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# Influence des conditions de Robin sur le spectre du Laplacien magnétique

Rayan Fahs

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# THÈSE DE DOCTORAT

DE  
L'UNIVERSITÉ D'ANGERS  
SOUS LE SCEAU DE  
LA COMUE ANGERS-LE MANS

ÉCOLE DOCTORALE N° 641  
*Mathématiques et Sciences et Technologies  
de l'Information et de la Communication*  
Spécialité : *Mathématiques et leurs interactions*

Par

**Rayan FAHS**

## **Influence des conditions de Robin sur le spectre du Laplacien magnétique**

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## **Influence des conditions de Robin sur le spectre du Laplacien magnétique**

**Résumé :** Cette thèse concerne l'étude spectrale dans le régime semi-classique de l'opérateur de Schrödinger en présence d'un champ magnétique sur un domaine borné et régulier avec condition de Robin au bord. L'objectif est de décrire l'influence des conditions de Robin sur le spectre magnétique lorsque le paramètre semi-classique tend vers zéro et lorsque le champ magnétique joue un rôle majeur en concurrence avec la condition de Robin. La description précise des éléments propres passe par la compréhension de différents opérateurs modèles.

Dans la première partie, lorsque le champ magnétique est d'intensité modérée et dans la limite du couplage fort, on montre que les valeurs propres du Laplacien de Robin sont bien approchées par celles d'un opérateur de Schrödinger périodique 1D sur la frontière. Lorsque la courbure a un unique maximum non dégénéré, on estime l'écart spectral entre les valeurs propres successives. Dans le cas du disque, on révèle explicitement la contribution du champ magnétique sur la première valeur propre.

Dans la deuxième partie, on étudie l'effet de Stark en toutes les dimensions. On considère l'opérateur de Laplace avec un champ électrique fort et uniforme et des conditions de Dirichlet sur le bord d'un domaine borné et régulier. Dans la limite du champ électrique fort, on dérive, sous certaines conditions de convexité locale, une description précise des valeurs propres et des fonctions propres associées.

Dans la troisième partie, lorsque le champ magnétique est plus intense, et dans la limite semi-classique, en utilisant la méthode de Grushin et les outils d'analyse microlocale, on établit une description uniforme du spectre du Laplacien magnétique de Robin situé entre les niveaux de Landau, incluant les Laplaciens de Neumann et Dirichlet. De plus, on examine les propriétés de localisation des états fondamentaux. Les estimations ressemblent à celles d'Agmon, mais elles ne sont pas obtenues par des moyens variationnels comme c'est le cas dans de nombreux contextes magnétiques.

**Mots-clés :** Théorie spectrale, Analyse microlocale, Limite semiclassique, Laplacien magnétique, Condition de Robin, Estimation d'Agmon.



## **Influence of Robin's conditions on the spectrum of the magnetic Laplacian**

**Abstract :** This thesis concerns the spectral study in the semiclassical regime of the Schrödinger operator in the presence of a magnetic field on a bounded and regular domain with the Robin condition at the boundary. The aim is to describe the influence of the Robin conditions on the magnetic spectrum when the semiclassical parameter tends to zero and when the magnetic field plays an important role in competition with the Robin condition. The precise description of the eigenpairs requires an understanding of effective operators.

In the first part, when the magnetic field is of moderate intensity, we show that the eigenvalues of the Robin Laplacian are well approximated by those of a 1D periodic Schrödinger operator on the boundary. When the curvature has a unique maximum that is not degenerate, we estimate the spectral gap between successive eigenvalues. In the case of the disk, we explicitly display the contribution of the magnetic field to the first eigenvalue.

In the second part, we study the Stark effect in any dimension. We consider the Laplace operator with a strong and uniform electric field and Dirichlet conditions at the boundary of a bounded and regular domain. In the limit of the strong electric field, under certain local convexity conditions, we derive an asymptotic expansion of the low-lying eigenvalues at any order and also study the localization of the ground states.

In the third part, when the magnetic field is stronger, and in the semiclassical limit, we establish a uniform description of the spectrum between the Landau levels, of the magnetic Laplacians including the Neumann and Dirichlet Laplacians, using the Grushin method and microlocal analysis tools. We also investigate the localization properties of the ground states. The estimates look like Agmon's estimates, but they are not obtained via variational methods, as is the case in many magnetic settings.

**Keywords :** Spectral theory, Microlocal analysis, Semiclassical limit, Magnetic Laplacian, Robin condition, Agmon estimates.



“ L'apprentissage est la seule chose que l'esprit n'épuise jamais, ne craint jamais et ne regrette jamais. ”

*Léonard de Vinci*

“ Nous devons savoir, et nous saurons. ”

*David Hilbert*

*À mes parents,*

*À ma famille,*

*À Mahdi,*





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# Introduction générale

## 1.1 Motivation physique et présentation générale

### 1.1.1 Première motivation : théorie de la supraconductivité

Une première motivation qui peut conduire à l'analyse spectrale du Laplacien magnétique découle de la théorie mathématique de la supraconductivité. L'analyse des problèmes mathématiques liés à la théorie de la supraconductivité a été intensivement développée au cours des 30 dernières années (voir [FH10b]). Le comportement macroscopique des matériaux supraconducteurs placés dans un ouvert  $\Omega$  est décrit par les minimiseurs  $(\psi, A_0)$  dans  $H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$  de la fonctionnelle de Ginzburg-Landau  $\mathcal{G}$  (cf. [dG66, BS98, BPT98]) :

$$\begin{aligned} \mathcal{G}(\psi, A_0) = & \int_{\Omega} \left\{ |(\nabla - i\kappa b A_0)\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right\} dx + \kappa^2 b^2 \int_{\mathbb{R}^2} |\operatorname{curl} A_0 - 1|^2 dx \\ & + \int_{\partial\Omega} \beta |\psi(x)|^2 ds(x). \end{aligned} \quad (1.1)$$

Ici  $\Omega \subset \mathbb{R}^2$  est un ouvert borné régulier,  $\psi$  est appelée le paramètre d'ordre,  $|\psi|^2$  est la densité des paires de Cooper,  $A_0$  est appelé le potentiel magnétique et  $\operatorname{curl} A_0 = \nabla \times A_0$  est le champ magnétique induit dans un échantillon supraconducteur  $\Omega$ . Le paramètre  $b > 0$  est l'intensité du champ magnétique appliqué à l'échantillon. Le paramètre  $\kappa > 0$  est une caractéristique du matériau et il est appelé le paramètre de Ginzburg-Landau. Dans la littérature physique, on fait généralement la distinction entre les matériaux de type I et les matériaux de type II. Le matériau est dit de type I si  $\kappa$  est suffisamment petit et il est dit de type II, lorsque  $\kappa$  est grand. C'est ce dernier cas que nous considérons ici.

Les minimiseurs de la fonctionnelle de Ginzburg-Landau  $\mathcal{G}$  décrivent les propriétés du matériau lorsqu'il est soumis au champ magnétique externe. On distingue traditionnellement trois types de minimiseurs possibles. Nous disons qu'un minimiseur  $(\psi, A)$  est supraconducteur si  $\psi$  ne s'annule jamais, normal si  $\psi$  est identiquement 0, et mixte si  $\psi$  a des zéros, mais ne s'annule pas identiquement. La fonctionnelle  $\mathcal{G}$  a un point critique du type  $(0, A)$  "un état normal", avec  $\operatorname{curl} A = 1$ , qui satisfait le système de Ginzburg-Landau (cf. [GP99]). Il est naturel d'analyser quand cette solution normale est un minimum local de la fonctionnelle. Cela conduit naturellement à la question de la positivité du Hessien de  $\mathcal{G}$ , qui conduit à examiner la dérivée seconde en  $\psi$  de  $\mathcal{G}$  (voir par exemple [FH10b, LP99, LP00, dPFS00, HP03]).

La positivité du Hessien est immédiatement liée à la positivité de l'opérateur

$$(\nabla - i\kappa b A_0)^2 - \kappa^2,$$

avec la condition au bord suivante dite de de Gennes <sup>1</sup> (cf. [dG66, Chapitre 3]) :

$$\nu \cdot (\nabla - i\kappa b A_0)\psi + \beta\psi = 0, \quad \text{sur } \partial\Omega \quad (1.2)$$

où  $\nu$  est le vecteur normal extérieur de  $\partial\Omega$  et  $\beta \in \mathbb{R}$  est appelée dans la littérature physique le paramètre de de Gennes <sup>2</sup>. Par suite, cela amène à analyser l'état fondamental de l'opérateur de Schrödinger avec champ magnétique

$$(\nabla - i\kappa b A_0)^2,$$

avec la condition de Robin (1.2) au bord.

Lorsque  $\kappa$  est grand, cela devient un problème semi-classique <sup>3</sup> après le changement de paramètre  $h = \frac{1}{\kappa b}$  et le problème ramène ainsi à l'étude spectrale du Laplacien magnétique de Robin

$$\mathcal{N}_{h,\beta} = -(h\nabla - iA_0)^2, \quad (1.3)$$

défini sur

$$\text{Dom}(\mathcal{N}_{h,\beta}) = \{ u \in H^2(\Omega) : \nu \cdot (h\nabla - iA_0)u + \beta hu = 0 \text{ sur } \partial\Omega \}, \quad (1.4)$$

lorsque le paramètre  $h$  tend vers 0. Le régime semi-classique  $h \rightarrow 0$  est maintenant équivalent au  $\kappa \rightarrow +\infty$  et/ou  $b \rightarrow +\infty$ . Nous souhaitons établir des asymptotiques spectrales précises pour le Laplacien magnétique de Robin  $\mathcal{N}_{h,\beta}$  dans les régimes où le champ magnétique joue un rôle majeur, en concurrence avec la condition de Robin (pour  $\beta = \gamma h^{\frac{-1}{2}}$ , avec  $\gamma$  une constante réelle).

Lorsque  $\kappa = 1$ , l'étude du phénomène de la supraconductivité ramène à l'analyse spectrale du Laplacien magnétique

$$-(\nabla - ibA_0)^2, \quad (1.5)$$

avec les conditions au bord de type Robin, comme (1.2). Une partie du travail présenté ici a pour objet l'étude spectrale de cet opérateur dans le régime de couplage fort ( $\beta \rightarrow \infty$ ) et  $b \rightarrow +\infty$  simultanément. L'analyse spectrale de ces opérateurs fait l'objet des articles [Fah23a, FLTRVuN23].

## 1.1.2 Une autre motivation

Vers la fin du XIX<sup>ème</sup> siècle, on pensait que l'évolution des systèmes physiques pouvait être décrite dans le cadre de la mécanique classique. Cependant, certaines expériences physiques ont donné des résultats qui ne peuvent pas être expliqués par la mécanique classique et qui semblaient même contradictoires entre eux. Les résultats de l'expérience de Bragg (*i.e.* la réflexion de Bragg et la diffraction du rayonnement), la dualité onde-corpuscule de la lumière et la radioactivité du corps noir de Gustav Kirchoff (cf. [Jac07]) sont des phénomènes pour lesquels la mécanique classique ne présente pas d'explications suffisantes. Sur cette base, une nouvelle mécanique a été créée : la mécanique quantique <sup>4</sup>. Le lecteur ou la lectrice intéressée par la construction de la théorie quantique et de son modèle mathématique, peut consulter par exemple les livres [Rob87, Introduction] ou [LL66, FH10a, CTDL21, LL77].

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1. La condition au bord (1.2) est appelée dans d'autres domaines de la théorie des équations aux dérivées partielles condition de Robin, mais qu'elle a été introduite pour la première fois par Fourier.

2. Le physicien de Gennes (cf. [dG66]) a introduit le paramètre  $\beta$  dans le but de modéliser les interfaces entre des matériaux supraconducteurs et normaux et dans ce contexte,  $\beta$  est une constante strictement positive. Les valeurs négatives de  $\beta$  sont aussi considérées dans la littérature physique (cf. [FJ69]) pour modéliser la situation de supraconducteur à côté d'un autre supraconducteur ayant une plus haute température critique. Pour plus de détails, voir par exemple [Kac06a, Kac06b].

3. Le lecteur ou la lectrice intéressé par l'analyse semi-classique, peut consulter par exemple les livres [Zwo12, Mar02, DS99].

4. La mécanique quantique a permis d'expliquer de nombreuses propriétés physiques jusqu'alors mystérieuses. Les principes fondamentaux de la mécanique quantique ont été établis principalement entre 1922 et 1927 par les physiciens Bohr, Dirac, de Broglie, Heisenberg, Jordan, Pauli et Schrödinger.

Une équation fondamentale de la mécanique quantique non-relativiste est la célèbre "équation de Schrödinger"<sup>5</sup> dépendant du temps :

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \mathcal{P} \psi(x, t), \quad (1.6)$$

où,  $\psi$  est la fonction d'onde avec  $\|\psi(x)\|^2$  est la densité de probabilité de trouver la particule en  $x$ . Un exemple très classique de Hamiltonien quantique associé à l'énergie totale du système est :

$$\mathcal{P} = \frac{-\hbar^2}{2m} \Delta + V(x), \quad (1.7)$$

et il décrit une particule non relativiste<sup>6</sup> de masse  $m$  (ex :  $m_{\text{electron}} \approx 10^{-31} \text{Kg}$ ) qui se déplace dans un potentiel  $V$ , où  $V(x)$  est l'énergie potentielle de la particule au point  $x$ , et  $\hbar$  est la constante de Planck divisée par  $2\pi$ , et elle vaut  $\hbar \approx 1.054\,571\,818 \times 10^{-34} \text{ Js}$ .

La recherche de solutions harmoniques en temps, *i.e.* de la forme

$$\psi(x, t) = e^{-\frac{iEt}{\hbar}} \psi(x, 0),$$

conduit à

$$\frac{-\hbar^2}{2m} \Delta \psi(x, 0) + V(x) \psi(x, 0) = E \psi(x, 0),$$

où  $E$  est l'énergie associée à l'état propre  $\psi(x, 0)$ .

La densité de probabilité de présence de la particule est donc indépendante de temps et elle satisfait

$$|\psi(x, t)|^2 = |\psi(x, 0)|^2.$$

C'est d'ailleurs la raison pour laquelle ces états sont qualifiés de "stationnaires".

Notons enfin, pour comprendre comment la particule évolue, *i.e.* étudier l'évolution temporelle de la fonction d'onde, il faut étudier le spectre de l'opérateur non-borné  $\mathcal{P}$  sur  $L^2(\Omega)$ ,  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ . C'est pourquoi le spectre de  $\mathcal{P}$  joue un rôle tout particulier dans l'étude de (1.6). Plus généralement, n'importe quel opérateur auto-adjoint  $\mathcal{H}$  définit un système dynamique quantique par (1.6).

Dans cette thèse, nous nous intéressons à d'autres types de Hamiltonien  $\mathcal{P}$ , notamment magnétiques ou/et sur des domaines de  $\mathbb{R}^d$ . Nous cherchons à décrire le spectre et les fonctions propres de  $\mathcal{P}$  dans certains régimes asymptotiques (grand champ magnétique, grand champ électrique).

### 1.1.2.1 Hamiltonien de Stark

Considérons une particule chargée soumise à un champ électrique uniforme  $\mathbf{F}$ . Dans ce cas, le Hamiltonien qui nous intéresse est alors (agissant sur  $L^2(\Omega)$  et défini avec des conditions de bord de Dirichlet sur  $\partial\Omega$ )

$$\mathcal{H} = \frac{-\hbar^2}{2m} \Delta + q\mathbf{F} \cdot \mathbf{x}, \quad (1.8)$$

où

- $\hbar$  est la constante de Planck divisée par  $2\pi$ ,
- $m$  est la masse effective de la particule,
- $q$  est la charge de la particule.

5. Erwin Schrödinger(1887-1961) a introduit sa célèbre équation pour la première fois dans l'article [Sch26].

6. L'équation de Schrödinger ne tient pas compte des effets de relativité restreinte. Lorsque ces effets sont importants, il faut alors utiliser l'équation de Dirac (1928).

Une partie de cette thèse concerne l’analyse des propriétés spectrales du *Hamiltonien de Stark*  $\mathcal{H}$  défini en (1.8), sur un domaine ouvert, borné, régulier et connexe de  $\mathbb{R}^d$ ,  $d \geq 2$ , lorsque l’intensité du champ électrique est grande. Nous nous intéressons à étudier le comportement asymptotique des valeurs propres de  $\mathcal{H}$ , et les propriétés de localisation des fonctions propres associées ainsi que des approximations pour ces fonctions. L’analyse spectrale de l’opérateur  $\mathcal{H}$  fait l’objet de mon article [Fah23b].

### 1.1.2.2 Laplacien magnétique

Considérons une particule soumise à un champ purement magnétique qui est mathématiquement modélisé par le *Laplacien magnétique*

$$\mathcal{P}_\beta^b = -(\nabla - ibA_0)^2, \quad (1.9)$$

où  $A_0$  est un potentiel magnétique et  $b$  est l’intensité du champ magnétique appliqué. Le domaine de l’opérateur  $\mathcal{P}_\beta^b$  est

$$\text{Dom}(\mathcal{P}_\beta^b) = \{u \in H^2(\Omega) : \nu \cdot (\nabla - ibA_0)u + \beta u = 0 \text{ sur } \partial\Omega\}.$$

Une grande partie de cette thèse concerne l’analyse spectrale de cette famille de laplaciens magnétiques défini en (1.9) sur un domaine ouvert, borné, régulier et simplement connexe de  $\mathbb{R}^2$  dans divers régimes “grand champ magnétique” et/ou “grande condition de Robin”. L’analyse spectrale de l’opérateur  $\mathcal{P}_\beta^b$  font l’objet de mes articles [Fah23a, FLTRVuN23].

## 1.2 Présentation détaillée : objectifs de la thèse et résultats connus

### 1.2.1 Laplacien magnétique de Robin

La théorie spectrale du Laplacien magnétique de Robin a suscité beaucoup d’intérêt ces dernières années, notamment dans le régime de couplage fort (c’est-à-dire quand le paramètre de Robin est grand). De nombreux auteurs se sont intéressés à l’analyse asymptotique de cet opérateur. Nous allons maintenant définir cet opérateur. Soit  $\Omega$  un ouvert borné connexe de  $\mathbb{R}^2$ . Nous supposons que la frontière  $\Gamma = \partial\Omega$  est  $C^\infty$ -régulière. Considérons le problème spectral

$$\mathcal{P}_\beta^b u := -(\nabla - ibA_0)^2 u = Eu \quad \text{sur } \Omega, \quad (1.10)$$

$$\nu \cdot (\nabla - ibA_0)u + \beta u = 0 \quad \text{sur } \Gamma, \quad (1.11)$$

où

- $A_0$  est le champ de vecteur,
- $\nu$  est le vecteur normal unitaire extérieur de  $\Gamma$ ,
- $\beta \in \mathbb{R}$  est le paramètre de Robin,
- $b$  est l’intensité du champ magnétique appliqué,
- $E$  est une valeur propre et  $u$  une fonction propre non nulle associée.

Le domaine de l’opérateur  $\mathcal{P}_\beta^b$  est

$$\text{Dom}(\mathcal{P}_\beta^b) = \{u \in H^2(\Omega) : \nu \cdot (\nabla - ibA_0)u + \beta u = 0 \text{ sur } \Gamma\}.$$

La condition (1.11) est appelée condition de Robin. Elle est dite attractive si  $\beta < 0$  et répulsive si  $\beta > 0$ . Elle peut être considérée comme celle de Neumann lorsque  $\beta = 0$  et celle de Dirichlet lorsque  $\beta = \infty$ .

Le but est d'étudier le comportement asymptotique des valeurs propres lorsque le paramètre  $\beta$  est grand. Le potentiel magnétique défini par

$$A_0(x_1, x_2) = (A_{0_1}, A_{0_2}) = \frac{1}{2}(-x_2, x_1),$$

génère un champ magnétique constant :

$$B := \nabla \times A_0 = \partial_{x_1} A_{0_2} - \partial_{x_2} A_{0_1} = 1.$$

Comme  $\operatorname{div}(A_0) = \nabla \cdot A_0 = \partial_{x_1} A_{0_1} + \partial_{x_2} A_{0_2} = 0$ , alors l'opérateur  $\mathcal{P}_\beta^b$  s'écrit comme suit :

$$\begin{aligned} \mathcal{P}_\beta^b &= -(\nabla - ibA_0)^2 \\ &= -(\nabla - ibA_0) \cdot (\nabla - ibA_0) \\ &= -\Delta + 2ibA_0 \cdot \nabla + ib \operatorname{div}(A_0) + b^2 \|A_0\|^2 \\ &= -\Delta + 2ibA_0 \cdot \nabla + b^2 \|A_0\|^2. \end{aligned}$$

Pour être plus précis, l'opérateur de Schrödinger magnétique  $\mathcal{P}_\beta^b$  est défini via le théorème d'extension de Friedrich (voir par exemple [LM54, Thms. 2.1 & 2.2] et [Hel13, Ch. 4]), à partir de la forme quadratique fermée semi-bornée inférieurement, définie sur  $H^1(\Omega)$  par la formule

$$\forall u \in H^1(\Omega), \quad \mathcal{Q}_\beta^b(u) := \int_\Omega |(\nabla - ibA_0)u(x)|^2 dx + \beta \int_\Gamma |u(x)|^2 ds(x),$$

où  $ds$  est la mesure de surface standard sur la frontière. Notons que, par l'inégalité dans [BEL14, Lem. 2.6], la trace de  $u$  est bien définie comme un élément de  $H^{\frac{1}{2}}(\Gamma)$ , et ensuite la forme quadratique  $\mathcal{Q}_\beta^b$  est bien définie.

**Lemme 1.2.1.** *La forme quadratique  $\mathcal{Q}_\beta^b$  est bornée inférieurement.*

*Démonstration.* Rappelons que d'après l'inégalité diamagnétique [LL01, Thm. 7.21], pour tout  $u \in H^1(\Omega)$ , on a

$$\|\nabla|u|\|_{L^2(\Omega)}^2 \leq \|(\nabla - ibA_0)u\|_{L^2(\Omega)}^2. \quad (1.12)$$

En combinant (1.12) avec l'inégalité dans [BEL14, Lem. 2.6], on obtient que pour tout  $\varepsilon > 0$ , il existe une constante  $C(\varepsilon) > 0$  telle que, pour tout  $u \in H^1(\Omega)$ , on a

$$\|u\|_{\partial\Omega}^2 \leq \varepsilon \|(\nabla - ibA_0)u\|_{L^2(\Omega)}^2 + C(\varepsilon) \|u\|_{L^2(\Omega)}^2. \quad (1.13)$$

- Si  $\beta \geq 0$ , alors  $\mathcal{Q}_\beta^b(u) \geq \|(\nabla - ibA_0)u\|^2 = \mathcal{Q}_0^b(u)$ ,
- Si  $\beta < 0$ , alors  $\mathcal{Q}_\beta^b(u) \geq (1 + \varepsilon\beta) \|(\nabla - ibA_0)u\|^2 + C(\varepsilon) \|u\|_{L^2(\Omega)}^2 = (1 + \varepsilon\beta) \mathcal{Q}_0^b(u) + C(\varepsilon) \|u\|_{L^2(\Omega)}^2$ , où  $\mathcal{Q}_0^b$  est la forme quadratique correspondant au Laplacien magnétique de Neumann sur  $\Omega$  avec le champ magnétique constant. La conclusion s'ensuit puisque  $\mathcal{Q}_0^b$  est bornée inférieurement par 0.  $\square$

Comme  $\Omega$  est borné et régulier, par [CR21, Thm. 4.20] et [Gri11, Section 1.4.4], il existe  $C > 0$  tel que

$$\left( \underbrace{\operatorname{Dom}(\mathcal{Q}_\beta^b)}_{=H^1(\Omega)}, \sqrt{\mathcal{Q}_\beta^b(\cdot) + (C+1)\|\cdot\|_{L^2(\Omega)}} \right) \xrightarrow{\text{compacte}} \left( L^2(\Omega), \|\cdot\|_{L^2(\Omega)} \right),$$

ainsi, par [CR21, Proposition 4.24] l'opérateur  $\mathcal{P}_\beta^b + (C+1)Id_{L^2(\Omega)}$  est à résolvante compacte et ensuite l'opérateur  $\mathcal{P}_\beta^b$  est à résolvante compacte. Et donc, voir e.g. [RS78, Thm. XIII.64], son spectre est purement

discret et comme  $\mathcal{P}_\beta^b$  est auto-adjoint et borné inférieurement, il est formé d'une suite de valeurs propres <sup>7</sup>

$$\lambda_1(b, \beta) \leq \lambda_2(b, \beta) \leq \dots \leq \lambda_n(b, \beta) \leq \dots \rightarrow +\infty$$

où chaque valeur propre est répétée selon sa multiplicité finie, et les fonctions propres associées forment une base orthonormale de  $L^2(\Omega)$ .

Si  $\Omega$  est simplement connexe, la valeur propre  $\lambda_n(b, \beta)$  est indépendante du choix du potentiel vecteur  $A_0$  du champ magnétique. Ceci est une conséquence de l'invariance par changement de jauge ; si  $A \in H^1(\Omega; \mathbb{R}^2)$  et  $\text{curl} A = 1$ , alors  $A = A_0 + \nabla \phi$  pour une fonction  $\phi \in H^2(\Omega)$  (cf. [FH10b, Props D.1.1 and D.2.1]), et à leur tour

$$e^{ib\phi}(\nabla - ibA_0)^2 e^{-ib\phi} = (\nabla - ibA_0 - ib\nabla\phi)^2 = (\nabla - ibA)^2.$$

Outre son intérêt propre, l'étude du spectre de l'opérateur  $\mathcal{P}_\beta^b$  apparaît dans la modélisation de nombreux problèmes physiques comme l'équation de Schrödinger modélisant l'évolution d'une particule non relativiste (Section 1.1.2) ou encore l'équation des ondes décrivant les vibrations d'une membrane. Le problème aux valeurs propres de Robin (1.10)-(1.11) apparaît dans de nombreuses applications comme l'estimation de la température critique/les champs critiques de certains supraconducteurs entourés d'autres matériaux : [Kac06a, KS20] en dimension 2 et [Mir23, HKR17] en dimension 3, ou encore l'étude des équations de réaction-diffusion en temps longs [LOS98]. D'autre part, les asymptotiques de valeurs propres dans la limite singulière  $\beta \rightarrow -\infty$ , fournissent des contre-exemples dans le contexte de la géométrie spectrale [FK15, PP16], et ont des liens avec l'étude des valeurs propres de Steklov [HK22]. Le cas  $\beta = 0$  correspond au Laplacien magnétique de Neumann, qui a été étudié dans de nombreux articles [Ray17, FH10b]. L'étude des inégalités isopérimétriques a été étendue aux conditions de Robin à paramètre  $\beta$  négatif. L'asymptotique des valeurs propres peut également être utilisée pour discuter/généraliser la validité des inégalités isopérimétriques dans le régime  $\beta \rightarrow -\infty$ . Dans le travail récent de Kachmar-Lotoreichik [KL22], pour un paramètre de Robin négatif et un champ magnétique d'intensité modérée, ils ont trouvé que le disque maximise l'énergie de l'état fondamental  $\lambda_1(b, \beta)$  dans une classe admissible de domaines ayant le même périmètre que le disque.

### 1.2.1.1 Cas du champ magnétique modéré et condition de Robin attractive

En l'absence du champ magnétique ( $b = 0$ ), l'asymptotique des valeurs propres de l'opérateur  $\mathcal{P}_\beta^0$  a donné lieu à de nombreuses contributions (dans divers contextes géométriques) au cours des dernières années [EMP14, HK17, Ker18, LP08, PP16, Kac16, KKR16].

**Théorème 1.2.1** ([EMP14, Pan13, PP15]). *Pour tout  $n \in \mathbb{N}^*$ , on a*

$$\lambda_n(0, \beta) = -\beta^2 + \kappa_{\max}\beta + o(\beta) \quad \text{lorsque } \beta \rightarrow -\infty, \quad (1.14)$$

où  $\kappa_{\max}$  est la courbure maximale du bord  $\Gamma$ .

<sup>7</sup> La caractérisation variationnelle des valeurs propres par les principes du min-max (voir le Théorème A.1.7 dans l'appendice), appliqué à  $\mathcal{Q}_\beta^b$ , nous donne :

$$\lambda_n(b, \beta) = \sup_{\psi_1, \dots, \psi_{n-1}} \inf_{\substack{u \in H^1(\Omega), u \neq 0 \\ u \in \text{Vect}\{\psi_1, \dots, \psi_{n-1}\}^\perp}} \frac{\mathcal{Q}_\beta^b(u)}{\|u\|_{L^2(\Omega)}^2},$$

pour  $n = 1$

$$\lambda_1(b, \beta) = \inf_{u \in H^1(\Omega)} \frac{\mathcal{Q}_\beta^b(u)}{\|u\|_{L^2(\Omega)}^2}.$$

Notons que le premier terme de (1.14) avait déjà été obtenu précédemment (voir [LP08] et ses références). Nous remarquons que dans l'estimation (1.14), les propriétés géométriques du bord, et plus particulièrement la courbure, apparaissent, mais le numéro de la valeur propre n'est, par contre, pas visible dans les deux premiers termes. De ce fait, les plus petites valeurs propres sont indistinguables. L'article [HK17] permet de résoudre ce problème en obtenant un développement asymptotique complet de chaque valeur propre  $\lambda_n(0, \beta)$ , lorsque  $\beta \rightarrow -\infty$ . Paramétrons la frontière  $\partial\Omega$ , par la longueur de l'arc, que nous notons  $s$ . Soit  $\kappa(s)$  la courbure de  $\partial\Omega$  au point défini par la longueur d'arc  $s$ , avec la convention que  $\kappa(s) \geq 0$  quand  $\partial\Omega$  est convexe au voisinage de  $s$ , et négatif sinon.

**Hypothèse 1.** *Supposons que la courbure  $\kappa$  a un unique maximum atteint en un point  $s_0$  et le maximum est non dégénéré :  $\kappa''(s_0) < 0$ . La paramétrisation de la longueur d'arc de la frontière est définie de sorte que  $\kappa(0) = \kappa_{max}$  i.e.  $s_0 = 0$ .*

**Théorème 1.2.2** ([HK17]). *Sous l'Hypothèse 1, pour tout  $n \in \mathbb{N}^*$ , il existe une suite  $(\gamma_{j,n})_{j \geq 0}$  de nombres réels telle que, pour tout  $M \in \mathbb{N}$  positif, la valeur propre  $\lambda_n(0, \beta)$  a, comme  $\beta \rightarrow -\infty$ , le développement asymptotique*

$$\lambda_n(0, \beta) = -\beta^2 + \kappa_{max}\beta + (2n-1)\sqrt{\frac{-\kappa''(s_0)}{2}}|\beta|^{1/2} + \sum_{j=0}^M \gamma_{j,n}|\beta|^{-j/2} + o(|\beta|^{-M/2}). \quad (1.15)$$

En utilisant la localisation des fonctions propres près du point de courbure maximale, la construction de quasi-modes permet d'obtenir une approximation des valeurs propres grâce au théorème spectral. Dans la limite,  $\beta \rightarrow -\infty$ , les premiers termes de l'asymptotique sont donnés par l'approximation harmonique (cf. [Ray17, Sections 3.2 et 4.1.2] et [FH10b, Section 7.1]) de l'opérateur  $H^{\text{bnd}} = -\frac{d^2}{ds^2} - \beta^2 + \beta\kappa(s)$  sur le bord. L'opérateur réduit  $H^{\text{bnd}}$  est analysé en remplaçant le potentiel  $-\kappa(s)$  par son approximation quadratique au point de minimum non dégénéré  $s_0$  (par hypothèse). Les termes  $(2n-1)\sqrt{\frac{-\kappa''(s_0)}{2}}$  sont exactement les valeurs propres de l'oscillateur harmonique  $-\frac{d^2}{ds^2} + \frac{-\kappa''(s_0)}{2}s^2$  dans  $L^2(\mathbb{R})$ . Par rapport aux opérateurs de Schrödinger semi-classiques existants (voir [HS84]), la courbure agit comme un puits de potentiel. L'hypothèse sur la courbure équivaut au cas d'un seul puits. En cas de symétries, la détermination de l'effet tunnel entre les points de courbure maximale a donné lieu à des travaux. Dans [HKR17], ils ont étudié les valeurs propres basses du Laplacien de Robin dans un domaine symétrique par rapport à un axe. Dans le cas où la courbure de la frontière du domaine atteint son maximum exactement à deux points, ils ont établi une formule asymptotique explicite pour les deux premières valeurs propres. Il s'agit d'une dérivation rigoureuse de l'effet tunnel induit par la géométrie du domaine.

En supraconductivité, imposer une condition de Robin, appelée dans ce contexte condition de de Gennes, modélise un supraconducteur entouré d'un autre matériau normal/supraconducteur, (comme nous avons vu dans la Section 1.1.1). Dans ce contexte, nous sommes naturellement conduits à l'analyse du Laplacien de Robin avec un champ magnétique où différents régimes interviennent en fonction de la comparaison entre l'intensité du champ magnétique et le paramètre de Robin. Une question naturelle se pose :

« *Quel est l'effet du champ magnétique sur les valeurs propres basses, lorsque  $\beta$  tend vers  $-\infty$  et  $b$  ("dépend possiblement de  $\beta$ ") tend vers  $+\infty$  simultanément ?* »

L'inégalité de diamagnétique (A.1.3) donne, pour tout  $\beta \in \mathbb{R}$  et  $b > 0$



$$\lambda_1(b, \beta) - \lambda_1(0, \beta) \geq 0 \text{ }^8. \quad (1.16)$$

Lorsque l'intensité du champ magnétique est une constante fixe (c'est-à-dire que le paramètre  $b$  est indépendant de  $\beta$ ), l'inégalité diamagnétique et le principe min-max impliquent que le développement asymptotique dans (1.14) reste vrai pour  $\lambda_1(b, \beta)$ . Notre intérêt se porte uniquement sur la condition de Robin attractive et dans la suite le paramètre de Robin  $\beta$  est strictement négatif. Cependant, dans la limite  $\beta \rightarrow -\infty$ , il est naturel de permettre que  $b$  dépend de  $\beta$ .

Dans l'article [Kac16], il est prouvé une estimation asymptotique à deux termes pour l'énergie de l'état fondamental magnétique  $\lambda_1(b, \beta)$ , lorsque  $\beta$  tend vers  $-\infty$  et  $b$  tend vers  $+\infty$  simultanément.

**Théorème 1.2.3** ([Kac16]). *Soit  $0 \leq \eta < \frac{3}{2}$  et  $0 < c_1 < c_2$ . On suppose que*

$$c_1|\beta|^\eta \leq b \leq c_2|\beta|^\eta.$$

*Lorsque  $\beta$  tend vers  $-\infty$ ,*

$$\lambda_1(b, \beta) = -\beta^2 + \kappa_{\max}\beta + \beta o(1).$$

Dans ce cas, le diamagnétisme est faible et sa contribution est négligeable par rapport à celle du terme de correction de courbure (comparer avec (1.14) pour  $n = 1$ ). Nous remarquons que dans l'estimation du Théorème 1.2.3, les propriétés géométriques du bord apparaissent, mais le numéro de la valeur propre n'est, par contre, pas visible dans les deux premiers termes. Notre premier objectif est de résoudre ce problème en obtenant un développement asymptotique à trois termes de chaque valeur propre  $\lambda_n(b, \beta)$ , lorsque  $\beta \rightarrow -\infty$  et  $b = \mathcal{O}(|\beta|^\eta)$  pour  $0 \leq \eta < \frac{3}{2}$ . De plus, notre objectif est d'améliorer/compléter les estimations antérieures et également de clarifier la contribution du champ magnétique dans l'asymptotique spectrale. En particulier, dans le cas du disque unité, on décrit explicitement la contribution du champ magnétique sur l'asymptotique des basses valeurs propres. Cela fait l'objet du chapitre 3.

### Transformation en un problème semi-classique :

On va écrire l'opérateur (1.10)-(1.11) sous une forme semi-classique. On montre comment la limite singulière où le paramètre de Robin  $\beta$  tend vers l'infini équivaut à une limite semi-classique impliquant un petit paramètre semi-classique positif  $h$  défini par

$$\beta = \gamma h^{-\frac{1}{2}}, \quad (1.17)$$

avec  $\gamma \in \mathbb{R} \cup \{+\infty\}$  indépendant de  $h$ .

La limite  $\beta \rightarrow \pm\infty$  est maintenant équivalente à la limite semi-classique  $h \rightarrow 0$ . La forme quadratique peut s'écrire

$$\mathcal{Q}_\beta^b(u) = h^{-2} \left( \int_\Omega |(h\nabla - ihbA_0)u(x)|^2 dx + \gamma h^{\frac{3}{2}} \int_\Gamma |u(x)|^2 ds(x) \right) := h^{-2} \mathcal{Q}_{h,\gamma}^b(u).$$

Par conséquent, l'opérateur auto-adjoint dépendant de  $h$  associé à la forme quadratique  $\mathcal{Q}_{h,\gamma}^b$  est

$$\mathcal{L}_{h,\gamma}^b = -(h\nabla - ihbA_0)^2, \quad (1.18)$$

---

8. En termes physiques, cette inégalité fait référence au diamagnétisme, ce qui signifie simplement que l'introduction d'un champ magnétique augmente l'énergie de l'état fondamental.

défini sur le domaine

$$\text{Dom}(\mathcal{L}_{h,\gamma}^b) = \{ u \in H^2(\Omega) : h\nu \cdot (h\nabla - ibhA_0)u + \gamma h^{\frac{3}{2}}u = 0 \text{ sur } \Gamma \}. \quad (1.19)$$

Comme  $\mathcal{P}_\beta^b = h^{-2}\mathcal{L}_{h,\gamma}^b$ , alors les spectres des opérateurs  $\mathcal{P}_\beta^b$  et  $\mathcal{L}_{h,\gamma}^b$  sont liés par :

$$\text{Sp}(\mathcal{P}_\beta^b) = h^{-2}\text{Sp}(\mathcal{L}_{h,\gamma}^b). \quad (1.20)$$

L'opérateur auto-adjoint  $\mathcal{L}_{h,\gamma}^b$  est à résolvante compacte et borné inférieurement. Son spectre est purement discret et formé d'une suite croissante des valeurs propres  $(\mu_n(h, b, \gamma))_{n \in \mathbb{N}^*}$ . L'asymptotique semi-classique des valeurs propres de cet opérateur a fait l'objet d'études récentes.

Mon premier objectif est d'étudier les basses valeurs propres de  $\mathcal{L}_{h,\gamma}^b$  dans le régime semi-classique  $h \rightarrow 0$  et l'intensité du champ magnétique  $b \sim h^{\frac{-\eta}{2}}$  pour  $\eta < \frac{3}{2}$  et lorsque  $\gamma < 0$ . Une question naturelle se pose :

**« Quelle est l'influence d'un champ magnétique intense et d'une condition de Robin générale sur le bas du spectre de  $\mathcal{L}_{h,\gamma}^b$  ? »**

### 1.2.1.2 Cas du champ magnétique plus intense et condition de Robin générale

On considère un champ magnétique plus intense

$$b = h^{-1}b_0,$$

avec  $b_0 > 0$ . L'opérateur mis en jeu est alors

$$\mathcal{L}_{h,\gamma}^{b_0} = -(h\nabla - ib_0A_0)^2, \quad (1.21)$$

défini sur

$$\text{Dom}(\mathcal{L}_{h,\gamma}^{b_0}) = \{ u \in H^2(\Omega) : h\nu \cdot (h\nabla - ib_0A_0)u + \gamma h^{\frac{3}{2}}u = 0 \text{ sur } \Gamma \}. \quad (1.22)$$

Il s'agit de l'opérateur associé à la forme quadratique, définie pour tous  $\psi \in H^1(\Omega)$  par

$$\mathcal{Q}_{h,\gamma}^{b_0}(\psi) = \int_{\Omega} |(h\nabla - ib_0A_0)\psi(x)|^2 dx + \gamma h^{\frac{3}{2}} \int_{\Gamma} |\psi(x)|^2 ds(x). \quad (1.23)$$

On peut supposer sans perte de généralité, que  $b_0 = 1$  quitte à remplacer  $h$  par  $h/b_0$  et  $\gamma$  par  $\gamma/b_0^2$ . Le paramètre  $\gamma \in \mathbb{R} \cup \{+\infty\}$  sert à contrôler la "force" de la condition au bord dans (1.22). Par convention,  $\gamma = +\infty$  correspond à la condition au bord de Dirichlet ( $\psi \in H_0^1(\Omega)$ ).

**Notation 1.2.2.** On note par  $\mathcal{L}_{h,\gamma}$  l'opérateur  $\mathcal{L}_{h,\gamma}^{b_0}$  avec  $b_0 = 1$  et par  $\mathcal{Q}_{h,\gamma}$ , la forme quadratique  $\mathcal{Q}_{h,\gamma}^{b_0}$  avec  $b_0 = 1$ .

L'intérêt porté aux basses valeurs propres du Laplacien magnétique de Robin  $\mathcal{L}_{h,\gamma}$  avec paramètre  $\gamma$  négatif a été non négligeable ces dix dernières années. Cependant, jusqu'à présent, des résultats très précis ne sont disponibles que pour le Laplacien magnétique de Neumann (lorsque  $\gamma = 0$ ). Dans ce cas, les valeurs propres les plus basses ont été analysées en détail dans [FH06] et des estimations uniformes ont été récemment établies dans [BNHR22] où une formule d'effet tunnel purement magnétique a été prouvée. Lorsque  $\gamma \neq 0$ , les seuls résultats connus remontent aux travaux de Kachmar, voir [Kac06b], où seule la première valeur propre a été estimée. Dans le cas  $\gamma = +\infty$ , la localisation de la fonction propre près du bord a été observée

dans [GV21], conduisant à une description du spectre dans une fenêtre spectrale très mince, voir [GV21, Corollaire 2.7]. Cependant, la décroissance exponentielle n’a pas été établie. Une question naturelle que l’on peut se demander au vu de ces résultats est alors

**« Si on considère le spectre dans une fenêtre spectrale donnée (et pas seulement les basses valeurs propres), peut-on étudier les propriétés spectrales de  $\mathcal{L}_{h,\gamma}$  dans le régime semi-classique  $h \rightarrow 0$ , et pour  $\gamma \in \mathbb{R} \cup \{+\infty\}$ ? »**

Par un simple changement d’échelle, la limite semi-classique  $h \rightarrow 0$  se traduit en un régime quantique où l’intensité du champ magnétique tend vers l’infini. Dans la littérature physique des conducteurs minces ou des gaz d’électrons (approchés par des domaines 2D) soumis à un fort champ magnétique externe, il est bien connu que la présence d’une frontière (ou, plus généralement, d’un champ magnétique changeant brusquement le long d’une courbe) génère un courant le long du bord en raison de la présence de « l’effet rebond » classiquement localisé à une distance  $\sqrt{E}/B$  du bord ( $E$  est l’énergie cinétique et  $B$  l’intensité magnétique : dans notre cas  $B = 1$ ), voir par exemple [Hal82]. Ces “états de bord” existent dès que le niveau de Fermi du conducteur se situe strictement entre deux niveaux de Landau consécutifs, et produisent une dynamique balistique le long de la frontière. Si la frontière est compacte, cette dynamique est quantifiée et produit de nouveaux niveaux d’énergie discrets. Ce sont précisément les valeurs propres que nous souhaitons décrire.

D’un point de vue heuristique, la localisation au voisinage de  $\partial\Omega$  est souvent expliquée par les modes de rebond classiques mentionnés ci-dessus, mais elle est également facile à comprendre d’un point de vue quantique. En effet, si nous oublions la condition au bord, l’opérateur agit comme le Laplacien magnétique à champ magnétique constant, sur le plan euclidien,  $\mathcal{L}_{h,\gamma}^{\mathbb{R}^2}$ . Le spectre de cet opérateur est bien connu et constitué des fameux niveaux de Landau  $\{(2n-1)h, n \geq 1\}$ , qui sont des valeurs propres infiniment dégénérées. Ceci suggère que, si l’on considère des valeurs propres de  $\mathcal{L}_{h,\gamma}$  dans une fenêtre de la forme  $I_h = [ha, hb]$  avec  $2n-1 < a < b < 2n+1$  pour un entier  $n > 0$  (pour  $n = 0$ , on prend  $a = -\infty$ ), les fonctions propres correspondantes doivent être localisées près de la frontière. Ce phénomène a des applications physiques intéressantes comme l’effet Hall quantique. Une autre application est le confinement de particules dans de petits domaines, ou « quantum dots » (parfois appelés « anti-dots » car on prend  $B = 0$  à l’intérieur du domaine, et  $B = 1$  à l’extérieur), voir [RPM99, LSIC04].

Dans cette partie, nous traitons le cas général  $\gamma \in \mathbb{R} \cup \{+\infty\}$ . Physiquement, cela correspond à un domaine  $\Omega$  recouvert d’une très fine couche d’un matériau différent (voir par exemple [BL96]). Puisque  $\Omega$  est borné, le spectre dans  $I_h$  est toujours discret et une première estimation grossière montre que le nombre de valeurs propres se trouvant dans  $I_h$ , noté  $N(\mathcal{L}_{h,\gamma}, I_h)$ , satisfait

**Proposition 1.2.3.** *Il existe  $C > 0$ , tel que pour tous  $h > 0$  assez petits, on a*

$$N(\mathcal{L}_{h,\gamma}, I_h) \leq Ch^{-2}. \quad (1.24)$$

*Démonstration.* Cette estimée est très élémentaire à établir. Nous rappelons sa preuve brièvement.

– Pour  $\gamma \in \mathbb{R}$  : Grâce à l’inégalité de Young, on a, pour tout  $\psi \in H^1(\Omega)$ ,

$$\mathcal{Q}_{h,\gamma}(\psi) \geq \frac{h^2}{2} \|\nabla\psi\|^2 - 2\|A_0\|_\infty^2 \|\psi\|^2 + \gamma h^{\frac{3}{2}} \int_{\partial\Omega} |\psi|^2 ds.$$

– Lorsque  $\gamma \geq 0$ , on obtient que

$$\mathcal{Q}_{h,\gamma}(\psi) \geq \frac{h^2}{2} \|\nabla\psi\|^2 - 2\|A_0\|_\infty^2 \|\psi\|^2.$$

– Lorsque  $\gamma < 0$ , on utilise un théorème de trace classique : il existe  $C > 0$  tel que, pour tout  $\varepsilon > 0$ ,

$$\int_{\partial\Omega} |\psi|^2 ds \leq \varepsilon \|\nabla\psi\|^2 + C\varepsilon^{-1}\|\psi\|^2.$$

En choisissant  $\varepsilon = -\frac{\sqrt{h}}{4\gamma} > 0$ , on en déduit que

$$\mathcal{Q}_{h,\gamma}(\psi) \geq \frac{h^2}{4} \|\nabla\psi\|^2 - 2\|A_0\|_\infty^2 \|\psi\|^2 - 4\gamma^2 Ch \|\psi\|^2.$$

Dans les deux cas, il existe  $\tilde{C} > 0$  tel que, pour tout  $h \in (0, 1)$  et tout  $\psi \in H^1(\Omega)$ ,

$$\mathcal{Q}_{h,\gamma}(\psi) \geq \frac{h^2}{4} \|\nabla\psi\|^2 - \tilde{C}\|\psi\|^2.$$

Avec le principe min-max, cela montre que, pour tout  $\lambda$ ,

$$N(\mathcal{L}_{h,\gamma}, \lambda) \leq N\left(-\Delta^{\text{Neu}}, 4\frac{\lambda + \tilde{C}}{h^2}\right).$$

– Pour  $\gamma = +\infty$  : En appliquant la même stratégie que celle décrite ci-dessus, on obtient que, pour tout  $\lambda$ ,

$$N(\mathcal{L}_{h,\gamma}, \lambda) \leq N\left(-\Delta^{\text{Dir}}, 2\frac{\lambda + \tilde{C}}{h^2}\right).$$

La conclusion découle de l'asymptotique de Weyl pour le Laplacien de Neumann/Dirichlet, voir par exemple [NS05, Introduction].

□

Notre objectif est d'obtenir une description très précise, dans le régime semi-classique, des éléments spectraux correspondant à l'intervalle  $I_h$ , beaucoup plus précise que la Proposition 1.2.3. Cela inclut aussi la description de la localisation près de la frontière du domaine  $\Omega$  des fonctions propres correspondantes (qui sont des états de bord). C'est l'objet du chapitre 5.

## 1.2.2 Hamiltonien de Stark

Dans cette partie, nous nous intéressons à l'étude des basses valeurs propres de l'hamiltonien (en dimension  $N$ ) suivant restreint à un ensemble ouvert  $\Omega$  de  $\mathbb{R}^N$ ,  $N \geq 2$  (c'est l'opérateur défini en (1.8)) :

$$\mathcal{H} = \frac{-\hbar^2}{2m} \Delta + q\mathbf{F} \cdot \mathbf{x} = \frac{-\hbar^2}{2m} \sum_{i=1}^N \partial_{x_i}^2 + q\mathbf{F}x_1, \quad (1.25)$$

agissant sur un sous-espace dense de  $L^2(\Omega)$  avec des conditions de Dirichlet au bord. Noter que nous avons choisi des coordonnées telles que  $\mathbf{F}$  soit parallèle à l'axe  $x_1$ .

L'hamiltonien en (1.25) décrit un phénomène important en physique des semi-conducteurs<sup>9</sup> qui est l'*effet Stark confiné quantique*<sup>10</sup>. Les dispositifs semi-conducteurs les plus importants sont les puits quantiques,

9. Les semi-conducteurs jouent un rôle essentiel dans l'industrie et de nombreux appareils électroniques à base de semi-conducteurs sont présents dans notre quotidien : ordinateurs de bureau, tablettes, smartphones, écrans plats, capteurs...

10. Le terme "confinement quantique" apparaît principalement si les mouvements des particules d'un semi-conducteur sont limités dans une ou plusieurs dimensions. Physiquement, le confinement peut être réalisé en interfaçant le semi-conducteur avec un isolateur

les fils quantiques et les boîtes quantiques, où le matériau est confiné dans une, deux et trois directions orthogonales, respectivement (voir [HV16, Chap. 8]). L'effet Stark confiné quantique a été considéré dans de nombreux travaux, par exemple, dans [BMCE83, MCD<sup>+</sup>84] pour les puits quantiques, dans [SN90, WKH13, WWY07, VdCMS03, LS05, Har09] pour les fils quantiques, et dans [Ped19, WWY08] pour les boîtes quantiques. L'article récent [CKP<sup>+</sup>22b] fournit plus de détails sur les aspects physiques de ce phénomène.

Les plus petites valeurs propres de cet opérateur ont été étudiées, partiellement numériquement, pour différentes géométries dans plusieurs articles tels que dans [SN90, WKH13] pour les carrés, dans [WWY07] pour les rectangles, dans [VdCMS03, LS05, Ped17] pour les disques, et dans [Har09] pour les anneaux. Concernant l'analyse dans le cas des champs électriques imaginaires, le lecteur ou la lectrice pourra aussi considérer par exemple [GH18, AGH19, AGH18].

**Laplacien électrique de Stark.** Soit  $N \geq 1$  et  $\Omega \subset \mathbb{R}^N$  un ouvert borné régulier et connexe avec une frontière régulier. Nous travaillerons avec l'hypothèse suivante sur le domaine  $\Omega$ , voir Figure 1.1.

**Hypothèse 2.** On suppose qu'il existe un point unique  $\mathcal{A}_0 \in \partial\Omega$  tel que la première composante de  $\mathcal{A}_0$  soit donnée par :

$$x_{\min} = \inf_{(x_1, x_2, \dots, x_N) \in \Omega} x_1 = \min_{\mathbf{x} \in \overline{\Omega}} x_1.$$

Nous supposons également que  $\Omega$  est strictement convexe près de  $\mathcal{A}_0$ .

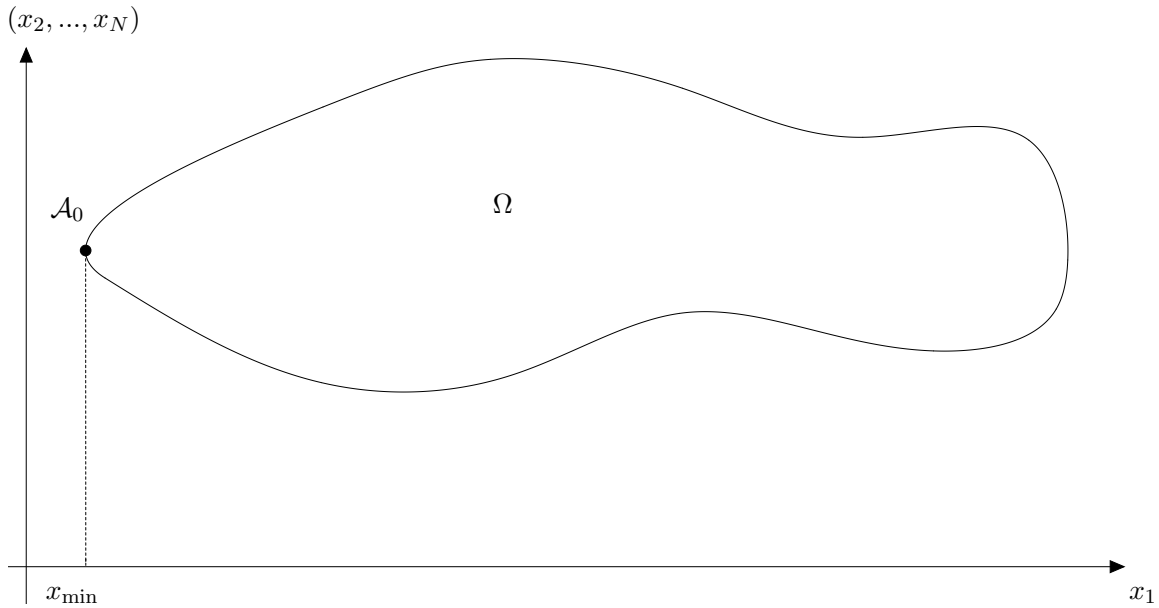


FIGURE 1.1 – Le domaine  $\Omega$ .

Noter qu'en factorisant  $q\mathbf{F} > 0$  dans l'expression de  $\mathcal{H}$  dans (1.25), nous avons le Hamiltonien de Stark équipé des conditions au bord de Dirichlet

$$\mathcal{L}_h := -\hbar^2 \Delta + x_1 = -\hbar^2 (\partial_{x_1}^2 + \dots + \partial_{x_N}^2) + x_1. \quad (1.26)$$

ou avec un autre semi-conducteur de plus grand gap [HK09, HV16].

Le domaine de  $\mathcal{L}_h$  est

$$\text{Dom}(\mathcal{L}_h) = H_0^1(\Omega) \cap H^2(\Omega).$$

Cet opérateur apparaît dans la physique des semi-conducteurs où  $h$  est donné par

$$h = \frac{\hbar}{\sqrt{2mq\mathbf{F}}}.$$

Pour être plus précis, l'opérateur  $\mathcal{L}_h$  est défini via le théorème d'extension de Friedrich (voir par exemple [LM54, Thms 2.1 & 2.2] et [Hel13, Chp.]), à partir de la forme quadratique fermée et semi-bornée

$$\mathcal{Q}_h(\varphi) = h^2 \int_{\Omega} |\nabla \varphi(\mathbf{x})|^2 \, d\mathbf{x} + \int_{\Omega} x_1 |\varphi(\mathbf{x})|^2 \, d\mathbf{x}, \quad \forall \varphi \in H_0^1(\Omega).$$

Puisque  $\Omega$  est borné, alors par [CR21, Thm. 4.19], le plongement de  $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$  est compact, donc l'opérateur auto-adjoint  $\mathcal{L}_h$  a une résolvante compacte. Son spectre est purement discret et formé d'une suite croissante de valeurs propres

$$\lambda_1(h) \leq \lambda_2(h) \leq \dots \leq \lambda_n(h) \leq \dots \rightarrow +\infty$$

où chaque valeur propre est répétée selon sa multiplicité.

Dans [CKP<sup>+</sup>22a], les auteurs ont trouvé un développement asymptotique à trois termes des valeurs propres  $\lambda_n(h)$  de l'opérateur  $\mathcal{L}_h$  défini en (1.26) sur un domaine  $\Omega$  de  $\mathbb{R}^2$  satisfaisant les conditions de l'Hypothèse 2.

**Théorème 1.2.4** ([CKP<sup>+</sup>22a]). *Pour tout  $n \in \mathbb{N}^*$ , on a*

$$\lambda_n(h) = x_{\min} + z_1 h^{\frac{2}{3}} + h(2n-1) \sqrt{\frac{\kappa_0}{2}} + \mathcal{O}(h^{\frac{4}{3}}) \quad \text{lorsque } h \rightarrow 0, \quad (1.27)$$

où  $z_1 \approx 2.338$  est la valeur absolue du plus petit zéro de la fonction Airy et  $\kappa_0$  est la courbure positive en  $\mathcal{A}_0$ .

Heuristiquement, le troisième terme du développement peut être expliqué comme suit : sous un fort champ électrique (petit paramètre semi-classique  $h$ ), la particule est poussée vers  $x_{\min}$  et se comporte comme un oscillateur harmonique dans la direction perpendiculaire au champ avec une constante élastique proportionnelle à la courbure en  $x_{\min}$ .

Notre but est d'établir l'influence du bord sur les valeurs propres  $\lambda_n(h)$  quelle que soit la dimension, c'est-à-dire de généraliser (1.27) à n'importe quelle dimension et de donner un développement asymptotique complet de  $\lambda_n(h)$ , lorsque  $h \rightarrow 0$  (dans la limite d'un fort champ électrique) et d'étudier également la localisation des états fondamentaux. En outre, généraliser nos techniques pour trouver un développement asymptotique complet des basses valeurs propres de l'opérateur plus général  $-h^2\Delta + V(\mathbf{x})$  où  $h$  est un petit paramètre et  $V$  est un potentiel réel régulier satisfaisant certaines conditions supplémentaires.

### 1.3 Résultats obtenus et idées de démonstrations

Les travaux de cette thèse portent sur les extensions auto-adjointes et la théorie spectrale semi-classique des opérateurs de Schrödinger agissant sur des domaines bornés, des opérateurs électromagnétiques de Laplace couplés à différentes conditions aux limites (Robin, Neumann et Dirichlet). L'étude spectrale de ces opérateurs de Schrödinger est basée sur une méthode de réduction de dimension, afin d'exprimer le spectre

en fonction de celui d'opérateurs effectifs définis sur le bord. Pour établir cette réduction, j'ai utilisé plusieurs techniques telles que : l'approximation de Born-Oppenheimer, les estimations de localisation de type Agmon, la construction de quasimodes, le théorème spectral, et le principe min-max (pour les chapitres 3 et 4). Dans le chapitre 5, j'ai utilisé une autre méthode de réduction plus avancée, la méthode de Grushin qui nécessite l'analyse microlocale et des techniques de calcul pseudo-différentiel.

Nous allons présenter les résultats par ordre de difficulté/complexité croissante.

### 1.3.1 Perturbations magnétiques du Laplacien de Robin dans la limite du couplage fort (chapitre 3)

L'asymptotique des valeurs propres de l'opérateur  $\mathcal{P}_\beta^0$  (sans champ magnétique) dans la limite de couplage fort  $\beta \rightarrow -\infty$  a fait l'objet d'études récentes ([EMP14, HK17, Ker18, LP08, PP16, Kac16, KKR16]), et suggère l'existence d'un opérateur effectif sur le bord, faisant intervenir l'opérateur de Laplace-Beltrami ainsi que la courbure moyenne de la frontière. Le problème de la dérivation d'un Hamiltonien effectif a également été abordé pour étudier l'effet tunnel semi-classique en présence de symétries dans [HKR17]. Dans le même esprit, nous étudierons le spectre du Laplacien de Robin  $\mathcal{P}_\beta^b$  sur un domaine borné avec la condition de Robin attractive au bord (problème (1.10)-(1.11)).

#### 1.3.1.1 Présentation des résultats

Notre principal résultat est le Théorème 1.3.1 qui implique un opérateur effectif que nous introduisons ci-dessous. On note  $|\Omega|$  l'aire du domaine  $\Omega$  et  $|\partial\Omega|$  la longueur de sa frontière. Paramétrons la frontière  $\partial\Omega$  par la longueur de l'arc, que nous notons  $s$ . Nous rappelons que  $\kappa(s)$  est la courbure de  $\partial\Omega$  au point défini par la longueur d'arc  $s$ , avec la convention que  $\kappa(s) \geq 0$  quand  $\partial\Omega$  est convexe au voisinage de  $s$ , et négatif sinon. Notre opérateur effectif est un opérateur auto-adjoint dans l'espace de Hilbert  $L^2(\partial\Omega)$ , agissant sur des fonctions périodiques, et défini comme suit

$$\begin{aligned} \mathcal{L}_{\beta, \gamma_0}^{\text{eff}} &= -\beta^{-2}(\partial_s - ib\gamma_0)^2 - 1 + \beta^{-1}\kappa(s) \\ \text{Dom}(\mathcal{L}_{\beta, \gamma_0}^{\text{eff}}) &= H^2(\mathbb{R}/|\partial\Omega|\mathbb{Z}) \end{aligned} \quad (1.28)$$

avec  $\gamma_0$  le flux du champ magnétique,

$$\gamma_0 = \frac{|\Omega|}{|\partial\Omega|}. \quad (1.29)$$

**Remarque 1.3.1.** *Par l'approximation semi-classique standard, on peut établir que*

$$\lambda_n(\mathcal{L}_{\beta, \gamma_0}^{\text{eff}}) = -1 + \kappa_{\max}\beta^{-1} + o(\beta^{-1}) \quad \text{lorsque } \beta \rightarrow -\infty, \quad (1.30)$$

où

$$\kappa_{\max} = \max_{\partial\Omega} \kappa(s). \quad (1.31)$$

Nous allons décrire les asymptotiques des valeurs propres négatives de  $\mathcal{P}_\beta^b$ , via celles de l'opérateur effectif sur le bord. Plus précisément, la  $n$ -ième valeur propre de l'opérateur de Laplace  $\mathcal{P}_\beta^b$  est approchée par la  $n$ -ième valeur propre des opérateurs  $\mathcal{L}_{\beta, \gamma_0}^{\text{eff}}$ . Nous utiliserons la notation suivante :

$$M = \tilde{\mathcal{O}}(|\beta|^\alpha) \quad \text{ssi } \forall \epsilon > 0, M = \mathcal{O}(|\beta|^{\alpha-\epsilon}). \quad (1.32)$$

**Théorème 1.3.1.** *Soit  $0 \leq \eta < \frac{3}{2}$  et  $0 < c_1 < c_2$ . On suppose que*

$$c_1|\beta|^\eta \leq b \leq c_2|\beta|^\eta.$$

Pour tout  $n \in \mathbb{N}^*$ , lorsque  $\beta$  tend vers  $-\infty$ , on a

$$|\lambda_n(b, \beta) - \beta^2 \lambda_n(\mathcal{L}_{\beta, \gamma_0}^{\text{eff}})| = \tilde{O}(\beta^{2(\eta-1)}),$$

où  $\lambda_n(\mathcal{L}_{\beta, \gamma_0}^{\text{eff}})$  est la  $n$ -ième valeur propre de l'opérateur effectif introduit en (1.28).

**Remarque 1.3.2.** Ce théorème est valable sur n'importe quel domaine régulier borné de  $\mathbb{R}^2$ . Aucune hypothèse n'est faite sur la géométrie du bord, en particulier sur les points de courbures maximales, en comparaison avec le résultat (1.15) par exemple.

Il résulte du Théorème 1.3.1 et (1.30),

$$\lambda_n(b, \beta) = -\beta^2 + \kappa_{\max} \beta + \tilde{O}(\beta^{2(\eta-1)}). \quad (1.33)$$

Lorsque  $b$  est une constante fixe (i.e.  $\eta = 0$ ), (1.33) est connu dans [KL22, Prop. 5.1] et en l'absence de champ magnétique ( $b = 0$ ) dans (1.14). Noter que le terme de reste est d'ordre inférieur tant que  $\eta < \frac{3}{2}$ . Si  $\eta = \frac{3}{2}$ , on a par [Kac16],

$$\lambda_n(b, \beta) = -\beta^2 + \left( \kappa_{\max} + \frac{b^2 \beta^{-3}}{4} \right) \beta + o(\beta).$$

Dans une situation générique où la courbure a un unique maximum non dégénéré, le Théorème 1.3.1 permet de déterminer le terme d'ordre dominant de l'écart spectral entre les valeurs propres successives. Ceci est valable sous l'Hypothèse 1. Rappelons cette hypothèse :

$$\begin{cases} \kappa \text{ atteint son maximum } \kappa_{\max} \text{ en un point unique;} \\ \text{le maximum est non dégénéré, i.e. } \kappa''(0) < 0, \end{cases}$$

où  $\kappa_{\max}$  est introduit en (1.31), désigne la courbure maximale le long de la frontière  $\partial\Omega$  et la paramétrisation de la longueur d'arc de la frontière est définie de sorte que  $\kappa(0) = \kappa_{\max}$ .

Sous l'Hypothèse 1, nous pouvons établir via l'approximation harmonique (cf. [Ray17, Sections 3.2 et 4.1.2] et [FH10b, Section 7.1]), l'asymptotique des valeurs propres de l'opérateur effectif :

$$\lambda_n(\mathcal{L}_{\beta, \gamma_0}^{\text{eff}}) = -1 + \kappa_{\max} \beta^{-1} + (2n-1) \sqrt{\frac{-\kappa''(0)}{2}} |\beta|^{-3/2} + o(|\beta|^{-3/2}). \quad (1.34)$$

Cela donne le corollaire suivant du Théorème 1.3.1 :

**Corollaire 1.3.1.** Soit  $0 \leq \eta < \frac{5}{4}$  et  $0 < c_1 < c_2$ . On suppose que

$$c_1 |\beta|^\eta \leq b \leq c_2 |\beta|^\eta.$$

Sous l'Hypothèse 1, pour tout  $n \in \mathbb{N}^*$ , quand  $\beta$  tend vers  $-\infty$ , on a

$$\lambda_n(b, \beta) = -\beta^2 + \kappa_{\max} \beta + (2n-1) \sqrt{\frac{-\kappa''(0)}{2}} |\beta|^{1/2} + o(|\beta|^{1/2}).$$

**Remarque 1.3.3.** La condition technique sur  $\eta$  apparaît comme suit. Après avoir inséré (1.34) dans l'estimation du Théorème 1.3.1, nous obtenons l'erreur  $\tilde{O}(\beta^{2(\eta-1)})$ , qui est d'ordre  $o(|\beta|^{1/2})$  si et seulement si  $\eta < \frac{5}{4}$ .

Dans l'asymptotique du Corollaire 1.3.1, la dépendance par rapport à l'étiquetage des valeurs propres,  $n$ , apparaît dans le troisième terme du développement. Cependant, dans les situations génériques décrites



par l’Hypothèse 1, la contribution du champ magnétique est cachée dans le terme de reste, et est d’ordre inférieur par rapport à celle de la courbure. Le Corollaire 1.3.1 découle de l’asymptotique spectrale des opérateurs effectifs 1D, qui est valide sous une hypothèse plus faible sur l’intensité du champ magnétique ( $b = \mathcal{O}(|\beta|^{5/4})$ ) par rapport à celui du Théorème 1.3.1 ( $b = \mathcal{O}(|\beta|^{3/2})$ ).

Dans le cas du disque, l’Hypothèse 1 n’est pas vérifiée, mais nous pouvons améliorer le Théorème 1.3.1 et exhiber l’influence du champ magnétique dans les asymptotiques des valeurs propres les plus petites.

**Théorème 1.3.2.** *Supposons que  $\Omega = D(0, 1)$  est le disque unité et que  $0 \leq \eta < 1$ . Nous avons, lorsque  $\beta \rightarrow -\infty$ ,*

$$\lambda_1(b, \beta) = -\beta^2 + \beta + \left( \inf_{m \in \mathbb{Z}} \left( m - \frac{b}{2} \right)^2 - \frac{1}{2} \right) + o(1),$$

uniformément par rapport à  $b$  jusqu’à  $b = \mathcal{O}(|\beta|^\eta)$ .

Le Théorème 1.3.2 était connu lorsque  $b$  est fixé [KS20, Thm. 1.1]. La preuve du Théorème 1.3.2 est basée sur le Théorème 1.3.1, en utilisant les séries de Fourier. Le spectre de l’opérateur  $\mathcal{P}_\beta^b$  sur le disque unité est approché modulo  $o(1)$  par le spectre de l’opérateur effectif suivant, qui est plus précis que celui de (1.28) (noter que dans le disque unité, la courbure est constante et égal à 1).

$$\mathcal{L}_{\beta, \text{disc}}^{\text{eff}} = -\beta^{-2}(\partial_s - ib\gamma_0)^2 - 1 + \beta^{-1} - \frac{1}{2}\beta^{-2}. \quad (1.35)$$

### 1.3.1.2 Stratégie de la démonstration

La stratégie pour la démonstration du Théorème 1.3.1 s’inspire de l’approximation de Born-Oppenheimer. Nous allons discuter ici la forme semi-classique des résultats, *i.e.* pour  $\beta = \gamma h^{-\frac{1}{2}}$  défini en (1.17), avec  $\gamma < 0$ . Pour simplifier, nous posons  $\gamma = -1$ .

Dans un premier temps, nous montrons, par l’estimation d’Agmon classique, que les fonctions propres de  $\mathcal{L}_{h, -1}^b$  (défini en (1.18) avec  $\gamma = -1$ ) associées à des valeurs propres plus petites que  $-\epsilon h$ , où  $\epsilon \in (0, 1)$ , sont uniformément localisées dans un voisinage d’ordre  $h^{1/2}$  du bord  $\partial\Omega$  (voir la Proposition 3.3.4). Nous pouvons donc nous ramener au Laplacien magnétique dans un voisinage tubulaire légèrement plus grossier du bord (voir la Section 3.3.3)

$$\Omega_\delta = \{ x \in \Omega : \text{dist}(x, \Gamma) < \delta \}, \quad \text{avec} \quad \delta = h^{\frac{1}{2}-\rho}, \quad \rho \in (0, 1/2).$$

On dispose alors, pour  $h \in (0, h_0)$  assez petit, d’un système de coordonnées  $(s, t) \in \mathbb{R}/|\partial\Omega|\mathbb{Z} \times (0, \delta)$ , défini près du bord. La signification précise des coordonnées  $(s, t)$  est la suivante :  $t$  mesure la distance au bord et  $s$  mesure la distance curviligne dans  $\partial\Omega$ . Ces coordonnées standard sont introduites précisément dans la Section 3.2. L’opérateur auto-adjoint dans ces coordonnées agissant sur  $L^2(\mathbb{R}/|\partial\Omega|\mathbb{Z} \times (0, \delta), \text{adsdt})$  est  $\tilde{\mathcal{L}}_{h, -1}^b$  défini par (voir la Section 3.3.4) :

$$\begin{cases} -h^2 a^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) a^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) - h^2 a^{-1} \partial_t a \partial_t, \\ \text{Dom}(\tilde{\mathcal{L}}_{h, -1}^b) = \{ u \in H^2(\text{adsdt}) : u(s, \delta) = 0 \quad \text{et} \quad \partial_t u(s, 0) = -h^{-1/2} u(s, 0) \}, \end{cases}$$

où  $a = 1 - t\kappa(s)$ .

Commençons par révéler la structure partiellement semi-classique de notre opérateur, ce qui aura aussi

pour effet de rendre la condition au bord indépendante de  $h$ . On considère le changement d'échelle

$$(s, \tau) = (s, h^{-1/2}t).$$

L'opérateur auto-adjoint dans les nouvelles coordonnées  $\hat{\mathcal{L}}_{h,-1}^b = h^{-1} \tilde{\mathcal{L}}_{h,-1}^b$  est définie sur  $L^2(\mathbb{R}/|\partial\Omega|\mathbb{Z} \times (0, h^{-\rho}), \hat{a} ds dt)$  par (voir la Section 3.3.5) :

$$\begin{cases} \hat{\mathcal{L}}_{h,-1}^b = -h\hat{a}^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -h^{\frac{1}{2}}\tau + h\frac{\tau^2}{2}\kappa(s) \right) \right) \hat{a}^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -h^{\frac{1}{2}}\tau + h\frac{\tau^2}{2}\kappa(s) \right) \right) \\ \quad - \hat{a}^{-1} \partial_\tau \hat{a} \partial_\tau, \\ \text{Dom}(\hat{\mathcal{L}}_{h,-1}^b) = \{u \in H^2(\hat{a} ds dt) : u(s, h^{-\rho}) = 0 \text{ et } \partial_\tau u(s, 0) = -u(s, 0)\}, \end{cases}$$

où  $\hat{a} = 1 - h^{1/2}\tau\kappa(s)$ .

On voit apparaître un opérateur unidimensionnel défini pour  $s$  fixé, par (voir la Section 3.4.1)

$$\begin{cases} \mathcal{H}_{\kappa(s),h} = -\hat{a}^{-1} \partial_\tau \hat{a} \partial_\tau = -\partial_\tau^2 + h^{1/2}\kappa(s)\hat{a}^{-1} \partial_\tau \\ \partial_\tau u(s, 0) = -u(s, 0) \text{ dans } L^2((0, h^{-\rho}), \hat{a} d\tau). \end{cases}$$

L'idée principale, due à Born et Oppenheimer dans [BO27], est de remplacer, pour  $s$  fixé, l'opérateur  $\mathcal{H}_{\kappa(s),h}$  par ses valeurs propres. Pour cette raison, en Section 3.4.2, on introduit la "projection de Feshbach"  $\Pi_s$  sur l'état fondamental normalisé de  $\mathcal{H}_{\kappa(s),h}$ , noté  $v_{\kappa(s),h}$ ,

$$\Pi_s \psi = \langle \psi, v_{\kappa(s),h} \rangle_{L^2((0, h^{-\rho}), \hat{a} d\tau)} v_{\kappa(s),h}.$$

Notons que l'idée de considérer une projection de Feshbach apparaît dans de nombreux contextes différents : analyse WKB (voir par exemple [BNHR16, Sections 2.4 & 3.2], [HK17] et [HKR17]), convergence en norme de la résolvante (voir par exemple [KR14, Section 4.2] et [KRRS18]) ou limites adiabatiques espace/temps (voir [Teu03, Chapitre 3]).

Enfin, nous voulons prouver des bornes supérieures et inférieures sur les valeurs propres de  $\hat{\mathcal{L}}_{h,-1}^b$ . Pour les bornes supérieures, nous insérons des fonctions de test presque explicites (produits quasi-tensoriels de fonctions sur la frontière et de l'état fondamental  $v_{\kappa(s),h}(\tau)$  de l'opérateur transverse  $\mathcal{H}_{\kappa(s),h}$ ). Pour la borne inférieure, on utilise la décomposition spectrale de l'opérateur transverse  $\mathcal{H}_{\kappa(s),h}$  pour décomposer  $\hat{\mathcal{L}}_{h,-1}^b$  en deux composantes orthogonales, modulo quelques restes faisant intervenir le commutateur entre l'état fondamental transverse (dépendant de  $s$ ) et la dérivée tangentielle  $\partial_s$ . Pour les détails, nous renvoyons le lecteur ou la lectrice aux Sections 3.4.4. et 3.4.5.

## 1.3.2 Analyse semi-classique des opérateurs de Schrödinger avec des potentiels électriques linéaires sur un domaine borné (chapitre 4)

### 1.3.2.1 Présentation des résultats

Dans [CKP<sup>+</sup>22a], en utilisant des fonctions tests et en appliquant le principe min-max, les auteurs ont trouvé un développement asymptotique à trois termes des valeurs propres  $\lambda_n(h)$  de l'opérateur  $\mathcal{L}_h$  (défini en (1.26)) sur un domaine  $\Omega$  de  $\mathbb{R}^2$  satisfaisant les conditions dans l'Hypothèse 2 (voir le Théorème 1.2.4). Notre but est de généraliser le résultat de [CKP<sup>+</sup>22a] à n'importe quelle dimension et de donner un développement asymptotique complet de  $\lambda_n(h)$ . Plus précisément, nous établissons un tel développement asymptotique en faisant intervenir l'endomorphisme de Weingarten de  $\partial\Omega$  en  $\mathcal{A}_0$  défini par (voir par exemple [Küh15, Chap.

3] et [Tho79, Chap. 9])

$$W_{\mathcal{A}_0} = d_{\mathcal{A}_0} \nu : T_{\mathcal{A}_0} \partial\Omega \rightarrow T_{\mathcal{A}_0} \partial\Omega \quad (1.36)$$

où

$$\nu : \partial\Omega \rightarrow S^{N-1} \subset \mathbb{R}^N$$

est l'application de Gauss de  $\partial\Omega$  et pour tout  $p \in \partial\Omega$ ,  $\nu(p)$  est la normale pointant vers l'extérieur à  $T_p \partial\Omega$ .

**Remarque 1.3.4.** L'Hypothèse 2 sur  $\Omega$  induit que la matrice de  $W_{\mathcal{A}_0}$  est définie positive. Ses valeurs propres positives  $(\kappa_i)_{i \geq 1}$  sont liées à la courbure de la frontière.

1. Les  $\kappa_i$  sont appelées courbures principales de  $\partial\Omega$  à  $\mathcal{A}_0$ .
2.  $\det W_{\mathcal{A}_0}$  est la courbure gaussienne de  $\partial\Omega$  à  $\mathcal{A}_0$ .
3.  $\frac{\text{Tr } W_{\mathcal{A}_0}}{N-1}$  est la courbure moyenne de  $\partial\Omega$  à  $\mathcal{A}_0$ .

Notre résultat principal est le suivant :

**Théorème 1.3.3.** Pour tout  $n \geq 1$ , il existe une suite réelle  $\{\zeta_{j,n}\}_{j \geq 0}$  telle que, pour tout  $M \in \mathbb{N}$ , on a

$$\lambda_n(h) \underset{h \rightarrow 0}{=} x_{\min} + z_1 h^{\frac{2}{3}} + \beta_n h + h^{4/3} \sum_{j=0}^M h^{j/6} \zeta_{j,n} + \mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}}),$$

avec  $\beta_n$  est la  $n$ -ième valeur propre de l'oscillateur harmonique de dimension  $N-1$

$$\sum_{i=1}^{N-1} -\partial_{u_i}^2 + \frac{\kappa_i}{2} u_i^2,$$

où  $\kappa_i$  sont les valeurs propres de l'endomorphisme de Weingarten de  $\partial\Omega$  en  $\mathcal{A}_0$ .

Ce résultat est essentiellement une généralisation de celui énoncé en (1.27), mais la preuve présentée ici, est basée sur une construction de quasimodes. L' "optimalité" de nos constructions est alors établie grâce à des estimations de décroissance et la réduction à des opérateurs modèles qui font intervenir un opérateur d'Airy sur  $\mathbb{R}_+$  et un oscillateur harmonique en dimension  $N-1$ .

**Remarque 1.3.5.**  $\beta_n$  est le  $n$ -ième élément compté avec une multiplicité de

$$\Sigma = \left\{ \sum_{i=1}^{N-1} \sqrt{\frac{\kappa_i}{2}} (2n_i - 1), n \in \mathbb{N}^{*N-1} \right\}.$$

Remarquer que pour  $N=2$ , on a

$$\Sigma = \left\{ (2n-1) \sqrt{\frac{\kappa_1}{2}}, n \in \mathbb{N}^* \right\},$$

et tous les éléments sont de multiplicité un.

**Remarque 1.3.6.** Notre stratégie peut être adaptée pour traiter un nombre fini de minima non dégénérés, et il serait même possible d'étudier l'effet tunnel lorsque ces minima présentent des symétries.

### 1.3.2.2 Stratégie de la démonstration

Dans un premier temps, nous montrons par une estimation d'Agmon, que les fonctions propres de  $\mathcal{L}_h$  ont une décroissance exponentielle en dehors de tout voisinage ne contenant pas  $\mathcal{A}_0$  (voir la Proposition 4.2.1). Par conséquent, la masse de toute fonction propre est essentiellement concentrée près du minimum du potentiel  $x_{\min}$ . Cela permet de travailler dans un voisinage tubulaire fin de  $\mathcal{A}_0$  (voir la Section 4.2.2).

Nous redressons localement la frontière en introduisant un système de coordonnées locales  $(s, t) \in \mathbb{R}^{N-1} \times \mathbb{R}_+$ , où  $t$  est la distance à la frontière et  $s$  est la variable tangentielle sur  $\partial\Omega$ . Ces coordonnées standard sont introduites précisément dans la Section 4.3.1. En Section 4.3.3, nous montrons que les fonctions propres sont localisées près de  $\mathcal{A}_0$  à une échelle  $h^{1/2}$  dans la direction  $s$  et à l'échelle  $h^{2/3}$  dans la direction  $t$ .

On rappelle une conséquence utile du théorème spectral (voir la Section A.1.3 présentée en Annexe), que si on trouve une bonne fonction  $\psi$  (appelée quasi-mode) et un nombre  $\lambda$ , tels que  $\|(\mathcal{L}_h - \lambda)\psi\| \leq \epsilon \|\psi\|$ , alors  $d(\lambda, \text{Sp}(\mathcal{L}_h)) \leq \epsilon$ . Cette méthode permet d'exhiber des nombres  $\lambda$  proches du spectre à  $\epsilon$  près. On construit des quasi-modes à partir des fonctions propres des opérateurs modèles (opérateur d'Airy et l'oscillateur harmonique en dimension  $N - 1$ ) qui sont introduits dans le chapitre 2.

En Section 4.4.1, on commence par effectuer un changement de variable local, grâce à l'estimation d'Agmon tubulaire

$$\sigma = h^{-\frac{1}{2}}s \quad \text{et} \quad \tau = h^{-\frac{2}{3}}t,$$

qui nous ramène (presque) au domaine  $\mathbb{R}^{N-1} \times \mathbb{R}_+$ , on peut développer l'opérateur obtenu (notée  $\widetilde{\mathcal{L}}_h$ ) en puissances de  $h$ , en un sens formel, sans parler de convergence (on utilise la notation  $\sim$  au lieu du signe = pour un tel développement) :

$$\widetilde{\mathcal{L}}_h \sim \sum_{j \geq 0} h^{j/6} \mathcal{L}_j,$$

avec

$$\begin{cases} \mathcal{L}_0 = x_{\min}, \\ \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3 = \mathcal{L}_5 = 0, \\ \mathcal{L}_4 = -\partial_\tau^2 + \tau, \\ \mathcal{L}_6 \text{ est unitairement équivalent à } \mathcal{H}_{\text{harm}} = \sum_{i=1}^{N-1} -\partial_{s_i}^2 + \frac{\kappa_i}{2} s_i^2, \end{cases}$$

où  $\kappa_i$  sont les valeurs propres de l'endomorphisme de Weingarten de  $\partial\Omega$  en  $\mathcal{A}_0$ , et l'on cherche des paires propres  $(\lambda, \psi)$

$$\begin{aligned} \lambda &\sim x_{\min} + h^{2/3}\alpha_0 + h\alpha_1 + \dots \\ \psi &\sim u(\tau)f(\sigma) + \dots \end{aligned}$$

vérifiant

$$\widetilde{\mathcal{L}}_h \psi \sim \lambda \psi.$$

\* Termes d'ordre  $h^{2/3}$ . On résout :

$$(\mathcal{L}_4 - \alpha_0)u(\tau)f(\sigma) = 0.$$

Il est naturel de choisir  $u$  la fonction propre de  $\mathcal{L}_4$  (c'est l'opérateur d'Airy défini en Section 2.2) avec  $\alpha_0$  est la valeur propre associée. On peut prendre

$$u(\tau) = Ai(\tau - z_1) \quad \text{et} \quad \alpha_0 = z_1,$$

où  $Ai$  est la fonction d'Airy normalisée  $L^2(\mathbb{R}^+)$  et  $-z_1$  est son premier zéro négatif.

\* Termes d'ordre  $h$ . On résout

$$\mathcal{L}_6 u(\tau) f(\sigma) = \alpha_1 u(\tau) f(\sigma).$$

On obtient alors que  $\alpha_1 \in \text{Sp}(\mathcal{L}_6)$  et que  $f$  est dans le sous espace propre associé. Comme  $\mathcal{L}_6$  est unitairement équivalent à  $\mathcal{H}_{\text{harm}}$ , on peut prendre  $\alpha_1$  est la  $n$ -ième valeur propre de  $\mathcal{H}_{\text{harm}}$ , notée  $\beta_n$ .

\* Termes d'ordre supérieur. Ce procédé peut s'appliquer à un ordre arbitrairement grand (voir la Section 4.4).

L'optimalité de nos constructions est alors établie grâce à une réduction aux opérateurs modèles. Les termes de reste sont contrôlés grâce aux estimés d'Agmon (voir la Section 4.5).

### 1.3.3 États de bord pour le Laplacien magnétique de Robin (chapitre 5)

Nous allons maintenant revenir au cœur de ce qui a motivé cette thèse et qui est un travail en collaboration avec L. Le Treust, N. Raymond et S. Vũ Ngọc (voir la Section 1.2.1.2 ci-avant).

#### 1.3.3.1 Opérateur de de Gennes avec condition de Robin

Nos résultats seront exprimés en termes de valeurs propres de l'opérateur de de Gennes avec condition de Robin au bord. Cet opérateur, qui apparaît naturellement dans l'étude des effets magnétiques près d'un bord [SJDG63, FH10b], est un opérateur différentiel d'ordre deux dépendant des paramètres réels  $\gamma$  et  $\sigma$  et agissant comme suit

$$H[\gamma, \sigma] = -\frac{d^2}{d\tau^2} + (\tau - \sigma)^2,$$

sur le domaine

$$\text{Dom}(H[\gamma, \sigma]) = \left\{ u \in B^1(\mathbb{R}_+) : \left( -\frac{d^2}{d\tau^2} + (\tau - \sigma)^2 \right) u \in L^2(\mathbb{R}_+), u'(0) = \gamma u(0) \right\},$$

où

$$B^1(\mathbb{R}_+) = \{ u \in H^1(\mathbb{R}_+) : \tau u \in L^2(\mathbb{R}_+) \}.$$

Il est bien connu que  $H[\gamma, \sigma]$  est un opérateur auto-adjoint à résolvante compacte. Ainsi, son spectre peut être écrit comme une suite croissante de valeurs propres  $(\mu_n(\gamma, \sigma))_{n \geq 1}$  (qui sont toutes simples grâce au théorème de Cauchy-Lipschitz). Nous désignons par  $u_n^{[\gamma, \sigma]}$  la suite normalisée des fonctions propres correspondantes. Nous posons

$$\Theta^{[n-1]}(\gamma) := \inf_{\sigma \in \mathbb{R}} \mu_n(\gamma, \sigma).$$

L'indice  $n - 1$  est compatible avec la notation utilisée dans le cas de l'opérateur de de Gennes (cas où  $\gamma = 0$ ), voir [FH10b, Section 3.2]. La famille  $(H[\gamma, \sigma])_{(\gamma, \sigma) \in \mathbb{R}^2}$  est analytique de type (B) (au sens de Kato, voir [Kat76, Chapitre VII, §4]), *i.e.*, le domaine de forme ne dépend pas des paramètres et la forme sesquilinéaire est analytique en fonction de  $\gamma$  ou  $\sigma$ . Par convention, nous notons par  $H[+\infty, \sigma]$  (*i.e.*  $\gamma = +\infty$ ) l'opérateur correspondant avec condition de Dirichlet  $u(0) = 0$ .

La proposition suivante regroupe les principales propriétés des fonctions  $\mu_n(\gamma, \cdot)$  (qui sont généralement appelées *courbes de dispersion*) qui seront utilisées dans cette section. La plupart d'entre elles ont été établies pour  $n = 1$  dans [Kac06b] (voir également [Kac07, Kac08], et [DBP99] dans le cas de Dirichlet).

**Proposition 1.3.2.** *Fixons  $n \geq 1$ . Lorsque  $\gamma \in \mathbb{R}$ , la fonction  $\mu_n(\gamma, \cdot)$  est analytique et*

$$\lim_{\sigma \rightarrow -\infty} \mu_n(\gamma, \sigma) = +\infty, \quad \lim_{\sigma \rightarrow +\infty} \mu_n(\gamma, \sigma) = 2n - 1. \quad (1.37)$$

De plus,  $\mu_n(\gamma, \cdot)$  a un minimum unique atteint en  $\sigma = \xi_{n-1}(\gamma)$ , mais non atteint à l'infini. Ce minimum est non dégénéré. La fonction  $\mu_n(\gamma, \cdot)$  est décroissante sur  $(-\infty, \xi_{n-1}(\gamma))$  et croissante sur  $(\xi_{n-1}(\gamma), +\infty)$ . De plus, on a, pour tout  $n \geq 2$ ,

$$2n - 3 < \Theta^{[n-1]}(\gamma) < 2n - 1. \quad (1.38)$$

Lorsque  $\gamma = +\infty$ , c'est-à-dire lorsque la condition de Robin est remplacée par la condition de Dirichlet,  $\mu_n(+\infty, \cdot)$  est toujours lisse, mais désormais décroissante de  $+\infty$  à  $2n - 1$ .

La preuve de la Proposition 1.3.2 sera donnée à la Section 2.4 (les méthodes de démonstration utilisées sont différentes de celles présentées en [Kac06b, Kac08]). Cette proposition a les conséquences élémentaires suivantes mais importantes pour notre analyse, qui sont illustrées dans la Figure 1.2 (réalisée par S. Vũ Ngọc).

**Corollaire 1.3.3.** Soit  $\gamma \in \mathbb{R} \cup \{+\infty\}$  fixé. Soit  $\Theta$  l'ensemble de toutes les valeurs critiques des fonctions  $\mu_k$  :

$$\Theta = \{\Theta^{[n-1]}(\gamma), \quad n \geq 1\}.$$

Soit  $\Lambda$  l'ensemble des points limites des fonctions  $\mu_k$  à l'infini :

$$\Lambda := \{2n - 1, \quad n \geq 1\}.$$

Soit  $[a, b] \subset \mathbb{R}$  un intervalle disjoint de  $\Lambda$ . Soit  $n = 1$  si  $a < 1$  ou  $n \geq 2$  tel que  $[a, b] \subset (2n - 3, 2n - 1)$ . (Dans le cas  $n = 1$  on prend  $a = -\infty$ ). Il découle de (1.37) que pour tout entier  $k \geq 1$ ,  $\mu_k^{-1}([a, b])$  est compact.

Soit  $p(k)$  le nombre de composantes connexes de  $\mu_k^{-1}([a, b])$ . Nous avons

$$\begin{cases} p(k) = 1 & \text{if } 1 \leq k < n \\ p(n) = 0 & \text{if } \gamma = +\infty \\ p(n) = 1 & \text{if } \gamma \in \mathbb{R} \text{ et } \Theta^{[n-1]} \in [a, b] \\ p(n) = 2 & \text{if } \gamma \in \mathbb{R} \text{ et } \Theta^{[n-1]} < a \\ p(n) = 0 & \text{if } \gamma \in \mathbb{R} \text{ et } b < \Theta^{[n-1]} \\ p(k) = 0 & \text{if } k > n. \end{cases}$$

Par conséquent, quand  $\gamma \in \mathbb{R}$ ,

$$N(\gamma, a, b) := \#\{k \geq 1 : \mu_k(\gamma, \cdot)^{-1}([a, b]) \neq \emptyset\} = \begin{cases} n & \text{si } b \geq \Theta^{[n-1]}(\gamma) \\ n - 1 & \text{sinon} \end{cases},$$

et si  $\gamma = +\infty$  (cas de Dirichlet) alors  $\mu_1(+\infty, \cdot)$  ne prend aucune valeur dans  $(-\infty, 1)$ , et  $N(\gamma, a, b) = n - 1$ .

