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Meriam El Mansour. Conditional random sets with applications in mathematical finance and optimization. Dynamical Systems [math.DS]. Université Paris sciences et lettres; Université de Tunis El Manar, 2023. English. NNT : 2023UPSLD023 . tel-04308348

HAL Id: tel-04308348

<https://theses.hal.science/tel-04308348>

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THÈSE DE DOCTORAT
DE L'UNIVERSITÉ PSL

Préparée à l'Université Paris-Dauphine
Dans le cadre d'une cotutelle avec La Faculté Des Sciences de Tunis

**Ensembles aléatoires conditionnels avec applications en
mathématiques financières et en optimisation.**

Conditional random sets with applications in mathematical
finance and optimization.

Soutenue par
Meriam EL MANSOUR
Le 25 mai 2023

Ecole doctorale n° ED 543
Ecole doctorale SDOSE

Spécialité
Mathématique

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CEREMADE, PARIS DAUPHINE UNIVERSITY

LATAO, FACULTÉ DES SCIENCES DE TUNIS

PHD THESIS

Conditional random sets with applications in
mathematical finance and optimization.

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Abstract

In this thesis we introduce two new types of conditional random sets taking values in a Banach space: the conditional interior and the conditional closure. The conditional interior is a version of the conditional core, as introduced by A. Truffert and recently developed by Lépinette and Molchanov, and may be seen as a measurable version of the topological interior. The conditional closure is a measurable version of the topological closure. These concepts are useful for applications in mathematical finance and conditional optimization as we shall see in the thesis.

In the second part of the thesis we apply the theoretical results established in the first part to solve the problem of super-hedging European or Asian options for discrete-time financial market models where executable prices are uncertain. The risky asset prices are not described by single-valued processes but measurable selections of random sets that allows to consider a large variety of models including bid-ask models with order books, but also models with a delay in the execution of the orders. We provide a numerical procedure to compute the infimum price under a weak no-arbitrage condition, the so-called AIP condition, under which the prices of any non negative European options are non negative. This condition is weaker than the existence of a risk-neutral martingale measure but it is sufficient to numerically solve the super-hedging problem. We illustrate our method by a numerical example.

In the last part we show that, in discrete-time, it is possible to evaluate the minimal super-hedging price when we restrict ourselves to integer-valued strategies. In fact, usual theory of asset pricing in finance assumes that the financial strategies, i.e. the quantity of risky assets to invest, are real-valued so that they are not integer-valued in general, as we can see in the Black and Scholes model for instance. This is clearly contrary to what it is possible to do in the real world. Surprisingly, it seems that there are few contributions in that direction in the literature (see the reference in the last version). To do so, we only consider terminal claims that are continuous piecewise affine functions of the underlying asset. We formulate a dynamic programming principle that can be directly implemented on an historical data, and which also provides the optimal integer-valued strategy.

Résumé

Dans cette thèse nous introduisons deux nouveaux types d'ensembles aléatoires conditionnels à valeurs dans un espace de Banach : l'intérieur conditionnel et la clôture conditionnelle des ensembles aléatoires. L'intérieur conditionnel est une version mesurable des ouverts topologiques, tel qu'introduit par A. Truffert et récemment développé par Lépinette et Molchanov, et peut être vu comme une version mesurable de l'intérieur topologique. La clôture conditionnelle est une généralisation de la notion de support conditionnel d'une variable aléatoire. Ces concepts sont utiles pour des applications en mathématique financière et en optimisation conditionnelle.

Dans la deuxième partie, nous appliquons les résultats théoriques établis dans la première partie pour résoudre le problème de sur-réplication des options européennes ou asiatiques pour les modèles de marchés financiers en temps discret où les prix exécutables sont incertains. Les prix des actifs risqués ne sont pas décrits par des processus à valeur unique mais des sélections mesurables d'ensembles aléatoires qui permettent de considérer une grande variété de modèles, y compris des modèles bid-ask avec carnets d'ordres, mais aussi des modèles avec un retard dans l'exécution des ordres. Nous proposons un procédé numérique pour calculer le prix minimal sous une condition faible de non-arbitrage, dite condition AIP, sous laquelle les prix positifs des options européennes sont positifs. Cette condition est plus faible que l'existence d'une probabilité risque neutre mais elle est suffisante pour résoudre numériquement le problème de super-couverture. Nous illustrons notre méthode par un exemple numérique.

Dans la dernière partie nous montrons que, dans un modèle de marché en temps discret, il est possible d'évaluer le prix minimal de sur-réplication lorsqu'on se limite aux stratégies à valeurs entières. En fait, la théorie habituelle de l'évaluation des options en finance suppose que les stratégies financières, c'est-à-dire la quantité d'actifs risqués à investir, sont à valeurs réelles de sorte qu'elles ne sont pas à valeurs entières en général, comme on peut le voir dans le modèle de Black et Scholes par exemple. Ceci est clairement contraire à ce qu'il est possible de faire dans le monde réel. Il semble qu'il y'ait peu de contribution en ce sens dans la littérature. Pour ce faire, nous ne considérons que des payoffs qui sont des fonctions continues affines par morceaux de l'actif sous-jacent. Nous formulons un principe de programmation dynamique qui peut être directement implémenté sur une donnée historique et qui fournit également la stratégie optimale à valeurs entières.

Acknowledgements

I want to thank my supervisor Pr. Emmanuel Lépinette who during these years of thesis, have managed this work with all the skills and human qualities that can be expected from a supervisor. He was very patient and I learned a lot from him. I would like to tell him how much his advices were beneficial to me. I am also grateful to Pr. Dorra Bourguiba for her help during this journey and her support. I also like to thank Pr. Yuri Kabanov and Pr. Serguei Pergamenchtkhikov for reporting this work. I am grateful to all the members of my thesis committee for accepting to participate in my thesis defense. I want to take a moment to express my sincerest gratitude to Mohamed Amine Ben Amor for his unwavering support throughout my academic journey. His guidance, mentorship, and encouragement have been invaluable to me. Also I want to thank Pr Youssef Azouzi for sharing his knowledge and expertise with me and for his advices. I am very grateful to my family, my friends for their support. Finally, i would like to express my gratitude to my mother, who has been my pillar of strength, my constant support and my source of inspiration.

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Chapter 1

General Introduction

1.1 Motivation

In mathematical finance, options play an important role. They are a powerful tool for investors to manage and hedge any potential risk. In fact, options are contracts that give the holder the right (but not the obligation) to buy or sell the underlying asset at a fixed price on a specified date. As a result, finding a fair price for both the buyer and the seller has been a challenging problem that several mathematicians tried to answer: The seller of the option will deliver a random amount at a specific time to the buyer. In exchange the seller receives from the buyer some amount of capital that we call price.

So, we introduce first the classical theory of arbitrage and pricing for financial markets without frictions. We recall the definition of arbitrage opportunity and then we present the fundamental theory of asset pricing based on convex duality and the dual characterization of super-hedging prices. We refer to [42] and [56] and the references therein for a general overview.

1.2 Framework

In order to determine the price of an asset, we consider a discrete financial market model defined by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$ where $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ is a filtration that represents the information that the investor holds at time t and such that $\mathcal{F}_T = \mathcal{F}$. Let $\mathcal{M}(\mathbb{R}^d, \mathcal{F}_t)$ be the set of \mathbb{R}^d -valued \mathcal{F}_t -measurable random variables. The random variables are defined up to negligible set, which means that we work directly with quotient spaces. We consider a price process $S := (S_t^0, S_t)_{0 \leq t \leq T}$ such that $S_t \in \mathcal{M}(\mathbb{R}_+^d, \mathcal{F}_t)$. A strategy $\hat{\theta} = (\theta_t^0, \theta_t)_{0 \leq t \leq T}$ is a stochastic process adapted to the filtration \mathbb{F} .

The liquidation value at time t using the strategy θ is given by:

$$V_t^{\hat{\theta}} := \theta_t^0 S_t^0 + \theta_t \cdot S_t = \theta_t^0 S_t^0 + \sum_{i=1}^d \theta_t^i S_t^i$$

Definition 1.2.1. *An European option is a financial contract between a buyer and a seller that gives the buyer the right to execute the contract at a fixed maturity time T and to get a terminal wealth ξ_T also known as the payoff. In return he should pay the price of the contract to the seller at time $t = 0$. For example the payoff ξ_T of a Call option is $(S_T - K)^+$ where $K \geq 0$ is a constant called strike.*

One of the interesting problems in mathematical finance is to find the price of an option. So we need to recall some definitions and theorems that will be useful.

Definition 1.2.2. *A portfolio process $(V_t^{\hat{\theta}})_{t \in \{0, \dots, T\}}$ is said to be self-financing if:*

$$\theta_{t-1}^0 S_t^0 + \theta_{t-1} \cdot S_t = \theta_t^0 S_t^0 + \theta_t \cdot S_t, \quad \forall t = 1, \dots, T.$$

We have the following equivalences:

1. $(V_t^{\hat{\theta}})_{t \in \{0, \dots, T\}}$ is said to be self-financing.
2. Let $\Delta V_t^{\hat{\theta}} = V_t^{\hat{\theta}} - V_{t-1}^{\hat{\theta}}$, and $\Delta S_t = S_t - S_{t-1}$, then

$$\Delta V_t^{\hat{\theta}} = \theta_{t-1}^0 \Delta S_t^0 + \theta_{t-1} \cdot \Delta S_t \quad \forall t = 1, \dots, T.$$

3. Let $\tilde{V}_t^{\hat{\theta}} = \frac{V_t^{\hat{\theta}}}{S_t^0}$, be the discounted value of $V_t^{\hat{\theta}}$ and $\tilde{S}_t = \frac{S_t}{S_t^0}$, the discounted value of S_t then

$$\Delta \tilde{V}_t^{\hat{\theta}} = \theta_{t-1} \cdot \Delta \tilde{S}_t \quad \forall t = 1, \dots, T.$$

In the sequel, we denote by \mathcal{D}_0^T the set of all the discounted terminal values $\tilde{V}_T^{\hat{\theta}}$ with initial value $\tilde{V}_0^{\hat{\theta}} = V_0 = 0$ that is:

$$\mathcal{D}_0^T = \left\{ \sum_{t=1}^T \theta_{t-1} \cdot \Delta \tilde{S}_t : \theta_t \in \mathcal{M}(\mathbb{R}, \mathcal{F}_t), t = 0, \dots, T-1 \right\}.$$

We denote also by $\mathcal{SH}_0^T = \mathcal{D}_0^T - \mathcal{M}(\mathbb{R}_+, \mathcal{F}_T)$ the set of super-hedgeable claims, i.e. $\xi_T \in \mathcal{SH}_0^T$ if and only if there exists $V_T^{\hat{\theta}} \in \mathcal{D}_0^T$ such that $V_T^{\hat{\theta}} \geq \xi_T$ a.s. A portfolio process with terminal value $V_T^{\hat{\theta}}$ super-replicates ξ_T at time T if $V_T^{\hat{\theta}} \geq \xi_T$, a.s. If $V_T^{\hat{\theta}} = \xi_T$ a.s then $V_T^{\hat{\theta}}$ replicates ξ_T .

Now, let's give the definition of a price:

Definition 1.2.3. *Let ξ_T be a payoff and $V^{\hat{\theta}}$ be a self-financing portfolio process. A price is any initial endowment $V_0^{\hat{\theta}}$ such that $V^{\hat{\theta}} \geq \xi_T$. We denote by $\mathcal{P}(\xi_T)$ the set of all prices for the payoff ξ_T .*

We can define now the super hedging price. Let x be the initial capital of a self-financing strategy denoted by $V^{x, \hat{\theta}}$. If ξ_T is a contingent claim and Θ is the set

of all strategies, then the (infimum) super-hedging price is denoted by $p(\xi_T)$, defined by:

$$p(\xi) := \inf \left\{ p \in \mathbb{R}, \exists \hat{\theta} \in \Theta \text{ s.t. } V_T^{p, \hat{\theta}} \geq \xi_T \text{ a.s.} \right\}.$$

Note that it is an infimum price and we are not sure that it is a minimal price. The classical approaches to solve the super-hedging problem is assuming the absence of arbitrage (NA). To do so, let us recall the definition of an arbitrage opportunity.

Definition 1.2.4. *An arbitrage opportunity is a financial strategy $\hat{\theta}$ starting from a zero initial endowment such that:*

- $V_T^{\hat{\theta}} \geq 0, \mathbb{P} \text{ a.s.}$
- $\mathbb{P}(V_T^{\hat{\theta}} > 0) > 0.$

Definition 1.2.5. *The (NA) condition holds if there is no arbitrage condition.*

This means that it is impossible to make a profit without taking some risk. The (NA) condition can be expressed in the set setting as follow:

- NA holds if $\mathcal{D}_0^T \cap \mathcal{M}(\mathbb{R}_+, \mathcal{F}_T) = \{0\}.$
- NA holds if $\mathcal{SH}_0^T \cap \mathcal{M}(\mathbb{R}_+, \mathcal{F}_T) = \{0\}.$

In the sequel we state the first fundamental theorem of asset pricing (FTAP) that gives a characterisation of the absence of arbitrage using the martingale criterion first proven by Harrison and Krep in [32] and Harrison and Pliska in [33] in the case of finite universe Ω while Dalang, Morton and Willinger in [21] gave the proof for an arbitrary Ω . There exist several proofs of this theorem based on the Hahn Banach theorem and some results of functional analysis. One can find in [56] another proof where the martingale measures are explicitly constructed using the Esscher conditional transformation. In their seminal work [39], Jouini.E and Kallel.H worked on arbitrage theory in presence of transaction costs and in [40] extend the result to the case of multi-asset models especially the case of the currency models by considering martingale density and solvency cones rather than measures.

Theorem 1.2.6. *The financial market is arbitrage free if and only if there exists a probability measure \mathbb{Q} equivalent to \mathbb{P} such that the discounted price process \tilde{S} is a \mathbb{Q} -martingale measure¹.*

In fact FTAP states that: NA holds if and only if $\mathcal{EMM}(S) \neq \emptyset$, where $\mathcal{EMM}(S)$ is the set of all equivalent martingale measures. The dual elements of the previous theorem allow to minimal super-replicating price.

¹A sequence $(X_t)_{0 \leq t \leq T}$ is said to be a martingale relative to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ if every X_t is integrable and $\mathbb{E}(X_t | \mathcal{F}_s) = X_s, \forall s < t.$

Theorem 1.2.7. *If NA holds then:*

$$p(\xi) = \sup_{\mathbb{Q} \in \mathcal{EMM}(S)} \mathbb{E}_{\mathbb{Q}}(\xi).$$

The second theorem of asset pricing is given by the following theorem:

Theorem 1.2.8. *If $\mathcal{EMM}(S) \neq \emptyset$, then the following holds: The market is complete if and only if $\mathcal{EMM}(S)$ is a singleton.*

The completeness of financial markets is an important condition. In fact, if the market is complete then we can compute the super-hedging price as the expectation of the contingent claim under the unique martingale measure which can be determined numerically using for example Monte Carlo methods. But in many cases, this condition doesn't hold true which complicates the task of determining the super-hedging price using the dual characterisation. In [16], the authors introduced a weak no-arbitrage condition called the absence of immediate profit (AIP) that ensures the possibility to price a European claim in a financial market.

Definition 1.2.9. *We say that a financial market satisfies the AIP condition if the super-hedging price of the zero claim is zero.*

Some contributions

In this thesis we introduce in a first place theoretical results that will be used to solve the super-hedging problem. Actually we define two new types of conditional random sets: Conditional interior and conditional closure sets and then we solve the super-hedging problem without any no-arbitrage condition, by considering the price process as measurable selections of random sets, contrarily to what it is usual to do. We then provide a numerical procedure to compute the infimum of the super-hedging price under AIP condition. We shall follow the idea of L.Carassus and E.Lépinette [16] by considering the infimum value of the set of super-hedging prices and then computing the super-hedging price by using the convex duality without any assumptions on the market. To do so, we use the notions of Fenchel-Legendre conjugate and biconjugate, the conditional essential supremum and the conditional support of random variables. Then we provide the optimal integer-valued strategy and we formulate a dynamic programming principle implemented on an historical data base. Several important notions are defined in the Appendix that will be used throughout the thesis. For further references see [42], [16], [54] and [2].

Chapter 2

Conditional interior and conditional closure of random sets

2.1 Introduction

The conditional essential supremum and infimum of a real-valued random variable have been introduced in [4]. A generalization is then proposed in [41] for vector-valued random variables with respect to random preference relations. Actually, these two concepts are related to the notion of conditional core as first introduced in [58] and developed in [45] for random sets in separable Banach spaces, with respect to a complete σ -algebra \mathcal{H} . A conditional core of a set-valued mapping $\Gamma(\omega)$, $\omega \in \Omega$, is defined as the largest \mathcal{H} -graph measurable random set $\Gamma'(\omega)$ such that $\Gamma'(\omega) \subset \Gamma(\omega)$. This concept provides a natural conditional risk measure, that generalizes the concept of essential infimum for multi-asset portfolios in mathematical finance. Applications are deduced for geometrical market models with transaction costs: see [46] and [45] and the theory with transaction costs developed in [42].

In this chapter, we first introduce the open version of the conditional core as proposed in [58], [45]. Precisely, if \mathcal{H} is a complete sub- σ -algebra on a probability space, the conditional interior (or open conditional core) of a set-valued mapping $\Gamma(\omega)$, $\omega \in \Omega$, is defined as the largest \mathcal{H} -measurable random open set $\Gamma'(\omega)$ such that $\Gamma'(\omega) \subset \Gamma(\omega)$ P-almost every $\omega \in \Omega$. It may be seen as a measurable version of the classical interior in topology. One of our main contribution is to show the existence and uniqueness of such a conditional interior for an arbitrary random set in a separable Banach space. Then, the dual concept, the conditional closure, is introduced as a generalization of the conditional support of a real-valued random variable to a family of vector-valued random variables. A numerical application of the latter is deduced in conditional random optimization: We show that an essential supremum is a pointwise supremum on a conditional closure.

The chapter is organized as follows. In Section 2.2, we recall the definition and usual properties of measurable random sets in Banach spaces. Then, we introduce the notion of conditional interior and show the existence of such sets in Banach

spaces. In Section 2.3, the conditional closure is introduced and an application in conditional optimization is formulated in the next section. At last, we present an application in mathematical finance. The definitions of the conditional support, the conditional essential supremum and the conditional essential infimum are postponed to the appendix.

2.2 Conditional Interior

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a complete sub- σ -algebra \mathcal{H} of \mathcal{F} . Set-valued mappings we consider take their values in the family of all subsets of a separable Banach space \mathcal{X} equipped with its Borel σ -algebra $\mathcal{B}(\mathcal{X})$. In the following, we first recall the concept of graph measurable random set. We then introduce the concept of conditional interior of a random set. Existence is deduced from the existence of the conditional core of a closed random set, see [45], even if it is not always non-empty.

Recall that a random set $\Gamma(\omega)$, $\omega \in \Omega$, is a set-valued mapping that assigns to each $\omega \in \Omega$ a subset $\Gamma(\omega)$ of \mathcal{X} . We say that Γ is \mathcal{H} -measurable if, for any open set $O \subseteq \mathcal{X}$,

$$\{\omega \in \Omega, \Gamma(\omega) \cap O \neq \emptyset\} \in \mathcal{H}.$$

We say that Γ is \mathcal{H} -graph measurable, if

$$\text{graph } \Gamma := \{(\omega, x) \in \Omega \times \mathcal{X} : x \in \Gamma(\omega)\} \in \mathcal{H} \otimes \mathcal{B}(\mathcal{X}).$$

When the sets $\Gamma(\omega)$ are closed (resp. open) \mathbb{P} -almost for all $\omega \in \Omega$, we shall say that Γ is closed (resp. open).

Remark 2.2.1. *If Γ is closed-valued and the σ -algebra is complete, then Γ is \mathcal{H} -graph measurable, if and only if it is \mathcal{H} -measurable: see [47, Definition 1.3.1].*

We say that a \mathcal{F} -measurable random variable $\xi : \Omega \rightarrow \mathcal{X}$ is an \mathcal{F} -measurable selection of Γ if $\xi(\omega) \in \Gamma(\omega)$ for almost all $\omega \in \Omega$. The set of such selections is denoted by $L^0(\Gamma, \mathcal{F}) \subseteq \mathcal{M}(\mathcal{X}, \mathcal{F})$ where the set $\mathcal{M}(\mathcal{X}, \mathcal{F})$ of all \mathcal{X} -valued \mathcal{F} -measurable random variables is equipped with the metric of convergence in probability.

The following result may be found in [34, Th. 4.4] and, when applied, it is usually mentioned as *measurable selection argument*, see also [47, Theorem 2.3].

Theorem 2.2.2 (Measurable selection argument). *If Γ is an \mathcal{F} -graph measurable random set which is non-empty and closed a.s., then $L^0(\Gamma, \mathcal{F}) \neq \emptyset$.*

The following result is proven in [45, Proposition 2.7] whenever the random set we consider is closed or not. We denote by $\text{cl}(X)$ the closure of any subset $X \subseteq \mathcal{X}$ with respect to the norm topology. The closure of any subset of $\mathcal{M}(\mathcal{X}, \mathcal{F})$ is taken with respect to the metric topology of $L^0(\mathcal{X}, \mathcal{F})$.

Proposition 2.2.3. *Suppose that \mathcal{F} is complete. Suppose that Γ is an \mathcal{F} -graph measurable random set which is non empty a.s.. Then, $\text{cl}(\Gamma)$ is \mathcal{F} -graph measurable and admits a Castaing representation, i.e., there exists a countable family $(\xi_n)_{n \geq 1}$ of measurable selections of Γ such that*

$$\text{cl}(\Gamma(\omega)) = \text{cl}(\xi_n(\omega) : n \geq 1), \quad \omega \in \Omega.$$

Moreover, we have $\text{cl}(L^0(\Gamma, \mathcal{F})) = L^0(\text{cl}(\Gamma), \mathcal{F})$.

Corollary 2.2.4. *If Γ is an \mathcal{F} -graph measurable random set, then its topological interior $\text{int}(\Gamma(\omega))$, defined for each $\omega \in \Omega$, and its boundary $\partial\Gamma(\omega)$ are also \mathcal{F} -graph measurable.*

Proof. Indeed, consider $Y(\omega) = \mathcal{X} \setminus \Gamma(\omega)$. It is clear that Y is \mathcal{F} -graph measurable. Moreover, we have

$$\text{int}(\Gamma(\omega)) = \text{int}(\mathcal{X} \setminus Y(\omega)) = \mathcal{X} \setminus \text{cl}(Y(\omega)),$$

where $\text{cl}(Y(\omega))$ is \mathcal{F} -graph measurable by the proposition above, hence the complement set $\text{int}(\Gamma(\omega))$ is \mathcal{F} -graph measurable. We also deduce that the boundary $\partial\Gamma(\omega) = \text{cl}(X(\omega)) \setminus \text{int}(X(\omega))$ is \mathcal{F} -graph measurable. \square

For $x \in \mathcal{X}$ and $r \geq 0$, $B(x, r)$ denotes the open ball in \mathcal{X} of center x and radius r and $\bar{B}(x, r)$ is its closure. For any $A \subseteq \mathcal{X}$ and $\lambda \in \mathbb{R}$, we use the convention that $\lambda \times A = \{\lambda a : a \in A\}$. In particular, $0 \times A = \{0\}$. Recall the following definition, see the paper by A. Truffert [58] and [45]:

Definition 2.2.5. *The \mathcal{H} -conditional core $\mathbf{m}(\Gamma|\mathcal{H})$, of a set-valued mapping Γ , is the largest \mathcal{H} -graph measurable random set Γ' such that $\Gamma'(\omega) \subseteq \Gamma(\omega)$ a.s..*

Note that $\mathbf{m}(\Gamma|\mathcal{H})$ is the largest subset in the sense that, if $\tilde{\Gamma}$ is another \mathcal{H} -graph measurable random set contained in Γ a.s., then $\tilde{\Gamma}(\omega) \subseteq \mathbf{m}(\Gamma|\mathcal{H})(\omega)$ P-almost all $\omega \in \Omega$. The following result is proved in [45] and is an extension of the primal result of [58]. Recall that, if Γ is a random set, $L^0(\Gamma, \mathcal{H}) \subseteq \mathcal{M}(\mathcal{X}, \mathcal{H})$ is the set of all \mathcal{H} -measurable random variables γ with values $\gamma(\omega) \in \Gamma(\omega)$ a.s..

Proposition 2.2.6. *Suppose that Γ is a closed \mathcal{F} -graph measurable set that may be empty. Then, the conditional core $\mathbf{m}(\Gamma|\mathcal{H})$ exists and we have*

$$L^0(\mathbf{m}(\Gamma|\mathcal{H}), \mathcal{H}) = L^0(\Gamma, \mathcal{H}).$$

If $\gamma \in \mathcal{M}(\mathcal{X}, \mathcal{H})$ and $r \in \mathcal{M}([0, \infty), \mathcal{F})$, $\mathbf{m}(\bar{B}(\gamma, r)|\mathcal{H}) = \bar{B}(\gamma, \text{ess inf}_{\mathcal{H}} r)$. The conditional core plays a role in mathematical finance as it naturally appears when considering the dynamics of a self-financing discret-time portfolio process $(V_t)_{t=0}^T$ of the form $V_{t-1} \in V_t + G_t$, $t \geq 1$, where G_t is the solvency set, see [42], i.e., $V_{t-1} \in \mathbf{m}(V_t + G_t|\mathcal{F}_{t-1})$, see [46]. When the σ -algebra is trivial, the conditional core becomes the set of fixed points of X ; it is also related to the essential intersection considered

in [35]. Notice that the \mathcal{H} -measurable conditional core is mainly independent of \mathcal{F} , i.e., instead of completing the underlying measurable space with respect to a single (or a dominated family of) probability measure(s), one may pass to the universal completion, and hence avoid postulating the existence of any reference measure.

We now introduce an open version of the conditional core:

Definition 2.2.7. *The \mathcal{H} -graph measurable interior $\mathbf{o}(\Gamma|\mathcal{H})$ of a set-valued mapping Γ is the largest \mathcal{H} -graph measurable random open set Γ' such that $\Gamma'(\omega) \subseteq \Gamma(\omega)$ a.s..*

Notice that such a conditional interior is necessarily unique by definition. We first show the existence of the conditional interior, when Γ is open-valued.

Theorem 2.2.8. *Let us consider an \mathcal{F} -graph measurable open random set \mathcal{O} . Then, there exists a unique \mathcal{H} -graph measurable interior of \mathcal{O} .*

Proof. By Proposition 2.2.3, $\text{cl } \mathcal{O}$ and $\partial \mathcal{O} := \text{cl } \mathcal{O} \setminus \mathcal{O}$ are closed \mathcal{F} -graph measurable random sets, see Corollary 2.2.4. When $\partial \mathcal{O}(\omega) = \emptyset$, we define $d(x, \partial \mathcal{O}(\omega)) = \infty$ for all $x \in \mathcal{X}$. Otherwise, $\partial \mathcal{O}$ admits a Castaing representation on $\{\partial \mathcal{O} \neq \emptyset\}$ by Proposition 2.2.3. We deduce that the random mapping $(\omega, x) \mapsto d(x, \partial \mathcal{O}(\omega))$ is $\mathcal{F} \otimes \mathcal{B}(\mathcal{X})$ -measurable. Therefore, the random sets

$$F^n := \{x : d(x, \partial \mathcal{O}) \geq 1/n\} \cap \text{cl } \mathcal{O}, \quad n \geq 1,$$

are closed \mathcal{F} -graph measurable random subsets of \mathcal{O} and we have $\mathcal{O} = \bigcup_n F^n$. Let us define the \mathcal{H} -graph measurable open random set (see Corollary 2.2.4 and Proposition 2.2.6):

$$\mathbf{o}(\mathcal{O}|\mathcal{H}) = \text{int} \left(\bigcup_n \mathbf{m}(F^n|\mathcal{H}) \right) \subseteq \mathcal{O}.$$

Let us show that this is the largest \mathcal{H} -graph measurable open random subset of \mathcal{O} . To do so, let $\mathcal{O}_{\mathcal{H}} \subseteq \mathcal{O}$ be a \mathcal{H} -graph measurable open subset of \mathcal{O} . As for \mathcal{O} , we may write $\mathcal{O}_{\mathcal{H}} = \bigcup_n H^n$ where $H^n := \{x : d(x, \partial \mathcal{O}_{\mathcal{H}}) \geq 1/n\} \cap \text{cl } \mathcal{O}_{\mathcal{H}}$ are \mathcal{H} -graph measurable closed subsets of $\mathcal{O}_{\mathcal{H}}$.

We claim that $H^n \subseteq F^n$. Indeed, if $x \in H^n$, it suffices to show that $d(x, \partial \mathcal{O}_{\mathcal{H}}) \leq d(x, \partial \mathcal{O})$ a.s.. In the contrary case, on a non-null set, there exists $o \in \partial \mathcal{O}$ such that $o \in B(x, d(x, \partial \mathcal{O}_{\mathcal{H}}))$. By Lemma 2.2.10, this implies that $o \in \mathcal{O}_{\mathcal{H}} \subseteq \mathcal{O}$ which yields a contradiction.

As H^n is \mathcal{H} -graph measurable and closed, $H^n \subseteq \mathbf{m}(F^n|\mathcal{H})$ for all n . We deduce that $\mathcal{O}_{\mathcal{H}} \subseteq \bigcup_n \mathbf{m}(F^n|\mathcal{H})$ hence $\mathcal{O}_{\mathcal{H}} \subseteq \mathbf{o}(\mathcal{O}|\mathcal{H})$. \square

We then deduce the general case:

Theorem 2.2.9. *For any \mathcal{F} -graph measurable random set Γ , the \mathcal{H} -graph measurable interior of Γ exists and $\mathbf{o}(\Gamma|\mathcal{H}) = \mathbf{o}(\text{int } \Gamma|\mathcal{H})$ is unique.*

The following lemma is recalled for the sake of completeness. It is used in the proof above.

Lemma 2.2.10. *Let \mathcal{O} be an open set in a separable Banach space. For every $x \in \mathcal{O}$, $B(x, d(x, \partial\mathcal{O})) \subseteq \mathcal{O}$.*

Proof. Let us consider $r^* = \sup R$ where R is the non empty set of all $r > 0$ such that $B(x, r) \subseteq \mathcal{O}$. It is trivial that $B(x, r^*) \subseteq \mathcal{O}$. So, for all $o \in \partial\mathcal{O}$, $d(x, o) \geq r^*$ hence $r^* \leq d(x, \partial\mathcal{O})$. Moreover, by definition of r^* , for all n there exists $z^n \in B(x, r^* + n^{-1})$ such that $z^n \notin \mathcal{O}$ hence $r^* \leq \|z^n - x\| \leq r^* + n^{-1}$. As the sequence $(z^n)_n$ is bounded, using the completeness of the Banach space we deduce the existence of a subsequence, that we denote $(z_n)_n$ converges to some z in the Banach space. Then, $\|z - x\| = r^*$ hence $z \in \text{cl } \mathcal{O}$. In the case where $z \in \mathcal{O}$, $z^n \in \mathcal{O}$ for n large enough since \mathcal{O} is open, which yields a contradiction. Therefore, $z \in \partial\mathcal{O}$. This implies that $r^* = d(x, z) \geq d(x, \partial\mathcal{O})$ and finally $r^* = d(x, \partial\mathcal{O})$. \square

2.3 Conditional Closure

We now introduce the concept of conditional closure. The existence is proved in the following theorem which is the second main contribution of this chapter. As previously, \mathcal{F} and \mathcal{H} are supposed to be complete with respect to some probability measure \mathbb{P} .

Definition 2.3.1. *The \mathcal{H} -graph measurable conditional closure $\text{cl}(\Gamma|\mathcal{H})$ of a set-valued mapping Γ is the smallest \mathcal{H} -graph measurable random closed set Γ' such that $\Gamma(\omega) \subseteq \Gamma'(\omega)$ a.s..*

Theorem 2.3.2. *For any \mathcal{F} -graph measurable random set Γ , the \mathcal{H} -graph measurable conditional closure $\text{cl}(\Gamma|\mathcal{H})$ of Γ exists and is unique. We have*

$$\text{cl}(\Gamma|\mathcal{H}) = \mathcal{X} \setminus \mathbf{o}(\mathcal{X} \setminus \Gamma|\mathcal{H}).$$

Moreover, for all measurable selection γ of $\text{cl}(\Gamma|\mathcal{H})$ and $\varepsilon \in \mathcal{M}(]0, \infty[, \mathcal{H})$, for all $H \in \mathcal{H}$ such that $\mathbb{P}(H) > 0$, we have $\mathbb{P}(\{\Gamma \cap B(\gamma, \varepsilon) \neq \emptyset\} \cap H) > 0$.

Proof. The first part is a direct consequence of Corollary 2.2.9. Indeed, we first observe that $\Gamma \subseteq \mathcal{X} \setminus \mathbf{o}(\mathcal{X} \setminus \Gamma|\mathcal{H})$. Moreover, if $\Gamma_{\mathcal{H}}$ is a closed-valued \mathcal{H} -graph measurable set containing Γ , then $\mathcal{X} \setminus \Gamma_{\mathcal{H}} \subseteq \mathcal{X} \setminus \Gamma$. We deduce that $\mathcal{X} \setminus \Gamma_{\mathcal{H}} \subseteq \mathbf{o}(\mathcal{X} \setminus \Gamma|\mathcal{H})$ and, finally, $\mathcal{X} \setminus \mathbf{o}(\mathcal{X} \setminus \Gamma|\mathcal{H}) \subseteq \Gamma_{\mathcal{H}}$. Suppose that $\mathbb{P}(\{\Gamma \cap B(\gamma, \varepsilon) \neq \emptyset\} \cap H) = 0$ for some $H \in \mathcal{H}$ and $\varepsilon \in \mathcal{M}((0, \infty), \mathcal{H})$. Therefore, by definition of the conditional closure as a smallest set, we have

$$\text{cl}(\Gamma|\mathcal{H}) = \text{cl}(\Gamma|\mathcal{H})1_{\Omega \setminus H} + \text{cl}(\Gamma|\mathcal{H}) \cap (\mathcal{X} \setminus B(\gamma, \varepsilon))1_H.$$

Indeed, the \mathcal{H} -graph measurable set in the r.h.s. above contains Γ by assumption hence it contains $\text{cl}(\Gamma|\mathcal{H})$. We get a contradiction since $\gamma \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$ a.s. by assumption. Uniqueness is clear as the conditional closure of Γ is the smallest closed set, up to a negligible set, in the sense that it is included in any other \mathcal{H} -measurable set containing Γ . \square

Remark 2.3.3. We may deduce the conditional support of a random variable $X \in L^0(\mathcal{X}, \mathcal{F})$. Precisely, there exists a smallest \mathcal{H} -graph measurable random closed set denoted by $\text{supp}_{\mathcal{H}}(X)$ such that $P(X \in \text{supp}_{\mathcal{H}}(X)) = 1$. It is given by $\text{supp}_{\mathcal{H}}(X) = \text{cl}(\{X\}|\mathcal{H})$ and is called the \mathcal{H} -conditional support of X . Moreover, for all $\gamma \in L^0(\text{supp}_{\mathcal{H}}(X), \mathcal{H})$ and $\varepsilon \in \mathcal{M}(]0, \infty[, \mathcal{H})$, for all $H \in \mathcal{H}$ such that $P(H) > 0$, we have $P(\{X \in B(\gamma, \varepsilon)\} \cap H) > 0$.

Notice that the conditional support is necessarily non empty a.s. hence it admits a measurable selection. In the following, we adopt the notation $kA = \{ka : a \in A\}$ for any subset $A \subseteq \mathcal{X}$ and $k \in A$.

Lemma 2.3.4. Let $H \in \mathcal{H}$. Then, $\text{cl}(\Gamma 1_H | \mathcal{H}) = 1_H \text{cl}(\Gamma | \mathcal{H})$, for every \mathcal{F} -graph measurable set Γ .

