot) \end{pmatrix} = \int_0^t \left[ \mathcal{A} \begin{pmatrix} w_1(s, \cdot) \\ w_2(s, \cdot) \end{pmatrix} + \mathcal{B}(u(s) + d(s)) \right] ds. \quad (3.37)$$

Now, let  $n \in \mathbb{N} - \{0\}$  fixed and we denote by  $\eta_i$  (resp.  $\gamma_i$ ) the  $i$ -th moment of  $w_1$  (resp.  $w_2$ ), with

$i = 0, \dots, n$ , i.e.,

$$\eta_i(t) := \int_0^L x^i w_1(t, x) dx \quad \text{and} \quad \gamma_i(t) := \int_0^L x^i w_2(t, x) dx \quad (3.38)$$

Since  $(w_1, w_2) \in C([0, \infty); \mathcal{I}) \cap \mathcal{H}_{\text{loc}}^1((0, \infty); D(\mathcal{A}^*)')$  then, for all  $i = 0, \dots, n$ ,  $\eta_i$  and  $\gamma_i$  are absolutely continuous. Therefore, by performing integrations by parts and using boundary conditions of (3.31), we can prove that the moments are the Carathéodory solutions<sup>3</sup> of the following system:

$$\left\{ \begin{array}{l} \dot{\eta}_n(t) = c_1 \gamma_n(t) - \lambda_1 L^n w_1(t, L) + n \lambda_1 \eta_{n-1}(t), \\ \dot{\eta}_{n-1}(t) = c_1 \gamma_{n-1}(t) - \lambda_1 L^{n-1} w_1(t, L) + (n-1) \lambda_1 \eta_{n-2}(t), \\ \vdots \\ \dot{\eta}_1(t) = c_1 \gamma_1(t) - \lambda_1 L w_1(t, L) + \lambda_1 \eta_0(t), \\ \dot{\eta}_0(t) = c_1 \gamma_0(t) - \lambda_1 w_1(t, L) + \lambda_1 (u(t) + d(t)), \\ \dot{\gamma}_n(t) = c_2 \eta_n(t) + a \lambda_2 L^n w_1(t, L) - n \lambda_2 \gamma_{n-1}(t), \\ \dot{\gamma}_{n-1}(t) = c_2 \eta_{n-1}(t) + a \lambda_2 L^{n-1} w_1(t, L) - (n-1) \lambda_2 \gamma_{n-2}(t), \\ \vdots \\ \dot{\gamma}_1(t) = c_2 \eta_1(t) + a \lambda_2 L w_1(t, L) - \lambda_2 \gamma_0(t), \\ \dot{\gamma}_0(t) = c_2 \eta_0(t) + a \lambda_2 w_1(t, L) - \lambda_2 w_2(t, 0) \end{array} \right. \quad (3.39)$$

which gives

$$\dot{Y}(t) = AY(t) + B \begin{pmatrix} u(t) + d(t) \\ \Gamma(t) \end{pmatrix}, \quad (3.40)$$

where

$$Y_i = \begin{cases} \eta_{n+1-i}, & i = 1, \dots, n+1, \\ \gamma_{2n+2-i}, & i = n+2, \dots, 2n+2. \end{cases} \quad (3.41)$$

and  $A \in \mathcal{M}_{2n+2}(\mathbb{R})$ ,  $B, \Gamma(t) \in \mathbb{R}^{2n+2}$  such that

$$\Gamma_i(t) = \begin{cases} -\lambda_1 L^{n+1-i} w_1(t, L), & \text{if } i = 1, \dots, n+1 \\ a \lambda_2 L^{2n+2-i} w_1(t, L), & \text{if } i = n+2, \dots, 2n+1, \\ a \lambda_2 w_1(t, L) - \lambda_2 w_2(t, 0), & \text{else,} \end{cases} \quad (3.42)$$

$$B_i = \begin{cases} \lambda_1, & \text{if } i = n+1 \\ 0, & \text{else,} \end{cases} \quad (3.43)$$

$$A_{i,j} = \begin{cases} (n+1-i) \lambda_1, & \text{if } j = i+1, \text{ and } i = 1, \dots, n+1 \\ -(2n+2-i) \lambda_2, & \text{if } j = i+1, \text{ and } i = n+2, \dots, 2n+1 \\ c_1, & \text{if } j = i+n+1, \text{ and } i = 1, \dots, n+1 \\ c_2, & \text{if } j = i-(n+1), \text{ and } i = n+2, \dots, 2n+2 \\ 0, & \text{else.} \end{cases} \quad (3.44)$$

The system (3.39) is the system under canonical form from where the disturbance is estimated as

<sup>3</sup>See e.g [26, Theorem 1.1, page 43]

detailed in the next section.

### Estimation of disturbance

Now, the idea is to estimate the disturbance  $d$  appearing in (3.31) from the system (3.39). Note that if  $c_2 = 0$  then the dynamics of  $\gamma_i$ ,  $i = 0, 1, \dots, n$ , do not depend on the disturbance  $d$ . Thus, for example, to estimate  $d$  from (3.39), we can assume that, for all  $t \geq 0$ , we measure  $\eta_n(t)$ ,  $w_1(t, L)$ ,  $w_2(t, 0)$  and  $\gamma_i(t)$ ,  $i = 0, 1, \dots, n$  and design a robust observer as in the previous section, based only on the dynamics of  $\eta_i$ ,  $i = 0, 1, \dots, n$ . This means that we measure  $n + 1$  moments of  $w_2$  and the  $n - th$  moment of  $w_1$  to achieve the goal. Note that this case is much better than to measure

$$\int_0^L \varphi_1(x) w_1(t, x) dx \quad \text{and} \quad \int_0^L \varphi_2(x) w_2(t, x) dx \quad (3.45)$$

with  $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in D(\mathcal{A}^*)$  because, for instance, if  $\varphi_1$  and  $\varphi_2$  are analytic then each integral of (3.45) is a kind of infinite sum of  $\eta_i$  and  $\gamma_i$  respectively. Moreover, if  $c_1 = c_2 = 0$  then it is enough to measure  $\eta_n(t)$  and  $w_1(t, L)$  to be able to estimate the disturbance  $d$  thanks to a robust observer for the dynamics of  $\eta_i$ ,  $i = 0, 1, \dots, n$ . Thus, it is similar to what was done in the previous section. Therefore we assume that  $c_1 \neq 0$  and  $c_2 \neq 0$ , which corresponds to a much more complicated case than before. Our objective is to measure less than  $n + 2$  moments to estimate the disturbance  $d$ . In order to do so, the following hypothesis is stated.

**Assumption 5** *The system (3.40) is controllable.*

**Remark 4** *In order to understand the importance of Assumption 5 and why it is far from trivial that such a property holds, let us talk about what happens when  $n = 1$ . In this case,*

$$A = \begin{pmatrix} 0 & \lambda_1 & c_1 & 0 \\ 0 & 0 & 0 & c_1 \\ c_2 & 0 & 0 & -\lambda_2 \\ 0 & c_2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ \lambda_1 \\ 0 \\ 0 \end{pmatrix}. \quad (3.46)$$

The Kalman matrix  $K(A, B)$  is given by

$$\begin{aligned} K(A, B) &= (B, AB, A^2B, A^3B) \\ &= \begin{pmatrix} 0 & \lambda_1^2 & 0 & 2c_1c_2\lambda_1^2 - c_1c_2\lambda_1\lambda_2 \\ \lambda_1 & 0 & c_1c_2\lambda_1 & 0 \\ 0 & 0 & c_2\lambda_1^2 - c_2\lambda_1\lambda_2 & 0 \\ 0 & c_2\lambda_1 & 0 & c_1c_2^2\lambda_1 \end{pmatrix}, \end{aligned} \quad (3.47)$$

and its determinant is given by

$$\det(K(A, B)) = \lambda_1^4 c_1 c_2^3 (\lambda_1 - \lambda_2)^2. \quad (3.48)$$

Thus, we can see that, if  $\lambda_1 = \lambda_2$ , then  $\det(K(A, B)) = 0$ ; in this case, the system is not controllable. Therefore, the system (3.40) is not controllable if  $\lambda_1 = \lambda_2$  and  $n = 1$ . However, when  $\lambda_1 \neq \lambda_2$ , the system is controllable. Note that, even if  $\lambda_1 \neq \lambda_2$ , it is possible that the system (3.40) is not controllable for a higher value of  $n$ .

From Assumption 5 and according to [123, Theorem 2.2.7], there exists a change of variable  $Z = MY$  that allows to write the system (3.40) in Brunovsky's normal form

$$\dot{Z}(t) = A_z Z(t) + B_z \left( u(t) + d(t) \right) + M\Gamma(t) \quad (3.49)$$

with

$$A_z = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{2n+1} \end{pmatrix} \quad \text{and} \quad B_z = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (3.50)$$

$$\begin{cases} M_{\cdot, 2n+2} = B, \\ M_{\cdot, i} = AM_{\cdot, i+1} + a_i B, \quad i = 1, \dots, 2n+1, \end{cases} \quad (3.51)$$

where  $M_{\cdot, i}$  is the  $i$ -th column of the matrix  $M \in \mathcal{M}_{2n+2}(\mathbb{R})$  and the coefficients  $a_i$  are those of the characteristic polynomial of  $A$ . We can prove that the characteristic polynomial  $P_A$  of  $A$  is defined by

$$P_A(x) = (-1)^{n+1} (c_1 c_2 - x^2)^{n+1} = x^{2n+2} + \sum_{i=0}^{2n+1} a_i x^i. \quad (3.52)$$

Moreover, from (3.52), we can deduce that for all  $q = 0, \dots, n$ ,

$$a_{2q+1} = 0. \quad (3.53)$$

Note that, with the Brunovsky output  $Z_1 = C_z Z$ , the system (3.49) is observable where  $C_z = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ . Thus, in order to estimate the disturbance  $d$ , we can design a robust observer or an extended state observer based on the system (3.49). For that, we assume that we also measure  $w_1(t, L)$  and  $w_2(t, 0)$  for all  $t \geq 0$ . Moreover, if  $n$  is odd, then

$$Z_1 = M_{1\cdot} Y \quad \text{and} \quad M_{1\cdot, i} = 0 \quad \forall i \in \llbracket 1, 2n+2 \rrbracket \setminus \left\{ n+2, n-2q; q \in \left[ \left[ 0, \left\lfloor \frac{n}{2} \right\rfloor \right] \right] \right\}, \quad (3.54)$$

else, we have

$$Z_1 = M_{1\cdot} Y \quad \text{and} \quad M_{1\cdot, i} = 0 \quad \forall i \in \llbracket 1, 2n+2 \rrbracket \setminus \left\{ n+2, n-2q; q \in \left[ \left[ 0, \left\lfloor \frac{n}{2} \right\rfloor - 1 \right] \right] \right\}, \quad (3.55)$$

where  $M_{1\cdot}$  is the first row of  $M$ . Therefore, if  $n$  is odd (resp.  $n$  is even)  $Z_1$  depends on  $\gamma_n$  and  $\eta_{2q+1}$ ,  $q = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$  (resp.  $q = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor - 1$ ). As a consequence, if  $n$  is odd (resp.  $n$  is even), we assume also that for all  $t \geq 0$ , we measure  $\gamma_n(t)$  and  $\eta_{2q+1}(t)$ ,  $q = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$  (resp.  $q = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor - 1$ ).

**Example 1** For  $n=1$ , we have  $\lfloor \frac{n}{2} \rfloor = 0$ . Since  $n = 1$  is odd then  $q = 0$ . Thus,  $Z_1$  depends on  $\gamma_1$  and  $\eta_1$ . On the other hand,  $P_A(x) = x^4 - 2c_1c_2x^2 + c_1^2c_2^2$  and  $M$  is given by

$$\begin{pmatrix} -c_1c_2\lambda_1\lambda_2 & 0 & \lambda_1^2 & 0 \\ 0 & -c_1c_2\lambda_1 & 0 & \lambda_1 \\ 0 & c_2\lambda_1^2 - c_2\lambda_1\lambda_2 & 0 & 0 \\ -c_1c_2^2\lambda_1 & 0 & c_2\lambda_1 & 0 \end{pmatrix}. \quad (3.56)$$

Thus, we have

$$\begin{aligned} Z_1 &= M_1 \cdot Y = \begin{pmatrix} -c_1c_2\lambda_1\lambda_2 & 0 & \lambda_1^2 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_0 \\ \gamma_1 \\ \gamma_0 \end{pmatrix} \\ &= -c_1c_2\lambda_1\lambda_2\eta_1 + \lambda_1^2\gamma_1. \end{aligned} \quad (3.57)$$

Then, we can see that  $Z_1$  depends on  $\gamma_1$  and  $\eta_1$ . It is therefore sufficient to assume the knowledge of  $\gamma_1$  and  $\eta_1$ .

Since the pair  $(A_z, C_z)$  is observable, then there exists  $G_z \in \mathbb{R}^{2n+2}$  such that the matrix  $A_0 = A_z - G_z C_z$  is Hurwitz, with  $C_z = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$ . Therefore, system (3.49) can be rewritten as follows

$$\dot{Z}(t) = A_0 Z(t) + G_z C_z Z(t) + B_z \begin{pmatrix} u(t) \\ d(t) \end{pmatrix} + M \Gamma(t). \quad (3.58)$$

Now, in order to estimate the disturbance through a system with fewer inputs, let us introduce the following auxiliary system

$$\dot{\bar{Z}}(t) = A_0 \bar{Z}(t) + G_z C_z Z(t) + B_z u(t) + M \Gamma(t). \quad (3.59)$$

Note that, system (3.59) depends only on the input and output of the original system (3.58). Thus it is completely known. As a consequence, the error  $Z - \bar{Z}$  system describing the dynamics of the error  $\tilde{Z} = Z - \bar{Z}$  is ideal for disturbance estimation.  $\tilde{Z}$ -dynamics reads as

$$\dot{\tilde{Z}}(t) = A_0 \tilde{Z}(t) + B_z d(t) \quad (3.60)$$

i.e

$$\begin{cases} \dot{\tilde{Z}}_i(t) &= \tilde{Z}_{i+1}(t) - g_i \tilde{Z}_1, \quad i = 1, \dots, 2n+1 \\ \dot{\tilde{Z}}_{2n+2}(t) &= -\sum_{i=1}^{2n+1} a_i \tilde{Z}_{i+1} - (a_0 + g_{2n+2}) \tilde{Z}_1 + d(t) \end{cases} \quad (3.61)$$

where the constants  $g_i, i = 1, \dots, 2n+2$ , are the components of  $G_z$ .

Now, we assume that the disturbance  $d$  is continuous and bounded. Furthermore, one supposes that there exists a known positive constant  $K_d$  such that, for all  $t \in \mathbb{R}_+$ ,

$$|d(t)| \leq K_d. \quad (3.62)$$

Since  $A_0$  is Hurwitz, then the solution of (3.61) is bounded. Then, we can propose the following observer for system (3.61)

$$\begin{cases} \dot{\hat{Z}}_{ii} = \hat{Z}_{i(i+1)} + \phi_i \left( \tilde{Z}_1, \hat{Z}_{12}, \hat{Z}_{23}, \dots, \hat{Z}_{(i-1)i} \right) - L_i k_1 \left( \hat{Z}_{ii} - \hat{Z}_{(i-1)i} \right) \\ \dot{\hat{Z}}_{i(i+1)} \in \hat{Z}_{(i+1)(i+2)} + \phi_{i+1} \left( \tilde{Z}_1, \hat{Z}_{12}, \hat{Z}_{23}, \dots, \hat{Z}_{i(i+1)} \right) - L_i^2 k_2 \left( \hat{Z}_{ii} - \hat{Z}_{(i-1)i} \right), \quad i = 1, \dots, 2n+1, \end{cases} \quad (3.63)$$

with

$$\begin{cases} k_1(s) := q(\kappa s) \\ k_2(s) := \text{sign}(s) + q(\kappa s) \\ \phi_i \left( \tilde{Z}_1, \hat{Z}_{12}, \hat{Z}_{23}, \dots, \hat{Z}_{(i-1)i} \right) = -g_i \tilde{Z}_1, \quad i = 1, \dots, 2n+1 \\ \phi_{2n+2} \left( \tilde{Z}_1, \hat{Z}_{12}, \hat{Z}_{23}, \dots, \hat{Z}_{(2n+1)(2n+2)} \right) = -\sum_{i=1}^{2n+1} a_i \hat{Z}_{i(i+1)} - (a_0 + g_{2n+2}) \tilde{Z}_1 \end{cases}, \quad q(s) := [s]^{\frac{1}{2}} + s. \quad (3.64)$$

and the conventions  $\hat{Z}_{01} = \tilde{Z}_1$  and  $\hat{Z}_{(2n+2)(2n+3)} = 0$ , and where  $\kappa$  is a positive real number and the  $L_i$  are positive real numbers that will be selected later on. The overall state dimension of the observer is  $4n+2$ . The indexes of the observer variables are selected with the convention that  $\hat{Z}_{ii}$  is the first state-component of the  $i$ -th block providing an estimate of the variable  $\tilde{Z}_i$ ,  $\hat{Z}_{i(i+1)}$  is the second state-component of the  $i$ -th block providing an estimate of the variable  $\tilde{Z}_{i+1}$ . As a consequence, for each variable  $\tilde{Z}_i$ , with  $i = 2, \dots, 2n+1$ , we have two different estimates. We define the estimation error as  $E_{ii} := \hat{Z}_{ii} - \tilde{Z}_i$  and  $E_{i(i+1)} := \hat{Z}_{i(i+1)} - \tilde{Z}_{i+1}$  for  $i = 1, \dots, 2n+1$ . The error dynamics is given by

$$\begin{cases} \dot{E}_{ii} = E_{i(i+1)} - L_i k_1 (E_{ii} - E_{(i-1)i}) \\ \dot{E}_{i(i+1)} \in E_{(i+1)(i+2)} - L_i^2 k_2 (E_{ii} - E_{(i-1)i}) + v_i, \quad i = 1, \dots, 2n+1, \end{cases} \quad (3.65)$$

with

$$v_i(t) = \begin{cases} 0, & i = 1, \dots, 2n \\ -\sum_{j=1}^{2n+1} a_j E_{j(j+1)}(t) - d(t), & i = 2n+1. \end{cases} \quad (3.66)$$

We can prove as in [1] that by selecting the gains  $L_i$  sufficiently large and in an appropriate way, observer (3.63) ensures finite time convergence of the estimate to the state of the system (3.61). More precisely, we have the following Proposition.

**Proposition 3** [1, Theorem 2 and Theorem 3] *There exist positive real numbers  $(\kappa, L_1, \dots, L_{2n+1}) > 0$ , such that observer (3.63) ensures finite time estimation of system (3.61), namely there exists a time  $t_r \geq 0$  such that*

$$\hat{Z}_{ii}(t) = \tilde{Z}_1(t), \hat{Z}_{i(i+1)}(t) = \hat{Z}_{(i+1)(i+1)}(t) = \tilde{Z}_{i+1}(t), \quad i = 1, \dots, 2n+1, \quad \forall t \geq t_r. \quad (3.67)$$

*In addition, for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any solution  $\tilde{Z}$  of (3.61) initialized in  $\mathcal{C}_0$  and any*

solution  $\hat{Z}$  of (3.63) initialized in  $\hat{C}_0$ , we have the implication  $\forall i = 1, \dots, 2n + 1$

$$\left| \hat{Z}_{ii}(0) - \tilde{Z}_i(0) \right| + \left| \hat{Z}_{i(i+1)}(0) - \tilde{Z}_{i+1}(0) \right| < \delta \quad \implies \quad \left| \hat{Z}_{ii}(t) - \tilde{Z}_i(t) \right| + \left| \hat{Z}_{i(i+1)}(t) - \tilde{Z}_{i+1}(t) \right| < \epsilon \quad \forall t \quad (3.68)$$

where  $\mathcal{C}_0 \subset \mathbb{R}^{2n+2}$  and  $\hat{C}_0 \subset \mathbb{R}^{4n+2}$  are compact sets.

*Proof.* The proof of the Proposition 3 is the proof of [1, Theorem 2 and Theorem 3] with,

$$\ell_1 = \max_{i=1, \dots, 2n+1} a_i$$

and  $\ell_0$  any positive real number.  $\square$

Then, from (3.65) with  $i = 2n + 1$  and Proposition 3, we have for all  $t \geq t_r$

$$0 \in -L_{2n+1}^2 \text{sign} \left( \hat{Z}_{(2n+1)(2n+1)}(t) - \hat{Z}_{2n(2n+1)}(t) \right) - d(t). \quad (3.69)$$

Since  $\text{sign} \left( \hat{Z}_{(2n+1)(2n+1)}(t) - \hat{Z}_{2n(2n+1)}(t) \right) = [-1, 1]$  for all  $t \geq t_r$ , then for all  $t \geq t_r$  the set  $-L_{2n+1}^2 \text{sign} \left( \hat{Z}_{(2n+1)(2n+1)}(t) - \hat{Z}_{2n(2n+1)}(t) \right)$  is closed, bounded, and upper semi-continuous. Thus, according to [49, Page 78] there exists a measurable function  $\hat{d}(t) \in -L_{2n+1}^2 \text{sign} \left( \hat{Z}_{(2n+1)(2n+1)}(t) - \hat{Z}_{2n(2n+1)}(t) \right)$  such that

$$\forall t \geq t_r, d(t) = \hat{d}(t). \quad (3.70)$$

Therefore, for all  $t \geq t_r$ ,  $\hat{d}(t)$  is an estimate of  $d(t)$ .

## Control design

In order to make the origin of (3.31) globally asymptotically stable, the feedback controller  $u$  is designed as follows:

$$u(t) := bw_2(t, 0) - \hat{d}(t). \quad (3.71)$$

It is clear from (3.71) that the first term is the control that makes the closed-loop system (3.31) exponentially stable without the disturbance [11, Corollary 5.5]. The term  $-\hat{d}$  is used to compensate the effect of

the disturbance. Under the feedback (3.71), the closed-loop system of (3.31) becomes

$$\left\{ \begin{array}{l} \partial_t w_1(t, x) + \lambda_1 \partial_x w_1(t, x) = c_1 w_2(t, x), \\ \partial_t w_2(t, x) - \lambda_2 \partial_x w_2(t, x) = c_2 w_1(t, x), \\ w_1(t, 0) = b w_2(t, 0) - \hat{d}(t) + d(t), \\ w_2(t, L) = a w_1(t, L), \\ \dot{\bar{Z}}(t) = B_z \left( b w_2(t, 0) - \hat{d}(t) \right) + A_0 \bar{Z}(t) + G_z C_z Z(t) + M \Gamma(t), \\ \dot{\hat{Z}}_{ii} = \hat{Z}_{i(i+1)} - g_i \tilde{Z}_1 - L_i k_1 \left( \hat{Z}_{ii} - \hat{Z}_{(i-1)i} \right), \\ \dot{\hat{Z}}_{i(i+1)} \in \hat{Z}_{(i+1)(i+2)} - g_{i+1} \tilde{Z}_1 - L_i^2 k_2 \left( \hat{Z}_{ii} - \hat{Z}_{(i-1)i} \right), \quad i = 1, \dots, 2n \\ \dot{\hat{Z}}_{(2n+1)(2n+1)} = \hat{Z}_{(2n+1)(2n+2)} - g_{2n+1} \tilde{Z}_1 - L_{2n+1} k_1 \left( \hat{Z}_{(2n+1)(2n+1)} - \hat{Z}_{2n(2n+1)} \right), \\ \dot{\hat{Z}}_{(2n+1)(2n+2)} = - \sum_{i=1}^{2n+1} a_i \hat{Z}_{i(i+1)} - (a_0 + g_{2n+2}) \tilde{Z}_1 + \hat{d}(t) - L_{2n+1}^2 q \left( \kappa \left( \hat{Z}_{(2n+1)(2n+1)} - \hat{Z}_{2n(2n+1)} \right) \right), \\ \hat{d}(t) \in -L_{2n+1}^2 \text{sign} \left( \hat{Z}_{(2n+1)(2n+1)}(t) - \hat{Z}_{2n(2n+1)}(t) \right). \end{array} \right. \quad (3.72)$$

We are now in position to state our two main results of this section, namely: an existence theorem and a stabilization theorem ensuring that the disturbance is rejected. We set  $\mathcal{J} = \mathcal{I} \times \mathbb{R}^{2n+2} \times \mathbb{R}^{4n+2}$ .

**Theorem 5** *Let parameters  $(\kappa, L_1, \dots, L_{2n+1})$  be fixed as in Proposition 3. Then, for any initial state  $(w_1(0, \cdot), w_2(0, \cdot), \bar{Z}(0), \hat{Z}(0)) \in \mathcal{J}$ , the closed-loop system (3.72) admits a mild solution  $(w_1, w_2, \bar{Z}, \hat{Z}) \in C(0, \infty; \mathcal{J})$ .*

**Theorem 6** *Let parameters  $(\kappa, L_1, \dots, L_{2n+1})$  be fixed as in Proposition 3. Then, there exists a  $\mathcal{KL}$ -function  $\beta$  such that, the following inequality*

$$\left\| \begin{pmatrix} w_1(t, \cdot) \\ w_2(t, \cdot) \end{pmatrix} \right\|_{\mathcal{I}} \leq \alpha \left( \left\| \begin{pmatrix} w_1(0, \cdot) \\ w_2(0, \cdot) \end{pmatrix} \right\|_{\mathcal{I}} + |\mathcal{E}(0)|, t \right) \quad (3.73)$$

*is satisfied for any initial state  $(w_1(0, \cdot), w_2(0, \cdot), \bar{Z}(0), \hat{Z}(0)) \in \mathcal{J}$ , for any  $t \geq 0$  and for all solution  $(w_1, w_2, \bar{Z}, \hat{Z})$  of (3.72), where  $\mathcal{E}$  is the vector of component  $\mathcal{E}_i = (E_{ii}, E_{i(i+1)})$  with  $i = 1, \dots, 2n+1$ .*

The proofs are given in the Appendix B.

## 3.4 Conclusion

In this chapter, we apply the ADRC method for stabilization of a linear hyperbolic system subject to a boundary disturbance. A robust observer is used to estimate disturbance. This requires measurement



of some moments of the state, which is better in terms of measurement. Then, the feedback control is designed, which contains two parts. The first part is used to compensate the disturbance and the second part is the nominal feedback. The existence of solutions of the closed-loop system is shown, and the global asymptotic stability of the closed-loop system is proven.

# ISS LYAPUNOV STRICTIFICATION VIA OBSERVER DESIGN AND INTEGRAL ACTION CONTROL FOR A KORTEWEG-DE VRIES EQUATION

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## 4.1 Introduction

This chapter deals with the construction of an ISS-Lyapunov functional via a strictification procedure and the output regulation of a Korteweg-de Vries (KdV) equation. The KdV equation, introduced in 1895 by Diederik J. Korteweg and Gustav de Vries, is a mathematical model of waves on shallow water surfaces (see e.g., [21] for a survey). Such an equation has been studied in [110, 23, 27] in the controllability context, in [22, 29, 86, 120, 87] in terms of stabilization, and in [121, 25] where some asymptotic analysis of the equilibrium point coinciding with the origin are given.