**Notation 1.3.4.** Dorénavant, nous désignons par  $N(\gamma, a, b) = N$  ce cardinal.

**Hypothèse 3.** Dans la suite,  $a$  et  $b$  sont autorisés à dépendre de  $h$ , dès lors qu'ils restent dans un intervalle indépendant de  $h$  compact  $(2n - 3, 2n - 1)$ .

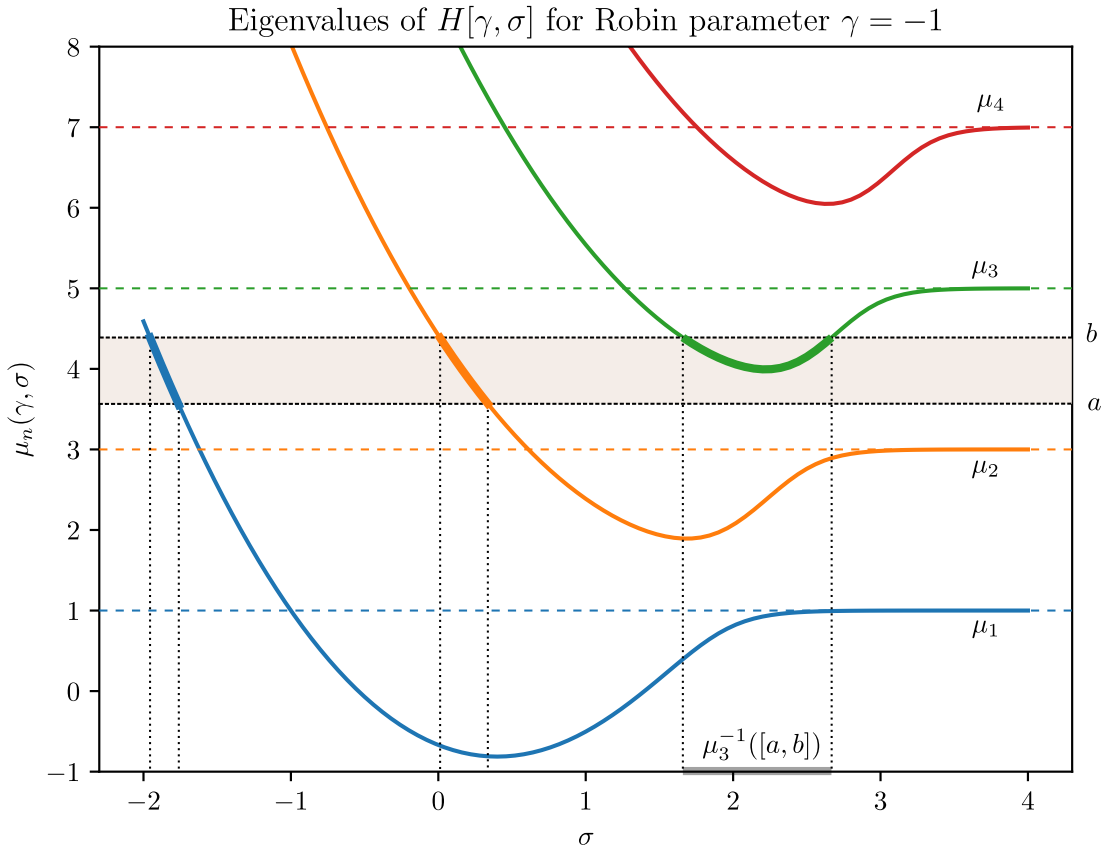


FIGURE 1.2 – Cette figure montre les courbes de dispersion  $\mu_k(\gamma, \cdot)$ , pour  $\gamma = -1$ . On visualise la préimage d'un intervalle donné  $[a, b] \subset (2n - 3, 2n - 1)$  avec  $n = 3$ . Les courbes sont obtenues par un schéma numérique standard basé sur les différences finies.

Avec cette image à l'esprit, pour chaque  $k \geq 1$ , nous pouvons construire une fonction régulière  $\overset{\circ}{\mu}_k$ , bornée (avec toutes ses dérivées), qui coïncide avec  $\mu_k$  dans un voisinage de  $\mu_k^{-1}([a, b])$ .

**Notation 1.3.5.** Soit  $\Xi_0 : \mathbb{R} \rightarrow \mathbb{R}$  une fonction régulière, bornée avec toutes ses dérivées, et croissante telle que pour tout  $k \in \{1, \dots, N\}$ ,  $\mu_k(\gamma, \Xi_0(\sigma)) = \mu_k(\gamma, \sigma)$  dans un voisinage de  $\mu_k^{-1}([a, b])$  et  $\mu_k \circ \Xi_0$  prend ses valeurs dans  $(-\infty, a) \cup (b, +\infty)$  loin de  $\mu_k^{-1}([a, b])$ . Nous considérons

$$\overset{\circ}{\mu}_k := \mu_k(\gamma, \Xi_0(\cdot)), \quad (1.39)$$

où nous omettons la référence au paramètre  $\gamma$  pour alléger la notation. Dans la suite, nous désignerons plus généralement par  $\overset{\circ}{\varphi}$ , la fonction  $\varphi$  après  $\Xi_0$ .

Les fonctions bornées  $\overset{\circ}{\mu}_k$  seront pratiques pour énoncer notre théorème principal, qui fait intervenir des opérateurs  $h^{\frac{1}{2}}$ -pseudo-différentiels avec des symboles dans la classe habituelle  $S_{\mathbb{R}^2}(1)$  donnée par

$$S_{\mathbb{R}^2}(1) = \{a \in \mathcal{C}^\infty(\mathbb{R}_{s,\sigma}^2) : \forall \alpha \in \mathbb{N}^2, \exists C_\alpha > 0 : |\partial^\alpha a| \leq C_\alpha\}.$$

Notons  $\hbar = h^{\frac{1}{2}}$ . Nous rappelons que la quantification de Weyl d'un symbole  $p$  est donnée par la formule :

$$(\text{Op}_h^W p)\psi(x) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(x-y)\eta/\hbar} p\left(\frac{x+y}{2}, \eta\right) \psi(y) dy d\eta, \quad \forall \psi \in \mathcal{S}(\mathbb{R}), \quad (1.40)$$

et que cette formule définit un opérateur borné de  $L^2(\mathbb{R})$  en  $L^2(\mathbb{R})$  si  $p \in S_{\mathbb{R}^2}(1)$ , par le théorème de Calderón-Vaillancourt. Pour abrégé la notation, nous écrirons parfois  $p^W$  au lieu de  $\text{Op}_h^W p$ .

Soit  $\mathbf{T}_{2L} = \mathbb{R}/2L\mathbb{Z}$ , et  $L^2(\mathbf{T}_{2L})$  le sous-ensemble de  $L^2_{\text{loc}}(\mathbb{R})$  des fonctions  $2L$ -périodiques, munies de la norme  $L^2$  habituelle sur  $[0, 2L]$ , où  $L$  est la demi-longueur de la frontière. On dit  $p \in S_{\mathbf{T}_{2L} \times \mathbb{R}}(1)$ , lorsque  $p \in S_{\mathbb{R}^2}(1)$  et qu'elle est  $2L$ -périodique dans sa première variable  $s$ . On désigne par  $e^{i\alpha \cdot}$  la fonction  $x \mapsto e^{i\alpha x}$ .

### 1.3.3.2 Présentation des résultats

Décrivons maintenant nos principaux résultats de cette section, qui seront exprimés en termes d'opérateurs pseudo-différentiels en une dimension. Puisque notre résultat principal décrit le spectre de  $\mathcal{L}_{h,\gamma}$  (défini en Notation 1.2.2) "modulo  $\mathcal{O}(h^\infty)$ ", nous devons préciser cette notion.

**Définition 1.3.6.** On dira que les spectres de deux opérateurs auto-adjoints  $T_1$  et  $T_2$  dépendant de  $h$  coïncident dans  $I_h$  modulo  $\mathcal{O}(h^\alpha)$ ,  $\alpha \in \mathbb{R} \sqcup \{+\infty\}$ , lorsqu'il existe  $C, h_0 > 0$  tel que, pour tout  $h \in (0, h_0)$ ,

- $T_1$  et  $T_2$  ont un spectre discret dans  $I_h + [-Ch^\alpha, Ch^\alpha]$ ,
- pour tout intervalle  $J_h \subset I_h$  on peut trouver un intervalle  $K_h$  tel que  $J_h \subset K_h$  avec  $d_H(K_h, J_h) \leq Ch^\alpha$  et

$$\text{rank } \mathbb{1}_{J_h}(T_1) \leq \text{rank } \mathbb{1}_{K_h}(T_2), \quad \text{rank } \mathbb{1}_{J_h}(T_2) \leq \text{rank } \mathbb{1}_{K_h}(T_1),$$

où  $d_H$  représente la distance de Hausdorff :

$$d_H(A, B) = \sup_{(a,b) \in A \times B} \max(d(a, B), d(b, A)).$$

**Remarque 1.3.7.** — Notons que, si les spectres de  $T_1$  et  $T_2$  coïncident dans  $I_h$  modulo  $\mathcal{O}(h^\alpha)$ , alors, pour tout  $\tilde{I}_h \subset I_h$ , les spectres de  $T_1$  et  $T_2$  coïncident dans  $\tilde{I}_h$  modulo  $\mathcal{O}(h^\alpha)$ .

- Si les spectres de  $T_1$  et  $T_2$  coïncident dans  $I_h$  modulo  $\mathcal{O}(h^\alpha)$ , on a

$$d_H(\text{Sp}(T_1) \cap I_h, \text{Sp}(T_2) \cap I_h) = \mathcal{O}(h^\alpha).$$

- La notion décrite dans la Définition 1.3.6 apparaît déjà sous diverses formes dans la littérature (voir, par exemple, le point de vue dans [HS84, Section 1] et [Hel88, §4]).

Nous pouvons maintenant énoncer notre résultat principal, où nous utilisons, entre autres, les valeurs propres  $\mu_k(\gamma, \sigma)$  et les fonctions propres  $u_k^{[\gamma, \sigma]}$  de l'opérateur de de Gennes (Section 1.3.3.1), l'entier  $N$  défini dans (1.3.4), et la notation introduite dans (1.39).

**Théorème 1.3.4.** Sous l'hypothèse 2, le spectre de  $\mathcal{L}_{h,\gamma}$  dans  $[ha, hb]$  coïncide avec celui de  $h\mathfrak{M}_h$  modulo  $\mathcal{O}(h^2)$ , où

$$\mathfrak{M}_h := \begin{bmatrix} m_1^W & 0 & \cdots & 0 \\ 0 & m_2^W & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & m_N^W \end{bmatrix}$$



est un opérateur borné agissant diagonalement sur  $e^{i\alpha(h)} L^2(\mathbf{T}_{2L})^N$ . Ici

$$\alpha(h) = \frac{|\Omega|}{|\partial\Omega|h},$$

et chaque  $m_k^W$  est un opérateur pseudodifférentiel  $h^{\frac{1}{2}}$  avec un symbole dans  $S_{\mathbf{T}_{2L} \times \mathbb{R}}(1)$ . Dénotons par  $(s, \sigma)$  les variables (canoniques) dans  $\mathbf{T}_{2L} \times \mathbb{R}$ . Alors nous avons :

- le symbole principal de  $m_k^W$  est  $\overset{\circ}{\mu}_k(\sigma)$ ;
- le sous-symbole principal est  $-\kappa(s)\overset{\circ}{C}_k(\sigma)$  avec

$$C_k(\sigma) = \langle ((\tau - \sigma)\tau^2 - \partial_\tau - 2\tau(\sigma - \tau)^2) u_k^{[\gamma, \sigma]}(\tau), u_k^{[\gamma, \sigma]}(\tau) \rangle_{L^2(\mathbb{R}_+)}, \quad (1.41)$$

et  $\kappa(s)$  est la courbure de la frontière au point d'abscisse curviligne  $s$ .

**Remarque 1.3.8.** On peut vérifier que, pour tout  $j \geq 1$ ,  $C_j(\xi_{j-1}(\gamma))$  a le même signe que  $\gamma_0^{[j-1]} - \gamma$ , voir la Proposition 5.6.2 où le seuil  $\gamma_0^{[j-1]}$  est discuté. La Proposition 5.6.2 corrige également une erreur dans [Kac06b, Lemma II.3 & (2.24)], où il est indiqué que  $C_1(\xi_0(\gamma))$  est toujours positif.

Il est important de noter que le Théorème 1.3.4 est en fait un résultat de diagonalisation puisqu'il réduit l'analyse spectrale de  $\mathcal{L}_{h, \gamma}$  à celle d'une famille d'opérateurs pseudo-différentiels en une dimension : le spectre de  $\mathfrak{M}_h$  est la superposition (en comptant les multiplicités) des spectres de  $m_k^W$ ,  $k = 1, \dots, N$ . Il s'avère que le spectre de chacun de ces opérateurs pseudo-différentiels peut être complètement décrit en utilisant des raffinements de résultats connus. En effet, on remarque que les symboles principaux  $\overset{\circ}{\mu}_j := \mu_j \circ \Xi_0$  ont des particularités :

- Ils ne dépendent que de la variable de fréquence  $\sigma$ .
- En tant que fonctions de  $\sigma$ , ils ont un point critique unique, qui est un minimum non dégénéré (Proposition 1.3.2).

Ainsi, d'un point de vue microlocal, seules deux situations sont à considérer :

- $E$  est une valeur régulière de  $\overset{\circ}{\mu}_j$ , et alors les règles bien connues de Bohr-Sommerfeld s'appliquent (voir, par exemple, [Roz75, HR82, HR84, Dui74]).
- $E$  est une valeur critique de  $\overset{\circ}{\mu}_j$ , auquel cas le hamiltonien  $(s, \sigma) \mapsto \overset{\circ}{\mu}_j(\sigma)$  admet un minimum transversal sur un cercle, et l'étude récente [DV21] s'applique.

Notre première application concerne le cas où l'intervalle  $[a, b]$  consiste en des valeurs régulières de tous les  $\mu_k$ . Nous utiliserons le résultat spectral bien connu, une extension à tous les ordres des règles de Bohr-Sommerfeld (voir, par exemple, [Roz75, HR82, HR84, Dui74]), que nous prouvons au chapitre 5 (voir la Section 5.5). Nous obtenons le résultat suivant, où nous utilisons la notation du Corollaire 1.3.3 et du Théorème 1.3.4.

**Corollaire 1.3.7** (Spectre à des valeurs régulières). *Soit  $[a, b]$  un intervalle disjoint de  $\Theta$  et  $\Lambda$ . Pour chaque  $k = 1, \dots, N$ , pour chaque  $q = 1, \dots, p(k)$ , soit  $\Sigma_{k,q} \subset \mathbb{R}$  un intervalle tel que  $\mu_k(\gamma, \cdot)$  est un difféomorphisme de  $\Sigma_{k,q}$  vers un voisinage de  $[a, b]$ , de telle sorte que tous les  $\Sigma_{k,q}$  sont disjoints deux à deux et que  $\bigcup_{q=1}^{p(k)} \Sigma_{k,q}$  contienne  $\mu_k(\gamma, \cdot)^{-1}([a, b])$  (c'est-à-dire  $\mu_k$  doit être monotone sur  $\Sigma_{k,q}$ ). Alors, il existe une fonction régulière  $\Sigma_{k,q} \ni \sigma \mapsto f_{k,q}(\sigma, \hbar) \in \mathbb{R}$  avec un développement asymptotique*

$$f_{k,q}(\sigma, \hbar) \sim f_{k,q,0}(\sigma) + \hbar f_{k,q,1}(\sigma) + \hbar^2 f_{k,q,2}(\sigma) + \dots$$

tel que le spectre de  $\mathcal{L}_{h,\gamma}$  dans  $[ha, hb]$  coïncide, modulo  $\mathcal{O}(h^2)$ , avec l'union disjointe

$$\left( \bigsqcup_{k=1}^N \bigsqcup_{q=1}^{p(k)} \left\{ h f_{k,q}(\sigma, h^{\frac{1}{2}}), \sigma \in h^{\frac{1}{2}} \left( \frac{\pi}{L} \mathbb{Z} + \alpha(h) \right) \cap \Sigma_{k,q} \right\} \right) \cap [ha, hb].$$

De plus, nous avons, lorsque  $\sigma \in \Sigma_{k,q}$ ,

$$f_{k,q,0}(\sigma) = \mu_k(\gamma, \sigma) \quad (1.42)$$

$$f_{k,q,1}(\sigma) = -\langle \kappa \rangle C_k(\sigma) \quad (1.43)$$

où  $\langle \kappa \rangle$  est la courbure moyenne :

$$\langle \kappa \rangle = \frac{1}{2L} \int_0^{2L} \kappa(s) ds.$$

Puisque les premiers termes (1.42) et (1.43) ne dépendent pas de  $q$  (en dehors du domaine de définition  $\Sigma_{k,q}$ ) nous obtenons que le spectre de  $\mathcal{L}_{h,\gamma}$  dans  $[ha, hb]$  coïncide, modulo  $\mathcal{O}(h^2)$ , avec l'union disjointe

$$\bigsqcup_{k=1}^N \left\{ h \mu_k(\gamma, \sigma) - h^{\frac{3}{2}} \langle \kappa \rangle \mathring{C}_k(\sigma), \sigma \in h^{\frac{1}{2}} \left( \frac{\pi}{L} \mathbb{Z} + \alpha(h) \right) \right\} \cap [ha, hb].$$

Comme application de ce corollaire, nous obtenons une formule très précise pour le nombre de valeurs propres de  $\mathcal{L}_{h,\gamma}$  dans  $[ha, hb]$ , qui est bien inférieure à l'estimation (1.24) :

**Théorème 1.3.5** (Formule de Weyl précise). *Soit  $I_h = [ha, hb]$  où  $[a, b]$  est un intervalle disjoint de  $\Theta$  et  $\Lambda$ . Alors, le nombre de valeurs propres de  $\mathcal{L}_{h,\gamma}$  dans  $I_h$  est*

$$N(\mathcal{L}_{h,\gamma}, I_h) = \left[ \frac{L}{\pi h^{1/2}} \sum_{k,q} \delta_{k,q}^{[0]} + \frac{L \langle \kappa \rangle}{\pi} \sum_{k,q} \delta_{k,q}^{[1]} + \mathcal{O}(h^{1/2}) \right],$$

où nous utilisons la notation  $\sum_{k,q} := \sum_{k=1}^N \sum_{q=1}^{p(k)}$ , et

$$\delta_{k,q}^{[0]} := |\alpha_{k,q} - \beta_{k,q}|, \quad \delta_{k,q}^{[1]} := \frac{C_k(\beta_{k,q})}{|\mu'_k(\beta_{k,q})|} - \frac{C_k(\alpha_{k,q})}{|\mu'_k(\alpha_{k,q})|},$$

avec  $\alpha_{k,q} := \mu_{k,q}^{-1}(a)$ ,  $\beta_{k,q} := \mu_{k,q}^{-1}(b)$ .

Dans cet énoncé, nous avons noté  $\mu_{k,q} := \mu_k(\gamma, \cdot)|_{\Sigma_{k,q}}$ . Remarquons que, comme le reste  $\mathcal{O}(h^{1/2})$  tend vers 0, on obtient que, lorsque  $h$  est suffisamment petit,  $N(\mathcal{L}_{h,\gamma}, I_h)$  est égal à la partie entière de  $\frac{L}{\pi h^{1/2}} \sum_{k,q} \delta_{k,q}^{[0]} + \frac{L \langle \kappa \rangle}{\pi} \sum_{k,q} \delta_{k,q}^{[1]}$ , ou ce plus ou moins 1.

**Remarque 1.3.9.** *Le Corollaire 1.3.7 donne une description beaucoup plus précise du spectre que l'estimation dans la Proposition 1.2.3 en nous fournissant des approximations explicites des valeurs propres dans  $[ha, hb]$  modulo  $\mathcal{O}(h^2)$ . Lorsque  $\gamma = +\infty$  (i.e. dans le cas Dirichlet), cela améliore également la description donnée dans [GV21, Corollaire 2.7] concernant une fenêtre spectrale mince contenant une valeur régulière. De plus, bien que nos résultats soient formulés en termes d'approximation des valeurs propres, la stratégie, basée sur les projections microlocales, conduisant au Théorème 1.3.4 peut également être utilisée pour décrire les espaces propres de  $\mathcal{L}_{h,\gamma}$  à l'aide de ceux de  $\mathfrak{M}_h$ .*

Notre théorème principal s'applique également au cas où la fenêtre spectrale contient une valeur critique, *i.e.* un élément de  $\Theta$ , voir Corollaire 1.3.3 (une telle valeur critique est le minimum global unique minimum global non dégénéré d'une courbe de dispersion unique, voir Proposition 1.3.2). Pour illustrer cela, concentrons-nous sur les valeurs propres basses. Notre deuxième corollaire améliore le [Kac06b, Théorème I.5,  $\alpha = \frac{1}{2}$ ] en établissant l'asymptotique spectrale des valeurs propres les plus basses et en présentant des écarts spectraux d'ordre  $h^{\frac{7}{4}}$  (au lieu de  $h^{\frac{3}{2}}$  dans le cas de valeurs régulières) pour chaque courbe de dispersion donnée. Il étend également à tout paramètre de Robin le résultat obtenu par Fournais et Helffer dans [FH06] lorsque  $\gamma = 0$ .

Une fois que le Théorème 1.3.4 est appliqué et a réduit l'analyse à un seul opérateur  $h^{\frac{1}{2}}$ -pseudo-différentiel, ce corollaire devient essentiellement une application de [DV21, Proposition 6.8], cf. Section 5.5.2 du chapitre 5 où nous donnons quelques détails.

**Corollaire 1.3.8.** *Considérons  $\gamma \neq \gamma_0^{[0]}$  avec  $\gamma_0^{[0]}$  défini dans la Remarque 1.3.8, et soit  $\epsilon = \text{sign}(\gamma_0^{[0]} - \gamma) = \text{sign}(C_1(\xi_0(\gamma)))$ . Supposons que  $\epsilon\kappa$  admet un maximum unique en  $s_{\max}$ , lequel est non dégénéré. Alors, pour tout  $j \geq 1$ , uniformément quand  $j h^{\frac{1}{4}} = o(1)$ ,*

$$\lambda_j(\gamma, h) = \Theta^{[0]}(\gamma)h - \kappa_{\max} C_1(\xi_0(\gamma))h^{\frac{3}{2}} + \left( (2j-1) \sqrt{\frac{C_1(\xi_0(\gamma))k_2\mu_1''(\gamma, \xi_0(\gamma))}{4}} \right) h^{\frac{7}{4}} + o(h^{\frac{7}{4}}),$$

avec  $k_2 = -\kappa''(s_{\max}) > 0$ ,  $\lambda_j(\gamma, h)$  est la  $j$ -ième valeur propre de  $\mathcal{L}_{h,\gamma}$  et où l'on rappelle que  $\xi_0(\gamma)$  est donné dans la Proposition 1.3.2.

**Remarque 1.3.10.** *Terminons la description de nos résultats par quelques commentaires sur les conséquences et les extensions découlant de notre approche.*

- (i) *Le Corollaire 1.3.8 décrit les valeurs propres basses avec une certaine uniformité dans  $j$  (ce qui n'était pas le cas dans [FH06]), dans un intervalle de la forme  $(-\infty, \Theta^{[0]}(\gamma)h + Ch^{3/2}]$ . D'autre part, le Corollaire 1.3.7 donne le spectre dans tout intervalle de la forme  $[ha, hb]$  avec  $a > \Theta^0(\gamma)$  et  $b < 1$ . On a donc un intervalle spectral entre ces deux régimes que nous ne décrivons pas ici. En fait, en utilisant des résultats spectraux raffinés pour les opérateurs pseudodifférentiels 1D, et en particulier la stratégie de [DV21] dans le cas où  $\kappa$  est une fonction de Morse, il serait également possible d'établir un résultat similaire au Corollaire 1.3.7. Ce résultat dépendrait du signe de  $\gamma - \gamma_0^{[n-1]}$  (qui change la nature de l'asymptotique spectrale, voir ci-dessous).*
- (ii) *Lorsque  $\gamma > \gamma_0^{[0]}$  et sous l'hypothèse que la courbure a un minimum unique, qui est non dégénéré, nous pouvons prouver un développement similaire à celle du Corollaire 1.3.8. Dans ce cas, notre analyse peut être utilisée pour prouver que les fonctions propres (associées aux valeurs propres basses) sont concentrées près des points de courbure minimale. Ceci contraste avec le cas de Neumann où les points de courbure maximale jouent le rôle de puits d'attraction. Ce phénomène n'avait pas été observé auparavant, voir la Remarque 1.3.8.*
- (iii) *Le cas  $\gamma = \gamma_0^{[0]}$  est critique puisque  $C_1(\xi_0(\gamma)) = 0$ . Cependant, notre analyse peut encore être utilisée en calculant des termes sous-principaux supplémentaires dans notre opérateur effectif. Un phénomène similaire a récemment été observé dans l'étude de l'opérateur de Dirac magnétique [BLTRS23, Section 8] et également dans l'analyse de l'opérateur de Schrödinger magnétique avec des champs magnétiques discontinus [FHKR22]. Dans ce cas, nous avons, pour tout  $j \geq 1$ ,*

$$\lambda_j(\gamma, h) = \Theta^{[0]}(\gamma)h + h^2 \lambda_j(\mathcal{A}_h) + o(h^2),$$

où

$$\mathcal{A}_h = \frac{\partial_\sigma^2 \mu(\gamma, \xi_0(\gamma))}{2} (D_s + \alpha(h) - h^{-\frac{1}{2}} \xi_0(\gamma))^2 + C_\gamma \kappa^2(s),$$

pour un certain  $C_\gamma \in \mathbb{R}$ . Dans ce régime de transition, l'opérateur effectif n'est pas semi-classique.

- (iv) Lorsque la courbure  $\kappa$  est constante, dans le cas où  $\gamma \in \mathbb{R}$ , nous sommes dans une situation dégénérée assez similaire au cas où  $\gamma = \gamma_0^{[0]}$ . En ce qui concerne les opérateurs  $m_k^W$  du Théorème 5.1.7, ce cas correspond à [DV21, Proposition 6.4]. Nous pouvons prouver un développement sous la forme

$$\lambda_j(\gamma, h) = \Theta^{[0]}(\gamma)h - \kappa C_1(\xi_0(\gamma))h^{\frac{3}{2}} + h^2 \lambda_j(\mathcal{A}_h) + o(h^2).$$

Ici, les valeurs propres de  $\mathcal{A}_h$  génèrent des oscillations magnétiques, voir [DV21, Théorème 2.2;  $k = 0$ ]. Lorsque  $\gamma = 0$  et  $j = 1$ , une estimation similaire est décrite dans [FH06, Theorem 5.3.1].

### 1.3.3.3 Stratégie de la démonstration

Décrivons les grandes lignes de la preuve du Théorème 1.3.4.

Dans un premier temps, nous prouvons que les fonctions propres associées aux valeurs propres de  $\mathcal{L}_{h,\gamma}$  dans  $[ha, hb]$  sont exponentiellement localisées près du bord de  $\Omega$ , voir la Proposition 5.2.1. Les estimations ressemblent à celles d'Agmon, mais elles ne sont pas obtenues par les moyens variationnels habituels (voir par exemple, [Hel88] ou [Ray17, Prop. 4.7]), comme c'est le cas dans de nombreux contextes magnétiques. Cela échoue ici, puisque nous voulons considérer des valeurs propres entre deux niveaux de Landau consécutifs. Pour surmonter ce problème, notre stratégie, qui généralise la méthode variationnelle, est basée sur la bijectivité du Laplacien magnétique entre des espaces  $L^2$  à poids exponentiels.

La localisation exponentielle près de la frontière à l'échelle d'ordre  $h^{1/2}$  nous invite à utiliser les coordonnées tubulaires classiques  $(s, t)$  près de la frontière. Nous notons par  $\widetilde{\mathcal{L}}_{h,\gamma}$ , l'opérateur  $\mathcal{L}_{h,\gamma}$  en coordonnées  $(s, t)$ , défini dans la Section 5.3.1. Après avoir montré une estimation d'Agmon pour  $\widetilde{\mathcal{L}}_{h,\gamma}$  (voir la Section 5.6.3.1), nous établissons dans l'Appendice 5.6.3, que les spectres de  $\mathcal{L}_{h,\gamma}$  et  $\widetilde{\mathcal{L}}_{h,\gamma}$  coïncident modulo  $\mathcal{O}(h^\infty)$ . Maintenant, le problème est que nous ne pouvons pas réduire directement le problème à un problème près du bord. En fait, si nous considérons l'opérateur dans un voisinage tubulaire avec des conditions de Dirichlet ( $\gamma = +\infty$ ) sur le bord extérieur et intérieur de ce voisinage, et par suite par les estimations d'Agmon, nous ne pouvons pas localiser le problème dans ce voisinage tubulaire. Pour ce faire, nous allons introduire une fonction de troncature en  $t$  près de 0, notée  $\zeta$ .

Ensuite, afin d'obtenir une condition au bord indépendante de  $h$ , on se ramène à un problème partiellement semi-classique par le changement d'échelle <sup>11</sup>

$$(s, t) = (s, h^{1/2}\tau).$$

Nous introduisons un opérateur modèle  $\mathcal{N}_h$  (voir la Section 5.3.2) dépendant du paramètre effectif semi-classique  $\hbar = h^{\frac{1}{2}}$ , agissant sur des fonctions  $2L$  périodiques et impliquant un terme de flux  $\mathfrak{f}_0$  ( $\mathfrak{f}_0 = \frac{|\Omega|}{|\partial\Omega|}$ )

$$\mathcal{N}_h = \hat{a}_h(s, \tau)^{-1} p_{s,\hbar}^{\Xi_0} \hat{a}_h(s, \tau)^{-1} p_{s,\hbar}^{\Xi_0} - \hat{a}_h(s, \tau)^{-1} \partial_\tau \hat{a}_h(s, \tau) \partial_\tau,$$

11. En Section 5.3.2, nous définissons  $\widehat{\mathcal{L}}_{h,\gamma}$ , l'opérateur après le changement de variable et après la troncature  $\zeta$  en  $t$ . Par des estimations d'Agmon (voir la Proposition 5.6.5), nous pouvons montrer que les spectres de  $\mathcal{L}_{h,\gamma}$  et  $\widehat{\mathcal{L}}_{h,\gamma}$  sont coïncident modulo  $\mathcal{O}(h^\infty)$ . Ceci nous permet de nous concentrer sur l'analyse spectrale de  $\mathcal{L}_{h,\gamma}$ . En fait, dans notre présentation, nous avons évité l'analyse spectrale de  $\widehat{\mathcal{L}}_{h,\gamma}$  (existence du spectre discret, etc.).

où

$$\begin{aligned} p_{s,\hbar}^{\Xi_0} &= \text{Op}_\hbar^W \Xi_0(\cdot + \theta) - \tau + \hbar \kappa(s) \frac{\hat{\tau}^2}{2}, \quad \theta = \hbar^{-1} \mathfrak{f}_0, \\ \hat{a}_\hbar(s, \tau) &= 1 - \hbar \hat{\tau} \kappa \quad \text{avec} \quad \hat{\tau} = \zeta(\hbar^{2\eta} \tau) \tau = \zeta_\eta(\tau) \tau, \end{aligned}$$

où  $\zeta$  est une fonction régulier égale à 1 près de 0 et  $0 < \eta < \frac{1}{2}$ .

$$\begin{aligned} \text{Dom}(\mathcal{N}_\hbar) &= \{u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+) : -\partial_\tau^2 u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+), \\ &\quad \tau^2 u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+), \partial_\tau u(\cdot, 0) = \gamma u(\cdot, 0)\}. \end{aligned}$$

Nous montrons que les fonctions propres de  $\mathcal{L}_{\hbar,\gamma}$  associées aux valeurs propres dans  $[ah, bh]$  sont grossièrement microlocalisées dans un ensemble compact de l'espace des phases attaché à la frontière <sup>12</sup>, voir la Proposition 5.3.1. Ceci permet de prouver que le spectre de  $\mathcal{L}_{\hbar,\gamma}$  (entre les niveaux de Landau) est situé près de celui de  $\hbar \mathcal{N}_\hbar$ , voir Proposition 5.3.2. Cependant, on verra que la Proposition 5.3.2 n'est pas directement utile pour établir nos théorèmes principaux. Elle est plutôt un prétexte pour motiver l'introduction de  $\mathcal{N}_\hbar$  et décrire les estimations spectrales nécessaires pour prouver que les spectres coïncident modulo  $\mathcal{O}(\hbar^\infty)$ . En fait, on comparera directement le spectre de  $\mathcal{L}_{\hbar,\gamma}$  à celui d'un opérateur effectif sur la frontière de  $\Omega$  pour éviter l'analyse spectrale de  $\mathcal{N}_\hbar$  (existence du spectre discret, estimations d'Agmon, etc.).

Nous construisons un problème de Grushin afin d'inverser l'opérateur (pseudo)différentiel  $\mathfrak{N}_\hbar$  (qui agit comme  $\mathcal{N}_\hbar$  avec  $\mathfrak{f}_0$  remplacé par 0). Cette méthode, inspirée des travaux de Martinez et Sjöstrand (et adaptée aux opérateurs magnétiques par Keraval, voir [Ker18]), a récemment montré son efficacité pour décrire les valeurs propres de divers opérateurs magnétiques (voir, par exemple, [FHKR22] et [BNHR22]). La nouveauté ici est de l'utiliser pour s'attaquer à la description de valeurs propres plus grandes pour les opérateurs magnétiques de Schrödinger à bord, lorsque plusieurs courbes de dispersion sont impliquées (voir Figure 1.2), et pas seulement la première comme dans [FHKR22] ou [BNHR22]. Afin d'utiliser cette méthode, nous écrivons un développement semiclassique de  $\mathcal{N}_\hbar$  (voir la Proposition 5.4.4) :

$$\mathfrak{N}_\hbar = \mathfrak{n}_0 + \hbar \mathfrak{n}_1 + \dots,$$

où, les  $\mathfrak{n}_j$  sont donnés par  $\mathfrak{n}_j = \text{Op}_\hbar^W n_j$  avec

$$\begin{aligned} n_0 &= -\partial_\tau^2 + (\Xi_0(\sigma) - \tau)^2, \\ n_1 &= \kappa(s) \left[ (\Xi_0(\sigma) - \tau) \zeta_\mu^2 \tau^2 + \zeta_\mu \partial_\tau + 2\zeta_\mu \tau (\Xi_0(\sigma) - \tau)^2 \right]. \end{aligned}$$

En particulier, nous pouvons écrire  $\mathfrak{N}_\hbar = \text{Op}_\hbar^W(n_\hbar)$  avec un symbole  $n_\hbar$  satisfaisant

$$n_\hbar = n_0 + \hbar n_1 + \dots$$

Le symbole principal est l'opérateur de de Gennes (avec la condition de Robin), qui peut être inversé dans la fenêtre spectrale  $[a, b]$  quitte à considérer une matrice augmentée  $\mathcal{P}_0$  impliquant les fonctions propres de l'opérateur de de Gennes (voir le lemme 5.4.5).

12. Nous avons montré une estimation d'Agmon compatible avec le multiplicateur de Fourier. Bien que nous ne l'ayons pas utilisé dans notre présentation, il convient de noter qu'il s'agit d'un nouveau résultat sur la décroissance exponentielle compatible avec le multiplicateur de Fourier (voir la Section 5.6.4).

Pour tout  $z \in [a, b]$ ,

$$\mathcal{P}_0(z) = \begin{pmatrix} n_0(s, \sigma) - z & \Pi^* \\ \Pi & 0 \end{pmatrix},$$

est bijectif avec inverse

$$\mathcal{Q}_0(z) = \begin{pmatrix} q_0 & \Pi^* \\ \Pi & z - M_0(\sigma) \end{pmatrix}, \quad q_0 = (n_0(s, \sigma) - z)^{-1} (\Pi^* \Pi)^\perp,$$

où  $M_0(\sigma)$  est la matrice diagonale  $N \times N$  dont la diagonale est  $(\overset{\circ}{\mu}_1, \dots, \overset{\circ}{\mu}_N)$ ,  $\Pi^*(\alpha) = \sum_{j=1}^N \alpha_j \overset{\circ}{u}_j$  et  $\Pi\psi = (\langle \psi, \overset{\circ}{u}_j \rangle)_{1 \leq j \leq N}$ .

Cela permet de construire un inverse approché  $\text{Op}_h^W \mathcal{Q}_h$  d'une version augmentée de  $\mathfrak{N}_h$  notée  $\text{Op}_h^W \mathcal{P}_h$ ,

$$\text{Op}_h^W \mathcal{P}_h = \begin{pmatrix} \mathfrak{N}_h - z & \mathfrak{P}^* \\ \mathfrak{P} & 0 \end{pmatrix}, \quad \mathfrak{P} = \text{Op}_h^W \Pi.$$

Cherchons un inverse approché tel que

$$\begin{cases} \text{Op}_h^W \mathcal{Q}_h \text{Op}_h^W \mathcal{P}_h(z) = \text{Id} + \mathcal{O}(\hbar^2), \\ \text{Op}_h^W \mathcal{P}_h(z) \text{Op}_h^W \mathcal{Q}_h = \text{Id} + \mathcal{O}(\hbar^2). \end{cases}$$

Comme dans [Ker18, Section 3.1.3], nous sommes amenés à choisir  $\mathcal{Q}_h$  sous la forme :

$$\mathcal{Q}_h = \begin{pmatrix} q_h & q_h^+ \\ q_h^- & q_h^\pm \end{pmatrix}.$$

Avec ce choix, le coefficient inférieur droit, noté  $q_h^\pm$ , de la matrice  $\mathcal{Q}_h$  est

$$q_h^\pm = z - M_0(\sigma) - \hbar \Pi n_1 \Pi^*.$$

Ceci invite à considérer l'opérateur pseudo-différentiel matriciel effectif dont le symbole est

$$M_h = M_0(\sigma) + \hbar \Pi n_1 \Pi^*.$$

On considère  $\psi$  une fonction propre de  $\mathcal{L}_{h,\gamma}$  associée à  $\lambda \in [ah, bh]$ . Pour  $z = h^{-1}\lambda$ , Grâce à ces quasi inverses, on obtient

$$\text{Op}_h^W \mathcal{Q}_h \text{Op}_h^W \mathcal{P}_h \begin{pmatrix} \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} + \mathcal{O}(\hbar^2) \|\varphi\|,$$

où  $\varphi$  désigne la fonction  $\psi$  après multiplication par une fonction de troncature en  $t$ .

Nous en déduisons que

$$\begin{cases} \text{Op}_h^W q_h^+ \mathfrak{P} \varphi = \varphi + \mathcal{O}(\hbar^2) \|\varphi\|, \\ (z - \text{Op}_h^W M_h) \mathfrak{P} \varphi = \mathcal{O}(\hbar^2) \|\varphi\|. \end{cases}$$

Puisque  $\text{Op}_h^W q_h^+$  est borné uniformément en  $\hbar$ , la première relation implique que

$$\|\varphi\| \leq C \|\mathfrak{P} \varphi\|.$$

Ensuite, de la deuxième relation et du théorème spectral, nous déduisons que

$$\text{dist}(z, \text{Sp}(\text{Op}_h^W M_h)) \leq C\hbar^2.$$

Cela signifie que le spectre de  $h^{-1}\mathcal{L}_{h,\gamma}$  dans la fenêtre  $[ah, bh]$  est à une distance d'ordre  $\hbar^2$  du spectre de l'opérateur effectif  $\text{Op}_h^W M_h$ .

En Section 5.5, nous effectuons l'analyse spectrale de  $\text{Op}_h^W M_h$  en utilisant le fait que le symbole principal matriciel  $M_0$  est diagonal avec des écarts uniformes entre les entrées diagonales. On en déduit que les spectres de  $\text{Op}_h^W M_h$  et  $\text{Op}_h^W M_0 - \hbar \text{diag} \left( \text{Op}_h^W \kappa(s) C_j \circ \Xi_0(\sigma) \right)$  coïncident modulo  $\mathcal{O}(\hbar^2)$ .

Nous déduisons les Corollaires 1.3.7 et 1.3.8.

## 1.4 Sommaire

Dans le chapitre 2, nous discutons de certaines propriétés spectrales liées aux opérateurs modèles en dimension 1, qui apparaissent dans le voisinage de la frontière et qui interviennent dans notre stratégie de réduction de la dimension.

Dans le chapitre 3, nous étudions l'analyse spectrale semi-classique de l'opérateur de Schrödinger auto-adjoint avec un champ magnétique fort et une condition de Robin attractive au bord d'un domaine régulier et borné dans les régimes où le champ magnétique joue un rôle majeur, en concurrence avec la condition de Robin. Par les estimations d'Agmon habituelles, nous montrons pourquoi les fonctions propres correspondant aux plus petites valeurs propres sont localisées près de la frontière. En utilisant l'approximation de Born-Oppenheimer, nous dérivons un opérateur effectif sur la frontière dont les valeurs propres décrivent simultanément les valeurs propres du Laplacien magnétique de Robin. Nous remarquons que cette approximation est valable sans aucune hypothèse supplémentaire sur le domaine. Dans une situation générique où la courbure a un maximum unique non dégénéré, nous estimons l'écart spectral et trouvons que le champ magnétique ne contribue pas au développement asymptotique à trois termes des valeurs propres basses. Dans le cas des domaines de disques, nous montrons la contribution du champ magnétique dans l'asymptotique des valeurs propres les plus basses. Ce chapitre est issu de [Fah23a].

Dans le chapitre 4, nous établissons un développement asymptotique complet des basses valeurs propres de l'hamiltonien de Stark en dimension  $N \geq 1$ , en impliquant les valeurs propres de l'endomorphisme de Weingarten, et nous montrons que leurs fonctions propres associées sont localisées près du potentiel minimal. En outre, nous appliquons nos techniques pour trouver un développement asymptotique complet des valeurs propres de l'opérateur plus général  $-h^2\Delta + V(\mathbf{x})$  où  $h$  est un petit paramètre et  $V$  est un potentiel réel régulier satisfaisant certaines conditions. Ces résultats sont issus de [Fah23b].

Dans le chapitre 5, nous présentons des résultats obtenus en collaboration avec N. Raymond, L. Le Treust et S. Vũ Ngọc [FLTRVuN23]. Pour  $\gamma \in \mathbb{R} \cup \{+\infty\}$ , dans la limite semi-classique, en utilisant la méthode de Grushin et les outils d'analyse microlocale, nous réduisons l'analyse spectrale de  $\mathcal{L}_{h,\gamma}$  à celle d'une famille d'opérateurs pseudo-différentiels en une dimension. Ensuite, nous établissons une description uniforme du spectre de  $\mathcal{L}_{h,\gamma}$  situé entre les niveaux de Landau. Cela permet d'obtenir une description uniforme du spectre dans une large classe de Laplaciens magnétiques incluant les Laplaciens de Neumann et de Dirichlet. Les fonctions propres correspondantes, appelées *états de bord*, sont exponentiellement localisées près de la frontière. Les estimations ressemblent à celles d'Agmon, mais elles ne sont pas obtenues par des moyens variationnels comme c'est le cas dans de nombreux contextes magnétiques. Au moyen d'une réduction dimensionnelle microlocale, notre approche nous permet d'affiner simultanément d'anciens résultats sur les

valeurs propres basses dans le cas de Robin et des résultats récents sur les états de bord dans le cas de Dirichlet, en établissant les asymptotiques spectrales des valeurs propres les plus basses et en montrant les écarts spectraux.





# Analysis of auxiliary operators

## Abstract

In this chapter, we discuss some spectral properties related to model operators in dimension one (a couple of known results are recalled and useful new ones are established). These model operators appear naturally in our dimensional reduction strategy. In particular, it will appear in the neighborhood of the boundary and will play an important role in the following chapters.

## Résumé

Dans ce chapitre, nous discutons de certaines propriétés spectrales liées aux opérateurs modèles en dimension un (certains résultats connus sont rappelés et de nouveaux résultats utiles sont établis). Ces opérateurs modèles apparaissent naturellement dans notre stratégie de réduction dimensionnelle. En particulier, ils apparaissent dans le voisinage de la frontière et joueront un rôle important dans les chapitres suivants.

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## 2.1 Robin Laplacians

The purpose of this section is to recall some spectral properties related to the Robin Laplacian in dimension one. Most of them have been established, for example in [HK17, HKR17, KS20, KKR16, Ray17, LB03].

### 2.1.1 Robin Laplacian on a half line

Consider the operator

$$\mathcal{H}_0 := -\frac{d^2}{d\tau^2} \quad \text{in } L^2(\mathbb{R}_+),$$

with domain

$$\text{Dom}(\mathcal{H}_0) = \{u \in H^2(\mathbb{R}_+) : u'(0) = -u(0)\}.$$

This operator is self-adjoint but does not have a compact resolvent.

**Lemma 2.1.1.** *The discrete spectrum ( $\text{Sp}_{\text{dis}}$ ) of  $\mathcal{H}_0$  is  $\{-1\}$  and its essential spectrum ( $\text{Sp}_{\text{ess}}$ ) is  $[0, +\infty[$ . The subspace associated with the single eigenvalue  $-1$  is generated by the  $L^2$ -normalized eigenfunction*

$$u_0(\tau) = \sqrt{2} \exp(-\tau).$$

*Proof.* For the proof, see for example [Ray17, Prop. 2.8]. □

### 2.1.2 Robin Laplacian on an interval

The following operator arises naturally in the analysis of the Robin Laplacian without magnetic field (see for instance [EMP14, HK17, HKR17, LP08, PP16]). Let  $T > 0$ . Consider the operator

$$\mathcal{H}_0^{\{T\}} := -\frac{d^2}{d\tau^2} \quad \text{in } L^2(0, T),$$

with domain

$$\text{Dom}(\mathcal{H}_0^{\{T\}}) = \{u \in H^2(0, T) : u'(0) = -u(0) \text{ and } u(T) = 0\}.$$

Note that this operator is associated with the quadratic form

$$V_h \ni u \mapsto \int_0^T |u'(\tau)|^2 d\tau - |u(0)|^2,$$

with

$$V_h := \{u \in H^1(0, T) : u(T) = 0\}.$$

The operator  $\mathcal{H}_0^{\{T\}}$  is self-adjoint and has a compact resolvent. Its spectrum is therefore purely discrete and consists of an increasing sequence of eigenvalues denoted by  $(\lambda_n(\mathcal{H}_0^{\{T\}}))_{n \in \mathbb{N}^*}$ .

The following lemma gives the behavior of the lowest eigenvalues  $\lambda_1(\mathcal{H}_0^{\{T\}})$  and  $\lambda_2(\mathcal{H}_0^{\{T\}})$  when  $T \rightarrow +\infty$ .

**Proposition 2.1.2.** *As  $T \rightarrow +\infty$ , there holds*

$$\lambda_1(\mathcal{H}_0^{\{T\}}) = -1 + \mathcal{O}(\exp(-2T)), \quad \text{and} \quad \lambda_2(\mathcal{H}_0^{\{T\}}) \geq 0. \quad (2.1)$$

*Proof.* For the proof, see for example [HK17, Lemma 4.1]. □

### 2.1.3 Robin Laplacian in a weighted space

In order to deal with curvature effects, one will need a slightly more sophisticated operator. Let  $B \in \mathbb{R}$ ,  $T > 0$  such that  $|B|T < \frac{1}{3}$ . In the weighted space  $L^2((0, T); (1 - B\tau) d\tau)$ , we introduce the self-adjoint operator

$$\mathcal{H}_B^{\{T\}} = -\frac{d^2}{d\tau^2} + \frac{B}{1 - B\tau} \frac{d}{d\tau},$$

with domain

$$\text{Dom}(\mathcal{H}_B^{\{T\}}) = \{u \in H^2(0, T) : u'(0) = -u(0) \text{ and } u(T) = 0\}.$$

The operator  $\mathcal{H}_B^{\{T\}}$  is associated with the following quadratic form:

$$q_B^{\{T\}}(u) = \int_0^T |u'(\tau)|^2 (1 - B\tau) d\tau - |u(0)|^2,$$

with domain

$$\text{Dom}(q_B^{\{T\}}) = \{u \in H^1(0, T) : u(T) = 0\},$$

in the ambient space  $L^2((0, T); (1 - B\tau)d\tau)$ . The operator  $\mathcal{H}_B^{\{T\}}$  has a compact resolvent. Hence, its spectrum  $\text{Sp}(\mathcal{H}_B^{\{T\}}) \subseteq \mathbb{R}$  is purely discrete and consists of a strictly increasing sequence of eigenvalues  $(\lambda_n(\mathcal{H}_B^{\{T\}}))_{n \in \mathbb{N}^*}$ .

The following proposition states a two-term asymptotic expansion of the eigenvalue  $\lambda_1(\mathcal{H}_B^{\{T\}})$ .

**Proposition 2.1.3** (Asymptotic of  $\lambda_1(\mathcal{H}_B^{\{T\}})$ ). *There exist constants  $C > 0$  and  $T_0 > 0$ , such that for all  $T \geq T_0$ , and  $B \in (-1/(3T), 1/(3T))$ , we have:*

$$|\lambda_1(\mathcal{H}_B^{\{T\}}) - (-1 - B)| \leq CB^2.$$

*Proof.* For the proof, see for example [HK17, Prpo. 4.5] or [Ray17, Prop. 2.11].  $\square$

Let us use the change of function  $u = (1 - B\tau)^{-\frac{1}{2}} \tilde{u}$ , since the new Hilbert space becomes  $L^2((0, T), d\tau)$ , the form domain is still independent from the parameter, and the expression of the operator depends on  $B$  analytically:

$$\tilde{\mathcal{H}}_B^{\{T\}} = -\frac{d^2}{d\tau^2} - \frac{B^2}{4(1 - B\tau)^2},$$

with the new Robin condition at 0 given by  $\tilde{u}'(0) = \left(-1 - \frac{B}{2}\right) \tilde{u}(0)$  and  $\tilde{u}(T) = 0$ .

The price to pay is that the domain of the operators depends on  $B$  through the  $B$ -dependent boundary condition. Note that this operator is associated with quadratic form defined on  $H^1(0, T)$  by

$$\tilde{q}_B^{\{T\}}(\psi) = \int_0^T |\partial_\tau \psi|^2 d\tau - \int_0^T \frac{B^2}{4(1 - B\tau)^2} |\psi|^2 d\tau - \left(1 + \frac{B}{2}\right) |\psi(0)|^2.$$

Thus, the form domain does not depend on the parameter. For further use, we will need the decay estimate of the ground state  $\tilde{u}_B^{\{T\}}$ . As far as we know, this estimate was not established in the above mentioned works.

**Proposition 2.1.4.** *There exist constants  $C > 0$ ,  $\alpha > 0$  and  $T_0 > 0$  such that, for all  $T \geq T_0$ , and  $B \in (-1/(3T), 1/(3T))$ , we have:*

$$\|e^{\alpha\tau} \tilde{u}_B^{\{T\}}\|_{L^2((0, T), d\tau)} \leq C,$$

where  $\tilde{u}_B^{\{T\}}$  is the normalized eigenfunction associated with  $\lambda_1(\mathcal{H}_B^{\{T\}})$ .

*Proof.* Let  $\phi = \alpha \tau$  with  $\alpha$  a real positive constant. We have the identity

$$\left\langle \tilde{\mathcal{H}}_B^{\{T\}} \tilde{u}_B^{\{T\}}, e^{2\phi} \tilde{u}_B^{\{T\}} \right\rangle = \tilde{q}_B^{\{T\}} \left( e^\phi \tilde{u}_B^{\{T\}} \right) - \left\| \phi' e^\phi \tilde{u}_B^{\{T\}} \right\|^2,$$

which follows by integration by parts (see for instance [Ray17, Prop. 4.7]). Then, by using that  $\tilde{u}_B^{\{T\}}$  is an eigenfunction of  $\mathcal{H}_B^{\{T\}}$  associated with  $\lambda_1(\mathcal{H}_B^{\{T\}})$ , we get

$$\begin{aligned} \lambda_1(\mathcal{H}_B^{\{T\}}) \left\| e^\phi \tilde{u}_B^{\{T\}} \right\|^2 &= \int_0^T \left| \partial_\tau \left( e^{2\phi} \tilde{u}_B^{\{T\}} \right) \right|^2 d\tau - \int_0^T \frac{B^2}{4(1-B\tau)^2} e^{2\phi} \left| \tilde{u}_B^{\{T\}} \right|^2 d\tau \\ &\quad - \left( 1 + \frac{B}{2} \right) e^{2\phi(0)} \left| \tilde{u}_B^{\{T\}}(0) \right|^2 - \left\| \phi' e^\phi \tilde{u}_B^{\{T\}} \right\|^2. \end{aligned}$$

We have

$$\left| \tilde{u}_B^{\{T\}}(0) \right|^2 = -2 \int_0^T \partial_\tau \tilde{u}_B^{\{T\}}(\tau) \tilde{u}_B^{\{T\}}(\tau) d\tau,$$

and then, there are constants  $C > 0$  such that, for all  $\epsilon > 0$ , we have:

$$\left| \tilde{u}_B^{\{T\}}(0) \right|^2 \leq C\epsilon \left\| \partial_\tau \tilde{u}_B^{\{T\}} \right\|_{L^2(0,T)}^2 + C\epsilon^{-1} \left\| \tilde{u}_B^{\{T\}} \right\|_{L^2(0,T)}^2. \quad (2.2)$$

Inserting (2.2) in  $\tilde{q}_B^{\{T\}}$ , there exist constants  $C > 0$  such that:

$$\left( 1 - C\epsilon - \frac{C\epsilon B}{2} \right) \left\| \partial_\tau \tilde{u}_B^{\{T\}} \right\|_{L^2(0,T)}^2 \leq \left( \lambda_1(\mathcal{H}_B^{\{T\}}) + C\epsilon^{-1} \left( 1 + \frac{B}{2} \right) + C \right) \left\| \tilde{u}_B^{\{T\}} \right\|_{L^2(0,T)}^2.$$

For  $B \rightarrow 0$ ,  $\epsilon = \frac{1}{3C}$  and from Proposition 2.1.3, we obtain:

$$\left\| \tilde{u}_B^{\{T\}} \right\|_{H^1(0,T)} \leq C,$$

thanks to (2.2), we get

$$\left| \tilde{u}_B^{\{T\}}(0) \right|^2 \leq C.$$

This implies the simple estimate

$$- \int_0^T \frac{B^2}{4(1-B\tau)^2} e^{2\phi} \left| \tilde{u}_B^{\{T\}} \right|^2 d\tau - \left( 1 + \frac{B}{2} \right) C - \int_0^T |\phi'|^2 e^{2\phi} \left| \tilde{u}_B^{\{T\}} \right|^2 d\tau \leq \lambda_1(\mathcal{H}_B^{\{T\}}) \left\| e^\phi \tilde{u}_B^{\{T\}} \right\|^2,$$

and then:

$$\int_0^T \left[ -\frac{B^2}{4(1-B\tau)^2} - \lambda_1(\mathcal{H}_B^{\{T\}}) - |\phi'|^2 \right] e^{2\phi} \left| \tilde{u}_B^{\{T\}} \right|^2 d\tau \leq C.$$

As  $T \rightarrow +\infty$ , we have  $-1 \leq B \leq 1$ , and

- $\frac{B^2}{4(1-B\tau)^2} \leq \frac{B^2}{4(1-BT)^2}$ ,
- $\lambda_1(\mathcal{H}_B^{\{T\}}) \leq -1 - B + C_1 B^2$ .

We choose  $\alpha$  such that

$$-\frac{B^2}{4(1-BT)^2} - 1 - B + C_1 B^2 - \alpha^2 \geq C,$$

therefore,

$$\int_0^T e^{2\alpha\tau} |\tilde{u}_B^{\{T\}}|^2 d\tau \leq C.$$

□

**Lemma 2.1.5.** *There exist constants  $C > 0$  and  $T_0 > 0$ , such that, for all  $T \geq T_0$  and  $B \in (-1/(3T), 1/(3T))$ , we have:*

$$|\lambda_1(\mathcal{H}_B^{\{T\}})| \leq C \quad \text{and} \quad \|\partial_B \tilde{u}_B^{\{T\}}\|_{L^2((0,T), d\tau)} \leq C.$$

*Proof.* For the proof, see for example [HKR17, Lemma 7.3].

□

## 2.2 Airy operator on a half-line

In the study of the Stark Hamiltonian, we will encounter the famous Airy operator. The Airy operator  $\mathcal{A}$  with domain

$$\text{Dom}(\mathcal{A}) = \{f \in H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+), tf \in L^2(\mathbb{R}_+)\},$$

is defined by

$$\mathcal{A}f = (-\partial_t^2 + t)f.$$

The Airy operator  $\mathcal{A}$  is injective, self-adjoint and has a compact resolvent. Hence, its spectrum is discrete and consists of a sequence of real eigenvalue  $\mu_j$  tending to  $+\infty$ . The corresponding eigenfunctions can be obtained in the following way.

We call Airy's function, and we denote by  $Ai$  the inverse Fourier transform of  $\xi \mapsto e^{i\xi^3/3}$ . It is positive on  $[0, +\infty[$ , and its zeros form a decreasing sequence tending towards  $-\infty$ . See Figure 2.1.

Let  $-\mu$  be one of the zeros of the Airy function (recall that they are negative). Then  $t \mapsto Ai(t - \mu)$  is an eigenfunction of the operator  $\mathcal{A}$ , with eigenvalue  $\mu$ . These eigenvalues are simple (see for instance [Hel13, Section 14.5.1], [DR12, Section 2.2.2] and [OB14, Section 2.3.1]).

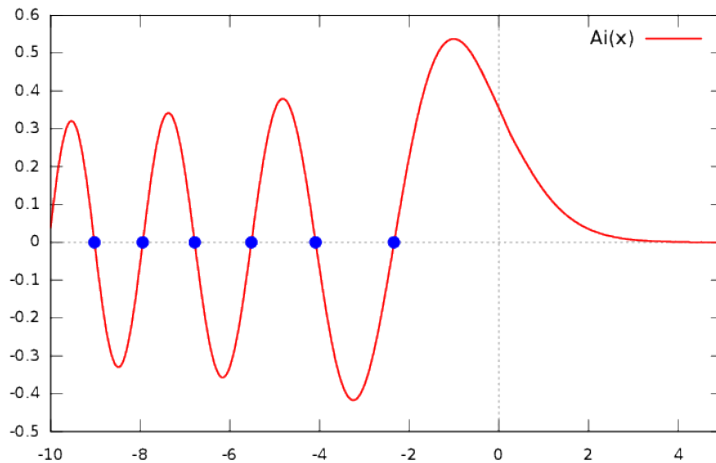


Figure 2.1 – Airy's function  $Ai$ . Source: Figure in [Rou15].

## 2.3 Harmonic oscillator

### 2.3.1 A 1-dimensional harmonic oscillator

We recall the spectrum of the harmonic oscillator, which we will encounter several times in this thesis. Consider the self-adjoint realization on  $L^2(\mathbb{R})$  of :

$$\mathcal{H}_{\text{harm}}^a = -\partial_x^2 + \frac{a}{2} x^2, \quad a > 0. \quad (2.3)$$

This operator can be defined as the self-adjoint operator associated with the quadratic form

$$\mathfrak{Q}_{\text{harm}}^a(\psi) = \|\psi'\|^2 + \frac{a}{2} \|x\psi\|^2, \quad \psi \in \mathbb{B}^1(\mathbb{R}), \quad (2.4)$$

where

$$\mathbb{B}^1(\mathbb{R}) = \left\{ \psi \in L^2(\mathbb{R}) : \psi' \in L^2(\mathbb{R}), x\psi \in L^2(\mathbb{R}) \right\}.$$

Since  $\mathbb{B}^1(\mathbb{R})$  is compactly embedded in  $L^2(\mathbb{R})$ , then the self-adjoint operator  $\mathcal{H}_{\text{harm}}^a$  has compact resolvent. Its spectrum is a sequence of eigenvalues that tends to  $+\infty$ .

The  $n$ -th eigenvalue of  $\mathcal{H}_{\text{harm}}^a$  is  $(2n-1)\sqrt{\frac{a}{2}}$ ,  $n \in \mathbb{N}^*$ . They are simple and associated with the normalized Hermite's functions denoted by  $f_n(x)$ . They can be determined by recursion as follows

$$f_1(x) = c_1 \exp^{-\sqrt{\frac{a}{2}} \frac{x^2}{2}}, \quad f_n(x) = c_n (L^+)^{n-1} f_1(x) \quad (n \geq 1),$$

where  $L^+ = -\frac{d}{dx} + \sqrt{\frac{a}{2}}x$ , and, for every  $n \geq 1$ , the constant  $c_n$  normalizes  $f_n(x)$  in  $L^2(\mathbb{R})$ . For more details on the spectral study of  $\mathcal{H}_{\text{harm}}^a$ , see for example [Hel13, Section 1.3] and [Ray17, Section 2.1].

### 2.3.2 A $d$ -dimensional harmonic oscillator

In dimension  $d \geq 1$ , a  $d$ -dimensional harmonic oscillator is an operator of the following form

$$\sum_{i=1}^d -\partial_{x_i}^2 + \frac{a_i}{2} u_i^2,$$

where  $a_i > 0$ . We deduce from the case  $d = 1$ , that the spectrum of this operator is given by

$$\left\{ \sum_{i=1}^d \sqrt{\frac{a_i}{2}} (2n_i - 1), \quad n = (n_1, \dots, n_d) \in \mathbb{N}^d \right\}.$$

The corresponding normalized eigenfunctions are  $f_n(x) = f_{n_1}(x_1) \dots f_{n_d}(x_d)$ , where the function  $f_{n_i}$  are the Hermite functions.

## 2.4 De Gennes operator

In this section, we are interested in the spectral analysis of the self-adjoint realization on  $L^2(\mathbb{R}_+)$  of the de Gennes operator, which is a differential operator of order two depending on the real parameters  $\gamma$  and  $\xi$  and acting as

$$H[\gamma, \sigma] = -\frac{d^2}{dt^2} + (t - \sigma)^2,$$

associated with the Robin boundary condition:

$$u'(0) = \gamma u(0).$$

The spectral analysis of the de Gennes operator has been the subject of several works, see [Bol92, DH93b, DH93a, Inc26, Uch77, Kac08, Kac06b]. This operator appears naturally in the study of the magnetic effects induced at the boundaries [SJdG63, FH10b].

The quadratic form associated to  $H[\gamma, \sigma]$  is defined by:

$$q[\gamma, \sigma](u) = \int_0^\infty \left[ |u'(t)|^2 + |(t - \sigma)u(t)|^2 \right] dt + \gamma |u(0)|^2, \quad (2.5)$$

with domain

$$B^1(\mathbb{R}_+) = \left\{ u \in H^1(\mathbb{R}_+) : tu(t) \in L^2(\mathbb{R}_+) \right\}.$$

The domain of  $H[\gamma, \sigma]$  is defined by:

$$\text{Dom}(H[\gamma, \sigma]) = \left\{ u \in B^1(\mathbb{R}_+) : \left( -\frac{d^2}{dt^2} + (t - \sigma)^2 \right) u \in L^2(\mathbb{R}_+), u'(0) = \gamma u(0) \right\}.$$

Note that the domain of  $H[\gamma, \sigma]$  is independent of  $\sigma$  and that the domain of the quadratic form is independent of  $\sigma$  and  $\gamma$ . Moreover, the operator  $H[\gamma, \sigma]$  has compact resolvent. Its spectrum can be written as a non-decreasing sequence of eigenvalues  $(\mu_n(\gamma, \sigma))_{n \geq 1}$  (which are all simple due to the Cauchy-Lipschitz theorem). We denote by  $u_n^{[\gamma, \sigma]}$  the normalized sequence of the corresponding eigenfunctions and we denote by  $\Theta^{[n-1]}(\gamma)$  the quantity:

$$\Theta^{[n-1]}(\gamma) := \inf_{\sigma \in \mathbb{R}} \mu_n(\gamma, \sigma).$$

The family  $(H[\gamma, \sigma])_{(\gamma, \sigma) \in \mathbb{R}^2}$  is analytic of type (B) (in the sense of Kato, see [Kat76, Chapter VII, §4]), *i.e.*, the form domain does not depend on the parameters and the sesquilinear form is analytic as a function of  $\gamma$  or  $\sigma$ .

Note that the case  $\gamma = 0$  corresponds to the Neumann realization, denoted  $H^N[\sigma]$ . We denote by  $q^N[\sigma]$  the quadratic form associated to  $H^N[\sigma]$ :

$$q^N[\sigma] = q[0, \sigma] = \int_0^\infty \left[ |u'(t)|^2 + |(t - \sigma)u(t)|^2 \right] dt.$$

We also denote by  $H^D[\sigma]$  the Dirichlet realization, with domain:

$$\text{Dom}(H^D[\sigma]) = \left\{ u \in B^1(\mathbb{R}_+) : \left( -\frac{d^2}{dt^2} + (t - \sigma)^2 \right) u \in L^2(\mathbb{R}_+), u(0) = 0 \right\}.$$

The quadratic form associated to  $H^D[\sigma]$  is given by

$$q^D[\sigma](u) = \int_0^{+\infty} \left[ |u'(t)|^2 + |(t - \sigma)u(t)|^2 \right] dt,$$

with domain

$$\text{Dom}(q^D[\sigma]) = B^1(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+).$$

The operators  $H^D[\sigma]$  and  $H^N[\sigma]$  have compact resolvent. Their spectrum is an increasing sequence of eigenvalues denoted by  $\mu_n^N(\sigma)$  and  $\mu_n^D(\sigma)$  respectively.



By Kato’s perturbation theory [Kat76], we get that the functions:

$$\sigma \mapsto \mu_n^D(\sigma), \quad \sigma \mapsto \mu_n^N(\sigma), \quad \sigma \mapsto \mu_n(\gamma, \sigma),$$

are analytic.

Moreover, we recall the following two theorems obtained by adapting the proof in Dauge-Helffer [DH93a], Reed-Simon [RS78], Fournais-Helffer [FH10b] and Raymond [Ray17], for the case  $n = 1$ .

**Theorem 2.4.1.** *The function  $\sigma \mapsto \mu_n^D(\sigma)$  is non-constant, analytic, decreasing and verifying*

$$\lim_{\sigma \rightarrow -\infty} \mu_n^D(\sigma) = +\infty, \quad \lim_{\sigma \rightarrow +\infty} \mu_n^D(\sigma) = 2n - 1.$$

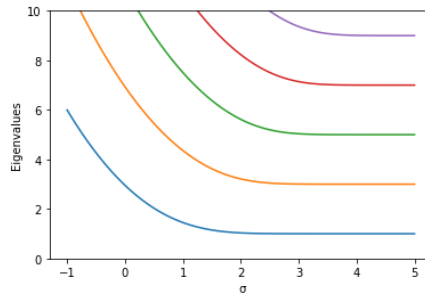


Figure 2.2 –  $\sigma \mapsto \mu_k^D(\sigma)$ , for  $k = 1, 2, 3, 4, 5$ .

**Theorem 2.4.2.** *The function  $\sigma \mapsto \mu_n^N(\sigma)$  is non-constant, analytic and verifying:*

$$\lim_{\sigma \rightarrow -\infty} \mu_n^N(\sigma) = +\infty, \quad \lim_{\sigma \rightarrow +\infty} \mu_n^N(\sigma) = 2n - 1.$$

In addition,  $\mu_n^N(\sigma)$  admits a unique minimum, which is non-degenerate and such that

$$2n - 3 < \Theta^{[n-1]} = \inf_{\sigma \in \mathbb{R}} \mu_n^N(\sigma) < 2n - 1,$$

and

$$0 < \Theta^{[0]} = \inf_{\sigma \in \mathbb{R}} \mu_1^N(\sigma) < 1.$$

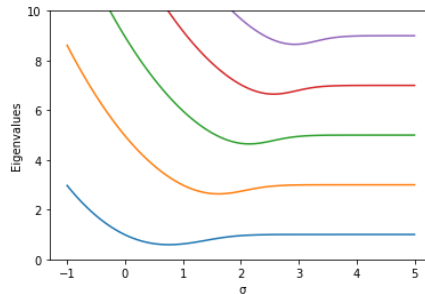


Figure 2.3 –  $\sigma \mapsto \mu_k^N(\sigma)$ , for  $k = 1, 2, 3, 4, 5$ .

We will show that the behavior of  $\mu_n(\gamma, \sigma)$  is similar to that of  $\mu_n^N(\sigma)$  given in Theorem 2.4.2.

**Theorem 2.4.3.** *Let us fix  $n \geq 1$ . When  $\gamma \in \mathbb{R}$ , the function  $\mu_n(\gamma, \cdot)$  is analytic and*

$$\lim_{\sigma \rightarrow -\infty} \mu_n(\gamma, \sigma) = +\infty, \quad \lim_{\sigma \rightarrow +\infty} \mu_n(\gamma, \sigma) = 2n - 1.$$

Moreover,  $\mu_n(\gamma, \cdot)$  has a unique minimum. It is non degenerate and attained at  $\sigma = \xi_{n-1}(\gamma)$ , but not attained at infinity. The function  $\mu_n(\gamma, \cdot)$  is decreasing on  $(-\infty, \xi_{n-1}(\gamma))$  and increasing on  $(\xi_{n-1}(\gamma), +\infty)$ . In addition, we have, for all  $n \geq 2$

$$2n - 3 < \Theta^{[n-1]}(\gamma) < 2n - 1, \quad (2.6)$$

and

$$\Theta^{[0]}(\gamma) < 1.$$

**Remark 2.4.1.** *Theorem 2.4.3 was essentially obtained by Kachmar in [Kac06b, Kac08], but without the lower bound in (2.6). The demonstration methods used are different from those presented in [Kac06b, Kac08].*

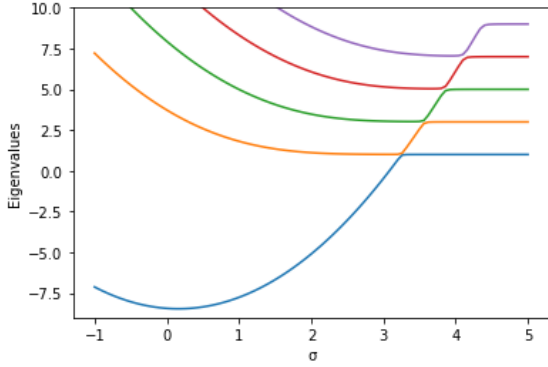


Figure 2.4 –  $\sigma \mapsto \mu_k(-3, \sigma)$ , for  $k = 1, \dots, 5$ .

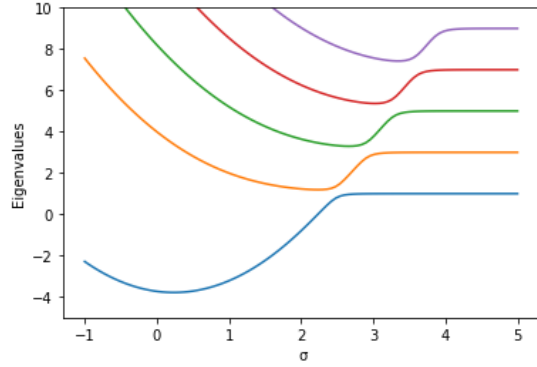


Figure 2.5 –  $\sigma \mapsto \mu_k(-2, \sigma)$ , for  $k = 1, \dots, 5$ .

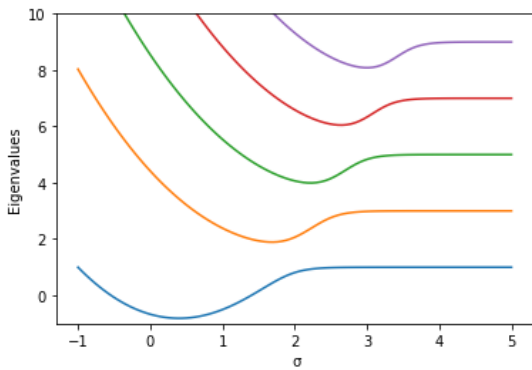


Figure 2.6 –  $\sigma \mapsto \mu_k(-1, \sigma)$ , for  $k = 1, \dots, 5$ .

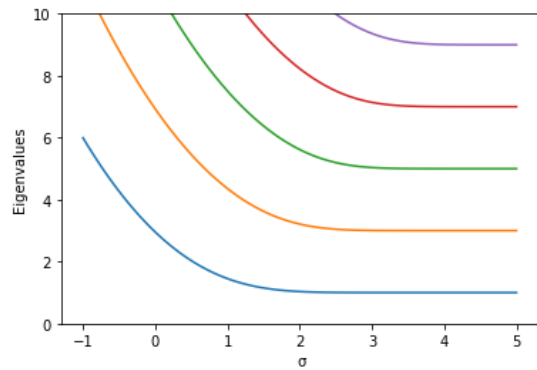


Figure 2.7 –  $\sigma \mapsto \mu_k(1000, \sigma)$ , for  $k = 1, \dots, 5$ .

### 2.4.1 Proof of Theorem 2.4.3

Theorem 2.4.3 is the consequence of Lemmas 2.4.8 and 2.4.10.

**Lemma 2.4.4.** *Given  $\varepsilon \in ]0, 1]$  and  $\gamma \in \mathbb{R}$ , we have*

$$\mu_n(\gamma, \sigma) \geq (1 - \varepsilon)\mu_n^N(\sigma) - \frac{\gamma_-^2}{\varepsilon}, \quad \forall \gamma, \sigma \in \mathbb{R}, \quad (2.7)$$

where  $\gamma_- = \max(-\gamma, 0)$ . In particular,

$$\lim_{\sigma \rightarrow -\infty} \mu_n(\gamma, \sigma) = +\infty. \quad (2.8)$$

*Proof.* From a direct comparison of the quadratic forms, the estimate (2.7) is obviously true when  $\gamma \geq 0$ . Let us assume that  $\gamma < 0$ .

Using the density of  $C_0^\infty(\overline{\mathbb{R}_+})$  in  $H^1(\mathbb{R}_+)$ , we get for any function  $u \in H^1(\mathbb{R}_+)$  :

$$|u(0)|^2 = -2 \int_{\mathbb{R}_+} u'(t) u(t) dt. \quad (2.9)$$

By the Cauchy-Schwarz inequality, we get for any  $\alpha > 0$  :

$$|u(0)|^2 \leq 2 \|u'\| \|u\| \leq \alpha \|u'\|^2 + \frac{1}{\alpha} \|u\|^2.$$

Taking  $\alpha = -\frac{\varepsilon}{\gamma}$  with  $\gamma < 0$ , we get :

$$\gamma |u(0)|^2 \geq -\varepsilon \|u'\|^2 - \frac{\gamma_-^2}{\varepsilon} \|u\|^2.$$

We deduce that, for all  $\gamma \in \mathbb{R}$ , we have

$$\gamma |u(0)|^2 \geq -\varepsilon \|u'\|^2 - \frac{\gamma_-^2}{\varepsilon} \|u\|^2,$$

where  $\gamma_- = \max(-\gamma, 0)$ .

Substituting this estimate into the expression for  $q[\gamma, \sigma]$  given in (2.5), we obtain, for all  $0 < \varepsilon < 1$

$$q[\gamma, \sigma](u) \geq (1 - \varepsilon) \|u'\|^2 + \|(t - \sigma)u\|^2 - \frac{\gamma_-^2}{\varepsilon} \|u\|^2 \geq (1 - \varepsilon) q^N[\sigma](u) - \frac{\gamma_-^2}{\varepsilon} \|u\|^2. \quad (2.10)$$

The min-max principle now gives

$$\mu_n(\gamma, \sigma) \geq (1 - \varepsilon)\mu_n^N(\sigma) - \frac{\gamma_-^2}{\varepsilon}, \quad \forall \gamma, \sigma \in \mathbb{R}.$$

Applying Theorem 2.4.2, we get (2.8). □

**Lemma 2.4.5.** *Given  $\gamma \in \mathbb{R}$ , we have*

$$\lim_{\sigma \rightarrow +\infty} \mu_n(\gamma, \sigma) = 2n - 1. \quad (2.11)$$

*Proof.* For the upper bound, we observe that, for all  $u \in B^1(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$

$$q[\gamma, \sigma](u) = \int_0^\infty \left[ |u'(t)|^2 + |(t - \sigma)u(t)|^2 \right] dt = q^D[\sigma](u).$$

A direct application of the min max principle allows us to deduce, from Theorem 2.4.1, that

$$\mu_n(\gamma, \sigma) \leq \mu_n^D(\sigma) \xrightarrow{\sigma \rightarrow +\infty} 2n - 1. \quad (2.12)$$

Let us investigate the opposite inequality. Let us prove that in some sense that the normalized eigenfunction  $u_n^{[\gamma, \sigma]}$  associated with  $\mu_n[\gamma, \sigma]$  is small near 0 when  $\sigma \gg 1$ .

From (2.10), for  $u = u_j^{[\gamma, \sigma]}$ , we get

$$\mu_j(\gamma, \sigma) \geq \left\| (t - \sigma)u_j^{[\gamma, \sigma]} \right\|^2 - \frac{\gamma^2}{\varepsilon}. \quad (2.13)$$

By (2.12), and by the monotonicity of the Dirichlet eigenvalue with respect to the domain, we have, for all  $\sigma > 0$

$$\mu_j(\gamma, \sigma) \leq \mu_j^D(\sigma) \leq \lim_{\sigma \rightarrow +\infty} \mu_{j+1}^D(\sigma) = 2j + 1. \quad (2.14)$$

Then,

$$\int_0^{+\infty} (t - \sigma)^2 |u_j^{[\gamma, \sigma]}|^2 dt \leq 2j + 1 + \frac{\gamma^2}{\varepsilon}.$$

This implies that, there exists  $C > 0$  such that:

$$\int_0^1 |u_j^{[\gamma, \sigma]}|^2 dt \leq \frac{C}{(\sigma - 1)^2}, \quad \sigma \geq 2. \quad (2.15)$$

Let  $E$  be the subspace of  $\text{Dom}(q[\gamma, \xi])$  spanned by the family  $(u_j^{[\gamma, \sigma]})_{j \in \{1, \dots, n\}}$ . Let us introduce the family of test function  $(\chi(t)u_j^{[\gamma, \sigma]})_{1 \leq j \leq n}$ , with  $\chi$  supported in  $(0, +\infty)$  and equal to 1 for  $t \geq 1$ . We have, for all  $\varphi_{\gamma, \sigma} \in E$ ,

$$\begin{aligned} q[\gamma, \sigma](\chi\varphi_{\gamma, \sigma}) &= \int_0^\infty \left[ |(\chi\varphi_{\gamma, \sigma})'(t)|^2 + |(t - \sigma)\chi(t)\varphi_{\gamma, \sigma}(t)|^2 \right] dt + \gamma |(\chi\varphi_{\gamma, \sigma})(0)|^2 \\ &= \int_0^\infty \left[ |(\chi\varphi_{\gamma, \sigma})'(t)|^2 + |(t - \sigma)\chi(t)\varphi_{\gamma, \sigma}(t)|^2 \right] dt \\ &= \int_{\mathbb{R}} \left[ |(\chi\varphi_{\gamma, \sigma})'(t)|^2 + |(t - \sigma)\chi(t)\varphi_{\gamma, \sigma}(t)|^2 \right] dt. \end{aligned}$$

A simple change of variable ( $t \rightarrow t + \sigma$ ), gives

$$\begin{aligned} q[\gamma, \sigma](\chi\varphi_{\gamma, \sigma}) &= \int_{\mathbb{R}} \left[ |(\chi\varphi_{\gamma, \sigma})'(t + \sigma)|^2 + t^2 |(\chi\varphi_{\gamma, \sigma})(t + \sigma)|^2 \right] dt \\ &= \mathfrak{Q}_{\text{harm}}^2(\chi(\cdot + \sigma)\varphi_{\gamma, \sigma}(\cdot + \sigma)), \end{aligned} \quad (2.16)$$

where  $\mathfrak{Q}_{\text{harm}}^2$  is defined in (2.4). We notice that, from (2.15), we have  $\dim(\chi E) = n$ . We have

$$\begin{aligned} q[\gamma, \sigma](\chi\varphi_{\gamma, \sigma}) &= \int_0^\infty \left[ |(\chi\varphi_{\gamma, \sigma})'(t)|^2 + |(t - \sigma)\chi(t)\varphi_{\gamma, \sigma}(t)|^2 \right] dt + \gamma |(\chi\varphi_{\gamma, \sigma})(0)|^2 \\ &= \langle H[\gamma, \sigma]\chi\varphi_{\gamma, \sigma}, \chi\varphi_{\gamma, \sigma} \rangle. \end{aligned}$$

A simple integration by part, we get

$$q[\gamma, \sigma](\chi\varphi_{\gamma, \sigma}) = \langle \chi H[\gamma, \sigma]\varphi_{\gamma, \sigma}, \chi\varphi_{\gamma, \sigma} \rangle + \|\chi'\varphi_{\gamma, \sigma}\|^2. \quad (2.17)$$

Now, we are going to calculate  $\langle \chi H[\gamma, \sigma] \varphi_{\gamma, \sigma}, \chi \varphi_{\gamma, \sigma} \rangle$ . We recall that  $\varphi_{\gamma, \sigma}$  in  $E$ , so it can be written as follows

$$\varphi_{\gamma, \sigma} = \alpha_1 u_1^{[\gamma, \sigma]} + \dots + \alpha_n u_n^{[\gamma, \sigma]},$$

where  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ . Then

$$\begin{aligned} \langle \chi H[\gamma, \sigma] \varphi_{\gamma, \sigma}, \chi \varphi_{\gamma, \sigma} \rangle &= \left\langle \chi H[\gamma, \sigma] \left( \alpha_1 u_1^{[\gamma, \sigma]} + \dots + \alpha_n u_n^{[\gamma, \sigma]} \right), \chi \left( \alpha_1 u_1^{[\gamma, \sigma]} + \dots + \alpha_n u_n^{[\gamma, \sigma]} \right) \right\rangle \\ &= \left\langle \chi \left( \alpha_1 \mu_1(\gamma, \sigma) u_1^{[\gamma, \sigma]} + \dots + \alpha_n \mu_n(\gamma, \sigma) u_n^{[\gamma, \sigma]} \right), \chi \left( \alpha_1 u_1^{[\gamma, \sigma]} + \dots + \alpha_n u_n^{[\gamma, \sigma]} \right) \right\rangle. \end{aligned}$$

For all  $i \neq j \in \{1, \dots, n\}$

$$\begin{aligned} \langle |\chi|^2 u_i^{[\gamma, \sigma]}, u_j^{[\gamma, \sigma]} \rangle &= \int_0^1 |\chi(t)|^2 u_i^{[\gamma, \sigma]}(t) u_j^{[\gamma, \sigma]}(t) dt + \int_1^\infty |\chi(t)|^2 u_i^{[\gamma, \sigma]}(t) u_j^{[\gamma, \sigma]}(t) dt \\ &= \int_0^1 |\chi(t)|^2 u_i^{[\gamma, \sigma]}(t) u_j^{[\gamma, \sigma]}(t) dt + \int_1^\infty u_i^{[\gamma, \sigma]}(t) u_j^{[\gamma, \sigma]}(t) dt. \end{aligned}$$

As, the vectors  $u_i^{[\gamma, \sigma]}$  and  $u_j^{[\gamma, \sigma]}$  are orthogonal, then

$$\int_1^\infty u_i^{[\gamma, \sigma]}(t) u_j^{[\gamma, \sigma]}(t) dt = - \int_0^1 u_i^{[\gamma, \sigma]}(t) u_j^{[\gamma, \sigma]}(t) dt.$$

We deduce that, from (2.15)

$$\langle |\chi|^2 u_i^{[\gamma, \sigma]}, u_j^{[\gamma, \sigma]} \rangle = \int_0^1 (|\chi(t)|^2 - 1) u_i^{[\gamma, \sigma]}(t) u_j^{[\gamma, \sigma]}(t) dt \leq \frac{C}{(\sigma - 1)^2}, \quad \sigma \geq 2.$$

Therefore,

$$\begin{aligned} \langle \chi H[\gamma, \sigma] \varphi_{\gamma, \sigma}, \chi \varphi_{\gamma, \sigma} \rangle &\leq \mu_1(\gamma, \sigma) \left\| \alpha_1 \chi u_1^{[\gamma, \sigma]} \right\|^2 + \dots + \mu_n(\gamma, \sigma) \left\| \alpha_n \chi u_n^{[\gamma, \sigma]} \right\|^2 + \frac{C}{(\sigma - 1)^2} \\ &\leq \mu_n(\gamma, \sigma) \left[ \left\| \alpha_1 \chi u_1^{[\gamma, \sigma]} \right\|^2 + \dots + \left\| \alpha_n \chi u_n^{[\gamma, \sigma]} \right\|^2 \right] + \frac{C}{(\sigma - 1)^2} \\ &\leq \mu_n(\gamma, \sigma) \|\chi \varphi_{\gamma, \sigma}\|^2 + \frac{C}{(\sigma - 1)^2}. \end{aligned}$$

Then, (2.16) and (2.17) gives

$$\mathfrak{Q}_{\text{harm}}^2(\chi(\cdot + \sigma) \varphi_{\gamma, \sigma}(\cdot + \sigma)) = q[\gamma, \sigma](\chi \varphi_{\gamma, \sigma}) \leq \mu_n(\gamma, \sigma) \|\chi \varphi_{\gamma, \sigma}\|^2 + \frac{C}{(\sigma - 1)^2} + \|\chi' \varphi_{\gamma, \sigma}\|^2. \quad (2.18)$$

By application of the min-max principle, we obtain

$$\mu_n(\mathcal{H}_{\text{harm}}^2) \|\chi \varphi_{\gamma, \sigma}\|_{L^2(\mathbb{R})}^2 \leq \mu_n(\gamma, \sigma) \|\chi \varphi_{\gamma, \sigma}\|^2 + \frac{C}{(\sigma - 1)^2} + \|\chi' \varphi_{\gamma, \sigma}\|^2. \quad (2.19)$$

where  $\mu_n(\mathcal{H}_{\text{harm}}^2) = 2n - 1$  is the  $n$ -th eigenvalue of the harmonic oscillator  $\mathcal{H}_{\text{harm}}^2$  defined in (2.3). We infer that, for  $\sigma$  large enough, (2.15) and (2.19) yields,

$$2n - 1 \leq \mu_n(\gamma, \sigma) + \frac{C}{(\sigma - 1)^2}.$$

This completes the proof of (2.11).  $\square$

**Lemma 2.4.6** (Feynman-Hellmann formula). *For all  $\gamma, \sigma \in \mathbb{R}$ , we have*

$$\partial_\sigma \mu_n(\gamma, \sigma) = -2 \int_{t>0} (t - \sigma) \left| u_n^{[\gamma, \sigma]}(t) \right|^2 dt. \quad (2.20)$$

*Proof.* We have

$$H[\gamma, \sigma] u_n^{[\gamma, \sigma]} = \mu_n(\gamma, \sigma) u_n^{[\gamma, \sigma]}.$$

Taking the derivative with respect to  $\sigma$ , we get:

$$(\partial_\sigma H[\gamma, \sigma] - \partial_\sigma \mu_n(\gamma, \sigma)) u_n^{[\gamma, \sigma]} = (\mu_n(\gamma, \sigma) - H[\gamma, \sigma]) \partial_\sigma u_n^{[\gamma, \sigma]}.$$

Taking the scalar product with  $u_n^{[\gamma, \sigma]}$  in  $L^2(\mathbb{R}_+)$ , we obtain

$$\left\langle \partial_\sigma H[\gamma, \sigma] u_n^{[\gamma, \sigma]}, u_n^{[\gamma, \sigma]} \right\rangle - \partial_\sigma \mu_n(\gamma, \sigma) = \left\langle (\mu_n(\gamma, \sigma) - H[\gamma, \sigma]) \partial_\sigma u_n^{[\gamma, \sigma]}, u_n^{[\gamma, \sigma]} \right\rangle.$$

The operator  $H[\gamma, \sigma]$  is self-adjoint on  $L^2(\mathbb{R}_+)$ , then

$$\left\langle (\mu_n(\gamma, \sigma) - H[\gamma, \sigma]) \partial_\sigma u_n^{[\gamma, \sigma]}, u_n^{[\gamma, \sigma]} \right\rangle = \left\langle \partial_\sigma u_n^{[\gamma, \sigma]}, (\mu_n(\gamma, \sigma) - H[\gamma, \sigma]) u_n^{[\gamma, \sigma]} \right\rangle = 0.$$

From the previous equality, we get

$$\partial_\sigma \mu_n(\gamma, \sigma) = \left\langle \partial_\sigma H[\gamma, \sigma] u_n^{[\gamma, \sigma]}, u_n^{[\gamma, \sigma]} \right\rangle.$$

We have

$$\partial_\sigma H[\gamma, \sigma] = -2(t - \sigma),$$

and we obtain the desired formula.  $\square$

**Lemma 2.4.7.** *For all  $\gamma, \sigma \in \mathbb{R}$ , we have:*

$$\partial_\sigma \mu_n(\gamma, \sigma) = -(\mu_n(\gamma, \sigma) - \sigma^2 + \gamma^2) \left| u_n^{[\gamma, \sigma]}(0) \right|^2. \quad (2.21)$$

*Proof.* To obtain (2.21), we use the Feynman-Hellmann formula (2.20). We can write that

$$2(t - \sigma) = \frac{d}{dt} (t - \sigma)^2.$$

From (2.20), we get

$$\partial_\sigma \mu_n(\gamma, \sigma) = -2 \int_{t>0} (t - \sigma) \left| u_n^{[\gamma, \sigma]}(t) \right|^2 dt = - \int_{t>0} \frac{d}{dt} (t - \sigma)^2 \left| u_n^{[\gamma, \sigma]}(t) \right|^2 dt.$$

By integration by parts, we have

$$\partial_\sigma \mu_n(\gamma, \sigma) = 2 \int_0^{+\infty} (t - \sigma)^2 u_n^{[\gamma, \sigma]}(t) \partial_t u_n^{[\gamma, \sigma]}(t) dt + \sigma^2 \left| u_n^{[\gamma, \sigma]}(0) \right|^2. \quad (2.22)$$

We have:

$$-\partial_t^2 u_n^{[\gamma, \sigma]} + (t - \sigma)^2 u_n^{[\gamma, \sigma]} = \mu_n(\gamma, \sigma) u_n^{[\gamma, \sigma]}.$$

Taking the scalar product with  $\partial_t u_n^{[\gamma, \sigma]}$  in  $L^2(\mathbb{R}_+)$ , we obtain

$$\int_0^{+\infty} (t - \sigma)^2 u_n^{[\gamma, \sigma]}(t) \partial_t u_n^{[\gamma, \sigma]}(t) dt = \int_0^{+\infty} \partial_t^2 u_n^{[\gamma, \sigma]}(t) \partial_t u_n^{[\gamma, \sigma]}(t) dt + \mu_n(\gamma, \sigma) \int_0^{+\infty} u_n^{[\gamma, \sigma]}(t) \partial_t u_n^{[\gamma, \sigma]}(t) dt. \quad (2.23)$$

By integration by parts, we get:

$$\int_0^{+\infty} \partial_t^2 u_n^{[\gamma, \sigma]}(t) \partial_t u_n^{[\gamma, \sigma]}(t) dt = - \int_0^{+\infty} \partial_t u_n^{[\gamma, \sigma]}(t) \partial_t^2 u_n^{[\gamma, \sigma]}(t) dt - \left| \partial_t u_n^{[\gamma, \sigma]}(0) \right|^2,$$

and

$$\int_0^{+\infty} u_n^{[\gamma, \sigma]}(t) \partial_t u_n^{[\gamma, \sigma]}(t) dt = - \int_0^{+\infty} u_n^{[\gamma, \sigma]}(t) \partial_t^2 u_n^{[\gamma, \sigma]}(t) dt - \left| u_n^{[\gamma, \sigma]}(0) \right|^2.$$

We deduce that:

$$2 \int_0^{+\infty} \partial_t^2 u_n^{[\gamma, \sigma]}(t) \partial_t u_n^{[\gamma, \sigma]}(t) dt = - \left| \partial_t u_n^{[\gamma, \sigma]}(0) \right|^2,$$

and

$$2 \int_0^{+\infty} u_n^{[\gamma, \sigma]}(t) \partial_t u_n^{[\gamma, \sigma]}(t) dt = - \left| u_n^{[\gamma, \sigma]}(0) \right|^2.$$

We put the previous equality in (2.22) and (2.23) and deduce that:

$$\partial_\sigma \mu_n(\gamma, \sigma) = -\mu_n(\gamma, \sigma) \left| u_n^{[\gamma, \sigma]}(0) \right|^2 - \left| \partial_t u_n^{[\gamma, \sigma]}(0) \right|^2 + \sigma^2 \left| u_n^{[\gamma, \sigma]}(0) \right|^2.$$

Remembering that

$$\partial_t u_n^{[\gamma, \sigma]}(0) = \gamma u_n^{[\gamma, \sigma]}(0),$$

we get (2.21). □

**Lemma 2.4.8.** For each  $\gamma \in \mathbb{R}$ ,  $\Theta^{[n-1]}(\gamma) < 2n - 1$ . The function  $\sigma \mapsto \mu_n(\gamma, \sigma)$  attains its minimum at a unique positive point  $\xi_{n-1}(\gamma) > 0$  and we have:

$$\Theta^{[n-1]}(\gamma) = \mu_n(\gamma, \xi_{n-1}(\gamma)) \quad \text{and} \quad \xi_{n-1}(\gamma) = \Theta^{[n-1]}(\gamma) + \gamma^2.$$

Moreover, the minimum is non-degenerate.