Proof. First, observe that $P(\Gamma 1_H \subseteq 1_H \text{cl}(\Gamma | \mathcal{H})) = 1$ as $1_H \text{cl}(\Gamma | \mathcal{H}) = \{0\}$ on $\Omega \setminus H$. We deduce that $\text{cl}(\Gamma 1_H | \mathcal{H}) \subseteq 1_H \text{cl}(\Gamma | \mathcal{H})$ a.s. and $\text{cl}(\Gamma 1_H | \mathcal{H}) = \{0\}$ on $\Omega \setminus H$. Let us define $Z = \text{cl}(\Gamma 1_H | \mathcal{H}) 1_H + \text{cl}(\Gamma | \mathcal{H}) 1_{\Omega \setminus H}$. As $\Gamma \subseteq \Gamma 1_H$ on H , we deduce that $\Gamma \subseteq Z$ a.s. hence $\text{cl}(\Gamma | \mathcal{H}) \subseteq Z$ a.s.. Therefore, we deduce that $\text{cl}(\Gamma | \mathcal{H}) 1_H \subseteq Z 1_H$ and finally $\text{cl}(\Gamma | \mathcal{H}) 1_H \subseteq \text{cl}(\Gamma 1_H | \mathcal{H})$ as the latter set is $\{0\}$ on $\Omega \setminus H$. \square

Example 2.3.5. Let $\mathcal{X} = \mathbb{R}$ and $\gamma_1, \gamma_2 \in \mathcal{M}(\mathcal{X}, \mathcal{F})$ be such that $\gamma_1 \leq \gamma_2$ a.s.. Consider $\Gamma = [\gamma_1, \gamma_2]$. Then, $\text{cl}(\Gamma | \mathcal{H}) = [\text{ess inf}_{\mathcal{H}} \gamma_1, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R}$. Indeed, first observe that $\Gamma \subseteq [\text{ess inf}_{\mathcal{H}} \gamma_1, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R}$ a.s.. Consider a \mathcal{H} -measurable closed set $\Gamma_{\mathcal{H}}$ containing Γ and let us show that

$$[\text{ess inf}_{\mathcal{H}} \gamma_1, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R} \subseteq \Gamma_{\mathcal{H}}.$$

To do so, consider $\xi \in L^0([\text{ess inf}_{\mathcal{H}} \gamma_1, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R}, \mathcal{H})$ and suppose that $P(\xi \notin \Gamma_{\mathcal{H}}) > 0$. We suppose w.l.o.g. that $P(\xi \notin \Gamma_{\mathcal{H}}) = 1$. As $\Gamma \subseteq \Gamma_{\mathcal{H}}$, we necessarily have $\xi \in [\text{ess inf}_{\mathcal{H}} \gamma_1, \gamma_1] \cap \mathbb{R}$ or $\xi \in [\gamma_2, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R}$. Let us consider the case where $\xi \in [\text{ess inf}_{\mathcal{H}} \gamma_1, \gamma_1] \cap \mathbb{R}$, the other case being similar. If $\text{ess inf}_{\mathcal{H}} \gamma_1 = \gamma_1$, then $\xi = \gamma_1 \in \Gamma_{\mathcal{H}}$, i.e., a contradiction. Therefore, $\text{ess inf}_{\mathcal{H}} \gamma_1 < \gamma_1$ and, since ξ is \mathcal{H} -measurable and $\xi \leq \gamma_1$ a.s., we deduce that $\xi \leq \text{ess inf}_{\mathcal{H}} \gamma_1$. It follows that $\xi = \text{ess inf}_{\mathcal{H}} \gamma_1 \in \mathbb{R}$. By definition of the essential infimum, there is a non-null set on which $\text{ess inf}_{\mathcal{H}} \gamma_1 + \frac{1}{2}d(\xi, \Gamma_{\mathcal{H}}) > \gamma_1$ where $d(\xi, \Gamma_{\mathcal{H}}) > 0$ as $\xi \notin \Gamma_{\mathcal{H}}$. We may suppose that $\xi + \frac{1}{2}d(\xi, \Gamma_{\mathcal{H}}) \in [\gamma_1, \gamma_2]$ even if we have to change $d(\xi, \Gamma_{\mathcal{H}})$ by a smaller \mathcal{F} -measurable term. Therefore, $\xi + \frac{1}{2}d(\xi, \Gamma_{\mathcal{H}}) \in \Gamma_{\mathcal{H}}$ and $d(\xi, \Gamma_{\mathcal{H}}) \leq d(\xi, \xi + \frac{1}{2}d(\xi, \Gamma_{\mathcal{H}})) \leq \frac{1}{2}d(\xi, \Gamma_{\mathcal{H}})$. We get a contradiction and, finally, $\text{cl}(\Gamma | \mathcal{H}) = [\text{ess inf}_{\mathcal{H}} \gamma_1, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R}$.

Example 2.3.6. Let $\gamma \in \mathcal{M}(\mathcal{X}, \mathcal{H})$ and $r \in \mathcal{M}((0, \infty), \mathcal{F})$. Consider Γ the random closed ball $\overline{B}(\gamma, r)$ of center γ and radius r . We shall prove that

$$\text{cl}(\Gamma | \mathcal{H}) = \overline{B}(\gamma, \text{ess sup}_{\mathcal{H}} r).$$

First observe that $\Gamma \subseteq \overline{B}(\gamma, \text{ess sup}_{\mathcal{H}} r)$ a.s. Moreover, consider $\Gamma_{\mathcal{H}}$ a \mathcal{H} -graph measurable random closed set containing Γ a.s. Suppose that there is

$$\xi \in L^0(\overline{B}(\gamma, \text{ess sup}_{\mathcal{H}} r), \mathcal{H})$$

such that $P(\xi \notin \Gamma_{\mathcal{H}}) > 0$. We first consider the case where $\text{ess sup}_{\mathcal{H}} r < \infty$. On the set $A_{\mathcal{H}} = \{\xi \notin \Gamma_{\mathcal{H}}\} \in \mathcal{H}$, $d(\xi, \Gamma_{\mathcal{H}}) > 0$. Note that $d(\xi, \Gamma_{\mathcal{H}}) < \infty$. There is a non-null \mathcal{F} -measurable subset of $A_{\mathcal{H}}$ on which $\text{ess sup}_{\mathcal{H}} r - d(\xi, \Gamma_{\mathcal{H}}) < r$ by definition of the essential supremum. Let us define $\tilde{\xi} = \gamma + \alpha(\xi - \gamma)$ where $\alpha = r/(r + d(\xi, \Gamma_{\mathcal{H}}))$. As we have $\|\xi - \gamma\| \leq r + d(\xi, \Gamma_{\mathcal{H}})$ when $\text{ess sup}_{\mathcal{H}} r - d(\xi, \Gamma_{\mathcal{H}}) < r$, we deduce that $\tilde{\xi} \in \Gamma$ and $\tilde{\xi}$ is only \mathcal{F} -measurable. Therefore, $\tilde{\xi} \in \Gamma_H$ and $d(\xi, \Gamma_{\mathcal{H}}) \leq \|\tilde{\xi} - \xi\|$ where we may show that $\|\tilde{\xi} - \xi\| < d(\xi, \Gamma_{\mathcal{H}})$ hence a contradiction. Consider now the case where $\text{ess sup}_{\mathcal{H}} r = \infty$. We need to show that $\Gamma_{\mathcal{H}} = \mathcal{X}$. In the contrary case, on a non-null set, we may construct a \mathcal{H} -measurable selection ζ of $\mathcal{X} \setminus \Gamma_{\mathcal{H}}$. Moreover, on a smaller non-null set, $r \geq \|\zeta - \gamma\|$ as $\text{ess sup}_{\mathcal{H}} r = \infty$. It follows that $\zeta \in \Gamma$ hence $\zeta \in \Gamma_{\mathcal{H}}$, i.e., a contradiction.

2.4 Applications

2.4.1 Conditional Optimization

The second main contribution is the following. It allows one to compute numerically an essential supremum as a pointwise supremum on the conditional closure. This result is a generalization of [3, Proposition 2.7] and is useful in robust finance dynamic programming, see our example below. In this section we suppose that $\mathcal{X} = \mathbb{R}^d$. We recall that an integrand $h(\omega, x)$, $(\omega, x) \in \Omega \times \mathcal{X}$, is a jointly measurable function, which is lower semi-continuous in x and takes values in the extended real line $\mathbb{R} \cup \{+\infty\}$, see [54, Corollary 14.34].

Lemma 2.4.1. *Let $h(\omega, x)$, $(\omega, x) \in \Omega \times \mathcal{X}$, be an $\mathcal{H} \otimes \mathcal{B}(\mathcal{X})$ -measurable integrand. Let Γ be a non-empty closed \mathcal{F} -graph measurable set of \mathcal{X} . Then, $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})(\omega)} h(x) = \sup_n h(\gamma_n)$ where $(\gamma_n)_{n \in \mathbb{N}}$ is a Castaing representation of $\text{cl}(\Gamma|\mathcal{H})$.*

Proof. As $\text{cl}(\Gamma|\mathcal{H})$ is \mathcal{H} -graph measurable and closed-valued, it admits a Castaing representation $\text{cl}(\Gamma|\mathcal{H})(\omega) = \text{cl}\{\gamma_n(\omega) : n \in \mathbb{N}\}$ a.s. where, for all n , $\gamma_n \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$, by Proposition 2.2.3. Notice that we may adjust the values on a set of measure zero, and therefore assume that the equality holds everywhere on Ω , see the proof of [42, Proposition 5.4.4].

As $(\gamma_n)_n \subset \text{cl}\{\gamma_n : n \in \mathbb{N}\} = \text{cl}(\Gamma|\mathcal{H})$, $h(\gamma_n) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$. Therefore, $\sup_n h(\gamma_n) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$. Moreover, if $x \in \text{cl}(\Gamma|\mathcal{H})(\omega)$, we may write almost surely $x = \lim_n \gamma_n(\omega)$ by the Castaing representation $(\gamma_n)_n$ of $\text{cl}(\Gamma|\mathcal{H})$. By the lower semicontinuity of h , we get that $h(x) \leq \liminf_n h(\gamma_n)$. Thus, $h(x) \leq \sup_n h(\gamma_n)$ and $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})(\omega)} h(x) \leq \sup_n h(\gamma_n)$ on Ω . The equality is then deduced. \square

Theorem 2.4.2. *Let $h(\omega, x)$, $(\omega, x) \in \Omega \times \mathcal{X}$, be an $\mathcal{H} \otimes \mathcal{B}(\mathcal{X})$ -measurable integrand. Let Γ be a non-empty closed \mathcal{F} -graph measurable set of \mathcal{X} . Then, with the notation $h(\gamma)(\omega) = h(\omega, \gamma(\omega))$, we have*

$$\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} = \sup_{x \in \text{cl}(\Gamma|\mathcal{H})(\omega)} h(x), \text{ a.s..}$$

Proof. As $\text{cl}(\Gamma|\mathcal{H})$ is \mathcal{H} -graph measurable and closed-valued, it admits a Castaing representation $\text{cl}(\Gamma|\mathcal{H})(\omega) = \text{cl}\{\gamma_n(\omega) : n \in \mathbf{N}\}$ where, for all n , $\gamma_n \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$, by Proposition 2.2.3, see also the lemma above.

Step 1. We show that

$$\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \text{ess sup}_{\mathcal{H}} \{h(\gamma), \gamma \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}.$$

To see it, notice that by Lemma 2.4.1,

$$\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n) \leq \text{ess sup}_{\mathcal{H}} \{h(\gamma), \gamma \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}.$$

If $\gamma \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$, then $\gamma \in \text{cl}\{\gamma_n : n \in \mathbf{N}\}$ i.e., $\gamma = \lim_n \gamma_n$ for a subsequence and, by lower semi-continuity,

$$h(\gamma) = \liminf_n h(\gamma_n) \leq \sup_n h(\gamma_n) = \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x).$$

Since the family $\{h(\gamma), \gamma \in L^0(\text{cl}(\Gamma|\mathcal{H}))\}$ is directed upward, we also deduce that $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \lim_n \uparrow h(\gamma_n)$ where $\gamma_n \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$.

Step 2. We show that $\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} = \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$ a.s.. First notice that $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n)$ is \mathcal{H} -measurable, since h is $\mathcal{H} \otimes \mathcal{B}(\mathcal{X})$ -measurable. Moreover, as $\Gamma(\omega) \subseteq \text{cl}(\Gamma|\mathcal{H})(\omega)$ a.s., we deduce that $h(\gamma) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$, for every $\gamma \in L^0(\Gamma, \mathcal{F})$. We deduce that

$$\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) \text{ a.s..}$$

To show the reverse inequality, consider any \mathcal{H} -measurable selection γ of $\text{cl}(\Gamma|\mathcal{H})$ and a deterministic sequence $\varepsilon_n > 0$ with $\lim_n \varepsilon_n = 0$. Let us define

$$\Lambda_n = \{\Gamma \cap \bar{B}(\gamma, \varepsilon_n) \neq \emptyset\} \in \mathcal{F}.$$

By Theorem 2.3.2, $P(\Lambda_n|\mathcal{H}) > 0$ a.s.. Indeed, in the contrary case, on a non-null \mathcal{H} -measurable set $\tilde{\Lambda}_n$, we have $P(\tilde{\Lambda}_n \cap \Lambda_n|\mathcal{H}) = 0$ hence $P(\tilde{\Lambda}_n \cap \Lambda_n) = 0$, i.e., a contradiction with Theorem 2.3.2.

By a measurable selection argument, we consider $\hat{\gamma}_n \in L^0(\Gamma, \mathcal{F})$ such that $\hat{\gamma}_n \in \bar{B}(\gamma, \varepsilon_n)$ on Λ_n . We define $\hat{\Lambda}_n = \{\hat{\gamma}_n \in \bar{B}(\gamma, \varepsilon_n)\}$. Since $\Lambda_n \subseteq \hat{\Lambda}_n$, we have $P(\hat{\Lambda}_n|\mathcal{H}) > 0$ a.s.. Moreover, using the conditional expectation, we have

$$\begin{aligned} \text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} 1_{\hat{\Lambda}_n} &\geq h(\hat{\gamma}_n) 1_{\hat{\Lambda}_n}, \\ \text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} P(\hat{\Lambda}_n|\mathcal{H}) &\geq \mathbb{E}(h(\hat{\gamma}_n) 1_{\hat{\Lambda}_n} | \mathcal{H}), \text{ a.s..} \end{aligned} \quad (2.4.1)$$

Moreover, we have

$$\mathbb{E}(h(\hat{\gamma}_n) 1_{\hat{\Lambda}_n} | \mathcal{H}) \geq \mathbb{E}(\inf_{z \in \bar{B}(\gamma, \varepsilon_n)} h(z) 1_{\hat{\Lambda}_n} | \mathcal{H}).$$

Notice that $\inf_{z \in \bar{B}(\gamma, \varepsilon_n)} h(z) = \inf_{z \in \mathcal{X}} \bar{h}(z)$ where $\bar{h} = h$ on $\bar{B}(\gamma, \varepsilon_n)$ and $\bar{h} = +\infty$ otherwise. As \bar{h} is also an integrand, we deduce by [54, Theorem 14.37] that $\inf_{z \in \bar{B}(\gamma, \varepsilon_n)} h(z)$ is \mathcal{H} -measurable. Therefore,

$$\mathbb{E}(\inf_{z \in \bar{B}(\gamma, \varepsilon_n)} h(z) 1_{\hat{\Lambda}_n} | \mathcal{H}) = \inf_{z \in \bar{B}(\gamma, \varepsilon_n)} h(z) \mathbb{P}(\hat{\Lambda}_n | \mathcal{H}).$$

As $\mathbb{P}(\hat{\Lambda}_n | \mathcal{H}) > 0$ a.s., we deduce by (2.4.1) that

$$\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} \geq \inf_{z \in \bar{B}(\gamma, \varepsilon_n)} h(z), \text{ a.s.}$$

for every $n \geq 1$. Therefore,

$$\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} \geq \lim_n \inf_{z \in \bar{B}(\gamma, \varepsilon_n)} h(z), \text{ a.s..}$$

Since $\bar{B}(\gamma, \varepsilon_n)$ is a.s. compact and h is a.s. lower semi-continuous, we deduce that $\inf_{z \in \bar{B}(\gamma, \varepsilon_n)} h(z) = h(z_n)$ where $z_n \in \bar{B}(\gamma, \varepsilon_n)$ converges pointwise to γ as $n \rightarrow \infty$. We finally deduce by lower semi-continuity that

$$\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} \geq \liminf_n h(z_n) \geq h(\gamma).$$

This inequality holds for any selection γ of $\text{cl}(\Gamma | \mathcal{H})$. Therefore, we get that

$$\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} \geq \text{ess sup}_{\mathcal{H}} \{h(\gamma), \gamma \in L^0(\text{cl}(\Gamma | \mathcal{H}), \mathcal{H})\}.$$

The conclusion of the lemma follows. \square

Recall that a set Λ of measurable random variables is said \mathcal{F} -decomposable if for any finite partition $(F_i)_{i=1}^n \subseteq \mathcal{F}$ of Ω , and for every family $(\gamma_i)_{i=1}^n$ of Λ , we have $\sum_{i=1}^n \gamma_i 1_{F_i} \in \Lambda$. Decomposability was initially introduced by Rockafellar: see also [36]. In the following, we denote by $\Sigma(\Lambda)$ the \mathcal{F} -decomposable envelope of Λ , i.e., the smallest \mathcal{F} -decomposable family containing Λ . Notice that

$$\Sigma(\Lambda) = \left\{ \sum_{i=1}^n \gamma_i 1_{F_i} : n \geq 1, (\gamma_i)_{i=1, \dots, n} \subseteq \Lambda, (F_i)_{i=1, \dots, n} \subseteq \mathcal{F} \text{ s.t. } \sum_{i=1}^n F_i = \Omega \right\}.$$

The closure $\bar{\Sigma}(\Lambda)$ in probability of $\Sigma(\Lambda)$ is decomposable even if Λ is not decomposable. By [42, Proposition 5.4.3], there exists an \mathcal{F} -graph measurable closed random set $\sigma(\Lambda)$ such that $\bar{\Sigma}(\Lambda)$ coincides with $L^0(\sigma(\Lambda), \mathcal{F})$, the set of all measurable selectors of $\sigma(\Lambda)$.

Theorem 2.4.3. *Let $h(\omega, x)$, $x \in \mathcal{X}$, be an $\mathcal{H} \otimes \mathcal{B}(\mathcal{X})$ -measurable integrand. Let us consider a family Λ of measurable random variables so that*

$$\bar{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$$

is the set of all measurable selectors of some \mathcal{F} -graph measurable random closed set $\sigma(\Lambda)$. Then,

$$\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in \Lambda\} = \text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in \bar{\Sigma}(\Lambda)\} = \sup_{x \in \text{cl}(\sigma(\Lambda) | \mathcal{H})} h(x).$$

Proof. Notice that for any finite partition $(F_i)_{i=1}^n \subseteq \mathcal{F}$ of Ω , $n \geq 1$, and for every family $(\gamma_i)_{i=1}^n$ of Λ , we have

$$h\left(\sum_{i=1}^n \gamma_i 1_{F_i}\right) = \sum_{i=1}^n h(\gamma_i) 1_{F_i}.$$

Therefore, as $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} \geq h(\gamma)$ a.s. for any $\gamma \in \Lambda$, we deduce that $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} \geq h(\gamma)$ a.s. for any $\gamma \in \Sigma(\Lambda)$. Since h is l.s.c. and any $\gamma \in \overline{\Sigma}(\Lambda)$ is a limit of elements of $\Sigma(\Lambda)$, we get that the inequality also holds for any $\gamma \in \overline{\Sigma}(\Lambda)$. Taking the essential supremum over all $\gamma \in \overline{\Sigma}(\Lambda)$, we deduce that

$$\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} \geq \text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \overline{\Sigma}(\Lambda)\}$$

and, finally, the equality holds since $\Lambda \subseteq \overline{\Sigma}(\Lambda)$. The last equality of the corollary is deduced from Theorem 2.4.1. \square

2.4.2 Application in Finance: Robust Super-Hedging of an European or Asian Option

We consider a financial market in discrete time defined by a complete stochastic basis $(\Omega, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. We suppose that there is a non-risky asset whose price is $S^0 = 1$, without loss of generality. The (discounted) prices are modeled by a vector-valued stochastic process $(S_t)_{t=0}^T$ adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$ with values in \mathbb{R}^d , $d \geq 1$.

We consider the one step super-hedging problem between two dates $t-1$ and t with $t \geq 1$. We suppose that after time $t-1$ but strictly before time t the portfolio manager observes the price S_{t-1} , as a consequence of her/his order. We denote by S_0^t the price process $(S_u)_{u \leq t}$, for all $0 \leq t \leq T$. More precisely, the portfolio manager knows S_0^{t-2} at time $t-1$ and sends an order at time $t-1$ which is executed with a delay so that the executed price S_{t-1} is only observed strictly after $t-1$.

Let us consider, for each $t \leq T$, $\Lambda_t \subseteq L^0(\mathbb{R}_+^d, \mathcal{F}_t)$ an \mathcal{F}_t -measurable random set representing the possible prices for the risky assets at time t . We suppose that, at time t , the set Λ_t may depend on the observed prices before time $t-1$, i.e. to each vector of prices S_0^{t-1} , we associate a set $\Lambda_t = \Lambda_t(S_0^{t-1})$ representing the possible next prices at time t given that we have observed the executed prices S_0^{t-1} . Therefore:

Definition 2.4.4. *A price process is an $(\mathcal{F}_t)_{t=0}^T$ -adapted non-negative process $(S_t)_{t=-1}^T$ such that $S_t \in \Lambda_t(S_0^{t-1})$ for all $t = 1, \dots, T$ and $S_{-1} \in \mathbb{R}^d$ is given .*

Note that S_t represents the prices (S_t^1, \dots, S_t^d) of the risky assets proposed by the market to the portfolio manager when selling or buying. A typical case could be $\Lambda_t = L^0(I_t, \mathcal{F}_t)$ with

$$I_t = \Pi_{j=1}^d [S_t^{bj}, S_t^{aj}],$$

where $(S_t^{bj})_{j=1}^d$ and $(S_t^{aj})_{j=1}^d$ are respectively the bid and the ask price processes observed in the market at time t that may depend on S_0^{t-1} . They are not necessary

the best bid/ask prices as, in practice, the real transaction price may be a convex combination of bid and ask prices. Indeed, a transaction is the result of an agreement between sellers and buyers but it also depends on the traded volume. Clearly, the portfolio manager does not benefit from the last price observed in the market when sending an order. On the contrary, he should face an uncertain price S_t which depends on the type of order (which may be not executed) but it also depends on some random events he does not control, e.g. slippage. A simple way to model this phenomenon is to suppose that the executed prices obtained by the manager belong to random intervals.

Another interesting case could be when Λ_t coincides with a parametrized family $\{S_t^\theta : \theta \in \Theta\}$ of random variables. For instance, consider fixed processes $(\xi_u)_{u \leq T}$ and $(m_u)_{u \leq T}$ adapted to $(\mathcal{F}_t)_{t=0}^T$ and independent of \mathcal{F}_{t-1} . Let C be a compact set and suppose that S_{-1} is given. We define recursively

$$\Lambda_t(S_0^{t-1}) = \{S_{t-1} \exp(\sigma \xi_t + m_t) : S_{t-1} \in \Lambda_{t-1}, \sigma \in C\}, \quad t \leq T.$$

In this model, there is an uncertainty on prices because of the unknown parameter (e.g. volatility) σ .

In the following, we consider the σ -algebra $\mathcal{F}_{t-1} = \sigma(S_u : u \leq t-1)$ for all $t \geq 1$. Let us consider a random function g_t defined on \mathbb{R}^t , $t \geq 1$. We assume that the mapping $(\omega, z) \mapsto g_t(S_0(\omega), \dots, S_{t-1}(\omega), z)$ is $\mathcal{F}_{t-1} \times \mathcal{B}(\mathbb{R})$ -measurable and $z \mapsto g_t(S_0, S_1, \dots, S_{t-1}, z)$ is lower-semicontinuous (l.s.c.) almost surely whatever the price process S_0^{t-1} . Our goal is to characterise the set \mathcal{P}_{t-1} of all $V_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_{t-1})$ such that

$$V_{t-1} + \theta_{t-1} \Delta S_t \geq g_t(S_1, \dots, S_t), \text{ a.s. for all } S_t \in \Lambda_t(S_0^{t-1}), \quad (2.4.2)$$

for some $\theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$. We observe that, by lower-semicontinuity, (3.3.2) holds if and only if

$$V_{t-1} + \theta_{t-1} \Delta S_t \geq g_t(S_1, \dots, S_t), \quad \text{for all } S_t \in \overline{\Sigma}(\Lambda_t(S_0^{t-1})). \quad (2.4.3)$$

Recall that $\overline{\Sigma}(\Lambda_t(S_0^{t-1}))$ is defined in the previous section. This means that we may suppose w.l.o.g. that $\overline{\Sigma}(\Lambda_t(S_0^{t-1})) = \Lambda_t(S_0^{t-1})$. In the following, we denote by $I_t(S_0^{t-1})$ the \mathcal{F}_t -measurable closed random set such that $\overline{\Sigma}(\Lambda_t(S_0^{t-1})) = L^0(I_t(S_0^{t-1}), \mathcal{F}_t)$: see [45, Theorem 2.4].

By Theorem 2.4.2, we deduce that (3.3.3) is equivalent to $V_{t-1} \geq p_{t-1}$ where $p_{t-1} = p_{t-1}(S_0^{t-1}, \theta_{t-1})$ is given by

$$\begin{aligned} p_{t-1} &= \theta_{t-1} S_{t-1} + \sup_{z \in \text{cl}(I_t(S_0^{t-1}) | \mathcal{F}_{t-1})} (g_t(S_1, \dots, S_{t-1}, z) - \theta_{t-1} z), \\ &= \theta_{t-1} S_{t-1} + f_{t-1}^*(-\theta_{t-1}). \end{aligned}$$

In the formula above, $f_{t-1}^*(y) = \sup_{z \in \mathbb{R}^d} (yz - f_{t-1}(z))$ is the Fenchel-Legendre conjugate function of f_{t-1} defined as

$$f_{t-1}(z) := -g_t(S_1, \dots, S_{t-1}, z) + \delta_{\text{cl}(I_t(S_0^{t-1}) | \mathcal{F}_{t-1})}(z),$$

where $\delta_{\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_{t-1})} \in \{0, \infty\}$ is infinite on the complimentary of $\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_{t-1})$ and 0 otherwise. Notice that f_{t-1}^* is convex and l.s.c. as a supremum (on $\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_{t-1})$) of convex and l.s.c. functions. Moreover, by Theorem 2.4.2, $(\omega, y) \mapsto f_{t-1}^*(\omega, y)$ is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. Therefore, $\text{Dom } f_{t-1}^* := \{y : f_{t-1}^*(\omega, y) < \infty\}$ is an \mathcal{F}_{t-1} -measurable random set. We deduce that the \mathcal{F}_{t-1} -measurable prices at time $t - 1$ are given by

$$\begin{aligned} \mathcal{P}_{t-1}(S_0^{t-1}) &= \{\theta_{t-1}S_{t-1} + f_{t-1}^*(-\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})\} \\ &\quad + L^0(\mathbb{R}_+, \mathcal{F}_{t-1}). \end{aligned}$$

The second step is to determine the infimum super-hedging price as

$$p_{t-1}(S_0^{t-1}) = \text{ess inf}_{\mathcal{F}_{t-1}} \mathcal{P}_{t-1}(S_0^{t-1}).$$

To do so, we use the arguments of [3, Theorem 2.8] and we obtain that:

$$\begin{aligned} p_{t-1}(S_0^{t-1}) &= \text{ess inf}_{\mathcal{F}_{t-1}} \{\theta_{t-1}S_{t-1} + f_{t-1}^*(-\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})\}, \\ &= \text{ess inf}_{\mathcal{F}_{t-1}} \{-\theta_{t-1}S_{t-1} + f_{t-1}^*(\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})\}, \\ &= -\text{ess sup}_{\mathcal{F}_{t-1}} \{\theta_{t-1}S_{t-1} - f_{t-1}^*(\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})\}, \\ &= -\text{ess sup}_{\mathcal{F}_{t-1}} \{\theta_{t-1}S_{t-1} - f_{t-1}^*(\theta_{t-1}) : \theta_{t-1} \in \text{Dom } f_{t-1}^*\}, \\ &= -\sup_{z \in \overline{\text{Dom } f_{t-1}^*}} (zS_{t-1} - f_{t-1}^*(z)), \\ &= -\sup_{z \in \mathbb{R}^d} (zS_{t-1} - f_{t-1}^*(z)), \\ &= -f_{t-1}^{**}(S_{t-1}). \end{aligned}$$

In the following, we suppose that, for all price process S_0^{t-1} , there exists $\alpha_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$ and $\beta_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_{t-1})$ such that

$$g_t(S_0, S_1, \dots, S_{t-1}, x) \leq \alpha_{t-1}x + \beta_{t-1}, \quad \forall x \in \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_{t-1}).$$

This is the case for instance for Asian options whose payoff is of the form $k(S_0 + S_1 + \dots + S_t - K)^+$, $k > 0$. By [3, Theorem 2.8], we then deduce that

$$\begin{aligned} p_{t-1}(S_0^{t-1}) &= \\ &\inf \{\alpha S_{t-1} + \beta : \alpha x + \beta \geq g_t(S_0, \dots, S_{t-1}, x), \forall x \in \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_{t-1})\}. \end{aligned}$$

In this section, we have solved the super-hedging problem without any no-arbitrage condition, contrarily to what it is usual to do.

2.5 Conclusions

In section 2.4.2, we have solved the general super-hedging problem in one step for an Asian option. The next step is to repeat the whole procedure to deduce backwardly

the infimum prices and the associated super-hedging strategy from the maturity date T to the starting date $t = 0$. It will show the relevance of the conditional closure.

The conditional closure could be also useful more generally for robust finance dynamic programming as in the paper [28]. We conjecture that it is possible to solve a discrete-time stochastic control problem through random set conditioning.

At last, some interesting problems leave open in the direction of conditional topologies: see [27]. A deeper study of the basic properties of the conditional closure and interior of random sets may be interesting with a comparison to the classical results of topology but also with the paper by Truffert [58]. This also allows to consider new types of martingales, see [36], and, in continuous time, new problems should arise.

Chapter 3

Robust discrete-time super-hedging strategies under AIP condition and under price uncertainty

3.1 Introduction

As observed in practice, the executed value of an asset may depend on the order sent by the trader and, also, on the quantities available in the order book. Among the possible causes of the well-known slippage phenomenon, delays in the execution of the orders, liquidity disorders, market impacts, or transaction costs may influence the executed value. An approach to overcome this difficulty is to assume that we do not know in advance the traded prices. In that case, as proposed in the chapter, the order that the trader sends is a mapping that associates to each possible price available in the market a quantity to sell or buy. This is exactly what we generally observe in practice, in a presence of an order book for example, since there is no single price.

On the contrary, it is traditional in mathematical finance to suppose that we first observe a (new) single market price and, then, we choose almost instantaneously the number of assets to sell or buy in order to revise the portfolio. This means that the last traded price is kept constant long enough in the order book. Moreover, it coincides with a bid and ask price so that the buy and sell orders are executed at the same value.

In real life, there may be delayed information, see the recent paper [1] or [50], [55] among others on stochastic control. The delayed information in the problem of pricing is sometimes modeled through incomplete or restricted information as in [38], [43], [25], [20] or using a two filtrations setting as in [19] or in [26].

Another type of uncertainty is due to the choice of the model supposed to approximate the real financial market [8]. Model risk may lead to price miscalculations

that are studied in recent papers, in the growing field of robust finance. Since the seminal work of Knight [44], it is now broadly accepted that uncertainty may be described by a parametrized family of models, instead of considering only one model, if there is a lack of information on the parameters, see [51], [15], [48], [7], [6], [30], [57]. Other models consider that the market is driven by a family of probability measures in such a way that uncertainty stems from the existence of several possible reference probability measures determining which events are negligible, see [52], [37], [18], [13], [9], [14], [49], [17].

In any case, uncertainty is taken into account in the literature by considering either several probabilistic structures, e.g. a family of reference probability measures and filtrations for the same price process or a family of price process models on the same stochastic basis. In the recent paper [53], the choice is made to fix only one filtered probability space on which a collection of stochastic processes describes the possible dynamics of the stock prices. We follow this alternative approach. Precisely, we consider a unique stochastic basis but we suppose that, in discrete time, the next stock prices at any time are not modeled by a unique vector-valued random variable as it is usual to do. Instead, we assume that the next stock prices belong to a collection of possible processes. The approach we adopt is slightly different from [53] in the sense that the collections of possible prices we consider are connected from time to time in such a way that it is possible to represent them through measurable random sets.

Moreover, a less common type of uncertainty is introduced in this chapter. Recall that it is usual in the literature, even in the recent papers on robust finance, to suppose that the transactions are executed at a price which is known in advance. For example, in the Black and Scholes model, the delta-hedging strategy for the European Call option at time t is a function $\Phi(t, S_t)$ of the single price S_t observed at time t . In practice, the strategy is discretized at some dates $(t_i)_{i=0, \dots, n}$ with $n \rightarrow +\infty$ so that the number of stocks to trade at time t_i is $\Delta\Phi_{t_i} = \Phi(t_i, S_{t_i}) - \Phi(t_{i-1}, S_{t_{i-1}})$. In the case where $\Delta\Phi_{t_i} < 0$, the executed price at time t_i should be a bid price in the order book and an ask price otherwise, i.e. there should be at least two possible prices.

We take into account this ambiguity or uncertainty in our chapter by assuming that there may be several possible executable prices at the next instant. This means in particular that we do not know in advance the price when we send an order to be executed. Precisely, an executed price S_t at time t is only \mathcal{F}_{t+1} -measurable where \mathcal{F}_t describes the market information available at time t . This is illustrated in our numerical example where the stock price is modeled by a pair of bid and ask prices.

This chapter addresses the super-hedging problem of European or Asian options under uncertainty and may be easily adapted to American options in discrete time. Here the uncertainty mainly refers to the uncertainty in executed prices due to the delay, which is modeled by random sets, and there is one single physical probability measure. Moreover, uncertainty may also refer to the presence of an order book so

that several prices may exist and depend on the traded volumes.

CARNET D'ORDRES 

ORDRES	QTÉ	ACHAT	VENTE	QTÉ	ORDRES
1	100	104.100	104.980	50	1
11	155	104.000	105.000	6	1
1	11	103.960	106.000	1147	4
1	1	103.900	106.400	25	2
1	50	103.800	106.500	16	2
1	125	103.600	106.600	22	1
1	1	103.500	106.760	3	1
1	29	103.400	106.800	470	2
6	107	103.000	106.900	127	4
1	50	102.900	106.980	21	1
25	629	TOTAL	TOTAL	1 887	19

Figure 3.1: Vinci order book

CARNET D'ORDRES 

ORDRES	QTÉ	ACHAT	VENTE	QTÉ	ORDRES
1	3	124.700	125.500	605	2
1	2	124.580	126.000	370	14
3	25	124.500	126.140	2	1
1	4	124.400	126.360	20	1
1	9	124.000	126.480	4	1
1	10	123.320	126.500	1 388	32
1	1	123.200	126.560	7	1
1	500	123.100	126.580	2	1
8	384	123.000	126.600	277	6
1	30	122.900	126.620	20	1
19	968	TOTAL	TOTAL	2 695	60

Figure 3.2: Airbus order book

Figure 3.3: Examples of order books

The advantage of the approach we consider is its flexibility, including a large variety of possible models, e.g. with transaction costs or limit order books. Contrarily to the classical approach, we do not suppose the existence of a risk-neutral probability measure but we work under the AIP condition of [3], [16], i.e. we suppose that the super-hedging prices of the non-negative European claims are non-negative, as it is easily observed in the real financial market. We recall that the AIP condition is weaker than the usual NA condition but it is sufficient to deduce numerically tractable pricing estimations, as illustrated in our numerical example.

The chapter first focuses on the one-period case, see Section 3.3.1, and the multi-period case is automatically obtained by (measurably) paste all periods together. The one-period hedging problem can be described as:

$$V_{t-1} + \theta_{t-1} \Delta S_t \geq g_t(S_0, \dots, S_t), \text{ a.s. for all } S_t \in \Lambda_t(S_0^{t-1}).$$

Here S_t is a possible executed price which is \mathcal{F}_{t+1} -measurable, θ_{t-1} is a trading strategy which is made at time $t-1$ and its outcome is revealed at the same time t as S_{t-1} due to execution delay and, thus, V_{t-1} , which models the portfolio value at time $t-1$, is also \mathcal{F}_t -measurable; g is an Asian option to be hedged while $\Lambda_t(S_0^{t-1})$ represents the set of all possible prices S_t that can be traded strictly after time t . We recall that S_0^t denotes the price process $(S_u)_{u \leq t}$, for all $0 \leq t \leq T$. The problem is essentially converted to the one without delay by taking supremum conditioned on \mathcal{F}_t in the above equation, and the (minimal) super-hedging price is provided in Theorem 3.3.1 in terms of the concave envelope of some related function restricted on the conditional closure of $\Lambda_t(S_0^{t-1})$, see [29]. Properties of the hedging price,

including continuity, convexity, and measurability are analyzed in Section 3.3.2. These properties are important to deduce backwardly the multi-period case which involves a measurable pasting.