The system considered here can be written as follows:

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w + w \partial_x w = d_1(t, x), \\ w(t, 0) = w(t, L) = 0, \\ \partial_x w(t, L) = d_2(t), \\ w(0, x) = w_0(x), \end{cases} \quad (4.1)$$

where  $(t, x) \in \mathbb{R}_+ \times [0, L]$ ,  $L > 0$ ,  $d_1$  and  $d_2$  denote external inputs that might be seen, for instance, as disturbances, and its associated linearized dynamics around the origin described by

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w = d_1(t, x), \\ w(t, 0) = w(t, L) = 0, \\ \partial_x w(t, L) = d_2(t), \\ w(0, x) = w_0(x). \end{cases} \quad (4.2)$$

where  $(t, x) \in \mathbb{R}_+ \times [0, L]$ .

Inspired by a strictification methodology recently introduced in the finite-dimensional context, we will construct an ISS-Lyapunov functional for the KdV equations (4.1) and (4.2) thanks to the use of an

observer which is designed following the backstepping approach [81]. This will allow us to conclude that the KdV equations (4.1) and (4.2) satisfy an ISS property with respect to the disturbances  $d_1, d_2$ . Then, we will consider the regulation problem of KdV equations (4.1) and (4.2) in which the disturbance  $d_2$  is considered as a control input acting at the boundary condition, and the output  $y(t) = \partial_x w(t, 0)$  has to be regulated at a certain desired constant reference  $r$  in presence of unknown distributed constant disturbances  $d_1$ . Thanks to the previous ISS-Lyapunov functional, we apply the forwarding approach [89, 7] in order to solve the desired output regulation problem.

This chapter is organized as follows. Section 4.2 presents the strictification methodology in the finite-dimensional context. In Section 4.3, we formulate the problem and state the results about the construction of the ISS Lyapunov functional. Section 4.4 states and proves some regulation results for the KdV equation. Finally, Section 4.5 collects concluding remarks.

## 4.2 Strictification of Lyapunov function

Lyapunov functions are known as a very efficient tool to study stability. But, there does not exist a systematic method to find a suitable Lyapunov function for any system except linear finite-dimensional ones. There are two kinds of Lyapunov functions: weak Lyapunov function and strict Lyapunov function. The weak Lyapunov functions are distinguished by negative semi-definite time derivatives along all trajectories of the system, while the strict Lyapunov functions have negative definite time derivatives along all trajectories of the system. Note that there is no systematic method to build strict Lyapunov function either for nonlinear ordinary differential equations or (linear or nonlinear) partial differential equations. In contrast with strict Lyapunov functions, in many situations, a weak Lyapunov function exists. Often, it also coincides with the energy of the system. Moreover, it is well known that the weak Lyapunov functions are sufficient to prove stability in the sense of Lyapunov, but they are not sufficient when it comes to proving asymptotic stability or carrying out a stability analysis of a system in the presence of disturbance. Strict Lyapunov functions can be used to solve several important robustness and stabilization problems. For example, it plays a very important role in quantifying the convergence rates of all trajectories of the system towards an equilibrium point or to quantify the effects of uncertainties, since they can be used to prove input-to state stability. This motivated a lot of meaningful research on methods to explicitly build strict Lyapunov functions [82, 118, 107, 112, 36, 108]. In particular, in the finite-dimensional context, a lot of attention has been put in the *strictification* of weak Lyapunov functions, namely the conception of systematic procedures to modify a weak Lyapunov function in order to make it strict, see, for instance, [82, 107]. To the best of our knowledge, in the infinite-dimensional context, such an approach has been applied only to certain classes of hyperbolic systems [108].

The first contribution of this chapter is the construction of an ISS-Lyapunov functional for the KdV equation via a strictification procedure. The methodology we propose to achieve this goal is inspired on [107] and is based on the design of an observer. In order to understand the construction of our ISS-Lyapunov functional, we are going to briefly present in the next subsection the strictification procedure of [107].

### 4.2.1 Observers to the aid of strictification of Lyapunov functions

Let  $\mathcal{O}$  be an open neighborhood of the origin in  $\mathbb{R}^n$ . We consider a system the dynamics of which on  $\mathcal{O}$  are:

$$\dot{w} = f(w) \quad (4.3)$$

with  $w \in \mathcal{O}$  and  $f : \mathcal{O} \rightarrow \mathbb{R}^n$  a sufficiently many times differentiable function which vanishes at 0. We assume the knowledge of a weak Lyapunov function  $V : \mathcal{O} \rightarrow \mathbb{R}_+$ . We look for a function  $h : \mathcal{O} \rightarrow \mathbb{R}^p$  satisfying:

$$\langle \nabla V(w), f(w) \rangle_{\mathbb{R}^n} \leq -\alpha(|h(w)|) \quad (4.4)$$

where  $\alpha$  is a class  $\mathcal{K}$  function, and such that the system:

$$\dot{w} = f(w), \quad y = h(w)$$

is observable in such a way that we can design an observer. What we mean by we can design an observer is that there exist continuous functions  $\tau : \mathcal{O} \rightarrow \mathbb{R}^m$  and  $\varphi : h(\mathcal{O}) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  such that:

1. the function  $x \mapsto (h(w), \tau(w))$  is injective on  $\mathcal{O}$ .
2. We have:

$$\nabla \tau(w) f(w) = \varphi(h(w), \tau(w)), \quad \tau(0) = 0. \quad (4.5)$$

3. For the augmented system:

$$\dot{w} = f(w), \quad \dot{\hat{w}} = \varphi(h(w), \hat{w}) \quad (4.6)$$

we know a  $C^1$  function  $\mathcal{V} : \mathcal{O} \times \mathbb{R}^m \rightarrow \mathbb{R}_+$  satisfying:

$$\mathcal{V}(x, \tau(w)) = 0$$

and for all  $(w, \hat{w})$  such that  $\hat{w} \neq \tau(w)$ ,

$$\mathcal{V}(w, \hat{w}) > 0, \quad \frac{\partial \mathcal{V}}{\partial w}(w, \hat{w}) f(w) + \frac{\partial \mathcal{V}}{\partial \hat{w}}(w, \hat{w}) \varphi(h(w), \hat{w}) = -\mathcal{W}(w, \hat{w}) < 0 \quad (4.7)$$

This does give an observer since, on one hand, when  $\hat{w}$  is in the image  $\tau(\mathcal{O})$ , and the value  $h(w)$  is known, there is a unique  $w$  satisfying:

$$\hat{w} = \tau(w),$$

and, on the other hand, the solution  $(X(w, t), Z((w, \hat{w}), t))$  of (4.6), issued from  $(w, \hat{w})$  at time 0, is such that  $Z((w, \hat{w}), t)$  converges to  $\tau(X(w, t))$  as  $t$  goes to infinity.

By evaluating at  $(0, \tau(w))$  the functions involved in (4.7), we obtain:

$$\frac{\partial \mathcal{V}}{\partial \hat{w}}(0, \tau(w)) \varphi(0, \tau(w)) = -\mathcal{W}(0, \tau(w)) < 0 \quad \forall w \in \mathcal{O} : \tau(w) \neq 0. \quad (4.8)$$

This allow to define the auxiliary function  $V_a$  as:

$$V_a(w) = \mathcal{V}(0, \tau(w)).$$

In view of (4.5), we have the decomposition:

$$\langle \nabla V_a(w), f(w) \rangle_{\mathbb{R}^n} = -\mathcal{W}(0, \tau(w)) + \frac{\partial \mathcal{V}}{\partial \hat{w}}(0, \tau(w))[\varphi(h(w), \tau(w)) - \varphi(0, \tau(w))].$$

Here,  $h$  plays the role of a disturbance. We assume that we are in an ISS-like context, i.e. there exist a class  $\mathcal{K}^\infty$  function  $\gamma$  and a continuous positive definite function  $\alpha_a : \tau(\mathcal{O}) \rightarrow \mathbb{R}_+$  satisfying :

$$\frac{\partial \mathcal{V}}{\partial \hat{w}}(0, \tau(w))\varphi(h(w), \tau(w)) \leq -\omega\alpha_a(V_a(w)) + \gamma(|h(w)|) \quad (4.9)$$

with  $\omega > 1$ . Then, we obtain

$$L_f V_a(w) \leq -W_a(w) + \gamma(|h(w)|) \quad (4.10)$$

with  $W_a : \mathcal{O} \rightarrow \mathbb{R}_+$  such that the function  $x \mapsto W_a(w) + |h(w)|$  is positive definite. In this case, the function

$$V_s(w) = \mu(V(w)) + \mu_a(V_a(w)), \quad (4.11)$$

where  $\mu$  and  $\mu_a$  are appropriately chosen  $C^1$  class  $\mathcal{K}^\infty$  functions, is a good candidate for being a strict Lyapunov function.

Finally, we only keep the functions  $\tau$  and  $\mathcal{V}$  of the procedure and we need explicit expressions for them. To give the reader a better understanding of the procedure, we propose the following example.

**Example 2** Consider the following system

$$\dot{w} = Aw \quad (4.12)$$

with  $A$  a matrix. Let us assume there is  $P$  positive definite matrix such that

$$PA + A^\top P \leq -C^\top C$$

where  $Q$  is only semidefinite positive matrix, namely  $Q \geq 0$ . Then, the function

$$V(w) = w^\top Pw$$

is a weak Lyapunov function since

$$\dot{V} \leq -(Cw)^\top Cw.$$

Let us suppose that the pair  $(A, C)$  is observable. Then there exists  $K$  such that  $A - LC$  is Hurwitz, namely

$$\hat{w} = A\hat{w} + L(Cw - C\hat{w}), \quad (4.13)$$

is an observer. Furthermore, since  $A - LC$  is Hurwitz, there exists a positive definite matrix  $S$  satisfying

$$S(A - LC) + (A - LC)^\top S = -I_{\mathbb{R}^n}.$$

Thus, we can show that the system (4.13) is an observer for the system (4.12) by using

$$U(w, \hat{w}) = (w - \hat{w})^\top S(w - \hat{w})$$

which satisfies

$$\dot{U} = -|w - \hat{w}|^2.$$

Then, we can deduce that  $\mathcal{V} = U$  and  $\tau$  is the identity function. Now, let us consider the Lyapunov function

$$W(w) = V(w) + bU(x, 0) = w^\top Pw + bw^\top Sw$$

with  $b > 0$  to be chosen. Its derivative satisfies

$$\begin{aligned} \dot{W} &\leq -w^\top Q^\top Qw + 2bw^\top S(A - KQ + KQ)w \\ &\leq -w^\top Q^\top Qw - b|w|^2 + 2bw^\top SKQw \\ &\leq -w^\top Q^\top Qw + \frac{1}{2}w^\top Q^\top Qw - (1 - 2b|S||K||Q|)|w|^2 \\ &\leq -\frac{1}{2}w^\top Q^\top Qw - (1 - 2b|S||K||Q|)|w|^2. \end{aligned}$$

Hence, by selecting  $b$  small enough, in particular such that

$$b|S||K||Q| = \frac{1}{4}$$

then we obtain

$$\dot{W} \leq -\frac{1}{2}(w^\top Q^\top Qw + |w|^2).$$

As a consequence,  $W$  is a strict Lyapunov function.

In the following section, this procedure is used to construct a coercive ISS Lyapunov function for systems (4.1) and (4.2).

### 4.3 Construction of an ISS Lyapunov functional

The objective of this section is to study the ISS properties of the KdV models (4.1) and (4.2) through the existence of a strict ISS-Lyapunov functional. The construction of this ISS-Lyapunov functional is based on the strictification procedure described in the previous section. More precisely, let us consider system (4.1) with no inputs, namely  $d_1 = d_2 = 0$ . A formal computation shows that the time derivative of the energy  $E$  defined as

$$E(w) := \int_0^L w(t, x)^2 dx \tag{4.14}$$

yields along solutions

$$\dot{E}(w) := \frac{d}{dt} \int_0^L w(t, x)^2 dx = -|\partial_x w(t, 0)|^2. \tag{4.15}$$

These computations are sufficient to establish that the origin is Lyapunov stable but not to conclude stronger properties (such as asymptotic stability or an ISS property if we re-introduce the effect of the disturbances in the computation of the derivative of the energy along the trajectories of (4.1). In other words, the energy  $E$  is a weak-Lyapunov functional. From [110, Proposition 3.3],  $\partial_x w(t, 0)$  is an exactly observable output as soon as  $L \notin \mathcal{N}$  with

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}} : k, l \in \mathbb{N} \right\},$$

then, following the previous section, our strategy consists in designing an observer with the output  $\partial_x w(t, 0)$ . Such an observer is obtained by using the *backstepping* approach (see, e.g., [81]) and the Fredholm transformation (see, e.g., [29] or [52]). The proposed observer differs from the works in [85, 86, 120] in the same context of KdV equations because a different measured output is considered. Finally, by combining the Lyapunov functional derived from the observer analysis and the energy  $E$ , we obtain a strict Lyapunov functional, that will be used to establish the desired ISS properties for systems (4.1) and (4.2) with respect to the inputs  $d_1$  and  $d_2$ .

Note that we will not provide further discussions on the well-posedness of (4.2) and (4.1), since it is not the main topic of this section. Interested readers may refer to [110, 33, 16] for more information on this issue. Nevertheless, we recall the two results below. The first result deals with mild solutions. The statement reads as follows.

**Proposition 4** [23, Theorem 9] *For any  $T, L > 0$ , for any  $w_0 \in L^2(0, L)$ , for any  $d_1 \in L^1([0, T]; L^2(0, L))$  and  $d_2 \in L^2(0, T)$ , systems (4.1) and (4.2) admit a unique mild solution*

$$w \in C^0([0, T]; L^2(0, L)) \cap L^2(0, T; \mathcal{H}^1(0, L)).$$

The second result deals with more regular solutions. The statement reads as follows.

**Proposition 5** [16, Theorem 1.3][33, Proposition 7] *For any  $T, L > 0$ , for any  $w_0 \in \mathcal{H}_L^3(0, L)$ , for any  $d_1 \in C^1([0, T], L^2(0, L))$  and  $d_2 \in C^2([0, T])$ , systems (4.1) and (4.2) admit a unique classical solution*

$$w \in C(0, T; \mathcal{H}^3(0, L)) \cap C^1(0, T; L^2(0, L))$$

with  $\mathcal{H}_L^3(0, L) := \{w \in \mathcal{H}^3(0, L) : w(0) = w(L) = 0, w'(L) = d_2(0)\}$ .

Next, we state the following definition of input-to-state stability for systems (4.1) and (4.2).

**Definition 5** *System (4.1) (resp. (4.2)) is said to be (exponentially) input-to-state stable (ISS), if there exist positive constants  $c_0, c_1, c_2, \mu$ , such that any solution  $w \in C^0(\mathbb{R}_+; L^2(0, L)) \cap L^2(\mathbb{R}_+; \mathcal{H}^1(0, L))$  to (4.1) (respectively to (4.2)) satisfies for all  $t \geq 0$*

$$\|w(t, \cdot)\|_{L^2} \leq c_0 e^{-\mu t} \|w_0\|_{L^2} + c_1 \int_0^t e^{-\mu(t-s)} \|d_1(s, \cdot)\|_{L^2} ds + c_2 \int_0^t e^{-\mu(t-s)} |d_2(s)| ds, \quad (4.16)$$

for any initial condition  $w_0 \in L^2(0, L)$ ,  $d_1 \in L^2([0, t]; L^2(0, L))$  and  $d_2 \in L^2(0, t)$ . Furthermore, if there

exists  $\delta > 0$  such that (4.16) holds only with  $w_0, d_1, d_2$  satisfying

$$\|w_0\|_{L^2} + \lim_{t \rightarrow \infty} \int_0^t e^{-\mu(t-s)} \left( \|d_1(s, \cdot)\|_{L^2} + |d_2(s)| \right) ds \leq 3\delta$$

then the system (4.1) (resp. (4.2)) is said to be locally (exponentially) input-to-state stable (LISS).

In the literature, the definition (4.16) is related to the notion of the ‘‘Fading Memory Input-to-State Stability’’, see e.g [66, Chapter 7], due to the presence of weighting exponential functions used in the norms characterizing the gain of the signals  $d_1$  and  $d_2$ . Thus, with some abuse of language, we call it Input-to-State Stability in this chapter. Also, it is important to underline that such a definition allows to consider a large class of disturbances  $d_1, d_2$ , which includes, among others, constant and periodic signals.

In general, proving the ISS property defined above needs the knowledge of the trajectories of the system, which is not an easy task. Therefore, in practice, ISS Lyapunov functionals are used to prove the desired ISS properties. To this end, we recall the result in [90, Theorem 3], showing that the existence of an ISS Lyapunov functional is sufficient to establish the ISS properties of Definition 5.

Before stating the definition of such Lyapunov functionals, we recall now which type of derivatives we are going to use in this chapter. Indeed, if any Lyapunov functional  $V$  for solutions to (4.2) or (4.1) is Fréchet differentiable, one has the following equality:

$$\dot{V}_d(w_0) = \left. \frac{d}{dt} V(w(t)) \right|_{t=0} = D_V(w_0) \partial_t w(0), \quad \forall w_0 \in \mathcal{H}^3(0, L), \quad (4.17)$$

where  $D_V(w_0)$  denotes the Fréchet derivative at  $w_0$  (see for instance [37, Definition A.5.33] for the definition). The proof of this equality follows the same path than the one given in [37, Lemma 11.2.5]. For example, the energy  $E(w) = \|w\|_{L^2}^2$  is Fréchet differentiable. Indeed, for all  $w, h \in L^2(0, L)$  we have

$$E(w + h) = E(w) + 2 \int_0^L w(x)h(x)dx + E(h).$$

Thus

$$\lim_{\|h\|_{L^2} \rightarrow 0} \frac{|E(w + h) - E(w) - 2 \int_0^L w(x)h(x)dx|}{\|h\|_{L^2}} = \lim_{\|h\|_{L^2} \rightarrow 0} \frac{E(h)}{\|h\|_{L^2}} = \lim_{\|h\|_{L^2} \rightarrow 0} \|h\|_{L^2} = 0.$$

Therefore, this means that, the energy  $E$  is Fréchet differentiable and the Fréchet differential  $D_E(w)$  at  $w$  is defined by  $h \in L^2(0, L) \mapsto D_E(w)h = 2 \int_0^L w(x)h(x)dx$ . Then, the time derivative of  $E$  along classical solution of (4.2) can be computed as

$$\dot{E}(w) = D_E(w) \partial_t w = 2 \int_0^L (-\partial_x w - \partial_{xxx} w + d_1) w dx, \quad (4.18)$$

and, for time derivative along classical solution of (4.1):

$$\dot{E}(w) = D_E(w) \partial_t w = 2 \int_0^L (-\partial_x w - \partial_{xxx} w - w \partial_x w + d_1) w dx, \quad (4.19)$$



showing that the time does not play any role when using the Fréchet derivative. This is why the time will disappear when differentiating Lyapunov functionals in the rest of the chapter.

We are now in position to state the following definition of ISS Lyapunov functional.

**Definition 6** *A Fréchet differentiable function  $V : L^2(0, L) \rightarrow \mathbb{R}$  is said to be an exponentially ISS Lyapunov functional for the system (4.1) (resp. (4.2)), if there exist positive constants  $\underline{\alpha}, \bar{\alpha}, \alpha, \sigma_1, \sigma_2$  such that:*

(i) *For all  $w \in L^2(0, L)$ ,*

$$\underline{\alpha}\|w\|_{L^2}^2 \leq V(w) \leq \bar{\alpha}\|w\|_{L^2}^2. \quad (4.20)$$

(ii) *The time derivative of  $V$  along the trajectories of (4.1) (resp. (4.2)) satisfies*

$$\dot{V}(w) \leq -\alpha\|w\|_{L^2}^2 + \sigma_1\|d_1\|_{L^2}^2 + \sigma_2|d_2|^2. \quad (4.21)$$

*for any  $w \in L^2(0, L)$ ,  $d_1 \in L^2(0, L)$  and  $d_2 \in \mathbb{R}$ . If there exists  $\delta > 0$  such that (ii) holds only if  $\|w\|_{L^2} + \|d_1\|_{L^2} + |d_2| \leq 3\delta$  then  $V$  is said to be a locally exponentially ISS Lyapunov functional for the system (4.1).*

As explained above, for any  $L \notin \mathcal{N}$ , the energy function in  $L^2$ -norm defined in (4.14) is a weak Lyapunov functional for the system (4.1) (resp. (4.2)) in view of (4.15). Indeed, on the the right hand side of the inequality, we have a function which depends only on a part the state  $w(t, x)$ , i.e.,  $\partial_x w(t, 0)$ . From (4.15), one can deduce that the origin of the system of (4.1) with  $d_1 = d_2 = 0$  is Lyapunov stable. In order to show also the exponential stability properties of the origin, one can follow [110, Proposition 3.3], by using the fact that  $\partial_x w(t, 0)$  is exactly observable as soon as  $L \notin \mathcal{N}$ : indeed, using the related observability inequality, and integrating (4.15) between 0 and  $T$ , exponential stability can be established as illustrated in [21, §4.1.]. However, nothing can be easily said in the presence of disturbances. As a consequence, in order to show the desired ISS properties of the system (4.1) (resp. (4.2)), we follow a different approach here: we aim at constructing a strict ISS Lyapunov functional, which is a new result, to the best of our knowledge. Using the observability of the output  $\partial_x w(t, 0)$ , we can follow the methodology described in the previous section and that can be decomposed as follows. First, we design an observer for the output  $\partial_x w(t, 0)$ . Second, we consider the sum of the Lyapunov functional coming from the latter observer design and the natural energy, and we prove that this sum boils down to be a strict Lyapunov functional. Third, thanks to this strict Lyapunov functional, we deduce ISS properties for systems (4.1) and (4.2). These properties are written more precisely in the following theorem, which is the first main result of this chapter.

**Theorem 7** *Suppose that  $L \notin \mathcal{N}$ . Then, there exists a functional  $W : L^2(0, L) \rightarrow \mathbb{R}_+$  such that, the functional  $V(w) := W(w) + E(w)$  with  $E$  being the energy in  $L^2$ -norm defined in (4.14), is*

(a) *an exponentially ISS Lyapunov functional for the system (4.2);*

(b) *a locally exponentially ISS Lyapunov functional for the system (4.1).*

*Moreover, the functional  $W$  is given by  $W(w) := \|\Pi(w)\|_{L^2}^2$  with  $\Pi$  being a continuous linear operator from  $L^2(0, L)$  to  $L^2(0, L)$  with a continuous inverse.*

The proof of Theorem 7 is postponed to Appendix C. In particular, in the next section, we will first show how to design an ISS observer for the linearized system (4.2) by means of the output  $\partial_x w(t, 0)$ . The proposed design is based on the backstepping method, see, e.g., [81] and on the Fredholm transformation, see, e.g., [52, 29]. Then, in the proof of Theorem 7, we will use the ISS-Lyapunov functional associated to such an observer to build the functional  $W$  claimed in the statement of Theorem 7.

The following result will be also useful when dealing with the regulation problem. It is an ISS result for a perturbed version of (4.1) with non-constant (small) coefficients. Its proof is omitted for compactness since it follows the same path used in the proof of Theorem 7, item (b).

**Corollary 1** *Suppose  $L \notin \mathcal{N}$ . There exists positive real numbers  $\bar{a}, \bar{b}$  such that, for any  $a \in C([0, L])$ ,  $b \in C^1([0, L])$  satisfying  $\|a\|_\infty \leq \bar{a}$  and  $\|b\|_{W^{1,\infty}} \leq \bar{b}$ , the Lyapunov functional  $V$  established in Theorem 7 is a locally exponentially ISS Lyapunov function for the following system*

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w + w \partial_x w = a(x)w + b(x)\partial_x w, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x w(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L]. \end{cases} \quad (4.22)$$

*Moreover, it is an exponential ISS Lyapunov functional for the linearized dynamics of (4.22), i.e. in absence of the term  $w \partial_x w$ .*

### 4.3.1 Observer design for a Linear KdV equation

In this section, we design an observer for the linear KdV equation (4.2) with  $y(t) = \partial_x w(t, 0)$  defined as the output function. In particular, we consider the following system

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w = d_1, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x w(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L], \\ y(t) = \partial_x w(t, 0), & t \in \mathbb{R}_+, \end{cases} \quad (4.23)$$

and we design an observer with a distributed correction term of the form

$$\begin{cases} \partial_t \hat{w} + \partial_x \hat{w} + \partial_{xxx} \hat{w} + p(x)[y(t) - \partial_x \hat{w}(t, 0)] = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \hat{w}(t, 0) = \hat{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x \hat{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \hat{w}(0, x) = \hat{w}_0(x), & x \in [0, L], \end{cases} \quad (4.24)$$

where  $p$  is an output injection gain to be designed. Note that the well-posedness of system (4.24) can be proved by following the same approach as in [85]. We define now the estimation error coordinates as follows

$$\hat{w} \mapsto \tilde{w} := w - \hat{w}$$

mapping system (4.24) into

$$\begin{cases} \partial_t \tilde{w} + \partial_x \tilde{w} + \partial_{xxx} \tilde{w} - p(x) \partial_x \tilde{w}(t, 0) = d_1, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x \tilde{w}(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ \tilde{w}(0, x) = \tilde{w}_0(x), & x \in [0, L]. \end{cases} \quad (4.25)$$

The objective of this section is to show that the gain  $p$  can be selected so that to guarantee the system (4.25) to be ISS with respect to the disturbances  $d_1, d_2$ . This, in turns, guarantees the convergence of the solutions of the observer (4.24) towards the trajectories of the observed plant (4.23) in the unperturbed case ( $d_1 = 0, d_2 = 0$ ), and desirable bounded-input bounded-output properties otherwise. This is established in the next theorem claiming the existence of an ISS-Lyapunov functional for the system (4.25) under an appropriate choice of the function  $p$ .

**Theorem 8** *Suppose that  $L \notin \mathcal{N}$ . For any  $\lambda > 0$ , there exist a non-zero function  $p \in L^2(0, L)$ , a Lyapunov functional  $U : L^2(0, L) \rightarrow \mathbb{R}$  and positive constants  $\underline{c}, \bar{c}, \varrho_1, \varrho_2$  satisfying the following properties.*

(i) For all  $\tilde{w} \in L^2(0, L)$

$$\underline{c} \|\tilde{w}\|_{L^2}^2 \leq U(\tilde{w}) \leq \bar{c} \|\tilde{w}\|_{L^2}^2. \quad (4.26)$$

(ii) The time derivative of  $U$  along the trajectories of (4.25) satisfies, for all  $w \in L^2(0, L)$ ,  $d_1 \in L^2(0, L)$  and  $d_2 \in \mathbb{R}$ ,

$$\dot{U}(\tilde{w}) \leq -\lambda U(\tilde{w}) + \varrho_1 \|d_1\|_{L^2}^2 + \varrho_2 |d_2|^2. \quad (4.27)$$

Moreover, the functional  $U$  is given by  $U(w) := \|\bar{\Pi}^{-1}(w)\|_{L^2}^2$ , with  $\bar{\Pi}$  being a continuous linear operator from  $L^2(0, L)$  to  $L^2(0, L)$  with continuous inverse.

*Proof.* The proof of Theorem 8 is divided into two parts. The first step consists in proving the existence of  $p \in L^2(0, L)$  such that the origin of (4.25), in the unperturbed case  $d_1 = 0, d_2 = 0$ , is exponentially stable. The second step is to show the existence of a Lyapunov functional  $U$  satisfying the inequalities (4.26) and (4.27).