*Proof.* We notice that, if  $\gamma > 0$ , we have

$$(\mu_n(\gamma, \sigma) - \sigma^2 + \gamma^2)|_{\sigma=0} = \mu_n(\gamma, 0) + \gamma^2 > 0.$$

This result is true for  $\gamma \leq 0$ . In fact, using (2.7), and the Theorem 2.4.2,

$$\mu_n(\gamma, \sigma) > -\frac{\gamma^2}{\varepsilon},$$

for  $\varepsilon = 1$ , we obtain:

$$\mu_n(\gamma, \sigma) + \gamma^2 > 0.$$

Using Lemma 2.4.5, we have

$$\lim_{\sigma \rightarrow +\infty} [\mu_n(\gamma, \sigma) - \sigma^2 + \gamma^2] = -\infty.$$

Using the intermediate value theorem, this gives the existence of a positive point  $\xi_{n-1}(\gamma)$  of  $\mu_n(\gamma, \sigma)$  verifying:

$$\mu_n(\gamma, \xi_{n-1}(\gamma)) - \xi_{n-1}(\gamma)^2 + \gamma^2 = 0.$$

Remembering the formula (2.21), we obtain that  $\xi_{n-1}(\gamma)$  is a critical point of  $\mu_n(\gamma, \sigma)$ . Note that, for any positive critical point  $\xi_c$  of  $\mu_n(\gamma, \sigma)$ , we have

$$\partial_\sigma^2 \mu_n(\gamma, \sigma)|_{\sigma=\xi_c} = 2\xi_c \left| u_n^{[\gamma, \sigma]}(0) \right|^2 > 0. \quad (2.24)$$

This shows that any positive critical point is a local minimum of  $\mu_n(\gamma, \sigma)$ . Thus, there can be at most one positive critical point. It also follows from (2.24) that any negative critical point will be a local maximum. However,  $\lim_{\sigma \rightarrow -\infty} \mu_n(\gamma, \sigma) = +\infty$ . Combining these two pieces of information, we find that no negative critical point exists. Therefore, the minimum of  $\sigma \mapsto \mu_n(\gamma, \sigma)$  is attained at a unique point  $\xi_{n-1}(\gamma) > 0$  and the function  $\sigma \mapsto \mu_n(\gamma, \sigma)$  is strictly increasing on  $[\xi_{n-1}(\gamma), +\infty[$ . This proves in particular:

$$\Theta^{[n-1]}(\gamma) = \mu_n(\gamma, \xi_{n-1}(\gamma)) < \lim_{\sigma \rightarrow +\infty} \mu_n(\gamma, \sigma) = 2n - 1.$$

□

**Lemma 2.4.9.** For  $\gamma \in \mathbb{R}$ , we have

$$\mu_n(\gamma, \sigma) > -\gamma_-^2,$$

where  $\gamma_- = \max(-\gamma, 0)$ .

*Proof.* Using (2.7), and Theorem 2.4.2, we have, for all  $\varepsilon \in (0, 1]$ ,

$$\mu_n(\gamma, \sigma) > -\frac{\gamma_-^2}{\varepsilon}.$$

This ends the proof. □

**Lemma 2.4.10.** For each  $\gamma \in \mathbb{R}$ ,  $n \geq 2$ , we have

$$\mu_n(\gamma, \sigma) > 2n - 3.$$

In particular, we have

$$\Theta^{[n-1]}(\gamma) > 2n - 3.$$

*Proof.* Let  $n \geq 2$ , from the Sturm–Liouville theory,  $u_n^{[\gamma, \sigma]}$  admits  $(n - 1)$  zeros on  $\mathbb{R}_+$ . We denote by  $z_{n,1}(\gamma, \sigma)$  the first zero. We consider the function

$$\varphi_{\gamma, \sigma}^{[n]}(t) = u_n^{[\gamma, \sigma]}(t + z_{n,1}(\gamma, \sigma)),$$

which satisfies

$$H^D[\sigma - z_{n,1}(\gamma, \sigma)]\varphi_{\gamma, \sigma}^{[n]} = \mu_n(\gamma, \sigma)\varphi_{\gamma, \sigma}^{[n]},$$

with Dirichlet boundary condition

$$\varphi_{\gamma, \sigma}^{[n]}(0) = 0,$$

and has exactly  $n - 2$  zeros on  $\mathbb{R}_+$ . By the Sturm's oscillation theorem,  $\varphi_{\gamma, \sigma}^{[n]}$  is the  $(n - 1)$ -th eigenfunction of  $H^D[\sigma - z_{n,1}(\gamma, \sigma)]$ . Therefore, we have

$$\mu_n(\gamma, \sigma) = \mu_{n-1}^D(\sigma - z_{n,1}(\gamma, \sigma)).$$



Moreover, from Theorem 2.4.1, for all  $\sigma \in \mathbb{R}$ ,

$$\mu_{n-1}^D(\sigma) > 2n - 3.$$

This proved the desired conclusion.  $\square$

## 2.4.2 Some important relations

We end this section by presenting some relations that we will use in this thesis.

**Proposition 2.4.11.** *For each  $\gamma \in \mathbb{R}$ , the following relations hold:*

$$\int_0^{+\infty} (t - \xi_{n-1}(\gamma)) \left| u_n^{[\gamma, \xi_{n-1}(\gamma)]}(t) \right|^2 dt = 0, \quad (2.25)$$

$$\int_0^{+\infty} (t - \xi_{n-1}(\gamma))^3 \left| u_n^{[\gamma, \xi_{n-1}(\gamma)]}(t) \right|^2 dt = \frac{1}{6} [1 + 2\gamma \xi_{n-1}(\gamma)] \left| u_n^{[\gamma, \xi_{n-1}(\gamma)]}(0) \right|^2. \quad (2.26)$$

*Proof.* We let

$$u_n^{[\gamma]} = u_n^{[\gamma, \xi_{n-1}(\gamma)]}.$$

Let us consider the differential operator:

$$L = -\partial_t^2 + (t - \xi_{n-1}(\gamma))^2 - \Theta^{[n-1]}(\gamma).$$

Note that for any polynomial  $p$ , we have:

$$Lv = \left( p^{(3)} - 4 \left[ (t - \xi_{n-1}(\gamma))^2 - \Theta(\gamma)^{[n-1]} \right] p' - 4(t - \xi_{n-1}(\gamma))p \right) u_n^{[\gamma]}, \quad (2.27)$$

and

$$\int_0^{+\infty} u_n^{[\gamma]}(t)(Lv)(t)dt = \int_0^{+\infty} Lu_n^{[\gamma]}(t)v(t)dt + (v'(0) - \gamma v(0)) u_n^{[\gamma]}(0), \quad (2.28)$$

for  $v = 2p[u_n^{[\gamma]}]' - p'u_n^{[\gamma]}$ . Taking  $p = 1$ , we get

$$-4 \int_0^{+\infty} (t - \xi_{n-1}(\gamma)) \left| u_n^{[\gamma]}(t) \right|^2 dt = 2 \left( \xi_{n-1}(\gamma)^2 - \gamma^2 - \Theta^{[n-1]}(\gamma) \right) \left| u_n^{[\gamma]}(0) \right|^2.$$

Recalling Lemma 2.4.8, the above formula proves (2.25). To prove (2.26), we take  $p = (t - \xi_{n-1}(\gamma))^2$ . Then, we have

$$v'(0) - \gamma v(0) = -2(2\gamma \xi_{n-1}(\gamma) + 1) u_n^{[\gamma]}(0).$$

We get now from (2.26) and (2.27)

$$-12 \int_0^{+\infty} (t - \xi_{n-1}(\gamma))^3 \left| u_n^{[\gamma]}(t) \right|^2 dt = -2(2\gamma \xi_{n-1}(\gamma) + 1) \left| u_n^{[\gamma]}(0) \right|^2.$$

$\square$

**Remark 2.4.2.** *Proposition 2.4.11 corrects a mistake in [Kac06b, Lemma II.3 & (2.21)], where it is stated that*

$$\int_0^{+\infty} (t - \xi_{n-1}(\gamma))^3 \left| u_n^{[\gamma, \xi_{n-1}(\gamma)]}(t) \right|^2 dt = \frac{1}{6} [1 - 2(\gamma \xi_{n-1}(\gamma))^2] \left| u_n^{[\gamma, \xi_{n-1}(\gamma)]}(0) \right|^2.$$

In fact, the numerical simulations suggest that the behavior of  $\mu_1(\gamma, \cdot)$  differs from that of  $\mu_n(\gamma, \cdot)$  for  $n \geq 2$ . This can be quantified in the limit  $\gamma \rightarrow -\infty$ , as we will see in the following lemma.

**Lemma 2.4.12.** *Let us consider a compact  $K$ . We have, uniformly for  $\sigma \in K$ ,*

$$\mu_1(\gamma, \sigma) \underset{\gamma \rightarrow -\infty}{=} -\gamma^2 + \sigma^2 - \frac{\sigma}{|\gamma|} + \mathcal{O}(|\gamma|^{-2}).$$

*Proof.* By rescaling  $t = |\gamma|^{-1}u$ , we see that  $\gamma^{-2}H[\gamma, \sigma]$  is unitarily equivalent to

$$-\partial_u^2 + \left(|\gamma|^{-1}\sigma - |\gamma|^{-2}u\right)^2.$$

Subtracting the constant term, we consider the operator

$$\tilde{H}[\gamma, \sigma] = -\partial_u^2 + \left(|\gamma|^{-1}\sigma - |\gamma|^{-2}u\right)^2 - |\gamma|^{-2}\sigma^2$$

which can be expanded as follows

$$\tilde{H}[\gamma, \sigma] = -\partial_u^2 - 2\frac{\sigma u}{|\gamma|^3} + \mathcal{O}(|\gamma|^{-4}u^2).$$

We look for a quasimode of the form

$$f(u) = f_0(u) + |\gamma|^{-3}f_1(u),$$

and a quasi-eigenvalue

$$\lambda = \lambda_0 + |\gamma|^{-3}\lambda_1.$$

This invites to take  $f_0$  and  $\lambda_0$  such that

$$-f_0'' = \lambda_0 f_0, \quad f_0'(0) = -f_0(0).$$

From Section 2.1.1, we deduce that

$$\lambda_0 = -1 \quad \text{and} \quad f_0(u) = \sqrt{2}e^{-u}.$$

Considering the terms of order  $|\gamma|^{-3}$ , we get

$$(-\partial_u^2 + 1)f_1 = (\lambda_1 + 2\sigma u)f_0. \tag{2.29}$$

This equation has a solution if and only if

$$\lambda_1 = -\langle 2\sigma u f_0, f_0 \rangle = -4\sigma \int_0^{+\infty} u e^{-2u} du = -\sigma.$$

With this choice for  $\lambda_1$ , we consider the unique  $f_1$  solving (2.29) and orthogonal to  $f_0$ .

We get

$$\left\| \left( \tilde{H}[\gamma, \sigma] - \left( -1 - |\gamma|^{-3}\sigma \right) \right) (f_0 + |\gamma|^{-3}f_1) \right\| \leq C|\gamma|^{-4}.$$

By using the rescaling, the spectral theorem, and by observing that  $\mu_n(\gamma, \sigma) > 1$  for  $n \geq 2$  (see Lemma 2.4.10), we deduce that  $\mu_1(\gamma, \sigma)$  is the only possible eigenvalue close to  $\sigma^2 - \gamma^2 - \frac{\sigma}{|\gamma|}$  at a distance of order  $|\gamma|^{-2}$ .  $\square$



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# Magnetic perturbations of the Robin Laplacian in the strong coupling limit

## Abstract

This chapter is devoted to the asymptotic analysis of the eigenvalues of the Laplace operator with a strong magnetic field and Robin boundary condition on a smooth planar domain and with a negative boundary parameter. We study the singular limit when the Robin parameter tends to infinity which is equivalent to a semi-classical limit involving a small positive semi-classical parameter. The main result is a comparison between the spectrum of the magnetic Robin Laplacian with an effective operator defined on the boundary of the domain via the Born-Oppenheimer approximation. More precisely, the low-lying eigenvalues of the magnetic Robin Laplacian are well approximated by those of a 1D periodic Schrödinger operator on the boundary. When the curvature has a unique non-degenerate maximum, we estimate the spectral gap and find that the magnetic field does not contribute to the three-term expansion of the eigenvalues. In the case of the disc domains, the eigenvalue asymptotics displays the contribution of the magnetic field explicitly.

## Résumé

Ce chapitre est consacré à l'analyse asymptotique des valeurs propres de l'opérateur de Laplace avec un champ magnétique fort et une condition de Robin au bord d'un domaine borné régulier de  $\mathbb{R}^2$  et avec un paramètre de Robin négatif. Nous étudions la limite singulière lorsque le paramètre de Robin tend vers l'infini, ce qui est équivalent à une limite semi-classique impliquant un petit paramètre semi-classique positif. Le résultat principal est une comparaison entre le spectre du Laplacien magnétique de Robin et un opérateur effectif défini sur la frontière du domaine via l'approximation de Born-Oppenheimer. Plus précisément, les basses valeurs propres du Laplacien magnétique de Robin sont bien approximées par celles d'un opérateur de Schrödinger périodique sur la frontière. Lorsque la courbure a un maximum unique non dégénéré, nous estimons l'écart spectral et trouvons que le champ magnétique ne contribue pas au développement à trois termes des valeurs propres. Dans le cas du disque unité, l'asymptotique des valeurs propres montre explicitement la contribution du champ magnétique.

*” Celui qui trouve sans chercher  
est celui qui a longtemps  
cherché sans trouver. ”*

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*Gaston Bachelard*

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## 3.1 Introduction

### 3.1.1 Magnetic Robin Laplacian

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^2$ . We assume that the boundary  $\Gamma = \partial\Omega$  is  $C^\infty$ -smooth. We study the low-lying spectrum of the Laplacian in  $L^2(\Omega)$  with a strong mixed (Robin) “attractive” boundary condition on  $\Gamma$ , together with the strong magnetic field. Our investigation is partially motivated by recent works on the Robin Laplacian, in the context of superconductivity (see Section 1.1.1) and spectral geometry initiated in [KS20, HK17, Kac16]. The operator is

$$\mathcal{P}_\beta^b = -(\nabla - ibA_0)^2,$$

with domain

$$\text{Dom}(\mathcal{P}_\beta^b) = \{u \in H^2(\Omega) : \nu \cdot (\nabla - ibA_0)u + \beta u = 0 \text{ on } \Gamma\},$$

where

- $A_0$  is the vector potential,
- $\nu$  is the unit outward normal vector of  $\Gamma$ ,
- $\beta < 0$  is the Robin parameter,
- $b > 0$  is the intensity of the applied magnetic field,
- $\nu \cdot (\nabla - ibA_0)u + \beta u = 0$  on  $\Gamma$  is the Robin boundary condition.

The magnetic potential defined by

$$A_0(x_1, x_2) = (A_{0_1}, A_{0_2}) := \frac{1}{2}(-x_2, x_1),$$

generates a constant magnetic field:

$$B := \nabla \times A_0 = \partial_{x_1}A_{0_2} - \partial_{x_2}A_{0_1} = 1.$$

Since  $\text{div}(A_0) = \nabla \cdot A_0 = \partial_{x_1}A_{0_1} + \partial_{x_2}A_{0_2} = 0$ , then the operator  $\mathcal{P}_\beta^b$  is written as follows:

$$\begin{aligned} \mathcal{P}_\beta^b &= -(\nabla - ibA_0)^2 \\ &= -\Delta + 2ibA_0 \cdot \nabla + ib \text{div}(A_0) + b^2 \|A_0\|^2 \\ &= -\Delta + 2ibA_0 \cdot \nabla + b^2 \|A_0\|^2. \end{aligned}$$

To be more precise, the magnetic Robin Laplacian  $\mathcal{P}_\beta^b$  is defined by Friedrichs extension theorem, from the closed semi-bounded quadratic form (see for example [LM54, Thms. 2.1 & 2.2] and [Hel13, Ch. 4])

$$H^1(\Omega) \ni u \mapsto \mathcal{Q}_\beta^b(u) := \int_\Omega |(\nabla - ibA_0)u(x)|^2 dx + \beta \int_\Gamma |u(x)|^2 ds(x),$$

where  $ds$  is the standard surface measure on the boundary. Note that, by the inequality in [BEL14, Lem. 2.6], the trace of  $u$  is well defined as an element of  $H^{\frac{1}{2}}(\Gamma)$ , and combining with the diamagnetic inequality in [LL01, Thm. 7.21], we obtain that, the quadratic form  $\mathcal{Q}_\beta^b$  is well defined and bounded from below.

Since  $\Omega$  is bounded and regular, by [CR21, Thm. 4.20] and [Gri11, Section 1.4.4], the embedding of  $H^1(\Omega)$  into  $L^2(\Omega)$  is compact, therefore, by [CR21, Proposition 4.24], the operator  $\mathcal{P}_\beta^b$  has a compact resolvent. Its spectrum is purely discrete, and since  $\mathcal{P}_\beta^b$  is self-adjoint and bounded from below, it consists of an

increasing sequence of eigenvalues. Our aim is to examine the magnetic effects on the low-lying eigenvalues  $(\lambda_n(b, \beta))_{n \in \mathbb{N}^*}$ , when the Robin parameter  $\beta$  tends to  $-\infty$  and  $b$  tends to  $+\infty$  simultaneously.

If  $\Omega$  is simply connected, the eigenvalue  $\lambda_n(b, \beta)$  is independent of the choice of the vector potential  $A_0$  of the magnetic field. This is a consequence of invariance under gauge transformations; if  $A \in H^1(\Omega; \mathbb{R}^2)$  and  $\text{curl } A = 1$ , then  $A = A_0 + \nabla\phi$  for a function  $\phi \in H^2(\Omega)$  (cf. [FH10b, Props D.1.1 and D.2.1]), and in turn

$$e^{ib\phi}(\nabla - ibA_0)^2 e^{-ib\phi} = (\nabla - ibA_0 - ib\nabla\phi)^2 = (\nabla - ibA)^2.$$

Apart from its own interest, the study of the spectrum of the operator  $\mathcal{P}_\beta^b$  arises in several contexts, where both situations subject to magnetic fields,  $b > 0$ , or without magnetic fields,  $b = 0$ , occur. Indeed, this operator appears in the modeling of many physical problems such as the heat equation, which describes the evolution of the temperature of a solid, the Schrödinger equation modeling the evolution of a particle non-relativistic (see Section 1.1.2) or even the wave equation describing the vibrations of a membrane. The eigenvalue of  $\mathcal{P}_\beta^b$  appears in many applications such as the estimation of the critical temperature/critical fields of some superconductors surrounded by other materials [Kac06a, KS20] or the study of long-time reaction-diffusion equations [LOS98]. On the other hand, the eigenvalue asymptotics in the singular limit  $\beta \rightarrow -\infty$ , provide counterexamples in the context of the spectral geometry [FK15, PP16], and have links with the study of Steklov eigenvalues [HK22]. The case  $\beta = 0$  corresponds to Neumann's magnetic Laplacian, which has been studied in numerous articles [Ray17, FH10b].

### 3.1.2 Main results

The large coupling asymptotics of the lowest Robin eigenvalue in the absence of a magnetic field has been studied recently by many authors, see *e.g.* [EMP14, HK17, Ker18, LP08, PP16, Kac16, KKR16]. In particular, the derivation of an effective operator on the boundary, involving the Laplace-Beltrami operator as well as the mean curvature of the boundary. In the same spirit, we will study the negative spectrum of the Robin's Laplacian on a bounded domain with Robin condition at the boundary. Our aim is to improve/complement earlier estimates and also clarify the magnetic field's contribution in the spectral asymptotic.

The main contribution of this chapter is Theorem 3.1.1 below, which involves an effective operator that we introduce as follows. We denote by  $|\Omega|$  the area of the domain  $\Omega$  and by  $|\partial\Omega|$  the arc-length of its boundary. Let us parameterize the boundary,  $\partial\Omega$ , by the arc-length, which we denote by  $s$ . Let  $\kappa(s)$  be the curvature of  $\partial\Omega$  at the point defined by the arc-length  $s$ , with the convention that  $\kappa(s) \geq 0$  when  $\partial\Omega$  is convex in a neighborhood of  $s$ , and negative otherwise. The effective operators are self-adjoint operators in the Hilbert space  $L^2(\partial\Omega)$ , acting on periodic functions, and defined as follows

$$\begin{aligned} \mathcal{L}_{\beta, \gamma_0}^{\text{eff}} &= -\beta^{-2}(\partial_s - ib\gamma_0)^2 - 1 + \beta^{-1}\kappa(s) \\ \text{Dom}(\mathcal{L}_{\beta, \gamma_0}^{\text{eff}}) &= H^2(\mathbb{R}/|\partial\Omega|\mathbb{Z}) \end{aligned} \quad (3.1)$$

with  $\gamma_0$  the circulation of the magnetic field,

$$\gamma_0 = \frac{|\Omega|}{|\partial\Omega|}. \quad (3.2)$$

**Remark 3.1.1.** *By the min-max principle, and standard semi-classical approximation, we can establish that*

$$\lambda_n(\mathcal{L}_{\beta, \gamma_0}^{\text{eff}}) = -1 + \kappa_{\max}\beta^{-1} + o(\beta^{-1}), \quad (3.3)$$

where

$$\kappa_{\max} = \max_{\partial\Omega} \kappa(s). \quad (3.4)$$

We will describe the asymptotics of the negative eigenvalues of  $\mathcal{P}_\beta^b$ , via those of the effective operator on the boundary. More precisely, the  $n$ -th eigenvalue for the Laplace operator  $\mathcal{P}_\beta^b$ , is approximated by the  $n$ -th eigenvalue of the operators  $\mathcal{L}_{\beta,\gamma_0}^{\text{eff}}$ . We will use the following notation:

$$M = \tilde{\mathcal{O}}(|\beta|^\alpha) \quad \text{iff} \quad \forall \epsilon > 0, M = \mathcal{O}(|\beta|^{\alpha-\epsilon}). \quad (3.5)$$

**Theorem 3.1.1.** *Let  $0 \leq \eta < \frac{3}{2}$  and  $0 < c_1 < c_2$ . Suppose that*

$$c_1 |\beta|^\eta \leq b \leq c_2 |\beta|^\eta.$$

As  $\beta$  tends to  $-\infty$ , for any  $n \in \mathbb{N}^*$ , we have

$$|\lambda_n(b, \beta) - \beta^2 \lambda_n(\mathcal{L}_{\beta,\gamma_0}^{\text{eff}})| = \tilde{\mathcal{O}}(\beta^{2(\eta-1)}),$$

where  $\lambda_n(\mathcal{L}_{\beta,\gamma_0}^{\text{eff}})$  is the  $n$ -th eigenvalue of the effective operator introduced in (3.1).

It results from Theorem 3.1.1 and (3.3),

$$\lambda_n(b, \beta) = -\beta^2 + \kappa_{\max} \beta + \tilde{\mathcal{O}}(\beta^{2(\eta-1)}). \quad (3.6)$$

When  $b$  is a fixed constant (i.e.  $\eta = 0$ ), (3.6) is known in [KL22, Prop. 5.1] and in the absence of the magnetic field ( $b = 0$ ) [HK17, Thm. 1.1] and [PP15, Thm. 1]. Notice that, the remainder term is of lower order as long as  $\eta < \frac{3}{2}$ .

If  $\eta = \frac{3}{2}$ , by [Kac16] we have

$$\lambda_n(b, \beta) = -\beta^2 + \left( \kappa_{\max} + \frac{b^2 \beta^{-3}}{4} \right) \beta + o(\beta).$$

In a generic situation where the curvature has a unique non-degenerate maximum, Theorem 3.1.1 allows us to determine the leading order term of the spectral gap between the successive eigenvalues. This is valid under the following assumption:

**Assumption A**  $\left\{ \begin{array}{l} \kappa \text{ attains its maximum } \kappa_{\max} \text{ at a unique point;} \\ \text{the maximum is non-degenerate, i.e. } \kappa''(0) < 0, \end{array} \right.$

where  $\kappa_{\max}$  is introduced in (3.4), denotes the maximal curvature along the boundary  $\partial\Omega$  and the arc-length parametrization of the boundary is selected so that  $\kappa(0) = \kappa_{\max}$ .

Under **Assumption A**, we study the three terms of the eigenvalue asymptotics of the effective operator and find that

$$\lambda_n(\mathcal{L}_{\beta,\gamma_0}^{\text{eff}}) = -1 + \kappa_{\max} \beta^{-1} + (2n-1) \sqrt{\frac{-\kappa''(0)}{2}} |\beta|^{-3/2} + o(|\beta|^{-3/2}). \quad (3.7)$$

This yields the following corollary of Theorem 3.1.1:



**Corollary 3.1.2.** *Let  $0 \leq \eta < \frac{5}{4}$  and  $0 < c_1 < c_2$ . Suppose that*

$$c_1|\beta|^\eta \leq b \leq c_2|\beta|^\eta.$$

*Under the Assumption A, as  $\beta$  tends to  $-\infty$ , for any  $n \in \mathbb{N}^*$ , we have*

$$\lambda_n(b, \beta) = -\beta^2 + \kappa_{\max}\beta + (2n-1)\sqrt{\frac{-\kappa''(0)}{2}}|\beta|^{1/2} + o(|\beta|^{1/2}).$$

The technical condition on  $\eta$  appears as follows. After inserting (3.7) into the estimate of Theorem 3.1.1, we get the error  $\tilde{\mathcal{O}}(\beta^{2(\eta-1)})$ , which is of order  $o(|\beta|^{1/2})$  if and only if  $\eta < \frac{5}{4}$ .

In the asymptotics of Corollary 3.1.2, the dependence with respect to the labeling of the eigenvalues,  $n$ , appears in the third term of the expansion. However, in the generic situations described by Assumption A, the contribution of the magnetic field is hidden in the remainder term, and is of lower order compared to that of the curvature. Corollary 3.1.2 follows from the spectral asymptotics of the 1D effective operators, which is valid under a weaker assumption on the strength of the magnetic field ( $b = \mathcal{O}(|\beta|^{5/4})$ ) when compared to the one in Theorem 3.1.1 ( $b = \mathcal{O}(|\beta|^{3/2})$ ).

In the case of disc domains, Assumption A fails, but we can improve Theorem 3.1.1 and display the influence of the magnetic field in the lowest eigenvalue asymptotics.

**Theorem 3.1.3.** *Assume that  $\Omega = D(0, 1)$  is the unit disc and that  $0 \leq \eta < 1$ . We have, as  $\beta \rightarrow -\infty$ ,*

$$\lambda_1(b, \beta) = -\beta^2 + \beta + \left( \inf_{m \in \mathbb{Z}} \left( m - \frac{b}{2} \right)^2 - \frac{1}{2} \right) + o(1),$$

*uniformly with respect to  $b$  up to  $b = \mathcal{O}(|\beta|^\eta)$ .*

Theorem 3.1.3 was known when  $b$  is fixed in [KS20, Thm. 1.1]. The proof of Theorem 3.1.3 relies on the derivation of the following explicitly solvable effective operator, which is more accurate than the one in (3.1) (note that in the unit disc, the curvature is constant and equal to 1),

$$\mathcal{L}_{\beta, \text{disc}}^{\text{eff}} = -\beta^{-2}(\partial_s - ib\gamma_0)^2 - 1 + \beta^{-1} - \frac{1}{2}\beta^{-2}. \quad (3.8)$$

### 3.1.3 Organization of the chapter

The chapter is organized as follows. In Section 3.3.1, we introduce an effective semiclassical parameter, we discuss a semi-classical version of the operator and we recall why the eigenfunctions are localized, via Agmon estimates near the boundary. As a consequence, we replace the initial problem by a problem on a thin tubular neighborhood of the boundary. In Section 3.4, by using the Born-Oppenheimer approximation, we derive an effective operator whose eigenvalues simultaneously describe the eigenvalues of the magnetic Robin Laplacian. In particular, we show that the effectiveness of the reduction to a boundary operator is determined by the estimate of the Born-Oppenheimer correction. This correction is an explicit quantity related to dimension one. It appears in physics and, for instance, in the contributions [Teu03, MS02, PST03, PST07, WT14] in the context of time evolution or in the works [KMSW92, MM94, LT13] related to resonance and eigenvalue problems (see also the review [Jec14] where both aspects are considered). In Section 3.5, we show that the bound states for the effective operator decay exponentially away from points of maximal curvature. Then, we reduce the study to a free-flux operator, which is a perturbation of the harmonic oscillator. Moreover, we estimate the eigenvalues for the effective operator with a large magnetic field, thereby proving

Corollary 3.1.2. Finally, in Section 3.6, in the case of the disc domains, we describe the term that determines the influence of the magnetic field on the spectrum of the Robin Laplacian, thereby proving Theorem 3.1.3.

## 3.2 Boundary coordinates

The key to proving Theorem 3.1.1 is a reduction to the boundary. Indeed, the eigenfunctions associated with low-lying eigenvalues concentrate exponentially near the boundary (cf. Proposition 3.3.4). To single out the influence of the boundary curvature, we need a special coordinate system displaying the arc-length along the boundary and the normal distance to the boundary. We will refer to such as boundary coordinates. In this section, we introduce the necessary notation to use these coordinates (cf. [HM01, FH10b]).

Let

$$\mathbb{R}/(|\partial\Omega|\mathbb{Z}) \ni s \longmapsto M(s) \in \partial\Omega$$

be the arc-length parametrization of  $\partial\Omega$ . We will always work with  $|\partial\Omega|$ -periodic functions sometimes restricted to the interval

$$\left] -\frac{|\partial\Omega|}{2}, \frac{|\partial\Omega|}{2} \right] = ] -L, L].$$

At the point  $M(s) \in \partial\Omega$ ,  $T(s) = M'(s)$  is the unit tangent vector and  $\nu(s)$  is the unit normal vector such that

$$\forall s \in \mathbb{R}/(|\partial\Omega|\mathbb{Z}), \quad \det(T(s), \nu(s)) = 1.$$

The curvature  $\kappa(s)$  at point  $M(s)$  is then defined as follows

$$T'(s) = \kappa(s)\nu(s).$$

The smoothness of the boundary yields the existence of a constant  $t_0 > 0$  such that, upon defining

$$V_{t_0} = \{x \in \Omega : \text{dist}(x, \partial\Omega) < t_0\}$$

the change of coordinates

$$\Phi : \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (0, t_0) \ni (s, t) \mapsto x = M(s) - t\nu(s) \in V_{t_0}$$

becomes a diffeomorphism. Let us note that, for  $x \in V_{t_0}$ , one can write

$$\Phi^{-1}(x) := (s(x), t(x)) \in \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (0, t_0)$$

where  $t(x) = \text{dist}(x, \partial\Omega)$  and  $s(x)$  is the coordinate of the point  $M(s(x)) \in \mathbb{R}/(|\partial\Omega|\mathbb{Z})$  satisfying the relation  $\text{dist}(x, \partial\Omega) = |x - M(s(x))|$ .

The inverse of  $\Phi$  defines a coordinate system on a tubular neighborhood of  $\partial\Omega$  in  $\bar{\Omega}$  that can be used locally. Now we express various integrals in the new coordinates  $(s, t)$ .

The determinant of the Jacobian matrix of the transformation  $\Phi$  is given by:

$$\det J_\Phi(s, t) = 1 - t\kappa(s).$$

In the new coordinates, the components of the vector field  $A_0$  are given as follows,

$$\begin{aligned} \tilde{A}_1(s, t) &= (1 - t\kappa(s))A_0(\Phi(s, t)) \cdot M'(s), \\ \tilde{A}_2(s, t) &= A_0(\Phi(s, t)) \cdot \nu(s). \end{aligned}$$

The new magnetic potential  $\tilde{A}_0 = (\tilde{A}_1, \tilde{A}_2)$  satisfies

$$\left[ \frac{\partial \tilde{A}_2}{\partial s}(s, t) - \frac{\partial \tilde{A}_1}{\partial t}(s, t) \right] ds \wedge dt = \text{curl } A_0(\Phi^{-1}(s, t)) dx \wedge dy = (1 - t\kappa(s)) ds \wedge dt.$$

For all  $u \in L^2(V_{t_0})$ , we assign the pull-back function  $\tilde{u}$  defined in the new coordinates as follows

$$\tilde{u} = u \circ \Phi.$$

Consequently, for all  $u \in H^1(V_{t_0})$ , we have

$$\int_{V_{t_0}} |(\nabla - ibA_0)u(x)|^2 dx = \int \left[ (1 - t\kappa(s))^{-2} |(\partial_s - ib\tilde{A}_1)\tilde{u}|^2 + |(\partial_t - ib\tilde{A}_2)\tilde{u}|^2 \right] (1 - t\kappa(s)) dt ds,$$

$$\int_{V_{t_0}} |u(x)|^2 dx = \int_{\Phi^{-1}(V_{t_0})} |\tilde{u}|^2 (1 - t\kappa(s)) dt ds,$$

and

$$\int_{V_{t_0} \cap \partial\Omega} |u(x)|^2 dx = \int |\tilde{u}(s, t=0)|^2 ds.$$

Finally, we recall a useful gauge transformation that we borrow from [HM01, FH10b]. For  $t_0 > 0$  small enough, there exists a gauge transformation  $\varphi(s, t)$  on  $\mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (0, t_0)$  such that  $\bar{A}_0$  defined by

$$\bar{A}_0 = \tilde{A}_0 - \nabla_{(s,t)}\varphi = \begin{pmatrix} \bar{A}_1(s, t) \\ \bar{A}_2(s, t) \end{pmatrix} = \begin{pmatrix} \gamma_0 - t + \frac{t^2}{2}\kappa(s) \\ 0 \end{pmatrix}, \quad (3.9)$$

with

$$\gamma_0 = \frac{1}{|\partial\Omega|} \int_{\Omega} \text{curl} A_0 dx = \frac{|\Omega|}{|\partial\Omega|},$$

and for all  $u \in H^1(V_{t_0})$ , we have

$$\int_{V_{t_0}} |(\nabla - ibA_0)u(x)|^2 dx = \int_{\Phi^{-1}(V_{t_0})} \left( (1 - tk(s))^{-2} |(\partial_s - ib\bar{A}_1)w|^2 + |\partial_t w|^2 \right) (1 - tk(s)) ds dt,$$

where  $w = e^{i\varphi}\tilde{u}$  and  $\tilde{u} = u \circ \Phi$ , the pull-back function.

### 3.3 Preliminaries

#### 3.3.1 Transformation into a semi-classical problem

We will prove the results of Section 3.1.2 in a semiclassical framework. Let us consider the semi-classical parameter

$$h = \beta^{-2}.$$

The limit  $\beta \rightarrow -\infty$  is now equivalent to the semi-classical limit  $h \rightarrow 0^+$ . The quadratic form can be written as

$$\mathcal{Q}_\beta^b(u) = h^{-2} \left( \int_{\Omega} |(h\nabla - ibhA_0)u(x)|^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |u(x)|^2 ds(x) \right) := h^{-2} q_h^b(u).$$

Consequently, we obtain the self-adjoint operator depending on  $h$

$$\mathcal{L}_h^b = -(h\nabla - ihbA_0)^2,$$

with domain

$$\text{Dom}(\mathcal{L}_h^b) = \{ u \in H^2(\Omega) : \nu \cdot (\nabla - ibA_0)u - h^{-\frac{1}{2}}u = 0 \text{ on } \Gamma \}.$$

**Remark 3.3.1.**  $\mathcal{L}_h^b$  is defined via the Lax-Milgram theorem from the quadratic form  $q_h^b$ .

Clearly,

$$\mathcal{P}_\beta^b = h^{-2}\mathcal{L}_h^b,$$

and thus the relation between the spectra of the operators  $\mathcal{P}_\beta^b$  and  $\mathcal{L}_h^b$  is displayed as follows :

$$\text{Sp}(\mathcal{P}_\beta^b) = h^{-2}\text{Sp}(\mathcal{L}_h^b). \quad (3.10)$$

The self-adjoint operator  $\mathcal{L}_h^b$  has a compact resolvent and is bounded from below. Its spectrum is purely discrete and consists of a non-decreasing sequence of eigenvalues. Let  $(\mu_n(h, b))_{n \in \mathbb{N}^*}$  be the sequence of eigenvalues of the operator  $\mathcal{L}_h^b$ . Theorem 3.1.1, Corollary 3.1.2, and Theorem 3.1.3 can be reformulated in semi-classical form as follows.

**Theorem 3.3.1.** Let  $n \in \mathbb{N}^*$ ,  $0 \leq \eta < \frac{3}{2}$  and  $0 < c_1 < c_2$ . There exist  $h_0 > 0$ , such that, if  $h \in (0, h_0)$  and  $b$  satisfies

$$c_1 h^{-\frac{\eta}{2}} \leq b \leq c_2 h^{-\frac{\eta}{2}},$$

then we have :

$$\left| \mu_n(h, b) - h \lambda_n(\mathcal{L}_{h, \gamma_0}^{\text{eff}}) \right| = \tilde{O}(h^{3-\eta}),$$

with

$$\mathcal{L}_{h, \gamma_0}^{\text{eff}} = -h(\partial_s - ib\gamma_0)^2 - 1 - \kappa(s)h^{\frac{1}{2}}.$$

By the method of harmonic approximation, we deduce the following corollary:

**Corollary 3.3.2.** Let  $n \in \mathbb{N}^*$ ,  $0 \leq \eta < \frac{5}{4}$  and  $0 < c_1 < c_2$ . Under the Assumption A, there exists  $h_0 > 0$ , such that for all  $h \in (0, h_0)$  and  $b$  satisfying

$$c_1 h^{-\frac{\eta}{2}} \leq b \leq c_2 h^{-\frac{\eta}{2}},$$

we have :

$$\mu_n(h, b) = -h - \kappa_{\max} h^{3/2} + (2n-1) \sqrt{\frac{-\kappa''(0)}{2}} h^{7/4} + o(h^{7/4}).$$

In the case of the disc, we derive the following effective operator

$$\mathcal{L}_{h, \text{disc}}^{\text{eff}} = -h(\partial_s - ib\gamma_0)^2 - 1 - h^{\frac{1}{2}} - \frac{h}{2}. \quad (3.11)$$

As a consequence, we get the following theorem.

**Theorem 3.3.3.** Assume that  $\Omega = D(0, 1)$  is the unit disc,  $0 \leq \eta < 1$  and that  $0 < c_1 < c_2$ . There exists  $h_0 > 0$ , such that for all  $h \in (0, h_0)$  and  $b$  satisfying

$$c_1 h^{-\frac{\eta}{2}} \leq b \leq c_2 h^{-\frac{\eta}{2}},$$

we have :

$$\mu_1(h, b) = -h - h^{\frac{3}{2}} + \left( \inf_{m \in \mathbb{Z}} \left( m - \frac{b}{2} \right)^2 - \frac{1}{2} \right) h^2 + o(h^2).$$

### 3.3.2 Agmon estimates

The goal of this section is to show that the eigenfunctions associated with eigenvalues less than  $-\epsilon h$ , with  $0 < \epsilon < 1$  a fixed constant, are concentrated exponentially near the boundary at the scale  $h^{1/2}$ . These localizations are quantified by the following proposition.

**Proposition 3.3.4.** *Let  $M \in (-1, 0)$ . For all  $\alpha < \sqrt{-M}$ , there exist constants  $C > 0$  and  $h_0 \in (0, 1)$  such that, for  $h \in (0, h_0)$ , if  $u_{h,b}$  is a normalized ground state of  $\mathcal{L}_h^b$  with eigenvalue  $\mu(h, b)$  such that  $\mu(h, b) < M h$ , then*

$$\int_{\Omega} \left( |u_{h,b}(x)|^2 + h |(\nabla - ibA_0)u_{h,b}(x)|^2 \right) \exp \left( \frac{2\alpha \text{dist}(x, \Gamma)}{h^{1/2}} \right) dx \leq C. \quad (3.12)$$

For the proof of the Proposition 3.3.4, we will first show two technical lemmas.

**Lemma 3.3.5.** *Let  $\phi \in C^\infty(\Omega)$  a real function, such that  $\phi$  is constant on  $\Gamma$ . We set:*

$$q_h^\phi(u) = \int_{\Omega} \left( |h\nabla - ibhA_0 u \phi|^2 - h^2 |\nabla \phi|^2 |u|^2 \right) dx - h^{\frac{3}{2}} \int_{\Gamma} |\phi u|^2 ds(x).$$

If  $u_{h,b}$  is an eigenfunction of  $\mathcal{L}_h^b$  with eigenvalue  $\mu(h, b)$ , then

$$q_h^\phi(u_{h,b}) = \mu(h, b) \|u_{h,b} \phi\|_{L^2(\Omega)}^2. \quad (3.13)$$

*Proof.* We have the simple identity:

$$\left\langle \phi^2 u_{h,b}, \mathcal{L}_h^b u_{h,b} \right\rangle = \mu(h, b) \|u_{h,b} \phi\|_{L^2(\Omega)}^2. \quad (3.14)$$

On the other hand,

$$\begin{aligned} q_h^\phi(u_{h,b}) &= h^2 Q_\beta^b(\phi u_{h,b}) - h^2 \int_{\Omega} |\nabla \phi|^2 |u_{h,b}|^2 dx \\ &= \left\langle \mathcal{L}_h^b \phi u_{h,b}, \phi u_{h,b} \right\rangle - h^2 \int_{\Omega} |\nabla \phi|^2 |u_{h,b}|^2 dx. \end{aligned}$$

Let's explain  $\mathcal{L}_h^b \phi u_{h,b}$  : we have

$$\begin{aligned} \mathcal{L}_h^b \phi u_{h,b} &= -h^2 \left( \Delta(\phi u_{h,b}) - 2ibA_0 \cdot \nabla(\phi u_{h,b}) - b^2 |A_0|^2 \phi u_{h,b} \right) \\ &= -h^2 \left( \phi(\Delta u_{h,b} - 2ibA_0 \cdot \nabla u_{h,b} - b^2 |A_0|^2 u_{h,b}) + \Delta \phi u_{h,b} + 2 \nabla \phi \cdot \nabla u_{h,b} \right. \\ &\quad \left. - 2ibA_0 \cdot \nabla \phi u_{h,b} \right) \\ &= \phi \mathcal{L}_h^b u_{h,b} - h^2 \left( \Delta \phi u_{h,b} + 2 \nabla \phi \cdot \nabla u_{h,b} - 2ibA_0 \cdot \nabla \phi u_{h,b} \right). \end{aligned}$$

Thus,

$$q_h^\phi(u_{h,b}) = \left\langle \phi \mathcal{L}_h^b u_{h,b} - h^2 \Delta \phi u_{h,b} - 2h^2 \nabla \phi \cdot \nabla u_{h,b} - 2ibh^2 A_0 \cdot \nabla \phi u_{h,b}, \phi u_{h,b} \right\rangle - h^2 \int_{\Omega} |\nabla \phi|^2 |u_{h,b}|^2 dx.$$

Consequently

$$q_h^\phi(u_{h,b}) = \langle \mathcal{L}_h^b u_{h,b}, \phi^2 u_{h,b} \rangle - h^2 \left( \int_{\Omega} (\Delta \phi u_{h,b} + 2 \nabla \phi \cdot \nabla u_{h,b}) \phi \overline{u_{h,b}} + |\nabla \phi|^2 |u_{h,b}|^2 + 2ib\phi |u_{h,b}|^2 A_0 \cdot \nabla \phi dx \right). \quad (3.15)$$

If we show

$$\int_{\Omega} \left( (\Delta \phi u_{h,b} + 2 \nabla \phi \cdot \nabla u_{h,b}) \phi \overline{u_{h,b}} + |\nabla \phi|^2 |u_{h,b}|^2 + 2ib\phi |u_{h,b}|^2 A_0 \cdot \nabla \phi \right) dx = 0, \quad (3.16)$$

we obtain, from (3.14):

$$q_h^\phi(u_{h,b}) = \langle \mathcal{L}_h^b u_{h,b}, \phi^2 u_{h,b} \rangle = \mu(h,b) \|u_{h,b} \phi\|_{L^2(\Omega)}^2.$$

Let's show (3.16). By Green's formula:

$$\begin{aligned} \int_{\Omega} \phi \overline{u_{h,b}} \Delta \phi u_{h,b} dx &= - \int_{\Omega} \nabla(\phi |u_{h,b}|^2) \cdot \nabla \phi dx + \int_{\Gamma} \phi |u_{h,b}|^2 \frac{\partial \phi}{\partial \nu} ds(x) \\ &= - \int_{\Omega} \nabla(\phi |u_{h,b}|^2) \cdot \nabla \phi dx \\ &= - \int_{\Omega} \left( \phi \nabla \phi \cdot \nabla |u_{h,b}|^2 + |\nabla \phi|^2 |u_{h,b}|^2 \right) dx, \end{aligned}$$

consequently,

$$\int_{\Omega} \left( (\Delta \phi u_{h,b} + 2 \nabla \phi \cdot \nabla u_{h,b}) \phi \overline{u_{h,b}} + |\nabla \phi|^2 |u_{h,b}|^2 \right) dx = \int_{\Omega} \phi \nabla \phi \cdot \left( 2 \overline{u_{h,b}} \nabla u_{h,b} - \nabla |u_{h,b}|^2 \right) dx. \quad (3.17)$$

On the other hand, using (3.15), we have

$$\begin{aligned} &\int_{\Omega} \left( (\Delta \phi u_{h,b} + 2 \nabla \phi \cdot \nabla u_{h,b}) \phi \overline{u_{h,b}} + |\nabla \phi|^2 |u_{h,b}|^2 + 2ib\phi |u_{h,b}|^2 A_0 \cdot \nabla \phi \right) dx \\ &= \frac{1}{h^2} \left( -q_h^\phi(u_{h,b}) + \langle \mathcal{L}_h^b u_{h,b}, \phi^2 u_{h,b} \rangle \right) \in \mathbb{R}. \end{aligned}$$

Since the integral of a real part is the part real of the integral, we get by (3.17):

$$\begin{aligned} &\int_{\Omega} \left( (\Delta \phi u_{h,b} + 2 \nabla \phi \cdot \nabla u_{h,b}) \phi \overline{u_{h,b}} + |\nabla \phi|^2 |u_{h,b}|^2 + 2ib\phi |u_{h,b}|^2 A_0 \cdot \nabla \phi \right) dx \\ &= \int_{\Omega} \phi \operatorname{Re} \left( \nabla \phi \cdot \left( 2 \overline{u_{h,b}} \nabla u_{h,b} - \nabla |u_{h,b}|^2 \right) + 2ib |u_{h,b}|^2 A_0 \cdot \nabla \phi \right) dx \\ &= 0, \end{aligned}$$

where  $\operatorname{Re}(z)$  is the *real* part of  $z$ . □

**Lemma 3.3.6** (Localization Formula). *If  $(\chi_{i,h})_{1 \leq i \leq 2}$  a partition of the unit of  $\mathbb{R}$ , such that*

$$\chi_{1,h}^2 + \chi_{2,h}^2 = 1,$$

*then there exist two quadratic forms  $q_{1,h}^\phi$  and  $q_{2,h}^\phi$  such that, if  $u_{h,b}$  is an eigenfunction of  $\mathcal{L}_h^b$  with an eigenvalue  $\mu(h, b)$ , then*

$$q_h^\phi(u_{h,b}) = q_{1,h}^\phi(u_{h,b}) + q_{2,h}^\phi(u_{h,b}), \quad (3.18)$$

*where  $\phi(x) = \exp\left(\frac{\alpha t(x)}{h^{1/2}}\right)$ ,  $t(x) = \text{dist}(x, \partial\Omega)$  and  $q_h^\phi$  defined in the Lemma 3.3.5, with*

$$q_{1,h}^\phi(u_{h,b}) = \int_{\Omega} \left( |(h\nabla - ibhA_0)\chi_{1,h} \phi u_{h,b}|^2 - h^2 |\nabla(\phi \chi_{1,h})|^2 |u_{h,b}|^2 \right) dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{1,h} \phi u_{h,b}|^2 ds(x), \quad (3.19)$$

*and*

$$q_{2,h}^\phi(u_{h,b}) = \int_{\Omega} \left( |(h\nabla - ibhA_0)\chi_{2,h} \phi u_{h,b}|^2 - h^2 |\nabla(\phi \chi_{2,h})|^2 |u_{h,b}|^2 \right) dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{2,h} \phi u_{h,b}|^2 ds(x). \quad (3.20)$$

*Proof.* We have

$$q_{1,h}^\phi(u_{h,b}) = q_h^{\chi_{1,h}\phi}(u_{h,b}) \stackrel{(3.13)}{=} \mu(h, b) \|u_{h,b} \phi \chi_{1,h}\|_{L^2(\Omega)}^2,$$

and

$$q_{2,h}^\phi(u_{h,b}) = q_h^{\chi_{2,h}\phi}(u_{h,b}) \stackrel{(3.13)}{=} \mu(h, b) \|u_{h,b} \phi \chi_{2,h}\|_{L^2(\Omega)}^2.$$

Hence

$$q_{1,h}^\phi(u_{h,b}) + q_{2,h}^\phi(u_{h,b}) = \mu(h, b) \|u_{h,b} \phi\|_{L^2(\Omega)}^2 = q_h^\phi(u_{h,b}).$$

□

### 3.3.2.1 Proof of the Proposition 3.3.4

The proof of the Proposition 3.3.4 is formed by two steps. We use in the proof the notations of quadratic forms of the Lemma 3.3.6.

We consider a partition of the unit of  $\mathbb{R}$ ,

$$\chi_1^2 + \chi_2^2 = 1,$$

such that  $\chi_1 = 1$  in  $(-\infty, 1)$ ,  $\text{supp}\chi_1 \subset (-\infty, 2)$ ,  $\chi_1 \geq 0$  and  $\chi_2 \geq 0$  in  $\mathbb{R}$ . We define

$$\chi_{j,h} = \chi_j \left( \frac{t(x)}{h^{1/2}} \right), \quad j \in \{1, 2\},$$

with  $\chi_{2,h}$  is a truncation inside and  $\chi_{1,h}$  is a truncation near the boundary.

The proof is a direct adaptation of the cas of non-magnetic field, see [HK17, Theorem 5.1].

1. **First step.** Let us show that, there exist  $c > 0$ , such that

$$q_{1,h}^\phi(u_{h,b}) \geq -c h. \quad (3.21)$$

We have, for  $j \in \{1, 2\}$ ,

$$\nabla \phi(x) = \frac{\alpha \nabla t(x)}{h^{\frac{1}{2}}} \phi(x),$$

and

$$\nabla \chi_{j,h}(x) = \frac{\nabla t(x)}{h^{1/2}} \chi_j' \left( \frac{t(x)}{h^{1/2}} \right),$$

with  $|\nabla t(x)| = 1$ . There exist constants  $c > 0$  such that

$$\begin{aligned} |\nabla(\phi \chi_{j,h})(x)|^2 &= |\chi_{j,h}(x) \nabla \phi(x) + \phi(x) \nabla \chi_{j,h}(x)|^2 \\ &\leq 2|\chi_{j,h}(x) \nabla \phi(x)|^2 + 2|\phi(x) \nabla \chi_{j,h}(x)|^2 \\ &\leq 2\frac{\alpha^2}{h} |\phi(x)|^2 |\chi_{j,h}(x)|^2 + 2\frac{c}{h} |\phi(x)|^2 \left| \chi_j' \left( \frac{t(x)}{h^{1/2}} \right) \right|^2, \end{aligned}$$

hence

$$|\nabla(\phi \chi_{j,h})(x)|^2 \leq \frac{2\alpha^2}{h} |\phi(x)|^2 |\chi_{j,h}(x)|^2 + \frac{2c}{h} |\phi(x)|^2 \left| \chi_j' \left( \frac{t(x)}{h^{1/2}} \right) \right|^2. \quad (3.22)$$

According to (3.22), there exist constants  $c > 0$  such that

$$\begin{aligned} q_{1,h}^\phi(u_{h,b}) &\geq \int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |(h\nabla - ibhA_0)\chi_{1,h} \phi u_{h,b}|^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{1,h} \phi u_{h,b}|^2 ds(x) \\ &\quad - 2ch \int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |\phi|^2 |u_{h,b}|^2 dx - 2\alpha^2 h \int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |\phi|^2 |\chi_{1,h} u_{h,b}|^2 dx \\ &\geq \int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |(h\nabla - ibhA_0)\chi_{1,h} \phi u_{h,b}|^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{1,h} \phi u_{h,b}|^2 ds(x) \\ &\quad - 2(c + \alpha^2) h \int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |\phi|^2 |u_{h,b}|^2 dx. \end{aligned}$$

Since  $u_{h,b}$  is a normalized eigenfunction of  $\mathcal{L}_h^b$  in  $L^2(\Omega)$  and that  $\phi(x) \leq \exp(2\alpha)$ ,  $\forall x \in \{t(x) \leq 2h^{\frac{1}{2}}\}$ , we have:

$$- \int_{\Omega} h |\chi_{1,h} \phi u_{h,b}|^2 dx \geq - \int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} h |\phi u_{h,b}|^2 dx \geq -ch. \quad (3.23)$$

We deduce that

$$q_{1,h}^\phi(u_{h,b}) \geq \int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |(h\nabla - ibhA_0)\chi_{1,h} \phi u_{h,b}|^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{1,h} \phi u_{h,b}|^2 ds(x) - ch.$$

Let us show that

$$\int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |(h\nabla - ibhA_0)\chi_{1,h} \phi u_{h,b}|^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{1,h} \phi u_{h,b}|^2 ds(x) \geq -ch. \quad (3.24)$$

According to Theorem A.1.3, we have

$$\begin{aligned} &\int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |(h\nabla - ibhA_0)\chi_{1,h} \phi u_{h,b}|^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{1,h} \phi u_{h,b}|^2 ds(x) \\ &\geq \int_{\{t(x) \leq 2h^{\frac{1}{2}}\}} |h\nabla |\chi_{1,h} \phi u_{h,b}||^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{1,h} \phi u_{h,b}|^2 ds(x). \end{aligned}$$



In the tubular coordinates of Section 3.2, the inequality becomes:

$$\begin{aligned}
 & \int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |(h\nabla - ibhA_0)\chi_{1,h} \phi u_{h,b}|^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{1,h} \phi u_{h,b}|^2 ds(x) \\
 & \geq \int_{-\frac{|\partial\Omega|}{2}}^{\frac{|\partial\Omega|}{2}} \int_0^{2h^{\frac{1}{2}}} h^2 \left[ (1 - t\kappa(s))^{-2} |\partial_s w|^2 + |\partial_t w|^2 \right] (1 - t\kappa(s)) dt ds - h^{\frac{3}{2}} \int_{-\frac{|\partial\Omega|}{2}}^{\frac{|\partial\Omega|}{2}} |w(s, t=0)|^2 ds \\
 & \geq \int_{-\frac{|\partial\Omega|}{2}}^{\frac{|\partial\Omega|}{2}} \left( \int_0^{2h^{\frac{1}{2}}} h^2 |\partial_t w|^2 (1 - t\kappa(s)) dt - h^{\frac{3}{2}} |w(s, t=0)|^2 \right) ds,
 \end{aligned}$$

with  $w(s, t) = |\chi_{1,h} \phi u_{h,b}(s, t)|$ .

Consequently,

$$\begin{aligned}
 & \int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |(h\nabla - ibhA_0)\chi_{1,h} \phi u_{h,b}|^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{1,h} \phi u_{h,b}|^2 ds(x) \\
 & \geq (1 - ch^{\frac{1}{2}}) \int_{-\frac{|\partial\Omega|}{2}}^{\frac{|\partial\Omega|}{2}} \int_0^{2h^{\frac{1}{2}}} h^2 |\partial_t w|^2 dt ds - \int_{-\frac{|\partial\Omega|}{2}}^{\frac{|\partial\Omega|}{2}} h^{\frac{3}{2}} |w(s, t=0)|^2 ds.
 \end{aligned} \tag{3.25}$$

Let  $\frac{1}{2} < \rho < 1$ ,  $\tau = (1 - ch^{\frac{1}{2}})h^{-1}t$ , we introduce the function  $v(s, \tau) = w(s, t)$ . The function  $v$  vanishes if  $\tau \geq ch^{-\frac{1}{2}}$  and for  $h$  small enough  $2h^{-\frac{1}{2}} \leq h^{-\rho}$ . We have

$$\begin{aligned}
 (1 - ch^{\frac{1}{2}}) \int_0^{2h^{\frac{1}{2}}} h^2 |\partial_t w|^2 dt - h^{\frac{3}{2}} |w(s, t=0)|^2 &= h \left( \int_0^{h^{-\rho}} |\partial_{\tau} v|^2 d\tau - h^{\frac{1}{2}} |v(s, t=0)|^2 \right) \\
 &\geq h\lambda_1(\mathcal{H}_{0,h}) \int_0^{h^{-\rho}} |v|^2 d\tau.
 \end{aligned}$$

From Section 2.1.2, we get

$$(1 - ch^{\frac{1}{2}}) \int_0^{2h^{\frac{1}{2}}} h^2 |\partial_t w|^2 dt - h^{\frac{3}{2}} |w(s, t=0)|^2 \geq -2h \int_0^{h^{-\rho}} |v|^2 d\tau.$$

Consequently,

$$\begin{aligned}
 & \int_{\{x \in \Omega : t(x) \leq 2h^{\frac{1}{2}}\}} |(h\nabla - ibhA_0)\chi_{1,h} \phi u_{h,b}|^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |\chi_{1,h} \phi u_{h,b}|^2 ds(x) \\
 & \geq \int_{-\frac{|\partial\Omega|}{2}}^{\frac{|\partial\Omega|}{2}} \left( -2h \int_0^{h^{-\rho}} |v|^2 d\tau \right) ds \\
 & = -2h(1 - ch^{\frac{1}{2}}) \int_{-\frac{|\partial\Omega|}{2}}^{\frac{|\partial\Omega|}{2}} \int_0^{2h^{\frac{1}{2}}} |\chi_{1,h} \phi u_{h,b}|^2 dt ds \\
 & \geq -ch \int_{\Omega} |\chi_{1,h} \phi u_{h,b}|^2 dx \\
 & \stackrel{(3.23)}{\geq} -ch.
 \end{aligned}$$

This shows the point (3.24) and we deduce that:

$$q_{1,h}^{\phi}(u_{h,b}) \geq -ch.$$

2. **Second step.** Let's show that

$$\int_{\Omega} \left( h |(\nabla - ibA_0) \chi_{2,h} \phi u_{h,b}|^2 - (M + \alpha^2) |\phi \chi_{2,h} u_{h,b}|^2 \right) dx \leq c. \quad (3.26)$$

We have, from (3.13) et (3.18)

$$q_h^\phi(u_{h,b}) = q_{1,h}^\phi(u_{h,b}) + q_{2,h}^\phi(u_{h,b}) = \mu(h, b) \|u_{h,b} \phi\|_{L^2(\Omega)}^2,$$

and so from (3.21) and (3.23), we have

$$\begin{aligned} q_{2,h}^\phi(u_{h,b}) &\leq \mu(h, b) \|u_{h,b} \phi\|_{L^2(\Omega)}^2 + ch \\ &= \mu(h, b) \|\chi_{1,h} u_{h,b} \phi\|_{L^2(\Omega)}^2 + \mu(h, b) \|\chi_{2,h} u_{h,b} \phi\|_{L^2(\Omega)}^2 + ch \\ &\leq M h \|u_{h,b} \chi_{2,h} \phi\|_{L^2(\Omega)}^2 + ch. \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_{\Omega} |(h\nabla - ibhA_0) \chi_{2,h} \phi u_{h,b}|^2 dx \\ &\leq M h \|u_{h,b} \chi_{2,h} \phi\|_{L^2(\Omega)}^2 + ch + h^2 \int_{\Omega} |\nabla(\phi \chi_{2,h})|^2 |u_{h,b}|^2 dx \\ &\leq M h \|u_{h,b} \chi_{2,h} \phi\|_{L^2(\Omega)}^2 + ch + ch \int_{\Omega} |u_{h,b}|^2 dx + \alpha^2 h \int_{\Omega} |\phi \chi_{2,h} u_{h,b}|^2 dx \\ &\leq (M + \alpha^2) h \int_{\Omega} |\phi u_{h,b} \chi_{2,h}|^2 dx + ch, \end{aligned}$$

and we deduce the formula (3.26).

3. **Conclusion.** Since  $(M + \alpha^2) < 0$ , then from (3.26) we have:

$$\int_{\Omega} |\phi u_{h,b} \chi_{2,h}|^2 dx \leq c,$$

which suffices to prove that, for  $h$  sufficiently small,

$$\int_{\Omega} |\phi u_{h,b}|^2 dx \leq c.$$

Moreover, we have

$$\begin{aligned} \|\chi_{2,h} \phi (\nabla - ibA_0) u_{h,b}\|_{L^2(\Omega)}^2 &= \|(\nabla - ibA_0) \chi_{2,h} \phi u_{h,b} - u_{h,b} \nabla(\chi_{2,h} \phi)\|_{L^2(\Omega)}^2 \\ &\leq 2 \|(\nabla - ibA_0) \chi_{2,h} \phi u_{h,b}\|_{L^2(\Omega)}^2 + 2 \|u_{h,b} \nabla(\chi_{2,h} \phi)\|_{L^2(\Omega)}^2 \\ &\stackrel{(3.26)}{\leq} \frac{c}{h} + 2 \|u_{h,b} \nabla(\chi_{2,h} \phi)\|_{L^2(\Omega)}^2. \end{aligned}$$

Since  $\Omega$  is bounded, the function  $x \mapsto t(x)$  is bounded and  $\phi$  is bounded. From (3.22), we have

$$\|\chi_{2,h} \phi (\nabla - ibA_0) u_{h,b}\|_{L^2(\Omega)}^2 \leq \frac{c}{h},$$

which suffices to prove that, for  $h$  sufficiently small,

$$h \|\phi (\nabla - ibA_0) u_{h,b}\|_{L^2(\Omega)}^2 \leq c.$$

We deduce that

$$\int_{\Omega} \left( |u_{h,b}(x)|^2 + h |(\nabla - ibA_0)u_{h,b}(x)|^2 \right) \exp\left(\frac{2\alpha \text{dist}(x, \Gamma)}{h^{\frac{1}{2}}}\right) dx \leq c,$$

for  $h \rightarrow 0_+$ .

We record the following simple corollary of Proposition 3.3.4.

**Corollary 3.3.7.** *Let  $M \in (-1, 0)$ ,  $\rho \in (0, \frac{1}{2})$  and  $c > 0$ . For all  $0 < \alpha < \sqrt{-M}$ , there exists  $C > 0$  and  $h_0 \in (0, 1)$  such that, if  $u_{h,b}$  is a normalized ground state of  $\mathcal{L}_h^b$  with eigenvalue  $\mu(h, b)$  such that  $\mu(h, b) < Mh$ , then for all  $h \in (0, h_0)$ ,*

$$\int_{\text{dist}(x, \Gamma) \geq ch^{\frac{1}{2}-\rho}} \left( |u_{h,b}(x)|^2 + h |(\nabla - ibA_0)u_{h,b}(x)|^2 \right) dx \leq C \exp(-2\alpha ch^{-\rho}). \quad (3.27)$$

From Corollary 3.3.7, we deduce that for  $h \rightarrow 0_+$  we have,

$$\int_{\text{dist}(x, \Gamma) < ch^{\frac{1}{2}-\rho}} |u_{h,b}(x)|^2 dx = 1 - \int_{\text{dist}(x, \Gamma) > ch^{\frac{1}{2}-\rho}} |u_{h,b}(x)|^2 dx \geq 1 - C \exp(-2\alpha ch^{-\rho}). \quad (3.28)$$

As a consequence, for small  $h$ , the ground states of the operator  $\mathcal{L}_h^b$  are concentrated near the boundary of  $\Omega$ . This will allow us to work in a tubular neighborhood of  $\partial\Omega$ .

### 3.3.3 Spectral reduction

Let  $\rho \in (0, 1/2)$ , we introduce the  $\delta$ -neighborhood of the boundary

$$\Omega_{\delta} = \{x \in \Omega : \text{dist}(x, \Gamma) < \delta\}, \quad \delta = h^{\frac{1}{2}-\rho}.$$

The quadratic form, defined on the variational space

$$W_{\delta} = \{u \in H^1(\Omega_{\delta}) : u(x) = 0, \forall x \in \Omega \text{ such that } \text{dist}(x, \Gamma) = \delta\},$$

is given by the formula

$$q_h^{b,\rho}(u) = h^2 \int_{\Omega_{\delta}} |(\nabla - ibA_0)u(x)|^2 dx - h^{\frac{3}{2}} \int_{\partial\Omega_{\delta}} |u(x)|^2 ds(x). \quad (3.29)$$

Note again that the trace on  $\partial\Omega_{\delta}$  is well-defined by a classical trace theorem. The associated self-adjoint operator is:

$$\mathcal{L}_h^{b,\rho} = -(h\nabla - ihbA_0)^2,$$

with domain

$$\text{Dom}(\mathcal{L}_h^{b,\rho}) = \{u \in H^2(\Omega_{\delta}) : \nu \cdot (\nabla - ibA_0)u - h^{-\frac{1}{2}}u = 0 \text{ on } \Gamma \text{ and } u(x) = 0, \forall x \in \Omega \text{ such that } \text{dist}(x, \Gamma) = \delta\}.$$

That is, we consider the realization with mixed boundary conditions (Robin and Dirichlet conditions). Let  $(\mu_n(h, b, \rho))_{n \in \mathbb{N}^*}$  be the sequence of eigenvalues of the operator  $\mathcal{L}_h^{b,\rho}$ . It is then standard to deduce from the min-max principle and the Agmon estimates of Corollary 3.3.7 the following proposition.

**Proposition 3.3.8.** *Let  $\epsilon > 0$  and  $\alpha \in (0, \sqrt{\epsilon})$ , there exist constants  $C > 0$ ,  $h_0 \in (0, 1)$  such that, for all  $h \in (0, h_0)$ ,  $n \geq 1$  and  $\mu_n(h, b) < -\epsilon h$ ,*

$$\mu_n(h, b, \rho) \leq \mu_n(h, b) + C \exp(-\alpha h^{-\rho}).$$

Moreover, for all  $n \geq 1$  and  $h > 0$ , we have

$$\mu_n(h, b) \leq \mu_n(h, b, \rho).$$

*Proof.* The inequality  $\mu_n(h, b) \leq \mu_n(h, b, \rho)$  is not asymptotic. Let  $(v_{k,h})_{1 \leq k \leq n}$  be a family of eigenfunction of  $\mathcal{L}_h^{b,\rho}$  associated with the eigenvalues  $(\mu_k(h, b, \rho))_{1 \leq k \leq n}$ . We define the function

$$u_{n,h} = \begin{cases} v_{n,h} & \text{if } x \in \Omega_\delta \\ 0 & \text{if } x \in \Omega \setminus \Omega_\delta. \end{cases}$$

Let  $E$  be the vector subspace generated by the family  $(u_{k,h})_{1 \leq k \leq n}$ . We will find an upper bound of the quadratic form  $q_h^b(u)$  for all  $u \in E$ . By inserting  $u$  in the quadratic form  $q_h^{b,\rho}(u)$ , we get

$$q_h^b(u) = q_h^{b,\rho}(v) = \langle \mathcal{L}_h^{b,\rho} v, v \rangle \leq \mu_n(h, b, \rho) \|v\|^2 = \mu_n(h, b, \rho) \|u\|^2.$$

By the min-max theorem, we get the non-asymptotic inequality

$$\mu_n(h, b) \leq \mu_n(h, b, \rho).$$

Now we determine an upper bound of  $\mu_n(h, b, \rho)$  in terms of  $\mu_n(h, b)$ . Let  $(u_{k,h})_{1 \leq k \leq n}$  an orthonormal family of eigenfunctions associated with the eigenvalues  $(\mu_k(h, b))_{1 \leq k \leq n}$ . Let the truncation function define on  $\mathbb{R}$  as follows:

$$0 \leq \chi \leq 1, \chi = 1 \text{ on } ]-\infty, 1/2] \text{ and } \chi = 0 \text{ on } [1, +\infty[.$$

We define the function

$$\chi_1(x) = \chi\left(\frac{t(x)}{\delta}\right),$$

with  $\delta = h^{\frac{1}{2}-\rho}$ . For  $k = 1, \dots, n$ , we let

$$v_{k,h} = \chi_1 u_{k,h}.$$

Let  $F_h$  be the vector subspace of  $\text{Dom}(\mathcal{L}_h^{b,\rho})$  generated by the family  $(v_{k,h})_{1 \leq k \leq n}$  and  $w_h \in F_h$  which is written as follows:

$$w_h = \sum_{k=1}^n \beta_k v_{k,h}.$$

The functions  $(v_{k,h})$  are quasi-orthonormal (using Proposition 3.3.4). We notice that  $\dim F_h = n$ , and

$$\|w_h\|^2 = \sum_{k=1}^n |\beta_k|^2 + \mathcal{O}(\exp(-\alpha h^{-\rho})).$$

Now we have to find an upper bound of  $q_h^{b,\rho}(w_h)$  by making  $\mu_n(h, b)$  appear. We have the following decomposition:

$$q_h^{b,\rho}(w_h) = \langle \mathcal{L}_h^{b,\rho} w_h, w_h \rangle = \sum_{j,k=1}^n \beta_j \beta_k \langle \mathcal{L}_h^{b,\rho} v_{j,h}, v_{k,h} \rangle.$$

For  $j, k$  fixed and for  $h$  small enough, we have

$$\begin{aligned} \langle \mathcal{L}_h^{b,\rho} v_{j,h}, v_{k,h} \rangle &= \langle \chi_1 \mathcal{L}_h^b u_{j,h}, \chi_1 u_{k,h} \rangle - h^2 \langle u_{j,h} \Delta \chi_1, \chi_1 u_{k,h} \rangle - 2h^2 \langle \nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h}, \chi_1 u_{k,h} \rangle \\ &= \mu_j(h, b) \langle \chi_1 u_{j,h}, \chi_1 u_{k,h} \rangle - h^2 \langle u_{j,h} \Delta \chi_1, \chi_1 u_{k,h} \rangle - 2h^2 \langle \nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h}, \chi_1 u_{k,h} \rangle \\ &= \delta_{j,k} \mu_j(h, b) - h^2 \langle u_{j,h} \Delta \chi_1, \chi_1 u_{k,h} \rangle - 2h^2 \langle \nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h}, \chi_1 u_{k,h} \rangle \\ &\quad + \mathcal{O}(\exp(-\alpha h^{-\rho})). \end{aligned}$$

We have

$$\nabla \chi_1(x) = h^{\rho-1/2} \nabla t(x) \chi'(h^{\rho-1/2} t(x)),$$

and

$$\Delta \chi_1(x) = h^{2\rho-1} |\nabla t(x)|^2 \chi''(h^{\rho-1/2} t(x)) + h^{\rho-1/2} \Delta t(x) \chi'(h^{\rho-1/2} t(x)).$$

According to the Hölder inequality, we have

$$\begin{aligned} h^2 |\langle u_{j,h} \Delta \chi_1, \chi_1 u_{k,h} \rangle| &= h^2 \left| \int_{\{t(x) \geq \delta/2\}} u_{j,h} \Delta \chi_1 \chi_1 \overline{u_{k,h}} \, dx \right| \\ &\leq h^2 \left( \int_{\{t(x) \geq \delta/2\}} |u_{j,h} \Delta \chi_1|^2 \, dx \right)^{1/2} \left( \int_{\{t(x) \geq \delta/2\}} |u_{k,h} \chi_1|^2 \, dx \right)^{1/2}, \end{aligned}$$

and

$$\begin{aligned} &h^2 |\langle \nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h}, \chi_1 u_{k,h} \rangle| \\ &= h^2 \left| \int_{\{t(x) \geq \delta/2\}} \nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h} \chi_1 \overline{u_{k,h}} \, dx \right| \\ &\leq h^2 \left( \int_{\{t(x) \geq \delta/2\}} |\nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h}|^2 \, dx \right)^{1/2} \left( \int_{\{t(x) \geq \delta/2\}} |u_{k,h} \chi_1|^2 \, dx \right)^{1/2}. \end{aligned}$$

Since  $h \cdot h^{\rho-1/2} < 1$  and  $h \cdot h^{2\rho-1} < 1$  and by Corollary 3.3.7, there exist constants  $c > 0$  such that :

$$\begin{aligned} h^2 |\langle u_{j,h} \Delta \chi_1, \chi_1 u_{k,h} \rangle| &\leq c \left( \int_{\{t(x) \geq \delta/2\}} |u_{j,h}|^2 \, dx \right)^{1/2} \left( \int_{\{t(x) \geq \delta/2\}} |u_{k,h}|^2 \, dx \right)^{1/2} \\ &\leq c \exp(-\alpha h^{-\rho}), \end{aligned}$$

and

$$\begin{aligned} h^2 |\langle \nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h}, \chi_1 u_{k,h} \rangle| &\leq c \left( \int_{\{t(x) \geq \delta/2\}} h |\nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h}|^2 \, dx \right)^{1/2} \left( \int_{\{t(x) \geq \delta/2\}} |u_{k,h}|^2 \, dx \right)^{1/2} \\ &\leq c \exp(-\alpha h^{-\rho}). \end{aligned}$$

Hence

$$h^2 \langle u_{j,h} \Delta \chi_1, \chi_1 u_{k,h} \rangle + 2h^2 \langle \nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h}, \chi_1 u_{k,h} \rangle = \mathcal{O}(\exp(-\alpha h^{-\rho})).$$

Consequently,

$$\begin{aligned} q_h^{b,\rho}(w_h) &= \sum_{j,k=1}^n \beta_j \beta_k \langle \mathcal{L}_h^{b,\rho} v_{j,h}, v_{k,h} \rangle = \sum_{j=1}^n \mu_j(h, b) |\beta_j|^2 + \mathcal{O}(\exp(-\alpha h^{-\rho})) \sum_{j,k=1}^n \beta_j \beta_k \\ &\leq (\mu_n(h, b) + C \exp(-\alpha h^{-\rho})) \|w_h\|^2. \end{aligned}$$

By the min-max theorem, we have

$$\mu_n(h, b, \rho) = \inf_{\substack{E \subset W_\delta \\ \dim E = n}} \sup_{\substack{u \in E \\ u \neq 0}} \frac{q_h^{b,\rho}(u)}{\int_{\Omega_h} |u|^2 dx},$$

therefore,

$$\mu_n(h, b, \rho) \leq \mu_n(h, b) + C \exp(-\alpha h^{-\rho}).$$

□

Proposition 3.3.8 leads us to replace the initial problem by a new Robin-Dirichlet in a  $\delta$ -neighborhood of the boundary  $\partial\Omega$ , and then reduces the analysis to the operator  $\mathcal{L}_h^{b,\rho}$ .

### 3.3.4 The Robin Laplacian in a tubular neighborhood

From (3.9), the new quadratic form is expressed in tubular coordinates and is written as follows:

$$\begin{aligned} \tilde{q}_h^{b,\rho}(u) &= h^2 \int_{-L}^L \int_0^\delta |(\partial_s - ib\bar{A}_1)\tilde{u}|^2 (1 - t\kappa(s))^{-1} dt ds \\ &\quad + h^2 \int_{-L}^L \int_0^\delta |(\partial_t - ib\bar{A}_2)\tilde{u}|^2 (1 - t\kappa(s)) dt ds - h^{3/2} \int_{-L}^L |\tilde{u}(s, t=0)|^2 ds, \end{aligned}$$

with  $\tilde{u} = e^{i\varphi} u \circ \Phi$  and  $L = \frac{|\partial\Omega|}{2}$ . Here  $\Phi$  is the transformation map that transforms Cartesian coordinates to boundary coordinates in a neighborhood of the edge and  $\varphi$  is the gauge function needed to assume that  $\bar{A}_2 = 0$ . These functions are defined in Section 3.2.

We replace  $\bar{A}_0 = (\bar{A}_1, \bar{A}_2)$  by its value given in (3.9), we obtain:

$$\begin{aligned} \tilde{q}_h^{b,\rho}(u) &= h^2 \int_{-L}^L \int_0^\delta \left| \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) \tilde{u} \right|^2 (1 - t\kappa(s))^{-1} dt ds \\ &\quad + h^2 \int_{-L}^L \int_0^\delta |\partial_t \tilde{u}|^2 (1 - t\kappa(s)) dt ds - h^{3/2} \int_{-L}^L |\tilde{u}(s, t=0)|^2 ds, \end{aligned}$$

and the ambient Hilbert space is  $L^2((1 - t\kappa(s)) dt ds)$ , the integration domain  $]-L, L] \times (0, \delta)$  and the natural scalar product

$$\langle f, g \rangle_{L^2((1 - t\kappa(s)) dt ds)} = \int_{-L}^L \int_0^\delta f \bar{g} (1 - t\kappa(s)) dt ds,$$

with

$$\gamma_0 = \frac{1}{|\partial\Omega|} \int_{\Omega} \operatorname{curl} A_0 dx = \frac{|\Omega|}{|\partial\Omega|}.$$

We would like to describe the spectrum of the operator associated with the quadratic form  $\tilde{q}_h^{b,\rho}$ . To see what happens, we write the operator  $\mathcal{L}_h^{b,\rho}$  in tubular coordinates  $(s, t)$ . The operator  $\mathcal{L}_h^{b,\rho}$  is unitarily equivalent to  $\tilde{\mathcal{L}}_h^{b,\rho}$  the self-adjoint realization on  $L^2([-L, L] \times (0, \delta), a dt ds)$ , of the differential operator, with  $2L$ -periodic boundary condition with respect to  $s$ ,

$$-h^2(1-t\kappa(s))^{-1}(\partial_s - ib\bar{A}_1)(1-t\kappa(s))^{-1}(\partial_s - ib\bar{A}_1) - h^2(1-t\kappa(s))^{-1}\partial_t(1-t\kappa(s))\partial_t.$$

We replace  $\bar{A}_0 = (\bar{A}_1, \bar{A}_2)$  by its explicit value given by (3.9), the operator  $\mathcal{L}_h^{b,\rho}$  becomes :

$$\tilde{\mathcal{L}}_h^{b,\rho} = -h^2 a^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) a^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) - h^2 a^{-1} \partial_t a \partial_t,$$

where  $a = 1 - t\kappa(s)$ .

**Lemma 3.3.9.** *In boundary coordinates, the Robin condition becomes*

$$\partial_t u(s, t = 0) = -h^{-1/2} u(s, t = 0). \quad (3.30)$$

*Proof.*  $\tilde{\mathcal{L}}_h^{b,\rho}$  is the self-adjoint operator associated with the quadratic form  $\tilde{q}_h^{b,\rho}$ . Let the sesquilinear form associated with  $\tilde{q}_h^{b,\rho}$  be defined for all  $u, v \in H^1((-L, L) \times (0, \delta))$  and  $u(s, \delta) = v(s, \delta) = 0$  by

$$\begin{aligned} Q_h^{b,\rho}(u, v) &= h^2 \int_{-L}^L \int_0^\delta \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) u \overline{\left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) v \right)} a^{-1} dt ds \\ &\quad + h^2 \int_{-L}^L \int_0^\delta \partial_t u \overline{\partial_t v} a dt ds - h^{3/2} \int_{-L}^L u(s, t = 0) \overline{v(s, t = 0)} ds. \end{aligned}$$

By definition of the operator  $\tilde{\mathcal{L}}_h^{b,\rho}$ , for all  $u \in \text{Dom}(\tilde{\mathcal{L}}_h^{b,\rho})$ ,  $v \in H^1((-L, L) \times (0, \delta))$  and  $v(s, t = \delta) = 0$ , we have

$$Q_h^{b,\rho}(u, v) = \left\langle \tilde{\mathcal{L}}_h^{b,\rho} u, v \right\rangle_{L^2(a dt ds)}.$$

From this equality, we find the Robin condition in the tubular coordinates.

First, we calculate  $\left\langle \tilde{\mathcal{L}}_h^{b,\rho} u, v \right\rangle_{L^2(a dt ds)}$ . We have

$$\begin{aligned} &\left\langle \tilde{\mathcal{L}}_h^{b,\rho} u, v \right\rangle_{L^2(a dt ds)} \\ &= -h^2 \int_{-L}^L \int_0^\delta a^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) a^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) u \bar{v} a dt ds \\ &\quad - h^2 \int_{-L}^L \int_0^\delta \partial_t (a \partial_t u) \bar{v} dt ds. \end{aligned}$$

Using an integration by parts, we obtain

$$\begin{aligned} &\left\langle \tilde{\mathcal{L}}_h^{b,\rho} u, v \right\rangle \\ &= h^2 \int_{-L}^L \int_0^\delta a^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) u \left( \partial_s + ib\gamma_0 + ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) \bar{v} dt ds \\ &\quad - h^2 \int_0^\delta a^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) u \bar{v} \Big|_{-L}^L dt + h^2 \int_{-L}^L \int_0^\delta a \partial_t u \partial_t \bar{v} dt ds + h^2 \int_{-L}^L \partial_t u \bar{v} \Big|_{t=0} ds. \end{aligned}$$

Since  $u$  and  $v$  are  $2L$ -periodic with respect to the variable  $s$ , we have

$$h^2 \int_0^\delta a^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) u \bar{v} \Big|_{-L}^L dt = 0.$$

Then,

$$\begin{aligned} \langle \tilde{\mathcal{L}}_h^{b,\rho} u, v \rangle &= h^2 \int_{-L}^L \int_0^\delta a^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) u \left( \partial_s + ib\gamma_0 + ib \left( -t + \frac{t^2}{2} \kappa(s) \right) \right) \bar{v} dt ds \\ &\quad + h^2 \int_{-L}^L \int_0^\delta a \partial_t u \partial_t \bar{v} dt ds + h^2 \int_{-L}^L \partial_t u \bar{v} \Big|_{t=0} ds \\ &= Q_h^{b,\rho}(u, v). \end{aligned}$$

By comparison, we get for all  $v \in H^1((-L, L) \times (0, \delta))$  and  $v(s, t = \delta) = 0$

$$\begin{aligned} h^2 \int_{-L}^L \partial_t u \bar{v} \Big|_{t=0} ds &= -h^{\frac{3}{2}} \int_{-L}^L u(s, t = 0) \bar{v}(s, t = 0) ds \\ \Rightarrow \int_{-L}^L \left( \partial_t u(s, t = 0) + h^{-1/2} u(s, t = 0) \right) \bar{v}(s, t = 0) ds &= 0. \end{aligned}$$

Hence

$$\partial_t u(s, t = 0) = -h^{-1/2} u(s, t = 0).$$

□

We introduce,

$$\begin{aligned} \tilde{\Omega}_\delta &= \{(s, t) : s \in [-L, L[ \text{ and } 0 < t < \delta\}, \\ \text{Dom}(\tilde{q}_h^{b,\rho}) &= \{u \in H^1(\tilde{\Omega}_\delta) : u(s, \delta) = 0\}, \\ \text{Dom}(\tilde{\mathcal{L}}_h^{b,\rho}) &= \{u \in H^2(\tilde{\Omega}_\delta) : u(s, \delta) = 0 \text{ and } \partial_t u(s, t = 0) = -h^{-1/2} u(s, t = 0)\}. \end{aligned}$$

### 3.3.5 The rescaled operator

We know that the eigenfunctions are localized near the boundary, at the scale  $h^{\frac{1}{2}}$  (cf. Prop 3.3.4) and in order to obtain a Robin condition independent of  $h$ , we get a partially semiclassical problem by changing the variable

$$(s, t) = (s, h^{\frac{1}{2}} \tau).$$

This change of variable transforms the above expression of  $\tilde{\mathcal{L}}_h^{b,\rho}$  into an operator as follows

$$\begin{aligned} h \left[ -h \hat{a}^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -h^{\frac{1}{2}} \tau + h \frac{\tau^2}{2} \kappa(s) \right) \right) \hat{a}^{-1} \left( \partial_s - ib\gamma_0 - ib \left( -h^{\frac{1}{2}} \tau + h \frac{\tau^2}{2} \kappa(s) \right) \right) \right. \\ \left. - \hat{a}^{-1} \partial_\tau \hat{a} \partial_\tau \right], \end{aligned}$$

where the new weight

$$\hat{a}(s, \tau) = 1 - h^{\frac{1}{2}} \tau \kappa(s).$$



The boundary condition becomes

$$\partial_\tau u(s, \tau = 0) = -u(s, \tau = 0).$$

We consider rather the operator

$$\widehat{\mathcal{L}}_h^{b,\rho} = h^{-1} \widetilde{\mathcal{L}}_h^{b,\rho},$$

and the domain of integration becomes

$$[-L, L] \times (0, \delta/h^{1/2}) = [-L, L] \times (0, h^{-\rho}).$$

The associated quadratic form is

$$\begin{aligned} \widehat{q}_h^{b,\rho}(\psi) &= \int_{-L}^L \int_0^{h^{-\rho}} \left| \left( h^{1/2} \partial_s - ibh^{1/2} \gamma_0 - ibh \left( -\tau + h^{1/2} \frac{\tau^2}{2} \kappa(s) \right) \right) \psi \right|^2 \widehat{a}^{-1} d\tau ds \\ &+ \int_{-L}^L \int_0^{h^{-\rho}} |\partial_\tau \psi|^2 \widehat{a} d\tau ds - \int_{-L}^L |\psi(s, \tau = 0)|^2 ds. \end{aligned} \quad (3.31)$$

We let

$$\begin{aligned} \widehat{\Omega}_\delta &= \{(s, \tau) : s \in [-L, L[ \text{ and } 0 < \tau < h^{-\rho}\}, \\ \text{Dom}(\widehat{q}_h^{b,\rho}) &= \{u \in H^1(\widehat{\Omega}_\delta) : u(s, h^{-\rho}) = 0\}, \\ \text{Dom}(\widehat{\mathcal{L}}_h^{b,\rho}) &= \{u \in H^2(\widehat{\Omega}_\delta) : u(s, h^{-\rho}) = 0 \text{ and } \partial_\tau u(s, \tau = 0) = -u(s, \tau = 0)\}. \end{aligned} \quad (3.32)$$

Let  $\widehat{\mu}_n(h, b, \rho)$  be the  $n$ -th eigenvalue of the self-adjoint operator  $\widehat{\mathcal{L}}_h^{b,\rho}$ . Since  $\widehat{\mathcal{L}}_h^{b,\rho} = h^{-1} \widetilde{\mathcal{L}}_h^{b,\rho}$ , then

$$\text{Sp}(\widehat{\mathcal{L}}_h^{b,\rho}) = h^{-1} \text{Sp}(\widetilde{\mathcal{L}}_h^{b,\rho}).$$

Therefore,

$$\widehat{\mu}_n(h, b, \rho) = h^{-1} \widetilde{\mu}_n(h, b, \rho) = h^{-1} \mu_n(h, b, \rho), \quad (3.33)$$

with  $\widetilde{\mu}_n(h, b, \rho)$  is the  $n$ -th eigenvalue of the self-adjoint operator  $\widetilde{\mathcal{L}}_h^{b,\rho}$ .

Below is a table that represents the different operators that appeared in this work, their associated quadratic forms, and their eigenvalues.

Operator	Quadratic form	Eigenvalue
$\mathcal{P}_\beta^b$	$\mathcal{Q}_\beta^b$	$\lambda_n(b, \beta)$
$\mathcal{L}_h^b$	$q_h^b$	$\mu_n(h, b)$
$\mathcal{L}_h^{b,\rho}$	$q_h^{b,\rho}$	$\mu_n(h, b, \rho)$
$\widetilde{\mathcal{L}}_h^{b,\rho}$	$\widetilde{q}_h^{b,\rho}$	$\widetilde{\mu}_n(h, b, \rho)$
$\widehat{\mathcal{L}}_h^{b,\rho}$	$\widehat{q}_h^{b,\rho}$	$\widehat{\mu}_n(h, b, \rho)$

### 3.4 Asymptotics of the eigenvalues of Robin's Laplacian

The aim of this section is to prove the following result, which implies Theorem 3.1.1.

**Theorem 3.4.1.** *Let  $n \in \mathbb{N}^*$ ,  $0 \leq \eta < \frac{3}{2}$ ,  $\alpha > 0$  and  $0 < c_1 < c_2$ . There exist  $c_{\pm} > 0$  and  $h_0 > 0$ , such that, if  $h \in (0, h_0)$  and  $b$  satisfies*

$$c_1 h^{-\frac{\eta}{2}} \leq b \leq c_2 h^{-\frac{\eta}{2}},$$

then, we have :

$$\lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},-}) + \mathcal{O}(h^{2-\alpha-\eta}) \leq \widehat{\mu}_n(h, b, \rho) \leq \lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},+}) + \mathcal{O}(h^{2-\alpha-\eta}),$$

where  $\lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},\pm})$  is the  $n$ -th eigenvalue of  $\mathcal{L}_{h,\gamma_0}^{\text{eff},\pm}$  defined by

$$\mathcal{L}_{h,\gamma_0}^{\text{eff},\pm} = -h(1 \pm c_{\pm} h^r)(\partial_s - ib\gamma_0)^2 - 1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h \pm c_{\pm} h^{\frac{3}{2}},$$

and

$$r = \min\left(\alpha, \frac{1}{2}\right).$$

**Remark 3.4.1.** *We distinguish two important cases of  $\eta$  :*

- If  $0 \leq \eta < \frac{5}{4}$ , we choose  $\alpha = \frac{1}{2}(\frac{5}{4} - \eta)$ , then  $\mathcal{O}(h^{2-\alpha-\eta}) = o(h^{\frac{3}{4}})$ .
- If  $\frac{5}{4} \leq \eta < \frac{3}{2}$ , we choose  $\alpha = \frac{1}{2}(\frac{3}{2} - \eta)$ , then  $\mathcal{O}(h^{2-\alpha-\eta}) = o(h^{\frac{1}{2}})$ .

