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Jose Luis Robles Magdaleno

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Jose Luis Robles Magdaleno. Design of filtered high gain observers for a class of non uniformly observable systems. Automatic. Normandie Université; Institut Technologique de Culiacán, 2023. English. NNT: 2023NORMC226 . tel-04315839

**HAL Id: tel-04315839**

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Normandie Université



UNIVERSITÉ  
CAEN  
NORMANDIE

## THÈSE

Pour obtenir le diplôme de doctorat

Spécialité AUTOMATIQUE, SIGNAL, PRODUCTIQUE, ROBOTIQUE

Préparée au sein de l'Université de Caen Normandie

En cotutelle internationale avec Institut Technologique de Culiacán , MEXIQUE

### Design of filtered high gain observers for a class of non uniformly observable systems

Présentée et soutenue par  
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Thèse soutenue le 11/05/2023  
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To my parents, who always supported me,  
my sisters, who gave me so much love,  
Fernando, for being by my side,  
and my PhD friends, who without them, it wouldn't have been the same.



## Acknowledgements

The present thesis was co-tutored at the University of Caen, France, under the direction of Dr. Mondher Farza and with the Technological University of Culiacan, Mexico, under the direction of Dr. Abraham Efrain Rodriguez Mata. Before presenting this thesis, I would like to thank all those who have contributed to the preparation and success of my thesis.

I would like to express my sincere gratitude to my thesis director, Dr. Mondher FARZA, professor at the University of Caen Normandy, who was kind enough to receive me and guide me for a whole year, to share his knowledge. I thank him for his availability, his valuable advice and all the support he has given me throughout my stay in France. Despite the inconveniences presented by the pandemic, I am eternally grateful and value your support. Thank you.

To Dr. Abraham Efrain Rodríguez-Mata, thank you for giving me the opportunity to work with him on my PhD and for giving me the opportunity to do a co-tutelle with France. I thank him for always supporting me and giving me the necessary knowledge.

Likewise, this project would not have been possible without the support of the University of Caen, France, and the Technological University of Culiacan, Mexico. Thank you for lending me your laboratories and facilities to successfully develop my thesis project. To the Consejo Nacional de Humanidades, Ciencias y Tecnologías (CONAHCYT) for giving me the financial support to complete my PhD.



# List of publications

## International Journal

1. J.L. Robles-Magdaleno, A.E. Rodríguez-Mata, M. Farza and M. M'Saad (2020). A filtered high gain observer for a class of non uniformly observable systems – Application to a phytoplanktonic growth model, Journal of Process Control, Vol. 87, pp. 68-78.

## International Conferences

1. M. Farza, A. Efraim Rodríguez-Mata , J.L. Robles Magdaleno and M. M'Saad (2019). A New Filtered High Gain Observer Design for the Estimation of the Components Concentrations in a Photobioreactor in Microalgae Culture, IFAC-PapersOnLine Volume 52 (1), Pages 904-909 (Presented at the 12th IFAC Symposium on Dynamics and Control of Process Systems, including Biosystems (DYCOPS-CAB), April, Florianópolis-SC, Brazil).
2. J.L. Robles Magdaleno, M. Farza, L.E. Amabilis-Sosa and A.E. Rodríguez-Mata (2021). A Lipchitz Observer Application in Photobioreactors for Microalgae Cultivation, in Proc. of the 18th IEEE International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE), pp. 1-7, 10-12 Nov., Mexico City, Mexico.



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# Chapter 1

## General introduction

In spite of the intensive research efforts throughout the last four decades, the observer design for nonlinear systems is still an open problem. Various approaches have been proposed to design state observers for different classes of nonlinear systems (see for instance [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and references therein) but none of them provides a general solution as in the linear time invariant case. The seminal approaches dealing with the observer design for nonlinear systems are based on appropriate coordinates transformations which lead to linear error dynamics up to an output injection allowing thereby to design a Luenberger structure based observer (see for instance [14, 15, 16, 17, 18, 19, 20, 21]). An intensive research activity has been devoted to the class of systems that are observable for any input, i.e. uniformly observable systems (see for instance [8, 9, 22]). In [8], the authors proposed a canonical form for single output systems in the control affine case. This canonical form is composed of a fixed linear dynamics together with a nonlinear triangular controlled one. A Standard High Gain Observer (SHGO) has been designed on the basis of this canonical form and this observer design has been subsequently extended in [9] to deal with the non-control affine case and in [22] for a class of MIMO uniformly observable systems characterized by a normal form where the nonlinearities are triangular. Several extensions of the SHGO design have been proposed for some particular classes of MIMO uniformly observable systems. The underlying observers are characterized by a constant gain which is often issued from an algebraic Lyapunov equation. In the case of non-uniformly observable systems, there is no systematic approach to deal with the observer design since these systems may admit inputs that render them unobservable. The available contributions are rather genuine extensions of the observer design approach adopted for uniformly observable systems up to some sufficient conditions on the system inputs which allow the systems to be sufficiently observable to perform an appropriate observer design. These conditions are generally referred to as persistent excitation conditions which require the positive definiteness of the

system observability Gramian. In [23], the authors introduced the notion of local regular inputs providing thereby sufficient conditions to characterize those systems that can be immersed under higher dimension normal form composed of an affine part depending on the input and output and a triangular controlled nonlinear part. A high gain observer involving a Lyapunov ODE has then been designed on the basis of the normal form. The class of systems considered in [23] has been revisited in [24] where the authors enlarged the class of locally regular inputs to the class of regular inputs.

A particular attention has been paid to the Standard High Gain Observer (SHGO) design for fundamental as well as simplicity purposes ([9]). The simplicity feature is obtained thanks to the involved Luenberger observer structure together with the parametrization of the observer gain by a unique design parameter, i.e. a positive scalar commonly denoted  $\theta$ . It is worth noticing that this design parameter is generally taken high enough to ensure the underlying observation error convergence and intervenes with positive powers ranged from one to the system dimension in the observer gain expression. Concerning the fundamental feature, it follows essentially from the fact that SHGO design has been appropriately extended to delay systems and extensively used for designing output feedback nonlinear control systems (see [25, 26] and references therein). The case of arbitrary long time delays has been addressed using an appropriate cascade observer design approach (see [27] and references therein). Otherwise, a genuine approach has been proposed to redesign any SHGO when the output measurements are available only at sampling instants (see [28] and reference list therein). Such a design has been subsequently extended to tackle the case of multirate sampling with multiple time-varying output delays using a high gain observer chained to a cascade predictor (See [29]). Nevertheless, the applicability of SHGO suffers from two main limitations. The first one consists in its intrinsic sensibility to the measurement noise, while the second one is the peaking phenomenon occurring during the transient periods. As pointed out in [30], several solutions have been recently proposed to alleviate the sensitivity to measurement noise of SHGO ([31, 32, 33, 34, 35, 36]). A redesigned version of SHGO has been proposed in [34] for a class of nonlinear systems of dimension  $n$ . Though the observer dimension is equal to  $2(n - 1)$ , the underlying observer gain is parameterized by a scalar design parameter the power of which is limited to 2. Such a design feature allows to enhance the observer performance with respect to the ubiquitous measurement noise. The same objective has been investigated in [33] using a cascade observer the dimension of which is equal to  $2n$ . More specifically, the observer consists in cascading two subsystems of the same dimension as the original system. The first subsystem is nothing than a copy of the original system with a simple correction term which linearly depends on the state of the second subsys-

tem, while the second subsystem is a linear filter driven by the output observation error, namely the error between the output measurement and an output prediction provided by the first subsystem. In [37, 38], the authors discussed the use of low-pass filters in high gain observers and proposed some filters coupled to SHGO to improve the sensitivity to high frequency signals of the observer.

The peaking phenomenon is a challenging problem that has been recently investigated ([39, 40, 41, 42, 43]). A suitable redesign of the SHGO has been proposed in [40] leading to an observer of the same dimension as the original system where the power of the observer gain is limited to one. The observer design proposed in [34] has been reconsidered in [41] to deal with the peaking phenomenon by properly using nested saturation functions. In [30], a redesigned high gain observer with filtering capabilities and where the peaking phenomenon is significantly reduced has been proposed. The proposed observer is constituted by the cascade of two subsystems where the first one acts as a linear filter the input of which is the noisy output observation error and it provides a filtered version of this error which is used by the second system the structure of which is similar to that of a high gain observer. Moreover, each equation of the overall system involve a corrective term with nested saturation functions depending on the filtered output error. Though the structure of the proposed observer is quite simple, the related convergence analysis is somewhat elaborate and it is not clear how the adopted approach can be extended to more general class of systems i.e. systems with sampled/delayed outputs or non uniformly observable systems.

The ultimate motivation of this thesis is to propose redesigned versions of SHGO in order to improve its sensitivity to high frequency signals. Indeed, two redesigned versions shall be proposed. The first one [44] is proposed for a class of non uniformly observable systems and it can be interpreted as an extension of the filtered observer proposed in [33] for a class of uniformly observable systems. The second redesign is very recent and is under submission. It is first proposed for a class of uniformly observable system and then extended to a class of non uniformly observable systems. It should be emphasized that the second redesign approach is achieved in a unifying context that accounts for all the scenarios under which the output measurements may be available, i.e. continuously in time with eventually an output delay or under samples with the eventual presence of output delays.

This thesis is organized under the form of four chapters amongst which the actual general introduction and the conclusion. In chapter 2, the first contribution dealing with the

design of an observer for a class of non uniformly observable systems is detailed. In chapter 3, a new redesign of a Filtered High Gain Observer (FHGO) is put forward. First the redesign is detailed for a class of Uniformly Observable Systems (UOS). Then its extension to a class of Non Uniformly Observable Systems (NUOS) is considered. For each redesigned observer, simulation results dealing with an academic example as well as with a bioreactor are given in order to put forward the performance and the main properties of the underlying observer. Finally some concluding remarks and perspectives are given in the conclusion.

# Chapter 2

## Filtered High Gain Observer Design

Since the eighties, we have witnessed the emergence of nonlinear state observers performing an admissible estimation of the component concentrations inside bioreactors (see e.g. [45] and references therein). These observers have been designed using various approaches that have been progressively developed for specific classes of nonlinear systems using various design principles (See for instance [46, 47, 48, 49, 50]). The high gain principle has shown to provide an appealing approach to design observers for nonlinear systems as pointed out in [49]. Such an attraction is motivated by implementation simplicity and fundamental considerations. Indeed, the Standard High Gain Observer (SHGO) structure is very simple since it consists in adding a copy of the system dynamics with a corrective involving an observer gain which is essentially parameterized by a positive design parameter  $\theta$ . Of a fundamental interest, one can naturally recover the genuine separation principle for nonlinear systems when designing output feedback control systems incorporating a SHGO (see [51] and references therein). However, the design parameter  $\theta$  has to be taken high enough for observer convergence purposes and it intervenes with positive powers in the observer gain, leading thereby to two well known issues. The first issue is related to the intrinsic sensitivity of SHGO to the non avoidable output measurement noise while the second one consists in the peaking phenomenon of the state variable estimates, the magnitude of which is proportional to an appropriate power of  $\theta$ , leading to poor transient behavior of SHGO.

An important research activity has been recently devoted to alleviate the SHGO sensitivity to measurement noise using an appropriate filtering. The seminal contribution has been proposed in [52] for a class of single output nonlinear systems of dimension  $n \geq 3$ . The filtering feature is mainly used to limit the observer gain power to 2, reducing thereby the amplification of noise measurements. Two design features are worth to be emphasized. Firstly, the dimension of the proposed observer is equal to  $2(n - 1)$ . Secondly, the

observer provides an estimate of the first and last components of the system state whereas the intermediary states are provided by a couple of estimates. This contribution has been subsequently reconsidered in [53] and [54] where redesigned versions involving nested saturations have been proposed to deal with the peaking phenomena. A nice contribution has been proposed in [51], it consists in a redesigned SHGO having the same dimension as the original system with two important design features, namely the power of the observer design parameter is limited to one and the peaking phenomena is well reduced thanks to nested saturation functions. Further developments related to the design of SHGO with filtering capabilities have been carried out in [55, 56] for a class of systems with dimension  $n \geq 2$ . The underlying observers consist in two cascaded sub-systems each one of which has the same dimension as the original system. The first subsystem is a copy of the original system with a simple correction term which (linearly) depends on the state of the second subsystem. The latter subsystem is a linear filter driven by the output observation error, namely the error between the output measurements and an output prediction provided by the first subsystem. As in [52], the power of the observer gain is limited to 2.

It is worth noticing that all the high gain observers incorporating filtering capabilities, cited above, were proposed for uniformly observable systems, i.e. systems which are observable for any input. Up to the authors best knowledge, no equivalent solution has been proposed for non uniformly observable systems, i.e. systems where the considered inputs have to satisfy an adequate persistent excitation condition to ensure their observability. In this paper, one aims at providing a new Filtered High Gain Observer (FHGO) for a class of Multi-Input/Multi-Output (MIMO) non uniformly observable nonlinear systems. The underlying design is quite different from those given in [55, 56] for uniformly observable systems, namely the observer gain is issued from the resolution of a Riccati or Lyapunov Ordinary Differential Equation (ODE).

The chapter is organized as follows. A comprehensive presentation of the class of systems to be considered together with an appropriate SHGO are given in section 2. Section 3 is devoted to the FHGO design assuming that the output measurements are continuously available with a particular emphasis on a certain persistent excitation condition required for the observer convergence. The redesigned version of the FHGO dealing with sampled outputs is given in section 4 with a particular emphasis on the admissible sampling process. Section 5 is dedicated to an application, namely an admissible estimation of the component concentrations within a bioreactor in a realistic simulation framework. A particular attention is paid on the FHGO performance with respect to the SHGO in the presence of noise measurements. For clarity purposes, the convergence analysis related to

the FHGO with continuous output measurements as well as with sampled ones are given in Appendix.

Throughout this chapter as well as the next one, for any positive integers  $k$  and  $m$ ,  $I_k$  and  $0_k$  denote the  $k$ -dimensional identity and zero matrices respectively,  $0_{k \times m}$  is the  $k \times m$  zero matrix,  $\|\cdot\|$  denotes the euclidian norm and for any Symmetric Positive Definite (SPD) time-varying matrix  $Q(t)$ ,  $\lambda_M(Q(t))$  (resp.  $\lambda_m(Q(t))$ ) will be used to denote the largest (resp. the smallest) eigenvalue of  $Q(t)$  and  $\bar{\lambda}_M(Q(t)) = \sup_{t \geq t_0} \lambda_M(Q(t))$ ,  $\underline{\lambda}_m(Q(t)) = \min_{t \geq t_0} \lambda_m(Q(t))$  where  $t_0$  is any fixed non negative real number. Moreover, the arguments will be omitted when clear from the context.

## 2.1 Preliminaries

Recall that the ultimate motivation consists in addressing the SHGO sensitivity with respect to the ubiquitous noise measurements for a class of non uniformly observable systems. More specifically, we will consider the following class of nonlinear MIMO systems

$$\begin{cases} \dot{x}(t) = F(u(t), x(t)) x(t) + \varphi(u(t), x(t)) + B\varepsilon(t) \\ y(t) = Cx(t) + w(t) \end{cases} \quad (2.1)$$

with

$$x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(q)} \end{pmatrix} \quad \text{with } x^{(k)} \in \mathbb{R}^{n_k} \text{ and } \sum_{k=1}^q n_k = n, \quad (2.2)$$

$$F(u, x) = \begin{pmatrix} 0 & F_1(u, x) & 0 & 0 \\ \vdots & & F_2(u, x) & \\ \vdots & & & 0 \\ 0 & & 0 & F_{q-1}(u, x) \\ 0 & & & 0 \end{pmatrix} \quad \text{with} \quad (2.3)$$

$$F_k(u, x) = F_k(u, x^{(1)}, \dots, x^{(k)}),$$

$$\varphi(u, x) = \begin{pmatrix} \varphi^{(1)}(u, x) \\ \varphi^{(2)}(u, x) \\ \vdots \\ \varphi^{(q-1)}(u, x) \\ \varphi^{(q)}(u, x) \end{pmatrix} \in \mathbb{R}^n \quad \text{with} \quad (2.4)$$

$$\varphi^{(k)}(u, x) = \varphi^{(k)}(u, x^{(1)}, \dots, x^{(k)}),$$

$$B = \begin{pmatrix} 0_{n_1 \times n_q} \\ \vdots \\ 0_{n_{q-1} \times n_q} \\ I_{n_q} \end{pmatrix}, C = \begin{pmatrix} I_{n_1} & 0_{n_1 \times n_2} & \cdots & 0_{n_1 \times n_q} \end{pmatrix}, \quad (2.5)$$

where the state  $x(t) \in \mathbb{R}^n$ , each  $F_k(u, x)$  is a  $n_k \times n_{k+1}$  matrix which is triangular w.r.t.  $x$ , i.e.  $F_k(u, x) = F_k(u, x^{(1)}, \dots, x^{(k)})$ ,  $k = 1, \dots, q-1$ ,  $\varphi(x(t), u(t))$  is a nonlinear vector function that has a triangular structure w.r.t.  $x$ ;  $u \in \mathbb{R}^s$  denotes the system input,  $y \in \mathbb{R}^p$  is the system output;  $\varepsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_q}$ , is an unknown function which denotes the system uncertainties and  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_q}$  is the output noise.

The observer design will be performed under the following assumptions.

**A1.** The state  $x(t)$  and the control  $u(t)$  are bounded, i.e.  $x(t) \in X$  and  $u(t) \in U$  where  $X \subset \mathbb{R}^n$  and  $U \subset \mathbb{R}^s$  are compact sets. One sets  $x_M = \sup_{x \in X} \|x\|$ .

**A2.** Each  $\varphi^k(u, x)$ ,  $k = 1, \dots, q$  and each  $F_k(u, x)$ ,  $k = 1, \dots, q-1$ , is Lipschitz on  $X$  with respect to  $x$  uniformly in  $u$ , i.e.

$$\exists L_\varphi > 0 / \forall u \in U; \forall (x, \bar{x}) \in X \times X, \text{ one has } \|\varphi^{(k)}(u, x) - \varphi^{(k)}(u, \bar{x})\| \leq L_\varphi \|x - \bar{x}\|,$$

$$\exists L_F > 0 / \forall u \in U; \forall (x, \bar{x}) \in X \times X, \text{ one has } \|F_k(u, x) - F_k(u, \bar{x})\| \leq L_F \|x - \bar{x}\|.$$

**A3.** The unknown uncertainty  $\varepsilon$  and the noise signal  $w$  are essentially bounded functions, i.e.

$$\exists (\delta_\varepsilon, \delta_w) \in \mathbb{R}^+ \times \mathbb{R}^+ / \|\varepsilon\|_\infty \triangleq \text{ess sup}_{t \geq 0} \|\varepsilon(t)\| \leq \delta_\varepsilon, \|w\|_\infty \triangleq \text{ess sup}_{t \geq 0} \|w(t)\| \leq \delta_w. (2.6)$$

**Remark 2.1.1** *Since the state is bounded, one can construct global Lipschitz prolongations of the nonlinearities using saturation functions ([26]). To avoid symbol redundancy, one shall assume as in [57] that such prolongations have been achieved and that the functions  $F$  and  $\varphi$  result from these prolongations.*

System (2.1) has been already considered in [58] in the free uncertainties and noise measurements case and a SHGO has been designed under a set of assumptions amongst which **A1** to **A3**. Similarly, this system belongs to the class of systems considered in [59] and [60] where a SHGO has been proposed in the absence of noise measurements. For comparison purposes and in order to highlight the main steps of the FHGO design which shall

be detailed later, let us recall the equations of a SHGO proposed in the above references ([59, 60])

$$\dot{\hat{x}}(t) = F(u, \hat{x})\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta\Delta_\theta^{-1}P(t)C^T(\hat{x}(t) - y(t)), \quad (2.7)$$

with

$$\Delta_\theta = \text{diag} \left[ I_{n_1} \quad \frac{1}{\theta}I_{n_2} \quad \cdots \quad \frac{1}{\theta^{q-1}}I_{n_q} \right], \quad (2.8)$$

where  $\hat{x} = \begin{pmatrix} \hat{x}^{(1)} \\ \vdots \\ \hat{x}^{(q)} \end{pmatrix} \in \mathbb{R}^n$ , with  $x^{(k)} \in \mathbb{R}^{n_k}$ ,  $k = 1, \dots, q$ , denotes the state estimate and

$P(t)$  is a  $n \times n$  symmetric matrix governed by the following Riccati ODE

$$\begin{cases} \dot{P}(t) &= \theta(P(t) + F(u, \hat{x})P(t) + P(t)F^T(u, \hat{x}) - P(t)C^T C P(t)), \\ P(0) &= P^T(0) > 0. \end{cases} \quad (2.9)$$

Recall that the convergence of the underlying observation error has been established under Assumptions **A1** to **A3**, together with an appropriate persistent excitation condition similar to the one which shall be considered later when designing the FHGO.

## 2.2 Motivation of the proposed design

Before giving the equations of the proposed FHGO, one shall recall some results related on the design of a FHGO for a class of uniformly observable systems proposed in [33]. Indeed, the design that shall be proposed later is a tentative to extend the design considered in [56] to the class of non uniformly observable systems given by (2.1). The FHGO design considered in [33] deals with the following class of Single Input Single Output (SISO) nonlinear systems

$$\begin{cases} \dot{x}(t) = A_n x(t) + \varphi(u(t), x(t)) + B_n \varepsilon(t) \\ y(t) = C_n x(t) + w(t) = x_1(t) + v(t) \end{cases} \quad (2.10)$$

where  $x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^T \in \mathbb{R}^n$  denotes the state of the system with  $x_i \in \mathbb{R}$ ,  $i = 1, \dots, n$ ;  $u(t) \in \mathbb{R}$  denotes the system input and  $y(t) \in \mathbb{R}$  denotes the output of the system,  $w(t)$  is the output noise and the matrices  $A_n$ ,  $B_n$  and  $C_n$  are given by

$$A_n = \begin{bmatrix} 0_{n-1,1} & I_{n-1} \\ 0 & 0_{1,n-1} \end{bmatrix}, \quad B_n = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}^T \in \mathbb{R}^n, \quad C_n = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{1 \times n} \quad (2.11)$$

The function  $\varepsilon : [0, +\infty[ \mapsto \mathbb{R}$  denotes the system uncertainties and may depend on the input and uncertain parameters. It shall be treated as an unknown function which explicitly depends on time  $t$  for  $t \geq 0$ ; finally,  $\varphi(u, x) \in \mathbb{R}^n$  denotes a nonlinear vector field which assumes a triangular structure with respect to  $x$ , i.e.  $\varphi_i(u, x) = \varphi_i(u, x_1, \dots, x_i)$ ,

$i = 1, \dots, n$ .

The observer design has been performed under assumptions similar to (A1), (A2) and (A3) considered above for the design of the FHGO that shall be proposed later.

The equations of the observer proposed for system (2.10) in [33] are

$$\dot{\hat{x}}(t) = A_n \hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta \Delta_n^{-1}(\theta) G_n (C_n \hat{x}(t) - y(t)) \quad (2.12)$$

with  $\hat{x} \in \mathbb{R}^n$  denotes the state estimate,

$$\Delta_n = \text{diag}(1, 1/\theta, \dots, 1/\theta^{n-1}), \quad (2.13)$$

$$G_n = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \in \mathbb{R}^n, \quad (2.14)$$

where  $G_n$  is chosen such that the matrix  $\bar{A}_n \triangleq A_n - G_n C_n$  is Hurwitz and  $\theta \geq 1$  is a scalar design parameter. Notice that, since the matrix  $\bar{A}_n = A_n - G_n C_n$  is Hurwitz, there exist a  $n \times n$  SPD matrix  $P$  and a positive real  $\mu$  such that

$$P \bar{A}_n + \bar{A}_n^T P \leq -2\mu I_n \quad (2.15)$$

The analysis of the convergence of the observation error related to the SHGO is performed using a Lyapunov approach. More precisely, the Lyapunov function used throughout the convergence analysis is  $V(\bar{x}) = \bar{x}^T P \bar{x}$  with  $\bar{x} = \theta^{n-1} \Delta_\theta \tilde{x}$  where  $\Delta_\theta$  is given by (2.13),  $\tilde{x} = \hat{x} - x$  and  $P = P^T$  is the SPD matrix given by 2.15. In particular, it has been shown that

$$\begin{aligned} & \exists \theta_0 > 0; \forall \theta \geq \theta_0; \forall u \in U; \forall \hat{x}(0) \in \mathbb{R}^n; \text{ one has} \\ & \|\tilde{x}(t)\| \leq \sigma \theta^{n-1} e^{-(\mu\theta)/(2\lambda_M)t} \|\tilde{x}(0)\| + 2\lambda_M \sigma / \mu (\delta_\varepsilon / \theta + \theta^{n-1} \|G_n\| \delta_w) \end{aligned} \quad (2.16)$$

where  $\mu > 0$  is a positive real given by equation (2.15) and  $\delta_\varepsilon$  and  $\delta_w$  are the upper essential bounds of  $\|\varepsilon(t)\|$  and  $\|w(t)\|$  given in Assumption **A3** with  $\theta_0 = \max(1, 2L_\varphi \sqrt{n} \lambda_M / \mu)$  where the  $L_\varphi$  is the Lipschitz constants given by Assumption **A2** and  $\sigma$  is the conditioning number of the matrix  $P$  given by (2.15).

The observer design in [33] is in particular based on the following lemma.

**Lemma 2.2.1** *Let  $M_n = \begin{pmatrix} A_n & -K_n \\ C_n^T C_n & -D_n + A_n^T \end{pmatrix}$  where  $n \geq 2$  is a positive integer,  $A_n$  and  $C_n$  are given by (2.11), and  $D_n, K_n$  are the following diagonal matrices*

$$D_n = \text{diag}(\delta_1, \delta_2, \dots, \delta_n) \quad \text{and} \quad K_n = \text{diag}(k_1, k_2, \dots, k_n),$$

where each  $\delta_i$  (resp.  $k_i$ ),  $i = 1, \dots, n$ , is a real constant (resp. a non zero real constant). There exists a choice of  $\delta_i$  and  $k_i$ ,  $i = 1, \dots, n$  such that the eigenvalues of  $M_n$  can be arbitrarily assigned.

The arbitrary assignment of the eigenvalues of  $M_n$  can be achieved by using an iterative procedure provided in [56] and it requires the determination of the roots of  $n$  polynomials.

The FHGO proposed in [33] for the class of systems (2.1) is constituted by two subsystems each one of which has the same dimension as the original system (2.1). The first subsystem is a copy of the original system, with a corrective term that linearly depends on the state of the second subsystem. The latter is a linear filter the input of which is the difference between the output predicted by the observer and the output measurement, namely the output observation error. More precisely, the proposed observer is given by the following set of equations

$$\begin{cases} \dot{\hat{x}}(t) &= A_n \hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta K_n e(t), \\ \dot{e}(t) &= -\theta D_n e(t) + \theta^2 A_n^T e(t) + \theta C_n^T (C_n \hat{x}(t) - y(t)) \text{ with } e(0) = 0, \end{cases} \quad (2.17)$$

where  $\hat{x} = \begin{pmatrix} \hat{x}_1 & \dots & \hat{x}_n \end{pmatrix}^T \in \mathbb{R}^n$ ,  $e = \begin{pmatrix} e_1 & \dots & e_n \end{pmatrix}^T \in \mathbb{R}^n$  with  $\hat{x}_i, e_i \in \mathbb{R}$ ,  $\theta > 0$  is a positive real design parameter, and where  $D_n$  and  $K_n$  are the following diagonal matrices

$$D_n = \text{diag}(\delta_1, \dots, \delta_n), \quad K_n = \text{diag}(k_1, \dots, k_n). \quad (2.18)$$

The coefficients  $\delta_i$  and  $k_i$ ,  $i = 1, \dots, n$ , are real constants and are chosen such that the matrix

$$M_n = \begin{pmatrix} A_n & -K_n \\ C_n^T C_n & -D_n + A_n^T \end{pmatrix} \quad (2.19)$$

is Hurwitz. Notice that such a choice is possible according to Lemma 2.2.1 and it can be achieved by pursuing the proposed iterative procedure. Now, since the matrix  $M_n$  is Hurwitz, there exist a SPD matrix  $P = P^T > 0$  and a positive real  $\mu > 0$  such that

$$M_n^T P + P M_n \leq -2\mu I_{2n}. \quad (2.20)$$

The convergence analysis of the above FHGO is achieved by using a Lyapunov approach and the underlying Lyapunov function is based on the matrix  $P$  given by (2.20).

Let us compare the structure of the FHGO (2.17) with that of the SHGO (2.12) in order to put forward the rational behind the proposed design. Let us first focus on the observer

state  $e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$ . According to the ODE governing  $e(t)$  in (2.17), The first component  $e_1$  is a filtered version of the output observation error  $\hat{x}_1 - y$  with a filter of order 1 and the static gain of which does not depend on  $\theta$ . For  $i \geq 2$ , the component  $e_i$  is also a filtered version of  $e_{i-1}$  with a filter of order 1 and the static gain of which is proportional to  $\theta$ . The component  $e_i$  is therefore a filtered version of  $\hat{x}_1 - y$  with a filter of order  $i$  and the static gain of which is proportional to  $\theta^{i-1}$ . Now, the ODE associated to the state  $\hat{x}_i$  in the FHGO (2.17) is obtained from that given by the SHGO (2.12) by substituting the corrective term  $\theta^{i-1}\gamma_i(\hat{x}_1 - y)$  by its filtered version, namely  $k_i e_i$ . Two features are worth to be mentioned. The first one deals with the fact that the power of the design parameter  $\theta$  does not exceed 2 in the FHGO. The second feature deals with the fact the substitution of the output observation error by an appropriate filtered version in the state estimate equations will definitely improve the behavior of the FHGO with respect to the output measurements noise.

For clarity purposes and in order to put forward in a more clearer manner the rationale behind the proposed filtering process, one shall consider a general example where the dimension of the original system is fixed to  $n = 3$  and compare the structures of the underlying SHGO and FHGO. To simplify and to emphasize the filtering process of high frequency signals, one shall consider the disturbance free case, i.e.  $w(t) \equiv 0$ . The equations of the SHGO can be written under the following developed form

$$\begin{cases} \dot{\hat{x}}_1(t) &= \hat{x}_2 + \varphi_1(u, \hat{x}_1) - \gamma_1\theta(\hat{x}_1 - y(t)) \\ \dot{\hat{x}}_2(t) &= \hat{x}_3 + \varphi_2(u, \hat{x}_1, \hat{x}_2) - \gamma_2\theta^2(\hat{x}_1 - y(t)) \\ \dot{\hat{x}}_3(t) &= \varphi_3(u, \hat{x}) - \gamma_3\theta^3(\hat{x}_1 - y(t)) \end{cases} \quad (2.21)$$

with  $y(t) = x_1 + v(t)$  and where  $\gamma_i > 0$ ,  $i = 1, \dots, 3$  and  $\theta > 1$ .

Let  $\tilde{x} = \hat{x} - x$  be the underlying observation error. Then, one has

$$\begin{cases} \dot{\tilde{x}}_1(t) &= \tilde{x}_2 + \tilde{\varphi}_1(u, \hat{x}_1, x_1) - \gamma_1\theta(\tilde{x}_1 - v(t)) \\ \dot{\tilde{x}}_2(t) &= \tilde{x}_3 + \tilde{\varphi}_2(u, \hat{x}_1, \hat{x}_2, x_1, x_2) - \gamma_2\theta^2(\tilde{x}_1 - v(t)) \\ \dot{\tilde{x}}_3(t) &= \tilde{\varphi}_3(u, \hat{x}, x) - \gamma_3\theta^3(\tilde{x}_1 - v(t)) \end{cases} \quad (2.22)$$

where  $\tilde{\varphi}_i(u, \hat{x}, x) = \varphi(u, \hat{x}) - \varphi_i(u, x)$ ,  $i = 1, \dots, 3$ .

The system of error equations (2.22) can be written under the following compact form

$$\dot{\tilde{x}}(t) = A_3\tilde{x}(t) + \tilde{\varphi}(u, \hat{x}, x) - \theta\Delta_3^{-1}(\theta)G_3C_3\tilde{x} + \theta\Delta_\theta^{-1}G_3v(t) \quad (2.23)$$

where  $(A_3, C_3)$ ,  $\Delta_3$  and  $G_3$  are respectively given by (2.11), (2.13) and (2.14) with  $n = 3$ . More precisely, one has

$$\left\{ \begin{array}{l} A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, G_3 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \\ \Delta_3(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\theta} & 0 \\ 0 & 0 & \frac{1}{\theta^2} \end{pmatrix}. \end{array} \right.$$

Recall that the coefficients of the vector  $G_3$ , i.e. the  $\gamma_i$ 's,  $i = 1, 2, 3$  are chosen to assign the poles of  $\bar{A}_3 = A_3 - K_3 C_3$  at predefined desired values.

The stability analysis of the error system (2.23) can be fairly achieved by considering the following (classical) change of variables:  $\bar{x} = \theta^2 \Delta_3 \tilde{x}$ . Indeed, one can show that

$$\dot{\tilde{x}}(t) = \theta \bar{A}_3 \bar{x}(t) + \theta^2 \Delta_3 \tilde{\varphi}(u, \hat{x}, x) + \theta^3 G_3 v(t).$$

The remaining of the stability analysis is carried out by using a Lyapunov approach which is based on the exponential stability of the linear part  $(\theta \bar{A}_3 \bar{x})$  of the above ODE.

The error equations associated to the FHGO (2.17) specialize for  $n = 3$  as follows

$$\left\{ \begin{array}{l} \dot{\tilde{x}}_1 = \tilde{x}_2 + \tilde{\varphi}_1(u, \hat{x}_1, x_1) - k_1 \theta e_1 \\ \dot{\tilde{x}}_2 = \tilde{x}_3 + \tilde{\varphi}_2(u, \hat{x}_1, \hat{x}_2, x_1, x_2) - k_2 \theta e_2 \\ \dot{\tilde{x}}_3 = \tilde{\varphi}_3(u, \hat{x}, x) - k_3 \theta e_3 \\ \dot{e}_1 = -\theta \delta_1 e_1 + \theta(\tilde{x}_1 - v) \\ \dot{e}_2 = -\theta \delta_2 e_2 + \theta^2 e_1 \\ \dot{e}_3 = -\theta \delta_3 e_3 + \theta^2 e_2 \end{array} \right. \quad (2.24)$$

The above system can be written under the following compact form

$$\left\{ \begin{array}{l} \dot{\tilde{x}}(t) = A_3 \tilde{x} + \tilde{\varphi}(u, \hat{x}, x) - \theta K_3 e \\ \dot{e}(t) = -\theta D_3 e + \theta^2 A_3^T e + \theta(C_3^T C_3 \tilde{x} - v) \end{array} \right. \quad (2.25)$$

where  $K_3$  and  $D_3$  are the following diagonal matrices

$$K_3 = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, D_3 = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix}.$$

Recall that the coefficients of the above matrices can be chosen through an appropriate algorithm to assign the eigenvalues of the following matrix

$$M_3 = \begin{pmatrix} A_3 & -K_3 \\ C_3^T C_3 & -D_3 + A_3^T \end{pmatrix}, \quad (2.26)$$

to predefined values.

The stability analysis of the above error system (2.25) can be fairly performed by considering the following change of variables:  $\bar{x} = \theta^2 \Delta_3 \tilde{x}$  and  $\bar{e} = \theta^2 \Delta_3 e$ . Indeed, one can show that

$$\begin{cases} \dot{\bar{x}}(t) &= \theta A_3 \bar{x} - \theta K_3 \bar{e} + \theta^2 \Delta_3(\theta) \tilde{\varphi}(u, \hat{x}, x) \\ \dot{\bar{e}}(t) &= \theta C_3^T C_3 \bar{x} - \theta (D_3 + A_3^T) \bar{e} - \theta^3 \Delta_3(\theta) v \end{cases} \quad (2.27)$$

Again, the above system can be written under the following more condensed form

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{e}} \end{pmatrix} = \theta \underbrace{\begin{pmatrix} A_3 & -K_3 \\ C_3^T C_3 & -D_3 + A_3^T \end{pmatrix}}_{M_3} \begin{pmatrix} \bar{x} \\ \bar{e} \end{pmatrix} + \begin{pmatrix} \theta^2 \Delta_3(\theta) \tilde{\varphi}(u, \hat{x}, x) \\ -\theta^3 \Delta_3(\theta) v \end{pmatrix}, \quad (2.28)$$

Using the fact that the linear part ( $\theta M_3$ ) of the above error system is exponentially stable, a Lyapunov based approach is used to conclude to the stability of the above error system.