The benefit of our approach is its easy implementation as illustrated in Section 3.4. Indeed, roughly speaking, our main results state that we only need to know the range of the future price values in terms of the observed prices to deduce the strategy θ_t to be followed. This can be achieved from a historical data. The strategy depends at time t on the price S_t , i.e. $\theta_t = \theta_t(S_t)$ where S_t is only revealed at time $t + 1$ so that the order at time t is the \mathcal{F}_t -measurable mapping $z \mapsto \theta_t(z)$ and not $\theta_t(S_t)$. Note that the executed price S_t will depend on the model, e.g. S_t may be one of the several bid and ask prices, and the delayed observation of S_t at time $t + 1$ allows to deduce the quantity $\theta_t(S_t)$ to hold in the portfolio.

3.2 Formulation of the problem

Let $(\Omega, (\mathcal{F}_t)_{t \in \{0, \dots, T+1\}}, \mathcal{F}_T, P)$ be a filtered complete probability space where T is the time horizon. We suppose that \mathcal{F}_0 is the trivial σ -algebra and the σ -algebra \mathcal{F}_t represents the information available on the market at time t . The financial market we consider is composed of d risky assets and a bond S^0 . We assume without loss of generality that $S^0 = 1$.

In the following, we shall consider random subsets A of \mathbb{R}^d , i.e. $A = A(\omega)$ may depend on $\omega \in \Omega$. We then denote by $L^0(A, \mathcal{F}_t)$ the set of all random variables X_t which are \mathcal{F}_t -measurable and satisfies $X_t(\omega) \in A(\omega)$ a.s.. At last, \mathbb{R}_+^d is the set of all $x = (x_i)_{i=1}^d \in \mathbb{R}^d$ such that $x_i \geq 0$ for all $i = 1, \dots, d$.

Let us consider, for each $t \leq T + 1$, $\Lambda_t \subseteq L^0(\mathbb{R}_+^d, \mathcal{F}_{t+1})$ a collection of \mathcal{F}_{t+1} -measurable random variables representing the possible executable prices for the risky assets between time t and time $t + 1$. We suppose that, at time t , the set Λ_t may depend on the observed traded prices before time t , i.e. to each vector of prices S_0^{t-1} , we associate a set $\Lambda_t = \Lambda_t(S_0^{t-1})$ representing the possible next prices S_t after time t given that we have observed the executed prices S_0^{t-1} at time t . We adopt the financial principle that the executed price S_t is only known strictly after the order is sent at time t but before time $t + 1$.

Definition 3.2.1. *A price process is an $(\mathcal{F}_{t+1})_{t=-1, \dots, T-1}$ -measurable non-negative process $(S_t)_{t=-1, \dots, T}$ such that $S_t \in \Lambda_t(S_0^{t-1})$ is \mathcal{F}_{t+1} measurable for all $t = 0, \dots, T$ and $S_{-1} \in \mathbb{R}$ is given.*

Example 3.2.2. Recall that S_t represents the prices (S_t^1, \dots, S_t^d) of $d \geq 1$ risky assets proposed by the market to the portfolio manager when selling or buying. A typical case could be $\Lambda_t = L^0(I_t, \mathcal{F}_{t+1})$ with $\mathcal{F}_{t+1} = \sigma(S_u, u \leq t)$ for all $t \geq 1$ and

$$I_t = \Pi_{j=1}^d [S_t^{bj}, S_t^{aj}],$$

where $(S^{bj})_{j=1, \dots, d}$ and $(S^{aj})_{j=1, \dots, d}$ are respectively the bid and the ask price processes observed in the market between time t and $t + 1$ that may depend on S_0^{t-1} .

They are not necessarily the best bid/ask prices as, in practice, the real transaction price may be a convex combination of bid and ask prices. Indeed, a transaction is generally the result of an agreement between sellers and buyers but it also depends on the traded volume. Clearly, the portfolio manager does not benefit in general from the last traded price observed in the market when sending an order. On the contrary, he should face an uncertain price S_t that depends on the type of order (and may be not executed) but it also depends on some random events he does not control, e.g. slippage. A simple way to model this phenomenon is to suppose that the executable prices obtained by the manager belong to random intervals.

Example 3.2.3. Another interesting case is when $\Lambda_t = \{S_t^\theta : \theta \in \Theta\}$ is a parametrized family of random variables. For instance, consider fixed processes $(\xi_u)_{u \leq T}$ and $(m_u)_{u \leq T}$ adapted to $(\mathcal{F}_{t+1})_{t=0, \dots, T}$ and independent of \mathcal{F}_t . Let C be a compact subset of \mathbb{R} and suppose that S_{-1} is given. We define recursively

$$\Lambda_t(S_0^{t-1}) = \{S_{t-1} \exp(\sigma \xi_t + m_t) : \sigma \in C\}, S_{t-1} \in \Lambda_{t-1}, t \leq T,$$

where $\Theta = \{\sigma \in C\}$. In this model, there is an uncertainty on prices because of the unknown parameter (volatility) σ . This is a classical problem in robust finance, see for example [48].

A portfolio strategy is an $(\mathcal{F}_{t+1})_{t=-1, \dots, T}$ -adapted process $\hat{\theta} = (\theta^0, \theta)$ where, for all $t = 0, \dots, T$, $\theta_t \in \mathbb{R}^d$ (resp. $\theta_t^0 \in \mathbb{R}$) describes the quantities of risky assets (resp. the bond) held in the portfolio between time t and time $t + 1$. Since the strategies are not supposed to be adapted to $(\mathcal{F}_t)_{t=0, \dots, T}$ but only adapted to $(\mathcal{F}_{t+1})_{t=0, \dots, T}$, the manager is not supposed to control the quantity of assets he wants to sell or buy. This is what happens in practice because the orders are not necessarily executed, for instance in the case of limit stock market orders. Precisely, the portfolio manager may send an \mathcal{F}_t -measurable order at time t that depends on the uncertain price S_t which is only \mathcal{F}_{t+1} measurable. For instance, such an order could be *Buy at most 1000 units at a price less than or equal to 145 euros* so that the strategies and the executed prices are linked. In the example, the executed quantity should be deduced from an order book as the minimum between 1000 and the number of assets we may obtain for a price less than 145. Then, the executed price is a weighted average of all prices available for less than 145 in the order book.

For such a strategy $\hat{\theta} = (\theta^0, \theta)$, we define the portfolio process with initial endowment $V_0 \in L^0(\mathbb{R}, \mathcal{F}_1)$, as the liquidation value

$$V^{\hat{\theta}} = \theta^0 + \theta S = \theta^0 + \sum_{i=1}^d \theta^i S^i.$$

Recall that S_t is observed strictly after the portfolio manager sends an order for θ_t at time t . In the super-hedging problem we solve, we expect orders which are mapping $x \mapsto \Delta \theta_t(x) = \theta_t(x) - \theta_{t-1}(S_0^{t-1})$ where $\Delta \theta_t(x)$ is \mathcal{F}_t -measurable and the

executed quantity $\Delta\theta_t(S_t)$ is only \mathcal{F}_{t+1} -measurable since S_t is \mathcal{F}_{t+1} -measurable. Here the notation xy is used to designate the Euler scalar product between two vectors x, y of \mathbb{R}^d .

In the following, we only consider self-financing portfolio processes $V^{\hat{\theta}}$, i.e. they satisfy by definition:

$$\Delta V_t^{\hat{\theta}} := V_t^{\hat{\theta}} - V_{t-1}^{\hat{\theta}} = \theta_{t-1} \Delta S_t,$$

where $\Delta S_t := S_t - S_{t-1}$. Indeed, this dynamics holds if and only if we have $-(\theta_t^0 - \theta_{t-1}^0)S_t^0 = (\theta_t - \theta_{t-1})S_t$. This means that the cost of the new portfolio allocation (θ_t^0, θ_t) , i.e. buying or selling the quantities $(|\theta_t^i - \theta_{t-1}^i|)_{i=0}^d$, at the executed price S_t is charged to the cash account. Therefore,

$$V_t^{\hat{\theta}} = V_0 + \sum_{u=1}^t \theta_{u-1} \Delta S_u. \quad (3.2.1)$$

It is then natural by (3.2.1) to write $V^\theta = V^{\hat{\theta}}$.

The aim of the chapter is to solve the following problem: Construct the minimal super-hedging strategy of an Asian option whose payoff is $g(S_0, \dots, S_T)$ for some convex deterministic function g on $(\mathbb{R}^d)^{T+1}$. Because of price uncertainty, this means that we shall construct a self-financing strategy θ and we shall determine the minimal initial endowment $V_0 = V_0^\theta$ such that we have $V_T^\theta \geq g(S_0, S_1, \dots, S_T)$ independently of the value of the executable prices $S_t \in \Lambda_t(S_0^{t-1})$ are for $t \leq T$. Note that V_t is \mathcal{F}_{t+1} -measurable hence one more step is necessary to deduce the initial endowment P_0 at time $t = 0$ we need for initiating a super-hedging portfolio process V , i.e. $P_0 \geq V_0$. Indeed, P_0 should be \mathcal{F}_0 -measurable, i.e. a constant, or equivalently $P_0 \geq \text{esssup}_{\mathcal{F}_0}(V_0)$. We refer to [16] for the definitions of conditional essential supremum and infimum.

3.3 The super-hedging problem

3.3.1 The one time step resolution

We first introduce the basic tools and theoretical results we need in this section. A set Λ of measurable random variables is said \mathcal{F} -decomposable if for any finite partition $(F_i)_{i=1, \dots, n} \subseteq \mathcal{F}$ of Ω , and for every family $(\gamma_i)_{i=1, \dots, n}$ of Λ , we have $\sum_{i=1}^n \gamma_i 1_{F_i} \in \Lambda$. In the following, we denote by $\Sigma(\Lambda)$ the \mathcal{F} -decomposable envelope of Λ , i.e. the smallest \mathcal{F} -decomposable family containing Λ . Notice that

$$\Sigma(\Lambda) = \left\{ \sum_{i=1}^n \gamma_i 1_{F_i} : n \geq 1, (\gamma_i)_{i=1, \dots, n} \subseteq \Lambda, (F_i)_{i=1, \dots, n} \subseteq \mathcal{F} \text{ s.t. } \sum_{i=1}^n F_i = \Omega \right\}.$$

The closure $\overline{\Sigma}(\Lambda)$ in probability of $\Sigma(\Lambda)$ is decomposable even if Λ is not decomposable. By [45, Theorem 2.4], there exists a \mathcal{F} -measurable closed random set $\sigma(\Lambda)$ such that $\overline{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$ is the set of all \mathcal{F} -measurable selectors of $\sigma(\Lambda)$.

We now introduce the general one step problem between the dates $t - 1$ and t for $t \geq 1$. To do so, we suppose that after time $t - 1$ but strictly before time t the portfolio manager observes the price S_{t-1} , as a consequence of her/his order, see Definition 3.2.1. More precisely, the portfolio manager knows S_0^{t-2} at time $t - 1$ and sends an order at time $t - 1$ which is executed with a delay so that the executed price $S_{t-1} \in \Lambda_{t-1}(S_0^{t-2})$ is only observed strictly after $t - 1$, i.e. S_{t-1} is \mathcal{F}_t -measurable.

In the following, we consider the σ -algebra $\mathcal{F}_t = \sigma(S_u : u \leq t - 1)$ for all $t \geq 1$. Let us consider a random function g_t defined on $(\mathbb{R}^d)^{t+1}$, $t \geq 1$. We assume that the mapping $(\omega, z) \mapsto g_t(S_0(\omega), \dots, S_{t-1}(\omega), z)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and $z \mapsto g_t(S_0, S_1, \dots, S_{t-1}, z)$ is lower-semicontinuous (l.s.c.) almost surely independently the price process S_0^{t-1} is. The first goal is to characterise the set \mathcal{P}_{t-1} of all $V_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_t)$ that depend on S_0^{t-1} such that:

$$V_{t-1} + \theta_{t-1} \Delta S_t \geq g_t(S_0, \dots, S_t), \text{ a.s. for all } S_t \in \Lambda_t(S_0^{t-1}), \quad (3.3.2)$$

for some $\theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ ¹. As θ_{t-1} is only \mathcal{F}_t -measurable, we also expect a dependence between θ_{t-1} and S_0^{t-1} as we shall see later. Nevertheless, we do not suppose an explicit dependence of $\Lambda_t(S_0^{t-1})$ with respect to θ_{t-1} , which is an open problem. We observe by lower-semicontinuity that (3.3.2) holds if and only if

$$V_{t-1} + \theta_{t-1} \Delta S_t \geq g_t(S_0, \dots, S_t), \text{ a.s. for all } S_t \in \bar{\Sigma}(\Lambda_t(S_0^{t-1})). \quad (3.3.3)$$

This means that we may suppose w.l.o.g. that $\bar{\Sigma}(\Lambda_t(S_0^{t-1})) = \Lambda_t(S_0^{t-1})$. In the following, we denote by $I_t(S_0^{t-1})$ the \mathcal{F}_{t+1} -measurable closed random set such that $\bar{\Sigma}(\Lambda_t(S_0^{t-1})) = L^0(I_t(S_0^{t-1}), \mathcal{F}_{t+1})$, see [45, Theorem 2.4].

By [29, Theorem 4.2], we deduce that (3.3.2) is equivalent to $V_{t-1} \geq p_{t-1}$ where $p_{t-1} = p_{t-1}(S_0^{t-1}, \theta_{t-1})$ is given by

$$\begin{aligned} p_{t-1} &= \theta_{t-1} S_{t-1} + \sup_{z \in \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)} (g_t(S_0, \dots, S_{t-1}, z) - \theta_{t-1} z), \\ &= \theta_{t-1} S_{t-1} + f_{t-1}^*(-\theta_{t-1}). \end{aligned}$$

In the formula above, $\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$ is the conditional closure of $I_t(S_0^{t-1})$, i.e. the smallest \mathcal{F}_t -measurable closed random set which contains $I_t(S_0^{t-1})$ almost surely. We refer the readers to [29, Theorem 3.1] for the existence and uniqueness of such conditional random set. Moreover, $f_{t-1}^*(y) = \sup_{z \in \mathbb{R}^d} (yz - f_{t-1}(z))$ is the Fenchel-Legendre conjugate function of f_{t-1} defined as

$$f_{t-1}(z) := -g_t(S_0, \dots, S_{t-1}, z) + \delta_{\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)}(z), \quad (3.3.4)$$

where $\delta_{\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)} \in \{0, \infty\}$ is infinite on the complimentary of $\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$ and 0 otherwise. Notice that f_{t-1}^* is convex and l.s.c. as a supremum (on $\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$) of convex and l.s.c. functions. Moreover, by [29, Theorem 4.2], $(\omega, y) \mapsto f_{t-1}^*(\omega, y)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. Therefore, $\text{Dom } f_{t-1}^* :=$

¹Note that the condition $V_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_t)$ is not sufficient for the portfolio manager to observe it when $t = 1$ as V_0 is not \mathcal{F}_0 -measurable.

$\{y : f_{t-1}^*(\omega, y) < \infty\}$ is an \mathcal{F}_t -measurable random set. We deduce that the \mathcal{F}_t -measurable prices at time $t - 1$ are given by the Minkowski sum

$$\begin{aligned} \mathcal{P}_{t-1}(S_0^{t-1}) &= \{\theta_{t-1}S_{t-1} + f_{t-1}^*(-\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_t)\} \\ &\quad + L^0(\mathbb{R}_+, \mathcal{F}_t). \end{aligned} \quad (3.3.5)$$

The second step is to determine the infimum super-hedging price as

$$p_{t-1}(S_0^{t-1}) = \text{essinf}_{\mathcal{F}_t} \mathcal{P}_{t-1}(S_0^{t-1}). \quad (3.3.6)$$

To do so, we use the arguments of [16, Theorem 2.8] and we obtain our first main result:

Theorem 3.3.1. *Suppose that the mapping $(\omega, z) \mapsto g_t(S_0(\omega), \dots, S_{t-1}(\omega), z)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and $z \mapsto g_t(S_0, S_1, \dots, S_{t-1}, z)$ is lower-semicontinuous (l.s.c.) almost surely whatever the price process S_0^{t-1} is. Let us consider the function f_t defined by (3.3.4) and the set of all prices given by (3.3.5). Then, the infimum price given by (3.3.6), satisfies $p_{t-1}(S_0^{t-1}) = -f_{t-1}^{**}(S_{t-1})$.*

Proof. This is a consequence of the following chain of equalities:

$$\begin{aligned} p_{t-1}(S_0^{t-1}) &= \text{essinf}_{\mathcal{F}_t} \{\theta_{t-1}S_{t-1} + f_{t-1}^*(-\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_t)\}, \\ &= \text{essinf}_{\mathcal{F}_t} \{-\theta_{t-1}S_{t-1} + f_{t-1}^*(\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_t)\}, \\ &= -\text{esssup}_{\mathcal{F}_t} \{\theta_{t-1}S_{t-1} - f_{t-1}^*(\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_t)\}, \\ &= -\text{esssup}_{\mathcal{F}_t} \{\theta_{t-1}S_{t-1} - f_{t-1}^*(\theta_{t-1}) : \theta_{t-1} \in L^0(\text{Dom } f_{t-1}^*, \mathcal{F}_t)\}, \\ &= -\sup_{z \in \overline{\text{Dom } f_{t-1}^*}} (zS_{t-1} - f_{t-1}^*(z)), \\ &= -\sup_{z \in \mathbb{R}^d} (zS_{t-1} - f_{t-1}^*(z)), \\ &= -f_{t-1}^{**}(S_{t-1}). \end{aligned} \quad (3.3.7)$$

□

Note that we do not need to suppose no-arbitrage conditions to establish the very general pricing formula above. It is only based on the lower-semicontinuity and measurability assumptions satisfied by the payoff g .

3.3.2 Main properties satisfied by the one time step infimum super-hedging price

In this section we introduce the main contribution. They are needed to propagate the one time step pricing procedure of Section 3.3.1 to the multi-period case. In the following, we suppose that, for all price process S_0^{t-1} , there exists $\alpha_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ and $\beta_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_t)$ that may depend on S_0^{t-1} such that

$$g_t(S_0, \dots, S_{t-1}, x) \leq \alpha_{t-1}x + \beta_{t-1}, \quad \forall x \in \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t). \quad (3.3.8)$$

This is the case for Asian options whose payoffs are for example of the form $k(S_0 + S_1 + \dots + S_t - K)^+$, $k \geq 0$. By [16][Theorem 2.8], we know that

$$p_{t-1}(S_0^{t-1}) = \inf \{ \alpha S_{t-1} + \beta : \alpha x + \beta \geq g_t(S_0, \dots, S_{t-1}, x), \forall x \in \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t) \}. \quad (3.3.9)$$

We first establish the following result:²

Proposition 3.3.2. *Let S_0^{t-1} be a price process. Suppose that the mapping $(\omega, z) \mapsto g_t(S_0(\omega), \dots, S_{t-1}(\omega), z)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and the function $z \mapsto g_t(S_0, \dots, S_{t-1}, z)$ is l.s.c. almost surely. If $S_{t-1} \notin \overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$, then $p_{t-1}(S_0^{t-1}) = -\infty$. Moreover, $p_{t-1}(S_0^{t-1}) \geq g_t(S_0, \dots, S_{t-1}, S_{t-1})$ if $S_{t-1} \in \overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$. At last, if $g_t(S_0, \dots, S_{t-1}, \cdot)$ is bounded from below by $m_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_t)$ on $\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$, then we have $p_{t-1}(S_0^{t-1}) \geq m_{t-1}$ if $S_{t-1} \in \overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$.*

Proof. Suppose that $S_{t-1} \notin \overline{\text{conv}} \text{cl}(I_t|\mathcal{F}_t)$ where $I_t = I_t(S_0^{t-1})$. By the Hahn-Banach separation theorem and a measurable selection argument, there exists a non null α_{t-1}^* in $L^0(\mathbb{R}^d \setminus \{0\}, \mathcal{F}_t)$ and $c_{t-1}^1, c_{t-1}^2 \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ such that we have the inequality $\alpha_{t-1}^* y < c_{t-1}^1 < c_{t-1}^2 < \alpha_{t-1}^* S_{t-1}$ for all $y \in \text{cl}(I_t|\mathcal{F}_t)$. Multiplying the inequality by a sufficiently large positive multiplier, we may suppose that $\alpha_{t-1}^*(S_{t-1} - y) \geq n$ where $n \in \mathbb{N}$ is arbitrarily chosen. Let us introduce $\tilde{\alpha}_{t-1} = \alpha_{t-1} - \alpha_{t-1}^*$ and $\tilde{\beta}_{t-1}^n = \beta_{t-1} + \alpha_{t-1}^* S_{t-1} - n$, $n \geq 1$. By construction, $\alpha_{t-1} x + \beta_{t-1} \leq \tilde{\alpha}_{t-1} x + \tilde{\beta}_{t-1}^n$ for all $x \in \text{cl}(I_t|\mathcal{F}_t)$, where $\alpha_{t-1}, \beta_{t-1}$ are given in (3.3.8). It follows that $\tilde{\alpha}_{t-1} x + \tilde{\beta}_{t-1}^n \geq g_t(S_0, \dots, S_{t-1}, x)$, for every $x \in \text{cl}(I_t|\mathcal{F}_t)$. By (3.3.9), we deduce that $p_{t-1} \leq \tilde{\alpha}_{t-1} S_{t-1} + \tilde{\beta}_{t-1}^n$, i.e. $p_{t-1} \leq \alpha_{t-1} + \beta_{t-1} - n$. As $n \rightarrow \infty$, we deduce that $p_{t-1} = -\infty$.

Suppose that $z \mapsto g_t(S_0, \dots, S_{t-1}, z)$ is a.s. convex and, furthermore, $S_{t-1} \in \overline{\text{conv}} \text{cl}(I_t|\mathcal{F}_t)$. By (3.3.9),

$$p_{t-1}(S_0^{t-1}) \geq g_t(S_0, \dots, S_{t-1}, S_{t-1}).$$

At last, suppose that $z \mapsto g_t(S_0, S_1, \dots, S_{t-1}, z)$ is bounded from below by $m_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_t)$ on $\text{cl}(I_t|\mathcal{F}_t)$ and $S_{t-1} \in \overline{\text{conv}} \text{cl}(I_t|\mathcal{F}_t)$. Then, $S_{t-1} = \lim_{n \rightarrow \infty} S_n$ where $S_n \in \text{conv} \text{cl}(I_t|\mathcal{F}_t)$, i.e. $S_n = \sum_{i=1}^{J_n} \lambda_{i,n} x_{i,n}$ where $\lambda_{i,n} \geq 0$ with $\sum_{i=1}^{J_n} \lambda_{i,n} = 1$ and $x_{i,n} \in \text{cl}(I_t|\mathcal{F}_t)$ for all i, n . Consider (α, β) such that $\alpha x + \beta \geq g_t(S_0, \dots, S_{t-1}, x)$

²The notation $\overline{\text{conv}}(A)$ designates the closed convex hull of A , i.e. the smallest convex closed set containing A .

for all $x \in \text{cl}(I_t|\mathcal{F}_t)$. Then, $\alpha S_{t-1} + \beta = \lim_{n \rightarrow \infty} (\alpha S_n + \beta)$ with

$$\begin{aligned} \alpha S_n + \beta &= \sum_{i=1}^{J_n} \lambda_{i,n} (\alpha x_{i,n} + \beta) \geq \sum_{i=1}^{J_n} \lambda_{i,n} g_t(S_1, \dots, S_{t-1}, x_{i,n}) \\ &\geq m_{t-1}. \end{aligned}$$

We deduce that $\alpha S_{t-1} + \beta \geq m_{t-1}$ hence $p_{t-1} \geq m_{t-1}$ by (3.3.9). \square

Corollary 3.3.3. *Let S_0^{t-1} be a price process. Suppose that the mapping $(\omega, z) \mapsto g_t(S_0(\omega), \dots, S_{t-1}(\omega), z)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and the function $z \mapsto g_t(S_0, \dots, S_{t-1}, z)$ is l.s.c. a.s. and convex or bounded from below by $m_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_t)$ on $\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$. Then, $p_{t-1}(S_0^{t-1}) \neq -\infty$ if and only if $S_{t-1} \in \overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$. In particular, the infimum super-hedging price of any non negative payoff function is finite if and only if it is non negative or equivalently if $S_{t-1} \in \overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$.*

As studied in [16], the non negativity of the prices for the zero claim or more generally for non negative European call options corresponds to a weak no arbitrage condition (AIP) which is naturally observed in practice. Adapted to our setting, we introduce the following definition:

Definition 3.3.4. *We say that condition AIP holds between $t-1$ and t if the prices at time $t-1$ of the time t zero claim is non negative for every price process $(S_u)_{u \leq t-1}$ of the model. Moreover, we say that the condition AIP holds when AIP holds at any time step.*

As observed in [16] and above, when AIP fails, the infimum of the zero claim, and more generally of non negative payoffs, may be $-\infty$. In that case, the numerical procedure we develop in this chapter is still valid but unrealistic and non-implementable in practice. By Corollary 3.3.3, we have:

Corollary 3.3.5. *The condition AIP holds between $t-1$ and t if and only if $S_{t-1} \in \overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$ for any price process S_0^{t-1} , i.e.*

$$I_{t-1}(S_0^{t-2}) \subseteq \overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t), \quad t \geq 1.$$

In the following, if g is a function defined on \mathbb{R}^d and D is a subset of \mathbb{R}^d , we denote by $\text{conc}(g, D)$ the (relative) concave envelope of g on D , i.e. the smallest concave function defined on \mathbb{R}^d which dominates g only on D . Observe that $g \leq h$ on D is equivalent to $g - \delta_D \leq h$ on \mathbb{R}^d . Therefore, $\text{conc}(g, D)$ always exists as soon as g is dominated by an affine function on D .

The following result allows us to compute the infimum price rather easily.

Lemma 3.3.6. *Let S_0^{t-1} be a price process. Suppose that the mapping $(\omega, z) \mapsto g_t(S_0(\omega), \dots, S_{t-1}(\omega), z)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and the function $z \mapsto g_t(S_0, \dots, S_{t-1}, z)$ is l.s.c. almost surely. Consider the concave envelope*

$$h_{t-1}(x) = \text{conc}(g_t(S_0, \dots, S_{t-1}, \cdot), \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t))(x).$$

Then,

$$\begin{aligned} p_{t-1}(S_0^{t-1}) & \\ &= \inf \{ \alpha S_{t-1} + \beta : \alpha x + \beta \geq h_{t-1}(x), \text{ for all } x \in \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t) \}. \end{aligned} \quad (3.3.10)$$

Proof. By definition, h_{t-1} is the smallest concave function which dominates g . We deduce that the set of all affine functions dominating g coincides with the set of all affine functions dominating h_{t-1} . By (3.3.9) we deduce that (3.3.10) holds. \square

The following result provides a criterion under which the infimum price is a price:

Proposition 3.3.7. *Suppose that AIP holds. Let S_0^{t-1} be a price process. Suppose that the mapping $(\omega, z) \mapsto g_t(S_0(\omega), \dots, S_{t-1}(\omega), z)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and $z \mapsto g_t(S_0, \dots, S_{t-1}, z)$ is l.s.c. almost surely. Moreover, suppose that there exists $\alpha_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ and $\beta_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_t)$ such that $g_t(S_0, \dots, S_{t-1}, z) \leq \alpha_{t-1}z + \beta_{t-1}$ for all $z \in \overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$ and consider the concave envelope*

$$h_{t-1}(x) = \text{conc}(g_t(S_0, \dots, S_{t-1}, \cdot), \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t))(x). \quad (3.3.11)$$

We have $p_{t-1}(S_0^{t-1}) \in [g_t(S_0, \dots, S_{t-1}, S_{t-1}), \alpha_{t-1}S_{t-1} + \beta_{t-1}]$. Moreover, if the super-differential $\partial h_{t-1}(S_{t-1}) \neq \emptyset$, then $p_{t-1}(S_0^{t-1}) = h_{t-1}(S_{t-1})$ is a price, i.e. $p_{t-1}(S_0^{t-1}) \in \mathcal{P}_{t-1}(S_0^{t-1})$ with the super-replicating strategies $\theta_{t-1} \in \partial h_{t-1}(S_{t-1})$.

Proof. It is clear by Lemma 3.3.6 that $p_{t-1}(S_0^{t-1}) \geq h(S_{t-1})$ when S_{t-1} belongs to $\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$. By definition, for all $r_{t-1} \in \partial h(S_{t-1}) \neq \emptyset$, for all $x \in \overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$,

$$h(x) \leq h(S_{t-1}) + r_{t-1}(x - S_{t-1}) =: \delta(r_{t-1}, x). \quad (3.3.12)$$

Therefore, $p_{t-1}(S_0^{t-1}) \leq \delta(r_{t-1}, S_{t-1}) = h(S_{t-1})$, and finally

$$p_{t-1}(S_0^{t-1}) = h(S_{t-1}).$$

At last, applying (3.3.12) with $x = S_t \in I_t(S_0^{t-1}) \subseteq \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$, we deduce that

$$p_{t-1}(S_0^{t-1}) + r_{t-1}\Delta S_t \geq h(S_t) \geq g_t(S_0, \dots, S_{t-1}, S_t).$$

Since $x \mapsto g_t(S_0, \dots, S_{t-1}, x)$ is l.s.c., we consider the following random set:

$$\begin{aligned} G_t &:= \{(\omega, r_{t-1}) : \delta(r_{t-1}, x) \geq g_t(S_0, \dots, S_{t-1}, x), \forall x \in \overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)\}, \\ &= \{(\omega, r_{t-1}) : \delta(r_{t-1}, \gamma_t^n) \geq g_t(S_0, \dots, S_{t-1}, \gamma_t^n), \forall n \in \mathbb{N}\}, \end{aligned}$$

where $(\gamma_t^n)_{n \geq 1}$ is a Castaing representation of $\overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$. Since G_t is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and $G_t \neq \emptyset$ a.s., it admits a measurable selection which is a measurable strategy θ_t for the price $p_{t-1}(S_0^{t-1})$. \square

Remark 3.3.8. *As the function h_{t-1} in (3.3.11) is concave and finite a.s. on the conditional closure $\overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$, see proof of Proposition 3.3.2, the super-differential $\partial h(S_{t-1})$ of h_{t-1} at the point S_{t-1} is not empty when S_{t-1} belongs to the interior of $\overline{\text{conv}} \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$.*

The following result proves the measurability of the infimum super-hedging price $p_{t-1}(S_0^{t-1})$ with respect to S_0^{t-1} . To do so, we suppose the existence of a Castaing representation, see [54], [45].

Proposition 3.3.9. *Suppose that $\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$ admits a Castaing representation $(\xi_t^m)_{m \geq 1}$ where $\xi_t^m = x^m(S_0^{t-1})$, for all $m \geq 1$, and x^m are Borel functions on $(\mathbb{R}^d)^t$ independent of S_0^{t-1} . Then, there exist a Borel function ϕ_{t-1} on $(\mathbb{R}^d)^t$ such that $p_{t-1}(S_0^{t-1}) = \phi_{t-1}(S_0^{t-1})$.*

Proof. Let S_0^{t-1} be a price process. We denote by

$$\mathcal{S}^{(t-1)} = S_0^{t-1} \text{ and } \mathcal{I}_{t-1} = \text{cl}(I_t(\mathcal{S}^{(t-1)})|\mathcal{F}_t).$$

Recall that

$$p_{t-1}(\mathcal{S}^{(t-1)}) = \inf_{(\alpha, \beta)} \{ \alpha S_{t-1} + \beta : \alpha x + \beta \geq g_t(\mathcal{S}^{(t-1)}, x), \text{ for all } x \in \mathcal{I}_{t-1} \}.$$

By assumption x^m is a Borel function on $(\mathbb{R}^d)^t$ independent of the price process S_0^{t-1} . So:

$$\begin{aligned} p_{t-1}(\mathcal{S}^{(t-1)}) &= \inf_{(\alpha, \beta)} \{ \alpha S_{t-1} + \beta : \alpha x^m(\mathcal{S}^{(t-1)}) + \beta \geq g_t(\mathcal{S}^{(t-1)}, x^m(\mathcal{S}^{(t-1)})), \forall m \} \\ &= \inf_{\alpha} \{ \alpha S_{t-1} + f_{t-1}^*(-\alpha, \mathcal{S}^{(t-1)}) \} \end{aligned}$$

such that $f_{t-1}^*(-\alpha, \mathcal{S}^{(t-1)}) = \sup_m [g_t(\mathcal{S}^{(t-1)}, x^m(\mathcal{S}^{(t-1)})) - \alpha x^m(\mathcal{S}^{(t-1)})]$.

Let us denote $\mathbf{Q}^d = \{ \alpha^n = (\alpha_1^n, \dots, \alpha_d^n), n \geq 1, \alpha_i^n \in \mathbf{Q} \}$ and define the real-valued mapping ϕ_{t-1} as $\phi_{t-1}(\mathcal{S}^{(t-1)}) = \inf_n \{ \alpha^n S_{t-1} + f_{t-1}^*(-\alpha^n, \mathcal{S}^{(t-1)}) \}$. We claim that

$$p_{t-1}(\mathcal{S}^{(t-1)}) = \phi_{t-1}(\mathcal{S}^{(t-1)}). \quad (3.3.13)$$

It is clear that $p_{t-1}(\mathcal{S}^{(t-1)}) \leq \phi_{t-1}(\mathcal{S}^{(t-1)})$. Conversely, let $\alpha \in \mathbb{R}^d$, and $\alpha^n \in \mathbf{Q}^d$ a sequence such that for arbitrary fixed $\epsilon \in \text{int}(\mathbb{R}_+^d)$, we have $\alpha^n \geq \alpha$ and $\alpha > \alpha^n - \epsilon$ componentwise. Then, by definition of f_{t-1}^* , we have:

$$\begin{aligned} f_{t-1}^*(-\alpha, \mathcal{S}^{(t-1)}) &\geq g_t(\mathcal{S}^{(t-1)}, x^m(\mathcal{S}^{(t-1)})) - \alpha x^m(\mathcal{S}^{(t-1)}), \forall m \geq 1 \\ &\geq g_t(\mathcal{S}^{(t-1)}, x^m(\mathcal{S}^{(t-1)})) - \alpha^n x^m(\mathcal{S}^{(t-1)}) \\ &\quad + (\alpha^n - \alpha) x^m(\mathcal{S}^{(t-1)}), \forall m \geq 1. \end{aligned}$$

Notice that $x^m(\mathcal{S}^{(t-1)}) \in \mathbb{R}_+^d$ because $x^m(\mathcal{S}^{(t-1)}) \in \mathcal{I}_{t-1}$. So,

$$\begin{aligned} f_{t-1}^*(-\alpha, \mathcal{S}^{(t-1)}) &\geq g_t(\mathcal{S}^{(t-1)}, x^m(\mathcal{S}^{(t-1)})) - \alpha^n x^m(\mathcal{S}^{(t-1)}), \forall m \geq 1, \forall n \geq 1 \\ &\geq f_{t-1}^*(-\alpha^n, \mathcal{S}^{(t-1)}), \forall n \geq 1. \end{aligned}$$

Hence,

$$\begin{aligned}
\alpha S_{t-1} + f_{t-1}^*(-\alpha) &\geq \alpha S_{t-1} + f_{t-1}^*(-\alpha^n), \quad \forall n \geq 1 \\
&\geq \alpha^n S_{t-1} + f_{t-1}^*(-\alpha^n) - \epsilon S_{t-1}, \quad \forall n \geq 1 \\
&\geq \alpha^n S_{t-1} + f_{t-1}^*(-\alpha^n) - \epsilon S_{t-1}, \quad \forall n \geq 1 \\
&\geq \phi_{t-1}(\mathcal{S}^{(t-1)}) - \epsilon S_{t-1}.
\end{aligned}$$

As $\epsilon \rightarrow 0$, we get $\alpha S_{t-1} + f_{t-1}^*(-\alpha) \geq \phi_{t-1}(\mathcal{S}^{(t-1)})$. Therefore, we deduce that $p_{t-1}(\mathcal{S}^{(t-1)}) \geq \phi_{t-1}(\mathcal{S}^{(t-1)})$. Hence, the equality (3.3.13) holds, which proves that the infimum super-hedging price $p_{t-1}(S_0^{t-1})$ is measurable with respect to the argument S_0^{t-1} . \square

The rest of this section aims to prove that, under some technical conditions, the mapping $S_0^{t-1} \mapsto p_{t-1}(S_0^{t-1})$ is lower-semicontinuous, which is needed to propagate backwardly the numerical procedure of Theorem 3.3.6 in the multi-step model.