### 3.4.1 Robin Laplacian in a weighted space

Following the writing of the magnetic Robin Laplacian in tubular coordinates in Section 3.3.4, there appears a one-dimensional operator defined, for  $s$  fixed, by

$$\begin{cases} -\hat{a}^{-1}\partial_{\tau}\hat{a}\partial_{\tau} = -\partial_{\tau}^2 + h^{1/2}\kappa(s)\hat{a}^{-1}\partial_{\tau} \\ \partial_{\tau}u(s, \tau = 0) = -u(s, \tau = 0) \text{ in } L^2((0, h^{-\rho}), (1 - \tau h^{1/2}\kappa(s)) d\tau), \end{cases}$$

where  $\hat{a}(s, \tau) = 1 - h^{\frac{1}{2}}\tau\kappa(s)$ . This operator is denoted by

$$\mathcal{H}_{\kappa(s),h} = \mathcal{H}_B^{\{T\}},$$

with

$$T = h^{-\rho} \text{ and } B = h^{1/2}\kappa(s),$$

and where  $\mathcal{H}_B^{\{T\}}$  is defined in Section 2.1.3. The associated quadratic form is:

$$q_{\kappa(s),h}(u) = \int_0^{h^{-\rho}} |\partial_{\tau}u(\tau)|^2 \hat{a} d\tau - |u(0)|^2. \quad (3.34)$$

Let  $v_{\kappa(s),h}$  be a normalized groundstate of  $\mathcal{H}_{\kappa(s),h}$  associated with  $\lambda_1(\mathcal{H}_{\kappa(s),h})$ . According to the Lemma 2.1.5, there exists a constant  $C > 0$ , such that

$$\|\partial_s v_{\kappa(s),h}\|_{L^2((0, h^{-\rho}), (1 - h^{1/2}\kappa(s)\tau) d\tau)} \leq Ch^{1/2}. \quad (3.35)$$

**Lemma 3.4.2.** *Under the assumptions of Proposition 2.1.3, for any  $k \in \mathbb{N}$ , there exists a constant  $C_k > 0$  such that the function  $v_{\kappa(s),h}$  satisfies when  $h \rightarrow 0_+$  :*

$$\int_0^{h^{-\rho}} \tau^k |v_{\kappa(s),h}|^2 (1 - h^{1/2}\kappa(s)\tau) d\tau \leq C_k.$$

*Proof.* The demonstration comes simply from the following inequality:

$$\forall l \geq 0, \quad \forall k \in \mathbb{N}, \quad l^k \leq k! e^l.$$

This inequality gives us for  $l = 2\alpha\tau$  where  $\alpha$  is the constant in Proposition 2.1.4 :

$$2^k \alpha^k \tau^k \leq k! e^{2\alpha\tau}.$$

From the Proposition 2.1.4, we obtain:

$$\int_0^{h^{-\rho}} \tau^k |v_{\kappa(s),h}|^2 (1 - h^{1/2} \kappa(s) \tau) \, d\tau \leq \int_0^{h^{-\rho}} \frac{k!}{2^k \alpha^k} e^{2\alpha\tau} |v_{\kappa(s),h}|^2 (1 - h^{1/2} \kappa(s) \tau) \, d\tau \leq C_k.$$

□

By [KS20, Lemma 2.5], we have the asymptotic estimate of the first and second eigenvalue of the dominate operator  $\mathcal{H}_{\kappa(s),h}$

$$\lambda_1(\mathcal{H}_{\kappa(s),h}) = -1 - \kappa(s) h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2} h + \mathcal{O}(h^{3/2}), \quad (3.36)$$

and for  $C > 0$

$$\lambda_2(\mathcal{H}_{\kappa(s),h}) \geq -Ch^{1/2-\rho}. \quad (3.37)$$

### 3.4.2 The corrected Feshbach projection

For  $s \in [-L, L]$ , we introduce the Feshbach projection  $\Pi_s$  on the normalized ground state  $v_{\kappa(s),h}$  of  $\mathcal{H}_{\kappa(s),h}$

$$\Pi_s \psi = \langle \psi, v_{\kappa(s),h} \rangle_{L^2((0, h^{-\rho}), \hat{a} \, d\tau)} v_{\kappa(s),h}, \quad \forall \psi \in L^2([-L, L] \times (0, h^{-\rho}), \hat{a} \, ds \, d\tau). \quad (3.38)$$

We also let

$$\Pi_s^\perp = \text{Id} - \Pi_s,$$

and

$$R_h(s) = \|\partial_s v_{\kappa(s),h}\|_{L^2((0, h^{-\rho}), \hat{a} \, d\tau)}^2.$$

The quantity  $R_h$  is sometimes called ‘‘Born-Oppenheimer correction’’. It measures the commutation defect between  $\partial_s$  and  $\Pi_s$ .

**Remark 3.4.2.** *Note that the idea of considering a Feshbach projection appears in many different contexts: WKB analysis (see for example [BNHR16, Sections 2.4 & 3.2], [HK17] and [HKR17]), convergence resolvent norm (see for example [KR14, Section 4.2] and [KRRS18]) or space/time adiabatic limits (see [Teu03, Chapter 3]). The reader might also consider to read [LT13, p. 35], where the authors tackle a similar problem of finding a corrected Feshbach projection in order to decouple the variable of their fiber operator and the semiclassical variable (his called  $\varepsilon$  in their paper and they call ‘‘horizontal variable’’ our variables). In particular, they emphasize that, without such a correction, the decoupling appears modulo a remainder  $\mathcal{O}(h)$  larger than the gap between the eigenvalues.*

**Lemma 3.4.3.** *There exist constants  $C > 0$  and  $h_0 > 0$  such that, for all  $\psi \in \text{Dom}(\hat{q}_h^{b,\rho})$  and  $h \in (0, h_0)$ , we have:*

$$\|[\Pi_s, \partial_s] \psi\|_{L^2(\hat{a} \, d\tau)} = \|[\Pi_s^\perp, \partial_s] \psi\|_{L^2(\hat{a} \, d\tau)} \leq \left[ 2R_h(s)^{\frac{1}{2}} + ch^{1/2} \right] \|\psi\|_{L^2(\hat{a} \, d\tau)}.$$

*Proof.* We estimate the commutator:

$$\begin{aligned}
 [\Pi_s, \partial_s]\psi &= -[\Pi_s^\perp, \partial_s]\psi \\
 &= \langle \partial_s \psi, v_{\kappa(s),h} \rangle_{L^2(\hat{a}d\tau)} v_{\kappa(s),h} - \partial_s \left( \langle \psi, v_{\kappa(s),h} \rangle_{L^2(\hat{a}d\tau)} v_{\kappa(s),h} \right) \\
 &= -\langle \psi, \partial_s v_{\kappa(s),h} \rangle_{L^2(\hat{a}d\tau)} v_{\kappa(s),h} - \langle \psi, v_{\kappa(s),h} \rangle_{L^2(\hat{a}d\tau)} \partial_s v_{\kappa(s),h} \\
 &\quad + h^{1/2} \kappa'(s) \left( \int_0^{h^{-\rho}} \psi v_{\kappa(s),h} \tau d\tau \right) v_{\kappa(s),h}.
 \end{aligned}$$

We get, thanks to the Cauchy-Schwarz inequality and the Agmon estimates (see Lemma 3.4.2)

$$\|\llbracket \Pi_s, \partial_s \rrbracket \psi\|_{L^2(\hat{a}d\tau)} \leq \left[ 2R_h(s)^{\frac{1}{2}} + ch^{1/2} \right] \|\psi\|_{L^2(\hat{a}d\tau)}.$$

□

**Lemma 3.4.4.** *Let  $\psi \in \text{Dom}(\mathcal{H}_{\kappa(s),h})$ , we have the orthogonal decomposition:*

$$q_{\kappa(s),h}(\psi) = q_{\kappa(s),h}(\Pi_s \psi) + q_{\kappa(s),h}(\Pi_s^\perp \psi),$$

with  $q_{\kappa(s),h}$  is the quadratic form associated to the operator  $\mathcal{H}_{\kappa(s),h}$  defined in (3.34).

*Proof.* We have

$$\begin{aligned}
 q_{\kappa(s),h}(\psi) &= \langle \mathcal{H}_{\kappa(s),h} \psi, \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} \\
 &= \langle \mathcal{H}_{\kappa(s),h}(\Pi_s \psi + \Pi_s^\perp \psi), \Pi_s \psi + \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} \\
 &= \langle \mathcal{H}_{\kappa(s),h} \Pi_s \psi, \Pi_s \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} + \langle \mathcal{H}_{\kappa(s),h} \Pi_s^\perp \psi, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} \\
 &\quad + \langle \mathcal{H}_{\kappa(s),h} \Pi_s \psi, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} + \langle \Pi_s \psi, \mathcal{H}_{\kappa(s),h} \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)}.
 \end{aligned}$$

Since  $\mathcal{H}_{\kappa(s),h}$  is self-adjoint on  $L^2((0, h^{-\rho}), \hat{a}d\tau)$ , then

$$\langle \mathcal{H}_{\kappa(s),h} \Pi_s \psi, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} = \langle \Pi_s \psi, \mathcal{H}_{\kappa(s),h} \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)}.$$

Therefore

$$\begin{aligned}
 q_{\kappa(s),h}(\psi) &= \langle \mathcal{H}_{\kappa(s),h} \Pi_s \psi, \Pi_s \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} + \langle \mathcal{H}_{\kappa(s),h} \Pi_s^\perp \psi, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} \\
 &\quad + 2\langle \mathcal{H}_{\kappa(s),h} \Pi_s \psi, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)}.
 \end{aligned}$$

We have

$$\Pi_s \psi = \langle \psi, v_{\kappa(s),h} \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} v_{\kappa(s),h} = \varphi v_{\kappa(s),h}.$$

Therefore

$$\begin{aligned}
 \langle \mathcal{H}_{\kappa(s),h} \Pi_s \psi, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} &= \langle \mathcal{H}_{\kappa(s),h} \varphi v_{\kappa(s),h}, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} \\
 &= \varphi(s) \langle \mathcal{H}_{\kappa(s),h} v_{\kappa(s),h}, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} \\
 &= \varphi(s) \langle \lambda_1(\mathcal{H}_{\kappa(s),h}) v_{\kappa(s),h}, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} \\
 &= \lambda_1(\mathcal{H}_{\kappa(s),h}) \langle \Pi_s \psi, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)} \\
 &= 0.
 \end{aligned}$$

Consequently,

$$q_{\kappa(s),h}(\psi) = \langle \mathcal{H}_{\kappa(s),h} \Pi_s \psi, \Pi_s \psi \rangle_{L^2((0, h^{-\rho}), \hat{\alpha} d\tau)} + \langle \mathcal{H}_{\kappa(s),h} \Pi_s^\perp \psi, \Pi_s^\perp \psi \rangle_{L^2((0, h^{-\rho}), \hat{\alpha} d\tau)},$$

hence

$$q_{\kappa(s),h}(\psi) = q_{\kappa(s),h}(\Pi_s \psi) + q_{\kappa(s),h}(\Pi_s^\perp \psi). \quad (3.39)$$

□

### 3.4.3 Approximation of the norm on the weighted space

In this section, we show several lemmas that are approximations of the weight.

**Lemma 3.4.5.** *Let  $\psi \in L^2(\hat{\alpha} d\tau ds)$ , and  $0 < \rho < \frac{1}{4}$ . There exist  $c > 0$  such that :*

$$\|\psi\|_{L^2(\hat{\alpha}^{-1} d\tau ds)}^2 \leq (1 + ch^{1/2}) \|\Pi_s \psi\|_{L^2(\hat{\alpha} d\tau ds)}^2 + (1 + ch^{1/4}) \|\Pi_s^\perp \psi\|_{L^2(\hat{\alpha} d\tau ds)}^2, \quad (3.40)$$

and

$$\|\psi\|_{L^2(\hat{\alpha}^{-1} d\tau ds)}^2 \geq (1 - ch^{1/2}) \|\Pi_s \psi\|_{L^2(\hat{\alpha} d\tau ds)}^2 + (1 - ch^{1/4}) \|\psi\|_{L^2(\hat{\alpha} d\tau ds)}^2. \quad (3.41)$$

In particular,

$$\|\psi\|_{L^2(\hat{\alpha}^{-1} d\tau ds)}^2 \leq (1 + ch^{1/4}) \|\psi\|_{L^2(\hat{\alpha} d\tau ds)}^2.$$

*Proof.* Let  $\psi \in L^2(\hat{\alpha} d\tau ds)$ . There exist constants  $C > 0$  such that :

$$\begin{aligned} \left| \int_{-L}^L \int_0^{h^{-\rho}} |\psi|^2 \hat{\alpha} d\tau ds - \int_{-L}^L \int_0^{h^{-\rho}} |\psi|^2 \hat{\alpha}^{-1} d\tau ds \right| &= \left| \int_{-L}^L \int_0^{h^{-\rho}} (\hat{\alpha}^{-2} - 1) |\psi|^2 \hat{\alpha} d\tau ds \right| \\ &\leq C \int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\psi|^2 \hat{\alpha} d\tau ds \\ &\leq C \int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\Pi_s \psi + \Pi_s^\perp \psi|^2 \hat{\alpha} d\tau ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \int_{-L}^L \int_0^{h^{-\rho}} |\psi|^2 \hat{\alpha} d\tau ds - \int_{-L}^L \int_0^{h^{-\rho}} |\psi|^2 \hat{\alpha}^{-1} d\tau ds \right| \\ \leq C \left[ \int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\Pi_s \psi|^2 \hat{\alpha} d\tau ds + \int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\Pi_s^\perp \psi|^2 \hat{\alpha} d\tau ds \right]. \end{aligned} \quad (3.42)$$

By the Lemma 3.4.2, we have:

$$\begin{aligned} \int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\Pi_s \psi|^2 \hat{\alpha} d\tau ds &= \int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\langle \psi, v_{\kappa(s),h} \rangle|^2 |v_{\kappa(s),h}|^2 \hat{\alpha} d\tau ds \\ &= h^{1/2} \int_{-L}^L |\langle \psi, v_{\kappa(s),h} \rangle|^2 \left[ \int_0^{h^{-\rho}} \tau |v_{\kappa(s),h}|^2 \hat{\alpha} d\tau \right] ds \\ &\leq c h^{1/2} \int_{-L}^L |\langle \psi, v_{\kappa(s),h} \rangle|^2 ds \\ &= c h^{1/2} \|\Pi_s \psi\|_{L^2(\hat{\alpha} d\tau ds)}^2. \end{aligned}$$

Using that  $h^{1/2-\rho} \leq h^{1/4}$ , we have

$$\int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\Pi_s^\perp \psi|^2 \tilde{a} \, d\tau ds \leq h^{1/2-\rho} \int_{-L}^L \int_0^{h^{-\rho}} |\Pi_s^\perp \psi|^2 \hat{a} \, d\tau ds \leq h^{1/4} \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2.$$

From the equation (3.42), we have

$$\left| \int_{-L}^L \int_0^{h^{-\rho}} |\psi|^2 \hat{a} \, d\tau ds - \int_{-L}^L \int_0^{h^{-\rho}} |\psi|^2 \hat{a}^{-1} \, d\tau ds \right| \leq c h^{1/2} \|\Pi_s \psi\|_{L^2(\hat{a}d\tau ds)}^2 + c h^{1/4} \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2.$$

Consequently,

$$\begin{aligned} \|\psi\|_{L^2(\hat{a}^{-1}d\tau ds)}^2 &\geq \|\psi\|_{L^2(\hat{a}d\tau ds)}^2 - c h^{1/2} \|\Pi_s \psi\|_{L^2(\hat{a}d\tau ds)}^2 - c h^{1/4} \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2 \\ &\geq \|\Pi_s \psi + \Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2 - c h^{1/2} \|\Pi_s \psi\|_{L^2(\hat{a}d\tau ds)}^2 - c h^{1/4} \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2 \\ &\geq (1 - c h^{1/2}) \|\Pi_s \psi\|_{L^2(\hat{a}d\tau ds)}^2 + (1 - c h^{1/4}) \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2, \end{aligned}$$

hence

$$\begin{aligned} \|\psi\|_{L^2(\hat{a}^{-1}d\tau ds)}^2 &\leq (1 + c h^{1/2}) \|\Pi_s \psi\|_{L^2(\hat{a}d\tau ds)}^2 + (1 + c h^{1/4}) \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2 \\ &\leq (1 + c h^{1/4}) \|\psi\|_{L^2(\hat{a}d\tau ds)}^2. \end{aligned}$$

□

**Corollary 3.4.6.** *Let  $\psi \in L^2(\hat{a}d\tau ds)$ . There exist  $c > 0$  such that :*

$$\|\Pi_s \psi\|_{L^2(\hat{a}^{-1}d\tau ds)}^2 \leq (1 + c h^{1/2}) \int_{-L}^L |\varphi(s)|^2 \, ds,$$

with  $\varphi = \langle \psi, v_{\kappa(s), h} \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)}$ .

*Proof.* This corollary is a direct result of the equation (3.40) and the Lemma 3.4.2. Indeed,

$$\Pi_s \Pi_s \psi = \Pi_s \psi \quad \text{and} \quad \Pi_s \Pi_s^\perp \psi = 0.$$

□

### 3.4.4 Upper bound

To separate the variables, we consider the function of the form :

$$\psi(s, \tau) = \varphi(s) \cdot v_{\kappa(s), h}(\tau), \tag{3.43}$$

with  $\varphi \in H^1(\mathbb{R}/2L\mathbb{Z})$ .

The following proposition provides an upper bound of the quadratic form on a subspace.

**Proposition 3.4.7.** *Let  $\rho \in (0, 1/4)$ ,  $0 \leq \eta < \frac{3}{2}$ ,  $\alpha > 0$  and  $0 < c_1 < c_2$ . There exist constants  $c > 0$ ,  $h_0 > 0$  such that, for all  $h \in (0, h_0)$  and  $b$  satisfying*

$$c_1 h^{\frac{-\eta}{2}} \leq b \leq c_2 h^{\frac{-\eta}{2}},$$

we have for all  $\psi$  defined in (3.43)

$$\widehat{q}_h^{b,\rho}(\psi) \leq q_{h,\gamma_0}^{\text{eff},+}(\varphi) + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2,$$

where, for all  $\varphi \in H^1(\mathbb{R}/2L\mathbb{Z})$

$$q_{h,\gamma_0}^{\text{eff},+}(\varphi) = \int_{-L}^L \left( -1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h + ch^{\frac{3}{2}} \right) |\varphi(s)|^2 ds + h \left[ 1 + ch^{\min(\alpha, \frac{1}{2})} \right] \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds.$$

*Proof.* We write

$$\widehat{q}_h^{b,\rho}(\psi) = q^{\text{tang}}(\psi) + q^{\text{trans}}(\psi),$$

where

$$q^{\text{trans}}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| \left( h^{\frac{1}{2}}\partial_s - ibh^{\frac{1}{2}}\gamma_0 - ibh \left( -\tau + h^{\frac{1}{2}}\frac{\tau^2}{2}\kappa(s) \right) \right) \psi \right|^2 \widehat{a}^{-1} d\tau ds,$$

and

$$q^{\text{tang}}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} |\partial_\tau \psi|^2 \widehat{a} d\tau ds - \int_{-L}^L |\psi(s, \tau = 0)|^2 ds.$$

• **Upper bound of  $q^{\text{tang}}(\psi)$**

Using the min-max principle and (3.36), we obtain an upper bound of  $q^{\text{tang}}$

$$\begin{aligned} q^{\text{tang}}(\psi) &= \int_{-L}^L \left[ \int_0^{h^{-\rho}} |\partial_\tau(\varphi(s)v_{\kappa(s),h}(\tau))|^2 \widehat{a} d\tau - |\varphi(s)|^2 |v_{\kappa(s),h}(0)|^2 \right] ds \\ &= \int_{-L}^L |\varphi(s)|^2 \left[ \int_0^{h^{-\rho}} |\partial_\tau v_{\kappa(s),h}(\tau)|^2 \widehat{a} d\tau - |v_{\kappa(s),h}(0)|^2 \right] ds \\ &= \int_{-L}^L q_{\kappa(s),h}(v_{\kappa(s),h}) |\varphi(s)|^2 ds \\ &= \int_{-L}^L \lambda_1(\mathcal{H}_{\kappa(s),h}) |\varphi(s)|^2 ds \\ &\leq \int_{-L}^L \left( -1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h \right) |\varphi(s)|^2 ds + ch^{3/2} \|\varphi\|^2. \end{aligned}$$

• **Upper bound of  $q^{\text{trans}}(\psi)$**

Using the following classical inequality, for all  $a, b \in \mathbb{R}$ ,  $h > 0$ , and  $\alpha > 0$ , we have

$$|a + b|^2 \leq (1 + h^\alpha) |a|^2 + (1 + h^{-\alpha}) |b|^2,$$

we have

$$q^{\text{trans}}(\psi) \leq (1 + h^\alpha) q_1^{\text{trans}}(\psi) + (1 + h^{-\alpha}) q_2^{\text{trans}}(\psi),$$

where

$$q_1^{\text{trans}}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| \left( h^{\frac{1}{2}}\partial_s - ibh^{\frac{1}{2}}\gamma_0 \right) \psi \right|^2 \widehat{a}^{-1} d\tau ds,$$

and

$$q_2^{trans}(\psi) = b^2 h^2 \int_{-L}^L \int_0^{h^{-\rho}} \left| \left( -\tau + h^{\frac{1}{2}} \frac{\tau^2}{2} \kappa(s) \right) \psi \right|^2 \hat{a}^{-1} d\tau ds.$$

Find an upper bound for  $q_1^{trans}(\psi)$  and  $q_2^{trans}(\psi)$ .

• For  $q_2^{trans}(\psi)$ :

By the Lemma 3.4.5, there exist constants  $c > 0$  :

$$\int_{-L}^L \int_0^{h^{-\rho}} \left| \left( -\tau + h^{\frac{1}{2}} \frac{\tau^2}{2} \kappa(s) \right) \psi \right|^2 \hat{a}^{-1} d\tau ds \leq (1 + c h^{1/4}) \int_{-L}^L \int_0^{h^{-\rho}} \left| \left( -\tau + h^{\frac{1}{2}} \frac{\tau^2}{2} \kappa(s) \right) \psi \right|^2 \hat{a} d\tau ds.$$

Therefore

$$q_2^{trans}(\psi) \leq b^2 h^2 (1 + c h^{1/4}) \int_{-L}^L |\varphi(s)|^2 \left[ \int_0^{h^{-\rho}} \left| \left( -\tau + h^{\frac{1}{2}} \frac{\tau^2}{2} \kappa(s) \right) v_{\kappa,h} \right|^2 \hat{a} d\tau \right] ds.$$

By the Lemma 3.4.2, and since the curvature is bounded, there exists a constant  $c > 0$ , such that:

$$\int_0^{h^{-\rho}} \left| \left( -\tau + h^{\frac{1}{2}} \frac{\tau^2}{2} \kappa(s) \right) v_{\kappa,h} \right|^2 \hat{a} d\tau \leq c.$$

Consequently,

$$q_2^{trans}(\psi) \leq c(1 + c h^{1/4}) b^2 h^2 \|\varphi\|^2.$$

Hence,

$$(1 + h^{-\alpha}) q_2^{trans}(\psi) \leq c b^2 h^2 (1 + c h^{1/4}) (1 + h^{-\alpha}) \|\varphi\|^2 \leq c b^2 h^{2-\alpha} \|\varphi\|^2.$$

• For  $q_1^{trans}(\psi)$ :

$$\begin{aligned} q_1^{trans}(\psi) &= h \int_{-L}^L \int_0^{h^{-\rho}} |v_{\kappa(s),h} (\partial_s - ib\gamma_0)\varphi + \varphi \partial_s v_{\kappa(s),h}|^2 \hat{a}^{-1} d\tau ds \\ &\leq h(1 + h^\alpha) \int_{-L}^L \int_0^{h^{-\rho}} |v_{\kappa(s),h} (\partial_s - ib\gamma_0)\varphi|^2 \hat{a}^{-1} d\tau ds \\ &\quad + h(1 + h^{-\alpha}) \int_{-L}^L \int_0^{h^{-\rho}} |\partial_s v_{\kappa(s),h}|^2 |\varphi(s)|^2 \hat{a}^{-1} d\tau ds. \end{aligned}$$

By the Lemma 3.4.5 and the Corollary 3.4.6, there exist constants  $c > 0$ , such that:

$$\int_{-L}^L \int_0^{h^{-\rho}} |v_{\kappa(s),h} (\partial_s - ib\gamma_0)\varphi|^2 \hat{a}^{-1} d\tau ds \leq (1 + c h^{1/2}) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds,$$

and

$$\int_{-L}^L \int_0^{h^{-\rho}} |\partial_s v_{\kappa(s),h}|^2 |\varphi(s)|^2 \hat{a}^{-1} d\tau ds \leq (1 + c h^{1/4}) \int_{-L}^L \int_0^{h^{-\rho}} |\partial_s v_{\kappa(s),h}|^2 |\varphi(s)|^2 \hat{a} d\tau ds.$$



Thus,

$$\begin{aligned} q_1^{trans}(\psi) &\leq h(1+h^\alpha)(1+ch^{1/2}) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds \\ &\quad + h(1+h^{-\alpha})(1+ch^{1/4}) \int_{-L}^L |\varphi(s)|^2 \left[ \int_0^{h^{-\rho}} |\partial_s v_{\kappa(s),h}|^2 \hat{a} d\tau \right] ds. \end{aligned}$$

From (3.35), we have

$$q_1^{trans}(\psi) \leq h(1+h^\alpha)(1+ch^{1/2}) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds + ch^2(1+h^{-\alpha})(1+ch^{1/4}) \int_{-L}^L |\varphi(s)|^2 ds.$$

We deduce that

$$\begin{aligned} (1+h^\alpha)q_1^{trans}(\psi) &\leq h \left[ 1 + ch^{\min(\alpha, \frac{1}{2})} \right] \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds + \mathcal{O}(h^{2-\alpha}) \|\varphi\|^2 \\ &\leq h \left[ 1 + ch^{\min(\alpha, \frac{1}{2})} \right] \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|^2. \end{aligned}$$

Hence, there exists  $c > 0$  such that:

$$\begin{aligned} \hat{q}_h^{b,\rho}(\psi) &\leq \int_{-L}^L \left( -1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h + ch^{3/2} \right) |\varphi(s)|^2 ds \\ &\quad + h(1+ch^{\min(\alpha, \frac{1}{2})}) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|^2 \\ &= q_{h,\gamma_0}^{\text{eff},+}(\varphi) + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|^2, \end{aligned}$$

with

$$q_{h,\gamma_0}^{\text{eff},+}(\varphi) = \int_{-L}^L \left( -1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h + ch^{\frac{3}{2}} \right) |\varphi(s)|^2 ds + h \left[ 1 + ch^{\min(\alpha, \frac{1}{2})} \right] \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds.$$

□

### 3.4.5 Lower bound

The following proposition provides a lower bound of the quadratic form  $\hat{q}_h^{b,\rho}$ .

**Proposition 3.4.8.** *Let  $\rho \in (0, 1/4)$ ,  $0 \leq \eta < \frac{3}{2}$ ,  $0 < \alpha < 2 - \eta$  and  $0 < c_1 < c_2$ . There exist constants  $c > 0$  and  $h_0 > 0$  such that, for all  $h \in (0, h_0)$  and  $b$  satisfying*

$$c_1 h^{\frac{-\eta}{2}} \leq b \leq c_2 h^{\frac{-\eta}{2}},$$

we have

$$\hat{q}_h^{b,\rho}(\psi) \geq q_{h,\gamma_0}^{\text{eff},-}(\varphi) + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2 - o(1) \|\Pi_s^\perp \psi\|_{L^2(\hat{a} d\tau ds)}^2,$$

where

- $\hat{q}_h^{b,\rho}$  is defined on  $\text{Dom}(\hat{q}_h^{b,\rho})$  in (3.31).
- $\psi \in \text{Dom}(\hat{q}_h^{b,\rho})$ .

- $\varphi \in H^1(\mathbb{R}/2L\mathbb{Z})$  defined by

$$\varphi(s) := \langle \psi(s, \cdot), v_{\kappa(s), h} \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)},$$

and

$$\begin{aligned} q_{h, \gamma_0}^{\text{eff}, -}(\varphi) &= \int_{-L}^L \left( -1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h - ch^{\frac{3}{2}} \right) |\varphi(s)|^2 ds \\ &\quad + h \left[ 1 - ch^{\min(\alpha, \frac{1}{2})} \right] \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds. \end{aligned}$$

We write

$$\hat{q}_h^{b, \rho}(\psi) = q^{\text{tang}}(\psi) + q^{\text{trans}}(\psi),$$

where

$$q^{\text{trans}}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| \left( h^{\frac{1}{2}}\partial_s - ibh^{\frac{1}{2}}\gamma_0 - ibh \left( -\tau + h^{\frac{1}{2}}\frac{\tau^2}{2}\kappa(s) \right) \right) \psi \right|^2 \hat{a}^{-1} d\tau ds,$$

and

$$q^{\text{tang}}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} |\partial_\tau \psi|^2 \hat{a} d\tau ds - \int_{-L}^L |\psi(s, \tau = 0)|^2 ds.$$

### 3.4.5.1 Lower bound of the tangential form

**Proposition 3.4.9.** *There exists  $C > 0$ ,  $h_0$  such that, for all  $\psi \in \text{Dom}(\hat{q}_h^{b, \rho})$  and  $h \in (0, h_0)$ , we have*

$$q^{\text{tang}}(\psi) \geq \int_{-L}^L \left( -1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h \right) |\varphi(s)|^2 ds + \mathcal{O}(h^{3/2})\|\varphi\|^2 - Ch^{1/2-\rho}\|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2,$$

with

$$\varphi(s) := \langle \psi(s, \cdot), v_{\kappa(s), h} \rangle_{L^2((0, h^{-\rho}), \hat{a}d\tau)}.$$

*Proof.* Let  $\psi \in \text{Dom}(\hat{q}_h^{b, \rho})$ , using the orthogonal decomposition in Lemma 3.4.4, we have:

$$\begin{aligned} q^{\text{tang}}(\psi) &= \int_{-L}^L \left[ \int_0^{h^{-\rho}} |\partial_\tau \psi|^2 \tilde{a} d\tau - |\psi(s, \tau = 0)|^2 \right] ds \\ &= \int_{-L}^L q_{\kappa(s), h}(\psi) ds \\ &= \int_{-L}^L \left[ q_{\kappa(s), h}(\varphi v_{\kappa(s), h}) + q_{\kappa(s), h}(\Pi_s^\perp \psi) \right] ds \\ &= \int_{-L}^L q_{\kappa(s), h}(\varphi v_{\kappa(s), h}) ds + \int_{-L}^L q_{\kappa(s), h}(\Pi_s^\perp \psi) ds, \end{aligned}$$

according to the principle of min-max, we obtain

$$q^{\text{tang}}(\psi) \geq \int_{-L}^L \lambda_1(\mathcal{H}_{\kappa(s), h}) \|\varphi v_{\kappa(s), h}\|_{L^2((0, h^{-\rho}), \hat{a}d\tau)}^2 ds + \int_{-L}^L \lambda_2(\mathcal{H}_{\kappa(s), h}) \|\Pi_s^\perp \psi\|_{L^2((0, h^{-\rho}), \hat{a}d\tau)}^2 ds.$$

The conclusion flows, using (3.36) and (3.37), which give:

1. For all  $s \in [-L, L]$ , we have  $\|\varphi v_{\kappa(s), h}\|_{L^2((0, h^{-\rho}), \hat{a}d\tau)}^2 = |\varphi|^2$ .

2. For all  $s \in [-L, L]$ , we have  $\lambda_1(\mathcal{H}_{\kappa(s),h}) = -1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h + \mathcal{O}(h^{3/2})$ .
3. For all  $s \in [-L, L]$ , we have  $\lambda_2(\mathcal{H}_{\kappa(s),h}) \geq -Ch^{1/2-\rho}$ .

□

### 3.4.5.2 Lower bound of the transverse form

**Proposition 3.4.10.** *Let  $\rho \in (0, 1/4)$ ,  $0 \leq \eta < \frac{3}{2}$ ,  $0 < \alpha < 2 - \eta$  and  $0 < c_1 < c_2$ . There exist constants  $c > 0$ ,  $h_0 > 0$  such that, for any  $h \in (0, h_0)$  and  $b$  satisfying*

$$c_1 h^{-\frac{\eta}{2}} \leq b \leq c_2 h^{-\frac{\eta}{2}},$$

we have

$$\begin{aligned} q^{trans}(\psi) &\geq h[1 - ch^{\min(\alpha, \frac{1}{2})}] \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2 \\ &\quad - o(1) \|\Pi_s^\perp \psi\|_{L^2(\hat{a} d\tau ds)}^2, \end{aligned}$$

with

$$q^{trans}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| \left( h^{1/2} \partial_s - ibh^{1/2} \gamma_0 - ibh\alpha_s(\tau) \right) \psi \right|^2 \hat{a}^{-1} d\tau ds,$$

where

$$\alpha_s(\tau) = -\tau + h^{1/2} \frac{\tau^2}{2} \kappa(s).$$

*Proof.* From the Lemma 3.4.5, we have

$$\begin{aligned} q^{trans}(\psi) &\geq (1 - ch^{1/2}) \int_{-L}^L \int_0^{h^{-\rho}} \left| \Pi_s \left( h^{1/2} \partial_s - ibh^{1/2} \gamma_0 - ibh\alpha_s(\tau) \right) \psi \right|^2 \hat{a} d\tau ds \\ &\quad + (1 - ch^{1/4}) \int_{-L}^L \int_0^{h^{-\rho}} \left| \Pi_s^\perp \left( h^{1/2} \partial_s - ibh^{1/2} \gamma_0 - ibh\alpha_s(\tau) \right) \psi \right|^2 \hat{a} d\tau ds \\ &= (1 - ch^{1/2}) \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{1/2} \Pi_s \partial_s \psi - ibh^{1/2} \gamma_0 \Pi_s \psi - ibh \Pi_s(\alpha_s \psi) \right|^2 \hat{a} d\tau ds \\ &\quad + (1 - ch^{1/4}) \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{1/2} \Pi_s^\perp \partial_s \psi - ibh^{1/2} \gamma_0 \Pi_s^\perp \psi - ibh \Pi_s^\perp(\alpha_s \psi) \right|^2 \hat{a} d\tau ds. \end{aligned}$$

Write  $\Pi_s \partial_s \psi = \partial_s \Pi_s \psi + [\Pi_s, \partial_s] \psi$  and  $\Pi_s^\perp \partial_s \psi = \partial_s \Pi_s^\perp \psi + [\Pi_s^\perp, \partial_s] \psi$ , we obtain

$$\begin{aligned} q^{trans}(\psi) &\geq (1 - ch^{1/2}) \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{1/2} \partial_s \Pi_s \psi - ibh^{1/2} \gamma_0 \Pi_s \psi - ibh \Pi_s(\alpha_s \psi) + h^{1/2} [\Pi_s, \partial_s] \psi \right|^2 \hat{a} d\tau ds \\ &\quad + (1 - ch^{1/4}) \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{1/2} \partial_s \Pi_s^\perp \psi - ibh^{1/2} \gamma_0 \Pi_s^\perp \psi - ibh \Pi_s^\perp(\alpha_s \psi) + h^{1/2} [\Pi_s^\perp, \partial_s] \psi \right|^2 \hat{a} d\tau ds. \end{aligned}$$

Use the following classical inequality, for all  $a, b \in \mathbb{R}$ ,  $h > 0$ , and  $\alpha > 0$ , we have

$$|a + b|^2 \geq (1 - h^\alpha) |a|^2 - h^{-\alpha} |b|^2, \quad (3.44)$$

we obtain

$$q^{trans}(\psi) \geq (1 - ch^{1/2})(1 - h^\alpha)q_1^{trans}(\psi) - (1 - ch^{1/2})h^{-\alpha}q_2^{trans}(\psi) + (1 - ch^{1/4})(1 - h^\alpha)q_3^{trans}(\psi) - (1 - ch^{1/4})h^{-\alpha}q_2^{trans}(\psi), \quad (3.45)$$

with

- $q_1^{trans}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{1/2} \partial_s \Pi_s \psi - ibh^{1/2} \gamma_0 \Pi_s \psi - ibh \Pi_s(\alpha_s \psi) \right|^2 \hat{a} \, d\tau ds,$
- $q_2^{trans}(\psi) = h \int_{-L}^L \int_0^{h^{-\rho}} \left| [\Pi_s, \partial_s] \psi \right|^2 \hat{a} \, d\tau ds = h \int_{-L}^L \int_0^{h^{-\rho}} \left| [\Pi_s^\perp, \partial_s] \psi \right|^2 \hat{a} \, d\tau ds,$
- $q_3^{trans}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{1/2} \partial_s \Pi_s^\perp \psi - ibh^{1/2} \gamma_0 \Pi_s^\perp \psi - ibh \Pi_s^\perp(\alpha_s \psi) \right|^2 \hat{a} \, d\tau ds.$

**For  $q_1^{trans}$ :** Using (3.44), we have:

$$q_1^{trans}(\psi) \geq (1 - h^\alpha) \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{1/2} \partial_s \Pi_s \psi - ibh^{1/2} \gamma_0 \Pi_s \psi \right|^2 \hat{a} \, d\tau ds - h^{-\alpha} \int_{-L}^L \int_0^{h^{-\rho}} \left| bh \Pi_s(\alpha_s \psi) \right|^2 \hat{a} \, d\tau ds.$$

According to the Section 3.4.4 and with the same type of reasoning, there exist constants  $c > 0$  such that:

$$q_1^{trans}(\psi) \geq h(1 - ch^{\min(\alpha, \frac{1}{2})}) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 \, ds + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2 - \mathcal{O}(b^2 h^{2-\alpha}) \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2. \quad (3.46)$$

**For  $q_2^{trans}$ :** From the Lemma 3.4.3 and the inequality (3.35), we get:

$$\|[\Pi_s, \partial_s] \psi\|_{L^2(\hat{a}d\tau ds)} \leq \int_{-L}^L \left[ 2R_h(s)^{\frac{1}{2}} + ch^{1/2} \right] \|\psi\|_{L^2(\hat{a}d\tau)} \, ds \leq ch^{1/2} \|\psi\|_{L^2(\hat{a}d\tau ds)}.$$

Thus, we obtain:

$$\begin{aligned} \|[\Pi_s, \partial_s] \psi\|_{L^2(\hat{a}d\tau ds)}^2 &= \|[\Pi_s^\perp, \partial_s] \psi\|_{L^2(\hat{a}d\tau ds)}^2 \\ &\leq ch \|\psi\|_{L^2(\hat{a}d\tau ds)}^2 \\ &\leq ch \|\varphi\|^2 + ch \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2. \end{aligned}$$

Therefore,

$$q_2^{trans}(\psi) \leq ch^2 \|\varphi\|^2 + ch^2 \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2. \quad (3.47)$$

We use the inequality (3.47), (3.45), (3.46) and as  $q_3^{trans} \geq 0$ , we get:

$$q^{trans}(\psi) \geq h \left[ 1 - ch^{\min(\alpha, \frac{1}{2})} \right] \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 \, ds + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2 - \mathcal{O}(b^2 h^{2-\alpha}) \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2.$$

For  $0 < \alpha < 2 - \eta$ , we have  $-\eta + 2 - \alpha > 0$  and  $h^{-\eta+2-\alpha} = o(1)$ . Then, we have the conclusion of the proposition.  $\square$

### 3.4.5.3 Proof of Proposition 3.4.8

From the Propositions 3.4.9 and 3.4.10, we have:

$$q^{trans}(\psi) \geq h[1 - c h^{\min(\alpha, \frac{1}{2})}] \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|^2 - o(1) \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2,$$

and

$$q^{tang}(\psi) \geq \int_{-L}^L \left( -1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h - ch^{3/2} \right) |\varphi(s)|^2 ds - ch^{1/2-\rho} \|\Pi_s^\perp \psi\|_{L^2(\hat{a}d\tau ds)}^2.$$

For  $\rho \in (0, 1/4)$ , we have  $h^{1/2-\rho} = o(1)$ , therefore we obtain the proof of Proposition 3.4.8.

### 3.4.6 Proof of the main theorem

We now have everything we need to complete the proof of the theorem 3.4.1. The self-adjoint operator associated to the quadratic form  $q_{h,\gamma_0}^{\text{eff},\pm}$  is :

$$\mathcal{L}_{h,\gamma_0}^{\text{eff},\pm} = -h \left( 1 \pm c_\pm h^{\min(\alpha, \frac{1}{2})} \right) (\partial_s - ib\gamma_0)^2 - 1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h \pm c_\pm h^{3/2}$$

acting on  $L^2(\mathbb{R}/(|\partial\Omega|\mathbb{Z})) = L^2(\mathbb{R}/2L\mathbb{Z})$ , where  $c_\pm$  is a constant independent of  $h$ . The operator  $\mathcal{L}_{h,\gamma_0}^{\text{eff},\pm}$  has a compact resolvent and it is bounded from below. Its spectrum is purely discrete, and it is formed by a non-decreasing sequence of eigenvalues  $\lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},\pm})$ .

**Corollary 3.4.11.** *Let  $n \in \mathbb{N}^*$ ,  $\rho \in (0, 1/4)$ ,  $0 \leq \eta < \frac{3}{2}$ . There exists  $h_0 > 0$ , such that, if  $h \in (0, h_0)$  and  $b$  satisfies*

$$c_1 h^{-\frac{\eta}{2}} \leq b \leq c_2 h^{-\frac{\eta}{2}},$$

we have:

$$\hat{\mu}_n(h, b, \rho) \leq \lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},+}) + \mathcal{O}(h^{2-\alpha-\eta}).$$

*Proof.* Let  $(\varphi_j)_{1 \leq j \leq n}$  be an orthonormal family of eigenfunctions associated with eigenvalues  $(\lambda_j(\mathcal{L}_{h,\beta_0}^{\text{eff},+}))_{1 \leq j \leq n}$ .

Let  $E$  be the subspace of  $\text{Dom}(q_{h,\gamma_0}^{\text{eff},+})$  generated by the family  $(\varphi_j)_{1 \leq j \leq n}$ . By the Proposition 3.4.7, we have:

$$\hat{q}_h^{b,\rho}(v_{k(s),h} \varphi) \leq q_{h,\beta_0}^{\text{eff},+}(\varphi) + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|^2, \quad \forall \varphi \in E.$$

By applying the min-max theorem, we deduce

$$q_{h,\gamma_0}^{\text{eff},+}(\varphi) \leq \lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},+}) \|\varphi\|^2.$$

Consequently,

$$\begin{aligned} \hat{q}_h^{b,\rho}(v_{k(s),h} \varphi) &\leq \lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},+}) \|\varphi\|^2 + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|^2 \\ &\leq \left( \lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},+}) + \mathcal{O}(h^{2-\alpha-\eta}) \right) \|v_{k(s),h} \varphi\|^2. \end{aligned}$$

For all  $g \in v_{k(s),h}E$ , we have :

$$\widehat{q}_h^{b,\rho}(g) \leq \left( \lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},+}) + \mathcal{O}(h^{2-\alpha-\eta}) \right) \|g\|^2.$$

From the normalization of  $v_{k(s),h}$ , we can see that  $\dim(v_{k(s),h}E) = n$ . Thus, by applying the min-max principle, we obtain

$$\widehat{\mu}_n(h, b, \rho) \leq \lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},+}) + \mathcal{O}(h^{2-\alpha-\eta}).$$

□

**Corollary 3.4.12.** *Let  $n \in \mathbb{N}^*$ ,  $\rho \in (0, 1/4)$ ,  $0 \leq \eta < \frac{3}{2}$ . There exists  $h_0 > 0$ , such that, if  $h \in (0, h_0)$  and  $b$  satisfies*

$$c_1 h^{-\frac{\eta}{2}} \leq b \leq c_2 h^{-\frac{\eta}{2}},$$

we have:

$$\widehat{\mu}_n(h, b, \rho) \geq \lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},-}) + \mathcal{O}(h^{2-\alpha-\eta}).$$

*Proof.* Using the Proposition 3.4.8, there exist constants  $c > 0$ ,  $C_h$  and  $\epsilon_0 \in (0, 1)$  such that

$$\widehat{q}_h^{b,\rho}(\psi) \geq Q_{h,\gamma_0}^{\text{tens}}(\langle \psi, v_{\kappa(s),h} \rangle, \Pi^\perp \psi), \quad (3.48)$$

where, for all  $(\varphi, f) \in \text{Dom}(q_{h,\gamma_0}^{\text{eff},-}) \times \text{Dom}(\widehat{q}_h^{b,\rho})$

$$\begin{aligned} Q_{h,\gamma_0}^{\text{tens}}(\varphi, f) &= q_{h,\gamma_0}^{\text{eff},-}(\varphi) + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2 - \frac{\epsilon_0}{2} \|f\|_{L^2(\hat{a}d\tau ds)}^2 \\ &= Q_{h,\gamma_0}^{\text{eff}}(\varphi) - \frac{\epsilon_0}{2} \|f\|_{L^2(\hat{a}d\tau ds)}^2, \end{aligned}$$

with

$$Q_{h,\gamma_0}^{\text{eff}}(\varphi) = q_{h,\gamma_0}^{\text{eff},-}(\varphi) + \mathcal{O}(h^{2-\alpha-\eta}) \|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2.$$

According to the theorem of min-max and (3.48), we have :

$$\widehat{\mu}_n(h, b, \rho) \geq \inf_{\substack{G \subset \text{Dom}(\widehat{q}_h^{b,\rho}) \\ \dim G = n}} \sup_{\substack{\psi \in G \\ \psi \neq 0}} \frac{Q_{h,\gamma_0}^{\text{tens}}(\langle \psi, v_{\kappa(s),h} \rangle, \Pi^\perp \psi)}{\|\Pi\psi\|^2 + \|\Pi^\perp \psi\|^2}. \quad (3.49)$$

We define the linear injection:

$$\begin{aligned} \mathcal{J} : \text{Dom}(\widehat{q}_h^{b,\rho}) &\longrightarrow \text{Dom}(q_{h,\gamma_0}^{\text{eff},-}) \times \text{Dom}(\widehat{q}_h^{b,\rho}) \\ \psi &\longmapsto (\langle \psi, v_{\kappa(s),h} \rangle, \Pi^\perp \psi) \end{aligned}$$

We have:

$$\inf_{\substack{G \subset \text{Dom}(\widehat{q}_h^{b,\rho}) \\ \dim G = n}} \sup_{\substack{\psi \in G \\ \psi \neq 0}} \frac{Q_{h,\gamma_0}^{\text{tens}}(\langle \psi, v_{\kappa(s),h} \rangle, \Pi^\perp \psi)}{\|\Pi\psi\|^2 + \|\Pi^\perp \psi\|^2} = \inf_{\substack{\widehat{G} \subset \mathcal{J}(\text{Dom}(\widehat{q}_h^{b,\rho})) \\ \dim \widehat{G} = n}} \sup_{(\varphi, f) \in \widehat{G}} \frac{Q_{h,\gamma_0}^{\text{tens}}(\varphi, f)}{\|\varphi\|^2 + \|f\|^2},$$

and

$$\inf_{\substack{\widehat{G} \subset \mathcal{J}(\text{Dom}(\widehat{q}_h^{b,\rho})) \\ \dim \widehat{G} = n}} \sup_{(\varphi, f) \in \widehat{G}} \frac{Q_{h,\gamma_0}^{\text{tens}}(\varphi, f)}{\|\varphi\|^2 + \|f\|^2} \geq \inf_{\substack{\widehat{G} \subset \text{Dom}(q_{h,\gamma_0}^{\text{eff},-}) \\ \times \text{Dom}(\widehat{q}_h^{b,\rho}) \\ \dim \widehat{G} = n}} \sup_{(\varphi, f) \in \widehat{G}} \frac{Q_{h,\gamma_0}^{\text{tens}}(\varphi, f)}{\|\varphi\|^2 + \|f\|^2}.$$

Therefore,

$$\inf_{\substack{G \subset \text{Dom}(\widehat{q}_h^{b,\rho}) \\ \dim G = n}} \sup_{\substack{\psi \in G \\ \psi \neq 0}} \frac{Q_{h,\gamma_0}^{\text{tens}}(\langle \psi, v_{\kappa(s),h} \rangle, \Pi^\perp \psi)}{\|\Pi \psi\|^2 + \|\Pi^\perp \psi\|^2} \geq \inf_{\substack{\widehat{G} \subset \text{Dom}(q_{h,\gamma_0}^{\text{eff},-}) \\ \times \text{Dom}(\widehat{q}_h^{b,\rho}) \\ \dim \widehat{G} = n}} \sup_{(\varphi, f) \in \widehat{G}} \frac{Q_{h,\gamma_0}^{\text{tens}}(\varphi, f)}{\|\varphi\|^2 + \|f\|^2}. \quad (3.50)$$

By the inequalities (3.49) and (3.50), we have the comparison of the Rayleigh quotients:

$$\widehat{\mu}_n(h, b, \rho) \geq \widehat{\mu}_n^{\text{tens}}(h). \quad (3.51)$$

We note that the self-adjoint operator associated with the quadratic form  $Q_{h,\gamma_0}^{\text{eff}}$  is  $\mathcal{L}_{h,\gamma_0}^{\text{eff}}$ , and the self-adjoint operator associated with the quadratic form  $Q_{h,\gamma_0}^{\text{tens}}$  is  $\mathcal{L}_{h,\gamma_0}^{\text{tens}}$ .

It is easy to see that the spectrum of  $\mathcal{L}_{h,\gamma_0}^{\text{tens}}$  which lies below  $-\epsilon_0$  is coincident with the spectrum of  $\mathcal{L}_{h,\gamma_0}^{\text{eff}}$ . Then, for all  $n \in \mathbb{N}^*$ ,  $\widehat{\mu}_n^{\text{tens}}(h)$  is the  $n$ -th eigenvalue of  $\mathcal{L}_{h,\gamma_0}^{\text{tens}}$  and  $\widehat{\mu}_n^{\text{eff}}(h)$  is the  $n$ -th eigenvalue of  $\mathcal{L}_{h,\gamma_0}^{\text{eff}}$  have satisfied

$$\widehat{\mu}_n^{\text{tens}}(h) = \widehat{\mu}_n^{\text{eff}}(h) = \lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff},-}) + \mathcal{O}(h^{2-\alpha-n}). \quad (3.52)$$

Therefore, (3.51) and (3.52) give the result of the corollary.  $\square$

### 3.5 Spectrum of the effective operator

Let  $\lambda_n(\mathcal{L})$  be the  $n$ -th eigenvalue of an operator  $\mathcal{L}$ . Fix a constant  $c \in \mathbb{R}$ . The effective operators  $\mathcal{L}_{h,\gamma_0}^{\text{eff},\pm}$  are special cases of the following operator:

$$\mathcal{L}_{h,\alpha,0}^{\text{eff},b} = -h \left( 1 - c h^{\min(\alpha, \frac{1}{2})} \right) (\partial_s - ib\gamma_0)^2 - 1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h - ch^{\frac{3}{2}},$$

with  $b > 0$  and  $\alpha > 0$ . The associate quadratic form on  $L^2(\mathbb{R}/2L\mathbb{Z})$  is

$$q_{h,\alpha,0}^{\text{eff},b}(\varphi) = \int_{-L}^L \left( -1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h - ch^{\frac{3}{2}} \right) |\varphi(s)|^2 ds + h \left( 1 - c h^{\min(\alpha, \frac{1}{2})} \right) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds.$$

In the following, we will explain how the spectrum of the operator  $\mathcal{L}_{h,\alpha,0}^{\text{eff},b}$  reduces to the study of that stated in Theorem 3.3.1. The operator that we are going to study the spectrum is a perturbation of the harmonic oscillator. Then, the calculations are based on the context of the harmonic approximation (cf. [Ray17, Sections 3.2 and 4.1.2] and [FH10b, Section 7.1]), with a small adjustment, since the operator is defined on a periodic space and with a phase term  $ib\gamma_0$ .

**Lemma 3.5.1.** *Let the effective operator be of the form:*

$$\mathcal{L}_{h,\alpha,1}^{\text{eff},b} = -h \left( 1 - c h^{\min(\alpha, \frac{1}{2})} \right) (\partial_s - ib\gamma_0)^2 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h - ch^{\frac{3}{2}},$$

and the quadratic form on  $L^2(\mathbb{R}/2L\mathbb{Z})$  associated be:

$$q_{h,\alpha,1}^{\text{eff},b}(\varphi) = \int_{-L}^L \left( -\kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h - ch^{\frac{3}{2}} \right) |\varphi(s)|^2 ds + h \left( 1 - c h^{\min(\alpha, \frac{1}{2})} \right) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds.$$

Then,

$$\lambda_n(\mathcal{L}_{h,\alpha,0}^{\text{eff},b}) = -1 + \lambda_n(\mathcal{L}_{h,\alpha,1}^{\text{eff},b}). \quad (3.53)$$

*Proof.* Let  $\varphi \in H^1(\mathbb{R}/2L\mathbb{Z})$ , we have

$$q_{h,\alpha,0}^{\text{eff},b}(\varphi) - q_{h,\alpha,1}^{\text{eff},b}(\varphi) = -\|\varphi\|^2,$$

then

$$\frac{q_{h,\alpha,0}^{\text{eff},b}(\varphi)}{\|\varphi\|^2} = -1 + \frac{q_{h,\alpha,1}^{\text{eff},b}(\varphi)}{\|\varphi\|^2},$$

By definition of the spectrum, we have

$$\lambda_n(\mathcal{L}_{h,\alpha,0}^{\text{eff},b}) = -1 + \lambda_n(\mathcal{L}_{h,\alpha,1}^{\text{eff},b}).$$

□

Let the effective operator be of the form :

$$\mathcal{L}_{h,\alpha,2}^{\text{eff},b} = -h^{1/2} \left(1 - c h^{\min(\alpha, \frac{1}{2})}\right) (\partial_s - ib\gamma_0)^2 - \kappa(s) - \frac{\kappa(s)^2}{2} h^{1/2} - ch,$$

and the quadratic form on  $L^2(\mathbb{R}/2L\mathbb{Z})$  associated be:

$$q_{h,\alpha,2}^{\text{eff},b}(\varphi) = \int_{-L}^L \left(-\kappa(s) - \frac{\kappa(s)^2}{2} h^{1/2} - ch\right) |\varphi(s)|^2 ds + h^{1/2} \left(1 - c h^{\min(\alpha, \frac{1}{2})}\right) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds.$$

Since  $\mathcal{L}_{h,\alpha,1}^{\text{eff},b} = h^{1/2} \mathcal{L}_{h,\alpha,2}^{\text{eff},b}$ , then the following equality quantifies the relation between the spectrum of these operators

$$\text{Sp}(\mathcal{L}_{h,\alpha,1}^{\text{eff},b}) = h^{1/2} \text{Sp}(\mathcal{L}_{h,\alpha,2}^{\text{eff},b}),$$

therefore

$$\lambda_n(\mathcal{L}_{h,\alpha,1}^{\text{eff},b}) = h^{1/2} \lambda_n(\mathcal{L}_{h,\alpha,2}^{\text{eff},b}). \quad (3.54)$$

We introduce the new semi-classical parameter

$$\hbar = h^{1/4}.$$

**Lemma 3.5.2.** *Let the effective operator be of the form:*

$$\mathcal{L}_{h,\alpha,3}^{\text{eff},b} = -\hbar^2 \left(1 - c \hbar^{\min(4\alpha, 2)}\right) (\partial_s - ib\gamma_0)^2 - \kappa(s),$$

and the associated quadratic form on  $L^2(\mathbb{R}/2L\mathbb{Z})$  be:

$$q_{h,\alpha,3}^{\text{eff},b}(\varphi) = -\int_{-L}^L \kappa(s) |\varphi(s)|^2 ds + \hbar^2 \left(1 - c \hbar^{\min(4\alpha, 2)}\right) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds.$$

Then,

$$\lambda_n(\mathcal{L}_{h,\alpha,2}^{\text{eff}}) = \lambda_n(\mathcal{L}_{h,\alpha,3}^{\text{eff}}) + \mathcal{O}(\hbar^2). \quad (3.55)$$

*Proof.* Let  $\varphi \in H^1(\mathbb{R}/2L\mathbb{Z})$ , we have

$$\left| q_{h,\alpha,2}^{\text{eff},b}(\varphi) - q_{h,\alpha,3}^{\text{eff},b}(\varphi) \right| = \hbar^2 \left| \int_{-L}^L \left[ \frac{\kappa(s)^2}{2} + c\hbar^2 \right] |\varphi(s)|^2 ds \right|,$$

we use that  $\Omega$  is a bounded open, then the curvature  $\kappa$  is also bounded, hence there exists  $c > 0$  independent



of  $s$ , such that

$$\frac{\kappa(s)^2}{2} \leq c,$$

therefore

$$\left| q_{h,\alpha,2}^{\text{eff},b}(\varphi) - q_{h,\alpha,3}^{\text{eff},b}(\varphi) \right| \leq c \hbar^2 \|\varphi\|^2.$$

Thus

$$\left| \frac{q_{h,\alpha,2}^{\text{eff},b}(\varphi)}{\|\varphi\|^2} - \frac{q_{h,\alpha,3}^{\text{eff},b}(\varphi)}{\|\varphi\|^2} \right| \leq c \hbar^2,$$

according to the min-max theorem, we have

$$\left| \lambda_n(\mathcal{L}_{h,\alpha,2}^{\text{eff},b}) - \lambda_n(\mathcal{L}_{h,\alpha,3}^{\text{eff},b}) \right| \leq c \hbar^2.$$

Consequently,

$$\lambda_n(\mathcal{L}_{h,\alpha,2}^{\text{eff},b}) = \lambda_n(\mathcal{L}_{h,\alpha,3}^{\text{eff},b}) + \mathcal{O}(\hbar^2).$$

□

From (3.53), (3.54) and (3.55) we have

$$\lambda_n(\mathcal{L}_{h,\alpha,0}^{\text{eff},b}) = -1 + \hbar^2 \lambda_n(\mathcal{L}_{h,\alpha,3}^{\text{eff},b}) + \mathcal{O}(\hbar^4). \quad (3.56)$$

Thus, to study the spectrum of the effective operator  $\mathcal{L}_{h,\alpha,0}^{\text{eff},b}$ , it suffices to study the spectrum of the operator  $\mathcal{L}_{h,\alpha,3}^{\text{eff},b}$ , introduced in Lemma 3.5.2.

### 3.5.1 Location near the point of maximum curvature

Let  $\kappa_{\max}$  be the maximum curvature along the boundary  $\Gamma$ . Let the effective operator be of the form:

$$\mathcal{L}_{h,\alpha,4}^{\text{eff},b} = -\hbar^2 \left( 1 - c \hbar^{\min(4\alpha,2)} \right) (\partial_s - ib\gamma_0)^2 + \kappa_{\max} - \kappa(s),$$

and the associated quadratic form on  $L^2(\mathbb{R}/2L\mathbb{Z})$  be

$$q_{h,\alpha,4}^{\text{eff},b}(\varphi) = \int_{-L}^L (\kappa_{\max} - \kappa(s)) |\varphi(s)|^2 ds + \hbar^2 \left( 1 - c \hbar^{\min(4\alpha,2)} \right) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds.$$

Then,

$$\lambda_n(\mathcal{L}_{h,3}^{\text{eff},b}) = \lambda_n(\mathcal{L}_{h,4}^{\text{eff},b}) - \kappa_{\max}. \quad (3.57)$$

**Proposition 3.5.3.** *Consider the effective operator of the form*

$$\mathcal{L}_{h,4}^{\text{eff},b} = -\hbar^2 (\partial_s - ib\gamma_0)^2 + \kappa_{\max} - \kappa(s),$$

and the associated quadratic form on  $L^2(\mathbb{R}/2L\mathbb{Z})$  be

$$q_{h,4}^{\text{eff},b}(\varphi) = \int_{-L}^L (\kappa_{\max} - \kappa(s)) |\varphi(s)|^2 ds + \hbar^2 \int_{-L}^L |(\partial_s - ib\beta_0)\varphi|^2 ds.$$

Then, we have

$$\lambda_n(\mathcal{L}_{h,\alpha,4}^{\text{eff},b}) = \left( 1 + \mathcal{O}(\hbar^{\min(4\alpha,2)}) \right) \lambda_n(\mathcal{L}_{h,4}^{\text{eff},b}).$$

*Proof.* Let  $\varphi \in H^1(\mathbb{R}/2L\mathbb{Z})$ . We have:

$$\begin{aligned} |q_{h,\alpha,4}^{\text{eff},b}(\varphi) - q_{h,4}^{\text{eff},b}(\varphi)| &= |c|\hbar^{\min(4\alpha,2)}\hbar^2 \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds \\ &\leq |c|\hbar^{\min(4\alpha,2)} \left[ \hbar^2 \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds + \int_{-L}^L (\kappa_{\max} - \kappa(s))|\varphi(s)|^2 ds \right] \\ &\leq |c|\hbar^{\min(4\alpha,2)} q_{h,4}^{\text{eff},b}(\varphi). \end{aligned}$$

The conclusion of the lemma is now a simple application of the min-max principle.  $\square$

From (3.56), (3.57) and the Proposition 3.5.3, we obtain

$$\lambda_n(\mathcal{L}_{h,\alpha,0}^{\text{eff},b}) = -1 - \hbar^2 \kappa_{\max} + \hbar^2 \left(1 + \mathcal{O}(\hbar^{\min(4\alpha,2)})\right) \lambda_n(\mathcal{L}_{h,4}^{\text{eff},b}) + \mathcal{O}(\hbar^4).$$

In the following, we show that  $\lambda_n(\mathcal{L}_{h,4}^{\text{eff},b}) = \mathcal{O}(\hbar)$ , then we get (recall that  $\hbar = h^{1/4}$ )

$$\lambda_n(\mathcal{L}_{h,\alpha,0}^{\text{eff},b}) = -1 - \hbar^2 \kappa_{\max} + \hbar^2 \lambda_n(\mathcal{L}_{h,4}^{\text{eff},b}) + o(\hbar^3) = \lambda_n(\mathcal{L}_{h,\gamma_0}^{\text{eff}}) + \tilde{\mathcal{O}}(h^{2-\eta}). \quad (3.58)$$

Note that the Theorem 3.4.1 and equality (3.58), give the proof of the Theorem 3.3.1 which is a reformulation of the Theorem 3.1.1.

The goal is to show that the  $\mathcal{L}_{h,4}^{\text{eff},b}$  eigenfunctions associated with eigenvalues of order  $\hbar$  concentrate exponentially near the point of maximum curvature .

**Proposition 3.5.4.** *For  $\epsilon \in (0, 1)$ , there exist  $C > 0$  and  $\hbar_0 > 0$  such that, for all  $\hbar \in (0, \hbar_0)$  and for all eigenfunctions  $\psi$  of  $\mathcal{L}_{h,4}^{\text{eff},b}$ , associated with the eigenvalues of orders  $\hbar$ , we have:*

$$\|e^{\epsilon\phi_0/\hbar} \psi\|^2 \leq C \|\psi\|^2, \quad q_{h,4}^{\text{eff},b}(e^{\epsilon\phi_0/\hbar} \psi) \leq C\hbar \|\psi\|^2,$$

where

$$\phi_0(s) = \min_{k \in \mathbb{Z}} \left| \int_0^{s+2kL} \sqrt{\kappa_{\max} - \kappa(y)} dy \right|.$$

*Proof.* The proof of Proposition 3.5.4 is a standard application of the method of Agmon estimates (see for example [Ray17, Prop. 4.10]). The proof is formed by three steps.

Let  $\epsilon \in (0, 1)$ . Consider an eigenvalue  $\lambda = \mathcal{O}(\hbar)$  of  $\mathcal{L}_{h,4}^{\text{eff},b}$ , and an associated eigenfunction  $\psi$ .

### 1. First step.

Let  $\phi$  be a regular and  $2L$  periodic function, with real value. We have:

$$q_{h,4}^{\text{eff},b}(e^\phi \psi) = \lambda \|e^\phi \psi\|^2 + \hbar^2 \|\partial_s e^\phi \psi\|^2. \quad (3.59)$$

Indeed,

$$\begin{aligned} \mathcal{L}_{h,4}^{\text{eff},b}(e^\phi \psi) &= -\hbar^2 (\partial_s - ib\gamma_0)^2 (e^\phi \psi) + (\kappa_{\max} - \kappa(s)) e^\phi \psi \\ &= -\hbar^2 \left( \psi \partial_s^2 e^\phi + e^\phi \partial_s^2 \psi + 2\partial_s e^\phi \partial_s \psi - 2ib\gamma_0 (\psi \partial_s e^\phi + e^\phi \partial_s \psi) - b^2 \gamma_0^2 e^\phi \psi \right) \\ &\quad + (\kappa_{\max} - \kappa(s)) e^\phi \psi \\ &= e^\phi \mathcal{L}_{h,4}^{\text{eff},b}(\psi) - \hbar^2 \left( \psi \partial_s^2 e^\phi + 2\partial_s e^\phi \partial_s \psi - 2ib\gamma_0 \psi \partial_s e^\phi \right). \end{aligned}$$

Hence,

$$\begin{aligned} q_{h,4}^{\text{eff},b}(e^\phi \psi) &= \langle \mathcal{L}_{h,4}^{\text{eff},b}(e^\phi \psi), e^\phi \psi \rangle \\ &= \langle e^\phi \mathcal{L}_{h,4}^{\text{eff},b}(\psi), e^\phi \psi \rangle - \hbar^2 \int_{-L}^L \left[ \psi \partial_s^2 e^\phi + 2\partial_s e^\phi \partial_s \psi - 2ib\gamma_0 \psi \partial_s e^\phi \right] \bar{\psi} e^\phi ds. \end{aligned}$$

Therefore, as  $\psi$  is the eigenfunction of  $\mathcal{L}_{h,4}^{\text{eff},b}$  associated with the eigenvalue  $\lambda$ , we get

$$q_{h,4}^{\text{eff},b}(e^\phi \psi) = \lambda \|e^\phi \psi\|^2 - \hbar^2 \int_{-L}^L \left[ \psi \partial_s^2 e^\phi + 2\partial_s e^\phi \partial_s \psi - 2ib\gamma_0 \psi \partial_s e^\phi \right] \bar{\psi} e^\phi ds. \quad (3.60)$$

Let's calculate  $\int_{-L}^L \left[ \psi \partial_s^2 e^\phi + 2\partial_s e^\phi \partial_s \psi - 2ib\gamma_0 \psi \partial_s e^\phi \right] \bar{\psi} e^\phi ds := E$ .

By integration by part, we have:

$$\begin{aligned} \int_{-L}^L \psi \partial_s^2 e^\phi \bar{\psi} e^\phi ds &= \int_{-L}^L e^\phi |\psi|^2 \partial_s^2 e^\phi ds \\ &= - \int_{-L}^L \partial_s (e^\phi |\psi|^2) \partial_s e^\phi ds + e^\phi |\psi|^2 \partial_s e^\phi \Big|_{-L}^L \\ &= - \int_{-L}^L \partial_s e^\phi \left[ |\psi|^2 \partial_s e^\phi + e^\phi \partial_s |\psi|^2 \right] ds + e^\phi |\psi|^2 \partial_s e^\phi \Big|_{-L}^L \\ &= - \|\partial_s e^\phi \psi\|^2 - \int_{-L}^L e^\phi \partial_s e^\phi \partial_s |\psi|^2 ds + e^\phi |\psi|^2 \partial_s e^\phi \Big|_{-L}^L. \end{aligned}$$

Since  $\phi$  is  $2L$  periodic, then

$$\begin{aligned} &\int_{-L}^L \left[ \psi \partial_s^2 e^\phi + 2\partial_s e^\phi \partial_s \psi - 2ib\gamma_0 \psi \partial_s e^\phi \right] \bar{\psi} e^\phi ds \\ &= - \|\partial_s e^\phi \psi\|^2 - \int_{-L}^L e^\phi \partial_s e^\phi \partial_s |\psi|^2 ds + 2 \int_{-L}^L \partial_s e^\phi \partial_s \psi \bar{\psi} e^\phi ds - 2ib\gamma_0 \int_{-L}^L \psi \partial_s e^\phi \bar{\psi} e^\phi ds \\ &= - \|\partial_s e^\phi \psi\|^2 + \int_{-L}^L e^\phi \partial_s e^\phi \left[ -\partial_s |\psi|^2 + 2\partial_s \psi \bar{\psi} - 2ib\gamma_0 |\psi|^2 \right] ds. \end{aligned}$$

Therefore,

$$E = - \|\partial_s e^\phi \psi\|^2 + \int_{-L}^L e^\phi \partial_s e^\phi \left[ -\partial_s |\psi|^2 + 2\partial_s \psi \bar{\psi} - 2ib\gamma_0 |\psi|^2 \right] ds. \quad (3.61)$$

From (3.60), we have:

$$\hbar^2 E = -q_{h,4}^{\text{eff},b}(e^\phi \psi) + \lambda \|e^\phi \psi\|^2 \in \mathbb{R}.$$

Consequently, (3.61) gives

$$E = - \|\partial_s e^\phi \psi\|^2. \quad (3.62)$$

From (3.60) and (3.62), we obtain:

$$q_{h,4}^{\text{eff},b}(e^\phi \psi) = \lambda \|e^\phi \psi\|^2 + \hbar^2 \|\partial_s e^\phi \psi\|^2.$$

## 2. Second step.

Let us show that:

$$q_{h,4}^{\text{eff},b}(e^{\epsilon\phi_0/h} \psi) \leq C\hbar \|e^{\epsilon\phi_0/h} \psi\|^2, \quad (3.63)$$

and

$$\int_{-L}^L (\kappa_{\max} - \kappa(s)) e^{2\epsilon\phi_0/h} |\psi|^2 ds \leq C\hbar \|e^{\epsilon\phi_0/h} \psi\|^2. \quad (3.64)$$

For  $\phi = \frac{\epsilon\phi_0}{\hbar}$ , with  $\phi_0(s) = \min_{k \in \mathbb{Z}} \left| \int_0^{s+2kL} \sqrt{\kappa_{\max} - \kappa(y)} dy \right|$ . We notice that

$$|\phi'(s)|^2 = \frac{\epsilon^2}{\hbar^2} |\phi_0'(s)|^2 = \frac{\epsilon^2}{\hbar^2} (\kappa_{\max} - \kappa(s)). \quad (3.65)$$

We recall that

$$q_{h,4}^{\text{eff},b}(\varphi) = \int_{-L}^L (\kappa_{\max} - \kappa(s)) |\varphi(s)|^2 ds + \hbar^2 \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 ds,$$

consequently,

$$q_{h,4}^{\text{eff},b}(\varphi) \geq \int_{-L}^L (\kappa_{\max} - \kappa(s)) |\varphi(s)|^2 ds. \quad (3.66)$$

From (3.59) and (3.66), we have:

$$\begin{aligned} \int_{-L}^L (\kappa_{\max} - \kappa(s)) e^{2\phi} |\psi|^2 ds &\leq \lambda \|e^\phi \psi\|^2 + \hbar^2 \|\partial_s e^\phi \psi\|^2 \\ \Rightarrow \int_{-L}^L [(\kappa_{\max} - \kappa(s)) - \lambda - \hbar^2 |\phi'|^2] e^{2\phi} |\psi|^2 ds &\leq 0. \end{aligned}$$

By (3.65), we get

$$\begin{aligned} \int_{-L}^L [\kappa_{\max} - \kappa(s) - \lambda - \epsilon^2 (\kappa_{\max} - \kappa(s))] e^{2\phi} |\psi|^2 ds &\leq 0 \\ \Rightarrow \int_{-L}^L [1 - \epsilon^2] (\kappa_{\max} - \kappa(s)) e^{2\phi} |\psi|^2 ds &\leq \lambda \|e^\phi \psi\|^2. \end{aligned}$$

Therefore,

$$\int_{-L}^L [1 - \epsilon^2] (\kappa_{\max} - \kappa(s)) e^{2\phi} |\psi|^2 ds \leq C\hbar \|e^\phi \psi\|^2. \quad (3.67)$$

We notice that for any  $\epsilon \in (0, 1)$ , there exists a constant  $C > 0$ , such that

$$1 - \epsilon^2 \geq C.$$

We deduce the inequality (3.64)

$$\int_{-L}^L (\kappa_{\max} - \kappa(s)) e^{2\phi} |\psi|^2 ds \leq C\hbar \|e^\phi \psi\|^2.$$

On the other hand, by (3.59) and (3.65), we have:

$$\begin{aligned} q_{h,4}^{\text{eff},b}(e^\phi \psi) &\leq \lambda \|e^\phi \psi\|^2 + \hbar^2 \|\partial_s e^\phi \psi\|^2 \\ &\leq \lambda \|e^\phi \psi\|^2 + \epsilon^2 \int_{-L}^L (\kappa_{\max} - \kappa(s)) e^{2\phi} |\psi|^2 ds. \end{aligned}$$

We use (3.64), to deduce the inequality (3.63).

### 3. Third step.

Show that:

$$\int_{-L}^L e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \leq C \|\psi\|^2, \quad (3.68)$$

and

$$q_{h,4}^{\text{eff},b}(e^{\epsilon\phi_0/\hbar} \psi) \leq C\hbar \|\psi\|^2. \quad (3.69)$$

We use that  $\kappa_{\max} - \kappa(s)$  admits a unique non-degenerate minimum, we deduce that there exists  $c > 0$  such that for all  $C_0 > 0$ , there exist constants  $C, \hbar_0$  such that  $\forall \hbar \in (0, \hbar_0)$ ,

$$\int_{|s| \leq C_0 \hbar^{1/2}} (\kappa_{\max} - \kappa(s)) e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \leq C\hbar \|\psi\|^2, \quad (3.70)$$

$$\int_{|s| \leq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \leq C \|\psi\|^2, \quad (3.71)$$

and

$$\int_{|s| \geq C_0 \hbar^{1/2}} (\kappa_{\max} - \kappa(s)) e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \geq c C_0^2 \hbar \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds. \quad (3.72)$$

By the inequality (3.71), to show (3.68), it is enough to show the following inequality:

$$\int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \leq C \|\psi\|^2. \quad (3.73)$$

Let's fix  $C_0 > 0$  large enough,

$$\begin{aligned} \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds &\leq \frac{1}{c C_0^2 \hbar} \int_{|s| \geq C_0 \hbar^{1/2}} (\kappa_{\max} - \kappa(s)) e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \\ &\leq \frac{1}{c C_0^2 \hbar} \int_{-L}^L (\kappa_{\max} - \kappa(s)) e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds. \end{aligned}$$

From the inequality (3.64), we get

$$\begin{aligned} \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds &\leq \frac{C}{c C_0^2} \int_{-L}^L e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \\ &\leq \frac{C}{c C_0^2} \int_{|s| \leq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds + \frac{C}{c C_0^2} \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds. \end{aligned}$$

Thus,

$$\left(1 - \frac{C}{c C_0^2}\right) \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \leq \frac{C}{c C_0^2} \int_{|s| \leq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds.$$

From the inequality (3.71), we obtain (3.73) and therefore we show the point (3.68).

From (3.63), we get

$$q_{h,4}^{\text{eff},b}(e^{\epsilon\phi_0/\hbar} \psi) \leq C\hbar \|e^{\epsilon\phi_0/\hbar} \psi\|^2,$$

and from the inequality (3.68), we obtain the inequality (3.69).  $\square$

### 3.5.2 Reduction to a flux-free operator

Consider the Dirichlet effective operator of the form:

$$\mathcal{L}_h^{\text{eff}} = -\hbar^2 \partial_s^2 + \kappa_{\max} - \kappa(s),$$

with domain

$$\text{Dom}(\mathcal{L}_h^{\text{eff}}) = \{ u \in H^2([-L, L]), u(-L) = u(L) = 0 \},$$

and the associated quadratic form on  $L^2([-L, L])$  is

$$q_h^{\text{eff}}(\varphi) = \int_{-L}^L (\kappa_{\max} - \kappa(s)) |\varphi(s)|^2 ds + \hbar^2 \int_{-L}^L |\partial_s \varphi|^2 ds.$$

Let  $(\lambda_n(\mathcal{L}_h^{\text{eff}}))_{n \in \mathbb{N}^*}$  be the sequence of eigenvalues of the operator  $\mathcal{L}_h^{\text{eff}}$ . As a consequence of Proposition 3.5.4, for small  $\hbar$ , the eigenfunctions of  $\mathcal{L}_h^{\text{eff}}$ , associated with the eigenvalues of orders  $\hbar$  concentrate exponentially near the point of maximum curvature at the scale  $\hbar^{1/2}$ . A standard application of the min-max principle and the Agmon estimates of Proposition 3.5.4, is the reduction to analysis to the flux-free operator  $\mathcal{L}_h^{\text{eff}}$  (Proposition 3.5.6).

**Notation 3.5.5.** The notation  $\mathcal{O}(\hbar^\infty)$  indicates a quantity satisfying that, for all  $N \in \mathbb{N}$ , there exists  $C_N > 0$  and  $\hbar_N > 0$  such that, for all  $\hbar \in (0, \hbar_N)$ ,  $|\mathcal{O}(\hbar^\infty)| \leq C_N \hbar^N$ .

**Proposition 3.5.6.** Let  $n \in \mathbb{N}^*$  there exist constant  $\hbar_0 \in (0, 1)$  such that, for all  $\hbar \in (0, \hbar_0)$  and  $\lambda_n(\mathcal{L}_{h,4}^{\text{eff},b}) = \mathcal{O}(\hbar)$ ,

$$\lambda_n(\mathcal{L}_h^{\text{eff}}) \leq \lambda_n(\mathcal{L}_{h,4}^{\text{eff},b}) + \mathcal{O}(\hbar^\infty).$$

Moreover, we have, for all  $n \geq 1$ ,  $\hbar > 0$

$$\lambda_n(\mathcal{L}_{h,4}^{\text{eff},b}) \leq \lambda_n(\mathcal{L}_h^{\text{eff}}).$$

*Proof.* For all  $N \geq 1$ , we can take an orthonormal family eigenfunctions  $(\psi_{n,\hbar})_{n=1,\dots,N}$  associated to the eigenvalues  $(\lambda_n(\mathcal{L}_{h,4}^{\text{eff},b}))_{n=1,\dots,N}$ . Let us consider the cut off function  $\chi$  define on  $\mathbb{R}$  as follows:

$$0 \leq \chi \leq 1, \quad \chi = 1 \text{ on } \left[ -\frac{L}{2}, \frac{L}{2} \right] \quad \text{and} \quad \chi = 0 \text{ on } \mathbb{R} \setminus ]-L, L[.$$

For  $k = 1, \dots, N$ , we define the function

$$\Phi_{k,\hbar}(s) = e^{-ib\gamma_0 s} \chi(s) \psi_{j,\hbar}(s).$$

We notice that

$$\Phi_{k,\hbar}(-L) = \Phi_{k,\hbar}(L) = 0 \quad \text{and} \quad \Phi_{k,\hbar} \in H^2([-L, L]).$$

Let  $\mathcal{E}_N(\hbar)$  be the vector subspace of  $\text{Dom}(\mathcal{L}_h^{\text{eff}})$  spanned by the family  $(\Phi_{n,\hbar})_{n=1,\dots,N}$ , and  $u_\hbar \in F_\hbar$  which is written as follows:

$$u_\hbar = \sum_{k=1}^N \beta_k \Phi_{k,\hbar}.$$

The functions  $(\Phi_{n,h})$  are almost orthonormal. We note that  $\dim(\mathcal{E}_N(\hbar)) = N$  and

$$\|u_h\|^2 = \sum_{k=1}^N |\beta_k|^2 + \mathcal{O}(\hbar^\infty).$$

Insertion of  $u_h$  in the quadratic form  $q_h^{\text{eff}}(u_h)$

$$q_h^{\text{eff}}(u_h) = \langle \mathcal{L}_h^{\text{eff}} u_h, u_h \rangle = \sum_{j,k=1}^N \beta_j \beta_k \langle \mathcal{L}_h^{\text{eff}} \Phi_{j,h}, \Phi_{k,h} \rangle.$$

For  $j$  fixed and for  $h$  small enough, we have

$$\begin{aligned} \mathcal{L}_h^{\text{eff}} \Phi_{j,h} &= e^{-ib\gamma_0 s} \left[ -\hbar^2 (\partial_s - ib\gamma_0)^2 \chi \psi_{j,h} + (\kappa_{\max} - \kappa(s)) \chi \psi_{j,h} \right] \\ &= e^{-ib\gamma_0 s} \left[ -\hbar^2 \chi(s) (\partial_s - ib\gamma_0)^2 \psi_{j,h} + (\kappa_{\max} - \kappa(s)) \chi \psi_{j,h} - 2\hbar^2 \partial_s \chi \partial_s \psi_{j,h} - \hbar^2 \psi_{j,h} \partial_s^2 \chi \right. \\ &\quad \left. + 2\hbar^2 ib\gamma_0 \psi_{j,h} \partial_s \chi \right] \\ &= e^{-ib\gamma_0 s} \left[ -\hbar^2 \chi (\partial_s - ib\gamma_0)^2 \psi_{j,h} + (\kappa_{\max} - \kappa(s)) \chi \psi_{j,h} - \hbar^2 (2\partial_s \chi [\partial_s - ib\gamma_0] \psi_{j,h} + \psi_{j,h} \partial_s^2 \chi) \right] \\ &= e^{-ib\gamma_0 s} \left[ \chi \mathcal{L}_{h,4}^{\text{eff},b} \psi_{j,h} - \hbar^2 (2\partial_s \chi [\partial_s - ib\gamma_0] \psi_{j,h} + \psi_{j,h} \partial_s^2 \chi) \right]. \end{aligned}$$

Therefore, for  $j, k$  fixed and for  $h$  small enough, we have

$$\begin{aligned} \langle \mathcal{L}_h^{\text{eff}} \Phi_{j,h}, \Phi_{k,h} \rangle &= \langle \chi \mathcal{L}_{h,4}^{\text{eff},b} \psi_{j,h}, \chi \psi_{k,h} \rangle - \hbar^2 \langle (2\partial_s \chi [\partial_s - ib\gamma_0] \psi_{j,h} + \psi_{j,h} \partial_s^2 \chi), \chi \psi_{k,h} \rangle \\ &= \delta_{j,k} \lambda_j(\mathcal{L}_{h,4}^{\text{eff},b}) - \hbar^2 \langle (2\partial_s \chi [\partial_s - ib\gamma_0] \psi_{j,h} + \psi_{j,h} \partial_s^2 \chi), \chi \psi_{k,h} \rangle + \mathcal{O}(\hbar^\infty). \end{aligned}$$

From the Hölder inequality and the Proposition 3.5.4, we obtain

$$\hbar^2 |\langle (2\partial_s \chi [\partial_s - ib\gamma_0] \psi_{j,h} + \psi_{j,h} \partial_s^2 \chi), \chi \psi_{k,h} \rangle| = \mathcal{O}(\hbar^\infty).$$

From the previous inequality, we obtain

$$\langle \mathcal{L}_h^{\text{eff}} \Phi_{j,h}, \Phi_{k,h} \rangle = \delta_{j,k} \lambda_j(\mathcal{L}_{h,4}^{\text{eff},b}) + \mathcal{O}(\hbar^\infty).$$

Consequently,

$$q_h^{\text{eff}}(u_h) = \sum_{j=1}^N \lambda_j(\mathcal{L}_{h,4}^{\text{eff},b}) |\beta_j|^2 + \mathcal{O}(\hbar^\infty) \sum_{j,k=1}^N \beta_j \beta_k \leq \left( \lambda_N(\mathcal{L}_{h,4}^{\text{eff},b}) + \mathcal{O}(\hbar^\infty) \right) \|u_h\|^2.$$

By the min-max theorem, we have

$$\lambda_N(\mathcal{L}_h^{\text{eff}}) \leq \lambda_N(\mathcal{L}_{h,4}^{\text{eff},b}) + \mathcal{O}(\hbar^\infty).$$

Now we determine an upper bound of  $\lambda_n(\mathcal{L}_h^{\text{eff}})$  in terms of  $\lambda_n(\mathcal{L}_{h,4}^{\text{eff},b})$ . For all  $N \geq 1$ , we can take an orthonormal family eigenfunctions  $(u_{n,h})_{n=1,\dots,N}$  associated to the eigenvalues  $(\lambda_n(\mathcal{L}_h^{\text{eff}}))_{n=1,\dots,N}$ . For  $k = 1, \dots, N$ , we define the function

$$\Psi_{k,h}(s) = e^{ib\gamma_0 s} u_{k,h}(s).$$

We notice that

$$\Psi_{k,h}(-L) = \Psi_{k,h}(L) = 0 \quad \text{and} \quad \Psi_{k,h} \in H^2([-L, L]).$$

Let  $\mathcal{E}_N(\hbar)$  the  $N$ -dimensional vector subspace of  $\text{Dom}(\mathcal{L}_{\hbar,4}^{\text{eff},b})$  spanned by the family  $(\Psi_{n,\hbar})_{n=1,\dots,N}$ . Let  $\Psi \in \mathcal{E}_N(\hbar)$  which is written as follows:

$$\Psi = \sum_{k=1}^N \beta_k \Psi_{k,\hbar} = \sum_{k=1}^N \beta_k e^{ib\gamma_0 s} u_{k,\hbar} = e^{ib\gamma_0 s} u, \quad u = \sum_{k=1}^N \beta_k u_{k,\hbar}.$$

We have

$$\mathcal{L}_{\hbar,4}^{\text{eff},b} \Psi = e^{ib\gamma_0 s} \mathcal{L}_{\hbar}^{\text{eff}} u,$$

then

$$\langle \mathcal{L}_{\hbar,4}^{\text{eff},b} \Psi, \Psi \rangle = \langle \mathcal{L}_{\hbar}^{\text{eff}} u, u \rangle \leq \lambda_N(\mathcal{L}_{\hbar}^{\text{eff}}) \|u\|^2 = \lambda_N(\mathcal{L}_{\hbar}^{\text{eff}}) \|\Psi\|^2.$$

By the min-max theorem, we get the non-asymptotic inequality

$$\lambda_N(\mathcal{L}_{\hbar,4}^{\text{eff},b}) \leq \lambda_N(\mathcal{L}_{\hbar}^{\text{eff}}).$$

□

### 3.5.3 End of the proof

In this section we will construct trial states  $(\Phi_n)$  in the domain of the effective operator  $\mathcal{L}_{\hbar}^{\text{eff}}$ , such that, for every fixed  $n \in \mathbb{N}^*$ , we have

$$\left\| \mathcal{L}_{\hbar}^{\text{eff}} \Phi_n - (2n-1)\hbar \sqrt{\frac{-\kappa''(0)}{2}} \Phi_n \right\|_{L^2([-L, L])} \leq C\hbar^{3/2}. \quad (3.74)$$

We know that, the eigenfunctions are localized near the boundary at the scale  $\hbar^{\frac{1}{2}}$  (cf. Prop 3.5.4, the same proof can be adapted for  $\mathcal{L}_{\hbar}^{\text{eff}}$ ). It suggests to introduce the rescaling

$$s = \hbar^{1/2} \sigma. \quad (3.75)$$

The effective operator  $\mathcal{L}_{\hbar}^{\text{eff}}$  becomes

$$\mathcal{L}_{\hbar}^{\text{eff}} = -\hbar \partial_{\sigma}^2 + \kappa_{\max} - \kappa(\hbar^{\frac{1}{2}} \sigma).$$

Recall that the value of the maximum curvature is  $\kappa_{\max} = \kappa(0)$ . We can use a Taylor expansion of  $\kappa$  near 0, as follows

$$\kappa(\hbar^{\frac{1}{2}} \sigma) = \kappa_{\max} + \frac{\hbar \sigma^2}{2} \kappa''(0) + \hbar^{\frac{3}{2}} q(\sigma),$$

where the functions  $q$  satisfy for  $|\sigma| = \mathcal{O}(\hbar^{-\frac{1}{2}})$

$$|q(\sigma)| \leq C|\sigma|^3. \quad (3.76)$$

Consequently, we write the effective operator  $\mathcal{L}_{\hbar}^{\text{eff}}$  as

$$\hbar^{-1} \mathcal{L}_{\hbar}^{\text{eff}} = -\partial_{\sigma}^2 - \frac{\sigma^2}{2} \kappa''(0) - \hbar^{\frac{1}{2}} q(\sigma). \quad (3.77)$$



Since  $\kappa(0)$  is the non-degenerate maximum of  $\kappa$ , we have  $\kappa''(0) < 0$ . The eigenvalues of the harmonic oscillator  $H_{\text{harm}}$

$$-\partial_\sigma^2 + \frac{-\kappa''(0)}{2}\sigma^2 \text{ in } L^2(\mathbb{R}) \quad (3.78)$$

are  $(2n - 1)\sqrt{\frac{-\kappa''(0)}{2}}$  with  $n \in \mathbb{N}^*$ . The corresponding normalized eigenfunctions are denoted by  $f_n(\sigma)$ . They have the form

$$f_n(\sigma) = h_n(\sigma) \exp\left(-\sqrt{\frac{-\kappa''(0)}{2}}\frac{\sigma^2}{2}\right), \quad (3.79)$$

where the  $(h_n(\sigma))_{n \geq 1}$  are the rescaled Hermite polynomials.

