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Integral Motives and perverse Sheaves: t-structures and weak Artin vanishing property

Raphaël Ruimy

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THÈSE

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Raphaël RUIMY

Motifs entiers et faisceaux pervers d'Artin : t-structures et propriété de Lefschetz faible

Après l'avis de :

M. Joseph AYOUB	Professeur	Université de Zürich	Rapporteur
M. Jakob SCHOLBACH	Professeur associé	Université de Padoue	Rapporteur

Devant le jury composé de :

M. Jakob SCHOLBACH	Professeur associé	Université de Padoue	Rapporteur
M. Olivier BENOIST	CR CNRS	École Normale Supérieure	Examinateur
Mme Sophie MOREL	DR CNRS	ENS de Lyon	Examinatrice
Mme Wieslawa NIZIOL	DR CNRS	IMJ-PRG	Examinatrice
M. Fabrice ORGOGOZO	CR CNRS	IMJ-PRG	Examinateur
M. Frédéric DEGLISE	DR CNRS	ENS de Lyon	Directeur de thèse

Résumé

Dans cette thèse, on s'intéresse principalement à l'existence de t-structures sur les sous-catégories des objets d'Artin de catégories motiviques. Ces derniers sont les objets qui proviennent des motifs de schémas finis sur le schéma de base. On étudie d'abord le cas ℓ -adique où l'on dispose de la t-structure ordinaire et de la t-structure perverse que l'on cherche à restreindre aux objets d'Artin. Dans un deuxième temps, on s'intéresse au cas des motifs entiers ou rationnels où l'on ne dispose plus *a priori* de t-structures ; il faut alors les construire par des méthodes directes. Dans ces deux études, on se penche dans un premier temps sur les objets lisses que l'on relie à des systèmes locaux étales ou pro-étales, puis dans un deuxième temps sur la t-structure ordinaire qui existe toujours. Dans le cas ℓ -adique, on démontre ensuite que la t-structure perverse induit une t-structure sur les objets d'Artin lorsque le schéma de base est de dimension au plus 2 et pour certains schémas de dimension 3. Ce résultat tombe cependant en défaut pour les schémas de dimension 3 généraux. Dans le cas des coefficients \mathbb{Q}_ℓ et lorsque le schéma de base est de type fini sur un corps fini, on construit également une t-structure dite *homotopique perverse* qui est la meilleure approximation possible d'une t-structure perverse sur les complexes ℓ -adiques d'Artin. La même approche permet de construire une t-structure sur les motifs d'Artin à coefficients rationnels. Les cœurs de ces deux t-structures ont des propriétés similaires à celles de la catégorie des faisceaux pervers et contiennent le motif d'Ayoub-Zucker ou sa réalisation. Enfin, on construit une t-structure motivique perverse sur les motifs d'Artin à coefficients entiers lorsque le schéma de base est de dimension au plus 2 et on montre que ce résultat tombe en défaut en dimension 4. Cette construction repose en particulier sur un analogue faible du théorème de Lefschetz affine.

Abstract

In this thesis, we are mainly interested in the existence of t-structures on the subcategories of Artin objects of motivic categories. The latter are the objects that arise from motives of finite schemes on the base scheme. We first study the ℓ -adic setting, where we have the ordinary t-structure and the perverse t-structure, which we seek to restrict to Artin objects. Next, we look at the case of integral or rational motives, where t-structures are no longer available *a priori* and have to be constructed by direct methods. In both studies, we first focus on smooth objects, which we link to local systems, either étales or pro-étales, and then on the ordinary t-structure, the existence of which we prove in all cases. In the ℓ -adic case, we then show that the perverse t-structure induces a t-structure on Artin objects when the base scheme is of dimension at most 2 and for some base schemes of dimension 3. This result falls short, however, for general 3-dimensional schemes. In the case of \mathbb{Q}_ℓ coefficients and when the base scheme is of finite type over a finite field, we also construct a so-called *perverse homotopy* t-structure which is the best possible approximation to a perverse t-structure on Artin ℓ -adic complexes. The same approach is used to construct a t-structure on Artin motives with rational coefficients. The hearts of both t-structures have properties similar to those of the category of perverse sheaves and contain the Ayoub-Zucker motive or its realization. Finally, we construct a perverse motivic t-structure on Artin motives with integer coefficients when the base scheme is of dimension at most 2 and show that this result falls short in dimension 4. This construction relies notably on a weak analogue of Artin's vanishing theorem.

Table des matières

Introduction	9
A Des conjectures de Weil aux motifs	10
B Premières approximations des motifs : les catégories ℓ -adiques	14
C Catégories motiviques	20
D Présentation de la thèse	24
1 Motifs étales	35
1.1 Rappels catégoriques	35
1.1.1 Fondations	35
1.1.2 Présentabilité	41
1.1.3 Stabilité	44
1.1.4 Catégories monoidales	49
1.1.5 Hyperfaisceaux	55
1.2 La catégorie stable des motifs étales	57
1.2.1 Formalisme général des catégories motiviques	57
1.2.2 Catégories stables de motifs étales	64
1.2.3 La réalisation ℓ -adique et le formalisme de Bhatt-Scholze	69
2 Artin Perverse Sheaves	73
2.1 Preliminaries	78
2.1.1 Notions of abelian and stable subcategories generated by a set of objects	79
2.1.2 Induced t-structures	81
2.1.3 Finite permutation resolutions	83
2.1.4 Artin representations and representations of Artin origin	83
2.1.5 Categories of ℓ -adic sheaves and complexes	86
2.1.6 Reminders on cdh-descent and Mayer-Vietoris for closed immersions	89
2.1.7 Perverse sheaves	90
2.2 Artin ℓ -adic complexes	92
2.2.1 Artin ℓ -adic sheaves and complexes	92
2.2.2 Description of smooth Artin ℓ -adic complexes	93
2.2.3 The trace of the six functors on Artin ℓ -adic complexes	94
2.2.4 Artin ℓ -adic complexes as stratified complexes of representations	96
2.3 The perverse t-structure	97
2.3.1 The smooth Artin case	97
2.3.2 General properties	98
2.3.3 Construction of Artin perverse sheaves over schemes of dimension 2 or less	100