Definition 3.3.10. *We say that the mapping*

$$I_t : S_0^{t-1} \mapsto \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$$

is lower-semicontinuous if the following property holds: For all sequence of price processes $((S_u^n)_{u \leq t-1})_{n \geq 1}$ converging a.s. to a process S_0^{t-1} , and for all $z \in \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$, there exists a sequence $(z^n)_{n \geq 1}$ such that $\lim_n z^n = z$ and $z^n \in \text{cl}(I_t((S_u^n)_{u \leq t-1})|\mathcal{F}_t)$ for all $n \geq 1$.

Example 3.3.11. Suppose that $d = 1$ and

$$\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t) = [m_{t-1}S_{t-1}, M_{t-1}S_{t-1}]$$

where $m_{t-1}, M_{t-1} \in L^0(\mathbb{R}_+, \mathcal{F}_t)$ and $m_{t-1} \leq M_{t-1}$.

Consider $z \in \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$, i.e. $z = \alpha_t m_{t-1} S_{t-1} + (1 - \alpha_t) M_{t-1} S_{t-1}$ where $\alpha_t \in L^0([0, 1], \mathcal{F}_t)$. Let us define $z^n = \alpha_t m_{t-1} S_{t-1}^n + (1 - \alpha_t) M_{t-1} S_{t-1}^n$ for all $n \geq 1$. Then, $z^n \in \text{cl}(I_t((S_u^n)_{u \leq t-1})|\mathcal{F}_t)$ and

$$|z^n - z| \leq 2M_{t-1}|S_{t-1}^n - S_{t-1}|$$

hence $\lim_n z^n = z$.

In the following, we define the closed convex random sets

$$E_{t-1}^\epsilon(S_0^{t-1}, z) = \bar{B}(0, \epsilon) \cap (\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t) - z),$$

where $\bar{B}(0, \epsilon)$ is the closed ball of center $z = 0$ and radius $\epsilon > 0$. We say that the mapping $z \mapsto E_{t-1}^\epsilon(S_0^{t-1}, z)$ is convex if, for all $\alpha \in [0, 1]$, and $z_1, z_2 \in \mathbb{R}^d$, we have

$$E_{t-1}^\epsilon(S_0^{t-1}, \alpha z_1 + (1 - \alpha) z_2) \subseteq \alpha E_{t-1}^\epsilon(S_0^{t-1}, z_1) + (1 - \alpha) E_{t-1}^\epsilon(S_0^{t-1}, z_2).$$

Note that this convexity property above is automatically satisfied if $d = 1$.

Proposition 3.3.12. Consider a payoff function g_t defined on $(\mathbb{R}^d)^{t+1}$ such that, there exists $\alpha_{t-1} \in L^0((\mathbb{R}^d)^{t+1}, \mathcal{F}_t)$ such that $g_t(x) - g_t(y) \geq \alpha_{t-1}(x - y)$, $x, y \in (\mathbb{R}^d)^{t+1}$. Suppose that $I_t : S_0^{t-1} \mapsto \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$ is lower-semicontinuous and that $z \mapsto E_{t-1}^\epsilon(S_0^{t-1}, z)$ is convex for all S_0^{t-1} . Then, $S_0^{t-1} \mapsto p_{t-1}(S_0^{t-1})$ is lower-semicontinuous, i.e. $p_{t-1}(S_0^{t-1}) \leq \liminf_n p_{t-1}((S_u^n)_{u \leq t-1})$ if $((S_u^n)_{u \leq t-1})_{n \geq 1}$ converges a.s. to S_0^{t-1} .

Proof. Suppose that $((S_u^n)_{u \leq t-1})_{n \geq 1}$ converges a.s. to S_0^{t-1} . By assumption, we know that for all $z \in \text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$, there exists a sequence $z^n \in \text{cl}(I_t((S_u^n)_{u \leq t-1})|\mathcal{F}_t)$ such that $\lim_n z^n = z$. We may suppose that $|z - z^n| \leq \epsilon$ where $\epsilon > 0$ is arbitrarily fixed. By assumption, for all $\tilde{z} \in \text{cl}(I_t((S_u^n)_{u \leq t-1})|\mathcal{F}_t)$ in the ball $\bar{B}(z, \epsilon)$ of center z and radius ϵ , we have:

$$\begin{aligned} g_t(S_0^{t-1}, z) &\leq g_t((S_u^n)_{u \leq t-1}, \tilde{z}) + |\alpha_{t-1}| \times |(S_0^{t-1}, z) - ((S_u^n)_{u \leq t-1}, \tilde{z})|, \\ g_t(S_0^{t-1}, z) &\leq g_t((S_u^n)_{u \leq t-1}, \tilde{z}) + |\alpha_{t-1}| \sup_{u \leq t-1} |S_u^n - S_u| + |\alpha_{t-1}| \epsilon, \\ g_t(S_0^{t-1}, z) &\leq h^{(n)}(\tilde{z}) + |\alpha_{t-1}| \sup_{u \leq t-1} |S_u^n - S_u| + |\alpha_{t-1}| \epsilon, \end{aligned} \quad (3.3.14)$$

where $h^{(n)}$ is an arbitrary affine function satisfying $h^{(n)} \geq g_t((S_u^n)_{u \leq t-1}, \cdot)$ on $\text{cl}(I_t((S_u^n)_{u \leq t-1})|\mathcal{F}_t)$. Let us define

$$\bar{h}^{(n)}(z) = \inf_{\tilde{z} \in \bar{B}(z, \epsilon) \cap \text{cl}(I_t((S_u^n)_{u \leq t-1})|\mathcal{F}_{t-1})} h^{(n)}(\tilde{z}) + |\alpha_{t-1}| \sup_{u \leq t-1} |S_u^n - S_u| + |\alpha_{t-1}| \epsilon.$$

By convention, we set $\inf \emptyset = -\infty$. Let us show that $\bar{h}^{(n)}$ is concave. To see it, observe that $\tilde{z} \in \bar{B}(z, \epsilon) \cap \text{cl}(I_t((S_u^n)_{u \leq t-1})|\mathcal{F}_t)$ if and only if $\tilde{z} = z + u$ where $u \in E^n(z) = \bar{B}(0, \epsilon) \cap (\text{cl}(I_t((S_u^n)_{u \leq t-1})|\mathcal{F}_t) - z)$. Therefore,

$$\bar{h}^{(n)}(z) = \inf_{u \in E^n(z)} h^{(n)}(z + u) + |\alpha_{t-1}| \sup_{u \leq t-1} |S_u^n - S_u| + |\alpha_{t-1}| \epsilon.$$

Let $z = \lambda z_1 + (1 - \lambda) z_2$. We only need to consider the case where $E^n(z_1) \neq \emptyset$ and $E^n(z_2) \neq \emptyset$. We deduce that $E^n(z) \neq \emptyset$. Moreover, by assumption, any $u \in E^n(z)$ may be written as $u = \alpha u_1 + (1 - \alpha) u_2$ where $u_i \in E^n(z_i)$, $i = 1, 2$. Therefore,

$$\begin{aligned} h^{(n)}(z + u) &= \alpha h^{(n)}(z_1 + u_1) + (1 - \alpha) h^{(n)}(z_2 + u_2), \\ &\geq \alpha \bar{h}^{(n)}(z_1) + (1 - \alpha) \bar{h}^{(n)}(z_2). \end{aligned}$$

Taking the infimum in the left hand side of the inequality above, we deduce that $\bar{h}^{(n)}(\lambda z_1 + (1 - \lambda) z_2) \geq \alpha \bar{h}^{(n)}(z_1) + (1 - \alpha) \bar{h}^{(n)}(z_2)$, i.e. $\bar{h}^{(n)}$ is concave.

By (3.3.14), we deduce that $p_{t-1}(S_0^{t-1}) \leq \bar{h}^{(n)}(S_t)$ for all $h^{(n)}$. As $S_{t-1}^n \in E^n(S_{t-1})$, for n large enough, under AIP, we deduce that

$$p_{t-1}(S_0^{t-1}) \leq h^{(n)}(S_{t-1}^n) + |\alpha_{t-1}| \sup_{u \leq t-1} |S_u^n - S_u| + |\alpha_{t-1}| \epsilon.$$

Taking the infimum over all affine functions $h^{(n)}$, we get that for n large enough:

$$p_{t-1}(S_0^{t-1}) \leq p_{t-1}((S_u^n)_{u \leq t-1}) + |\alpha_{t-1}| \sup_{u \leq t-1} |S_u^n - S_u| + |\alpha_{t-1}| \epsilon.$$

As ϵ is arbitrarily chosen, we may conclude that

$$p_{t-1}(S_0^{t-1}) \leq \liminf_n p_{t-1}((S_u^n)_{u \leq t-1}).$$

□

3.3.3 Case of a convex payoff function

We shall prove that $p_{t-1}(S_0^{t-1})$ is a convex function of the price process S_0^{t-1} if so Λ_{t-1} is. In the following, we say that the mapping

$$\Lambda_{t-1} : S_0^{t-1} \longmapsto \Lambda_{t-1}(S_0^{t-1}) := \overline{\text{conv}}(\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t))$$

is convex for the inclusion if, for $\lambda \in [0, 1]$,

$$\Lambda_{t-1}((\lambda(S_0^{t-1}) + (1 - \lambda)(\tilde{S}_u)_{u \leq t-1}) \subseteq \lambda \Lambda_{t-1}(S_0^{t-1}) + (1 - \lambda) \Lambda_{t-1}((\tilde{S}_u)_{u \leq t-1}),$$

for all price process $S_0^{t-1}, (\tilde{S}_u)_{u \leq t-1}$.

Proposition 3.3.13. *Suppose that the mapping*

$$(\omega, z) \mapsto g_t(S_0, S_1(\omega), \dots, S_{t-1}(\omega), z) \text{ is } \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d) \text{ measurable,}$$

non negative and

$$z \mapsto g_t(S_0, S_1, \dots, S_{t-1}, z) \text{ is lower semi-continuous and convex almost surely}$$

and suppose that the mapping $\Lambda_{t-1} : S_0^{t-1} \longmapsto \Lambda_{t-1}(S_0^{t-1})$ is convex. Then, the mapping $S_0^{t-1} \mapsto p_{t-1}(S_0^{t-1})$ is convex .

Proof. Let $\tilde{S}_0^{t-1}, \bar{S}_0^{t-1}$ be two price processes. Let us define the following price process $S_0^{t-1} = \lambda \bar{S}_0^{t-1} + (1 - \lambda) \tilde{S}_0^{t-1}$ for $\lambda \in [0, 1]$. We consider the following random sets:

$$\begin{aligned} \Lambda_{t-1} &= \overline{\text{conv}}(\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)), t \geq 1, \\ \tilde{\Lambda}_{t-1} &= \overline{\text{conv}}(\text{cl}(I_t(\tilde{S}_0^{t-1})|\mathcal{F}_t)), t \geq 1, \\ \bar{\Lambda}_{t-1} &= \overline{\text{conv}}(\text{cl}(I_t(\bar{S}_0^{t-1})|\mathcal{F}_t)), t \geq 1. \end{aligned}$$

By assumption, we have $\Lambda_{t-1} \subseteq \lambda \bar{\Lambda}_{t-1} + (1 - \lambda) \tilde{\Lambda}_{t-1}$ for $\lambda \in [0, 1]$. Let \bar{h} and \tilde{h} be two affine functions such that:

$$\bar{h}(\bar{x}) \geq g_t(\bar{S}_0^{t-1}, \bar{x}), \forall \bar{x} \in \bar{\Lambda}_{t-1}.$$

$$\tilde{h}(\tilde{x}) \geq g_t(\tilde{S}_0^{t-1}, \tilde{x}), \quad \forall \tilde{x} \in \tilde{\Lambda}_{t-1}.$$

Thus, for $\lambda \in]0, 1[$, we have

$$\begin{aligned} \lambda \bar{h}(\bar{x}) + (1 - \lambda) \tilde{h}(\tilde{x}) &\geq \lambda g_t(\bar{S}_0^{t-1}, \bar{x}) + (1 - \lambda) g_t(\tilde{S}_0^{t-1}, \tilde{x}) \\ &\geq g_t(\lambda \bar{S}_0^{t-1} + (1 - \lambda) \tilde{S}_0^{t-1}, \lambda \bar{x} + (1 - \lambda) \tilde{x}). \end{aligned}$$

Let $x \in \Lambda_{t-1}$ such that $x = \lambda \bar{x} + (1 - \lambda) \tilde{x}$. By above, we have:

$$\lambda \bar{h}(\bar{x}) + (1 - \lambda) \tilde{h}(\tilde{x}) \geq g_t(S_0^{t-1}, x) =: \hat{g}_t(x).$$

Now, let us consider

$$E_x = \left\{ \frac{\lambda - 1}{\lambda} \tilde{\Lambda}_{t-1} + \frac{1}{\lambda} x, \lambda \in]0, 1[\right\} \cap \bar{\Lambda}_{t-1}.$$

Observe that $\alpha E_{x_1} + (1 - \alpha) E_{x_2} = E_{\alpha x_1 + (1 - \alpha) x_2}$ for all $\alpha \in [0, 1]$, and $x_1, x_2 \in \mathbb{R}^d$. Then, with $x = \alpha x_1 + (1 - \alpha) x_2$, any $\bar{x} \in E_x$ may be written as $\bar{x} = \alpha \bar{x}_1 + (1 - \alpha) \bar{x}_2$, where $\bar{x}_i \in E_{x_i}$, $i = 1, 2$. As $(x, \bar{x}) \mapsto \tilde{h}(\frac{1}{1 - \lambda}(x - \lambda \bar{x}))$ is affine, we deduce that

$$\begin{aligned} \lambda \bar{h}(\bar{x}) + (1 - \lambda) \tilde{h}\left(\frac{1}{1 - \lambda}(x - \lambda \bar{x})\right) &\geq \alpha \left(\lambda \bar{h}(\bar{x}_1) + (1 - \lambda) \tilde{h}\left(\frac{1}{1 - \lambda}(x_1 - \lambda \bar{x}_1)\right) \right) \\ &\quad + (1 - \alpha) \left(\lambda \bar{h}(\bar{x}_2) + (1 - \lambda) \tilde{h}\left(\frac{1}{1 - \lambda}(x_2 - \lambda \bar{x}_2)\right) \right), \end{aligned}$$

$$\lambda \bar{h}(\bar{x}) + (1 - \lambda) \tilde{h}\left(\frac{1}{1 - \lambda}(x - \lambda \bar{x})\right) \geq \alpha \hat{h}(x_1) + (1 - \alpha) \hat{h}(x_2),$$

where $\hat{h}(x) = \inf_{\bar{x} \in E_x} \{ \lambda \bar{h}(\bar{x}) + (1 - \lambda) \tilde{h}(\frac{1}{1 - \lambda}(x - \lambda \bar{x})) \}$. Therefore, taking the infimum in the right side of the inequality above, we deduce that \hat{h} is a (non negative) concave function with finite values. So, it is continuous and we have $\hat{h}(x) \geq \hat{g}_t(x)$ for all $x \in \Lambda_{t-1}$. We deduce that

$$\begin{aligned} p_{t-1}(S_0^{t-1}) &\leq \hat{h}(S_{t-1}) \\ &\leq \lambda \bar{h}(\bar{S}_{t-1}) + (1 - \lambda) \tilde{h}(\tilde{S}_{t-1}), \quad \forall \bar{S}_{t-1} \in \bar{\Lambda}_{t-1}, \tilde{S}_{t-1} \in \tilde{\Lambda}_{t-1}. \end{aligned}$$

Taking the infimum over all the affine functions \bar{h} and \tilde{h} , we deduce that

$$p_{t-1}(S_0^{t-1}) \leq \lambda p_{t-1}(\bar{S}_0^{t-1}) + (1 - \lambda) p_{t-1}(\tilde{S}_0^{t-1})$$

and the conclusion follows. \square

Remark 3.3.14. Suppose that the AIP condition holds and that (3.3.8) holds. Consider $\phi_{t-1}(u) = \inf_n \{ \alpha^n u_{t-1} + f_{t-1}^*(-\alpha^n, u) \}$, $u = (u_0, \dots, u_{t-1}) \in (\mathbb{R}^d)^t$, where $f_{t-1}^*(-\alpha, u) = \sup_m [g_t(u, x^m(u)) - \alpha x^m(u)]$. Recall that, by Proposition 3.3.9,

$$p_{t-1}(S_0^{t-1}) = \phi_{t-1}(S_0^{t-1}).$$

When g_t is convex, then ϕ_{t-1} is convex by Proposition 3.3.13. Moreover, if $g_t \geq 0$, $0 \leq \phi_{t-1} < \infty$ by Proposition 3.3.7. Then, $\text{dom } \phi_{t-1} = (\mathbb{R}^d)^t$ and we deduce that ϕ_{t-1} is continuous on $(\mathbb{R}^d)^t$.

Remark 3.3.15. Consider the case $d = 1$. By a measurable selection argument, we may show that there exists $m_{t-1}, M_{t-1} \in L^0([0, \infty], \mathcal{F}_t)$ such that

$$\overline{\text{conv}}(\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)) = [m_{t-1}, M_{t-1}].$$

By Lemma 3.3.6, we deduce that under (AIP)

$$\begin{aligned} p_{t-1}(S_0^{t-1}) &= g_t(S_0, \dots, S_{t-1}, m_{t-1}) \\ &+ \frac{g_t(S_0, \dots, S_{t-1}, M_{t-1}) - g_t(S_0, \dots, S_{t-1}, m_{t-1})}{M_{t-1} - m_{t-1}}(S_{t-1} - m_{t-1}). \end{aligned} \quad (3.3.15)$$

Moreover, the strategy is given by

$$\theta_{t-1} = \frac{g_t(S_0, \dots, S_{t-1}, M_{t-1}) - g_t(S_0, \dots, S_{t-1}, m_{t-1})}{M_{t-1} - m_{t-1}}.$$

If we suppose that $m_{t-1} = k_{t-1}^d S_{t-1}$ and $M_{t-1} = k_{t-1}^u S_{t-1}$ as in [3], where k_{t-1}^d and k_{t-1}^u are deterministic coefficients, then $p_{t-1}(S_0^{t-1}) = g_{t-1}(S_0^{t-1})$ with

$$g_{t-1}(x_0, \dots, x_{t-1}) = \lambda_{t-1} g_t(x_0, \dots, x_{t-1}, k_{t-1}^d x_{t-1}) + (1 - \lambda_{t-1}) g_t(x_0, \dots, x_{t-1}, k_{t-1}^u x_{t-1}),$$

where $\lambda_{t-1} = \frac{k_{t-1}^u - 1}{k_{t-1}^u - k_{t-1}^d}$ and g_T is the payoff function.

At last, the order to be sent at time t is given by the deterministic mapping defined on \mathbb{R}^t by

$$\theta_{t-1}(s_0, \dots, s_{t-1}) = \frac{g_t(s_0, \dots, s_{t-1}, k_{t-1}^u s_{t-1}) - g_t(s_0, \dots, s_{t-1}, k_{t-1}^d s_{t-1})}{(k_{t-1}^u - k_{t-1}^d) s_{t-1}}.$$

Remark 3.3.16 (Market impact). It is possible in our model to include a market impact. Indeed, it suffices to make the order (demand) mapping $D_t(x) = \theta_t(x) - \theta_{t-1}(S_{t-1})$ coincided at time t with the supply mapping $O_t(x)$, i.e. the available quantity we may buy or sell at price x in the order book. By convention, O_t is negative for bid prices and positive for ask prices. It is an increasing function on \mathbb{R}_+ starting from $O_t(0+) = -\infty$ at price 0 (we can sell as many assets as we want to the market at price 0) and ending up with $O_t(+\infty) = +\infty$, i.e. we can buy as many assets as we want to the market at price $+\infty$. As soon as D_t is bounded, there exists executable bid prices S_t^b in the order book such that $D_t(S_t^b) \geq O_t(S_t^b)$ when $D_t(S_t^b) \leq 0$, i.e. the order may be executed at price S_t^b as the quantity $|D_t(S_t^b)| \leq |O_t(S_t^b)|$. The executed bid price is naturally the best one among all possible. Similarly, there exists executable ask prices S_t^a in the order book such that $D_t(S_t^a) \leq O_t(S_t^a)$ when $D_t(S_t^a) \geq 0$ and the order may be executed at price S_t^a for the quantity $D_t(S_t^a) \leq O_t(S_t^a)$. Note that the executed bid price may be closed to 0 while the executed ask price may be very large. This liquidity phenomenon is then taken into account in the model through the conditional supports allowing to compute the strategy in our approach.

3.3.4 The multistep backward procedure

The main results of Section 3.3.2 for the one step model may be applied recursively, starting from time T , as the payoff function g_T is known.

Consider the case where the conditional support $\text{cl}(I_t(S_0^{t-1})|\mathcal{F}_t)$ admits a Castaing representation $(\xi^m)_{m \geq 1}$ where $\xi^m = x^m(S_0^{t-1})$, for all $m \geq 1$, and x^m are Borel functions on $(\mathbb{R}^d)^t$. Then, by Proposition 3.3.9, we know that the infimum price at time $T-1$ is a Borel function g_{T-1} of the prices S_0, \dots, S_{T-1} . Then, we may repeat the procedure if we are in position to verify that g_{T-1} is also l.s.c. This is the case by Proposition 3.3.13 and Remark 3.3.14, under convexity conditions.

Many questions could be investigated for future research, e.g. sensitivity to modeling assumptions, but also how to calibrate such a model from statistical estimations. Mainly, we need to estimate conditional supports. This is illustrated in the numerical example that we propose in the next section. A technical question is also to consider discontinuous payoff functions even if this is less usual in finance where g is generally a convex function. Actually, by Lemma 3.3.6, we may replace the payoff function by its concave envelope. Note that our analysis is general enough to consider a lot of models, e.g. with order books.

3.4 Numerical illustration

3.4.1 Formulation of the problem with $d = 1$

In this section we consider the example of the European call option at time $T = 2$, i.e. with the payoff function $g(S_2) = (S_2 - K)^+$, $K > 0$. Let $(S_t)_{t=0,1,2}$ be the executed price process. Recall that S_t belongs to the random set Λ_t , for $t = 0, 1, 2$, respectively. We suppose that the risk-free asset is given by $S^0 = 1$. Recall that there exist \mathcal{F}_{t+1} -measurable closed random sets $I_t = I_t(S_0^{t-1})$ such that:

$$\overline{\Sigma}(\Lambda_t(S_0^{t-1})) = L^0(I_t(S_0^{t-1}), \mathcal{F}_{t+1}), \quad t = 0, 1, 2.$$

We may suppose that $\Lambda = \overline{\Sigma}(\Lambda)$ so that $S_t \in I_t$ a.s. for $t = 0, 1, 2$. At each step, we shall apply the procedure we have developed in the sections above. In particular, we seek for the strategy θ and we deduce the portfolio value V associated to the executed price process S . Then, we may estimate the error between the terminal value of V_2 and the payoff $g_2(S_2)$ that we denote by $\epsilon_2 = V_2 - g_2(S_2)$.

We start from a known price S_{-1} at time $t = 0$, which corresponds to the last traded price. We suppose that $I_t = I_t(S_0^{t-1}) = [S_{t-1}m_t, S_{t-1}M_t]$, $t = 0, 1, 2$, where the two random variables m_t and M_t are independent of S_{t-1} and are uniformly distributed as $m_t \sim \mathcal{U}[0.7, 1]$ and $M_t = m_t + spr_t$ such that $spr_t \sim \mathcal{U}[0, 0.4]$ is independent of m_t . Observe that $m_t^- = 0.7$ and $M_t^+ = 1.4$.

At time $t = 0$, we choose in our model to pick randomly S_0 in the interval I_0 . Precisely, $S_0 = S_{-1}m_0 + k_0S_{-1}(M_0 - m_0)$, where k_0 is a random variable such that $k_0 \sim \mathcal{U}[0, 1]$. We make this choice for simplicity and that corresponds to the case

where the bid and ask prices of the market coincide with the mid price S_0 . The order we sent is of the form *buy or sell the quantity $\theta_0(z)$ at the price z* .

At time $t = 1$, we choose to model bid and ask prices S_1^{bid} , S_1^{ask} respectively as: $S_1^{bid} = S_0 m_1$ and $S_1^{ask} = S_0 M_1$ where S_0 is the last executed price. Notice that the order of buying or selling depends on the bid-ask values, see Figure 3.4. We define S_1^* such that $\Delta\theta_1(S_1^*) = 0$. If $S_1^{bid} \leq S_1^{ask} \leq S_1^*$ (in the green zone $\{S_1 : \Delta\theta_1(S_1) \leq 0\}$), then $S_1 = S_1^{bid}$ since $\Delta\theta_1 \leq 0$. If $S_1^* \leq S_1^{bid} \leq S_1^{ask}$, (the yellow zone,) then $S_1 = S_1^{ask}$ as $\Delta\theta_1 > 0$. Otherwise, if $S_1^{bid} < S_1^* < S_1^{ask}$, we may arbitrarily choose $S_1 = S_1^{ask}$ or $S_1 = S_1^{bid}$. In our model, we make the (arbitrary) choice that, if $|S_1^* - S_1^{bid}| \leq |S_1^* - S_1^{ask}|$, then $S_1 = S_1^{ask}$ and $S_1 = S_1^{bid}$ otherwise.

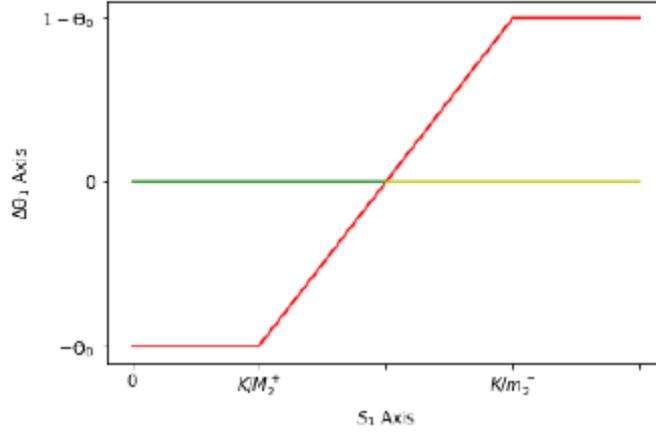


Figure 3.4

At last, we choose $S_2 = S_2^{ask} = S_2^{bid} \in I_2 = [m_2 S_1, M_2 S_1]$ accordingly to the formula $S_2 = S_1 m_2 + k_2 S_1 (M_2 - m_2)$ where k_2 a uniform random variable in the interval $[0, 1]$.

Note that the mapping $s_1 \mapsto \Delta\theta_1(s_1)$ is the \mathcal{F}_1 -measurable order we send at time $t = 1$, see Figure 3.4. The later depends on S_0 , which is \mathcal{F}_1 -measurable.

3.4.2 Explicit computation of the strategy

We deduce the portfolio value and the strategy value at any time by dominating the payoff function by the smallest affine function on the conditional support of S , as mentioned in (3.3.9). We consider the terminal payoff function $g(S_T) = (S_T - K)^+$ for several strikes.

The strategy at time $t = 1$

Recall that $S_2 \in \Lambda_2(S_1) \sim I_2 = [S_1 m_2, S_1 M_2]$. In order to compute the strategy $\theta_1 = \theta_1(S_1)$ we first compute the function φ_1 given by (3.3.9) which dominates the the pay-off function g_2 on the conditional support $\text{cl}(I_2(S_1)|\mathcal{F}_2) = [S_1 m_2^-, S_1 M_2^+]$.

1st case: $K \in [S_1 m_2^-, S_1 M_2^+] \Leftrightarrow S_1 \in [\frac{K}{M_2^+}, \frac{K}{m_2^-}]$.

The dominating affine function φ_1 , see Figure 3.5, is given by:

$$\varphi_1(x) = \frac{(S_1 M_2^+ - K)(x - S_1 m_2^-)}{S_1(M_2^+ - m_2^-)}.$$

So,

$$V_1(S_1) = p_1(S_1) = \varphi_1(S_1) = \frac{(S_1 M_2^+ - K)(1 - m_2^-)}{M_2^+ - m_2^-} =: g_1(S_1),$$

and

$$\theta_1(S_1) = \frac{S_1 M_2^+ - K}{S_1(M_2^+ - m_2^-)}.$$

A simple computation shows that:

$$V_2 = V_1(S_1) + \theta_1(S_1)(S_2 - S_1) = \varphi_1(S_2) \geq g_2(S_2).$$

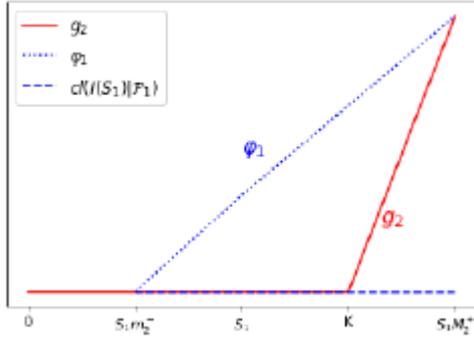


Figure 3.5

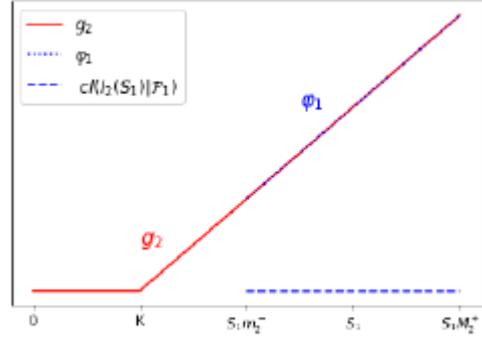


Figure 3.6

2nd case: $K \leq S_1 m_2^- \Leftrightarrow S_1 \geq \frac{K}{m_2^-}$.

In this case, we have $\varphi_1(x) = (x - K)^+$ for all $x \in [S_1 m_2^-, S_1 M_2^+]$, see Figure 3.6. Hence, $V_1(S_1) = (S_1 - K)^+ =: g_1(S_1)$ and $\theta_1(S_1) = 1$.

3rd case: $K \geq S_1 M_2^+ \Leftrightarrow S_1 \leq \frac{K}{M_2^+}$.

Observe that the dominating affine function φ_1 coincides with the x-axis on the support $[S_1 m_2^-, S_1 M_2^+]$, see Figure 3.7. Therefore, $V_1(S_1) = g_1(S_1) := 0$ and we deduce that $\theta_1(S_1) = 0$.

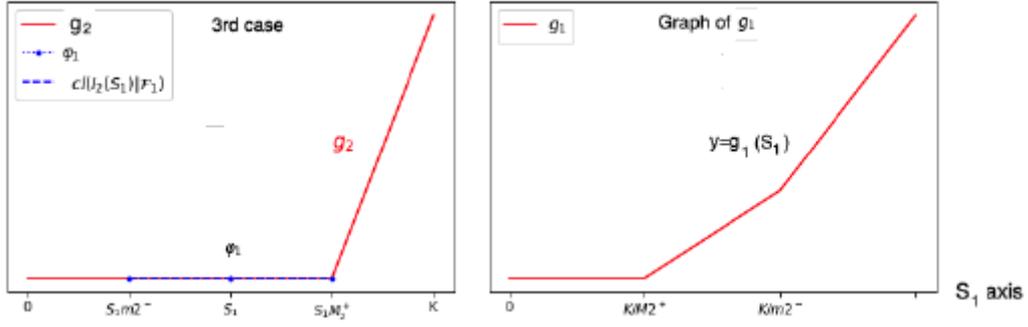


Figure 3.7

We finally deduce that

$$g_1(x) = \frac{(xM_2^+ - K)(1 - m_2^-)}{M_2^+ - m_2^-} 1_{\left[\frac{K}{M_2^+}, \frac{K}{m_2^-}\right]}(x) + (x - K)^+ 1_{\left[\frac{K}{m_2^-}, \infty\right)}(x).$$

The graph of the payoff function g_1 is represented in Figure 3.7.

The strategy at time $t = 0$

In order to determine the strategy θ_0 , we compute the smallest affine function φ_0 that dominates g_1 on the conditional support $cl(I_1(S_0)|\mathcal{F}_0)$.

1st case: $S_0 M_1^+ \leq \frac{K}{M_2^+}$, i.e. $S_0 \leq \frac{K}{M_1^+ M_2^+}$.

We have $V_0(S_0) = g_0(S_0) = 0$ and $\theta_0(S_0) = 0$, see Figure 3.8.

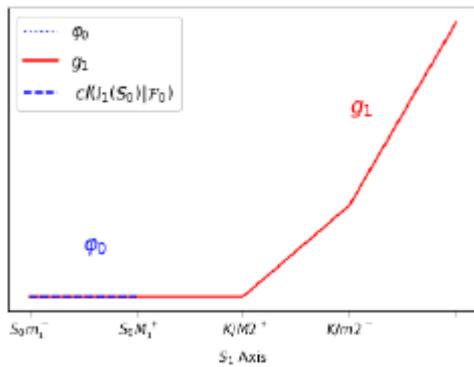


Figure 3.8

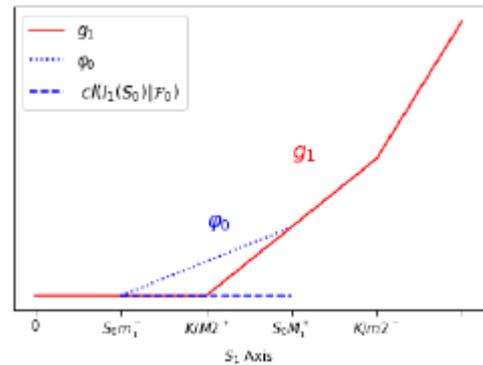


Figure 3.9

2nd case: $S_0 m_1^- \leq \frac{K}{M_2^+}$ and $S_0 M_1^+ \in [\frac{K}{M_2^+}, \frac{K}{m_2^-}]$, i.e. $S_0 \in [\frac{K}{M_1^+ M_2^+}, \frac{K}{m_1^- M_2^+} \wedge \frac{K}{m_2^- M_1^+}]$.

We find that (see Figure 3.9):

$$\varphi_0(x) = \frac{(S_0 M_1^+ M_2^+ - K)(1 - m_2^-)}{S_0(M_1^+ - m_1^-)(M_2^+ - m_2^-)}(x - S_0 m_1^-).$$

So,

$$V_0(S_0) = \varphi_0(S_0) = \frac{(S_0 M_1^+ M_2^+ - K)(1 - m_2^-)(1 - m_1^-)}{(M_2^+ - m_2^-)(M_1^+ - m_1^-)} =: g_0(S_0),$$

and

$$\theta_0(S_0) = \frac{(S_0 M_1^+ M_2^+ - K)(1 - m_2^-)}{S_0(M_2^+ - m_2^-)(M_1^+ - m_1^-)}.$$

3rd case: $S_0 m_1^- \leq \frac{K}{M_2^+}$ and $S_0 M_1^+ \geq \frac{K}{m_2^-}$, i.e. $S_0 \in [\frac{K}{m_2^- M_1^+}, \frac{K}{m_1^- M_2^+}]$.

We have, see Figure 3.10:

$$\varphi_0(x) = \frac{S_0 M_1^+ - K}{S_0(M_1^+ - m_1^-)}(x - S_0 m_1^-).$$

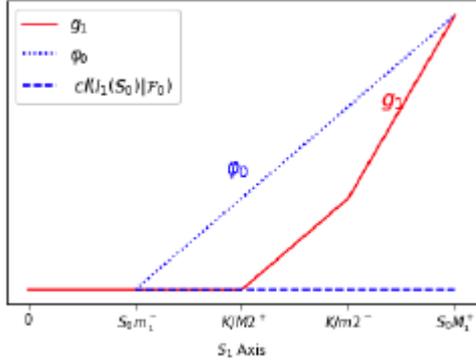


Figure 3.10

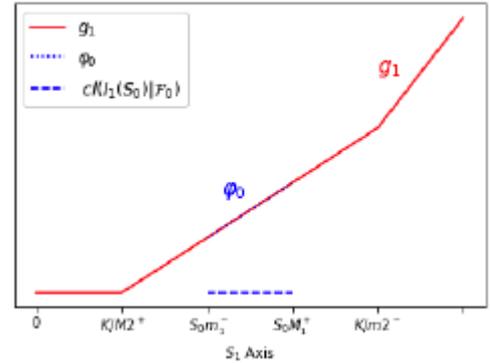


Figure 3.11

So,

$$V_0(S_0) = \varphi_0(S_0) = \frac{(S_0 M_1^+ - K)(1 - m_1^-)}{M_1^+ - m_1^-} =: g_0(S_0), \quad \theta_0(S_0) = \frac{S_0 M_1^+ - K}{S_0(M_1^+ - m_1^-)}.$$

4th case: $S_0 m_1^- \in [\frac{K}{M_2^+}, \frac{K}{m_2^-}]$ and $S_0 M_1^+ \in [\frac{K}{M_2^+}, \frac{K}{m_2^-}]$, i.e. $S_0 \in [\frac{K}{m_1^- M_2^+}, \frac{K}{m_2^- M_1^+}]$.