Define the trial state  $\Phi_n$  in  $L^2(]-L, L[)$  as follows:

$$\Phi_n(s) = \hbar^{-1/4} \chi\left(\frac{s}{2L}\right) f_n(\hbar^{-1/2}s) = \hbar^{-1/4} u_n(s), \quad (3.80)$$

where  $\chi$  is a cut-off function defined on  $\mathbb{R}$  by:

$$0 \leq \chi \leq 1, \quad \chi = 1 \text{ in } ]-1/4, 1/4[ \text{ and } \chi = 0 \text{ in } \mathbb{R} \setminus ]-1/2, 1/2[,$$

and

$$u_n(s) = \chi\left(\frac{s}{2L}\right) f_n(\hbar^{-1/2}s).$$

Using the exponential decay of  $f_n$  at infinity, it is easy to verify

$$\|\Phi_n\|_{L^2(]-L, L[)} = 1 + \mathcal{O}(\hbar^\infty). \quad (3.81)$$

Then, we observe that after a change of variables and by construction of  $\Phi_n$ , (3.77) yields

$$\begin{aligned} \hbar^{-1} \mathcal{L}_\hbar^{\text{eff}} \Phi_n(s) &= -\hbar^{-1/4} \left[ \partial_\sigma^2 u_n(\sigma) + \frac{\sigma^2}{2} \kappa''(0) u_n(\sigma) + \hbar^{1/2} q(\sigma) u_n(\sigma) \right] \\ &= -\hbar^{-1/4} \left[ \chi\left(\frac{\hbar^{1/2}\sigma}{2L}\right) \partial_\sigma^2 f_n(\sigma) + 2\hbar^{1/2} \chi'\left(\frac{\hbar^{1/2}\sigma}{2L}\right) f_n'(\sigma) + \hbar \chi''\left(\frac{\hbar^{1/2}\sigma}{2L}\right) f_n(\sigma) \right. \\ &\quad \left. + \frac{\sigma^2}{2} \kappa''(0) u_n(\sigma) + \hbar^{1/2} q(\sigma) u_n(\sigma) \right] \\ &= -\hbar^{-1/4} \left[ -(2n-1) \sqrt{\frac{-\kappa''(0)}{2}} u_n(\sigma) + 2\hbar^{1/2} \chi'\left(\frac{\hbar^{1/2}\sigma}{2L}\right) f_n'(\sigma) + \hbar \chi''\left(\frac{\hbar^{1/2}\sigma}{2L}\right) f_n(\sigma) \right. \\ &\quad \left. + \hbar^{1/2} q(\sigma) u_n(\sigma) \right] \\ &= (2n-1) \sqrt{\frac{-\kappa''(0)}{2}} \Phi_n(s) - \hbar^{-1/4} \left[ \left( 2\hbar^{1/2} \chi'\left(\frac{\hbar^{1/2}\sigma}{2L}\right) f_n'(\sigma) + \hbar \chi''\left(\frac{\hbar^{1/2}\sigma}{2L}\right) f_n(\sigma) \right) \right. \\ &\quad \left. + \hbar^{1/2} q(\sigma) u_n(\sigma) \right]. \end{aligned} \quad (3.82)$$

Using the exponential decay of  $f_n$  at infinity, we infer from (3.82) that

$$\left\| \hbar^{-1} \mathcal{L}_\hbar^{\text{eff}} \Phi_n - (2n-1) \sqrt{\frac{-\kappa''(0)}{2}} \Phi_n \right\| \leq C \hbar^{1/2}.$$

As a consequence of the spectral theorem, (3.74) yields that, for every fixed  $n \in \mathbb{N}^*$ , there exists an eigenvalue  $\tilde{\lambda}_n(\mathcal{L}_\hbar^{\text{eff}})$  of the operator  $\mathcal{L}_\hbar^{\text{eff}}$  such that

$$\tilde{\lambda}_n(\mathcal{L}_\hbar^{\text{eff}}) = (2n-1) \hbar \sqrt{\frac{-\kappa''(0)}{2}} + \mathcal{O}(\hbar^{3/2}). \quad (3.83)$$

Now, looking for lower bound.

**Proposition 3.5.7.** *For all  $n \geq 1$ , there exist  $\hbar_0 > 0$  such that, for all  $\hbar \in (0, \hbar_0)$*

$$\lambda_n(\mathcal{L}_\hbar^{\text{eff}}) \geq (2n-1) \hbar \sqrt{\frac{-\kappa''(0)}{2}} + \mathcal{O}(\hbar^{3/2}).$$

*Proof.* For all  $N \geq 1$ , we can take a family of eigenvalues and eigenfunctions  $(\lambda_n(\mathcal{L}_\hbar^{\text{eff}}), \psi_{n,\hbar})_{n=1,\dots,N}$  such that  $(\psi_{n,\hbar})_{n=1,\dots,N}$  is an orthonormal family. Let  $\mathcal{E}_N(\hbar)$  the vector subspace of  $\text{Dom}(\mathcal{L}_\hbar^{\text{eff}})$  of dimension  $N$  spanned by the family  $(\psi_{n,\hbar})_{n=1,\dots,N}$ .

It is rather easy to observe that, for  $\psi \in \mathcal{E}_N(\hbar)$

$$\langle \mathcal{L}_\hbar^{\text{eff}} \psi, \psi \rangle \leq \lambda_N(\mathcal{L}_\hbar^{\text{eff}}) \|\psi\|^2.$$

Since  $\psi(-L) = \psi(L) = 0$ , the following function is in  $H^1(\mathbb{R})$

$$\tilde{\psi} = \begin{cases} \psi & \text{if } s \in ]-L, L[ \\ 0 & \text{if } s \in \mathbb{R} \setminus ]-L, L[. \end{cases}$$

For  $N \geq 1$ , let

$$\tilde{\mathcal{E}}_N(\hbar) = \text{Span}\{\tilde{\psi}_{n,\hbar}\}_{n=1,\dots,N}.$$

Since  $\dim(\mathcal{E}_N(\hbar)) = N$ , then  $\dim(\tilde{\mathcal{E}}_N(\hbar)) = N$ . We notice that, for all  $\psi \in \mathcal{E}_N(\hbar)$ ,  $\psi$  also satisfies the Agmon estimates (Proposition 3.5.4). Using the change of variable introduced in (3.75), we easily obtain that, there exists  $C > 0$  such that, for all  $\psi \in \mathcal{E}_N(\hbar)$

$$\int_{-\hbar^{-\frac{1}{2}}L}^{\hbar^{-\frac{1}{2}}L} |\sigma|^3 |\psi(\sigma)|^2 d(\hbar^{1/2} \sigma) \leq C \|\psi\|^2.$$

Consequently, we get

$$\begin{aligned} \lambda_N(\mathcal{L}_\hbar^{\text{eff}}) \|\psi\|^2 &\geq \langle \mathcal{L}_\hbar^{\text{eff}} \psi, \psi \rangle \\ &\geq \int_{\mathbb{R}} \hbar \frac{-\kappa''(0)}{2} \sigma^2 |\tilde{\psi}(\hbar^{1/2} \sigma)|^2 d(\hbar^{1/2} \sigma) + \hbar \int_{\mathbb{R}} |\partial_\sigma \tilde{\psi}|^2 d(\hbar^{1/2} \sigma) - C \hbar^{3/2} \|\psi\|^2 \\ &\geq \hbar \left[ \int_{\mathbb{R}} \left( |\partial_\sigma \tilde{\psi}(\hbar^{1/2} \sigma)|^2 + \frac{-\kappa''(0)}{2} \sigma^2 |\tilde{\psi}(\hbar^{1/2} \sigma)|^2 \right) d(\hbar^{1/2} \sigma) \right] - C \hbar^{3/2} \|\psi\|^2. \end{aligned}$$

This becomes

$$\hbar \left[ \int_{\mathbb{R}} \left( |\partial_{\sigma} \tilde{\psi}(\hbar^{1/2} \sigma)|^2 + \frac{-\kappa''(0)}{2} \sigma^2 |\tilde{\psi}(\hbar^{1/2} \sigma)|^2 \right) d(\hbar^{1/2} \sigma) \right] \leq \lambda_N(\mathcal{L}_h^{\text{eff}}) \|\psi\|^2 + \mathcal{O}(\hbar^{3/2}) \|\psi\|^2.$$

We deduce that

$$\max_{\tilde{\psi} \in \tilde{\mathcal{E}}_N(\hbar)} \frac{\langle H_{\text{harm}} \tilde{\psi}, \tilde{\psi} \rangle_{L^2(\mathbb{R})}}{\|\tilde{\psi}\|_{L^2(\mathbb{R})}^2} \leq \lambda_N(\mathcal{L}_h^{\text{eff}}) + \mathcal{O}(\hbar^{3/2}).$$

Thus, by the min-max principle, we have the comparison of the Rayleigh quotients

$$\begin{aligned} \lambda_N(\mathcal{L}_h^{\text{eff}}) &\geq \lambda_N(H_{\text{harm}}) + \mathcal{O}(\hbar^{3/2}) \\ &\geq (2N-1)\hbar \sqrt{\frac{-\kappa''(0)}{2}} + \mathcal{O}(\hbar^{3/2}). \end{aligned}$$

□

Finally, from (3.83), the asymptotic expansion of the  $n$ -th eigenvalue of  $\mathcal{L}_h^{\text{eff}}$  is:

$$\lambda_n(\mathcal{L}_h^{\text{eff}}) = (2n-1)\hbar \sqrt{\frac{-\kappa''(0)}{2}} + \mathcal{O}(\hbar^{3/2}).$$

Now, by inequalities (3.58) and Proposition 3.5.6, we deduce the asymptotic expansion of the  $n$ -th eigenvalue of the effective operator  $\mathcal{L}_{h,\alpha,0}^{\text{eff},b}$

$$\lambda_n(\mathcal{L}_{h,\alpha,0}^{\text{eff},b}) = -1 - \kappa_{\max} \hbar^2 + (2n-1) \sqrt{\frac{-\kappa''(0)}{2}} \hbar^3 + \mathcal{O}(\hbar^{7/2}). \quad (3.84)$$

From Theorem 3.4.1 and Proposition 3.3.8 and the equality (3.33), we have shown Corollary 3.3.2, which was a reformulation of Corollary 3.1.2.

### 3.6 Unit disc case

In this section we analyse the magnetic Robin Laplacian with a negative boundary parameter on the disc. There by proving of Theorem 3.3.3. We will study the spectrum of the effective operator  $\mathcal{L}_{h,\text{disc}}^{\text{eff},\pm}$ , where  $\Omega = D(0, 1)$ ,  $0 < c_1 < c_2$ ,  $0 < \alpha$  and  $0 \leq \eta < 1$ . We have

$$L = \frac{|\partial\Omega|}{2} = \pi, \quad \gamma_0 = \frac{|\Omega|}{|\partial\Omega|} = \frac{1}{2}, \quad c_1 h^{\frac{-\eta}{2}} \leq b \leq c_2 h^{\frac{-\eta}{2}} \quad \text{and} \quad \kappa = 1.$$

Thus, the effective operators can be written as follows

$$\mathcal{L}_{h,\text{disc}}^{\text{eff},\pm} = -h(1 \pm c_{\pm} h^{\min(\alpha, \frac{1}{2})}) \left( \partial_s - i \frac{b}{2} \right)^2 - 1 - h^{\frac{1}{2}} - \frac{h}{2} \pm c_{\pm} h^{\frac{3}{2}},$$

in  $L^2(\mathbb{R}/2\pi\mathbb{Z})$ , where  $c_{\pm}$  are constants independent of  $h$ . The associated quadratic form on  $H^1(\mathbb{R}/2\pi\mathbb{Z})$  be

$$q_{h,\text{disc}}^{\text{eff},\pm}(\varphi) = \left( -1 - h^{\frac{1}{2}} - \frac{h}{2} \pm c_{\pm} h^{\frac{3}{2}} \right) \int_{-\pi}^{\pi} |\varphi(s)|^2 ds + h \left( (1 \pm c_{\pm} h^{\min(\alpha, \frac{1}{2})}) \int_{-\pi}^{\pi} \left| \left( \partial_s - i \frac{b}{2} \right) \varphi \right|^2 ds \right).$$

The spectrum of  $-\left(\partial_s - i\frac{b}{2}\right)^2$  in  $L^2(\mathbb{R}/2\pi\mathbb{Z})$  can be explicitly computed by using Fourier series. In fact, for all  $m \in \mathbb{Z}$ ,  $e_m(b) := (m - \frac{b}{2})^2$  is an eigenvalue with the eigenfunction  $v_m(s) := e^{ims}/\sqrt{2\pi}$ . Since  $\{v_m\}_{m \in \mathbb{Z}}$  is a Hilbertian basis of  $L^2(\mathbb{R}/2\pi\mathbb{Z})$ , we get that the spectrum of  $-\left(\partial_s - i\frac{b}{2}\right)^2$  is  $\{e_m(b)\}_{m \in \mathbb{Z}}$  and its lowest eigenvalue is  $\inf_{m \in \mathbb{Z}} e_m(b)$ .

We deduce the asymptotic expansion of the first eigenvalue of the effective operator  $\mathcal{L}_{h,\text{disc}}^{\text{eff},\pm}$

$$\begin{aligned} \lambda_1(\mathcal{L}_{h,\text{disc}}^{\text{eff},\pm}) &= \inf_{m \in \mathbb{Z}} \lambda_1(\mathcal{L}_{h,\text{disc},m}^{\text{eff},\pm}) = -1 - h^{\frac{1}{2}} - \frac{h}{2} \pm c_{\pm} h^{\frac{3}{2}} + h \left(1 \pm c_{\pm} h^{\min(\alpha, \frac{1}{2})}\right) \inf_{m \in \mathbb{Z}} \left(m - \frac{b}{2}\right)^2 \\ &= -1 - h^{\frac{1}{2}} - \frac{h}{2} + h \inf_{m \in \mathbb{Z}} \left(m - \frac{b}{2}\right)^2 + o(h). \end{aligned}$$

According to Theorem 3.4.1 and Proposition 3.3.8 and the equality (3.33), for  $\eta < 1$ , we have

$$\mu_1(h, b) = -h - h^{\frac{3}{2}} + \left( \inf_{m \in \mathbb{Z}} \left(m - \frac{b}{2}\right)^2 - \frac{1}{2} \right) h^2 + o(h^2).$$

Then, we complete the proof of the Theorem 3.3.3 which was a reformulation of Theorem 3.1.3. Note that, in the case with constant magnetic field, we recover by a very simple proof of [KS20, Theorem 1.1].



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# On the semi-classical analysis of Schrödinger operators with linear electric potentials in a bounded domain.

## Abstract

This chapter aims to improve the asymptotic expansion of the eigenvalues of the Laplace operator, with strong uniform electric field and Dirichlet boundary conditions on a smooth bounded domain of  $\mathbb{R}^N$ ,  $N \geq 2$ . This work aims at generalizing the recent results of Cornean, Krejčířík, Pedersen, Raymond, and Stockmeyer in dimension 2. More precisely, in dimension  $N$ , in the strong electric field limit, we derive, under certain local convexity conditions, a full asymptotic expansion of the low-lying eigenvalues. To establish our main result, we perform the construction of quasi-modes. The "optimality" of our constructions is then established thanks to a reduction to model operators and localization estimates. In addition, we apply our techniques to find a full asymptotic expansion of the low-lying eigenvalues of the more general operator  $-h^2\Delta + V(\mathbf{x})$  where  $h$  is a small parameter and  $V$  is a smooth real potential satisfies some additional conditions.

## Résumé

Le but de ce chapitre est d'améliorer le développement asymptotique des valeurs propres de l'opérateur de Laplace, avec un champ électrique uniforme fort et des conditions de Dirichlet au bord sur un domaine borné régulier de  $\mathbb{R}^N$ ,  $N \geq 2$ . Ce travail vise à généraliser les résultats récents de Cornean, Krejčířík, Pedersen, Raymond et Stockmeyer en dimension 2. Plus précisément, en dimension  $N$ , dans la limite du champ électrique fort, nous dérivons, sous certaines conditions de convexité locale, un développement asymptotique complet des basses valeurs propres. Pour établir notre résultat principal, nous effectuons la construction de quasi-modes. L'"optimalité" de nos constructions est ensuite établie grâce à une réduction aux opérateurs modèles et aux estimations de localisation. En outre, nous appliquons nos techniques pour trouver un développement asymptotique complet des valeurs propres de l'opérateur plus général  $-h^2\Delta + V(\mathbf{x})$  où  $h$  est un petit paramètre et  $V$  est un potentiel réel et régulier satisfaisant certaines conditions supplémentaires.

*” La mathématique est l’art  
de donner le même nom à  
des choses différentes. ”*

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*Henri Poincaré*

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## 4.1 Introduction

### 4.1.1 A Stark electric Laplacian

The study of the Laplacian in the usual Euclidean space  $\mathbb{R}^N$ , and more generally of the Schrödinger operator, is of course very present in physics, for example in quantum mechanics. Consider a system of one particle moving in a uniform electric field  $\mathbf{F}$ . In a simplified model, one may consider independently electrons or holes. The total energy Stark Hamiltonian (Schrödinger operator) for such a system takes the following form:

$$\frac{-\hbar^2}{2m}\Delta + q\mathbf{F} \cdot \mathbf{x}, \quad (4.1)$$

where

- $\hbar$  is Planck constant divided by  $2\pi$ ,
- $m > 0$  is the effective mass of the particle (electron or hole),
- $-q$  is the charge of the particle (electron or hole).

The Hamiltonian in (4.1) describes an important phenomenon in semiconductor physics which is the *quantum confined Stark effect*. The most prominent semiconductor devices are the so-called quantum wells, quantum wires, and quantum dots, where the material is confined in one, two, and three orthogonal directions, respectively [HV16, Chap. 8]. The quantum confined Stark effect modeled with Dirichlet boundary condition has been considered, for example, in [BMCE83, MCD<sup>+</sup>84] for quantum wells, in [SN90, WKH13, WWY07, VdCMS03, LS05, Har09] for quantum wires, and in [Ped19, WWY08] for quantum dots. The recent paper [CKP<sup>+</sup>22b] provides more details on the physical orientation of this phenomenon.

In this chapter, we are interested in the study of the low-lying eigenvalues of the following  $N$ -dimensional Hamiltonian restricted to an open set  $\Omega$  of  $\mathbb{R}^N$ ,  $N \geq 2$ :

$$\mathcal{L}_h := -h^2\Delta + x_1 = -h^2(\partial_{x_1}^2 + \dots + \partial_{x_N}^2) + x_1, \quad (4.2)$$

acting on a dense subspace of the square integral functions  $L^2(\Omega)$  with Dirichlet boundary conditions. This operator appears in the physics semi-conductors where  $h$  is given by

$$h = \frac{\hbar}{\sqrt{2mqF}}.$$

Notice that we have chosen coordinates such that  $\mathbf{F}$  is parallel to the  $x_1$ -axis.

The low-lying eigenvalues of this operator have been studied, partially numerically, for different geometries in several papers such as in [SN90, WKH13] for squares, in [WWY07] for rectangles, in [VdCMS03, LS05, Ped17] for disks, and in [Har09] for annuli. Concerning the analysis in the case of imaginary electric fields, the reader might also want to consider for example [GH18, AGH19, AGH18].

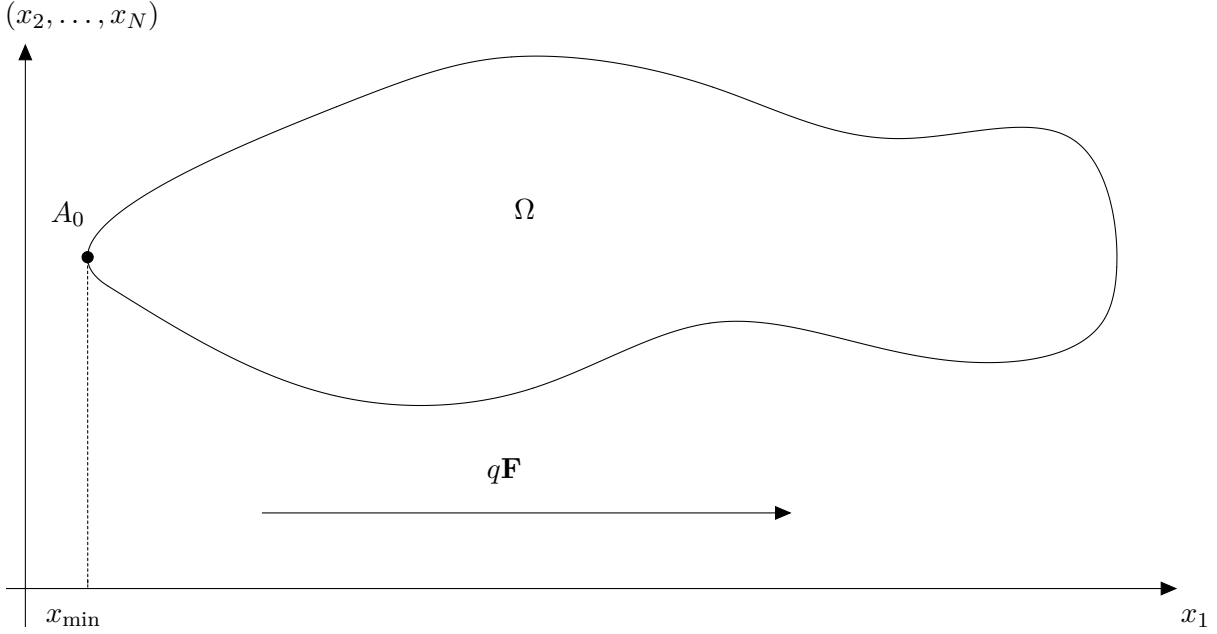
For now, we will work with domains satisfying the following conditions, see Figure 4.1.

**Assumption 1.** *The set  $\Omega$  be a smooth open bounded and connected domain of  $\mathbb{R}^N$  with a smooth boundary. We assume that there exists a unique point  $A_0 \in \partial\Omega$  such that the first component of  $A_0$  is given by:*

$$x_{\min} = \inf_{(x_1, x_2, \dots, x_N) \in \Omega} x_1 = \min_{\mathbf{x} \in \Omega} x_1.$$

*We also assume that  $\Omega$  strictly convex domain near  $A_0$ .*




 Figure 4.1 – The domain  $\Omega$ .

To be more precise, the operator  $\mathcal{L}_h$  is defined by Friedrichs extension theorem, from the closed and semi-bounded quadratic form

$$\mathcal{Q}_h(\varphi) = h^2 \int_{\Omega} |\nabla \varphi(\mathbf{x})|^2 \, d\mathbf{x} + \int_{\Omega} x_1 |\varphi(\mathbf{x})|^2 \, d\mathbf{x}, \quad \forall \varphi \in H_0^1(\Omega).$$

The domain of  $\mathcal{L}_h$  is contained in  $H_0^1(\Omega)$  and  $\mathcal{L}_h$  acts as in (4.2). This is the Dirichlet realization of the Stark Hamiltonian. If  $\mathcal{Q}_h(\cdot, \cdot)$  is the bilinear form associated with the quadratic form  $\mathcal{Q}_h(\cdot)$ , then

$$\text{Dom}(\mathcal{L}_h) = \{\varphi \in H_0^1(\Omega) : \psi \mapsto \mathcal{Q}_h(\varphi, \psi) \text{ is continuous for the } L^2(\Omega)\text{-topology}\}.$$

Since  $\Omega$  is bounded, then the embedding of  $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$  is compact, therefore the self-adjoint operator  $\mathcal{L}_h$  has a compact resolvent. Its spectrum is purely discrete formed by a non-decreasing sequence of eigenvalues denoted by  $(\lambda_n(h))_{n \geq 1}$ , where each eigenvalue is repeated according to its multiplicity. The purpose of this paper is to understand the behavior as  $h \rightarrow 0$  (in the limit of the strong electric field) of the eigenvalues  $\lambda_n(h)$ .

### 4.1.2 Main result

In [CKP<sup>+</sup>22a], by constructing suitable test functions, and applying the min-max principle, the authors found a three-term asymptotic expansion of the low-lying eigenvalues. It is proved that, in  $\mathbb{R}^2$ , for every fixed  $n \in \mathbb{N}^*$ ,

$$\lambda_n(h) = x_{\min} + z_1 h^{\frac{2}{3}} + h(2n-1) \sqrt{\frac{\kappa_1}{2}} + \mathcal{O}(h^{\frac{4}{3}}) \quad \text{as } h \rightarrow 0, \quad (4.3)$$

where  $z_1 \approx 2.338$  is the absolute value of the smallest zero of the Airy function and  $\kappa_1$  is the positive curvature at  $A_0$ .

This chapter aims to improve the asymptotic expansion in (4.3) and to give a full asymptotic expansion of  $\lambda_n(h)$ . More precisely, we establish a full asymptotic expansion of the low-lying eigenvalues, by involving the eigenvalues of the Weingarten map of  $\partial\Omega$  at  $A_0$  defined by

$$W_{A_0} = d_{A_0}\nu : T_{A_0}\partial\Omega \rightarrow T_{A_0}\partial\Omega \quad (4.4)$$

where

$$\nu : \partial\Omega \rightarrow S^{N-1} \subset \mathbb{R}^N$$

is a Gauss map of  $\partial\Omega$  and for all  $p \in \partial\Omega$ ,  $\nu(p)$  is the outward-pointing normal to  $T_p\partial\Omega$ .

**Remark 4.1.1.** *Some properties of Weingarten map (see for example [Küh15, Chap. 3] and [Tho79, Chap. 9]).*

1. *If  $(U, \phi)$  is a parametrization of  $\partial\Omega$ , with  $A_0 \in \phi(U)$ , then, the map  $\nu$  can be written as*

$$\nu = n \circ \phi^{-1},$$

*where  $n : U \rightarrow S^{N-1}$ , such that  $n(s)$  is the unit outward normal field at the point  $\phi(s)$ .*

2. *The differential of  $\nu$  at  $A_0$  is the linear map  $d_{A_0}\nu : T_{A_0}\partial\Omega \rightarrow \mathbb{R}^N$  defined by*

$$d_{A_0}\nu.X = (\nu \circ \gamma)'(0),$$

*where  $\gamma$  is a derivable arc drawn on  $\partial\Omega$  such that  $\gamma(0) = A_0$  and  $\gamma'(0) = X$ .*

*We have the following properties:*

- a. *This means that the endomorphism of Weingarten does not depend on the choice of  $(U, \phi)$ .*
- b. *To get that  $d_{A_0}\nu.X$  does not depend on the choice of  $\gamma$ , we use the coordinates  $(U, \phi)$ , such that  $A_0 = \phi(s_0)$ . We have*

$$\nu \circ \gamma = \nu \circ \phi \circ \phi^{-1} \circ \gamma = n \circ (\phi^{-1} \circ \gamma),$$

*then,*

$$(\nu \circ \gamma)'(0) = d_{(\phi^{-1} \circ \gamma)(0)}n.(\phi^{-1} \circ \gamma)'(0) = d_{s_0}n.(\phi^{-1} \circ \gamma)'(0).$$

*Moreover,*

$$X = \gamma'(0) = d_{(\phi^{-1} \circ \gamma)(0)}\phi.(\phi^{-1} \circ \gamma)'(0) = d_{s_0}\phi.(\phi^{-1} \circ \gamma)'(0).$$

*This gives*

$$d_{A_0}\nu.X = d_{s_0}n.[d_{s_0}\phi]^{-1}X. \quad (4.5)$$

*All mention of  $\gamma$  has disappeared.*

3. *The Weingarten's endomorphism is symmetrical and linear endomorphism. But does not mean that its matrix in any base is symmetrical (only its writing in an orthonormal basis). On the other hand, it is diagonalizable.*

**Remark 4.1.2.** *From Section 4.7.2, the assumption on  $\Omega$  induces that the matrix of  $W_{A_0}$  is positive definite. Their positive eigenvalues  $(\kappa_i)_{i \geq 1}$  are related to the curvature of the boundary.*

1. *The  $\kappa_i$  are called principal curvatures of  $\partial\Omega$  at  $A_0$ .*
2.  *$\det W_{A_0}$  is the Gaussian curvature from  $\partial\Omega$  at  $A_0$ .*
3.  *$\frac{\text{Tr } W_{A_0}}{N-1}$  is the mean curvature from  $\partial\Omega$  at  $A_0$ .*

The main result in this chapter is:

**Theorem 4.1.1.** For any positive  $n$ , there exists a real sequence  $\{\zeta_{j,n}\}_{j \geq 0}$  such that, for any positive  $M \in \mathbb{N}$ , we have

$$\lambda_n(h) \underset{h \rightarrow 0}{=} x_{\min} + z_1 h^{\frac{2}{3}} + \beta_n h + h^{4/3} \sum_{j=0}^M h^{j/6} \zeta_{j,n} + \mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}}),$$

where  $\beta_n$  is the  $n$ -th eigenvalue of the harmonic oscillator in dimension  $N - 1$

$$\sum_{i=1}^{N-1} -\partial_{u_i}^2 + \frac{\kappa_i}{2} u_i^2,$$

where  $\kappa_i$  are the eigenvalues of the Weingarten map of  $\partial\Omega$  at  $A_0$ .

This result is essentially a generalization of those states in (4.3), but the proof presented here, is based on the construction of quasimodes. The "optimality" of our constructions is then established thanks to decay estimates and the reduction to model operators involving a 1-dimensional Airy operator in  $\mathbb{R}_+$  and a harmonic oscillator in dimension  $N - 1$ . One of the technical difficulties of the proof is that, there are eigenvalues of the harmonic oscillator with multiplicity greater than 1, see Section 4.6.

**Remark 4.1.3.**  $\beta_n$  is the  $n$ -th element counted with multiplicity of

$$\Sigma = \left\{ \sum_{i=1}^{N-1} \sqrt{\frac{\kappa_i}{2}} (2n_i - 1), n \in \mathbb{N}^{*N-1} \right\}.$$

Notice that for  $N = 2$  we have

$$\Sigma = \left\{ (2n - 1) \sqrt{\frac{\kappa_0}{2}}, n \in \mathbb{N}^* \right\},$$

and all the elements are of multiplicity one.

**Notation 4.1.2.** The notation  $\mathcal{O}(h^\infty)$  indicates a quantity satisfying that, for all  $N \in \mathbb{N}$ , there exists  $C_N > 0$  and  $h_N > 0$  such that, for all  $h \in (0, h_N)$ ,  $|\mathcal{O}(h^\infty)| \leq C_N h^N$ .

**Remark 4.1.4.** Since Theorem 4.1.1 is valid for all  $N \geq 2$ , it would be instructive to mention what happens in dimension one. In dimension 1, the operator involved is

$$\mathcal{L}_h = -h^2 \partial_x^2 + x,$$

acting on  $L^2((0, T))$  with Dirichlet boundary conditions, and  $T > 0$ . Using a simple change of variable  $t = h^{-\frac{2}{3}} x$ , for  $h$  small enough, we deduce that  $\mathcal{L}_h$  is unitarily equivalent to  $h^{\frac{2}{3}} \mathcal{A}_h$ , where

$$\mathcal{A}_h = -\partial_t^2 + t,$$

acting on  $L^2((0, h^{-\frac{2}{3}} T))$  with Dirichlet boundary conditions. It is then standard to deduce from the min-max principle and the exponentially decreasing of the Airy operator at  $+\infty$  (see for instance [Hel13, Section 14.5.1] and [DR12, Section 2.2.2]), the following asymptotic expansion of the low-lying eigenvalue:

$$\lambda_n(h) \underset{h \rightarrow 0}{=} z_n h^{\frac{2}{3}} + \mathcal{O}(h^\infty),$$

where  $z_n$  are the absolute value of the zeros of the Airy function.

**Remark 4.1.5.** – Our strategy can be adapted to deal with a finite number of non-degenerate minima, and it would even be possible to study the tunnel effects when these minima have symmetries.

– It is possible to prove asymptotic expansions of the low-lying eigenvalues of the Hamiltonian  $\mathcal{L}_h$ , if we replace the potential  $x_1$  by  $V(\mathbf{x})$ , where  $V$  is a smooth real potential satisfies some additional conditions (see the perspectives).

### 4.1.3 Organization of the chapter

This chapter is organized as follows: In Section 4.2.1, we show that low energy eigenfunctions and their derivatives are exponentially well localized around the potential minimum  $x_{\min}$ . This will allow us to work in a thin tubular neighborhood of  $A_0$ . As a consequence, we find an approximated operator  $\mathcal{L}_{h,\delta}$ , whose low energy eigenvalues are those of  $\mathcal{L}_h$  up to an exponentially small error, see Section 4.2.2. In Section 4.3, we locally straighten the boundary by introducing a system of local coordinates: "tubular coordinates". The operator  $\mathcal{L}_{h,\delta}$  is expressed in these coordinates and acts on a tiny domain around the origin with Dirichlet boundary conditions. In Section 4.4 and 4.5, we construct quasimodes, thanks to the reduction to model operators and localization estimates, we provide asymptotic upper and lower bounds for the eigenvalues of  $\mathcal{L}_{h,\delta}$ , see Section 4.6. Eventually, we deduce the full asymptotic expansion of the low-lying eigenvalues stated in Theorem 4.1.1. Section 4.7 is dedicated to the analysis of the simplified model which shall be used as local approximations for operator  $\mathcal{L}_{h,\delta}$ .

## 4.2 Preliminaries

### 4.2.1 Concentration of bound states near the potential minimum

The following proposition states that the eigenfunctions associated with the low-lying eigenvalues are localized in  $x_1$  near  $x_{\min}$  at the scale  $h^{2/3}$ . For more details on the proofs of the proposition and corollary of this section, see [CKP<sup>+</sup>22a, Section 2].

**Proposition 4.2.1.** *Let  $M > 0$ . There exist  $\varepsilon, C, h_0 > 0$  such that, for all  $h \in (0, h_0)$ , for all eigenvalues  $\lambda$  such that  $\lambda \leq x_{\min} + Mh^{2/3}$ , and all corresponding eigenfunctions  $\psi$ ,*

$$\int_{\Omega} e^{\varepsilon|x_1 - x_{\min}|^{3/2}/h} |\psi|^2 \, d\mathbf{x} \leq C \|\psi\|^2, \quad (4.6)$$

and

$$\int_{\Omega} e^{\varepsilon|x_1 - x_{\min}|^{3/2}/h} |h \nabla \psi|^2 \, d\mathbf{x} \leq Ch^{2/3} \|\psi\|^2. \quad (4.7)$$

*Proof.* For all  $\psi \in \text{Dom}(\mathcal{L}_h)$  and all bounded Lipschitz functions  $\Phi$ , we have

$$\langle \mathcal{L}_h \psi, e^{2\Phi/h} \psi \rangle_{L^2(\Omega)} = \lambda \|e^{\Phi/h} \psi\|_{L^2(\Omega)}^2. \quad (4.8)$$

By an integration by parts, we get the useful identity:

$$\langle \mathcal{L}_h \psi, e^{2\Phi/h} \psi \rangle_{L^2(\Omega)} = \mathcal{Q}_h(e^{\Phi/h} \psi) - \|e^{\Phi/h} \nabla \Phi \psi\|_{L^2(\Omega)}^2. \quad (4.9)$$

From (4.8) and (4.9), we write the Agmon formula (see for example [Ray17, Proposition 4.7])

$$\int_{\Omega} h^2 |\nabla e^{\Phi/h} \psi|^2 + (x_1 - \lambda - |\nabla \Phi|^2) |e^{\Phi/h} \psi|^2 \, d\mathbf{x} = 0, \quad (4.10)$$

and thus

$$\int_{\Omega} h^2 |\nabla e^{\Phi/h} \psi|^2 + (x_1 - x_{\min} - Mh^{2/3} - |\nabla \Phi|^2) |e^{\Phi/h} \psi|^2 \, d\mathbf{x} \leq 0. \quad (4.11)$$

Now, we choose  $\Phi = \varepsilon |x_1 - x_{\min}|^{3/2}$ , we obtain

$$\int_{\Omega} \left( \left(1 - \frac{9}{4}\varepsilon^2\right) (x_1 - x_{\min}) - Mh^{2/3} \right) |e^{\Phi/h}\psi|^2 d\mathbf{x} \leq 0.$$

We take  $\varepsilon$  such that  $1 - \frac{9}{4}\varepsilon^2 > 0$  and fix  $R > 0$  such that

$$\left(1 - \frac{9}{4}\varepsilon^2\right) R - M = 1. \quad (4.12)$$

Using (4.12), we get

$$\begin{aligned} h^{2/3} \int_{|x_1 - x_{\min}| \geq Rh^{2/3}} |e^{\Phi/h}\psi|^2 d\mathbf{x} &\leq \int_{|x_1 - x_{\min}| \geq Rh^{2/3}} \left( \left(1 - \frac{9}{4}\varepsilon^2\right) (x_1 - x_{\min}) - Mh^{2/3} \right) |e^{\Phi/h}\psi|^2 d\mathbf{x} \\ &\leq Mh^{2/3} \int_{|x_1 - x_{\min}| < Rh^{2/3}} e^{2\Phi/h} |\psi|^2 d\mathbf{x} \leq Ch^{2/3} \|\psi\|^2. \end{aligned}$$

This proves (4.6). Next we prove (4.7). First observe that repeating the calculation above without dropping the first term in (4.11) we get that

$$\int_{\Omega} h^2 |\nabla e^{\Phi/h}\psi|^2 dx dy \leq Ch^{2/3} \|\psi\|^2.$$

Using the triangle inequality, (4.6), (4.10) to complete the proof of (4.7).  $\square$

Let  $\eta > 0$ , we denote by  $\complement B(A_0, \eta)$  the complement of the open ball  $B(A_0, \eta)$ .

**Corollary 4.2.2.** *There exist  $c_\eta, C_\eta, h_0 > 0$  such that, for all  $h \in (0, h_0)$ , for all eigenvalues  $\lambda$  such that  $\lambda \leq x_{\min} + Mh^{\frac{2}{3}}$ , and all corresponding eigenfunctions  $\psi$ ,*

$$\int_{\Omega \cap \complement B(A_0, \eta)} |\psi|^2 d\mathbf{x} \leq C_\eta e^{-c_\eta/h} \|\psi\|^2,$$

and

$$\int_{\Omega \cap \complement B(A_0, \eta)} |\nabla \psi|^2 d\mathbf{x} \leq C_\eta e^{-c_\eta/h} \|\psi\|^2.$$

*Proof.* Since  $\Omega$  is bounded then the set  $\bar{\Omega} \cap \complement B(A_0, \eta)$  is compact. The map  $\bar{\Omega} \cap \complement B(A_0, \eta) \ni (x_1, \dots, x_N) \mapsto x_1 - x_{\min}$  is continuous and positive. By compactness it has a positive minimum. The conclusion follows from Proposition 4.2.1.  $\square$

As a consequence, for small  $h$ , the ground states of the operator  $\mathcal{L}_h$  are concentrated around  $x_{\min}$  (cf. Corollary 4.2.2). We can now reduce our investigation to a neighborhood of  $A_0$ .

## 4.2.2 Reduction near the boundary

Let  $\delta > 0$  small enough, we introduce the  $\delta$ -neighborhood of the boundary

$$\Omega_\delta = \{ \mathbf{x} = (x_1, \dots, x_N) \in \Omega : \text{dist}(\mathbf{x}, A_0) < \delta \}.$$

Consider the quadratic form, defined on the variational space

$$W_\delta = \{ u \in H_0^1(\Omega_\delta) : u(\mathbf{x}) = 0, \forall \mathbf{x} \in \Omega \text{ such that } \text{dist}(\mathbf{x}, A_0) = \delta \},$$

by the formula

$$\mathcal{Q}_{h,\delta}(\varphi) = h^2 \int_{\Omega_\delta} |\nabla \varphi(\mathbf{x})|^2 \, d\mathbf{x} + \int_{\Omega_\delta} x_1 |\varphi(\mathbf{x})|^2 \, d\mathbf{x}. \quad (4.13)$$

Let the self-adjoint operator associated with the quadratic form  $\mathcal{Q}_{h,\delta}$  be defined on  $L^2(\Omega_\delta)$  and with the Dirichlet condition on the boundary of  $\Omega_\delta$ ,

$$\mathcal{L}_{h,\delta} = -h^2 \Delta + x_1.$$

Let  $(\lambda_n(h, \delta))_{n \in \mathbb{N}^*}$  be the sequence of min-max values of the operator  $\mathcal{L}_{h,\delta}$ . It is standard to deduce from the min-max principle and the Agmon estimates of Proposition 4.2.1 the following proposition (see [Hel88] or Proposition 3.3.8). It is an important result that compares the eigenvalues and gives an approximation of  $\lambda_n(h)$ .

**Proposition 4.2.3.** *Let  $n \geq 1$  and  $\delta > 0$  small enough. There exist constants  $C_\delta, k_\delta > 0, h_0 \in (0, 1)$  such that, for all  $h \in (0, h_0)$  and  $\lambda_n(h) \leq x_{\min} + Mh^{\frac{2}{3}}$ ,*

$$\lambda_n(h, \delta) \leq \lambda_n(h) + C_\delta \exp(-k_\delta/h). \quad (4.14)$$

Moreover, we have, for all  $n \geq 1, h > 0$

$$\lambda_n(h) \leq \lambda_n(h, \delta). \quad (4.15)$$

*Proof.* The inequality (4.15) follows from the fact that  $\Omega_\delta \subset \Omega$ , the Dirichlet condition and the min-max principle.

The inequality (4.14) follows from the Agmon estimates. Let  $(u_{k,h})_{1 \leq k \leq n}$  an orthonormal family of eigenfunctions associated with the eigenvalues  $(\lambda_k(h))_{1 \leq k \leq n}$ . Let the truncation function define on  $\mathbb{R}_+$  as follows:

$$0 \leq \chi \leq 1, \chi = 1 \text{ on } [0, 1/2[ \text{ and } \chi = 0 \text{ on } [1, +\infty[.$$

We define the function

$$\chi_\delta(\mathbf{x}) = \chi\left(\frac{\text{dist}(\mathbf{x}, A_0)}{\delta}\right).$$

Let

$$\mathcal{C}_n(h, \delta) = \text{span}_{1 \leq k \leq n} \chi_\delta u_{k,h}.$$

Thus  $\mathcal{C}_n(h, \delta) \subset W_\delta$  and using Proposition 4.2.1, we notice that  $\dim \mathcal{C}_n(h, \delta) = n$ . Consider  $\varphi \in \mathcal{C}_n(h, \delta)$  and write

$$\varphi = \chi_\delta \psi,$$

where  $\psi \in \text{span}_{1 \leq k \leq n} u_{k,h}$ .

We have

$$\begin{aligned} \mathcal{Q}_{h,\delta}(\chi_\delta \psi) &= h^2 \int_{\Omega} |\psi \nabla \chi_\delta + \chi_\delta \nabla \psi|^2 \, d\mathbf{x} + \int_{\Omega} x_1 |\chi_\delta \psi(\mathbf{x})|^2 \, d\mathbf{x} \\ &\leq h^2 \int_{\Omega} |\nabla \psi|^2 \, d\mathbf{x} + 2h^2 \|\psi \nabla \chi_\delta\| \|\chi_\delta \nabla \psi\| + h^2 \int_{\Omega} |\nabla \chi_\delta \psi|^2 \, d\mathbf{x} + \int_{\Omega} x_1 |\psi(\mathbf{x})|^2 \, d\mathbf{x} \\ &\leq \mathcal{Q}_h(\psi) + 2h^2 \|\psi \nabla \chi_\delta\| \|\chi_\delta \nabla \psi\|_{L^2(\Omega \setminus \Omega_{\delta/2})} + h^2 \int_{\Omega} |\nabla \chi_\delta \psi|^2 \, d\mathbf{x}. \end{aligned}$$

Since the  $(u_{k,h})_{1 \leq k \leq n}$  are orthogonal eigenfunctions, we get

$$\mathcal{Q}_h(\psi) \leq \lambda_n(h) \|\psi\|^2.$$

From Proposition 4.2.1, there exist constants  $C, C_\delta, k_\delta > 0$ , such that

$$h \|\psi \nabla \chi_\delta\| \leq C_\delta \exp(-k_\delta/h),$$

and

$$\|\psi \nabla \chi_\delta\|_{L^2(\Omega \setminus \Omega_{\delta/2})} \leq Ch^{1/3} \exp(-k_\delta/h).$$

Thus, the conclusion follows from the min-max principle.  $\square$

### 4.3 Local boundary coordinates

In this section, we introduce local coordinates in the neighborhood of  $\mathbf{x}_0 \in \partial\Omega$ , in order to straighten a portion of the boundary. Throughout this section, we mainly refer to [FH10b, App. F] and [HM04, Pan04], although these coordinates were also used in [AH14] in a two-dimensional framework.

#### 4.3.1 The coordinates and the metric

We fix  $\epsilon > 0$  such that the distance function

$$t(\mathbf{x}) = \text{dist}(\mathbf{x}, \partial\Omega)$$

is smooth on  $\Omega_\epsilon = \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) < \epsilon\}$ .

Let  $\mathbf{x}_0 \in \partial\Omega$  be fixed and choose a chart  $\phi : \mathcal{V}_{\mathbf{x}_0} \mapsto \phi(\mathcal{V}_{\mathbf{x}_0})$  such that  $\mathbf{x}_0 \in \phi(\mathcal{V}_{\mathbf{x}_0})$  and  $\mathcal{V}_{\mathbf{x}_0}$  is an open set of  $\mathbb{R}^{N-1}$ . We set

$$s_0 = \phi^{-1}(\mathbf{x}_0) \quad \text{and} \quad W_0 = \phi(\mathcal{V}_{\mathbf{x}_0}).$$

Then, in these coordinates, the metric on the surface  $\phi(\mathcal{V}_{\mathbf{x}_0})$  induced by the Euclidian metric  $g_0$  of  $\mathbb{R}^N$  writes

$$G = \sum_{1 \leq i, j \leq N-1} G_{ij} ds_i \otimes ds_j,$$

where

$$\forall i, j = 1, \dots, N-1, \quad G_{ij}(s_1, \dots, s_{N-1}) = \partial_{s_i} \phi(s_1, \dots, s_{N-1}) \cdot \partial_{s_j} \phi(s_1, \dots, s_{N-1}).$$

Now, we define local coordinates in a neighborhood of  $\mathbf{x}_0$  in  $\Omega$ . For  $\epsilon > 0$  small enough, we consider

$$\mathcal{V}_{\mathbf{x}_0} \times ]0, \epsilon[ \ni (s_1, \dots, s_{N-1}, t) \mapsto \Phi(s, t) = \mathbf{x} = \phi(s_1, \dots, s_{N-1}) - \nu(\phi(s_1, \dots, s_{N-1})) t, \quad (4.16)$$

where  $\nu(\mathbf{x})$  is the unit outward pointing normal vector at the point  $\mathbf{x} \in \partial\Omega$ .

This defines a diffeomorphism of  $\mathcal{V}_{\mathbf{x}_0} \times ]0, \epsilon[$  onto  $\Phi(\mathcal{V}_{\mathbf{x}_0} \times ]0, \epsilon[)$  in  $\Omega_\epsilon$  and its inverse gives local coordinates on  $\mathcal{V}_{\mathbf{x}_0}$ . We set

$$U_0 = \Phi(\mathcal{V}_{\mathbf{x}_0} \times ]0, \epsilon[).$$

Note that  $t$  denotes the normal variable in the sense that for a point  $\mathbf{x} \in U_0$  such that

$$t(\mathbf{x}) = \text{dist}(\mathbf{x}, \phi(s_1, \dots, s_{N-1})) = \text{dist}(\mathbf{x}, \partial\Omega).$$

In particular,  $t = 0$  is the equation of the surface  $U_0 \cap \partial\Omega$ .

In the following we use the notation  $(s_1, \dots, s_{N-1}, t) = (s, t)$ ,  $\partial_j = \partial_{s_j}$  if  $j \in \{1, \dots, N-1\}$ , and  $\partial_N = \partial_t$ . The matrix of the metric  $g_0$  in these new coordinates  $(s, t)$  is  $d\Phi^T d\Phi$ . The matrix of  $d\Phi$  in the canonical basis of  $\mathbb{R}^N$  is

$$d\Phi = \begin{pmatrix} \partial_1 \Phi_1 & \dots & \partial_{N-1} \Phi_1 & \partial_N \Phi_1 \\ \vdots & \dots & \vdots & \vdots \\ \partial_1 \Phi_N & \dots & \partial_{N-1} \Phi_N & \partial_N \Phi_N \end{pmatrix} \in M_N(\mathbb{R}).$$

From (4.16), for all  $i \in \{1, \dots, N\}$ , for all  $j \in \{1, \dots, N-1\}$ , we have

$$\partial_j \Phi_i = \partial_j \phi_i - t d\nu_i(\partial_j \phi) = [\partial_j \phi]_i - t [d\nu(\partial_j \phi)]_i = [\partial_j \phi - t d\nu(\partial_j \phi)]_i = [(Id - t d\nu)(\partial_j \phi)]_i,$$

and

$$\partial_N \Phi_i = -\nu_i(\phi).$$

We deduce that

$$d\Phi = \begin{pmatrix} (Id - t d\nu)(\partial_1 \phi) & \dots & (Id - t d\nu)(\partial_{N-1} \phi) & -\nu(\phi) \end{pmatrix} \in M_N(\mathbb{R}),$$

where

$$\nu(\phi) = \begin{pmatrix} \nu_1(\phi) \\ \vdots \\ \nu_N(\phi) \end{pmatrix} \in M_{N \times 1}(\mathbb{R}) \quad \text{and} \quad (Id - t d\nu)(\partial_i \phi) = \begin{pmatrix} \partial_i \phi_1 - t d\nu_1(\partial_i \phi) \\ \vdots \\ \partial_i \phi_N - t d\nu_N(\partial_i \phi) \end{pmatrix} \in M_{N \times 1}(\mathbb{R}).$$

Let us calculate the symmetric matrix  $d\Phi^T d\Phi$ .

$$d\Phi^T d\Phi = \begin{pmatrix} [(Id - t d\nu)(\partial_1 \phi)]^T \\ \vdots \\ [(Id - t d\nu)(\partial_{N-1} \phi)]^T \\ [-\nu(\phi)]^T \end{pmatrix} \cdot \begin{pmatrix} (Id - t d\nu)(\partial_1 \phi) & \dots & (Id - t d\nu)(\partial_{N-1} \phi) & -\nu(\phi) \end{pmatrix},$$

we multiply the two matrices, we get

$$d\Phi^T d\Phi = \left( \begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{B}^T & [\nu(\phi)]^T \cdot \nu(\phi) \end{array} \right),$$

where

$$\mathcal{A} = (a_{ij}) = \left( [(Id - t d\nu)(\partial_i \phi)]^T (Id - t d\nu)(\partial_j \phi) \right) \in M_{N-1}(\mathbb{R}),$$



and

$$\mathcal{B} = \begin{pmatrix} -[(Id - t d\nu)(\partial_1 \phi)]^T \cdot \nu(\phi) \\ \vdots \\ -[(Id - t d\nu)(\partial_{N-1} \phi)]^T \cdot \nu(\phi) \end{pmatrix} \in M_{(N-1) \times 1}(\mathbb{R}).$$

Since  $\partial_i \phi(s)$  and  $\partial_i \nu(\phi(s))$  are tangent to  $\partial\Omega$ , we have, for all  $i = 1, \dots, N-1$

$$\begin{aligned} -[(Id - t d\nu)(\partial_i \phi)]^T \cdot \nu(\phi) &= \sum_{j=1}^N \left[ -\partial_i \phi_j \nu_j(\phi) + t \partial_i \nu_j(\phi) \nu_j(\phi) \right] \\ &= -\langle \partial_i \phi, \nu(\phi) \rangle + t \langle \partial_i \nu(\phi), \nu(\phi) \rangle \\ &= 0, \end{aligned}$$

then,

$$\mathcal{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We have also,

$$[\nu(\phi)]^T \cdot \nu(\phi) = \|\nu(\phi)\|^2 = 1.$$

Now we are going to make explain what is the matrix  $\mathcal{A}$ . Besides, for all  $i, j \in \{1, \dots, N-1\}$ , we have

$$\begin{aligned} a_{ij}(s, t) &= [(Id - t d\nu)(\partial_i \phi)]^T (Id - t d\nu)(\partial_j \phi) \\ &= [\partial_i \phi(s) - t \partial_i \nu(\phi(s))] \cdot [\partial_j \phi(s) - t \partial_j \nu(\phi(s))] \\ &= G_{ij}(s) - t[\partial_i \nu(\phi(s)) \cdot \partial_j \phi(s) + \partial_i \phi(s) \cdot \partial_j \nu(\phi(s))] + t^2 \partial_i \nu(\phi(s)) \cdot \partial_j \nu(\phi(s)), \end{aligned}$$

then, we obtain

$$a_{ij}(s, t) = G_{ij}(s) - 2K_{ij}(s)t + t^2 L_{ij}(s), \quad (4.17)$$

where  $K_{ij} = \partial_i \phi \cdot \partial_j \nu(\phi)$  and  $L_{ij} = \partial_i \nu(\phi) \cdot \partial_j \nu(\phi)$  are the coefficients of the second and third fundamental form of  $\partial\Omega$ .

We conclude that

$$d\Phi^T d\Phi = \left( \begin{array}{c|c} (G_{ij}(s) - 2K_{ij}(s)t + t^2 L_{ij}(s))_{i,j=1,\dots,n-1} & 0_{\mathbb{R}^{N-1}} \\ \hline 0_{\mathbb{R}^{N-1}}^T & 1 \end{array} \right).$$

Finally, we conclude that

$$\begin{aligned} g_0 &= dt \otimes dt + \sum_{1 \leq i, j \leq N-1} [G_{ij}(s) - 2tK_{ij}(s) + t^2 L_{ij}(s)] ds_i \otimes ds_j \\ &= dt \otimes dt + G - 2tK + t^2 L, \end{aligned}$$

where we denote by  $K$  and  $L$  the second and the third fundamental forms on  $\partial\Omega$ . In the coordinates  $y$  and with respect to the canonical basis, their matrices are given by

$$K = \sum_{1 \leq i, j \leq N-1} K_{ij} ds_i \otimes ds_j \quad \text{and} \quad L = \sum_{1 \leq i, j \leq N-1} L_{ij} ds_i \otimes ds_j.$$

We denote by  $g = (g_{ij})_{i,j}$  the matrix of the metric  $g_0$  in the  $(s, t)$  coordinates; the determinant of  $g$  is denoted by  $|g|$ . We have

$$\begin{aligned} |g|^{1/2} &= |d\Phi^T d\Phi|^{1/2} = [\det (G - 2tK + t^2L)]^{1/2} \\ &= [\det G(I - 2tG^{-1}K + t^2G^{-1}L)]^{1/2} \\ &= |G|^{1/2} [\det (I - 2tG^{-1}K + t^2G^{-1}L)]^{1/2}. \end{aligned}$$

By the differential calculus of the determinant and since  $t$  close to zero, we get

$$|g|^{1/2} = \left(1 - t\text{tr}(G^{-1}K) + t^2p(s, t)\right) |G|^{1/2}, \tag{4.18}$$

where  $p$  is a bounded function in the neighborhood  $\mathcal{V}_{x_0} \times ]0, \epsilon[$ .

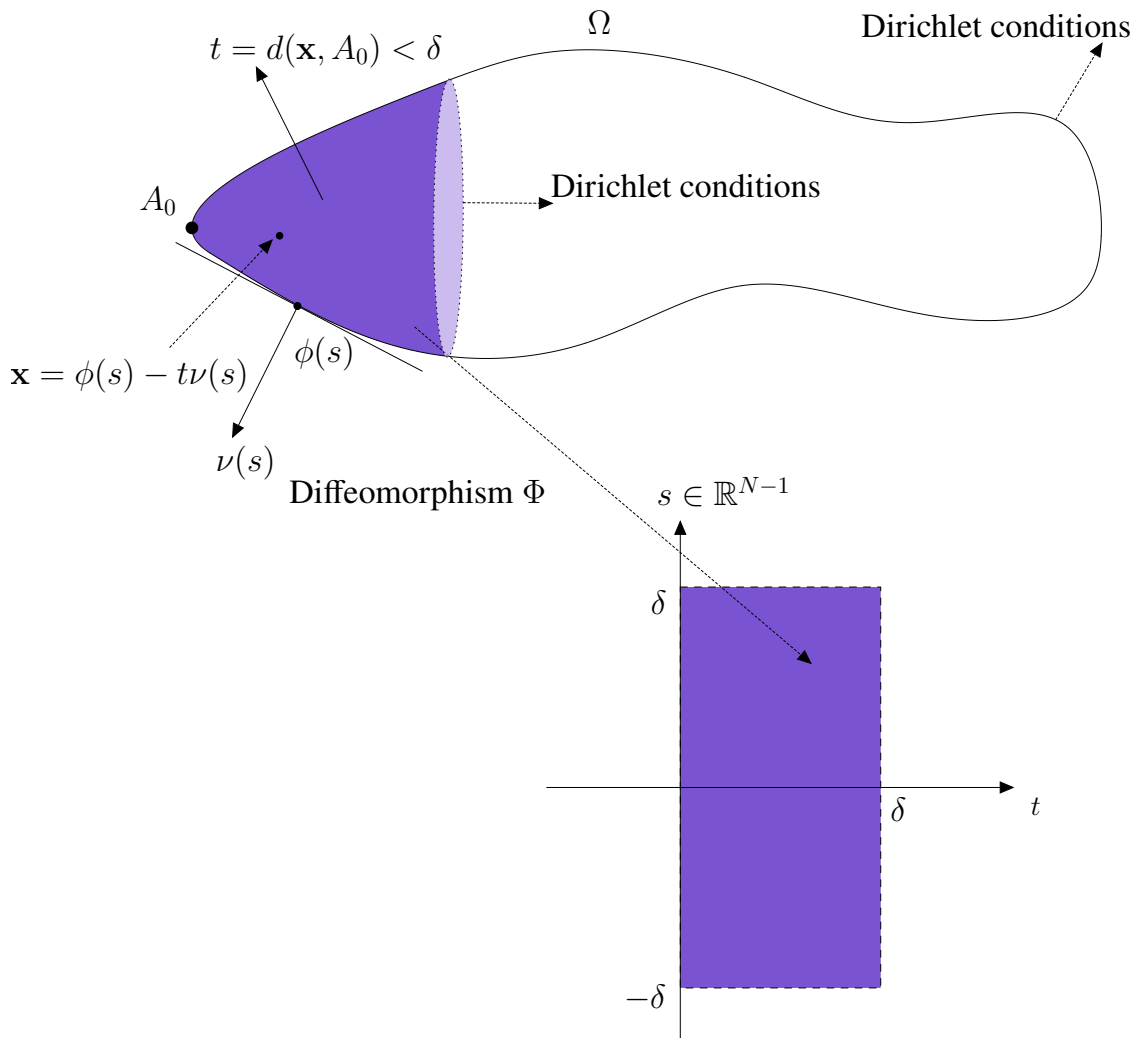


Figure 4.2 – The local tubular coordinates

### 4.3.2 The operator in boundary coordinates

For  $\varphi \in H_0^1(\Phi^{-1}(\Omega_\delta))$ , the quadratic form is written in the new coordinates  $(s, t)$  as follows

$$\begin{aligned} \mathcal{Q}_{h,\delta}(\varphi) &= h^2 \int_{\mathcal{V}_{A_0} \times ]0, \delta[} |\partial_t \tilde{\varphi}|^2 |g|^{1/2} ds_1 \dots ds_{N-1} dt \\ &\quad + h^2 \sum_{1 \leq i, j \leq N-1} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} g^{ij} \partial_{s_i} \tilde{\varphi} \overline{\partial_{s_j} \tilde{\varphi}} |g|^{1/2} ds_1 \dots ds_{N-1} dt \\ &\quad + \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \Phi_1(s, t) |\tilde{\varphi}|^2 |g|^{1/2} ds_1 \dots ds_{N-1} dt, \end{aligned}$$

where  $(g^{ij})$  are the coefficients of the inverse matrix of  $(g_{ij})$  and  $\tilde{\varphi} = \varphi \circ \Phi$ . The associated operator is

$$\widetilde{\mathcal{L}}_{h,\delta} = -h^2 |g|^{-1/2} \partial_t (|g|^{1/2} \partial_t) - h^2 \sum_{1 \leq i, j \leq N-1} |g|^{-1/2} \partial_{s_i} (g^{ij} |g|^{1/2} \partial_{s_j}) + \Phi_1(s, t),$$

acting on  $L^2(\mathcal{V}_{A_0} \times ]0, \delta[; |g|^{1/2} ds dt)$ .

The operator  $\mathcal{L}_{h,\delta}$  is unitarily equivalent to the Dirichlet realization of  $\widetilde{\mathcal{L}}_{h,\delta}$ . Let us describe  $\Phi_1$  near zero.

**Lemma 4.3.1.** *For  $(s, t) \in \mathcal{V}_{A_0} \times ]0, \delta[$ , we have*

$$\Phi_1(s, t) = x_{\min} + \frac{1}{2} \text{Hess}_{(0)} \Phi_1(s)(s) + t + \mathcal{O}(|s|^3 + t|s|^2).$$

*Proof.* From (4.16), we have

$$\Phi_1(s, t) = \phi_1(s) - t \nu_1(\phi(s)), \quad (4.19)$$

where  $\nu = (\nu_1, \dots, \nu_N)$  and  $\phi = (\phi_1, \dots, \phi_N)$ .

After a dilation and a translation of the  $s$  coordinates of the Section 4.3.1, we may assume that  $A_0 = \phi(0)$ .

Since  $x_1(A_0) = x_{\min}$  is the minimum, then

$$\partial_{s_j} \phi_1(0) = 0 \quad \forall j = 1, \dots, N-1.$$

By Taylor expansion near  $s = 0$ , we get

$$\begin{aligned} \phi_1(s) &= \phi_1(0) + \frac{1}{2} \text{Hess}_{(0)} \phi_1(s)(s) + \mathcal{O}(|s|^3) \\ &= x_{\min} + \frac{1}{2} \text{Hess}_{(0)} \phi_1(s)(s) + \mathcal{O}(|s|^3), \end{aligned}$$

and we deduce that

$$\nu_1(\phi(s)) = \nu_1(\phi(0)) + \mathcal{O}(|s|^2).$$

We notice that

$$\nu_1(\phi(0)) = \nu_1(A_0).$$

From (4.19), we have

$$\Phi_1(0, t) = x_1 = \phi_1(0) - t \nu_1(\phi(0)) = x_{\min} - t \nu_1(A_0),$$

or,

$$x_{\min} = \inf_{(x_1, x_2, \dots, x_N) \in \Omega} x_1,$$

then

$$\nu_1(A_0) \leq 0.$$

Since, for all  $j \in \{2, \dots, N\}$

$$\Phi_j(0, t) = \phi_j(0),$$

then, from (4.19)

$$\nu_j(A_0) = 0.$$

As  $\nu(A_0)$  is the unit outward pointing normal vector at the point  $A_0 \in \partial\Omega$ , then we get

$$\nu_1(A_0) = -1. \quad (4.20)$$

In particular,

$$\Phi_1(s, t) = x_{\min} + \frac{1}{2} \text{Hess}_{(0)} \phi_1(s)(s) + t + \mathcal{O}(|s|^3 + t|s|^2). \quad (4.21)$$

□

### 4.3.3 Agmon Estimates in Tubular Variables

The following proposition is a slight adaptation in coordinates  $(s, t)$  of Proposition 4.2.1. The estimates state that the eigenfunctions associated with the low-lying eigenvalues are localized near  $A_0$  at a scale  $h^{1/2}$  in the  $s_i$  direction for  $i \in \{1, \dots, N-1\}$ , and at a scale  $h^{2/3}$  in the  $t$  direction.

**Proposition 4.3.2.** *Let  $M > 0$ . There exist  $\epsilon, C, h_0 > 0$  such that, for all  $h \in (0, h_0)$ , and for all eigenfunctions  $\psi$  of  $\widetilde{\mathcal{L}}_{h,\delta}$  corresponding to eigenvalues  $\lambda$  with  $\lambda \leq x_{\min} + Mh^{2/3}$ , we have:*

$$\int_{\mathcal{V}_{A_0} \times ]0, \delta[} e^{\epsilon t^{3/2}/h} |\psi|^2 |g|^{1/2} \, ds dt \leq C \|\psi\|^2,$$

$$\int_{\mathcal{V}_{A_0} \times ]0, \delta[} e^{\epsilon t^{3/2}/h} |h \nabla_{s,t} \psi|^2 |g|^{1/2} \, ds dt \leq Ch^{2/3} \|\psi\|^2,$$

where  $\nabla_{s,t}$  is the differential in the tubular coordinates  $(s, t)$ , defined as follows

$$|\nabla_{s,t} \psi|^2 = |\partial_t \psi|^2 + \sum_{1 \leq i, j \leq N-1} g^{ij} \partial_{s_i} \psi \overline{\partial_{s_j} \psi}.$$

*Proof.* For all  $\psi \in \text{Dom}(\mathcal{L}_{h,\delta})$  and all bounded Lipschitz functions  $\varphi$ , we write the Agmon formula (see for example [Ray17, Proposition 4.7])

$$\int_{\Omega_\delta} h^2 \left| \nabla e^{\varphi/h} \psi \right|^2 + \left( x_1 - \lambda - |\nabla \varphi|^2 \right) \left| e^{\varphi/h} \psi \right|^2 \, dx = 0. \quad (4.22)$$

From (4.22), we use the quadratic form in tubular coordinates, and for

$$\varphi = \epsilon t^{3/2},$$

we get

$$\int_{\mathcal{V}_{A_0} \times ]0, \delta[} \left( -\frac{9}{4} \epsilon^2 t + \Phi_1 - x_{\min} - Mh^{2/3} \right) |e^{\varphi/h} \psi|^2 |g|^{1/2} \, ds dt \leq 0. \quad (4.23)$$

From (4.21), as  $\Omega$  strictly convex domain near  $s = 0$ , then  $\text{Hess}_{(0)}\phi_1$  is a positive definite matrix, we obtain

$$\Phi_1(s, t) \geq x_{\min} + t,$$

by (4.23), we deduce

$$\int_{\mathcal{V}_{A_0} \times ]0, \delta[} \left( \left[ 1 - \frac{9}{4}\varepsilon^2 \right] t - Mh^{2/3} \right) |e^{\phi/h}\psi|^2 |g|^{1/2} \, ds dt \leq 0. \quad (4.24)$$

We can directly use the strategy of Proposition 4.2.1 to get the proof.  $\square$

Therefore, the operator  $\widetilde{\mathcal{L}}_{h,\delta}$  can be replaced by

$$\widehat{\mathcal{L}}_{h,\delta} = -h^2 |g|^{-1/2} \partial_t (|g|^{1/2} \partial_t) - h^2 \sum_{1 \leq i, j \leq N-1} |g|^{-1/2} \partial_{s_i} (g^{ij} |g|^{1/2} \partial_{s_j}) + \Phi_1(s, t),$$

with Dirichlet boundary conditions, acting on  $L^2(\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[; |g|^{1/2} \, ds dt)$ , for some  $\eta \in (0, 1/3)$ .

Let  $\mu_n(h)$  be the associated eigenvalues of  $\widehat{\mathcal{L}}_{h,\delta}$ . The decay estimates of Proposition 4.3.2 are still satisfied by the eigenfunctions of  $\widehat{\mathcal{L}}_{h,\delta}$  with eigenvalues  $\lambda \leq x_{\min} + Mh^{\frac{2}{3}}$ . By using this exponential decay, there exist  $C, h_0 > 0$  such that, for all  $h \in (0, h_0)$ .

$$\mu_n(h) - \frac{1}{C} e^{-Ch^{-3\eta/2}} \leq \lambda_n(h, \delta) \leq \mu_n(h). \quad (4.25)$$

Thus, modulo an exponentially small error, the asymptotic analysis of  $\lambda_n(h)$  is reduced to that of  $\mu_n(h)$ .

By shrinking the spectral window, we can even get a localization with respect to the  $s$  variable, as stated in the next proposition.

**Proposition 4.3.3.** *Let  $M > 0$  and  $\eta \in (0, 1/3)$ . There exist  $\varepsilon, C, h_0 > 0$  such that, for all  $h \in (0, h_0)$ , and for all eigenfunctions  $\psi$  of  $\widehat{\mathcal{L}}_{h,\delta}$  corresponding to eigenvalues  $\lambda$  with  $\lambda \leq x_{\min} + h^{2/3}z_1 + Mh$ , we have:*

$$\int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} e^{\varepsilon|s|^2/h} |\psi|^2 |G|^{1/2} \, ds dt \leq C \|\psi\|^2,$$

and

$$\int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} e^{\varepsilon|s|^2/h} |h \nabla_s \psi|^2 |G|^{1/2} \, ds dt \leq Ch \|\psi\|^2,$$

where  $\nabla_s$  is the differential on  $\mathcal{V}_{A_0}$  seen through the metric  $G$ .

*Proof.* We let  $\varphi(s) = \varepsilon|s|^2/2$  and write the Agmon formula (see for example [Ray17, Proposition 4.7]):

$$\left\langle \widehat{\mathcal{L}}_{h,\delta} \psi, e^{2\varphi/h} \psi \right\rangle = \left\langle \widehat{\mathcal{L}}_{h,\delta} e^{\varphi/h} \psi, e^{\varphi/h} \psi \right\rangle - \|e^{\varphi/h} \nabla_{s,t} \varphi \psi\|^2. \quad (4.26)$$

Since  $\psi$  is the eigenfunction of  $\widehat{\mathcal{L}}_{h,\delta}$  corresponding to an eigenvalue  $\lambda$ , we get

$$\int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} \left[ |h \nabla_{s,t} e^{\varphi/h} \psi|^2 + \Phi_1(s, t) |e^{\varphi/h} \psi|^2 - (\lambda + |\nabla_{s,t} \varphi|^2) |e^{\varphi/h} \psi|^2 \right] |g|^{1/2} \, ds dt = 0,$$

with  $\nabla_{s,t}$  is the differential in the tubular coordinates  $(s, t)$ .

First, we drop the tangential derivative:

$$\int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} \left[ |h\partial_t e^{\varphi/h} \psi|^2 + \Phi_1(s, t) |e^{\varphi/h} \psi|^2 - (\lambda + |\nabla_{s,t} \varphi|^2) |e^{\varphi/h} \psi|^2 \right] |g|^{1/2} \, ds dt \leq 0.$$

We observe from (4.21), as  $s = 0$  is a non degenerate minimum, then  $\text{Hess}_{(0)} \phi_1$  is a positive definite matrix, then there exists  $C^* > 0$ , such that

$$\text{Hess}_{(0)} \phi_1(s)(s) \geq C^* |s|^2,$$

from (4.21), we obtain

$$\Phi_1(s, t) \geq x_{\min} + t + C^* |s|^2.$$

Introducing the last inequality in the above integral, we get:

$$\begin{aligned} & \int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} \left[ |h\partial_t e^{\varphi/h} \psi|^2 + t |e^{\varphi/h} \psi|^2 \right] |g|^{1/2} \, ds dt \\ & + \int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} \left[ (-\lambda + x_{\min} + C^* |s|^2 - |\nabla_{s,t} \varphi|^2) |e^{\varphi/h} \psi|^2 \right] |g|^{1/2} \, ds dt \leq 0. \end{aligned}$$

From (4.18), on  $\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[$ , for sufficiently small  $h$ , there exists  $C > 0$  such that

$$|g|^{1/2} \geq (1 - Ch^{\frac{2}{3}-\eta}) |G|^{1/2}.$$

By using this in the above integrals, we have that

$$\begin{aligned} & (1 - Ch^{\frac{2}{3}-\eta}) \int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} \left[ |h\partial_t e^{\varphi/h} \psi|^2 + t |e^{\varphi/h} \psi|^2 - h^{\frac{2}{3}} z_1 |e^{\varphi/h} \psi|^2 \right] |G|^{1/2} \, ds dt \\ & + \int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} (-\lambda + x_{\min} + C^* |s|^2 + h^{\frac{2}{3}} z_1 - |\nabla_{s,t} \varphi|^2 - Ch^{\frac{4}{3}-\eta}) |e^{\varphi/h} \psi|^2 |g|^{1/2} \, ds dt \leq 0. \end{aligned}$$

By using the min-max principle, we notice that

$$\int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} \left[ |h\partial_t e^{\varphi/h} \psi|^2 + t |e^{\varphi/h} \psi|^2 \right] |G|^{1/2} \, ds dt \geq h^{\frac{2}{3}} z_1 \int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} |e^{\varphi/h} \psi|^2 |G|^{1/2} \, ds dt.$$

Therefore,

$$\int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} (-\lambda + x_{\min} + C^* |s|^2 + h^{\frac{2}{3}} z_1 - |\nabla_{s,t} \varphi|^2 - Ch^{\frac{4}{3}-\eta}) |e^{\varphi/h} \psi|^2 |g|^{1/2} \, ds dt \leq 0.$$

So that, using the assumption of the location of  $\lambda$  and for  $h$  small enough, then:

$$\int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} (-2Mh + C^* |s|^2 - |\nabla_{s,t} \varphi|^2) |e^{\varphi/h} \psi|^2 |G|^{1/2} \, ds dt \leq 0.$$

We have

$$|\nabla_{s,t} \varphi|^2 = |\partial_t \varphi|^2 + \sum_{1 \leq i, j \leq N-1} g^{ij} \partial_{s_i} \varphi \overline{\partial_{s_j} \varphi} = \varepsilon^2 \sum_{1 \leq i, j \leq N-1} g^{ij} s_i s_j,$$

where  $(g^{ij})$  is the inverse matrix of  $(g_{ij})$ . There exists  $c > 0$ , such that

$$\varepsilon^2 \sum_{1 \leq i, j \leq N-1} g^{ij} s_i s_j \leq C\varepsilon^2 |s|^2,$$

It follows:

$$\int_{\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[} (-2Mh + (C^* - C\varepsilon^2)|s|^2) |e^{\varphi/h}\psi|^2 |G|^{1/2} ds dt \leq 0.$$

For  $\varepsilon$  small enough, the conclusion follows as in the proof of Proposition 4.2.1.  $\square$

This shows that the eigenfunctions of "low energy" are localized near  $A_0$  at a scale  $h^{1/2}$  in the  $s_i$  direction for  $i \in \{1, \dots, N-1\}$ .

## 4.4 Construction of quasimodes

This section aims to explain how the following proposition follows. For the proof, we will follow the same strategy as in [HK17].

**Proposition 4.4.1.** *For any positive  $n$ , there exists a sequence  $(\zeta_{j,n})_{j \geq 0}$  such that, for any positive  $M \in \mathbb{N}$ , we have, as  $h \rightarrow 0$*

$$\|\widehat{\mathcal{L}}_{h,\delta} \Phi_{n,M} - h^{2/3} \lambda_{n,M} \Phi_{n,M}\| \leq Ch^{\frac{4}{3} + \frac{M+1}{6}} \|\Phi_{n,M}\|,$$

with  $z_1, \beta_{1,n}$  are defined in Theorem 4.1.1,  $\lambda_{n,M}$  is defined in (4.44) and  $\Phi_{n,M}$  a family of quasimodes introduced in (4.49).

### 4.4.1 Rescaling operator

We recall that

$$\widehat{\mathcal{L}}_{h,\delta} = -h^2 |g|^{-1/2} \partial_t (|g|^{1/2} \partial_t) - h^2 \sum_{1 \leq i, j \leq N-1} |g|^{-1/2} \partial_{s_i} (g^{ij} |g|^{1/2} \partial_{s_j}) + \Phi_1(s, t),$$

with Dirichlet boundary conditions, acting on  $L^2(\mathcal{V}_{A_0} \times ]0, h^{\frac{2}{3}-\eta}[; |g|^{1/2} ds dt)$ , also we recall that

$$\Phi_1(s, t) = x_{\min} + \frac{1}{2} \text{Hess}_{(0)} \phi_1(s)(s) + t + \mathcal{O}(|s|^3 + t|s|^2).$$

We know that, the eigenfunctions are localized near the boundary at the scale  $h^{\frac{1}{2}}$  in the  $s_i$  direction and  $h^{2/3}$  in the  $t$  direction (cf. Proposition 4.3.2 and 4.3.3). This suggests to use the rescaling:

$$(s, t) = (h^{\frac{1}{2}} \sigma, h^{\frac{2}{3}} \tau).$$

Then, we have

$$\Phi_1(h^{\frac{1}{2}} \sigma, h^{\frac{2}{3}} \tau) = x_{\min} + \frac{h}{2} \text{Hess}_{(0)} \phi_1(\sigma)(\sigma) + h^{2/3} \tau + \mathcal{O}(h^{3/2} |\sigma|^3 + h^{5/3} \tau |\sigma|^2).$$

This change of variables transforms the above expression of  $\widehat{\mathcal{L}}_{h,\delta}$  into

$$\begin{aligned} \widehat{\mathcal{H}}_{h,\delta} &= -h^{2/3} \left[ |g|^{-1/2} \partial_\tau (|g|^{1/2} \partial_\tau) - \tau \right] \\ &+ h \left[ - \sum_{1 \leq i, j \leq N-1} |g|^{-1/2} \partial_{\sigma_i} (g^{ij} |g|^{1/2} \partial_{\sigma_j}) + \frac{1}{2} \text{Hess}_{(0)} \phi_1(\sigma)(\sigma) \right] + x_{\min} + h^{4/3} h^{1/6} q_{1,h}(\sigma, \tau), \end{aligned}$$

where the function  $q_{1,h}$  satisfies

$$|q_{1,h}(\sigma, \tau)| \leq C(|\sigma|^3 + h^{1/6}\tau|\sigma|^2).$$

#### 4.4.2 Formal expansion

We have the formal expansion of the operator  $\widehat{\mathcal{H}}_{h,\delta}$  (for the details see Lemma 4.4.2)

$$h^{-2/3}\widehat{\mathcal{H}}_{h,\delta} = x_{\min}h^{-2/3} + \mathcal{L}_0 + h^{1/3}\mathcal{L}_1 + h^{2/3}Q_h, \quad (4.27)$$

where

$$\begin{aligned} \mathcal{L}_0 &= -\partial_\tau^2 + \tau, \\ \mathcal{L}_1 &= -\Delta^{G_0} + \frac{1}{2}\text{Hess}_{(0)}\phi_1(\sigma)(\sigma), \quad G_0 = G(0), \\ Q_h &= h^{1/6}q_{1,h}(\sigma, \tau) + q_{2,h}(\sigma, \tau)\partial_\tau + h^{1/6}\sum_{1 \leq i, j \leq N-1} \partial_{\sigma_i}(q_{3,h}(\sigma, \tau)\partial_{\sigma_j}), \end{aligned} \quad (4.28)$$

where the Laplace-Beltrami operator on  $\partial\Omega$  in the neighborhood of  $A_0$  is

$$\Delta^{G_0} = \sum_{1 \leq i, j \leq N-1} |G_0|^{-1/2}\partial_{\sigma_i}(G_0^{ij}|G_0|^{1/2}\partial_{\sigma_j}), \quad (4.29)$$

with the functions  $q_{j,h}$ , for  $j \in \{2, 3\}$  satisfy

$$|q_{2,h}(\sigma, \tau)| \leq C(\tau + |\sigma|) \quad \text{and} \quad |q_{3,h}(\sigma, \tau)| \leq C(h^{1/6}\tau + |\sigma|).$$

**Lemma 4.4.2.** For  $0 < h < \frac{1}{2}$ ,  $0 \leq \tau \leq h^{-\eta}$  and  $\sigma_i = \mathcal{O}(h^{-1/2})$ , we have the following identities:

$$|g|^{-1/2}\partial_\tau(|g|^{1/2}\partial_\tau) = \partial_\tau^2 + (\partial_\tau|g|^{1/2})|g|^{-1/2}\partial_\tau = \partial_\tau^2 + h^{2/3}q_{2,h}(\sigma, \tau)\partial_\tau, \quad (4.30)$$

and

$$|g|^{-1/2}\partial_{\sigma_i}(g^{ij}|g|^{1/2}\partial_{\sigma_j}) = |G_0|^{-1/2}\partial_{\sigma_i}(G_0^{ij}|G_0|^{1/2}\partial_{\sigma_j}) + h^{1/2}\partial_{\sigma_i}(q_{3,h}(\sigma, \tau)\partial_{\sigma_j}), \quad (4.31)$$

where the functions  $q_{j,h}$ , for  $j \in \{2, 3\}$  satisfy

$$|q_{2,h}(\sigma, \tau)| \leq C(\tau + |\sigma|) \quad \text{and} \quad |q_{3,h}(\sigma, \tau)| \leq C(h^{1/6}\tau + |\sigma|).$$

*Proof.* We have the following asymptotic expansions:

$$\begin{aligned} M(\sigma, \tau) &:= |g|^{1/2}(\sigma, \tau) \\ &= \left(1 - h^{\frac{2}{3}}\tau\text{tr}(G^{-1}K)(h^{\frac{1}{2}}\sigma) + h^{\frac{4}{3}}\tau^2p(h^{\frac{1}{2}}\sigma, h^{\frac{2}{3}}\tau)\right) |G|^{1/2}(h^{\frac{1}{2}}\sigma) \\ &= M(0, 0) + \tau\frac{\partial M}{\partial\tau}(0, 0) + \sum_{i=1}^{N-1} \sigma_i\frac{\partial M}{\partial\sigma_i}(0, 0) + hF_1(\sigma, \tau), \end{aligned}$$

where

$$— M(0, 0) = |G|^{1/2}(0),$$



$$\begin{aligned} - \frac{\partial M}{\partial \tau}(0, 0) &= -h^{2/3} \text{tr}(G^{-1}K)(0) |G|^{1/2}(0), \\ - \frac{\partial M}{\partial \sigma_i}(0, 0) &= h^{1/2} \frac{\partial |G|^{1/2}}{\partial \sigma_i}(0). \end{aligned}$$

Then, we have

$$|g|^{1/2}(\sigma, \tau) = |G|^{1/2}(0) - h^{2/3} \tau \text{tr}(G^{-1}K)(0) |G|^{1/2}(0) + h^{1/2} \sum_{1 \leq i \leq N-1} \sigma_i \frac{\partial |G|^{1/2}}{\partial \sigma_i}(0) + hF_1(\sigma, \tau).$$

From the previous inequality, we have

$$|g|^{-1/2}(\sigma, \tau) = |G|^{-1/2}(0) + h^{1/2} F_2(\sigma, \tau). \quad (4.32)$$

This gives us the following identities:

$$(\partial_\tau |g|^{1/2}) |g|^{-1/2}(\sigma, \tau) = -h^{2/3} F_3(\sigma, \tau). \quad (4.33)$$

From Section 4.3.1, we have

$$g_{ij}(\sigma, \tau) = G_{ij}(h^{1/2}\sigma) - 2h^{2/3} \tau K_{ij}(h^{1/2}\sigma) + h^{4/3} \tau^2 L_{ij}(h^{1/2}\sigma) = G_{ij}(0) + h^{1/2} F_4(\sigma, \tau), \quad (4.34)$$

and thus,

$$g^{ij}(\sigma, \tau) = G^{ij}(0) + h^{1/2} F_5(\sigma, \tau), \quad (4.35)$$

where, for  $0 < h < \frac{1}{2}$ ,  $0 \leq \tau \leq h^{-\eta}$  and  $\sigma_i = \mathcal{O}(h^{-1/2})$ , the functions  $F_j$ ,  $j = 1, \dots, 5$ , satisfy

$$F_1(\sigma, \tau) \leq C(h^{1/3} \tau^2 + |\sigma|^2), \quad F_3(\sigma, \tau) \leq C(\tau + |\sigma|) \quad \text{and}$$

$$F_j(\sigma, \tau) \leq C(h^{1/6} \tau + |\sigma|), \quad \forall j = 2, 4, 5.$$

This gives us the following identities:

$$|g|^{-1/2} \partial_\tau (|g|^{1/2} \partial_\tau) = \partial_\tau^2 + (\partial_\tau |g|^{1/2}) |g|^{-1/2} \partial_\tau = \partial_\tau^2 + h^{2/3} q_{2,h}(\sigma, \tau) \partial_\tau, \quad (4.36)$$

and

$$|g|^{-1/2} \partial_{\sigma_i} (g^{ij} |g|^{1/2} \partial_{\sigma_j}) = |G|^{-1/2}(0) \partial_{\sigma_i} (G^{ij}(0) |G|^{1/2}(0) \partial_{\sigma_j}) + h^{1/2} \partial_{\sigma_i} (q_{3,h}(\sigma, \tau) \partial_{\sigma_j}), \quad (4.37)$$

where the functions  $q_{i,h}$ , for  $i \in \{2, 3\}$  satisfy for  $0 < h < \frac{1}{2}$ ,  $0 \leq \tau \leq h^{-\eta}$  and  $|\sigma| = \mathcal{O}(h^{-1/2})$

$$|q_{2,h}(\sigma, \tau)| \leq C(\tau + |\sigma|) \quad \text{and} \quad |q_{3,h}(\sigma, \tau)| \leq C(h^{1/6} \tau + |\sigma|).$$

□

### 4.4.3 Construction of trial states

In this section, we construct trial states that will give together with the lower bounds, an accurate expansion of the eigenvalues of the operator  $\widehat{\mathcal{L}}_{h,\delta}$  valid to any order. The key to the construction is to expand the operator  $Q_h$  in powers of  $h^{1/6}$ .

The functions  $q_{1,h}$ ,  $q_{2,h}$  and  $q_{3,h}$  in (4.28) admit Taylor expansions that can be rearranged in the form:

$$\begin{aligned} h^{1/6}q_{1,h}(\sigma, \tau) &= \sum_{k=0}^{\infty} q_{1,k}(\sigma, \tau)h^{k/6}, \\ q_{2,h}(\sigma, \tau) &= \sum_{k=0}^{\infty} q_{2,k}(\sigma, \tau)h^{k/6}, \\ h^{1/6}q_{3,h}(\sigma, \tau) &= \sum_{k=0}^{\infty} q_{3,k}(\sigma, \tau)h^{k/6}, \end{aligned} \quad (4.38)$$

with the functions  $q_{1,k}(\sigma, \tau)$ ,  $q_{2,k}(\sigma, \tau)$  and  $q_{3,k}(\sigma, \tau)$  being polynomial functions of  $(\sigma, \tau)$ . The operator  $Q_h$  in (4.28) admits the formal expansion

$$Q_h = \sum_{k=0}^{\infty} Q_k h^{k/6}, \quad (4.39)$$

where

$$Q_k = q_{1,k}(\sigma, \tau) + q_{2,k}(\sigma, \tau)\partial_\tau + \sum_{1 \leq i, j \leq N-1} \partial_{\sigma_i}(q_{3,k}(\sigma, \tau)\partial_{\sigma_j}). \quad (4.40)$$

Furthermore, for every  $k \geq 0$  and  $f \in \mathcal{S}(\mathbb{R}^{N-1} \times \overline{\mathbb{R}_+})$ ,  $Q_k f \in \mathcal{S}(\mathbb{R}^{N-1} \times \overline{\mathbb{R}_+})$ .

For all  $M \geq 0$ , we introduce two operators

$$\widehat{\mathcal{H}}_{h,M} = x_{\min}h^{-2/3} + \mathcal{L}_0 + h^{1/3}\mathcal{L}_1 + h^{2/3}F_M \quad \text{and} \quad F_M = \sum_{k=0}^M Q_k h^{k/6}. \quad (4.41)$$

Thanks to (4.27) and (4.28), for every  $f \in \mathcal{S}(\mathbb{R}^{N-1} \times \overline{\mathbb{R}_+})$ , there exists a constant  $C > 0$  such that, for all  $h \in (0, 1)$ ,

$$\|h^{-2/3}\widehat{\mathcal{H}}_{h,\delta}f - \widehat{\mathcal{H}}_{h,M}f\|_{L^2(\mathbb{R}^{N-1} \times \overline{\mathbb{R}_+})} \leq Ch^{\frac{2}{3} + \frac{M+1}{6}}. \quad (4.42)$$

Let  $n \in \mathbb{N}$ . We will construct a sequence of real numbers  $(\zeta_{n,j})_{j=0}^{\infty}$  and two sequences of real-valued Schwartz functions  $(v_{n,j})_{j=1}^{\infty} \subset \mathcal{S}(\mathbb{R}^{N-1})$ ,  $(g_{n,j})_{j=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^{N-1} \times \overline{\mathbb{R}_+})$  such that

$$\forall j, \quad g_{n,j}|_{\tau=0} = 0,$$

and for all  $M \in \mathbb{N}$ , the function

$$\Psi_{n,M}(\sigma, \tau) = u_0(\tau)f_n(\sigma) + \sum_{j=1}^{M+1} h^{\frac{j+1}{6}}u_0(\tau)v_{n,j}(\sigma) + h^{2/3}\sum_{j=0}^M h^{j/6}g_{n,j}(\sigma, \tau),$$

is in the Schwartz space  $\mathcal{S}(\mathbb{R}^{N-1} \times \overline{\mathbb{R}_+})$  and satisfies (for all  $h \in (0, h_{n,M})$ )

$$\|\widehat{\mathcal{H}}_{h,M}\Psi_{n,M} - \lambda_{n,M}\Psi_{n,M}\|_{L^2(\mathbb{R}^{N-1} \times \overline{\mathbb{R}_+})} \leq C_{n,M}h^{\frac{2}{3} + \frac{M+1}{6}}, \quad (4.43)$$

where

$$\lambda_{n,M} = x_{\min}h^{-2/3} + \beta_{0,n} + h^{1/3}\beta_{1,n} + h^{2/3}\sum_{j=0}^M h^{j/6}\zeta_{j,n}, \quad (4.44)$$

$C_{n,M} > 0$  and  $h_{n,M}$  are two constants determined by the values of  $n$  and  $M$  solely, and  $\beta_{0,n}, \beta_{1,n}, u_0$  and  $f_n$  to be determined.

**For  $M = 0$ :** We want to find a real numbers  $\zeta_{0,n}, \beta_{0,n}, \beta_{1,n}$  and two functions  $v_{n,1}(\sigma), g_{n,0}(\sigma, \tau), u_0(\tau), f_n(\sigma)$  such that

$$\Psi_{n,0}(\sigma, \tau) = u_0(\tau)f_n(\sigma) + h^{1/3}u_0(\tau)v_{n,1}(\sigma) + h^{2/3}g_{n,0}(\sigma, \tau),$$

and

$$\lambda_{n,0} = x_{\min}h^{-2/3} + \beta_{0,n} + h^{1/3}\beta_{1,n} + h^{2/3}\zeta_{0,n}.$$

In order that (4.43) is satisfied for  $M = 0$ , it is sufficient to select  $\zeta_{0,n}, g_{n,0}(\sigma, \tau), u_0(\tau), f_n(\sigma)$  and  $v_{n,1}(\sigma)$  as follows:

$$\begin{cases} (\mathcal{L}_0 - \beta_{0,n})u_0(\tau)f_n(\sigma) = 0, \\ (\mathcal{L}_0 - \beta_{0,n})u_0(\tau)v_{n,1}(\sigma) + (\mathcal{L}_1 - \beta_{1,n})u_0(\tau)f_n(\sigma) = 0, \\ (\mathcal{L}_0 - \beta_{0,n})g_{n,0}(\sigma, \tau) + (\mathcal{L}_1 - \beta_{1,n})u_0(\tau)v_{n,1}(\sigma) + (F_0 - \zeta_{0,n})u_0(\tau)f_n(\sigma) = 0. \end{cases}$$

• For the first equation, the natural choice is then to choose  $u_0$  the eigenfunction of  $\mathcal{L}_0$ , with  $\beta_{0,n}$  is the corresponding eigenvalue, therefore

$$u_0(\tau) = Ai(\tau - z_1) \quad \text{and} \quad \beta_{0,n} = z_1, \quad (4.45)$$

where  $Ai$  is the  $L^2(\mathbb{R}^+)$ -normalized Airy function and  $-z_1$  is its first negative zero.

• The second equation is equivalent to

$$(\mathcal{L}_1 - \beta_{1,n})u_0(\tau)f_n(\sigma) = 0,$$

then, we choose  $\beta_{1,n}$  is the  $n$ -th eigenvalue on  $L^2(\mathbb{R}^{N-1}, |G_0|^{1/2}d\sigma)$  of  $\mathcal{L}_1$  determined in Section 4.7.1 associated with eigenfunction  $f_n$ .