2.3.4	The case of 3-folds with real closed or separably closed closed points.	111
2.3.5	A counter-example	113
2.4	The perverse homotopy t-structure	114
2.4.1	Cohomological ℓ -adic complexes and the six functors for them	115
2.4.2	Definition for Ind-Artin ℓ -adic complexes	115
2.4.3	The smooth Artin case	117
2.4.4	Gluing formalism	118
2.4.5	The perverse homotopy and the perverse t-structures	119
2.4.6	Comparison of the Artin truncation functor with the weightless truncation functor for schemes of finite type over a finite field and with coefficients \mathbb{Q}_ℓ	120
2.4.7	The perverse homotopy t-structure for schemes of finite type over a finite field and with coefficients \mathbb{Q}_ℓ	123
2.4.8	Weightless perverse t-structure	125
2.4.9	Artin intermediate extension and simple Artin homotopy perverse sheaves	129
2.5	Explicit description of Artin perverse sheaves over schemes of dimension 1	132
2.5.1	Perverse sheaves as gluing of representations over schemes of dimension 1	132
2.5.2	Artin perverse sheaves over schemes of dimension 1	136
3	Abelian categories of Artin motives	139
3.1	Preliminaries	149
3.1.1	Categorical notions and lemmas	149
3.1.2	Reminders and complements on étale motives	154
3.1.3	Torsion étale motives and torsion étale sheaves	162
3.1.4	Reminders and complements on Artin motives	171
3.1.5	Artin representations and lisse étale sheaves	176
3.1.6	Reminders on the ℓ -adic theory	184
3.1.7	Some computations of étale cohomology and étale motivic cohomology	189
3.2	Description of smooth Artin motives and applications	191
3.2.1	Equivalence between smooth Artin motives and lisse étale sheaves over a regular basis	191
3.2.2	The conservativity conjecture for Artin motives	196
3.2.3	Constructible Artin motives as stratified representations	198
3.3	Computations around the Artin truncation functor	199
3.3.1	The Artin truncation over a field	199
3.3.2	Artin residues over 1-dimensional schemes	206
3.4	The ordinary homotopy t-structure on Artin motives	211
3.4.1	Definition for non-constructible Artin motives	211
3.4.2	The ordinary homotopy t-structure on constructible Artin motives and the t-exactness of the ℓ -adic realization	214
3.5	The perverse homotopy t-structure on Artin motives	219
3.5.1	Definition for non-constructible Artin motives	219
3.5.2	Locality and consequences	222
3.5.3	Induced t-structure on constructible motives with rational coefficients and weak Lefschetz properties	227

3.5.4	Induced t-structure on constructible Artin motives over a base scheme of dimension at most 2	238
3.6	The abelian category of perverse Artin étale motives	247
3.6.1	Definition and first properties	247
3.6.2	Intermediate extension, simple objects and the Ayoub-Zucker weightless intersection complex (rational coefficients)	248
3.6.3	Description of perverse Artin motives and t-exactness of the realization	250
3.6.4	Explicit description of perverse Artin motives on schemes of dimension 1	255
	Bibliography	264

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Introduction

On expose dans cette introduction l'émergence historique du programme des motifs. On s'y intéresse d'abord aux conjectures de Weil qui en sont la source principale. On explique comment ce problème a fait naître l'idée des *cohomologies de Weil* laquelle a conduit Grothendieck à esquisser le programme conjectural des *motifs*. On présente ensuite les premières approximations des motifs que sont les catégories de *faisceaux ℓ -adiques* et de *faisceaux pervers* ainsi que le *formalisme des six foncteurs de Grothendieck* qui relie entre elles les catégories de motifs. On évoque alors les *catégories triangulées de motifs mixtes* dont l'introduction par Voevodsky constitue une avancée majeure dans la réalisation du programme motivique de Grothendieck. On décrit enfin le problème de la *t -structure motivique*, dont la résolution compléterait la théorie, ainsi que ses variantes relatives pour les motifs d'Artin qui seront l'objet principal des chapitres suivants. On ouvre l'introduction sur la présentation des principaux résultats de la thèse.

Notations

Par convention, tous les anneaux sont supposés être commutatifs et unitaires.

- \mathbb{Z} désigne l'ensemble des nombres entiers.
- \mathbb{Q} désigne l'ensemble des nombres rationnels.
- \mathbb{C} désigne l'ensemble des nombres complexes.
- Si q est une puissance d'un nombre premier, on note \mathbb{F}_q le corps fini à q éléments.
- Si ℓ est un nombre premier, on note \mathbb{Q}_ℓ le corps des nombres ℓ -adiques.
- On note \mathbf{Ab} la catégorie des groupes abéliens.
- On note \mathbf{Set} la catégorie des ensembles.
- Si \mathbf{K} est un corps, on note $(\mathbf{K} - \mathbf{ev})^{\mathbb{Z}}$ la catégorie des \mathbf{K} -espaces vectoriels \mathbb{Z} -gradués.
- Si K est un corps de nombres, on note \mathcal{O}_K son anneau des entiers.
- Si R est un anneau (commutatif), si G est un groupe et si E est un G -ensemble fini, on note $R[E]$ la représentation R -linéaire de base E (on prolonge l'action de G sur E à $R[E]$ par linéarité).
- Si K est un corps, et si n est un entier positif, on note $\mathbb{P}^n(K)$ l'espace projectif de dimension n associé à K défini comme l'ensemble des droites vectorielles dans K^{n+1} .