Let us now focus on the developed form of the error equations system associated to the SHGO (2.22) on one hand and on the equations of the error state observation error involved in (2.24) on the other hand. Indeed, the error equation associated to  $\tilde{x}_1$  in (2.22) is

$$\dot{\tilde{x}}_1 = \tilde{x}_2 + \tilde{\varphi}_1(u, \hat{x}, x) - \underbrace{\gamma_1 \theta (\tilde{x}_1 - v)}_{\text{corrective term}}. \quad (2.29)$$

while this error equation in the error system associated to the FHGO (2.24) is

$$\dot{\tilde{x}}_1 = \tilde{x}_2 + \tilde{\varphi}_1(u, \hat{x}, x) - \underbrace{k_1 \theta e_1}_{\text{corrective term}}. \quad (2.30)$$

It is clear that the sole difference between the above two equations, i.e. (2.29) and (2.30), is that the dynamics of  $\tilde{x}_1$  in the FHGO can be obtained from its dynamics in the SHGO by substituting the term involving the noisy output observation error, i.e.  $\gamma_1(\tilde{x}_1 - v)$  by  $k_1 e_1$  which is a filtered version of this error obtained from the following first order linear filter (see the forth equation of system (2.24))

$$\dot{e}_1 = -\theta \delta_1 e_1 + \theta(\tilde{x}_1 - v).$$

Notice that the relative degree of the transfer function between  $e_1$  and  $\tilde{x}_1$  is equal to 1 and the underlying static gain is independent of  $\theta$ .

Let us now focus on the dynamics of  $\tilde{x}_2$ . According to (2.22), its dynamics issued from the SHGO is

$$\dot{\tilde{x}}_2 = \tilde{x}_3 + \tilde{\varphi}_2(u, \hat{x}, x) - \underbrace{\gamma_2 \theta^2 (\tilde{x}_1 - v)}_{\text{corrective term}}. \quad (2.31)$$

while it is updated in the FHGO as follows (see (2.24))

$$\dot{\tilde{x}}_2 = \tilde{x}_3 + \tilde{\varphi}_2(u, \hat{x}, x) - \underbrace{k_2 \theta e_2}_{\text{corrective term}}. \quad (2.32)$$

Hence, the term  $\gamma_2 \theta^2 \tilde{x}_1$  is substituted by  $k_2 \theta e_2$ . According to (2.24),  $e_2$  is a filtered version of  $e_1$ , i.e.

$$\dot{e}_2 = -\theta \delta_2 e_2 + \theta^2 e_1.$$

It is clear that the relative degree of the transfer function between  $e_2$  and  $e_1$  is equal to 1 and the underlying static gain is of the order of  $\theta$ . Hence, one can easily deduce that, the relative degree of the transfer function between  $e_2$  and  $\tilde{x}_1$  is equal to 2 and the underlying static gain is of the order of  $\theta$ .

In a similar manner, one can easily deduce that  $e_3$  is a filtered version of  $\tilde{x}_1$  and the relative degree of the transfer function between  $e_3$  and  $\tilde{x}_1$  is equal to 2 with a static gain of the order of  $\theta^2$ . This explains why the term  $-\gamma_3 \theta^3 \tilde{x}_1$  constituting the corrective term in  $\dot{\tilde{x}}_3$  in the SHGO (2.22) has been substituted by  $-k_3 \theta e_3$  in the FHGO (2.24).

## 2.2.1 Tentative to extend the design to non uniformly observable systems

A direct extension of the design considered in [33] has been first envisaged to the following class of SISO non uniformly observable system

$$\begin{cases} \dot{x}(t) = F_n(u(t))x(t) + \varphi(u(t), x(t)) + B_n \varepsilon(t) \\ y(t) = C_n x(t) + w(t) = x_1(t) + v(t) \end{cases} \quad (2.33)$$

where

$$F_n(u(t)) = \begin{pmatrix} 0 & f_1(u) & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & f_{n-1}(u) \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}. \quad (2.34)$$

where the  $f_i(t)$ ,  $i = 1, \dots, n$  are time-varying functions and all the other variables keep the same meaning as in system (3.84).

The first prerequisite to perform the FHGO design extension proposed in [33] to the class of system (3.84) is to look for a counterpart of the result provided by lemma 2.2.1. However, the main result of lemma 2.2.1 is the exhibition of a Hurwitz (constant) matrix. The checking of the Hurwitz property of the matrix is performed by checking the real parts of its eigenvalues. Such a procedure is no longer true when the coefficients of the matrix are time-varying. More precisely, the counterpart of the problem solved by lemma 2.2.1 can be formulated as follows. Could one choose diagonal matrices  $D_n(t)$  and  $K_n(t)$  (with eventually time-varying coefficients) such that the following matrix

$$M_n = \begin{pmatrix} F_n(u) & -K_n(t) \\ C_n^T C_n & -D_n(t) + A_n^T \end{pmatrix} \quad (2.35)$$

is stable in the sense that the state of the following linear time-varying system  $\dot{x}(t) = M_n(t)x(t)$  with state  $x$  is stable. Notice that the response to the above question is not easy. This explains why the idea to exactly extend the above design was subscribed and a similar but different design shall be considered.

Before detailing the proposed design, one shall put forward the rational behind it. Indeed, in the single output case, one shall look for a filtered observer, where the matrix  $D_n(t)$  is no longer diagonal but reduced to a scalar constant matrix, i.e.  $D_n(t) = \delta I_n$  where  $\delta$  is a positive real. Since, the problem has been significantly simplified, one shall consider the design for a class of systems with multiple outputs given by system (2.1).

## 2.3 The FHGO design

Recall that one aims at alleviating the sensitivity of the SHGO (2.12) to the ubiquitous noise measurements. A suitable filtering process is used to this end by cascading two subsystems. The first subsystem corresponds to a copy of the original system, with a (time-varying) corrective term that linearly depends on the state of the second subsystem. The latter is a linear filter the input of which is the difference between the output predicted by the observer and the output measurement, namely the output observation error. The proposed observer is given by the following set of equations

$$\begin{cases} \dot{\hat{x}}(t) &= F(u, \hat{x})\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta K(t)\eta(t), \\ \dot{\eta}(t) &= \delta\theta \{-\eta(t) + \theta A^T \eta(t) + C_{q_n}^T (C\hat{x}(t) - y(t))\}, \quad \eta(0) = 0, \end{cases} \quad (2.36)$$

where  $\hat{x} = \begin{pmatrix} \hat{x}^{(1)} \\ \vdots \\ \hat{x}^{(q)} \end{pmatrix} \in \mathbb{R}^n$  with  $x^{(k)} \in \mathbb{R}^{n_k}$ ,  $k = 1, \dots, q$ ,  $\eta = \begin{pmatrix} \eta^{(1)} \\ \vdots \\ \eta^{(q)} \end{pmatrix} \in \mathbb{R}^{qn_1}$  with  $\eta^{(k)} \in \mathbb{R}^{n_1}$  and  $A$  is the  $qn_1 \times qn_1$  anti-shift matrix, i.e.

$$A = \begin{bmatrix} 0_{n_1} & I_{n_1} & 0_{n_1} & 0_{n_1} \\ \vdots & \ddots & \ddots & 0_{n_1} \\ 0_{n_1} & \cdots & \cdots & I_{n_1} \\ 0_{n_1} & \cdots & \cdots & 0_{n_1} \end{bmatrix}, \quad C_{qn_1}^T = \begin{bmatrix} I_{n_1} \\ 0_{n_1} \\ \vdots \\ 0_{n_1} \end{bmatrix}. \quad (2.37)$$

The terms  $\delta$  and  $\theta > 0$  are positive real design parameters and finally,  $K(t)$  is the following diagonal matrix

$$K(t) = \text{diag}(P(t)C^T) \triangleq \text{diag}(K^{(1)}(t) \dots K^{(q)}(t)), \quad (2.38)$$

where each  $K^{(i)}$  is  $n_i \times n_1$  matrix and  $P(t)$  is a  $n \times n$  symmetric matrix governed by the following Riccati ODE

$$\begin{cases} \dot{P}(t) &= \theta (P(t) + F(u, \hat{x})P(t) + P(t)F^T(u, \hat{x}) - P(t)C^T C P(t)), \\ P(0) &= P^T(0) > 0. \end{cases} \quad (2.39)$$

Besides Assumptions **A1** to **A3**, the following assumption is needed for the observer design:

**A4.** The Riccati ODE (2.39) has a positive definite solution that satisfies the following property

$$\exists \rho^* > 0; \exists \theta^* > 0; \quad \forall \theta > \theta^*; \exists t^* \geq 0; \forall t \geq t^*, (1/\bar{\lambda}_M(P(t))) \geq (\rho^*/\alpha(\theta))I_n, \quad (2.40)$$

where  $\alpha(\theta)$  is a positive function satisfying

$$\forall \theta > 0, \alpha(\theta) \geq 1 \quad \text{and} \quad \exists 0 < \chi \leq 1, \quad \lim_{\theta \rightarrow \infty} \frac{\alpha(\theta)}{\theta^{2\chi}} = 0. \quad (2.41)$$

Notice that the dynamics of the Riccati equation (2.39) involved in Assumption A4 depends on the state estimate  $\hat{x}$  as in [61, 62] and not on the state  $x$ . At a first glance, it would seem more natural to express Assumption A4 using the state rather than its estimate. But this would lead to a non checkable assumption since the state is not available.

Considering the observer states instead of the system states, as formulated in Assumption A4, would allow to check it at least on-line, even though not *a priori*. Indeed, Assumption A4 can be checked on-line by simply computing the inverse of the largest instantaneous eigenvalue of  $P(t)$ .

On other aspects, Assumption A4, which is similar to that considered in [61, 62], is of a primary importance for the stability of the observer. Indeed, as noted in [61], this assumption is satisfied for uniformly observable systems, i.e. systems which are observable for any input. For non uniformly observable systems, the characterization of the class of inputs which satisfy Assumption A4 is still an open problem despite that some interesting results dealing with some subclasses of systems (2.1) have already been obtained. Indeed, the authors in [23] introduced the notion of local regular inputs which are defined as those satisfying Assumption A4 with  $\alpha(\theta) \equiv 1$  for all  $\theta$ . However, these inputs were defined for a class of systems included in (2.1), i.e. the matrices  $F_k$  do not depend on the state. In [24], the class of systems considered in [23] was revisited and the authors enlarged the class of locally regular inputs to the class of regular inputs. In the spirit of the persistent excitation conditions proposed in [23, 24], a similar condition which allows Assumption **A4** to be satisfied has been formulated in [59, 60] where some classes of systems similar to (2.1) have been considered with a view to observer design in the free noise case. As in [23, 24], the underlying formulation used the transition matrix  $\Phi_{u,\hat{x}}(t, s)$  of the following state affine system  $\dot{\xi}_{u,\hat{x}}(t) = A(u, \hat{x})\xi_{u,\hat{x}}(t)$  where  $\xi_{u,\hat{x}} \in \mathbb{R}^n$  is the system state,  $u$  and  $\hat{x}$  are the inputs of the system and they respectively correspond to the input of system (2.1) and to the state of the dynamical system (2.36). Recall that the matrix  $\Phi_{u,\hat{x}}(t, s)$  is defined as follows

$$\begin{cases} \frac{d}{dt} (\Phi_{u,\hat{x}}(t, s)) &= A(u(t), \hat{x}(t))\Phi_{u,\hat{x}}(t, s), \\ \Phi_{u,\hat{x}}(s, s) &= I_n. \end{cases} \quad (2.42)$$

The persistent excitation condition was formulated as follows in [59, 60]

- $\exists \theta_0 > 0; \exists \rho_0 > 0; \exists T^* > 0; \forall \theta \geq \theta_0; \forall t \geq \frac{T^*}{\theta}$ , one has

$$\int_{t-\frac{T^*}{\theta}}^t \Phi_{u,\hat{x}}^T(s, t) C^T C \Phi_{u,\hat{x}}(s, t) ds \geq \frac{\rho_0}{\theta \alpha(\theta)} \Delta_\theta^2, \quad (2.43)$$

where the function  $\alpha(\theta)$  satisfies (3.66) and  $\Delta_\theta$  is given by (2.8).

It has been shown in [59, 60] that Assumption **A4** is satisfied under the persistent exci-

tation condition (3.88)-(2.43) and one has

$$\theta^* = \theta_0, \quad t^* = \frac{T^*}{\theta} \quad \text{and} \quad \rho^* = e^{-T^*} \rho_0.$$

Before stating the theorem where the main properties of observer (2.36) are summarized, one shall compare the structure of this observer with that of the SHGO (2.12) in order to put forward the rational behind the proposed design. Let us first focus on the observer

sub-state  $\eta = \begin{pmatrix} \eta^{(1)} \\ \vdots \\ \eta^{(q)} \end{pmatrix}$ . The first component  $\eta^{(1)}$  is a filtered version of the output observation error  $\hat{x}^{(1)} - y$  with a filter of order 1 and the static gain of which does not depend on  $\theta$ . For  $i \geq 2$ , the component  $\eta^{(i)}$  is also a filtered version of  $\eta^{(i-1)}$  with a filter of order 1 and the static gain of which is proportional to  $\theta$ . The component  $\eta^{(i)}$  is therefore a filtered version of  $\hat{x}^{(1)} - y$  with a filter of order  $i$  and the static gain of which is proportional to  $\theta^{i-1}$ .

Now, the ODE associated to the state  $\hat{x}_i$  in the FHGO (2.36) is obtained from that given by the SHGO (2.12) by replacing the corrective term  $\theta^i K^{(i)}(\hat{x}_1 - y)$  by its filtered version, namely  $\theta K^{(i)} \eta_i$ . Two features are worth to be mentioned. The first one deals with the fact that the power of the design parameter  $\theta$  does not exceed 2 in the FHGO. The second feature deals with the fact the substitution of the output observation error by an appropriate filtered version in the state estimate equations will definitely improve the behavior of the FHGO with respect to the output measurements noise.

Now, the main properties of the proposed observer (2.36) are summarized in the following theorem.

**Theorem 2.3.1** *Consider system (2.1) subject to assumptions **A1** to **A3**. Then, for every bounded input satisfying **A4**, there exist  $\delta^*, \theta^*$ , such that for all  $\theta \geq \theta^*$  and for all  $\delta > \max(\delta^*, \theta^*)$ , one has for all  $t \geq t^*$ ,*

$$\begin{aligned} \|\tilde{x}(t)\| &\leq \beta_f \theta^{q-1} \sqrt{\alpha(\theta)} e^{-\nu \frac{\theta}{2} \mu(\theta)(t-t^*)} \left( (1 + \sqrt{n}) \|\tilde{x}(t^*)\| + \|\eta(t^*)\| \right) \\ &\quad + 2 \frac{\beta_f}{\nu \mu(\theta)} \left( \sqrt{\frac{\alpha(\theta)}{\theta^2}} \delta_\varepsilon + \sqrt{\alpha(\theta)} \delta \theta^{q-1} \delta_w \right). \end{aligned} \quad (2.44)$$

where  $\tilde{x} = \hat{x} - x$  with  $\hat{x}$  being any trajectory of system (2.36),  $\delta_w$  and  $\delta_\varepsilon$  are the ultimate bounds of the uncertainties given by Assumption **A3**,  $\beta_f$  and  $\nu$  are positive constants independent of  $\theta$  and  $\delta$ ; the positive reals  $t^*, \rho^*, \chi$  and the function  $\alpha$  are given by Assumption **A4** and finally  $\mu(\theta) > 0$  is such that  $\lim_{\theta \rightarrow \infty} \mu(\theta) = 1$ .

**Proof of Theorem 2.3.1.** Set  $\tilde{x} = \hat{x} - x$ ,  $\bar{x} = \theta^{q-1} \Delta_\theta \tilde{x}$  and  $\bar{\eta} = \theta^{q-1} D_\theta \eta$  where  $\Delta_\theta$  is the diagonal matrix given by (2.8) and  $D_\theta$  is the following  $qn_1 \times qn_1$  diagonal matrix

$$D_\theta = \text{diag}(I_{n_1}, \frac{1}{\theta}I_{n_1}, \dots, \frac{1}{\theta^{q-1}}I_{n_1}). \quad (2.45)$$

Notice that one can easily check the following equalities

$$\Delta_\theta F(u, \hat{x})\Delta_\theta^{-1} = \theta F(u, \hat{x}); \quad \Delta_\theta K(t) = K(t)D_\theta; \quad D_\theta C_{q n_1}^T C = C_{q n_1}^T C \Delta_\theta. \quad (2.46)$$

From (2.1) and (2.36), one has

$$\dot{\hat{x}} = F(u, \hat{x})\hat{x} + \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) - \theta K(t)\eta(t) - B\varepsilon(t). \quad (2.47)$$

Now, using (2.46), one gets

$$\begin{aligned} \dot{\hat{x}} &= \theta F(u, \hat{x})\bar{x} + \theta^{q-1}\Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q \Delta_\theta K(t)\eta(t) - \theta^{q-1}\Delta_\theta B\varepsilon(t) \\ &= \theta F(u, \hat{x})\bar{x} + \theta^{q-1}\Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q K(t)D_\theta \eta(t) - B\varepsilon(t) \\ &= \theta F(u, \hat{x})\bar{x} + \theta^{q-1}\Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta K(t)\bar{\eta}(t) - B\varepsilon(t). \end{aligned} \quad (2.48)$$

Similarly, from (2.1) and (2.36) and using (2.46), one gets

$$\begin{aligned} \dot{\bar{\eta}} &= -\theta\delta \left( (I_{q n_1} - A^T)\bar{\eta} + \theta^{q-1}D_\theta C_{q n_1}^T C(\tilde{x} - w(t)) \right) \\ &= -\theta\delta \left( (I_{q n_1} - A^T)\bar{\eta} + \theta^{q-1}C_{q n_1}^T C \Delta_\theta(\tilde{x} - w(t)) \right) \\ &= -\theta\delta \left( (I_{q n_1} - A^T)\bar{\eta} + C_{q n_1}^T C\bar{x} + \theta^{q-1}C_{q n_1}^T C w(t) \right). \end{aligned} \quad (2.49)$$

Adding and subtracting the term  $\theta P(t)C^T C\bar{x}$  to the right side of equation (2.48), one gets

$$\begin{aligned} \dot{\hat{x}} &= \theta \left( F(u, \hat{x}) - P(t)C^T C \right) \bar{x} + \theta^{q-1}\Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) \\ &\quad - B\varepsilon(t) - \theta \left( K(t)\bar{\eta} - P(t)C^T C\bar{x} \right). \end{aligned} \quad (2.50)$$

Now, since  $K(t) = \text{diag}(P(t)C^T)$ , one has  $K(t)U = P(t)C^T C$  where

$$U \triangleq \begin{pmatrix} I_{n_1} & 0_{n_1, n_2} & \dots & 0_{n_1, n_q} \\ \vdots & \vdots & \vdots & \vdots \\ I_{n_1} & 0_{n_1, n_2} & \dots & 0_{n_1, n_q} \end{pmatrix} \in \mathbb{R}^{q n_1 \times n}. \quad (2.51)$$

Hence, equation (2.50) can be rewritten as follows

$$\begin{aligned}\dot{\hat{x}} &= \theta (F(u, \hat{x}) - P(t)C^T C) \bar{x} + \theta^{q-1} \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) \\ &\quad - B\varepsilon(t) - \theta K(t)\bar{z},\end{aligned}\tag{2.52}$$

where

$$\bar{z} = \bar{\eta} - U\bar{x}.\tag{2.53}$$

Let now derive the time derivative of  $\bar{z}$ . Indeed, using (2.48), one gets

$$\begin{aligned}\dot{\bar{z}} &= \dot{\bar{\eta}} - U\dot{\hat{x}} \\ &= -\theta\delta \left( (I_{q_{n_1}} - A^T)\bar{\eta} - C_{q_{n_1}}^T C\bar{x} + \theta^{q-1} C_{q_{n_1}}^T Cw(t) \right) - \theta U F(u, \hat{x})\bar{x} \\ &\quad - \theta^{q-1} U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) + \theta U K(t)\bar{\eta} + U B\varepsilon(t) \\ &= \theta \left( -\delta(I_{q_{n_1}} - A^T) + U K(t) \right) \bar{\eta} + \theta \left( \delta C_{q_{n_1}}^T C - U F(u, \hat{x}) \right) \bar{x} \\ &\quad - \theta^{q-1} U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q \delta C_{q_{n_1}}^T Cw(t) \\ &= \theta \left( -\delta(I_{q_{n_1}} - A^T) + U K(t) \right) (\bar{z} + U\bar{x}) + \theta \left( \delta C_{q_{n_1}}^T C - U F(u, \hat{x}) \right) \bar{x} \\ &\quad - \theta^{q-1} U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q \delta C_{q_{n_1}}^T Cw(t) \\ &= \theta \left( -\delta(I_{q_{n_1}} - A^T) + U K(t) \right) \bar{z} \\ &\quad + \theta \left( \delta C_{q_{n_1}}^T C - U F(u, \hat{x}) - \delta(I_{q_{n_1}} - A^T)U + U K(t)U \right) \bar{x} \\ &\quad - \theta^{q-1} U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q \delta C_{q_{n_1}}^T Cw(t).\end{aligned}$$

Now, one can check that  $(I_{q_{n_1}} - A^T)U = C_{q_{n_1}}^T C$  and the above last equation can therefore be written as follows

$$\begin{aligned}\dot{\bar{z}} &= \theta\delta \left( -(I_{q_{n_1}} - A^T) + \frac{1}{\delta} U K(t) \right) \bar{z} \\ &\quad + \theta \left( -U F(u, \hat{x}) + U K(t)U \right) \bar{x} \\ &\quad - \theta^{q-1} U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q \delta C_{q_{n_1}}^T Cw(t).\end{aligned}\tag{2.54}$$

According to Assumption **A4**, the matrix  $P(t)$  governed by (2.39) is SPD. Let  $S(t) = P^{-1}(t)$ ; one can show that  $S(t)$  is governed by the following Lyapunov ODE

$$\begin{cases} \dot{S}(t) &= -\theta (S(t) + F^T(u, \hat{x})S(t) + S(t)F(u, \hat{x}) - C^T C), \\ S(0) &= S^T(0) > 0. \end{cases}\tag{2.55}$$

Again, according to Assumption **A4**, one has

$$\forall t \geq t^*, \quad \underline{\lambda}_m(S(t)) \geq \frac{\rho^*}{\alpha(\theta)},\tag{2.56}$$

where  $\alpha(\theta)$  satisfies (3.66).

Now, proceeding as in [58, 59], one can show that the matrix  $S(t)$  is bounded and the underlying upper bound (or equivalently  $\bar{\lambda}_M(S(t))$ ) is independent of  $\theta$ . Let  $V_1(\bar{x}, t) = \bar{x}^T S(t) \bar{x}$ ; proceeding as in [58, 59], one can show that for  $\theta \geq 1$ ,

$$\begin{aligned} \dot{V}_1(\bar{x}, t) &\leq -\theta\mu(\theta)V_1(\bar{x}, t) + 2\sqrt{\bar{\lambda}_M(S(t))}\sqrt{V_1(\bar{x}, t)}\|\varepsilon(t)\| - 2\theta\bar{x}^T S(t)K(t)\bar{z}(t) \\ &\leq -\theta\mu(\theta)V_1(\bar{x}, t) + 2\sqrt{\bar{\lambda}_M(S(t))}\sqrt{V_1(\bar{x}, t)}\|\varepsilon(t)\| \\ &\quad + 2\theta\sqrt{\bar{\lambda}_M(S(t))}\sqrt{V_1(\bar{x}, t)}\|K(t)\|\|\bar{z}(t)\|, \end{aligned} \tag{2.57}$$

where

$$\mu(\theta) = 1 - \gamma\sqrt{\frac{\alpha(\theta)}{\theta^2}} \text{ with } \gamma = 2\sqrt{n}(L_F x_M + L_\varphi)\sqrt{\frac{\bar{\lambda}_M(S(t))}{\rho^*}}, \tag{2.58}$$

where  $x_M$  is given by Assumption **A1**,  $L_F$  and  $L_\varphi$  by Assumption **A2** and finally  $\rho^*$  and  $\alpha(\theta)$  by Assumption **A4**.

Similarly, set  $V_2(\bar{z}) = \bar{z}^T \bar{z} = \|\bar{z}\|^2$ . Then, one has

$$\begin{aligned} \dot{V}_2 &= -\theta\delta\bar{z}^T(2I_{q_{n_1}} - (A^T + A))\bar{z} + 2\theta\bar{z}^T U K(t)\bar{z} + 2\theta\bar{z}^T (-UF(u, \hat{x}) + UK(t)U)\bar{x} \\ &\quad - 2\theta^{q-1}\bar{z}^T U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - 2\theta^q \delta \bar{z}^T C_{q_{n_1}}^T C w(t) \\ &\leq -\theta\delta\kappa\|\bar{z}\|^2 + 2\theta\|U\|\|K_1(t)\|\|\bar{z}\|^2 + 2\theta\|U\|(\|F_1(u, \hat{x})\| + \|K_1(t)\|)\|\bar{x}\|\|\bar{z}\| \\ &\quad - 2\theta^{q-1}\bar{z}^T U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - 2\theta^q \delta \bar{z}^T C_{q_{n_1}}^T C w(t), \end{aligned}$$

where  $\kappa$  is the smallest eigenvalue of the SPD matrix  $2I_{q_{n_1}} - (A + A^T)$ .

According to Assumption **A2**, one can show that

$$\begin{aligned} 2\theta^{q-1}\bar{z}^T U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) &\leq 2\|\bar{z}\|\|U\|\sqrt{n}(L_F x_M + L_\varphi)\|\bar{x}\| \\ &\leq 2\|\bar{z}\|\|U\|\sqrt{n}(L_F x_M + L_\varphi)\frac{\sqrt{V_1(\bar{x}, t)}}{\sqrt{\bar{\lambda}_m(S(t))}} \\ &\leq 2n(L_F x_M + L_\varphi)\sqrt{\frac{\alpha(\theta)}{\rho^*}}\|\bar{z}\|\sqrt{V_1(\bar{x}, t)} \\ &= 2n(L_F x_M + L_\varphi)\sqrt{\frac{\alpha(\theta)}{\rho^*}}\sqrt{V_1(\bar{x}, t)}\sqrt{V_2(\bar{z})}. \end{aligned}$$

Similarly, one has

$$\begin{aligned} 2\theta\|U\|(\|F_1(u, \hat{x})\| + \|K_1(t)\|)\|\bar{x}\|\|\bar{z}\| &\leq 2\theta\sqrt{n}(F_M + K_M)\frac{\sqrt{V_1(\bar{x}, t)}}{\sqrt{\bar{\lambda}_m(S(t))}}\sqrt{V_2(\bar{z})} \\ &\leq 2\theta\sqrt{n}(F_M + K_M)\sqrt{\frac{\alpha(\theta)}{\rho^*}}\sqrt{V_1(\bar{x}, t)}\sqrt{V_2(\bar{z})}, \end{aligned}$$

where  $K_M = \sup_{t \geq 0} \|K_1(t)\|$ .

Hence,

$$\begin{aligned} \dot{V}_2 \leq & -\theta\delta\kappa \left(1 - 2\frac{K_M\sqrt{n}}{\kappa\delta}\right) V_2 + 2\theta^q\delta\sqrt{V_2}\|w(t)\| \\ & + 2\sqrt{n} \left\{ \sqrt{n}(L_F x_M + L_\varphi) + \theta(F_M + K_M) \right\} \sqrt{\frac{\alpha(\theta)}{\rho^*}} \sqrt{V_1(\bar{x}, t)} \sqrt{V_2(\bar{z})}. \end{aligned} \quad (2.59)$$

Now, choose  $\delta$  such that  $\left(1 - 2\frac{K_M\sqrt{n}}{\kappa\delta}\right) \geq \frac{1}{2}$ , i.e.  $\delta \geq 2\sqrt{n}\frac{K_M}{\kappa}$  and choose  $\theta$  such that  $\theta \geq 1$ . Then, inequality (2.59) leads to

$$\dot{V}_2 \leq -\frac{\theta\delta\kappa}{2}V_2 + c_2\theta\sqrt{\alpha(\theta)}\sqrt{V_1(\bar{x}, t)}\sqrt{V_2(\bar{z})} + 2\theta^q\delta\sqrt{V_2}\|w(t)\|,$$

where  $c_2 = 2\sqrt{\frac{n}{\rho^*}} \left\{ \sqrt{n}(L_F x_M + L_\varphi) + (F_M + K_M) \right\}$  is a constant independent of  $\theta$  and  $\delta$ . Using inequality (2.57), one gets

$$\dot{V}_1 \leq -\theta\mu(\theta)V_1 + c_1\theta\sqrt{V_1}\sqrt{V_2} + 2\sqrt{\bar{\lambda}_M(S)}\sqrt{V_1}\|\varepsilon(t)\|, \quad (2.60)$$

where  $c_1 = 2K_M\sqrt{\bar{\lambda}_M(S)}$  is a positive constant independent of  $\theta$  and  $\delta$ .

Set

$$V_1^* = \theta\mu(\theta)V_1, \quad V_2^* = \frac{\theta\delta\kappa}{2}V_2 \quad \text{and let } V^* = V_1^* + V_2^*. \quad (2.61)$$

Notice that one has

$$V_1^* \leq V^* \quad \text{and} \quad V_2^* \leq V^*. \quad (2.62)$$

Hence,

$$\begin{aligned} \dot{V}_1 \leq & -V_1^* + c_1\theta\sqrt{\frac{V_1^*}{\theta\mu(\theta)}}\sqrt{\frac{V_2^*}{\frac{\theta\delta\kappa}{2}}} + 2\sqrt{\bar{\lambda}_M(S)}\sqrt{V_1}\|\varepsilon(t)\| \\ = & -V_1^* + \frac{c_1}{\sqrt{\delta}}\sqrt{\frac{2}{\kappa\mu(\theta)}}\sqrt{V_1^*}\sqrt{V_2^*} + 2\sqrt{\bar{\lambda}_M(S)}\sqrt{V_1}\|\varepsilon(t)\|. \end{aligned}$$

Using (2.62), one gets

$$\dot{V}_1 \leq -V_1^* + \frac{c_1}{\sqrt{\delta}}\sqrt{\frac{2}{\kappa\mu(\theta)}}V^* + 2\sqrt{\bar{\lambda}_M(S)}\sqrt{V_1}\|\varepsilon(t)\|. \quad (2.63)$$

Similarly, one has

$$\begin{aligned}
\dot{V}_2 &\leq -V_2^* + c_2\theta\sqrt{\alpha(\theta)}\sqrt{\frac{V_1^*}{\theta\mu(\theta)}}\sqrt{\frac{V_2^*}{\frac{\theta\delta\kappa}{2}}} + 2\theta^q\delta\sqrt{V_2}\|w(t)\| \\
&= -V_2^* + c_2\sqrt{\frac{\alpha(\theta)}{\delta}}\sqrt{\frac{2}{\kappa\mu(\theta)}}\sqrt{V_1^*}\sqrt{V_2^*} + 2\theta^q\delta\sqrt{V_2}\|w(t)\| \\
&\leq -V_2^* + c_2\sqrt{\frac{\alpha(\theta)}{\delta}}\sqrt{\frac{2}{\kappa\mu(\theta)}}V^* + 2\theta^q\delta\sqrt{V_2}\|w(t)\|. \tag{2.64}
\end{aligned}$$

Now, choose  $\delta$  such that  $\delta \geq \theta^\chi$ , i.e.  $\frac{1}{\delta} \leq \frac{1}{\theta^\chi}$  where  $\chi$  is given by (3.66). Then, inequality (2.64) becomes

$$\dot{V}_2 \leq -V_2^* + c_2\sqrt{\frac{\alpha(\theta)}{\theta^{2\chi}}}\sqrt{\frac{2}{\kappa\mu(\theta)}}V^* + 2\theta^q\delta\sqrt{V_2}\|w(t)\|.$$

Now, let  $V(\bar{x}, \bar{z}, t) = V_1(\bar{x}, t) + V_2(\bar{z})$  be the candidate Lyapunov function. Using (2.63) and (2.65) and from the facts that  $\sqrt{V_1} \leq V$  and  $\sqrt{V_2} \leq V$ , one gets

$$\begin{aligned}
\dot{V} &\leq -V^* + \sqrt{\frac{2}{\kappa\mu(\theta)}}\left(\frac{c_1}{\delta} + c_2\sqrt{\frac{\alpha(\theta)}{\theta^{2\chi}}}\right)V^* \\
&\quad + 2\left(\sqrt{\bar{\lambda}_M(S)}\|\varepsilon(t)\| + \delta\theta^q\|w(t)\|\right)\sqrt{V}. \tag{2.65}
\end{aligned}$$

Now, for  $\theta$  and  $\delta$  sufficiently high and according to (3.66), one has

$$\lim_{\rho \rightarrow \infty} \frac{\alpha(\theta)}{\theta^{2\chi}} = \lim_{\delta \rightarrow \infty} \frac{1}{\delta} = 0, \tag{2.66}$$

and as a result, there exists  $0 < \nu < 1$  such that

$$1 - \sqrt{\frac{2}{\kappa\mu(\theta)}}\left(\frac{c_1}{\delta} + c_2\sqrt{\frac{\alpha(\theta)}{\theta^{2\chi}}}\right) > \nu. \tag{2.67}$$

Similarly, for such values of  $\theta$  and  $\delta$ , one has  $\theta\mu(\theta) \leq \frac{\theta\delta\kappa}{2}$  which implies that

$$\theta\mu(\theta)V \leq V^* \leq \frac{\theta\delta\kappa}{2}V. \tag{2.68}$$

Hence, combining (2.65), (2.67) and (2.68), one gets

$$\dot{V} \leq -\nu\theta\mu(\theta)V + 2 \left( \sqrt{\bar{\lambda}_M(S)} \|\varepsilon(t)\| + \theta^q \delta \|w(t)\| \right) \sqrt{V}. \quad (2.69)$$

Using the comparison lemma, one gets for  $t \geq t^* \triangleq \frac{T^*}{\theta}$ ,

$$\sqrt{V(\bar{x}(t), \bar{z}(t), t)} \leq e^{-\nu\frac{\theta}{2}\mu(\theta)(t-t^*)} \sqrt{V(\bar{x}(t^*), \bar{z}(t^*), t^*)} + \frac{2}{\nu\theta\mu(\theta)} \left( \sqrt{\bar{\lambda}_M(S)} \delta_\varepsilon + \theta^q \delta \delta_w \right).$$

Coming back to  $\bar{x}$  and from the fact that  $V_1 \leq V$ , one gets

$$\|\bar{x}(t)\| \leq \sqrt{\frac{\lambda_M}{\underline{\lambda}_m(S(t))}} e^{-\nu\frac{\theta}{2}\mu(\theta)(t-t^*)} \|\bar{\xi}(t^*)\| + \sqrt{\frac{\lambda_M}{\underline{\lambda}_m(S(t))}} \frac{2}{\nu\theta\mu(\theta)} (\delta_\varepsilon + \theta^q \delta \delta_w),$$

where  $\bar{\xi}(t) = \begin{pmatrix} \bar{x}(t) \\ \bar{z}(t) \end{pmatrix}$  and  $\lambda_M = \max(\bar{\lambda}_M(S), 1)$ .