• For the third equation, we can select  $\zeta_{0,n}$  such that

$$(F_0 - \zeta_{0,n})u_0(\tau)f_n(\sigma) \perp u_0(\tau)f_n(\sigma) \quad \text{in } L^2(\mathbb{R}^{N-1} \times \mathbb{R}_+).$$

This is given by the formula

$$\zeta_{0,n} = \langle F_0(u_0 \otimes f_n), u_0 \otimes f_n \rangle_{L^2(\mathbb{R}^{N-1} \times \mathbb{R}_+)}. \quad (4.46)$$

Consequently,

$$h_1(\sigma) := - \int_0^\infty u_0(\tau) (F_0 - \zeta_{0,n}) u_0(\tau) f_n(\sigma) d\tau \perp f_n(\sigma) \quad \text{in } L^2(\mathbb{R}^{N-1}).$$

Since  $\mathcal{L}_1 - \beta_{1,n}$  can be inverted on the orthogonal complement of the  $n$ -th eigenfunction  $f_n(\sigma)$ , we may select  $v_{n,1}(\sigma)$  to be

$$v_{n,1}(\sigma) = (\mathcal{L}_1 - \beta_{1,n})^{-1} h_1(\sigma).$$

As a consequence of the choice of  $\zeta_{0,n}$  and  $v_{n,1}(\sigma)$ , we get that, for all  $\sigma$ ,

$$\begin{aligned} w_1(\sigma, \cdot) &:= -u_0(\cdot) (\mathcal{L}_1 - \beta_{1,n}) v_{n,1}(\sigma) - (F_0 - \zeta_{0,n}) u_0(\cdot) f_n(\sigma) \\ &= -u_0(\cdot) h_1(\sigma) - (F_0 - \zeta_{0,n}) u_0(\cdot) f_n(\sigma) \perp u_0(\cdot) \quad \text{in } L^2(\mathbb{R}_+). \end{aligned}$$

The operator  $\mathcal{L}_0 - \beta_{0,n}$  can be inverted on the orthogonal complement of  $\{u_0(\tau)\}$ , and the inverse is an

operator in  $L^2(\mathbb{R}_+)$  which sends  $\mathcal{S}(\overline{\mathbb{R}_+})$  into itself. The proof of this is standard and follows the same argument as in [FH06, Lemma A.5]. In that way, we may select  $g_{0,n}(\sigma, \tau)$  to be

$$g_{n,0}(\sigma, \cdot) = (\mathcal{L}_0 - \beta_{0,n})^{-1}(w_1(\sigma, \cdot)).$$

**The iteration process.** Here we suppose that we have selected  $(\zeta_{0,n}, \dots, \zeta_{M,n}; \Psi_{n,0}, \dots, \Psi_{n,M})$  such that the following induction hypothesis is satisfied

$$(\widehat{\mathcal{H}}_{h,M} - \lambda_{n,M})\Psi_{n,M} = h^{\frac{2}{3} + \frac{M+1}{6}} \sum_{j=0}^{N_M} h^{j/6} r_{M,j}(\sigma, \tau), \quad (4.47)$$

where  $r_{M,j}$  are Schwartz functions. We have to select  $\zeta_{M+1,n}$  and two functions  $v_{n,M+1}(\sigma)$ ,  $g_{n,M+1}(\sigma, \tau)$  such that

$$\begin{aligned} \Psi_{n,M+1} &= \Psi_{n,M} + h^{\frac{M+3}{6}} u_0(\tau) v_{n,M+2}(\sigma) + h^{\frac{2}{3} + \frac{M+1}{6}} g_{n,M+1}(\sigma, \tau) \quad \text{and} \\ \lambda_{n,M+1} &= \lambda_{n,M} + h^{\frac{2}{3} + \frac{M+1}{6}} \zeta_{M+1,n} \end{aligned}$$

satisfy (4.43) and (4.47) for  $M$  replaced by  $M+1$  (and some collection of Schwartz functions  $r_{M+1,j}(\sigma, \tau)$ ). Notice that we have

$$\widehat{\mathcal{H}}_{h,M+1} = \widehat{\mathcal{H}}_{h,M} + h^{\frac{2}{3} + \frac{M+1}{6}} Q_{M+1},$$

and

$$\begin{aligned} &(\widehat{\mathcal{H}}_{h,M+1} - \lambda_{n,M+1})\Psi_{n,M+1} \\ &= (\widehat{\mathcal{H}}_{h,M} - \lambda_{n,M})\Psi_{n,M} \\ &+ h^{\frac{2}{3} + \frac{M+1}{6}} [(\mathcal{L}_0 - \beta_{0,n})g_{n,M+1}(\sigma, \tau) + (\mathcal{L}_1 - \beta_{1,n})u_0(\tau)v_{n,M+2}(\sigma) + (Q_{M+1} - \zeta_{M+1,n})u_0(\tau)f_n(\sigma)] \\ &+ h^{\frac{2}{3} + \frac{M+1}{6}} [(Q_{M+1} - \zeta_{M+1,n})(\Psi_{n,M} - u_0(\tau)f_n(\sigma))] \\ &+ h^{\frac{2}{3} + \frac{M+1}{6}} [(h^{1/3}\mathcal{L}_1 + h^{2/3}F_{M+1} - (\lambda_{n,M} - \beta_{0,n}))g_{n,M+1}] \\ &+ h^{\frac{2}{3} + \frac{M+1}{6}} [h^{1/3}F_{M+1} - (\lambda_{n,M+1} - \beta_{0,n} - \beta_{1,n}h^{1/3})u_0(\tau)v_{n,M+2}(\sigma)]. \end{aligned} \quad (4.48)$$

It is sufficient to select  $(\zeta_{M+1,n}, v_{n,M+2}, g_{n,M+1})$  such that

$$\begin{aligned} &(\mathcal{L}_0 - \beta_{0,n})g_{n,M+1}(\sigma, \tau) + (\mathcal{L}_1 - \beta_{1,n})u_0(\tau)v_{n,M+2}(\sigma) \\ &+ (Q_{M+1} - \zeta_{M+1,n})u_0(\tau)f_n(\sigma) + r_{M,0}(\sigma, \tau) = 0. \end{aligned}$$

We select  $\zeta_{M+1,n}$  such that

$$(Q_{M+1} - \zeta_{M+1,n})u_0(\tau)f_n(\sigma) + r_{M,0}(\sigma, \tau) \perp u_0(\tau)f_n(\sigma) \quad \text{in } L^2(\mathbb{R}^{N-1} \times \mathbb{R}_+),$$

i.e.

$$\zeta_{M+1,n} = \langle Q_{M+1}(u_0 \otimes f_n), u_0 \otimes f_n \rangle_{L^2(\mathbb{R}^{N-1} \times \mathbb{R}_+)} + \langle r_{M,0}, u_0 \otimes f_n \rangle_{L^2(\mathbb{R}^{N-1} \times \mathbb{R}_+)}.$$

Consequently,

$$h_{M+2}(\sigma) := - \int_0^\infty u_0(\tau) ((Q_{M+1} - \zeta_{M+1,n})u_0(\tau)f_n(\sigma) + r_{M,0}(\sigma, \tau)) d\tau \perp f_n(\sigma)$$

in  $L^2(\mathbb{R}^{N-1})$ . We can select  $v_{n,M+2}$  as follows:

$$v_{n,M+2}(\sigma) = (\mathcal{L}_1 - \beta_{1,n})^{-1} h_{M+2}(\sigma).$$

As a consequence of the choice of  $\zeta_{M+1,n}$  and  $v_{n,M+2}(\sigma)$ , we get that, for all  $\sigma$

$$\begin{aligned} w_{M+2}(\sigma, \cdot) &:= -u_0(\cdot) (\mathcal{L}_1 - \beta_{1,n}) v_{n,M+2}(\sigma) \\ &\quad - (Q_{M+1} - \zeta_{M+1,n}) u_0(\tau) f_n(\sigma) - r_{M,0}(\sigma, \tau) \perp u_0(\cdot) \quad \text{in } L^2(\mathbb{R}_+). \end{aligned}$$

Finally, we may select  $g_{n,M+1}(\sigma, \tau)$  as follows:

$$g_{n,M+1}(\sigma, \cdot) = (\mathcal{L}_0 - \beta_{0,n})^{-1} (w_{M+2}(\sigma, \cdot)).$$

#### 4.4.4 End of the proof

Consider a smooth function  $\chi_h$  with compact support equal to 1 near  $(s, t) = (0, 0)$  on a scale of order  $h^{1/2-\theta}$  in the  $s$  direction, and on a scale of order  $h^{2/3-\theta}$  in the  $t$  direction, for some  $\theta \in (0, \eta)$ . Let us introduce the following family of test functions

$$\Phi_{n,M}(s, t) = h^{\frac{-1-(N-1)}{3}-\frac{(N-1)}{4}} \underbrace{\chi\left(\frac{t}{h^{2/3-\theta}}\right) \chi\left(\frac{s}{h^{1/2-\theta}}\right)}_{\chi_h(s,t)} \Psi_{n,M}(h^{-1/2}s, h^{-2/3}t). \quad (4.49)$$

We notice that the support of  $\chi - 1$  and the supports of the derivative of  $\chi$  are located in a region where  $h^{-1/2}|s| \geq Ch^{-\theta}$  and  $h^{-2/3}t \geq Ch^{-\theta}$ , thus all integrals which contain such derivatives will be of order  $h^{-\infty}$  due to the exponential localization of  $\Psi_{n,M}$ .

Thanks to (4.42) and (4.43), we may write

$$\|\widehat{\mathcal{L}}_{h,\delta} \Phi_{n,M} - h^{2/3} \lambda_{n,M} \Phi_{n,M}\| \leq Ch^{\frac{4}{3} + \frac{M+1}{6}} \|\Phi_{n,M}\|.$$

### 4.5 Lower bound

The following proposition provides a lower bound of the eigenvalue  $\mu_n(h)$ , as  $h \rightarrow 0$ .

**Proposition 4.5.1.** *For any positive  $n$ , the eigenvalue  $\mu_n(h)$  has, as  $h \rightarrow 0$ , the lower bound*

$$\mu_n(h) \geq x_{\min} + z_1 h^{2/3} + \beta_{1,n} h + \mathcal{O}(h^{4/3}),$$

where  $\beta_{1,n}$  is defined in Theorem 4.1.1.

*Proof.* Let  $k \geq 1$ , and consider a family of eigenfunctions  $(\psi_{n,h})_{n=1,\dots,k}$  of  $\widehat{\mathcal{L}}_{h,\delta}$  associated with the eigenvalues  $(\mu_n(h))_{n=1,\dots,k}$ . Let  $\mathcal{E}_k(h)$  be the vector subspace of  $\text{Dom}(\widehat{\mathcal{L}}_{h,\delta})$  spanned by the family  $(\psi_{n,h})_{n=1,\dots,k}$ . For all  $\psi \in \mathcal{E}_k(h)$ ,  $\psi$  satisfies the same decay estimates as in Propositions 4.3.2 and 4.3.3. Let  $\psi \in \mathcal{E}_k(h)$ , since  $|g|^{1/2} \leq (1 + C_* t) |G|^{1/2}$ , using the decay estimates of Proposition 4.3.2, we have the important inequality ( $\delta = h^{\frac{2}{3}-\eta}$ )

$$\int_{\mathcal{V}_{A_0} \times ]0, \delta[} |\psi|^2 |g|^{1/2} \, ds dt \leq \int_{\mathcal{V}_{A_0} \times ]0, \delta[} |\psi|^2 |G|^{1/2} \, ds dt + Ch^{2/3} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} |\psi|^2 |g|^{1/2} \, ds dt, \quad (4.50)$$

where  $C_*$ ,  $C$  are constants independent of  $s$  and  $t$  (the value of  $C$  might change from one formula to another). We also have

$$\begin{aligned} \langle \widehat{\mathcal{L}}_{h,\delta}\psi, \psi \rangle &= \int_{\mathcal{V}_{A_0} \times ]0, \delta[} h^2 |\partial_t \psi|^2 |g|^{1/2} \, ds dt + h^2 \sum_{1 \leq i, j \leq N-1} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} g^{ij} \partial_{s_i} \psi \overline{\partial_{s_j} \psi} |g|^{1/2} \, ds dt \\ &\quad + \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \Phi_1(s, t) |\psi|^2 |g|^{1/2} \, ds dt. \end{aligned}$$

We recall that  $|g|^{1/2} = (1 - t \operatorname{tr}(G^{-1}K) + t^2 p(s, t)) |G|^{1/2}$  so that, using the decay estimates of Proposition 4.3.2, there exists  $C > 0$  such that

$$\begin{aligned} \langle \widehat{\mathcal{L}}_{h,\delta}\psi, \psi \rangle &\geq \int_{\mathcal{V}_{A_0} \times ]0, \delta[} |h \partial_t \psi|^2 |G|^{1/2} \, ds dt + h^2 \sum_{1 \leq i, j \leq N-1} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} g^{ij} \partial_{s_i} \psi \overline{\partial_{s_j} \psi} |G|^{1/2} \, ds dt \\ &\quad + \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \Phi_1(s, t) |\psi|^2 |g|^{1/2} \, ds dt - Ch^{4/3} \|\psi\|^2. \end{aligned}$$

From (4.21), we have

$$\Phi_1(s, t) \geq x_{\min} + \frac{1}{2} \operatorname{Hess}_{(0)} \phi_1(s)(s) + t - \tilde{C}(|s|^3 + t|s|^2).$$

We can deduce that

$$\begin{aligned} \langle \widehat{\mathcal{L}}_{h,\delta}\psi, \psi \rangle &\geq \int_{\mathcal{V}_{A_0} \times ]0, \delta[} |h \partial_t \psi|^2 |G|^{1/2} \, ds dt + h^2 \sum_{1 \leq i, j \leq N-1} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} g^{ij} \partial_{s_i} \psi \overline{\partial_{s_j} \psi} |G|^{1/2} \, ds dt \\ &\quad + \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \left( x_{\min} + \frac{1}{2} \operatorname{Hess}_{(0)} \phi_1(s)(s) + t \right) |\psi|^2 |g|^{1/2} \, ds dt \\ &\quad - Ch^{4/3} \|\psi\|^2 - \tilde{C} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} (|s|^3 + t|s|^2) |\psi|^2 |g|^{1/2} \, ds dt. \end{aligned}$$

The last integral is of order  $h^{4/3}$  (by using the exponential decay in the  $s$  and  $t$  variables; Propositions 4.3.2 and 4.3.3). Then, there exists  $C > 0$  such that

$$\begin{aligned} \langle \widehat{\mathcal{L}}_{h,\delta}\psi, \psi \rangle &\geq \int_{\mathcal{V}_{A_0} \times ]0, \delta[} [|h \partial_t \psi|^2 + t|\psi|^2] |G|^{1/2} \, ds dt + h^2 \sum_{i,j} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} g^{ij} \partial_{s_i} \psi \overline{\partial_{s_j} \psi} |G|^{1/2} \, ds dt \\ &\quad + \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \frac{1}{2} \operatorname{Hess}_{(0)} \phi_1(s)(s) |\psi|^2 |G|^{1/2} \, ds dt + x_{\min} \|\psi\|^2 - Ch^{4/3} \|\psi\|^2. \end{aligned} \tag{4.51}$$

Thus, by the min max principle, we have

$$\int_{\mathcal{V}_{A_0} \times ]0, \delta[} [|h \partial_t \psi|^2 + t|\psi|^2] |G|^{1/2} \, ds dt \geq z_1 h^{2/3} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} |\psi|^2 |G|^{1/2} \, ds dt,$$

using the inequalities in (4.50), we have

$$\int_{\mathcal{V}_{A_0} \times ]0, \delta[} [|h \partial_t \psi|^2 + t|\psi|^2] |G|^{1/2} \, ds dt \geq z_1 h^{2/3} \|\psi\|^2 - Ch^{4/3} \|\psi\|^2. \tag{4.52}$$

By Taylor expansion near  $s = 0$ , we get

$$\left| |G(s)|^{1/2} - |G_0|^{1/2} \right| \leq C |s|, \quad G_0 = G(0). \quad (4.53)$$

From (4.53), there exists constants  $C > 0$  such that

$$\begin{aligned} & \left| \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \text{Hess}_{(0)} \phi_1(s)(s) |\psi|^2 |G|^{1/2} \, ds dt - \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \text{Hess}_{(0)} \phi_1(s)(s) |\psi|^2 |G(0)|^{1/2} \, ds dt \right| \\ & \leq \left| \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \text{Hess}_{(0)} \phi_1(s)(s) |\psi|^2 [|G(s)|^{1/2} - |G(0)|^{1/2}] \, ds dt \right| \\ & \leq C \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \text{Hess}_{(0)} \phi_1(s)(s) |s| |\psi|^2 \, ds dt \\ & \leq C \int_{\mathcal{V}_{A_0} \times ]0, \delta[} |s|^3 |\psi|^2 \, ds dt \leq C \int_{\mathcal{V}_{A_0} \times ]0, \delta[} |s|^3 |\psi|^2 |G|^{1/2} \, ds dt. \end{aligned}$$

From Proposition 4.3.3, we get

$$\begin{aligned} & \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \text{Hess}_{(0)} \phi_1(s)(s) |\psi|^2 |G|^{1/2} \, ds dt \\ & \geq \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \text{Hess}_{(0)} \phi_1(s)(s) |\psi|^2 |G(0)|^{1/2} \, ds dt - Ch^{3/2} \|\psi\|^2. \end{aligned} \quad (4.54)$$

The same idea proves that

$$\begin{aligned} & \sum_{1 \leq i, j \leq N-1} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} g^{ij} h \partial_{s_i} \psi \overline{h \partial_{s_j} \psi} |G|^{1/2} \, ds dt \\ & \geq \sum_{1 \leq i, j \leq N-1} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} g^{ij} h \partial_{s_i} \psi \overline{h \partial_{s_j} \psi} |G(0)|^{1/2} \, ds dt - Ch^{3/2} \|\psi\|^2. \end{aligned} \quad (4.55)$$

Observing that,

$$\left| g^{ij}(s, t) - G_0^{ij} \right| \leq C(|s| + t), \quad g^{ij}(0, 0) = G_0^{ij}. \quad (4.56)$$

From Propositions 4.3.2 and 4.3.3, we get

$$\begin{aligned} & \sum_{1 \leq i, j \leq N-1} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} g^{ij} h \partial_{s_i} \psi \overline{h \partial_{s_j} \psi} |G(0)|^{1/2} \, ds dt \\ & \geq \sum_{1 \leq i, j \leq N-1} \int_{\mathcal{V}_{A_0} \times ]0, \delta[} G_0^{ij} h \partial_{s_i} \psi \overline{h \partial_{s_j} \psi} |G(0)|^{1/2} \, ds dt - Ch^{4/3} \|\psi\|^2. \end{aligned} \quad (4.57)$$

From (4.54), (4.55) and (4.57), we can easily see that

$$\begin{aligned} & \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \left[ \sum_{1 \leq i, j \leq N-1} h^2 g^{ij} \partial_{s_i} \psi \overline{\partial_{s_j} \psi} + \frac{1}{2} \text{Hess}_{(0)} \phi_1(s)(s) |\psi|^2 \right] |G|^{1/2} \, ds dt \\ & \geq \int_{\mathcal{V}_{A_0} \times ]0, \delta[} \left[ \sum_{1 \leq i, j \leq N-1} h^2 G_0^{ij} \partial_{s_i} \psi \overline{\partial_{s_j} \psi} + \frac{1}{2} \text{Hess}_{(0)} \phi_1(s)(s) |\psi|^2 \right] |G_0|^{1/2} \, ds dt - Ch^{4/3} \|\psi\|^2. \end{aligned} \quad (4.58)$$

Now, using the inequalities in (4.51), (4.52), and (4.58). By the min-max principle, we have

$$\begin{aligned} \mu_k(h) \|\psi\|^2 &\geq \langle \widehat{\mathcal{L}}_{h,\delta} \psi, \psi \rangle \\ &\geq \int_{\mathcal{V}_{A_0} \times ]0,\delta[} \left[ \sum_{1 \leq i,j \leq N-1} h^2 G_0^{ij} \partial_{s_i} \psi \overline{\partial_{s_j} \psi} + \frac{1}{2} \text{Hess}_{(0)} \phi_1(s)(s) |\psi|^2 \right] |G_0|^{1/2} \, ds dt \\ &\quad + x_{\min} \|\psi\|^2 + z_1 h^{2/3} \|\psi\|^2 - Ch^{4/3} \|\psi\|^2. \end{aligned} \quad (4.59)$$

Therefore, we have

$$\begin{aligned} &\int_{\mathcal{V}_{A_0} \times ]0,\delta[} \left[ \sum_{1 \leq i,j \leq N-1} h^2 G_0^{ij} \partial_{s_i} \psi \overline{\partial_{s_j} \psi} + \frac{1}{2} \text{Hess}_{(0)} \phi_1(s)(s) |\psi|^2 \right] |G_0|^{1/2} \, ds dt \\ &\leq (\mu_k(h) - x_{\min} - z_1 h^{2/3} + Ch^{4/3}) \|\psi\|^2. \end{aligned} \quad (4.60)$$

On the left hand side of the above identity, we recognize the quadratic form associated to the operator

$$\mathbb{1} \otimes \mathcal{H}_h,$$

where  $\mathcal{H}_h = -h^2 \Delta^{G_0} + \frac{1}{2} \text{Hess}_{(0)} \phi_1(s)(s)$  on  $L^2(\mathbb{R}^{N-1}, |G_0|^{1/2} ds)$ .

From Sections 4.7.1 and 4.7.2, we show that, for  $h$  small enough,  $\mathcal{H}_h$  is unitarily equivalent to

$$\mathcal{M}_h = \sum_{i=1}^{N-1} -h^2 \partial_{u_i}^2 + \frac{1}{2} \lambda_i u_i^2$$

on  $L^2(\mathbb{R}^{N-1}, du)$ , where  $\lambda_i$  are the eigenvalues of the Weingarten map of  $\partial\Omega$  at  $A_0$ . Their spectrum is given by

$$\text{Sp}(\mathcal{H}_h) = \text{Sp}(\mathcal{M}_h) = \{h\beta_{1,n}, n \geq 1\},$$

where  $\beta_{1,n}$  is the  $n$ -th eigenvalue of  $\mathcal{N}_h = \sum_{i=1}^{N-1} -\partial_{v_i}^2 + \frac{1}{2} \lambda_i v_i^2$  on  $L^2(\mathbb{R}^{N-1}, dv)$ .

The decay estimates, given that  $\dim(\mathcal{E}_k(h))=k$  on  $L^2(\mathbb{R}^{N-1}, |G_0|^{1/2} ds)$ . Notice that the set  $\mathcal{E}_k(h)$  is contained in the form domain of  $\mathcal{H}_h$ . Thus, by the min-max principle, we have

$$\begin{aligned} h\beta_{1,k} &\leq \sup_{\psi \in \mathcal{E}_k(h)} \frac{\langle (\mathbb{1} \otimes \mathcal{H}_h) \psi, \psi \rangle_{L^2(\mathcal{V}_{A_0} \times ]0,\delta[, |G_0|^{1/2} ds dt)}}{\|\psi\|_{L^2(\mathcal{V}_{A_0} \times ]0,\delta[, |G_0|^{1/2} ds dt)}^2} \\ &\leq (\mu_k(h) - x_{\min} - z_1 h^{2/3} + Ch^{4/3}) \frac{\|\psi\|^2}{\|\psi\|_{L^2(|G_0|^{1/2} ds dt)}^2}. \end{aligned} \quad (4.61)$$

From (4.50), (4.53) and Proposition 4.3.3, we obtain

$$\begin{aligned} \int_{\mathcal{V}_{A_0} \times ]0,\delta[} |\psi|^2 |G_0|^{1/2} \, ds dt &\geq \int_{\mathcal{V}_{A_0} \times ]0,\delta[} |\psi|^2 |G|^{1/2} \, ds dt - C \int_{\mathcal{V}_{A_0} \times ]0,\delta[} |s| |\psi|^2 \, ds dt \\ &\geq (1 - Ch^{1/2}) \|\psi\|^2. \end{aligned} \quad (4.62)$$

Using the last inequality in (4.61), we get

$$\mu_k(h) - x_{\min} - z_1 h^{2/3} + Ch^{4/3} \geq h\beta_{1,k} \frac{\|\psi\|_{L^2(|G_0|^{1/2} ds dt)}^2}{\|\psi\|^2} \geq h\beta_{1,k} - Ch^{3/2}.$$



This implies the desired lower bound and concludes the proof of Proposition 4.5.1.  $\square$

## 4.6 Full asymptotic expansion of the eigenvalues

*Proof of Theorem 4.1.1:* From Proposition 4.5.1, we have the following lower bound of  $\mu_n(h)$  :

$$\mu_1(h) \geq x_{\min} + z_1 h^{2/3} + \beta_{1,1} h + \mathcal{O}(h^{4/3}).$$

Thanks to Proposition 4.4.1, we know that, from spectral theorem, there exists an eigenvalue  $\tilde{\mu}_1(h)$  of the operator  $\widehat{\mathcal{L}}_{h,\delta}$  satisfying, as  $h \rightarrow 0_+$ ,

$$\tilde{\mu}_1(h) = x_{\min} + h^{2/3} z_1 + \beta_{1,1} h + h^{4/3} \sum_{j=0}^M h^{j/6} \zeta_{j,1} + \mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}}).$$

Since  $\beta_{1,1}$  is a simple eigenvalue of the harmonic oscillator ( $\beta_{1,1} < \beta_{1,2}$ , see Section 4.7.1), we deduce that

$$\mu_1(h) = x_{\min} + h^{2/3} z_1 + \beta_{1,1} h + h^{4/3} \sum_{j=0}^M h^{j/6} \zeta_{j,1} + \mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}}).$$

Let  $n \geq 2$  and  $N \in \mathbb{N}$ . Thanks to Proposition 4.4.1, we know that, from spectral theorem, there exists a sequence of quasi eigenvalues  $(\tilde{\mu}_n(h))_{n \geq 2}$  of the operator  $\widehat{\mathcal{L}}_{h,\delta}$  satisfying, as  $h \rightarrow 0_+$ ,

$$\tilde{\mu}_n(h) = x_{\min} + h^{2/3} z_1 + \beta_{1,n} h + h^{4/3} \sum_{j=0}^N h^{j/6} \zeta_{j,n} + \mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}}).$$

Now, we will arrange the sequence of quasi eigenvalues  $\tilde{\mu}_n(h)$  in increasing order modulo  $\mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}})$  and we note them  $\bar{\mu}_n(h)$ . We have

$$\bar{\mu}_n(h) = x_{\min} + h^{2/3} z_1 + \beta_{1,n} h + h^{4/3} \sum_{j=0}^M h^{j/6} \alpha_{j,n} + \mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}}),$$

where  $\alpha_{j,n}$  are a sequence determined from  $\zeta_{j,n}$ .

Imagine that, there exist  $C > 0$ ,  $k \in \mathbb{N}$ , such that

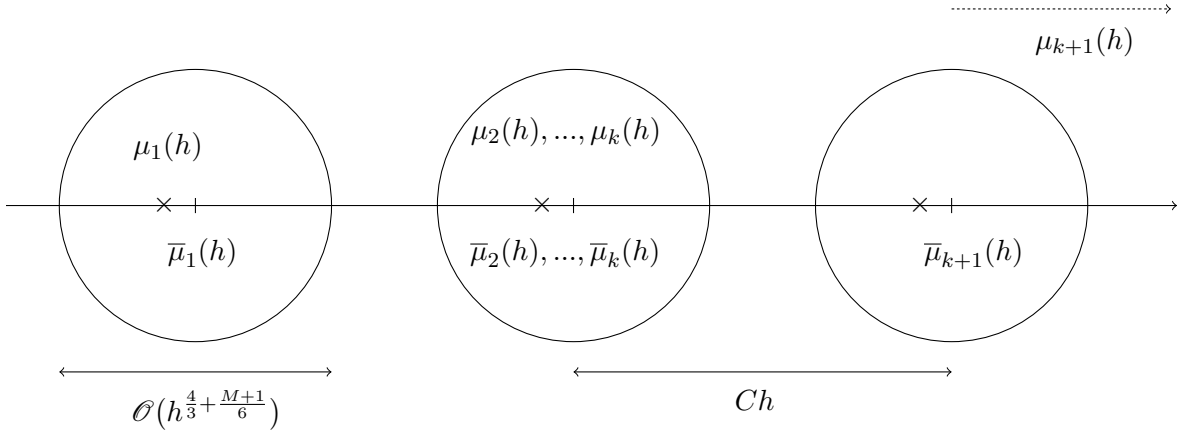
$$\begin{cases} \bar{\mu}_{i+1}(h) = \bar{\mu}_i(h) + o(h), \forall i = 2, \dots, k-1 \\ \bar{\mu}_{k+1}(h) - \bar{\mu}_k(h) \geq Ch. \end{cases} \quad (4.63)$$

That is

$$\begin{cases} \beta_{1,2} = \dots = \beta_{1,k} \\ \beta_{1,k+1} > \beta_{1,k}. \end{cases}$$

There are actually two possibilities:

1.  $\bar{\mu}_2(h), \dots, \bar{\mu}_k(h)$  coincide modulo  $\mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}})$ . From spectral theorem, the rank of the spectral projector near these values is at least  $k-1$ . Then, there exists an eigenvalue very close, modulo  $\mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}})$  to  $\bar{\mu}_i(h)$  for all  $i \in \{2, \dots, k\}$  : it can only be  $\mu_2(h), \dots, \mu_k(h)$ , since the other eigenvalues are elsewhere (see (4.63) and Proposition 4.5.1).

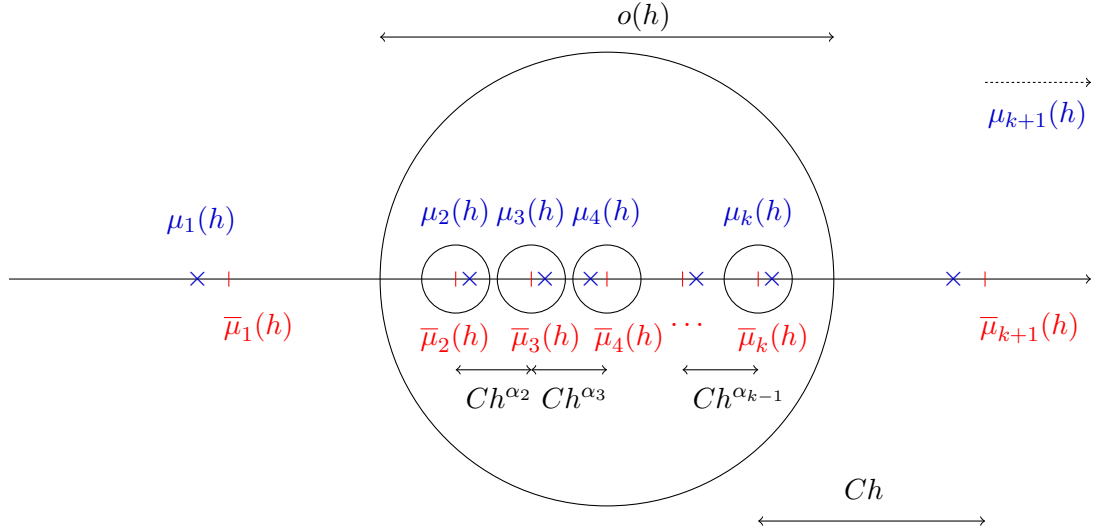


2. They differ modulo  $h^\alpha$  with  $\alpha < \frac{4}{3} + \frac{M+1}{6}$ . There are actually three possibilities:

- There exist  $C > 0$ ,  $k \in \mathbb{N}$ ,  $\alpha_i < \frac{4}{3} + \frac{M+1}{6}$ ,  $\forall i = 2, \dots, k-1$ , such that

$$\begin{cases} \bar{\mu}_{i+1}(h) - \bar{\mu}_i(h) \geq Ch^{\alpha_i}, \forall i = 2, \dots, k-1 \\ \bar{\mu}_{k+1}(h) - \bar{\mu}_k(h) \geq Ch. \end{cases} \quad (4.64)$$

The construction of quasimode implies that there exists an eigenvalue very close, modulo  $\mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}})$ , to  $\bar{\mu}_i(h)$  for all  $i \in \{2, \dots, k\}$  (it cannot be the same). Since the eigenvalues are increasing, then it can't be more than  $\mu_2(h), \dots, \mu_{k-1}(h)$ , and  $\mu_k(h)$  respectively, because the other eigenvalues are somewhere else (see (4.64) and Proposition 4.5.1).

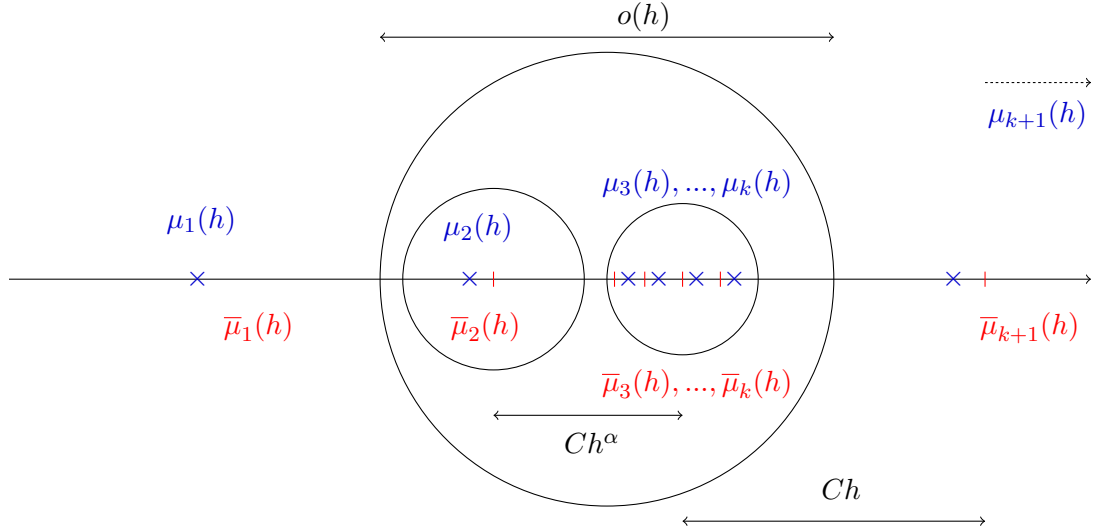


- There exist  $C > 0$ ,  $k \in \mathbb{N}$ , such that

$$\begin{cases} \bar{\mu}_3(h) - \bar{\mu}_2(h) \geq Ch^\alpha, \\ \bar{\mu}_{i+1}(h) = \bar{\mu}_i(h) + \mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}}), \forall i = 3, \dots, k-1 \\ \bar{\mu}_{k+1}(h) - \bar{\mu}_k(h) \geq Ch. \end{cases} \quad (4.65)$$

The rank of the spectral projector near  $\{\bar{\mu}_i(h)\}_{i=3, \dots, k}$  is at least  $k-2$ . Thus, there exists an eigenvalue

very close, modulo  $\mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}})$ , to  $\bar{\mu}_i(h)$  for all  $i \in \{3, \dots, k\}$ . The construction of quasimode implies that there exists an eigenvalue very close, modulo  $\mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}})$  to  $\bar{\mu}_2(h)$  (and it cannot be the same). It can't be more than  $\mu_2(h)$  near  $\bar{\mu}_2(h)$  and  $\mu_3(h), \dots, \mu_k(h)$  near  $\{\bar{\mu}_i(h)\}_{i=3, \dots, k}$ , because the other eigenvalues are somewhere else (see (4.65) and Proposition 4.5.1).



- There exist  $C > 0$ ,  $k_0, k \in \mathbb{N}$  and  $k_0 \leq k$  such that

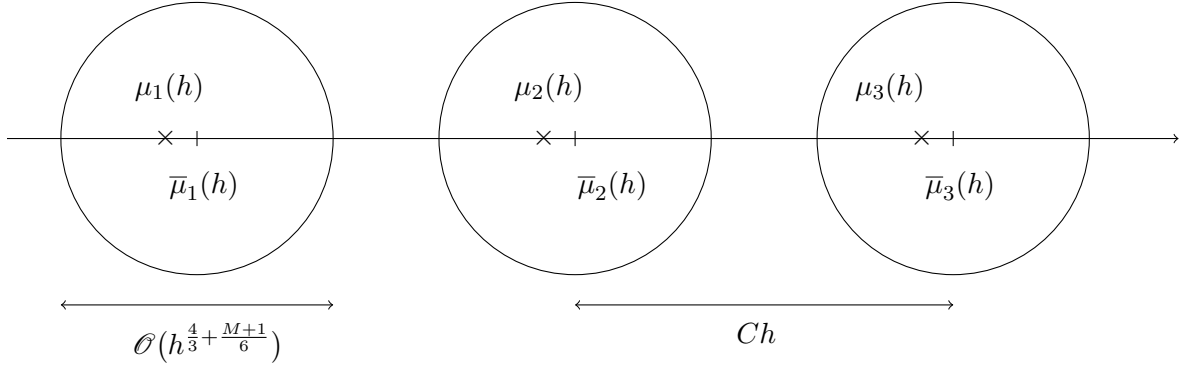
$$\begin{cases} \bar{\mu}_{i+1}(h) = \bar{\mu}_i(h) + \mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}}), \forall i = 2, \dots, k_0 - 1 \\ \bar{\mu}_{k_0+1}(h) - \bar{\mu}_{k_0}(h) \geq Ch^\alpha, \\ \bar{\mu}_{i+1}(h) = \bar{\mu}_i(h) + \mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}}), \forall i = k_0 + 1, \dots, k - 1 \\ \bar{\mu}_{k+1}(h) - \bar{\mu}_k(h) \geq Ch. \end{cases} \quad (4.66)$$

The rank of the spectral projector near  $\{\bar{\mu}_i(h)\}_{i=2, \dots, k_0}$  is at least  $k_0 - 1$  and near  $\{\bar{\mu}_i(h)\}_{i=k_0+1, \dots, k}$  is at least  $k - k_0$ . Then, there exists an eigenvalues very close, modulo  $\mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}})$ , to  $\bar{\mu}_i(h)$  for all  $i \in \{2, \dots, k_0\}$ , but also another very close, modulo  $\mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}})$  to  $\bar{\mu}_i(h)$  for all  $i \in \{k_0 + 1, \dots, k\}$  (and it cannot be the same). It can't be more than  $\mu_2(h), \dots, \mu_{k_0}(h)$  near  $\{\bar{\mu}_i(h)\}_{i=2, \dots, k_0}$  and  $\mu_{k_0+1}(h), \dots, \mu_k(h)$  near  $\{\bar{\mu}_i(h)\}_{i=k_0+1, \dots, k}$ , because the other eigenvalues are somewhere else (see (4.66) and Proposition 4.5.1).

Now, if (4.63) is not satisfied:

$$\bar{\mu}_3(h) - \bar{\mu}_2(h) \geq Ch.$$

As a consequence of the spectral theorem, there exists an eigenvalue very close modulo  $\mathcal{O}(h^{\frac{4}{3} + \frac{M+1}{6}})$  to  $\bar{\mu}_2(h)$ , but also another very close to  $\bar{\mu}_3(h)$  (and it cannot be the same). Since the eigenvalues are indeed separated, it can't be more than  $\mu_2(h)$  and  $\mu_3(h)$  respectively.



Thus, by (4.25) and Proposition 4.2.3, we have shown Theorem 4.1.1.

## 4.7 Analysis of the effective operator

### 4.7.1 The model operator

In this section, we want to study the spectrum of the operator defined by

$$\mathcal{L} = -\Delta^{G_0} + \frac{1}{2}H(\sigma)(\sigma), \quad H = \text{Hess}_{(0)}\phi_1, \quad G_0 = G(0),$$

where  $\phi_1$  is defined in Section 4.3.1. The associated quadratic form on  $H^1(\mathbb{R}^{N-1}; |G_0|^{1/2} d\sigma)$  be

$$q_1(\psi) = \langle \mathcal{L}\psi, \psi \rangle_{L^2(|G_0|^{1/2} d\sigma)} = \langle -\Delta^{G_0}\psi, \psi \rangle_{L^2(|G_0|^{1/2} d\sigma)} + \frac{1}{2} \int \langle H\sigma, \sigma \rangle |\psi|^2 |G_0|^{1/2} d\sigma.$$

By replacing  $\Delta^{G_0}$  by its expression given by (4.29), we obtain

$$\langle -\Delta^{G_0}\psi, \psi \rangle_{L^2(|G_0|^{1/2} d\sigma)} = - \sum_{1 \leq i, j \leq N-1} \int \partial_{\sigma_i} (G_0^{ij} |G_0|^{1/2} \partial_{\sigma_j}) \psi \bar{\psi} d\sigma, \quad G_0^{ij} = G^{ij}(0).$$

By integration by parts, we have

$$\langle -\Delta^{G_0}\psi, \psi \rangle_{L^2(|G_0|^{1/2} d\sigma)} = \sum_{1 \leq i, j \leq N-1} \int G_0^{ij} \partial_{\sigma_j} \psi \overline{\partial_{\sigma_i} \psi} |G_0|^{1/2} d\sigma = \int \langle G_0^{-1} \nabla_{\sigma} \psi, \nabla_{\sigma} \psi \rangle |G_0|^{1/2} d\sigma.$$

Consequently, we have

$$q_1(\psi) = \int \langle G_0^{-1} \nabla_{\sigma} \psi, \nabla_{\sigma} \psi \rangle |G_0|^{1/2} d\sigma + \frac{1}{2} \int \langle H\sigma, \sigma \rangle |\psi|^2 |G_0|^{1/2} d\sigma. \quad (4.67)$$

We introduce the change of variable

$$\sigma = G_0^{-1/2} z.$$

Thus,

$$\nabla_{\sigma} = G_0^{1/2} \nabla_z \quad \text{and} \quad d\sigma = |G_0|^{-1/2} dz.$$

By replacing the previous equality in (4.67), and we use that  $G_0$  is self-adjoint, we get

$$\begin{aligned} q_1(\psi) &= \int \langle G_0^{-1} G_0^{1/2} \nabla_z \bar{\psi}, G_0^{1/2} \nabla_z \bar{\psi} \rangle dz + \frac{1}{2} \int \langle H G_0^{-1/2} z, G_0^{-1/2} z \rangle |\bar{\psi}|^2 dz \\ &= \int \langle \nabla_z \bar{\psi}, \nabla_z \bar{\psi} \rangle dz + \frac{1}{2} \int \langle G_0^{-1/2} H G_0^{-1/2} z, z \rangle |\bar{\psi}|^2 dz, \end{aligned}$$

with  $\bar{\psi}(z) = \psi(\sigma)$ .

We define a quadratic form on  $H^1(\mathbb{R}^{N-1})$  by

$$q_2(\psi) = \int \langle \nabla_z \psi, \nabla_z \psi \rangle dz + \frac{1}{2} \int \langle G_0^{-1/2} H G_0^{-1/2} z, z \rangle |\psi|^2 dz.$$

The associated operator is unitarily equivalent to  $\mathcal{L}$ , and defined by

$$\mathcal{L}_2 = -\Delta_z + \frac{1}{2} \langle G_0^{-1/2} H G_0^{-1/2} z, z \rangle.$$

We set  $S = G_0^{-1/2} H G_0^{-1/2}$ . As  $H$  is a positive definite matrix, then  $S$  is also a positive definite matrix. Consequently, there exists an orthogonal matrix  $Q$  such that

$$Q^{-1} S Q = B,$$

where  $B$  is a diagonal matrix formed by the positive eigenvalues of  $S$ . Notice that

$$B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_{N-1} \end{pmatrix},$$

where  $\lambda_i$  are the eigenvalues of  $S$ .

Considering  $z = Qu$ , we get

$$\begin{aligned} q_2(\psi) &= \int |\nabla_u \psi|^2 du + \frac{1}{2} \int \langle Q^{-1} S Q u, u \rangle |\psi|^2 du \\ &= \int |\nabla_u \psi|^2 du + \frac{1}{2} \int \langle B u, u \rangle |\psi|^2 du := q_3(\psi). \end{aligned}$$

The operator associated with the quadratic form  $q_3$  is unitarily equivalent to  $\mathcal{L}$  and defined by

$$\mathcal{L}_3 = -\Delta_u + \frac{1}{2} \langle B u, u \rangle = -\sum_{i=1}^{N-1} \partial_{u_i}^2 + \frac{1}{2} \sum_{i=1}^{N-1} \lambda_i u_i^2 = \sum_{i=1}^{N-1} \left( -\partial_{u_i}^2 + \frac{1}{2} \lambda_i u_i^2 \right) = \sum_{i=1}^{N-1} \mathcal{H}_i,$$

where,  $\mathcal{H}_i$  is the harmonic oscillator

$$\mathcal{H}_i = -\partial_{u_i}^2 + \frac{1}{2} \lambda_i u_i^2.$$

The  $n$ -th eigenvalues of  $\mathcal{H}_i$  are given by

$$(2n-1) \sqrt{\frac{\lambda_i}{2}}.$$

Consequently, the  $n$ -th eigenvalues of  $\mathcal{L}_3$  is the same of  $\mathcal{L}_1$ , it is the  $n$ -th element of the following set, by grouping the elements in increasing order

$$\left\{ \beta_{1,n} = \sum_{i=1}^{N-1} \sqrt{\frac{\lambda_i}{2}} (2n_i - 1), n \in \mathbb{N}^{N-1} \right\}. \quad (4.68)$$

Since the matrices  $S$  and  $G_0^{-1}H$  are similar, then they have the same spectrum and therefore  $\lambda_i$  are the eigenvalues of  $G_0^{-1}H$ .

## 4.7.2 Relation to the Weingarten map

We recall from Section 4.3.1, the coefficients of the first and second fundamental form, in the base  $(\partial_1\phi, \dots, \partial_{N-1}\phi)$ , are given respectively by the formulas

$$\forall i, j = 1, \dots, N-1, \quad G_{ij}(s) = \partial_i\phi(s) \cdot \partial_j\phi(s) \quad \text{and} \quad K_{ij} = \partial_i\phi \cdot \partial_j\nu(\phi),$$

where  $\phi$  is a parameterization of  $\partial\Omega$  defined in Section 4.3.1, such that  $\phi(0) = A_0$  and  $\nu(A_0) = (-1, 0, \dots, 0)$ . We get

$$\begin{aligned} K_{ij}(s) &= \partial_i\phi(s) \cdot \partial_j\nu(\phi)(s) \\ &= \partial_i\phi(s) \cdot d_s\nu(\phi)(e_j) \\ &= \partial_i\phi(s) \cdot d_{\phi(s)}\nu[\partial_j\phi(s)] \\ &= \partial_i\phi(s) \cdot W_{\phi(s)}(\partial_j\phi(s)). \end{aligned}$$

Let  $L(s) := (l_{ij}(\phi(s)))_{1 \leq i, j \leq N-1}$  the matrix of  $W_{\phi(s)}$  in the base  $(\partial_1\phi(s), \dots, \partial_{N-1}\phi(s))$ . Then

$$W_{\phi(s)}(\partial_j\phi(s)) = \sum_{i=1}^{N-1} l_{ij}(\phi(s))\partial_i\phi(s).$$

Therefore, we can show that

$$\begin{aligned} K_{ij}(s) &= \partial_i\phi(s) \cdot W_{\phi(s)}(\partial_j\phi(s)) \\ &= \partial_i\phi(s) \cdot \sum_{k=1}^{N-1} l_{kj}(\phi(s))\partial_k\phi(s) \\ &= \sum_{k=1}^{N-1} l_{kj}(\phi(s))G_{ik}(s). \end{aligned}$$

We deduce that

$$K(s) = G(s)L(s). \quad (4.69)$$

On the other hand

$$K_{ij}(s) = \partial_i\phi(s) \cdot \partial_j\nu(\phi)(s) = -\partial_i\partial_j\phi(s) \cdot \nu(\phi)(s).$$

In  $s = 0$ , we get

$$K_{ij}(0) = -\partial_i\partial_j\phi(s)|_{s=0} \cdot \nu(A_0) = \partial_i\partial_j\phi_1(s)|_{s=0} = h_{ij},$$

where  $H = \text{Hess}_{(0)}\phi_1 = (h_{ij})_{1 \leq i, j \leq N-1}$ .

Thanks to (4.69), we obtain

$$H = G(0)L(0) = G_0L(0).$$

Consequently,

$$L(0) = G_0^{-1}H.$$

From the previous section,  $\lambda_i$  are the eigenvalues of  $G_0^{-1}H$  and by the last equality, we deduce that  $\lambda_i$  are the eigenvalues of the Weingarten map  $W_{A_0}$  of  $\partial\Omega$  at  $A_0$ .

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# Edge states for the Robin magnetic Laplacian

## Abstract

This chapter tackles the spectral analysis of the Robin Laplacian on a smooth bounded two-dimensional domain in the presence of a constant magnetic field. In the semi-classical limit, a uniform description of the spectrum located between the Landau levels is obtained. The corresponding eigenfunctions, called the edge states, are exponentially localized near the boundary. The estimates look like Agmon's estimates, but there are not obtained via variational means as it is the case in many magnetic settings. Here, they follow from resolvent estimates using the distance to the Landau levels. By means of a microlocal dimensional reduction, our unifying approach allows on the one hand to derive a very precise Weyl law, and on the other hand to refine simultaneously old results about the low-lying eigenvalues in the Robin case and recent ones about the edge states in the Dirichlet case. This is also the opportunity to revisit the Neumann case, by establishing more uniform asymptotic expansions, with slightly more general boundary conditions.

## Résumé

Ce chapitre aborde l'analyse spectrale du Laplacien de Robin sur un domaine bidimensionnel régulier et borné en présence d'un champ magnétique constant. Dans la limite semi-classique, une description uniforme du spectre situé entre les niveaux de Landau est obtenue. Les fonctions propres correspondantes, appelées états de bord, sont exponentiellement localisées près de la frontière. Les estimations ressemblent à celles d'Agmon, mais elles ne sont pas obtenues par des moyens variationnels comme c'est le cas dans de nombreux contextes magnétiques. Ici, elles découlent par des estimations du résolvant utilisant la distance aux niveaux de Landau. Au moyen d'une réduction dimensionnelle microlocale, notre approche unificatrice permet d'une part de dériver une loi de Weyl très précise, et d'autre part d'affiner simultanément des résultats anciens sur les valeurs propres de basse basse dans le cas de Robin et des résultats récents sur les états de bord dans le cas de Dirichlet. C'est aussi l'occasion de revisiter le cas de Neumann, en établissant des développements asymptotiques plus uniformes, avec des conditions aux bord plus générales.



*” Un jour, à force de fouiller l’atome,  
un savant expliquera peut-être  
la joie et la paix de l’esprit  
par des formules mathématiques.”*

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*Bernard Moitessier*

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## 5.1 Motivations and results

### 5.1.1 About the magnetic Robin Laplacian

We want to describe the spectrum of the semiclassical magnetic Laplacian

$$\mathcal{L}_h = (-ih\nabla - \mathbf{A})^2,$$

on a smooth, bounded, and simply connected open Euclidean domain  $\Omega \subset \mathbb{R}^2$ , with boundary conditions of Robin type. The vector potential  $\mathbf{A} : \bar{\Omega} \rightarrow \mathbb{R}^2$  is supposed to be smooth and generating a constant magnetic field of intensity 1:

$$\partial_1 A_2 - \partial_2 A_1 = 1.$$

The magnetic Robin boundary conditions are enforced by defining the operator  $\mathcal{L}_h = \mathcal{L}_{h,\gamma}$  to be the selfadjoint operator associated with the quadratic form defined for all  $\psi \in H^1(\Omega)$  by:

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) = \int_{\Omega} |(-ih\nabla - \mathbf{A})\psi|^2 dx + \gamma h^{\frac{3}{2}} \int_{\partial\Omega} |\psi|^2 ds, \quad (5.1)$$

where  $\gamma \in \mathbb{R} \cup \{+\infty\}$ , and  $ds$  is the length measure of the boundary induced by the Euclidean metric. By convention,  $\gamma = +\infty$  corresponds to the Dirichlet boundary condition  $\psi \in H_0^1(\Omega)$ . In the whole chapter, our estimates will be uniform when  $\gamma \in [-\gamma_0, +\infty]$  for an arbitrary fixed  $\gamma_0 > 0$ . When  $\gamma \in \mathbb{R}$ , the domain of  $\mathcal{L}_h$  is given by

$$\text{Dom}(\mathcal{L}_h) = \{\psi \in H^1(\Omega) : (-ih\nabla - \mathbf{A})^2\psi \in L^2(\Omega), -ih\mathbf{n} \cdot (-ih\nabla - \mathbf{A})\psi = \gamma h^{\frac{3}{2}}\psi \text{ on } \partial\Omega\},$$

where  $\mathbf{n}$  is the outward pointing normal to the boundary.

**Remark 5.1.1.** *Note that a change of gauge can be used to ensure that  $\mathbf{A} \cdot \mathbf{n} = 0$ . In this case, the magnetic Robin condition becomes a usual Robin condition:*

$$-\mathbf{n} \cdot \nabla\psi = \gamma h^{-\frac{1}{2}}\psi. \quad (5.2)$$

We would like to establish accurate spectral asymptotics for the magnetic Robin Laplacian  $\mathcal{L}_h$  in regimes where the magnetic field plays a major role, competing with the Robin condition (this is the origin, as we will see, of the factor  $h^{\frac{3}{2}}$  in the Robin condition). Until now, very accurate results are available only for the Neumann magnetic Laplacian (when  $\gamma = 0$ ). In this case, the lowest eigenvalues have been analyzed in detail in [FH06] and uniform estimates have been recently established in [BNHR22] where a purely magnetic tunnelling effect formula has been proved. When  $\gamma \neq 0$ , the only known results go back to the works by Kachmar, see [Kac06b], where only the smallest eigenvalue has been estimated. In all these situations (except when  $\gamma = +\infty$ , where the first eigenvalue is asymptotic to  $h$  times the magnetic intensity — here, 1), one can show that the first eigenvalue becomes smaller than  $h$  as soon as  $h$  is small enough. This energy bound is usually associated with a localization behavior near the boundary of the eigenfunctions, which can be quantified by semiclassical Agmon estimates.

By a simple rescaling, the semiclassical limit  $h \rightarrow 0$  translates into a quantum regime where the intensity of the magnetic field tends to infinity. In the physics literature of thin conductors or electron gases (approximated by 2D domains) subject to a strong external magnetic field, it is well known that the presence of a boundary (or, more generally, of an abruptly changing magnetic field along a curve) generates a current along the boundary due to the presence of “bouncing modes” classically localized at a distance  $\sqrt{E}/B$  to the boundary ( $E$  is the kinetic energy and  $B$  is the magnetic intensity: in this work  $B = 1$ ), see for

instance [Hal82]. These so-called “edge states” or “boundary states” exist as soon as the Fermi level of the conductor lies strictly in between two consecutive Landau levels, and produce ballistic dynamics along the boundary. If the boundary  $\partial\Omega$  is compact, this dynamics is quantized and produces new discrete energy levels. These are precisely the eigenvalues that we wish to describe in this chapter.

Heuristically, the localization near  $\partial\Omega$  is often explained by the classical bouncing modes alluded to above, but it is also easy to understand from a quantum perspective. Indeed, if we forget the boundary condition,  $\mathcal{L}_h$  acts as the magnetic Laplacian with constant magnetic field, on the Euclidean plane,  $\mathcal{L}_h^{\mathbb{R}^2}$ . The spectrum of this so-called “bulk” operator is well-known and made of the famous Landau levels  $\{(2n - 1)h, n \geq 1\}$ , which are infinitely degenerate eigenvalues. This suggests that, if one considers potential eigenvalues of  $\mathcal{L}_h$  in a window of the form  $I_h = [ha, hb]$  with  $2n - 1 < a < b < 2n + 1$  for some integer  $n \geq 0$  (for  $n = 0$ , we take  $a = -\infty$ ), they cannot correspond to any bulk state, and hence the corresponding eigenfunctions should be localized near the boundary. This phenomenon has interesting physical applications; a famous one is the quantum Hall effect, when the domain is not simply connected, which expresses the collective effects of several boundaries on the total net current. Another application is the confinement of particles in small domains, or “quantum dots” (sometimes called “anti-dots” because one takes  $B = 0$  inside the domain, and  $B = 1$  outside), see [RPM99, LSIC04].

On the mathematics side, the existence of edge currents in a half-plane with Dirichlet boundary condition was shown in [DBP99]. In a compact setting, the eigenfunction localization at the boundary has been observed (again in the Dirichlet case  $\gamma = +\infty$ , which is usually chosen in physics) in [GV21], leading to a description of the spectrum in a thin spectral window, see [GV21, Corollary 2.7]. However the *exponential* decay was not established. In fact, as we will see, this decay does not really follow from the usual Agmon estimates.

In this work we treat the general case  $\gamma \in \mathbb{R} \cup \{+\infty\}$ . This corresponds, physically to a domain  $\Omega$  coated with a very thin layer of a different material (see for instance [BL96]). Since  $\Omega$  is bounded, the spectrum in  $I_h$  is always discrete and a first rough estimate shows that the number of eigenvalues lying in  $I_h$ , denoted by  $N(\mathcal{L}_h, I_h)$ , satisfies

$$N(\mathcal{L}_h, I_h) \leq Ch^{-2}, \quad (5.3)$$

for some  $C > 0$  and all  $h > 0$  small enough (see Appendix 5.6.1 where we recall the origin of this estimate).

Our goal is to obtain a very precise description, in the semiclassical regime, of the spectral elements corresponding to the interval  $I_h$ , much more accurate than (5.3). This includes the localization behavior near the boundary of the domain  $\Omega$  of the corresponding eigenfunctions. For instance, when  $\gamma \in \mathbb{R}$ , a consequence of our main result Theorem 5.1.7 is the appearance of a quite interesting phenomenon: for a given (low) energy, one can have boundary quasimodes corresponding to classical currents flowing in opposite directions, leading to magnetic oscillations of eigenvalues.

This work is also an opportunity to revisit the Neumann case analyzed in [HM01, FH06] (see also [BNHR22]) by establishing more uniform asymptotic expansions, with slightly more general boundary conditions.

### 5.1.2 De Gennes operator with Robin condition

Our results will be expressed in terms of the eigenvalues of the de Gennes operator with Robin boundary condition. This operator, which appears naturally in the study of boundary induced magnetic effects [SJDG63, FH10b], is a differential operator of order two depending on the real parameters  $\gamma$  and  $\sigma$  and acting as

$$H[\gamma, \sigma] = -\frac{d^2}{dt^2} + (t - \sigma)^2,$$

on the domain

$$\text{Dom}(H[\gamma, \sigma]) = \left\{ u \in B^1(\mathbb{R}_+) : \left( -\frac{d^2}{dt^2} + (t - \sigma)^2 \right) u \in L^2(\mathbb{R}_+), u'(0) = \gamma u(0) \right\},$$

where

$$B^1(\mathbb{R}_+) = \{ u \in H^1(\mathbb{R}_+) : [t \mapsto tu(t)] \in L^2(\mathbb{R}_+) \}.$$

It is well-known that  $H[\gamma, \sigma]$  is a self-adjoint elliptic operator with compact resolvent.

**Notation 5.1.1.** Its spectrum can be written as a non-decreasing sequence of eigenvalues  $(\mu_n(\gamma, \sigma))_{n \geq 1}$  (which are all simple due to the Cauchy-Lipschitz theorem). We denote by  $u_n^{[\gamma, \sigma]}$  the normalized sequence of the corresponding eigenfunctions (with  $u_n^{[\gamma, \sigma]}(0) > 0$ ).

We let

$$\Theta^{[n-1]}(\gamma) := \inf_{\sigma \in \mathbb{R}} \mu_n(\gamma, \sigma).$$

The index  $n - 1$  is compatible with the notation used in the case of the de Gennes operator (case when  $\gamma = 0$ ), see [FH10b, Section 3.2]. The family  $(H[\gamma, \sigma])_{(\gamma, \sigma) \in \mathbb{R}^2}$  is analytic of type (B) (in the sense of Kato, see [Kat76, Chapter VII, §4]), *i.e.*, the form domain does not depend on the parameters and the sesquilinear form is analytic as a function of  $\gamma$  or  $\sigma$ .

The following proposition gathers the main properties of the functions  $\mu_n(\gamma, \cdot)$  (which are usually called *dispersion curves*) that will be used in this chapter. Most of them have been established in [Kac06b] (see also [Kac08], and [DBP99] in the Dirichlet case).

**Proposition 5.1.2.** *Let us fix  $n \geq 1$ . When  $\gamma \in \mathbb{R}$ , the function  $\mu_n(\gamma, \cdot)$  is analytic and*

$$\lim_{\sigma \rightarrow -\infty} \mu_n(\gamma, \sigma) = +\infty, \quad \lim_{\sigma \rightarrow +\infty} \mu_n(\gamma, \sigma) = 2n - 1. \quad (5.4)$$

*Moreover,  $\mu_n(\gamma, \cdot)$  has a unique minimum attained at  $\sigma = \xi_{n-1}(\gamma)$ , but not attained at infinity. This minimum is non-degenerate. The function  $\mu_n(\gamma, \cdot)$  is decreasing on  $(-\infty, \xi_{n-1}(\gamma))$  and increasing on  $(\xi_{n-1}(\gamma), +\infty)$ . In addition, we have, for all  $n \geq 2$ ,*

$$2n - 3 < \Theta^{[n-1]}(\gamma) < 2n - 1. \quad (5.5)$$

*When  $\gamma = +\infty$ , that is when the Robin condition is replaced by the Dirichlet condition,  $\mu_n(+\infty, \cdot)$  is still smooth, but now decreasing from  $+\infty$  to  $2n - 1$ .*

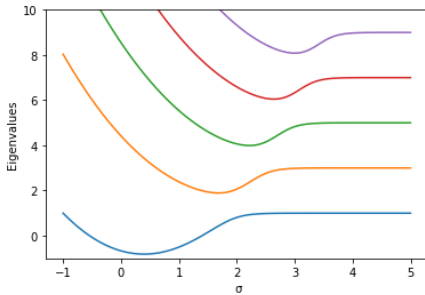


Figure 5.1 –  $\sigma \mapsto \mu_k(-1, \sigma)$ , for  $k = 1, \dots, 5$ .

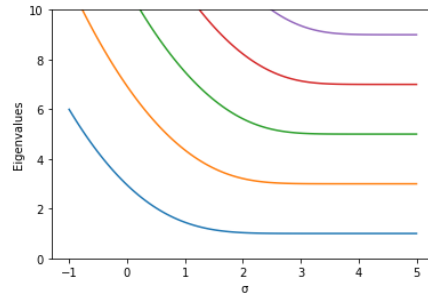


Figure 5.2 –  $\sigma \mapsto \mu_k(+\infty, \sigma)$ , for  $k = 1, \dots, 5$ .

The proof of this proposition, is established in Section 2.4 of Chapter 2. This proposition has the following elementary but important consequences for our analysis, which are illustrated in Figure 5.3.

**Corollary 5.1.3.** *Let  $\gamma \in \mathbb{R} \cup \{+\infty\}$  be fixed. Let  $\Theta$  be the set of all critical values of the functions  $\mu_n$ : we have*

$$\Theta = \{\Theta^{[n-1]}(\gamma), \quad n \geq 1\}.$$

Let  $\Lambda$  be the set of limit points of the functions  $\mu_n$  at infinity:

$$\Lambda := \{2n - 1, \quad n \geq 1\}.$$

Let  $[a, b] \subset \mathbb{R}$  be an interval disjoint from  $\Lambda$ . Let either  $n = 1$  if  $a < 1$  or  $n \geq 2$  be such that  $[a, b] \subset (2n - 3, 2n - 1)$ . (In the case  $n = 1$  we allow  $a = -\infty$ .) It follows from (5.4) that for any integer  $k \geq 1$ ,  $\mu_k^{-1}([a, b])$  is compact.

Let  $p(k)$  be the number of connected components of  $\mu_k^{-1}([a, b])$ : we have

$$\begin{cases} p(k) = 1 & \text{if } 1 \leq k < n \\ p(n) = 0 & \text{if } \gamma = +\infty \\ p(n) = 1 & \text{if } \gamma \in \mathbb{R} \text{ and } \Theta^{[n-1]} \in [a, b] \\ p(n) = 2 & \text{if } \gamma \in \mathbb{R} \text{ and } \Theta^{[n-1]} < a \\ p(n) = 0 & \text{if } \gamma \in \mathbb{R} \text{ and } b < \Theta^{[n-1]} \\ p(k) = 0 & \text{if } k > n. \end{cases}$$

Therefore, when  $\gamma \in \mathbb{R}$ ,

$$N(\gamma, a, b) := \#\{k \geq 1 : \mu_k(\gamma, \cdot)^{-1}([a, b]) \neq \emptyset\} = \begin{cases} n & \text{if } b \geq \Theta^{[n-1]}(\gamma) \\ n - 1 & \text{otherwise} \end{cases},$$

and if  $\gamma = +\infty$  (Dirichlet case) then  $\mu_1(+\infty, \cdot)$  does not take any value in  $(-\infty, 1)$ , and  $N(\gamma, a, b) = n - 1$ .

**Notation 5.1.4.** From now on, we denote by  $N(\gamma, a, b) = N$  this cardinal.

**Assumption 2.** In the following,  $a$  and  $b$  are allowed to depend on  $h$ , as soon as they stay in an  $h$ -independent compact interval inside  $(2n - 3, 2n - 1)$ .

With this picture in mind, for each  $k \geq 1$ , we may construct a smooth function  $\overset{\circ}{\mu}_k$ , bounded with all its derivatives, which coincide with  $\mu_k$  in a neighborhood of  $\mu_k^{-1}([a, b])$ .

**Notation 5.1.5.** Indeed, let  $\Xi_0 : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth, bounded with all its derivatives, and increasing function such that for all  $k \in \{1, \dots, N\}$ ,  $\mu_k(\gamma, \Xi_0(\sigma)) = \mu_k(\gamma, \sigma)$  in a neighborhood of  $\mu_k^{-1}([a, b])$  and  $\mu_k \circ \Xi_0$  takes its values in  $(-\infty, a) \cup (b, +\infty)$  away from it. We consider

$$\overset{\circ}{\mu}_k := \mu_k(\gamma, \Xi_0(\cdot)), \tag{5.6}$$

where we omit the reference to the parameter  $\gamma$  to lighten the notation. In the following, we will more generally denote by  $\overset{\circ}{\varphi}$ , the function  $\varphi$  after composition with  $\Xi_0$ .

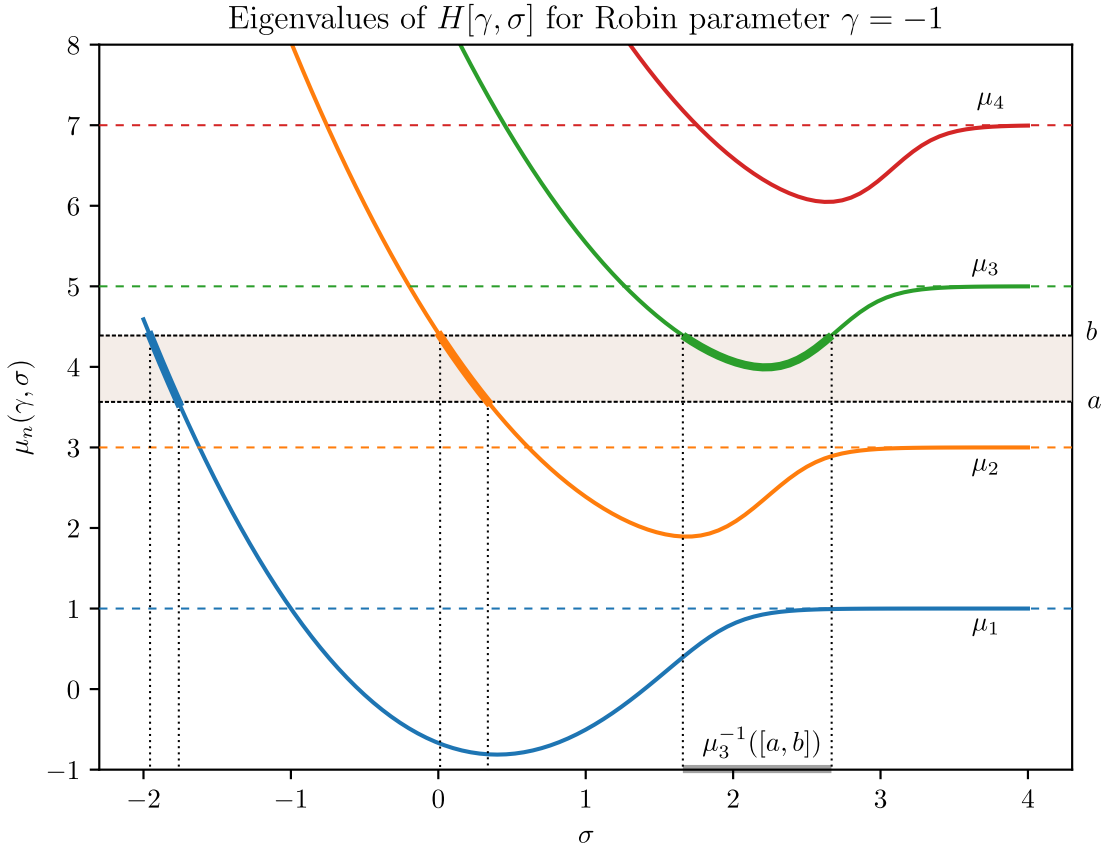


Figure 5.3 – This figure shows the dispersion curves  $\mu_k(\gamma, \cdot)$ , for  $\gamma = -1$ . We visualise the preimage of a given interval  $[a, b] \subset (2n - 3, 2n - 1)$  with  $n = 3$ . The curves are obtained by a standard finite difference numerical scheme.

### 5.1.3 Results

Let us now describe the main results of our article, which will be expressed in terms of pseudo-differential operators in one dimension.

#### 5.1.3.1 A pseudo-differential framework

The bounded functions  $\overset{\circ}{\mu}_k$  will be convenient to state our main theorem, which involves  $h^{\frac{1}{2}}$ -pseudo-differential operators with symbols in the usual class  $S_{\mathbb{R}^2}(1)$  given by

$$S_{\mathbb{R}^2}(1) = \{a \in \mathcal{C}^\infty(\mathbb{R}_{s,\sigma}^2) : \forall \alpha \in \mathbb{N}^2, \exists C_\alpha > 0 : |\partial^\alpha a| \leq C_\alpha\}.$$

As we said before, the eigenfunctions of  $\mathcal{L}_h$  will be localized near the boundary of  $\Omega$ , which is a closed smooth curve with length  $2L$ . Our main result describes their distribution with the help of an  $h^{\frac{1}{2}}$ -pseudo-differential operator on the boundary (see for instance [FHKR22, Section 4.1] where similar considerations have been done in the context of discontinuous magnetic fields). Let us denote  $\hbar = h^{\frac{1}{2}}$ . We recall that the

Weyl quantization of a symbol  $p$  is given by the formula:

$$(\text{Op}_h^W p)\psi(x) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(x-y)\eta/\hbar} p\left(\frac{x+y}{2}, \eta\right) \psi(y) dy d\eta, \quad \forall \psi \in \mathcal{S}(\mathbb{R}), \quad (5.7)$$

and that this formula defines a bounded operator from  $L^2(\mathbb{R})$  to  $L^2(\mathbb{R})$  if  $p \in S_{\mathbb{R}^2}(1)$ , by the Calderón-Vaillancourt theorem. To shorten the notation, we will sometimes write  $p^W$  instead of  $\text{Op}_h^W p$ .

Let  $\mathbf{T}_{2L} = \mathbb{R}/2L\mathbb{Z}$ , and  $L^2(\mathbf{T}_{2L})$  be the subset of  $L^2_{\text{loc}}(\mathbb{R})$  of  $2L$ -periodic functions, equipped with the usual  $L^2$  norm on  $[0, 2L]$ . When  $p \in S_{\mathbf{T}_{2L} \times \mathbb{R}}(1)$ , i.e.  $p \in S_{\mathbb{R}^2}(1)$  and is  $2L$ -periodic in its first variable  $s$ , then for any  $\theta \in \mathbb{R}$ , the operator given in (5.7) induces a bounded operator from  $e^{i\theta \cdot} L^2(\mathbf{T}_{2L})$  to  $e^{i\theta \cdot} L^2(\mathbf{T}_{2L})$  — we denote by  $e^{i\theta \cdot}$  the function  $x \mapsto e^{i\theta x}$ .

**Remark 5.1.2.** *The space  $e^{i\theta \cdot} L^2(\mathbf{T}_{2L})$  only depends on the class of  $\theta$  modulo  $\frac{\pi}{L}$ ; it is equal to the subspace of functions in  $L^2_{\text{loc}}(\mathbb{R})$  equipped with the Floquet boundary condition  $\psi(x+2L) = e^{i2L\theta} \psi(x)$ . The operator  $\text{Op}_h^W p$  acting on  $e^{i\theta \cdot} L^2(\mathbf{T}_{2L})$  is unitarily equivalent to  $\text{Op}_h^W p(x, \eta + \hbar\theta)$  acting on  $L^2(\mathbf{T}_{2L})$ .*

### 5.1.3.2 Main theorem

Since our main result describes the spectrum of  $\mathcal{L}_h$  “modulo  $\mathcal{O}(h^\infty)$ ”, we need to make this notion precise.

**Definition 5.1.6.** *we will say that the spectra of two self-adjoint operators  $T_1$  and  $T_2$  depending on  $h$  coincide in  $I_h$  modulo  $\mathcal{O}(h^\alpha)$ ,  $\alpha \in \mathbb{R} \sqcup \{+\infty\}$ , when there exists  $C, h_0 > 0$  such that, for all  $h \in (0, h_0)$ ,*

- $T_1$  and  $T_2$  have discrete spectrum in  $I_h + [-Ch^\alpha, Ch^\alpha]$ ,
- for all interval  $J_h \subset I_h$  we can find an interval  $K_h$  such that  $J_h \subset K_h$  with  $d_H(K_h, J_h) \leq Ch^\alpha$  and

$$\text{rank } \mathbf{1}_{J_h}(T_1) \leq \text{rank } \mathbf{1}_{K_h}(T_2), \quad \text{rank } \mathbf{1}_{J_h}(T_2) \leq \text{rank } \mathbf{1}_{K_h}(T_1),$$

where  $d_H$  denotes the Hausdorff distance:

$$d_H(A, B) = \sup_{(a,b) \in A \times B} \max(d(a, B), d(b, A)).$$

This definition translates to discrete subsets of  $\mathbb{R}$  as follows: for each discrete subset  $S \subset \mathbb{R}$ , we associate the sum of Dirac masses  $\delta_S := \sum_{s \in S} \delta_s$ , and consider the corresponding self-adjoint operator whose spectral measure is  $\delta_S$ . Then we say that two discrete subsets  $A_1$  and  $A_2$  coincide modulo  $\mathcal{O}(h^\alpha)$  when the spectra of the corresponding operators coincide modulo  $\mathcal{O}(h^\alpha)$  in the above sense. In order to deal with multiplicities, we will, by convention, associate with the disjoint union  $S \sqcup S'$  the operator corresponding to the spectral measure  $\delta_S + \delta_{S'}$ .

**Remark 5.1.3.** — *The relation “the spectra of  $T_1$  and  $T_2$  coincide in  $I_h$  modulo  $\mathcal{O}(h^\alpha)$ ” is an equivalence relation. It is obviously symmetric and reflexive (taking  $K_h = J_h$ ). The transitivity follows from the triangle inequality for  $d_H$ .*

- *If the spectra of  $T_1$  and  $T_2$  coincide in  $I_h$  modulo  $\mathcal{O}(h^\alpha)$ , then, for all  $\tilde{I}_h \subset I_h$ , the spectra of  $T_1$  and  $T_2$  coincide in  $\tilde{I}_h$  modulo  $\mathcal{O}(h^\alpha)$ .*
- *If the spectra of  $T_1$  and  $T_2$  coincide in  $I_h$  modulo  $\mathcal{O}(h^\alpha)$ , we have*

$$d_H(\text{Sp}(T_1) \cap I_h, \text{Sp}(T_2) \cap I_h) = \mathcal{O}(h^\alpha).$$

- If the endpoints of the interval  $I_h$  stay away from an  $h^\beta$ -neighborhood of the spectrum, with  $\beta < \alpha$ , then for  $h$  small enough  $T_1$  and  $T_2$  have exactly the same number of eigenvalues inside  $I_h$ , counted with multiplicities.
- The notion described in Definition 5.1.6 already appears under various forms in the literature (see, for instance, the view point in [HS84, Section 1] and [Hel88, §4]).

We can now state our main result, where we use, among others, the eigenvalues  $\mu_k(\gamma, \sigma)$  and eigenfunctions  $u_k^{[\gamma, \sigma]}$  of the de Gennes operator (Section 5.1.2), the integer  $N$  defined in Notation 5.1.4, and the notation introduced in (5.6).

**Theorem 5.1.7.** *Under Assumption 2, the spectrum of  $\mathcal{L}_h$  in  $[ha, hb]$  coincides with that of  $h\mathfrak{M}_h$  modulo  $\mathcal{O}(h^2)$ , where*

$$\mathfrak{M}_h := \begin{bmatrix} m_1^{\mathbb{W}} & 0 & \cdots & 0 \\ 0 & m_2^{\mathbb{W}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & m_N^{\mathbb{W}} \end{bmatrix}$$

is a bounded operator acting diagonally on  $e^{i\alpha(h)} \cdot L^2(\mathbf{T}_{2L})^N$ . Here

$$\alpha(h) = \frac{|\Omega|}{|\partial\Omega|h},$$

and each  $m_k^{\mathbb{W}}$  is an  $h^{\frac{1}{2}}$ -pseudodifferential operator with symbol in  $S_{\mathbf{T}_{2L} \times \mathbb{R}}(1)$ . Let us denote by  $(s, \sigma)$  the (canonical) variables in  $\mathbf{T}_{2L} \times \mathbb{R}$ . Then we have:

- the principal symbol of  $m_k^{\mathbb{W}}$  is  $\mu_k^\circ(\sigma)$ ;
- its subprincipal symbol is  $-\kappa(s)\mathring{C}_k(\sigma)$  with

$$C_k(\sigma) = \left\langle \left( (\tau - \sigma)\tau^2 - \partial_\tau - 2\tau(\sigma - \tau)^2 \right) u_k^{[\gamma, \sigma]}(\tau), u_k^{[\gamma, \sigma]}(\tau) \right\rangle_{L^2(\mathbb{R}_+)}, \quad (5.8)$$

and  $\kappa(s)$  is the curvature of the boundary at the point of curvilinear abscissa  $s$ .

**Remark 5.1.4.** *Note that  $N$  will be constant as soon as  $a, b$  are chosen carefully enough in Assumption 2. Nevertheless, it is not necessary in our result.*

**Remark 5.1.5.** *We can check that, for all  $j \geq 1$ ,  $C_j(\xi_{j-1}(\gamma))$  has the same sign as  $\gamma_0^{[j-1]} - \gamma$ , see Proposition 5.6.2 where the threshold  $\gamma_0^{[j-1]}$  is discussed. Proposition 5.6.2 also corrects a mistake in [Kac06b, Lemma II.3 & (2.24)], where it is stated that  $C_1(\xi_0(\gamma))$  is always positive.*

It is important to notice that Theorem 5.1.7 is actually a diagonalization result since it reduces the spectral analysis of  $\mathcal{L}_h$  to that of a family of pseudo-differential operators in one dimension: the spectrum of  $\mathfrak{M}_h$  is the superposition (counting multiplicities) of the spectra of the  $m_k^{\mathbb{W}}$ ,  $k = 1, \dots, N$ . As it turns out, the spectrum of each of these pseudo-differential operators can be completely described using (refinements of) old and new results in the literature. Indeed, notice that the principal symbols  $\mu_k^\circ$  have a special feature: they depend only on the frequency variable  $\sigma$ , and, as functions of  $\sigma$ , they have at most a unique critical point, which is a nondegenerate minimum (Proposition 5.1.2). Hence, from a microlocal viewpoint, only two situations must be considered. Let  $E \in [a, b]$ , either  $E$  is a regular value of  $\mu_k^\circ$  (or  $\mu_k(\gamma, \cdot)$ , equivalently), and then the well-known Bohr-Sommerfeld rules apply, or  $E$  is a critical value of  $\mu_k^\circ$ , in which case the Hamiltonian  $(s, \sigma) \mapsto \mu_k^\circ(\sigma)$  admits a transversally non-degenerate minimum on a circle, and the recent study [DV21] of folded quantum action variables applies.



### 5.1.3.3 Eigenvalues in a regular spectral window

Our first application concerns the case where the interval  $[a, b]$  consists of regular values of all  $\mu_k$ . We will use the following well-known spectral result, an extension to all orders of the Bohr-Sommerfeld rules (see, for instance, [Roz75, HR82, HR84, Dui74]), which we prove in Section 5.5.2.1.

**Proposition 5.1.8.** *Consider an  $\hbar$ -pseudo-differential operator  $\mathbf{P}_\hbar \in \text{Op}_\hbar^W(S_{\mathbb{R}^2}(1))$  with symbol  $2L$ -periodic with respect to  $s$  and with principal symbol  $(s, \sigma) \mapsto \mu(\sigma)$  and subprincipal symbol  $(s, \sigma) \mapsto -\kappa(s)C(\sigma)$ . We consider its realization on  $e^{is\alpha(\hbar^2)}L^2(\mathbf{T}_{2L})$ . Let  $E$  be a regular value of  $\mu$  for which  $\mu^{-1}(E)$  is a finite set of points  $\sigma_1^E, \dots, \sigma_p^E$ .*

*Then, there exists  $\varepsilon > 0$  such that  $[E - \varepsilon, E + \varepsilon]$  is a set of regular values of  $\mu$ , and  $\mu^{-1}([E - \varepsilon, E + \varepsilon])$  is the disjoint union  $\Sigma_1 \sqcup \dots \sqcup \Sigma_p$  where each  $\Sigma_q \subset \mathbb{R}$  is a compact interval containing  $\sigma_q^E$  in its interior. Let  $\varepsilon > 0$  be any value satisfying the above conditions. For each  $q = 1, \dots, p$ , let  $\tilde{\Sigma}_q$  be an open interval containing  $\Sigma_q$  such that the  $\tilde{\Sigma}_q$ 's are pairwise disjoint. Then the following holds.*

*For each  $q = 1, \dots, p$ , there exists a smooth map  $\tilde{\Sigma}_q \ni \sigma \mapsto f_q(\sigma, \hbar) \in \mathbb{R}$  with an asymptotic expansion*

$$f_q(\sigma, \hbar) \sim f_{q,0}(\sigma) + \hbar f_{q,1}(\sigma) + \hbar^2 f_{q,2}(\sigma) + \dots$$

*depending only on the symbol of  $\mathbf{P}_\hbar$  in the cylinder  $\mathbf{T}_{2L} \times \Sigma_q$ , such that the spectrum of  $\mathbf{P}_\hbar$  inside  $[E - \varepsilon, E + \varepsilon]$  coincides, modulo  $\mathcal{O}(\hbar^\infty)$ , with the disjoint union*

$$\left( \bigsqcup_{q=1}^p \left\{ f_q(\sigma, \hbar), \sigma \in \hbar \left( \frac{\pi}{L} \mathbb{Z} + \alpha(\hbar^2) \right) \cap \tilde{\Sigma}_q \right\} \right) \cap [E - \varepsilon, E + \varepsilon],$$

*see Definition 5.1.6. Moreover, we have*

$$f_{q,0}(\sigma) = \mu(\sigma)|_{\Sigma_q} \tag{5.9}$$

$$f_{q,1}(\sigma) = \frac{-C(\sigma)|_{\Sigma_q}}{2L} \int_0^{2L} \kappa(s) ds. \tag{5.10}$$

Combining Proposition 5.1.8 and Theorem 5.1.7, we get the following result, where we use the notation of Corollary 5.1.3 and Theorem 5.1.7.

**Corollary 5.1.9 (Spectrum at Regular Values).** *Let  $[a, b]$  be an interval disjoint from  $\Theta$  and  $\Lambda$ . For each  $k = 1, \dots, N$ , for each  $q = 1, \dots, p(k)$ , let  $\Sigma_{k,q} \subset \mathbb{R}$  be an interval such that  $\mu_k(\gamma, \cdot)$  is a diffeomorphism from  $\Sigma_{k,q}$  to a neighborhood of  $[a, b]$  (i.e.  $\mu_k$  must be monotone on  $\Sigma_{k,q}$ ), in such a way that all  $\Sigma_{k,q}$  are pairwise disjoint and  $\bigcup_{q=1}^{p(k)} \Sigma_{k,q}$  contains  $\mu_k(\gamma, \cdot)^{-1}([a, b])$ . Then there exists a smooth map  $\Sigma_{k,q} \ni \sigma \mapsto f_{k,q}(\sigma, \hbar) \in \mathbb{R}$  with an asymptotic expansion*

$$f_{k,q}(\sigma, \hbar) \sim f_{k,q,0}(\sigma) + \hbar f_{k,q,1}(\sigma) + \hbar^2 f_{k,q,2}(\sigma) + \dots$$

*such that the spectrum of  $\mathcal{L}_\hbar$  in  $[ha, hb]$  coincides, modulo  $\mathcal{O}(\hbar^2)$ , with the disjoint union*

$$\left( \bigsqcup_{k=1}^N \bigsqcup_{q=1}^{p(k)} \left\{ \hbar f_{k,q}(\sigma, \hbar^{\frac{1}{2}}), \sigma \in \hbar^{\frac{1}{2}} \left( \frac{\pi}{L} \mathbb{Z} + \alpha(h) \right) \cap \Sigma_{k,q} \right\} \right) \cap [ha, hb].$$

Moreover, we have, when  $\sigma \in \Sigma_{k,q}$ ,

$$f_{k,q,0}(\sigma) = \mu_k(\gamma, \sigma) \quad (5.11)$$

$$f_{k,q,1}(\sigma) = -\langle \kappa \rangle C_k(\sigma) \quad (5.12)$$

where  $\langle \kappa \rangle$  is the average curvature:

$$\langle \kappa \rangle = \frac{1}{2L} \int_0^{2L} \kappa(s) ds.$$

Since the leading terms (5.11) and (5.12) do not depend on  $q$  (apart from the domain of definition  $\Sigma_{k,q}$ ) we obtain that the spectrum of  $\mathcal{L}_h$  in  $[ha, hb]$  coincides, modulo  $\mathcal{O}(h^2)$ , with the disjoint union

$$\bigsqcup_{k=1}^N \left\{ h\mu_k(\gamma, \sigma) - h^{\frac{3}{2}} \langle \kappa \rangle \mathring{C}_k(\sigma), \quad \sigma \in h^{\frac{1}{2}} \left( \frac{\pi}{L} \mathbb{Z} + \alpha(h) \right) \right\} \cap [ha, hb].$$

**Remark 5.1.6.** Corollary 5.1.9 gives a very accurate description of the spectrum of  $\mathcal{L}_h$  by providing us with explicit approximations of the eigenvalues in  $[ha, hb]$  modulo  $\mathcal{O}(h^2)$ . When  $\gamma = +\infty$  (i.e. in the Dirichlet case), it also improves the description given in [GV21, Corollary 2.7] concerned with a thin spectral window containing a regular value. Moreover, although our results are formulated in terms of approximation of the eigenvalues, the strategy, based on microlocal projections, leading to Theorem 5.1.7 can also be used to describe the eigenspaces of  $\mathcal{L}_h$  in terms of those of  $\mathfrak{M}_h$ .

As a first application of this corollary, we obtain a very accurate formula for the number of eigenvalues of  $\mathcal{L}_h$  in  $[ha, hb]$ , which is much less than the crude estimate (5.3):

**Theorem 5.1.10** (Precise Weyl formula). *Let  $I_h = [ha, hb]$  where  $[a, b]$  is an interval disjoint from  $\Theta$  and  $\Lambda$ . Then the number of eigenvalues of  $\mathcal{L}_h$  in  $I_h$  is*

$$N(\mathcal{L}_h, I_h) = \left\lfloor \frac{L}{\pi h^{1/2}} \sum_{k,q} \delta_{k,q}^{[0]} + \frac{L \langle \kappa \rangle}{\pi} \sum_{k,q} \delta_{k,q}^{[1]} + \mathcal{O}(h^{1/2}) \right\rfloor,$$

where we use the notation  $\sum_{k,q} := \sum_{k=1}^N \sum_{q=1}^{p(k)}$ , and

$$\delta_{k,q}^{[0]} := |\alpha_{k,q} - \beta_{k,q}|, \quad \delta_{k,q}^{[1]} := \frac{C_k(\beta_{k,q})}{|\mu'_k(\beta_{k,q})|} - \frac{C_k(\alpha_{k,q})}{|\mu'_k(\alpha_{k,q})|},$$

with  $\alpha_{k,q} := \mu_{k,q}^{-1}(a)$ ,  $\beta_{k,q} := \mu_{k,q}^{-1}(b)$ .

In this statement we have denoted  $\mu_{k,q} := \mu_k(\gamma, \cdot)|_{\Sigma_{k,q}}$ . Notice that, since the remainder term  $\mathcal{O}(h^{1/2})$  tends to 0, we obtain that, when  $h$  is small enough,  $N(\mathcal{L}_h, I_h)$  is equal to the integer part of  $\frac{L}{\pi h^{1/2}} \sum_{k,q} \delta_{k,q}^{[0]} + \frac{L \langle \kappa \rangle}{\pi} \sum_{k,q} \delta_{k,q}^{[1]}$ , or this plus or minus 1.

### 5.1.3.4 Critical values

Our main theorem also applies to the case when the spectral window contains a critical value, i.e. an element of  $\Theta$ , see Corollary 5.1.3 (such a critical value is the unique non-degenerate global minimum of a unique dispersion curve, see Proposition 5.1.2). To illustrate this, let us focus on the low-lying eigenvalues. The following corollary improves [Kac06b, Theorem I.5,  $\alpha = \frac{1}{2}$ ] by establishing the spectral asymptotics of the lowest eigenvalues and by exhibiting spectral gaps of order  $h^{\frac{7}{4}}$  instead of  $h^{\frac{3}{2}}$  in the case of regular values

for each given dispersion curve. It also extends to any Robin parameter the result obtained by Fournais and Helffer in [FH10b] when  $\gamma = 0$ .

Once Theorem 5.1.7 is applied and reduces the analysis to a single  $\hbar$ -pseudo-differential operator, this corollary becomes essentially an application of [DV21, Proposition 6.8], see details in Section 5.5.2.2.

**Corollary 5.1.11.** *Consider  $\gamma \neq \gamma_0^{[0]}$  with  $\gamma_0^{[0]}$  defined in Remark 5.1.5, and let  $\epsilon = \text{sign}(\gamma_0^{[0]} - \gamma) = \text{sign}(C_1(\xi_0(\gamma)))$ . Assume that  $\epsilon\kappa$  admits a unique maximum at  $s_{\max}$ , which is non-degenerate. Then, for all  $j \geq 1$ , uniformly when  $j\hbar^{\frac{1}{4}} = o(1)$ ,*

$$\lambda_j(\gamma, h) = \Theta^{[0]}(\gamma)h - \kappa_{\max}C_1(\xi_0(\gamma))h^{\frac{3}{2}} + \left( (2j-1)\sqrt{\frac{C_1(\xi_0(\gamma))k_2\mu_1''(\gamma, \xi_0(\gamma))}{4}} \right) h^{\frac{7}{4}} + o(h^{\frac{7}{4}}),$$

with  $k_2 = -\kappa''(s_{\max}) > 0$ , and where we recall that  $\xi_0(\gamma)$  is given in Proposition 5.1.2.