A Des conjectures de Weil aux motifs

Comptage de solutions d'équations polynomiales

Un des buts principaux de la géométrie arithmétique est de comprendre le comportement des solutions de systèmes d'équations de la forme

$$(S) \begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_r(x_1, \dots, x_n) = 0 \end{cases}$$

où f_1, \dots, f_r sont des polynômes à coefficients entiers en n variables. Les solutions recherchées sont des n -uplets (x_1, \dots, x_n) de nombres entiers, ou de nombres rationnels, d'éléments d'un corps de nombres quelconque ou encore de son anneau des entiers.

Si A est un anneau, on notera $S(A)$ l'ensemble des solutions au système (S) dans A^n . Fixons un corps de nombres K . Le problème énoncé ci-dessus se spécialise en celui de l'étude des ensembles $S(K)$ et $S(\mathcal{O}_K)$. Ces derniers sont contenus dans l'ensemble $S(\mathbb{C})$, lequel est une sous-variété (non-nécessairement lisse) de \mathbb{C}^n que l'on peut donc étudier d'un point de vue analytique et dont on peut calculer des groupes d'homologie ou d'homotopie. Les groupes d'homologie de $S(\mathbb{C})$ contraignent très fortement les ensembles $S(K)$ et $S(\mathcal{O}_K)$: à titre d'exemple, Faltings a montré dans [Fal83] que si $S(\mathbb{C})$ est une surface lisse de genre $g > 2$, l'ensemble $S(K)$ est fini.

Dans le but d'étudier $S(\mathcal{O}_K)$, il est souvent plus aisé d'étudier l'ensemble des solutions du système (S) modulo un idéal premier de \mathcal{O}_K . Celles-ci sont alors des n -uplets d'éléments d'un corps fini k . L'ensemble k^n étant fini, l'étude de $S(k)$ est de nature combinatoire. Soit \bar{k} une clôture algébrique de k et soit

$$F: \bar{k} \rightarrow \bar{k}$$

l'endomorphisme de Frobenius. L'action de F sur \bar{k} induit une action sur $S(\bar{k})$ dont l'ensemble des points fixes est $S(k)$. Weil s'inspire du théorème suivant afin de formuler ses conjectures à la fin des années 1940 à la suite de son article [Wei49].

Théorème. ([Lef26]) *Soit X une variété complexe compacte de dimension d et $\sigma: X \rightarrow X$ une application biholomorphe dont l'ensemble $\text{Fix}(\sigma)$ des points fixes est fini. Pour chaque point fixe x , notons $m(x)$ la multiplicité de x comme point fixe de σ . Alors, on a*

$$\sum_{x \in \text{Fix}(\sigma)} m(x) = \sum_{i=0}^{2d} (-1)^i \text{tr} \left(\sigma_*: H^i(X, \mathbb{Q}) \rightarrow H^i(X, \mathbb{Q}) \right).$$

Pour espérer démontrer un résultat analogue dans le cas de $S(\bar{k})$, il semble d'abord nécessaire de rendre cette dernière compacte, c'est-à-dire de considérer les solutions du système précédent dans $\mathbb{P}^n(\bar{k})$. Plus précisément, notons (\bar{S}) le système formé par les équations homogènes en $n+1$ variables associées à (S) et pour tout corps L , notons $\bar{S}(L)$ le sous-ensemble de $\mathbb{P}^n(L)$ formé des solutions de (\bar{S}) dans $\mathbb{P}^n(L)$. Dans la suite de ce texte, on s'intéresse à la combinatoire de $\bar{S}(k)$.

Comme dans le théorème ci-dessus les variétés complexes considérées sont lisses, il semble également nécessaire de rajouter une condition supplémentaire sur le système S qui tienne compte de cette contrainte. La lissité d'une variété complexe définie par un système d'équations se vérifie au moyen d'un critère dit *jacobien* sur les dérivées partielles de ces mêmes équations. Il semble par conséquent raisonnable de se restreindre aux système \bar{S} qui vérifient le *critère jacobien de lissité* c'est-à-dire la condition analogue pour $\bar{S}(\bar{k})$.

Cohomologies de Weil

Une des idées clefs de Weil est d'attacher à chaque système \overline{S} vérifiant le critère jacobien de lissité des groupes de cohomologie dans le but de démontrer ensuite une formule de Lefschetz pour ces groupes de cohomologie. Pour cela, on associe à \overline{S} une variété algébrique projective lisse définie sur le corps k . Notons PSm_k la catégorie des variétés projectives lisses sur k . Dans le but de formuler un Théorème de Lefschetz sur les corps finis, Weil remarque que la cohomologie de Betti fournit un foncteur contravariant

$$H_B^* : \text{PSm}_{\mathbb{C}} \rightarrow (\mathbb{Q}\text{-ev})^{\mathbb{Z}}$$

ayant de remarquables propriétés analogues aux axiomes ci-dessous et conjecture l'existence d'un foncteur contravariant

$$H_W^* : \text{PSm}_k \rightarrow (\mathbf{K}\text{-ev})^{\mathbb{Z}}$$

défini pour un certain corps \mathbf{K} de caractéristique nulle et tel que les condition suivantes sont vérifiées.