Let us start the proof of the first step. Inspired by [29, equation (1.8)], consider the change of coordinates

$$\tilde{w} \mapsto \gamma := \bar{\Pi}^{-1} \tilde{w} \quad (4.28)$$

where the functional  $\bar{\Pi}$  is defined thanks to the following Fredholm integral transformation

$$\tilde{w}(x) = \bar{\Pi}(\gamma)(x) := \gamma(x) - \int_0^L P(x, z) \gamma(z) dz, \quad (4.29)$$

for all  $x \in [0, L]$ , where  $\tilde{w}$  satisfies (4.25) with  $d_1 = 0$  and  $d_2 = 0$ ,  $P$  is a function to be defined and  $\gamma$  is

the solution to the following system

$$\begin{cases} \partial_t \gamma + \partial_x \gamma + \partial_{xxx} \gamma + \lambda \gamma = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \gamma(t, 0) = \gamma(t, L) = \partial_x \gamma(t, L) = 0, & t \in \mathbb{R}_+, \\ \gamma(0, x) = \gamma_0(x), & x \in [0, L], \end{cases} \quad (4.30)$$

with  $\lambda > 0$ . Note that using an integration by parts and the boundary conditions of (4.30), one immediately obtains

$$\frac{d}{dt} \int_0^L |\gamma(t, x)|^2 dx \leq -2\lambda \int_0^L |\gamma(t, x)|^2 dx$$

from which it is straightforward to deduce the exponential stability in the  $L^2$ -norm of  $\gamma$ . As a consequence, the main idea of the proof consists in selecting the function  $p$  such that (4.29) holds. To do so, we need to find the kernel  $P$  such that  $\tilde{w}(t, x) = \bar{\Pi}(\gamma)(t, x)$  satisfies (4.25) when  $d_1 = 0$  and  $d_2 = 0$ . Furthermore, we have also to ensure that the corresponding transformation is invertible and continuous. To this end, we first formally differentiate with respect to the time and with respect to the space the change of coordinates (4.29). We obtain the following identities

$$\partial_t \tilde{w}(t, x) = \partial_t \gamma(t, x) + \int_0^L P(x, z) (\lambda \gamma(t, z) + \gamma_z(t, z) + \gamma_{zzz}(t, z)) dz, \quad (4.31)$$

$$\partial_x \tilde{w}(t, x) = \partial_x \gamma(t, x) - \int_0^L \partial_x P(x, z) \gamma(t, z) dz, \quad (4.32)$$

$$\partial_{xxx} \tilde{w}(t, x) = \partial_{xxx} \gamma(t, x) - \int_0^L \partial_{xxx} P(x, z) \gamma(t, z) dz, \quad (4.33)$$

in which (4.31) has been obtained by using the  $\gamma$ -dynamics in (4.30). After some integrations by parts, (4.31) gives

$$\begin{aligned} \partial_t \tilde{w}(t, x) &= \partial_t \gamma(t, x) - P(x, 0) \gamma(t, 0) + P(x, L) \gamma(t, x) + P(x, L) \partial_{xx} \gamma(t, L) - P(x, 0) \partial_{xx} \gamma(t, 0) \\ &\quad - \int_0^L \left( -\lambda P(x, z) + \partial_z P(x, z) + \partial_{zzz} P(x, z) \right) \gamma(t, z) dz - \partial_z P(x, L) \partial_x \gamma(t, L) + \partial_{zz} P(x, L) \gamma(t, L) \\ &\quad - \partial_{zz} P(x, 0) \gamma(t, 0) + \partial_z P(x, 0) \partial_x \gamma(t, 0). \end{aligned} \quad (4.34)$$

Then, by adding on both sides the terms  $\partial_x \tilde{w}$ ,  $\partial_{xxx} \tilde{w}$  and  $-p(x) \partial_x \tilde{w}(t, 0)$  and using (4.25), (4.30) and the previous identities (4.32), (4.33), we further obtain

$$\begin{aligned} \partial_t \tilde{w}(t, x) + \partial_x \tilde{w}(t, x) + \partial_{xxx} \tilde{w}(t, x) - p(x) \partial_x \tilde{w}(t, 0) &= \\ &= \partial_t \gamma(t, x) + \partial_x \gamma(t, x) + \partial_{xxx} \gamma(t, x) + \lambda \gamma(t, x) - \int_0^L \left( -\lambda P + \partial_z P + \partial_{zzz} P + \partial_{xxx} P + \partial_x P \right) \gamma(t, z) dz \\ &\quad - \lambda \gamma(t, x) + P(x, L) \partial_{xx} \gamma(t, L) + \partial_z P(x, 0) \partial_x \gamma(t, 0) - P(x, 0) \partial_{xx} \gamma(t, 0) \\ &\quad - p(x) \left[ \partial_x \gamma(t, 0) - \int_0^L \partial_x P(0, z) \gamma(t, z) dz \right]. \end{aligned}$$

Then, using the identity

$$-\lambda\gamma(t, x) = \int_0^L \lambda\delta(x - z)\gamma(t, z)dz,$$

where  $\delta(x - z)$  denotes the Dirac measure on the diagonal of the square  $[0, L] \times [0, L]$ , the previous equation gives

$$\begin{aligned} & \partial_t \tilde{w}(t, x) + \partial_x \tilde{w}(t, x) + \partial_{xxx} \tilde{w}(t, x) - p(x)\partial_x \tilde{w}(t, 0) \\ &= \partial_t \gamma(t, x) + \partial_x \gamma(t, x) + \partial_{xxx} \gamma(t, x) + \lambda\gamma(t, x) - P(x, 0)\partial_{xx} \gamma(t, 0) + P(x, L)\partial_{xx} \gamma(t, L) \\ & \quad - \int_0^L \left( -\lambda P + \partial_z P + \partial_{zzz} P + \partial_x P + \partial_{xxx} P - \lambda\delta(x - z) \right) \gamma(t, z) dz \\ & \quad + p(x) \int_0^L \partial_x P(0, z) \gamma(t, z) dz - \partial_x \gamma(t, 0) [p(x) - \partial_z P(x, 0)]. \end{aligned} \quad (4.35)$$

From equation (4.35), we finally obtain the following conditions for the functions  $P$  and  $p$ .

- (a) The identity  $-\lambda P + \partial_z P + \partial_{zzz} P + \partial_x P + \partial_{xxx} P = \lambda\delta(x - z)$  is satisfied for all  $(x, z) \in [0, L] \times [0, L]$ .
- (b) The boundary conditions  $P(x, 0) = P(x, L) = \partial_x P(0, z) = 0$  are satisfied for all  $(x, z) \in [0, L] \times [0, L]$ .
- (c) An appropriate choice of  $p$  is given by  $p(x) := \partial_z P(x, 0)$ , for all  $x \in [0, L]$ .

Moreover, note also that the following.

- (d) By setting  $x = 0$  and  $x = L$  in (4.29), we need:  $P(0, z) = P(L, z) = 0$  for all  $z \in [0, L]$ .
- (e) By setting  $x = L$  in (4.32), we need:  $\partial_x P(L, z) = 0$  for all  $z \in [0, L]$ .

Therefore, collecting the conditions (a)-(e), we impose the function  $P$  to satisfy the following PDE

$$\begin{cases} -\lambda P + \partial_z P + \partial_{zzz} P + \partial_x P + \partial_{xxx} P = \lambda\delta(x - z), \\ P(x, 0) = P(x, L) = 0, \\ P(L, z) = P(0, z) = 0, \\ \partial_x P(L, z) = \partial_x P(0, z) = 0, \end{cases} \quad (4.36)$$

where  $(x, z) \in [0, L] \times [0, L]$  and  $\delta(x - z)$  denotes the Dirac measure on the diagonal of the square  $[0, L] \times [0, L]$ . Now, in order to show the existence of a solution to (4.36), let us make the following change of variable:

$$\begin{pmatrix} z \\ x \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} := \begin{pmatrix} L - z \\ L - x \end{pmatrix},$$

and define  $G(\bar{x}, \bar{z}) := -P(x, z)$ . From (4.36) it is obtained

$$\begin{cases} \lambda G + \partial_{\bar{z}} G + \partial_{\bar{z}\bar{z}\bar{z}} G + \partial_{\bar{x}} G + \partial_{\bar{x}\bar{x}\bar{x}} G = \lambda\delta(\bar{x} - \bar{z}), \\ G(\bar{x}, 0) = G(\bar{x}, L) = 0, \\ G(L, \bar{z}) = G(0, \bar{x}) = 0, \\ \partial_{\bar{z}} G(\bar{x}, 0) = \partial_{\bar{z}} G(\bar{x}, L) = 0, \end{cases} \quad (4.37)$$

with  $(\bar{x}, \bar{z})$  belonging to  $[0, L] \times [0, L]$ . Note that in [29, Lemma 2.1], it has been proved that, for any  $L \notin \mathcal{N}$ , the system (4.37) admits a unique solution  $G \in \mathcal{H}_0^1((0, L) \times (0, L))$ . Therefore, we can conclude that the kernel  $P$  exists. Then according to [29, Lemma 3.1], the transformation  $\bar{\Pi}$  is invertible and continuous on  $L^2(0, L)$  and its inverse is also continuous. As a consequence, we have shown that, for an appropriate choice of the function  $p \in L^2(0, L)$ , the system (4.25) is transformed into the system (4.30) via a linear change of coordinates which is invertible with a continuous inverse. Since the origin of system (4.30) is exponentially stable, we conclude that the same is true for the origin of (4.25) in the non-perturbed case (i.e.,  $d_1 = 0, d_2 = 0$ ). Note also that  $p$  is non-zero. Indeed, if  $p = 0$  then, in view of the condition (c), we would have  $\partial_z P(x, 0) = 0$ . Therefore, the system (4.36) would have seven boundary conditions. But then, because of the degree of the first equation of (4.36), the system (4.36) would have no solution. This concludes the first part of the proof.

We want now to prove the existence of a Lyapunov functional which satisfies the inequalities (4.26) and (4.27) in presence of  $d_1, d_2$ . To this end, we choose the following candidate Lyapunov functional  $U : L^2(0, L) \rightarrow \mathbb{R}$

$$U(w) := \|\bar{\Pi}^{-1}(w)\|_{L^2}^2 \quad (4.38)$$

Since  $\bar{\Pi}^{-1}$  exists, then  $U$  is well defined in  $L^2(0, L)$ . Moreover, according to the continuity of  $\bar{\Pi}^{-1}$  and  $\bar{\Pi}$  in  $L^2(0, L)$ , there exist two positive constants  $\underline{c}$  and  $\bar{c}$  satisfying inequality (4.26) for all  $w \in L^2(0, L)$ . Note that the functional  $w \in L^2(0, L) \mapsto U(w) \in \mathbb{R}_+$  is equivalent to the standard norm on the space  $L^2(0, L)$  according to (4.26). It only remains to prove that  $U$  satisfies the inequality (4.27). To this end, we show inequality (4.27) for  $\tilde{w}_0 \in \mathcal{H}_L^3(0, L)$ ,  $d_2 \in C^2([0, T])$  and  $d_1 \in C^1([0, T], L^2(0, L))$ . The result follows for all  $\tilde{w}_0 \in L^2(0, L)$ ,  $d_1 \in L^1([0, T]; L^2(0, L))$  and  $d_2 \in L^2(0, T)$ , by a standard density argument similar to the one used in [83, Lemma 1]. Now, consider again the transformation defined in (4.28), (4.29). Similar computations can be used to show that its inverse transformation is defined by

$$\gamma(x) := \bar{\Pi}^{-1}(\tilde{w})(x) = \tilde{w} + \int_0^L Q(x, z)\tilde{w}(z)dz, \quad (4.39)$$

where  $Q \in \mathcal{H}_0^1((0, L) \times (0, L))$  is now the solution of the following system

$$\begin{cases} \lambda Q + \partial_z Q + \partial_{zzz} Q + \partial_x Q + \partial_{xxx} Q = \lambda \delta(x - z), \\ Q(x, 0) = Q(x, L) = 0, \\ Q(L, z) = Q(0, z) = 0, \\ \partial_x Q(L, z) = 0, \end{cases} \quad (4.40)$$

and satisfies  $p(x) + \int_0^L p(z)Q(x, z)dz = \partial_z Q(x, 0)$  for all  $x \in [0, L]$ . Now, consider the solution  $\tilde{w}$  of system (4.25) with  $d_1, d_2$  possibly different from zero. Then, applying the change of coordinates  $\gamma = \bar{\Pi}^{-1}(\tilde{w})$  defined in (4.39), (4.40), we obtain

$$\begin{cases} \partial_t \gamma + \partial_x \gamma + \partial_{xxx} \gamma + \lambda \gamma = \bar{\Pi}^{-1}(d_1) + \partial_z Q(x, L)d_2, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \gamma(t, 0) = \gamma(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x \gamma(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ \gamma(0, x) = \gamma_0(x), & x \in [0, L]. \end{cases} \quad (4.41)$$

The derivative of  $U$  along the trajectory of (4.25), or equivalently on the trajectory of (4.41), yields

$$\begin{aligned}\dot{U}(w) &= -2 \int_0^L \gamma \left( \partial_{xxx} \gamma + \partial_x \gamma + \lambda \gamma - \bar{\Pi}^{-1}(d_1) - \partial_z Q(x, L) d_2 \right) dx \\ &= -2\lambda \int_0^L |\gamma|^2 dx + 2 \int_0^L \partial_x \gamma \partial_{xx} \gamma dx + 2 \int_0^L \bar{\Pi}^{-1}(d_1) \gamma dx + 2d_2 \int_0^L \partial_z Q(x, L) \gamma dx \\ &\leq -2\lambda \|\gamma\|_{L^2}^2 + 2 \left| \int_0^L \bar{\Pi}^{-1}(d_1) \gamma dx \right| + d_2^2 - (\partial_x \gamma(0))^2 + 2 \left| d_2 \int_0^L \partial_z Q(x, L) \gamma dx \right|,\end{aligned}\quad (4.42)$$

where, in the second equation, we have used an integration by parts to compute

$$2 \int_0^L \partial_x \gamma \partial_{xx} \gamma dx = \left[ (\partial_x \gamma(x))^2 \right]_0^L = d_2^2 - (\partial_x \gamma(0))^2.$$

Using first Cauchy-Schwarz's inequality and then Young's inequality  $2ab \leq \nu a^2 + \frac{1}{\nu} b^2$ , for any  $\nu > 0$ , from (4.42) we finally obtain

$$\begin{aligned}\dot{U}(w) &\leq -2\lambda \|\gamma\|_{L^2}^2 + 2\|\gamma\|_{L^2} \|\bar{\Pi}^{-1}(d_1)\|_{L^2} + 2|d_2| \|\gamma\|_{L^2} \|\partial_z Q(\cdot, L)\|_{L^2} + |d_2|^2 \\ &\leq -\lambda \|\gamma\|_{L^2}^2 + \frac{2}{\lambda} \|\bar{\Pi}^{-1}(d_1)\|_{L^2}^2 + \left( 1 + \frac{2}{\lambda} \|\partial_z Q(\cdot, L)\|_{L^2}^2 \right) |d_2|^2.\end{aligned}$$

Using the inequality (4.26) on the term depending on  $d_1$ , we finally obtain

$$\dot{U}(w) \leq -\lambda \|\gamma\|_{L^2}^2 + \frac{2\bar{c}}{\lambda} \|d_1\|_{L^2}^2 + \left( 1 + \frac{2}{\lambda} \|\partial_z Q(\cdot, L)\|_{L^2}^2 \right) |d_2|^2 \quad (4.43)$$

showing the inequality (4.27) with  $\varrho_1 = \frac{2\bar{c}}{\lambda}$ ,  $\varrho_2 = 1 + \frac{2}{\lambda} \|\partial_z Q(\cdot, L)\|_{L^2}^2$ . This completes the proof.  $\square$

From the existence of the ISS Lyapunov functional established in Theorem 8, one can immediately deduce the following property for the observer (4.24).

**Corollary 2** *For any  $\lambda > 0$ , there exists a function  $p \in L^2(0, L)$  such that the observer (4.24) is an ISS exponential convergent observer for system (4.23) with convergence rate  $\lambda$ , namely, there exist some  $c_0, c_1, c_2 > 0$  such that the following inequality holds*

$$\|\widehat{w}(t, \cdot) - w(t, \cdot)\|_{L^2} \leq c_0 e^{-\lambda t} \|\widehat{w}_0 - w_0\|_{L^2} + c_1 \int_0^t e^{-\lambda(t-s)} \|d_1(s, \cdot)\|_{L^2} ds + c_2 \int_0^t e^{-\lambda(t-s)} |d_2(s)| ds, \quad (4.44)$$

*for any initial conditions  $w_0, \widehat{w}_0 \in L^2(0, L)$ , any  $d_1 \in L^2([0, t]; L^2(0, L))$ , any  $d_2 \in L^2(0, t)$  and for all  $t \geq 0$ .*

*Proof.* The proof can be directly inherited from Theorem 8 by applying Grönwall's lemma to inequality (4.27).  $\square$

As a conclusion of this section, we note that, taking into account the exponential stability properties of the observer (4.24), a local observer may also be designed for the non-linear KdV model (4.1). In

particular, selecting the gain  $p$  as in Corollary 2 it is possible to show that the following system

$$\begin{cases} \partial_t \widehat{w} + \partial_x \widehat{w} + \partial_{xxx} \widehat{w} + \widehat{w} \partial_x \widehat{w} + p(x)[y(t) - \partial_x \widehat{w}(t, 0)] = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \widehat{w}(t, 0) = \widehat{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x \widehat{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \widehat{w}(0, x) = \widehat{w}_0(x), & x \in [0, L], \end{cases}$$

is a locally exponentially ISS observer for system (4.1), namely inequality (4.44) holds for all  $w_0, \widehat{w}_0, d_1, d_2$  satisfying

$$\|\widehat{w}_0\|_{L^2} + \|w_0\|_{L^2} + \lim_{t \rightarrow \infty} \int_0^t e^{-\mu(t-s)} \left( \|d_1(s, \cdot)\|_{L^2} + |d_2(s)| \right) ds \leq \delta$$

for some  $\delta$  small enough. The proof can be derived by combining the arguments of the proof of Theorem 8 with the robustness result established in Corollary 1.

In the next section, the proposed ISS-Lyapunov functional in Theorem 7 will be used in order to design an output feedback integral action controller.

## 4.4 Output regulation problem

In this section we consider the regulation problem of a KdV equation in which the disturbance  $d_2$  is considered as a control input acting at the boundary, and the output  $y(t) = \partial_x w(t, 0)$  has to be regulated at a certain desired constant reference  $r$  in presence of unknown distributed constant disturbances  $d_1$ . Roughly speaking, the output regulation problem consists in designing a feedback-law such that the output converges asymptotically towards a desired reference and such that disturbances are rejected, possibly in spite of some “small” model uncertainties. Following the celebrated *internal-model principle*, a solution to such a problem exists when references and disturbance (denoted generically as exosignals) are generated by a known autonomous dynamical system (denoted as exosystem), and a copy of such a system is embedded in the controller dynamics, see, e.g. [50, 99]. A well known example is the use of integral action for tracking and rejecting constant references and disturbances.

Output regulation is an old topic in the finite dimensional context, but many results remain to be found in the context of nonlinear systems (see e.g., [7, 53] for recent results in this field), and many further research lines have to be followed when dealing with time-varying references. See, for instance, [5] where a finite-dimensional system is regulated by adding a transport equation for the case of periodic exosignals. For infinite dimensional systems, even if one can mention some old results such as [60], the topic is still very active. A generalization of internal-model principle has been proposed in [99], but the use of integral action to achieve output regulation in the presence of constant references/perturbations for infinite dimensional systems has been initiated earlier in [104]. Since then, several methods to design an integral action have been developed for linear dynamics following, for instance, a spectral approach in [103, 135, 99], by using operator and semi-group methods in [80, 136], based on frequency domain methods with Laplace transform in [13, 30] or by relying on Lyapunov techniques in [76], [43, 124]. We may also mention [40, 41] which propose to regulate an output towards time-varying references that are generated by a known linear dynamical system. In the context of nonlinear PDEs, we recall also the



works [92, 65, 138].

Among all these techniques, we are particularly interested in Lyapunov techniques. Indeed, such a methodology has been proved to be efficient to deal with nonlinear systems. Among these techniques, we aim at using the forwarding methodology that has been first introduced for finite dimensional systems in cascade form [89, 7] and then extended to some hyperbolic systems [122] in the regulation context, and to abstract systems [84] in the stabilization context. In [122], it is shown that a *strict* Lyapunov functional is needed for open-loop stable systems that we aim at regulating. In other words, before adding an integral action, we should be able to show that a strict Lyapunov functional for the open-loop dynamics does exist (or can be obtained after employing a preliminary stabilizing state-feedback, see, e.g. [7] in the finite dimensional context). Such Lyapunov functionals are known for hyperbolic systems [12], but it is not the case for the KdV equation. In addition to the existence of this Lyapunov functional, some ISS properties are needed to apply the forwarding method.

To achieve the output regulation problem, we extend the plant with an integral action processing the error  $\partial_x w(t, 0) - r$  and we show how to design an output-feedback law. The gain of the controller is obtained via the forwarding technique which is employed to construct a strict Lyapunov functional built upon the ISS-Lyapunov functional obtained in the section 4.3. Global stability properties are established for the linear model (4.2) while only local ones are proved for the nonlinear one (4.1). Note that, in both cases, we prove pointwise convergence of the tracking error i.e  $\lim_{t \rightarrow \infty} |\partial_x w(t, 0) - r| = 0$ . This is in contrast with the results in [109, Theorem 1.1] where the convergence towards 0 of the tracking error has been proved only in the norm of the space  $L_\alpha^2([0, +\infty))$ , for some  $\alpha < 0$ . However, from a practical point of view, it is more interesting to have a pointwise convergence. In the context of output regulation of nonlinear PDEs as in (4.1), there exist very few results. Let us mention for instance [138] which studies quasilinear hyperbolic systems. We recall also [92, 65] in which the local problem is solved for regular linear operators perturbed by nonlinearities satisfying a Lipschitz condition. Note, however, that these results do not directly apply to the KdV nonlinear model because the nonlinearity  $w \partial_x w$  is not Lipschitz in the right space. In this section, we are able to solve the local regulation problem for (4.1) thanks to the strict Lyapunov functional that we established in section 4.3.

In the section 4.4.1, we focus on the linearized version of the KdV model (4.2). Then, in section 4.4.2, we will show a local result for the nonlinear system (4.1).

#### 4.4.1 Regulation of linear KdV equation by means of the forwarding method

Consider the following system

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w = d(x), & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+ \\ \partial_x w(t, L) = u(t), & t \in \mathbb{R}_+ \\ w(0, x) = w_0(x), & x \in [0, L] \\ y(t) = \partial_x w(t, 0), & t \in \mathbb{R}_+ \end{cases} \quad (4.45)$$

where  $d \in L^2(0, L)$  is a constant perturbation,  $u \in \mathbb{R}$  is the control input, and  $y \in \mathbb{R}$  is the output to be regulated at a certain desired constant reference  $r$ . We define the regulated output error  $e = y - r$  and

we defined our regulation objective as

$$\lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} y(t) - r = 0. \quad (4.46)$$

To this end, we follow the standard set-up of output regulation [7, 122] and we extend system (4.45) with an integral action processing the desired error to be regulated. In other words, we consider a dynamical feedback law of the form

$$\dot{\eta} = y - r, \quad u = k\eta, \quad (4.47)$$

where  $\eta \in \mathbb{R}$  is the state of the controller and  $k$  is a positive constant to be selected small enough, as shown later. The closed-loop system (4.45), (4.47) can be seen as an augmented system, i.e. a PDE system (whose state is  $w$ ) coupled with an ODE (whose state is  $\eta$ ), which reads

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w = d(x), & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x w(t, L) = k\eta(t), & t \in \mathbb{R}_+, \\ \dot{\eta}(t) = \partial_x w(t, 0) - r, & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), \eta(0) = \eta_0, & x \in [0, L]. \end{cases} \quad (4.48)$$

We define the space  $X := \mathbb{R} \times L^2(0, L)$ , that is the state space of (4.48). It is a Hilbert space as the Cartesian product of two Hilbert spaces. In the rest of the section, we will show the following properties for the closed-loop system (4.48): it is well posed, it admits a unique equilibrium which is exponentially stable, and the regulation objective (4.46) is achieved when considering sufficiently regular solutions.

To this end, we introduce now the following two linear operators  $\mathcal{S}$  and  $\mathcal{A}$  that will be used in the rest of the section. In particular, we denote with  $\mathcal{S}$  the operator associated with the linear KdV equation (4.2). The operator  $\mathcal{S}$  and its domain  $D(\mathcal{S}) \subset L^2(0, L)$  are defined as

$$\mathcal{S}w = -w' - w''', \quad D(\mathcal{S}) := \{w \in \mathcal{H}^3(0, L) : w(0) = w(L) = w'(L) = 0\}. \quad (4.49)$$

Then, we define the operator  $\mathcal{A}$  in order to describe the closed-loop system (4.48) in the following abstract form

$$\frac{d}{dt}\zeta = \mathcal{A}\zeta + \Gamma, \quad \zeta(0) = \tilde{\zeta}_0, \quad \zeta := \begin{pmatrix} \eta \\ w \end{pmatrix}, \quad \mathcal{A}(\eta, w) := \begin{bmatrix} w'(0) \\ -w' - w''' \end{bmatrix}, \quad \Gamma := \begin{bmatrix} -r \\ d \end{bmatrix}, \quad (4.50)$$

with the domain of  $\mathcal{A}$  defined as  $D(\mathcal{A}) := \{(\eta, w) \in \mathbb{R} \times \mathcal{H}^3(0, L) \mid w(0) = w(L) = 0, w'(L) = k\eta\} \subset X$ . We start by proving the existence and uniqueness of an equilibrium for system (4.48) in the following lemma.

**Lemma 1** *For any  $k \neq 0$  and  $(d, r) \in L^2(0, L) \times \mathbb{R}$  there exist a unique equilibrium state  $(\eta_\infty, w_\infty) \in X$  to system (4.48).*

The proof of Lemma 1 is postponed to Appendix C

Next, we show the following well-posedness result for the closed-loop system (4.48). In the proof, we will also introduce a strict Lyapunov functional for the closed-loop system (4.48). Such a Lyapunov functional is obtained via the forwarding methodology similarly to [122, 84] and it is based on the ISS-Lyapunov established in Theorem 7.

**Lemma 2** *Let  $L \notin \mathcal{N}$ . There exist  $k_0^* > 0$  such that for any  $k \in (0, k_0^*)$ , for any  $(d, r) \in L^2(0, L) \times \mathbb{R}$  and for any initial condition  $(\eta_0, w_0) \in X$  (resp.  $D(\mathcal{A})$ ), there exists a unique weak solution  $(\eta, w) \in C^0(\mathbb{R}_+; X)$  (resp. strong solution in  $C^1(\mathbb{R}_+; X) \cap C^0(\mathbb{R}_+; D(\mathcal{A}))$ ) to system (4.48).*

The proof of Lemma 2 is postponed to Appendix C.

Finally, the next result deals with the exponential stability of equilibrium state  $(\eta_\infty, w_\infty)$  and with the related output regulation objective (4.46).

**Theorem 9 (Stabilization and regulation)** *Let  $L \notin \mathcal{N}$  and consider system (4.48). For any  $k \in (0, k_0^*)$ , with  $k_0^*$  given by Lemma 2, there exist  $b_0, \nu_0 > 0$ , and for any  $(d, r) \in L^2(0, L) \times \mathbb{R}$  there exists  $(\eta_\infty, w_\infty) \in X$ , computed according to Lemma 1, such that any solution to system (4.48) with initial condition  $(\eta_0, w_0) \in X$  satisfies*

$$\|(\eta(t), w(t, \cdot)) - (\eta_\infty, w_\infty)\|_X \leq b_0 e^{-\nu_0 t} \|(\eta_0, w_0) - (\eta_\infty, w_\infty)\|_X. \quad (4.51)$$

*for all  $t \geq 0$ . Moreover, for any strong solution to (4.48), and in particular, for any  $(\eta_0, w_0) \in D(\mathcal{A})$ , the output  $y$  is asymptotically regulated at the reference  $r$ , namely (4.46) is satisfied.*

The proof of Theorem 9 is postponed to Appendix C.