Now, using (2.56), the last inequality becomes

$$\begin{aligned} \|\bar{x}(t)\| &\leq \sqrt{\frac{\lambda_M}{\rho^*}} \sqrt{\alpha(\theta)} e^{-\nu\frac{\theta}{2}\mu(\theta)(t-t^*)} \|\bar{\xi}(t^*)\| + \sqrt{\frac{\lambda_M \alpha(\theta)}{\rho^*}} \frac{2}{\nu\theta\mu(\theta)} (\delta_\varepsilon + \theta^q \delta \delta_w) \\ &\leq \beta_f \sqrt{\alpha(\theta)} e^{-\nu\frac{\theta}{2}\mu(\theta)(t-t^*)} \|\bar{\xi}(t^*)\| \\ &\quad + 2 \frac{\beta_f}{\nu\mu(\theta)} \left( \sqrt{\frac{\alpha(\theta)}{\theta^2}} \delta_\varepsilon + \sqrt{\alpha(\theta)} \theta^{q-1} \delta \delta_w \right), \end{aligned} \quad (2.70)$$

where

$$\beta_f = \sqrt{\frac{\lambda_M}{\rho^*}}. \quad (2.71)$$

Now, set  $\xi = \begin{pmatrix} \tilde{x}(t) \\ \tilde{z}(t) \end{pmatrix}$  where  $\tilde{z}$  is such that  $\bar{z} = \theta^{q-1} \Delta_\theta^{-1} \tilde{z}(t)$ . It is easy to check that

$$\|\tilde{\xi}(t)\| \leq \|\bar{\xi}(t)\| \leq \theta^{q-1} \|\tilde{\xi}(t)\|, \quad \forall t \geq t^* \text{ and } \forall \theta \geq 1. \quad (2.72)$$

Besides, one can show that  $\tilde{z} = \eta - U\tilde{x}$  and therefore, one has for all  $t \geq t^*$

$$\|\tilde{\xi}(t)\| \leq \|\tilde{x}(t)\| + (\|e(t)\| + \|U\| \|\tilde{x}(t)\|) \leq (1 + \sqrt{n}) \|\tilde{x}(t)\| + \|\eta(t)\|. \quad (2.73)$$

Using (2.72) and (2.73), inequality (2.70) leads to (2.44). The proof of Theorem 2.3.1 is ended.  $\square$ .

**Remark 2.3.1** Notice that in the uncertainties free case and in the absence of measurement noise, the observation error exponentially converges to zero. Moreover, in the presence of uncertainties and in the free noise case, the asymptotic estimation error can be made as small as desired by choosing  $\theta$  high enough.

## 2.4 The FHGO redesigned version with sampled outputs

The observer designed above assumes that the outputs are available in a continuous manner. This is rarely the case in practice since these outputs are generally available only at sampling time instants  $0 \leq t_0 < \dots < t_k < t_{k+1} < \dots$  with (time-varying) sampling intervals  $\tau_k = t_{k+1} - t_k$  and  $\lim_{k \rightarrow \infty} t_k = +\infty$ . In the sequel, one shall propose a redesigned version of the above observer that accounts for the output sampling process. For this sake, one naturally assumes that the time intervals  $\tau_k$ 's are bounded away from zero by  $\tau_m$  and are upperly bounded by the upper bound of the sampling partition diameter  $\tau_M$ , i.e.

$$0 < \tau_m \leq \tau_k = t_{k+1} - t_k \leq \tau_M, \quad \forall k \geq 0. \quad (2.74)$$

The approach which shall be adopted to redesign the continuous time output FHGO is similar to that proposed in [57]. One needs the following additional hypothesis on the boundedness of the noise samples  $w(t_k)$ .

**A5.** For all  $t_k$ , the samples  $w(t_k)$  are bounded by  $\delta_w$  where  $\delta_w$  is the essential bound given by Assumption A3.

The redesigned observer takes the following form

$$\begin{cases} \dot{\hat{x}}(t) &= F(u, \hat{x})\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta K(t)\eta(t), \\ \dot{\eta}(t) &= \delta\theta (-\eta(t) + \theta A^T \eta(t) + C_{q_{n_1}}^T (C\hat{x}(t_k) - y(t_k) + \alpha(t))), \\ \dot{\alpha}(t) &= -\theta CP(t)C^T C_{q_{n_1}} \eta(t) = -\delta\theta K^{(1)}(t)\eta^{(1)}(t), \quad t \in [t_k, t_{k+1}[), \\ \eta(0) &= 0 \text{ and } \alpha(t_k) = 0, k \geq 0. \end{cases} \quad (2.75)$$

Notice that, the above continuous-discrete time observer involves a new state  $\alpha$  governed by an ODE which is re-initialized at zero at each sampling instant  $t_k$ .

The main properties of the proposed observer (2.75) are summarized in the following theorem.

**Theorem 2.4.1** Consider system (2.1) subject to assumptions **A1**, **A2**, **A3** and **A5**. Then, for every bounded input satisfying **A4**, there exist  $\delta^*, \theta^*$ , such that for all  $\theta \geq \theta^*$  and for all  $\delta > \max(\delta^*, \theta^\chi)$ , there exist positive constants  $\varrho_\theta > 0$ ,  $\eta_\theta > 0$  and  $N_\theta$  such that if the upper bound of the sampling partition diameter  $\tau_M$  is such that  $\tau_M < \varrho_\theta$ , then for every  $u \in U$  and every  $\hat{x}(0) \in X$ , one has for all  $t \geq t^*$ ,

$$\begin{aligned} \|\tilde{x}(t)\| \leq & \beta_f \sqrt{\alpha(\theta)} \theta^{q-1} e^{-\zeta_\theta(t-t^*)} \left( (1 + \sqrt{n}) \|\tilde{x}(t^*)\| + \|\eta(t^*)\| \right) \\ & + \beta_f N_\theta(\tau_m, \tau_M) \left( \sqrt{\frac{\alpha(\theta)}{\theta^2}} \delta_\varepsilon + \sqrt{\alpha(\theta)} \delta \theta^{q-1} \delta_w \right), \end{aligned} \quad (2.76)$$

where  $\tilde{x} = \hat{x} - x$  with  $\hat{x}$  being any trajectory of system (2.75),  $\delta_w, \delta_\varepsilon, \beta_f, \nu, t^*, \rho^*, \chi, \alpha(\theta)$  and  $\mu(\theta) > 0$  are as given in Theorem 2.3.1; the expressions  $\varrho_\theta, \zeta_\theta$  and  $N_\theta$  are

$$\begin{aligned} \varrho_\theta &= \frac{\beta_f \sqrt{\alpha(\theta)} (\theta F_M + L_\varphi)}{2\nu\theta\mu(\theta)}, \quad \zeta_\theta = a_\theta \left(1 - \frac{\tau_M}{\varrho_\theta}\right) e^{-a_\theta \tau_m}, \\ a_\theta &= \frac{\nu\theta\mu(\theta)}{2}, \quad N_\theta(\tau_m, \tau_M) = \theta \tau_M \frac{2 - e^{-\zeta_\theta \tau_m}}{1 - e^{-\zeta_\theta \tau_m}}, \end{aligned} \quad (2.77)$$

where  $F_M = \sup_{x \in X, u \in U} F(u, x)$  and  $L_\varphi$  is the Lipschitz constant of  $\varphi$  given by Assumption **A2**.

**Remark 2.4.1** Notice that, for a fixed value of  $\tau_M$ , the tuning parameter  $\theta$  cannot be taken arbitrarily and has to satisfy

$$\tau_M < \frac{\beta_f \sqrt{\alpha(\theta)} (\theta F_M + L_\varphi)}{2\nu\theta\mu(\theta)}. \quad (2.78)$$

**Proof of Theorem 2.4.1** Set  $\tilde{x} = \hat{x} - x$ ,  $\bar{x} = \theta^{q-1} \Delta_\theta \tilde{x}$  and  $\bar{\eta} = \theta^{q-1} D_\theta \eta$ . Proceeding as in the continuous output case, one gets (compare the equations below with (2.52) and (2.49), respectively),

$$\begin{cases} \dot{\tilde{x}}(t) &= \theta (F(u, \hat{x}) - P(t)C^T C) \bar{x} - B\varepsilon(t) - \theta K(t)\bar{z} \\ &+ \theta^{q-1} \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) \\ \dot{\bar{\eta}}(t) &= -\theta\delta \left( (I_{q_{n_1}} - A^T) \bar{\eta}(t) + C_{q_{n_1}}^T C \bar{x}(t) + \theta^{q-1} C_{q_{n_1}}^T C w(t_k) \right) \\ &- \theta\delta C_{q_{n_1}}^T \bar{\gamma}(t), \end{cases}$$

where  $\bar{z}(t)$  is defined as in (2.53) and

$$\bar{\gamma} = C(\bar{x}(t) - \bar{x}(t_k)) - \theta^{q-1} \alpha(t) \quad (2.79)$$

The dynamics of  $\bar{z}$  is given by (compare with (2.54))

$$\begin{aligned}\dot{\bar{z}} &= \theta\delta \left( -(I_{q_{n_1}} - A^T) + \frac{1}{\delta}UK(t) \right) \bar{z} \\ &\quad + \theta(-UF(u, \hat{x}) + UK(t)U) \bar{x} \\ &\quad - \theta^{q-1}U\Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q\delta C_{q_{n_1}}^T Cw(t) - \theta\delta C_{q_{n_1}}^T \bar{\gamma}(t). \quad (2.80)\end{aligned}$$

Notice that the first equation in (2.79) is similar to (2.52) while (2.80) differs from (2.54) by the presence of the term involving  $\bar{\gamma}$ . Hence, the forthcoming development will be very similar to those carried in the continuous out case. The sole difference consist in the accounting for the additional term  $\bar{\gamma}$ . Hence, let us derive a bound of  $\bar{\gamma}$ . One has

$$\begin{aligned}\dot{\bar{\gamma}} &= \theta F_1(u, \hat{x}^{(1)})\bar{x}^{(2)} + \theta^{q-1}\varphi^{(1)}(u, x^{(1)}) - \theta K^{(1)}(t)\bar{\eta}^{(1)}(t) - \theta^{q-1}\dot{\alpha}(t) \\ &= \theta F_1(u, \hat{x}^{(1)})\bar{x}^{(2)} + \theta^{q-1}\varphi^{(1)}(u, x^{(1)}) - \theta^{(q-1)}(\dot{\alpha}(t) + \theta K^{(1)}(t)\eta^{(1)}(t)) \\ &= \theta F_1(u, \hat{x}^{(1)})\bar{x}^{(2)} + \theta^{q-1}\varphi^{(1)}(u, x^{(1)}). \quad (2.81)\end{aligned}$$

Notice that the last inequality comes from the fact that  $\alpha(t)$  is governed by the ODE given by (2.79). Now, since  $\alpha(t_k) = 0$ , one can easily check that  $\bar{\gamma}(t_k) = 0$  and integrating the last equation from  $t^* \leq t$  to  $t$  gives

$$\bar{\gamma}(t) = \int_{t^*}^t (\theta F_1(u, \hat{x}^{(1)})\bar{x}^{(2)} + \theta^{q-1}\varphi^{(1)}(u, x^{(1)})) ds,$$

and hence

$$\|\bar{\gamma}(t)\| \leq (\theta F_M + L_\varphi) \int_{t^*}^t \|\bar{x}(s)\| ds,$$

where  $F_M = \sup_{x \in X, u \in U} F(u, x)$  and  $L_\varphi$  is the Lipschitz constant of  $\varphi$  given by Assumption **A2**.

Now, as in the continuous output case, let  $V_1(\bar{x}, t) = \bar{x}^T S(t)\bar{x}$ ,  $V_2(\bar{z}) = \|\bar{z}\|^2$  and  $V = V_1 + V_2$ . Again, proceeding as in the continuous output case, one can show that (compare with (2.69)),

$$\begin{aligned}\dot{V} &\leq -\nu\theta\mu(\theta)V + 2\sqrt{\bar{\lambda}_M(S)}\sqrt{V_2}(\theta F_M + L_\varphi) \int_{t^*}^t \|\bar{x}(s)\| ds \\ &\quad + 2 \left( \sqrt{\bar{\lambda}_M(S)}\|\varepsilon(t)\| + \theta^q\delta\|w(t)\| \right) \sqrt{V}.\end{aligned}$$

Using (2.56), one gets

$$\begin{aligned}\dot{V} \leq & -\nu\theta\mu(\theta)V + 2\beta_f\sqrt{\alpha(\theta)}(\theta F_M + L_\varphi)\sqrt{V} \int_{t^*}^t \sqrt{V(s)}ds \\ & + 2\lambda_M(\|\varepsilon(t)\| + \theta^q\delta\|w(t)\|)\sqrt{V},\end{aligned}$$

where  $\lambda_M = \max(\bar{\lambda}_M(S), 1)$  and  $\beta_f$  is as in (2.71).

The last inequality leads to

$$\begin{aligned}\frac{d}{dt}\sqrt{V} \leq & -\frac{1}{2}\nu\theta\mu(\theta)\sqrt{V} + \beta_f\sqrt{\alpha(\theta)}(\theta F_M + L_\varphi) \int_{t^*}^t \sqrt{V(s)}ds \\ & + \lambda_M\left(\sqrt{\bar{\lambda}_M(S)}\|\varepsilon(t)\| + \theta^q\delta\|w(t)\|\right).\end{aligned}$$

Now, let  $a_\theta = \frac{1}{2}\nu\theta\mu(\theta)$ ,  $b_\theta = \beta_f\sqrt{\alpha(\theta)}(\theta F_M + L_\varphi)$  and  $p_\theta(t) = \sqrt{\bar{\lambda}_M}(\|\varepsilon(t)\| + \theta^q\delta\|w(t)\|)$  and assume that the upper diameter of the sampling partition  $\tau_M$  satisfies the following condition

$$\tau_M < \varrho_\theta \triangleq \frac{a_\theta}{b_\theta} = \frac{\beta_f\sqrt{\alpha(\theta)}(\theta F_M + L_\varphi)}{2\nu\theta\mu(\theta)}, \quad (2.82)$$

then, according to lemma 2.1 in [57], one has

$$\sqrt{V(\bar{x}(t), \bar{z}(t), t)} \leq e^{-\zeta_\theta(t-t^*)}\sqrt{V(\bar{x}(t^*), \bar{z}(t^*), t^*)} + c_\theta\tau_M \frac{2 - e^{-\zeta_\theta\tau_M}}{1 - e^{-\zeta_\theta\tau_M}},$$

where

$$\text{ess sup}_{t \geq 0} p_\theta(t) \leq c_\theta = \sqrt{\bar{\lambda}_M}(\delta_\varepsilon + \delta\theta^q\delta_w), \quad \zeta_\theta(\tau_M) = a_\theta\left(1 - \frac{\tau_M}{\varrho_\theta}\right)e^{-a_\theta\tau_M}. \quad (2.83)$$

Coming back to  $\bar{x}$  and proceeding as in the continuous output case leads to (2.76). The proof of Theorem 2.4.1 is ended.  $\square$

The following remark provides insights about the observer dynamics and the sampling process specification according to the above fundamental result.

**Remark 2.4.2** *The bound of the estimation error related to the continuous-discrete time observer, i.e. the right side of inequation (2.76) is very similar to that derived in the continuous output case, i.e. the right side of inequality (2.44), with the following two differences:*

(i) The rate of the exponential decay is equal to  $a_\theta = \nu\theta\mu(\theta)/2$  in the continuous output case while it is equal to  $\zeta_\theta$  when the output is sampled.

(ii) The term  $2/(\nu\mu(\theta))$  involved in the expression of the ultimate bound in the continuous output case is replaced by  $N_\theta(\tau_m, \tau_M)$  when the output is sampled.

Proceeding as in [57], one can show that in the case of a constant sampling period, i.e.  $\tau_M = \tau_m = T_s$ ,  $\zeta_\theta(T_s)$  is a decreasing function of  $T_s$  and  $\lim_{T_s \rightarrow 0} \zeta_\theta(T_s) = a_\theta$ . One can also show that  $N_\theta(T_s)$  is a non decreasing function of  $T_s$  and that  $\lim_{T_s \rightarrow 0} N_\theta(T_s) = 2/(\nu\mu(\theta))$ . Hence, the results obtained in the sampled output case are in accordance with those derived in the continuous output one: the decreasing to zero of the observation error is inversely proportional to the magnitude of the sampling period while the value of the ultimate bound is proportional to this magnitude. More specifically, when the sampling period tends to zero, the expressions for the decay rate and ultimate bound become identical to those derived in the continuous output case.

## 2.5 Academic example

One shall highlight the performance and main properties of the FHGO (3.103) through the following academic example

$$\begin{cases} \dot{x}_1(t) &= f_1(t)x_2(t) - 0.1x_1(t), \\ \dot{x}_2(t) &= f_2(t) - 0.1x_2(t) + \tanh(x_1(t)x_2(t)), \\ \dot{x}_3(t) &= -0.1x_3(t) + \tanh(x_1(t)x_2(t)) + 5 \sin(0.5\pi t) + \varepsilon(t) \\ y(t) &= x_1(t) + v(t) \end{cases} \quad (2.84)$$

where  $f_1, f_2$  are expressed as  $f_1(t) = \sin(0.5t)$ ,  $f_2 = \sin(2.7t)$ . The expression of the unknown disturbance used in simulation is  $\varepsilon(t) = 0.1\sin(0.1t)$  and the noise signal affecting the output is chosen as the sum of two high frequency signals, namely

$$v(t) = 0.25 \sin(387t) + 0.5 \sin(987t). \quad (2.85)$$

Two scenarios depending on the availability of the output measurements shall be considered. In the first one, the output measurements are assumed to be available continuously. In the second scenario, these measurements are assumed to be available at equally spaced sampling instants. In each case, the obtained estimates are compared with those obtained from a SHGO. These results are detailed hereafter.

### 2.5.1 Simulation results with continuous output measurements

An observer of the form (2.36) has been designed in order to estimate the state of the system. The output measurements issued from simulation are corrupted by the noise

signal given by (2.85). The corresponding time evolution is given in Figure(2.1). The

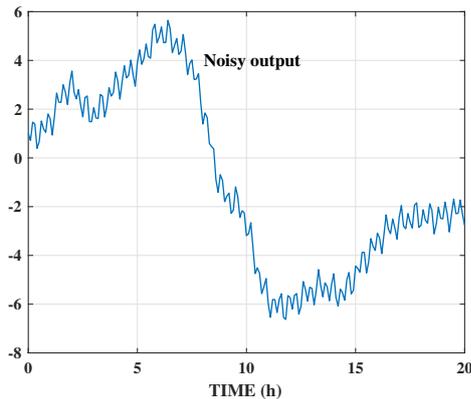


Figure 2.1: Noisy time-continuous output measurements

estimates of the states  $x_i, i = 1, \dots, 3$  are given in figure 2.2 where each of these estimates is compared to its true time evolution issued from the simulation of system (3.135). Recall that the expression of the disturbance  $\varepsilon(t)$  is unknown by the observer. The values of  $\theta$  and  $\delta$  have been set to 5 and 3, respectively. The initial state values of the system are set  $x_i = 1, \dots, 3$  and those of the observer to  $\eta = \hat{x} = 0$ .

The obtained results are quite smooth and accurate enough. In order to highlight the improvement in term of sensitivity to noise of the observer, a SHGO of the form (3.20) has been designed to derive the state estimate. The obtained results are given in fig 2.3 where as for the FHGO, each estimate is compared to its true value. It is clear that the obtained estimates are relatively noisy, in particular those related to  $x_2$  and  $x_3$  and this clearly confirms the significant improvement obtained with the FHGO.

## 2.5.2 Simulation results with sampled output measurements

Two sets of simulation results will be presented. In each case the output measurements are assumed to be available at sampling equally spaced time instants with a sampling period equal to  $T_e$ . In the first set of simulation results, the sampling period  $T_e$  has been set to a relatively small value, namely  $T_e = 0.02$ . The value of the tuning parameter  $\theta$  was set to 2 and that of  $\delta$  to 3. The estimates provided by the FHGO are compared to their true time evolutions in Figure 2.4. Those provided by the SHGO are reported in Figure 2.5. The obtained estimates, for the FHGO as well as for the SHGO, clearly shows that for small sampling periods and small amplitude for the output delay, the behaviours of the observers and in particular their sensitivity to the noise measurements are similar to the case where the output measurements are available continuously in time. In order to put forward the fact, that these properties still be inherited by the observers even for

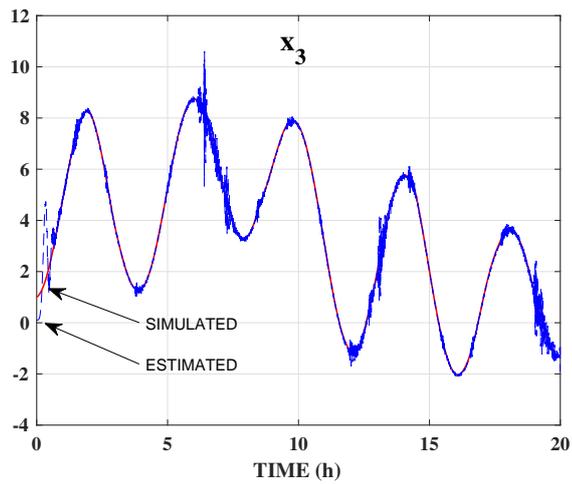
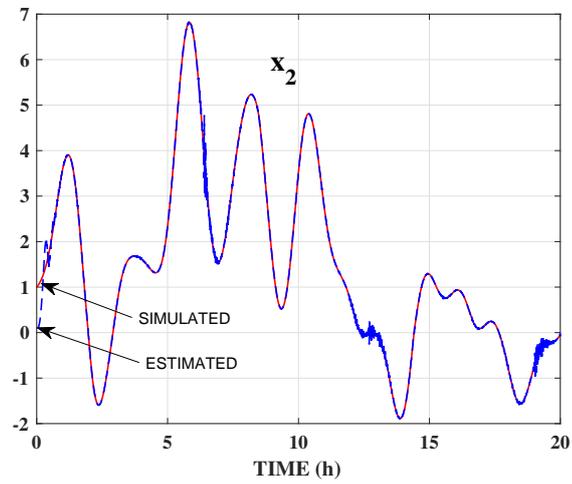
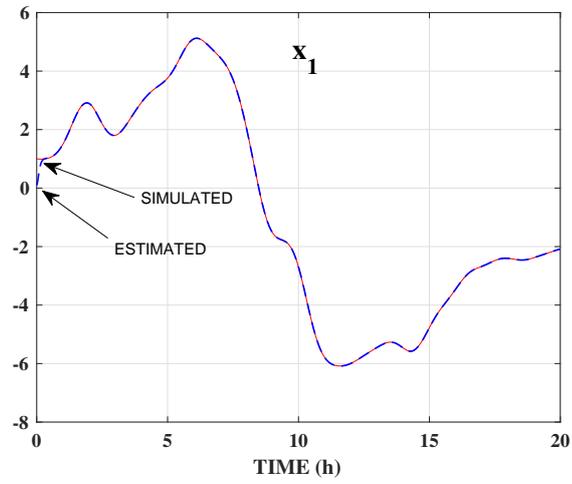


Figure 2.2: Estimates of  $x_i, i = 1, 2, 3$  provided by the FHGO

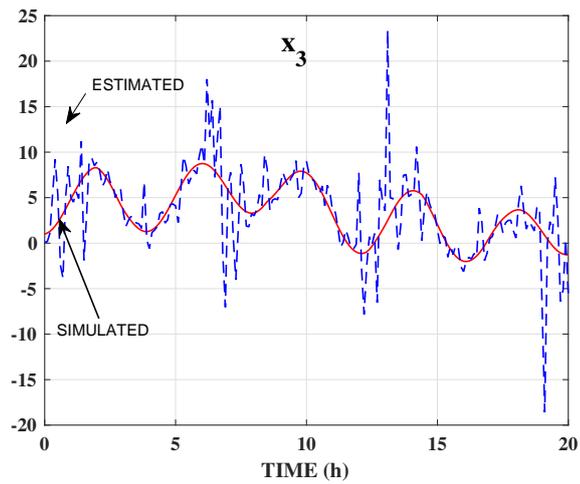
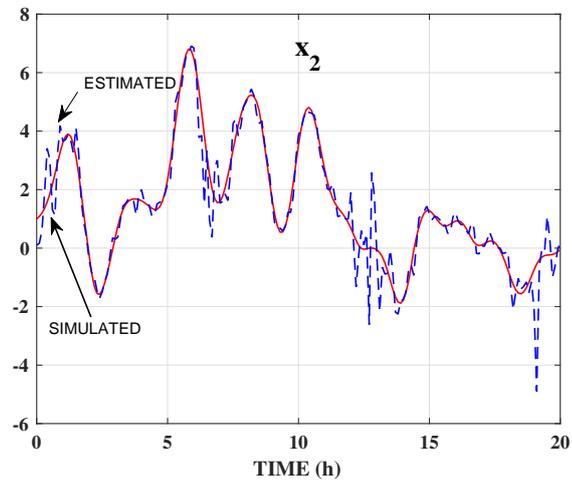
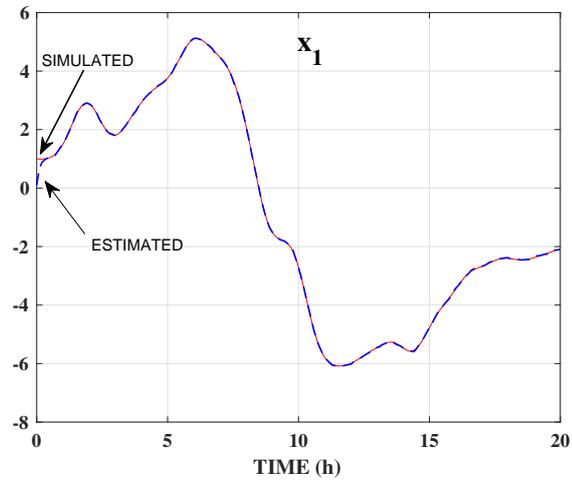


Figure 2.3: Estimates of  $x_i, i = 1, 2, 3$  provided by the SHGO

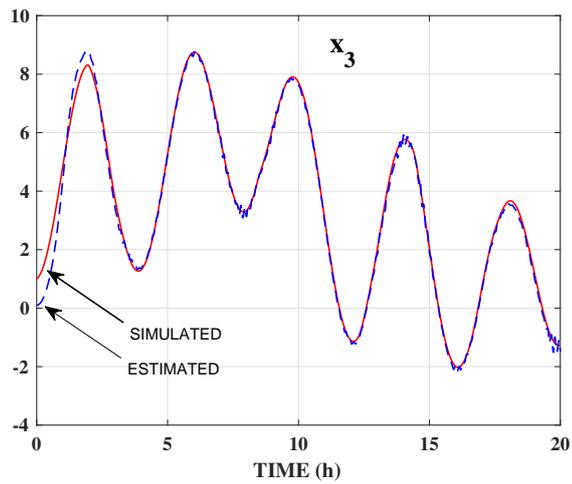
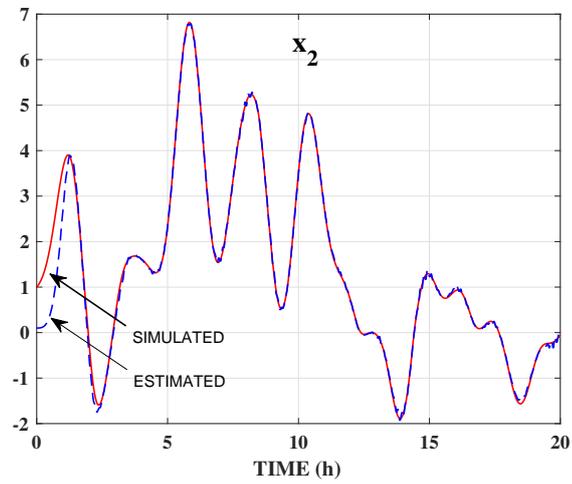
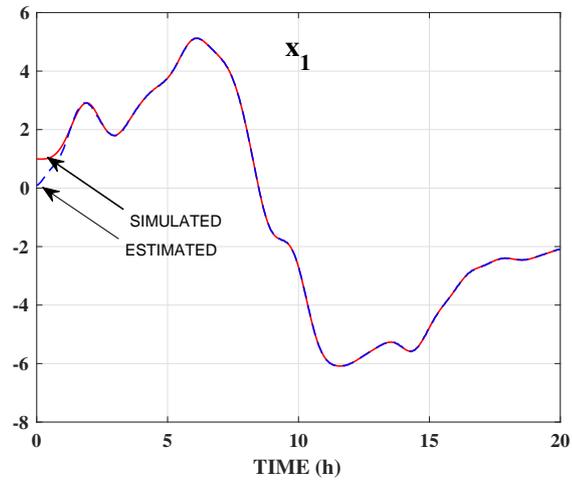


Figure 2.4: Estimates of  $x_i, i = 1, 2, 3$  provided by the FHGO with  $T_e = 0.02, \theta = 2, \delta = 3$

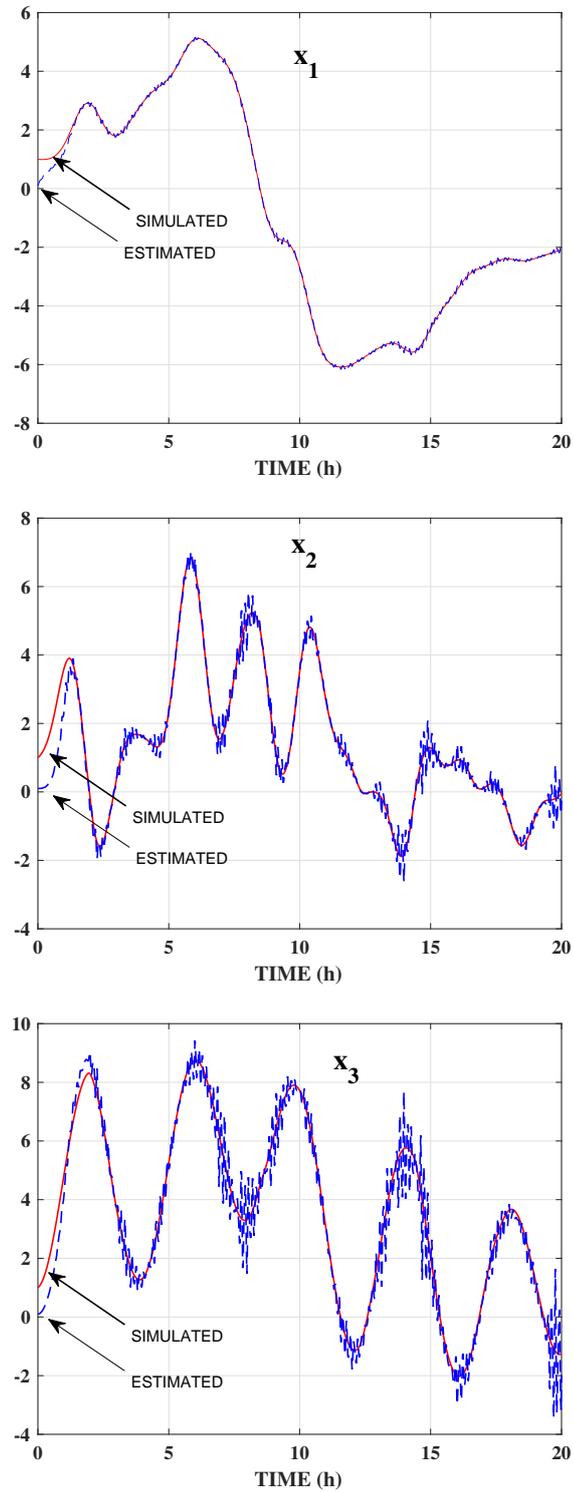


Figure 2.5: Estimates of  $x_i, i = 1, 2, 3$  provided by the SHGO with  $T_e = 0.02, \theta = 2$

more high values of  $T_e$ , another simulation experiment was carried out with  $T_e = 0.1$ . The value of  $\theta$  was set to 2. The obtained estimates are given in Figure 2.6 for the FHGO and in Figure 2.7 for the SHGO.

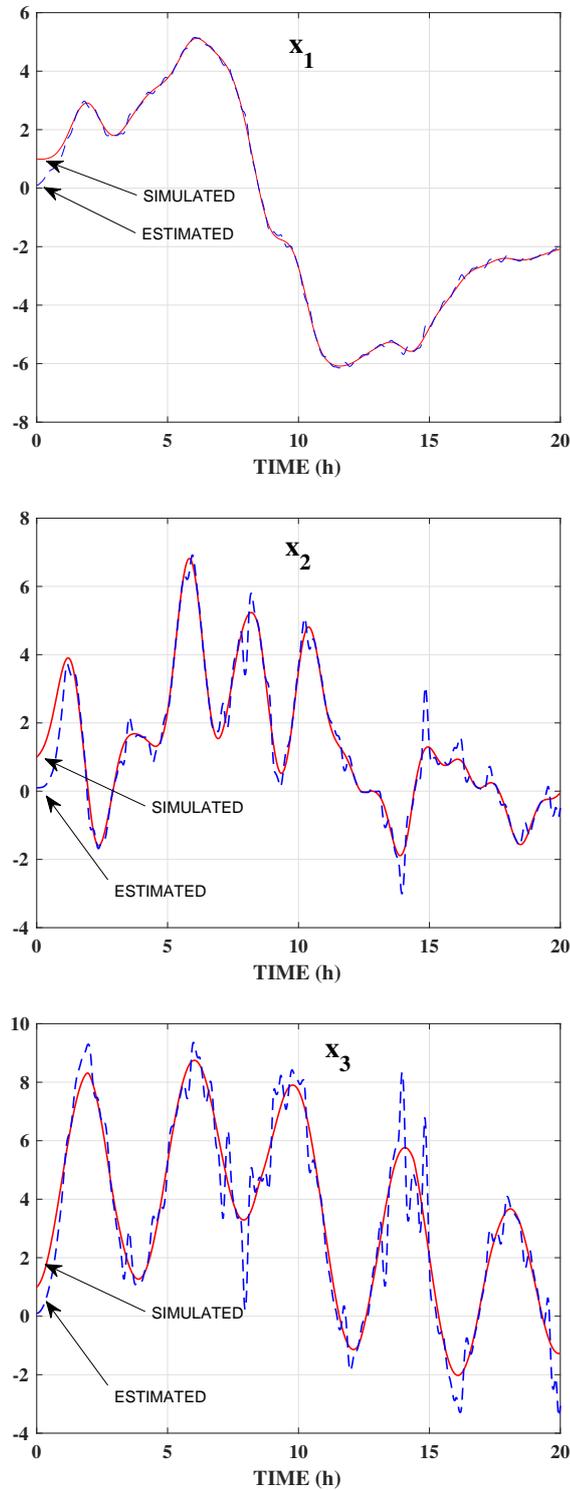


Figure 2.6: Estimates of  $x_i, i = 1, 2, 3$  provided by the FHGO with  $T_e = 0.2, \theta = 2, \delta = 3$

Again, the obtained results clearly confirm the the significance of the improvement ob-

tained with respect the sensitivity to noise through the redesigned version given by the FHGO.