We have $\varphi_0(x) = g_1(x)$, for all $x \in cl(I_1(S_0)|\mathcal{F}_0)$, see Figure 3.11. Therefore,

$$V_0(S_0) = \varphi_0(S_0) = \frac{(S_0 M_2^+ - K)(1 - m_2^-)}{M_2^+ - m_2^-} =: g_0(S_0), \quad \theta_0(S_0) = \frac{M_2^+(1 - m_2^-)}{M_2^+ - m_2^-}.$$

5th case: $S_0 m_1^- \in [\frac{K}{M_2^+}, \frac{K}{m_2^-}]$ and $S_0 M_1^+ \geq \frac{K}{m_2^-}$, i.e. $S_0 \in [\frac{K}{m_1^- M_2^+} \vee \frac{K}{m_2^- M_1^+}, \frac{K}{m_1^- m_2^-}]$.

We obtain that (see Figure 3.12):

$$\begin{aligned} \varphi_0(x) = & \frac{(S_0 M_1^+ - K)(M_2^+ - m_2^-) - (S_0 m_1^- M_2^+ - K)(1 - m_2^-)}{S_0(M_1^+ - m_1^-)(M_2^+ - m_2^-)} x \\ & + \frac{-m_1^-(S_0 M_1^+ - K)(M_2^+ - m_2^-) + M_1^+(S_0 m_1^- M_2^+ - K)(1 - m_2^-)}{(M_1^+ - m_1^-)(M_2^+ - m_2^-)}. \end{aligned}$$

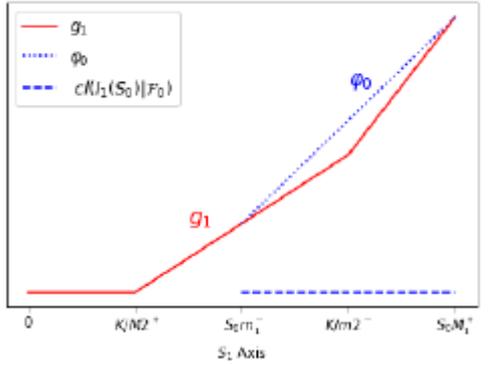


Figure 3.12

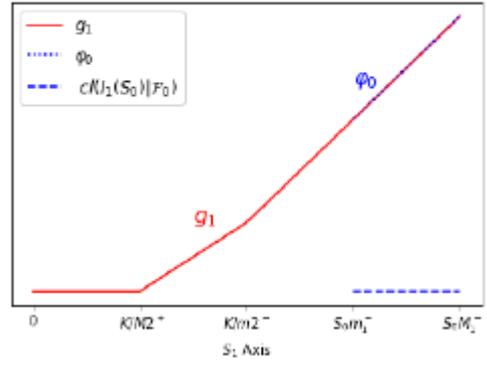


Figure 3.13

Then,

$$\begin{aligned} V_0(S_0) &= \varphi_0(S_0) =: g_0(S_0) \\ &= \frac{(S_0 M_1^+ - K)(M_2^+ - m_2^-)(1 - m_1^-) - (S_0 m_1^- M_2^+ - K)(1 - m_2^-)(1 - M_1^+)}{(M_1^+ - m_1^-)(M_2^+ - m_2^-)} \end{aligned}$$

and

$$\theta_0(S_0) = \frac{(S_0 M_1^+ - K)(M_2^+ - m_2^-) - (S_0 m_1^- M_2^+ - K)(1 - m_2^-)}{S_0(M_2^+ - m_2^-)(M_1^+ - m_1^-)}.$$

6th case: $S_0 m_1^- \geq \frac{K}{m_2^-}$ and $S_0 M_1^+ \geq \frac{K}{m_2^-}$, i.e. $S_0 \geq \frac{K}{m_2^- m_1^-}$.

We have $V_0(S_0) = (S_0 - K)^+ =: g_0(S_0)$ and $\theta_0(S_0) = 1$, see Figure 3.13.

3.4.3 Empirical results

For an observed price S_{-1} at time $t = 0$ (which corresponds to the last traded price), and for different strike values K , we test the infimum super-hedging strategy

by computing the relative error ϵ_R from a data set of 10^6 simulated prices S_t for $t \in 0, 1, 2$. To do so, we wrote a script in Python. The relative error is given by

$$\epsilon_R = \frac{V_2 - (S_2 - K)^+}{S_2}.$$

In the following table 3.14, empirical results are presented for different values of the strike K and a sample of 10^6 scenarios.

K	50	75	100	125	150
$E(S_0)$	95.002	94.983	95.006	94.98	95.001
$E(S_1)$	99.56	94.94	87.085	82.104	81.736
$E(S_2)$	94.56	90.180	82.716	78.01	77.664
$E(V_0)$	46.503	29.357	16.960	11.244	6.7
$\max V_0$	89.677	66.72	49.726	33.05	22.562
$E(V(S_0)/S_{-1})$	0.465	0.294	0.170	0.112	0.067
$E(V(S_0)/S_0)$	0.483	0.300	0.173	0.114	0.066
$\min(V(S_0)/S_0)$	0.359	0.163	0.098	0.032	0
$\max(V(S_0)/S_0)$	0.642	0.479	0.358	0.237	0.162
$E(\epsilon_R)$	0.017	0.077	0.076	0.064	0.039
$\sigma(\epsilon_R)$	0.024	0.045	0.04	0.037	0.0317
$\min \epsilon_R$	0	$2.23 * 10^{-6}$	$1,9 * 10^{-7}$	$5.975 * 10^{-8}$	0
$\max(\epsilon_R)$	0.18	0.19	0.195	0.187	0.187
$E(\theta_0 S_0/V_0)$	199%	255%	322%	333%	313%
$E(\theta_1 S_1/V_1)$	205%	230%	134%	32%	3%

Figure 3.14: The empirical results.

We observe that the executed prices depend on the strike $K > 0$, i.e. there is a market impact of the orders on the prices. Indeed, as expected, the orders we send depend on the payoff function. As K increases, the payoff decreases and, as expected, the option price V_0 decreases. The distribution of S_1 admits two regimes as seen in Figure 3.16 that correspond to the bid and ask prices.

Notice that the proportion of the portfolio value invested in the risky assets at time $t = 1$ decreases as the payoff decreases. We also observe that this proportion decreases (resp. increases) when the price S decreases (resp. increases) between time $t = 0$ and $t = 1$, i.e. when $\Delta S_1 < 0$ (resp. $\Delta S_1 \geq 0$). At last, the empirical results obtained for the relative error confirm the efficiency of the super-hedging strategy, see Figure 3.18.

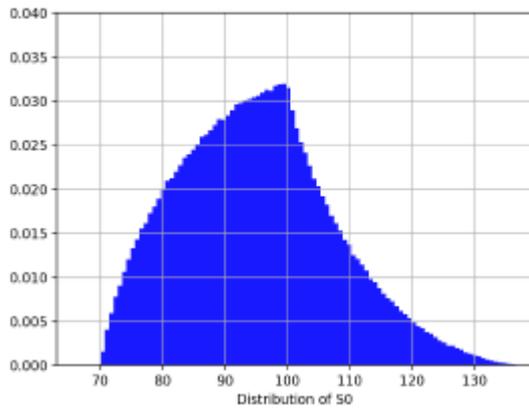


Figure 3.15

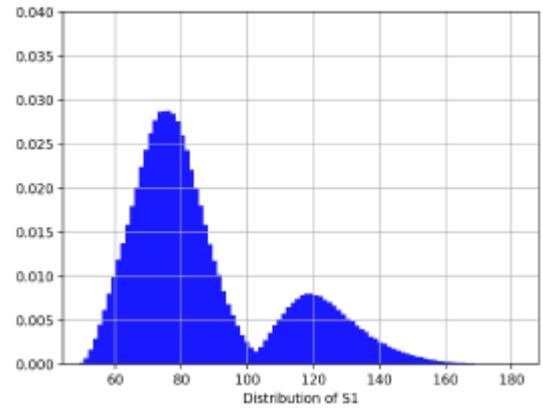


Figure 3.16: $K=100$.

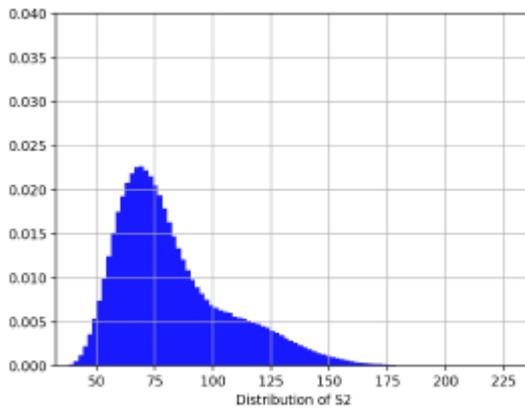


Figure 3.17: $K=100$.

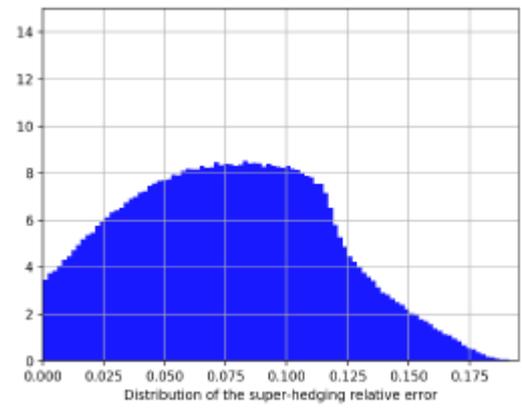


Figure 3.18: $K=100$.

Chapter 4

Super-hedging an arbitrary number of European options with integer-valued strategies

4.1 Introduction

The problem of super-hedging a European claim, such as a call option, is very classical in mathematical finance but has only been solved for real-valued strategies so that the optimal strategy, corresponding to the minimal hedging or super-hedging price, is not integer-valued contrary to what it is allowed to do in the real life. In this joint work with D.Cherif and E.Lepinette we propose to solve the problem of super-hedging a European claim with integer-valued financial strategies.

Actually, the main contribution in the literature for integer-valued financial strategies is the paper [31] where a finite set of states Ω is considered. In this setting, the authors adapt the usual theory for the real-valued strategies to the integer-valued ones, i.e. they introduce a no-arbitrage condition which is equivalent to the existence of a risk-neutral probability measure and show that the minimal super-hedging price may be characterized through the martingale measures similarly to the usual case with real-valued strategies. See also [59] where Ω is finite. The general case with an arbitrary state space Ω is still an open problem. Also, portfolio optimization problems of Markowitz type are considered in [10], [12], [5].

Let us recall that the usual approach of pricing assumes that the financial market model satisfies a no-arbitrage condition NA, which is equivalent to the existence of a risk-neutral probability measure Q under which the discounted asset prices are martingales, see the Dalang-Morton-Willinger theorem [21]. Under NA, we may show that there exists a minimal super-hedging price $P_0^*(\xi_T)$ for the European claim $\xi_T \geq 0$ given by

$$P_0^*(\xi_T) = \sup_{Q \in \mathcal{M}(P)} E_Q(\xi_T), \quad (4.1.1)$$

where $\mathcal{M}(P)$ is the set of all risk-neutral probability measures equivalent to the initial probability measure P of the model. Here, we suppose that the risk-free interest rate of the model is $r = 0$. Recall that the formula above holds in discrete time but also in continuous time with extra-conditions on the model. Indeed, the no-arbitrage condition needs to be strengthened and it is only equivalent to the existence of $Q \sim P$ under which discounted asset prices are local martingales, see [22], [23], [24].

In any case, the optimal strategy that achieves the minimal super-hedging price (4.1.1) is not, in general, integer-valued. The typical example is the continuous-time Black and Scholes model where the so-called delta-hedging strategy for the European call is explicit and lies in the set $[0, 1] \setminus \{0, 1\}$, see [11].

Clearly, a new approach is necessary to compute the super-hedging prices for only integer-valued financial strategies. We follow the ideas developed in [16] where the problem is initially solved without any no-arbitrage conditions. Then, a no-arbitrage condition AIP naturally appears and means that the infimum price of the zero claim (non negative claims more generally) is not $-\infty$. This condition is clearly necessary for numerical purposes. Actually, it is shown that AIP is equivalent to the property that the infimum super-hedging price of any non negative claim is non negative, as observed in the real markets. In this chapter, we do not explicitly suppose such a no-arbitrage condition but the form of the conditional supports of the asset price we assume implies this condition for the model with integer-valued strategies.

Our aim is to compute super-hedging prices with only integer-valued financial strategies. We restrict ourselves to payoffs which are piecewise affine functions of the underlying asset and we assume specific conditional supports for the asset prices. Problems such as characterizations of the no-arbitrage condition NA with only integer-valued strategies or generalization of our work to arbitrary conditional supports of the asset prices remains open if Ω is not finite.

Notations If $A \subset \mathbb{R}$ is a random subset of \mathbb{R} and \mathcal{F} is a σ -algebra, we denote by $\mathbb{L}^0(A, \mathcal{F})$ the family of all \mathcal{F} -measurable random variables X on the probability space (Ω, \mathcal{F}, P) such that $X \in A$ a.s..

4.2 Formulation of the problem

We consider $n \geq 1$ options that we want to super-replicate in discrete time $t = \{0, \dots, T\}$. Let $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T})$ be a stochastic basis where $(\mathcal{F}_t)_{0 \leq t \leq T}$ is supposed to be complete. We consider a financial market model composed of two assets. We suppose, without loss of generality, that the risk-free asset is $S_t^0 = 1$ for all $t \in \{0, \dots, T\}$, while the risky asset price is described by a stochastic process $S = (S_t)_{0 \leq t \leq T}$. Recall that a self-financing portfolio process $(V_t)_{0 \leq t \leq T}$ satisfies by definition:

$$\Delta V_t = V_t - V_{t-1} = \theta_{t-1} \Delta S_t, \quad t = 1, \dots, T,$$

where θ_{t-1} is \mathcal{F}_{t-1} -measurable and represents the number of risky assets of the portfolio.

In this chapter, we consider European options whose payoffs are of the form $\xi_T = g(S_T) \in \mathbb{L}^0(\mathbb{R}, \mathcal{F}_T)$, where g is a continuous piecewise affine function. The typical example is the European call option, i.e. $g(x) = (x - K)^+$, $K > 0$. Our goal is to compute the set of all super-hedging prices of ξ_T , i.e the set of all V_0 , initial values of self-financing portfolio processes $(V_t)_{0 \leq t \leq T}$, such that $V_T \geq \xi_T$ almost surely. Contrarily to what it is usual to do in the literature, we restrict ourselves to the case of integer-valued strategies, i.e $\theta_t \in \mathbf{Z}$ almost surely, for all $t \in \{0, \dots, T\}$, where $\mathbf{Z} = \mathbb{N} \cup (-\mathbb{N})$ and \mathbb{N} is the set of all non negative integers. In the case of super-hedging an arbitrary number of options $n \geq 1$, the problem reads as $V_T \geq n\xi_T$, a.s. and it is clearly interesting to analyse the impact of n on the strategies and the infimum prices, as linearity is not necessarily preserved with respect to the quantity n of claims.

To solve this problem, we follow the approach of [16], [3] that we adapt to integer-valued strategies. To do so, we first solve backwardly the super-hedging problem between two dates $t - 1$ and t , and we show that the procedure may be propagated backwardly as the minimal super-hedging price we obtain at time $t - 1$ is still a continuous piecewise affine payoff function of the underlying asset. It is then possible to deduce the minimal super-hedging price at time $t = 0$ by induction.

4.3 The super-hedging problem

Let $t \leq T$ and g_t be a continuous piecewise affine function, i.e. there exists a subdivision $0 = a_0 < a_1 < \dots < a_{n-1} < a_n = \infty$ of $[0, \infty]$ such that g_t is an affine function for all $x \in [a_i, a_{i+1})$, $\forall i \in \{0, \dots, n - 1\}$. As the asset prices are non negative, we suppose without loss of generality that $a_0 = 0$. We first solve the one step problem: find V_{t-1} and the strategy $\theta_{t-1} \in \mathbb{L}^0(\mathbb{R}, \mathcal{F}_{t-1})$ such that:

$$V_{t-1} + \theta_{t-1} \Delta S_t \geq g_t(S_t), \text{ a.s.}$$

This is equivalent to:

$$\begin{aligned} V_{t-1} &\geq g_t(S_t) - \theta_{t-1} \Delta S_t, \\ \Leftrightarrow V_{t-1} &\geq g_t(S_t) - \theta_{t-1} S_t + \theta_{t-1} S_{t-1}, \\ \Leftrightarrow V_{t-1} &\geq \text{esssup}_{\mathcal{F}_{t-1}}(g_t(S_t) - \theta_{t-1} S_t) + \theta_{t-1} S_{t-1}. \end{aligned}$$

Equivalently, we have:

$$V_{t-1} \geq V_{t-1}(\theta_{t-1}) := \sup_{x \in \text{supp}_{\mathcal{F}_{t-1}}(S_t)} (g_t(x) - \theta_{t-1} x) + \theta_{t-1} S_{t-1}, \quad (4.3.2)$$

where $\text{supp}_{\mathcal{F}_{t-1}}(S_t)$ is the conditional support of S_t knowing \mathcal{F}_{t-1} , see [16] and [29] for the definition and the proof of the inequality above.

In the following, we suppose that there exist two deterministic numbers $k_{t-1}^d \in (0, 1)$ and $k_{t-1}^u \in (1, \infty)$ such that $\text{supp}_{\mathcal{F}_{t-1}}(S_t) = [k_{t-1}^d S_{t-1}, k_{t-1}^u S_{t-1}]$. This model may be seen as a generalization of the Binomial model and the conditions imposed on the coefficients k_{t-1}^d and k_{t-1}^u are equivalent to a no-arbitrage condition, see [16]. In particular, we have:

$$V_{t-1}(\theta_{t-1}) = \sup_{x \in [k_{t-1}^d S_{t-1}, k_{t-1}^u S_{t-1}]} (g_t(x) - \theta_{t-1}x) + \theta_{t-1}S_{t-1} \in \mathbb{L}^0(\mathbb{R}, \mathcal{F}_{t-1}).$$

We define V_{t-1}^* as the infimum of all the superhedging prices at time $t - 1$ over all integer-valued strategies in \mathbf{Z} , i.e.

$$V_{t-1}^* := \text{ess inf}_{\theta_{t-1} \in \mathbb{L}^0(\mathbf{Z}, \mathcal{F}_{t-1})} V_{t-1}(\theta_{t-1}).$$

Lemma 4.3.1. *We have*

$$V_{t-1}^* = \inf_{\theta \in \mathbf{Z}} V_{t-1}(\theta). \quad (4.3.3)$$

Proof. Let us define $\gamma = \inf_{\theta \in \mathbf{Z}} V_{t-1}(\theta) \in L^0(\mathbb{R}, \mathcal{F}_{t-1})$, see [16]. As $V_{t-1}^* \leq V_{t-1}(\theta)$, for all $\theta \in \mathbf{Z}$, we get that $V_{t-1}^* \leq \inf_{\theta \in \mathbf{Z}} V_{t-1}(\theta) = \gamma$. On the other hand, if $\theta_{t-1} \in \mathbb{L}^0(\mathbf{Z}, \mathcal{F}_{t-1})$, then:

$$\begin{aligned} \theta_{t-1} &= \sum_{\theta \in \mathbf{Z}} \theta 1_{\{\theta_{t-1} = \theta\}}, \\ V_{t-1}(\theta_{t-1}) &= \sum_{\theta \in \mathbf{Z}} V_{t-1}(\theta) 1_{\{\theta_{t-1} = \theta\}} \geq \sum_{\theta \in \mathbf{Z}} \gamma 1_{\{\theta_{t-1} = \theta\}} = \gamma. \end{aligned}$$

We deduce that $V_{t-1}^* \geq \gamma$ and the conclusion follows. \square

Theorem 4.3.2 (One step problem). *Let us consider $t \in \{1, \dots, T\}$ and suppose that $\xi_t = g_t(S_t)$ where g_t is a continuous piecewise affine function. Moreover, we assume that there exists two deterministic numbers $k_{t-1}^d \in (0, 1)$ and $k_{t-1}^u \in (1, \infty)$ such that*

$$\text{supp}_{\mathcal{F}_{t-1}}(S_t) = [k_{t-1}^d S_{t-1}, k_{t-1}^u S_{t-1}].$$

Then, $V_{t-1}^ = g_{t-1}(S_{t-1})$ where g_{t-1} is a continuous piecewise linear function.*

Proof. By assumption, there exist a subdivision $(a_i)_{i=0, \dots, n}$ of $[0, \infty]$, with $a_0 = 0 < a_1 < \dots < a_{n-1} < a_n = \infty$, such that g_t is an affine function on each interval. Let us define

$$x_i(S_{t-1}) = (k_{t-1}^d S_{t-1} \vee a_i) \wedge k_{t-1}^u S_{t-1}, \quad i = 0, \dots, n.$$

It is straightforward that

$$V_{t-1}(\theta_{t-1}) = \sup_{i=0, \dots, n} [g_t(x_i(S_{t-1})) - \theta_{t-1}x_i(S_{t-1})] + \theta_{t-1}S_{t-1}.$$

Note that $x_0(S_{t-1}) = k^d S_{t-1}$ and $x_n(S_{t-1}) = k^u S_{t-1}$ and some terms of the sequence $(x_i)_i$ may coincide. Let us define the functions

$$h^i(\theta_{t-1}, S_{t-1}) = g_t(x_i(S_{t-1})) + \theta_{t-1}(S_{t-1} - x_i(S_{t-1})), \quad i = 0, \dots, n.$$

The slopes of the affine functions $\theta_{t-1} \mapsto h^i(\theta_{t-1}, S_{t-1})$ are given by the non decreasing sequence $(S_{t-1} - x_i(S_{t-1}))_{i=n, n-1, \dots, 0}$ such that $S_{t-1} - x_n(S_{t-1}) < 0$ and $S_{t-1} - x_0(S_{t-1}) > 0$.

By ordering the indices in the decreasing order, we obtain $(n+1)$ affine functions $\theta_{t-1} \mapsto h^i(\theta_{t-1})$ for $i \in \{n, n-1, \dots, 1, 0\}$ with increasing slopes $(S_{t-1} - x_i)_{i \in \{n, n-1, \dots, 1, 0\}}$, such that: $S_{t-1} - x_n < 0$ and $S_{t-1} - x_0 > 0$. Therefore, the mapping $V_{t-1} : \theta_{t-1} \mapsto \sup_{i=n, \dots, 0} h^i(\theta_{t-1}, S_{t-1})$ is a piecewise affine function, i.e. there exists a subdivision:

$$-\infty = \alpha_0 < \alpha_1(S_{t-1}) \leq \dots \leq \alpha_{m-1}(S_{t-1}) < \alpha_m = \infty$$

such that V_{t-1} is an affine function of θ_{t-1} on each interval $[\alpha_i(S_{t-1}), \alpha_{i+1}(S_{t-1})]$, $i = 0, \dots, m-1$. Note that the function V_{t-1} is convex in θ_{t-1} and the elements of the partition define the intersection points between two distinct and successive graphs of the affine functions $h^{i+1}(\theta_{t-1}), h^i(\theta_{t-1})$. So, there exists $\theta_{t-1}^* \in [\alpha_1(S_{t-1}) - 1, \dots, \alpha_{m-1}(S_{t-1}) + 1] \cap \mathbf{Z}$ such that:

$$\inf_{\theta_{t-1} \in \mathbf{Z}} V_{t-1}(\theta_{t-1}) = V_{t-1}(\theta_{t-1}^*).$$

It remains to evaluate $\alpha_1(S_{t-1})$ and $\alpha_{m-1}(S_{t-1})$. To do so, let us solve the equations $h^i(\alpha) = h^j(\alpha)$, $i, j = 0, \dots, m$ and $x_i \neq x_j$. Since we suppose that $x_i - x_j \neq 0$, we get that

$$\alpha = \frac{g_t(x_i) - g_t(x_j)}{x_i - x_j}.$$

We deduce that $|\alpha| \leq L_t$ where $L_t > 0$ is a Lipschitz constant of the piecewise affine function g_t . We deduce that $\theta_{t-1}^* \in [-L_t - 1, L_t + 1] \cap \mathbf{Z}$ and

$$g_{t-1}(S_{t-1}) = V_{t-1}(\theta_{t-1}^*(S_{t-1})) = \min_{\theta_{t-1} \in [-L_t - 1, L_t + 1] \cap \mathbf{Z}} \sup_{i=0, \dots, n} h^i(\theta_{t-1}, S_{t-1}).$$

We conclude that g_{t-1} is a continuous piecewise affine function as a finite minimum of continuous piecewise affine functions. \square

Corollary 4.3.3. *(The multi-period super-hedging problem) Suppose that, at time $T > 0$, the payoff is $\xi_T = g_T(S_T)$ where g_T is a continuous piecewise affine function. Moreover, we assume that there exists deterministic numbers $k_{t-1}^d \in (0, 1)$ and $k_{t-1}^u \in (1, \infty)$ for each $t = 1, \dots, T$ such that we have $\text{supp}_{\mathcal{F}_{t-1}}(S_t) = [k_{t-1}^d S_{t-1}, k_{t-1}^u S_{t-1}]$. Then, there exists a minimal super-hedging portfolio process $(V_t^*)_{t=0, \dots, T}$ such that*

$V_T^* \geq \xi_T$. We have $V_t^* = g(t, S_t)$ where $g(t, \cdot)$ is a continuous piecewise affine function given by

$$g(t, s) = \min_{\theta \in [-L_{t+1}-1, L_{t+1}+1] \cap \mathbf{Z}} \sup_{i=0, \dots, n^{(t+1)}} \left(g(t+1, x_i^{(t+1)}(s)) + \theta(s - x_i^{(t+1)}(s)) \right),$$

$$x_i^{(t+1)}(s) = (k_t^d s \vee a_i^{(t+1)}) \wedge k_t^u s, \quad i = 0, \dots, n^{(t+1)},$$

where L_{t+1} is any Lipschitz constant of $g(t+1, \cdot)$ and $(a_i^{(t+1)})_{i=0, \dots, n^{(t+1)}}$ is any partition such that $a_0^{(t+1)} = 0 < a_1^{(t+1)} < \dots < a_{n^{(t+1)}-1}^{(t+1)} < a_{n^{(t+1)}}^{(t+1)} = \infty$ and $g(t+1, \cdot)$ is an affine function on $[a_i^{(t+1)}, a_{i+1}^{(t+1)}]$, $i \leq n^{(t+1)} - 1$. The associated super-hedging strategy θ^* is given by the argmin of the minimisation problem defining $g(t, \cdot)$ in the expression above.

4.3.1 Example in the one step problem: the case of the Call option

At time $t = T$, suppose that the payoff is $\xi_T^n = ng(S_T)$ where $n \geq 1$ and $g(x) = (x - K)^+$, $K = 500$. We suppose that $\text{supp}_{\mathcal{F}_{T-1}}(S_T) = [k^d S_{T-1}, k^u S_{T-1}]$ for some constants k^d, k^u such that $0 < k^d < 1 < k^u$. Precisely, we suppose that $k^d = 0.9$ and $k^u = 1.2$. Observe that the super-hedging problem $V_{T-1} + \theta_{T-1} \Delta S_T \geq ng(S_T)$ is equivalent to

$$V_{T-1} \geq V_{T-1}(\theta_{T-1}) = \max_{k \in \{k^d, k^u\}} [ng(kS_{T-1}) - \theta_{T-1} k S_{T-1}] + \theta_{T-1} S_{T-1}.$$

In the following we give the explicit expression of $V_{T-1}(\theta_{T-1}) = V_{T-1}^n(\theta_{T-1})$.

If $k^u \leq K/S_{T-1}$, i.e. $S_{T-1} \leq K/k^u$, then

$$V_{T-1}(\theta_{T-1}) = \begin{cases} \theta_{T-1} S_{T-1} (1 - k^u) & \text{if } \theta_{T-1} \leq 0, \\ \theta_{T-1} S_{T-1} (1 - k^d) & \text{if } \theta_{T-1} \geq 0. \end{cases}$$

Therefore, $\theta_{T-1}^{*,n}(S_{T-1}) = 0$, and $V_{T-1}^{*,n}(S_{T-1}) = V_{T-1}(\theta_{T-1}^{*,n}) = 0$.

If $k^d \geq K/S_{T-1}$, i.e. $S_{T-1} \geq K/k^d$,

$$V_{T-1}(\theta_{T-1}) = \begin{cases} \theta_{T-1} S_{T-1} (1 - k^u) + nk^u S_{T-1} - nK & \text{if } \theta_{T-1} \leq n, \\ \theta_{T-1} S_{T-1} (1 - k^d) + nk^d S_{T-1} - nK & \text{if } \theta_{T-1} \geq n. \end{cases}$$

We conclude that $\theta_{T-1}^{*,n}(S_{T-1}) = n$, and $V_{T-1}^{*,n}(S_{T-1}) = n(S_{T-1} - K)$.

If $k^d \leq K/S_{T-1} \leq k^u$, i.e. $S_{T-1} \in [K/k^u, K/k^d]$,

$$V_{T-1}(\theta_{T-1}) = \begin{cases} \theta_{T-1} S_{T-1} (1 - k^u) + nk^u S_{T-1} - nK & \text{if } \theta_{T-1} \leq \frac{nk^u S_{T-1} - nK}{S_{T-1}(k^u - k^d)}, \\ \theta_{T-1} S_{T-1} (1 - k^d) & \text{if } \theta_{T-1} \geq \frac{nk^u S_{T-1} - nK}{S_{T-1}(k^u - k^d)}. \end{cases}$$

Let us define

$$\begin{aligned}\alpha_{T-1}^n(S_{T-1}) &:= \frac{nk^u S_{T-1} - nK}{S_{T-1}(k^u - k^d)}, \\ f^n(x, S_{T-1}) &:= xS_{T-1}(1 - k^d)1_{\{x \geq \alpha_{T-1}(S_{T-1})\}} \\ &\quad + (xS_{T-1}(1 - k^u) + nk^u S_{T-1} - nK)1_{\{x < \alpha_{T-1}(S_{T-1})\}}.\end{aligned}$$

We denote by $\lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor$ the lower integer part of α_{T-1}^n . Then,

$$\theta_{T-1}^{*,n}(S_{T-1}) = \begin{cases} \lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor & \text{if } f^n(\lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor) \leq f^n(\lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor + 1), \\ \lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor + 1 & \text{otherwise.} \end{cases}$$

So, we have:

$$\begin{aligned}V_{T-1}^{*,n}(S_{T-1}) &= (\lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor S_{T-1}(1 - k^u) + nk^u S_{T-1} - nK)1_{G_{T-1}^n}(S_{T-1}) \\ &\quad + (\lfloor \alpha_{T-1}^n(S_{T-1}) \rfloor + 1)S_{T-1}(1 - k^d)1_{(G_{T-1}^n)^c}(S_{T-1}),\end{aligned}$$

where

$$\begin{aligned}G_{T-1}^n &:= \{S : f^n(\lfloor \alpha_{T-1}^n(S) \rfloor) \leq f^n(\lfloor \alpha_{T-1}^n(S) \rfloor + 1)\} = \{\lfloor \alpha_{T-1}^n(S) \rfloor \leq \beta_{T-1}^n(S)\}, \\ \beta_{T-1}^n(S) &:= \alpha_{T-1}^n(S) + \frac{1 - k^d}{k^d - k^u}.\end{aligned}$$

A graphic illustration of $V_{T-1}^{*,n}/n$ as a function of S_{T-1} is given in Figure 4.1.

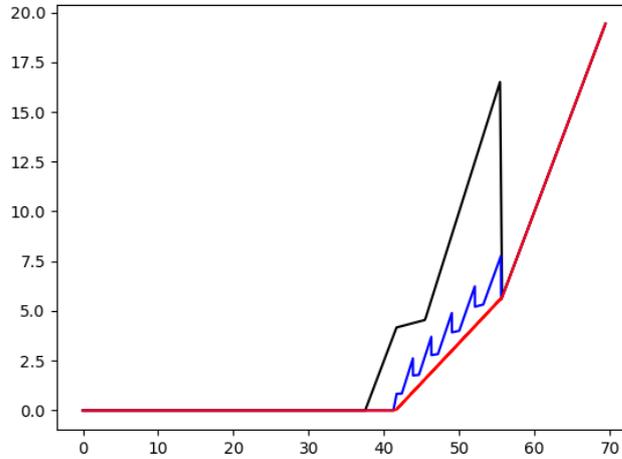


Figure 4.1: The function $x = S_{T-1} \mapsto g(T-1, x, n)/n = V_{T-1}^{*,n}/n, K = 500$, for $n = 1$ (black), $n = 5$ (blue), $n = 100$ (red).

We observe that $V_{T-1}^{*,n}$ is not a convex function of S_{T-1} even if the payoff function is and, moreover, $g(T-1, x, n) \neq ng(T-1, x, 1)$.

4.4 Numerical illustration

In this section, we illustrate the method developed above when the underlying asset S is the french CAC 40 index and the European claim is the Call option. The historical data is composed of daily observations of the CAC 40 values between the 6th of June 2019 and the 16th of June 2021 (Appendix C[4.4]). We use the two first years of the data set to calibrate the model while we implement the model on the third year. Here, we suppose that $\text{supp}_{\mathcal{F}_t} S_{t+1} = [k_t^d S_t, k_t^u S_t]$ where k_t^d and k_t^u are estimated as follows:

$$\begin{aligned} k_t^d &= \min_{i=j, \dots, N} S_{t+1}^{(j)} / S_t^{(j)}, \\ k_t^u &= \max_{i=j, \dots, N} S_{t+1}^{(j)} / S_t^{(j)}, \end{aligned}$$

where N is the number of training periods and $S_t^{(j)}$ are the observed values at time t during the j -th periods. The algorithms are written in Python, see Appendix C [4.4]. The main difficulty is to write a code whose execution time is reasonable. Indeed, recall that the price function $g(t, x)$ is computed backwardly from $g(t+1, x)$. If this function $g(t, x)$ is naively coded from $g(t+1, x)$, then the computation may take more than two weeks ! So it is better to approximate, at each step, the function $g(t, x)$ as a numpy array consisting of discretized values following a grid $(x_i)_{i=0}^{N_t}$ where $x_i = \text{step} * i$. Here, we choose $\text{step} = 0.1$ and N_t is chosen so that $x_{N_t} \leq S_0^{\max}$ where S_0^{\max} is the maximal value for S_0 that we observe in our data. At last, $x_{N_t} \leq S_0^{\max} * (\max_{r \leq t} k_r^u)^t$.

The relative hedging error is defined as $\epsilon_T = 100 * (V_T - g(T, S_T)) / S_T$ where $(V_t^*)_{t=0, \dots, T}$ is the optimal super-hedging portfolio process whose initial value is the minimal super-hedging price, as computed in the last section. We present in Figure 4.2.1 the distribution of ϵ_T when $n = 1$. Of course, we expect that $\epsilon_T \geq 0$ a.s. and this is confirmed on our test data set. Note that, we could have observed some negative values as the model is calibrated from data values anterior to the test data set.

Let us denote by $g(t, x, n)$ the price function at time t of the optimal portfolio process, i.e. $V_t^{*,n} = g(t, S_t, n)$ such that $V_T^{*,n} \geq \xi_T^n$ a.s., when the European claim is $\xi_T^n := n * (S_T - K)^+$. The natural question is the following: Do we have $g(t, x, n) = n g(t, x, 1)$? The answer is yes when real-valued strategies are allowed since the hedging problem is then linear with respect to the number of claims.