We remark that the results of Lemma 2 and Theorem 9 can be alternatively proven according to [109, Theorem 1.1] and standard properties of well-posed linear systems (see, e.g., [126, Proposition 4.6]). These proofs, however, cannot be easily extended to the context of nonlinear systems. In this article, instead, we propose alternative proofs which are based on Lyapunov arguments. As we shall see in the next section, such proofs are instrumental to the design of an integral action control of the form (4.47) for the nonlinear model (4.1).

#### 4.4.2 Regulation of nonlinear KdV equation by means of the forwarding method

In this section, we consider the regulation problem for a nonlinear KdV equation (4.1). In particular, we consider the system

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w + w \partial_x w = d(x), & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x w(t, L) = u(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L], \\ y(t) = \partial_x w(t, 0), & t \in \mathbb{R}_+, \end{cases} \quad (4.52)$$

where  $d \in L^2(0, L)$  is a constant perturbation,  $u \in \mathbb{R}$  is the control input and  $y(t) \in \mathbb{R}$  is the output to be regulated to a constant reference  $r$  as in (4.46). Following the design proposed in Section 4.4.1 for the linear model (4.2), we consider the same output-feedback integral control (4.47) and we compactly write the closed-loop system (4.52), (4.47) as

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w + w \partial_x w = d(x), & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x w(t, L) = k\eta(t), & t \in \mathbb{R}_+, \\ \dot{\eta}(t) = \partial_x w(t, 0) - r, & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), \eta(0) = \eta_0, & x \in [0, L]. \end{cases} \quad (4.53)$$

In the following, we will show that for sufficiently small perturbations  $d$  and references  $r$  the closed-loop system (4.53) is well posed and it admits a unique equilibrium which is locally exponentially stable. Furthermore, for solutions which are sufficiently regular, the regulation objective (4.46) is satisfied. We start by showing the existence and uniqueness of an equilibrium.

**Lemma 3** *There exist  $\bar{d} > 0$  and  $\bar{r} > 0$  such that, for any  $(d, r) \in L^2(0, L) \times \mathbb{R}$  satisfying  $\|d\|_{L^2} \leq \bar{d}$  and  $|r| \leq \bar{r}$ , there exists a unique equilibrium state  $(\eta_\infty, w_\infty) \in X$  to system (4.53). Furthermore there exists  $\bar{w} > 0$  such that for any  $\bar{w}_0 \in (0, \bar{w}]$ , there exists  $d_0 > 0$  and  $r_0 > 0$  so that, for any  $(d, r) \in L^2(0, L) \times \mathbb{R}$  satisfying  $\|d\|_{L^2} \leq d_0$  and  $|r| \leq r_0$  then  $\|w_\infty\|_{\mathcal{H}^3} \leq \bar{w}_0$ .*

The proof of Lemma 3 is postponed to Appendix C.

Now, given  $(d, r) \in L^2(0, L) \times \mathbb{R}$  satisfying the assumptions of Lemma 3, let  $(\eta_\infty, w_\infty)$  be the corresponding equilibrium to system (4.53) and consider the following change of coordinates

$$(w, \eta) \mapsto (\tilde{w}, \tilde{\eta}) := (w - w_\infty, \eta - \eta_\infty).$$

The  $(\tilde{w}, \tilde{\eta})$ -dynamics is given by

$$\begin{cases} \partial_t \tilde{w} + \partial_x \tilde{w} + \partial_{xxx} \tilde{w} + \tilde{w} \partial_x \tilde{w} = -w'_\infty \tilde{w} - w_\infty \partial_x \tilde{w}, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x \tilde{w}(t, L) = k\tilde{\eta}(t), & t \in \mathbb{R}_+, \\ \dot{\tilde{\eta}}(t) = \partial_x \tilde{w}(t, 0), & t \in \mathbb{R}_+, \\ \tilde{w}(0, x) = \tilde{w}_0(x), \tilde{\eta}(0) = \tilde{\eta}_0, & x \in [0, L], \end{cases} \quad (4.54)$$

where  $\tilde{w}_0(x) = w_0(x) - w_\infty(x) \in \mathcal{H}^3(0, L)$  and  $\tilde{\eta}_0 = \eta_0 - \eta_\infty \in \mathbb{R}$ . In the new coordinates, the regulation objective (4.46) for system (4.54) reads

$$\lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} \partial_x \tilde{w}(t, 0) = 0 \quad (4.55)$$

Note that showing the well-posedness of system (4.54) is equivalent to prove the well-posedness of system (4.53) in the original coordinates  $(w, \eta)$ . As a consequence, in the rest of the section, we will focus on the system (4.54) in the new coordinates  $(\tilde{w}, \tilde{\eta})$ .

**Lemma 4** For any  $\bar{w}_\infty$ , there exists  $k_1^* > 0$  such that, for any  $k \in (0, k_1^*]$ , for any  $w_\infty \in \mathcal{H}^3(0, L)$  satisfying  $\|w_\infty\|_{\mathcal{H}^3} \leq \bar{w}_\infty$ , and any initial condition  $(\tilde{\eta}_0, \tilde{w}_0) \in D(\mathcal{A})$ , there exists  $\tau > 0$  such that the Cauchy problem (4.54) is well-posed in the space  $C^1(0, \tau) \times (C([0, \tau]; \mathcal{H}^3(0, L)) \cap L^2([0, \tau]; \mathcal{H}^4(0, L)))$ .

The proof of Lemma 4 is postponed to Appendix C.

Note that we have established the existence of unique classical solution of (4.54) locally in time. However, the Lyapunov functional introduced in the Section 4.4.1 needed to establish Lemma 2 and Theorem 9 can be used to deduce the existence of unique solution global in time. Indeed, since the time derivative of the Lyapunov functional will be proved to be non-increasing, this shows that the solution cannot explode for large time, proving thus that the solution exists for any positive time, as soon as the initial conditions are small enough. The next result deals with the local exponential stability of the origin of system (4.54).

**Theorem 10 (Local Exponential Stability)** There exist positive real number  $k_2^*, \bar{w}_\infty$ , such that, for any  $k \in (0, k_2^*)$  there exist positive real numbers  $\Delta, \nu_1, b_1$ , such that for any solution to system (4.54) with  $w_\infty$  satisfying  $\|w_\infty\|_{\mathcal{H}^3} \leq \bar{w}_\infty$  and initial conditions  $(\tilde{\eta}_0, \tilde{w}_0) \in D(\mathcal{A})$  satisfying  $|\tilde{\eta}_0| + \|\tilde{w}_0\|_{L^2} \leq 2\Delta$ , the following inequality holds  $\|(\tilde{\eta}(t), \tilde{w}(t))\|_X \leq b_1 e^{-\nu_1 t} \|(\tilde{\eta}_0, \tilde{w}_0)\|_X$  for all  $t \geq 0$ . Moreover the regulation objective defined in (4.55) is satisfied.

The proof of Theorem 10 is postponed to Appendix C.

Finally, by combining the statement of Lemma 3 and Theorem 10 we have the following output regulation result for the system (4.53) in the original coordinates  $w, \eta$ .

**Corollary 3 (Output Regulation)** There exist positive real numbers  $k_2^*, \bar{d}, \bar{r}$ , such that, for any disturbance  $d$  and reference  $r$  satisfying  $\|d\|_{L^2} \leq \bar{d}$  and  $|r| \leq \bar{r}$  and for any  $k \in (0, k_2^*)$  the output  $y$  is asymptotically regulated at the reference  $r$ , namely (4.46) is satisfied, for any solution to system (4.53), with initial conditions  $(\eta_0, w_0) \in D(\mathcal{A})$  sufficiently small in the norm  $\mathbb{R} \times L^2(0, L)$ .

## 4.5 Conclusion

In this chapter, we have solved the output regulation problem by means of an integral action for a Korteweg-de-Vries (KdV) equation controlled at the boundary and subject to some distributed constant disturbances so that to regulate a boundary output to a given constant reference. For this, we have followed a Lyapunov approach. We have first designed an ISS Lyapunov functional which is obtained by strictifying the energy associated to the system. In particular, the energy is modified by adding a second term which is obtained from the design of an observer built with the backstepping technique. Then, thanks to this ISS Lyapunov functional, we have applied the forwarding method to achieve our goal in the context of output regulation. In particular, we extended the system with an integral action and

we designed an output feedback controller acting at the boundary. We show that if the selected gain is sufficiently small then the solutions of the closed-loop system converge to an equilibrium. Furthermore, for strong solutions, point-wise convergence of the regulated output is achieved. Similar results hold locally for the nonlinear model of the KdV, namely in presence of small references and perturbations and with a local domain of attraction.



# CONCLUSION

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In this thesis, some problems related to the robust control of infinite-dimensional systems have been investigated. To be more precise, we have addressed Questions 1, 2 and 3 stated in the Introduction using various techniques from finite and infinite-dimensional system theory.

## Question 1

How to propose a systematic methodology for the design of sliding variables for linear infinite-dimensional systems?

**Answer to Question 1.** In Chapter 2, linear abstract control systems with an unbounded linear control operator have been studied. Assuming that

- the linear operator  $A$  generates a strongly continuous semigroup,
- the linear control operator  $B$  is admissible,
- there exists a stabilizing linear feedback law for unperturbed system,

we followed the sliding mode strategy by adding to the already known feedback law, a second feedback law that compensates the effect of the disturbance and makes the origin of the closed-loop system globally asymptotically stable in the presence of the disturbance. The sliding variable is defined as the scalar product of the state and an eigenfunction of the adjoint operator of the unperturbed closed-loop system. The semigroup theory and Filippov theory have been used to conclude on the existence of the solution of the closed-loop system. The finite-time stability in the context of the finite-dimensional system combined with semigroup theory have been used to prove the asymptotic stability of the origin of the closed-loop system.

## Question 2

Is it possible to maintain the asymptotic stability of linear infinite-dimensional systems subjected to disturbances by using the measurement of a few moments of the state of the system?



**Answer to Question 2.** Chapter 3 follows the active disturbance rejection control strategy to answer Question 2. This strategy was used to design a robust control for a linear hyperbolic system. The semi-group theory has been used to conclude on the well-posedness of the closed-loop system and Lyapunov techniques have been applied to tackle the asymptotic stability of the origin of the closed-loop system.

**Question 3**

Is it possible to design a PI controller for Korteweg-de Vries equation using a Lyapunov method?

**Answer to Question 3.** In Chapter 4, we have first designed an ISS Lyapunov functional for a Korteweg-de-Vries equation. This functional was obtained by strictifying the energy associated to the system. In particular, the energy was modified by adding a second term which was obtained from the design of an observer built with the backstepping technique. Then, thanks to this ISS Lyapunov functional, we have applied the forwarding method to achieve the output regulation problem.

**Perspectives** The main challenges that still need to be addressed are the following:

- extending our results for the case where the operator  $A$  is nonlinear in Chapter 2. This will allow the treatment of nonlinear PDEs. Many notions will need to be adapted such as the admissibility;
- design of adaptive sliding mode control for abstract linear or nonlinear control systems with matched disturbance. Adaptive in the sense of time-dependent gains. As a result, it is no longer necessary to know the bound of disturbance.
- it might also be interesting to investigate the case where the disturbance does not match with the control as it has been done for ODEs in [20, 133, 45, 106];
- extending our strictification approach to other classes of PDEs for which a strict Lyapunov functional is not yet known.

## A Proof of the main results of Chapter 2

### Proof of Theorem 1

The proof of Theorem 1 is divided into two parts. In the first part, the proof of the Theorem 1 is presented in the case of system (2.37). The second part deals with the proof of Theorem 1 in the case of system (2.47).

Let us start the proof of the first part.

#### Sliding mode control

We consider the following ODE

$$\begin{cases} \dot{\gamma}(t) \in B^*\varphi\left(d - \frac{K}{B^*\varphi} \text{sign}(\gamma(t))\right), & t \in \mathbb{R}_+, \\ \gamma(0) = \gamma_0 \in \mathbb{R}, \end{cases} \quad (\text{A.1})$$

The system (A.1) is understood in the sense of Filippov [48]. In the next lemma, we state that there exists a unique solution to (A.1) and that (A.1) is stabilized in finite-time.

**Lemma 5** *Assume that Assumption 2 hold. Then, the ODE (A.1) admits a unique Filippov solution. Moreover, there exists  $t_r > 0$  such that, for any Filippov solution  $\gamma$  of (A.1),*

$$\gamma(t) = 0, \forall t \geq t_r,$$

with

$$t_r \leq \frac{|\gamma(0)|}{K - K_d|B^*\varphi|}.$$

Lemma 5 is an immediate consequence of the general Filippov theory [48, Chapter 2] (for the real case), [131, Theorem 2.8] (for the complex case), when applied to the particular case of (A.1). The finite-time stability can be deduced easily by Lyapunov arguments.

Let  $\gamma$  be the Filippov solution of (A.1) with initial condition  $\gamma(0) = \langle \varphi, w_0 \rangle_H$ . We consider the following system

$$\begin{cases} \frac{d}{dt}\phi = A_L\phi + \frac{1}{B^*\varphi}B(\dot{\gamma} - \lambda\gamma), \\ \phi(0) = \phi_0 \in H. \end{cases} \quad (\text{A.2})$$

If  $B$  is an admissible operator for  $\mathbb{S}$  and  $\dot{\gamma} - \lambda\gamma \in L^2_{loc}([0, \infty); \mathbb{K})$ , then system (A.2) admits a unique mild solution, where  $(\mathbb{S}(t))_{t \geq 0}$  is the strongly continuous semigroup associated with the operator  $A_L$ . This is what we will prove in the next Lemma, which says that there exists a unique solution in the sense of [125, Definition 4.1.5].

**Lemma 6** *For all  $\phi_0 \in H$ , the system (A.2) admits a unique mild solution  $\phi \in C([0, \infty); H) \cap \mathcal{H}^1_{loc}([0, \infty); D(A^*)'$ .*

*Proof.* Let  $\gamma$  be a Filippov solution of (A.1). Then, according to Lemma 5,  $\gamma$  is absolutely continuous. Moreover,  $\dot{\gamma}$  is bounded and measurable according to Assumption 2. Thus, we have  $\dot{\gamma} - \lambda\gamma \in L^2_{loc}([0, \infty); \mathbb{K})$ . On the other hand, according to Item (ii) of Assumption 1,  $B$  is admissible for  $(\mathbb{T}(t))_{t \geq 0}$ , then according to [62, Proposition 4.2],  $B$  is an admissible control operator for  $(\mathbb{S}(t))_{t \geq 0}$ . Then, according to [125, Proposition 4.2.5], the statement of Lemma 6 holds, achieving the proof.  $\square$

Now, the aim is to prove that the mild solution  $\phi$  to (A.2) with initial condition  $w_0$  is a mild solution to (2.37). To that end, we will show that the following function

$$y(t) = \langle \varphi, \phi(t) \rangle_H, \quad (\text{A.3})$$

with  $\phi$  the solution of (A.2), is equal to  $\gamma$ , for any  $t > 0$ .

**Lemma 7** *For all  $w_0 \in H$ ,  $y$  is a Carathéodory solution to*

$$\begin{cases} \dot{y}(t) = \lambda y + \dot{\gamma}(t) - \lambda\gamma(t), & \text{for a.e } t \geq 0, \\ y(0) = \langle \varphi, w_0 \rangle_H \end{cases} \quad (\text{A.4})$$

*i.e.,  $y$  is an absolutely continuous map such that, for all  $t \geq 0$*

$$y(t) - y(0) = \int_0^t (\lambda y(s) + \dot{\gamma}(s) - \lambda\gamma(s)) ds. \quad (\text{A.5})$$

*Proof.* Let  $\phi$  be the mild solution of (A.2). Since  $\varphi \in D(A_L^*)$ , and using Item (ii) of Assumption 1, then according to [125, Remark 4.2.6], we obtain for that, every  $t \geq 0$ ,

$$\begin{aligned} \langle \varphi, \phi(t) - w_0 \rangle_H &= \int_0^t \left( \langle A_L^* \varphi, \phi(s) \rangle_H + \frac{1}{B^* \varphi} B^* \varphi (\dot{\gamma}(s) - \lambda\gamma(s)) \right) ds \\ &= \int_0^t \left( \lambda \langle \varphi, \phi(s) \rangle_H + \dot{\gamma}(s) - \lambda\gamma(s) \right) ds, \end{aligned} \quad (\text{A.6})$$

because  $A_L^* \varphi = \lambda\varphi$ . Then, using (A.3), one has, for all  $t \geq 0$ ,

$$y(t) - y(0) = \int_0^t (\lambda y(s) + \dot{\gamma}(s) - \lambda\gamma(s)) ds. \quad (\text{A.7})$$

This concludes the proof.  $\square$

We introduce the function  $g$  defined by  $g(t) = y(t) - \gamma(t)$ . From (A.1) and (A.4) with  $\gamma(0) = \langle \varphi, w_0 \rangle_H$ ,

$g$  is solution of

$$\begin{cases} \dot{g}(t) = \lambda g(t) \\ g(0) = 0 \end{cases} \quad (\text{A.8})$$

Thus, for any  $t \in \mathbb{R}$ ,  $g(t) = 0$ . By definition of  $g$ , we deduce that, for any  $t \in \mathbb{R}$ ,  $y(t) = \gamma(t)$ . Therefore, according to (A.1) we have, for a.e.  $t \geq 0$ ,

$$\frac{1}{B^*\varphi} \dot{\gamma}(t) \in -\frac{K}{B^*\varphi} \text{sign}(y(t)) + d(t). \quad (\text{A.9})$$

Thus, according to Lemma 6 and (A.9),  $\phi$  satisfies Definition 2. Then, we conclude that, for any Filippov solution  $\gamma$  of (A.1) with initial condition  $\gamma(0) = \langle \varphi, w_0 \rangle_H$ , the associated mild solution  $\phi$  of (A.2) is a mild solution of (2.37). This concludes the proof of Theorem 1 in the case of system (2.37).  $\square$

### Super-twisting control

Let  $w_0 \in H$ ,  $z_0 \in \mathbb{R}$ . Consider the following ODE

$$\begin{cases} \dot{\rho}(t) = -\alpha|\rho(t)|^{\frac{1}{2}} \text{sign}(\rho(t)) + \eta(t), & t \in \mathbb{R}_+, \\ \dot{\eta}(t) \in B^*\varphi \dot{d}(t) - \beta \text{sign}(\rho(t)), & t \in \mathbb{R}_+ \\ \rho(0) = \rho_0, \eta(0) = z_0. \end{cases} \quad (\text{A.10})$$

The system (A.10) is understood in the sense of Filippov [48]. In the next lemma, we state that there exists a solution to (A.10).

**Lemma 8** *Assume that (2.46) holds. Then, there exists an absolutely continuous map  $(\rho, \eta)$  that satisfies (A.10) for almost every  $t \geq 0$ .*

*Proof.* We consider the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$f(\rho, \eta) = \begin{cases} f^+(\rho, \eta) = (-\alpha\sqrt{\rho} + \eta, -\beta) & \text{if } \rho > 0, \\ f^-(\rho, \eta) = (\alpha\sqrt{-\rho} + \eta, \beta) & \text{if } \rho < 0 \end{cases} \quad (\text{A.11})$$

and let  $F_d : (\rho, \eta) \in \mathbb{R}^2 \mapsto F_d(\rho, \eta)$  be the set-valued map defined by

$$F_d(\rho, \eta) = \bar{B}(0, |B^*\varphi|C) + \begin{cases} \{f(\rho, \eta)\} & \text{if } \rho \neq 0, \\ \overline{\text{conv}}\{f^+(\rho, \eta), f^-(\rho, \eta)\} & \text{if } \rho = 0 \end{cases} \quad (\text{A.12})$$

where  $\overline{\text{conv}}$  denotes convex closure and  $\bar{B}(0, |B^*\varphi|C)$  is a closed ball of  $\mathbb{R}^2$  centered at 0 and of radius  $|B^*\varphi|C$ . Since  $f$  is continuous on  $\mathbb{R} \setminus \{0\} \times \mathbb{R}$ , then the function  $F_d$  is non-empty, compact, convex and upper semi-continuous. Then according to [15, Theorem 3.6], there exists at least one solution of the differential inclusion

$$\dot{\zeta} \in F_d(\zeta) \quad (\text{A.13})$$

where  $\zeta = (\rho, \eta)$ . Since  $F_d$  is the Filippov's construction (as in [48, Chapter 2]) associated with (A.10), then, there exists an absolutely continuous map that satisfies (A.10) for almost every  $t \geq 0$ , concluding

therefore the proof.  $\square$

Let  $(\rho, \eta)$  be a solution of (A.10) with initial condition  $\rho(0) = \langle \varphi, w_0 \rangle_H$ . We consider the following system

$$\begin{cases} \frac{d}{dt}\psi = A_L\psi + \frac{1}{B^*\varphi}B(\dot{\rho} - \lambda\rho), \\ \psi(0) = w_0 \in H. \end{cases} \quad (\text{A.14})$$

Since  $\rho$  and  $\eta$  are continuous then, according to the first line of (A.10), we deduce that  $\dot{\rho}$  is also continuous. Moreover, since  $\rho$  and  $\dot{\rho}$  are continuous, then  $\dot{\rho} - \lambda\rho \in L^2_{loc}([0, \infty); \mathbb{R})$ . Thus, according to Lemma 6, the system (A.14) admits a unique mild solution  $\psi \in C([0, \infty); H) \cap \mathcal{H}^1_{loc}([0, \infty); D(A^*)'$ .

As in the previous case, the aim is now to prove that the solution  $(\psi, \eta)$  is a mild solution to (2.47). For this purpose, we are going to show that the following function

$$\theta(t) = \langle \varphi, \psi(t) \rangle_H, \quad (\text{A.15})$$

with  $\psi$  the solution of (A.14), is equal to  $\rho$  for any  $t > 0$ .

**Lemma 9** *For all  $w_0 \in H$ ,  $\theta$  is a Carathéodory solution of*

$$\begin{cases} \dot{\theta}(t) = \lambda\theta + \dot{\rho}(t) - \lambda\rho(t), & t \geq 0, \\ \theta(0) = \langle \varphi, w_0 \rangle_H. \end{cases} \quad (\text{A.16})$$

The proof of Lemma 9 is similar to the proof of Lemma 7. We therefore omit the proof of Lemma 9.

Now, according to Lemma 8,  $\theta$  is absolutely continuous map. Moreover, if we set  $\kappa = \theta - \rho$ , then  $\kappa$  satisfies (A.8). Thus,  $\kappa(t) = 0$  for all  $t \in \mathbb{R}$ . This mean that,  $\theta(t) = \rho(t)$  for any  $t \in \mathbb{R}$ . As a consequence, according to (A.10), we have, for a.e  $t \geq 0$ ,

$$\dot{\eta}(t) \in B^*\varphi\dot{d}(t) - \beta \text{sign}(\theta(t)). \quad (\text{A.17})$$

Thus, according to Lemma 8 and (A.17),  $\eta$  is absolutely continuous map and satisfies (2.51). Therefore,  $(\psi, \eta)$  satisfies the Definition 3. This, mean that  $(\psi, \eta)$  is a mild solution of (2.47). This concludes the proof of Theorem 1 in the case of system (2.47).  $\square$

## Proof of Theorem 2

Like the proof of Theorem 1, the proof of Theorem 2 is divided into two parts. In the first part, the proof of the Theorem 2 is presented in the case of the system (2.37). The second part deals with the proof of Theorem 2 in the case of system (2.47).

Let us start the proof of the first part.