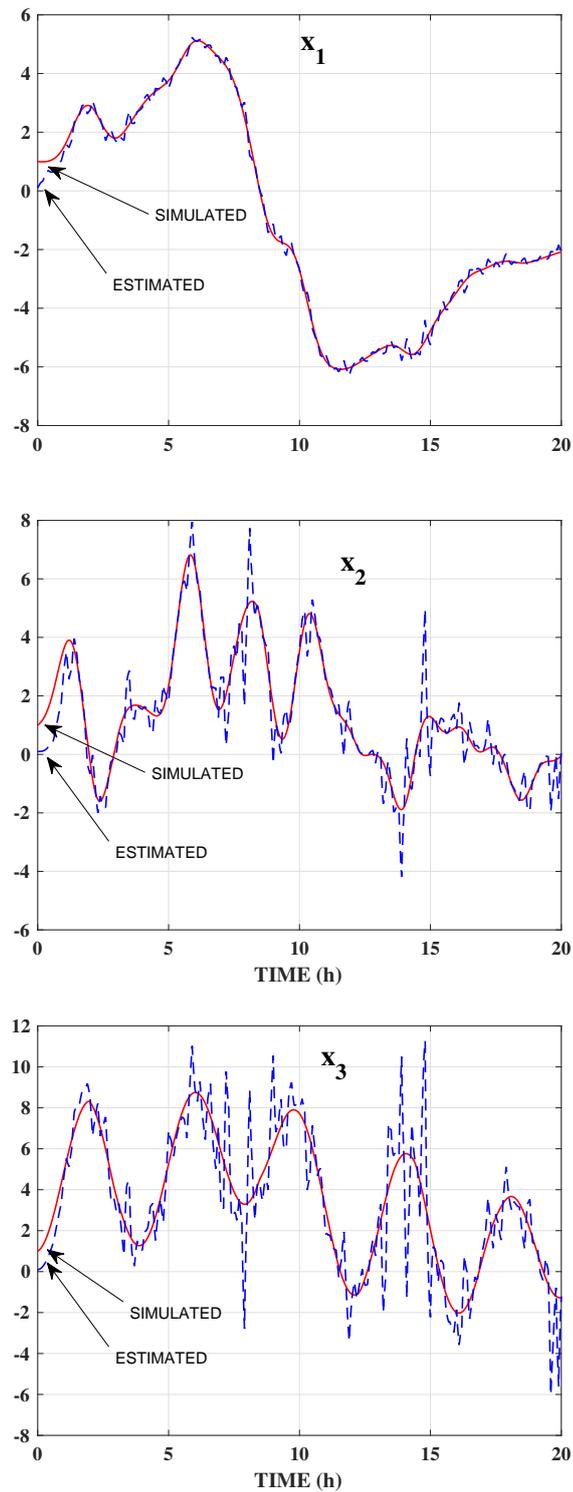


Figure 2.7: Estimates of  $x_i$ ,  $i = 1, 2, 3$  provided by the SHGO with  $T_e = 0.2, \theta = 2$

## 2.6 Application to a phytoplanktonic growth model

This section is devoted to the use of the proposed FHGO in a realistic simulation framework involving a phytoplanktonic growth model that has been validated in [63] through two high gain observers one of which is similar to (2.12) that uses continuous-time output measurements.

### 2.6.1 The phytoplanktonic growth model

One considers a bioreactor where a biomass (phytoplanktonic cells) with a concentration  $N$  is growing by consuming a substrate with a concentration  $S$ . The bioreactor is functioning under a continuous mode with a dilution rate  $D$  and an input substrate concentration  $S_{in}$ . A mathematical model which accounts for the biomass and substrate concentrations as well as the cell quota of assimilated nutrient,  $Q$ , has been considered in [63] and it can be written as follows

$$\begin{cases} \dot{N}(t) &= \mu_m \left(1 - \frac{K_Q}{Q(t)}\right) N(t) - D(t)N(t) \\ \dot{Q}(t) &= \rho_m \frac{S(t)}{K_\rho + S(t)} - \mu_m(Q(t) - K_Q) \\ \dot{S}(t) &= -\rho_m \frac{S(t)N(t)}{K_\rho + S(t)} + D(t)(S_{in} - S(t)) \end{cases} \quad (2.86)$$

where  $\mu_m, K_Q, \rho_m, K_\rho$  are constant kinetic parameters.

The biomass concentration is measured at equally spaced sampling instants and the observation objective is to provide continuous estimates of the biomass and substrate concentrations  $N$  and  $S$  and in particular the cell quota  $Q$  which can only be estimated with manual sampling and high cost measurements.

As in [63] and for writing convenience, one first adimensionalizes the state variables by considering the following changes

$$x_1 = \frac{\rho_m N}{S_{in}}, \quad x_2 = \frac{Q}{K_Q}, \quad x_3 = \frac{S}{S_{in}}, \quad a_1 = \frac{K_\rho}{S_{in}}, \quad a_2 = \mu_m, \quad a_3 = \frac{\rho_m}{K_Q}. \quad (2.87)$$

The resulting model which shall be used for the observer design specifies as follows

$$\begin{cases} \dot{x}_1 &= -a_2 x_1 \frac{1}{x_2} + (a_2 - u)x_1 \\ \dot{x}_2 &= -a_1 a_3 \frac{1}{a_1 + x_3} + a_3 - a_2(x_2 - 1) \\ \dot{x}_3 &= -\frac{x_1 x_3}{a_1 + x_3} + u(1 - x_3) \\ y(t_k) &= x_1(t_k) \end{cases} \quad (2.88)$$

where  $u \triangleq D$  is the input of the system.

Now, one shall consider the following change of coordinates which puts system (2.88)

under the normal form (2.1),  $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$  with

$$z_1 = x_1, \quad z_2 = \frac{1}{x_2}, \quad z_3 = \frac{1}{a_1 + x_3}. \quad (2.89)$$

Indeed, one can show that the above change of coordinates puts system (2.88) under form (2.1) with

$$x^{(k)} = z_k, \quad k = 1, \dots, 3, \text{ i.e. } n_k = 1, \dots, 3,$$

$$F_1(u, z) = -a_2 z_1, \quad F_2(u, z) = a_1 a_3 z_2^2, \quad \varphi(u, z) = \begin{pmatrix} (a_2 - u)z_1 \\ a_2 z_2 - (a_2 + a_3)z_2^2 \\ -a_1 z_1 z_3^3 + z_3^2(z_1 - u(1 + a_1)) + u z_3 \end{pmatrix}.$$

Note that, the original coordinates can be deduced from the new ones as follows

$$x_1 = z_1, \quad x_2 = \frac{1}{z_2}, \quad x_3 = \frac{1}{z_3} - a_1. \quad (2.90)$$

A discrete-continuous time FHGO observer of the form (2.75) has been designed for the resulting system in the new coordinates  $z$  to estimate the relevant state variables of the process. Notice that the observer equations could be derived in the original coordinates by considering the inverse of the transformation jacobian. Nevertheless, since the original dynamical system has already been expressed in the new coordinates  $z$  and the original variables can be easily deduced from the new ones through equations (2.90), one has simulated the underlying discrete-continuous time observer in the new coordinates  $z$ .

## 2.6.2 Estimation results

Two sets of simulation results are given. In each one, the measurements of the biomass are assumed to be available at equally spaced sampling instants with a sampling period equal to  $T_s$  which has been set to 0.02 *Day* in the first experiment and to 0.0625 *Day* (i.e. 1.5 *h* as in [63]) in the second one. Moreover and before being used by the observer, the output  $z_1(t_k)$  has been corrupted by a noise signal  $\mathcal{N}(t_k)$  to generate a noisy output  $y(t_k) = z_1(t_k)(1 + \mathcal{N}(t_k))$ . The noise signal  $\mathcal{N}(t_k)$  has been generated by considering the sum of 26 high frequency sinusoidal signals where the underlying frequencies are equally spaced between  $0.95/(2T_s)$  and  $1/(2T_s)$  (see Fig. 2.8). In each set of simulation results, the estimates provided by the continuous-discrete version of the FHGO are compared with their counterpart estimates provided by a continuous-discrete version of the SHGO

(see [58] for more details). The values of the model parameters used in simulation are [63]

$$a_1 = 0.02 \text{ g.g}^{-1}, \quad a_2 = 4.18 \text{ Day}^{-1}, \quad a_3 = 5.32 \text{ Day}^{-1}.$$

The dilution rate  $D = u$  was chosen as a sinusoidal signal, i.e.  $u = 1.71 \left( 1 + 0.6 \sin\left(\frac{2\pi}{T_0}t\right) \right)$  with  $T_0 = 1/3 \text{ Day}$ .

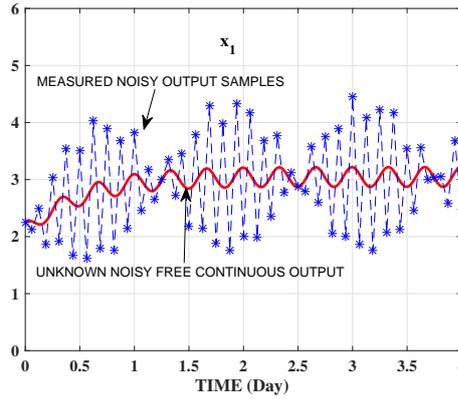


Figure 2.8: Noisy measurements of the output, with  $T_s = 0.0625 \text{ Day}$

Figure 2.9 shows estimates provided by the FHGO with  $T_s = 0.02 \text{ Day}$  together with their true values issued from the simulation model. The design parameters  $\theta$  and  $\delta$  have been set to 5 and 7, respectively. Figure 2.10 shows the estimates provided by the SHGO when the design parameter  $\theta$  has been set to 5 as for the SHGO. The fundamental results have been corroborated, namely the smooth behavior of the estimates provided by the FHGO is mainly due to its filtering capability. Figures 2.11 and 2.12 show the performance of the FHGO and SHGO, respectively during the second experiment with  $T_s = 0.0625 \text{ Day}$ . The underlying values of the design parameters are  $\theta = 3$  and  $\delta = 5$  for the FHGO and  $\theta = 3$  for the SHGO. Again, the FHGO outperforms the SHGO from the noise sensitivity point of view.

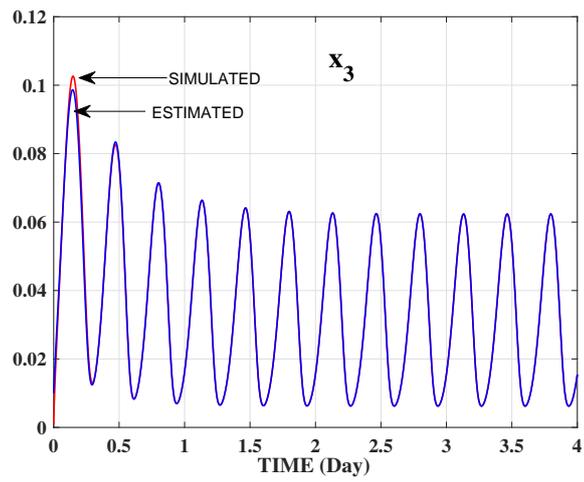
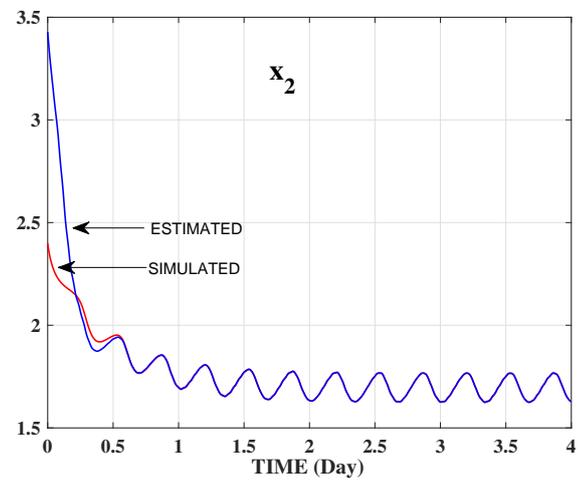
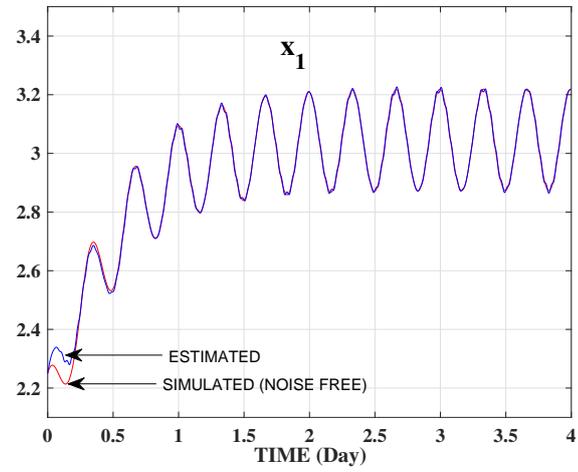


Figure 2.9: State estimation with FHGO, with  $T_s = 0.02 \text{ Day}$  ( $\theta = 5, \delta = 7$ )

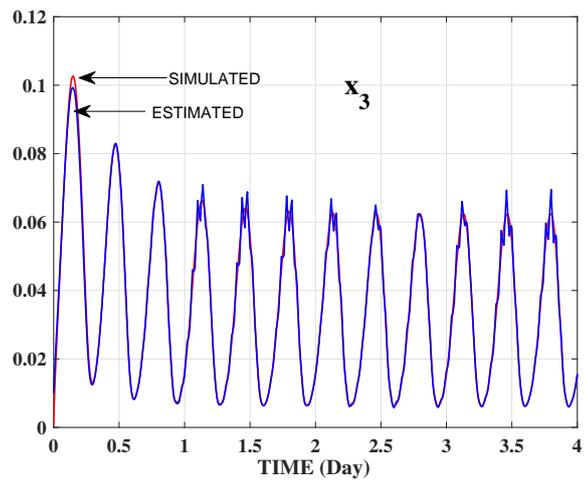
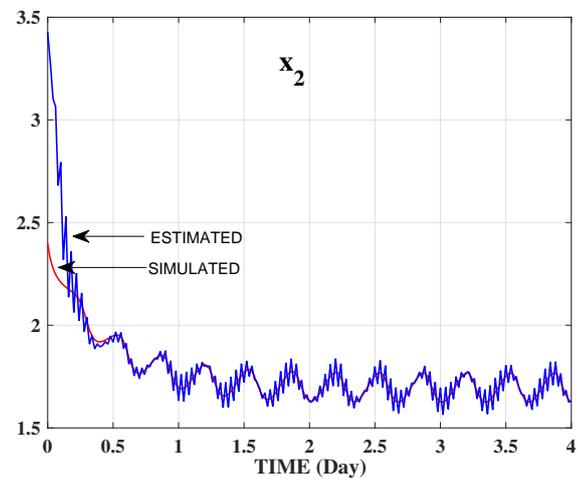
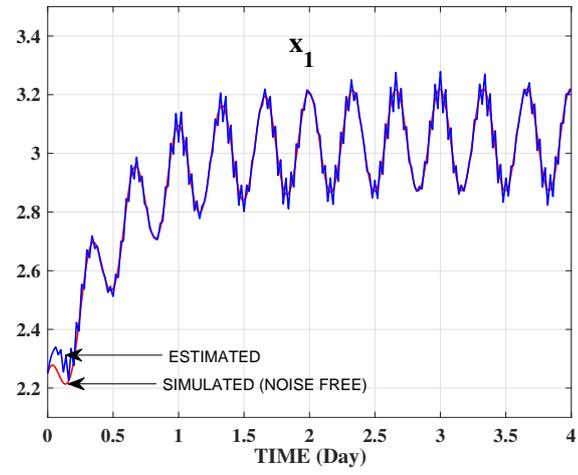


Figure 2.10: State estimation with SHGO, with  $T_s = 0.02 \text{ Day}$  ( $\theta = 5$ )

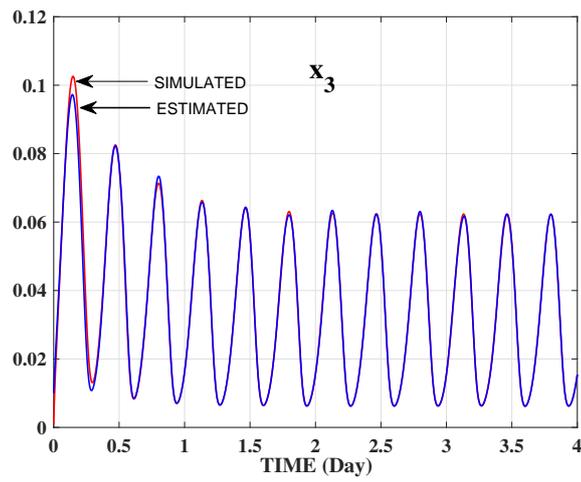
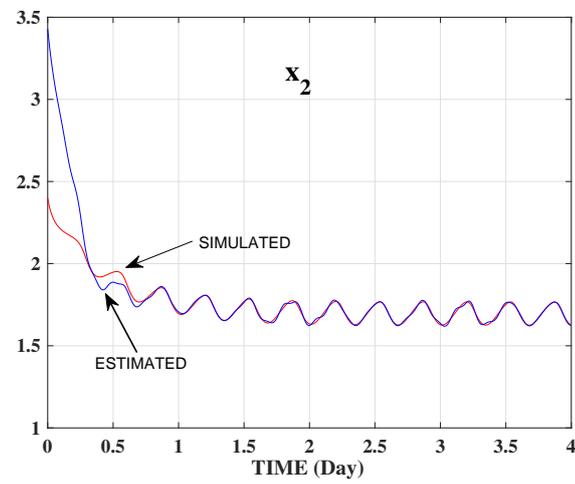
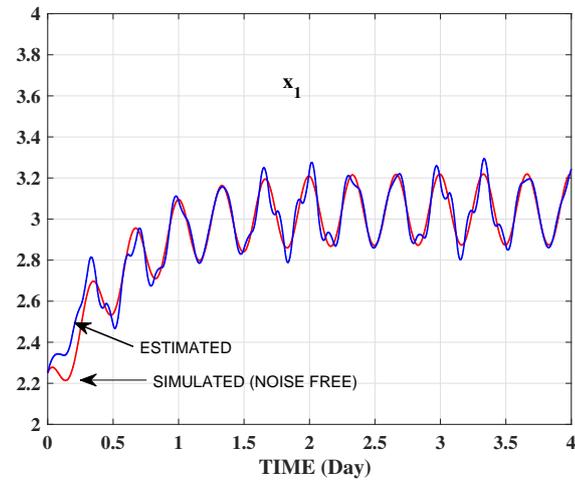


Figure 2.11: State estimation with FHGO, with  $T_s = 0.0625$  Day ( $\theta = 3, \delta = 5$ )

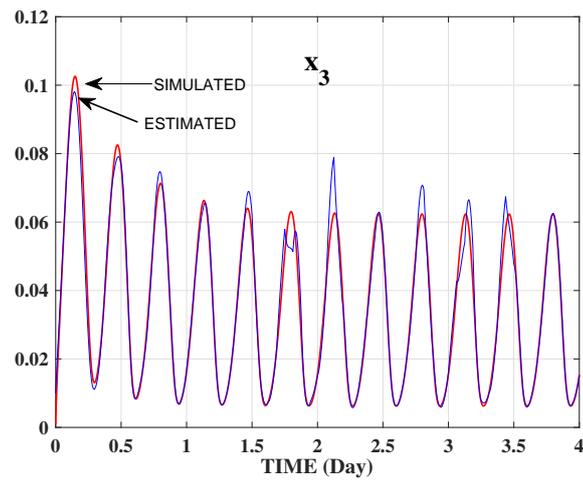
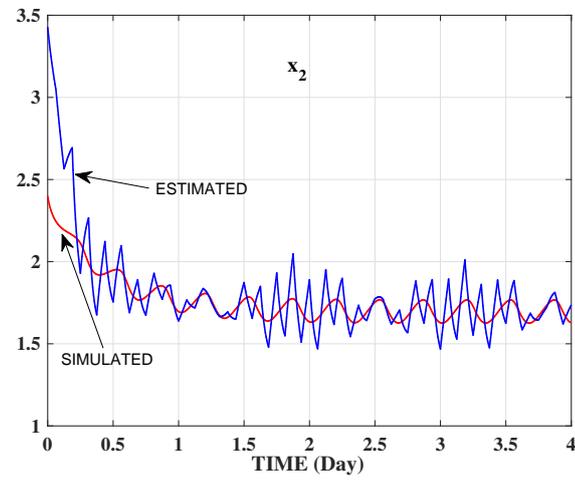
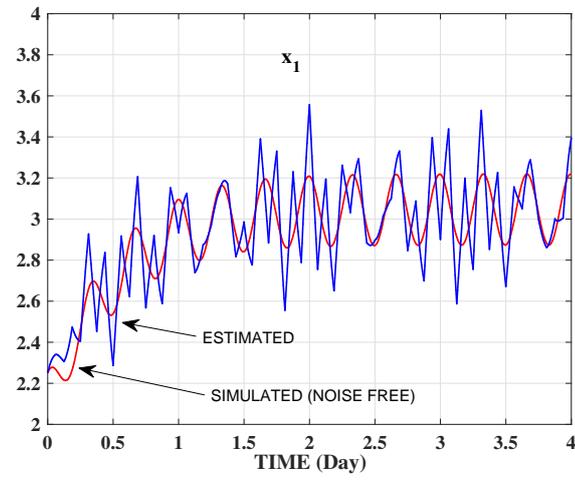


Figure 2.12: State estimation with SHGO, with  $T_s = 0.0625$  Day ( $\theta = 3$ )

## 2.7 Conclusion

A FHGO has been proposed for a class of non uniformly observable systems. This FHGO has been first designed assuming that the output measurements are continuously available and subsequently appropriately redesigned to account for the sampling process. A particular attention has been paid to the observer convergence analysis while emphasizing the admissible sampling process. The feasibility and performance of the proposed FHGO have been highlighted through an illustrative academic example. The estimates obtained by the filtered observer have been compared to their counterparts provided by a continuous-discrete time version of a SHGO which were rather noisy, in particular when the sampling periods are relatively high.



# Chapter 3

## Filtered High Gain Observer design with continuous, sampled and delayed output measurements

Though the redesigned observer, referred to FHGO, proposed in the previous chapter allowed a significant improvement of the sensitivity to noise of the standard high gain observer, its calibration requires the tuning of an additional parameter  $\delta$ . Moreover, the proposed design does not account for the case where the outputs are available with a delay.

In this chapter, one shall propose a general redesign of the SHGO in order to derive a filtered version that improves its sensitivity with respect to high frequency signal. The main novelty of the proposed design with respect to the one proposed in the previous chapter lies in the following facts

1. The proposed design does not need the introduction of any additional design parameter, i.e. the tuning of the resulting observer shall be achieved through the choice of the parameter  $\theta$  as for the SHGO.
2. The proposed design accounts for all possible scenarios related to the availability of the output measurements: continuously in time or under the form of samples. Moreover, in both cases, these measurements may be available with an output delay.

For clarity purposes, the proposed redesign shall be detailed in a first step for a class of UOS. Moreover, one shall first consider the case where the output measurements are available in a continuous manner. Then, one shall show how the design of the proposed filtered observer can be extended to the case where these output measurements are available under the form of samples and eventually with a delay. In a second step, a class

of NUOS shall be considered and one shall extend the proposed design to this class of systems.

One now defines some variables that will be used throughout this chapter. Indeed, one defines the following.

$$A_1 = 0, A_k = \begin{pmatrix} 0 & I_{k-1} \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{k \times k} \text{ for } k \geq 2, \quad (3.1)$$

$$B_k = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}^T, C_k = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^k \text{ for } k \geq 1, \quad (3.2)$$

$$G_k = \begin{pmatrix} \gamma_1 & \dots & \gamma_k \end{pmatrix}^T, \bar{A}_k \triangleq A_k - G_k C_k \text{ for } k \geq 1, \quad (3.3)$$

$$\Delta_k(\theta) = \text{diag}(1, 1/\theta, \dots, 1/\theta^{k-1}) \text{ for } k \geq 1. \quad (3.4)$$

The  $\gamma_i$ 's for  $k \geq 1$  and  $i = 1, \dots, k$  involved in the expression of the vectors  $G_k$  are positive reals and are chosen such that each matrix  $\bar{A}_k$  is Hurwitz; the parameter  $\theta$  involved in the expression of  $\Delta_k(\theta)$  is a real positive number.

On other aspects, let  $n \geq 1$  be a positive integer and let  $\xi^T = \begin{pmatrix} \xi_1 & \dots & \xi_n \end{pmatrix} \in \mathbb{R}^n$ ; then for any integer  $1 \leq k \leq n$ , one defines  $\underline{\xi}_k \in \mathbb{R}^k$  as follows

$$\underline{\xi}_k^T = \begin{pmatrix} \xi_1 & \dots & \xi_k \end{pmatrix} \in \mathbb{R}^k, k = 1, \dots, n. \quad (3.5)$$

Finally, one recalls an useful property: If  $\mathcal{A}_k$  is a  $n \times n$  Hurwitz matrix, then there exist a  $k \times k$  SPD matrix  $P_k$  and a positive real  $\mu_k$  such that

$$P_k \mathcal{A}_k + \mathcal{A}_k^T P_k \leq -2\mu_k I_k. \quad (3.6)$$

As stated above, the proposed redesign shall first focus of a class of UOS. Then, its extension to a class of NUOS shall be detailed.

### 3.1 The problem formulation

One aims at providing a FHGO for a class of disturbed nonlinear systems described by the following equations

$$\begin{cases} \dot{x}(t) = A_n x(t) + \varphi(u(t), x(t)) + B_n \varepsilon(t) \\ y(t) = C_n x(t) + v(t) \end{cases} \quad (3.7)$$

where  $x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^T \in \mathbb{R}^n$  denotes the state of the system with  $x_i \in \mathbb{R}$  for  $i = 1, \dots, n$ ;  $u(t) \in U \subset \mathbb{R}$  denotes the system input and  $y(t) \in \mathbb{R}$  denotes the actual output of the system, the matrices  $A_n$ ,  $B_n$  and  $C_n$  are defined as in (3.1) and (3.2) with

$k = n$ ,  $v(t)$  is the output noise, the function  $\varepsilon : [0, +\infty[ \mapsto \mathbb{R}$  denotes the system uncertainties and may depend on the input and uncertain parameters. It shall be treated as an unknown function which explicitly depends on time  $t$  for  $t \geq 0$ ; finally,  $\varphi(u, x) \in \mathbb{R}^n$  denotes a nonlinear vector field and each of its  $n$  components has a triangular structure with respect to  $x$ , i.e.  $\varphi_i(u, x) = \varphi_i(u, \underline{x}_i)$  for  $i = 1, \dots, n$  where  $\underline{x}_i$  is the sub-vector of  $x$  as defined by (3.5).

The observer design will be performed under the following usual high gain observer design assumptions.

**A1.** The state  $x(t)$  and the control  $u(t)$  are bounded, i.e.  $x(t) \in X$  and  $u(t) \in U$  where  $X \subset \mathbb{R}^n$  and  $U \subset \mathbb{R}$  are compact sets. More precisely, there exists a positive constant  $\rho_M$ , such that  $\forall t \geq 0$ ,  $|x_i(t)| \leq \rho_M, i = 1, \dots, n$ .

**A2.** The functions  $\varphi_k(u, \underline{x})$  for  $k = 1, \dots, n$  are Lipschitz on  $X$  with respect to  $x$  uniformly in  $u$ , i.e. for  $k = 1, \dots, n$ , one has

$$\exists L_k > 0; \forall u \in U; \forall (\hat{x}, x) \in X \times X, |\varphi_k(u, \hat{\underline{x}}_k) - \varphi_k(u, \underline{x}_k)| \leq L_k \sum_{i=1}^k |\tilde{x}_i|, \quad (3.8)$$

where  $\tilde{x}_i = \hat{x}_i - x_i$ .

**A3.** The unknown function  $\varepsilon$  is essentially bounded, i.e.

$$\exists \delta_\varepsilon > 0; |\varepsilon|_\infty \stackrel{\Delta}{=} \text{ess sup}_{t \geq 0} |\varepsilon(t)| \leq \delta_\varepsilon. \quad (3.9)$$

**A4.** The output noise  $w$  is essentially bounded, i.e.

$$\exists \delta_v > 0; |v|_\infty \stackrel{\Delta}{=} \text{ess sup}_{t \geq 0} |v(t)| \leq \delta_v. \quad (3.10)$$

Furthermore, let us consider the following saturation function

$$\forall z \in \mathbb{R}, \text{sat}_r(z) = r \text{sat}(z/r) = \begin{cases} z & \text{if } |z| \leq r \\ r \text{sign}(z) & \text{if } |z| > r \end{cases}, \quad (3.11)$$

where  $r > 0$  is a positive real and  $\text{sign}(\cdot)$  is the usual signum function. This allows to saturate the system nonlinearities  $\varphi_k$ 's for  $k = 1, \dots, n$  as follows

$$\varphi_i^s(u, x_1, \dots, x_i) = \varphi_i(u, \text{sat}_R(x_1), \dots, \text{sat}_R(x_i)). \quad (3.12)$$

with

$$R > \rho_M \quad (3.13)$$

where  $\rho_M$  is the positive real defined in Assumption A1. Two properties of the saturated nonlinearities can be easily checked ([30]). The first property consists in the following inequalities that hold for  $k = 1, \dots, n$ ,

$$\forall u \in U; \forall (\hat{x}, x) \in \mathbb{R}^n \times \mathbb{R}^n, |\varphi_k^s(u, \hat{x}_k) - \varphi_k^s(u, \underline{x}_k)| \leq L_k \sum_{i=1}^k |\tilde{x}_i|, \quad (3.14)$$

$$|\varphi_k^s(u, \hat{x}_k) - \varphi_k^s(u, \underline{x}_k)| \leq 2L_k \sum_{i=1}^k R \leq 2nL_\varphi R. \quad (3.15)$$

where

$$L_\varphi = \max_{1 \leq k \leq n} L_k. \quad (3.16)$$

The second property concerns the following bounds for the norm of the error vector of the saturated nonlinearities  $\tilde{\varphi}^s(u, \hat{x}, x) \triangleq \varphi^s(u, \hat{x}) - \varphi^s(u, x)$

$$\|\tilde{\varphi}^s(u, \hat{x}, x)\| \leq nL_\varphi \|\tilde{x}\|, \quad (3.17)$$

$$\|\tilde{\varphi}^s(u, \hat{x}, x)\| \leq \sqrt{n} \max_{1 \leq k \leq n} |\tilde{\varphi}_k^s(u, \hat{x}, x)| \leq 2n\sqrt{n}L_\varphi R. \quad (3.18)$$

These properties allow to postulate that the observer redesign can be handled for the following class of systems.

$$\begin{cases} \dot{x}(t) = A_n x(t) + \varphi^s(u(t), x(t)) + B_n \varepsilon(t) \\ y(t) = C_n x(t) = x_1(t) + v(t) \end{cases} \quad (3.19)$$

where  $\varphi^s(u, x) = \left( \varphi_1^s(u, x_1) \ \dots \ \varphi_n^s(u, x) \right)^T$ . Indeed, since system (3.7) coincides with system (3.19) on  $X$ , the observer that shall be designed for system (3.19) could be used in order to estimate the trajectories of system (3.7) which always lie in  $X$ . One now recalls some results related to the design of a SHGO for system (3.19) and to its main properties given in ([64]).

## 3.2 The underlying SHGO design

It has been shown in [64] that under Assumptions **A1** to **A4**, a SHGO can be designed for (3.19) and the underlying equations are

$$\dot{\hat{x}}(t) = A_n \hat{x}(t) + \varphi^s(u(t), \hat{x}(t)) - \theta \Delta_n^{-1}(\theta) G_n (C_n \hat{x}(t) - y(t)), \hat{x}(t_0) \in X, \quad (3.20)$$

where  $\hat{x} \in \mathbb{R}^n$  denotes the state estimate,  $G_n \in \mathbb{R}^n$  is defined as in (3.3) with  $k = n$  and the underlying coefficients  $\gamma_i$ 's,  $i = 1, \dots, n$  are chosen such that the matrix  $\bar{A}_n = A_n - \Gamma_n C_n$  is Hurwitz;  $\Delta_n(\theta)$  is a diagonal matrix defined as in (3.4) with  $k = n$  and  $\theta \geq 1$ .

Let  $\tilde{x}(t) = \hat{x}(t) - x(t)$  be the observation error associated to the SHGO (3.20). From (3.19) and (3.20), one has

$$\dot{\tilde{x}}(t) = A_n \tilde{x}(t) + \tilde{\varphi}^s(u(t), \hat{x}, x) - \theta \Delta_n^{-1}(\theta) G_n C_n \tilde{x} - B_n \varepsilon(t) + \theta \Delta_n^{-1}(\theta) \Gamma_n v(t), \quad (3.21)$$

where  $\tilde{\varphi}^s(u, \hat{x}, x) = \varphi^s(u(t), \hat{x}) - \varphi^s(u, x)$ . It has been shown in [30] that each component  $\tilde{x}_i(t), i = 1, \dots, n$ , of the vector of the observation error satisfies, for  $\theta$  sufficiently high, the following property

$$\begin{aligned} & \exists \theta_0; \forall \theta > \theta_0; \forall u \in U; \forall \hat{x}(t_0) \in \mathbb{R}^n; \\ & \forall t \geq t_0, |\tilde{x}_i(t)| \leq \chi_{P_n} \left( \theta^{i-1} e^{-\beta_n \theta(t-t_0)} \|\tilde{x}(t_0)\| + \frac{1}{\beta_n} (\theta^{i-1-n} \delta_\varepsilon + \theta^{i-1} \|G_n\| \delta_v) \right), \end{aligned} \quad (3.22)$$

with

$$\beta_n = \frac{\mu_n}{2\lambda_{P_n}^{(M)}}, \quad (3.23)$$

and where the SPD matrix  $P_n$  and the positive real  $\mu_n$  are given by equation (3.6) with  $k = n$  and  $\mathcal{A}_n = \bar{A}_n$  given by (3.3),  $\delta_\varepsilon$  is the essential upper bound of  $|\varepsilon(t)|$  given in Assumption **A3**,  $\delta_v$  is the essential upper bound of  $|w(t)|$  given in Assumption **A4** and  $\chi_{P_n}$  is the conditioning number of the matrix  $P_n$ .

According to inequality(3.22), in the noise-free case, i.e.  $\delta_v = 0$ , the asymptotic observation errors related to  $\tilde{x}_i, i = 1, \dots, n$ , can be made as small as desired by choosing  $\theta$  sufficiently high. Indeed, in this case, the underlying asymptotic ultimate bounds is equal to  $\theta^{i-1-n} \delta_\varepsilon / \beta_n$ . Since  $i \leq n$ , this bound is always lower or equal than  $\theta^{-1} \delta_\varepsilon / \beta_n$  which indeed tends to 0 when  $\theta$  goes to  $\infty$ . In the case where  $\delta_v \neq 0$ , the above property is no longer true unless an additional constraint is made on  $\delta_v$ . Indeed, if  $\delta_v \neq 0$ , then one can easily check that the resulting asymptotic ultimate bound can be made as small as desired if the following condition on  $\delta_v$  is satisfied

$$\exists \bar{\delta}_v; \exists \theta_0; \forall \theta \geq \theta_0, \delta_v \leq \frac{\bar{\delta}_v}{\theta^n}. \quad (3.24)$$

More precisely, under condition (3.24), inequality (3.22) specializes as follows

$$\begin{aligned} & \exists \theta_0; \forall \theta > \theta_0; \forall u \in U; \forall \hat{x}(t_0) \in \mathbb{R}^n; \forall t \geq t_0, \\ & |\tilde{x}_i(t)| \leq \chi_{P_n} \theta^{i-1} \left( e^{-\beta_n \theta(t-t_0)} \|\tilde{x}(t_0)\| + \frac{\delta_\varepsilon + \|G_n\| \bar{\delta}_v}{\beta_n \theta^n} \right). \end{aligned} \quad (3.25)$$

A low-pass filtering behaviour of the SHGO (3.20) has been pointed out in [65] by modelling the output measurement as

$$v(t) = \sum_{i=1}^N \nu_i \sin \left( \frac{\omega_i}{\epsilon} t + \phi_i \right), \quad (3.26)$$

where  $N > 0$  is a positive integer,  $\nu_i, \omega_i$  respectively characterize the amplitude and the phase of each component of  $v(t)$  and  $\epsilon \in ]0, 1[$  allows the parametrization of the frequencies of  $v(t)$ . The signal  $v(t)$  can be considered as the output of the following autonomous system

$$\epsilon \dot{w} = Sw, \quad v = Rw, \quad (3.27)$$

where

$$\begin{cases} w &= \left( w^{(1)T}, \dots, w^{(N)T} \right)^T \in \mathbb{R}^{2N}, \quad w^{(i)} \in \mathbb{R}^2, i = 1, \dots, N, \\ R &= \begin{pmatrix} (0 \ 1) & \dots & (0 \ 1) \end{pmatrix} \in \mathbb{R}^{2N}, \\ S &= \text{diag} \left( S^{(1)} \ \dots \ S^{(N)} \right) \text{ with } S^{(i)} = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}, \end{cases}$$

and  $w(0) \in W$  a compact subset of  $\mathbb{R}^{2N}$  which is invariant under system (3.27).

It has been shown in [65] that in the disturbance-free case, i.e.  $\delta_\epsilon = 0$ , the asymptotic behaviour of the SHGO is characterized as follows

$$\limsup_{t \rightarrow \infty} |\tilde{x}_i(t)| \leq (\epsilon \theta) \rho \theta^{i-1} \delta_v \quad (3.28)$$

where  $\rho$  is a positive constant independent of  $\theta$ .