1. Si X est une variété projective lisse de dimension d sur k ,
 - (a) pour tout entier i , le \mathbf{K} -espace vectoriel $H_W^i(X)$ est de dimension finie.
 - (b) pour tout entier $i \notin [0, 2d]$, on a $H_W^i(X) = 0$.
 - (c) le \mathbf{K} -espace vectoriel $H_W^{2d}(X)$ est de dimension 1.
 - (d) *Dualité de Poincaré* : on a un accouplement parfait

$$H_W^i(X) \otimes_{\mathbf{K}} H_W^{2d-i}(X) \rightarrow H_W^{2d}(X).$$

2. *Isomorphisme de Künneth* : le diagramme

$$\begin{array}{ccc} \text{PSm}_k \times \text{PSm}_k & \xrightarrow{H_W^* \times H_W^*} & (\mathbf{K}\text{-ev})^{\mathbb{Z}} \times (\mathbf{K}\text{-ev})^{\mathbb{Z}} \\ \downarrow \times & & \downarrow \otimes \\ \text{PSm}_k & \xrightarrow{H_W^*} & (\mathbf{K}\text{-ev})^{\mathbb{Z}} \end{array}$$

est commutatif à une équivalence naturelle près.

3. *Application classe de cycle* : si X est une variété projective lisse et si i est un entier, notons $Z^i(X)$ le groupe des cycles algébriques de codimension i , c'est-à-dire le groupe abélien libre engendré par les sous-variétés lisses fermées et irréductibles de X de codimension i ; on a un morphisme de groupes

$$\gamma : Z^i(X) \rightarrow H_W^{2i}(X)$$

appelé *application classe de cycle* qui vérifie des conditions de compatibilité avec le foncteur H^* et avec le produit induit par l'isomorphisme de Künneth et tel que si X est réduit à un point, le morphisme $Z^0(X) \rightarrow H_W^0(X)$ s'identifie au moyen de l'accouplement parfait ci-dessus à l'inclusion $\mathbb{Z} \subseteq \mathbf{K}$.

Les conjectures de Weil

Postulant implicitement l'existence d'une telle théorie cohomologique à coefficients rationnels, Weil conjecture le résultat suivant qui sera partiellement démontré par Dwork et Grothendieck, puis complété par Deligne en 1974.

Théorème. ([\[Del74a\]](#)) Soit k un corps fini, soit q son cardinal et soit X une variété projective lisse de dimension d sur k . Pour tout entier strictement positif N , notons a_N le nombre de points de X dans l'unique extension de corps de degré N de k . Soit

$$\zeta_X(t) = \exp \left(\sum_{N=1}^{+\infty} \frac{a_N}{N} t^N \right).$$

Alors,

1. Formule des traces : La fonction ζ_X est une fraction rationnelle de la forme

$$\frac{P_1 P_3 \cdots P_{2d-1}}{P_0 P_2 \cdots P_{2d}}$$

où chaque P_i est un polynôme à coefficients entiers et

$$P_0 = 1 - T \text{ et } P_{2d} = 1 - q^d T.$$

2. Dualité de Poincaré : Pour chaque entier $0 \leq i \leq 2d$, notant b_i le degré du polynôme P_i , il existe un entier c_i tel que

$$P_{2d-i} = c_i T^{b_i} P_i \left(\frac{1}{q^d T} \right)$$

De plus, on a

$$\zeta_X \left(\frac{1}{q^d t} \right) = \pm q^{\frac{\chi}{2}} t^\chi \zeta_X(t)$$

où $\chi = \sum_{i=0}^{2d} (-1)^i b_i$.

3. Hypothèse de Riemann : Pour chaque entier $0 \leq i \leq 2d$, les racines de P_i sont de module $q^{\frac{i}{2}}$.
4. Changement de base propre : S'il existe un corps de nombre K muni d'un morphisme surjectif $\mathcal{O}_K \rightarrow k$ et une variété \mathcal{X} définie sur \mathcal{O}_K telle que X est le tiré en arrière de \mathcal{X} à k , alors

$$b_i = \dim_{\mathbb{Q}}(H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q})).$$

Des cohomologies de Weil aux motifs purs

L'existence d'une cohomologie de Weil à coefficients dans le corps \mathbb{Q}_ℓ des nombres ℓ -adiques a été démontrée par Grothendieck, Verdier, Artin, Bucur, Houzel et Jouanolou en 1965 dans [\[G⁺77\]](#) pour tout nombre premier ℓ distinct de la caractéristique de k . En revanche, Serre a démontré qu'il n'existe pas de cohomologie de Weil à coefficients rationnels. Pour contourner ce problème et pour comparer entre elles les différentes cohomologies ℓ -adiques, Grothendieck

suggère de généraliser la notion de théorie cohomologique. Il conjecture alors l'existence d'une catégorie $M(k)$ des motifs purs sur k qui soit *tannakienne*, c'est-à-dire équivalente à la catégorie des représentations \mathbb{Q} -linéaires d'un certain groupe algébrique G_k^{mot} appelé *groupe de Galois motivique* de k , et d'un foncteur contravariant

$$h: \text{PSm}_k \rightarrow M(k)^{\mathbb{Z}}$$

tel que pour toute cohomologie de Weil H_W^* à coefficients dans un corps \mathbf{K} de caractéristique nulle, on ait un diagramme commutatif

$$\begin{array}{ccc} \text{PSm}_k & \xrightarrow{H_W^*} & (\mathbf{K} - ev)^{\mathbb{Z}} \\ h \downarrow & \nearrow & \\ M(k)^{\mathbb{Z}} & & \end{array} .$$