### Sliding-mode control

Let us consider  $w$  a mild solution of (2.37) with initial condition  $w_0 \in H$ . Then, according Definition 2, there exists  $h \in L^2_{loc}([0, \infty); \mathbb{K})$  such that  $h$  satisfies (2.49) and  $w$  satisfies (2.48). Therefore, since

$\varphi \in D(A_L^*)$ , and using Item (ii) of Assumption 1, then according to [125, Remark 4.2.6],  $w$  satisfies, for every  $t \geq 0$ ,

$$\langle \varphi, w(t) - w_0 \rangle_H = \int_0^t \left( \langle A_L^* \varphi, w(s) \rangle_H + B^* \varphi h(s) \right) ds = \int_0^t \left( \lambda \langle \varphi, w(s) \rangle_H + B^* \varphi h(s) \right) ds, \quad (\text{A.18})$$

because  $A_L^* \varphi = \lambda \varphi$ . Using (2.35), one has, for every  $t \geq 0$ ,

$$\sigma(t) - \sigma(0) = \int_0^t (\lambda \sigma(s) + B^* \varphi h(s)) ds. \quad (\text{A.19})$$

As a consequence,  $\sigma$  defined in (2.35) is a Carathéodory solution to

$$\begin{cases} \dot{\sigma}(t) = \lambda \sigma(t) + B^* \varphi h(t), \\ \sigma(0) = \langle \varphi, w_0 \rangle_H. \end{cases} \quad (\text{A.20})$$

Since  $h \in -\frac{1}{B^* \varphi} \left( \lambda \sigma + K \text{sign}(\sigma) \right) + d$ , then  $\sigma$  is a Filippov solution of (A.1) with initial condition  $\langle \varphi, z_0 \rangle_H$ . From Lemma 5, there exists a finite-time  $t_r$  such that

$$\sigma(t) = 0 \text{ for any } t > t_r.$$

Therefore,  $\dot{\sigma}(t) = 0$  for any  $t > t_r$ . As a consequence, from (A.20), for any  $t > t_r$ ,  $h(t) = 0$ . Thus, for any  $t > t_r$ , the system (2.37) is equivalent to the system (2.34) and hence is asymptotically stable in  $H$  from the Item (iii) of Assumption 1. Therefore, to conclude the proof of Theorem 2 in the case of system (2.37), it is just necessary to prove that the system (2.37) depends continuously on initial conditions on the time interval  $[0, t_r]$ . For this purpose, we consider  $w$  a mild solution of (2.37) with initial condition  $w_0 \in H$  on the interval  $[0, t_r]$ . Then, using the Definition 2, there exists  $C_0 > 0$  such that, for all  $t \in [0, t_r]$ , we have

$$\|w(t)\|_H \leq C_0 \|w_0\|_H + \left\| \int_0^t \mathbb{S}(t-s) B h(s) ds \right\|_H. \quad (\text{A.21})$$

Since  $(\mathbb{S}(t))_{t \geq 0}$  is exponentially stable and  $B$  is an admissible operator for  $(\mathbb{S}(t))_{t \geq 0}$ , then according to [125, Proposition 4.3.3], there exists  $C_1 > 0$  independent of  $t_r$  such that, for all  $t \in [0, t_r]$

$$\|w(t)\|_H \leq C_1 \left( \|w_0\|_H + \|h\|_{L^2(0, t_r)} \right). \quad (\text{A.22})$$

Moreover, since  $h \in -\frac{1}{B^* \varphi} (\lambda \sigma + K \text{sign}(\sigma)) + d$ , then according to Assumption 2,  $h$  is bounded. Therefore, there exists  $C_2 > 0$  such that

$$\|h\|_{L^2(0, t_r)} \leq C_2 t_r^{\frac{1}{2}}. \quad (\text{A.23})$$

Moreover, according to Lemma 5,  $t_r \leq \frac{|\langle \varphi, w_0 \rangle_H|}{K - \|B^* \varphi\| \|d\|_{L^\infty(\mathbb{R}_+)}}$ . Thus, using Cauchy-Schwarz's inequality, we have

$$t_r \leq \frac{\|\varphi\|_H}{K - \|B^* \varphi\| \|d\|_{L^\infty(\mathbb{R}_+)}} \|w_0\|_H. \quad (\text{A.24})$$

As a consequence, according to (A.22), (A.23) and (A.24), there exists  $C_3 > 0$  (independent of  $t_r$ ) such that for all  $t \in [0, t_r]$ ,

$$\|w(t)\|_H \leq C_3 \left( \|w_0\|_H + \sqrt{\|w_0\|_H} \right). \quad (\text{A.25})$$

This concludes the proof of Theorem 2 in the case of system (2.37).  $\square$

**Remark 5** *In contrast with many stabilization techniques, we do not need here to compute time-derivative of Lyapunov functionals for the infinite-dimensional system. More precisely, classical techniques rely on the existence of strong solutions for which one computes time derivative of a suitable Lyapunov functional, and one concludes then on the stability for weak solution by a density argument.*

### Super-twisting control

Let us consider  $(w, z)$  a mild solution of (2.47) with initial condition  $(w_0, z_0) \in H \times \mathbb{R}$ . Then, according Definition 3, there exists  $\tilde{z} \in L^1_{loc}([0, \infty); \mathbb{R})$  with  $\tilde{z}(t) \in \text{sign}(\sigma(t))$  such that, for a.e  $t \geq 0$ ,  $\dot{z}(t) = B^* \varphi d(t) - \beta \tilde{z}$  and  $w$  satisfies (2.50). Replacing  $h$  by  $\omega$  in (A.19), then  $\sigma$  satisfies (A.19). Then, according to (2.52), (A.19) we obtain, for a.e  $t \in [0, T]$

$$\begin{cases} \dot{\sigma}(t) = -\alpha |\sigma(t)|^{\frac{1}{2}} \text{sign}(\sigma(t)) + z(t), \\ \dot{z}(t) = B^* \varphi d(t) - \beta \tilde{z}(t). \end{cases} \quad (\text{A.26})$$

Since  $\tilde{z} \in \text{sign}(\sigma(t))$ , then  $(\sigma, z)$  is a Filippov solution of (A.10) with initial condition  $(\langle \varphi, w_0 \rangle_H, z_0)$ . According to Proposition 1, there exists a finite-time such that

$$\sigma(t) = 0 \text{ and } z(t) = 0$$

for any  $t > t_r$ . Then, for any  $t > t_r$ , the solution  $z$  to system (2.47) is solution to system (2.34) and hence is asymptotically stable in  $H$  from Item (iii) of Assumption 1. Therefore, as in the previous part, we just need to prove that the system (2.47) depends continuously on initial conditions on the time interval  $[0, t_r]$  to conclude the proof of Theorem 2. For this purpose, we consider  $(w, z)$  a mild solution of (2.47) with initial condition  $(w_0, z_0) \in H \times \mathbb{R}$  on the interval  $[0, t_r]$ . Then, like in the previous part, using Definition 3, there exists  $C_0 > 0$  such that, for all  $t \in [0, t_r]$ , we have

$$\|w(t)\|_H \leq C_0 \left( \|w_0\|_H + \|\omega\|_{L^2(0, t_r)} \right). \quad (\text{A.27})$$

Since  $w$  is continuous on  $[0, t_r]$ , then, according to (2.35),  $\sigma$  is also continuous. Therefore,  $\sigma$  is bounded on  $[0, t_r]$ . Moreover,  $z$  is an absolutely continuous map. Thus,  $z$  is bounded on  $[0, t_r]$ . Then, the function

$$\omega(\cdot) := \frac{1}{B^* \varphi} \left( -\lambda \sigma(\cdot) - \alpha |\sigma(\cdot)|^{\frac{1}{2}} \text{sign}(\sigma(\cdot)) + z(\cdot) \right)$$

is also bounded on  $[0, t_r]$ . Therefore, there exists  $C_1 > 0$  such that

$$\|\omega\|_{L^2(0, t_r)} \leq C_1 t_r^{\frac{1}{2}}. \quad (\text{A.28})$$

Now, according to [91, Theorem 2], there exist positive constants  $C_2, C_3$  such that

$$\begin{cases} t_r < C_2 (|\sigma(0)| + |z_0|), \\ |z(t)| \leq C_3 |z_0|. \end{cases} \quad (\text{A.29})$$

Using Cauchy-Schwarz's inequality, we obtain

$$|\sigma(0)| = |\langle \varphi, w_0 \rangle_H| \leq \|\varphi\|_H \|w_0\|_H. \quad (\text{A.30})$$

As a consequence, according to (A.27), (A.28), (A.29) and (A.30), there exists  $C_4 > 0$  such that, for all  $t \in [0, t_r]$ ,

$$\|w(t)\|_H + |z(t)| \leq C_4 \left( \|w_0\|_H + |z_0| + \sqrt{\|w_0\|_H + |z_0|} \right). \quad (\text{A.31})$$

This concludes the proof of Theorem 2.  $\square$

## B Proof of the main results of Chapter 3

### Proof of Theorem 3

Let's consider the operator  $\mathcal{A} : \phi \in D(\mathcal{A}) \subset L^2(0, L) \mapsto \mathcal{A}\phi \in L^2(0, L)$  defined as

$$\begin{cases} \mathcal{A}\phi = -\lambda\phi', \\ D(\mathcal{A}) = \{\phi \in \mathcal{H}^1(0, L) \mid \phi(0) = a\phi(L)\} \end{cases} \quad (\text{B.1})$$

where  $\lambda$  is given in system (3.13). According to the proof of [11, Theorem A.1], it generates a  $C_0$ -semigroup  $(\mathbb{T}(t))_{t \geq 0}$  of contractions in  $L^2(0, L)$ . Also, consider the operator  $\mathcal{B}$  defined as  $\lambda \langle \varphi, \mathcal{B}v \rangle_{D(\mathcal{A}^*), D(\mathcal{A}^*)'} = \lambda \varphi(0)v$  for all  $v \in \mathbb{R}$  and  $\varphi \in D(\mathcal{A}^*)$  where  $\mathcal{A}^*$  is the adjoint operator of  $\mathcal{A}$  and  $\langle \cdot, \cdot \rangle_{D(\mathcal{A}^*), D(\mathcal{A}^*)'}$  is the dual product. Now, let us prove that  $\mathcal{B}$  is admissible<sup>1</sup> for  $(\mathbb{T}(t))_{t \geq 0}$ . To do so, consider the system

$$\begin{cases} \frac{d}{dt}w = \mathcal{A}^*w, \\ \gamma = \mathcal{B}^*w. \end{cases} \quad (\text{B.2})$$

where  $\mathcal{A}^* : \varphi \in D(\mathcal{A}^*) \subset L^2(0, L) \mapsto \mathcal{A}^*\varphi \in L^2(0, L)$  and  $\mathcal{B}^* : \varphi \in D(\mathcal{A}^*) \mapsto \mathcal{B}^* : \varphi \in \mathbb{R}$  are given by

$$\begin{cases} \mathcal{A}^*\varphi = \lambda\varphi', \\ D(\mathcal{A}^*) = \{\varphi \in \mathcal{H}^1(0, L) \mid \varphi(L) = a\varphi(0)\}, \\ \mathcal{B}^* : \varphi \in D(\mathcal{A}^*) \mapsto \lambda\varphi(0). \end{cases} \quad (\text{B.3})$$

For all  $w_0 \in D(\mathcal{A}^*)$ , the function

$$w(t) = \mathbb{T}^*(t)w_0 \quad (\text{B.4})$$

<sup>1</sup>See e.g [125, Definition 4.2.1]



defines the unique classical solution of (B.2) where  $\mathbb{T}^*(t)$ <sup>2</sup> is a  $C_0$ -semigroup with infinitesimal generator  $\mathcal{A}^*$  on  $L^2(0, L)$ . Now, consider the following function

$$E(t) = \int_0^L (w(t, x))^2 dx.$$

The time derivative of  $E$  along the trajectories of (B.2) reads as, for all  $t \geq 0$ ,

$$\begin{aligned} \dot{E}(t) &= 2 \int_0^L \partial_t w(t, x) w(t, x) dx = 2\lambda \int_0^L \partial_x w(t, x) w(t, x) dx = \lambda(|w(t, L)|^2 - |w(t, 0)|^2) \\ &= -\lambda|w(t, 0)|^2(1 - a^2) \leq 0. \end{aligned} \quad (\text{B.5})$$

Then, from (B.5), one deduces that, for all  $T > 0$

$$\begin{aligned} \int_0^T |\gamma(t)|^2 dt &= \lambda^2 \int_0^T |w(t, 0)|^2 dt = \frac{\lambda}{a^2 - 1} \int_0^T \dot{E}(t) dt = \frac{\lambda}{1 - a^2} (E(0) - E(T)) \\ &\leq \frac{\lambda}{1 - a^2} E(0) = \frac{\lambda}{1 - a^2} \|w(0, \cdot)\|_{L^2(0, L)}^2 \end{aligned} \quad (\text{B.6})$$

where  $\gamma$  come from (B.2). Then

$$\int_0^T |\gamma(t)|^2 dt \leq \frac{\lambda}{1 - a^2} \|w(0, \cdot)\|_{L^2(0, L)}^2. \quad (\text{B.7})$$

Then according to [125, Definition 4.3.1 and Theorem 4.4.3.], this proves that  $\mathcal{B}$  is admissible for the  $C_0$ -semigroup  $(\mathbb{T}(t))_{t \geq 0}$ .

Now, let  $(y_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$  such that  $\hat{\eta}^0 \neq \eta^0$ . Then, there exist  $t_0 > 0$  such that for all  $t \in [0, t_0]$ , we have  $\hat{\eta}(t) \neq \eta(t)$ . Therefore, for all  $t \in [0, t_0]$ ,  $\text{sign}(\hat{\eta}_n(t) - \eta_n(t)) = \pm 1$ . Thus, for all  $t \in [0, t_0]$

$$\tilde{d} = \mp \frac{k_0 K_d}{\lambda}. \quad (\text{B.8})$$

As a consequence, for all  $t \in [0, t_0]$ , the system (3.26) is equivalent to the following system

$$\begin{cases} \partial_t w(t, x) + \lambda \partial_x w(t, x) = 0, \\ w(t, 0) = aw(t, L) + d(t) - \tilde{d}(t), \\ \dot{\hat{\eta}}_i(t) = -\lambda L^i w(t, L) + i\lambda \hat{\eta}_{i-1}(t) - k_i K_d^{\frac{n+1-i}{n+1}} [\hat{\eta}_n - \eta_n]^{\frac{i}{n+1}}, \quad i = 1, \dots, n \\ \dot{\hat{\eta}}_0(t) = \lambda w(t, L)(a - 1) \\ w(0, x) = w_0(x), \\ \hat{\eta}(0) = \hat{\eta}^0 \in \mathbb{R}^{n+1}. \end{cases} \quad (\text{B.9})$$

Since the function  $\hat{d} : t \mapsto d(t) - \tilde{d}(t)$  is bounded on  $[0, t_0]$  and  $\mathcal{B}$  is admissible for  $(\mathbb{T}(t))_{t \geq 0}$ , then, according to [125, Proposition 4.2.5], there exist a unique mild solution  $w \in C([0, t_0]; L^2(0, L))$  of (B.9). As a consequence, the functions  $w(\cdot, L)$  and  $\eta_n(\cdot)$  are continuous on  $[0, t_0]$ . Then, the right hand side of

<sup>2</sup>See e.g [125, Proposition 2.8.5]

the system

$$\begin{cases} \dot{\hat{\eta}}_i(t) &= -\lambda L^i w(t, L) + i\lambda \hat{\eta}_{i-1}(t) - k_i K_d^{\frac{n+1-i}{n+1}} [\hat{\eta}_n - \eta_n]^{\frac{i}{n+1}}, \quad i = 1, \dots, n \\ \dot{\hat{\eta}}_0(t) &= \lambda w(t, L)(a-1) \\ \hat{\eta}(0) &= \hat{\eta}^0 \in \mathbb{R}^{n+1}. \end{cases} \quad (\text{B.10})$$

is continuous on  $[0, t_0] \times \mathbb{R}^{n+1}$ . Thus, according to [26, Theorem 1.2], the system (B.10) admits a solution  $\hat{\eta} \in C^1([0, t_0])$ . As a consequence, the system (3.26) admits a mild solution on  $[0, t_0]$ . Moreover, for all  $t \in [0, t_0]$ ,  $w(t, \cdot) \in D(\mathcal{A})$  and satisfies the following equations in  $D(\mathcal{A}^*)'$

$$w(t, \cdot) - w_0 = -\lambda \int_0^t \partial_x w(s, \cdot) ds + \int_0^t \mathcal{B} \hat{d}(s) ds, \quad \forall t \in [0, t_0]. \quad (\text{B.11})$$

Thus, for all  $t \in [0, t_0]$ ,  $w(t, \cdot) \in D(\mathcal{A})$  and for all  $i = 0, 1, \dots, n$ , for all  $(t, x) \in [0, t_0] \times [0, L]$ ,  $y$  satisfies

$$x^i w(t, x) - x^i w_0(x) = -\lambda \int_0^t x^i \partial_x w(s, x) ds + \int_0^t x^i \mathcal{B} \hat{d}(s) ds. \quad (\text{B.12})$$

As a consequence, we have for all  $i = 0, 1, \dots, n$ , and for all  $t \in [0, t_0]$

$$\int_0^L x^i w(t, x) dx - \int_0^L x^i w_0(x) dx = -\lambda \int_0^t \int_0^L x^i \partial_x w(s, x) dx ds + \int_0^t \int_0^L x^i \mathcal{B} dx \hat{d}(s) ds. \quad (\text{B.13})$$

Since  $\mathcal{B}$  is the product of the delta function at  $x = 0$  with  $\lambda$ , then for all  $i = 1, \dots, n$

$$\int_0^L x^i \mathcal{B} dx = 0 \quad (\text{B.14})$$

and

$$\int_0^L \mathcal{B} dx = \lambda, \quad \text{because } 0 \in [0, L]. \quad (\text{B.15})$$

Thus, for all  $i = 1, \dots, n$ , we have

$$\int_0^L x^i w(t, x) dx - \int_0^L x^i w_0(x) dx = -\lambda \int_0^t \int_0^L x^i \partial_x w(s, x) dx ds, \quad \forall t \in [0, t_0] \quad (\text{B.16})$$

and

$$\int_0^L w(t, x) dx - \int_0^L w_0(x) dx = -\lambda \int_0^t \int_0^L \partial_x w(s, x) dx ds + \lambda \int_0^t \hat{d}(s) ds, \quad \forall t \in [0, t_0].$$

Using an integration by parts, one immediately obtains

$$\int_0^L w(t, x) dx - \int_0^L w_0(x) dx = -\lambda \int_0^t \left( w(s, L) - w(s, 0) - \hat{d}(s) \right) ds, \quad \forall t \in [0, t_0] \quad (\text{B.17})$$

and for all  $i = 1 \dots, n$ , for all  $t \in [0, t_0]$

$$\int_0^L x^i w(t, x) dx - \int_0^L x^i w_0(x) dx = i\lambda \int_0^t \int_0^L x^{i-1} w(s, x) dx ds - \lambda \int_0^t L^i w(s, L) ds.$$

Using (3.16) and the fact that, for all  $t \in [0, t_0]$ ,  $w(t, \cdot) \in D(\mathcal{A})$ , we obtain, for all  $t \in [0, t_0]$

$$\eta_0(t) - \eta_0(0) = -\lambda \int_0^t \left( w(s, L)(1-a) - \hat{d}(s) \right) ds \quad (\text{B.18})$$

and for all  $i = 1 \dots, n$ , for all  $t \in [0, t_0]$

$$\eta_i(t) - \eta_i(0) = \int_0^t \left( i\lambda \eta_{i-1}(s) - \lambda L^i w(s, L) \right) ds. \quad (\text{B.19})$$

This prove that, for all  $i = 0, 1 \dots, n$ , the  $i$ -th moment  $\eta_i$  of  $w$  is a Carathéodory solution to (3.17) on  $[0, t_0]$ .

Now, we assume without loss of generality that  $t_0 < t_r$ , where  $t_r$  is given in Proposition 2. Thus, according to Proposition 2,  $\hat{\eta}(t) \neq \eta(t)$  for all  $t \in [t_0, t_r[$ . As a result, the same reasoning at time interval  $[0, t_0]$  is step-by-step applied to time interval  $[t_0, t_r[$ , by considering  $w(t_0, \cdot)$  as the initial condition. As a consequence, the system (3.26) admits a solution on  $[0, t_r[$ .

Moreover, on one hand, according to Proposition 2,  $\hat{\eta}(t) = \eta(t)$  and  $d(t) - \tilde{d}(t) = 0$  for all  $t \geq t_r$ . Thus, for all  $t \geq t_r$ , the system (3.26) is equivalent to the following system

$$\begin{cases} \partial_t w(t, x) + \lambda \partial_x w(t, x) = 0, \\ w(t, 0) = aw(t, L), \\ \dot{\hat{\eta}}_i(t) = -\lambda L^i w(t, L) + i\lambda \hat{\eta}_{i-1}(t), \quad i = 1, \dots, n \\ \dot{\hat{\eta}}_0(t) = \lambda w(t, L)(a-1) \\ w(0, x) = w_0(x), \\ \hat{\eta}(0) = \hat{\eta}^0 \in \mathbb{R}^{n+1}. \end{cases} \quad (\text{B.20})$$

It can be proved as before that the system (B.20) admits a solution on  $[t_r, \infty[$ . Then, in conclusion, for all  $(w_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$  such that  $\hat{\eta}^0 \neq \eta^0$ , the system (3.26) admits a solution on  $[0, \infty[$ .

Let  $(w_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$  such that  $\hat{\eta}^0 = \eta^0$ . Then, there exist  $t_0 > 0$  such that, we have  $\hat{\eta}(t) = \eta(t)$ . Thus, for all,  $t \in [0, t_0[$  the system (3.26) is equivalent to the following system

$$\begin{cases} \partial_t w(t, x) + \lambda \partial_x w(t, x) = 0, \\ w(t, 0) = aw(t, L) + d(t) - \tilde{d}(t), \\ \dot{\hat{\eta}}_i(t) = -\lambda L^i w(t, L) + i\lambda \hat{\eta}_{i-1}(t), \quad i = 1, \dots, n \\ \dot{\hat{\eta}}_0(t) = \lambda w(t, L)(a-1) \\ \tilde{d}(t) \in \left[ -\frac{k_0 K_a}{\lambda}, \frac{k_0 K_a}{\lambda} \right] \\ w(0, x) = w_0(x), \\ \hat{\eta}(0) = \hat{\eta}^0 \in \mathbb{R}^{n+1}, \tilde{d}(0) = \tilde{d}_0 \in \mathbb{R}. \end{cases} \quad (\text{B.21})$$

Since the function  $t \mapsto d(t) - \tilde{d}(t)$  is bounded on  $[0, t_0[$  then we conclude as before that the system (B.21)

admits a solution on  $[0, t_0[$  and the  $i$ -th moment  $\eta_i$  of  $w$  is a Carathéodory solution to (3.17) on  $[0, t_0[$ .

However, since  $\widehat{\eta}(t) = \eta(t)$  for all  $t \in [0, t_0[$ , then according to Proposition 2,  $t_0 > t_r$ . Thus, for all  $t \geq t_r$ , the system (3.26) is equivalent to system (B.20). As a consequence  $t_0 = \infty$ .

This concludes the proof of Theorem 3 □

## Proof of Theorem 4

Let  $(w_0, \widehat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$ . Using the error variable  $e$  defined in (3.19), we can write the equivalent system of (3.26) as follows:

$$\begin{cases} \partial_t w(t, x) + \lambda \partial_x w(t, x) = 0, \\ w(t, 0) = aw(t, L) - \frac{1}{\lambda} \dot{e}_0(t), \\ \dot{e}_i(t) = i\lambda e_{i-1}(t) - k_i K_d \frac{n+1-i}{n+1} [e_n]^{\frac{i}{n+1}}, \quad i = 1, \dots, n \\ \dot{e}_0(t) = -\lambda d(t) + \lambda \tilde{d}(t), \\ \tilde{d} \in -\frac{k_0 K_d}{\lambda} \text{sign}(e_n) \end{cases} \quad (\text{B.22})$$

Then, according to Proposition 2 and Remark 3, there exists a finite-time  $t_r$  such that, for all  $t > t_r$ , the solution  $w$  of (B.22) is equivalent to the system (3.27) and hence is globally exponentially stable in  $L^2(0, L)$  from [11, Theorem 2.1]. Therefore, to conclude the proof of Theorem 4, it is just necessary to prove that the system (B.22) depends continuously on initial conditions on the time interval  $[0, t_r]$ . It is stated in the following Lemma.

**Lemma 10** *There exists a  $\mathcal{K}$ -function  $\beta$  such that for all  $(w_0, e^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$ , for all  $t \in [0, t_r]$ ,*

$$\|w(t, \cdot)\|_{L^2(0, L)} + |e|_{\mathbb{R}^{n+1}} \leq \beta(\|w_0\|_{L^2(0, L)} + |e^0|_{\mathbb{R}^{n+1}}). \quad (\text{B.23})$$

for all solution  $(w, e)$  of (B.22).

*Proof.* Let  $(w_0, e^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$  and we consider  $(w, e)$  a solution of (B.22) associated  $(w_0, e^0)$ . Then, according to [125, Proposition 2.1.2], there exists  $K_0 > 0$  such that, for all  $t \in [0, t_r]$ , we have

$$\|w(t, \cdot)\|_{L^2(0, L)} \leq K_0 \|w_0(\cdot)\|_{L^2(0, L)} + \left\| \frac{1}{\lambda} \int_0^t \mathbb{T}(t-s) \mathcal{B} \dot{e}_0(t) ds \right\|_{L^2(0, L)}. \quad (\text{B.24})$$

As a consequence, since  $(\mathbb{T}(t))_{t \geq 0}$  is exponentially stable and  $\mathcal{B}$  is admissible operator for  $(\mathbb{T}(t))_{t \geq 0}$ , then we have according to [125, Proposition 4.4.5], that there exists  $K_1 > 0$  independent of  $t_r$  such that

$$\|w(t, \cdot)\|_{L^2(0, L)} \leq K_1 \left( \|w_0(\cdot)\|_{L^2(0, L)} + \|\dot{e}_0(\cdot)\|_{L^2(0, t_r)} \right). \quad (\text{B.25})$$

Since  $d$  and the sign function are bounded then according to (3.19),  $\dot{e}_0$  is also bounded. Therefore, there exists  $K_2 > 0$  such that

$$\|\dot{e}_0\|_{L^2((0, t_r), \mathbb{R})}^2 \leq K_2 t_r. \quad (\text{B.26})$$

Now, according to [34, Theorem 1], there are positive constants  $K_3, K_4$  (dependent on the bound of  $d$ ) such that

$$\begin{cases} t_r < K_3 |e^0|_{\mathbb{R}^{n+1}}, \\ |e|_{\mathbb{R}^{n+1}} \leq K_4 |e^0|_{\mathbb{R}^{n+1}}. \end{cases} \quad (\text{B.27})$$

As a consequence, according to (B.25), (B.26) and (B.27), there exists  $C_1 > 0$  (independent of  $t_r$ ) such that, for all  $t \in [0, t_r]$ ,

$$\|w(t, \cdot)\|_{L^2(0,L)} + |e|_{\mathbb{R}^{n+1}} \leq \beta (\|w_0\|_{L^2(0,L)} + |e^0|_{\mathbb{R}^{n+1}}). \quad (\text{B.28})$$

where  $\beta$  is given by  $\beta : s \in \mathbb{R}_+ \mapsto C_1(s + \sqrt{s})$ . This concludes the proof of Lemma 10.  $\square$

Since for all  $t > t_r$ , the solution  $w$  of system (B.22) is globally exponentially stable in  $L^2(0, L)$ , then according to Lemma 10 and Proposition 2, there exists a  $\mathcal{KL}$ -function  $\alpha$  such that, for any  $w_0 \in L^2(0, L)$ , for any  $\hat{\eta}^0 \in \mathbb{R}^{n+1}$  and for any  $t \geq 0$ :

$$\|w(t, \cdot)\|_{L^2(0,L)} + |e(t)|_{\mathbb{R}^{n+1}} \leq \alpha (\|w_0\|_{L^2(0,L)} + |e^0|_{\mathbb{R}^{n+1}}, t) \quad (\text{B.29})$$

for all solution  $(w, e)$  of (B.22). This concludes the proof of Theorem 4  $\square$ .

## Proof of Theorem 5

Let  $(w_1(0, \cdot), w_2(0, \cdot), \bar{Z}(0), \hat{Z}(0; r)) \in \mathcal{J}$ . Using the error dynamics defined in (3.65), we can write the equivalent system of (3.72) as follows

$$\left\{ \begin{array}{l} \partial_t w_1(t, x) + \lambda_1 \partial_x w_1(t, x) = c_1 w_2(t, x), \\ \partial_t w_2(t, x) - \lambda_2 \partial_x w_2(t, x) = c_2 w_1(t, x), \\ w_1(t, 0) = b w_2(t, 0) - L_i^2 q \left( \kappa \left( E_{(2n+1)(2n+1)}(t) - E_{2n(2n+1)}(t) \right) \right) - \sum_{j=1}^{2n+1} a_j E_{j(j+1)}(t) - \dot{E}_{(2n+1)(2n+2)}(t), \\ w_2(t, L) = a w_1(t, L), \\ \dot{\tilde{Z}}(t) = A_0 \tilde{Z}(t) + B_z d(t), \\ \dot{E}_{ii}(t) = E_{i(i+1)}(t) - L_i k_1 \left( E_{ii}(t) - E_{(i-1)i}(t) \right) \\ \dot{E}_{i(i+1)}(t) \in E_{(i+1)(i+2)}(t) - L_i^2 k_2 \left( E_{ii}(t) - E_{(i-1)i}(t) \right), \quad i = 1, \dots, 2n, \\ \dot{E}_{(2n+1)(2n+1)}(t) = E_{(2n+1)(2n+2)}(t) - L_{2n+1} k_1 \left( E_{(2n+1)(2n+1)}(t) - E_{2n(2n+1)}(t) \right) \\ \dot{E}_{(2n+1)(2n+2)}(t) = -L_{2n+1}^2 q \left( \kappa \left( E_{(2n+1)(2n+1)}(t) - E_{2n(2n+1)}(t) \right) \right) - \sum_{j=1}^{2n+1} a_j E_{j(j+1)}(t) + \hat{d}(t) - d(t), \\ \hat{d}(t) \in -L_{2n+1}^2 \text{sign} \left( E_{(2n+1)(2n+1)}(t) - E_{2n(2n+1)}(t) \right). \end{array} \right. \quad (\text{B.30})$$

Then, we can see that the dynamics of  $\tilde{Z}$ ,  $E_{ii}$  and  $E_{i(i+1)}$  do not depend on  $w_1, w_2$ . Thus, we can prove the well-posedness  $\tilde{Z}$  and  $E$  of (B.30) separately.