**Remark 3.2.1** Notice that a redesigned version of the SHGO has been proposed in ([66]) in order to deal with sampled/delayed output measurements. The equations of the underlying observer can be written under the following form

$$\begin{cases} \dot{\hat{x}}(t) &= A_n \hat{x}(t) + \varphi^s(u(t), \hat{x}(t)) \\ &\quad - \theta \Delta_n^{-1}(\theta) G_n (C_n \hat{x}(t - d(t)) - y(t - d(t)) + \alpha(t) - \alpha(t - d(t))), \\ \dot{\alpha}(t) &= -\theta \gamma_1 (C_n \hat{x}(t - d(t)) - y(t - d(t)) + \alpha(t) - \alpha(t - d(t))). \end{cases} \quad (3.29)$$

where  $d(t)$  a piecewise continuous function and it corresponds to the output delay (including the sampled output case) and  $\gamma_1$  is the first component of the gain  $G_n$ .

### 3.3 Design of the FHGO

Many approaches dealing with the observer designs have been proposed to reduce the underlying observers sensitivity to noise measurements either by appropriately updating the observer design parameter  $\theta$  [67, 36, 35] or by substituting in the observer corrective term the output observation error by an appropriate filtered version provided by a linear filter cascaded with the original system [38, 33, 68, 69]. The solution that will be suggested

belongs to the second approach i.e. by filtering the noisy output observation error. In order to put forward the novelty of the proposed approach with respect to the existing ones, one shall briefly recall some of the latter. In [33], the proposed observer with filtering capabilities takes the following form

$$\begin{cases} \dot{\eta} &= (-\theta D_n + \theta^2 A_n^T)\eta + \theta C_n^T(C_n \hat{x} - y) \\ \dot{\hat{x}} &= A_n \hat{x} + \varphi^s(u, \hat{x}) - \theta K_n \eta \end{cases} \quad (3.30)$$

where  $\eta = \begin{pmatrix} \eta_1 & \dots & \eta_n \end{pmatrix} \in \mathbb{R}^n$  with  $\eta_i \in \mathbb{R}, i = 1, \dots, n$ ,  $D = \text{diag}(d_1, \dots, d_n)$  and  $K = \text{diag}(k_1, \dots, k_n)$  are  $n \times n$  diagonal matrices and their entries  $d_i, k_i$  are chosen such that the matrix  $\begin{pmatrix} -D_n + A_n & C_n^T C_n \\ -K_n & A_n \end{pmatrix}$  is Hurwitz. An appropriate algorithm is provided for the determination of such constants. Notice that the idea behind the design of observer (3.30) is to substitute in the equation of the SHGO (3.20) the corrective term  $G^{(SH)} \triangleq \theta \Delta_\theta(C_n \hat{x} - y(t))$  by the filtered version  $G^{(FH)} \triangleq \theta K_n \eta$ . More precisely, the  $i$ 'th component  $G_i^{(SH)} = \gamma_i \theta^i (C_n \hat{x} - y(t)), i = 1, \dots, n$  of  $G^{(SH)}$  is substituted by its filtered version  $G_i^{(FH)} = \theta k_i \eta$ . This can be justified by the fact that the static gain of the filter the input of which is  $(C_n \hat{x} - y(t))$  and the output of which is  $\eta_i$  is of the order of  $\theta^{i-1}$ . The main motivation of such substitution is to improve the sensitivity of the observer with respect to the noise measurement. Indeed, in the SHGO, the relative degree of  $\hat{x}_i$  with respect to  $y$  is equal to one for  $i = 1, \dots, n$ . However, in the redesigned observer (3.30) and since the relative degree between  $\eta_i$  and  $y$  is equal to  $i$ , the relative degree between  $\hat{x}_i$  and  $\eta_i$  is equal to the one between  $\eta_i$  and  $y$ , i.e.  $i$ . It has been noticed in [33] that the structure of the filter considered in the observer proposed in [34] is similar to the one given by (3.30).

The sensitivity of the observer can be improved further if the output observation error is substituted by the component of  $\eta$  that has the higher relative degree with respect to  $y$ , i.e;  $\eta_1$ . Indeed, the authors in [38] proposes the following filtered observer

$$\begin{cases} \dot{\eta} &= \frac{\theta}{\rho} ((-I_r + A_r^T)\eta + C_r^T(C_n \hat{x} - y)) \\ \dot{\hat{x}} &= A_n \hat{x} + \varphi^s(u, \hat{x}) - \theta K_n C_r \eta \end{cases} \quad (3.31)$$

where  $\eta = \begin{pmatrix} \eta_1 & \dots & \eta_r \end{pmatrix} \in \mathbb{R}^r$  with  $\eta_i \in \mathbb{R}, i = 1, \dots, r$ ,  $A_r$  and  $C_r$  are respectively defined as in (3.1) and (3.2) with  $k = n$  and  $\rho > 0$  is a positive design parameter of the observer. Notice that the state  $\eta$  of the filter is of of an arbitrarily dimension  $r > 0$  and not fixed to  $n$  as in (3.30). Moreover, only one additional scalar design parameter is introduced by the filter, i.e.  $\rho$  which has to be chosen such that the matrix  $F(\rho) \triangleq \begin{pmatrix} -\frac{1}{\rho}(I_r - A_r^T) & \frac{1}{\rho}C_r^T C_n \\ -K_n C_r & A_n \end{pmatrix}$  is Hurwitz. It has been shown in [38] that for each  $\theta > 0$ , there exists  $\rho^*$  such for all  $\rho \in [0, \rho^*$ , the matrix  $F(\rho)$  is Hurwitz. Another

observer where the component of the filter state having the highest relative degree has been used in the observer equation is proposed in [30]. The redesign approach is inspired by that given in [69, 68] and it consists in augmenting the original system (3.19) by an integrator of order  $r$  of the corrupted output. The so obtained augmented system keeps the triangular structure of the original one and hence by considering the integral of order  $r$  of the output as the output of the system, a SHGO can be designed for this system. The resulting observer takes the following form

$$\begin{cases} \dot{\eta} &= A_r \eta + B_r (C_n \hat{x} - y) - \theta \Delta_r^{-1}(\theta) K_r C_r \eta \\ \dot{\hat{x}} &= A_n \hat{x} + \varphi^s(u, \hat{x}) - \theta^{r+1} \Delta_n^{-1}(\theta) K_n C_r \eta \end{cases} \quad (3.32)$$

where  $K_r = \begin{pmatrix} k_1 & \dots & k_r \end{pmatrix}^T \in \mathbb{R}^r$  and  $K_n = \begin{pmatrix} k_{r+1} & \dots & k_{r+n} \end{pmatrix}^T$  are chosen such that the matrix  $\tilde{A}_{n+r} = A_{n+r} - K_{r+n} C_{r+n}$  is Hurwitz with  $K_{r+n} = \begin{pmatrix} K_r^T & K_n^T \end{pmatrix}^T$  and  $A_{r+n}$  and  $C_{r+n}$  are respectively defined as in (3.1) and (3.2) with  $k = r + n$ . Notice that the choice of the gain  $K_{n+r}$  is simple since it can be achieved through standard matlab functions (place or acker) by assigning the eigenvalue of the matrix  $A_{n+r}$ . The matrices  $\Delta_r(\theta)$  and  $\Delta_n(\theta)$  are defined as in (3.4) with  $k = r$  and  $k = n$ , respectively. On other aspects and on the contrary of observers (3.30) and (3.31) where the design parameter  $\theta$  intervenes with a power that does not exceed 2, the power of  $\theta$  is ranged from 1 to  $n + r$  in the observer (3.32). This might increase the level of the peaking of the observer state variables. However and as noticed in ([30]), the peaking phenomenon has been dealt with through a redesign procedure where nested saturations functions have been considered and hence the resulting proposed observer significantly reduces the peaking phenomenon.

The new filtered observer is presented hereafter.

### 3.3.1 Equations of the FHGO

The equations of the proposed observer take the following form

$$\begin{cases} \dot{\eta} &= \theta ((A_r - K_r C_r) \eta + B_r (C_n \hat{x} - y)) \\ \dot{\hat{x}} &= A_n \hat{x} + \varphi^s(u, \hat{x}) - \theta \Delta_n^{-1}(\theta) K_n C_r \eta, \quad \eta(t_0) = 0 \text{ and } \hat{x}(t_0) \in \mathbb{R}^n, \end{cases} \quad (3.33)$$

where  $K_r$  and  $K_n$  are defined as in observer (3.32) and they have to be chosen such that  $\tilde{A}_{n+r} = A_{n+r} - K_{r+n} C_{r+n}$  is Hurwitz with  $K_{r+n} = \begin{pmatrix} K_r^T & K_n^T \end{pmatrix}^T$  and  $A_{r+n}$  and  $C_{r+n}$  are respectively defined as in (3.1) and (3.2) with  $k = r + n$ . As for observer (3.32), such a choice can be fairly achieved through standard matlab functions (place or acker) by assigning the eigenvalue of the matrix  $A_{n+r}$ .

Before stating the main properties of the proposed FHGO, one shall motivate the structure and put forward the rationale behind the proposed design. Indeed, a main property related to the structure of the FHGO (3.33) lies in the fact that  $\hat{x}$  in this observer is governed by almost the same ODE as in the SHGO (3.20) and the only difference is the substitution in the equation related to the SHGO of the noisy observation error  $G_n(C_n\hat{x} - y)$  by a filtered version  $K_n C_r \eta$ . In order to motivate such a substitution, one shall give a natural interpretation of the state  $\eta$  governed by the ODE in (3.33). Indeed,  $\eta$  can be interpreted as the state of a filter of dimension  $r$ , which is arbitrary, the entry of which is the noisy output observation error  $(C_n\hat{x} - y)$ . Notice that the equations of the filter given by (3.33) are written in a canonical form and hence it is easy to check that the relative degree of the transfer function between the filter output, i.e.  $\eta_1 = C_r \eta$  and the filter entry, i.e.  $C_n\hat{x} - y$  is equal to  $r$  and the underlying static gain does not depend on  $\theta$ . Hence, the filter output  $C_r \eta$  is a filtered version (of order  $r$ ) of the noisy output observation and its use in the corrective term of the FHGO will improve the sensitivity of observer to the high frequency signals.

Notice that the equations of the filter in the FHGO (3.33) are given before the ODE governing  $\hat{x}$  while there were given after the underlying ODE in the FHGO (2.36) proposed in the preceding chapter. The inversion of the order of the two subsystems is simply motivated by clarity purposes since the above form (3.33) is more appropriate for the analysis of the stability of the underlying observation error equations.

The main property of observer (3.33) is stated in the following theorem.

**Theorem 3.3.1** *Consider system (3.19) subject to Assumption A1 to A4 together with observer (3.33) and assume that the essential bound  $\delta_v$  satisfies (3.24). Then, the components of the observation error vector satisfy the following property,*

$$\begin{aligned} & \exists \theta_0 > 0; \forall \theta \geq \theta_0; \forall u \in U; \forall \hat{x}(t_0) \in \mathbb{R}^n; \forall t \geq t_0, \\ & \left\{ \begin{array}{l} \left\| \begin{array}{l} \eta(t) \\ \tilde{x}_1(t) \end{array} \right\| \leq \chi_{P_{r+n}} \left( e^{-\beta_{r+n}\theta(t-t_0)} \|\tilde{x}(t_0)\| + \frac{\bar{\delta}_v + \delta_\varepsilon}{\beta_{r+n}\theta^n} \right), \\ |\tilde{x}_i(t)| \leq \chi_{P_{r+n}} \left( \theta^{i-1} e^{-\beta_{r+n}\theta(t-t_0)} \|\tilde{x}(t_0)\| + \frac{\bar{\delta}_v + \delta_\varepsilon}{\beta_{r+n}\theta^{n+1-i}} \right), \quad i = 2, \dots, n. \end{array} \right. \end{array} \quad (3.34)$$

with

$$\beta_{n+r} = \frac{\mu_{r+n}}{2\bar{\lambda}_{P_{r+n}}}, \quad (3.35)$$

where  $P_{r+n}$  and  $\mu_{r+n}$  are given by equation (3.6) with  $k = n + r$ ,  $\bar{\lambda}_{P_{r+n}}$  and  $\chi_{P_{r+n}}$  respectively denote the largest eigenvalue and the conditioning number of  $P_{r+n}$ ,  $\bar{\delta}_v$  is the bound of  $v$  as defined in (3.24) and  $\delta_\varepsilon$  is the essential bound of  $\varepsilon$  given by Assumption A4.

**Proof of Theorem 3.3.1.** Set  $\tilde{x} = \hat{x} - x$ ,  $\bar{x} = \theta^{n-1}\Delta_n\tilde{x}$  and  $\bar{\eta} = \theta^{n-1}\eta$ . Using (3.33), one gets

$$\begin{cases} \dot{\bar{\eta}} &= \theta \left( (A_r - K_r C_r)\bar{\eta} + B_r C_n \bar{x} \right) - \theta^n B_r v(t) \\ \dot{\tilde{x}} &= \theta A_n \bar{x} + \theta^{n-1} \Delta_n \tilde{\varphi}^s(u, \hat{x}, x) - \theta K_n C_r \bar{\eta} - B_n \varepsilon(t), \end{cases} \quad (3.36)$$

where  $\tilde{\varphi}^s(u, \hat{x}, x) = \varphi^s(u, \hat{x}) - \varphi^s(u, x)$ .

One can check that

$$\begin{pmatrix} A_r & B_r C_n \\ A_n & 0 \end{pmatrix} = A_{r+n}, \quad (3.37)$$

where  $A_{r+n}$  is the anti-shift matrix given by (3.1) with  $k = n$ . By setting  $z = \begin{pmatrix} \eta \\ \tilde{x} \end{pmatrix}$  and  $\bar{z} = \begin{pmatrix} \bar{\eta} \\ \bar{x} \end{pmatrix}$ , the error system (3.36) can be written under the following compact form

$$\dot{\bar{z}} = \theta \tilde{A}_{n+r} \bar{z} + \tilde{\Phi}(u, \hat{x}, x) - \theta^n v(t) \bar{B}_{r+n} - B_{r+n} \varepsilon(t), \quad (3.38)$$

where

$$\tilde{A}_{n+r} = A_{n+r} - K_{r+n} C_{r+n}, \quad (3.39)$$

with  $K_{r+n} = \begin{pmatrix} K_r^T & K_n^T \end{pmatrix}^T$ ,  $C_{r+n}$  and  $B_{r+n}$  are given by (3.2) with  $k = r+n$ ,  $\tilde{\Phi}(u, \hat{x}, x) = \begin{pmatrix} 0 \\ \theta^{n-1} \Delta_n \tilde{\varphi}^s(u, \hat{x}, x) \end{pmatrix}$  and  $\bar{B}_{r+n} \in \mathbb{R}^{r+n}$  is a column vector with all its entries are equal to zero excepting the one located at the  $r$ 'th row which is equal to 1. Recall that  $K_r$  and  $K_n$  are chosen such that  $\tilde{A}_{r+n}$  is Hurwitz. Hence, there exist a SPD matrix  $P_{r+n}$  and a positive real  $\mu_{r+n}$  such

$$\tilde{A}_{r+n}^T P_{r+n} + P_{r+n} \tilde{A}_{r+n} = -2\mu_{r+n} I_{r+n}. \quad (3.40)$$

Let  $V(\bar{z}) = \bar{z}^T P_{r+n} \bar{z}$  be the Lyapunov candidate function for the error system (3.38). Then, using (3.38) and (3.40), one gets

$$\begin{aligned} \dot{V}(\bar{z}) &= 2\theta \bar{z}^T P_{r+n} \tilde{A}_{r+n} \bar{z} + 2\bar{z}^T P_{r+n} \tilde{\Phi}(u, \hat{x}, x) - 2\theta^n v(t) \bar{z}^T P_{r+n} \bar{B}_{r+n} - 2\varepsilon(t) \bar{z}^T P_{r+n} B_{r+n} \\ &= -2\mu_{r+n} \theta \|\bar{z}\|^2 + 2\bar{z}^T P_{r+n} \tilde{\Phi}(u, \hat{x}, x) - 2\theta^n v(t) \bar{z}^T P_{r+n} \bar{B}_{r+n} \\ &\quad - 2\varepsilon(t) \bar{z}^T P_{r+n} B_{r+n}. \end{aligned} \quad (3.41)$$

Using the triangular structure and the Lipschitz assumption on  $\varphi$ , one gets for  $\theta \geq 1$  (see [64]),

$$\|\theta^{n-1} \Delta_n \tilde{\varphi}^s(u, \hat{x}, x)\| \leq c_1 \|\bar{x}\| \leq c_1 \|\bar{z}\|,$$

where  $c_1 > 0$  is a positive constant independent of  $\theta$  and hence, one gets

$$2z^T P_{r+n} \tilde{\Phi}(u, \hat{x}, x) \leq 2c_1 \bar{\lambda}_{P_{r+n}} \|\bar{z}\|^2. \quad (3.42)$$

where  $\bar{\lambda}_{P_{r+n}}$  is the largest eigenvalue of  $P_{r+n}$ . By choosing  $\theta$  high enough and in particular satisfying  $-2\mu_{r+n}\theta + 2c_1\bar{\lambda}_{P_{r+n}} \leq -\mu_{r+n}\theta$ , i.e.  $\theta \geq 2c_1 \frac{\bar{\lambda}_{P_{r+n}}}{\mu_{r+n}}$ , equality (3.41) leads to

$$\dot{V}(\bar{z}) \leq -\frac{\mu_{r+n}\theta}{\bar{\lambda}_{P_{r+n}}} V(\bar{z}) + 2\sqrt{\bar{\lambda}_{P_{r+n}}} \sqrt{V(\bar{z})} (\theta^n v(t) + \varepsilon(t)), \quad (3.43)$$

or equivalently

$$\frac{d}{dt} \sqrt{V(\bar{z})} \leq -\beta_{r+n}\theta \sqrt{V(\bar{z})} + \sqrt{\bar{\lambda}_{P_{r+n}}} (\theta^n v(t) + \varepsilon(t)).$$

where  $\beta_{n+r}$  is given as in (3.35).

Using the comparison lemma, one gets

$$\sqrt{V(\bar{z}(t))} \leq e^{-\beta_{r+n}\theta(t-t_0)} \sqrt{V(\bar{z}(t_0))} + \frac{\sqrt{\bar{\lambda}_{P_{r+n}}}}{\beta_{r+n}} (\theta^{n-1} \delta_v + \frac{\delta_\varepsilon}{\theta}).$$

Using inequality (3.24), the last inequality becomes

$$\sqrt{V(\bar{z}(t))} \leq e^{-\beta_{r+n}\theta(t-t_0)} \sqrt{V(\bar{z}(t_0))} + \frac{\sqrt{\bar{\lambda}_{P_{r+n}}}}{\beta_{r+n}} \frac{\bar{\delta}_v + \delta_\varepsilon}{\theta}.$$

Coming back to  $\|\bar{z}(t)\|$ , one gets

$$\|\bar{z}(t)\| \leq \chi_{P_{r+n}} \left( e^{-\beta_{r+n}\theta(t-t_0)} \|\bar{z}(t_0)\| + \frac{1}{\beta_{r+n}} (\frac{\bar{\delta}_v + \delta_\varepsilon}{\theta}) \right), \quad (3.44)$$

where  $\chi_{P_{r+n}}$  is the conditioning number of  $P_{r+n}$ .

According to the definition of  $\bar{z}$ , one has

$$\begin{cases} \bar{z}_i &= \bar{\eta}_i = \theta^{n-1} \eta_i, i = 1, \dots, r, \\ \bar{z}_{i+r} &= \bar{x}_i = \theta^{n-i} \tilde{x}_i, i = 1, \dots, n. \end{cases} \quad (3.45)$$

Hence, for  $\theta \geq 1$ , one has  $\|\bar{z}(t_0)\| \leq \theta^{n-1} \|z(t_0)\|$ , and

$$\left\| \begin{array}{c} \eta(t) \\ \tilde{x}_1(t) \end{array} \right\| \leq \theta^{1-n} \|\bar{z}(t)\|, \quad |\tilde{x}_i(t)| = \theta^{i-n} |\bar{z}_{i+r}(t)| \leq \theta^{i-n} \|\bar{z}(t)\|, \quad i = 2, \dots, n. \quad (3.46)$$

Combining (3.44) with (3.46) and accounting for the fact that  $\|z(t_0)\| = \|\tilde{x}(t_0)\|$  (since  $\eta(t_0) = 0$ ) lead to (3.34). This ends the proof of Theorem 3.3.1.

Note that the upper bound of the observation estimation error provided by the second inequality of (3.34) is very similar to that given by (3.22). In particular, the improvement of the sensitivity to high gain frequency signals of the filtered version cannot be appreciated by this bound. In order to point out this improvement, one shall proceed as in ([65, 38]).

### 3.3.2 Improvement of the sensitivity to high frequency signals

Assume that the measurement noise  $v(t)$  is modelled as in (3.27), then by proceeding as in ([38]), one can derive an upper bound for the observation error which puts forward the high frequency filtering capabilities of the observer. Indeed, one states the following theorem the proof of which borrows from that of Theorem 2 in ([38]).

**Theorem 3.3.2** *Consider system (3.19) with observer (3.33) and suppose that Assumption A1, A2 and A4 hold,  $\varepsilon(t) \equiv 0$  for all  $t \geq t_0$  and  $v(t)$  generated by (3.27) with the frequencies  $\omega_i$  being pairwise incommensurable. Then, there exists  $\epsilon^*(\theta) > 0$  such that for any  $\epsilon \in ]0, \epsilon^*]$ , one has*

$$\lim_{t \rightarrow +\infty} \sup |\tilde{x}_i(t)| \leq \rho(\epsilon \theta)^{r+1} k_{r+i} \theta^{i-1} \delta_v, i = 1, \dots, n, \quad (3.47)$$

where  $\rho$  is a constant independent of  $\theta$  and  $\epsilon$ .

**Proof of Theorem 3.3.2.** The proof borrows from that of Theorem 2 in [38] where only the linear case  $\varphi(u, x) = \Phi x$  where  $\Phi$  is a lower triangular matrix with  $\|\Phi\| \leq L$  is considered. As pointed out in [38], it has been shown in [65] that the obtained bounds can be extended to the nonlinear context.

Set  $z = \begin{pmatrix} \eta \\ \tilde{x} \end{pmatrix}$  where  $\eta$  and  $\tilde{x} = \hat{x}_x$  are the state of the error equation system (3.33). For  $\varepsilon(t) \equiv 0$ , system (3.36) can be written under the following compact form

$$\dot{z} = \Lambda z + \Gamma R w, \quad (3.48)$$

where

$$\Lambda = \begin{pmatrix} \theta(A_r - K_r C_r) & \theta B_r C_n \\ -\theta \Delta_n^{-1}(\theta) K_n C_r & A_n + \Phi \end{pmatrix}, \quad \Gamma = \begin{pmatrix} -\theta B_r \\ 0_{n \times 1} \end{pmatrix}. \quad (3.49)$$

The matrix  $\Lambda$  can be factorized as follows

$$\Lambda = \theta \Upsilon(\theta) \bar{\Lambda} \Upsilon^{-1}(\theta) \text{ with } \Upsilon(\theta) = \begin{pmatrix} I_r & 0_{r \times n} \\ 0_{n \times r} & \Delta_n^{-1}(\theta) \end{pmatrix} \text{ and} \quad (3.50)$$

$$\bar{\Lambda} = \begin{pmatrix} A_r - K_r C_r & B_r C_n \\ -K_n C_r & A_n + \frac{1}{\theta} \Delta_n(\theta) \Phi \Delta_n^{-1}(\theta) \end{pmatrix} \quad (3.51)$$

Notice that, the matrix  $\bar{\Lambda}$  can be expressed as follows

$$\bar{\Lambda} = \tilde{A}_{n+r} + \text{diag}(0_{r \times r}, \Theta(\theta)) \text{ with } \Theta(\theta) = \frac{1}{\theta} \Delta_n(\theta) \Phi \Delta_n^{-1}(\theta). \quad (3.52)$$

where  $\tilde{A}_{n+r}$  is the Hurwitz matrix given by (3.39).

Bearing in mind that the constant matrix  $\Phi$  is triangular, it is easy to check that each

non zero entry of the matrix  $\theta\Theta(\theta)$  is a polynomial in  $\frac{1}{\theta}$  and hence for  $\theta$  high enough the matrix  $\bar{\Lambda}$  is Hurwitz. Combining this fact with the decomposition (3.49), one concludes that the matrix  $\Lambda$  is Hurwitz and hence the steady state of the linear system (3.48) is given by  $z_{ss} = \Pi_\epsilon w$  where  $\Pi_\epsilon$  is the unique solution of the following Sylvester equation

$$\Pi_\epsilon S = \epsilon \left( \Lambda \Pi_\epsilon + \Gamma R \right). \quad (3.53)$$

and it can be expressed as follows (see [38] for more details)

$$\Pi_\epsilon = \prod_{k=1}^{+\infty} \epsilon^k \Lambda^{k-1} \Gamma R S^{-k}. \quad (3.54)$$

Now, one can check the following equalities

$$\begin{cases} \Lambda^{k-1} \Gamma = -\theta^k \begin{pmatrix} A_r^{k-1} B_r \\ 0_{n \times 1} \end{pmatrix}, \quad k = 1, \dots, r, \\ \Lambda^r \Gamma = \theta^{r+1} \begin{pmatrix} K_r \\ \Delta_n^{-1}(\theta) K_n \end{pmatrix}. \end{cases} \quad (3.55)$$

For  $i = r + 1, \dots, r + n$ , let  $e_i$  be the  $i$ 'th column of identity matrix of order  $r + n$ . One clearly has  $\tilde{x}_i = e_i^T z$ . Now, since  $\limsup_{t \rightarrow \infty} (z - \Pi_\epsilon w(t)) = 0$ , one has

$$\limsup_{t \rightarrow \infty} |\tilde{x}_i(t)| = \limsup_{t \rightarrow \infty} |e_i^T \Pi_\epsilon w(t)|, \quad i = 1, \dots, n. \quad (3.56)$$

Using (3.54), one gets

$$\begin{aligned} |e_i^T \Pi_\epsilon| &= \left| e_i^T \left( \prod_{k=1}^{r+1} \epsilon^k \Lambda^{k-1} \Gamma R S^{-k} + \prod_{k=r+2}^{\infty} \epsilon^k \Lambda^{k-1} \Gamma R S^{-k} \right) \right| \\ &= \left| e_i^T \left( \epsilon^{r+1} \Lambda^r \Gamma R S^{-(r+1)} + \prod_{k=r+2}^{\infty} \epsilon^k \Lambda^{k-1} \Gamma R S^{-k} \right) \right| \\ &= \epsilon^{r+1} \theta^{r+i} k_{r+i} |R S^{-(r+1)}| + \prod_{k=r+2}^{\infty} \epsilon^k |e_i^T \Lambda^{k-1} \Gamma R S^{-k}|. \end{aligned} \quad (3.57)$$

The same arguments provided in [38] are used to conclude. More precisely, for  $\epsilon$  small enough, i.e.  $\epsilon \in ]0, \epsilon^*(l)[$ , the dominant terms in (3.57) are the ones with smallest power of  $\epsilon$ . Since  $R, S$  and  $\Lambda$  are bounded and do not depend on  $\epsilon$ , there exists  $\rho > 0$  satisfying (3.47). This ends the proof of Theorem 3.3.2.

### 3.4 Redesign of the FHGO for sampled/delayed output measurements

The above design deals with the case where the outputs measurements are available under a continuous form. One proposes in what follows to redesign observer (3.33) in

order to derive an appropriate version that accounts for general scenarios under which the outputs may be available: delayed continuous outputs, sampled outputs and sampled delayed outputs. Notice that the design of state observers for nonlinear systems with sampled and delayed outputs has received a growing interest over the last two decades ([70, 71, 72, 73, 74, 75, 25, 76, 77, 78, 79, 80]). In many works, the proposed observers in the case of sampled/delayed outputs are issued from a redesign of observers working with continuous outputs and the convergence of the observation error associated to the so derived observers is guaranteed provided that the maximum value of the partition sampling diameters (or the upper bound of the output delays) is lower than a constant which generally depends on some observer design parameters(s). Notice that in most cases, the case of sampling outputs is treated as a particular case of delayed output with a time-varying delay. When the output delay is long, the proposed observers generally assume a cascade structure constituted by a chain of subsystems. One of these subsystems (the head or the tail of the cascade) is an observer which provides a delayed estimate of the system state where the underlying delay is smaller than the original one. Then, the so obtained delayed state estimates are exploited to predict the state of the systems over small intervals in such a way that the state of the last subsystem (the tail or the head of the cascade) provides an estimate of the system actual state ([25, 76, 77, 78]). In spite of the existing results, there is not a general approach that allows to redesign in a systematic manner an available observer working with outputs that are available continuously in order to generate an appropriate version working with sampled or/and delayed outputs while inheriting the main properties of the original observer. In the sequel, one shall generalize the FHGO design proposed previously by accounting for many scenarios under which the output measurements may be available and one shall show how the properties of the observer with continuous outputs are inherited by the redesigned version where the measurements are available under other forms.

### 3.4.1 Class of systems with different scenarios for the output measurements availability

The class of systems (3.19) is generalized to the following one

$$\begin{cases} \dot{x}(t) = A_n x(t) + \varphi^s(u(t), x(t)) + B_n \varepsilon(t) \\ y_d(t) = C_n x(t - d(t)) + v(t) = x_1(t - d(t)) + v(t) \end{cases} \quad (3.58)$$

where the function  $d : \mathbb{R}^+ \rightarrow [0, d_M]$ , where  $d_M \geq 0$  is a non negative constant, is a piecewise continuous function and it denotes the output delay. The other variables keep the same meaning as in (3.19). The function  $\varepsilon : [0, +\infty[ \mapsto \mathbb{R}$  denotes the system uncertainties and may depend on the input and uncertain parameters. It shall be treated as

an unknown function which explicitly depends on time  $t$  for  $t \geq 0$ ; finally,  $\varphi(u, x) \in \mathbb{R}^n$  denotes a nonlinear vector field and each of its  $n$  components has a triangular structure with respect to  $x$ , i.e.  $\varphi_i(u, x) = \varphi_i(u, \underline{x}_i)$   $i = 1, \dots, n$  where  $\underline{x}_i \in \mathbb{R}^i$  is the sub-vector of  $x$  defined  $x_i = [x_1 \dots x_i]$ .

The observer design will be performed under the the same four assumptions as for observer (3.33) with the following additional one:

**A5.** The known output delay  $d(t)$  is piecewise continuous and bounded, i.e.

$$\exists d_M \geq 0 ; \forall t \geq 0, |d(t)| \leq d_M. \quad (3.59)$$

Notice that the fact that the output delay may be time-varying which varies from zero to  $d_M$  allows to recover the following two practical situations related to the availability of the output measurements. The first one corresponds to the case where the output measurements are available continuously in time either with no delay or with a delay which may be constant or time-varying. The second situation deals with the case where the output measurements are available at sampling instants  $t_k$  either with no delay or with a delay,  $d_1$ , which may be constant or time-varying. More precisely, each measurement is available at the time instant  $t_k$  and it corresponds to the measurement of the output at the time instant  $t_k - d_1(t_k)$ . In such a case, one has  $y_d(t) = y(t_k - d_1(t_k)) = y(t - (t - t_k + d_1(t_k)))$  where  $y_d(t)$  and  $y(t)$  correspond to the delayed and actual output measurement, respectively. In this situation, the output time-varying delay can be expressed as  $d(t) = t - t_k + d_1(t_k)$ .

Before proposing an observer for system (3.7), let us state recall the following technical lemma [81].

**Lemma 3.4.1** *Consider a differentiable function  $v : t \in [t_0 - \delta, +\infty[ \mapsto v(t) \in \mathbb{R}^+$  with  $t_0, \delta \geq 0$ , satisfying the following inequality*

$$\dot{v}(t) \leq -\alpha v(t) + \beta \int_{t-\delta}^t v(s) ds + p(t), \quad \forall t \geq t_0, \quad (3.60)$$

where  $\alpha > 0$ ,  $\beta \geq 0$  and  $p(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an essentially bounded function with  $|p|_\infty = \text{Ess sup}_{t \geq t_0} p(t) \leq \delta_p$ . If  $\frac{\beta\delta}{\alpha} < 1$ , then,  $\forall t \geq t_0$ , the function  $v$  satisfies

$$v(t) \leq \left(1 + \delta(\alpha - \eta(\delta))\right) e^{-\eta(\delta)(t-t_0)} \max_{\nu \in [t_0 - \delta, t_0]} v(\nu) + \frac{\delta_p}{\eta(\delta)}, \quad (3.61)$$

with

$$0 < \eta(\delta) = (\alpha - \beta\delta)e^{-\alpha\delta} = \alpha\left(1 - \frac{\beta\delta}{\alpha}\right)e^{-\alpha\delta} \leq \alpha. \quad (3.62)$$

Before dealing with the design of a filtered high gain observer, one recall that the design of the SHGO (3.20) has been extended in ([66, 81]) to derive a redesigned version that accounts for all possible forms under which the output measurements may be available. The equations of the redesigned version can be written as follows

$$\begin{cases} \dot{\hat{x}}(t) &= A\hat{x}(t) + \varphi(u, \hat{x}) - \theta\Delta_n(\theta)K (g(t) - g(t-d(t)) + C\hat{x}(t-d(t)) - y_d(t)) \\ \dot{g}(t) &= -\theta k_1 (g(t) - g(t-d(t)) + C\hat{x}(t-d(t)) - y_d(t)) \end{cases} \quad (3.63)$$

where  $\hat{x} \in \mathbb{R}^n$  is the state estimate,  $K = \begin{pmatrix} k_1 & \dots & k_n \end{pmatrix}^T \in \mathbb{R}^n$  with  $k_i \in \mathbb{R}, i = 1, \dots, n$  is chosen such that  $A - KC$  is Hurwitz,  $d_M$  is the upper bound of  $d(t)$  provided by Assumption **A5**, and  $\Delta_n$  is as defined in (3.4) with  $k = n$ .

The above observer delay differential equations (3.63) are initialized as follows

$$\hat{x}(t_0) \in X \text{ and } g(s) = \psi(s), \forall s \in [t_0 - d_M, t_0],$$

where  $X$  is the subset defined in Assumption **A1** and  $\psi(s)$  is any arbitrarily continuous function on  $[t_0 - d_M, t_0]$ .

### 3.4.2 Equations of the redesigned FHGO

The equations of the proposed redesigned observers can be written as follows

$$\begin{cases} \dot{\eta} &= \theta \left( (A_r - K_r C_r)\eta + B_r \left( C_n \hat{x}(t-d(t)) - y_d(t) - \theta k_{r+1} \int_{t-d(t)}^t \eta_1(s) ds \right) \right) \\ \dot{\hat{x}} &= A_n \hat{x} + \varphi^s(u, \hat{x}) - \theta \Delta_n^{-1}(\theta) K_n C_r \eta, \quad \eta(t_0) = 0 \text{ and } \hat{x}(t_0) \in \mathbb{R}^n, \end{cases} \quad (3.64)$$

where all the variables have the same meaning as in observer (3.33). Recall that the coefficient  $k_{r+1}$  is the first component of the gain  $K_n$ .

It should be emphasized that the equations of the redesigned observer (3.64) are very similar to those of the original FHGO (3.33) with the sole difference that the term  $(C_n \hat{x}(t) - y(t))$  involved in the equations of (3.33) is substituted by  $(C_n \hat{x}(t-d(t)) - y_d(t) - \theta k_{r+1} \int_{t-d(t)}^t \eta_1(s) ds)$  in the equations of (3.64). Moreover, the integral term involving the integral in the equations of (3.64), i.e.  $-\theta k_{r+1} \int_{t-d(t)}^t \eta_1(s) ds$  can be computed using the following appropriate form

$$-\theta k_{r+1} \int_{t-d(t)}^t \eta_1(s) ds = \alpha(t) - \alpha(t-d(t)) \quad (3.65)$$

where the function  $\alpha$  is governed by the following delay differential equation

$$\dot{\alpha}(t) = -\theta k_{r+1} \eta_1(t) \text{ with } \alpha(s) = \psi(s), \forall s \in [t_0 - d_M, t_0], \quad (3.66)$$

where  $d_M$  is the upper bound of the output delay given by Assumption **A5** and  $\psi$  is any arbitrary continuous function on  $[-d_M, 0]$ . Notice that, since  $\alpha(t)$  intervenes in the

filter equations and in order to reduce the transient period of the filter, one can choose  $\psi(s) = 0, \forall s \in [t_0 - d_M, t_0]$ .