Au delà des conjectures de Weil qui sont désormais prouvées, l'existence d'une telle catégorie de motifs sur un corps arbitraire et plus particulièrement sur un corps de nombre ou même sur le corps \mathbb{C} aurait de nombreuses implications arithmétiques et géométriques. En effet, ce problème est étroitement lié à de subtiles questions autour des cycles algébriques. De plus, le groupe de Galois motivique d'un corps de nombre \square est un des ingrédients essentiels du *programme conjectural des périodes*. Il y joue le rôle d'un groupe de Galois pour les *périodes*, c'est-à-dire les nombres qui s'écrivent comme intégrales de formes différentielles algébriques sur des cycles algébriques. Une des conséquences de ce programme serait l'indépendance algébrique de la famille formée par π et les valeurs de la fonction ζ de Riemann aux entiers impairs.

Vers les motifs mixtes : la filtration par le poids

Grothendieck propose une définition pour la catégorie des motifs : la catégorie $M_{\text{num}}(k)$ des motifs numériques. On renvoie le lecteur à [And04, 1.1.3] pour une définition précise. Le défaut de cette catégorie est que l'on ne sait pas définir de foncteur de réalisation, c'est-à-dire de foncteur

$$M(k) \rightarrow (\mathbf{K} - ev)^{\mathbb{Z}}$$

qui factorise une cohomologie de Weil donnée et ce même dans le cas de la cohomologie singulière. Cette catégorie a la propriété très agréable d'être semi-simple d'après [Jan92], c'est-à-dire que tout objet s'écrit comme somme directe d'objets simples. Tout motif est alors décomposable de manière unique en des motifs élémentaires.

La cohomologie de Betti étant définie pour les variétés complexes quelconques, il est naturel de chercher à définir une notion de motifs pour les variétés quelconques. De tels motifs ne seront plus semi-simples. Les travaux de Deligne sur la théorie de Hodge [Del71, Del74b] et sur les conjectures de Weil [Del80] suggère toutefois que ces motifs sont munis d'une filtration appelée *filtration par le poids* dont les gradués sont des motifs purs au sens précédent. De tels motifs sont appelés *motifs mixtes* et on note $MM(k)$ la catégorie conjecturale des motifs mixtes.

¹Le groupe de Galois défini plus haut est le groupe de fondamental des motifs purs. Dans le programme des périodes, il faut plutôt utiliser le groupe de Galois des motifs mixtes introduit par Nori (on renvoie le lecteur à [Lev05, 3.3] pour un compte-rendu de l'approche de Nori) et par Ayoub dans [Ayo14a, Ayo14b].

B Premières approximations des motifs : les catégories ℓ -adiques

La cohomologie ℓ -adique : les origines

La première cohomologie de Weil construite dans le cas d'un corps de base k général est la cohomologie ℓ -adique. L'idée de la construction est due à Serre et Grothendieck. Pour calculer la cohomologie singulière d'un CW-complexe X , on peut considérer un recouvrement ouvert $\mathcal{U} = \{U_i \mid i \in I\}$ de X tel que toute intersection $U_{i_1} \cap \cdots \cap U_{i_n}$ d'un ensemble fini d'éléments de \mathcal{U} soit contractile ou vide. On associe alors à \mathcal{U} un ensemble simplicial, le nerf de Čech, dont les n -simplexes sont donnés par les sous-ensembles \mathcal{V} de \mathcal{U} de cardinal n tels que l'intersection $\bigcap_{V \in \mathcal{V}} V$ soit non vide. La cohomologie de cet ensemble simplicial donne alors la cohomologie singulière de X . Pour définir une cohomologie de Weil d'une variété sur un corps k , Serre propose de travailler en premier lieu à coefficients dans $\mathbb{Z}/\ell^m\mathbb{Z}$ où ℓ est un nombre premier distinct de la caractéristique de k et où m est un entier et de remplacer dans la construction précédente les recouvrements ouverts par les *recouvrements étales* c'est à dire les recouvrements combinant à la fois extensions de corps, revêtements non ramifiés et recouvrements ouverts pour la topologie de Zariski [2]. On prend ensuite la limite quand m tend vers l'infini que l'on tensorise par \mathbb{Q} au-dessus de \mathbb{Z} . Pour $X = k$, on retrouve par exemple la cohomologie galoisienne de k et si X est une courbe elliptique, le premier groupe de cohomologie est le module de Tate de X .

Pour formuler la définition de la cohomologie ℓ -adique, il est en fait nécessaire de travailler avec le formalisme développé par Grothendieck et ses élèves dans les années 1950 et 1960. On a alors besoin des outils suivants.

- La théorie des schémas [Gro60] qui généralise le cadre des variétés sur un corps et permet de considérer entre autres des variétés sans corps de base et des variétés avec multiplicité.
- La théorie des topos de Grothendieck [AGV73] qui permet d'étendre la notion d'espace topologique ; on définit ensuite la cohomologie dans un topos puis on définit le topos étale $S_{\text{ét}}$ d'un schéma S et la cohomologie étale de S .