**Remark 5.1.7.** *Let us end the description of our results with a few comments about consequences and extensions following from our approach.*

- (i) *Corollary 5.1.11 describes the low-lying eigenvalues with some uniformity in  $j$  (which was not the case in [FH06]), in an interval of the form  $(-\infty, \Theta^{[0]}(\gamma)h + Ch^{3/2}]$ . On the other hand, Corollary 5.1.9 gives the spectrum in any interval of the form  $[ha, hb]$  with  $a > \Theta^0(\gamma)$  and  $b < 1$ . Hence we have a spectral interval between these two regimes which we don't describe here. But actually, by using refined spectral results for 1D pseudodifferential operators, and in particular the strategy of [DV21] in the case where  $\kappa$  is a Morse function, it should also be possible to close this gap. However, this would require an analysis of the hyperbolic singularities arising from the minima of  $\epsilon\kappa$ , where we expect both a concentration of the eigenfunctions and a higher density of eigenvalues.*
- (ii) *When  $\gamma > \gamma_0^{[0]}$ , Corollary 5.1.11 shows that the eigenfunctions (associated with the low-lying eigenvalues) are concentrated near the points of minimal curvature. This contrasts with the Neumann case when the points of maximal curvature play the role of attractive wells. This phenomenon was not observed before, see Remark 5.1.5.*
- (iii) *The case  $\gamma = \gamma_0^{[0]}$  is critical since  $C_1(\xi_0(\gamma)) = 0$ . However, our analysis can still be used by computing additional subprincipal terms in our effective operator method. A similar phenomenon has recently been observed in the study of the magnetic Dirac operator [BLTRS23, Section 8] and also in the analysis of the magnetic Schrödinger operator with discontinuous magnetic fields [FHKR22]. In this case, we have, for all  $j \geq 1$ ,*

$$\lambda_j(\gamma, h) = \Theta^{[0]}(\gamma)h + h^2\lambda_j(\mathcal{A}_h) + o(h^2),$$

where

$$\mathcal{A}_h = \frac{\partial_\sigma^2 \mu(\gamma, \xi_0(\gamma))}{2}(D_s + \alpha(h) - h^{-\frac{1}{2}}\xi_0(\gamma))^2 + C_\gamma \kappa^2(s),$$

for some  $C_\gamma \in \mathbb{R}$ . In this transition regime, the effective operator is not semiclassical.

- (iv) *When the curvature  $\kappa$  is constant, in the case  $\gamma \in \mathbb{R}$ , we are in a degenerate situation rather similar to the case when  $\gamma = \gamma_0^{[0]}$ . Concerning the operators  $m_k^W$  of Theorem 5.1.7, this case corresponds to [DV21, Proposition 6.4]. We can prove an expansion in the form*

$$\lambda_j(\gamma, h) = \Theta^{[0]}(\gamma)h - \kappa C_1(\xi_0(\gamma))h^{\frac{3}{2}} + h^2\lambda_j(\mathcal{A}_h) + o(h^2).$$

Here, the eigenvalues of  $\mathcal{A}_h$  will generate magnetic oscillations, see [DV21, Theorem 2.2;  $k = 0$ ]. When  $\gamma = 0$  and  $j = 1$ , a similar estimate is described in [FH06, Theorem 5.3.1].

### 5.1.4 Organization of the chapter

In Section 5.2, we prove that the eigenfunctions associated with eigenvalues of  $\mathcal{L}_h$  in  $[ha, hb]$  are exponentially localized near the boundary of  $\Omega$ , see Proposition 5.2.1. Note that the strategy used to derive this localization deviates from the usual variational method (see, for instance, [Hel88] or [Ray17, Prop. 4.7]), which fails since we want to consider eigenvalues between two consecutive Landau levels. To overcome this issue, our strategy, which eventually generalizes the variational method, is based on establishing the bijectivity of the magnetic Laplacian between exponentially weighted  $L^2$  spaces. In Section 5.3, by means of tubular coordinates  $(s, t)$  near the boundary and a rescaling  $t = h^{\frac{1}{2}}\tau$ , we introduce a model operator  $\mathcal{N}_h$  depending on the effective semiclassical parameter  $\hbar = h^{\frac{1}{2}}$ , acting on  $2L$ -periodic functions and involving a flux term  $f_0$ , see (5.28) and (5.27). We also show that the eigenfunctions of  $\mathcal{L}_h$  are roughly microlocalized in a compact set of the phase space attached to the boundary, see Proposition 5.3.1. This allows to prove that the spectrum of  $\mathcal{L}_h$  (between the Landau levels) is located near that of  $\mathcal{N}_h$ , see Proposition 5.3.2. However, one will see that Proposition 5.3.2 is not directly useful to establish our main theorems. It is rather a pretext to motivate the introduction of  $\mathcal{N}_h$  and to describe the spectral estimates required to prove that spectra coincide modulo  $\mathcal{O}(h^\infty)$ . Actually, one will compare directly the spectrum of  $\mathcal{L}_h$  to that of an effective operator on the boundary of  $\Omega$ . For that purpose, in Section 5.4, we construct a Grushin problem in order to invert the (pseudo)differential operator  $\mathfrak{N}_h$  (which acts as  $\mathcal{N}_h$  with  $f_0$  replaced by 0). This method, inspired by the works of Martinez and Sjöstrand (and adapted to magnetic operators by Keraval, see [Ker18]), has recently shown its efficiency to describe the low-lying eigenvalues of various magnetic operators (see, for instance, [BNHR22] and [FHKR22]). The novelty in the present paper is to use it to tackle the description of larger eigenvalues for magnetic Schrödinger operators with boundaries, when several dispersion curves are involved (see Figure 5.3), and not only the first one as in [BNHR22] or [FHKR22]. In order to use this method, we write a semiclassical expansion of  $\mathcal{N}_h$ , see Proposition 5.4.4. The principal operator symbol is the de Gennes operator (with Robin condition), which can be inverted in the spectral window  $[a, b]$  up to considering an augmented matrix involving the eigenfunctions of the de Gennes operator, see Lemma 5.4.5. This allows to build an approximate inverse of an augmented version of  $\mathfrak{N}_h$  denoted by  $\text{Op}_h^W \mathcal{P}_h$ , see (5.43) (and the left and right quasi inverses (5.44) and (5.45)). Thanks to these quasi inverses, the bijectivity of  $\mathcal{L}_h - z$  is reduced to that of a pseudodifferential operator on  $\mathbf{T}_{2L}$  whose matrix symbol is  $M_h$ , modulo some remainders, see Proposition 5.4.6 where the eigenfunctions of  $\mathcal{L}_h$  are directly used as quasimodes for  $M_h^W$ . In Section 5.5, we perform the spectral analysis of  $M_h^W$  by using that the principal matrix symbol  $M_0$  is diagonal with uniform gaps between the diagonal entries. We deduce Proposition 5.1.8 and Corollary 5.1.11. In Appendix 5.6.1, we recall the origin of the estimate (5.3). Appendix 5.6.2 is devoted to the de Gennes operator with Robin conditions: a couple of known results are recalled and useful new ones are established.

## 5.2 Exponential localization near the boundary and consequences

Let us consider a smooth function  $\Phi_0 : \overline{\Omega} \rightarrow \mathbb{R}_+$  that coincides with  $\text{dist}(x, \partial\Omega)$  near  $\partial\Omega$ . We assume, as we may, that  $\Phi_0$  is extended to a smooth function on  $\mathbb{R}^2$  that also belongs to  $W^{\infty,2}(\mathbb{R}^2)$ . The following proposition states that the eigenfunctions associated with eigenvalues in  $I_h$  are localized near the boundary of  $\Omega$ . The estimates look like Agmon's estimates, but there are not obtained via variational means as it is the case in many magnetic settings. Here, they follow from resolvent estimates using the distance to the Landau levels.

**Proposition 5.2.1.** *There exist  $\alpha > 0, C > 0, h_0 > 0$  such that for all  $h \in (0, h_0)$  and all eigenfunctions  $\psi$  associated with an eigenvalue in  $I_h$ , we have*

$$\int_{\Omega} e^{2\alpha\Phi_0(x)/h^{\frac{1}{2}}} |\psi|^2 dx \leq C \|\psi\|^2, \quad (5.13)$$

and

$$\int_{\Omega} e^{2\alpha\Phi_0(x)/h^{1/2}} |(-ih\nabla - \mathbf{A})\psi|^2 dx \leq Ch\|\psi\|^2. \quad (5.14)$$

### 5.2.1 Preliminaries

In the following,  $\mathcal{L}_h^{\mathbb{R}^2}$  denotes the operator  $(-ih\nabla - \mathbf{A})^2$  acting on the Hilbert space  $L^2(\mathbb{R}^2)$ . By using the gauge invariance, we assume in the whole section that  $\mathbf{A} = \frac{1}{2}(-x_2, x_1)$ . Due to our choice of  $\lambda$ , we deduce that  $\mathcal{L}_h^{\mathbb{R}^2} - \lambda$  is bijective and that there exists  $C > 0$  such that, for all  $h > 0$ ,

$$\|(\mathcal{L}_h^{\mathbb{R}^2} - \lambda)^{-1}\| \leq Ch^{-1}.$$

More precisely, we can take  $C = \min(|2n - 3 - a|, |2n - 1 - b|)^{-1}$ . We let  $\Phi = \alpha\Phi_0$ , with  $\alpha > 0$  to be determined, and consider the conjugated operator

$$\mathcal{L}_h^{\Phi} := e^{\Phi/h^{1/2}} \mathcal{L}_h^{\mathbb{R}^2} e^{-\Phi/h^{1/2}} = (-ih\nabla - \mathbf{A} + ih^{1/2}\nabla\Phi)^2 \quad (5.15)$$

$$= \mathcal{L}_h^{\mathbb{R}^2} + 2ih^{1/2}\nabla\Phi \cdot (-ih\nabla - \mathbf{A}) - h|\nabla\Phi|^2 - ih^{3/2}\Delta\Phi. \quad (5.16)$$

The following lemma tells us that the invertibility is preserved for  $\mathcal{L}_h^{\Phi} - \lambda$  if  $\alpha$  is small enough.

**Lemma 5.2.2.** *There exists  $C > 0$  such that for all  $h > 0$  and all  $\alpha > 0$ ,*

$$h^{1/2}\|\nabla\Phi \cdot (-ih\nabla - \mathbf{A})(\mathcal{L}_h^{\mathbb{R}^2} - \lambda)^{-1}\| \leq C\alpha. \quad (5.17)$$

*In particular,  $\mathcal{L}_h^{\Phi} - \lambda$  is bijective as soon as  $\alpha$  is chosen small enough. With such a choice of  $\alpha$ , there exists  $C > 0$  such that, for all  $h > 0$ ,*

$$\left\| (\mathcal{L}_h^{\Phi} - \lambda)^{-1} \right\| \leq \frac{C}{h}. \quad (5.18)$$

*Proof.* Consider  $v \in L^2(\mathbb{R}^2)$  and let  $u = (\mathcal{L}_h^{\mathbb{R}^2} - \lambda)^{-1}v$ . We have

$$(\mathcal{L}_h^{\mathbb{R}^2} - \lambda)u = v,$$

so that, by taking the scalar product with  $u$  and using that  $\lambda \leq Ch$ ,

$$\|(-ih\nabla - \mathbf{A})u\|^2 \leq Ch\|u\|^2 + \|u\|\|v\|.$$

Therefore, since  $\nabla\Phi_0 \in L^\infty$ , there is a new constant  $C' > 0$  such that

$$h^{1/2}\|\nabla\Phi \cdot (-ih\nabla - \mathbf{A})u\| \leq C'\alpha h\|u\| + C'h^{1/2}\alpha\|u\|^{1/2}\|v\|^{1/2}.$$

Since  $\|u\| = \|(\mathcal{L}_h^{\mathbb{R}^2} - \lambda)^{-1}v\| \leq Ch^{-1}\|v\|$ , we see that

$$h^{1/2}\|\nabla\Phi \cdot (-ih\nabla - \mathbf{A})(\mathcal{L}_h^{\mathbb{R}^2} - \lambda)^{-1}v\| \leq \tilde{C}\alpha\|v\|,$$

which gives (5.17).

Let us now deal with the bijectivity. We have

$$\mathcal{L}_h^{\Phi} - \lambda = \mathcal{L}_h^{\mathbb{R}^2} - \lambda + B,$$

with

$$B := 2ih^{\frac{1}{2}} \nabla \Phi \cdot (-ih\nabla - \mathbf{A}) - h|\nabla \Phi|^2 - ih^{\frac{3}{2}} \Delta \Phi.$$

Since  $\nabla \Phi_0$  and  $\Delta \Phi_0$  are bounded, we deduce from (5.17) that

$$\|B(\mathcal{L}_h^{\mathbb{R}^2} - \lambda)^{-1}\| \leq C\alpha + C_1\alpha + h^{1/2}C_2\alpha^2 \leq \tilde{C}\alpha,$$

when  $\alpha$  is small enough. On the other hand,

$$\mathcal{L}_h^\Phi - \lambda = \left( \text{Id} + B(\mathcal{L}_h^{\mathbb{R}^2} - \lambda)^{-1} \right) (\mathcal{L}_h^{\mathbb{R}^2} - \lambda).$$

For  $\alpha$  small enough, we deduce that  $\text{Id} + B(\mathcal{L}_h^{\mathbb{R}^2} - \lambda)^{-1}$  is invertible, and thus so is  $\mathcal{L}_h^\Phi - \lambda$ .  $\square$

In order to prove Proposition 5.2.1, we need to localize on an  $h^{1/2}$ -neighborhood of  $\partial\Omega$ . For this purpose, we introduce two functions  $\chi_h \in \mathcal{C}_0^\infty(\Omega)$  and  $\tilde{\chi}_h \in \mathcal{C}^\infty(\bar{\Omega})$  as follows.

$$\chi_h: \begin{cases} \Omega \longrightarrow [0, 1] \\ x \longmapsto g(\Phi_0(x)/h^{\frac{1}{2}}) \end{cases} \quad \text{and} \quad \tilde{\chi}_h: \begin{cases} \bar{\Omega} \longrightarrow [0, 1] \\ x \longmapsto 1 - g(\Phi_0(x)/2h^{\frac{1}{2}}) \end{cases}$$

where  $g$  is a smooth non-decreasing function on  $\mathbb{R}$ , valued in  $[0, 1]$ , equal to 0 on  $(-\infty, 1)$  and to 1 on  $(2, +\infty)$ . In particular,

$$\text{supp}(\chi_h) \cap \bar{\Omega} \subset \{x \in \Omega, h^{-\frac{1}{2}}\Phi_0(x) \geq 1\}, \quad (5.19)$$

and

$$\text{supp}(\nabla \chi_h) \cap \bar{\Omega} \subset \{x \in \Omega : h^{-\frac{1}{2}}\Phi_0(x) \in [1, 2]\} \subset \{x \in \Omega : \tilde{\chi}_h(x) = 1\}. \quad (5.20)$$

Note that the following properties hold:

- $\chi_h = 1$  away from an  $h^{1/2}$ -neighborhood  $\partial\Omega$ ,
- $\nabla \chi_h$  is supported in an  $h^{1/2}$ -neighborhood of  $\partial\Omega$ ,
- $\mathbb{1}_{\text{supp} \nabla \chi_h} \leq \tilde{\chi}_h$ ,
- $\tilde{\chi}_h = 0$  away from an  $h^{1/2}$ -neighborhood of  $\partial\Omega$ .

## 5.2.2 Proof of Proposition 5.2.1

Let us consider  $\lambda \in [ha, hb] \cap \text{Sp}(\mathcal{L}_h)$  and an associated eigenfunction  $\psi \in \text{Dom}(\mathcal{L}_h)$ . We have

$$\left( (-ih\nabla - \mathbf{A})^2 - \lambda \right) \psi = 0.$$

Let  $\varphi = e^{\Phi/h^{1/2}} \psi$ . Using (5.16) in  $\Omega$ , the equation becomes

$$\left( \mathcal{L}_h^\Phi - \lambda \right) \varphi = 0. \quad (5.21)$$

Then, we have

$$\begin{aligned}
 (\mathcal{L}_h^\Phi - \lambda)(\chi_h \varphi) &= [\mathcal{L}_h^\Phi, \chi_h] \varphi \\
 &= e^{\Phi/h^{1/2}} [\mathcal{L}_h, \chi_h] e^{-\Phi/h^{1/2}} \varphi \\
 &= e^{\Phi/h^{1/2}} \left( -h^2 \Delta \chi_h - 2ih \nabla \chi_h \cdot (-ih \nabla - \mathbf{A}) \right) e^{-\Phi/h^{1/2}} \varphi \\
 &= \left( -h^2 \Delta \chi_h - 2ih \nabla \chi_h \cdot (-ih \nabla - \mathbf{A} + ih^{\frac{1}{2}} \nabla \Phi) \right) \varphi. \tag{5.22}
 \end{aligned}$$

We have  $\|h^2(\Delta \chi_h) \varphi\| \leq Ch \|\tilde{\chi}_h \varphi\|$ . (Here and in the rest of the paper,  $C$  denotes a constant that is independent on  $h$  but that can vary from line to line.) Let us explain how to deal with the last term. We have

$$h \|\nabla \chi_h \cdot (-ih \nabla - \mathbf{A} + ih^{\frac{1}{2}} \nabla \Phi) \varphi\| \leq h \|\nabla \chi_h \cdot (-ih \nabla - \mathbf{A}) \varphi\| + Ch \|\tilde{\chi}_h \varphi\|.$$

Let us temporarily admit that, for  $\alpha$  small enough,

$$h \|\nabla \chi_h \cdot (-ih \nabla - \mathbf{A}) \varphi\| \leq Ch \|\tilde{\chi}_h \varphi\|. \tag{5.23}$$

We then immediately deduce from (5.22) that

$$\|(\mathcal{L}_h^\Phi - \lambda)(\chi_h \varphi)\| \leq \tilde{C} h \|\tilde{\chi}_h \varphi\|. \tag{5.24}$$

Since  $\chi_h \varphi \in \text{Dom}(\mathcal{L}_h^{\mathbb{R}^2})$  we obtain from (5.18) that

$$\|\chi_h \varphi\| \leq C \|\tilde{\chi}_h \varphi\|,$$

which implies that

$$\|\varphi\| \leq \tilde{C} (\|\tilde{\chi}_h \varphi\| + \|(1 - \chi_h) \varphi\|),$$

showing that  $\varphi$  is localized near  $\partial\Omega$ . More precisely, recalling that  $\varphi = e^{\Phi/h^{\frac{1}{2}}} \psi$ , using that  $\Phi_0(x) = \text{dist}(x, \partial\Omega)$  near the boundary, and the fact that the supports of  $\tilde{\chi}_h$  and  $1 - \chi_h$  lie in neighborhood of the boundary of size  $h^{\frac{1}{2}}$ , we deduce (5.13).

Let us now deal with (5.14). We have the Agmon identity

$$\text{Re} \langle \mathcal{L}_h \psi, e^{2\Phi/h^{1/2}} \psi \rangle = \mathcal{Q}_{h, \mathbf{A}}(e^{\Phi/h^{1/2}} \psi) - h \|e^{\Phi/h^{1/2}} \psi \nabla \Phi\|^2,$$

which follows from (5.16) where we see that  $\text{Re} \mathcal{L}_h^\Phi = \mathcal{L}_h - h |\nabla \Phi|^2$  and we notice that  $\text{Re} \langle \mathcal{L}_h \psi, e^{2\Phi/h^{1/2}} \psi \rangle = \langle (\text{Re} \mathcal{L}_h^\Phi) e^{\Phi/h^{1/2}} \psi, e^{\Phi/h^{1/2}} \psi \rangle$ . Recall also that, when  $u \in \text{Dom}(\mathcal{L}_h)$ , then  $\langle \mathcal{L}_h u, u \rangle = \mathcal{Q}_{h, \mathbf{A}}(u)$ , see (5.1).

Then, by using that  $\psi$  is an eigenfunction, we get

$$\int_{\Omega} \left| (-ih \nabla - \mathbf{A}) \left( e^{\Phi/h^{1/2}} \psi \right) \right|^2 dx + \gamma h^{\frac{3}{2}} \int_{\partial\Omega} |e^{\Phi/h^{1/2}} \psi|^2 ds - h \|e^{\Phi/h^{1/2}} \psi \nabla \Phi\|^2 = \lambda \|e^{\Phi/h^{1/2}} \psi\|^2.$$

With (5.13), we find

$$\int_{\Omega} \left| (-ih \nabla - \mathbf{A}) \left( e^{\Phi/h^{1/2}} \psi \right) \right|^2 dx + \gamma h^{\frac{3}{2}} \int_{\partial\Omega} |e^{\Phi/h^{1/2}} \psi|^2 ds \leq Ch \|\psi\|^2.$$

From a classical trace theorem, there exists  $C > 0$  such that for all  $\varepsilon > 0$ , we have

$$\int_{\partial\Omega} |\varphi|^2 ds \leq C \left( \varepsilon^{-1} \|\varphi\|^2 + \varepsilon \|\nabla|\varphi|\|^2 \right).$$

With the diamagnetic inequality (Theorem A.1.3), we deduce that

$$h^2 \int_{\partial\Omega} |\varphi|^2 ds \leq C \left( h^2 \varepsilon^{-1} \|\varphi\|^2 + \varepsilon \|(-ih\nabla - \mathbf{A})\varphi\|^2 \right),$$

and then

$$h^{\frac{3}{2}} \int_{\partial\Omega} |\varphi|^2 ds \leq C \left( h^{\frac{3}{2}} \varepsilon^{-1} \|\varphi\|^2 + \varepsilon h^{-\frac{1}{2}} \|(-ih\nabla - \mathbf{A})\varphi\|^2 \right).$$

Taking  $\varepsilon = \frac{h^{\frac{1}{2}}}{2|c|C}$  implies that

$$\int_{\Omega} \left| (-ih\nabla - \mathbf{A}) \left( e^{\Phi/h^{1/2}} \psi \right) \right|^2 dx \leq \tilde{C} h \|\psi\|^2.$$

Computing a commutator gives (5.14).

It remains to explain why (5.23) holds. From (5.16) we can write

$$\mathcal{L}_h^\Phi = L_1 + ih^{\frac{1}{2}} L_2, \quad (5.25)$$

with

$$L_1 = (-ih\nabla - \mathbf{A})^2 - h|\nabla\Phi|^2, \quad L_2 = 2\nabla\Phi \cdot (-ih\nabla - \mathbf{A}) - ih\Delta\Phi.$$

From (5.25) and (5.21), we get

$$(L_1 - \lambda + ih^{\frac{1}{2}} L_2)\varphi = 0.$$

For  $j = 1, 2$ , we have

$$\operatorname{Re}\langle (L_1 - \lambda)\varphi, (\partial_j \chi_h)^2 \varphi \rangle - h^{\frac{1}{2}} \Im\langle L_2 \varphi, (\partial_j \chi_h)^2 \varphi \rangle = 0.$$

Thanks to the classical localization formula (see, for instance, [Ray17, Prop. 4.2]), we have

$$\begin{aligned} & \operatorname{Re}\langle (L_1 - \lambda)\varphi, (\partial_j \chi_h)^2 \varphi \rangle \\ &= \|(-ih\nabla - \mathbf{A})[(\partial_j \chi_h)\varphi]\|^2 - h \int_{\Omega} |\nabla\Phi|^2 |(\partial_j \chi_h)\varphi|^2 dx - \lambda \|\partial_j \chi_h \varphi\|^2 - h^2 \|\nabla(\partial_j \chi_h)\varphi\|^2. \end{aligned}$$

Moreover,

$$\begin{aligned} & |\Im\langle L_2 \varphi, (\partial_j \chi_h)^2 \varphi \rangle| \\ &= |\Im\langle (\partial_j \chi_h) L_2 \varphi, (\partial_j \chi_h)\varphi \rangle| \\ &\leq |\Im\langle L_2((\partial_j \chi_h)\varphi), (\partial_j \chi_h)\varphi \rangle| + |\langle [L_2, \partial_j \chi_h]\varphi, (\partial_j \chi_h)\varphi \rangle| \\ &\leq Ch \|(\partial_j \chi_h)\varphi\|^2 + C\alpha \|(-ih\nabla - \mathbf{A})(\partial_j \chi_h)\varphi\| \|\partial_j \chi_h \varphi\| + Ch \|\nabla(\partial_j \chi_h)\varphi\| \|\partial_j \chi_h \varphi\|. \end{aligned}$$

Due to the properties of  $\chi_h$ , we have

$$\begin{aligned} |\Im\langle L_2 \varphi, (\partial_j \chi_h)^2 \varphi \rangle| &\leq C \|\tilde{\chi}_h \varphi\| + Ch^{-\frac{1}{2}} \alpha \|(-ih\nabla - \mathbf{A})(\partial_j \chi_h)\varphi\|^2 + Ch^{-\frac{1}{2}} \alpha \|\tilde{\chi}_h \varphi\|^2 \\ &\quad + Ch^{-\frac{1}{2}} \|\tilde{\chi}_h \varphi\|^2. \end{aligned}$$



Therefore,

$$\|(-ih\nabla - \mathbf{A})[(\partial_j \chi_h)\varphi]\|^2 \leq C\|\tilde{\chi}_h\varphi\|^2 + C\alpha\|(-ih\nabla - \mathbf{A})(\partial_j \chi_h)\varphi\|^2.$$

Taking  $\alpha$  small enough, we get

$$\|(-ih\nabla - \mathbf{A})[(\partial_j \chi_h)\varphi]\|^2 \leq C\|\tilde{\chi}_h\varphi\|^2.$$

Computing a commutator, we get (5.23).

## 5.3 An operator on a semi-cylinder

### 5.3.1 A model operator

The exponential localization near the boundary at a scale of order  $h^{\frac{1}{2}}$  given by Proposition 5.2.1 invites us to use the classical tubular coordinates  $(s, t)$  near the boundary. We recall that these coordinates are defined thanks to the application

$$\Gamma : \mathbb{R}/2L\mathbb{Z} \times (0, t_0) \ni (s, t) \mapsto \Gamma(s) - t\mathbf{n}(s),$$

which is injective if  $t_0$  is small enough. Its Jacobian is  $a(s, t) = 1 - t\kappa(s)$ , where  $\kappa$  is the curvature of the boundary at the point  $\Gamma(s)$ . Here  $\Gamma$  is a counterclockwise parametrization by the curvilinear abscissa. Thus,  $\Gamma$  induces a smooth diffeomorphism between  $\mathbb{R}/2L\mathbb{Z} \times (0, t_0)$  and  $\Omega_{t_0} := \Gamma(\mathbb{R}/2L\mathbb{Z} \times (0, t_0))$ .

By using [FH10b, Appendix F], we can check that the magnetic Laplacian acts locally near the boundary in these coordinates as

$$\begin{aligned} \tilde{\mathcal{L}}_h = a(s, t)^{-1} \left( -ih\partial_s - t + f_0 + \kappa(s)\frac{t^2}{2} \right) a(s, t)^{-1} \left( -ih\partial_s - t + f_0 + \kappa(s)\frac{t^2}{2} \right) \\ - h^2 a(s, t)^{-1} \partial_t a(s, t) \partial_t, \end{aligned}$$

in the ambient Hilbert space  $L^2(ad_s dt)$ . Here  $f_0 = \frac{|\Omega|}{|\partial\Omega|}$ . The boundary condition (5.2) becomes

$$\partial_t \psi(s, 0) = \gamma h^{-\frac{1}{2}} \psi(s, 0).$$

Of course the operator  $\tilde{\mathcal{L}}_h$  is only defined near  $t = 0$ . We would like to consider a global operator, whose eigenfunctions satisfy Agmon type estimates with respect to  $t$ . This can be done by inserting cutoff functions with respect to  $t$ . We let  $\check{t} = t\zeta(h^{-\frac{1}{2}+\eta}t)$  with  $\eta \in (0, \frac{1}{2})$  and  $\zeta$  a smooth cutoff function equal to 1 near 0.

**Remark 5.3.1.** *We cannot directly reduce the problem to a problem near the boundary, as we have seen in previous chapters. In fact, if we consider the problem in a tubular neighborhood with Dirichlet condition on the outer and inner edge and since we have shown that the eigenfunctions are localized near the  $\partial\Omega$ , this also works on the edges of the tubular neighborhood, then we cannot localize the problem in this tubular neighborhood. To do this, we introduce the cutoff function  $\zeta$ .*

Let us consider the differential operator acting as

$$\begin{aligned} \tilde{\mathcal{L}}_h = a(s, \check{t})^{-1} \left( -ih\partial_s - t + f_0 + \kappa(s)\frac{\check{t}^2}{2} \right) a(s, \check{t})^{-1} \left( -ih\partial_s - t + f_0 + \kappa(s)\frac{\check{t}^2}{2} \right) \\ - h^2 a(s, \check{t})^{-1} \partial_t a(s, \check{t}) \partial_t, \end{aligned}$$

on the domain

$$\begin{aligned} \text{Dom}(\widetilde{\mathcal{L}}_h) = \{u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+) : -\partial_t^2 u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+), \\ (-ih\partial_s - t + \mathfrak{f}_0)^2 u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+), \partial_t u(\cdot, 0) = \gamma h^{-\frac{1}{2}} u(\cdot, 0)\}. \end{aligned}$$

The ambient Hilbert space is  $L^2(a(s, \check{t})dsdt) = L^2(dsdt)$ , with  $2L$ -periodic condition with respect to  $s$ .

### 5.3.2 The rescaled operator

The exponential localization at the scale  $h^{\frac{1}{2}}$  near the boundary suggests to consider the partial rescaling

$$(s, t) = (s, \hbar\tau),$$

where  $\hbar = h^{\frac{1}{2}}$ . We consider the new operator, acting in the ambient Hilbert space  $L^2(\hat{a}_\hbar dsdt) = L^2(dsdt)$ ,

$$\widehat{\mathcal{L}}_\hbar = \hat{a}_\hbar(s, \tau)^{-1} p_{s, \hbar} \hat{a}_\hbar(s, \tau)^{-1} p_{s, \hbar} - \hat{a}_\hbar(s, \tau)^{-1} \partial_\tau \hat{a}_\hbar(s, \tau) \partial_\tau, \quad (5.26)$$

with

$$p_{s, \hbar} = -i\hbar\partial_s - \tau + \theta + \hbar\kappa(s) \frac{\hat{\tau}^2}{2}, \quad \theta = \hbar^{-1}\mathfrak{f}_0, \quad (5.27)$$

and where  $\hat{a}_\hbar(s, \tau) = 1 - \hbar\hat{\tau}\kappa$  with  $\hat{\tau} = \zeta(\hbar^{2\eta}\tau)\tau$ . In the following, we let  $\mu = \hbar^{2\eta}$  and  $\zeta_\mu(\tau) = \zeta(\hbar^{2\eta}\tau)\tau$ . This will be convenient when expanding the operator in powers of  $\hbar$  ( $\mu$  will be considered a parameter).

The boundary condition becomes

$$\partial_\tau \psi(s, 0) = \gamma \psi(s, 0).$$

The domain is given by

$$\begin{aligned} \text{Dom}(\widehat{\mathcal{L}}_\hbar) = \{u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+) : -\partial_\tau^2 u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+), \\ (-i\hbar\partial_s - \tau + \hbar^{-1}\mathfrak{f}_0)^2 u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+), \partial_\tau u(\cdot, 0) = \gamma u(\cdot, 0)\}. \end{aligned}$$

In fact, it will even be more convenient to deal with the following operator

$$\mathcal{N}_\hbar = \hat{a}_\hbar(s, \tau)^{-1} p_{s, \hbar}^{\Xi_0} \hat{a}_\hbar(s, \tau)^{-1} p_{s, \hbar}^{\Xi_0} - \hat{a}_\hbar(s, \tau)^{-1} \partial_\tau \hat{a}_\hbar(s, \tau) \partial_\tau, \quad (5.28)$$

where

$$p_{s, \hbar}^{\Xi_0} = \Xi_0(\cdot + \theta)^W - \tau + \hbar\kappa(s) \frac{\hat{\tau}^2}{2}. \quad (5.29)$$

$$\begin{aligned} \text{Dom}(\mathcal{N}_\hbar) = \{u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+) : -\partial_\tau^2 u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+), \\ \tau^2 u \in L^2(\mathbf{T}_{2L} \times \mathbb{R}_+), \partial_\tau u(\cdot, 0) = \gamma u(\cdot, 0)\}. \end{aligned}$$

### 5.3.3 Microlocalization of the eigenfunctions of $\mathcal{L}_h$

In fact, we can prove that the eigenfunctions of  $\mathcal{L}_h$  associated with eigenvalues in  $[ha, hb]$  are roughly microlocalized with respect to  $\sigma + \hbar^{-1}\mathfrak{f}_0$ , the (shifted) dual variable of  $s$ . In order to quantify this, we consider the compact set

$$K = \bigcup_{j \geq 1} \{\sigma \in \mathbb{R} : \mu_j(\sigma) \in [a, b]\} \subset [\sigma_{\min}, \sigma_{\max}] =: \tilde{K}. \quad (5.30)$$

Note that  $K$  is indeed compact due to the properties of the  $\mu_j$  (tending to  $+\infty$  at  $-\infty$ ) and to the choice of  $[a, b]$ , which does not contain Landau levels (the limits of the  $\mu_j$  in  $+\infty$ ).

The following result establishes a rather rough microlocalization result (with respect to  $\sigma$ ) for the eigenfunctions: it tells us that the eigenfunctions are microlocalized in the compact set  $\tilde{K}$ . To quantify this, we consider a smooth function  $\Xi$  with values in  $[0, 1]$  such that  $\Xi = 0$  near  $\tilde{K}$  and 1 away from  $\tilde{K}$ .

We let  $\hat{\lambda} = h^{-1}\lambda$ .

**Proposition 5.3.1.** *Let us consider the eigenvalue equation  $\mathcal{L}_h\psi = \lambda\psi$  for  $\lambda \in [ha, hb]$ . Then,*

$$\widehat{\mathcal{L}}_h\varphi = \hat{\lambda}\varphi + \mathcal{O}(h^\infty)\|\psi\|, \quad (5.31)$$

with  $\varphi = \hat{\chi}_h\hat{\psi}$ , where  $\hat{\psi} = \psi \circ \Gamma(s, \hbar\tau)$  and  $\hat{\chi}_h(\tau) = \chi(\hbar^\eta\tau)$  for a smooth cutoff function  $\chi$  equal to 0 away from  $\tau = 0$ .

Moreover,

$$\text{Op}_h^W(\Xi(\sigma + \hbar^{-1}\mathbf{f}_0))\varphi = \mathcal{O}(\hbar^\infty)\|\psi\|. \quad (5.32)$$

*Proof.* The estimate (5.31) follows from the Agmon estimates (Proposition 5.2.1).

Then, let us only prove that (5.32) holds when  $\Xi$  is 0 near  $(-\infty, \sigma_{\max} + \frac{\epsilon}{2})$  and 1 on  $(\sigma_{\max} + \epsilon, +\infty)$ , the estimate following from similar arguments on  $(-\infty, \sigma_{\min} - \epsilon)$ .

In order to lighten the notation, we will use a slight abuse of notation by writing

$$\Xi^W := \text{Op}_h^W(\Xi(\sigma + \hbar^{-1}\mathbf{f}_0)). \quad (5.33)$$

Then, we write

$$(\widehat{\mathcal{L}}_h - \hat{\lambda})\Xi^W\varphi = [\widehat{\mathcal{L}}_h, \Xi^W]\varphi + \mathcal{O}(\hbar^\infty)\|\psi\|.$$

Let us consider the commutator. Thanks to the explicit expression (5.26), one of the terms is

$$\begin{aligned} \left\| [\hat{a}_h^{-1}D_\tau\hat{a}_hD_\tau, \Xi^W]\varphi \right\| &= \left\| [\hat{a}_h^{-1}D_\tau\hat{a}_h, \Xi^W]D_\tau\varphi \right\| \\ &\leq \mathcal{O}(\hbar^\infty)\|D_\tau\varphi\| + \hbar\|\Xi^W D_\tau\varphi\|, \end{aligned}$$

where  $\Xi$  has a support slightly larger than the one of  $\Xi$ , and where we used classical results of composition of pseudo-differential operators. The other term is

$$\begin{aligned} &\left\| [a_h^{-1}p_{s,h}a_h^{-1}p_{s,h}, \Xi^W]\varphi \right\| \\ &= \left\| [a_h^{-1}(\hbar D_s - \tau + \hbar^{-1}\mathbf{f}_0 + \hbar\kappa(s)\frac{\hat{\tau}^2}{2})a_h^{-1}(\hbar D_s - \tau + \hbar^{-1}\mathbf{f}_0 + \hbar\kappa(s)\frac{\hat{\tau}^2}{2}), \Xi^W]\varphi \right\| \\ &\leq \mathcal{O}(\hbar^\infty)\|\psi\| + C\hbar\|\Xi^W\varphi\|. \end{aligned}$$

Thanks to the explicit expression (5.26) and the Agmon estimate, we get

$$\|[\widehat{\mathcal{L}}_h, \Xi^W]\varphi\| \leq C\hbar\|\Xi^W\varphi\| + C\hbar\|\Xi^W\partial_\tau\varphi\| + \mathcal{O}(\hbar^\infty)\|\psi\|, \quad (5.34)$$

and we can write, by using the support of  $\zeta(h^{-\frac{1}{2}+\eta}t)$ ,

$$\widehat{\mathcal{L}}_h = \widehat{\mathcal{L}}_0 + \mathcal{R}_h, \quad \widehat{\mathcal{L}}_0 = -\partial_\tau^2 + p_{s,h,0}^2, \quad p_{s,h,0} = -i\hbar\partial_s + \hbar^{-1}\mathbf{f}_0 - \tau, \quad (5.35)$$

where the remainder  $\mathcal{R}_h$  can be written as

$$\mathcal{R}_h = \hbar^{1-2\eta}R_{h,2}(s, \tau)p_{s,h,0}^2 + \hbar^{1-4\eta}R_{h,1}(s, \tau)p_{s,h,0} + \hbar^{2-8\eta}R_{h,3} + \hbar R_{h,4}\partial_\tau,$$

the  $R_{\hbar,j}$  being smooth functions, uniformly bounded in  $\hbar$ .

Then, we consider an increasing function  $\sigma \mapsto \tilde{\Xi}(\sigma) \in (\sigma_{\max} + \frac{\epsilon}{4}, +\infty)$  that coincides with Id on  $(\sigma_{\max} + \frac{\epsilon}{2}, +\infty)$ . We let

$$\widehat{\mathcal{L}}_0^{\text{cut}} = \text{Op}_{\hbar}^{\text{W}} \left( -\partial_{\tau}^2 + (\tilde{\Xi}(\sigma + \hbar^{-1}f_0) - \tau)^2 \right),$$

where the superscript "cut" refers to the replacement of  $-i\hbar\partial_s + \hbar^{-1}f_0$  by  $\tilde{\Xi}^{\text{W}}$  (with the same abuse of notation as in (5.33)). We notice that  $\widehat{\mathcal{L}}_0^{\text{cut}} - \hat{\lambda}$  is bijective (with an inverse uniformly bounded in  $\hbar$ ) due to the choice of  $\tilde{\Xi}$  and the definition of  $\sigma_{\max}$ . Moreover, we have  $\{\Xi \neq 0\} \subset \{\tilde{\Xi} = \text{Id}\}$  so that, with (5.35),

$$\left( \widehat{\mathcal{L}}_0^{\text{cut}} - \hat{\lambda} + \mathcal{R}_{\hbar}^{\text{cut}} \right) \Xi^{\text{W}} \varphi = [\widehat{\mathcal{L}}_{\hbar}, \Xi^{\text{W}}] \varphi + \mathcal{O}(\hbar^{\infty}) \|\psi\|,$$

which can be written as

$$\left( \text{Id} + \mathcal{R}_{\hbar}^{\text{cut}} (\widehat{\mathcal{L}}_0^{\text{cut}} - \hat{\lambda})^{-1} \right) (\widehat{\mathcal{L}}_0^{\text{cut}} - \hat{\lambda}) \Xi^{\text{W}} \varphi = [\widehat{\mathcal{L}}_{\hbar}, \Xi^{\text{W}}] \varphi + \mathcal{O}(\hbar^{\infty}) \|\psi\|.$$

The operator  $\text{Id} + \mathcal{R}_{\hbar}^{\text{cut}} (\widehat{\mathcal{L}}_0^{\text{cut}} - \hat{\lambda})^{-1}$  is bijective as soon as  $\hbar$  is small enough.

With (5.34), this provides us first with

$$\|\Xi^{\text{W}} \varphi\|^2 \leq C\hbar \|\Xi^{\text{W}} \varphi\|^2 + C\hbar \|\Xi^{\text{W}} \partial_{\tau} \varphi\|^2 + \mathcal{O}(\hbar^{\infty}) \|\psi\|^2,$$

and then

$$\|\Xi^{\text{W}} \partial_{\tau} \varphi\|^2 + \|\Xi^{\text{W}} \varphi\|^2 \leq C\hbar \|\Xi^{\text{W}} \varphi\|^2 + C\hbar \|\Xi^{\text{W}} \partial_{\tau} \varphi\|^2 + \mathcal{O}(\hbar^{\infty}) \|\psi\|^2.$$

The estimate (5.32) follows by induction on the size of the support of  $\Xi$ . □

### 5.3.4 First spectral estimates

The aim of the following proposition is to establish that the spectrum of  $\mathcal{L}_{\hbar}$  in  $I_{\hbar}$  is close to that of  $h\mathcal{N}_{\hbar}$  and thus that  $\mathcal{N}_{\hbar}$  is a nice auxilliary operator to describe the spectrum of  $\mathcal{L}_{\hbar}$ . In fact, we will see that this proposition is not necessary to prove our spectral estimates, but its proof is instructive.

**Proposition 5.3.2.** *There exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$  the following holds. Let us consider an interval  $J_{\hbar} \subset I_{\hbar}$ . Then, there exists an interval  $\hat{J}_{\hbar}$  such that  $J_{\hbar} \subset \hat{J}_{\hbar} \subset I_{\hbar}$  with  $d_{\text{H}}(J_{\hbar}, \hat{J}_{\hbar}) = \mathcal{O}(\hbar^{\infty})$  and*

$$\text{rank } \mathbf{1}_{J_{\hbar}}(\mathcal{L}_{\hbar}) \leq \text{rank } \mathbf{1}_{\hat{J}_{\hbar}}(h\mathcal{N}_{\hbar}). \quad (5.36)$$

Moreover, for all  $\lambda \in I_{\hbar} \cap \text{Sp}(\mathcal{L}_{\hbar})$ ,

$$\text{dist}(\lambda, h \text{Sp}(\mathcal{N}_{\hbar})) = \mathcal{O}(\hbar^{\infty}). \quad (5.37)$$

*Proof.* Let us start by proving (5.37). Let us consider an eigenvalue  $\lambda \in I_{\hbar}$  of  $\mathcal{L}_{\hbar}$ . We write the eigenvalue equation  $\mathcal{L}_{\hbar}\psi = \lambda\psi$ .

With Proposition 5.3.1, we can write (5.31). Then, with (5.32), we deduce that

$$h\mathcal{N}_{\hbar}\varphi = \lambda\varphi + \mathcal{O}(\hbar^{\infty}) \|\psi\|.$$

Thus, (5.37) follows from the spectral theorem.

Let us now consider (5.36), which deals with multiplicities. Let us write  $\text{Sp}(\mathcal{L}_{\hbar}) \cap J_{\hbar} = \{\lambda_1, \dots, \lambda_p\}$

(where the  $\lambda_j$  are distinct) and underline that these eigenvalues depend on  $h$  as well as  $p$ . Consider the associated eigenspaces  $(E_j)_{1 \leq j \leq p}$  and note that  $\dim \bigoplus_{j=1}^p E_j = \mathcal{O}(h^{-2})$  thanks to the Weyl estimate (5.3). With the same notation as above, we consider the spaces of quasimodes  $(\hat{\chi}_h \hat{E}_j)_{1 \leq j \leq p}$ . Thanks to Proposition 5.3.1 (and the rough Weyl estimate),  $\dim(\hat{\chi}_h \hat{E}_j) = \dim E_j$ , as soon as  $h$  is small enough. Moreover, we have

$$\|(\bigoplus_{j=1}^p h \mathcal{N}_h - \lambda)\varphi\| \leq \varepsilon_h \|\varphi\|, \quad \varepsilon_h = \mathcal{O}(h^\infty),$$

for all  $\varphi = (\varphi_1, \dots, \varphi_p) \in \bigoplus_{j=1}^p \hat{\chi}_h \hat{E}_j$  and where  $\lambda = (\lambda_1, \dots, \lambda_p)$ . We set  $J_h = [a_h, b_h]$  and  $\hat{J}_h = [a_h - \varepsilon_h, b_h + \varepsilon_h]$ . If  $\text{rank } \mathbb{1}_{\hat{J}_h}(h \mathcal{N}_h) < \text{rank } \mathbb{1}_{J_h}(\mathcal{L}_h)$ , then the projection

$$\Pi : \bigoplus_{j=1}^p \hat{\chi}_h \hat{E}_j \rightarrow \text{ran } \mathbb{1}_{\hat{J}_h}(\tilde{\mathcal{L}}_h)$$

could not be injective. Indeed, if  $\Pi$  is injective, then

$$\dim\left(\bigoplus_{j=1}^p \hat{\chi}_h \hat{E}_j\right) = \dim(\text{ran } \mathbb{1}_{\hat{J}_h}(h \mathcal{N}_h)) = \text{rank } \mathbb{1}_{\hat{J}_h}(h \mathcal{N}_h).$$

On the other hand,

$$\dim\left(\bigoplus_{j=1}^p \hat{\chi}_h \hat{E}_j\right) = \dim\left(\bigoplus_{j=1}^p E_j\right) \geq \text{rank } \mathbb{1}_{J_h}(\mathcal{L}_h),$$

which gives a contradiction. Since  $\Pi$  is not injective, then

$$\dim(\text{ran } \mathbb{1}_{\hat{J}_h}(h \mathcal{N}_h)) < \dim\left(\bigoplus_{j=1}^p \hat{\chi}_h \hat{E}_j\right).$$

Considering a non-zero  $\varphi \in \bigoplus_{j=1}^p \hat{\chi}_h \hat{E}_j$  (in the kernel of  $\Pi$ ), the spectral theorem (Proposition A.1.6) would give

$$\|(\bigoplus_{j=1}^p h \mathcal{N}_h - \lambda)\varphi\| > \varepsilon_h \|\varphi\|,$$

which is a contradiction when  $\varphi \neq 0$ . Therefore, (5.36) follows.  $\square$

## 5.4 A Grushin problem

### 5.4.1 Preliminaries

Let us prove that an operator given by (5.7) with a  $2L$  periodic symbol,  $p(s+2L, \sigma) = p(s, \sigma)$ , preserves  $2L$ -periodic distributions and also locally square integrable  $2L$ -periodic functions (is the subset of  $L^2_{loc}(\mathbb{R})$  of  $2L$ -periodic functions).

**Proposition 5.4.1.** *If  $p$  is  $2L$ -periodic, the operator given in (5.7) induces a bounded operator from  $e^{is\theta} L^2(\mathbf{T}_{2L})$  to  $e^{is\theta} L^2(\mathbf{T}_{2L})$ , where  $\mathbf{T}_{2L} = \mathbb{R}/2L\mathbb{Z}$  (in other words  $L^2(\mathbf{T}_{2L})$  is the subset of  $L^2_{loc}(\mathbb{R})$  of  $2L$ -periodic functions, equipped with the usual  $L^2$  norm on  $[0, 2L]$ ).*

We notice that  $e^{is\theta} L^2_{2L}(\mathbb{R})$  is the set of functions

$$e^{is\theta} L^2_{2L}(\mathbb{R}) = \{u \in L^2_{loc}(\mathbb{R}) : u(s+2L) = e^{i2L\theta} u(s)\},$$

equipped with  $L^2$ -norm on a period  $[0, 2L]$ .

*Proof.* We recall that the Weyl quantization of a symbol  $p$  is given by the formula:

$$\mathrm{Op}_h^W p u(s) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(s-y)\sigma/\hbar} p\left(\frac{s+y}{2}, \sigma\right) u(y) dy d\sigma, \quad \forall u \in \mathcal{S}(\mathbb{R}).$$

Let  $u \in e^{is\theta} L_{2L}^2(\mathbb{R})$ , we have

$$\begin{aligned} \mathrm{Op}_h^W p u(s+2L) &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(s+2L-y)\sigma/\hbar} p\left(\frac{s+2L+y}{2}, \sigma\right) u(y) dy d\sigma \\ &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(s-y)\sigma/\hbar} p\left(\frac{s+4L+y}{2}, \sigma\right) u(y+2L) dy d\sigma \\ &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(s-y)\sigma/\hbar} p\left(\frac{s+4L+y}{2}, \sigma\right) e^{i2L\theta} u(y) dy d\sigma. \end{aligned}$$

Using that  $p$  is  $2L$  periodic symbol, then

$$\mathrm{Op}_h^W p u(s+2L) = e^{i2L\theta} \mathrm{Op}_h^W p u(s). \quad (5.38)$$

From the composition theorem for pseudodifferential operators (see Theorem A.2.4), we see that  $\langle s \rangle^{-1} \mathrm{Op}_h^W p \langle s \rangle$  is a pseudo differential operator with symbol in  $S_{\mathbb{R}^2}(1)$ , and thus it is bounded on  $L^2(\mathbb{R})$  thanks to the Calderon-Vaillancourt theorem, see Theorem A.2.5. This shows that  $\mathrm{Op}_h^W p$  is bounded on  $L^2(\mathbb{R}, \langle s \rangle^{-2} ds)$ .

It is equivalent to:

For  $u \in L^2(\mathbb{R}, \langle s \rangle^{-2} ds)$ , there exist  $C > 0$ , such that  $\|\mathrm{Op}_h^W p\|_{L^2(\mathbb{R}, \langle s \rangle^{-2} ds)} \leq \|u\|_{L^2(\mathbb{R}, \langle s \rangle^{-2} ds)}$ . Since  $\mathrm{Op}_h^W p$  is bounded on  $L^2(\mathbb{R}, \langle s \rangle^{-2} ds)$  and the norms on  $L^2(\mathbb{R}, \langle s \rangle^{-2} ds)$  and  $L_{2L}^2(\mathbb{R})$  are equivalent, we deduce that  $\mathrm{Op}_h^W p$  is bounded on  $L_{2L}^2(\mathbb{R})$ . Then, the conclusion follows.

Let us explain why  $\|\cdot\|_{L_{2L}^2(\mathbb{R})}$  and  $\|\cdot\|_{L^2(\mathbb{R}, \langle s \rangle^{-2} ds)}$  are equivalent. Notice that there exists  $C_1(L) > 0$ ,  $C_2(L) > 0$  and  $C_3(L) > 0$  such that for all  $u \in L_{2L}^2(\mathbb{R})$ ,

$$\begin{aligned} \|u\|_{L^2(\mathbb{R}, \langle s \rangle^{-2} ds)}^2 &= \int_{\mathbb{R}} |u(s)|^2 \langle s \rangle^{-2} ds \\ &= \sum_{l \in \mathbb{Z}} \int_{(2l-1)L}^{(2l+1)L} |u(s)|^2 \langle s \rangle^{-2} ds \\ &\leq C_1(L) \sum_{l \in \mathbb{Z}} \langle l \rangle^{-2} \int_{(2l-1)L}^{(2l+1)L} |u(s)|^2 ds \\ &\leq C_2(L) \|u\|_{L_{2L}^2(\mathbb{R})}^2 \\ &\leq C_3(L) \|u\|_{L^2(\mathbb{R}, \langle s \rangle^{-2} ds)}^2. \end{aligned}$$

□

**Proposition 5.4.2.** *The operator  $\mathrm{Op}_h^W p$  acting on  $e^{is\theta} L_{2L}^2(\mathbb{R})$  is unitarily equivalent to  $\mathrm{Op}_h^W p(s, \sigma + \hbar\theta)$  acting on  $L_{2L}^2(\mathbb{R})$ .*

*Proof.* Let  $u \in e^{is\theta} L^2_{2L}(\mathbb{R})$ , then there exists  $v \in L^2_{2L}(\mathbb{R})$  such that  $u(s) = e^{is\theta} v(s)$ . We have

$$\begin{aligned} \text{Op}_h^W p u(s) &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(s-y)\sigma/\hbar} p\left(\frac{s+y}{2}, \sigma\right) u(y) dy d\sigma \\ &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(s-y)(\sigma+\hbar\theta)/\hbar} p\left(\frac{s+y}{2}, \sigma+\hbar\theta\right) u(y) dy d\sigma \\ &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(s-y)\sigma/\hbar} e^{i(s-y)\theta} p\left(\frac{s+y}{2}, \sigma+\hbar\theta\right) e^{iy\theta} v(y) dy d\sigma \\ &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(s-y)\sigma/\hbar} e^{is\theta} p\left(\frac{s+y}{2}, \sigma+\hbar\theta\right) v(y) dy d\sigma \\ &= e^{is\theta} \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(s-y)\sigma/\hbar} p\left(\frac{s+y}{2}, \sigma+\hbar\theta\right) e^{-iy\theta} u(y) dy d\sigma. \end{aligned}$$

We deduce that:

$$\text{Op}_h^W p = e^{is\theta} \text{Op}_h^W p(s, \sigma + \hbar\theta) e^{-is\theta},$$

and then the conclusion follows.  $\square$

## 5.4.2 A pseudodifferential operator with operator-valued symbol

Recalling Remark 5.1.2, we notice that the operator  $\mathcal{N}_h$  can be seen as a pseudo-differential operator acting as

$$\mathfrak{N}_h = \hat{a}_h(s, \tau)^{-1} \mathcal{T}_h \hat{a}_h(s, \tau)^{-1} \mathcal{T}_h - \hat{a}_h(s, \tau)^{-1} \partial_\tau \hat{a}_h(s, \tau) \partial_\tau,$$

on functions of the form  $e^{isf_0/h} L^2(\mathbf{T}_{2L} \times \mathbb{R}_+)$  and where

$$\mathcal{T}_h = \Xi_0^W - \tau + \hbar \frac{\kappa}{2} \hat{\tau}^2. \quad (5.39)$$

In fact, it will be convenient to see  $\mathfrak{N}_h$  as a pseudo-differential operator with operator-valued symbol. At a formal level, the principal symbol of  $\mathfrak{N}_h$  is  $n_0(s, \sigma) = -\partial_\tau^2 + (\sigma - \tau)^2$  equipped with the domain

$$\text{Dom}(n_0) = \{\psi \in B^2(\mathbb{R}_+) : \psi'(0) = \gamma\psi(0)\}.$$

The vector space  $B^2(\mathbb{R}_+)$  is equipped with the  $(s, \sigma)$ -independent norm

$$\|\psi\|_{B^2(\mathbb{R}_+)}^2 = \|\psi''\|^2 + \|\psi'\|^2 + \|\langle t \rangle^2 \psi\|^2.$$

With this convention, we may write that  $n_0 \in S(\mathbb{R}^2, \mathcal{L}(B^2(\mathbb{R}_+), L^2(\mathbb{R}_+)))$ .

We say that  $\Psi \in S(\mathbb{R}^2, \mathcal{L}(B^2(\mathbb{R}_+), L^2(\mathbb{R}_+)))$  when, for all  $\alpha \in \mathbb{N}^2$ , there exists  $C_\alpha > 0$  such that for all  $(s, \sigma) \in \mathbb{R}^2$ ,

$$\|\partial^\alpha \Psi\|_{\mathcal{L}(B^2(\mathbb{R}_+), L^2(\mathbb{R}_+))} \leq C_\alpha.$$

Such symbols might also depend on  $\hbar$ ; in this case, the constant  $C_\alpha$  is uniform in  $\hbar$ .

**Lemma 5.4.3.** *The operator  $\mathfrak{N}_h$  can be written as the Weyl quantization of a symbol in  $S(\mathbb{R}^2, \mathcal{L}(B^2(\mathbb{R}_+), L^2(\mathbb{R}_+)))$ .*

*Proof.* We can write

$$\mathcal{T}_h = \text{Op}_h^W \left( \Xi_0(\sigma) - \tau + \hbar \frac{\kappa}{2} \hat{\tau}^2 \right),$$

the symbol ( $2L$ -periodic with respect to  $s$ ) belonging to the class

$S(\mathbb{R}^2, \mathcal{L}(B^1(\mathbb{R}_+), L^2(\mathbb{R}_+))) \cap S(\mathbb{R}^2, \mathcal{L}(B^2(\mathbb{R}_+), B^1(\mathbb{R}_+)))$ . The functions  $a_h(s, \tau)$  and  $a_h(s, \tau)^{-1}$  are

bounded uniformly with respect to  $\hbar$  (and so are all their derivatives). Then, the conclusion follows from the composition theorem for pseudo-differential operators, see Theorem A.2.4.  $\square$

In fact, we can describe more accurately the symbol of  $\mathfrak{N}_\hbar$  by expanding it in powers of  $\hbar$ . An analogous description for a very similar operator can be found in great detail in [FHKR22, Section 4.2].

**Proposition 5.4.4.** *The operator  $\mathfrak{N}_\hbar$  can be written as follows:*

$$\mathfrak{N}_\hbar = \mathfrak{n}_0 + \hbar \mathfrak{n}_1 + \hbar^2 \mathcal{R}_\hbar^{(2)} + \hbar w_\hbar \partial_\tau, \quad (5.40)$$

where, for some  $N \in \mathbb{N}$ ,  $C, \hbar_0 > 0$ , we have, for all  $\hbar \in (0, \hbar_0)$ ,

- (i)  $w_\hbar$  is a smooth function supported in  $\{(s, \tau) : C^{-1}\hbar^{-2\eta} \leq \langle \tau \rangle \leq C\hbar^{-2\eta}\}$  and such that  $w_\hbar = \mathcal{O}(\langle \tau \rangle)$ ,
- (ii)  $\mathcal{R}_\hbar^{(2)}$  is a pseudodifferential operator whose symbol belongs to a bounded set in the space of symbols  $S(\mathbb{R}^2, \mathcal{L}(B^2(\mathbb{R}_+), L^2(\mathbb{R}_+, \langle \tau \rangle^{-N} d\tau)))$ .

Moreover, the  $\mathfrak{n}_j$  are given by  $\mathfrak{n}_j = \text{Op}_\hbar^W n_j$  with

$$\begin{aligned} n_0 &= -\partial_\tau^2 + (\Xi_0(\sigma) - \tau)^2, \\ n_1 &= \kappa(s) \left[ (\Xi_0(\sigma) - \tau) \zeta_\mu^2 \tau^2 + \zeta_\mu \partial_\tau + 2\zeta_\mu \tau (\Xi_0(\sigma) - \tau)^2 \right]. \end{aligned} \quad (5.41)$$

In particular, we can write  $\mathfrak{N}_\hbar = \text{Op}_\hbar^W(n_\hbar)$  with a symbol  $n_\hbar$  satisfying

$$n_\hbar = n_0 + \hbar n_1 + \hbar^2 r_\hbar^{(2)} + \hbar w_\hbar \partial_\tau,$$

where  $r_\hbar^{(2)}$  belongs to the class of operator symbols  $S(\mathbb{R}^2, \mathcal{L}(B^2(\mathbb{R}_+), L^2(\mathbb{R}_+, \langle \tau \rangle^{-N} d\tau)))$  uniformly in  $\hbar$ .

*Proof.* Let us provide a Taylor expansion of

$$\begin{aligned} \mathcal{N}_\hbar &= \hat{a}_\hbar(s, \tau)^{-1} \mathcal{T}_\hbar \hat{a}_\hbar(s, \tau)^{-1} \mathcal{T}_\hbar - \hat{a}_\hbar(s, \tau)^{-1} \partial_\tau \hat{a}_\hbar(s, \tau) \partial_\tau \\ &= (\hat{a}_\hbar(s, \tau)^{-1} \mathcal{T}_\hbar)^2 - \hat{a}_\hbar(s, \tau)^{-1} \partial_\tau \hat{a}_\hbar(s, \tau) \partial_\tau, \end{aligned}$$

where  $\mathcal{T}_\hbar$  is defined in (5.39) and can be rewritten in the form

$$\mathcal{T}_\hbar = \mathfrak{p}_0 + \hbar \frac{\kappa}{2} \zeta_\mu^2 \tau^2.$$

Now, we expand  $\hat{a}_\hbar^{-1}$  in powers of  $\hbar$ , we get

$$\hat{a}_\hbar^{-1} = 1 + \hbar \kappa \zeta_\mu \tau + \hbar^2 \frac{(\kappa \zeta_\mu \tau)^2}{1 - \hbar \kappa \zeta_\mu \tau},$$

so that

$$\hbar \hat{a}_\hbar^{-1} = \hbar + \hbar^2 \frac{\kappa \zeta_\mu \tau}{1 - \hbar \kappa \zeta_\mu \tau}.$$



We have the following expansion

$$\begin{aligned}\hat{a}_h^{-1}\mathcal{T}_h &= \hat{a}_h^{-1}\mathfrak{p}_0 + \hbar\hat{a}_h^{-1}\frac{\kappa}{2}\zeta_\mu\tau^2 \\ &= \left(1 + \hbar\kappa\zeta_\mu\tau + \hbar^2\frac{(\kappa\zeta_\mu\tau)^2}{1 - \hbar\kappa\zeta_\mu\tau}\right)\mathfrak{p}_0 + \left(\hbar + \hbar^2\frac{\kappa\zeta_\mu\tau}{1 - \hbar\kappa\zeta_\mu\tau}\right)\frac{\kappa}{2}\zeta_\mu\tau^2 \\ &= \mathfrak{p}_0 + \hbar\kappa\zeta_\mu\tau\left(\mathfrak{p}_0 + \frac{1}{2}\zeta_\mu\tau\right) + \hbar^2\frac{(\kappa\zeta_\mu\tau)^2}{1 - \hbar\kappa\zeta_\mu\tau}\left(\mathfrak{p}_0 + \frac{1}{2}\zeta_\mu\tau\right).\end{aligned}$$

The previous expression can be rearranged as follows

$$\hat{a}_h^{-1}\mathcal{T}_h = \mathfrak{p}_0 + \hbar\mathfrak{p}_1 + \hbar^2\mathcal{R}_h,$$

where

$$\mathfrak{p}_1 = \kappa\zeta_\mu\tau\left(\mathfrak{p}_0 + \frac{1}{2}\zeta_\mu\tau\right), \quad \mathcal{R}_h = \frac{(\kappa\zeta_\mu\tau)^2}{1 - \hbar\kappa\zeta_\mu\tau}\left(\mathfrak{p}_0 + \frac{1}{2}\zeta_\mu\tau\right).$$

We can also expand the operator in the transversal variable and get

$$\begin{aligned}-\hat{a}_h^{-1}\partial_\tau\hat{a}_h\partial_\tau &= -\partial_\tau^2 - (\hat{a}_h^{-1}\partial_\tau\hat{a}_h)\partial_\tau \\ &= -\partial_\tau^2 + \hbar\hat{a}_h^{-1}(\zeta_\mu\kappa + \tau\kappa\partial_\tau\zeta_\mu)\partial_\tau \\ &= -\partial_\tau^2 + \hbar\zeta_\mu\kappa\partial_\tau + \hbar w_{h,2}\partial_\tau + \hbar^2 v_h,\end{aligned}$$

where the functions  $v_h$  and  $w_{h,2}$  are smooth functions satisfy, uniformly with respect to  $s$ , and  $\hbar$ ,

$$v_h = \mathcal{O}(\langle\tau\rangle)\partial_\tau \quad \text{and} \quad w_{h,2} = \mathcal{O}(|\langle\tau\rangle\partial_\tau\zeta_\mu|).$$

We get this expansion of the operator

$$\begin{aligned}\mathcal{N}_h &= \left(-\partial_\tau^2 + \mathfrak{p}_0^2\right) + \hbar(\mathfrak{p}_0\mathfrak{p}_1 + \mathfrak{p}_1\mathfrak{p}_0 + \zeta_\mu\kappa\partial_\tau) + \hbar w_{h,2}\partial_\tau \\ &\quad + \hbar^2\left(v_h + \mathfrak{p}_0\mathcal{R}_h + \mathcal{R}_h\mathfrak{p}_0 + \mathfrak{p}_1^2 + \hbar(\mathfrak{p}_1\mathcal{R}_h + \mathcal{R}_h\mathfrak{p}_1) + \hbar^2\mathcal{R}_h^2\right),\end{aligned}$$

then, the conclusion follows from the composition theorem for pseudo-differential operators, see Theorem A.2.4.  $\square$

### 5.4.3 Dimensional reduction

The aim of this section is to analyse the spectrum of  $\mathcal{N}_h$ . This can be done thanks to a Grushin reduction. The principal symbol of  $\mathfrak{N}_h$  is the "de Gennes operator" with Robin boundary conditions. Explicitly, we have

$$n_0(s, \sigma) = -\partial_\tau^2 + (\Xi_0(\sigma) - \tau)^2.$$

The increasing sequence of its (simple) eigenvalues is  $(\mu_k(\Xi_0(\sigma)))_{k \geq 1}$ . We recall that the functions  $\mu_k$  are described in Proposition 5.1.2.

Now, consider the window  $[a, b] \subset (2n - 1, 2n + 1)$ .

**Lemma 5.4.5.** *For all  $z \in [a, b]$ , let us consider the matrix operator*

$$\mathcal{P}_0(z) = \begin{pmatrix} n_0(s, \sigma) - z & \Pi^* \\ \Pi & 0 \end{pmatrix} : B^2(\mathbb{R}_+) \times \mathbb{C}^N \longrightarrow L^2(\mathbb{R}_+) \times \mathbb{C}^N,$$

where  $\Pi^*(\alpha) = \sum_{j=1}^N \alpha_j \mathring{u}_j$  and  $\Pi\psi = (\langle \psi, \mathring{u}_j \rangle)_{1 \leq j \leq N}$ .

Then,  $\mathcal{P}_0(z)$  is bijective with inverse

$$\mathcal{Q}_0(z) = \begin{pmatrix} q_0 & \Pi^* \\ \Pi & z - M_0(\sigma) \end{pmatrix}, \quad q_0 = (n_0(s, \sigma) - z)^{-1}(\Pi^*\Pi)^\perp,$$

where  $M_0(\sigma)$  is the diagonal  $N \times N$  matrix whose diagonal is  $(\mathring{\mu}_1, \dots, \mathring{\mu}_N)$ .

*Proof.* Let  $g \in L^2(\mathbb{R}^+)$  and  $\beta \in \mathbb{C}^N$ . Let us look for  $f \in \text{Dom}(n_0)$  and  $\alpha \in \mathbb{C}^N$  such that

$$\mathcal{P}_0(z)(f \oplus \alpha) = g \oplus \beta.$$

In other words,

$$\begin{cases} (n_0(s, \sigma) - z)f + \Pi^*\alpha = g, \\ \Pi f = \beta. \end{cases}$$

Let  $E = \text{span}(\mathring{u}_1, \dots, \mathring{u}_N)$ , and  $F = E^\perp$ . We can write

$$f = f_E + f_F,$$

where

$$f_E = \sum_{j=1}^N \langle f, \mathring{u}_j \rangle \mathring{u}_j.$$

Or  $\Pi f = \beta$ , then  $\langle f, \mathring{u}_j \rangle = \beta_j$ . Therefore,

$$f_E = \Pi^*\beta = \Pi^*\Pi f, \quad f_F = (\Pi^*\Pi)^\perp f.$$

We have

$$\begin{aligned} (n_0(s, \sigma) - z)f_F &= -(n_0(s, \sigma) - z)f_E - \Pi^*\alpha + g \\ &= -(n_0(s, \sigma) - z) \sum_{j=1}^N \beta_j \mathring{u}_j - \Pi^*\alpha + g, \end{aligned}$$

so that

$$(n_0(s, \sigma) - z)f_F = - \sum_{j=1}^N \beta_j (\mathring{\mu}_j - z) \mathring{u}_j - \Pi^*\alpha + g. \quad (5.42)$$

The space  $F$  is stable by  $n_0(s, \sigma) - z$  in the sense that: For all  $u \in \text{Dom}(n_0) \cap F$ , we have

$$\langle (n_0(s, \sigma) - z)u, \mathring{u}_j \rangle = \langle u, (n_0(s, \sigma) - z)\mathring{u}_j \rangle = \langle u, \mathring{\mu}_j \mathring{u}_j - z\mathring{u}_j \rangle = 0.$$

The operator  $n_0(s, \sigma) - z$  stabilizes  $F$  and thus induces an operator on  $\text{Dom}(n_0) \cap F$ .

Moreover, thanks to the self-adjointness of  $n_0$ , the min-max principle and the fact that  $\min \mathring{\mu}_{N+1} \geq 2N + 1 > z$ , there exists  $c > 0$  such that, for all  $u \in \text{Dom}(n_0) \cap F$ ,

$$\langle (n_0 - z)u, u \rangle = \langle n_0 u, u \rangle - z\|u\|^2 \geq (\mathring{\mu}_{N+1} - z)\|u\|^2 \geq c\|u\|^2.$$

Thus, the operator  $(n_0 - z)|_F$  is injective with closed range and, by considering the adjoint, we deduce that

it is bijective. We also notice that

$$\|(n_0 - z)|_F^{-1}\| \leq (\mathring{\mu}_{N+1} - z)^{-1} \leq c^{-1}.$$

Then (5.42) has a solution if and only if the right-hand-side belongs to  $F$ , that is

$$\begin{aligned} & -\sum_{j=1}^N \beta_j (\mathring{\mu}_j - z) \mathring{u}_j - \Pi^* \alpha + g \in F \\ \Rightarrow & -\sum_{j=1}^N \beta_j (\mathring{\mu}_j - z) \mathring{u}_j - \sum_{j=1}^N \alpha_j \mathring{u}_j + g \in F, \end{aligned}$$

then,

$$\left\langle -\sum_{j=1}^N \beta_j (\mathring{\mu}_j - z) \mathring{u}_j - \sum_{j=1}^N \alpha_j \mathring{u}_j + g, \mathring{u}_k \right\rangle = 0, \quad \forall k = 1, \dots, N.$$

As,  $\mathring{u}_j$  is an orthonormal family, then

$$-\beta_k (\mathring{\mu}_k - z) - \alpha_k + \langle g, \mathring{u}_k \rangle = 0,$$

we deduce that,

$$\alpha_k = \beta_k (z - \mathring{\mu}_k) + \langle g, \mathring{u}_k \rangle,$$

and

$$\alpha = \begin{pmatrix} \beta_1 (z - \mathring{\mu}_1) + \langle g, \mathring{u}_1 \rangle \\ \vdots \\ \beta_N (z - \mathring{\mu}_N) + \langle g, \mathring{u}_N \rangle \end{pmatrix} = \Pi g + (z - M_0(\sigma)) \beta.$$

This unique solution is given by

$$\begin{aligned} f_F &= (\Pi^* \Pi)^\perp \left[ -\sum_{j=1}^N \beta_j (\mathring{\mu}_j - z) (n_0(s, \sigma) - z)^{-1} \mathring{u}_j - (n_0(s, \sigma) - z)^{-1} \Pi^* \alpha + (n_0(s, \sigma) - z)^{-1} g \right] \\ &= (\Pi^* \Pi)^\perp (n_0(s, \sigma) - z)^{-1} g. \end{aligned}$$

Therefore,

$$f = f_E + f_F = \Pi^* \beta + (\Pi^* \Pi)^\perp (n_0(s, \sigma) - z)^{-1} g.$$

□

Let us now consider the full symbol

$$\mathcal{P}_\hbar(z) := \begin{pmatrix} n_\hbar - z & \Pi^* \\ \Pi & 0 \end{pmatrix},$$

which may be expanded in powers of  $\hbar$  as

$$\mathcal{P}_\hbar(z) = \mathcal{P}_0(z) + \hbar \mathcal{P}_1(z) + \mathcal{R}_\hbar,$$

with

$$\mathcal{P}_0(z) = \begin{pmatrix} n_0(s, \sigma) - z & \Pi^* \\ \Pi & 0 \end{pmatrix}, \quad \mathcal{P}_1(z) = \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{R}_h(z) = \begin{pmatrix} r_h & 0 \\ 0 & 0 \end{pmatrix},$$

where  $r_h = \hbar^2 r_h^{(2)} + \hbar w_h \partial_\tau$ , see Proposition 5.4.4.

We notice that

$$\mathcal{P}_h^W = \begin{pmatrix} \mathfrak{N}_h - z & \mathfrak{P}^* \\ \mathfrak{P} & 0 \end{pmatrix}, \quad \mathfrak{P} = \Pi^W. \quad (5.43)$$

Since the principal symbol of  $\mathcal{P}_h^W$  is bijective, it is natural to try to construct an approximate inverse in the semiclassical limit. Let us look for an approximate inverse whose symbol is in the form

$$\mathcal{Q}_{h,1} = \mathcal{Q}_0(z) + \hbar \mathcal{Q}_1(z).$$

As in [Ker18], we are led to choose

$$\mathcal{Q}_1 = -\mathcal{Q}_0 \mathcal{P}_1 \mathcal{Q}_0 = - \begin{pmatrix} q_0 n_1 q_0 & q_0 n_1 \Pi^* \\ \Pi n_1 q_0 & \Pi n_1 \Pi^* \end{pmatrix}.$$

This choice is convenient since the composition theorem for pseudo-differential operators (see [Ker18]) implies that

$$\begin{aligned} \mathcal{Q}_{h,1}^W (\mathcal{P}_0(z) + \hbar \mathcal{P}_1)^W &= \text{Id} + \hbar \left( \frac{1}{i} \{ \mathcal{Q}_0, \mathcal{P}_0 \} + \mathcal{Q}_0 \mathcal{P}_1 + \mathcal{Q}_1 \mathcal{P}_0 \right)^W \\ &\quad + \mathcal{O}_{L^2(\mathbf{T}_{2L} \times \mathbb{R}_+, \langle \tau \rangle^N \text{dsd}\tau) \times L^2(\mathbf{T}_{2L}) \rightarrow L^2(\mathbf{T}_{2L} \times \mathbb{R}_+) \times L^2(\mathbf{T}_{2L})}(\hbar^2), \end{aligned}$$

where the remainder is estimated thanks to the Calderón-Vaillancourt theorem (see [Ker18, Theorem 2.1.16]) and the resolvent estimate in Lemma 5.6.3 (applied with an appropriate  $\alpha > 0$ ). The  $\hbar$ -term vanishes due the choice of  $\mathcal{Q}_1$  and that fact that the Poisson bracket is actually 0 since the principal symbol does not depend on  $s$ . With this choice, the bottom right coefficient, denoted by  $\mathcal{Q}_{h,1}^\pm$ , of the matrix  $\mathcal{Q}_{h,1}$  is

$$\mathcal{Q}_{h,1}^\pm = z - M_0(\sigma) - \hbar \Pi n_1 \Pi^*.$$

This invites to consider the effective matrix pseudo-differential operator whose symbol is

$$M_h = M_0(\sigma) + \hbar M_1(s, \sigma),$$

with

$$M_1(s, \sigma) = \kappa(s) \Pi \mathcal{C}(\tau, \chi_0(\sigma)) \Pi^*, \quad \mathcal{C}(\tau, \xi) = (\xi - \tau) \zeta_\mu^2 \tau^2 + \zeta_\mu \partial_\tau + 2\zeta_\mu \tau (\xi - \tau)^2.$$

Using again the composition theorem to deal with the remainder  $\mathcal{R}_h$ , we get

$$\mathcal{Q}_{h,1}^W \mathcal{P}_h^W = \text{Id} + \mathcal{O}_{L^2(\mathbf{T}_{2L} \times \mathbb{R}_+, \langle \tau \rangle^N \text{dsd}\tau) \times L^2(\mathbf{T}_{2L}) \rightarrow L^2(\mathbf{T}_{2L} \times \mathbb{R}_+) \times L^2(\mathbf{T}_{2L})}(\hbar^2) + \mathcal{Q}_{h,1}^W \begin{pmatrix} \hbar w_h \partial_\tau & 0 \\ 0 & 0 \end{pmatrix}^W. \quad (5.44)$$

Moreover, similar arguments show that  $\mathcal{Q}_{h,1}^W$  is also an approximate right inverse of  $\mathcal{P}_h^W$  in the sense that

$$\mathcal{P}_h^W \mathcal{Q}_{h,1}^W = \text{Id} + \mathcal{O}_{L^2(\mathbf{T}_{2L} \times \mathbb{R}_+, \langle \tau \rangle^N \text{dsd}\tau) \times L^2(\mathbf{T}_{2L}) \rightarrow L^2(\mathbf{T}_{2L} \times \mathbb{R}_+) \times L^2(\mathbf{T}_{2L})}(\hbar^2) + \begin{pmatrix} \hbar w_h \partial_\tau & 0 \\ 0 & 0 \end{pmatrix}^W \mathcal{Q}_{h,1}^W. \quad (5.45)$$

**Proposition 5.4.6.** *The spectrum of  $\mathcal{L}_h$  in  $[ah, bh]$  coincides (with multiplicity) with that of  $h\text{Op}_h^W M_h$  modulo  $\mathcal{O}(h^2)$ .*

*Proof.* First, we consider  $\psi$  an eigenfunction of  $\mathcal{L}_h$  associated with  $\lambda \in [ah, bh]$ . We use (5.44) with  $z = h^{-1}\lambda$  we get that

$$\mathcal{Q}_{h,1}^W \mathcal{P}_h^W \begin{pmatrix} \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} + \mathcal{O}(\hbar^2) \|\varphi\|,$$

where  $\varphi$  denotes the function  $\psi$  after multiplication by a cutoff function in  $t$  and rescaling as in Proposition 5.3.1. Note that we used the exponential decay in  $\tau$  of our quasimode  $\varphi$  (which comes from that of  $\psi$ ) to control the remainder term in (5.44). We infer that

$$\mathfrak{P}^* \mathfrak{P} \varphi = \varphi + \mathcal{O}(\hbar) \|\varphi\|, \quad (\hat{\lambda} - \text{Op}_h^W M_h) \mathfrak{P} \varphi = \mathcal{O}(\hbar^2) \|\varphi\|, \quad (5.46)$$

where we used that the principal symbol of the top right coefficient of  $\mathcal{Q}_{h,1}$  is  $\Pi^*$ . Since  $\mathfrak{P}^*$  is bounded uniformly in  $\hbar$  (as the quantization of a bounded symbol), the first relation implies that

$$\|\varphi\| \leq C \|\mathfrak{P} \varphi\|.$$

Then, from the second relation and the spectral theorem, we deduce that

$$\text{dist}(\hat{\lambda}, \text{Sp}(\text{Op}_h^W M_h)) \leq C \hbar^2.$$

This means that the spectrum of  $h^{-1}\mathcal{L}_h$  in the window  $[ah, bh]$  is at a distance of order  $\hbar^2$  to the spectrum of the effective operator  $\text{Op}_h^W M_h$ .

Let us now proceed as in the proof of Proposition 5.3.2 and keep the same notation. We have

$$\|(\oplus_{j=1}^p \mathcal{A}_h - \hat{\lambda}) \varphi\| \leq \varepsilon_h \|\varphi\|, \quad \varepsilon_h = \mathcal{O}(h^\infty),$$

for all  $\varphi = (\varphi_1, \dots, \varphi_p) \in \oplus_{j=1}^p \hat{\chi}_h \hat{E}_j$  and where  $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_p)$ .

Similarly as (5.46), we have

$$\|\varphi\| \leq C \|\mathfrak{P} \varphi\|, \quad (5.47)$$

and

$$(\oplus_{j=1}^p \text{Op}_h^W M_h - \hat{\lambda}) \mathfrak{P} \varphi = \mathcal{O}(\hbar^2) \|\mathfrak{P} \varphi\|,$$

where  $\mathfrak{P} \varphi = (\mathfrak{P} \varphi_1, \dots, \mathfrak{P} \varphi_p)$ . Due to (5.47), the action of the map  $\mathfrak{P}$  is injective on  $\oplus_{j=1}^p \hat{\chi}_h \hat{E}_j$ . Therefore, as in the proof of Proposition 5.3.2, the spectral theorem provides us with

$$\text{rank } \mathbb{1}_{J_h}(\mathcal{L}_h) \leq \text{rank } \mathbb{1}_{K_h}(hM_h^W),$$

where  $J_h \subset I_h$  and  $K_h$  is an interval such that  $J_h \subset K_h$  and  $d_H(J_h, K_h) = \mathcal{O}(h^2)$ .

Let us now prove the converse estimate. We use (5.45) with an eigenvalue  $z = \hat{\lambda}$  of  $M_h^W$  and for  $f$  a corresponding eigenfunction. We have

$$\mathcal{P}_h^W \mathcal{Q}_{h,1}^W \begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} + \mathcal{O}(\hbar^2) \|f\|, \quad (5.48)$$

where the remainder term involving  $w_h$  has been controlled by using the exponential decay of the eigenfunctions of the de Gennes - Robin operator  $n_0$ .

Then, the first line in (5.48) gives

$$(\mathcal{N}_h - \hat{\lambda}) \left( \mathcal{Q}_{h,1}^+ \right)^W f = \mathcal{O}(\hbar^2) \|f\|, \quad (5.49)$$

where as the second line gives

$$\mathfrak{P} \left( \mathcal{Q}_{h,1}^+ \right)^W f = \mathcal{O}(\hbar^2) \|f\|,$$

which leads to

$$\|f\| \leq C \left\| \left( \mathcal{Q}_{h,1}^+ \right)^W f \right\|.$$

With (5.49), we get

$$(\mathcal{N}_h - \hat{\lambda}) \left( \mathcal{Q}_{h,1}^+ \right)^W f = \mathcal{O}(\hbar^2) \left\| \left( \mathcal{Q}_{h,1}^+ \right)^W f \right\|.$$

Now, by using the exponential decay of  $\left( \mathcal{Q}_{h,1}^+ \right)^W f$  and the rough microlocalization of  $f$  in the support of  $\Xi_0$  (since the principal symbol of the scalar pseudodifferential operator  $M_h$  is  $n_0$ ), we get the quasimode estimate

$$(\mathcal{L}_h - \lambda) \psi^{\text{quasi}} = \mathcal{O}(\hbar^2) \|\psi^{\text{quasi}}\|,$$

with  $\psi^{\text{quasi}}(x) = \chi(t(x)/\hbar^{1-\gamma}) \Psi^{\text{quasi}} \circ \Gamma^{-1}(x)$  where

$$\Psi^{\text{quasi}}(s, t) = \left( \mathcal{Q}_{h,1}^+ \right)^W f(s, \hbar^{-1}t),$$

and  $\chi$  is a smooth cutoff function equal to 1 near 0 and 0 away from a neighborhood of  $t = 0$  and  $\gamma \in (0, 1)$  is chosen small enough so that  $t\zeta(\hbar^{-\frac{1}{2}+\rho}t) = t$  on the support of  $\chi(t/\hbar^{1-\gamma})$ .

The spectral theorem shows that  $\lambda$  is close to the spectrum of  $\mathcal{L}_h$  at a distance or order at most  $\mathcal{O}(\hbar^2)$ . The argument concerning the multiplicities can again be used (as above) by exchanging the roles of  $\mathcal{N}_h$  and  $M_h^W$ . The conclusion follows.  $\square$

**Remark 5.4.1.** In Proposition 5.3.2 we only proved one inclusion of spectra. In contrast, Proposition 5.4.6 is stronger, since it provides an equality  $\mathcal{O}(\hbar^2)$ , in the sense of Definition 5.1.6. Indeed, in the proof of Proposition 5.4.6, we only have to use quasimodes for  $\mathcal{N}_h$  and not necessarily the true eigenfunctions of  $\mathcal{N}_h$  (whose existence in the spectral window of interest is not obvious). Our presentation avoids the spectral analysis of  $\mathcal{N}_h$  (existence of the discrete spectrum, Agmon estimates, etc.) by comparing directly the spectra of  $\mathcal{L}_h$  and of the effective operator.

## 5.5 Analysis of the effective operator

This section is devoted to the spectral study of  $M_h^W$  in  $[a, b]$ . Let us diagonalize this operator, up to a remainder of order  $\mathcal{O}(\hbar^2)$ . Note that, by using the exponential decay of the eigenfunctions of  $n_0$ , we may (and so do we) replace  $\zeta_\mu$  by 1.

### 5.5.1 Asymptotic diagonalization and end of the proof of Theorem 5.1.7

We notice that the spectrum of

$$\mathcal{T}_h = \exp(\hbar A^W) M_h^W \exp(-\hbar A^W)$$

is the same as the one of  $M_h^W$ , as soon as  $A$  belongs to  $S(1)$  and is  $2L$ -periodic with respect to  $s$ . In this case, we recall that  $A^W$  is bounded from  $L^2(\mathbf{T}_{2L})$  to  $L^2(\mathbf{T}_{2L})$  (and thus its exponential is well-defined as an element of  $\mathcal{L}(L^2(\mathbf{T}_{2L}))$ ) thanks to the classical power series). Let us explain how to choose  $A$ . By expanding the exponential, we have

$$\mathcal{T}_h = (\text{Id} + \hbar A^W)M_h^W(\text{Id} - \hbar A^W) + \mathcal{O}(\hbar^2),$$

and thus

$$\mathcal{T}_h = M_h^W + \hbar[A^W, M_h^W] + \mathcal{O}(\hbar^2),$$

so that

$$\mathcal{T}_h = M_0^W + \hbar(M_1 + [A, M_0])^W + \mathcal{O}(\hbar^2).$$

Therefore,  $A$  should be chosen so that  $M_1 - [M_0, A]$  is diagonal. The map  $\text{Skew}_n(\mathbb{R}) \ni A \mapsto [M_0, A] \in \text{Sym}_n^0(\mathbb{R})$  is well-defined and an isomorphism since  $M_0$  is diagonal with distinct real entries, where  $\text{Skew}_n(\mathbb{R})$  is the vector space of skew-symmetric matrices and  $\text{Sym}_n^0(\mathbb{R})$  the space of symmetric matrices with null diagonal. It is actually easy to compute its inverse. Consider  $M$  a symmetric matrix with null diagonal. We want to find  $A \in \text{Skew}_n(\mathbb{R})$  such that  $[M_0, A] = M$ . For all  $j \in \{1, \dots, n\}$ , we have

$$(M_0 - \overset{\circ}{\mu}_j)Ae_j = Me_j,$$

and then, for all  $k \in \{1, \dots, n\}$ ,

$$(\overset{\circ}{\mu}_k - \overset{\circ}{\mu}_j)\langle Ae_j, e_k \rangle = \langle Me_j, e_k \rangle.$$

Thus, for all  $k \neq j$ ,

$$\langle Ae_j, e_k \rangle = (\overset{\circ}{\mu}_k - \overset{\circ}{\mu}_j)^{-1}\langle Me_j, e_k \rangle,$$

which determines a unique  $A \in \text{Skew}_n(\mathbb{R})$ .

Since there is a uniform gap between the  $\overset{\circ}{\mu}_j$  (with respect to  $\sigma$ ), we get the existence of a skew-symmetric  $A$  in  $S(1)$  such that

$$\underbrace{M_1 - \text{diag}(M_1)}_{\in \text{Sym}_n^0(\mathbb{R})} + [A, M_0] = 0.$$

With this choice, we get

$$\mathcal{T}_h = M_0^W + \hbar \text{diag}(M_1^W) + \mathcal{O}(\hbar^2).$$

Note that, for all  $j \in \{1, \dots, n\}$ , we have  $\langle M_1 e_j, e_j \rangle = \kappa(s)\langle \mathcal{C}(\tau, \Xi_0(\sigma))u_j^{[\Xi_0(\sigma), \gamma]}, u_j^{[\Xi_0(\sigma), \gamma]} \rangle$ . By the spectral theorem, we deduce that the spectra of  $M_h^W$  and  $M_0^W + \hbar \text{diag}(M_1^W)$  coincide modulo  $\mathcal{O}(\hbar^2)$ . This procedure can be continued at any order.

## 5.5.2 Spectral consequences

The aim of this last section is to prove Proposition 5.1.8 and Corollary 5.1.11.

### 5.5.2.1 Proof of Proposition 5.1.8

We could not find this particular statement in the literature, because a) we have to deal with non-connected level sets of the principal symbol, and b) we have Floquet periodic conditions, with  $\hbar$ -dependent Floquet exponent.. The first issue is treated with usual microlocal arguments: each connected component carries with itself a Bohr-Sommerfeld asymptotic series, as in [HR84], and the initial spectrum is obtained, modulo  $\mathcal{O}(\hbar^\infty)$ , by the superposition (with multiplicities) of all these series. The second one is easily included in

the general theory thanks to the “sheaf” approach of [VN00]. Indeed, near each point of the energy level curve  $\sigma = \text{const}$ , the operator  $P_h$  is microlocally a usual  $\hbar$ -pseudo-differential operator, and the quantum Darboux-Carathéodory normal form holds. Therefore, the Bohr-Sommerfeld cocycle of [VN00, Proposition 5.6] holds; the difference being that the condition for a global section should include the Floquet exponent  $\alpha$ . This gives a Bohr-Sommerfeld rule for quantized energies  $E$  (for each connected component) of the form

$$\mathcal{A}(E) + \hbar m(E) \frac{\pi}{2} + \hbar \mathcal{K}(E) + \mathcal{O}(\hbar^2) = 2\pi\hbar(\ell + \frac{L}{\pi}\alpha), \quad \ell \in \mathbb{Z}, \quad (5.50)$$

where  $\mathcal{A}(E)$  is the action integral (here  $\mathcal{A}(E) = 2L\sigma$  when  $E = \mu(\sigma)$ ),  $m(E)$  the Maslov index (which vanishes here, because the curves  $\sigma = \text{const}$  project diffeomorphically on the  $s$  variable), and  $\mathcal{K}(E)$  is the integral of the subprincipal form [VN00, Definition 3.2] along the energy level set. In order to compute  $\mathcal{K}$ , we notice that the Hamiltonian vector field of  $\mu(\sigma)$  is  $\mu'(\sigma) \frac{\partial}{\partial s}$  and hence the subprincipal form is  $\frac{-r}{\mu'(\sigma)} ds$ , where  $r$  is the subprincipal symbol of  $P_h$  (here  $r = -C(\sigma)\kappa(s)$ ). Hence, for  $E = \mu(\sigma)$ , we have

$$\mathcal{K}(E) = \frac{C(\sigma)}{\mu'(\sigma)|_{\Sigma_q}} \int_0^{2L} \kappa(s) ds.$$

Inverting the formal series (5.50), we get

$$E = \mu(\sigma) - \mathcal{K}(\mu(\sigma)) \frac{1}{2L} \mu'(\sigma) + \mathcal{O}(\hbar^2), \quad \sigma = \frac{\pi}{L} \hbar(\ell + \frac{L}{\pi}\alpha)$$

which gives (5.9) and (5.10).