Hence, using (3.66) to compute the integral term, the equations of the proposed redesigned observer (3.64) specialize as follows

$$\begin{cases} \dot{\eta}(t) &= \theta \left( (A_r - K_r C_r) \eta(t) + B_r \left( C_n \hat{x}(t - d(t)) - y_d(t) + \alpha(t) - \alpha(t - d(t)) \right) \right), \\ \dot{\hat{x}}(t) &= A_n \hat{x}(t) + \varphi^s(u, \hat{x}) - \theta \Delta_n^{-1}(\theta) K_n C_r \eta(t), \\ \dot{\alpha}(t) &= -\theta k_{r+1} \eta_1(t). \end{cases} \quad (3.67)$$

with the initial conditions  $\hat{x}(t_0) \in X$  and  $\eta(s) = \alpha(s) = 0, \forall s \in [t_0 - d_M, t_0]$ .

The main properties of observer (3.67) are stated in the following theorem.

**Theorem 3.4.1** *Consider system (3.58) subject to Assumption A1 to A5 together with observer (3.33) and assume that the essential bound  $\delta_v$  satisfies (3.24) and that the upper bound  $d_M$  of the delay satisfies the following condition*

$$d_M < \frac{\beta_{r+n}}{\chi_{P_{r+n}}(L_\varphi + \theta)}. \quad (3.68)$$

where  $\beta_{r+n}$  is given by (3.35),  $\chi_{P_{r+n}}$  is the conditioning number of the matrix  $P_{r+n}$  given by (3.40) and  $L_\varphi$  is the Lipschitz constant given by (3.16). Then, the components of the observation error vector satisfy the following property,

$$\begin{aligned} &\exists \theta_0 > 0; \forall \theta \geq \theta_0; \forall u \in U; \forall \hat{x}(t_0) \in \mathbb{R}^n; \forall t \geq t_0, \\ &\begin{cases} \left\| \begin{array}{l} \eta(t) \\ \tilde{x}_1(t) \end{array} \right\| &\leq \chi_{P_{r+n}} \left( e^{-\eta_\theta(d_M)(t-t_0)} \|\tilde{x}(t_0)\| + \frac{\bar{\delta}_v + \delta_\varepsilon}{\beta_{r+n} \theta^{n-1} \eta_\theta(d_M)} \right), \\ |\tilde{x}_i(t)| &\leq \chi_{P_{r+n}} \left( \theta^{i-1} e^{-\eta_\theta(d_M)(t-t_0)} \|\tilde{x}(t_0)\| + \frac{\bar{\delta}_v + \delta_\varepsilon}{\beta_{r+n} \eta_\theta(d_M) \theta^{n-i}} \right), \quad i = 2, \dots, n. \end{cases} \end{aligned} \quad (3.69)$$

where

$$\eta_\theta(d_M) = (\alpha_\theta - \beta_\theta d_M) e^{-\alpha_\theta d_M} \text{ with } \alpha_\theta = \beta_{r+n} \theta, \beta_\theta = \chi_{P_{r+n}} \theta (L_\varphi + \theta). \quad (3.70)$$

**Proof of Theorem 3.4.1.** Set  $\tilde{x} = \hat{x} - x, \bar{x} = \theta^{n-1} \Delta_n \tilde{x}, \bar{\eta} = \theta^{n-1} \eta$  and  $\bar{\alpha} = \theta^{n-1} \alpha$ .

Using (3.67), one gets

$$\begin{cases} \dot{\bar{\eta}}(t) &= \theta \left( (A_r - K_r C_r) \bar{\eta}(t) + B_r \left( C_n \bar{x}(t - d(t)) + \bar{\alpha}(t) - \bar{\alpha}(t - d(t)) \right) \right) - \theta^n B_r v(t) \\ \dot{\tilde{x}}(t) &= \theta A_n \tilde{x} + \theta^{n-1} \Delta_n \tilde{\varphi}^s(u, \hat{x}, x) - \theta K_n C_r \bar{\eta}(t) - B_n \varepsilon(t), \\ \dot{\bar{\alpha}}(t) &= -\theta k_{r+1} \bar{\eta}_1(t) \end{cases} \quad (3.71)$$

where  $\tilde{\varphi}^s(u, \hat{x}, x) = \varphi^s(u, \hat{x}) - \varphi^s(u, x)$ .

Adding and subtracting the term  $\theta B_r C_n \bar{x}(t)$  to the left side of the ODE governing  $\bar{\eta}$ , one gets

$$\dot{\bar{\eta}} = \theta \left( (A_r - K_r C_r) \bar{\eta}(t) + B_r C_n \bar{x}(t) \right) + \theta B_r \xi(t) - \theta^n B_r v(t) \quad (3.72)$$

where

$$\begin{aligned} \xi(t) &= C_n \left( \bar{x}(t - d(t)) - \bar{x}(t) \right) + \bar{\alpha}(t) - \bar{\alpha}(t - d(t)) \\ &= \bar{x}_1(t) - \bar{x}_1(t - d(t)) + \bar{\alpha}(t) - \bar{\alpha}(t - d(t)). \end{aligned} \quad (3.73)$$

Notice that the variable  $\xi(t)$  can be expressed as follows

$$\begin{aligned} \xi(t) &= \int_{t-d(t)}^t (\dot{\bar{x}}_1(s) - \dot{\bar{\alpha}}_1(s)) ds \\ &= \int_{t-d(t)}^t (\dot{\bar{x}}_1(s) + \theta k_{r+1} \theta \bar{\eta}_1(s)) ds. \end{aligned} \quad (3.74)$$

According to (3.71), one has

$$\dot{\bar{x}}_1(t) = \theta \bar{x}_2(t) + \theta^{n-1} \tilde{\varphi}(u, \hat{x}_1, x) - \theta k_{r+1} \theta \bar{\eta}_1(t). \quad (3.75)$$

Combining (3.74) and (3.75), one gets

$$\xi(t) = \int_{t-d(t)}^t (\theta \bar{x}_2(s) + \theta^{n-1} \tilde{\varphi}(u, \hat{x}_1, x)) ds \quad (3.76)$$

Using the Lipschitz assumption on the  $\varphi$ , one gets

$$|\xi(t)| \leq \int_{t-d(t)}^t (\theta + L_\varphi) \|\bar{x}(s)\| ds, \quad (3.77)$$

where  $L_\varphi$  is the Lipschitz constant constant by (3.16).

Now, set  $z = \begin{pmatrix} \eta \\ \tilde{x} \end{pmatrix}$  and  $\bar{z} = \begin{pmatrix} \bar{\eta} \\ \bar{x} \end{pmatrix}$ . Proceeding as in the proof of Theorem (3.3.1), the equation of the error system can be written under the following compact form (compare with (3.38))

$$\dot{\bar{z}} = \theta \tilde{A}_{n+r} z + \tilde{\Phi}(u, \hat{x}, x) - \bar{B}_{r+n} (\theta^n v(t) + \theta \xi(t)) - B_{r+n} \varepsilon(t), \quad (3.78)$$

where  $\xi$  is as defined in (3.73) and the other variables keep the same meaning as in (3.38).

Let  $V(\bar{z}) = \bar{z}^T P_{r+n} \bar{z}$  be the Lyapunov candidate function for the error system (3.78) where  $P_{r+n}$  is the SPD matrix given as in (3.40). Again, proceeding as in the proof of Theorem (3.3.1), one can show that (compare with (3.43))

$$\dot{V}(\bar{z}) \leq -\frac{\mu_{r+n} \theta}{\lambda_{P_{r+n}}} V(\bar{z}) + 2\sqrt{\lambda_{P_{r+n}}} \sqrt{V(\bar{z})} (\theta |\xi(t)| + \theta^n v(t) + \varepsilon(t)). \quad (3.79)$$

using the bound of  $\|\xi\|$  provided by (3.77), inequality (3.79) leads to

$$\begin{aligned}
\dot{V}(\bar{z}) &\leq -\frac{\mu_{r+n}\theta}{\bar{\lambda}_{P_{r+n}}}V(\bar{z}) + 2\chi_{P_{r+n}}\theta\sqrt{V(\bar{z})}(L_\varphi + \theta) \int_{t-d(t)}^t \sqrt{V(\bar{z}(s))}ds \\
&\quad + 2\sqrt{\bar{\lambda}_{P_{r+n}}}\sqrt{V(\bar{z})}(\theta^n v(t) + \varepsilon(t)) \\
&\leq -\frac{\mu_{r+n}\theta}{\bar{\lambda}_{P_{r+n}}}V(\bar{z}) + 2\chi_{P_{r+n}}\theta\sqrt{V(\bar{z})}(L_\varphi + \theta) \int_{t-d_M}^t \sqrt{V(\bar{z}(s))}ds \\
&\quad + 2\sqrt{\bar{\lambda}_{P_{r+n}}}\sqrt{V(\bar{z})}(\theta^n v(t) + \varepsilon(t))
\end{aligned} \tag{3.80}$$

where  $\chi_{P_{r+n}}$  is the conditioning number of  $P_{r+n}$  and  $d_M$  is the upper bound of the delay.

Inequality (3.80) can be rewritten as follows

$$\begin{aligned}
\frac{d}{dt}\sqrt{V(\bar{z})} &\leq -\beta_{r+n}\theta\sqrt{V(\bar{z})} + \chi_{P_{r+n}}\theta(L_\varphi + \theta) \int_{t-d_M}^t \sqrt{V(\bar{z}(s))}ds \\
&\quad + \sqrt{\bar{\lambda}_{P_{r+n}}}(\theta^n v(t) + \varepsilon(t)).
\end{aligned} \tag{3.81}$$

Set

$$\alpha_\theta = \beta_{r+n}\theta, \quad \beta_\theta = \chi_{P_{r+n}}\theta(L_\varphi + \theta) \text{ and } p_\theta = \sqrt{\bar{\lambda}_{P_{r+n}}}(\theta^n v(t) + \varepsilon(t)). \tag{3.82}$$

According to Assumptions **A3**, **A4** and inequality (3.24), the function  $p_\theta(t)$  is essentially bound and the underlying essential bound is equal to  $\sqrt{\bar{\lambda}_{P_{r+n}}}(\bar{\delta}_v + \delta_\varepsilon)$  where  $\delta\bar{\delta}_v$  and  $\delta_\varepsilon$  are the essential bounds provided by Assumptions **A3**, **A4** and inequality (3.24). Now assume that

$$\frac{\beta_\theta}{\alpha_\theta}d_M < 1, \tag{3.83}$$

Since  $p_\theta(t)$  is essentially bounded, then conditions of lemma 3.4.1 and one gets

$$\sqrt{V(\bar{z}(t))} \leq \left(1 + (\alpha_\theta - \eta_\theta(d_M))d_M\right) e^{-\eta_\theta(d_M)(t-t_0)} \max_{s \in [t_0-d_M, t_0]} \sqrt{V(\bar{z}(s))} + \frac{\sqrt{\bar{\lambda}_{P_{r+n}}}(\bar{\delta}_v + \delta_\varepsilon)}{\eta(d_M)}\delta_\varepsilon,$$

where  $\eta_\theta(d_M) = (\alpha_\theta - \beta_\theta d_M)e^{-\alpha_\theta d_M}$ .

Coming back to  $\bar{z}$ , one gets

$$\|\bar{z}(t)\| \leq \chi_{P_{r+n}} \left( \left(1 + (\alpha_\theta - \eta_\theta(d_M))d_M\right) e^{-\eta_\theta(d_M)(t-t_0)} \max_{s \in [t_0-d_M, t_0]} \|\bar{z}(s)\| + \frac{(\bar{\delta}_v + \delta_\varepsilon)}{\eta(d_M)}\delta_\varepsilon \right),$$

Coming back to  $\eta$  and  $\tilde{x}$  by using (3.46), one gets (3.69). This ends the proof of Theorem 3.4.1.

## 3.5 Extension of the design to the class of non uniformly observable systems

Contrarily to the class of Uniformly Observable Systems, there are a few results dealing with the redesign of the SHGO for the class of Non Uniformly Observable Systems (NUOS) to improve its sensitivity to high frequency signals. In [44], the authors proposed a FHGO for a class of NUOS. The proposed design is inspired from that given in [82] which has been proposed for a class of UOS. As in [82], the FHGO in [44] has been first proposed in the case where the output measurements are available in a continuous manner. Then, a redesigned version was proposed for the case where these measurements are available only at sampling instants. The gain of the FHGO is similar as that of the SHGO and it is issued from the resolution of a Lyapunov (or Riccati) ODE.

One proposes in the sequel to extend the design proposed for UOS to a class of NUOS. The proposed extension shall be achieved by considering the general framework related to the availability of the measurements outputs. Before detailing this design, one shall introduce the considered class of systems and adopt the required assumptions generally needed when adopting a high gain approach for the observer design.

### 3.5.1 Class of NUOS

Consider the class of NUOS which are diffeomorphic to the following form

$$\begin{cases} \dot{x}(t) = F_n(u(t))x(t) + \varphi^s(u(t), x(t)) + B_n\varepsilon(t) \\ y_d(t) = C_n x(t - d(t)) + v(t) = x_1(t - d(t)) + v(t) \end{cases} \quad (3.84)$$

where

$$F_n(u(t)) = \begin{bmatrix} 0 & f_1(u) & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & \dots & 0 & f_{n-1}(u) \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \quad (3.85)$$

and all other variables involved in (3.84) have the same meaning as in system (3.58). It is worth noticing that system (3.84) is very similar to system (3.58) and the only difference lies in the fact the constant anti-shift matrix  $A_n$  in system (3.58) is substituted by the time-varying anti-shift matrix  $F_n(u)$  in system (3.84). Hence, the observer design which shall be proposed later shall be achieved by considering the same assumptions as in the case of UOS, i.e. Assumptions **A1** to **A5**. Moreover, to design a SHGO for system (3.84), one needs to guarantee its observability over arbitrarily short time horizons. This observability property shall be achieved through the formulation of specific excitation conditions

allowing thereby to qualify the input behavior for small time horizons. This leads to the following additional assumption [23, 83, 59] which will also be adopted.

**A6** The input  $u$  is such that for any trajectory  $x(t)$  of system (3.84) starting from  $x(t_0) \in X$ ,  $\exists T^* > 0$ ,  $\exists \theta_0 > 0$ ,  $\exists \delta^* > 0$ ,  $\forall \theta \geq \theta_0$  and  $\forall t \geq t_0^* \triangleq T^*/\theta$ , the following persistent excitation condition is satisfied

$$\int_{t-T^*/\theta}^t (\Phi_u(s, t))^T C^T C \Phi_u(s, t) ds \geq \frac{\delta^*}{\theta} \Delta_n^2(\theta), \quad (3.86)$$

where  $\Delta_n$  is as in (3.4) with  $k = n$  and  $\Phi(t, s)$  denotes the state transition matrix of the state affine system

$$\dot{\xi}(t) = F_n(u(t))\xi(t). \quad (3.87)$$

Recall that the matrix  $\Phi_u(t, s)$  is defined as

$$\frac{d}{dt} \left( \Phi_u(t, s) \right) = F_n(u(t))\Phi_u(t, s) \quad \forall t \geq s \geq t_0 \quad \text{with} \quad \Phi_u(s, s) = I_n. \quad (3.88)$$

To avoid the redundance of variables and without loss of generality, one shall assume that the time instant  $t_0^*$  involved in Assumption **A5** is equal to  $t_0$ .

The design of a SHGO for system (3.84) has been proposed in [23] in the disturbance and noisy free case. A similar observer has been proposed [58, 59] in the presence of the disturbance  $\varepsilon$  and under assumptions **A1** to **A3** and **A6**.

Before giving the equations of the proposed observer, one proposes to introduce some variables and recall some results related to the design of SHGO for NUOS.

### 3.5.2 Preliminaries

In the case where the outputs are available under a continuous form, the equations of a SHGO for system (3.84) can be written as follows ([23, 83, 59])

$$\dot{\hat{x}}(t) = F_n(u)\hat{x}(t) + \varphi(u, \hat{x}) - \theta S_n^{-1}(t) C_n^T (C_n \hat{x}(t) - y(t)), \quad \hat{x}_0 \in X \quad (3.89)$$

where  $\Delta_n(\theta)$  is a diagonal matrix defined as in (3.4) with  $k = n$  and  $\theta \geq 1$  and  $S_n$  is a symmetric matrix solution of the following Lyapunov ordinary differential equation

$$\dot{S}_n(t) = -\theta \left( S_n(t) + F_n^T(t) S_n(t) + S_n(t) F_n(t) - C_n^T C_n \right). \quad (3.90)$$

Notice that the equations of observer (3.89) are very similar to those of observer (3.20) with the difference that the constant gain  $G_n$  involved in (3.20), which is chosen such

that  $An - KC$  is Hurwitz, is substituted by  $S_n^{-1}C^T$  in (3.89). One shall show that under Assumption **A6**, the matrix  $S_n$  governed by (3.90) is SPD. To this end, one first notices that the transition matrix, that shall be denoted  $\tilde{\Phi}_u$ , of the following state affine system

$$\dot{\tilde{\xi}}(t) = \theta F_n(u)\tilde{\xi}(t),$$

is given by

$$\tilde{\Phi}_u(t, s) = \Delta_\theta \Phi_u(t, s) \Delta_\theta^{-1}, \quad (3.91)$$

where  $\Phi_u$  is defined by (3.88). This allows to express the matrix  $S$  as follows

$$\begin{aligned} S_n(t) &= e^{-\theta t} \tilde{\Phi}_u^T(t_0, t) S_n(t_0) \tilde{\Phi}_u(t_0, t) + \theta \int_{t_0}^t e^{-\theta(t-s)} \tilde{\Phi}_u^T(s, t) C_n^T C_n \tilde{\Phi}_u(s, t) ds \\ &= e^{-\theta t} \Delta_\theta^{-1} \Phi_u^T(t_0, t) \Delta_\theta S_n(t_0) \Delta_\theta \Phi_u(t_0, t) \Delta_\theta^{-1} \\ &\quad + \theta \int_{t_0}^t e^{-\theta(t-s)} \Delta_\theta^{-1} \Phi_u^T(s, t) \Delta_\theta C_n^T C_n \Delta_\theta \Phi_u(s, t) \Delta_\theta^{-1} ds. \end{aligned} \quad (3.92)$$

Since  $\Delta_\theta^{-1} \Phi_u^T(t_0, t) \Delta_\theta S_n(t_0) \Delta_\theta \Phi_u(t_0, t) \Delta_\theta^{-1}$  is a SPD matrix and that the integral term in (3.93) is a symmetric non negative matrix, it follows that  $S_n(t)$  is a SPD matrix. From (3.92), one gets where  $\Phi_u$  is defined by (3.88). This allows to express the matrix  $S_n$  as follows

$$S_n(t) \geq \theta \int_{t_0}^t e^{-\theta(t-s)} \Delta_\theta^{-1} \Phi_u^T(s, t) \Delta_\theta C_n^T C_n \Delta_\theta \Phi_u(s, t) \Delta_\theta^{-1} ds. \quad (3.93)$$

Using the facts that  $C \Delta_\theta = C$  and  $S_n(t_0)$  is SPD, one gets for  $t \geq t_0 \triangleq T^*/\theta$

$$S_n(t) \geq \theta \int_{t_0}^t e^{-\theta(t-s)} \Delta_\theta^{-1} \Phi_u^T(s, t) C_n^T C_n \Phi_u(s, t) \Delta_\theta^{-1} ds \quad (3.94)$$

$$\geq \theta \int_{t - \frac{T^*}{\theta}}^t e^{-\theta(t-s)} \Delta_\theta^{-1} \Phi_u^T(s, t) C_n^T C_n \Phi_u(s, t) \Delta_\theta^{-1} ds$$

$$\geq \theta e^{-T^*} \int_{t - \frac{T^*}{\theta}}^t \Delta_\theta^{-1} \Phi_u^T(s, t) C_n^T C_n \Phi_u(s, t) \Delta_\theta^{-1} ds \quad (3.95)$$

$$\geq e^{-T^*} \delta_0 I_n \quad (3.96)$$

where  $T^*$  and  $\delta_0$  are given in assumption **A6**.

Set

$$\mathcal{F}(u) = \begin{bmatrix} A_r & B_r C_n \\ 0_{n,r} & F_n(u) \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C_r & 0_{1,n} \end{bmatrix}. \quad (3.97)$$

and consider the following augmented system

$$\begin{cases} \begin{pmatrix} \dot{\zeta} \\ \dot{x} \end{pmatrix} = \mathcal{F}(u(t)) \begin{pmatrix} \zeta \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ \varphi^s(u, x) \end{pmatrix} + B_{r+n} \varepsilon(t) \\ y(t) = \mathcal{C} \begin{pmatrix} \zeta \\ x \end{pmatrix} = C_r \zeta \end{cases} \quad (3.98)$$

Moreover, let  $\mathcal{S} = \mathcal{S}^T$  be the  $(n+r) \times (n+r)$  symmetric matrix governed by the consider the following Lyapunov ODE

$$\dot{\mathcal{S}} = -\theta \left( \mathcal{S} + \mathcal{S}\mathcal{F} + \mathcal{F}^T\mathcal{S} - \mathcal{C}^T\mathcal{C} \right). \quad (3.99)$$

Let  $\Psi_u(t, s)$  be the transition matrix of the following state affine system

$$\dot{\xi}(t) = \mathcal{F}(u(t))\xi(t). \quad (3.100)$$

where  $\xi \in \mathbb{R}^{n+r}$ .

The design of the FHGO requires the following Assumption

**A7** The input  $u$  is such that for any trajectory  $x(t)$  of system (3.98) starting from  $x(t_0) \in X$  and  $\zeta(t_0) = 0$ ,  $\exists \bar{T}^* > 0$ ,  $\exists \bar{\theta}_0 > 0$ ,  $\exists \bar{\delta}^* > 0$ ,  $\forall \theta \geq \bar{\theta}_0$  and  $\forall t \geq \bar{t}_0 \triangleq \bar{T}^*/\theta$ , the following persistent excitation condition is satisfied

$$\int_{t-\bar{T}^*/\theta}^t (\Psi_u(s, t))^T \mathcal{C}^T \mathcal{C} \Psi_u(s, t) ds \geq \frac{\bar{\delta}^*}{\theta} \Delta_{n+r}^2(\theta), \quad (3.101)$$

where  $\Delta_{n+m}$  is as in (3.4) with  $k = n+r$  and  $\Psi(t, s)$  is the transition matrix of the affine system (3.100).

Notice that under Assumption **A7**, one can show that the symmetric matrix  $\mathcal{S}$  governed by (3.99) is SPD. Moreover, it is not clear that inequality (3.101) of Assumption **A7** can be derived from Assumption **A6**. This remains an open question. However, in the case where where the  $f_i$ 's are constant i.e. for UOS, the response is positive, i.e. inequality (3.86) of Assumption **A6** does imply inequality (3.101) of Assumption **A7**.

Since  $\mathcal{S}(t)$  is SPD under Assumption **A7**, let  $K_r(t) = \begin{pmatrix} k_1(t) \\ \vdots \\ k_r(t) \end{pmatrix} \in \mathbb{R}^r$  and  $K_n(t) =$

$\begin{pmatrix} k_{r+1}(t) \\ \vdots \\ k_{r+n}(t) \end{pmatrix} \in \mathbb{R}^n$  be the subvector of  $\mathcal{S}^{-1}(t)\mathcal{C}^T$  defined as follows

$$\mathcal{S}^{-1}(t)\mathcal{C}^T = \begin{pmatrix} K_r(t) \\ K_n(t) \end{pmatrix}. \quad (3.102)$$

### 3.5.3 Equations of the FHGO for NUOS

The equations of the proposed FHGO can be written as follows

$$\begin{cases} \dot{\eta}(t) &= \theta \left( (A_r - K_r(t)C_r)\eta(t) + B_r \left( C_n \hat{x}(t-d(t)) - y_d(t) + \alpha(t) - \alpha(t-d(t)) \right) \right), \eta(t_0) = 0, \\ \dot{\hat{x}}(t) &= F_n(u(t))\hat{x}(t) + \varphi^s(u, \hat{x}) - \theta \Delta_n^{-1}(\theta) K_n(t) C_r \eta(t), \hat{x}(t_0) \in \mathbb{R}^n, \\ \dot{\alpha}(t) &= -\theta k_{r+1}(t) \eta_1(t), \alpha(s) = \psi(s), \forall s \in [t_0 - d_M, t_0]. \end{cases} \quad (3.103)$$

where  $K_r(t)$  and  $K_n(t)$  are the subvectors of  $\mathcal{S}^{-1}(t)\mathcal{C}^T$  defined as in (3.102).

Note that the equations of observer (3.103) are very similar to those of observer (3.67), the sole difference lies in the gains  $K_r(t)$  and  $K_n(t)$  which are no longer constant as in (3.67) but issued from the resolution of the ODE Lyapunov equation (3.99).

As one could expect, the main properties of observer (3.103) are similar to those of observer (3.67). They are stated in the following theorem.

**Theorem 3.5.1** *Consider system (3.58) subject to Assumption **A1** - **A5** and **A7** together with observer (3.103) and assume that the essential bound  $\delta_v$  satisfies (3.24) and that the upper bound  $d_M$  of the delay satisfies the following condition*

$$d_M < \frac{1}{\chi_{\mathcal{S}}(L_{\varphi} + \theta)}. \quad (3.104)$$

where  $\chi_{\mathcal{S}}$  is the conditioning number of the matrix  $\mathcal{S}$  given by (3.99) and  $L_{\varphi}$  is the Lipschitz constant given by (3.16). Then, the components of the observation error vector satisfy the following property,

$$\begin{cases} \exists \theta_0 > 0; \forall \theta \geq \theta_0; \forall u \in U; \forall \hat{x}(t_0) \in \mathbb{R}^n; \forall t \geq t_0, \\ \left\{ \begin{array}{l} \left\| \begin{array}{l} \eta(t) \\ \tilde{x}_1(t) \end{array} \right\| \leq \chi_{\mathcal{S}} \left( e^{-\eta_{\theta}(d_M)(t-t_0)} \|\tilde{x}(t_0)\| + \frac{\bar{\delta}_v + \delta_{\varepsilon}}{\theta^{n-1}\eta_{\theta}(d_M)} \right), \\ |\tilde{x}_i(t)| \leq \chi_{\mathcal{S}} \left( \theta^{i-1} e^{-\eta_{\theta}(d_M)(t-t_0)} \|\tilde{x}(t_0)\| + \frac{\bar{\delta}_v + \delta_{\varepsilon}}{\eta_{\theta}(d_M)\theta^{n-i}} \right), \quad i = 2, \dots, n. \end{array} \right. \end{cases} \quad (3.105)$$

where

$$\eta_{\theta}(d_M) = \theta \left( 1 - \chi_{\mathcal{S}}(L_{\varphi} + \theta)d_M \right) e^{-\theta d_M}. \quad (3.106)$$

**Proof of Theorem 3.5.1.** This proof is similar to that of Theorem 3.4.1. Indeed, set  $\tilde{x} = \hat{x} - x$ ,  $\bar{x} = \theta^{n-1}\Delta_n\tilde{x}$ ,  $\bar{\eta} = \theta^{n-1}\eta$  and  $\bar{\alpha} = \theta^{n-1}\alpha$ . Using (3.103), one gets

$$\begin{cases} \dot{\bar{\eta}}(t) = \theta \left( (A_r - K_r(t)C_r)\bar{\eta}(t) + B_r \left( C_n\bar{x}(t - d(t)) + \bar{\alpha}(t) - \bar{\alpha}(t - d(t)) \right) \right) - \theta^n B_r v(t) \\ \dot{\tilde{x}}(t) = \theta F_n(u(t))\tilde{x} + \theta^{n-1}\Delta_n\tilde{\varphi}^s(u, \hat{x}, x) - \theta K_n(t)C_r\bar{\eta}(t) - B_n\varepsilon(t), \\ \dot{\tilde{\alpha}}(t) = -\theta k_{r+1}(t)\bar{\eta}_1(t) \end{cases} \quad (3.107)$$

where  $\tilde{\varphi}^s(u, \hat{x}, x) = \varphi^s(u, \hat{x}) - \varphi^s(u, x)$ .

Adding and subtracting the term  $\theta B_r C_n \bar{x}(t)$  to the left side of the ODE governing  $\bar{\eta}$ , one gets

$$\dot{\bar{\eta}} = \theta \left( (A_r - K_r(t)C_r)\bar{\eta}(t) + B_r C_n \bar{x}(t) \right) + \theta B_r \xi(t) - \theta^n B_r v(t) \quad (3.108)$$

where

$$\begin{aligned}\xi(t) &= C_n \left( \bar{x}(t-d(t)) - \bar{x}(t) \right) + \bar{\alpha}(t) - \bar{\alpha}(t-d(t)) \\ &= \bar{x}_1(t) - \bar{x}_1(t-d(t)) + \bar{\alpha}(t) - \bar{\alpha}(t-d(t)).\end{aligned}\quad (3.109)$$

Notice that and as in the proof of Theorem 3.4.1, the variable  $\xi(t)$  can be expressed as follows (compare with (3.74))

$$\xi(t) = \int_{t-d(t)}^t (\dot{\bar{x}}_1(s) + \theta k_{r+1} \theta \bar{\eta}_1(s)) ds. \quad (3.110)$$

Again, proceeding as in the proof of Theorem 3.4.1, one can show that (compare with (3.77))

$$|\xi(t)| \leq \int_{t-d(t)}^t (\theta + L_\varphi) \|\bar{x}(s)\| ds, \quad (3.111)$$

where  $L_\varphi$  is the Lipschitz constant constant by (3.16).

Now, set  $z = \begin{pmatrix} \eta \\ \tilde{x} \end{pmatrix}$  and  $\bar{z} = \begin{pmatrix} \bar{\eta} \\ \bar{x} \end{pmatrix}$ . Proceeding as in the proof of Theorem (3.3.1), the equation of the error system can be written under the following compact form (compare with (3.78))

$$\dot{\bar{z}} = \theta \mathcal{F}(u) z + \tilde{\Phi}(u, \hat{x}, x) - \bar{B}_{r+n} (\theta^n v(t) + \theta \xi(t)) - B_{r+n} \varepsilon(t), \quad (3.112)$$

where  $\xi$  is as defined in (3.109) and the other variables keep the same meaning as in (3.78).

Let  $V(\bar{z}) = \bar{z}^T \mathcal{S} \bar{z}$  be the Lyapunov candidate function for the error system (3.112) where  $\mathcal{S}$  is the SPD matrix given as by (3.99). Using (3.99) and proceeding as in the proof of Theorem (3.4.1), one can show that (compare with (3.79))

$$\dot{V}(\bar{z}) \leq -\theta V(\bar{z}) + 2\sqrt{\lambda_{\mathcal{S}}} \sqrt{V(\bar{z})} (\theta |\xi(t)| + \theta^n v(t) + \varepsilon(t)). \quad (3.113)$$

using the bound of  $|\xi|$  provided by (3.111), inequality (3.113) leads to

$$\begin{aligned}\dot{V}(\bar{z}) &\leq -\theta V(\bar{z}) + 2\chi_{\mathcal{S}} \theta \sqrt{V(\bar{z})} (L_\varphi + \theta) \int_{t-d(t)}^t \sqrt{V(\bar{z}(s))} ds \\ &\quad + 2\sqrt{\lambda_{\mathcal{S}}} \sqrt{V(\bar{z})} (\theta^n v(t) + \varepsilon(t)) \\ &\leq -\theta V(\bar{z}) + 2\chi_{\mathcal{S}} \theta \sqrt{V(\bar{z})} (L_\varphi + \theta) \int_{t-d_M}^t \sqrt{V(\bar{z}(s))} ds \\ &\quad + 2\sqrt{\lambda_{\mathcal{S}}} \sqrt{V(\bar{z})} (\theta^n v(t) + \varepsilon(t))\end{aligned}\quad (3.114)$$

where  $\chi_{\mathcal{S}}$  is the conditioning number of  $\mathcal{S}$  and  $d_M$  is the upper bound of the delay.

Inequality (3.114) can be rewritten as follows

$$\begin{aligned} \frac{d}{dt} \sqrt{V(\bar{z})} &\leq -\theta \sqrt{V(\bar{z})} + \chi_{\mathcal{S}} \theta (L_{\varphi} + \theta) \int_{t-d_M}^t \sqrt{V(\bar{z}(s))} ds \\ &\quad + \sqrt{\lambda_{\mathcal{S}}} (\theta^n v(t) + \varepsilon(t)). \end{aligned} \quad (3.115)$$

Set

$$\alpha_{\theta} = \theta, \quad \beta_{\theta} = \chi_{\mathcal{S}} \theta (L_{\varphi} + \theta) \quad \text{and} \quad p_{\theta} = \sqrt{\lambda_{P_{r+n}}} (\theta^n v(t) + \varepsilon(t)). \quad (3.116)$$

According to Assumptions **A3**, **A4** and inequality (3.24), the function  $p_{\theta}(t)$  is essentially bound and the underlying essential bound is equal to  $\sqrt{\lambda_{P_{r+n}}} (\bar{\delta}_v + \delta_{\varepsilon})$  where  $\delta_{\delta_v}$  and  $\delta_{\varepsilon}$  are the essential bounds provided by Assumptions **A3**, **A4** and inequality (3.24). Now assume that

$$\frac{\beta_{\theta}}{\alpha_{\theta}} d_M < 1, \quad (3.117)$$

Since  $p_{\theta}(t)$  is essentially bounded, then conditions of lemma 3.4.1 and one gets

$$\sqrt{V(\bar{z}(t))} \leq \left( 1 + (\alpha_{\theta} - \eta_{\theta}(d_M)) d_M \right) e^{-\eta_{\theta}(d_M)(t-t_0)} \max_{s \in [t_0-d_M, t_0]} \sqrt{V(\bar{z}(s))} + \frac{\sqrt{\lambda_{\mathcal{S}}} (\bar{\delta}_v + \delta_{\varepsilon})}{\eta(d_M)} \delta_{\varepsilon},$$

where  $\eta_{\theta}(d_M) = (\alpha_{\theta} - \beta_{\theta} d_M) e^{-\alpha_{\theta} d_M} = \theta (1 - \chi_{\mathcal{S}} (L_{\varphi} + \theta) d_M)$ .

Coming back to  $\bar{z}$ , one gets

$$\|\bar{z}(t)\| \leq \chi_{\mathcal{S}} \left( \left( 1 + (\alpha_{\theta} - \eta_{\theta}(d_M)) d_M \right) e^{-\eta_{\theta}(d_M)(t-t_0)} \max_{s \in [t_0-d_M, t_0]} \|\bar{z}(s)\| + \frac{(\bar{\delta}_v + \delta_{\varepsilon})}{\eta(d_M)} \delta_{\varepsilon} \right),$$

Coming back to  $\eta$  and  $\tilde{x}$  by using (3.46), one gets (3.105). This ends the proof of Theorem 3.5.1.

### 3.5.4 Implementation of the FHGO for NUOS

Contrarily to the FHGO proposed for UOS the gain of which is constant, the gain FHGO (3.103) proposed for the class of NUOS is time-varying and it requires the resolution of the ODE (3.99) governing the matrix  $\mathcal{S}$ . The number of scalar ODE involved in (3.99) is equal to  $\frac{(n+r)(n+r)}{2}$ . Due to the particular structure of  $\mathcal{F}(u)$  and in particular to the fact that this matrix has as a bloc entry the constant matrix  $A_r$ , one shall show that the number of the scalar ODE can be reduced. This issue shall be detailed hereafter. Indeed, consider the following partition of the matrix  $\mathcal{S}(t)$

$$\mathcal{S} = \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix}, \quad (3.118)$$

where  $\mathcal{S}_1$ ,  $\mathcal{S}_2$  and  $\mathcal{S}_3$  are of dimensions  $r \times r$ ,  $r \times n$  and  $n \times n$ , respectively.