In the case of integer-valued strategies, the answer is not trivial and is actually negative, see the first example above. By definition of the infimum super-hedging price, we have $g(t, x, n) \leq n g(t, x, 1)$. As a first step, we have computed the relative infimum super-hedging prices per unit of claims, i.e. $V_0^{*,n} / n$ at time 0, for different values of n on each period of the test data set. Then, computing the average of the $V_0^{*,n} / n$ values over all the periods, we get that the empirical average of $V_0^{*,n} / n$ is

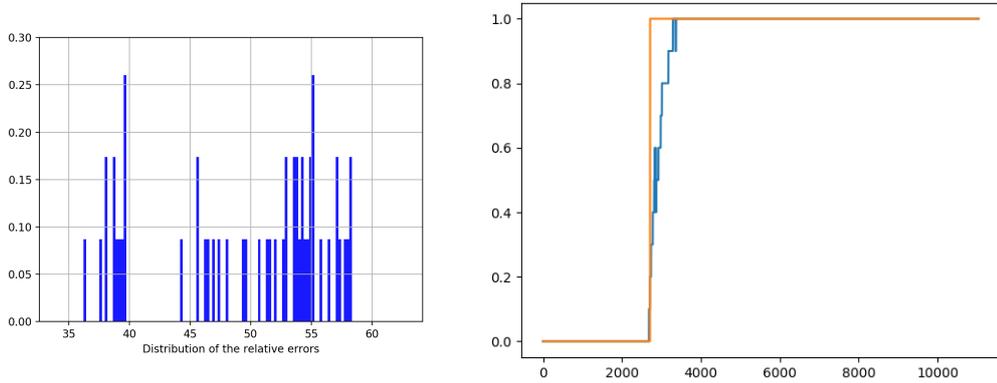


Figure 4.2: F1.1: Super-hedging errors with $n = 1$ and $K = 3000$. F1.2: Comparison of the optimal strategies per unit of claims for $n = 1$ and $n = 10$.

approximately equal to 49.48% for $n = 1, 5, 10, 15, 20$. Nevertheless, we observe that the price function per unit of unit of claims, i.e. $g(0, S_0, n)/n$ is non-increasing when n increases, see Figure 4.3. This implies that the equality $g(t, x, n) = ng(t, x, 1)$ does not hold. We conjecture that $g(t, x, n)/n$ converges to the price function $\hat{g}(t, x)$ of the model where real-valued strategies are allowed, see [16]. This is an open question we suggest. In Figure 4.3, we clearly observe the convergence of $x \mapsto g(0, x, n)/n$ as $n \rightarrow \infty$.

The same question arises for the optimal strategy associated to $V^{*,n}$, i.e. do we have $\theta^*(t, S_t, n) = n\theta^*(t, S_t, 1)$? Intuitively, this is a priori not the case as $\theta^*(t, S_t, 1) = \theta^*(t, S_t, n)/n$ could be not integer-valued. This is confirmed at time 0 when we compute the optimal strategy $\theta^*(0, S_0, n)/n$ per unit of claims. This is illustrated by Figure 4.2.2 where we compare $\theta^*(0, S_0, n)/n$ for $n = 10$ to $\theta^*(0, S_0, 1)$. We may observe that the optimal strategy per unit of claims $\theta^*(0, S_0, n)/n$ (blue graph) is smaller than $\theta^*(0, S_0, 1)$ for $n = 10$.

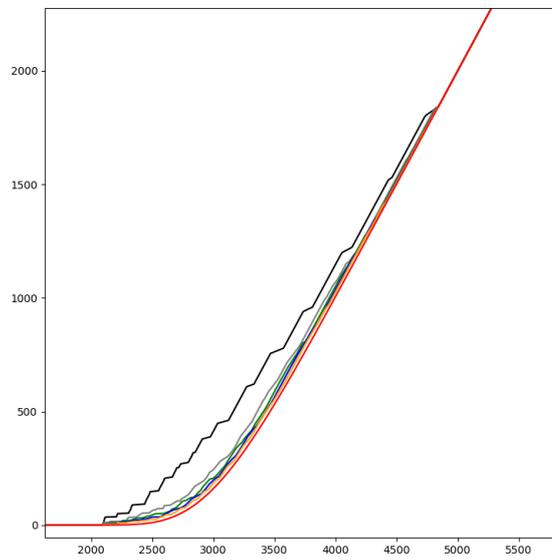


Figure 4.3: Super-hedging price mapping $x \mapsto g(0, x, n)/n$ of n units of call option per unit of claims for different values of $n = 1$ (black), $n = 3$ (grey), $n = 5$ (green), $n = 7$ (blue), $n = 10$ (orange), $n = 100$ (red).

Conclusion and future perspectives

In this thesis, the main goal is to explore new approaches of pricing where executable prices are uncertain. This chapter will discuss some potential ideas to explore in future research.

It was possible to determine and compute the infimum of the super-hedging price for general models by considering the conditional random sets and the essential supremum and the essential infimum operators under the AIP condition without any assumptions of no-arbitrage on the market. In fact the AIP condition was introduced to ensure the finiteness of prices. It is important to note that we only needed to know the conditional support of the returns $\frac{S_{t+1}}{S_t}$ to compute the infimum of the super-hedging prices. We suggested a straightforward model in the one-dimensional case. It is a generalisation of the binomial model as observed in [3] that works well with real data. However, it is interesting to consider the multivariate case, i.e the case of multiple assets. In this context, several questions arise :

1. How to define the conditional support for the multivariate case to get an efficient calibration on a historical data base both with fast and simple computation?
2. Moreover the advantage of the approach used in chapter 3 lies in its ability to include a large range of potential models, e.g. with transaction costs or limit order book. Therefore, the numerical aspect can be further explored and compared to other models.
3. Another natural question is: what happens if we increase the number of discrete dates?
4. It is also interesting to study the case where the payoff functions are not necessary convex.

On the other hand, regarding the super-hedging problem with integer valued-strategies, we can observe that the execution is large as soon as the number of dates increases (more then 6 dates). Also, we only considered simple payoff functions. Then, the multivariate case can be studied with more complex payoff, but we expect that the computation will be more time consuming.

Appendices

Appendix A

Conditional support and conditional essential supremum

Definition 4.4.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, \mathcal{H} a sub- σ algebra of \mathcal{F} and let $X \in L^0(\mathbb{R}^d, \mathcal{F})$. Then the conditional support of X with respect to \mathcal{H} is the random set:

$$\text{supp}_{\mathcal{H}}X = \cap\{A \subset \mathbb{R}^d, \text{ closed }, \mathbb{P}(X \in A \mid \mathcal{H}(\omega)) = 1\}.$$

We remark that if $\mathcal{H} = \{\emptyset, \Omega\}$, then $\text{supp}_{\mathcal{H}}X$ is the support of X .

Proposition 4.4.2. Let \mathcal{H} and \mathcal{F} be complete σ -algebra such that $\mathcal{H} \subset \mathcal{F}$ and $\Lambda = (\lambda_i)_{i \in I}$ be a family of real valued \mathcal{F} -measurable random variables. There exists a unique random variable \mathcal{H} -measurable $\lambda_{\mathcal{H}} \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$, denoted by $\text{ess sup}_{\mathcal{H}}(\Lambda)$ that satisfies:

1. $\lambda_{\mathcal{H}} \geq \lambda_i$ a.s , $\forall i \in I$.
2. If $\alpha \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$ such that $\alpha \geq \lambda_i$ a.s , $\forall i \in I$ then $\alpha \geq \lambda_{\mathcal{H}}$ a.s .

Symmetrically the essential infimum is defined as:

$$\text{ess inf}_{\mathcal{H}}(\Lambda) = -\text{ess sup}_{\mathcal{H}}(-\Lambda)$$

Lower semi-continuity

Definition 4.4.3. Let $f : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ be a function.

- We call the effective domain the set $\text{dom}(f)$ defined by:

$$\text{dom}(f) := \{x \in \mathbb{R}^d \mid f(x) < \infty\}$$

- We say that the function f is proper if:

1. There exists $x \in \mathbb{R}^d$, such that $f(x) < \infty$.

2. For all $x \in \mathbb{R}^d$, $f(x) > -\infty$.

- The epigraph of f is the set:

$$\text{epi}(f) := \{(x, \alpha) \in \mathbb{R}^d \times \mathbb{R} \mid \alpha \geq f(x)\}.$$

Definition 4.4.4. A function $f : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ is said to be lower semi-continuous (l.s.c) at $\bar{x} \in \mathbb{R}^d$ if

$$\forall \epsilon > 0, \exists \delta > 0 / f(\bar{x}) - \epsilon < f(x), \forall x \in B(\bar{x}, \delta).$$

The lower semi continuity can be characterised as follow:

Theorem 4.4.5. The following properties of a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ are equivalent:

1. f is lower semi continuous on \mathbb{R}^d .
2. The epigraph set $\text{epi}(f)$ is closed in $\mathbb{R}^d \times \mathbb{R}$.

Super-differential

Definition 4.4.6. [2]

- Let $f : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ be a convex function. We say that a vector $v \in \mathbb{R}^d$ is a subgradient of f at x if it satisfies the following:

$$f(y) \geq f(x) + v(y - x), \quad \forall y \in \mathbb{R}^d.$$

The set of subgradients at x is the sub-differential of f , denoted $\partial f(x)$.

- If f is concave function satisfies the reverse inequality, i.e:

$$f(y) \leq f(x) + v(y - x), \quad \forall y \in \mathbb{R}^d,$$

then we say that v is a supergradient of f at x . The set of supergradients is the super-differential and also denoted $\partial f(x)$.

Fenchel-Legendre conjugate

Definition 4.4.7. Let $f : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ be a function. The Fenchel-Legendre conjugate function is the function $f^* : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ defined by

$$f^*(y) := \sup_x \{ \langle y, x \rangle - f(x) \}.$$

The biconjugate function is given by $f^{**} = (f^*)^*$ defined by:

$$f^{**}(x) := \sup_y \{ \langle y, x \rangle - f^*(y) \}.$$

Theorem 4.4.8. *Let $f : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ a function. We denote by $\text{conv}(f)$ the convex hull of function f .¹ We suppose that $\text{conv}(f)$ is proper. Then f^* and f^{**} are proper, lsc and convex and we have:*

$$f^{**} = \text{cl } \text{conv}(f).$$

*Moreover, we have $f^{**} \leq f$, and if f is proper, lsc and convex then we have $f^{**} = f$.*

¹If $f : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ is a function, then $\text{conv}(f)$ is the greatest convex function majorized by f .

Appendix B

```
1
2 import pandas as pd
3 from statistics import stdev
4
5
6 def interval0(n0,N0,S_initial):
7     c0=1
8     A0=0.8+1.2*rand(n0)
9     B0=A0+rand(n0)
10    i=randint(n0)
11    a0=A0[i]
12    j=randint(n0)
13    if a0 < b0:
14        I0=S_initial*(a0+((b0-a0)*(rand(N0))))
15        k=randint(N0)
16
17        S0=I0[k]
18
19    else:
20
21        c0=a0
22        a0=b0
23        b0=c0
24        I0=a0+((b0-a0)*(S_initial*rand(N0)))
25        k=randint(N0)
26        S0=I0[k]
27
28
29 def interval1(n1,N1,S0):
30     c1=1
31     A1=0.8+1.2*rand(n1)
32     B1=A1+rand(n1)
33     i=randint(n1)
34     a1=A1[i]
35     j=randint(n1)
36     b1=B1[j]
37     return(a1,b1)
38
```

```

39
40
41
42
43
44 def vals1(S0,K,S1ask,S1bid):
45     m2=0.8
46     M2=2.2
47     m1=0.8
48     M1=2.2
49     if K/M2 <= S1ask <= K/m2: #1st case for S1
50         theta1_ask=(S1ask*M2-K)/(S1ask*(M2-m2))
51
52         if S0*M1 <= K/M2: #1st case for S0
53
54             theta0=0
55             S1star=K/M2
56             if S1ask <= S1star:
57                 S1=S1bid
58
59             elif S1bid >= S1star:
60                 S1=S1ask
61
62             elif S1bid <= S1star <= S1ask:
63                 if abs(S1star-S1bid) <= abs(S1star-S1ask)
64                     :
65                     S1=S1ask
66
67                 elif abs(S1star-S1ask) <= abs(S1star-
68                     S1bid):
69                     S1=S1bid
70
71             elif (S0*m1 <= K/M2) and (K/M2 <= S0*M1 <= K/m2):
72                 #2nd case of S0
73
74                 theta0= ( (S0*M1*M2-K)* (1-m2) ) / ( S0* (M2-
75                     m2) * (M1-m1))
76                 S1star=K/(M2-theta0*(M2-m2))
77
78             if S1ask <= S1star:
79                 S1=S1bid

```

```

78
79
80     elif S1bid >= S1star:
81         S1=S1ask
82
83
84     elif S1bid <= S1star <= S1ask:
85         if abs(S1star-S1bid) <= abs(S1star-S1ask)
86             :
87                 S1=S1ask
88
89         elif abs(S1star-S1ask) <= abs(S1star-
90             S1bid):
91             S1=S1bid
92
93
94     elif (S0*m1 <= K/M2) and ( S0*M1 >= K/m2):#3rd
95         case of S0
96
97         theta0= (S0*M1-K) / ( S0* (M1-m1))
98         S1star=K/(M2-theta0*(M2-m2))
99
100        if S1ask <= S1star:
101            S1=S1bid
102
103        elif S1bid >= S1star:
104            S1=S1ask
105
106        elif S1bid <= S1star <= S1ask:
107            if abs(S1star-S1bid) <= abs(S1star-S1ask)
108                :
109                    S1=S1ask
110
111            elif abs(S1star-S1ask) <= abs(S1star-
112                S1bid):
113                S1=S1bid
114
115
116     elif (K/M2 <= S0*m1 <= K/m2) and ( K/M2 <= S0*M1
117         <= K/m2):#4th case of S0

```

```

115
116     theta0= (S0*M2-K) / ( S0* (M2-m2))
117     S1star=K/(M2-theta0*(M2-m2))
118
119     if S1ask <= S1star:
120         S1=S1bid
121     elif S1bid >= S1star:
122         S1=S1ask
123
124     elif S1bid <= S1star <= S1ask:
125         if abs(S1star-S1bid) <= abs(S1star-S1ask)
126             :
127             S1=S1ask
128
129         elif abs(S1star-S1ask) <= abs(S1star-
130             S1bid):
131             S1=S1bid
132
133     elif (K/M2 <= S0*m1 <= K/m2) and ( K/m2 <= S0*M1
134         ):#5th case of S0
135
136         theta0= ((S0*M1-K)*(M2-m2)-(S0*m1*M2-K)*(1-
137             m2) )/ ( S0* (M2-m2)*(M1-m1))
138         S1star=K/(M2-theta0*(M2-m2))
139
140         if S1ask <= S1star:
141             S1=S1bid
142
143         elif S1bid >= S1star:
144             S1=S1ask
145
146         elif S1bid <= S1star <= S1ask:
147             if abs(S1star-S1bid) <= abs(S1star-S1ask)
148                 :
149                 S1=S1ask
150
151             elif abs(S1star-S1ask) <= abs(S1star-
152                 S1bid):
153                 S1=S1bid

```

```

152
153     elif (K/m2 <= S0*m1) and ( K/m2 <= S0*M1 ):#6th
154         case of S0
155             theta0=1
156             S1star=K/m2
157
158             if S1ask <= S1star:
159                 S1=S1bid
160
161             elif S1bid >= S1star:
162                 S1=S1ask
163
164             elif S1bid <= S1star <= S1ask:
165                 if abs(S1star-S1bid) <= abs(S1star-S1ask)
166                     :
167                         S1=S1ask
168
169                 elif abs(S1star-S1ask) <= abs(S1star-
170                     S1bid):
171                     S1=S1bid
172
173     elif 0 <= S1ask <= K/M2: #2nd case of S1
174         theta1=0
175         S1=S1bid
176
177
178     elif S1ask>=K/m2:#3rd case of S1
179         theta1 = 1
180         S1=S1ask
181
182
183     return S1
184
185
186
187 S0=interval0(100,100,100)
188 m=interval1(100,100,S0)
189 a1=m[0]
190 print('a1=',a1)
191 b1=m[1]

```

```

192 print('b1=', b1)
193 S1ask=S0*a1
194 S1bid=S0*b1
195 K=100
196 S1=vals1(S0,K,S1ask,S1bid)
197 print('S1=', S1)
198
199
200 def interval2(n2,N2,S1):
201     c2=1
202     A2=0.8+1.2*rand(n2)
203     B2=A2+rand(n2)
204     i=randint(n2)
205     a2=A2[i]
206     j=randint(n2)
207     b2=B2[j]
208     if a2 < b2:
209
210         I2=S1*(a2+((b2-a2)*(rand(N2))))
211         k=randint(N2)
212         S2=I2[k]
213
214     else:
215
216         c2=a2
217
218         a2=b2
219
220         b2=c2
221
222         I2=S1*(a2+((b2-a2)*(rand(N2))))
223
224         k=randint(N2)
225
226         S2=I2[k]
227     return(S2)
228
229
230
231
232 def test1(S0,K):
233     S1=vals1(S0,K,S1ask,S1bid)
234     if K<= S1*2.2 and S1*0.8<=K:

```

```

235     V1=((1-0.8)*((S1*2.2)-K))/(2.2-0.8)
236     #print('1st case V1=',V1)
237     theta1=(S1*2.2-K)/(S1*(2.2-0.8))
238     #print('1st case theta1=',theta1)
239     i=1
240     return theta1
241 elif S1*0.8 >= K:
242     V1=S1-K
243     #print('2nd case V1=',V1)
244     theta1=1
245     #print('2nd case theta1=',theta1)
246     i=2
247     return theta1
248 elif S1*2.2<= K :
249     V1=0
250     #print('3rd case V1=',V1)
251     theta1=0
252     #print('3rd case theta1=',theta1)
253     i=3
254     return theta1
255
256
257
258
259
260 def test0(n0,N0,S_initial,K,S2):
261     S0=interval0(n0,N0,S_initial)
262     S1=vals1(S0,K,S1ask,S1bid)
263     theta1=test1(S0,K)
264     m=max(0,S2-K)
265
266
267     if S0*2.2 <=K/2.2:
268         V0=0
269         #print('1st case V0=',V0)
270         theta0=0
271         #print('1st case theta0=',theta0)
272         V2=theta1*S2
273         #print('V2=',V2)
274         eps=V2-m
275         epsrel=eps/S2
276         #print('the error value is:',eps)
277         #return eps,epsrel,theta0

```

```

278     return epsrel,V0
279 elif S0*0.8 <= K/2.2 and K/2.2 <= S0*2.2 and S0*2.2
    <= K/0.8:
280     V0=((S0*2.2*2.2)-K)*(1-0.8)*(1-0.8)/((2.2-0.8)
        *(2.2-0.8))
281     #print('2nd case V0=',V0)
282     theta0=((S0*2.2*2.2)-K)*(1-0.8)/(S0*(2.2-0.8)
        *(2.2-0.8))
283     #print('2nd case theta0=',theta0)
284     V2=V0+theta0*(S1-S0)+theta1*S2
285     #print('V2=',V2)
286     eps=V2-m
287     epsrel=eps/S2
288     #print('the error value is:',eps)
289     #return eps,epsrel,theta0
290     return epsrel,V0
291 elif S0*0.8 <= K/2.2 and S0*2.2 >= K/0.8:
292     V0=((S0*2.2*2.2)-K)*(1-0.8)/(2.2-0.8)
293     #print('3rd case V0=',V0)
294     theta0=((S0*2.2*2.2)-K)/(S0*(2.2-0.8))
295     #print('3rd case theta0=',theta0)
296     V2=V0+theta0*(S1-S0)+theta1*S2
297     #print('V2=',V2)
298     eps=V2-m
299     epsrel=eps/S2
300     #print('the error value is:',eps)
301     #return eps,epsrel,theta0
302     return epsrel,V0
303 elif K/2.2 <= S0*0.8 and S0*0.8 <= K/0.8 and K/2.2 <=
    S0*2.2 and S0*2.2 <= K/0.8:
304     V0=((S0*2.2*2.2)-K)*(1-0.8)/(2.2-0.8)
305     #print('4th case V0=',V0)
306     theta0=((S0*2.2*2.2)-K)/(S0*(2.2-0.8))
307     #print('4th case theta0=',theta0)
308     V2=V0+theta0*(S1-S0)+theta1*S2
309     #print('V2=',V2)
310     eps=V2-m
311     epsrel=eps/S2
312     #print('the error value is:',eps)
313     #return eps,epsrel,theta0
314     return epsrel,V0
315 elif K/2.2 <= S0*0.8 and S0*0.8 <= K/0.8 and S0*2.2
    >= K/0.8:

```

```

316     V0=(((((S0*2.2)-K)*(2.2-0.8)*(1-0.8))+((S0
        *0.8*2.2-K)*(1-0.8)*(2.2-1)))/((2.2-0.8)
        *(2.2-0.8))
317     #print('5th case V0=',V0)
318     theta0=(((((S0*2.2)-K)*(2.2-0.8))-(((S0*0.8*2.2)-K
        *(1-0.8)))/(S0*(2.2-0.8)*(2.2-0.8))
319     #print('5th case theta0=',theta0)
320     V2=V0+theta0*(S1-S0)+theta1*S2
321     #print('V2=',V2)
322     eps=V2-m
323     epsrel=eps/S2
324     #print('the error value is:',eps)
325     #return eps,epsrel,theta0
326     return epsrel,V0
327 elif S0*0.8 >= K/0.8 and S0*2.2 >= K/0.8:
328     V0=S0-K
329     #print('6th case V0=',V0)
330     theta0=1
331     #print('6th case theta0=',theta0)
332     V2=V0+S1-S0+theta1*S2
333     #print('V2=',V2)
334     eps=V2-m
335     epsrel=eps/S2
336     #print('the error value is:',eps)
337     #return eps,epsrel,theta0
338     return epsrel,V0
339
340
341
342
343
344
345 S_initial=100
346 K=100
347 X=50 # nombre de traj
348 A=zeros(X)
349 Vzero=zeros(X)
350 Szero=zeros(X)
351 Sun=zeros(X)
352 vs=zeros(X)
353 vs1=zeros(X)
354 Sdeux=zeros(X)
355 for i in range(X):

```

```

356     S0=interval0(X,X,S_initial)
357     print('S0=',S0)
358     m=interval1(X,X,S0)
359     a1=m[0]
360     b1=m[1]
361     S1ask=S0*b1
362     print('S1ask=',S1ask)
363     S1bid=S0*a1
364     print('S1bid=',S1bid)
365     S1=vals1(S0,K,S1ask,S1bid)
366     print('S1=',S1)
367     S2=interval2(X,X,S1)
368     print('S2=',S2)
369     if S2 > K:
370
371         t0=test0(X,X,S_initial,K,S2)
372         epsrel=t0[0]
373         V0=t0[1]
374         A[i]=epsrel
375         Szero[i]=S0
376         Sun[i]=S1
377         Sdeux[i]=S2
378         Vzero[i]=V0
379         vs[i]=Vzero[i]/Szero[i]
380         vs1[i]=Vzero[i]/S_initial
381         print(vs1[i])
382         print('epsrel=',epsrel)
383
384     moyenne_V0,Sinit=mean(vs1)
385     print('la moyenne de V0 par Sinit est=',moyenne_V0,Sinit)
386     moyenne_V0,S0=mean(vs)
387     print('la moyenne de V0 par S0 est=',moyenne_V0,S0)
388     maximum_V0,S0=max(vs)
389     print('le maximum de V0 par S0 est=',maximum_V0,S0)
390     minimum_V0,S0=min(vs)
391     print('le minimum de V0 par S0 est=',minimum_V0,S0)
392     moyenne_S0=mean(Szero)
393     print('la moyenne de S0 est=',moyenne_S0)
394     Maximum_V0=max(Vzero)
395     print('la valeur maximale de V0 est=',Maximum_V0)
396     moyenne_V0=mean(Vzero)
397     print('la moyenne de V0 est=',moyenne_V0)
398     moyenne_S1=mean(Sun)

```

```

399 print('la moyenne de S1 est=',moyenne_S1)
400 moyenne_S2=mean(Sdeux)
401 print('la moyenne de S2 est=',moyenne_S2)
402 hist(Szero ,bins=100,width=2)
403 plt.title('Distribution of S0')
404 show()
405 hist(Sun ,bins=100,width=2)
406 plt.title('Distribution of S1')
407 show()
408 hist(Sdeux ,bins=100,width=2)
409 plt.title('Distribution of S2')
410 show()
411 ecarttype=stdev(A)
412 print('L ecart type de l erreur relatif=',ecarttype)
413 moyenne_err=mean(A)
414 print('la moyenne de l erreur=',mean(A))
415 Minimum_err=min(A)
416 print('la valeur minimale de l erreur est=',Minimum_err)
417 Maximum_err=max(A)
418 print('la valeur maximale de l erreur est=',Maximum_err)
419 hist(A,density=1,bins=100,width=10)
420 plt.title('Distribution of the super-hedging relative
         error')
421 show()

```


Appendix C

The data base presented below concerns the historical performance of CAC 40 and was collected from Boursorama website (06/07/23).

Date	Date	Date	Date	Date	Date
15/06/2022	6 030.13	+1.35%	6 062.62	5 972.31	6 022.84
14/06/2022	5 949.84	-1.20%	6 075.37	5 934.12	6 063.80
13/06/2022	6 022.32	-2.67%	6 122.26	6 006.28	6 088.05
10/06/2022	6 187.23	-2.69%	6 325.89	6 177.31	6 316.41
09/06/2022	6 358.46	-1.40%	6 456.91	6 342.99	6 412.58
08/06/2022	6 448.63	-0.80%	6 517.55	6 413.87	6 517.55
07/06/2022	6 500.35	-0.74%	6 523.60	6 459.92	6 523.16
06/06/2022	6 548.78	+0.98%	6 593.15	6 533.82	6 533.82
03/06/2022	6 485.30	-0.23%	6 547.82	6 473.74	6 545.00
02/06/2022	6 500.44	+1.27%	6 506.60	6 447.30	6 447.37
01/06/2022	6 418.89	-0.77%	6 510.26	6 414.22	6 509.19
31/05/2022	6 468.80	-1.43%	6 543.42	6 457.53	6 539.86
30/05/2022	6 562.39	+0.72%	6 582.05	6 536.77	6 547.86
27/05/2022	6 515.75	+1.64%	6 519.73	6 424.85	6 445.25
26/05/2022	6 410.58	+1.78%	6 424.88	6 298.45	6 301.49
25/05/2022	6 298.64	+0.73%	6 319.29	6 229.84	6 296.28
24/05/2022	6 253.14	-1.66%	6 322.74	6 243.84	6 282.32
23/05/2022	6 358.74	+1.17%	6 358.74	6 263.96	6 350.88
20/05/2022	6 285.24	+0.20%	6 377.58	6 264.84	6 305.46
19/05/2022	6 272.71	-1.26%	6 288.99	6 196.19	6 279.69
18/05/2022	6 352.94	-1.20%	6 455.80	6 342.67	6 440.30
17/05/2022	6 430.19	+1.30%	6 448.33	6 373.73	6 379.34
16/05/2022	6 347.77	-0.23%	6 369.05	6 286.02	6 315.68
13/05/2022	6 362.68	+2.52%	6 363.24	6 238.79	6 250.16
12/05/2022	6 206.26	-1.01%	6 238.25	6 098.22	6 130.69
11/05/2022	6 269.73	+2.50%	6 269.73	6 137.24	6 173.10
10/05/2022	6 116.91	+0.51%	6 206.56	6 116.91	6 151.52
09/05/2022	6 086.02	-2.75%	6 260.85	6 086.02	6 207.15
06/05/2022	6 258.36	-1.73%	6 322.43	6 213.64	6 318.66
05/05/2022	6 368.40	-0.43%	6 558.91	6 350.66	6 549.49
04/05/2022	6 395.68	-1.24%	6 494.99	6 395.68	6 482.26
03/05/2022	6 476.18	+0.79%	6 507.13	6 423.03	6 463.73
02/05/2022	6 425.61	-1.66%	6 474.81	6 308.47	6 469.31
29/04/2022	6 533.77	+0.39%	6 601.15	6 514.10	6 557.66
28/04/2022	6 508.14	+0.98%	6 575.95	6 435.75	6 520.53
27/04/2022	6 445.26	+0.48%	6 474.96	6 338.61	6 427.40
26/04/2022	6 414.57	-0.54%	6 535.92	6 412.49	6 520.01
25/04/2022	6 449.38	-2.01%	6 512.26	6 408.71	6 450.90
22/04/2022	6 581.42	-1.99%	6 654.64	6 572.87	6 618.69
21/04/2022	6 715.10	+1.36%	6 757.75	6 636.79	6 642.71
20/04/2022	6 624.91	+1.38%	6 636.55	6 542.21	6 557.96
19/04/2022	6 534.79	-0.83%	6 566.80	6 494.33	6 545.97
14/04/2022	6 589.35	+0.72%	6 607.55	6 541.90	6 557.66
13/04/2022	6 542.14	+0.07%	6 563.96	6 478.45	6 519.16
12/04/2022	6 537.41	-0.28%	6 570.75	6 424.97	6 454.25
11/04/2022	6 555.81	+0.12%	6 614.96	6 502.14	6 558.53
08/04/2022	6 548.22	+1.34%	6 581.56	6 494.63	6 540.20
07/04/2022	6 461.68	-0.57%	6 582.34	6 451.10	6 530.82
06/04/2022	6 498.83	-2.21%	6 643.75	6 442.44	6 634.83

05/04/2022	6 645.51	-1.28%	6 743.62	6 603.44	6 731.34
04/04/2022	6 731.37	+0.70%	6 745.22	6 641.61	6 709.60
01/04/2022	6 684.31	+0.37%	6 708.56	6 635.27	6 672.98
31/03/2022	6 659.87	-1.21%	6 776.13	6 658.74	6 759.42
30/03/2022	6 741.59	-0.74%	6 771.09	6 705.69	6 771.09
29/03/2022	6 792.16	+3.08%	6 829.40	6 658.78	6 675.15
28/03/2022	6 589.11	+0.54%	6 683.70	6 572.01	6 578.72
25/03/2022	6 553.68	-0.03%	6 615.90	6 538.80	6 569.45
24/03/2022	6 555.77	-0.39%	6 636.58	6 541.32	6 593.15
23/03/2022	6 581.43	-1.17%	6 684.70	6 559.31	6 674.68
22/03/2022	6 659.41	+1.17%	6 666.84	6 578.91	6 586.65
21/03/2022	6 582.33	-0.57%	6 642.46	6 574.07	6 608.79
18/03/2022	6 620.24	+0.12%	6 620.24	6 499.59	6 613.95
17/03/2022	6 612.52	+0.36%	6 635.31	6 530.79	6 617.23
16/03/2022	6 588.64	+3.68%	6 680.00	6 481.70	6 484.33
15/03/2022	6 355.00	-0.23%	6 383.75	6 206.05	6 294.72
14/03/2022	6 369.94	+1.75%	6 421.08	6 271.82	6 298.96
11/03/2022	6 260.25	+0.85%	6 466.12	6 165.41	6 233.12
10/03/2022	6 207.20	-2.83%	6 380.77	6 166.11	6 376.57
09/03/2022	6 387.83	+7.13%	6 387.83	6 122.80	6 131.28
08/03/2022	5 962.96	-0.32%	6 169.60	5 903.10	5 906.62
07/03/2022	5 982.27	-1.31%	6 115.23	5 756.38	5 887.83
04/03/2022	6 061.66	-4.97%	6 321.09	6 061.66	6 308.34
03/03/2022	6 378.37	-1.84%	6 543.77	6 361.60	6 488.25
02/03/2022	6 498.02	+1.59%	6 544.15	6 312.30	6 359.98
01/03/2022	6 396.49	-3.94%	6 662.25	6 396.49	6 646.16
28/02/2022	6 658.83	-1.39%	6 684.54	6 516.16	6 620.48
25/02/2022	6 752.43	+3.55%	6 762.78	6 521.56	6 571.95
24/02/2022	6 521.05	-3.83%	6 617.09	6 432.89	6 496.25
23/02/2022	6 780.67	-0.10%	6 904.81	6 768.38	6 801.66
22/02/2022	6 787.60	-0.01%	6 834.77	6 633.74	6 633.74
21/02/2022	6 788.34	-2.04%	6 986.83	6 744.13	6 984.31
18/02/2022	6 929.63	-0.25%	6 995.62	6 895.09	6 951.83
17/02/2022	6 946.82	-0.26%	7 017.07	6 905.69	6 999.13
16/02/2022	6 964.98	-0.21%	7 031.93	6 931.09	7 009.69
15/02/2022	6 979.97	+1.86%	6 983.14	6 823.45	6 830.74
14/02/2022	6 852.20	-2.27%	6 875.31	6 757.33	6 868.07
11/02/2022	7 011.60	-1.27%	7 071.01	6 992.09	7 025.50
10/02/2022	7 101.55	-0.41%	7 169.63	7 051.60	7 141.69
09/02/2022	7 130.88	+1.46%	7 146.31	7 086.34	7 086.34
08/02/2022	7 028.41	+0.27%	7 085.64	6 994.15	7 008.20
07/02/2022	7 009.25	+0.83%	7 025.59	6 929.71	6 987.17
04/02/2022	6 951.38	-0.77%	7 066.44	6 914.77	7 049.80
03/02/2022	7 005.63	-1.54%	7 125.58	6 995.41	7 115.30
02/02/2022	7 115.27	+0.22%	7 150.54	7 109.30	7 117.45
01/02/2022	7 099.49	+1.43%	7 105.23	7 035.91	7 052.30
31/01/2022	6 999.20	+0.48%	7 043.22	6 937.10	7 039.71
28/01/2022	6 965.88	-0.82%	7 053.00	6 846.22	7 043.43
27/01/2022	7 023.80	+0.60%	7 053.45	6 873.60	6 879.64
26/01/2022	6 981.96	+2.11%	7 024.14	6 901.74	6 901.74

25/01/2022	6 837.96	+0.74%	6 890.54	6 776.80	6 870.27
24/01/2022	6 787.79	-3.97%	7 070.92	6 754.24	7 014.84
21/01/2022	7 068.59	-1.75%	7 122.06	7 013.49	7 088.44
20/01/2022	7 194.16	+0.30%	7 199.97	7 123.62	7 190.02
19/01/2022	7 172.98	+0.55%	7 211.24	7 104.11	7 104.11
18/01/2022	7 133.83	-0.94%	7 175.46	7 098.19	7 172.43
17/01/2022	7 201.64	+0.82%	7 213.70	7 148.98	7 167.53
14/01/2022	7 143.00	-0.81%	7 171.40	7 119.01	7 132.59
13/01/2022	7 201.14	-0.50%	7 227.35	7 166.94	7 215.48
12/01/2022	7 237.19	+0.75%	7 249.88	7 183.38	7 235.47
11/01/2022	7 183.38	+0.95%	7 227.71	7 160.44	7 185.21
10/01/2022	7 115.77	-1.44%	7 250.24	7 105.55	7 245.21
07/01/2022	7 219.48	-0.42%	7 269.65	7 178.18	7 250.79
06/01/2022	7 249.66	-1.72%	7 316.06	7 240.88	7 264.00
05/01/2022	7 376.37	+0.81%	7 384.86	7 313.53	7 320.48
04/01/2022	7 317.41	+1.39%	7 332.21	7 250.24	7 273.57
03/01/2022	7 217.22	+0.90%	7 245.66	7 195.82	7 197.40
31/12/2021	7 153.03	-0.28%	7 163.73	7 139.60	7 153.66
30/12/2021	7 173.23	+0.16%	7 189.84	7 156.79	7 167.53
29/12/2021	7 161.52	-0.27%	7 201.65	7 137.35	7 179.11
28/12/2021	7 181.11	+0.57%	7 187.76	7 141.67	7 141.85
27/12/2021	7 140.39	+0.76%	7 149.14	7 062.40	7 070.14
24/12/2021	7 086.58	-0.28%	7 116.88	7 086.58	7 091.60
23/12/2021	7 106.15	+0.77%	7 119.87	7 049.92	7 071.93
22/12/2021	7 051.67	+1.24%	7 051.67	6 963.71	6 982.53
21/12/2021	6 964.99	+1.38%	6 967.93	6 888.52	6 935.66
20/12/2021	6 870.10	-0.82%	6 872.75	6 747.69	6 783.83
17/12/2021	6 926.63	-1.12%	7 011.61	6 884.91	6 982.07
16/12/2021	7 005.07	+1.12%	7 054.92	6 987.95	7 054.92
15/12/2021	6 927.63	+0.47%	6 949.92	6 904.94	6 923.52
14/12/2021	6 895.31	-0.69%	6 993.30	6 891.34	6 978.77
13/12/2021	6 942.91	-0.70%	7 029.38	6 929.13	7 004.76
10/12/2021	6 991.68	-0.24%	7 025.88	6 960.39	6 960.39
09/12/2021	7 008.23	-0.09%	7 044.99	6 983.33	7 035.93
08/12/2021	7 014.57	-0.72%	7 104.06	7 014.57	7 064.12
07/12/2021	7 065.39	+2.91%	7 066.95	6 934.68	6 934.68
06/12/2021	6 865.78	+1.48%	6 888.07	6 776.17	6 810.93
03/12/2021	6 765.52	-0.44%	6 872.79	6 730.63	6 860.73
02/12/2021	6 795.75	-1.25%	6 850.37	6 762.74	6 794.77
01/12/2021	6 881.87	+2.39%	6 902.46	6 750.65	6 774.80
30/11/2021	6 721.16	-0.81%	6 780.88	6 655.86	6 679.03
29/11/2021	6 776.25	+0.54%	6 841.79	6 760.31	6 831.41
26/11/2021	6 739.73	-4.75%	6 861.15	6 725.08	6 814.90
25/11/2021	7 075.87	+0.48%	7 079.49	7 047.86	7 067.61
24/11/2021	7 042.23	-0.03%	7 089.11	6 983.98	7 052.18
23/11/2021	7 044.62	-0.85%	7 112.49	7 002.56	7 064.38
22/11/2021	7 105.00	-0.10%	7 143.60	7 087.67	7 130.24
19/11/2021	7 112.29	-0.42%	7 183.08	7 068.43	7 172.72
18/11/2021	7 141.98	-0.21%	7 181.55	7 136.86	7 155.05
17/11/2021	7 156.85	+0.06%	7 167.80	7 147.01	7 149.30