Dans la suite de cette thèse, on utilisera librement les notions de la théorie des schémas et de la théorie des topos de Grothendieck. Si n est un nombre entier positif, le n -ème groupe de cohomologie ℓ -adique d'un k -schéma de type fini S est alors

$$H_{\text{ét}}^n(S_k, \mathbb{Q}_\ell) := \lim_m H_{\text{ét}}^n(S_k, \mathbb{Z}/\ell^m\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Celui-ci est de plus muni d'une action du groupe de Galois absolu G_k de k . On obtient donc un foncteur contravariant

$$\text{PSm}_k \rightarrow \text{Rep}(G_k, \mathbb{Q}_\ell)$$

où $\text{Rep}(G_k, \mathbb{Q}_\ell)$ est la catégorie des représentations continues de G_k à valeurs dans les \mathbb{Q}_ℓ -espaces vectoriels de dimension finie.

Cette catégorie des représentations ℓ -adiques est en fait une bonne approximation de la catégorie des motifs mixtes dont l'inconvénient principal est de dépendre du nombre premier ℓ . Si k est un corps fini, elle est par exemple munie d'une filtration par le poids et les gradués de la cohomologie de tout k -schéma de type fini vis-à-vis de cette filtration sont semi-simples.

²pour trouver la bonne définition de la cohomologie, il faut en fait travailler avec les *hyper-recouvrements* et non avec les recouvrements ordinaires.

Le formalisme des six foncteurs (première approche)

Dans le cas des espaces topologiques, on dispose en fait de 3 variantes de la cohomologie singulière : la cohomologie à support compact, l'homologie de Borel-Moore et l'homologie à support compact. On peut les construire en termes de catégories de faisceaux. En effet, notons $\mathrm{Sh}(X)$ la catégorie des faisceaux en \mathbb{Q} -espaces vectoriels sur un CW-complexe X donné ; si n est un entier positif, en notant $f: X \rightarrow \{*\}$ l'application vers le point, on peut obtenir le n -ième groupe de cohomologie singulière de X comme le \mathbb{Q} -espace vectoriel $R^n f_* f^* \mathbb{Q}$. De même, si le foncteur $f_!$ est donné par les sections à support compact du foncteur f_* , le \mathbb{Q} -espace vectoriel $R^n f_! f^* \mathbb{Q}$ est le n -ième groupe de cohomologie à support compact de X . De plus, le foncteur f_* permet de faire varier en famille les constructions cohomologiques.

On a vu précédemment que dans le cas des corps, on remplaçait la catégorie des \mathbb{Q} -espaces vectoriels par celle des motifs ; on peut envisager qu'il existe une catégorie des faisceaux de motifs ou des faisceaux motiviques sur un schéma quelconque et que la fibre d'un tel faisceaux sur un schéma X en un point x soit un motif sur le corps résiduel $k(x)$. On retrouverait alors comme ci-dessus les constructions cohomologiques à partir de foncteurs de type f_* , f^* , $f_!$ et $f^!$ reliant entre elles ces catégories de motifs relatifs.

On a vu lors du paragraphe précédent que la catégorie des représentations continues du groupe de Galois absolu d'un corps k à valeurs dans le corps \mathbb{Q}_ℓ était une bonne première approximation des motifs. Dans ce paragraphe, on s'intéresse aux premières constructions historiques de catégories motiviques dans le cadre des faisceaux de torsion qui ont été développées dans [G⁺77]. On fixe un nombre premier ℓ et un entier m strictement positif et on pose $\Lambda = \mathbb{Z}/\ell^m \mathbb{Z}$. Considérons un schéma noethérien X . Une première idée est d'envisager la catégorie $\mathrm{Sh}(X_{\text{ét}}, \Lambda)$ des faisceaux de Λ -module sur le petit site étale de X comme une catégorie de motifs relatifs sur un schéma X à coefficients dans Λ . Cette catégorie est malheureusement bien trop grosse : on a besoin d'une condition de finitude sur les faisceaux. En effet, dans le cas des corps, il faut par exemple retrouver la catégorie des Λ -modules de type fini. On introduit alors la catégorie $\mathrm{Sh}_c(X_{\text{ét}}, \Lambda)$ des faisceaux constructibles comme la catégorie des faisceaux F tels qu'il existe une stratification finie (X_i) de X par des sous-schémas X_i , qui sont constructibles et localement fermés, telle que la restriction de F à chaque X_i soit localement constante.

Ces catégories sont alors reliées entre elles par le formalisme des six foncteurs. Pour formuler ce formalisme de manière plus précise, il nous faut faire un détour catégorique.

- Une catégorie \mathcal{A} est une *catégorie abélienne* si elle possède certaines structures similaire à celles de la catégorie \mathbf{Ab} des groupe abéliens. On renvoie à [Gro57] (où la notion a été introduite) ou à [Lur17, 1.1.2.2] pour une définition précise.
- Étant donnée une catégorie abélienne \mathcal{A} , la *catégorie dérivée* de \mathcal{A} notée $D(\mathcal{A})$ est la catégorie des complexes d'objets de \mathcal{A} où deux complexes sont identifiés s'il existe un morphisme entre eux qui induit un isomorphisme sur tous les groupes de cohomologie. On renvoie le lecteur à [Ver96] (où la notion a été introduite) ou à [Sta23, 05QI] pour plus de précisions. La catégorie $D(\mathcal{A})$ est alors munie d'une structure supplémentaire : celle de *catégorie triangulée* (voir par exemple [Sta23, 05QK]). La notion naturelle de foncteur entre des catégories triangulées est la notion de *foncteur exact*.
- Une catégorie \mathcal{C} est *monoïdale (symétrique)* s'il existe un bifoncteur $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ qui ressemble au foncteur produit tensoriel sur une catégorie d'espaces vectoriels. Elle est

fermée s'il existe un bifoncteur $\underline{\text{Hom}}: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ tel que pour tout objet C de \mathcal{C} , on a une adjonction.

$$- \otimes C: \mathcal{C} \rightleftarrows \mathcal{C}: \underline{\text{Hom}}(C, -).$$

Cette notion a été introduite de manière indépendante dans [Mac63, Ben63]. On renvoie le lecteur à [Sta23, 0FFJ] pour une définition plus précise des catégories monoïdales. La notion naturelle de foncteur entre des catégories monoïdales est la notion de *foncteur monoïdal*.