The ODE (3.99) can be written in the following expanded form

$$\begin{aligned}
\begin{bmatrix} \dot{S}_1 & \dot{S}_2 \\ \dot{S}_2^T & \dot{S}_3 \end{bmatrix} &= -\theta \left( \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} + \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} \begin{bmatrix} A_r & B_r C_n \\ 0_{n,r} & F_n \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} A_r^T & 0_{r,n} \\ C_n^T B_r^T & F_n^T \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} - \begin{bmatrix} C_r^T C_r & 0_{r,n} \\ 0_{n,r} & 0_{n,n} \end{bmatrix} \right) \\
&= -\theta \left( \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} + \begin{bmatrix} S_1 A_r & S_1 B_r C_n + S_2 F_n \\ S_2^T A_r & S_2^T B_r C_n + S_3 F_n \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} S_1 A_r & S_1 B_r C_n + S_2 F_n \\ S_2^T A_r & S_2^T B_r C_n + S_3 F_n \end{bmatrix}^T - \begin{bmatrix} C_r^T C_r & 0_{r,n} \\ 0_{n,r} & 0_{n,n} \end{bmatrix} \right) \\
&= -\theta \begin{bmatrix} S_1 + S_1 A_r + A_r^T S_1 - C_r^T C_r & S_2 + S_2 F_n + A_r^T S_2 + S_1 B_r C_n \\ (S_2 + S_2 F_n + A_r^T S_2 + S_1 B_r C_n)^T & S_3 + S_3 F_n + F_n^T S_3 + S_2^T B_r C_n + C_n^T B_r^T S_2 \end{bmatrix}.
\end{aligned}$$

Hence, one gets

$$\begin{cases} \dot{S}_1 &= -\theta (S_1 + S_1 A_r + A_r^T S_1) + C_r^T C_r \\ \dot{S}_2 &= -\theta (S_2 + S_2 F_n + A_r^T S_2 + S_1 B_r C_n) \\ \dot{S}_3 &= -\theta (S_3 + S_3 F_n + F_n^T S_3 + S_2^T B_r C_n + C_n^T B_r^T S_2) \end{cases} \quad (3.119)$$

The matrix  $S$  can be decomposed as follows

$$\mathcal{S} = \begin{bmatrix} I_r & -\Upsilon \\ 0_{n,r} & I_n \end{bmatrix}^T \begin{bmatrix} S_1 & 0_{r,n} \\ 0_{n,r} & X \end{bmatrix} \begin{bmatrix} I_r & -\Upsilon \\ 0_{n,r} & I_n \end{bmatrix}, \quad (3.120)$$

where

$$\Upsilon = -S_1^{-1} S_2 \text{ and } X = S_3 + S_2^T \Upsilon. \quad (3.121)$$

Let us derive the ODE governing the matrix  $X$ . To this end, let us first derive the one governing  $\Upsilon$ . Indeed, set  $G = S_1^{-1}$ . one has

$$\dot{G} = \theta \left( G + A_r G + G A_r^T - G C_r^T C_r G \right). \quad (3.122)$$

Hence, one gets

$$\dot{\Upsilon} = -\dot{G} S_2 - G \dot{S}_2 \quad (3.123)$$

$$\begin{aligned}
&= -\theta \left( (G + A_r G + G A_r^T - G C_r^T C_r G) S_2 + G (S_2 + S_2 F_n + A_r^T S_2 + S_1 B_r C_n) \right) \\
&= -\left( (A_r G - G C_r^T C_r G) S_2 + G S_2 F_n + B_r C_n \right) \\
&= \theta \left( (A_r - S_1^{-1} C_r^T C_r) \Upsilon - \Upsilon F_n + B_r C_n \right). \quad (3.124)
\end{aligned}$$

According to (3.121), one has

$$\begin{aligned}
\dot{X} &= \dot{S}_3 + \dot{S}_2^T \Upsilon + S_2^T \dot{\Upsilon} \\
&= -\theta \left( S_3 + S_3 F_n + F_n^T S_3 + S_2^T B_r C_n + C_n^T B_r^T S_2 \right) \\
&\quad -\theta \left( S_2^T + F_n^T S_2^T + S_2^T A_r + C_n^T B_r^T S_1 \right) \Upsilon \\
&\quad +\theta S_2^T \left( \left( A_r - S^{-1} C_r^T C_r \right) \Upsilon - \Upsilon F_n + B_r C_n \right) \\
&= \theta \left( - \left( S_3 + S_3 F_n + F_n^T S_3 \right) - \left( S_2^T + F_n^T S_2^T \right) \Upsilon + S_2^T \left( -S^{-1} C_r^T C_r \Upsilon - \Upsilon F_n \right) \right) \\
&= \theta \left( - \left( S_3 + S_2^T \Upsilon \right) - \left( S_3 + S_2^T \Upsilon \right) F_n - F_n^T \left( S_3 + S_2^T \Upsilon \right) + \Upsilon^T C_r^T C_r \Upsilon \right) \\
&= -\theta \left( X + X F_n + F_n^T X - \Upsilon^T C_r^T C_r \Upsilon \right) \tag{3.125}
\end{aligned}$$

Considering the decomposition of  $\mathcal{S}$  under the form (3.120), its inverse can be expressed as follows

$$\begin{aligned}
\mathcal{S}^{-1}(t) &= \begin{bmatrix} I_r & \Upsilon(t) \\ 0_{n,r} & I_n \end{bmatrix} \begin{bmatrix} S_1^{-1}(t) & 0_{r,n} \\ 0_{n,r} & P(t) \end{bmatrix} \begin{bmatrix} I_r & 0_{r,n} \\ \Upsilon^T(t) & I_n \end{bmatrix} \\
&= \begin{bmatrix} S_1^{-1}(t) + \Upsilon(t)P(t)\Upsilon^T(t) & \Upsilon P(t) \\ P(t)\Upsilon^T & P(t) \end{bmatrix} \text{ with } P(t) \triangleq X^{-1}(t), \tag{3.126}
\end{aligned}$$

and hence

$$\mathcal{S}^{-1}(t)\mathcal{C}^T = \begin{bmatrix} (S_1^{-1}(t) + \Upsilon(t)P(t)\Upsilon^T(t)) C_r^T \\ P(t)\Upsilon^T C_r^T \end{bmatrix} \tag{3.127}$$

Combining (3.102) and the above equation (3.127), one gets

$$K_r(t) = (S_1^{-1}(t) + \Upsilon(t)P(t)\Upsilon^T(t)) C_r^T \text{ and } K_n(t) = P(t)\Upsilon^T C_r^T. \tag{3.128}$$

Notice that the first component of  $K_n(t)$  denoted  $K_{r+1}$  can be expressed as follows

$$k_{r+1}(t) = C_n P(t) \Upsilon^T C_r^T \text{ with } C_n = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}. \tag{3.129}$$

Notice that the ODE governing  $S_1(t)$  admits a stationary solution  $S_{1,\infty}$  which satisfies the following algebraic equation

$$S_{1,\infty} + A_r^T S_{1,\infty} + S_{1,\infty} A_r = C_r^T C_r. \tag{3.130}$$

It is well known (see e.g [8, 84]) that  $S_{1,\infty}$  is SPD and one has

$$S_{1,\infty}^{-1} C_r^T = \begin{pmatrix} C_r^1 \\ C_r^2 \\ \vdots \\ C_r^r \end{pmatrix}. \tag{3.131}$$

According to the Lyapunov ODE (3.125) governing  $X$ , one can show that the matrix  $P = X^{-1}$  is governed by the following Riccati ODE

$$\dot{P}(t) = \theta \left( P(t) + F_n(u(t))P(t) + P(t)F_n^T(u(t)) - P(t)\Upsilon^T(t)C_r^T C_r \Upsilon(t)P(t) \right). \quad (3.132)$$

According to the above developments, the FHGO for NUOS (3.103)-(3.99) can be implemented under the following more appropriate form, from the numerical point of view,

$$\begin{cases} \dot{\eta}(t) &= \theta \left( A_r \eta(t) + B_r \left( C_n \hat{x}(t - d(t)) - y_d(t) + \alpha(t) - \alpha(t - d(t)) \right) \right. \\ &\quad \left. - \left( S_{1,\infty}^{-1} C_r^T + \Upsilon(t)P(t)\Upsilon^T(t)C_r^T \right) C_r \eta \right), \\ \dot{\hat{x}}(t) &= F_n(u(t))\hat{x}(t) + \varphi^s(u, \hat{x}) - \theta \Delta_n^{-1}(\theta) K_n(t) C_r \eta(t), \\ \dot{\alpha}(t) &= -\theta C_n P(t) \Upsilon^T C_r^T C_r \eta(t), \quad \alpha(s) = \psi(s), \\ \dot{\Upsilon} &= \theta \left( (A_r - S_1^{-1} C_r^T C_r) \Upsilon - \Upsilon F_n + B_r C_n \right), \\ \dot{P}(t) &= \theta \left( P(t) + F_n(u(t))P(t) + P(t)F_n^T(u(t)) - P(t)\Upsilon^T(t)C_r^T C_r \Upsilon(t)P(t) \right), \end{cases} \quad (3.133)$$

with the following initial conditions

$$\hat{x}(t_0) \in X, \quad P(t_0) = P(t_0)^T > 0, \quad \Upsilon(t_0) = 0, \quad \eta(s) = \alpha(s) = 0, \quad \forall s \in [t_0 - d_M, t_0].$$

### 3.5.5 Further extensions

As it has been mentioned previously, the structure of the class of NUOS (3.84) is very similar to that of the class of UOS given by (3.7). Indeed, to obtain the class of NUOS system (3.84) from the class of UOS (3.7), it suffices to substitute in (3.7) the constant anti-shift matrix  $A_n$  (the non zero entries of which are all equal to 1) by the time-varying anti-shift matrix  $F_n(u(t))$  the non zero entries of which are constituted by the  $f_i(u(t))$ ,  $i = 1, \dots, n$ . However and in order to ensure the observability of the resulting NUOS, the persistent excitation condition (3.86) is required.

The design of the FHGO (3.103) can be extended forwardly to the class of NUOS where the functions  $f_i(u(t))$  depends not only on the input  $u$  but also on the state  $x$  but in a triangular manner. Otherwise, said, each  $f_i$ ,  $i = 1, \dots, n - 1$  has the following structure

$$f_i(u(t), x(t)) \triangleq f_i(u(t), x_1(t), \dots, x_i(t)), \quad i = 1, \dots, n - 1. \quad (3.134)$$

The equations of the underlying FHGO are similar to those of observer (3.103) but each  $f_i(u)$  is substituted by  $f_i(u, \hat{x})$  where  $\hat{x}$  is the state estimate.

## 3.6 Academic example

One shall highlight the performance and main properties of the FHGO (3.103) through the same example considered in the previous chapter. For clarity purposes, the underlying

equations are recalled

$$\begin{cases} \dot{x}_1(t) &= f_1(t)x_2(t) - 0.1x_1(t), \\ \dot{x}_2(t) &= f_2(t) - 0.1x_2(t) + \tanh(x_1(t)x_2(t)), \\ \dot{x}_3(t) &= -0.1x_3(t) + \tanh(x_1(t)x_2(t)) + 5\sin(0.5\pi t) + \varepsilon(t) \\ y(t) &= x_1(t - d_1(t)) + v(t) \end{cases} \quad (3.135)$$

where  $f_1, f_2$  are expressed as  $f_1(t) = \sin(0.5t)$ ,  $f_2 = \sin(2.7t)$ . The expression of the unknown disturbance used in simulation is  $\varepsilon(t) = 0.1\sin(0.1t)$  and the noise signal affecting the output is chosen as the sum of two high frequency signals, namely

$$v(t) = 0.25 \sin(387t) + 0.5 \sin(987t). \quad (3.136)$$

A filtered high gain observer under the form of (3.103) has been designed and simulated by considering two scenarios. In the first ones, the outputs measurements have been assumed to be available continuously and are free from delay. In the second scenario, these measurements are assumed to be available at equally spaced sampling instants with a time-varying delay. In each case, the obtained estimates are compared with those obtained from a SHGO. These results are detailed hereafter.

### 3.6.1 Simulation results with delay free continuous output measurements

The output measurements issued from simulation are corrupted by the noise signal given by (3.136). The corresponding time evolution is given in Figure 3.1.

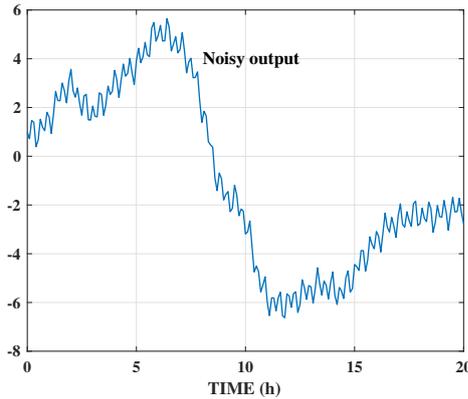


Figure 3.1: Noisy time-continuous output measurements

The estimates of the states  $x_i, i = 1, \dots, 3$  are given in Figure 3.2 where each of these estimates is compared to its true time evolution issued from the simulation of system

(3.135). Recall that the expression of the disturbance  $\varepsilon(t)$  is unknown by the observer. The value of  $\theta$  was set to 10. The initial state values of the system are set  $x_i = 1, \dots, 3$  and those of the observer to  $\eta = \hat{x} = 0$ .

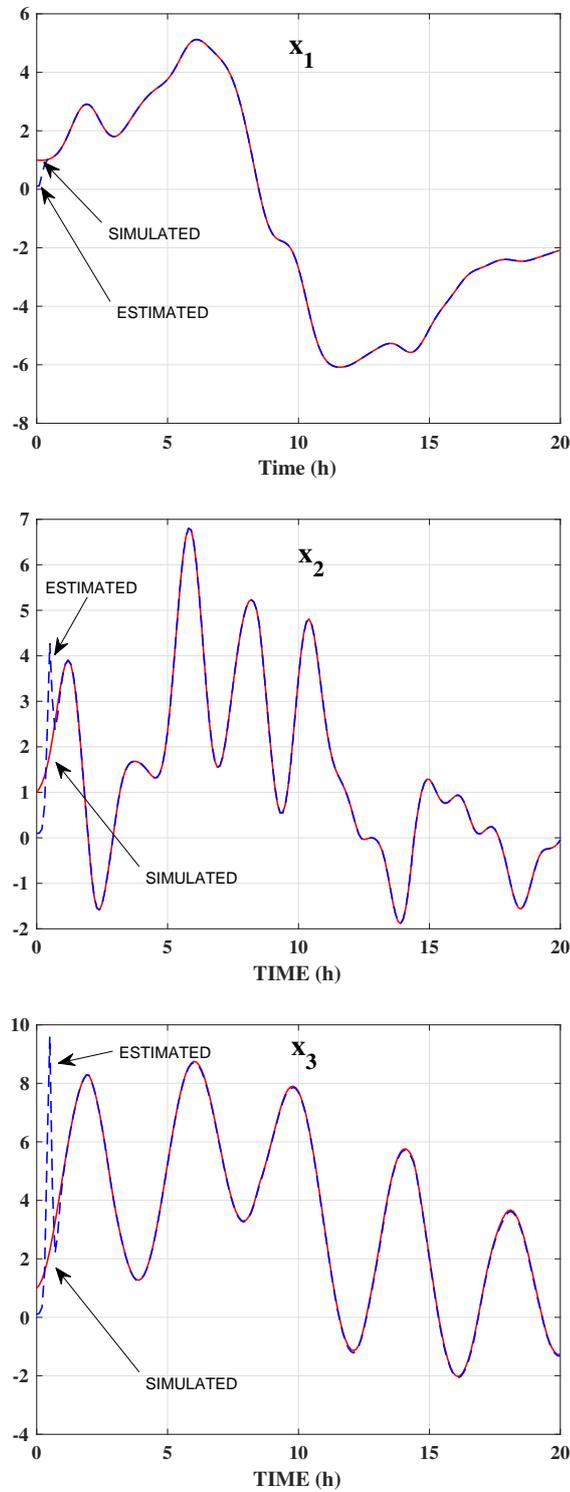


Figure 3.2: Estimates of  $x_i, i = 1, 2, 3$  provided by the FHGO

The obtained results are quite smooth and accurate enough. In order to highlight the improvement in term of sensitivity to noise of the observer, a SHGO of the form (3.20) has been designed to derive the state estimate. The obtained results are given in fig 3.3 where as for the FHGO, each estimate is compared to its true value. It is clear that the obtained estimates are relatively very noisy, in particular those related to  $x_2$  and  $x_3$  and this clearly confirms the significant improvement obtained with the FHGO.

### 3.6.2 Simulation results with sampled delayed output measurements

Two sets of simulation results will be presented. In each case the output measurements are assumed to be available at sampling equally spaced time instants with a sampling period equal to  $T_e$  with a time varying delay. Hence, by considering the sampling process as a particular case of a time-varying delay, the overall delay  $d(t)$  can be expressed as  $d(t) = (t - kT_e) + d_1(t)$ . The expression of the time-varying  $d_1(t)$  is

$$d_1(t) = d_M \cos(kT_e), k = 0, 1, \dots \quad (3.137)$$

where  $d_M > 0$  is a positive constant.

In the first set of simulation results, the sampling period  $T_e$  et the maximum value of the output delay,  $d_M$ , have been set to relatively small values, namely  $T_e = 0.01$  and  $d_M = 0.01$ . The value of the tuning parameter  $\theta$  was set to 0.5. The estimates provided by the FHGO are compared to their true time evolutions in Figure 3.4. The estimates provided by the SHGO are reported in Figure 3.5.

The obtained estimates, for the FHGO as well as for the SHGO, clearly shows that for small sampling periods and small amplitude for the output delay, the behaviours of the observers and in particular its sensitivity to the noise measurements are similar to the case where the output measurements are available continuously in time. In order to put forward the fact, that these properties still be inherited by the observers even for more high values of  $T_e$  and  $d_M$  others simulations experiments were carried out with  $T_e = 0.2$  and  $d_M = 0.1$ . The value of  $\theta$  was set to 2. The obtained estimates are given in Figure 3.6 for the FHGO and in Figure 3.7 for the SHGO. Again, the obtained results clearly confirm the the significance of the improvement obtained with respect the sensitivity to noise through the redesigned version given by the FHGO.

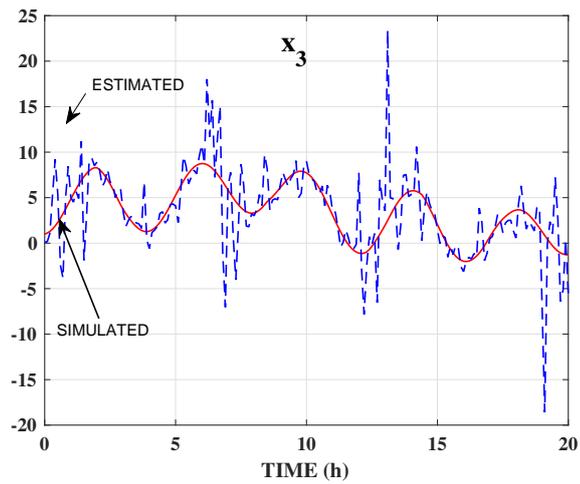
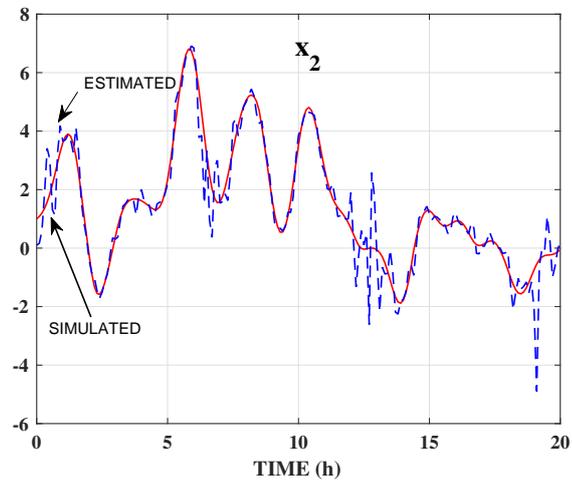
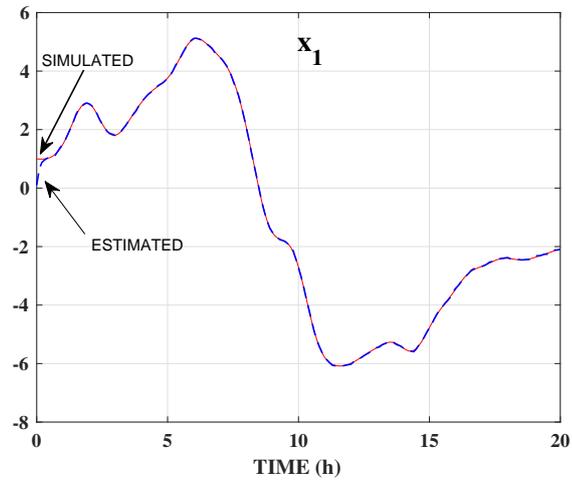


Figure 3.3: Estimates of  $x_i, i = 1, 2, 3$  provided by the FHGO

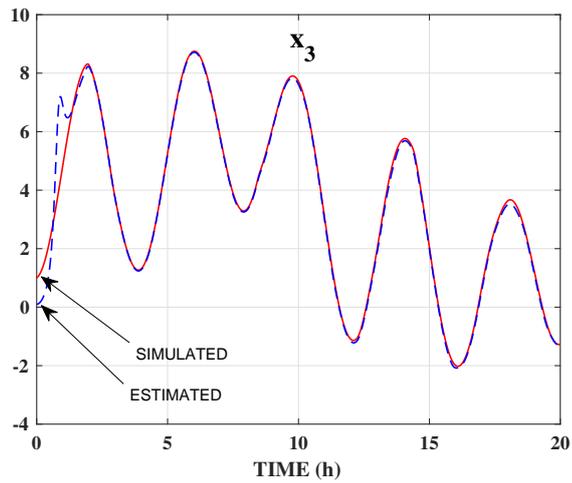
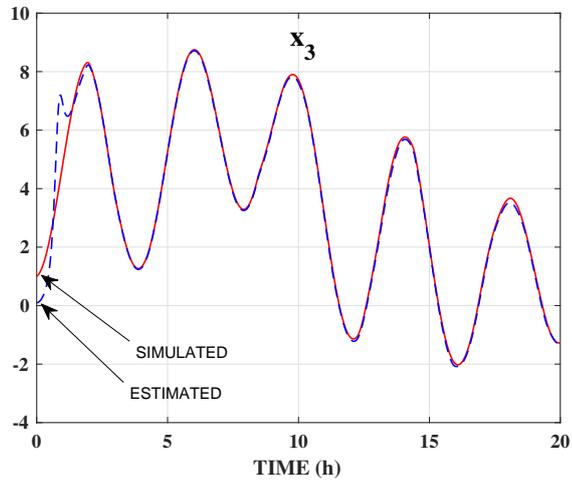
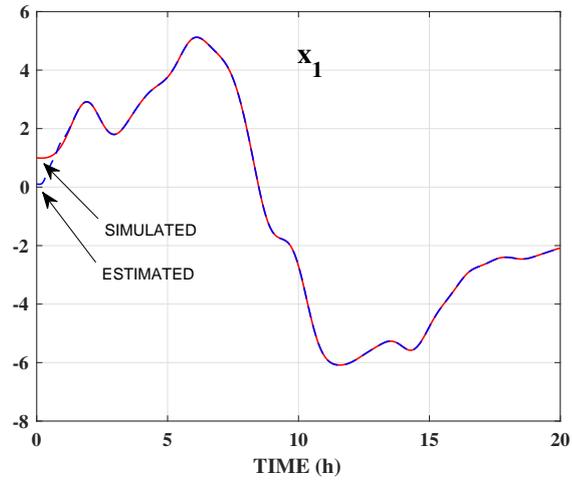


Figure 3.4: Estimates of  $x_i, i = 1, 2, 3$  provided by the FHGO with  $T_e = 0.01, d_M = 0.01, \theta = 5$

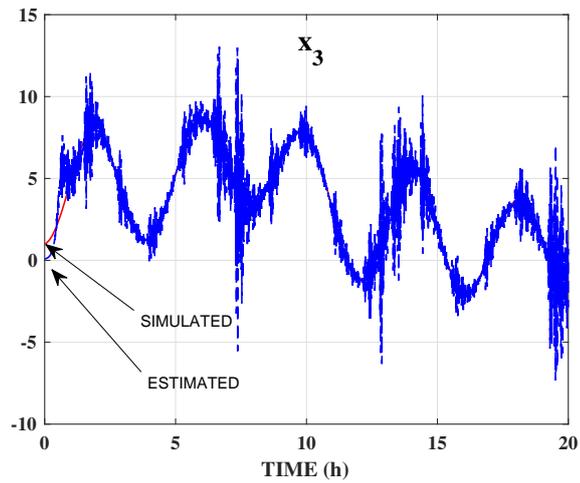
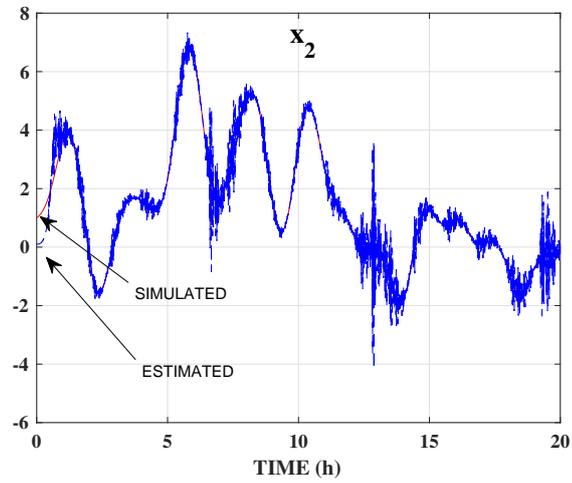
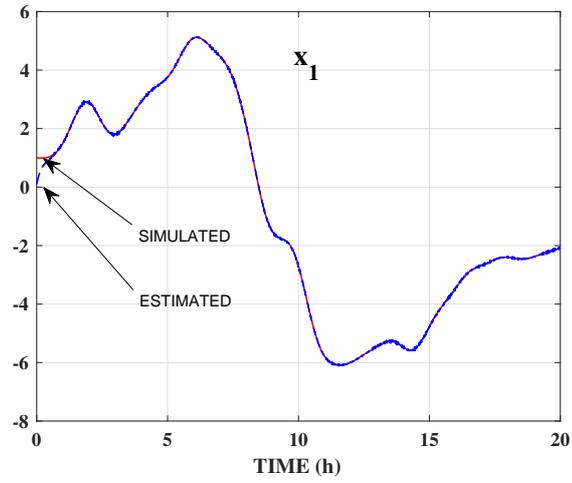


Figure 3.5: Estimates of  $x_i, i = 1, 2, 3$  provided by the SHGO with  $T_e = 0.01, d_M = 0.01, \theta = 5$

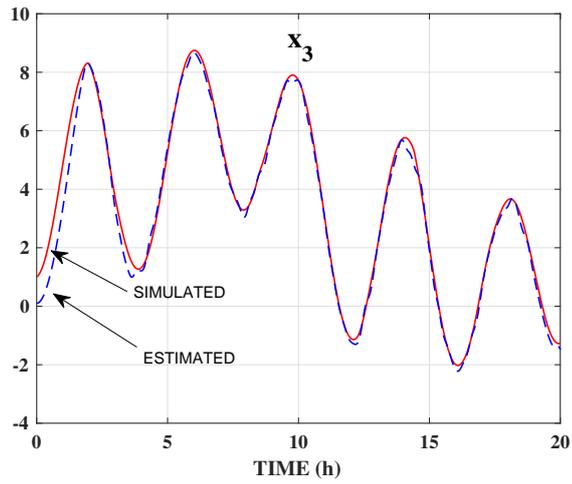
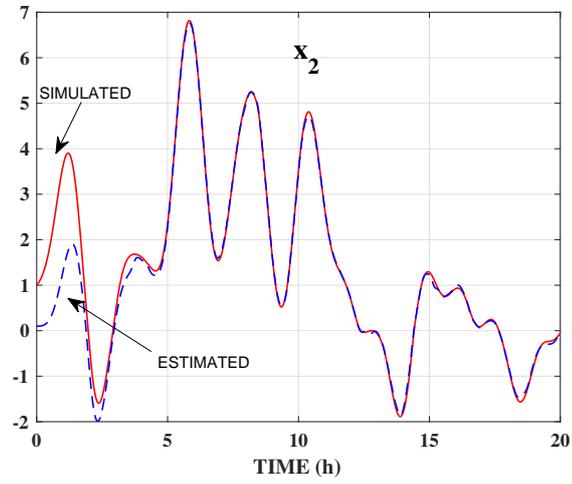
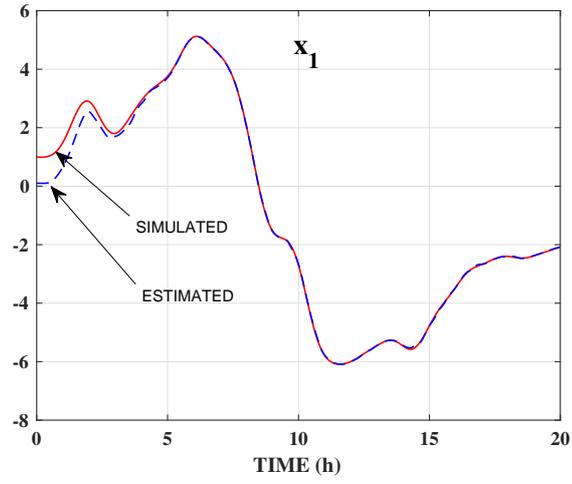


Figure 3.6: Estimates of  $x_i, i = 1, 2, 3$  provided by the FHGO with  $T_e = 0.2, d_M = 0.1, \theta = 2$

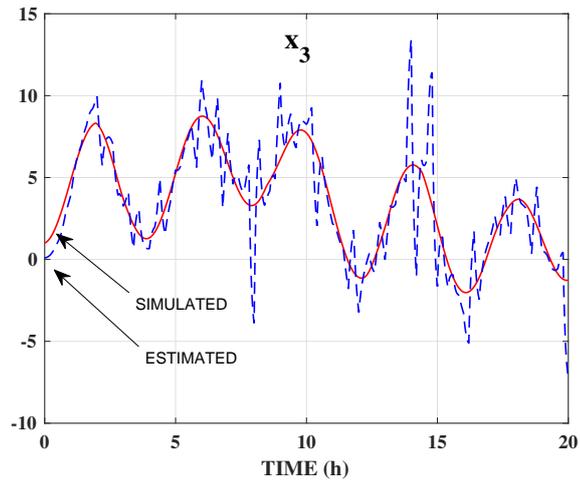
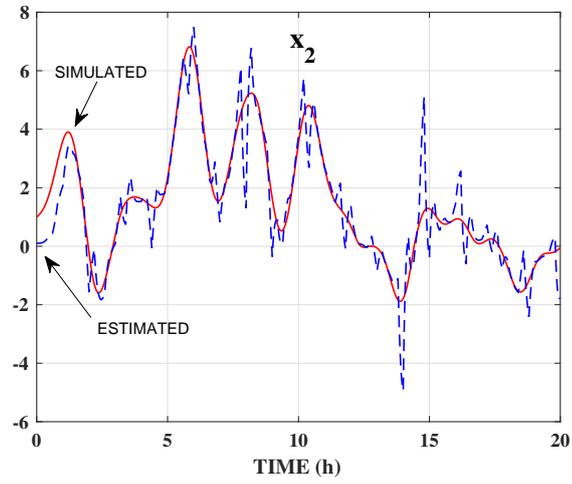
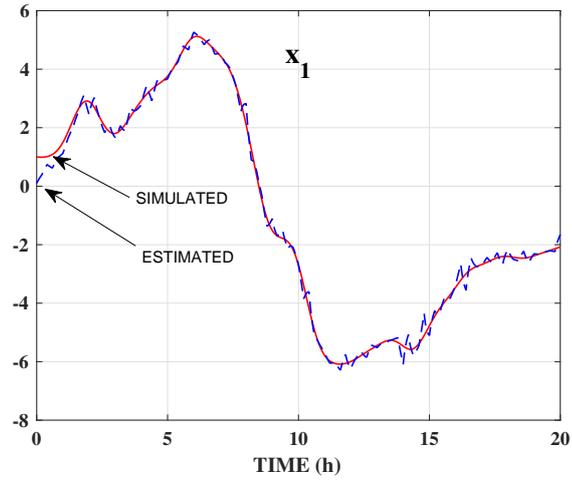


Figure 3.7: Estimates of  $x_i, i = 1, 2, 3$  provided by the SHGO with  $T_e = 0.2, d_M = 0.1, \theta = 2$

## 3.7 Application to a bioreactor

This section is devoted to the use of the proposed FHGO in a realistic simulation framework involving a typical bioreactor model. we consider a simple microbial culture involving a single biomass of concentration  $X$  denoted by  $x_1(t)$  growing on a single substrate of concentration  $S$ , denoted  $x_2(t)$ . The bioprocess is continuously operating with a dilution rate  $D(t)$  and a constant input substrate concentration  $s_{in}$  and the specific growth rate is assumed to follow the Contois model. The mathematical dynamical model of this process is hence constituted by the following two mass balance equations:

$$\begin{cases} \dot{x}_1(t) &= \mu^* x_1(t)x_2(t)/(k_c x_2(t) + x_2(t)) - D(t)x_1(t) \\ \dot{x}_2(t) &= -\mu^* x_1(t)x_2(t)/(k_c x_1(t) + x_2(t)) + D(t)(s_{in} - x_2(t)) \end{cases} \quad (3.138)$$

where  $\mu^*$  and  $k_c$  are the Contois law parameters. The biomass concentration,  $x_1$  is assumed to be measured at sampling instants with a constant sampling period and the objective is to estimate  $x_1(t)$  and  $x_2(t)$  continuously in time. System (3.138) has been already considered in [8] where the authors exhibited a compact set  $\Omega \subset \mathbb{R}^{++} \times \mathbb{R}^{++}$  which is positively invariant under the dynamics of (3.138). Moreover, it was shown that the transformation  $\Phi : (x_1, x_2)^T \in \Omega \longrightarrow z = (z_1 = x_1, z_2 = \mu^* x_1 x_2 / (k_c x_1 + x_2))^T$  is a diffeomorphism from  $\Omega$  onto its image which puts system (3.138) under the classical uniformly observable triangular form (3.7). Hence, a classical SHGO can be designed to estimate the states  $x_1$  and  $x_2$ . Since the output measurements are available under samples, one shall use the redesigned version (3.29) for comparison purposes. Indeed, one shall highlight in the sequel the performance of the proposed FHGO (3.67) through the bioreactor example by proceeding as follows

1. Two sets of simulations results shall be provided, each one corresponding to a value of the sampling period, namely  $T_e = 0.25h$  and  $T_e = 0.5h$ .
2. In each set, simulation results provided by the continuous-discrete time SHGO (3.29) are compared with their counterparts issued from the simulation of the FHGO (3.67). Moreover, three versions of the FHGO (3.67) are considered and they correspond to  $r = 1$ ,  $r = 2$  and  $r = 3$ .