Since  $d$  is bounded, then, for all initial value  $\tilde{Z}(0)$ , there exists a unique absolutely continuous map  $\tilde{Z}$  that satisfies  $\dot{\tilde{Z}}(t) = A_0 \tilde{Z}(t) + B_z d(t)$  for almost  $t \geq 0$ . Thus, for all  $i = 1, \dots, 2n+1$ ,  $\dot{\tilde{Z}}_{i+1} \in L_{\text{loc}}^1(0, \infty)$ .

Moreover, for each  $i = 1, \dots, 2n+1$ , there exists an absolutely continuous map  $(E_{ii}, E_{i(i+1)})$  that satisfies

$$\begin{cases} \dot{E}_{ii} = E_{i(i+1)} - L_i k_1 (E_{ii} - E_{(i-1)i}) \\ \dot{E}_{i(i+1)} \in E_{(i+1)(i+2)} - L_i^2 k_2 (E_{ii} - E_{(i-1)i}) + v_i, \end{cases} \quad (\text{B.31})$$

for almost  $t \geq 0$ , with

$$v_i(t) = \begin{cases} 0, & i = 1, \dots, 2n \\ -\sum_{j=1}^{2n+1} a_j E_{j(j+1)}(t) - d(t), & i = 2n+1. \end{cases} \quad (\text{B.32})$$

Now, in order to prove the well-posedness of the  $(w_1, w_2)$  part of (3.72), we consider the following operator

$$\begin{cases} \mathcal{A}_1 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 \varphi_1' + c_1 \varphi_2 \\ \lambda_2 \varphi_2' + c_2 \varphi_1 \end{pmatrix}, \\ D(\mathcal{A}_1) = \left\{ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathcal{H}^1(0, L, \mathbb{R}^2) \mid \varphi_1(0) = b\varphi_2(0), \varphi_2(L) = a\varphi_1(L) \right\}. \end{cases} \quad (\text{B.33})$$

We can prove as in the proof of [11, Theorem A.1] that,  $\mathcal{A}_1$  generates a  $C_0$ -semigroup  $(\mathcal{T}_1(t))_{t \geq 0}$  on  $\mathcal{I}$ . Also, consider the operator  $\mathcal{B}_1$  defined as

$$p_1 \lambda_1 \left\langle \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \mathcal{B}_1 v \right\rangle_{D(\mathcal{A}_1^*), D(\mathcal{A}_1^*)'} = p_1 \lambda_1 \varphi_1(0) v \quad (\text{B.34})$$

for all  $v \in \mathbb{R}$  and  $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in D(\mathcal{A}_1^*)$  where  $\mathcal{A}_1^*$  is the adjoint operator of  $\mathcal{A}$  and  $\langle \cdot, \cdot \rangle_{D(\mathcal{A}_1^*), D(\mathcal{A}_1^*)'}$  is the dual product. Now, let us prove that  $\mathcal{B}_1$  is admissible<sup>3</sup> for  $(\mathcal{T}_1(t))_{t \geq 0}$ . To do so, consider the system

$$\begin{cases} \frac{d}{dt} \begin{pmatrix} \varphi_1(t, \cdot) \\ \varphi_2(t, \cdot) \end{pmatrix} = \mathcal{A}_1^* \begin{pmatrix} \varphi_1(t, \cdot) \\ \varphi_2(t, \cdot) \end{pmatrix}, \\ y(t) = \mathcal{B}_1^* \begin{pmatrix} \varphi_1(t, \cdot) \\ \varphi_2(t, \cdot) \end{pmatrix} \end{cases} \quad (\text{B.35})$$

where  $\mathcal{A}_1^* : \varphi \in D(\mathcal{A}_1^*) \mapsto \mathcal{A}_1^* \varphi \in \mathcal{I}$  and  $\mathcal{B}_1^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in D(\mathcal{A}_1^*) \mapsto \mathcal{B}^* \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathbb{R}$  are given by

$$\begin{cases} \mathcal{A}_1^* \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \lambda_1(\varphi_1' - \mu\varphi_1) + \frac{c_2 p_2}{p_1} \varphi_2 e^{2\mu x} \\ -\lambda_2(\varphi_2' + \mu\varphi_2) + \frac{c_1 p_1}{p_2} \varphi_1 e^{-2\mu x} \end{pmatrix}, \\ D(\mathcal{A}_1^*) = \left\{ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathcal{H}^1(0, L, \mathbb{R}^2) \mid \begin{pmatrix} \varphi_1(L) \\ \varphi_2(0) \end{pmatrix} = \begin{pmatrix} 0 & \frac{p_2 \lambda_2}{p_1 \lambda_1} a e^{2\mu L} \\ \frac{p_1 \lambda_1}{p_2 \lambda_2} b & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(0) \\ \varphi_2(L) \end{pmatrix} \right\}, \\ \mathcal{B}_1^* : \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in D(\mathcal{A}_1^*) \mapsto p_1 \lambda_1 \varphi_1(0). \end{cases} \quad (\text{B.36})$$

<sup>3</sup>See e.g [125, Definition 4.2.1]

Since  $\mathcal{A}_1$  generates a  $C_0$ -semigroup  $(\mathcal{T}_1(t))_{t \geq 0}$  on  $\mathcal{I}$  then,  $\mathcal{A}_1^*$  generates a  $C_0$ -semigroup  $(\mathcal{T}_1^*(t))_{t \geq 0}$  on  $\mathcal{I}$ . Hence, for all  $\begin{pmatrix} \varphi_1^0 \\ \varphi_2^0 \end{pmatrix} \in D(\mathcal{A}_1^*)$ , the function

$$\begin{pmatrix} \varphi_1(t, \cdot) \\ \varphi_2(t, \cdot) \end{pmatrix} = \mathcal{T}_1^*(t) \begin{pmatrix} \varphi_1^0 \\ \varphi_2^0 \end{pmatrix}, \quad \forall t \geq 0 \quad (\text{B.37})$$

defines the unique classical solution of (B.35). Now, consider the following function

$$V(\varphi_1, \varphi_2) = \left\| \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right\|_{\mu} := p_1 \int_0^L \varphi_1(x)^2 e^{-\mu x} dx + p_2 \int_0^L \varphi_2(x)^2 e^{\mu x} dx \quad (\text{B.38})$$

The time derivative of  $V$  along the trajectories of (B.35) reads as, for all  $t \geq 0$ ,

$$\begin{aligned} \dot{V}(\varphi_1(t, \cdot), \varphi_2(t, \cdot)) &= -\frac{1}{2} \int_0^L \begin{pmatrix} \varphi_1(t, x) & \varphi_2(t, x) \end{pmatrix} \mathcal{P}(x) \begin{pmatrix} \varphi_1(t, x) \\ \varphi_2(t, x) \end{pmatrix} dx \\ &\quad - \frac{p_1 \lambda_1}{2} \varphi_1(t, 0)^2 \left( 1 - \frac{p_1 \lambda_1}{p_2 \lambda_2} b^2 \right) - \frac{p_2 \lambda_2}{2} \varphi_2(t, L)^2 \left( e^{\mu L} - \frac{p_2 \lambda_2}{p_1 \lambda_1} a^2 e^{3\mu L} \right) \end{aligned} \quad (\text{B.39})$$

According to (3.32) and (3.33), one can deduce from (B.39) that, for all  $t \geq 0$

$$\begin{cases} \dot{V}(\varphi_1(t, \cdot), \varphi_2(t, \cdot)) \leq 0, \\ \varphi_1(t, 0)^2 \leq -\frac{1}{\frac{p_1 \lambda_1}{2} \left( 1 - \frac{p_1 \lambda_1}{p_2 \lambda_2} b^2 \right)} \dot{V}(\varphi_1(t, \cdot), \varphi_2(t, \cdot)). \end{cases} \quad (\text{B.40})$$

Thus, from (B.40), we obtain for every  $T > 0$  and for all  $t \in [0, T]$

$$\begin{aligned} \int_0^T |y(t)|^2 dt &= p_1^2 \lambda_1^2 \int_0^T \varphi_1(t, 0)^2 dt \\ &\leq -\frac{p_1 \lambda_1}{\frac{1}{2} \left( 1 - \frac{p_1 \lambda_1}{p_2 \lambda_2} b^2 \right)} \int_0^T \dot{V}(\varphi_1(t, \cdot), \varphi_2(t, \cdot)) dt \\ &\leq \frac{p_1 \lambda_1}{\frac{1}{2} \left( 1 - \frac{p_1 \lambda_1}{p_2 \lambda_2} b^2 \right)} (V(\varphi_1(0, \cdot), \varphi_2(0, \cdot)) - V(\varphi_1(T, \cdot), \varphi_2(T, \cdot))) \\ &\leq \frac{p_1 \lambda_1}{\frac{1}{2} \left( 1 - \frac{p_1 \lambda_1}{p_2 \lambda_2} b^2 \right)} V(0) \end{aligned} \quad (\text{B.41})$$

where  $y$  come from (B.35). Then, for every  $T > 0$  we have

$$\int_0^T |y(t)|^2 dt \leq \frac{p_1 \lambda_1}{\frac{1}{2} \left( 1 - \frac{p_1 \lambda_1}{p_2 \lambda_2} b^2 \right)} \left\| \begin{pmatrix} \varphi_1(0, \cdot) \\ \varphi_2(0, \cdot) \end{pmatrix} \right\|_{\mu} \quad (\text{B.42})$$

Then according to [125, Definition 4.3.1 and Theorem 4.4.3.], this proves that  $\mathcal{B}_1$  is admissible for the  $C_0$ -semigroup  $(\mathcal{T}_1(t))_{t \geq 0}$ .

Now, since  $q$  is continuous and for each  $i = 1, \dots, 2n+1$ ,  $(E_{ii}$  and  $E_{i(i+1)})$  are absolutely continuous, then the function

$$\tilde{q} : t \in \mathbb{R}_+ \mapsto -L_i^2 q \left( \kappa \left( E_{(2n+1)(2n+1)}(t) - E_{2n(2n+1)}(t) \right) - \sum_{j=1}^{2n+1} a_j E_{j(j+1)}(t) - \dot{E}_{(2n+1)(2n+2)}(t) \right) \quad (\text{B.43})$$

belongs to  $L_{\text{loc}}^2(0, \infty)$ . As a consequence, since  $\mathcal{A}_1$  generates a  $C_0$ -semigroup  $(\mathcal{T}_1(t))_{t \geq 0}$  on  $\mathcal{I}$  and the operator  $\mathcal{B}_1$  is an admissible operator for the  $C_0$ -semigroup  $(\mathcal{T}_1(t))_{t \geq 0}$ , then for any initial value  $\begin{pmatrix} w_1(0, \cdot) \\ w_2(0, \cdot) \end{pmatrix} \in \mathcal{I}$ , there exists a unique mild solution  $(w_1, w_2) \in C([0, \infty); \mathcal{I}) \cap \mathcal{H}_{\text{loc}}^1((0, \infty); D(\mathcal{A}_1^*))'$  according to [125, Proposition 4.2.5]. This concludes the proof of Theorem 5.  $\square$

## Proof of Theorem 6

Let  $\begin{pmatrix} w_1(0, \cdot), w_2(0, \cdot), \bar{Z}(0), \hat{Z}(0) \end{pmatrix} \in \mathcal{J}$ . Then, according to Proposition 3, there exists a finite-time  $t_r$  such that, for all  $t \geq t_r$ , the solution  $(w_1, w_2)$  of (B.30) is equivalent to the solution of the following system

$$\begin{cases} \partial_t w_1(t, x) + \lambda_1 \partial_x w_1(t, x) = c_1 w_2(t, x), \\ \partial_t w_2(t, x) - \lambda_2 \partial_x w_2(t, x) = c_2 w_1(t, x), \\ w_1(t, 0) = u(t) + d(t), \\ w_2(t, L) = a w_1(t, L). \end{cases} \quad (\text{B.44})$$

Thus, according to [11, Corollary 5.5.], there exists a positive constant  $\xi$  such that

$$V(w_1(t, \cdot), w_2(t, \cdot)) \leq V(w_1(t_r, \cdot), w_2(t_r, \cdot)) e^{-\xi(t-t_r)} \quad (\text{B.45})$$

where  $V$  is given in (B.38). Therefore, to conclude the proof of Theorem 6, it is just necessary to prove that the solution  $(w_1, w_2)$  of system (B.30) depends continuously on initial conditions on the time interval  $[0, t_r]$ .

Now, according to [125, Proposition 2.1.2], there exists  $K_0 > 0$  such that, for all  $t \in [0, t_r]$ , we have

$$\left\| \begin{pmatrix} w_1(t, \cdot) \\ w_2(t, \cdot) \end{pmatrix} \right\|_{\mathcal{I}} \leq K_0 \left\| \begin{pmatrix} w_1(0, \cdot) \\ w_2(0, \cdot) \end{pmatrix} \right\|_{\mathcal{I}} + \left\| \int_0^t \mathbb{T}(t-s) \mathcal{B}_1 \tilde{q}(s) ds \right\|_{\mathcal{I}} \quad (\text{B.46})$$

where  $\tilde{q}$  is given in (B.43). As a consequence, since  $(\mathcal{T}_1(t))_{t \geq 0}$  is exponentially stable and  $\mathcal{B}_1$  is admissible operator for  $(\mathcal{T}_1(t))_{t \geq 0}$ , then we have according to [125, Proposition 4.4.5], that there exists  $K_1 > 0$  independent of  $t_r$  such that

$$\left\| \begin{pmatrix} w_1(t, \cdot) \\ w_2(t, \cdot) \end{pmatrix} \right\|_{\mathcal{I}} \leq K_1 \left( \left\| \begin{pmatrix} w_1(0, \cdot) \\ w_2(0, \cdot) \end{pmatrix} \right\|_{\mathcal{I}} + \|\tilde{q}(\cdot)\|_{L^2(0, t_r)} \right). \quad (\text{B.47})$$

Since  $q$  is continuous and for each  $i = 1, \dots, 2n+1$ ,  $(E_{ii}$  and  $E_{i(i+1)})$  are absolutely continuous, then  $\tilde{q}$



is continuous. Thus, there exists  $K_2 > 0$  such that

$$\|\tilde{q}(\cdot)\|_{L^2(0,t_r)}^2 \leq t_r. \quad (\text{B.48})$$

Now, according to [1, Theorem], there exists a positive constant  $K_3$  such that

$$t_r < K_3 |\mathcal{E}(0)|. \quad (\text{B.49})$$

As a consequence, according to (B.47), (B.48) and (B.49), there exists  $C > 0$  (independent of  $t_r$ ) such that, for all  $t \in [0, t_r]$ ,

$$\left\| \begin{pmatrix} w_1(t, \cdot) \\ w_2(t, \cdot) \end{pmatrix} \right\|_{\mathcal{I}} \leq C \left( \left\| \begin{pmatrix} w_1(0, \cdot) \\ w_2(0, \cdot) \end{pmatrix} \right\|_{\mathcal{I}} + \sqrt{|\mathcal{E}(0)|} \right). \quad (\text{B.50})$$

This concludes the proof of Theorem 6.  $\square$

## C Proof of the main results of Chapter 4

### Proof of Theorem 7

Let  $T > 0$ . We prove the statement of the Theorem 7 for  $w_0 \in \mathcal{H}_L^3(0, L)$ ,  $d_2 \in C_0^2([0, T])$  and  $d_1 \in C^1([0, T], L^2(0, L))$ , where we recall that  $\mathcal{H}_L^3(0, L)$  is defined in the Proposition 5. Since  $\mathcal{H}_L^3(0, L)$ ,  $C^2([0, T])$  and  $C^1([0, T], L^2(0, L))$  are dense in  $L^2(0, L)$ ,  $L^2(0, T)$  and  $L^1([0, T]; L^2(0, L))$ , respectively, the result follows for all  $w_0 \in L^2(0, L)$ ,  $d_1 \in L^1([0, T]; L^2(0, L))$  and  $d_2 \in L^2(0, T)$ , by a standard density argument similar to the one provided in [83, Lemma 1].

### Proof of item (a) of Theorem 7

The derivative of the Energy (4.14) gives along solutions of the linear KdV model (4.2) a negative term in  $\partial_x w(t, 0)$ . Moreover, Theorem 8 shows that using such a term in the  $w$ -dynamics, we are able to obtain an ISS-Lyapunov functional  $U$ . As a consequence, the main idea of this proof consists in adding and subtracting the term  $\partial_x w(t, 0)$ , multiplied by a coefficient  $p(x)$ , in the  $w$  dynamics: one term is used to obtain the negativity in the  $L^2$  norm of the full space as in (4.27), while the other is treated as a distributed disturbance  $d_1$  and compensated by the negativity of the Energy.

With the previous points in mind, fix  $\lambda = 1$  and consider the functions  $p$  and  $U$  given by Theorem 8. Set  $\bar{p} := \|p\|_{L^2}^2$ . Note that  $\bar{p} \neq 0$  because  $p$  is a non-zero function. We define the operator  $\Pi$  and the function  $W$  as follows

$$W(w) := \frac{1}{2\bar{p}\varrho_1} U(w) = \|\Pi(w)\|_{L^2}^2, \quad \Pi(w) := \frac{1}{\sqrt{2\bar{p}\varrho_1}} \bar{\Pi}^{-1}(w), \quad (\text{C.1})$$

for all  $w \in L^2(0, L)$ , where the operator  $\bar{\Pi}$  and the parameter  $\varrho_1$  are given by Theorem 8. We show that the statement of the theorem holds and in particular that the inequalities (4.20), (4.21) are satisfied.

First, in view of (4.26), we obtain

$$\frac{\underline{c}}{2\bar{p}\varrho_1} \|w\|_{L^2} \leq W(w) \leq \frac{\bar{c}}{2\bar{p}\varrho_1} \|w\|_{L^2}.$$

As a consequence, by recalling that  $E(w) = \|w\|_{L^2}^2$ , the inequality (4.20) is satisfied for the function  $V = E + W$  with  $\underline{\alpha} := 1 + \frac{\underline{c}}{2\bar{p}\varrho_1}$  and  $\bar{\alpha} := 1 + \frac{\bar{c}}{2\bar{p}\varrho_1}$ .

Then, in order to show the inequality (4.21) we compute the derivative of the functional  $V$  along the trajectories of the system (4.2). We first analyze the time derivative of the energy  $E$ . Using (4.15) and adding the effect of the perturbations  $d_1, d_2$ , we obtain

$$\begin{aligned} \dot{E}(w) &= -|\partial_x w(0)|^2 + 2 \int_0^L w(x) d_1(x) dx + |d_2|^2 \\ &\leq -|\partial_x w(0)|^2 + \frac{\underline{c}}{4\bar{p}\varrho_1} \|w\|_{L^2}^2 + \frac{4\bar{p}\varrho_1}{\underline{c}} \|d_1\|_{L^2}^2 + |d_2|^2 \end{aligned} \quad (\text{C.2})$$

where the second inequality has been obtained by using the Cauchy-Schwarz and Young inequalities, and with the parameters  $\underline{c}, \varrho_1$  given by Theorem 8. Next, we compute the derivative of  $W$  along the trajectories of system (4.2). To this end, we first add and subtract the term  $p(x)\partial_x w(t, 0)$  to the dynamics, obtaining

$$\begin{cases} \partial_t w + \partial_x w + \partial_{xxx} w - p(x)\partial_x w(t, 0) = -p(x)\partial_x w(t, 0) + d_1(t, x), & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x w(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L]. \end{cases} \quad (\text{C.3})$$

Applying the ISS-Lyapunov inequality (4.27) along solutions to (C.3), the derivative of  $U$  yields

$$\begin{aligned} \dot{U}(w) &\leq -U(w) + \varrho_1 \|d_1 - p\partial_x w(0)\|_{L^2}^2 + \varrho_2 |d_2|^2 \\ &\leq -\underline{c} \|w\|_{L^2}^2 + 2\varrho_1 \|d_1\|_{L^2}^2 + 2\varrho_1 \|p\partial_x w(0)\|_{L^2}^2 + \varrho_2 |d_2|^2, \end{aligned} \quad (\text{C.4})$$

where in the second inequality we used again the inequality (4.26). Finally, we can compute the derivative of the function  $V = E + W$ , with  $W$  defined in (C.1), by combining (C.4) and (C.2) and using the identity

$$-|\partial_x w(0)|^2 + \frac{1}{\bar{p}} \|p\partial_x w(0)\|_{L^2}^2 = 0. \quad (\text{C.5})$$

Simple computations give the inequality (4.21) with the choice  $\alpha := \frac{\underline{c}}{4\bar{p}\varrho_1}$ ,  $\sigma_1 = \frac{4\bar{p}\varrho_1}{\underline{c}} + \frac{1}{\bar{p}}$ ,  $\sigma_2 := 1 + \frac{\varrho_2}{2\bar{p}\varrho_1}$ . This concludes the proof of item (a) of Theorem 7.  $\square$

### Proof of the item (b) of Theorem 7

Consider again the function  $V = E + W$  with  $W$  defined in (C.1). The derivative of the energy (4.14) along the trajectories of the nonlinear system (4.1) is computed as in (C.2) because the contribution of the nonlinear term  $w\partial_x w$  is zero. Next, we compute the time derivative of  $W$ . However, due to the presence of the nonlinear term  $w\partial_x w$  we cannot apply off-the-shelf the inequality (4.27) by including such a term

in the disturbance  $d_1$ : it would not be bounded with the right norm. As a consequence, unfortunately, we need to revisit and adapt some steps of the proof of Theorem 8 and in particular we need to compute the change of coordinates defined in (4.28), (4.39). Recalling that we selected  $\lambda = 1$ , the  $\gamma$ -dynamics reads

$$\begin{cases} \partial_t \gamma + \partial_x \gamma + \partial_{xxx} \gamma + \gamma = -\bar{\Pi}^{-1}(p) \partial_x w(t, 0) + \bar{\Pi}^{-1}(d_1) - \bar{\Pi}^{-1}(w \partial_x w) + \partial_z Q(x, L) d_2, & (t, x) \in \Omega \\ \gamma(t, 0) = \gamma(t, L) = 0, & t \in \mathbb{R}_+ \\ \partial_x \gamma(t, L) = d_2(t), & t \in \mathbb{R}_+ \\ \gamma(0, x) = \gamma_0(x), & x \in [0, L]. \end{cases} \quad (\text{C.6})$$

where  $Q$  is defined in (4.39). With respect to system (4.41) we have two extra terms to analyse, that are the terms  $\bar{\Pi}^{-1}(p) \partial_x w(0)$  and  $\bar{\Pi}^{-1}(w \partial_x w)$ . As a consequence, we consider again the Lyapunov functional  $U(w) := \|\gamma\|_{L^2}^2$  as in (4.38), and we follow similar computations to those developed from (4.42) to (4.43). Also, as in the proof of item (a), we consider as a full disturbance the term  $d_1 - p \partial_x w(0)$ , see inequality (C.4). In particular, the derivative of  $U$  along the trajectories of system (C.6) satisfies, for all  $w \in L^2(0, L)$

$$\dot{U}(w) \leq -\|\gamma\|_{L^2}^2 + \varrho_1 \|d_1 - p \partial_x w(0)\|_{L^2}^2 + \varrho_2 |d_2|^2 + 2 \left| \int_0^L f(w \partial_x w) \gamma dx \right|$$

where the function  $f$  is defined as  $f(w \partial_x w)(x) := \bar{\Pi}^{-1}(w \partial_x w)(x) = w(x) \partial_x w(x) + \int_0^L Q(x, z) w(z) \partial_x w(z) dz$ . By using the same argument as in [29, Proof of Theorem 1.2, page 1111-1113], we can show the existence of positive constant  $\bar{f}$  that depends only on the function  $Q$ , such that

$$2 \left| \int_0^L f(w \partial_x w)(x) \gamma(x) dx \right| \leq \bar{f} \|\gamma\|_{L^2}^3 \quad \forall w \in L^2(0, L).$$

As a consequence, combining the previous inequalities and following the same computations in (C.4), we obtain, for all  $w \in L^2(0, L)$

$$\dot{U}(w) \leq -\left(1 - \bar{f} \|\gamma\|_{L^2}\right) \|\gamma\|_{L^2}^2 + 2\varrho_1 \|d_1\|_{L^2}^2 + 2\varrho_1 \|p \partial_x w(0)\|_{L^2}^2 + \varrho_2 |d_2|^2.$$

Therefore, using the inequality (4.26), we obtain

$$\dot{U}(w) \leq -\frac{\underline{c}}{2} \|w\|_{L^2}^2 + 2\varrho_1 \|d_1\|_{L^2}^2 + 2\varrho_1 \|p \partial_x w(0)\|_{L^2}^2 + \varrho_2 |d_2|^2$$

for all  $w$  satisfying  $\|w\|_{L^2} \leq \bar{\delta}$ , with  $\bar{\delta} = (2\sqrt{\underline{c}}\bar{f})^{-1}$ . Using the definition of the function  $W$  in (C.1) and following the same steps of item (a), we obtain the inequality (4.21) with the choice  $\alpha := \frac{\underline{c}}{8\bar{p}\varrho_1}$ ,  $\sigma_1 = \frac{4\bar{p}\varrho_1}{\underline{c}} + \frac{1}{\bar{p}}$ ,  $\sigma_2 := 1 + \frac{\varrho_2}{2\bar{p}\varrho_1}$ , and  $\delta = \frac{1}{3}\bar{\delta}$ . This concludes the proof of item (b) of Theorem 7.  $\square$

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### Proof of Lemma 1

Consider the following boundary value problem

$$\begin{cases} w'_\infty(x) + w'''_\infty(x) = d(x), & x \in [0, L], \\ w_\infty(0) = w_\infty(L) = 0, \\ w'_\infty(0) = r, \end{cases}$$

which represents the nonzero equilibrium state of (4.48), together with  $\eta_\infty = \frac{w'_\infty(L)}{k}$ . Consider the smooth function  $\phi(x) = \frac{rx(L-x)}{L}$ . It satisfies the boundary conditions  $\phi(0) = \phi(L) = 0$  and  $\phi'(0) = r$ . We set  $\psi = w_\infty - \phi$ . Then  $\psi$  satisfies the following system

$$\begin{cases} \psi'(x) + \psi'''(x) = j(x), & x \in [0, L], \\ \psi(0) = \psi(L) = 0, \\ \psi'(0) = 0, \end{cases}$$

where  $j(x) = d(x) - \phi'(x)$ . This system can be written in the operator form as  $\mathcal{S}^*\psi = j$ , where  $\mathcal{S}^*$ , is the adjoint operator of  $\mathcal{S}$  defined in (4.49). In particular,  $\mathcal{S}^*$ , is defined as  $\mathcal{S}^*\psi = \psi''' + \psi'$  with domain  $D(\mathcal{S}^*) := \{w \in \mathcal{H}^3(0, L) : w(0) = w(L) = w'(0) = 0\}$ . Following [88, Lemma 4], we can prove that the canonical embedding from  $D(\mathcal{S}^*)$ , equipped with the graph norm, into  $L^2(0, L)$ , is compact. Then, according to [24, Proposition 4.24],  $\mathcal{S}^*$  is an operator with compact resolvent. This implies that its spectrum consists only of eigenvalues. Moreover, 0 is not an eigenvalue of  $\mathcal{S}^*$ . Hence, there exists a unique solution  $\psi_\infty$  to the equation  $\mathcal{S}^*\psi = j$ . The equilibrium  $(\eta_\infty, w_\infty)$  can then be computed as  $w_\infty(x) = \psi_\infty + \phi(x)$  for all  $x \in [0, L]$  and  $\eta_\infty = \frac{w'_\infty(L)}{k}$ , with  $\phi$  being the function defined at the beginning of the proof.  $\square$

### Proof of Lemma 2

Given  $(d, r) \in L^2(0, L) \times \mathbb{R}$ , let  $(\eta_\infty, w_\infty)$  the corresponding equilibrium to (4.48) computed according to Lemma 1. Consider the following change of coordinates

$$(w, \eta) \mapsto (\tilde{w}, \tilde{\eta}) := (w - w_\infty, \eta - \eta_\infty). \quad (\text{C.7})$$

The  $(\tilde{w}, \tilde{\eta})$ -dynamics is given by

$$\begin{cases} \partial_t \tilde{w} + \partial_x \tilde{w} + \partial_{xxx} \tilde{w} = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x \tilde{w}(t, L) = k\tilde{\eta}(t), & t \in \mathbb{R}_+, \\ \dot{\tilde{\eta}}(t) = \partial_x \tilde{w}(t, 0), & t \in \mathbb{R}_+, \\ \tilde{w}(0, x) = \tilde{w}_0(x), \tilde{\eta}(0) = \tilde{\eta}_0, & x \in [0, L], \end{cases} \quad (\text{C.8})$$

where  $\tilde{w}_0(x) = w_0(x) - w_\infty(x)$  and  $\tilde{\eta}_0 = \eta_0 - \eta_\infty$ . System (C.8) can be rewritten, in the operator form, as

$$\frac{d}{dt} \tilde{\zeta} = \mathcal{A} \tilde{\zeta}, \quad \tilde{\zeta} = \zeta_0, \quad \tilde{\zeta} := \begin{pmatrix} \tilde{\eta} \\ \tilde{w} \end{pmatrix}$$

with  $\mathcal{A}$  and its domain  $D(\mathcal{A})$  defined as in (4.50). As a consequence, systems (4.48) and (C.8) are equivalent. Then, if one proves that the operator  $\mathcal{A}$  defined in (4.50) is a  $m$ -dissipative operator on  $(X, \|\cdot\|_X)$ , one can apply the result provided by [17, Theorem 3.1], and conclude that the statement of Lemma 2 holds. For that, we look for an equivalent norm and a related scalar product coming from a Lyapunov functional. We will prove then the dissipativity with respect to such a scalar product. This Lyapunov functional is built following the forwarding approach (see e.g [122]). To simplify the notation, in the rest of this proof, we will write  $(\eta, w)$  instead of  $(\tilde{\eta}, \tilde{w})$ .