- Une *2-catégorie* est une catégorie dont l'ensemble des morphismes forment une catégorie. On demande de plus des compatibilités entre les différentes manières de composer les 2-morphismes. Par exemple, la catégorie des catégories est une 2-catégorie. La notion naturelle de foncteur entre des 2-catégories est la notion de *2-foncteur*. On renvoie le lecteur à [Bén67] pour une définition précise.

On décrit maintenant, suivant [CD19, A.5], le formalisme des six foncteurs pour les faisceaux étales de torsion. Si X est un schéma, on note $D(X, \Lambda)$ la catégorie dérivée de la catégorie abélienne $\text{Sh}(X_{\text{ét}}, \Lambda)$. Les schémas considérés ci-dessous sont supposés être définis sur $\mathbb{Z}[1/\ell]$ et quasi-excellents. Si X est un schéma et si n est un entier positif, notons μ_{ℓ^n} le faisceau des racines ℓ^n -èmes de l'unité sur $X_{\text{ét}}$, le n -ème *twist de Tate* d'un faisceau F est le faisceau $F(n) = F \otimes \mu_{\ell^n}^{\otimes n}$.

Si X est un schéma noethérien, un complexe dans $D(X, \Lambda)$ est *constructible* s'il est borné et si ses faisceaux de cohomologie sont constructibles. On note $D_c^b(X, \Lambda)$ la sous-catégorie pleine de $D(X, \Lambda)$ formée des complexes constructibles. Les résultats suivants sont dus à Grothendieck *et al.* (voir [G⁺77]) dans le cas des schémas noethériens admettant une résolution des singularités et à Gabber (voir [LYF14]) dans le cas général des schémas noethériens quasi-excellents.

1. On a trois paires de foncteurs adjoints exacts comme ci-dessous.
 - $f^*: D_c^b(Y, \Lambda) \rightleftarrows D_c^b(X, \Lambda): f_*$ pour tout morphisme de schémas de type fini $f: X \rightarrow Y$. Le morphisme f^* est de plus bien défini pour les morphismes de schémas quelconques.
 - $f_!: D_c^b(Y, \Lambda) \rightleftarrows D_c^b(X, \Lambda): f^!$ pour tout morphisme séparé de type fini de schémas $f: X \rightarrow Y$.
 - $(\otimes, \underline{\text{Hom}})$ une structure monoïdale symétrique fermée sur chaque $D_c^b(X, \Lambda)$.

De plus les foncteurs de type f^* sont monoïdaux.

2. Les foncteurs f_* et $f_!$ (resp. f^* et $f^!$) fournissent des 2-foncteurs covariants (resp. contravariants) de la catégorie des schémas (de type fini sur une base fixée pour les foncteurs exceptionnels) vers la 2-catégorie des catégories triangulées.
3. Il existe un morphisme de 2-foncteurs

$$\alpha: f_! \rightarrow f_*$$

qui est une équivalence si f est propre.

4. *Changement de base* : Pour tout carré cartésien de schémas

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow g' & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

tel que f est séparé de type fini, on a des isomorphismes naturels

$$g^* f_! \rightarrow f'_!(g')^* \text{ et } g'_*(f')^! \rightarrow f^! g_*.$$

5. *Formule de projection* : Pour tout morphisme séparé de type fini $f: X \rightarrow Y$, on a des isomorphismes naturels

$$\begin{aligned} (f_! F) \otimes G &\rightarrow f_!(F \otimes f^* G) \\ \underline{\mathrm{Hom}}(f_! F, G) &\rightarrow f_* \underline{\mathrm{Hom}}(F, f^! G) \\ f^! \underline{\mathrm{Hom}}(F, G) &\rightarrow \underline{\mathrm{Hom}}(f^* F, f^! G). \end{aligned}$$

6. *Localisation* : Pour toute immersion fermée $i: F \rightarrow X$ de complémentaire $j: X \setminus F \rightarrow X$, on a des triangles exacts de transformations naturelles de la forme

$$\begin{aligned} j_! j^! &\rightarrow \mathrm{Id} \rightarrow i_* i^* \\ i_! i^! &\rightarrow \mathrm{Id} \rightarrow j_* j^* \end{aligned}$$

donnés par les unités et coïunités des adjonctions idoines.

7. *Pureté relative* : Pour tout morphisme lisse séparé $f: X \rightarrow S$ de dimension relative d , on a un isomorphisme

$$\mathfrak{p}_f: f^* \rightarrow f^!(-d)[-2d].$$

8. *Pureté absolue* : Si $i: F \rightarrow X$ est une immersion fermée purement de codimension c entre schémas réguliers, on a un isomorphisme canonique

$$\mathbb{1}_F(-c)[-2c] \rightarrow i^! \mathbb{1}_X.$$

9. *Dualité de Poincaré-Verdier* : Soit S un schéma régulier et K_S un objet \otimes -inversible de $\mathcal{D}_c^b(S, \Lambda)$. Si $f: X \rightarrow S$ est un morphisme séparé de type fini, on pose

$$D_{X/S} := \underline{\mathrm{Hom}}(-, f^! K_S).$$

On a alors les propriétés suivantes.