16/11/2021	7 152.60	+0.34%	7 164.27	7 136.30	7 146.37
15/11/2021	7 128.63	+0.53%	7 136.13	7 088.71	7 090.96
12/11/2021	7 091.40	+0.45%	7 097.46	7 056.93	7 062.54
11/11/2021	7 059.55	+0.20%	7 072.63	7 032.35	7 038.84
10/11/2021	7 045.16	+0.03%	7 053.27	7 014.94	7 046.17
09/11/2021	7 043.27	-0.06%	7 079.58	7 016.73	7 023.58
08/11/2021	7 047.48	+0.10%	7 067.10	7 037.67	7 041.60
05/11/2021	7 040.79	+0.76%	7 063.40	6 983.80	6 983.80
04/11/2021	6 987.79	+0.53%	6 993.01	6 962.01	6 977.62
03/11/2021	6 950.65	+0.34%	6 955.10	6 917.75	6 919.47
02/11/2021	6 927.03	+0.49%	6 933.35	6 879.69	6 883.06
01/11/2021	6 893.29	+0.92%	6 908.65	6 859.44	6 870.28
29/10/2021	6 830.34	+0.38%	6 830.34	6 748.53	6 778.79
28/10/2021	6 804.22	+0.75%	6 804.22	6 750.78	6 750.78
27/10/2021	6 753.52	-0.19%	6 771.19	6 731.86	6 755.63
26/10/2021	6 766.51	+0.80%	6 775.82	6 711.21	6 723.06
25/10/2021	6 712.87	-0.31%	6 752.98	6 707.32	6 752.98
22/10/2021	6 733.69	+0.71%	6 765.46	6 716.15	6 720.92
21/10/2021	6 686.17	-0.29%	6 699.21	6 658.95	6 673.14
20/10/2021	6 705.61	+0.54%	6 715.52	6 636.61	6 636.61
19/10/2021	6 669.85	-0.05%	6 692.14	6 650.84	6 690.95
18/10/2021	6 673.10	-0.81%	6 697.60	6 642.16	6 697.29
15/10/2021	6 727.52	+0.63%	6 746.81	6 703.26	6 714.26
14/10/2021	6 685.21	+1.33%	6 692.94	6 618.79	6 651.18
13/10/2021	6 597.38	+0.75%	6 606.36	6 510.27	6 539.61
12/10/2021	6 548.11	-0.34%	6 548.11	6 491.58	6 495.15
11/10/2021	6 570.54	+0.16%	6 574.22	6 522.55	6 546.07
08/10/2021	6 559.99	-0.61%	6 617.34	6 559.99	6 609.51
07/10/2021	6 600.19	+1.65%	6 616.53	6 545.74	6 563.24
06/10/2021	6 493.12	-1.26%	6 510.47	6 421.64	6 509.60
05/10/2021	6 576.28	+1.52%	6 577.29	6 481.72	6 481.72
04/10/2021	6 477.66	-0.61%	6 540.96	6 452.78	6 484.90
01/10/2021	6 517.69	-0.04%	6 535.46	6 412.70	6 421.10
30/09/2021	6 520.01	-0.62%	6 621.75	6 509.23	6 615.99
29/09/2021	6 560.80	+0.83%	6 590.81	6 538.11	6 549.41
28/09/2021	6 506.50	-2.17%	6 643.58	6 491.42	6 634.72
27/09/2021	6 650.91	+0.19%	6 688.76	6 647.08	6 683.85
24/09/2021	6 638.46	-0.95%	6 678.65	6 625.43	6 675.73
23/09/2021	6 701.98	+0.98%	6 718.80	6 667.36	6 691.08
22/09/2021	6 637.00	+1.29%	6 642.31	6 595.93	6 612.10
21/09/2021	6 552.73	+1.50%	6 570.13	6 513.75	6 513.75
20/09/2021	6 455.81	-1.74%	6 471.09	6 389.62	6 450.39
17/09/2021	6 570.19	-0.79%	6 697.08	6 551.62	6 679.45
16/09/2021	6 622.59	+0.59%	6 663.41	6 612.16	6 613.71
15/09/2021	6 583.62	-1.04%	6 659.27	6 577.02	6 654.83
14/09/2021	6 652.97	-0.36%	6 677.07	6 613.52	6 672.19
13/09/2021	6 676.93	+0.20%	6 722.00	6 670.85	6 688.77
10/09/2021	6 663.77	-0.31%	6 722.73	6 651.06	6 707.33
09/09/2021	6 684.72	+0.24%	6 708.33	6 605.09	6 618.82
08/09/2021	6 668.89	-0.85%	6 712.05	6 628.98	6 688.84

07/09/2021	6 726.07	-0.26%	6 749.14	6 721.35	6 731.36
06/09/2021	6 743.50	+0.80%	6 756.40	6 700.45	6 704.35
03/09/2021	6 689.99	-1.08%	6 758.38	6 667.39	6 758.38
02/09/2021	6 763.08	+0.06%	6 778.33	6 747.17	6 753.18
01/09/2021	6 758.69	+1.18%	6 784.66	6 730.17	6 736.44
31/08/2021	6 680.18	-0.11%	6 708.14	6 650.90	6 692.85
30/08/2021	6 687.30	+0.08%	6 697.52	6 675.32	6 690.28
27/08/2021	6 681.92	+0.24%	6 682.15	6 644.68	6 658.27
26/08/2021	6 666.03	-0.16%	6 673.94	6 618.69	6 646.37
25/08/2021	6 676.48	+0.18%	6 684.21	6 667.25	6 683.18
24/08/2021	6 664.31	-0.28%	6 717.77	6 634.69	6 717.77
23/08/2021	6 683.10	+0.86%	6 699.32	6 657.14	6 699.32
20/08/2021	6 626.11	+0.31%	6 633.19	6 567.93	6 602.95
19/08/2021	6 605.89	-2.43%	6 671.17	6 560.74	6 665.12
18/08/2021	6 770.11	-0.73%	6 828.33	6 759.74	6 824.64
17/08/2021	6 819.84	-0.28%	6 827.49	6 789.11	6 812.42
16/08/2021	6 838.77	-0.83%	6 863.74	6 817.43	6 849.68
13/08/2021	6 896.04	+0.20%	6 913.67	6 885.64	6 888.48
12/08/2021	6 882.47	+0.36%	6 891.29	6 854.99	6 861.44
11/08/2021	6 857.99	+0.55%	6 867.11	6 821.21	6 840.28
10/08/2021	6 820.21	+0.10%	6 833.12	6 811.77	6 817.63
09/08/2021	6 813.18	-0.06%	6 832.77	6 806.98	6 818.54
06/08/2021	6 816.96	+0.53%	6 832.94	6 763.01	6 772.49
05/08/2021	6 781.19	+0.52%	6 788.69	6 750.86	6 750.86
04/08/2021	6 746.23	+0.33%	6 766.61	6 730.60	6 751.15
03/08/2021	6 723.81	+0.72%	6 749.88	6 688.12	6 692.60
02/08/2021	6 675.90	+0.95%	6 690.80	6 639.31	6 657.95
30/07/2021	6 612.76	-0.32%	6 648.20	6 596.99	6 608.16
29/07/2021	6 633.77	+0.37%	6 671.12	6 633.77	6 647.08
28/07/2021	6 609.31	+1.18%	6 609.31	6 530.74	6 547.47
27/07/2021	6 531.92	-0.71%	6 578.04	6 516.57	6 565.19
26/07/2021	6 578.60	+0.15%	6 588.09	6 517.05	6 532.01
23/07/2021	6 568.82	+1.35%	6 574.86	6 510.27	6 518.60
22/07/2021	6 481.59	+0.26%	6 525.21	6 474.05	6 498.31
21/07/2021	6 464.48	+1.85%	6 471.65	6 381.94	6 382.36
20/07/2021	6 346.85	+0.81%	6 386.46	6 304.85	6 325.16
19/07/2021	6 295.97	-2.54%	6 403.51	6 253.25	6 398.84
16/07/2021	6 460.08	-0.51%	6 526.76	6 418.25	6 524.95
15/07/2021	6 493.36	-0.99%	6 554.79	6 471.07	6 538.41
14/07/2021	6 558.38	-0.00%	6 565.20	6 530.91	6 533.80
13/07/2021	6 558.47	-0.01%	6 569.37	6 536.45	6 561.74
12/07/2021	6 559.25	+0.46%	6 567.98	6 479.54	6 525.11
09/07/2021	6 529.42	+2.07%	6 532.56	6 429.67	6 435.29
08/07/2021	6 396.73	-2.01%	6 483.32	6 348.64	6 479.40
07/07/2021	6 527.72	+0.31%	6 550.95	6 497.75	6 529.49
06/07/2021	6 507.48	-0.91%	6 561.51	6 483.20	6 551.00
05/07/2021	6 567.54	+0.22%	6 582.87	6 520.38	6 545.98
02/07/2021	6 552.86	-0.01%	6 582.79	6 540.12	6 572.00
01/07/2021	6 553.82	+0.71%	6 588.14	6 502.05	6 549.82
30/06/2021	6 507.83	-0.91%	6 575.84	6 474.43	6 562.17

29/06/2021	6 567.43	+0.14%	6 599.88	6 561.09	6 564.07
28/06/2021	6 558.02	-0.98%	6 633.73	6 556.88	6 611.99
25/06/2021	6 622.87	-0.12%	6 641.90	6 604.86	6 637.68
24/06/2021	6 631.15	+1.22%	6 639.83	6 572.81	6 576.07
23/06/2021	6 551.07	-0.91%	6 619.02	6 551.07	6 616.73
22/06/2021	6 611.50	+0.14%	6 622.20	6 583.50	6 609.98
21/06/2021	6 602.54	+0.51%	6 607.99	6 511.93	6 525.03
18/06/2021	6 569.16	-1.46%	6 687.29	6 560.12	6 657.23
17/06/2021	6 666.26	+0.20%	6 674.37	6 631.63	6 636.70
16/06/2021	6 652.65	+0.20%	6 659.38	6 632.36	6 657.52
15/06/2021	6 616.35	+0.35%	6 655.66	6 634.45	6 643.42
14/06/2021	6 616.35	+0.24%	6 650.16	6 599.27	6 625.55
11/06/2021	6 600.66	+0.83%	6 607.61	6 549.83	6 550.30
10/06/2021	6 546.49	-0.26%	6 574.59	6 526.23	6 574.59
09/06/2021	6 563.45	+0.19%	6 570.82	6 533.37	6 561.93
08/06/2021	6 551.01	+0.11%	6 574.21	6 540.71	6 545.69
07/06/2021	6 543.56	+0.43%	6 560.39	6 485.80	6 509.98
04/06/2021	6 515.66	+0.12%	6 517.93	6 497.39	6 514.27
03/06/2021	6 507.92	-0.21%	6 522.23	6 473.67	6 518.58
02/06/2021	6 521.52	+0.49%	6 521.82	6 482.10	6 489.90
01/06/2021	6 489.40	+0.66%	6 521.57	6 464.71	6 470.42
31/05/2021	6 447.17	-0.57%	6 496.32	6 441.76	6 483.32
28/05/2021	6 484.11	+0.75%	6 493.72	6 448.87	6 453.12
27/05/2021	6 435.71	+0.69%	6 467.72	6 379.83	6 379.83
26/05/2021	6 391.60	+0.02%	6 413.72	6 375.14	6 403.95
25/05/2021	6 390.27	-0.28%	6 422.83	6 390.27	6 416.61
24/05/2021	6 408.49	+0.35%	6 408.49	6 382.71	6 395.96
21/05/2021	6 386.41	+0.68%	6 402.10	6 351.95	6 360.88
20/05/2021	6 343.58	+1.29%	6 343.58	6 267.55	6 296.25
19/05/2021	6 262.55	-1.43%	6 310.49	6 192.33	6 292.05
18/05/2021	6 353.67	-0.21%	6 412.91	6 343.55	6 406.18
17/05/2021	6 367.35	-0.28%	6 410.21	6 349.33	6 397.22
14/05/2021	6 385.14	+1.54%	6 385.67	6 301.19	6 335.82
13/05/2021	6 288.33	+0.14%	6 303.70	6 150.43	6 219.73
12/05/2021	6 279.35	+0.19%	6 299.24	6 233.96	6 259.63
11/05/2021	6 267.39	-1.86%	6 324.31	6 226.11	6 314.27
10/05/2021	6 385.99	+0.01%	6 395.60	6 367.04	6 395.38
07/05/2021	6 385.51	+0.45%	6 390.04	6 342.37	6 388.92
06/05/2021	6 357.09	+0.28%	6 377.58	6 321.72	6 348.90
05/05/2021	6 339.47	+1.40%	6 339.47	6 275.14	6 284.54
04/05/2021	6 251.75	-0.89%	6 355.87	6 238.79	6 319.49
03/05/2021	6 307.90	+0.61%	6 314.92	6 252.66	6 285.51
30/04/2021	6 269.48	-0.53%	6 320.14	6 268.86	6 316.32
29/04/2021	6 302.57	-0.07%	6 352.36	6 287.44	6 341.12
28/04/2021	6 306.98	+0.53%	6 320.11	6 283.62	6 293.21
27/04/2021	6 273.76	-0.03%	6 282.52	6 255.41	6 273.36
26/04/2021	6 275.52	+0.28%	6 288.79	6 240.99	6 256.04
23/04/2021	6 257.94	-0.15%	6 280.81	6 227.19	6 259.63
22/04/2021	6 267.28	+0.91%	6 274.19	6 225.59	6 230.21
21/04/2021	6 210.55	+0.74%	6 227.04	6 178.66	6 180.69

20/04/2021	6 165.11	-2.09%	6 285.29	6 154.03	6 281.10
19/04/2021	6 296.69	+0.15%	6 319.08	6 293.17	6 294.74
16/04/2021	6 287.07	+0.85%	6 299.56	6 229.30	6 229.65
15/04/2021	6 234.14	+0.41%	6 244.23	6 209.59	6 219.04
14/04/2021	6 208.58	+0.40%	6 217.74	6 194.55	6 214.49
13/04/2021	6 184.10	+0.36%	6 193.24	6 156.49	6 171.88
12/04/2021	6 161.68	-0.13%	6 183.02	6 151.52	6 167.32
09/04/2021	6 169.41	+0.06%	6 188.48	6 161.42	6 171.60
08/04/2021	6 165.72	+0.57%	6 171.45	6 146.22	6 153.36
07/04/2021	6 130.66	-0.01%	6 154.38	6 118.91	6 137.63
06/04/2021	6 131.34	+0.47%	6 159.10	6 125.74	6 155.22
01/04/2021	6 102.96	+0.59%	6 106.12	6 063.86	6 079.67
31/03/2021	6 067.23	-0.34%	6 097.74	6 060.24	6 085.02
30/03/2021	6 088.04	+1.21%	6 095.08	6 041.39	6 044.54
29/03/2021	6 015.51	+0.45%	6 029.59	5 985.36	5 995.00
26/03/2021	5 988.81	+0.61%	6 002.95	5 962.46	5 979.23
25/03/2021	5 952.41	+0.09%	5 952.41	5 886.00	5 911.04
24/03/2021	5 947.29	+0.03%	5 948.67	5 890.55	5 901.64
23/03/2021	5 945.30	-0.39%	5 968.56	5 921.83	5 943.31
22/03/2021	5 968.48	-0.49%	5 987.23	5 943.71	5 960.85
19/03/2021	5 997.96	-1.07%	6 047.65	5 983.61	6 021.13
18/03/2021	6 062.79	+0.13%	6 082.92	6 044.32	6 076.21
17/03/2021	6 054.82	-0.01%	6 062.36	6 037.25	6 048.31
16/03/2021	6 055.43	+0.32%	6 061.19	6 034.76	6 050.40
15/03/2021	6 035.97	-0.17%	6 089.20	6 018.55	6 069.09
12/03/2021	6 046.55	+0.21%	6 046.55	6 017.41	6 030.63
11/03/2021	6 033.76	+0.72%	6 033.76	5 995.41	6 004.16
10/03/2021	5 990.55	+1.11%	5 993.63	5 912.35	5 915.17
09/03/2021	5 924.97	+0.37%	5 937.81	5 895.33	5 903.26
08/03/2021	5 902.99	+2.08%	5 912.50	5 799.05	5 822.92
05/03/2021	5 782.65	-0.82%	5 837.55	5 755.60	5 794.48
04/03/2021	5 830.65	+0.01%	5 844.69	5 797.67	5 800.73
03/03/2021	5 830.06	+0.35%	5 871.43	5 788.66	5 841.20
02/03/2021	5 809.73	+0.29%	5 835.74	5 773.52	5 774.62
01/03/2021	5 792.79	+1.57%	5 804.35	5 765.80	5 770.89
26/02/2021	5 703.22	-1.39%	5 768.03	5 688.26	5 715.69
25/02/2021	5 783.89	-0.24%	5 834.36	5 783.89	5 824.72
24/02/2021	5 797.98	+0.31%	5 804.14	5 758.30	5 769.21
23/02/2021	5 779.84	+0.22%	5 804.90	5 721.92	5 781.24
22/02/2021	5 767.44	-0.11%	5 776.64	5 703.48	5 737.47
19/02/2021	5 773.55	+0.79%	5 784.90	5 733.84	5 737.19
18/02/2021	5 728.33	-0.65%	5 779.01	5 720.58	5 757.75
17/02/2021	5 765.84	-0.36%	5 790.27	5 752.85	5 762.01
16/02/2021	5 786.53	+0.00%	5 797.92	5 768.96	5 796.43
15/02/2021	5 786.25	+1.45%	5 801.38	5 722.16	5 732.30
12/02/2021	5 703.67	+0.60%	5 705.02	5 627.99	5 651.67
11/02/2021	5 669.82	-0.02%	5 692.00	5 656.80	5 684.94
10/02/2021	5 670.80	-0.36%	5 719.32	5 647.28	5 719.32
09/02/2021	5 691.54	+0.10%	5 703.53	5 674.43	5 703.53
08/02/2021	5 686.03	+0.47%	5 714.28	5 673.63	5 688.68

05/02/2021	5 659.26	+0.90%	5 673.03	5 627.84	5 638.74
04/02/2021	5 608.54	+0.82%	5 613.95	5 566.58	5 568.39
03/02/2021	5 563.05	-0.00%	5 626.41	5 554.09	5 610.59
02/02/2021	5 563.11	+1.86%	5 574.77	5 509.24	5 510.62
01/02/2021	5 461.68	+1.16%	5 481.67	5 429.78	5 441.43
29/01/2021	5 399.21	-2.02%	5 474.80	5 399.21	5 421.53
28/01/2021	5 510.52	+0.93%	5 525.95	5 379.22	5 407.74
27/01/2021	5 459.62	-1.16%	5 545.45	5 401.43	5 517.53
26/01/2021	5 523.52	+0.93%	5 555.28	5 471.09	5 473.25
25/01/2021	5 472.36	-1.57%	5 584.40	5 453.24	5 579.73
22/01/2021	5 559.57	-0.56%	5 573.71	5 511.44	5 568.88
21/01/2021	5 590.79	-0.67%	5 665.98	5 589.80	5 660.27
20/01/2021	5 628.44	+0.53%	5 646.06	5 602.49	5 615.41
19/01/2021	5 598.61	-0.33%	5 655.22	5 592.20	5 647.75
18/01/2021	5 617.27	+0.10%	5 618.42	5 584.59	5 584.85
15/01/2021	5 611.69	-1.22%	5 656.45	5 563.74	5 648.70
14/01/2021	5 681.14	+0.33%	5 690.49	5 656.75	5 662.17
13/01/2021	5 662.67	+0.21%	5 679.07	5 643.75	5 648.96
12/01/2021	5 650.97	-0.20%	5 679.06	5 636.75	5 672.20
11/01/2021	5 662.43	-0.78%	5 704.09	5 629.46	5 684.66
08/01/2021	5 706.88	+0.65%	5 721.89	5 676.15	5 711.58
07/01/2021	5 669.85	+0.70%	5 689.28	5 629.49	5 651.04
06/01/2021	5 630.60	+1.19%	5 648.42	5 553.39	5 601.01
05/01/2021	5 564.60	-0.44%	5 603.66	5 530.48	5 561.60
04/01/2021	5 588.96	+0.68%	5 656.42	5 567.97	5 614.04
31/12/2020	5 551.41	-0.86%	5 598.93	5 551.41	5 573.20
30/12/2020	5 599.41	-0.22%	5 625.60	5 594.11	5 603.72
29/12/2020	5 611.79	+0.42%	5 625.52	5 603.74	5 610.13
28/12/2020	5 588.38	+1.20%	5 601.00	5 547.90	5 562.04
24/12/2020	5 522.01	-0.10%	5 545.71	5 517.01	5 542.49
23/12/2020	5 527.59	+1.11%	5 539.16	5 472.03	5 472.14
22/12/2020	5 466.86	+1.36%	5 477.87	5 411.15	5 411.15
21/12/2020	5 393.34	-2.43%	5 422.10	5 306.58	5 392.55
18/12/2020	5 527.84	-0.39%	5 581.96	5 519.17	5 533.78
17/12/2020	5 549.46	+0.03%	5 585.51	5 547.59	5 578.79
16/12/2020	5 547.68	+0.31%	5 589.76	5 515.50	5 551.45
15/12/2020	5 530.31	+0.04%	5 565.52	5 517.06	5 521.79
14/12/2020	5 527.84	+0.37%	5 577.18	5 527.84	5 547.30
11/12/2020	5 507.55	-0.76%	5 532.81	5 466.56	5 528.14
10/12/2020	5 549.65	+0.05%	5 572.80	5 511.48	5 558.96
09/12/2020	5 546.82	-0.25%	5 599.41	5 545.94	5 590.19
08/12/2020	5 560.67	-0.23%	5 574.57	5 521.87	5 553.79
07/12/2020	5 573.38	-0.64%	5 595.83	5 535.42	5 594.50
04/12/2020	5 609.15	+0.62%	5 616.62	5 574.12	5 574.12
03/12/2020	5 574.36	-0.15%	5 584.10	5 546.29	5 581.53
02/12/2020	5 583.01	+0.02%	5 585.87	5 549.79	5 563.05
01/12/2020	5 581.64	+1.14%	5 589.68	5 529.25	5 542.46
30/11/2020	5 518.55	-1.42%	5 599.94	5 518.55	5 564.16
27/11/2020	5 598.18	+0.56%	5 612.76	5 556.62	5 563.04
26/11/2020	5 566.79	-0.08%	5 592.78	5 559.72	5 589.38

25/11/2020	5 571.29	+0.23%	5 589.78	5 545.01	5 577.97
24/11/2020	5 558.42	+1.21%	5 574.69	5 536.11	5 546.07
23/11/2020	5 492.15	-0.07%	5 555.83	5 492.15	5 543.83
20/11/2020	5 495.89	+0.39%	5 524.25	5 461.39	5 463.99
19/11/2020	5 474.66	-0.67%	5 490.18	5 449.45	5 458.02
18/11/2020	5 511.45	+0.52%	5 521.01	5 452.12	5 468.17
17/11/2020	5 483.00	+0.21%	5 487.31	5 441.26	5 468.29
16/11/2020	5 471.48	+1.70%	5 518.91	5 402.31	5 430.57
13/11/2020	5 380.16	+0.33%	5 411.12	5 343.25	5 343.32
12/11/2020	5 362.57	-1.52%	5 421.40	5 348.63	5 394.37
11/11/2020	5 445.21	+0.48%	5 462.95	5 408.67	5 433.66
10/11/2020	5 418.97	+1.55%	5 439.08	5 327.22	5 351.31
09/11/2020	5 336.32	+7.57%	5 387.49	5 021.33	5 036.92
06/11/2020	4 960.88	-0.46%	4 997.74	4 915.24	4 964.26
05/11/2020	4 983.99	+1.24%	4 998.19	4 942.18	4 956.02
04/11/2020	4 922.85	+2.44%	4 926.56	4 730.31	4 736.46
03/11/2020	4 805.61	+2.44%	4 812.29	4 730.23	4 743.86
02/11/2020	4 691.14	+2.11%	4 705.11	4 581.86	4 614.95
30/10/2020	4 594.24	+0.54%	4 601.74	4 519.37	4 519.37
29/10/2020	4 569.67	-0.03%	4 606.72	4 512.57	4 571.13
28/10/2020	4 571.12	-3.37%	4 640.63	4 521.61	4 635.19
27/10/2020	4 730.66	-1.77%	4 818.98	4 723.70	4 818.98
26/10/2020	4 816.12	-1.90%	4 896.47	4 814.99	4 844.64
23/10/2020	4 909.64	+1.20%	4 934.71	4 843.14	4 843.28
22/10/2020	4 851.38	-0.05%	4 873.90	4 783.65	4 830.44
21/10/2020	4 853.95	-1.53%	4 945.33	4 849.01	4 945.33
20/10/2020	4 929.28	-0.27%	4 968.61	4 914.49	4 932.87
19/10/2020	4 942.62	+0.14%	4 993.39	4 881.46	4 955.39
16/10/2020	4 935.86	+2.03%	4 957.60	4 875.76	4 904.10
15/10/2020	4 837.42	-2.11%	4 881.31	4 808.58	4 873.03
14/10/2020	4 941.66	-0.12%	4 965.32	4 928.43	4 956.69
13/10/2020	4 947.61	-0.64%	4 976.58	4 928.64	4 976.28
12/10/2020	4 979.29	+0.66%	4 998.51	4 944.00	4 952.47
09/10/2020	4 946.81	+0.71%	4 954.02	4 918.06	4 925.41
08/10/2020	4 911.94	+0.61%	4 936.04	4 877.10	4 910.49
07/10/2020	4 882.00	-0.27%	4 915.63	4 865.26	4 888.49
06/10/2020	4 895.46	+0.48%	4 925.71	4 850.36	4 881.80
05/10/2020	4 871.87	+0.97%	4 876.86	4 842.26	4 863.63
02/10/2020	4 824.88	+0.02%	4 824.88	4 757.22	4 769.43
01/10/2020	4 824.04	+0.43%	4 871.83	4 804.77	4 850.20
30/09/2020	4 803.44	-0.59%	4 854.02	4 782.69	4 793.19
29/09/2020	4 832.07	-0.23%	4 846.63	4 806.45	4 832.87
28/09/2020	4 843.27	+2.40%	4 849.97	4 788.18	4 806.14
25/09/2020	4 729.66	-0.69%	4 758.05	4 666.26	4 754.81
24/09/2020	4 762.62	-0.83%	4 806.83	4 744.04	4 746.10
23/09/2020	4 802.26	+0.62%	4 871.45	4 802.26	4 824.17
22/09/2020	4 772.84	-0.40%	4 827.00	4 772.84	4 809.57
21/09/2020	4 792.04	-3.74%	4 949.76	4 779.21	4 949.76
18/09/2020	4 978.18	-1.22%	5 041.33	4 978.18	5 035.94
17/09/2020	5 039.50	-0.69%	5 053.32	4 995.65	5 005.89

16/09/2020	5 074.42	+0.13%	5 091.52	5 029.56	5 065.75
15/09/2020	5 067.93	+0.32%	5 087.99	5 035.10	5 056.90
14/09/2020	5 051.88	+0.35%	5 087.52	5 034.91	5 071.87
11/09/2020	5 034.14	+0.20%	5 053.63	4 997.86	5 014.97
10/09/2020	5 023.93	-0.38%	5 062.89	5 005.84	5 039.48
09/09/2020	5 042.98	+1.40%	5 058.35	4 973.36	4 980.46
08/09/2020	4 973.52	-1.59%	5 061.60	4 935.38	5 052.88
07/09/2020	5 053.72	+1.79%	5 064.64	4 979.87	4 979.87
04/09/2020	4 965.07	-0.89%	5 068.88	4 928.79	4 974.69
03/09/2020	5 009.52	-0.44%	5 131.39	4 982.34	5 074.64
02/09/2020	5 031.74	+1.90%	5 062.01	4 959.64	4 969.62
01/09/2020	4 938.10	-0.18%	4 993.56	4 892.83	4 974.42
31/08/2020	4 947.22	-1.11%	5 067.55	4 942.18	5 041.34
28/08/2020	5 002.94	-0.26%	5 031.53	4 970.64	5 031.31
27/08/2020	5 015.97	-0.64%	5 052.26	5 005.46	5 052.26
26/08/2020	5 048.43	+0.80%	5 050.10	4 977.94	4 991.63
25/08/2020	5 008.27	+0.01%	5 073.61	5 008.27	5 023.06
24/08/2020	5 007.89	+2.28%	5 013.70	4 948.71	4 948.71
21/08/2020	4 896.33	-0.30%	4 939.25	4 839.08	4 927.61
20/08/2020	4 911.24	-1.33%	4 937.95	4 888.15	4 914.69
19/08/2020	4 977.23	+0.79%	4 977.23	4 917.53	4 934.79
18/08/2020	4 938.06	-0.68%	5 001.29	4 924.15	4 951.98
17/08/2020	4 971.94	+0.18%	4 995.55	4 937.72	4 972.59
14/08/2020	4 962.93	-1.58%	5 018.95	4 921.58	5 018.95
13/08/2020	5 042.38	-0.61%	5 079.02	5 037.55	5 055.42
12/08/2020	5 073.31	+0.90%	5 096.53	5 010.80	5 017.73
11/08/2020	5 027.99	+2.41%	5 052.12	4 949.05	4 952.41
10/08/2020	4 909.51	+0.41%	4 937.63	4 878.58	4 905.27
07/08/2020	4 889.52	+0.09%	4 896.21	4 843.90	4 873.10
06/08/2020	4 885.13	-0.98%	4 951.86	4 860.98	4 919.41
05/08/2020	4 933.34	+0.90%	4 954.60	4 907.41	4 916.96
04/08/2020	4 889.52	+0.28%	4 920.80	4 856.65	4 905.66
03/08/2020	4 875.93	+1.93%	4 899.52	4 763.60	4 797.06
31/07/2020	4 783.69	-1.43%	4 902.90	4 783.69	4 866.24
30/07/2020	4 852.94	-2.13%	4 952.38	4 801.02	4 952.08
29/07/2020	4 958.74	+0.60%	4 978.50	4 945.58	4 948.48
28/07/2020	4 928.94	-0.22%	4 950.01	4 893.63	4 930.10
27/07/2020	4 939.62	-0.34%	4 957.77	4 923.19	4 944.30
24/07/2020	4 956.43	-1.54%	4 982.12	4 925.41	4 973.73
23/07/2020	5 033.76	-0.07%	5 083.97	5 019.21	5 064.72
22/07/2020	5 037.12	-1.32%	5 099.29	5 028.48	5 098.48
21/07/2020	5 104.28	+0.22%	5 172.98	5 097.42	5 129.78
20/07/2020	5 093.18	+0.47%	5 100.79	5 015.11	5 058.48
17/07/2020	5 069.42	-0.31%	5 091.13	5 045.07	5 088.00
16/07/2020	5 085.28	-0.46%	5 105.12	5 048.32	5 076.52
15/07/2020	5 108.98	+2.03%	5 145.33	5 039.19	5 045.92
14/07/2020	5 007.46	-0.96%	5 007.46	4 941.73	4 990.48
13/07/2020	5 056.23	+1.73%	5 060.43	4 980.12	5 026.20
10/07/2020	4 970.48	+1.01%	4 974.37	4 882.50	4 891.61
09/07/2020	4 921.01	-1.21%	5 014.85	4 911.56	5 006.52

08/07/2020	4 981.13	-1.24%	5 040.52	4 969.83	5 004.82
07/07/2020	5 043.73	-0.74%	5 056.52	5 009.03	5 054.04
06/07/2020	5 081.51	+1.49%	5 121.72	5 059.07	5 112.50
03/07/2020	5 007.14	-0.84%	5 062.51	4 982.24	5 056.67
02/07/2020	5 049.38	+2.49%	5 072.71	4 956.66	4 963.13
01/07/2020	4 926.94	-0.18%	4 963.37	4 851.37	4 939.80
30/06/2020	4 935.99	-0.19%	4 967.35	4 901.96	4 958.94
29/06/2020	4 945.46	+0.73%	4 977.63	4 867.33	4 887.23
26/06/2020	4 909.64	-0.18%	5 014.83	4 908.15	4 965.72
25/06/2020	4 918.58	+0.97%	4 941.96	4 795.05	4 860.16
24/06/2020	4 871.36	-2.92%	5 004.04	4 871.36	4 985.63
23/06/2020	5 017.68	+1.39%	5 046.31	4 962.60	4 972.88
22/06/2020	4 948.70	-0.62%	5 006.40	4 902.06	4 928.01
19/06/2020	4 979.45	+0.42%	5 040.47	4 979.45	4 997.53
18/06/2020	4 958.75	-0.75%	5 017.18	4 908.60	4 978.30
17/06/2020	4 995.97	+0.88%	5 026.84	4 952.57	4 952.57
16/06/2020	4 952.46	+2.84%	5 006.79	4 887.56	4 922.79
15/06/2020	4 815.72	-0.49%	4 841.53	4 691.81	4 716.98
12/06/2020	4 839.26	+0.49%	4 940.79	4 759.17	4 774.11
11/06/2020	4 815.60	-4.71%	4 956.30	4 815.60	4 925.95
10/06/2020	5 053.42	-0.82%	5 152.45	5 043.57	5 119.86
09/06/2020	5 095.11	-1.55%	5 203.42	5 053.21	5 187.09
08/06/2020	5 175.52	-0.43%	5 213.67	5 137.20	5 159.30
05/06/2020	5 197.79	+3.71%	5 199.65	5 060.92	5 060.92
04/06/2020	5 011.98	-0.21%	5 052.75	4 966.82	4 994.18
03/06/2020	5 022.38	+3.36%	5 026.45	4 909.99	4 911.45
02/06/2020	4 858.97	+2.02%	4 880.15	4 795.28	4 799.88
01/06/2020	4 762.78	+1.43%	4 782.03	4 721.63	4 776.72
29/05/2020	4 695.44	-1.59%	4 768.85	4 695.36	4 733.98
28/05/2020	4 771.39	+1.76%	4 784.09	4 701.52	4 718.11
27/05/2020	4 688.74	+1.79%	4 711.99	4 614.41	4 621.77
26/05/2020	4 606.24	+1.46%	4 619.84	4 573.89	4 588.30
25/05/2020	4 539.91	+2.15%	4 539.91	4 451.56	4 482.54
22/05/2020	4 444.56	-0.02%	4 475.35	4 368.80	4 381.82
21/05/2020	4 445.45	-1.15%	4 498.08	4 429.27	4 443.92
20/05/2020	4 496.98	+0.87%	4 499.79	4 396.20	4 434.62
19/05/2020	4 458.16	-0.89%	4 536.07	4 421.11	4 536.07
18/05/2020	4 498.34	+5.16%	4 508.95	4 354.73	4 361.03
15/05/2020	4 277.63	+0.11%	4 335.77	4 255.19	4 314.97
14/05/2020	4 273.13	-1.65%	4 313.26	4 194.58	4 294.01
13/05/2020	4 344.95	-2.85%	4 422.74	4 336.56	4 420.50
12/05/2020	4 472.50	-0.39%	4 499.39	4 453.65	4 473.43
11/05/2020	4 490.22	-1.31%	4 569.76	4 460.24	4 561.19
08/05/2020	4 549.64	+1.07%	4 560.19	4 519.59	4 538.09
07/05/2020	4 501.44	+1.54%	4 503.94	4 440.98	4 456.44
06/05/2020	4 433.38	-1.11%	4 484.16	4 425.78	4 484.16
05/05/2020	4 483.13	+2.40%	4 492.52	4 419.95	4 455.74
04/05/2020	4 378.23	-4.24%	4 427.74	4 362.13	4 413.14
30/04/2020	4 572.18	-2.12%	4 719.74	4 554.91	4 693.28
29/04/2020	4 671.11	+2.22%	4 678.93	4 552.81	4 566.29