Now, by recalling the definition of the operator  $\mathcal{S}$  in given in (4.49), we define the operator  $\mathcal{M} : L^2(0, 1) \rightarrow \mathbb{R}$  as solution to the following Sylvester equation

$$\mathcal{M}\mathcal{S}w = \mathcal{C}w, \quad \forall w \in D(\mathcal{S}), \quad (\text{C.9})$$

where  $\mathcal{C} : f \in \mathcal{H}_0^1(0, L) \mapsto f'(0) \in \mathbb{R}$ . Since the strongly continuous semigroup generated by the operator  $\mathcal{S}$  is exponentially stable, the Sylvester equation (C.9) admits a unique solution, see [101, Lemma 22]. Moreover, since  $\mathcal{M}$  is a linear form, according to Riesz representation theorem [18, Theorem 4.11], the operator  $\mathcal{M}$  is uniquely defined as  $\mathcal{M}w = \int_0^L M(x)w(x)dx$ . In order to obtain an explicit solution, we write equation (C.9) in the explicit form

$$w'(0) = - \int_0^L M(x)[w'(x) + w'''(x)]dx \quad \forall w \in D(\mathcal{S}).$$

Using integration by parts we obtain

$$w'(0) = \int_0^L w(x)[M'(x) + M'''(x)]dx + M(0)w''(0) - M(L)w''(L) - M'(0)w'(0),$$

for all  $w \in D(\mathcal{S})$ . From the latter equation, we obtain the following boundary value problem

$$\begin{cases} M''' + M' = 0, \\ M(0) = M(L) = 0, \\ M'(0) = -1. \end{cases} \quad (\text{C.10})$$

It can be verified that the function

$$M : x \in \mathbb{R} \mapsto \frac{-2 \sin(\frac{x}{2}) \sin(\frac{L-x}{2})}{\sin(\frac{L}{2})} \quad (\text{C.11})$$

is a solution to (C.10). Computations are omitted for space reasons. Moreover, it is the unique solution to (C.10) and the operator  $\mathcal{M}$  defined above is the unique solution to the Sylvester equation (C.9). Then, the operator  $\mathcal{M} : L^2(0, L) \rightarrow \mathbb{R}$  can be expressed as  $\mathcal{M}\varphi = \int_0^L M(x)\varphi(x)dx$ .

With the operator  $\mathcal{M}$  so defined, consider the candidate Lyapunov functional  $\mathcal{V} : X \rightarrow \mathbb{R}$  defined as

$$\mathcal{V}(\eta, w) = V(w) + (\eta - \mathcal{M}w)^2, \quad (\text{C.12})$$

where  $V$  is the Lyapunov functional given by Theorem 7. By construction, the Lyapunov functional  $\mathcal{V}$  is

equivalent to the standard norm on the space  $X$ , and in particular, there exist positive constants  $\underline{\nu}, \bar{\nu}$  such that the following holds

$$\underline{\nu}\|(\eta, w)\|_X^2 \leq \mathcal{V}(\eta, w) \leq \bar{\nu}\|(\eta, w)\|_X^2, \quad \forall (\eta, w) \in X. \quad (\text{C.13})$$

To show this fact, note that, following similar arguments used in the proof of Proposition 4 of [122], for any  $\rho \in ]0, 1[$  we have

$$\rho \left( \frac{1}{2}\eta^2 - \|M\|_{L^2}^2 \|w\|_{L^2}^2 \right) \leq (\eta - \mathcal{M}w)^2 \leq 2(\eta^2 + \|M\|_{L^2}^2 \|w\|_{L^2}^2)$$

for all  $(\eta, w) \in X$ . Furthermore, according to Theorem 7, we know that  $V$  satisfies the inequality (4.20). Then we have

$$\rho \left( \frac{1}{2}\eta^2 - \|M\|_{L^2}^2 \|w\|_{L^2}^2 \right) + \underline{\alpha}\|w\|_{L^2}^2 \leq \mathcal{V}(w) \leq 2(\eta^2 + \|M\|_{L^2}^2 \|w\|_{L^2}^2) + \bar{\alpha}\|w\|_{L^2}^2.$$

Therefore, by selecting  $\rho$  sufficiently small, inequality (C.13) holds for some  $\bar{\nu} > \underline{\nu} > 0$ . By recalling that the functional  $V$  established in Theorem 7 is of the form  $V = E + W$ , where  $E$  and  $W$  are quadratic forms of the  $L^2$  norm of  $w$ , from the Lyapunov functional  $\mathcal{V}$  defined in (C.12), we can also deduce a scalar product, that we define as follows

$$\left\langle \begin{bmatrix} \eta_1 & w_1 \end{bmatrix}^\top, \begin{bmatrix} \eta_2 & w_2 \end{bmatrix}^\top \right\rangle_{\mathcal{V}} := (\eta_1 - \mathcal{M}w_1)(\eta_2 - \mathcal{M}w_2) + \langle w_1, w_2 \rangle_{L^2} + \langle \Pi w_1, \Pi w_2 \rangle_{L^2}, \quad (\text{C.14})$$

with  $\Pi$  being the linear operator given by Theorem 7. It is equivalent to the usual scalar product in  $X$ .

Now, we are in position to prove that  $\mathcal{A}$  is  $m$ -dissipative according to [100]. For this, we need to show that  $\mathcal{A}$  is dissipative and maximal. We begin with showing the dissipative properties. To this end, we use the scalar product given in (C.14). By using the definition of  $\mathcal{A}$  given in (4.50), we obtain, for all  $\zeta \in D(\mathcal{A})$ ,

$$\begin{aligned} \langle \mathcal{A}\zeta, \zeta \rangle_{\mathcal{V}} &= (w'(0) + \mathcal{M}(w''' + w'))(\eta - \mathcal{M}w) - \langle w' + w''', w \rangle_{L^2} - \langle \Pi(w''' + w'), \Pi w \rangle_{L^2} \\ &= \left( w'(0) + \int_0^L M(x)[w'(x) + w'''(x)]dx \right) (\eta - \mathcal{M}w) - \langle w' + w''', w \rangle_{L^2} - \langle \Pi(w''' + w'), \Pi w \rangle_{L^2}. \end{aligned} \quad (\text{C.15})$$

For the first term, it can be shown, after some integrations by parts, that

$$\int_0^L M(x)[w'(x) + w'''(x)]dx = -k\eta - w'(0) \quad (\text{C.16})$$

for all  $\zeta \in D(\mathcal{A})$ . Then, for the second term, we recall the ISS properties of the functional  $V$  stated in Theorem 7. In particular, applying the inequality (4.21) to the system (4.48), in which  $d$  is the distributed disturbance (thus having the role of  $d_1$ ) and  $k\eta$  is seen as a disturbance acting at the boundary condition

(thus having the role of  $d_2$ ), we obtain

$$-2\langle w' + w''', w \rangle_{L^2} - 2\langle \Pi(w'''' + w'), \Pi w \rangle_{L^2} \leq -\alpha \|w\|_{L^2}^2 + \sigma_2 k^2 \eta^2. \quad (\text{C.17})$$

for all  $\zeta \in D(\mathcal{A})$ . Hence, combining inequalities (C.15) with (C.16) and (C.17), we obtain

$$\begin{aligned} \langle \mathcal{A}\zeta, \zeta \rangle_{\mathcal{V}} &\leq -k\eta(\eta - \mathcal{M}w) - \frac{\alpha}{2} \|w\|_{L^2}^2 + \frac{\sigma_2}{2} k^2 \eta^2 \\ &\leq -k \left( 1 - \left( \frac{\sigma_2}{2} + \frac{\|M\|_{L^2}}{4\alpha} \right) k \right) \eta^2 - \frac{\alpha}{4} \|w\|_{L^2}^2 \end{aligned}$$

for all  $\zeta \in D(\mathcal{A})$ , where the second inequality has been obtained by using Young's inequality. As a consequence, we can select

$$k_0^* = \left( \frac{\sigma_2}{2} + \frac{\|M\|_{L^2}}{4\alpha} \right)^{-1}.$$

This implies that for any  $k \in (0, k_0^*)$  there exists  $\varepsilon > 0$  such that we have

$$\langle \mathcal{A}\zeta, \zeta \rangle_{\mathcal{V}} \leq -\varepsilon (\|\eta\|^2 + \|w\|_{L^2}^2) \quad (\text{C.18})$$

for all  $\zeta \in D(\mathcal{A})$ , which shows that the operator  $\mathcal{A}$  is dissipative.

Now, we want to show that  $\mathcal{A}$  is a maximal operator. According to Lümer-Phillips theorem [100, Theorem 4.3], proving that  $\mathcal{A}$  is maximal reduces to show that there exists a positive  $\lambda_0$  such that for all  $\zeta \in X$ , there exists  $\tilde{\zeta} \in D(\mathcal{A})$  such that  $(\lambda_0 I_X - \mathcal{A})\tilde{\zeta} = \zeta$ . Let  $(\eta, w) \in X$ . We look for a  $(\tilde{\eta}, \tilde{w}) \in D(\mathcal{A})$  satisfying

$$\begin{cases} \tilde{w}''' + \tilde{w}' + \lambda_0 \tilde{w} = w, & x \in [0, L], \\ \tilde{w}(0) = \tilde{w}(L) = 0, \\ \tilde{w}'(L) = k\tilde{\eta}, \\ \lambda_0 \tilde{\eta} - \tilde{w}'(0) = \eta, \end{cases} \quad (\text{C.19})$$

namely

$$\begin{cases} \tilde{w}''' + \tilde{w}' + \lambda_0 \tilde{w} = w, & x \in [0, L], \\ \tilde{w}(0) = \tilde{w}(L) = 0, \\ \tilde{w}'(L) = \frac{k}{\lambda_0} (\eta + \tilde{w}'(0)), \\ \lambda_0 \tilde{\eta} - \tilde{w}'(0) = \eta. \end{cases}$$

Now, we consider the following boundary value problem

$$\begin{cases} \tilde{w}''' + \tilde{w}' + \lambda_0 \tilde{w} = w, & x \in [0, L], \\ \tilde{w}(0) = \tilde{w}(L) = 0, \\ \tilde{w}'(L) = \frac{k}{\lambda_0} (\eta + \tilde{w}'(0)), \end{cases}$$

and the smooth function  $\tilde{\phi}(x) = \frac{k\eta x^2(x-L)}{\lambda_0 L^2}$  satisfying the boundary conditions

$$\tilde{\phi}(0) = \tilde{\phi}(L) = \tilde{\phi}'(0) = 0, \quad \tilde{\phi}'(L) = \frac{k}{\lambda_0} \eta.$$

We set  $\tilde{\psi} = \tilde{w} - \tilde{\phi}$ . Then  $\tilde{\psi}$  satisfies the following boundary value problem

$$\begin{cases} \tilde{\psi}' + \tilde{\psi}''' + \lambda_0 \tilde{\psi} = \tilde{j}(x), & x \in [0, L], \\ \tilde{\psi}(0) = \tilde{\psi}(L) = 0, \\ \tilde{\psi}'(L) = \frac{k}{\lambda_0} \tilde{\psi}'(0), \end{cases} \quad (\text{C.20})$$

where  $\tilde{j}(x) = w(x) - \tilde{\phi}'(x) - \tilde{\phi}'''(x) - \lambda_0 \tilde{\phi}$ . Now, we define the operator  $\widehat{\mathcal{S}}$  and its domain  $D(\widehat{\mathcal{S}}) \subset L^2(0, L)$  as

$$\widehat{\mathcal{S}}\psi = -\psi' - \psi''', \quad D(\widehat{\mathcal{S}}) := \left\{ \psi \in \mathcal{H}^3(0, L) : \psi(0) = \psi(L) = 0, \psi'(L) = \frac{k}{\lambda_0} \psi'(0) \right\}.$$

We define also its adjoint operator  $\widehat{\mathcal{S}}^*$  and its domain  $D(\widehat{\mathcal{S}}^*)$  as

$$\widehat{\mathcal{S}}^*\psi = \psi''' + \psi', \quad D(\widehat{\mathcal{S}}^*) := \left\{ \psi \in \mathcal{H}^3(0, L) : \psi(0) = \psi(L) = 0, \psi'(0) = \frac{k}{\lambda_0} \psi'(L) \right\}.$$

Note that  $\widehat{\mathcal{S}}$  and  $\widehat{\mathcal{S}}^*$  are dissipative. Indeed, by selecting  $\lambda_0 > k$ , we have

$$\begin{aligned} \int_0^L \psi \widehat{\mathcal{S}}\psi dx &= \left( \frac{k}{\lambda_0} - 1 \right) \psi'(0)^2 < 0, & \psi \in D(\widehat{\mathcal{S}}), \\ \int_0^L \psi \widehat{\mathcal{S}}^*\psi dx &= \left( \frac{k}{\lambda_0} - 1 \right) \psi'(L)^2 < 0, & \psi \in D(\widehat{\mathcal{S}}^*). \end{aligned}$$

Moreover,  $\widehat{\mathcal{S}}$  is closed and  $D(\widehat{\mathcal{S}})$  is dense in  $L^2(0, L)$ . Then, according to [100, Theorem 4.3 and Corollary 4.4]  $\widehat{\mathcal{S}}$  is  $m$ -dissipative operator. Finally, since  $\widehat{\mathcal{S}}$  is a  $m$ -dissipative operator then the system (C.20) admits a solution  $\tilde{\psi}$  in  $D(\widehat{\mathcal{S}})$ . As a consequence, there exist  $(\tilde{\eta}, \tilde{w}) \in D(\mathcal{A})$  solution of (C.19). This proves that  $\mathcal{A}$  is maximal and concludes the proof of Lemma 2.  $\square$

### Proof of Theorem 9

The first part of the proof is proved for any initial condition  $(\eta_0, w_0) \in D(\mathcal{A})$ . The result follows for all initial conditions in  $X$  by a standard density argument (see e.g. [83, Lemma 1]). Consider the equilibrium  $(\eta_\infty, w_\infty)$ , recall the change of coordinates defined in (C.7) and consider the error system (C.8). We show now that the its origin is exponentially stable. To this end, consider the Lyapunov functional  $\mathcal{V}$  defined in (C.12). According to the proof of dissipativity of  $\mathcal{A}$  of Lemma (2), for any  $k \in (0, k^*)$  the time derivative of  $\mathcal{V}$  along the strong solution to (C.8) satisfies (C.18). As a consequence, from (C.13) and Grönwall's lemma, there exist positive constants  $b_0, \nu_0$  such that, for all  $(\eta_0, w_0) \in D(\mathcal{A})$  and for all  $t \geq 0$

$$\|(\tilde{\eta}(t), \tilde{w}(t, \cdot))\|_X \leq b_0 e^{-\nu_0 t} \|(\tilde{\eta}_0, \tilde{w}_0)\|_X. \quad (\text{C.21})$$

By using the density of  $D(\mathcal{A})$  in  $X$ , and the change of coordinates (C.7), we conclude that (4.51) holds.

Now, we need to show that the regulation objective (4.46) is achieved for strong solutions. For this, note that if  $(\eta_0, w_0) \in D(\mathcal{A})$ , then  $(\tilde{\eta}_0, \tilde{w}_0) \in D(\mathcal{A})$ . Then  $(\tilde{\eta}, \tilde{w}) \in C^1(\mathbb{R}_+; X) \cap C^0(\mathbb{R}_+; D(\mathcal{A}))$ . Now, let us introduce the new variables  $v, \xi$  defined as follows

$$(\tilde{w}, \tilde{\eta}) \mapsto (v, \xi) := (\partial_t \tilde{w}, \dot{\tilde{\eta}}). \quad (\text{C.22})$$



The dynamics of  $(v, \xi)$  is given as

$$\begin{cases} \partial_t v + \partial_x v + \partial_{xxx} v = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ v(t, 0) = v(t, L) = 0, & t \in \mathbb{R}_+ \\ \partial_x v(t, L) = k\xi(t), & t \in \mathbb{R}_+ \\ \dot{\xi}(t) = v_x(t, 0), & t \in \mathbb{R}_+ \\ v(0, x) = v_0(x), \xi(0) = \xi_0, & x \in [0, L]. \end{cases} \quad (\text{C.23})$$

with

$$v_0(x) = -\tilde{w}'_0(x) - \tilde{w}'''_0(x), \quad x \in [0, L], \quad \xi_0 = \tilde{w}'_0(0). \quad (\text{C.24})$$

Since  $(v(0, \cdot), \xi(0)) \in X$ , then, according to the Lemma 2 and the first statement of Theorem 9, we have  $(v, \xi) \in C^0(\mathbb{R}_+; X)$  and

$$\|(\xi(t), v(t, \cdot))\|_X \leq b_0 e^{-\nu_0 t} \|(\xi(0), v(0, \cdot))\|_X, \quad \forall (v_0, \xi_0) \in X.$$

By definition of  $v$  and  $\xi$  and using (C.24), one can see that, once one considers  $(v_0, \xi_0) \in X$ , then this implies that  $(\eta_0, w_0) \in D(\mathcal{A})$ . Then, using the definition of the change of coordinates (C.22) and (C.24) we obtain

$$\|\partial_t \tilde{w}(t, \cdot)\|_{L^2} \leq \|(\dot{\tilde{\eta}}(t), \partial_t \tilde{w}(t, \cdot))\|_X \leq b_0 e^{-\nu_0 t} \|(-\tilde{w}'_0(x) - \tilde{w}'''_0(x), \tilde{w}'_0(0))\|_X, \quad \forall w_0 \in D(\mathcal{A}). \quad (\text{C.25})$$

Now, by multiplying the first equation of (C.8) by  $\tilde{w}$  and integrating by parts, we get after some computations

$$k^2 \tilde{\eta}(t)^2 - \partial_x \tilde{w}(t, 0)^2 = \int_0^L \tilde{w}(t, x) \partial_t \tilde{w}(t, x) dx.$$

Using Cauchy-Schwarz's inequality, from (C.21) and (C.25) we finally obtain

$$|\partial_x \tilde{w}(t, 0)|^2 \leq \|\tilde{w}(t, \cdot)\|_{L^2} \|\partial_t \tilde{w}(t, \cdot)\|_{L^2} + k^2 |\tilde{\eta}(t)|^2 \xrightarrow[t \rightarrow \infty]{} 0, \quad \forall (\eta_0, w_0) \in D(\mathcal{A}).$$

From the previous inequality we obtain  $\lim_{t \rightarrow \infty} |\partial_x \tilde{w}(t, 0)| = \lim_{t \rightarrow \infty} |\partial_x w(t, 0) - r| = 0$  for all  $(\eta_0, w_0) \in D(\mathcal{A})$ , and therefore (4.46), concluding the proof.  $\square$

### Proof of Lemma 3

Consider the following boundary value problem

$$\begin{cases} w'_\infty(x) + w'''_\infty(x) + w_\infty(x)w'_\infty(x) = d(x), & x \in [0, L], \\ w_\infty(0) = w_\infty(L) = 0, \\ w'_\infty(0) = r, \end{cases} \quad (\text{C.26})$$

which represents the nonzero equilibrium state of (4.48), with  $\eta_\infty = \frac{w'_\infty(L)}{k}$ . We prove that there exists a solution to system (C.26) by following a fixed-point strategy. We set

$$\mathcal{H}_r^3(0, L) := \left\{ w \in \mathcal{H}^3(0, L) : w(0) = w(L) = 0, w'(0) = r \right\},$$

and we introduce the operator  $\mathcal{T}_0 : \mathcal{H}_r^3(0, L) \rightarrow \mathcal{H}_r^3(0, L)$  defined by  $\mathcal{T}_0(w) = \varphi$  where  $\varphi$  is the solution to

$$\begin{cases} \varphi'(x) + \varphi'''(x) = d(x) - w_\infty(x)w'_\infty(x), & x \in [0, L], \\ \varphi(0) = \varphi(L) = 0, \\ \varphi'(0) = r, \end{cases} \quad (\text{C.27})$$

Note that the functional  $\|\cdot\|_{\mathcal{H}_r^3} : w \in \mathcal{H}^3(0, L) \mapsto \|w' + w'''\|_{L^2} \in \mathbb{R}_+$  is a semi-norm on the space  $\mathcal{H}^3(0, L)$ . Furthermore,  $\mathcal{H}_r^3(0, L) \subset \mathcal{H}_0^1(0, L)$ . Then, according to the Poincaré's inequality, the semi-norm  $\|\cdot\|_{\mathcal{H}_r^3}$  is a norm on the space  $\mathcal{H}_r^3(0, L)$  which is equivalent to the standard norm induced by  $\mathcal{H}^3(0, L)$ . In other words, there exists a positive constant  $\kappa$  such that

$$\|w\|_{\mathcal{H}_r^3} \leq \|w\|_{\mathcal{H}^3(0, L)} \leq \kappa \|w\|_{\mathcal{H}_r^3}, \quad \forall w \in \mathcal{H}_r^3(0, L). \quad (\text{C.28})$$

Now, we have,

$$\begin{aligned} \|\mathcal{T}_0(w)\|_{\mathcal{H}_r^3} &= \|d - ww'\|_{L^2} \\ &\leq \|d\|_{L^2} + \|ww'\|_{L^2} \\ &\leq \|d\|_{L^2} + \|w\|_{L^\infty} \|w'\|_{L^2}, \end{aligned}$$

for all  $w \in \mathcal{H}_r^3(0, L)$ . Denoting with the constant  $\ell$  the norm of the embedding  $\mathcal{H}^3(0, L)$  in  $L^\infty(0, L)$ , according to the Rellich-Kondrachov Theorem (see [18, Theorem 9.16]), we have

$$\begin{aligned} \|\mathcal{T}_0(w)\|_{\mathcal{H}_r^3} &\leq \|d\|_{L^2} + \ell \|w\|_{\mathcal{H}^3(0, L)} \|w'\|_{L^2} \\ &\leq \|d\|_{L^2} + \kappa \ell \|w\|_{\mathcal{H}_r^3}^2 \\ &\leq \bar{d} + \kappa \ell \|w\|_{\mathcal{H}_r^3}^2, \end{aligned}$$

for all  $w \in \mathcal{H}_r^3(0, L)$  and all  $d$  satisfying  $\|d\|_{L^2} \leq \bar{d}$ . Moreover, we have for all  $w_1, w_2 \in \mathcal{H}_r^3(0, L)$

$$\begin{aligned} \|\mathcal{T}_0(w_1) - \mathcal{T}_0(w_2)\|_{\mathcal{H}_r^3} &= \|w_1 w'_1 - w_2 w'_2\|_{L^2} \\ &\leq \|(w_1 - w_2)w'_1\|_{L^2} + \|w_2(w'_1 - w'_2)\|_{L^2} \\ &\leq \kappa \ell \|w_1 - w_2\|_{\mathcal{H}_r^3} \|w_1\|_{\mathcal{H}_r^3} + \kappa \ell \|w_1 - w_2\|_{\mathcal{H}_r^3} \|w_2\|_{\mathcal{H}_r^3} \\ &\leq \kappa \ell (\|w_2\|_{\mathcal{H}_r^3} + \|w_2\|_{\mathcal{H}_r^3}) \|w_1 - w_2\|_{\mathcal{H}_r^3}. \end{aligned}$$

We consider now the operator  $\mathcal{T}_0$  defined as in (C.27), restricted on the closed ball

$$B_{\bar{w}} := \left\{ w \in \mathcal{H}_r^3(0, L) : \|w\|_{\mathcal{H}_r^3} \leq \bar{w} \right\}$$

with  $\bar{w}$  to be chosen later. Then, collecting all the previous inequalities we have

$$\begin{aligned} \|\mathcal{T}_0(w)\|_{\mathcal{H}_r^3} &\leq \bar{d} + \kappa \ell \bar{w}^2, \\ \|\mathcal{T}_0(w_1) - \mathcal{T}_0(w_2)\|_{\mathcal{H}_r^3} &\leq 2\kappa \ell \bar{w} \|w_1 - w_2\|_{\mathcal{H}_r^3}, \end{aligned}$$

for all  $w, w_1, w_2 \in B_{\bar{w}}$ . Finally, we select  $\bar{d}$  and  $\bar{w}$  such that the following conditions hold

$$\bar{d} < \frac{1}{4\kappa\ell} \quad \text{and} \quad \frac{1 - \sqrt{1 - 4\bar{d}\kappa\ell}}{2\kappa\ell} \leq \bar{w} < \frac{1}{2\kappa\ell}. \quad (\text{C.29})$$

With such a choice, we obtain  $\|\mathcal{T}_0(w)\|_{\mathcal{H}_r^3} \leq \bar{w}$  for all  $w \in B_{\bar{w}}$  and  $\|\mathcal{T}_0(w_1) - \mathcal{T}_0(w_2)\|_{\mathcal{H}_r^3} < \|w_1 - w_2\|_{\mathcal{H}_r^3}$ , for all  $w_1, w_2 \in B_{\bar{w}}$ . This shows that the operator  $\mathcal{T}_0$  is an operator of contraction. Applying the Banach fixed point theorem [18, Theorem 5.7] we deduce that the operator  $\mathcal{T}_0$  admits a unique fixed point, and therefore that there exists a unique solution  $w_\infty \in B_{\bar{w}}$  to (C.26). Now, given  $\bar{w}$ , we deduce the value of  $\bar{r}$ . Indeed, since  $w_\infty \in \mathcal{H}^3(0, L)$  then we have  $w'_\infty \in \mathcal{H}^2(0, L)$ . Then, according to the embedding of  $\mathcal{H}^2(0, L)$  in  $C^1([0, L])$ , we have  $w'_\infty \in C^1([0, L])$ . Therefore, according to [142, Lemma 1] we have

$$(w'_\infty(0))^2 \leq \frac{2}{L} \|w'_\infty\|_{L^2}^2 + L \|w''_\infty\|_{L^2}^2 \leq \left(\frac{2}{L} + L\right) \|w_\infty\|_{\mathcal{H}^3(0, L)}^2. \quad (\text{C.30})$$