### 5.5.2.2 Proof of Corollary 5.1.11

Thanks to Proposition 5.4.6 and the considerations in Section 5.5.1, we know that the spectrum of  $\mathcal{L}_h$  in  $[ah, bh]$  coincides with that of  $hM_h^W$  modulo  $\mathcal{O}(\hbar^2)$ . In the present section, since we are interested in the low-lying eigenvalues, we take  $a = -\infty$  and  $b = \Theta_0(\gamma) + \varepsilon < 1$  (for  $\varepsilon > 0$  small enough). Therefore, we have  $n = 1$  and the matrix symbol  $M_h$  reduces to a scalar symbol:

$$M_h(s, \sigma) = \mu_1(\gamma, \sigma) + \hbar \kappa(s) \langle \mathcal{C}(\tau, \Xi_0(\sigma)) u_1^{[\Xi_0(\sigma), \gamma]}, u_1^{[\Xi_0(\sigma), \gamma]} \rangle.$$

We are interested in the spectrum of  $M_h^W$  (when acting on  $e^{isf_0/\hbar} L^2(\mathbf{T}_{2L})$ ). Thanks to the Weyl asymptotic formula for pseudodifferential operators (see, for instance, [Zwo12, Theorem 14.11]), the counting function  $N(M_h^W, \Theta_0(\gamma) + \varepsilon)$  (giving the number of eigenvalues less than  $\Theta_0(\gamma) + \varepsilon$ ) satisfies

$$\begin{aligned} N(M_h^W, \Theta_0(\gamma) + \varepsilon) &= \frac{1}{2\pi\hbar} \int_{\{(s, \sigma) : \mu_1(\sigma) \leq \Theta_0(\gamma) + \varepsilon\}} ds d\sigma + o(\hbar^{-1}) \\ &= \frac{L}{\pi\hbar} |\{\sigma : \mu_1(\sigma) \leq \Theta_0(\gamma) + \varepsilon\}| (1 + o(1)). \end{aligned}$$

Now, we take  $\varepsilon = \hbar^\eta$ , for some given  $\eta > 0$ .

Due to the non-degeneracy of the minimum of  $\sigma \mapsto \mu_1(\gamma, \Xi_0(\sigma))$ , the eigenfunctions associated with eigenvalues less than  $b$  are microlocalized in a neighborhood of  $\xi_0(\gamma)$  of size  $\hbar^{\eta/2}$  (and so are all the linear combinations of such eigenfunctions due to the Weyl estimate). This invites us to expand the symbol near  $\xi_0(\gamma)$ :

$$M_h(s, \sigma) = \Theta_0(\gamma) + \frac{\partial_\sigma^2 \mu(\gamma, \xi_0(\gamma))}{2} (\sigma - \xi_0(\gamma))^2 - \hbar \kappa(s) C_1(\xi_0(\gamma)) + \mathcal{O}(|\sigma - \xi_0(\gamma)|^3 + \hbar |\sigma - \xi_0(\gamma)|), \quad (5.51)$$



where  $C_1(\xi_0(\gamma)) > 0$ , as soon as  $\gamma < \gamma_0^{[0]}$  (see Remark 5.1.5). Therefore,  $M_h$  is relative perturbation of the symbol of a classical electric Schrödinger operator. The corresponding operator is

$$\mathcal{M}_h = \Theta_0(\gamma) + \frac{\partial_\sigma^2 \mu(\gamma, \xi_0(\gamma))}{2} (\hbar D_s - \xi_0(\gamma))^2 - \hbar \kappa(s) C_1(\xi_0(\gamma)).$$

The assumption that  $\kappa$  has a unique maximum, which is non-degenerate, allows to use the harmonic approximation near the maximum of  $\kappa$  (and even a Birkhoff normal form, see, for instance, [Ray17, Chapter 5]). The eigenvalues of  $\mathcal{M}_h$  satisfy

$$\lambda_j(\mathcal{M}_h) = \Theta_0(\gamma) - \kappa_{\max} C_1(\xi_0(\gamma)) \hbar + \left( (2j-1) \sqrt{\frac{C_1(\xi_0(\gamma)) k_2 \mu_1''(\gamma, \xi_0(\gamma))}{4}} \right) \hbar^{\frac{3}{2}} + o(\hbar^{\frac{3}{2}}),$$

uniformly in  $j \geq 1$  such that  $j \hbar^{\frac{1}{2}} = o(1)$ .

We recall that  $\hbar = h^{\frac{1}{2}}$ . We get Corollary 5.1.11 by noticing, thanks to a perturbative analysis using (5.51), that the spectra of  $hM_h^W$  and  $h\mathcal{M}_h$  below  $h(\Theta_0(\gamma) + \hbar^n)$  coincide modulo  $o(h^{\frac{7}{4}})$ .

### 5.5.2.3 Proof of Theorem 5.1.10

By Theorem 5.1.7, and Definition 5.1.6 we have, for  $\epsilon = \mathcal{O}(h)$ ,

$$N(h\mathfrak{M}_h, [h(a+\epsilon), h(b-\epsilon)]) \leq N(\mathcal{L}_h, [ha, hb]) \leq N(h\mathfrak{M}_h, [h(a-\epsilon), h(b+\epsilon)]). \quad (5.52)$$

From Corollary 5.1.9, for any interval  $[a', b']$  disjoint from  $\Theta$  and  $\Lambda$ , the number of eigenvalues of  $h\mathfrak{M}_h$  in  $[ha', hb']$  is bounded by  $C \frac{b'-a'}{h^{1/2}}$  for some constant  $C > 0$ . Applying this with  $(a', b')$  equal, respectively, to the four intervals  $(a, a+\epsilon)$ ,  $(b-\epsilon, b)$ ,  $(a-\epsilon, a)$ , and  $(b, b+\epsilon)$ , it follows from (5.52) that

$$N(\mathcal{L}_h, [ha, hb]) = N(h\mathfrak{M}_h, [ha, hb]) + \mathcal{O}(\epsilon h^{-1/2}) = N(h\mathfrak{M}_h, [ha, hb]) + \mathcal{O}(h^{1/2}).$$

Therefore, it is enough to estimate  $N(h\mathfrak{M}_h, [ha, hb])$ , for which we apply Corollary 5.1.9 (which is actually a description of the spectrum of  $\mathfrak{M}_h$ ). This corollary says that the number of eigenvalues of  $\mathfrak{M}_h$  inside  $[ha, hb]$ , including multiplicities, is given, modulo  $\mathcal{O}(h^2)$ , by the number of integers  $\ell \in \mathbb{Z}$  such that

$$h^{\frac{1}{2}} \left( \frac{\pi}{L} \ell + \alpha(h) \right) \in f_{k,q,h}^{-1}([a, b]), \quad (5.53)$$

for some admissible  $(k, q)$ , where  $f_{k,q,h}(\sigma) := f_{k,q}(\sigma, h^{1/2})$ .

To simplify notations, let us momentarily fix  $(k, q)$  and denote

$$f_h := f_{k,q,h} =: f_0 + h^{1/2} f_1 + \mathcal{O}(h),$$

where  $f_0$  and  $f_1$  are defined in (5.11) and (5.12). By assumption,  $f_0$  is monotonous on  $\Sigma_{k,q}$ , let us assume that it is increasing; the decreasing case is obtained by swapping  $(a, b)$ . For  $h$  small enough,  $f_h$  is also increasing and hence  $f_h^{-1}([a, b]) = [f_h^{-1}(a), f_h^{-1}(b)]$ . Therefore, the solutions to (5.53) are exactly the integers belonging to the interval

$$\frac{Lh^{-1/2}}{\pi} [f_h^{-1}(a), f_h^{-1}(b)] - \frac{L}{\pi} \alpha(h). \quad (5.54)$$

Let  $\sigma = f_h^{-1}(a)$ ; of course  $\sigma$  depends on  $h$ , but since  $\sigma \in \Sigma_{k,q}$ , it is bounded and we have  $\sigma = f_0^{-1}(a) +$

$\mathcal{O}(h^{1/2})$ . Therefore,  $f_1(\sigma) = f_1(f_0^{-1}(a)) + \mathcal{O}(h^{1/2})$ . According to the statement of Theorem 5.1.10, we denote  $\alpha := f_0^{-1}(a)$ . Writing

$$f_0(\sigma) = a - h^{1/2}f_1(\beta) + \mathcal{O}(h),$$

we get, by Taylor expansion,

$$\sigma = \alpha - h^{1/2}(f_0^{-1})'(a)f_1(\alpha) + \mathcal{O}(h) = \alpha + h^{1/2}\frac{\langle \kappa \rangle C_k(\alpha)}{\mu'_k(\alpha)} + \mathcal{O}(h).$$

Using the analogous formula for  $f_h^{-1}(b)$ , we may compute the difference  $f_h^{-1}(b) - f_h^{-1}(a)$  and obtain the length of the interval (5.54) (with  $\beta := f_0^{-1}(b)$ ):

$$\frac{Lh^{-1/2}}{\pi}(f_h^{-1}(b) - f_h^{-1}(a)) = \frac{Lh^{-1/2}}{\pi}(\beta - \alpha) + \frac{L\langle \kappa \rangle}{\pi} \left( \frac{C_k(\beta)}{\mu'_k(\beta)} - \frac{C_k(\alpha)}{\mu'_k(\alpha)} \right) + \mathcal{O}(h^{1/2}),$$

which gives Theorem 5.1.10 by summing over admissible  $(k, q)$ .

## 5.6 Appendix

### 5.6.1 A rough Weyl estimate

The aim of this section is recall why (5.3) holds.

– For  $\gamma \in \mathbb{R}$ : Thanks to the Young inequality, we have, for all  $\psi \in H^1(\Omega)$ ,

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geq \frac{h^2}{2} \|\nabla \psi\|^2 - 2\|A\|_\infty^2 \|\psi\|^2 + \gamma h^{\frac{3}{2}} \int_{\partial\Omega} |\psi|^2 ds.$$

– When  $\gamma \geq 0$ , we get that

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geq \frac{h^2}{2} \|\nabla \psi\|^2 - 2\|A\|_\infty^2 \|\psi\|^2.$$

– When  $\gamma < 0$ , we use a classical trace theorem: there exists  $C > 0$  such that, for all  $\varepsilon > 0$ ,

$$\int_{\partial\Omega} |\psi|^2 ds \leq \varepsilon \|\nabla \psi\|^2 + C\varepsilon^{-1} \|\psi\|^2.$$

By choosing  $\varepsilon = -\frac{\sqrt{h}}{4\gamma} > 0$ , we deduce that

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geq \frac{h^2}{4} \|\nabla \psi\|^2 - 2\|A\|_\infty^2 \|\psi\|^2 - 4\gamma^2 Ch \|\psi\|^2.$$

In both cases, there exists  $\tilde{C} > 0$  such that, for all  $h \in (0, 1)$  and all  $\psi \in H^1(\Omega)$ ,

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geq \frac{h^2}{4} \|\nabla \psi\|^2 - \tilde{C} \|\psi\|^2.$$

With the min-max principle, this shows that, for all  $\lambda$ ,

$$N(\mathcal{L}_h, \lambda) \leq N\left(-\Delta^{\text{Neu}}, 4\frac{\lambda + \tilde{C}}{h^2}\right).$$

The conclusion follows from the Weyl asymptotics for the Neumann Laplacian, which is the same at the

main order as in the Dirichlet case, see, for instance, [NS05, Introduction].

– For  $\gamma = +\infty$  : Thanks to the Young inequality, we have, for all  $\psi \in H_0^1(\Omega)$ ,

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geq \frac{h^2}{2} \|\nabla \psi\|^2 - 2\|A\|_\infty^2 \|\psi\|^2.$$

There exists  $\tilde{C} > 0$  such that, for all  $h \in (0, 1)$  and all  $\psi \in H_0^1(\Omega)$ ,

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geq \frac{h^2}{2} \|\nabla \psi\|^2 - \tilde{C} \|\psi\|^2.$$

With the min-max principle, this shows that, for all  $\lambda$ ,

$$N(\mathcal{L}_h, \lambda) \leq N\left(-\Delta^{\text{Dir}}, 2\frac{\lambda + \tilde{C}}{h^2}\right).$$

The conclusion follows from the Weyl asymptotics for the Dirichlet Laplacian, see, for instance, [NS05, Introduction].

## 5.6.2 Spectral analysis of de Gennes operator

**Lemma 5.6.1.** *We let*

$$u_j^{[\gamma]} = u_j^{[\gamma, \xi_{j-1}(\gamma)]}.$$

*We have*

$$\begin{aligned} C_j(\xi_{j-1}(\gamma)) &= \frac{1}{2} \left(u_j^{[\gamma]}(0)\right)^2 - \int_0^{+\infty} (t - \xi_{j-1}(\gamma))^3 \left(u_j^{[\gamma]}(t)\right)^2 dt \\ &= \frac{1}{3} (1 - \gamma \xi_{j-1}(\gamma)) \left(u_j^{[\gamma]}(0)\right)^2, \end{aligned}$$

where  $C_j$  is defined in (5.8).

*Proof.* We write

$$(\sigma - t)t^2 + 2t(\sigma - t)^2 = (t - \sigma)^3 - \sigma^2(t - \sigma).$$

We take  $\sigma = \xi_{j-1}(\gamma)$ . The conclusion follows from Proposition 2.4.11.  $\square$

**Proposition 5.6.2.** *Let us fix  $j \geq 1$ . When  $\gamma \in \mathbb{R}$ , there exists  $\gamma_0^{[j-1]} > 0$ , such that,  $C_j(\xi_{j-1}(\gamma))$  is positive if  $\gamma < \gamma_0^{[j-1]}$  and negative if  $\gamma > \gamma_0^{[j-1]}$ .*

*Proof.* We notice that, for  $\gamma \leq 0$ , we get  $C_j(\xi_{j-1}(\gamma)) < 0$ .

Now, for  $\gamma > 0$ . From Proposition 2.4.8, we can rewrite  $C_j(\xi_{j-1}(\gamma))$  as

$$C_j(\xi_{j-1}(\gamma)) = \frac{1}{3} \left(1 - \gamma \sqrt{\gamma^2 + \Theta^{[j-1]}(\gamma)}\right) \left(u_j^{[\gamma]}(0)\right)^2.$$

Since  $\left(u_j^{[\gamma]}(0)\right)^2 > 0$ , then, to study the sign of  $C_j(\xi_{j-1}(\gamma))$  it is sufficient to study the sign of the function

$f$  defined by  $f(\gamma) = 1 - \gamma\sqrt{\gamma^2 + \Theta^{[j-1]}(\gamma)}$ . We have

$$f'(\gamma) = -\sqrt{\gamma^2 + \Theta^{[j-1]}(\gamma)} - \gamma \frac{(\Theta^{[j-1]}(\gamma))' + 2\gamma}{\sqrt{\gamma^2 + \Theta^{[j-1]}(\gamma)}}.$$

We can use [Kac06b, Section B] (which can be adapted to  $j \geq 1$ ) to deduce that  $f'(\gamma) < 0$ . Therefore,  $f$  is increasing on  $[0, +\infty[$ .

Let us notice now that  $f(0) = 1$  and  $\lim_{\gamma \rightarrow +\infty} f(\gamma) = -\infty$ . This establishes the existence of a unique zero of  $f(\gamma)$ , denoted by  $\gamma_0^{[j-1]}$ .  $\square$

**Lemma 5.6.3.** *Let  $\alpha \in \mathbb{R}$  and  $\beta \in \mathbb{N}$ . Consider the interval  $[a, b]$ . We consider  $\Pi\psi = (\langle \psi, u_j^{[\gamma, \sigma]} \rangle)_{1 \leq j \leq n}$ , where  $n$  is the number of dispersion curves  $\mu_j(\gamma, \sigma)$  taking values in  $[a, b]$  (see the discussion at the beginning of Section 5.3.3). We consider  $\hat{K}$  a neighborhood of  $\hat{K}$ .*

*There exists  $C_{\alpha, \beta} > 0$  such that for all  $z \in [a, b]$  and all  $\sigma \in \hat{K}$ , the following holds. For all  $v \in L^2(\mathbb{R}_+)$  such that  $\langle t \rangle^\alpha v \in L^2(\mathbb{R}_+)$ , we have*

$$\|\langle t \rangle^{-\alpha} \partial_\sigma^\beta (H[\gamma, \sigma] - z)^{-1} (\Pi^* \Pi)^\perp (\langle t \rangle^\alpha v)\| \leq C_{\alpha, \beta} \|v\|,$$

where  $(\Pi^* \Pi)^\perp = Id - \Pi^* \Pi$ .

*Proof.* Let us only prove this estimate for  $\beta = 0$ . We consider  $z \in [a, b]$ . Let us consider  $v \in \mathcal{S}(\overline{\mathbb{R}}_+)$  and let  $u$  be the unique solution to the equation

$$(H[\gamma, \sigma] - z)u = (\Pi^* \Pi)^\perp (\langle \epsilon t \rangle_k^\alpha v) \tag{5.55}$$

that is orthogonal to  $(u_j^{[\gamma, \sigma]})_{1 \leq j \leq n}$ , with

$$\langle t \rangle_k = \left(1 + t^2 \chi_k^2\right)^{\frac{1}{2}},$$

where  $\chi_k$  is a smooth non-negative function equal to 0 on  $[0, 1]$  and to 1 on  $[2k, +\infty)$  and such that  $|\chi_k'| \leq k^{-1}$ . In particular, the weight is 1 near 0. Here  $\epsilon > 0$  is a parameter to be chosen small enough.

We have seen in Lemma 5.2.2 that

$$\begin{pmatrix} H[\gamma, \sigma] - z & \Pi^* \\ \Pi & 0 \end{pmatrix} : B^2(\mathbb{R}_+) \times \mathbb{C}^n \longrightarrow L^2(\mathbb{R}_+) \times \mathbb{C}^n$$

is bijective. Thus, the equation (5.55) is equivalent to

$$\begin{pmatrix} H[\gamma, \sigma] - z & \Pi^* \\ \Pi & 0 \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} (\Pi^* \Pi)^\perp (\langle t \rangle_k^\alpha v) \\ 0 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} \langle \epsilon t \rangle_k^{-\alpha} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} H[\gamma, \sigma] - z & \Pi^* \\ \Pi & 0 \end{pmatrix} \begin{pmatrix} \langle \epsilon t \rangle_k^\alpha & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} H[\gamma, \sigma] + R_{\epsilon, k} - z & \langle \epsilon t \rangle_k^{-\alpha} \Pi^* \\ \Pi(\cdot \langle \epsilon t \rangle_k^\alpha) & 0 \end{pmatrix},$$

where

$$R_{\epsilon,k} = -\alpha\epsilon^2(t^2\chi_k^2)' \frac{1}{1+\epsilon^2t^2\chi_k^2} \partial_t - \left( \frac{\epsilon^2\alpha}{2} \frac{(t^2\chi_k^2)''}{1+\epsilon^2t^2\chi_k^2} + \left( \frac{\alpha}{2} - 1 \right) \frac{\epsilon^4\alpha}{2} \frac{(t^2\chi_k^2)'^2}{(1+\epsilon^2t^2\chi_k^2)^2} \right).$$

With

$$u = \langle \epsilon t \rangle_k^\alpha \tilde{u},$$

we get

$$\mathcal{H}[\gamma, \sigma, \epsilon, k] \begin{pmatrix} \tilde{u} \\ 0 \end{pmatrix} = \begin{pmatrix} \langle \epsilon t \rangle_k^{-\alpha} (\Pi^* \Pi)^\perp (\langle \epsilon t \rangle_k^\alpha v) \\ 0 \end{pmatrix},$$

with

$$\mathcal{H}[\gamma, \sigma, \epsilon, k] = \begin{pmatrix} H[\gamma, \sigma] + R_{\epsilon,k} - z & \langle \epsilon t \rangle_k^{-\alpha} \Pi^* \\ \Pi(\cdot \langle \epsilon t \rangle_k^\alpha) & 0 \end{pmatrix}.$$

Thanks to the exponential decay of the  $u_j^{[\gamma, \sigma]}$  (which is uniform for  $\sigma \in \hat{K}$ ), we notice that  $\mathcal{H}[\gamma, \sigma, \epsilon, k]$  is bijective as soon as  $\epsilon$  is small enough and  $k$  large enough. Moreover,

$$\|\mathcal{H}[\gamma, \sigma, \epsilon, k]^{-1}\| \leq C.$$

This implies that

$$\|\tilde{u}\| \leq C \|\langle \epsilon t \rangle_k^{-\alpha} (\Pi^* \Pi)^\perp (\langle \epsilon t \rangle_k^\alpha v)\|,$$

and then (by using again the exponential decay of the eigenfunctions)

$$\|\langle \epsilon t \rangle_k^{-\alpha} u\| \leq C \|v\|.$$

Taking the limit  $k \rightarrow +\infty$ , the Fatou lemma gives

$$\|\langle \epsilon t \rangle^{-\alpha} u\| \leq C \|v\|.$$

This provides us with the desired estimate since  $\langle \epsilon t \rangle \langle t \rangle^{-1} \in [\epsilon, 1]$ .

For  $\beta \neq 0$ , simply note that  $(\partial^\beta u = \partial^\beta H[\gamma, \sigma] - z)(\Pi^* \Pi)^\perp (\langle \epsilon t \rangle_k^\alpha v)$  and follow the same demonstration steps, by replacing  $u$  with  $\partial^\beta u$ . □

### 5.6.3 Spectral analysis of an operator on a semi-cylinder

In this section, we will establish that the spectra of  $\mathcal{L}_h$  and  $\widetilde{\mathcal{L}}_h$  coincide in  $I_h$  modulo  $\mathcal{O}(h^\infty)$ . Then, we can now focus on the spectral analysis of  $\widetilde{\mathcal{L}}_h$ . But, in fact, as we have already seen, our presentation avoids the spectral analysis of  $\widetilde{\mathcal{L}}_h$ , but it is important to write the proof because we have used new techniques (in tubular coordinates) without going through the variational method like most magnetic works.

Thanks to the Agmon estimates of the eigenfunctions of  $\mathcal{L}_h$ , we can establish that the spectrum of  $\mathcal{L}_h$  in  $I_h$  is close to that of  $\widetilde{\mathcal{L}}_h$ .

**Proposition 5.6.4.** *There exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$  the following holds. Let us consider an interval  $J_h \subset I_h$ . Then, there exists an interval  $\hat{J}_h$  such that  $J_h \subset \hat{J}_h \subset I_h$  with  $d_{\mathbb{H}}(J_h, \hat{J}_h) = \mathcal{O}(h^\infty)$  and*

$$\text{rank } \mathbb{1}_{J_h}(\mathcal{L}_h) \leq \text{rank } \mathbb{1}_{\hat{J}_h}(\widetilde{\mathcal{L}}_h). \quad (5.56)$$

Moreover, for all  $\lambda \in I_h \cap \text{Sp}(\mathcal{L}_h)$ ,

$$\text{dist}(\lambda, \text{Sp}(\widetilde{\mathcal{L}}_h)) = \mathcal{O}(h^\infty). \quad (5.57)$$

*Proof.* Let us start by proving (5.57). Let us consider an eigenvalue  $\lambda \in I_h$  of  $\mathcal{L}_h$ . We write the eigenvalue equation

$$\mathcal{L}_h \psi = \lambda \psi.$$

Since  $\psi$  satisfies the Agmon estimates of Proposition 5.2.1, we see that

$$\|\chi \psi\| = \|\psi\| + \mathcal{O}(h^\infty) \quad \text{and} \quad \mathcal{L}_h(\chi \psi) = \lambda \chi \psi + \mathcal{O}(h^\infty),$$

where  $\chi$  is a smooth function equal to 1 near the boundary and 0 away from a small enough neighborhood of the boundary. On the support of  $\chi$ , we may use the tubular coordinates and we deduce that

$$\widetilde{\mathcal{L}}_h(\widetilde{\chi} \widetilde{\psi}) = \lambda \widetilde{\chi} \widetilde{\psi} + \mathcal{O}(h^\infty),$$

where  $\widetilde{\chi}, \widetilde{\psi}$  is the expression of  $\chi, \psi$  in the tubular coordinates  $(s, t)$ . Thus, (5.57) follows from the spectral theorem.

Let us now consider (5.56), which deals with multiplicities. Let us write  $\text{Sp}(\mathcal{L}_h) \cap J_h = \{\lambda_1, \dots, \lambda_p\}$  (where the  $\lambda_j$  are distinct) and underline that these eigenvalues depend on  $h$  as well as  $p$ . Consider the associated eigenspaces  $(E_j)_{1 \leq j \leq p}$  and note that  $\dim \bigoplus_{j=1}^p E_j = \mathcal{O}(h^{-2})$ , thanks to the Weyl estimate (5.3). With the same notation as above, we consider the spaces of quasimodes  $(\widetilde{\chi} \widetilde{E}_j)_{1 \leq j \leq p}$ . Thanks to the Agmon estimates (and the rough Weyl estimate),  $\dim(\widetilde{\chi} \widetilde{E}_j) = \dim E_j$ , as soon as  $h$  is small enough. Moreover, we have

$$\|(\bigoplus_{j=1}^p \widetilde{\mathcal{L}}_h - \lambda) \varphi\| \leq \varepsilon_h \|\varphi\|, \quad \varepsilon_h = \mathcal{O}(h^\infty),$$

for all  $\varphi = (\varphi_1, \dots, \varphi_p) \in \bigoplus_{j=1}^p \widetilde{\chi} \widetilde{E}_j$  and where  $\lambda = (\lambda_1, \dots, \lambda_p)$ .

We set  $J_h = [a_h, b_h]$  and  $\hat{J}_h = [a_h - \varepsilon_h, b_h + \varepsilon_h]$ . If  $\text{rank } \mathbb{1}_{\hat{J}_h}(\widetilde{\mathcal{L}}_h) < \text{rank } \mathbb{1}_{J_h}(\mathcal{L}_h)$ , then the projection

$$\Pi : \bigoplus_{j=1}^p \widetilde{\chi} \widetilde{E}_j \rightarrow \text{ran } \mathbb{1}_{\hat{J}_h}(\widetilde{\mathcal{L}}_h)$$

could not be injective. Considering a non-zero  $\varphi$  in its kernel, the spectral theorem would give

$$\|(\bigoplus_{j=1}^p \widetilde{\mathcal{L}}_h - \lambda) \varphi\| > \varepsilon_h \|\varphi\|,$$

which is a contradiction when  $\varphi \neq 0$ . Therefore, (5.56) follows.  $\square$

The aim of the following section is to prove the converse estimate. For this, one needs to prove that the eigenfunctions of  $\widetilde{\mathcal{L}}_h$  have an exponential decay with respect to  $t$ , and thus that they can serve a good quasimodes for  $\mathcal{L}_h$ .

### 5.6.3.1 Agmon estimates for $\widetilde{\mathcal{L}}_h$

In this section, we establish the following.

**Proposition 5.6.5.** *There exist  $\alpha > 0, C > 0, h_0 > 0$  such that for all  $h \in (0, h_0)$  and all eigenfunctions  $\psi$  of  $\mathcal{L}_h$  associated with an eigenvalue in  $I_h$ , we have*

$$\int_{[-L,L] \times \mathbb{R}_+} e^{2\alpha t/h^{\frac{1}{2}}} |\psi|^2 ds dt \leq C \|\psi\|^2, \quad (5.58)$$

and

$$\int_{[-L,L] \times \mathbb{R}_+} e^{2\alpha t/h^{\frac{1}{2}}} \left[ |(-ih\partial_s - t + f_0)\psi|^2 + h^2 |\partial_t \psi|^2 \right] ds dt \leq Ch \|\psi\|^2. \quad (5.59)$$

The proof will follow from the same strategy as that of Proposition 5.2.1. Let us consider  $\lambda \in [ah, bh] \cap \text{Sp}(\widetilde{\mathcal{L}}_h)$  and an associated eigenfunction  $\psi$ . We have

$$\left( \widetilde{\mathcal{L}}_h - \lambda \right) \psi = 0. \quad (5.60)$$

We let  $\varphi = e^{\Phi/h^{\frac{1}{2}}} \psi$ , where  $\Phi = \Phi_m(t) = \alpha t \chi(m^{-1}t)$ , with  $m \geq 1$  and  $\chi$  a smooth non-negative function equal to 1 near  $t = 0$ . Here  $\alpha > 0$  is to be determined. The parameter  $m$  will be sent to  $+\infty$  at the end. This gives that

$$\begin{aligned} \widetilde{\mathcal{L}}_h \psi &= \widetilde{\mathcal{L}}_h e^{-\Phi/h^{\frac{1}{2}}} \varphi \\ &= e^{-\Phi/h^{\frac{1}{2}}} a(s, \check{t})^{-1} \left( -ih\partial_s - t + f_0 + \kappa(s) \frac{\check{t}^2}{2} \right) a(s, \check{t})^{-1} \left( -ih\partial_s - t + f_0 + \kappa(s) \frac{\check{t}^2}{2} \right) \varphi \\ &\quad - h^2 a(s, \check{t})^{-1} \partial_t a(s, \check{t}) \partial_t (e^{-\Phi/h^{\frac{1}{2}}} \varphi) \\ &= e^{-\Phi/h^{\frac{1}{2}}} a(s, \check{t})^{-1} \left( -ih\partial_s - t + f_0 + \kappa(s) \frac{\check{t}^2}{2} \right) a(s, \check{t})^{-1} \left( -ih\partial_s - t + f_0 + \kappa(s) \frac{\check{t}^2}{2} \right) \varphi \\ &\quad - h^2 a(s, \check{t})^{-1} \partial_t \left[ a(s, \check{t}) \left( -\frac{\Phi'}{h^{\frac{1}{2}}} \varphi + \partial_t \varphi \right) e^{-\Phi/h^{\frac{1}{2}}} \right] \\ &= e^{-\Phi/h^{\frac{1}{2}}} \left[ a(s, \check{t})^{-1} \left( -ih\partial_s - t + f_0 + \kappa(s) \frac{\check{t}^2}{2} \right) a(s, \check{t})^{-1} \left( -ih\partial_s - t + f_0 + \kappa(s) \frac{\check{t}^2}{2} \right) \varphi \right. \\ &\quad \left. + a(s, \check{t})^{-1} (hD_t + ih^{\frac{1}{2}} \Phi') a(s, \check{t}) (hD_t + ih^{\frac{1}{2}} \Phi') \varphi \right]. \end{aligned}$$

The equations (5.60) becomes

$$a(s, \check{t})^{-1} P_{s,h} a(s, \check{t})^{-1} P_{s,h} \varphi + a(s, \check{t})^{-1} (hD_t + ih^{\frac{1}{2}} \Phi') a(s, \check{t}) (hD_t + ih^{\frac{1}{2}} \Phi') \varphi = \lambda \varphi,$$

where  $P_{s,h} = -ih\partial_s - t + f_0 + \kappa(s) \frac{\check{t}^2}{2}$ . Then, we obtain

$$\left( \widetilde{\mathcal{L}}_h^\Phi - \lambda \right) \varphi = 0,$$

where

$$\widetilde{\mathcal{L}}_h^\Phi = a(s, \check{t})^{-1} P_{s,h} a(s, \check{t})^{-1} P_{s,h} + a(s, \check{t})^{-1} P_{t,h}^\Phi a(s, \check{t}) P_{t,h}^\Phi, \quad P_{t,h}^\Phi = hD_t + ih^{\frac{1}{2}} \Phi'. \quad (5.61)$$

### 5.6.3.2 Preliminary estimates

Let us consider a function  $t \mapsto \chi_h(t)$  equal to 0 near  $t = 0$  on a neighborhood of size  $h^{\frac{1}{2}}$ . The following lemma is just a commutator estimate.

**Lemma 5.6.6.** *We have*

$$(\widetilde{\mathcal{L}}_h^\Phi - \lambda)(\chi_h \varphi) = -2ih\chi_h' P_{t,h}^\Phi \varphi - h^2 \chi_h'' \varphi - h^2 \chi_h' (a(s, \check{t})^{-1} \partial_t [a(s, \check{t})]) \varphi.$$

*Proof.* Since  $\chi_h$  does not depend on  $s$ , we have

$$\begin{aligned} (\widetilde{\mathcal{L}}_h^\Phi - \lambda)(\chi_h \varphi) &= [\widetilde{\mathcal{L}}_h^\Phi, \chi_h] \varphi \\ &= [a(s, \check{t})^{-1} P_{s,h} a(s, \check{t})^{-1} P_{s,h} + a(s, \check{t})^{-1} P_{t,h}^\Phi a(s, \check{t}) P_{t,h}^\Phi, \chi_h] \varphi \\ &= [a(s, \check{t})^{-1} P_{s,h} a(s, \check{t})^{-1} P_{s,h}, \chi_h] \varphi + [a(s, \check{t})^{-1} P_{t,h}^\Phi a(s, \check{t}) P_{t,h}^\Phi, \chi_h] \varphi \\ &= [a(s, \check{t})^{-1} P_{t,h}^\Phi a(s, \check{t}) P_{t,h}^\Phi, \chi_h] \varphi \\ &= a(s, \check{t})^{-1} [P_{t,h}^\Phi a(s, \check{t}) P_{t,h}^\Phi, \chi_h] \varphi \\ &= a(s, \check{t})^{-1} P_{t,h}^\Phi a(s, \check{t}) [P_{t,h}^\Phi, \chi_h] \varphi + a(s, \check{t})^{-1} [P_{t,h}^\Phi, \chi_h] a(s, \check{t}) P_{t,h}^\Phi \varphi. \end{aligned}$$

Note that  $[P_{t,h}^\Phi, \chi_h] = -ih\chi_h'$  and thus

$$\begin{aligned} (\widetilde{\mathcal{L}}_h^\Phi - \lambda)(\chi_h \varphi) &= -iha^{-1} P_{t,h}^\Phi (a\chi_h' \varphi) - ih\chi_h' P_{t,h}^\Phi \varphi \\ &= -2ih\chi_h' P_{t,h}^\Phi \varphi - h^2 a^{-1} \partial_t (a\chi_h') \varphi. \end{aligned}$$

□

From Lemma 5.6.6, we deduce the following estimate.

**Lemma 5.6.7.** *We have*

$$\|(\widetilde{\mathcal{L}}_h^\Phi - \lambda)(\chi_h \varphi)\| \leq Ch \|\tilde{\chi}_h \varphi\| + Ch \|\chi_h' P_{t,h}^\Phi \varphi\|,$$

where  $\tilde{\chi}_h$  is bounded uniformly in  $h$  and supported in a neighborhood of  $t = 0$  of size  $h^{\frac{1}{2}}$ .

**Lemma 5.6.8.** *We have*

$$h \|\chi_h' P_{t,h}^\Phi \varphi\| \leq Ch \|\tilde{\chi}_h \varphi\|.$$

*Proof.* First, we notice that

$$h \|\chi_h' P_{t,h}^\Phi \varphi\| = h \|\chi_h' (hD_t + ih^{\frac{1}{2}} \Phi') \varphi\| \leq h \|\chi_h' (hD_t) \varphi\| + Ch \|\tilde{\chi}_h \varphi\|. \quad (5.62)$$

Then, we write

$$\begin{aligned} 0 = \operatorname{Re} \langle (\widetilde{\mathcal{L}}_h^\Phi - \lambda) \varphi, \chi_h'^2 \varphi \rangle_{L^2(\operatorname{adsdt})} &= \int_{[-L, L] \times \mathbb{R}} a(s, \check{t})^{-1} |P_{s,h}(\chi_h' \varphi)|^2 \operatorname{dsdt} \\ &\quad + \operatorname{Re} \langle P_{t,h}^\Phi a P_{t,h}^\Phi \varphi, (\chi_h')^2 \varphi \rangle_{L^2(\operatorname{dsdt})} - \lambda \|\chi_h' \varphi\|^2, \end{aligned}$$

we obtain,

$$\operatorname{Re} \langle P_{t,h}^\Phi a P_{t,h}^\Phi \varphi, (\chi_h')^2 \varphi \rangle_{L^2(\operatorname{dsdt})} \leq \lambda \|\chi_h' \varphi\|^2. \quad (5.63)$$



By computing commutators (and using the Dirichlet condition due to  $\chi'_h$ ), we get

$$\begin{aligned}
 & \operatorname{Re}\langle P_{t,h}^\Phi a P_{t,h}^\Phi \varphi, (\chi'_h)^2 \varphi \rangle_{L^2(\operatorname{dsdt})} \\
 &= \operatorname{Re}\langle P_{t,h}^\Phi \varphi, P_{t,h}^{-\Phi} (\chi'_h)^2 \varphi \rangle_{L^2(\operatorname{adsdt})} \\
 &= \operatorname{Re}\langle \chi'_h P_{t,h}^\Phi (\varphi), P_{t,h}^{-\Phi} (\chi'_h \varphi) \rangle_{L^2(\operatorname{adsdt})} + \operatorname{Re}\langle P_{t,h}^\Phi \varphi, [P_{t,h}^{-\Phi}, \chi'_h] \chi'_h \varphi \rangle_{L^2(\operatorname{adsdt})} \\
 &= \operatorname{Re}\langle P_{t,h}^\Phi (\chi'_h \varphi), P_{t,h}^{-\Phi} (\chi'_h \varphi) \rangle + \operatorname{Re}\langle P_{t,h}^\Phi \varphi, [P_{t,h}^{-\Phi}, \chi'_h] \chi'_h \varphi \rangle + \operatorname{Re}\langle [\chi'_h, P_{t,h}^\Phi] \varphi, P_{t,h}^{-\Phi} (\chi'_h \varphi) \rangle \\
 &= \operatorname{Re}\langle P_{t,h}^\Phi (\chi'_h \varphi), P_{t,h}^{-\Phi} (\chi'_h \varphi) \rangle + \operatorname{Re}\langle \chi'_h P_{t,h}^\Phi \varphi, (-ih\chi''_h) \varphi \rangle + \operatorname{Re}\langle ih\chi''_h \varphi, P_{t,h}^{-\Phi} (\chi'_h \varphi) \rangle \\
 &= \operatorname{Re}\langle P_{t,h}^\Phi (\chi'_h \varphi), P_{t,h}^{-\Phi} (\chi'_h \varphi) \rangle + \operatorname{Re}\langle P_{t,h}^\Phi \chi'_h \varphi, (-ih\chi''_h) \varphi \rangle - \operatorname{Re}\langle [P_{t,h}^\Phi, \chi'_h] \varphi, (-ih\chi''_h) \varphi \rangle \\
 &\quad + \operatorname{Re}\langle ih\chi''_h \varphi, P_{t,h}^{-\Phi} (\chi'_h \varphi) \rangle.
 \end{aligned}$$

We get

$$\begin{aligned}
 \operatorname{Re}\langle P_{t,h}^\Phi a P_{t,h}^\Phi \varphi, (\chi'_h)^2 \varphi \rangle_{L^2(\operatorname{dsdt})} &= \operatorname{Re}\langle P_{t,h}^\Phi (\chi'_h \varphi), P_{t,h}^{-\Phi} (\chi'_h \varphi) \rangle + \operatorname{Re}\langle P_{t,h}^\Phi (\chi'_h \varphi), (-ih\chi''_h) \varphi \rangle \\
 &\quad + \operatorname{Re}\langle ih\chi''_h \varphi, P_{t,h}^{-\Phi} (\chi'_h \varphi) \rangle + \operatorname{Re}\langle ih\chi''_h \varphi, (-ih\chi''_h) \varphi \rangle,
 \end{aligned}$$

and thus

$$\operatorname{Re}\langle P_{t,h}^\Phi a P_{t,h}^\Phi \varphi, (\chi'_h)^2 \varphi \rangle_{L^2(\operatorname{dsdt})} \geq \|hD_t(\chi'_h \varphi)\|^2 - C\|\tilde{\chi}_h \varphi\|^2 - C\alpha\|\tilde{\chi}_h \varphi\| \|hD_t(\chi'_h \varphi)\|.$$

By using the Young inequality and (5.63), for  $\alpha$  small enough, this gives

$$\|hD_t(\chi'_h \varphi)\|^2 \leq C\|\tilde{\chi}_h \varphi\|^2.$$

Coming back to (5.62), the result follows.  $\square$

Let us now consider the following operator

$$\widehat{\mathcal{L}}_h^{\operatorname{Cyl}} = (-ih\partial_s - t + f_0)^2 - h^2\partial_t^2,$$

acting in the ambient Hilbert space  $L^2(\operatorname{Cyl}, \operatorname{dsdt})$ , with  $\operatorname{Cyl} = [-L, L] \times \mathbb{R}$ , with  $2L$ -periodic boundary conditions. As we will see,  $\widehat{\mathcal{L}}_h^\Phi$  is perturbation of  $\widehat{\mathcal{L}}_h^{\operatorname{Cyl}}$ .

We use the completeness of the orthonormal family  $\left\{ e^{\frac{im\pi s}{L}} / \sqrt{2L} \right\}_{m \in \mathbb{Z}}$  in  $L^2(\mathbb{R}/2L\mathbb{Z})$ . For all  $\varphi \in H^1(\mathbb{R}/2L\mathbb{Z} \times \mathbb{R})$ , we have

$$\varphi(s, t) = \sum_{m \in \mathbb{Z}} \widehat{\varphi}_t(m) e^{\frac{im\pi s}{L}} = \sum_{m \in \mathbb{Z}} \psi_m(s, t),$$

where

$$\widehat{\varphi}_t(m) = \frac{1}{2L} \int_{-L}^L \varphi(s, t) e^{-\frac{im\pi s}{L}} \operatorname{ds},$$

and

$$\psi_m(s, t) = \widehat{\varphi}_t(m) e^{\frac{im\pi s}{L}}.$$

We have

$$\widehat{\mathcal{L}}_h^{\operatorname{Cyl}} \varphi = \sum_{m \in \mathbb{Z}} \left[ \left( h \frac{\pi m}{L} - t + \gamma_0 \right)^2 - h^2 \partial_t^2 \right] \psi_m.$$

Therefore, we may write

$$\widetilde{\mathcal{L}}_h^{\text{Cyl}} = \bigoplus_{n \in \mathbb{Z}} \widetilde{\mathcal{L}}_{h,n}^{\text{Cyl}}, \quad \widetilde{\mathcal{L}}_{h,n}^{\text{Cyl}} = \left( h \frac{\pi n}{L} - t + \gamma_0 \right)^2 - h^2 \partial_t^2.$$

By translation invariance, we see that  $\widetilde{\mathcal{L}}_{h,n}^{\text{Cyl}}$  is unitarily equivalent to the harmonic oscillator and thus that its spectrum is  $\{(2p-1)h, p \in \mathbb{N} \setminus \{0\}\}$ . Due our choice of  $\lambda$ , we get that

$$\left\| (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1} \right\| \leq Ch^{-1}. \quad (5.64)$$

The following lemma shows in which sense  $\widetilde{\mathcal{L}}_h^\Phi$  is a perturbation of  $\widetilde{\mathcal{L}}_h^{\text{Cyl}}$ .

**Lemma 5.6.9.** *Let  $\eta < \frac{1}{2}$ . We have*

$$\widetilde{\mathcal{L}}_h^\Phi = \widetilde{\mathcal{L}}_h^{\text{Cyl}} + \mathcal{O}(h^{1-2\eta})P_h + \mathcal{O}(h^{\frac{1}{2}-\eta})P_h^2 + 2ih^{\frac{1}{2}}\Phi'(hD_t) + \mathcal{O}(h)(hD_t) - h\Phi'^2 + o(h),$$

where  $P_h = hD_s - t + \mathfrak{f}_0$ .

*Proof.* From (5.61), we have

$$\widetilde{\mathcal{L}}_h^\Phi = a^{-2}P_{s,h}^2 + a^{-1}[P_{s,h}, a^{-1}]P_{s,h} + (P_{t,h}^\Phi)^2 + a^{-1}[P_{t,h}^\Phi, a]P_{t,h}^\Phi.$$

We recall that  $a(s, \check{t}) = 1 - \check{t}k(s)$ , with  $\check{t} = t\zeta(h^{-\frac{1}{2}+\eta}t)$ , and  $\eta \in (0, \frac{1}{2})$ .

We get these estimates

$$a^{-2} = 1 + \mathcal{O}(h^{\frac{1}{2}-\eta}), \quad a^{-1}[P_{s,h}, a^{-1}] = \mathcal{O}(h), \quad a^{-1}[P_{t,h}^\Phi, a] = \mathcal{O}(h^{\frac{3}{2}-\eta}).$$

Then, we can write

$$\widetilde{\mathcal{L}}_h^\Phi = P_{s,h}^2 + (P_{t,h}^\Phi)^2 + \mathcal{O}(h^{\frac{1}{2}-\eta})P_{s,h}^2 + \mathcal{O}(h^{\frac{3}{2}-\eta})P_{t,h}^\Phi + \mathcal{O}(h)P_{s,h}.$$

Thus,

$$\begin{aligned} \widetilde{\mathcal{L}}_h^\Phi &= P_{s,h}^2 + h^2 D_t^2 + 2ih^{\frac{1}{2}}\Phi'(hD_t) - h\Phi'^2 + h^{\frac{3}{2}}\Phi'' \\ &\quad + \mathcal{O}(h^{\frac{1}{2}-\eta})P_{s,h}^2 + \mathcal{O}(h^{\frac{3}{2}-\eta})P_{t,h}^\Phi + \mathcal{O}(h)P_{s,h}, \end{aligned}$$

and then

$$\begin{aligned} \widetilde{\mathcal{L}}_h^\Phi &= \widetilde{\mathcal{L}}_h^{\text{Cyl}} + \mathcal{O}(h^{1-2\eta})(hD_s - t + \mathfrak{f}_0) + \mathcal{O}(h^{2-4\eta}) + 2ih^{\frac{1}{2}}\Phi'(hD_t) - h\Phi'^2 + h^{\frac{3}{2}}\Phi'' \\ &\quad + \mathcal{O}(h^{\frac{1}{2}-\eta})P_{s,h}^2 + \mathcal{O}(h^{\frac{3}{2}-\eta})P_{t,h}^\Phi + \mathcal{O}(h)P_{s,h}. \end{aligned}$$

We can choose  $\eta$  small enough to obtain the result.  $\square$

**Proposition 5.6.10.** *We have*

$$\left\| (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda + \mathcal{R}_h)(\chi_h \varphi) \right\| \leq Ch \|\tilde{\chi}_h \varphi\| + C\alpha^2 h \|\chi_h \varphi\| + o(h) \|\chi_h \varphi\|,$$

where  $\mathcal{R}_h$  is in the form

$$\mathcal{R}_h = \mathcal{O}(h^{1-2\eta})P_h + \mathcal{O}(h^{\frac{1}{2}-\eta})P_h^2 + 2ih^{\frac{1}{2}}\Phi'(hD_t) + \mathcal{O}(h)(hD_t).$$

*Proof.* From Lemmas 5.6.7 and 5.6.8, we get

$$\|(\widetilde{\mathcal{L}}_h^\Phi - \lambda)(\chi_h \varphi)\| \leq Ch \|\tilde{\chi}_h \varphi\|,$$

and with Lemma 5.6.9, we deduce the estimate.  $\square$

### 5.6.3.3 End of the proof of Agmon estimate

From Proposition 5.6.10, we see that we have to estimate the terms

$$P_h(\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1}, \quad P_h^2(\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1}, \quad hD_t(\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1}.$$

This is the aim of the following two lemmas.

**Lemma 5.6.11.** *We have*

$$\|P_h(\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1}\| + \|hD_t(\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1}\| \leq Ch^{-\frac{1}{2}}.$$

*Proof.* Consider  $v \in L^2([-L, L] \times \mathbb{R})$  and  $u = (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1}v$ . We write

$$(\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)u = v.$$

Taking the inner product with  $u$ , we get

$$\begin{aligned} (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)u &= v \\ \Rightarrow \langle (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)u, u \rangle &= \langle u, v \rangle \\ \Rightarrow \int_{\mathbb{R}} \int_{\mathbb{S}^1} |P_h u|^2 ds dt + \int_{\mathbb{R}} \int_{\mathbb{S}^1} |hD_t u|^2 ds dt &= \langle u, v \rangle + \lambda \|u\|^2. \end{aligned}$$

Using that  $\lambda \leq Ch$ ; from Young inequality, we obtain

$$\|P_h u\|^2 + \|hD_t u\|^2 \leq Ch \|u\|^2 + \|u\| \|v\|.$$

From (5.64) we see that

$$\|u\| = \left\| (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1}v \right\| \leq Ch^{-1} \|v\|,$$

then the conclusion follows.  $\square$

**Lemma 5.6.12.** *We have*

$$\|P_h^2(\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1}\| \leq C.$$

*Proof.* Consider  $v \in L^2([-L, L] \times \mathbb{R})$  and  $u = (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1}v$ . We have

$$(\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)u = v,$$

so that, by taking the scalar product with  $P_h^2 u$  we get

$$\begin{aligned} \langle P_h^2 u, (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)u \rangle &= \langle P_h^2 u, v \rangle \\ \Rightarrow \int_{\mathbb{R}} \int_{\mathbb{S}^1} |P_h^2 u|^2 ds dt - \lambda \|P_h u\|^2 - h^2 \langle (P_h)^2 u, \partial_t^2 u \rangle &= \langle P_h^2 u, v \rangle \\ \Rightarrow \|P_h^2 u\|^2 - \lambda \|P_h u\|^2 + h^2 \|P_h \partial_t u\|^2 - 2h^2 \langle \partial_t u, P_h u \rangle &= \langle P_h^2 u, v \rangle. \end{aligned}$$

Then, we obtain

$$\|P_h^2 u\|^2 \leq \lambda \|P_h u\|^2 + 2\langle h^{3/2} \partial_t u, h^{1/2} P_h u \rangle + \langle P_h^2 u, v \rangle.$$

From Young inequality, we obtain

$$\|P_h^2 u\|^2 \leq \lambda \|P_h u\|^2 + h \|h D_t u\|^2 + h \|P_h u\|^2 + \frac{1}{2} \|P_h^2 u\|^2 + \frac{1}{2} \|v\|^2.$$

The conclusion follows by using the proof of the previous lemma.  $\square$

We can now give the proof of Proposition 5.6.5. We see that

$$\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda + \mathcal{R}_h = \left( \text{Id} + \mathcal{R}_h (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1} \right) (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda).$$

Thanks to Lemmas 5.6.11 and 5.6.12, we deduce that  $\text{Id} + \mathcal{R}_h (\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda)^{-1}$  is bijective and thus so is  $\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda + \mathcal{R}_h$  as soon as  $\alpha$  and  $h$  are small enough and that

$$\|(\widetilde{\mathcal{L}}_h^{\text{Cyl}} - \lambda + \mathcal{R}_h)^{-1}\| \leq Ch^{-1}.$$

With Proposition 5.6.10, this implies that

$$\|\chi_h \varphi\| \leq C \|\tilde{\chi}_h \varphi\| + C\alpha^2 \|\chi_h \varphi\| + o(1) \|\chi_h \varphi\|.$$

Let us underline that the constant  $C$  does not depend on  $m$ . By proceeding as in the proof of Proposition 5.2.1, we deduce that

$$\int_{[-L, L] \times \mathbb{R}_+} e^{2\alpha t \chi_m(t)/h^{1/2}} |\psi|^2 ds dt \leq C \|\psi\|^2. \quad (5.65)$$

By using the Fatou lemma ( $m \rightarrow +\infty$ ), we get (5.58). The estimate (5.59) follows from (5.65) and the same considerations as at the end of the proof of Proposition 5.2.1. By using the same considerations as in Proposition 5.6.4, we get the following.

**Proposition 5.6.13.** *There exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$  the following holds. Let us consider an interval  $J_h \subset I_h$ . Then, there exists an interval  $\check{J}_h$  such that  $\check{J}_h \subset J_h \subset I_h$  with  $d_{\mathbb{H}}(J_h, \check{J}_h) = \mathcal{O}(h^\infty)$  and*

$$\text{rank } \mathbb{1}_{\check{J}_h}(\widetilde{\mathcal{L}}_h) \leq \text{rank } \mathbb{1}_{J_h}(\mathcal{L}_h). \quad (5.66)$$

Moreover, for all  $\lambda \in I_h \cap \text{Sp}(\widetilde{\mathcal{L}}_h)$ ,

$$\text{dist}(\lambda, \text{Sp}(\mathcal{L}_h)) = \mathcal{O}(h^\infty). \quad (5.67)$$

In particular, we have established that the spectra of  $\mathcal{L}_h$  and  $\widetilde{\mathcal{L}}_h$  coincide in  $I_h$  modulo  $\mathcal{O}(h^\infty)$ .

#### 5.6.4 Agmon estimate with Fourier multiplier

We will prove an Agmon estimates compatible with the Fourier multiplier  $\text{Op}_h^{\text{W}}(\Xi(\sigma + \hbar^{-1} f_0))$ . In order to lighten the notation, we will use a slight abuse of notation by writing

$$\Xi^{\text{W}} := \text{Op}_h^{\text{W}}(\Xi(\sigma + \hbar^{-1} f_0)). \quad (5.68)$$

The proof of the following proposition is an adaptation of that of Proposition 5.6.5. Although we did not use the following proposition in our presentation, it should be noted that it is a new result on exponential decay compatible with the Fourier multiplier.

**Proposition 5.6.14.** *There exist  $\alpha > 0, C > 0, \hbar_0 > 0$  such that for all  $\hbar \in (0, \hbar_0)$  and all eigenfunctions  $\psi$  of  $\widehat{\mathcal{L}}_\hbar$  associated with an eigenvalue in  $[a, b]$ , we have*

$$\int_{[-L, L] \times \mathbb{R}_+} e^{2\alpha\tau} |\Xi^W \psi|^2 ds d\tau \leq C(\|\Xi^W \psi\|^2 + \hbar^2 \|\Xi^W \psi\|^2 + \hbar^2 \|\Xi^W \partial_\tau \psi\|^2) + \mathcal{O}(\hbar^\infty) \|\psi\|^2, \quad (5.69)$$

and

$$\begin{aligned} \int_{[-L, L] \times \mathbb{R}_+} e^{2\alpha\tau} \left[ |(-i\hbar \partial_s - \tau + \hbar^{-1} f_0) \Xi^W \psi|^2 + |\partial_\tau \Xi^W \psi|^2 \right] ds d\tau \\ \leq C(\|\Xi^W \psi\|^2 + \hbar^2 \|\Xi^W \psi\|^2 + \hbar^2 \|\Xi^W \partial_\tau \psi\|^2) + \mathcal{O}(\hbar^\infty) \|\psi\|^2, \end{aligned} \quad (5.70)$$

where  $\Xi$  has the same properties as  $\Xi$  and is 1 on support of  $\Xi$ .

*Proof.* Let us consider  $\lambda \in [a, b] \cap \text{Sp}(\widehat{\mathcal{L}}_\hbar)$  and an associated eigenfunction  $\psi$ . We have

$$\left( \widehat{\mathcal{L}}_\hbar - \lambda \right) \psi = 0. \quad (5.71)$$

We let  $\varphi = e^\Phi \psi$  where  $\Phi = \alpha\tau$  and  $\alpha$  to be determined. Note that we know that the integrals in the left-hand-sides are finite (due to the Agmon estimates). The point is to deal with the microlocal support of the functions in the right-hand-side.

Similarly to (5.61), we can write  $(\widehat{\mathcal{L}}_\hbar^\Phi - \lambda)\varphi = 0$  with

$$\widehat{\mathcal{L}}_\hbar^\Phi = a(s, \hat{\tau})^{-1} p_{s, \hbar} a(s, \hat{\tau})^{-1} p_{s, \hbar} + a(s, \hat{\tau})^{-1} p_\tau^\Phi a(s, \hat{\tau}) p_\tau^\Phi, \quad (5.72)$$

where  $p_{s, \hbar}$  is given in (5.27) and

$$p_\tau^\Phi = D_\tau + i\Phi'.$$

Let us now consider a function  $t \mapsto \chi(t)$  equal to 0 near  $t = 0$  on a neighborhood of size 1.

Similarly to Lemma 5.6.6, we see that

$$(\widehat{\mathcal{L}}_\hbar^\Phi - \lambda)(\chi\varphi) = -2i\chi' p_\tau^\Phi \varphi - \chi'' \varphi - \chi'(a(s, \hat{\tau})^{-1} \partial_\tau [a(s, \hat{\tau})]) \varphi.$$

Then, we apply  $\Xi^W$  and get

$$\begin{aligned} (\widehat{\mathcal{L}}_\hbar^\Phi - \lambda)(\chi \Xi^W \varphi) = -2i\chi' p_\tau^\Phi \Xi^W \varphi - \chi'' \Xi^W \varphi - \Xi^W \left( \chi'(a(s, \hat{\tau})^{-1} \partial_\tau [a(s, \hat{\tau})]) \varphi \right) \\ + [\widehat{\mathcal{L}}_\hbar^\Phi, \Xi^W](\chi\varphi). \end{aligned}$$

Since  $\Phi'$  is bounded, this implies that

$$\begin{aligned} \|(\widehat{\mathcal{L}}_\hbar^\Phi - \lambda)(\chi \Xi^W \varphi)\| \leq C\|\chi' \Xi^W \varphi\| + \|\chi' \partial_\tau \Xi^W \varphi\| + \|\chi'' \Xi^W \varphi\| \\ + \|\Xi^W \left( \chi'(a(s, \hat{\tau})^{-1} \partial_\tau [a(s, \hat{\tau})]) \varphi \right)\| + \|[\widehat{\mathcal{L}}_\hbar^\Phi, \Xi^W](\chi\varphi)\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|(\widehat{\mathcal{L}}_h^\Phi - \lambda)(\chi \Xi^W \varphi)\| &\leq C \|\tilde{\chi} \Xi^W \varphi\| + \|\chi' \partial_\tau \Xi^W \varphi\| + \|\chi' (\Xi^W (a(s, \hat{\tau})^{-1} \partial_\tau [a(s, \hat{\tau})]) \varphi)\| \\ &\quad + \|[\widehat{\mathcal{L}}_h^\Phi, \Xi^W](\chi \varphi)\|. \end{aligned}$$

By commuting  $\Xi^W$  and  $a(s, \hat{\tau})^{-1} \partial_\tau [a(s, \hat{\tau})]$ , we get a pseudo-differential operator (say with respect to  $s$ ) of order  $\hbar$ , uniformly in  $\tau$ , and thus

$$\|(\widehat{\mathcal{L}}_h^\Phi - \lambda)(\chi \Xi^W \varphi)\| \leq C \|\tilde{\chi} \Xi^W \varphi\| + \|\chi' \partial_\tau \Xi^W \varphi\| + \|[\widehat{\mathcal{L}}_h^\Phi, \Xi^W](\chi \varphi)\|. \quad (5.73)$$

Let us estimate the right-hand-side of (5.73). This is the aim of the following two lemmas.

**Lemma 5.6.15.** *We have*

$$\|[\widehat{\mathcal{L}}_h^\Phi, \Xi^W](\chi \varphi)\| \leq C \hbar \|\Xi^W(\chi \varphi)\| + C \hbar \|\Xi^W \partial_\tau(\chi \varphi)\| + \mathcal{O}(\hbar^\infty) \|\varphi\|.$$

*Proof.* The proof is the same as (5.34) in the Proposition 5.3.1.  $\square$

**Lemma 5.6.16.** *We have*

$$\|\Xi^W \chi' \partial_\tau \varphi\| \leq C \|\tilde{\chi} \Xi^W \varphi\| + C \hbar \|\tilde{\chi} \Xi^W \varphi\| + C \hbar \|\tilde{\chi} \Xi^W \partial_\tau \varphi\| + \mathcal{O}(\hbar^\infty) \|\varphi\|.$$

*Proof.* The proof follows the same lines as for Lemma 5.6.8. As in the proof of Lemma 5.6.8, we can write

$$\begin{aligned} &\operatorname{Re} \langle (\widehat{\mathcal{L}}_h^\Phi - \lambda) \Xi^W \varphi, \chi'^2 \Xi^W \varphi \rangle_{L^2(\hat{a}_h \operatorname{dsdt})} \\ &= \int_{\operatorname{Cyl}} \hat{a}_h^{-1} |p_{s, \hbar}(\chi'_h(\Xi^W \varphi))|^2 \operatorname{dsd}\tau + \operatorname{Re} \langle p_\tau^\Phi \hat{a}_h p_\tau^\Phi(\Xi^W \varphi), (\chi')^2(\Xi^W \varphi) \rangle_{L^2(\operatorname{dsd}\tau)} - \lambda \|\chi' \Xi^W \varphi\|^2, \end{aligned} \quad (5.74)$$

and similar commutator estimates provide us with

$$\operatorname{Re} \langle p_\tau^\Phi \hat{a}_h p_\tau^\Phi(\Xi^W \varphi), (\chi')^2(\Xi^W \varphi) \rangle_{L^2(\operatorname{dsd}\tau)} \geq c \|D_\tau(\chi' \Xi^W \varphi)\|^2 - C \|\tilde{\chi} \Xi^W \varphi\|^2. \quad (5.75)$$

Let us now look at the left-hand-side of (5.74). Thanks to a commutator and by using the eigenvalue equation, we find that

$$\left| \operatorname{Re} \langle (\widehat{\mathcal{L}}_h^\Phi - \lambda) \Xi^W \varphi, \chi'^2 \Xi^W \varphi \rangle_{L^2(\hat{a}_h \operatorname{dsdt})} \right| \leq C \|\chi' [\widehat{\mathcal{L}}_h^\Phi, \Xi^W] \varphi\| \|\chi' \Xi^W \varphi\|.$$

Then, similarly as for Lemma 5.6.15, we have

$$\|\chi' [\widehat{\mathcal{L}}_h^\Phi, \Xi^W] \varphi\| \leq C \hbar \|\tilde{\chi} \Xi^W \varphi\| + C \hbar \|\tilde{\chi} \Xi^W \partial_\tau \varphi\| + \mathcal{O}(\hbar^\infty) \|\varphi\|.$$

With the Young inequality, (5.74) and (5.75) give that

$$\|\partial_\tau(\chi' \Xi^W \varphi)\|^2 \leq C \|\tilde{\chi} \Xi^W \varphi\|^2 + C \hbar^2 \|\tilde{\chi} \Xi^W \varphi\|^2 + C \hbar^2 \|\tilde{\chi} \Xi^W \partial_\tau \varphi\|^2 + \mathcal{O}(\hbar^\infty) \|\varphi\|^2,$$

which implies the desired estimate.  $\square$

Let us now come back to (5.73). Lemmas 5.6.15 and 5.6.16 imply that

$$\begin{aligned} \|(\widehat{\mathcal{L}}_h^\Phi - \lambda)(\chi \Xi^W \varphi)\| &\leq C \|\tilde{\chi} \Xi^W \varphi\| + C \hbar \|\chi \Xi^W \varphi\| + C \hbar \|\tilde{\chi} \Xi^W \varphi\| + C \hbar \|\chi \Xi^W \partial_\tau \varphi\| \\ &\quad + C \hbar \|\tilde{\chi} \Xi^W \partial_\tau \varphi\| + \mathcal{O}(\hbar^\infty) \|\varphi\|. \end{aligned} \quad (5.76)$$

Similarly as in Lemma 5.6.9 and Proposition 5.6.10 (in the original coordinates), our rescaled operator  $\widehat{\mathcal{L}}_h^\Phi - \lambda$  can be written as a perturbation of a bijective operator, whose inverse is bounded uniformly in  $\hbar$ . This shows that

$$\|\chi \Xi^W \varphi\| \leq C \|\tilde{\chi} \Xi^W \varphi\| + C\hbar \|\chi \Xi^W \varphi\| + C\hbar \|\tilde{\chi} \Xi^W \varphi\| + C\hbar \|\chi \Xi^W \partial_\tau \varphi\| + C\hbar \|\tilde{\chi} \Xi^W \partial_\tau \varphi\| + \mathcal{O}(\hbar^\infty) \|\varphi\|.$$

In fact, considering again (5.76) and  $\operatorname{Re}\langle (\widehat{\mathcal{L}}_h^\Phi - \lambda)(\chi \Xi^W \varphi), \chi \Xi^W \varphi \rangle$ , we infer that

$$\|\chi \Xi^W \varphi\| + \|\chi \Xi^W \partial_\tau \varphi\| \leq C \|\tilde{\chi} \Xi^W \varphi\| + C\hbar \|\tilde{\chi} \Xi^W \varphi\| + C\hbar \|\tilde{\chi} \Xi^W \partial_\tau \varphi\| + \mathcal{O}(\hbar^\infty) \|\varphi\|,$$

and then

$$\|\Xi^W \varphi\| + \|\Xi^W \partial_\tau \varphi\| \leq C \|\tilde{\chi} \Xi^W \varphi\| + C\hbar \|\tilde{\chi} \Xi^W \varphi\| + C\hbar \|\tilde{\chi} \Xi^W \partial_\tau \varphi\| + \mathcal{O}(\hbar^\infty) \|\varphi\|.$$

Recalling that  $\varphi = e^\Phi \psi$  (with  $\Phi'$  small enough) and also that  $\psi$  already satisfies Agmon type estimates, we get

$$\|e^\Phi \Xi^W \psi\| + \|e^\Phi \Xi^W \partial_\tau \psi\| \leq C \|\Xi^W \psi\| + C\hbar \|\Xi^W \psi\| + C\hbar \|\Xi^W \partial_\tau \psi\| + \mathcal{O}(\hbar^\infty) \|\psi\|.$$

□

# Appendix

## A.1 Spectral theory

### A.1.1 Spectrum of self-adjoint operators

In the following  $H$  denotes a Hilbert space over  $\mathbb{C}$  and  $T$  is a self-adjoint operator of domain  $\text{Dom}(T)$ .

**Definition A.1.1.** *The spectrum of the operator  $T$  is the set noted  $\text{Sp}(T)$  and defined by :*

$$\text{Sp}(T) = \mathbb{C} \setminus \rho(T),$$

where  $\rho(T) = \{z \in \mathbb{C}, (T - z) \text{ is bijective}\}$  denotes the resolvent set.

Recall that  $\lambda$  is an eigenvalue if there exists a nonzero vector  $u \in \text{Dom}(T)$  such that  $Tu = \lambda u$ . The geometric multiplicity of  $\lambda$  is the dimension of  $\text{Ker}(T - \lambda)$ . Recall the following characterization of the spectrum:

**Definition A.1.2.** *The discrete spectrum of the operator  $T$ , denoted  $\text{Sp}_{disc}(T)$  is the set of its isolated eigenvalues of finite multiplicity. The essential spectrum of  $T$ , denoted  $\text{Sp}_{ess}(T)$ , is given by  $\text{Sp}_{ess}(T) = \text{Sp}(T) \setminus \text{Sp}_{disc}(T)$ .*

### A.1.2 Diamagnetic inequality

**Theorem A.1.3** (Diamagnetic inequality). *It is assumed that  $f \in L^1_{loc}(\Omega)$  such that  $(\nabla - ibA)f \in L^1_{loc}(\Omega)$ . Then  $|f| \in H^1_{loc}(\Omega)$  and*

$$\nabla|f| \leq |(\nabla - ibA)f|,$$

almost everywhere, with

$$\nabla|f|(x) = \text{Re}\{\text{sign}(f(x))\nabla f(x)\},$$

for almost everywhere  $x \in \Omega$ . In particular,

$$\nabla|f| \leq |\nabla f|.$$

The proof of this theorem is in the book of Fournais-Helffer [FH10b].



### A.1.3 Spectral theorem

**Theorem A.1.4** (Spectral theorem). *Let  $T$  be a self-adjoint operator. For any element  $z$  of the resolvent set, we have:*

$$\|(T - z)^{-1}\| \leq \frac{1}{d(z, \text{Sp}(T))}.$$

This result gives the following corollary, useful in practice:

**Corollary A.1.5.** *Let  $T$  be a self-adjoint operator. For any  $\lambda \in \mathbb{C}$  and  $\psi \in \text{Dom}(T)$ , the distance between  $\lambda$  and the spectrum of  $T$  verifies the inequality :*

$$d(\lambda, \text{Sp}(T))\|\psi\| \leq \|(T - \lambda)\psi\|.$$

An elementary consequence which will be used quite often is that if we find a good normalized test function  $\psi$  (called quasimode) in  $\text{Dom}(T)$  and a complex number  $\lambda$  such that, for  $\epsilon > 0$ ,

$$\|(T - \lambda)\psi\| < \epsilon,$$

then

$$d(\lambda, \text{Sp}(T)) < \epsilon.$$

*Proof.* See for example in Cheverry-Raymond's book [CR21]. □

**Proposition A.1.6** (Generalized spectral theorem). *Let  $T$  be a self-adjoint operator and let  $\epsilon > 0$ . It is assumed that there exists  $\lambda \in \mathbb{C}$  such that for any  $\psi \in F \subset \text{Dom}(T)$  (where  $F$  is an  $N$ -dimensional vector subspace), the following inequality exists:*

$$\|(T - \lambda)\psi\| \leq \epsilon\|\psi\|.$$

Then, we have

$$\dim \text{Im}(\mathbb{1}_{[\lambda - \epsilon, \lambda + \epsilon]}) \geq N.$$

*Proof.* See for example the Ph.D thesis of Miqueu [Miq16, Prop. A.8]. □

### A.1.4 Min-Max principle

**Theorem A.1.7.** *Let  $T$  be a self-adjoint operator with domain  $\text{Dom}(T)$ . We assume that  $T$  is semi-bounded from below. Let  $Q$  the quadratic form associated to  $T$  with domain  $\text{Dom}(Q)$ . The Rayleigh quotients which are associated with  $T$  are defined for all positive natural numbers  $n \in \mathbb{N}^*$  by*

$$\mu_n = \sup_{u_1, \dots, u_{n-1} \in \text{Dom}(Q)} \left( \inf_{u \in [u_1, \dots, u_{n-1}]^\perp \setminus \{0\}} \frac{Q(u)}{\|u\|^2} \right) = \inf_{u_1, \dots, u_n \in \text{Dom}(Q)} \left( \sup_{u \in [u_1, \dots, u_n] \setminus \{0\}} \frac{Q(u)}{\|u\|^2} \right),$$

where  $[u_1, \dots, u_n]$  is the vector subspace generated by  $u_1, \dots, u_n$ . Then for any integer  $n \in \mathbb{N}^*$ , we have the following alternative:

- a. The operator  $T$  has at least  $n$  eigenvalues (counted with multiplicity) under its essential spectrum, and  $\mu_n$  is the  $n$ -th eigenvalue (counted with multiplicity).
- b. The quantity  $\mu_n$  corresponds to the bottom of the essential spectrum of the operator  $T$ . In this case  $\mu_n = \mu_{n+1} = \dots$  and the operator  $T$  has less than  $n$  eigenvalues (counted with multiplicity) under  $\mu_n$ .

## A.2 Pseudo-differential operators with operator valued symbols

We denote a point  $(x, \eta) \in \mathbb{R}^2$  by  $X$ . We shall consider a family of Hilbert spaces  $\mathcal{A}_X$ ,  $X \in \mathbb{R}^2$  satisfying:

$$\text{For all } X \in \mathbb{R}^2, \mathcal{A}_X = \mathcal{A}_0 (= \mathcal{A}) \text{ as vector space.} \quad (\text{A.1})$$

There exist  $N \geq 0$  and  $C \geq 0$  such that  $\|a\|_{\mathcal{A}_X} \leq C \langle X - Y \rangle^N \|a\|_{\mathcal{A}_Y}$  for all  $a \in \mathcal{A}$  and  $X, Y \in \mathbb{R}^2$ . (A.2)

Let  $\mathcal{B}_X$ , be a second family with the same properties.

**Definition A.2.1.**  $S(\mathbb{R}^2; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$  is the set of functions  $p \in \mathcal{C}^\infty(\mathbb{R}^2; \mathcal{L}(\mathcal{A}, \mathcal{B}))$  such that, for all  $\alpha \in \mathbb{N}^2$ , there exists  $C_\alpha$  such that

$$\|\partial^\alpha p(X)\|_{\mathcal{L}(\mathcal{A}_X, \mathcal{B}_X)} \leq C_\alpha, \quad X \in \mathbb{R}^2. \quad (\text{A.3})$$

If  $p$  depends on the additional parameter  $h \in ]0, h_0]$ ,  $h_0 > 0$ , the constant  $C_\alpha$  must be uniform in  $h$ .

This class can be thought as generalization of the standard class of scalar symbols

$$S_{\mathbb{R}^2}(1) = \{p \in \mathcal{C}^\infty(\mathbb{R}_{x,\eta}^2, \mathbb{C}) : \forall \alpha \in \mathbb{N}^2, \exists C_\alpha > 0 : |\partial^\alpha p(x, \eta)| \leq C_\alpha\}$$

to operator-valued symbols. More details can be found in [Ker18, Chapter 2].

**Remark A.2.1.** If the family of norms on  $\mathcal{A}$  and  $\mathcal{B}$  are constants, we denotes  $S(\mathbb{R}^2; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$  by  $S(\mathbb{R}^2; \mathcal{L}(\mathcal{A}, \mathcal{B}))$ .

**Definition A.2.2.** The Weyl quantization of a symbol  $p \in S(\mathbb{R}^2; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$  is the operator, acting on  $\mathcal{S}(\mathbb{R}; \mathcal{A})$ , given by the formula

$$\text{Op}_h^W(p) \psi(x) := \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(x-y)\eta/\hbar} p\left(\frac{x+y}{2}, \eta\right) \psi(y) dy d\eta.$$

We also define the standard quantization by the formula

$$\text{Op}_h(p) \psi(x) := \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{i(x-y)\eta/\hbar} p(x, \eta) \psi(y) dy d\eta.$$

For an introduction to pseudo-differential operators, the reader is referred for instance to [Ker18], where rigorous definitions are given and several fundamental properties are established.

**Notation A.2.3.** We sometimes note

$$\text{Op}_h^W(p) = p^W(x, \hbar D),$$

and

$$\text{Op}_h(p) = p(x, \hbar D).$$

**Remark A.2.2.** We recall that the operator and its Weyl symbol are related by the following exact formula (see for instance [Ker18, Theorems 2.1.14 & 2.1.15])

$$p(x, \eta) = e^{-i\frac{\hbar}{2}D_x D_\eta} \left[ e^{-ix\cdot\eta/\hbar} \text{Op}_h(p) \left( e^{i\cdot\eta/\hbar} \right) \right] (x, \eta),$$

where  $e^{-i\frac{\hbar}{2}D_x D_\eta}$  is defined as a Fourier multiplier thanks to the Fourier transform with respect to  $(x, \eta)$ .

**Theorem A.2.4** (Composition). *Let  $p \in S(\mathbb{R}^2; \mathcal{L}(\mathcal{B}_X, \mathcal{C}_X))$ , and  $q \in S(\mathbb{R}^2; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$ . Then*

$$\text{Op}_h^W(p) \circ \text{Op}_h^W(q) = \text{Op}_h^W(r),$$

where  $r \in S(\mathbb{R}^2; \mathcal{L}(\mathcal{A}_X, \mathcal{C}_X))$ , is given by the formula

$$r(X) = p \sharp q(X) := e^{i\frac{\hbar}{2}\sigma(D_X: D_Y)}(p(X)q(Y))|_{Y=X},$$

with  $\sigma$  denotes the usual symplectic form on  $\mathbb{R}^2$  defined by

$$\sigma(X : Y) := \xi y - x \eta,$$

for  $X = (x, \xi), Y = (y, \eta) \in \mathbb{R}^2$ .

Moreover, for all  $n \in \mathbb{N}^*$ , we have

$$r = \sum_{k=0}^N \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \sigma(D_X : D_Y)^k (p(X)q(Y))|_{Y=X} + h^{N+1} r_{N+1},$$

where  $r_{N+1} \in S(\mathbb{R}^2; \mathcal{L}(\mathcal{A}_X, \mathcal{C}_X))$ . In particular,

$$r = p \circ q + \frac{\hbar}{2i} \{p, q\} + \frac{\hbar^2}{8} (\{\partial_x p, \partial_\xi q\} - \{\partial_\xi p, \partial_x q\}) + h^3 r_3.$$

The  $L^2$ -boundedness can be established exactly as in the scalar case:

**Theorem A.2.5** (Calderón-Vaillancourt). *Let  $p \in S(\mathbb{R}^2; \mathcal{L}(\mathcal{A}, \mathcal{B}))$ . Then*

$$\text{Op}_h^W(p) : L^2(\mathbb{R}; \mathcal{A}) \rightarrow L^2(\mathbb{R}; \mathcal{B})$$

is uniformly bounded for  $\hbar \in (0, \hbar_0)$ .

More precisely, there exists  $C, M > 0$  such that

$$\|\text{Op}_h^W(p)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C \sum_{|\alpha| \leq M} h^{|\alpha|/2} \sup_{X \in \mathbb{R}^2} \|\partial^\alpha p(X)\|_{\mathcal{L}(\mathcal{A}, \mathcal{B})}.$$

**Remark A.2.3.** We can generalize the results of this section to symbols in more general spaces, as in [Ker18, Chapter 2].

# Perspectives

We discuss the different perspectives and questions raised by this thesis.

## I. Robin electro-magnetic Laplacian

Consider the semiclassical electromagnetic Schrödinger operator on a smooth, bounded, and simply connected open Euclidean domain  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$

$$\mathcal{L}_{h,V} = (-ih\nabla - \mathbf{A})^2 + hV,$$

with domain given by

$$\text{Dom}(\mathcal{L}_{h,V}) = \{\psi \in H^1(\Omega) : (-ih\nabla - \mathbf{A})^2\psi \in L^2(\Omega), -ih\mathbf{n} \cdot (-ih\nabla - \mathbf{A})\psi = \gamma h^{\frac{3}{2}}\psi \text{ on } \partial\Omega\},$$

where  $\mathbf{A}$  is a smooth real magnetic vector potential,  $V$  is a smooth *complex* scalar potential,  $\mathbf{n}$  is the outward pointing normal to the boundary, and  $\gamma$  is a smooth coefficient. In dimension 2, when  $V = 0$ , and with  $\gamma \in \mathbf{R} \cup \{+\infty\}$ , we established a uniform description of the spectrum of  $\mathcal{L}_{h,0}$  located between the Landau levels.

Many interesting questions that naturally arise in the theory of superconductivity could be the subject of future research.

- \* We can analyse the effect of an electric field and of the Robin condition on the behavior of the magnetic eigenvalues of  $\mathcal{L}_{h,V}$  and the associated eigenfunctions when the semiclassical parameter  $h$  tends to zero. Of course, some conditions must be imposed on the magnetic field  $\mathbf{B} = \text{curl}\mathbf{A}$  and the potential  $V$  to understand this question. We might find a pseudo-differential operator that would allow us to describe the spectrum. It is not obvious that our method [Fah23a, FLTRVuN23] can be directly adapted, but it opens up possibilities for such a study.
- \* In my research, we consider only the case of a smooth domain  $\Omega$ . The most physically relevant areas would instead contain singularities, such as corners. Using the strategy developed in [BP16] (in the non-magnetic Robin case), it would be quite interesting to analyze the behavior of the principal eigenvalue of  $\mathcal{L}_{h,V}$ , in domains with corners. One could also study the bottom of the essential spectrum on the associated tangent structures given by the cones. A natural question that does not seem obvious concerns the semi-continuity property of the eigenvalue.

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## II. Schrödinger operators with electric potentials

Let  $\Omega$  be a smooth open bounded and connected domain of  $\mathbb{R}^N$ ,  $N \geq 2$ . We consider the  $N$ -dimensional Stark Hamiltonian with Dirichlet boundary conditions in  $L^2(\Omega)$

$$\mathcal{L}_h := -h^2 \Delta + x_1.$$

In [Fah23b], thanks to decay estimates and the reduction to model operators, we established a full asymptotic expansion of the low-lying eigenvalues, and we also shown that their eigenfunctions are localized near the potential minimum.

It is possible to prove asymptotic expansions of the low-lying eigenvalues of the Hamiltonian  $\mathcal{L}_h$ , if we replace the potential  $x_1$  by  $V(\mathbf{x})$ , where  $V$  is a smooth real potential satisfies some additional conditions.

**Assumption 3.** *We assume the following :*

1.  $V$  has a unique minimum non degenerate  $\mathbf{x}_0 \in \partial\Omega$ .
2.  $\nabla V(\mathbf{x}_0) \cdot \nu(\mathbf{x}_0) = \frac{\partial V}{\partial \nu}(\mathbf{x}_0)$  (“normal derivative”)  $< 0$ .

Under these conditions, we can deduce :

**Theorem.** *For any positive  $n$ , there exists a real sequence  $\{\zeta_{j,n}\}_{j \geq 0}$  such that, for any positive  $M \in \mathbb{N}$ , we have*

$$\lambda_n(h) \underset{h \rightarrow 0}{=} V(\mathbf{x}_0) + z_1(\nabla V(\mathbf{x}_0) \cdot \nu(\mathbf{x}_0))^{\frac{2}{3}} h^{\frac{2}{3}} + \beta_n h + h^{7/6} \sum_{j=0}^M h^{j/6} \zeta_{j,n} + \mathcal{O}(h^{\frac{7}{6} + \frac{M+1}{6}}),$$

with  $\beta_n$  is the  $n$ -th eigenvalue of the harmonic oscillator in dimension  $N - 1$

$$\sum_{i=1}^{N-1} -\partial_{u_i}^2 + \frac{\Lambda_i}{2} u_i^2,$$

where  $\Lambda_i$  are the eigenvalues of the matrix  $G(0)^{-1}H$ ,  $G$  the induced metrics on  $\partial\Omega$ , and  $H = \text{Hess}_{s=0}V \circ \phi$  with  $\phi$  denoting the embedding of  $\partial\Omega$  in  $\mathbb{R}^N$  such that  $\phi(0) = \mathbf{x}_0$ .

Firstly, we can show that the low energy eigenfunctions will be localized around  $\mathbf{x}_0$ . We can quantify this thanks to classical Agmon estimates. We use this to find an effective Hamiltonian on a tiny domain around  $\mathbf{x}_0$ .

Secondly, we can introduce local boundary coordinates in the neighborhood of  $\mathbf{x}_0 \in \partial\Omega$  in order to straighten a portion of the boundary. Let  $\phi$  denote the embedding of  $\partial\Omega$  in  $\mathbb{R}^N$  and  $G$  the induced metrics on  $\partial\Omega$ . If  $\text{dist}(\mathbf{x}, \partial\Omega) \leq \delta$ , for  $\delta$  sufficiently small, we introduce the diffeomorphism  $\Phi$  defined by the formula

$$\mathbf{x} = \Phi(s, t) = \phi(s) - t\nu(\phi(s)),$$

where  $\nu(\mathbf{x})$  is the unit outward pointing normal vector at the point  $\mathbf{x} \in \partial\Omega$ ,  $t$  is the distance to the boundary and  $s = (s_1, \dots, s_{N-1})$  is the tangential variable on  $\partial\Omega$ . We assume that  $\mathbf{x}_0 = \Phi(0, 0)$ .

Thanks to the Agmon estimates in coordinates  $(s, t)$ , the operator  $\mathcal{L}_h$  is expressed formally in coordinates  $(s, t)$  as :

$$\mathcal{L}_h = V(\mathbf{x}_0) - h^2 \partial_t^2 - \nabla V(\mathbf{x}_0) \cdot \nu(\mathbf{x}_0) t - h^2 \Delta^{G(0)} + \frac{1}{2} \frac{d^2}{ds^2} V(\phi)|_{s=0}(s)(s) + \mathcal{O}(h^{\frac{7}{6}}),$$

---

where  $\Delta^{G(0)}$  is the Laplace-Beltrami operator on  $\partial\Omega$  in the neighborhood of  $\mathbf{x}_0$ .

We assume the following :

1.  $\nabla V(\mathbf{x}_0) \cdot \nu(\mathbf{x}_0) = \frac{\partial V}{\partial \nu}(\mathbf{x}_0) < 0$ .
2.  $\frac{d^2}{ds^2} V(\phi)|_{s=0} := H$  is positive definite matrix, *i.e.*  $\mathbf{x}_0$  is a non degenerate minimum of  $V$ .

Under these conditions, our analysis can be adapted to get

$$\lambda_n(h) \underset{h \rightarrow 0}{=} V(\mathbf{x}_0) + z_1(\nabla V(\mathbf{x}_0) \cdot \nu(\mathbf{x}_0))^{\frac{2}{3}} h^{\frac{2}{3}} + \beta_n h + \mathcal{O}(h^{\frac{7}{6}}),$$

where  $\beta_n$  is the  $n$ -th eigenvalue of the harmonic oscillator in dimension  $N - 1$

$$\sum_{i=1}^{N-1} -\partial_{u_i}^2 + \frac{\Lambda_i}{2} u_i^2,$$

where  $\Lambda_i$  are the eigenvalues of the matrix  $G(0)^{-1}H$ . The full asymptotic expansion can be computed, by constructing a sequence of trial states (quasimodes) as Section 4.4.

It would be interesting to study some related spectral questions.

- \* We could improve and generalize the results obtained in [Fah23b]. Moreover, under these conditions for  $V$  in the semiclassical limit, we could to apply other advanced methods of reduction (the so-called Grushin's method), microlocal analysis, and the use of pseudo differential calculus techniques used in [FLTRVuN23], and we could be able to find a uniform description of the spectrum of  $\mathcal{H}_h = -h^2\Delta + V(\mathbf{x})$  (not only the bottom of the spectrum) by establishing a Weyl estimate  $N(\mathcal{H}_h, h^{\frac{2}{3}}E)$ , which is the number of eigenvalues of  $\mathcal{H}_h$  below the energy level  $h^{\frac{2}{3}}E$  where  $E$  is close to the absolute value of the second zero of the Airy function.
- \* Our strategy can be adapted to deal with a finite number of non-degenerate minima, and it would even be possible to study the tunnel effects when these minima have symmetries. We could establish the asymptotic formula for the splitting of the first two eigenvalues (tunneling effect) and give an approximation of the eigenfunctions by the WKB expansions.
- \* Replacing the potential  $V$  by  $W$ , where  $W$  is a smooth complex electric potential, we have a non-selfadjoint operator  $-h^2\Delta + W$  (lack of a "spectral theorem"). Going from the self-adjoint case to the non-self-adjoint case is rarely anecdotal, and it requires doing really different things. Under some assumptions about  $W$ , by a uniform resolvent estimate, one could fully understand the behavior of the eigenvalues (not only the left margin of the spectrum as in [HA19]).



## Publication

Title : "*Magnetic perturbations of the Robin Laplacian in the strong coupling limit*".

Abstract : This paper is devoted to the asymptotic analysis of the eigenvalues of the Laplace operator with a strong magnetic field and Robin boundary condition on a smooth planar domain and with a negative boundary parameter. We study the singular limit when the Robin parameter tends to infinity which is equivalent to a semi-classical limit involving a small positive semi-classical parameter. The main result is a comparison between the spectrum of the Robin Laplacian with an effective operator defined on the boundary of the domain via the Born-Oppenheimer approximation. More precisely, the low-lying eigenvalue of the Robin Laplacian is approximated by the those of the effective operator. When the curvature has a unique non-degenerate maximum, we estimate the spectral gap and find that the magnetic field does not contribute to the three-term expansion of the eigenvalues. In the case of the disc domains, the eigenvalue asymptotics displays the contribution of the magnetic field explicitly.

Lien : <https://doi.org/10.1063/5.0101330>, published in the *Journal of Mathematical Physics*.

Title : "*On the semi-classical analysis of Schrödinger operators with linear electric potentials in a bounded domain*".

Abstract : The aim of this paper is to improve the asymptotic expansion of the eigenvalues of the Laplace operator, with strong uniform electric field and Dirichlet boundary conditions on a smooth bounded domain of  $\mathbb{R}^N$ . This work aims at generalizing the recent results of Cornean, Krejčířík, Pedersen, Raymond and Stockmeyer in dimension 2. More precisely, in dimension  $N$ , in the strong electric field limit, we derive, under certain local convexity conditions, a full asymptotic expansion of the low-lying eigenvalues. To establish our main result, we perform the construction of quasi-modes that will make it possible to obtain the upper bounds for the eigenvalues. The "optimality" of our constructions is then established thanks to reduction to model operators and localization estimates.

Lien : <https://doi.org/10.3233/ASY-231848>, in press, to appear in the journal *Asymptotic Analysis*.

Title : "*Edge states of the Robin magnetic Laplacian*".

Abstract : This article tackles the spectral analysis of the Robin Laplacian on a smooth bounded two-dimensional domain in the presence of a constant magnetic field. In the semiclassical limit, a uniform description of the spectrum located between the Landau levels is obtained. The corresponding eigenfunctions, called edge states, are exponentially localized near the boundary. By means of a microlocal dimensional reduction, our unifying approach allows to refine simultaneously old results about the low-lying eigenvalues in the Robin case and recent ones about edge states in the Dirichlet case.





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**Titre :** Influence des conditions de Robin sur le spectre du Laplacien magnétique.

**Mot clés :** théorie spectrale, analyse microlocale, limite semiclassique, Laplacien magnétique, condition de Robin, estimation d'Agmon.

**Résumé :** Cette thèse concerne l'étude spectrale dans le régime semi-classique de l'opérateur de Schrödinger en présence d'un champ magnétique sur un domaine borné et régulier avec condition de Robin au bord. L'objectif est de décrire l'influence des conditions de Robin sur le spectre magnétique lorsque le paramètre semi-classique tend vers zéro et lorsque le champ magnétique joue un rôle majeur en concurrence avec la condition de Robin. La description précise des éléments propres passe par la compréhension de différents opérateurs modèles.

Dans la première partie, lorsque le champ magnétique est d'intensité modérée et dans la limite du couplage fort, on montre que les valeurs propres du Laplacien de Robin sont bien approchées par celles d'un opérateur de Schrödinger périodique 1D sur la frontière. Lorsque la courbure a un unique maximum non dégénéré, on estime l'écart spectral entre les valeurs propres successives. Dans le cas du disque, on révèle explicitement la contribution du champ magnétique sur la

première valeur propre.

Dans la deuxième partie, on étudie l'effet de Stark en toutes les dimensions. On considère l'opérateur de Laplace avec un champ électrique fort et uniforme et des conditions de Dirichlet sur le bord d'un domaine borné et régulier. Dans la limite du champ électrique fort, on dérive, sous certaines conditions de convexité locale, une description précise des valeurs propres et des fonctions propres associées.

Dans la troisième partie, lorsque le champ magnétique est plus intense, et dans la limite semiclassique, en utilisant la méthode de Grushin et les outils d'analyse microlocale, on établit une description uniforme du spectre du Laplacien magnétique de Robin situé entre les niveaux de Landau, incluant les Laplaciens de Neumann et Dirichlet. De plus, on examine les propriétés de localisation des états fondamentaux. Les estimations ressemblent à celles d'Agmon, mais elles ne sont pas obtenues par des moyens variationnels comme c'est le cas dans de nombreux contextes magnétiques.

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**Title:** Influence of Robin's conditions on the spectrum of the magnetic Laplacian.

**Keywords:** spectral theory, microlocal analysis, semiclassical limit, magnetic Laplacian, Robin condition, Agmon estimates.

**Abstract:** This thesis concerns the spectral study in the semiclassical regime of the Schrödinger operator in the presence of a magnetic field on a bounded and regular domain with the Robin condition at the boundary. The aim is to describe the influence of the Robin conditions on the magnetic spectrum when the semiclassical parameter tends to zero and when the magnetic field plays an important role in competition with the Robin condition. The precise description of the eigenpairs requires an understanding of effective operators.

In the first part, when the magnetic field is of moderate intensity and in the strong coupling limit, we show that the eigenvalues of the Robin Laplacian are well approximated by those of a 1D periodic Schrödinger operator on the boundary. When the curvature has a unique maximum that is not degenerate, we estimate the spectral gap between successive eigenvalues. In the case of the disk, we explicitly display the contribution of the magnetic field to the first eigenvalue.

In the second part, we study the Stark effect in any dimension. We consider the Laplace operator with a strong and uniform electric field and Dirichlet conditions at the boundary of a bounded and regular domain. In the limit of the strong electric field, under certain local convexity conditions, we derive an asymptotic expansion of the low-lying eigenvalues at any order and also study the localization of the ground states.

In the third part, when the magnetic field is stronger, and in the semiclassical limit, we establish a uniform description of the spectrum between the Landau levels, of the magnetic Robin Laplacian including the Neumann and Dirichlet Laplacians, using the Grushin method and microlocal analysis tools. We also investigate the localization properties of the ground states. The estimates look like Agmon's estimates, but they are not obtained via variational methods, as is the case in many magnetic settings.