28/04/2020	4 569.79	+1.43%	4 601.94	4 493.51	4 507.42
27/04/2020	4 505.26	+2.55%	4 505.26	4 448.05	4 479.86
24/04/2020	4 393.32	-1.30%	4 448.73	4 366.09	4 387.03
23/04/2020	4 451.00	+0.89%	4 495.90	4 395.75	4 450.19
22/04/2020	4 411.80	+1.25%	4 417.57	4 361.44	4 382.20
21/04/2020	4 357.46	-3.77%	4 477.31	4 357.46	4 467.49
20/04/2020	4 528.30	+0.65%	4 531.34	4 426.24	4 531.34
17/04/2020	4 499.01	+3.42%	4 550.78	4 464.09	4 470.92
16/04/2020	4 350.16	-0.08%	4 418.65	4 318.90	4 402.38
15/04/2020	4 353.72	-3.76%	4 525.50	4 335.59	4 511.87
14/04/2020	4 523.91	+0.38%	4 577.84	4 497.56	4 553.55
09/04/2020	4 506.85	+1.44%	4 543.69	4 409.04	4 512.30
08/04/2020	4 442.75	+0.10%	4 442.75	4 333.09	4 397.36
07/04/2020	4 438.27	+2.12%	4 527.60	4 379.27	4 489.40
06/04/2020	4 346.14	+4.61%	4 353.21	4 268.52	4 298.50
03/04/2020	4 154.58	-1.57%	4 214.59	4 142.28	4 204.81
02/04/2020	4 220.96	+0.33%	4 265.56	4 143.16	4 243.84
01/04/2020	4 207.24	-4.30%	4 266.69	4 186.45	4 259.94
31/03/2020	4 396.12	+0.40%	4 468.52	4 309.72	4 437.78
30/03/2020	4 378.51	+0.62%	4 379.15	4 216.41	4 362.09
27/03/2020	4 351.49	-4.23%	4 471.37	4 288.66	4 433.95
26/03/2020	4 543.58	+2.51%	4 543.58	4 296.06	4 332.79
25/03/2020	4 432.30	+4.47%	4 453.01	4 221.32	4 339.71
24/03/2020	4 242.70	+8.39%	4 242.70	4 038.06	4 087.61
23/03/2020	3 914.31	-3.32%	4 097.82	3 851.17	3 869.01
20/03/2020	4 048.80	+5.01%	4 109.11	3 984.30	4 066.79
19/03/2020	3 855.50	+2.68%	3 909.15	3 691.08	3 833.99
18/03/2020	3 754.84	-5.94%	3 908.72	3 726.45	3 905.53
17/03/2020	3 991.78	+2.84%	4 042.46	3 759.00	4 041.33
16/03/2020	3 881.46	-5.75%	3 962.01	3 632.06	3 886.82
13/03/2020	4 118.36	+1.83%	4 438.51	4 055.19	4 234.37
12/03/2020	4 044.26	-12.28%	4 404.26	4 025.89	4 374.67
11/03/2020	4 610.25	-0.57%	4 766.00	4 603.05	4 716.22
10/03/2020	4 636.61	-1.51%	4 924.84	4 615.16	4 770.62
09/03/2020	4 707.91	-8.39%	4 863.34	4 691.20	4 845.27
06/03/2020	5 139.11	-4.14%	5 284.08	5 117.57	5 253.82
05/03/2020	5 361.10	-1.90%	5 494.16	5 329.24	5 490.52
04/03/2020	5 464.89	+1.33%	5 493.25	5 357.35	5 400.43
03/03/2020	5 393.17	+1.12%	5 509.28	5 371.66	5 408.15
02/03/2020	5 333.52	+0.44%	5 430.05	5 197.57	5 416.02
28/02/2020	5 309.90	-3.38%	5 376.05	5 229.56	5 310.82
27/02/2020	5 495.60	-3.32%	5 613.72	5 421.31	5 571.72
26/02/2020	5 684.55	+0.09%	5 707.31	5 526.14	5 646.14
25/02/2020	5 679.68	-1.94%	5 828.46	5 670.71	5 825.37
24/02/2020	5 791.87	-3.94%	5 884.86	5 765.17	5 875.86
21/02/2020	6 029.72	-0.54%	6 067.29	5 995.14	6 035.03
20/02/2020	6 062.30	-0.80%	6 110.95	6 062.30	6 105.17
19/02/2020	6 111.24	+0.90%	6 111.41	6 072.66	6 086.11
18/02/2020	6 056.82	-0.48%	6 079.50	6 039.94	6 042.38
17/02/2020	6 085.95	+0.27%	6 088.60	6 064.92	6 079.53

14/02/2020	6 069.35	-0.39%	6 096.07	6 067.14	6 096.07
13/02/2020	6 093.14	-0.19%	6 098.26	6 028.28	6 076.92
12/02/2020	6 104.73	+0.83%	6 104.73	6 062.97	6 064.25
11/02/2020	6 054.76	+0.65%	6 060.97	6 032.79	6 047.96
10/02/2020	6 015.67	-0.23%	6 018.31	5 993.91	6 008.48
07/02/2020	6 029.75	-0.14%	6 044.97	5 999.95	6 027.54
06/02/2020	6 038.18	+0.88%	6 050.94	6 008.55	6 045.04
05/02/2020	5 985.40	+0.85%	6 004.33	5 912.49	5 919.06
04/02/2020	5 935.05	+1.76%	5 935.05	5 862.82	5 863.36
03/02/2020	5 832.51	+0.45%	5 857.40	5 804.14	5 822.04
31/01/2020	5 806.34	-1.11%	5 894.69	5 799.04	5 891.71
30/01/2020	5 871.77	-1.40%	5 904.08	5 846.45	5 886.02
29/01/2020	5 954.89	+0.49%	5 969.73	5 916.23	5 921.97
28/01/2020	5 925.82	+1.07%	5 933.72	5 857.23	5 886.81
27/01/2020	5 863.02	-2.68%	5 942.83	5 851.00	5 924.06
24/01/2020	6 024.26	+0.88%	6 064.55	6 016.77	6 019.60
23/01/2020	5 971.79	-0.65%	6 024.32	5 961.87	5 992.47
22/01/2020	6 010.98	-0.58%	6 069.26	6 005.77	6 053.85
21/01/2020	6 045.99	-0.54%	6 045.99	5 994.17	6 034.24
20/01/2020	6 078.54	-0.36%	6 094.09	6 071.17	6 092.97
17/01/2020	6 100.72	+1.02%	6 109.81	6 066.29	6 066.29
16/01/2020	6 039.03	+0.11%	6 058.97	6 021.03	6 039.62
15/01/2020	6 032.61	-0.14%	6 053.01	6 011.28	6 042.72
14/01/2020	6 040.89	+0.08%	6 046.12	5 980.05	6 037.45
13/01/2020	6 036.14	-0.02%	6 058.69	6 018.33	6 040.91
10/01/2020	6 037.11	-0.09%	6 057.83	6 028.29	6 056.74
09/01/2020	6 042.55	+0.19%	6 071.66	6 034.15	6 066.75
08/01/2020	6 031.00	+0.31%	6 031.00	5 972.77	5 986.81
07/01/2020	6 012.35	-0.02%	6 065.74	6 000.00	6 033.22
06/01/2020	6 013.59	-0.51%	6 017.97	5 955.25	6 001.21
03/01/2020	6 044.16	+0.04%	6 044.16	5 994.59	6 007.96
02/01/2020	6 041.50	+1.06%	6 062.92	6 011.21	6 016.61
31/12/2019	5 978.06	-0.07%	5 987.22	5 958.25	5 970.59
30/12/2019	5 982.22	-0.91%	6 037.70	5 982.22	6 028.96
27/12/2019	6 037.39	+0.13%	6 065.00	6 027.72	6 039.95
24/12/2019	6 029.55	+0.00%	6 033.99	6 025.62	6 027.19
23/12/2019	6 029.37	+0.13%	6 035.95	6 005.96	6 013.56
20/12/2019	6 021.53	+0.82%	6 024.17	5 966.88	5 979.53
19/12/2019	5 972.28	+0.21%	5 972.28	5 942.88	5 963.87
18/12/2019	5 959.60	-0.15%	5 983.02	5 959.60	5 971.40
17/12/2019	5 968.26	-0.39%	5 989.48	5 955.36	5 989.48
16/12/2019	5 991.66	+1.23%	6 003.38	5 942.30	5 946.65
13/12/2019	5 919.02	+0.59%	5 972.17	5 907.22	5 968.86
12/12/2019	5 884.26	+0.40%	5 915.65	5 845.00	5 871.20
11/12/2019	5 860.88	+0.22%	5 865.88	5 825.42	5 844.02
10/12/2019	5 848.03	+0.18%	5 851.11	5 776.67	5 833.48
09/12/2019	5 837.25	-0.59%	5 870.79	5 836.14	5 865.73
06/12/2019	5 871.91	+1.21%	5 871.91	5 811.29	5 815.75
05/12/2019	5 801.55	+0.03%	5 849.78	5 801.46	5 808.26
04/12/2019	5 799.68	+1.27%	5 812.71	5 725.97	5 725.97

03/12/2019	5 727.22	-1.03%	5 800.93	5 697.06	5 787.14
02/12/2019	5 786.74	-2.01%	5 947.68	5 781.03	5 910.12
29/11/2019	5 905.17	-0.13%	5 929.79	5 885.66	5 895.89
28/11/2019	5 912.72	-0.24%	5 919.13	5 900.21	5 909.95
27/11/2019	5 926.84	-0.05%	5 947.30	5 921.92	5 943.45
26/11/2019	5 929.62	+0.08%	5 941.51	5 909.93	5 927.41
25/11/2019	5 924.86	+0.54%	5 933.75	5 908.29	5 917.84
22/11/2019	5 893.13	+0.20%	5 930.46	5 885.61	5 885.61
21/11/2019	5 881.21	-0.22%	5 895.57	5 834.26	5 860.55
20/11/2019	5 894.03	-0.25%	5 902.95	5 859.58	5 880.69
19/11/2019	5 909.05	-0.35%	5 966.79	5 895.91	5 936.99
18/11/2019	5 929.79	-0.16%	5 942.47	5 900.35	5 939.64
15/11/2019	5 939.27	+0.65%	5 947.62	5 907.95	5 933.07
14/11/2019	5 901.08	-0.10%	5 916.90	5 894.07	5 902.40
13/11/2019	5 907.09	-0.21%	5 917.57	5 874.06	5 908.16
12/11/2019	5 919.75	+0.44%	5 931.19	5 897.30	5 904.27
11/11/2019	5 893.82	+0.07%	5 906.33	5 871.71	5 873.74
08/11/2019	5 889.70	-0.02%	5 889.70	5 861.61	5 867.42
07/11/2019	5 890.99	+0.41%	5 894.87	5 871.18	5 888.84
06/11/2019	5 866.74	+0.34%	5 876.27	5 838.40	5 847.61
05/11/2019	5 846.89	+0.39%	5 850.01	5 818.94	5 825.60
04/11/2019	5 824.30	+1.08%	5 833.69	5 784.85	5 788.36
01/11/2019	5 761.89	+0.56%	5 778.29	5 739.70	5 749.48
31/10/2019	5 729.86	-0.62%	5 778.24	5 718.63	5 770.26
30/10/2019	5 765.87	+0.45%	5 765.87	5 720.27	5 749.69
29/10/2019	5 740.14	+0.17%	5 742.08	5 715.26	5 728.10
28/10/2019	5 730.57	+0.15%	5 747.10	5 711.10	5 720.73
25/10/2019	5 722.15	+0.67%	5 722.64	5 671.82	5 694.90
24/10/2019	5 684.33	+0.55%	5 691.50	5 659.95	5 678.26
23/10/2019	5 653.44	-0.08%	5 657.65	5 616.02	5 617.30
22/10/2019	5 657.69	+0.17%	5 658.65	5 612.86	5 644.98
21/10/2019	5 648.35	+0.21%	5 664.58	5 625.10	5 633.04
18/10/2019	5 636.25	-0.65%	5 667.59	5 621.87	5 653.08
17/10/2019	5 673.07	-0.42%	5 737.15	5 673.07	5 679.92
16/10/2019	5 696.90	-0.09%	5 708.91	5 677.96	5 703.39
15/10/2019	5 702.05	+1.04%	5 724.31	5 659.69	5 676.74
14/10/2019	5 643.08	-0.40%	5 649.67	5 600.11	5 646.33
11/10/2019	5 665.48	+1.73%	5 667.40	5 574.26	5 576.95
10/10/2019	5 569.05	+1.27%	5 581.07	5 487.17	5 520.98
09/10/2019	5 499.14	+0.78%	5 515.64	5 447.01	5 458.16
08/10/2019	5 456.62	-1.18%	5 531.29	5 451.72	5 526.01
07/10/2019	5 521.61	+0.61%	5 524.19	5 461.59	5 476.77
04/10/2019	5 488.32	+0.91%	5 488.32	5 426.00	5 456.83
03/10/2019	5 438.77	+0.30%	5 472.62	5 393.49	5 432.01
02/10/2019	5 422.77	-3.12%	5 589.20	5 422.77	5 587.14
01/10/2019	5 597.63	-1.41%	5 704.93	5 594.93	5 696.20
30/09/2019	5 677.79	+0.66%	5 678.20	5 624.94	5 636.15
27/09/2019	5 640.58	+0.36%	5 645.36	5 621.58	5 627.44
26/09/2019	5 620.57	+0.66%	5 632.15	5 572.41	5 573.63
25/09/2019	5 583.80	-0.79%	5 600.76	5 531.62	5 598.50

24/09/2019	5 628.33	-0.04%	5 648.46	5 625.24	5 644.46
23/09/2019	5 630.76	-1.05%	5 683.93	5 619.29	5 679.31
20/09/2019	5 690.78	+0.56%	5 696.25	5 647.53	5 655.37
19/09/2019	5 659.08	+0.68%	5 662.86	5 616.91	5 617.70
18/09/2019	5 620.65	+0.09%	5 634.79	5 601.65	5 609.92
17/09/2019	5 615.51	+0.24%	5 619.45	5 587.74	5 602.96
16/09/2019	5 602.23	-0.94%	5 632.74	5 602.23	5 626.61
13/09/2019	5 655.46	+0.22%	5 672.07	5 638.17	5 649.23
12/09/2019	5 642.86	+0.44%	5 667.46	5 596.37	5 633.95
11/09/2019	5 618.06	+0.44%	5 626.05	5 606.43	5 606.43
10/09/2019	5 593.21	+0.08%	5 596.94	5 555.51	5 586.88
09/09/2019	5 588.95	-0.27%	5 611.59	5 579.93	5 606.36
06/09/2019	5 603.99	+0.19%	5 610.70	5 581.54	5 592.07
05/09/2019	5 593.37	+1.11%	5 605.88	5 559.82	5 569.59
04/09/2019	5 532.07	+1.21%	5 537.10	5 508.50	5 518.92
03/09/2019	5 466.07	-0.49%	5 484.54	5 441.18	5 484.36
02/09/2019	5 493.04	+0.23%	5 502.58	5 479.82	5 483.43
30/08/2019	5 480.48	+0.56%	5 504.01	5 446.58	5 451.60
29/08/2019	5 449.97	+1.51%	5 452.90	5 354.55	5 359.00
28/08/2019	5 368.80	-0.34%	5 375.04	5 322.99	5 372.08
27/08/2019	5 387.09	+0.67%	5 399.18	5 321.88	5 349.85
26/08/2019	5 351.02	+0.45%	5 377.02	5 300.99	5 310.16
23/08/2019	5 326.87	-1.14%	5 421.28	5 326.87	5 413.41
22/08/2019	5 388.25	-0.87%	5 439.64	5 378.25	5 414.81
21/08/2019	5 435.48	+1.70%	5 441.35	5 360.03	5 360.03
20/08/2019	5 344.64	-0.50%	5 391.23	5 328.94	5 363.59
19/08/2019	5 371.56	+1.34%	5 378.74	5 325.72	5 333.13
16/08/2019	5 300.79	+1.22%	5 306.07	5 249.33	5 254.62
15/08/2019	5 236.93	-0.27%	5 278.07	5 170.77	5 274.90
14/08/2019	5 251.30	-2.08%	5 367.54	5 233.86	5 367.54
13/08/2019	5 363.07	+0.99%	5 402.99	5 268.47	5 292.94
12/08/2019	5 310.31	-0.33%	5 386.46	5 293.76	5 362.39
09/08/2019	5 327.92	-1.11%	5 374.90	5 317.91	5 368.84
08/08/2019	5 387.96	+2.31%	5 387.96	5 305.86	5 333.18
07/08/2019	5 266.51	+0.61%	5 312.31	5 226.39	5 252.73
06/08/2019	5 234.65	-0.13%	5 297.45	5 234.65	5 245.52
05/08/2019	5 241.55	-2.19%	5 315.60	5 230.46	5 296.08
02/08/2019	5 359.00	-3.57%	5 455.13	5 350.76	5 427.55
01/08/2019	5 557.41	+0.70%	5 563.65	5 499.88	5 500.91
31/07/2019	5 518.90	+0.14%	5 538.59	5 503.13	5 511.51
30/07/2019	5 511.07	-1.61%	5 611.00	5 496.77	5 610.73
29/07/2019	5 601.10	-0.16%	5 624.34	5 590.82	5 598.65
26/07/2019	5 610.05	+0.57%	5 614.25	5 568.83	5 572.72
25/07/2019	5 578.05	-0.50%	5 672.77	5 549.66	5 646.73
24/07/2019	5 605.87	-0.22%	5 627.08	5 583.14	5 617.86
23/07/2019	5 618.16	+0.92%	5 638.66	5 580.52	5 590.60
22/07/2019	5 567.02	+0.26%	5 575.74	5 545.61	5 550.30
19/07/2019	5 552.34	+0.03%	5 598.00	5 534.25	5 580.99
18/07/2019	5 550.55	-0.38%	5 585.81	5 533.47	5 538.03
17/07/2019	5 571.71	-0.76%	5 621.21	5 567.85	5 602.25

16/07/2019	5 614.38	+0.65%	5 626.79	5 573.36	5 574.18
15/07/2019	5 578.21	+0.10%	5 606.48	5 550.86	5 581.11
12/07/2019	5 572.86	+0.38%	5 587.84	5 552.58	5 559.16
11/07/2019	5 551.95	-0.28%	5 592.30	5 548.95	5 589.68
10/07/2019	5 567.59	-0.08%	5 606.18	5 557.66	5 562.44
09/07/2019	5 572.10	-0.31%	5 582.77	5 544.62	5 578.34
08/07/2019	5 589.19	-0.08%	5 598.97	5 572.77	5 580.26
05/07/2019	5 593.72	-0.48%	5 616.63	5 573.35	5 613.92
04/07/2019	5 620.73	+0.03%	5 629.79	5 613.03	5 622.80
03/07/2019	5 618.81	+0.75%	5 618.81	5 579.04	5 579.24
02/07/2019	5 576.82	+0.16%	5 582.91	5 556.76	5 578.38
01/07/2019	5 567.91	+0.52%	5 611.28	5 567.91	5 584.13
28/06/2019	5 538.97	+0.83%	5 545.06	5 486.28	5 491.17
27/06/2019	5 493.61	-0.13%	5 511.22	5 465.62	5 508.38
26/06/2019	5 500.72	-0.25%	5 531.45	5 497.22	5 504.67
25/06/2019	5 514.57	-0.13%	5 523.57	5 499.65	5 499.82
24/06/2019	5 521.71	-0.12%	5 547.11	5 514.14	5 534.35
21/06/2019	5 528.33	-0.13%	5 582.24	5 518.39	5 534.43
20/06/2019	5 535.57	+0.31%	5 564.13	5 535.57	5 551.03
19/06/2019	5 518.45	+0.16%	5 521.81	5 496.54	5 513.32
18/06/2019	5 509.73	+2.20%	5 516.06	5 376.12	5 386.38
17/06/2019	5 390.95	+0.43%	5 408.75	5 368.07	5 370.76
14/06/2019	5 367.62	-0.15%	5 369.38	5 342.15	5 367.10
13/06/2019	5 375.63	+0.01%	5 392.51	5 353.11	5 361.60
12/06/2019	5 374.92	-0.62%	5 394.85	5 358.81	5 379.15
11/06/2019	5 408.45	+0.48%	5 431.54	5 381.31	5 382.88
10/06/2019	5 382.50	+0.34%	5 397.35	5 369.82	5 382.29
07/06/2019	5 364.05	+1.62%	5 389.95	5 299.71	5 299.71
06/06/2019	5 278.43	-0.26%	5 332.69	5 264.62	5 293.72

```

1 import xlrd
2 import numpy as np
3 import matplotlib.pyplot as plt
4
5 Fichier=xlrd.open_workbook("Data.xlsx") # Import our data
6     set
7 data=Fichier.sheet_by_index(0)
8 S=[]
9
10 for i in range(778):
11     s=data.cell_value(778-i,1)
12     s=s.replace(' ','')
13     s=float(s)
14     S.append(s) # S is the historical data set
15
16 pas=0.1 #pas=step
17 S_max=max(S)
18
19 def mon_min(x,y):
20     return x*(x<=y)+y*(y<x)
21
22 def mon_max(x,y):
23     return x*(x>=y)+y*(y>x)
24
25 NP_test=52 # number of periods
26 NP_calibration=25 # number of periods for calibration
27 T=int(9)
28 S_test=[S[i] for i in range(10*NP_calibration,10*(
29     NP_calibration+NP_test))]
30 S_calibration=[S[i] for i in range(10*NP_calibration)]
31
32
33
34 kd=np.array([100.0 for i in range(9)])
35 ku=np.array([0.0 for i in range(9)])
36
37 for i in range(9):
38     for j in range(NP_calibration):
39         kd[i]=min(kd[i],S_calibration[10*j+i+1]/
40             S_calibration[10*j+i])

```

```

40         ku[i]=max(ku[i],S_calibration[10*j+i+1]/
41                   S_calibration[10*j+i])
42 kmax=ku.max()
43 s=[]
44
45 for t in range(10):
46     npas=int((S_max)*(kmax**t)/pas)
47     v_s=np.array([pas*i for i in range(npas+1)])
48     s.append(v_s) # s[t] is the set of values s we fix
49                   when computing g(t,s)
50 def subdivision(g,t,ecart):# output= sequence (a_i)_i
51     where the piecewise affine function g=g(t,.) breaks
52     ecart=float(ecart)
53     deriv1=(g[1:]-g[:-1])/pas
54     deriv2=(deriv1[1:]-deriv1[:-1])/pas
55     long=len(s[t])
56     liste=s[t][1:long-1]
57     subd=liste[np.abs(deriv2)>ecart]
58     return subd
59 def gcall(x,n,K):
60     aux=n*(x-K)
61     y=aux*(aux>=0)
62     return y
63
64 def gcall_vect(n,K):
65     y=gcall(s[9],n,K)
66     return y
67
68 def h(s,a,t,teta,g_tPlusUn):# output= quantity to
69     maximize as in Corollary 3.3.
70     t=int(t)
71     x_is=mon_min(mon_max(s*kd[t],a), s*ku[t])
72     x_is_aux=x_is/pas
73     type_x=type(x_is_aux)
74     if (type_x==float):
75         indic_x_is=int(x_is_aux) # index of x_is
76         valg_tPlusUn=g_tPlusUn[indic_x_is]
77     else:
78         indic_x_is=x_is_aux.astype(int)
79         valg_tPlusUn=np.array([g_tPlusUn[i] for i in

```

```

        indic_x_is])
79
80     y=valg_tPlusUn+teta*(s-x_is)
81     return y
82
83 def g(t,g_tplus,n):# output=g(t,S_t) computed from
    g_tplus = g(t+1,S_{t+1}) (Dynamic Programming
    Principle)
84     ecart=(t>=9)*1+(t<=8)*12*n
85     t=int(t)
86     a=subdivision(g_tplus,t+1,ecart)
87     s_max=s[t].max()
88     a=np.append(0,a)
89     a=np.append(a,s_max)
90     Lips=n
91     teta=np.arange(n+1)
92     Matr3d=h(s[t][:,np.newaxis,np.newaxis],a[np.newaxis
        :,np.newaxis],t,\
93             teta[np.newaxis,np.newaxis,:],g_tplus)
94     Matr2d=np.max(Matr3d,axis=1)
95     g_t=np.min(Matr2d,axis=1)
96     indice_strat=np.argmin(Matr2d,axis=1)
97     strat=np.array([teta[i] for i in indice_strat])
98
99     return g_t,strat #output=g(t,.) and $\theta(t,.)$
100
101 def Num2(K,n): #output=hedging prices per unit of claims
    at time 0
102     gcall=[]
103     gcall.append(gcall_vect(n,K))
104     g8=g(8,gcall[-1],n)
105     gcall.append(g8[0])
106     for j in range(8):
107         gcall_plus=gcall[-1]
108         gj=g(7-j,gcall_plus,n)
109         gcall.append(gj[0])
110     ret=gcall[-1]/n
111     return ret

```

Bibliography

- [1] N. Agram and B. Oksendal. A financial market with singular drift and no arbitrage. *Mathematics and Financial Economics*, 15:477–500, 2021.
- [2] Charalambos D. Aliprantis and Kim C. Border. *Infinite dimensional analysis*. Springer, Berlin, third edition, 2006. A hitchhiker’s guide.
- [3] J. Baptiste, L. Carassus, and E. Lépinette. Pricing without martingale measures. Preprint. <https://hal.archives-ouvertes.fr/hal-01774150>, 2020.
- [4] E.N. Barron, P. Cardaliaguet, and R. Jensen. Conditional essential suprema with applications. *Appl. Math. Optim.*, 48(3):229–253, 2003.
- [5] P. Baumann and N. Trautmann. Portfolio-optimization models for small investors. *Math. Methods Oper. Res.*, 77(3):345–356, 2013.
- [6] D. Becherer and K. Kentia. Hedging under generalized good-deal bounds and model uncertainty. *Mathematical Methods of Operations Research*, 86:171–241, 2017.
- [7] D. Becherer and K. Kentia. Good deal hedging and valuation under combined uncertainty about drift and volatility. *Probability, Uncertainty and Quantitative Risk*, 2:1–40, 2017.
- [8] D. Bertsimas, V. Gupta, and N. Kallus. Data-driven robust optimization. *Mathematical Programming*, 167:235–292, 2018.
- [9] S. Biagini, B. Bouchard, and C. Kardaras. Robust fundamental theorem for continuous processes. *Mathematical Finance*, 27:963–987, 2017.
- [10] D. Bienstock. Computational study of a family of mixed-integer quadratic programming problems. *Math. Programming*, 74(2, Ser. A):121–140, 1996.
- [11] F. Black and M. Scholes. The pricing of options and corporate liabilities. *J. Polit. Econ.*, 81(3):637–654, 1973.
- [12] P. Bonami and M.A Lejeune. An exact solution approach for portfolio optimization problems under stochastic and integer constraints. *Oper. Res.*, 57(3):650–670, 2009.

- [13] B. Bouchard and M. Nutz. Arbitrage in nondominated discrete-time models. *The Annals of Applied Probability*, 25:823–859, 2015.
- [14] M. Burzoni, M. Frittelli, and M. Maggis. Universal arbitrage aggregator in discrete-time markets under uncertainty. *Finance and Stochastics*, 20:1–50, 2016.
- [15] M. Burzoni, F. Riedel, and M.H. Soner. Viability and arbitrage under knightian uncertainty. *Swiss Finance Institute Research Paper*, 0:17–48, 2018.
- [16] L. Carassus and E. Lépinette. Pricing without no-arbitrage condition in discrete time. *J. Math. Anal. Appl.*, 505(1):Paper No. 125441, 21, 2022.
- [17] L. Carassus, J. Oblòj, and J. Wiesel. The robust superreplication problem: a dynamic approach. *SIAM J. Financial Math.*, 10:907–941, 2019.
- [18] P. Cheredito, M. Kupper, and L. Tangpi. Duality formulas for robust pricing and hedging in discrete time. *SIAM J. Financial Math.*, 8:738–765, 2017.
- [19] C. Cuchiero, I. Klein, and J. Teichmann. A fundamental theorem of asset pricing for continuous time large financial markets in a two filtration setting. *Theory of Probability and its Applications*, 65, 2020.
- [20] K.R. Dahl. Pricing of claims in discrete time with partial information. *Applied Mathematics and Optimization*, 68:145–155, 2013.
- [21] R.C Dalang, A. Morton, and W. Willinger. Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics Stochastics Rep.*, 29(2):185–201, 1990.
- [22] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300(3):463–520, 1994.
- [23] F. Delbaen and W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.*, 312(2):215–250, 1998.
- [24] F. Delbaen and W. Schachermayer. *The mathematics of arbitrage*. Springer Finance. Springer-Verlag, Berlin, 2006.
- [25] D. DeValière, C. Stricker, and Y. Kabanov. No-arbitrage properties for financial markets with transaction costs and incomplete information. *Finance and Stochastics*, 11:237–251, 2007.
- [26] D. DeValière, Y. Kabanov, and C. Stricker. No-arbitrage criteria for financial markets with transaction costs and incomplete information. *Finance Stoch.*, 11(2):237–251, 2007.

- [27] S. Drapeau, A. Jamneshan, M. Karliczek, and M. Kupper. The algebra of conditional sets and the concepts of conditional topology and compactness. *J. Math. Anal. Appl.*, 437(1):561–589, 2016.
- [28] N. ElKaraoui and T. Xiaolu. Capacities, measurable selection and dynamic programming part i: abstract framework. ArXiv:1310.3363, 2013.
- [29] M. ElMansour and E. Lépinette. Conditional interior and conditional closure of random sets. *J. Optim. Theory Appl.*, 187(2):356–369, 2020.
- [30] T. Fadina, A. Neufeld, and T. Schmidt T. Affine processes under parameter uncertainty. *Uncertainty and Quantitative Risk*, 4, 2019.
- [31] S. Gerhold and P. Krühner. Dynamic trading under integer constraints. *Finance Stoch.*, 22(4):919–957, 2018.
- [32] J. Harrison and M. Kreps. Martingales and arbitrage in multiperiod securities markets. *J. Econom. Theory*, 20(3):381–408, 1979.
- [33] J. Harrison and R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.*, 11(3):215–260, 1981.
- [34] C. Hess. Set-valued integration and set-valued probability theory: an overview. In *Handbook of measure theory, Vol. I, II*, pages 617–673. North-Holland, Amsterdam, 2002.
- [35] C. Hess, R. Seri, and C. Choirat. Essential intersection and approximation results for robust optimization. *J. Nonlinear Convex Anal.*, 15(5):979–1002, 2014.
- [36] F. Hiai and H. Umegaki. Integrals, conditional expectations, and martingales of multivalued functions. *J. Multivariate Anal.*, 7(1):149–182, 1977.
- [37] D.G. Hobson. Robust hedging of the lookback option. *Finance and Stochastics*, 2:329–347, 1998.
- [38] T. Ichiba and M. Mousavi. Option pricing with delayed information. https://www.researchgate.net/publication/318256675_Option_Pricing_with_Delayed_Information, 2017.
- [39] E. Jouini and H. Kallal. Martingales and arbitrage in securities markets with transaction costs. *J. Econom. Theory*, 66(1):178–197, 1995.
- [40] Y. Kabanov. Hedging and liquidation under transaction costs in currency markets. *Finance and Stochastics*, 3:237–248, 1999.
- [41] Y. Kabanov and E. Lépinette. Essential supremum with respect to a random partial order. *J. Math. Econom.*, 49(6):478–487, 2013.

- [42] Y. Kabanov and M. Safarian. *Markets with transaction costs*. Springer Finance. Springer-Verlag, Berlin, 2009. Mathematical theory.
- [43] Y. Kazmerchuk, A. Swishchuk, and J. Wu. The pricing of options for securities markets with delayed response. *Mathematics and computers in simulation*, 75:69–79, 2007.
- [44] F.H. Knight. Risk, uncertainty, and profits. In Schaffner Hart and Marx, editors, *Boston MA*. Houghton Mifflin, 1921.
- [45] E. Lépinette and I. Molchanov. Conditional cores and conditional convex hulls of random sets. *J. Math. Anal. Appl.*, 478(2):368–392, 2019.
- [46] E. Lépinette and I. Molchanov. Risk arbitrage and hedging to acceptability under transaction costs. *Finance Stoch.*, 25(1):101–132, 2021.
- [47] I. Molchanov. *Theory of random sets*. Probability and its Applications (New York). Springer-Verlag London, Ltd., London, 2005.
- [48] A. Neufeld and M. Nutz. Superreplication under volatility uncertainty for measurable claims. *Electronic Journal of Probability*, 18, 2013.
- [49] J. Oblój and J. Wiesel. Robust estimation of superhedging prices. *Annals of Statistics*, 49:508–530, 2021.
- [50] B. Oksendal, A. Sulem, and T. Zhang. Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations. *Advances in Applied Probability*, 43:572–596, 2011.
- [51] H. Pham, X. Wei, and C. Zhou. Portfolio diversification and model uncertainty: a robust dynamic mean-variance approach. <https://arxiv.org/abs/1809.01464>, 2018.
- [52] M-C. Quenez. Optimal portfolio in a multiple-priors model. In Basel Birkhuser, editor, *Seminar on Stochastic Analysis, Random Fields and Applications IV*. Progr. Probab., 2004.
- [53] M. Rásonyi and A. Meireles-Rodrigues. On utility maximisation under model uncertainty in discrete-time markets. *Mathematical Finance*, 31:149–175, 2021.
- [54] R. Rockafellar and R. Wets. *Variational analysis*, volume 317 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1998.
- [55] Y.F. Saporito and J. Zhang. Stochastic control with delayed information and related nonlinear master equation. *Siam Journal on Control and Optimization*, 57:693–717, 2019.

- [56] N. Shiryaev. *Essentials of stochastic finance*, volume 3 of *Advanced Series on Statistical Science & Applied Probability*. World Scientific Publishing Co., Inc., River Edge, NJ, 1999. Facts, models, theory, Translated from the Russian manuscript by N. Kruzhilin.
- [57] R. Tevzaze, T. Toronjadze, and T. Uzunashvili. Robust utility maximization for a diffusion market model with misspecified coefficients. *Finance and Stochastics*, 17:535–563, 2009.
- [58] A. Truffert. Conditional expectation of integrands and random sets. volume 30, pages 117–156. 1991. *Stochastic programming, Part I* (Ann Arbor, MI, 1989).
- [59] D. Xiaotie, L. Zhong-Fei, and W. Shou-Yang. Computational complexity of arbitrage in frictional security market. *International Journal of Foundations of Computer Science*, 13:681–684, 2002.

RÉSUMÉ

Dans cette thèse nous introduisons deux nouveaux types d'ensembles aléatoires conditionnels à valeurs dans un espace de Banach : l'intérieur conditionnel et la clôture conditionnelle des ensembles aléatoires.

Dans la deuxième partie, nous appliquons les résultats théoriques établis dans la première partie pour résoudre le problème de sur-réplication des options européennes ou asiatiques pour les modèles de marchés financiers en temps discret où les prix exécutables sont incertains. Nous illustrons notre méthode par un exemple numérique.

Dans la dernière partie nous montrons que, dans un modèle de marché en temps discret, il est possible d'évaluer le prix minimal de sur-réplication lorsqu'on se limite aux stratégies à valeurs entières.

Nous formulons un principe de programmation dynamique qui peut être directement implémenté sur une donnée historique et qui fournit également la stratégie optimale à valeurs entières.

MOTS CLÉS

Ensemble aléatoire conditionnel, optimisation conditionnelle, problème de sur-réplication, Option européenne, prix de sur-réplication, information différées, incertitude, condition AIP , stratégie à valeurs entières, principe de programmation dynamique.

ABSTRACT

In this thesis we introduce two new types of conditional random sets taking values in a Banach space: the conditional interior and the conditional closure.

In the second part of the thesis, we apply the theoretical results established in the first part to solve the problem of super-hedging European or Asian options for discrete-time financial market models where executable prices are uncertain. We illustrate our method by a numerical example.

In the last part we show that, in discrete-time, it is possible to evaluate the minimal super-hedging price when we restrict ourselves to integer-valued strategies. We formulate a dynamic programming principle that can be directly implemented on an historical data, and which also provides the optimal integer-valued strategy.

KEYWORDS

Conditional random set, conditional optimization, super-hedging problem, European option, super-hedging prices, delayed information, uncertainty, AIP condition, integer valued strategies, dynamic programming principle.