Since  $w_\infty \in B_{\bar{w}}$ , then according to (C.28) and (C.30), and to the definition of  $\mathcal{H}_r^3(0, L)$ , we obtain

$$r^2 = (w'_\infty(0))^2 \leq \kappa \left(\frac{2}{L} + L\right) \bar{w}^2. \quad (\text{C.31})$$

Finally, we can choose  $\bar{r} = \bar{w} \sqrt{\kappa \left(\frac{2}{L} + L\right)}$ . Therefore, according to (C.29) and (C.31), we deduce that for any  $\bar{w}_0 \in (0, \bar{w}]$ , there exists  $d_0 > 0$  and  $\bar{r}_0 > 0$  so that, for any  $(d, r) \in L^2(0, L) \times \mathbb{R}$  satisfying  $\|d\|_{L^2} \leq \bar{d}_0$  and  $|r| \leq \bar{r}_0$  then  $\|w_\infty\|_{\mathcal{H}^3} \leq \bar{w}_0$ . This concludes the proof of Lemma 3.  $\square$

#### Proof of Lemma 4

First, by writing the explicit solution of  $\tilde{\eta}$  along solutions, that is  $\tilde{\eta}(t) = \tilde{\eta}_0 + \int_0^t \partial_x \tilde{w}(s, 0) ds$ , we rewrite system (4.54) as follows

$$\begin{cases} \partial_t \tilde{w} + \partial_x \tilde{w} + \partial_{xxx} \tilde{w} + \tilde{w} \partial_x \tilde{w} + \partial_x (\tilde{w} w_\infty) = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \partial_x \tilde{w}(t, L) = k \left( \tilde{\eta}_0 + \int_0^t \partial_x \tilde{w}(s, 0) ds \right), & t \in \mathbb{R}_+, \\ \tilde{w}(0, x) = \tilde{w}_0(x), & x \in [0, L]. \end{cases} \quad (\text{C.32})$$

Now, given  $(s, \tau) \in \mathbb{N} \times \mathbb{R}$ , we introduce the space  $S^s(\tau) := C([0, \tau]; H^s(0, L)) \cap L^2([0, \tau]; H^{s+1}(0, L))$  equipped with the norm defined as

$$\|w\|_{S^s(\tau)}^2 := \|w\|_{C([0, \tau]; H^s(0, L))}^2 + \|w\|_{L^2([0, \tau]; H^{s+1}(0, L))}^2 + \|\partial_x w\|_{C([0, \tau]; L^2(0, L))}^2.$$

We consider the operator  $\mathcal{T}_1 : S^3(\tau) \rightarrow S^3(\tau)$  defined by  $\mathcal{T}_1(\tilde{w}) = \varphi$  where  $\varphi$  is the solution of

$$\begin{cases} \partial_t \varphi + \partial_x \varphi + \partial_{xxx} \varphi + \partial_x (w_\infty \varphi) = -\tilde{w} \partial_x \tilde{w}, & (t, x) \in [0, \tau] \times [0, L], \\ \varphi(t, 0) = \varphi(t, L) = 0, & t \in [0, \tau], \\ \partial_x \varphi(t, L) = k \left( \tilde{\eta}_0 + \int_0^t \partial_x \tilde{w}(s, 0) ds \right), & t \in [0, \tau], \\ \varphi(0, x) = \tilde{w}_0(x), & x \in [0, L] \end{cases} \quad (\text{C.33})$$

with  $\tau > 0$  and  $k > 0$  to be chosen later. With the operator  $\mathcal{T}_1$  so defined, we deduce that if  $\tilde{w}$  is a fixed point of  $\mathcal{T}_1$  then  $\tilde{w} \in S^3(\tau)$  is a solution of (C.32). To this end we will apply the Banach fixed-point Theorem. According to [16, Proposition 5.1], for any  $\bar{w}_\infty$  there exists  $C > 0$  such that

$$\|\mathcal{T}_1(\tilde{w})\|_{S^3(\tau)}^2 \leq C \left( \|\tilde{w}_0\|_{\mathcal{H}^3(0,L)}^2 + k^2 \left( \tau |\bar{\eta}_0|^2 + \int_0^\tau \left| \int_0^t \partial_x \tilde{w}(s,0) ds \right|^2 dt + \|\partial_x w(\cdot,0)\|_{L^2(0,\tau)}^2 \right) + \|\tilde{w} \partial_x \tilde{w}\|_{H^1(0,\tau;\mathcal{H}^1(0,L))}^2 \right)$$

for all  $\tilde{w} \in S^3(\tau)$  and any  $\|w_\infty\|_{\mathcal{H}^3} \leq \bar{w}_\infty$ . On the other hand, we have

$$\int_0^\tau \left| \int_0^t \partial_x \tilde{w}(s,0) ds \right|^2 dt \leq \int_0^\tau \int_0^t |\partial_x \tilde{w}(s,0)|^2 ds dt \leq \tau \|\partial_x \tilde{w}(\cdot,0)\|_{L^2(0,\tau)}^2.$$

Since,  $\tilde{w} \in S^3(\tau)$ , then for all  $t \in [0, \tau]$ ,  $\tilde{w}(t, \cdot) \in \mathcal{H}^3(0, L)$ , which imply  $\partial_x \tilde{w}(t, \cdot) \in \mathcal{H}^2(0, L)$  for all  $t \in [0, \tau]$ . Then, according to the embedding of  $\mathcal{H}^2(0, L)$  in  $C^1([0, L])$ , we have  $\partial_x \tilde{w}(t, \cdot) \in C^1([0, L])$  for all  $t \in [0, \tau]$ . Therefore, according to [142, Lemma 1], we obtain

$$(\partial_x \tilde{w}(t,0))^2 \leq \left( \frac{2}{L} + L \right) \|\tilde{w}(t, \cdot)\|_{\mathcal{H}^3(0,L)}^2$$

for all  $t \in [0, \tau]$ , which implies

$$\|\partial_x \tilde{w}(\cdot,0)\|_{L^2(0,\tau)}^2 \leq \left( \frac{2}{L} + L \right) \|\tilde{w}\|_{L^2(0,\tau;\mathcal{H}^3(0,L))}^2 \leq \left( \frac{2}{L} + L \right) \|\tilde{w}\|_{S^3(\tau)}^2. \quad (\text{C.34})$$

Then, we have

$$\int_0^\tau \left| \int_0^t \partial_x \tilde{w}(s,0) ds \right|^2 dt \leq \tau \left( \frac{2}{L} + L \right) \|\tilde{w}\|_{L^2(0,\tau;\mathcal{H}^3(0,L))}^2 \leq \tau \left( \frac{2}{L} + L \right) \|\tilde{w}\|_{S^3(\tau)}^2. \quad (\text{C.35})$$

Also, since  $\tilde{w} \in S^3(\tau)$ , then according to [16, Lemma 3.1], we deduce the existence of  $C > 0$  such that

$$\|\tilde{w} \partial_x \tilde{w}\|_{H^1(0,\tau;\mathcal{H}^1(0,L))}^2 = \|\tilde{w} \partial_x \tilde{w}\|_{L^2(0,\tau;\mathcal{H}^1(0,L))}^2 + \|(\tilde{w} \partial_x \tilde{w})_t\|_{L^2(0,\tau;\mathcal{H}^1(0,L))}^2 \leq C(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \|\tilde{w}\|_{S^3(\tau)}^4. \quad (\text{C.36})$$

As a consequence, from (C.34), (C.35) and (C.36), there exists a positive constant  $C > 0$  such that

$$\|\mathcal{T}_1(\tilde{w})\|_{S^3(\tau)}^2 \leq C \left( \|\tilde{w}_0\|_{\mathcal{H}^3(0,L)}^2 + k^2 \tau |\bar{\eta}_0|^2 + \left( (\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \|\tilde{w}\|_{S^3(\tau)}^2 + \left( \frac{2}{L} + L \right) (k^2 \tau + k^2) \right) \|\tilde{w}\|_{S^3(\tau)}^2 \right). \quad (\text{C.37})$$

Moreover, from (C.34), (C.35) and (C.36), we obtain

$$\begin{aligned} & \|\mathcal{T}_1(\tilde{w}^1) - \mathcal{T}_1(\tilde{w}^2)\|_{S^3(\tau)}^2 \leq \\ & \leq C \left( k^2 \int_0^\tau \left| \int_0^t (\partial_x \tilde{w}^1(s, 0) - \partial_x \tilde{w}^2(s, 0)) ds \right|^2 dt + \|\partial_x \tilde{w}^1(\cdot, 0) - \partial_x \tilde{w}^2(\cdot, 0)\|_{L^2(0, \tau)}^2 \right. \\ & \quad \left. + \|\tilde{w}^2(\partial_x \tilde{w}^2 - \partial_x \tilde{w}^1) + \partial_x \tilde{w}^1(\tilde{w}^2 - \tilde{w}^1)\|_{H^1(0, \tau; \mathcal{H}^1(0, L))}^2 \right) \\ & \leq C \left( \left(\frac{2}{L} + L\right) (k^2 \tau + k^2) + (\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \|\tilde{w}^1\|_{S^3(\tau)}^2 + (\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \|\tilde{w}^2\|_{S^3(\tau)}^2 \right) \|\tilde{w}^1 - \tilde{w}^2\|_{S^3(\tau)}^2 \end{aligned}$$

for all  $\tilde{w}^1, \tilde{w}^2 \in S^3(\tau)$ . We consider  $\mathcal{T}_1$  restricted to the closed ball  $B_\rho = \{\tilde{w} \in S^3(\tau) : \|\tilde{w}\|_{S^3(\tau)} \leq \rho\} \subset S^3(\tau)$  with  $\rho$  to be chosen later. Then

$$\|\mathcal{T}_1(\tilde{w})\|_{S^3(\tau)}^2 \leq C \left( \|\tilde{w}_0\|_{\mathcal{H}^3(0, L)}^2 + k^2 \tau |\tilde{\eta}_0|^2 + \rho^2 \left(\frac{2}{L} + L\right) k^2 + \left((\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \rho^2 + \left(\frac{2}{L} + L\right) k^2 \tau\right) \rho^2 \right)$$

and

$$\|\mathcal{T}_1(\tilde{w}^1) - \mathcal{T}_1(\tilde{w}^2)\|_{S^3(\tau)}^2 \leq C \left( k^2 \left(\frac{2}{L} + L\right) + \left(\frac{2}{L} + L\right) k^2 \tau + 2(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \rho^2 \right) \|\tilde{w}^1 - \tilde{w}^2\|_{S^3(\tau)}^2.$$

Finally, we select the constant  $\rho, k_1$  and  $\tau$  so that to obtain a contractive operator. For instance, we can select  $\rho = \sqrt{3C} \|\tilde{w}_0\|_{\mathcal{H}^3(0, L)}$  and

$$k_1^* = \sqrt{\frac{1}{6C} \left( \frac{L}{2 + L^2} \right)}$$

and  $\tau > 0$  such that the following inequalities are satisfied

$$\begin{aligned} & \tau (k_1^*)^2 |\tilde{\eta}_0|^2 < \|\tilde{w}_0\|_{\mathcal{H}^3(0, L)}^2, \\ & \left( (\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \rho^2 + \left(\frac{2}{L} + L\right) (k_1^*)^2 \tau \right) \rho^2 < \frac{1}{2} \|\tilde{w}_0\|_{\mathcal{H}^3(0, L)}^2, \\ & \left(\frac{2}{L} + L\right) (k_1^*)^2 \tau + 2(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \rho^2 < \frac{1}{2}. \end{aligned}$$

It follows that, for any  $k \in (0, k_1^*]$ ,  $\|\mathcal{T}_1(\tilde{w})\|_{S^3(\tau)} \leq \rho$  for any  $\tilde{w} \in B_\rho$  and  $\|\mathcal{T}_1(\tilde{w}^1) - \mathcal{T}_1(\tilde{w}^2)\|_{S^3(\tau)} < \|\tilde{w}^1 - \tilde{w}^2\|_{S^3(\tau)}$  for any  $\tilde{w}^1, \tilde{w}^2 \in B_\rho$ . Then,  $\mathcal{T}_1$  is a contraction operator from  $B_\rho$  to  $B_\rho$ . According to the Banach fixed-point theorem,  $\mathcal{T}_1$  admits a unique fixed point. Its unique fixed point is the desired solution of (C.32) for  $0 \leq t \leq \tau$ . This shows that  $\tilde{w}$  in (C.33) has a unique solution in  $S^3(\tau)$ . Since  $\partial_x \tilde{w}(t, 0)$  is continuous on  $[0, \tau]$  then  $\tilde{\eta}$  is in  $C^1(\tau)$  by definition of solution of an ODE. This concludes the proof of Lemma (4).  $\square$

## Proof of Theorem 10

The main idea of this proof is to extend the analysis developed for the linear KdV model in Section 4.4.1. In particular, following the main steps of the proof of Lemma 2, we aim at building a Lyapunov functional for the overall closed-loop system (4.53) by relying on Corollary 1. Indeed, setting  $a = -w'_\infty$ ,  $b = -w_\infty$  and  $d_2 = k\eta$ , system (4.54) is in the form (4.22). As a consequence, there exist  $\delta > 0$  and a Lyapunov functional  $V$  such that, for any  $\|w'_\infty\|_\infty \leq \bar{a}$  and  $\|w_\infty\|_{W^{1, \infty}} \leq \bar{b}$ , the derivative of  $V$  along the

trajectories of system (4.54) satisfies

$$\dot{V}(\tilde{w}) \leq -\alpha \|\tilde{w}\|_{L^2}^2 + \sigma_1 k^2 \tilde{\eta}^2 \quad \forall (\tilde{\eta}, \tilde{w}) \in D_\delta(\mathcal{A}), \quad (\text{C.38})$$

with  $D_\delta(\mathcal{A}) := \{(\tilde{\eta}, \tilde{w}) \in D(\mathcal{A}) : \|(\tilde{\eta}, \tilde{w})\|_X \leq \delta\}$ . Now, we consider the Lyapunov functional  $\mathcal{V}$  defined in (C.12). We want to show the local exponential stability of the origin of the system (4.54) with the functional  $\mathcal{V}$ . First, note that  $\mathcal{V}$  is uniformly bounded by the norm in the space  $X$  of  $(\tilde{\eta}, \tilde{w})$ , similar to inequality (C.13). Then, using (C.38), its derivative along solutions to (4.53) is given by, for any  $(\tilde{\eta}, \tilde{w}) \in D_\delta(\mathcal{A})$ .

$$\dot{\mathcal{V}}(\tilde{\eta}, \tilde{w}) \leq -\alpha \|\tilde{w}\|_{L^2}^2 + \sigma_1 k^2 \tilde{\eta}^2 + 2F(\tilde{\eta}, \tilde{w}) \quad (\text{C.39})$$

with  $F(\tilde{\eta}, \tilde{w}) := (\tilde{\eta} - \mathcal{M}\tilde{w})(\dot{\tilde{\eta}} - \mathcal{M}\partial_t \tilde{w})$ . After some integrations by parts, and recalling the property of  $\mathcal{M}$  in (C.16), we obtain

$$\dot{\tilde{\eta}} - \mathcal{M}\partial_t \tilde{w} = -k\tilde{\eta}(t) + \int_0^L M(x)\tilde{w}(x)w'_\infty(x)dx - \frac{1}{2} \int_0^L M'(x)\tilde{w}(x)^2 dx - \int_0^L (M(x)w_\infty(x))' \tilde{w}(x)dx,$$

from which we obtain

$$F(\tilde{\eta}, \tilde{w}) = -k\tilde{\eta}^2 + \tilde{\eta}k\mathcal{M}\tilde{w} + \tilde{\eta}\Phi(\tilde{w}) - \mathcal{M}\tilde{w}\Phi(\tilde{w})$$

with

$$\Phi(\tilde{w}) = \int_0^L M(x)\tilde{w}(x)w'_\infty(x)dx - \frac{1}{2} \int_0^L M'(x)\tilde{w}(x)^2 dx - \int_0^L (M(x)w_\infty(x))' \tilde{w}(x)dx$$

According to (C.11),  $M \in C^\infty([0, L])$ . Therefore  $M'$  is bounded on  $[0, L]$ . Then, using first Cauchy-Schwarz's inequality and then Young's inequality, we bound the terms in  $F$  as follows:

$$\begin{aligned} |k\tilde{\eta}\mathcal{M}\tilde{w}| &\leq \frac{2k^2\|M\|_{L^2}^2}{\alpha}\tilde{\eta}^2 + \frac{\alpha}{8}\|\tilde{w}\|_{L^2}^2, \\ |\tilde{\eta}\Phi(\tilde{w})| &\leq \frac{k}{2}\tilde{\eta}^2 + \frac{1}{2k}\left(4\|Mw'_\infty\|_{L^2}^2 + 4\|(Mw_\infty)'\|_{L^2}^2 + \|M'\|_\infty^2\|\tilde{w}\|_{L^2}^2\right)\|\tilde{w}\|_{L^2}^2, \\ |\mathcal{M}\tilde{w}\Phi(\tilde{w})| &\leq \frac{\alpha}{8}\|\tilde{w}\|_{L^2}^2 + \frac{2\|M\|_{L^2}^2}{\alpha}\left(4\|Mw'_\infty\|_{L^2}^2 + 4\|(Mw_\infty)'\|_{L^2}^2 + \|M'\|_\infty^2\|\tilde{w}\|_{L^2}^2\right)\|\tilde{w}\|_{L^2}^2. \end{aligned}$$

As a consequence, combining the previous bounds we further obtain

$$\begin{aligned} F(\tilde{\eta}, \tilde{w}) &\leq \left(\frac{\alpha}{4} + \left(\frac{2\|M\|_{L^2}^2}{\alpha} + \frac{1}{2k}\right)\left(4\|Mw'_\infty\|_{L^2}^2 + 4\|(Mw_\infty)'\|_{L^2}^2 + \|M'\|_\infty^2\|\tilde{w}\|_{L^2}^2\right)\right)\|\tilde{w}\|_{L^2}^2 \\ &\quad - \frac{k}{2}\left(1 - 4k\frac{\|M\|_{L^2}^2}{\alpha}\right)\tilde{\eta}^2 \\ &\leq \left(\frac{\alpha}{4} + \left(\frac{2\|M\|_{L^2}^2}{\alpha} + \frac{1}{2k}\right)\left(8\|M\|_{W^{1,\infty}}^2\|w_\infty\|_{\mathcal{H}^3}^2 + \|M'\|_\infty^2\|\tilde{w}\|_{L^2}^2\right)\right)\|\tilde{w}\|_{L^2}^2 \\ &\quad - \frac{k}{2}\left(1 - 4k\frac{\|M\|_{L^2}^2}{a_1}\right)\tilde{\eta}^2 \end{aligned}$$

where, in the second inequality, we have used  $\|Mw'_\infty\|_{L^2}^2 + \|(Mw_\infty)'\|_{L^2}^2 \leq 2\|M\|_{W^{1,\infty}}^2 \|w_\infty\|_{\mathcal{H}^3}^2$ . Using the previous inequality together with (C.39) yields

$$\begin{aligned} \dot{\mathcal{V}}(\tilde{\eta}, \tilde{w}) \leq & \left( -\frac{\alpha}{2} + \left( \frac{4\|M\|_{L^2}^2}{\alpha} + \frac{1}{k} \right) \left( 8\|M\|_{W^{1,\infty}}^2 \|w_\infty\|_{\mathcal{H}^3}^2 + \|M'\|_\infty^2 \|\tilde{w}\|_{L^2}^2 \right) \right) \|\tilde{w}\|_{L^2}^2 \\ & + \left( \sigma_1 k^2 - k \left( 1 - 4k \frac{\|M\|_{L^2}^2}{\alpha} \right) \right) \tilde{\eta}^2 \end{aligned} \quad (\text{C.40})$$

for all  $(\tilde{\eta}, \tilde{w}) \in D_\delta(\mathcal{A})$ . As a consequence, we can finally fix all the parameters. In particular, we select

$$k_2^* = \min \left\{ k_0^*, k_1^*, \left( \frac{\alpha}{\alpha\sigma_1 + 4\|M\|_{L^2}^2} \right) \right\}$$

with  $k_0^*$  given by Lemma 2, and  $k_1^*$  given by Lemma 4. Moreover, given any  $k \in (0, k_2^*)$ , select

$$\bar{w}_\infty = \min \left\{ \bar{a}, \bar{b}, \sqrt{\frac{\alpha}{64\|M\|_{W^{1,\infty}}^2 \left( \frac{4\|M\|_{L^2}^2}{\alpha} + \frac{1}{k} \right)}} \right\},$$

with  $\bar{a}, \bar{b}$  given by Corollary 1. Moreover, let us define  $\bar{\delta} \in (0, \delta)$  satisfying

$$\bar{\delta}^2 \leq \frac{\alpha}{8\|M'\|_\infty^2} \left( \frac{4\|M\|_{L^2}^2}{\alpha} + \frac{1}{k} \right)^{-1}.$$

Using all these bounds we can finally conclude the existence of a positive constant  $\varepsilon$  such that

$$\dot{\mathcal{V}} \leq -\varepsilon(\|\tilde{w}\|_{L^2}^2 + \tilde{\eta}^2) \quad \forall (\tilde{\eta}, \tilde{w}) \in D_{\bar{\delta}}(\mathcal{A}). \quad (\text{C.41})$$

Finally, standard Lyapunov arguments briefly recalled here allows to conclude the result of the proof. In particular, consider a  $c > 0$  small enough such that  $\Omega_c := \{(\tilde{\eta}, \tilde{w}) \in D(\mathcal{A}) : \mathcal{V}(\tilde{\eta}, \tilde{w}) \leq c\} \subset D_{\bar{\delta}}(\mathcal{A})$ . Now consider any solution to (4.53) starting inside  $\Omega_c$ . By Lemma 3 there exists  $\tau > 0$  such that such a solution exists on  $[0, \tau]$ . Let  $T \geq \tau$  be its maximal interval of time of existence. In view of (C.41), the derivative of  $\mathcal{V}$  is always negative, showing that such the solution cannot escape the level set  $\Omega_c$ . Hence, its maximal interval of existence is  $[0, \infty)$ . Moreover, we can conclude the existence of a positive constant  $\Delta \in (0, \bar{\delta})$  such that the set  $D_\Delta(\mathcal{A})$  is included in the domain of attraction of the origin of system (4.54). Combining the Fréchet derivative (4.17) and the Grönwall's lemma with (C.41) one can show the first part of the statement, that is  $\|(\tilde{\eta}(t), \tilde{w}(t))\|_X \leq b_1 e^{-\nu_1 t} \|(\tilde{\eta}_0, \tilde{w}_0)\|_X$ , for all  $t \geq 0$  and for all  $(\tilde{\eta}_0, \tilde{w}_0) \in D_\Delta(\mathcal{A})$ .

Finally, to prove the second part of the statement, we can use the same argument as in the proof of [111, Proposition 3.9] to deduce that there exists a continuous nonnegative function  $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and positive constants  $C, \mu$  such that, for all  $(\tilde{\eta}_0, \tilde{w}_0) \in D_\Delta(\mathcal{A})$

$$\|\partial_t \tilde{w}(t, \cdot)\|_{L^2} \leq C e^{-\mu t} \chi(\|\tilde{w}_0\|_{L^2}) \|\partial_t \tilde{w}(0, \cdot)\|_{L^2}, \quad \forall t \geq 0. \quad (\text{C.42})$$

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By multiplying the first equation of (4.54) by  $\tilde{w}$  and integrating by parts, we get after some computations

$$\begin{aligned} -k^2 \tilde{\eta}(t)^2 + \partial_x \tilde{w}(t, 0)^2 &= -2 \int_0^L \tilde{w}(t, x) \partial_t \tilde{w}(t, x) dx + \int_0^L |\tilde{w}(t, x)|^2 w'_\infty(x) dx \\ &\leq 2 \|\tilde{w}(t, \cdot)\|_{L^2} \|\partial_t \tilde{w}(t, \cdot)\|_{L^2} + \ell \|\tilde{w}(t, \cdot)\|_{L^2}^2 \|\tilde{w}_\infty\|_{\mathcal{H}^3(0, L)}. \end{aligned}$$

where  $\ell$  is the constant of the embedding of  $\mathcal{H}^3(0, L)$  in  $L^\infty(0, L)$ . Then, we can deduce  $\lim_{t \rightarrow \infty} |\partial_x \tilde{w}(t, 0)| = \lim_{t \rightarrow \infty} |\partial_x w(t, 0) - r| = 0$  for all  $(\tilde{\eta}_0, \tilde{w}_0) \in D_\Delta(\mathcal{A})$ , concluding the proof.





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## **Titre: Contributions à la théorie du contrôle des systèmes de dimension infinie soumis à des perturbations/incertitudes**

**Mot clés :** Contrôle par mode glissant, stabilisation, régulation de sortie, équations aux dérivées partielles, stabilité entrée-état, observateur.

**Resumé :** Dans cette thèse, nous étudions des problèmes de stabilisation et de régulation de sortie pour des systèmes de dimension infinie soumis à des perturbations. Tout d'abord, nous considérons le problème de la stabilisation d'un système dynamique abstrait linéaire de dimensions infinies avec un opérateur de contrôle non borné et soumis à une perturbation située au même endroit que le contrôle. Pour résoudre ce problème, nous suivons une stratégie de contrôle par mode glissant. Dans un second temps, nous considérons le problème de la stabilisation d'un système hyperbolique (une équation de transport et un système d'équations de transport) contrôlé au bord et soumis à une perturbation située au même endroit que le contrôle. L'objectif ici est de proposer pour ce cas particulier un contrôle qui exige moins pour ce qui est de la sortie. Pour résoudre ce problème, nous proposons un "active disturbance rejection control". Enfin, nous nous intéressons à la construction d'une fonctionnelle de Lyapunov permettant de prouver la stabilité entrée-état et de résoudre un problème de régulation de sortie d'une équation de Korteweg-de Vries.

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## **Title: Contributions to control theory for infinite-dimensional systems subjected to disturbances/uncertainties**

**Keywords :** Sliding mode control, stabilization, output regulation, partial differential equations, input-to-state stability, observer.

**Abstract:** In this thesis, we study problems of stabilization and output regulation for infinite-dimensional systems subjected to disturbances. First, we consider the problem of the stabilization of an abstract linear infinite-dimensional system with unbounded control operators and subject to a matched disturbance. To solve this problem, we follow a sliding mode control strategy. Secondly, we consider the problem of the boundary stabilization of a linear hyperbolic system (a transport equation and a system of transport equations) subjected to a matched disturbance. The objective here is to propose for this particular case a control which requires much less in terms of measurement than the design proposed before. To solve this problem, we propose an active disturbance rejection control. Finally, we are interested in the construction of an input-to-state stability Lyapunov functional and the output regulation of a Korteweg-de Vries equation.