### 3.7.1 Equations of the SHGO

Set

$$f(x_1, x_2) = \begin{pmatrix} \mu^* x_1(t)x_2(t)/(k_c x_2(t) + x_2(t)) - D(t)x_1(t) \\ -\mu^* x_1(t)x_2(t)/(k_c x_1(t) + x_2(t)) + D(t)(s_{in} - x_2(t)) \end{pmatrix}. \quad (3.139)$$

and let  $J$  be the jacobian of the transformation that puts system (3.138) under the uniformly observable canonical form, i.e.

$$J(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ \frac{\mu^* x_2^2}{(k_c x_1 + x_2)^2} & \frac{\mu^* k_c x_1^2}{(k_c x_1 + x_2)^2} \end{pmatrix}. \quad (3.140)$$

One also defines the following diagonal matrix  $\Delta_2$  as follows

$$\Delta_2(\theta) = \text{diag}([1 \ \theta]). \quad (3.141)$$

The equations of the SHGO (3.29) in the original coordinates specialize as follows

$$\begin{cases} \dot{\hat{x}}(t) &= f(\hat{x}_1, \hat{x}_2) - \theta_0 J^{-1}(\hat{x}_1, \hat{x}_2) \Delta_2^{-1}(\theta_0) \Gamma_2 \\ & (\hat{x}_1(t - d(t)) - x_1(t - d(t)) + \alpha_0(t) - \alpha_0(t - d(t))) \\ \dot{\alpha}_0(t) &= -\theta_0 \gamma_1 (\hat{x}_1(t - d(t)) - x_1(t - d(t)) + \alpha_0(t) - \alpha_0(t - d(t))) \end{cases} \quad (3.142)$$

where  $\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}$ ,  $\Delta_2(\theta_0)$  is as in (3.141) with  $\theta = \theta_0$ ,  $\Gamma_2 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$  and the delay  $d(t)$  specializes as follows

$$d(t) = t - kT_e, \quad (3.143)$$

where  $T_e$  is the sampling period.

### 3.7.2 Equations of the FHGO

As stated above, three FHGO observers shall be simulated and they differ by the value of  $r$  which is equal to 1 in the first observer, to 2 in the second one and to 3 in the last one.

#### FHGO with $r = 1$

The underlying filter is of order  $r = 1$  and the equations of the observer can be written as follows

$$\begin{cases} \dot{\eta}(t) &= \theta_1 (-K_{r_1} \eta + \hat{x}_1(t - d(t)) - x_1(t - d(t)) + \alpha_1(t) - \alpha_1(t - d(t))) \\ \dot{\hat{x}}(t) &= f(\hat{x}_1, \hat{x}_2) - \theta_1 J^{-1}(\hat{x}_1, \hat{x}_2) \Delta_2^{-1}(\theta_1) K^{(r_1)} \eta \\ \dot{\alpha}_1(t) &= -\theta_1 K_1^{(r_1)} \eta \end{cases} \quad (3.144)$$

where  $\eta \in \mathbb{R}$  is the state of the filter of order  $r = 1$ ,  $K_{r_1} \in \mathbb{R}$  and  $K^{(r_1)} = \begin{pmatrix} K_1^{(r_1)} \\ K_2^{(r_1)} \end{pmatrix} \in \mathbb{R}^2$  are the subvectors of  $K_3 = \begin{pmatrix} K_{r_1} \\ K^{(r_1)} \end{pmatrix}$  which is chosen such that the matrix  $\bar{A}_3 \triangleq A_3 - K_3 C_3$  is Hurwitz where  $A_3$  and  $C_3$  are as in (3.1) and (3.2), respectively with  $k = 3$ .

### FHGO with $r = 2$

The underlying filter is of order  $r = 2$  and the equations of the observer can be written as follows

$$\begin{cases} \dot{\eta}(t) &= \theta_2 \{ (A_2 - K_{r_2} C_2) \eta + B_2 (\hat{x}_1(t - d(t)) - x_1(t - d(t)) + \alpha_2(t) - \alpha_2(t - d(t))) \} \\ \hat{x}(t) &= f(\hat{x}_1, \hat{x}_2) - \theta_2 J^{-1}(\hat{x}_1, \hat{x}_2) \Delta_2^{-1}(\theta_2) K^{(r_2)} \eta_1 \\ \dot{\alpha}_2(t) &= -\theta_2 K_1^{(r_2)} \eta_1 \end{cases} \quad (3.145)$$

where  $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$  is the state of the filter of order  $r = 2$ ,  $K_{r_2} \in \mathbb{R}^2$  and  $K^{(r_2)} = \begin{pmatrix} K_1^{(r_2)} \\ K_2^{(r_2)} \end{pmatrix} \in \mathbb{R}^2$  are the subvectors of  $K_4 = \begin{pmatrix} K_{r_2} \\ K^{(r_2)} \end{pmatrix}$  which is chosen such that the matrix  $\bar{A}_4 \triangleq A_4 - K_4 C_4$  is Hurwitz where  $A_4$  and  $C_4$  are as in (3.1) and (3.2), respectively with  $k = 4$ , and finally  $A_2$  and  $B_2, C_2$  are as in (3.1) and (3.2), respectively with  $k = 2$ .

### FHGO with $r = 3$

The underlying filter is of order  $r = 3$  and the equations of the observer can be written as follows

$$\begin{cases} \dot{\eta}(t) &= \theta_3 \{ (A_3 - K_{r_3} C_3) \eta + B_3 (\hat{x}_1(t - d(t)) - x_1(t - d(t)) + \alpha_3(t) - \alpha_3(t - d(t))) \} \\ \hat{x}(t) &= f(\hat{x}_1, \hat{x}_2) - \theta_3 J^{-1}(\hat{x}_1, \hat{x}_2) \Delta_2^{-1}(\theta_3) K^{(r_3)} \eta_1 \\ \dot{\alpha}_3(t) &= -\theta_3 K_1^{(r_3)} \eta_1 \end{cases} \quad (3.146)$$

where  $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \in \mathbb{R}^3$  is the state of the filter of order  $r = 3$ ,  $K_{r_3} \in \mathbb{R}^3$  and  $K^{(r_3)} = \begin{pmatrix} K_1^{(r_3)} \\ K_2^{(r_3)} \end{pmatrix} \in \mathbb{R}^2$  are the subvectors of  $K_5 = \begin{pmatrix} K_{r_3} \\ K^{(r_3)} \end{pmatrix}$  which is chosen such that the matrix  $\bar{A}_5 \triangleq A_5 - K_5 C_5$  is Hurwitz where  $A_5$  and  $C_5$  are as in (3.1) and (3.2), respectively with  $k = 5$ , and finally  $A_3$  and  $B_3, C_3$  are as in (3.1) and (3.2), respectively with  $k = 3$ .

For each observer, SHGO and FHGO, its gain is computed such that the poles of the observers are located at  $(-1)$ . Hence, the following values of the gains have been used

1. For the SHGO (3.142),

$$\Gamma_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

2. For the FHGO (3.144) with  $r = 1$ ,

$$K_3 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}, \text{ i.e. } K_{r_1} = 3 \text{ and } K^{(r_1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

3. For the FHGO (3.145) with  $r = 2$ ,

$$K_4 = \begin{pmatrix} 4 \\ 6 \\ 4 \\ 1 \end{pmatrix}, \text{ i.e. } K_{r_2} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \text{ and } K^{(r_2)} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

3. For the FHGO (3.146) with  $r = 3$ ,

$$K_5 = \begin{pmatrix} 5 \\ 10 \\ 10 \\ 5 \\ 1 \end{pmatrix}, \text{ i.e. } K_{r_3} = \begin{pmatrix} 5 \\ 10 \\ 10 \end{pmatrix} \text{ and } K^{(r_3)} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

### 3.7.3 Simulation results

The simulation of model (3.138) and all the observers has been carried out by considering the following expression of the dilution rate  $D(t) = \mu^*(1 - \frac{1}{2}|\sin(t)|)$  and the following values of the kinetic parameters

$$\mu^* = 1.064 \text{ h}^{-1}, k_c = 4.39 \text{ g.g}^{-1}, \text{ Sin} = 20 \text{ g.L}^{-1}.$$

The initial conditions for model states and the initial conditions for all the sub-state observers  $\hat{x}$  are

$$x_1(0) = 1.5 \text{ g.L}^{-1}, x_2(1) = 15 \text{ g.L}^{-1}, \hat{x}_1(0) = 1.0 \text{ g.L}^{-1}, \hat{x}_2(0) = 14 \text{ g.L}^{-1}.$$

The initial values of all the filter states involved in all observers as well as the initial values of the variables  $\alpha_i(t), i = 0, \dots, 3$ , are set to zero.

The output measurements used by the observers have been first generated by simulating model (3.138). Then, they have been sampled and corrupted by an additive high frequency signal  $v(t)$  which is generated as follows

$$v(t) = 0.5 \left( \sin(2\pi 0.95 \frac{t}{2T_e}) + \sin(2\pi 0.97 \frac{t}{2T_e}) \right). \quad (3.147)$$

Before giving the set of simulation results, one recall that since the output measurements are available under the form of samples, the value of the observer design parameter  $\theta$  (denoted  $\theta_0$  for (3.142) and  $\theta_i$  for each FHGO with  $r = i$ ) cannot be chosen arbitrarily high as in the continuous output case but its value has to satisfy a condition similar to that given by (3.68) where the upper bound  $d_M$  involved in (3.68) corresponds in our case to  $T_e$ .

### Simulation results with $T_e = 0.25h$

The corrupted outputs measurements used by the observers are given in Fig. 3.8. One has reported in Fig. 3.9 the continuous estimation of the output provided by the SHGO (3.142). Notice that the value of  $\theta$  was set to 1 and higher values give rise to very noisy estimates.

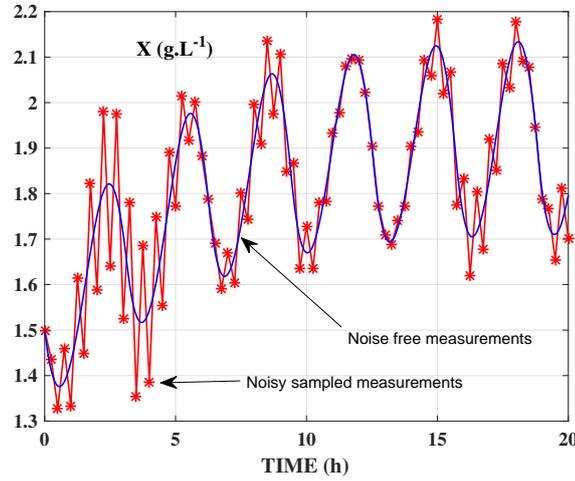


Figure 3.8: Noisy outputs samples with  $T_e = 0.25h$

The estimates of the biomass  $x_1$  provided by the filtered observers with  $r = 1, 2, 3$  are given in Fig 3.10 where each of these estimates is compared to the true sampled noise free values issued from simulation. It is clear that the sensitivity to noise of the observer is more and more improved when  $r$  grows. Notice that these estimates were obtained with  $\theta = 2$ . This explains why the estimates provided with  $r = 1$  is almost as noisy as those provided by the SHGO. However, when the latter was simulated with  $\theta = 2$ , the obtained estimates (not shown) were more noisy than those provided with  $r = 1$ .

Notice that the improvement of the sensitivity of the observer when changing the value of  $r$  from 2 to 3 is not easily perceptible on Figure 3.10. Such improvement can be more easily checked of Fig. 3.11 where the observation error related to  $x_1$  obtained with  $r = 2$  and  $r = 3$  are reported. Even though an improvement is obtained with  $r = 3$ ,

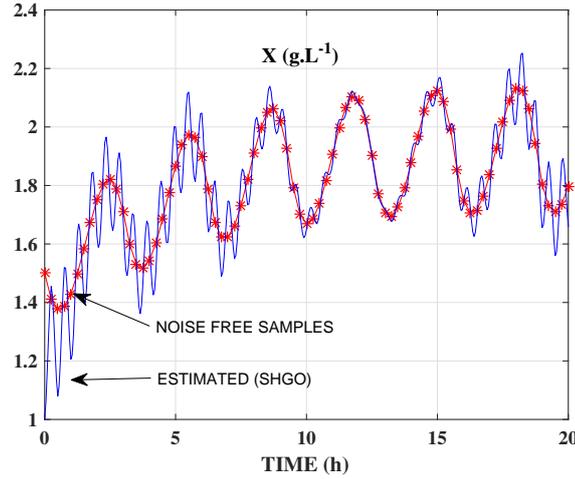


Figure 3.9: Estimates of  $x_1$  from SHGO  $\theta = 1$ ,  $T_e = 0.25h$

the estimates provided with  $r = 2$  are smooth enough and one can fix  $r = 2$  to gain in simplicity with respect to the observer implementation.

The estimate of  $x_2$  provided by the SHGO is reported in Fig 3.12 it is compared to their true unknown values issued from simulation. The relatively noisy estimates are to be emphasized.

One has reported in Fig 3.13 the estimates of  $x_2$  provided by the FHGO's with different values of  $r$ . Recall that the estimates obtained with  $r = 1$  seem more noisy than those provided by the SHGO by this is due to value of  $\theta$  which is set to 1 for the SHGO and to 2 in the FHGO. Notice that there is not a significant improvement with respect to the sensitivity to noise when changing the value of  $r$  from 2 to 3.

### Simulation results with $T_e = 0.5h$

In order to put forward more the improvement obtained with the proposed FHGO, the value of the sampling period has been increased by 100% and was set to  $0.5h$ . The underlying noisy sampled outputs are reported on Fig. 3.14.

One has reported in Fig 3.15 the continuous estimation of the output provided by the SHGO (3.142). The value of  $\theta$  was set to 1.

The estimates of the biomass  $x_1$  provided by the filtered observers with  $r = 1, 2, 3$  are given in Fig 3.16 where each of these estimates is compared to the true sampled noise free values issued from simulation. Notice that these estimates were obtained in this case with the same value of  $\theta$  used for the SHGO, i.e.  $\theta = 1$ . The improvement of the sensitivity to noise of the observer is clearly perceptible even with  $r = 1$ . One also notices that  $r_2$  and  $r = 3$  give raise to comparable results.

The estimate of  $x_2$  provided by the SHGO is reported in Fig 3.17 it is compared to their

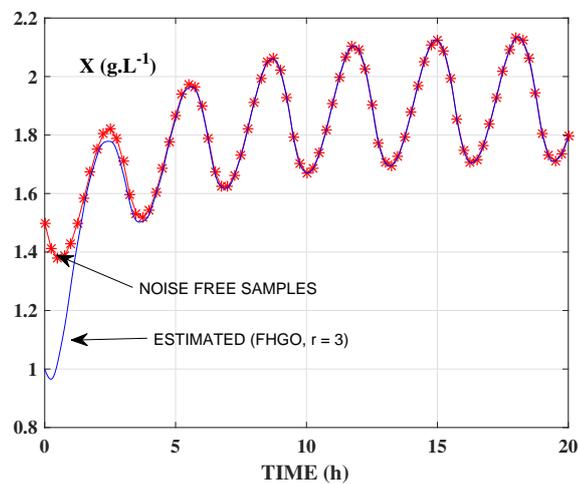
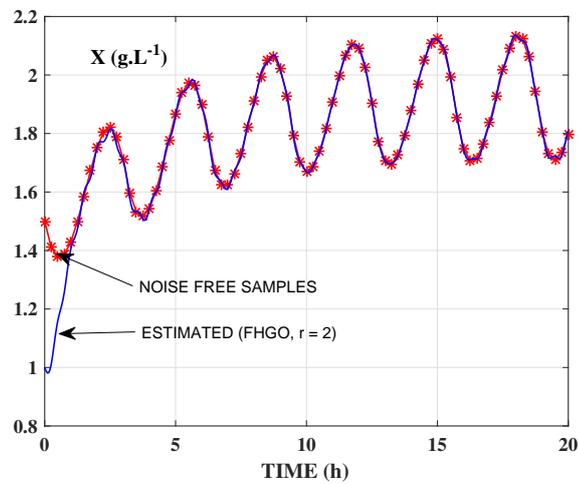
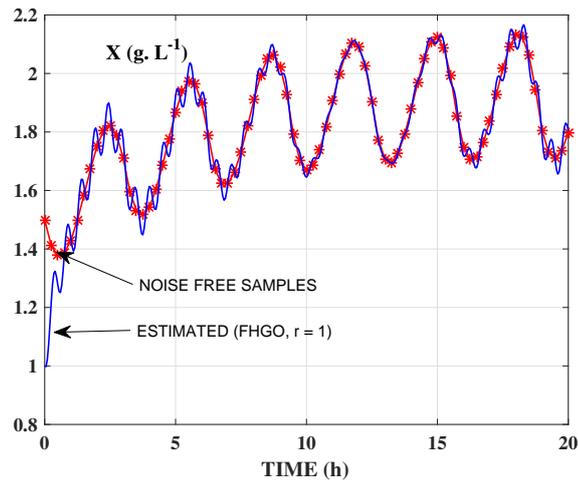


Figure 3.10: Estimates of  $x_1$  from the FHGO's  $\theta = 2$ ,  $T_e = 0.25h$

true unknown values issued from simulation. The relatively noisy estimates are to be emphasized.

One has reported in Fig 3.18 the estimates of  $x_2$  provided by the FHGO's with different values of  $r$ . Again, the improvement of the sensitivity to noise of the observer is clearly

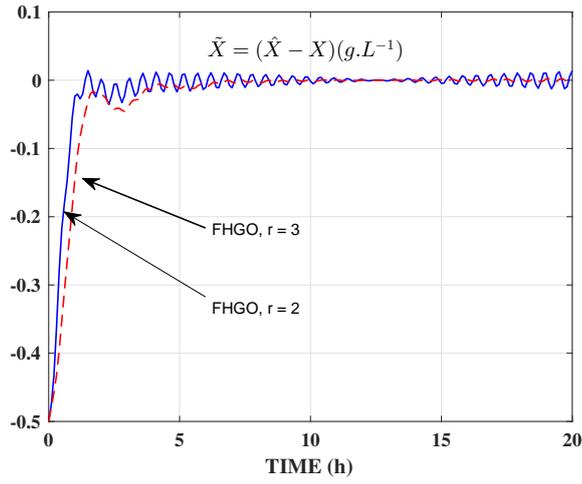


Figure 3.11: Observation error of  $x_1$  with  $r = 2$  and  $r = 3$   $\theta = 1$ ,  $T_e = 0.25h$

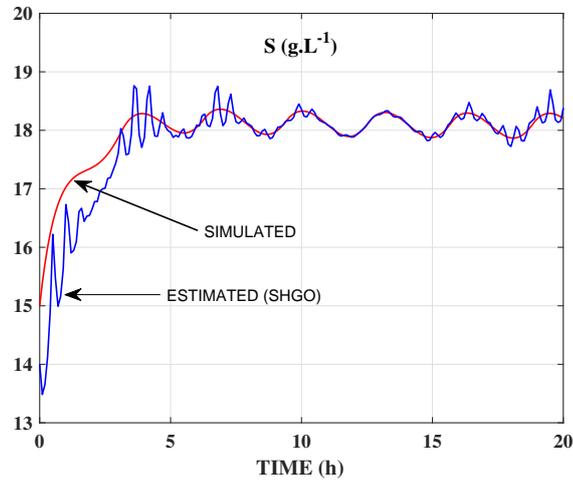


Figure 3.12: Estimates of  $x_2$  from SHGO  $\theta = 1$ ,  $T_e = 0.25h$

perceptible even with  $r = 1$ . Contrarily to  $x_1$ , the improvement with respect to the observer sensitivity is significant when changing the value of  $r$  from 2 to 3.

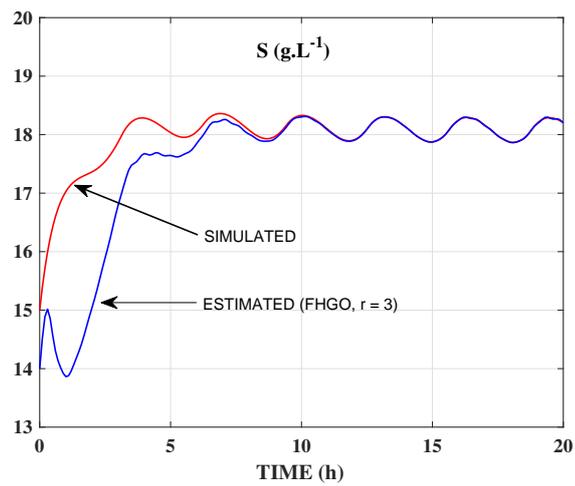
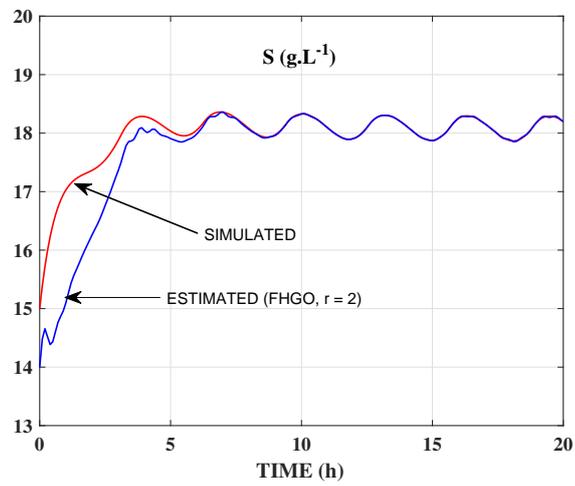
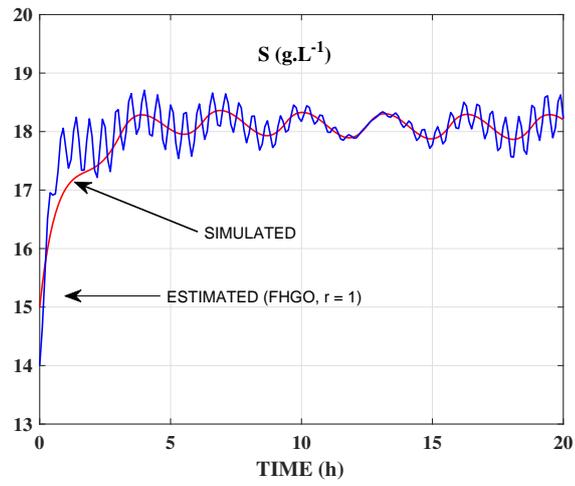


Figure 3.13: Estimates of  $x_2$  from the FHGO's  $\theta = 2$ ,  $T_e = 0.25h$

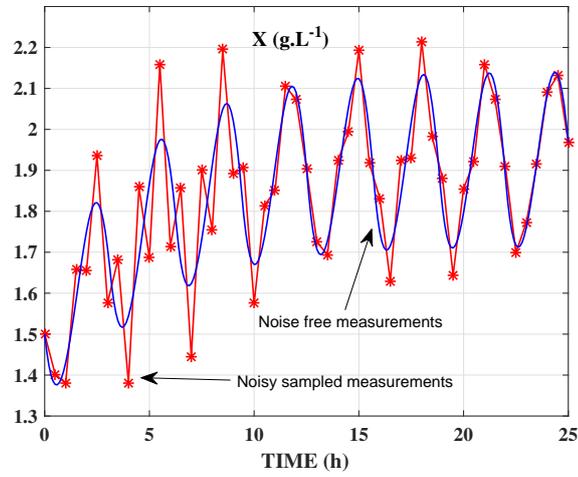


Figure 3.14: Noisy outputs samples with  $T_e = 0.5h$

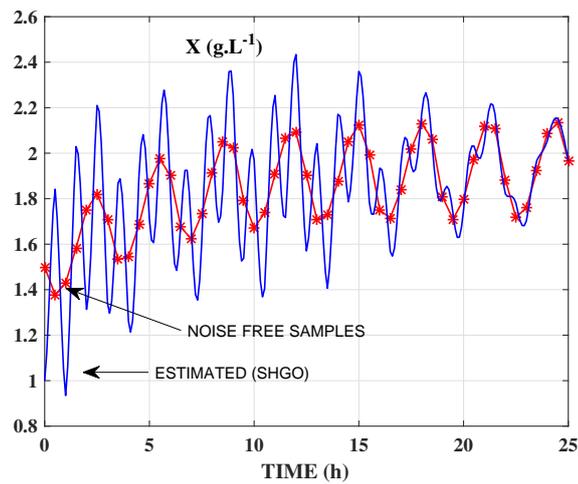


Figure 3.15: Estimates of  $x_1$  from SHGO  $\theta = 1$ ,  $T_e = 0.5h$

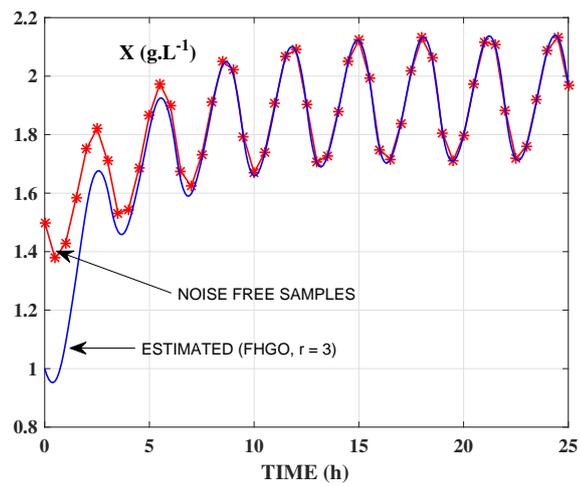
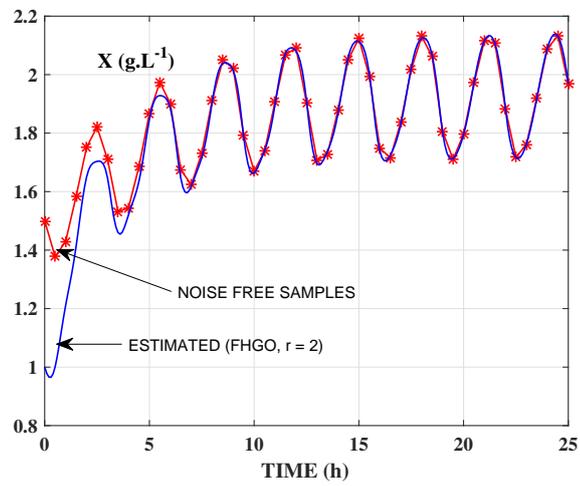
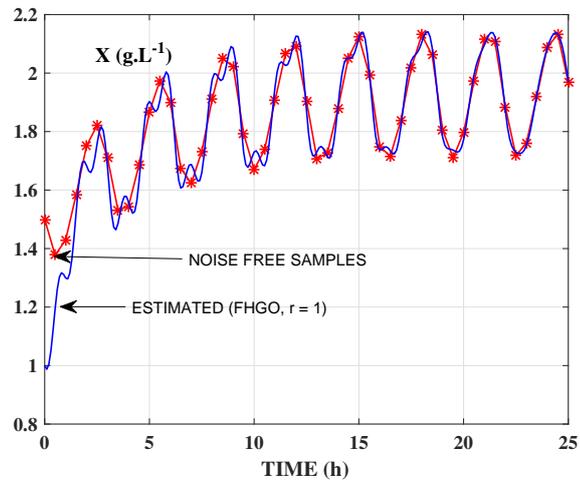


Figure 3.16: Estimates of  $x_1$  from the FHGO's  $\theta = 1$ ,  $T_e = 0.5h$

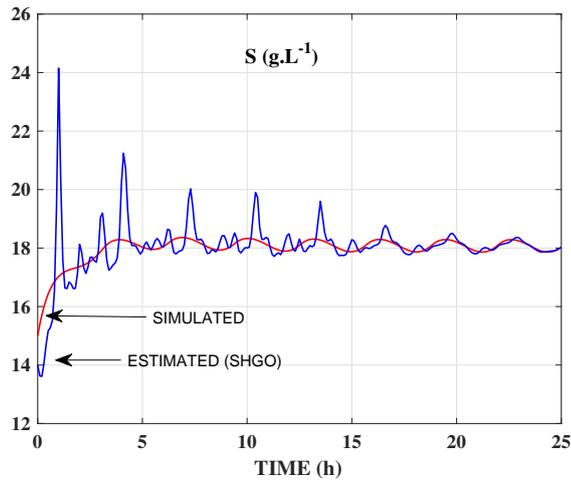


Figure 3.17: Estimates of  $x_2$  from SHGO  $\theta = 1$ ,  $T_e = 0.5h$

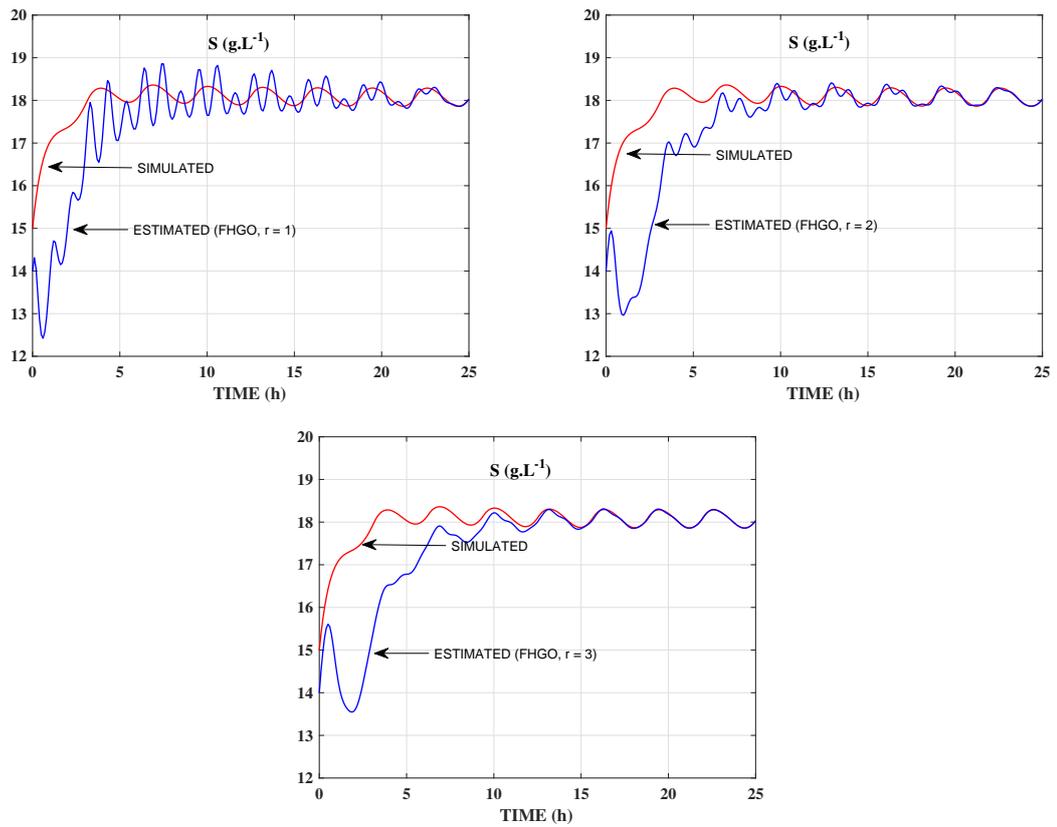


Figure 3.18: Estimates of  $x_2$  from the FHGO's  $\theta = 1$ ,  $T_e = 0.5h$

## 3.8 Conclusion

The main motivation of this chapter was to investigate one of the challenging issues of the high gain observer design, namely the measurement noise sensitivity, using a suitable filtering process. More specifically, a FHGO has been first proposed for a class of uniformly observable nonlinear systems bearing in mind the implementation simplicity and the convergence requirements. The former requirement has been achieved throughout an appropriate cascade of two subsystems of the same dimension as the considered system. The first subsystem is nothing else than a copy of the original system with a corrective term which linearly depends on the state of the second subsystem. The latter is an appropriate filter of the output observation error. The second requirement has been fulfilled using a suitable high gain observer design framework. The proposed observer was first presented in the case where the outputs measurements are available in a continuous way. Then, its design has been extended to account for the sampling process and the presence of delays in the outputs was derived. It has been shown that the redesigned version inherits the main properties of the original filtered observer working with continuous outputs. Further extensions of the design have been achieved by considering a class of non uniformly observable systems. The underlying extensions accounts for the continuous outputs as well as sampled/delayed ones. The effectiveness of the proposed observer has been illustrated through an academic example as well as an example dealing with a typical bioprocess. Further studies are under consideration in the proposed FHGO design framework, namely how to redesign the proposed FHGO in the presence of appropriate saturation functions to tackle the observer picking phenomenon.

# Chapter 4

## Conclusion and perspectives

The main contributions in this thesis deal with the redesign of the Standard High Gain Observer (SHGO) in order to derive redesigned versions where the sensitivity of the observer with respect to the output noise measurements is improved. Two redesigned observers involving filtering capabilities have been proposed. The first observer has been proposed for a class of non uniformly observable systems and it can be interpreted as an extension of an existing filtered observer that has been proposed for a class of uniformly observable systems. The main idea behind the observer design is to substitute in the SHGO corrective term, the noisy output observation error by filtered versions which are the states of a linear filter the entry of which is the noisy output observation error. The dimension of the linear filter is equal to the dimension of the original system in the single output case and the equations of the filter involve a new design parameter that has to be calibrated by a trial/error approach. The filtered observer design was first carried out in the case where the outputs measurements are available continuously in time before being extended to the case where these measurements are available only at sampling instants. Though this redesigned observer allowed a significant improvement of the sensitivity to noise of the standard high gain observer, its calibration requires the tuning of an additional design parameter. Moreover, the proposed design does not account for the case where the outputs are available with a delay. In order to overcome these problems, a second redesigned observer has been proposed. Three main novelties in the second design with respect to the first one are to be emphasized. Indeed, as for the first filtered observer, the idea behind the redesign is to substitute in the SHGO corrective term, the noisy output observation error, by a filtered version issued from the use of a linear filter. However, now, the dimension of the filter is not fixed as for the first redesigned observer but it can be arbitrarily chosen by the user. The second novelty is that the proposed redesign does not need the introduction of any additional design parameter, i.e. the tuning of the resulting observer shall be achieved through the choice of the parameter  $\theta$  as for the

SHGO. Finally, the third novelty lies in the fact the proposed redesign accounts for all possible scenarios related to the availability of the output measurements: continuously in time or under the form of samples. Moreover, in both cases, these measurements may be available with an output delay. The observer design has been first carried out for a class of uniformly observable systems. Then, it has been shown that this design can be extended almost in a straightforward manner to a class of non uniformly observable systems up to the adoption of an additional appropriate persistent excitation condition on the system inputs.

The performance and main properties of the proposed two redesigned observers have been highlighted through academic examples as well as through typical bioreactors models.

On other aspects, it has been shown that the proposed redesigned observers allowed to get a significant improvement of the observers sensitivity with respect to the output noise measurements since the underlying redesigned observers involve filtering capabilities. However, these observers still inherit from the SHGO the peaking phenomenon that occurs during the transient periods. We are now working on the redesign of the second observer in order to derive a version that appropriately deal with the peaking phenomenon. Such a redesign shall be based on the use of nested saturation functions of the filtered output observations errors as suggested in [30].

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## Abstract

In this thesis, the redesign of the Standard High Gain Observer is considered to improve its sensitivity to high frequency signals. In this context, two redesigned Filtered High Gain Observers, referred to as FHGO I and FHGO II, have been proposed. In particular, the design of the FHGO II accounts for all possible scenarios related to the availability of output measurements, namely continuously in time, under samples, with delays and with possible successions in time of all these cases. Moreover, this design has been first achieved for a class of uniformly observable systems before being extended to a class of non uniformly observable ones. The underlying convergence analysis is performed thanks to a comprehensive Lyapunov approach under a well-defined condition on the maximum value of the output delay together with the maximum sampling partition diameter in the case of sampled outputs. The effectiveness of the proposed FHGO's are highlighted through simulation results dealing with academic examples and typical bioreactor models.

## Résumé

Dans cette thèse, la resynthèse de l'observateur de type grand gain standard est considérée pour améliorer la sensibilité de l'observateur aux signaux à hautes fréquences. Dans ce contexte, deux observateurs à grand gain filtrés, auxquels on se réfère par FHGO I et FHGO II, ont été présentés. En particulier, la conception du FHGO II tient compte de tous les scénarios possibles liés à la disponibilité des mesures de la sortie, notamment continues dans le temps, sous forme d'échantillons, avec des retards et aussi avec des successions possibles dans le temps de tous ces cas. En outre, cette conception a d'abord été réalisée pour une classe de systèmes uniformément observables avant d'être étendue à une classe de systèmes non uniformément observables. L'analyse de convergence sous-jacente est réalisée grâce à une approche de Lyapunov sous des conditions bien définies sur la valeur maximale du retard de sortie ainsi que la valeur maximale de la période d'échantillonnage dans le cas des sorties échantillonnées. Les performances des observateurs proposés ont été illustrées à travers des résultats de simulation portant sur des exemples académiques et des modèles typiques de bioréacteurs.