(a) Pour tout S -schéma séparé de type fini X , la transformation naturelle

$$\mathrm{Id} \rightarrow D_{X/S} \circ D_{X/S}$$

est une équivalence.

(b) Pour tout S -schéma séparé de type fini X et toute paire (M, N) de complexes de faisceaux ℓ -adiques sur X , on a un isomorphisme canonique

$$D_{X/S}(M \otimes D_{X/S}(N)) \simeq \underline{\mathrm{Hom}}(M, N).$$

- (c) Pour tout morphisme $f: Y \rightarrow X$ entre S -schémas séparés de type fini, on a des isomorphismes canoniques

$$\begin{aligned} D_{Y/S} \circ f^* &\simeq f^! \circ D_{X/S} \\ f^* \circ D_{X/S} &\simeq D_{Y/S} \circ f^! \\ D_{X/S} \circ f_! &\simeq f_* \circ D_{Y/S} \\ f_! \circ D_{Y/S} &\simeq D_{X/S} \circ f_* \end{aligned}$$

Le formalisme ℓ -adique de Bhatt-Scholze

Passons maintenant au cas des coefficients \mathbb{Q}_ℓ . Dans le calcul de la cohomologie ℓ -adique, on a d'abord calculé la cohomologie de torsion avant de passer à la limite. Une première façon de construire des catégories ℓ -adiques de motifs relatifs est de prendre la limite des catégories motiviques à coefficients dans $\mathbb{Z}/\ell^n\mathbb{Z}$ quand n tend vers l'infini. C'est le point de vue de [Del77, Eke07, LYF14]. Nous adoptons ici un point de vue plus récent développé dans [BS15, HRS20] qui permet de faire la construction de manière plus directe.

Bhatt et Scholze définissent d'abord la topologie proétale. Un morphisme de schémas

$$f: X \rightarrow Y$$

est *faiblement étale* s'il est plat et si le morphisme diagonal $\Delta_f: X \rightarrow X \times_Y X$ est plat. Les limites projectives de morphismes étales sont faiblement étales. Cela permet donc par exemple de considérer comme un recouvrement le morphisme donné par la clôture algébrique d'un corps. Le site proétale $X_{\text{proét}}$ d'un schéma X est alors la catégorie des schémas faiblement étales sur X , munie d'une structure de site en décrétant qu'une famille $(X_i \rightarrow X)_{i \in I}$ est couvrante si pour tout ouvert affine U de X , il existe un ouvert affine de $\bigsqcup_{i \in I} X_i$ qui recouvre U .

Ils remarquent ensuite que si M est un espace topologique totalement discontinu, on peut définir sur tout schéma X un faisceau M_X sur $X_{\text{proét}}$ par la formule $U \mapsto \mathcal{C}^0(U, M)$. Si Λ est un anneau topologique, le faisceau Λ_X est un faisceau en anneaux. Soit alors Λ une extension finie de \mathbb{Q}_ℓ ou l'anneau des entiers d'une telle extension. Pour tout schéma X , Bhatt et Scholze définissent la catégorie $\text{Sh}(X, \Lambda)$ est alors la catégorie des faisceaux de Λ_X -modules sur le site $X_{\text{proét}}$. Si le schéma X est de plus noethérien, on dit comme dans le cas des coefficients de torsion qu'un faisceau F est *constructible* s'il existe une stratification finie (X_i) de X par des sous-schémas constructibles localement fermés telle que la restriction de F à chaque X_i soit localement constante pour la topologie proétale. On définit alors $D_c^b(X, \Lambda)$ comme la sous-catégorie pleine de $D(\text{Sh}(X, \Lambda))$ formée des complexes bornés dont les faisceaux de cohomologie sont constructibles. La catégorie $D_c^b(X, \Lambda)$ vérifie les points 1 à 8 et le point 9.(a) du formalisme des six foncteurs donné au paragraphe précédent d'après [BS15, 6.7]. Les items 9.(b) et 9.(c) sont vérifiés dans le cas où Λ est un corps ℓ -adique (par exemple d'après [CD19, 4.4.24]).

On retrouve alors la cohomologie ℓ -adique d'un schéma de type fini $f: X \rightarrow \text{Spec}(k)$, avec k un corps : celle-ci est donnée par la cohomologie du faisceau $f_* f^* \mathbb{Q}_{\ell, k}$.

Faisceaux pervers

Si $f: X \rightarrow \text{Spec}(k)$ est un morphisme lisse, la dualité de Poincaré pour la cohomologie étale fournit un accouplement parfait

$$H_{\text{ét}, c}^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes H_{\text{ét}}^{2d-n}(X_{\bar{k}}, \mathbb{Q}_\ell) \rightarrow \mathbb{Q}_\ell$$

où $H_{\text{ét},c}^n$ désigne la cohomologie à support compact. On peut interpréter ce résultat en termes du foncteur de dualité donné par le formalisme des six foncteurs. En effet, par dualité, on a

$$\underline{\text{Hom}}(p_!p^*\mathbb{Q}_\ell(d)[n], \mathbb{Q}_\ell) = \underline{\text{Hom}}(\mathbb{Q}_{\ell,X}, p^!\mathbb{Q}_\ell(-d)[-n]).$$

Ces deux objets vivent dans $D_c^b(k, \mathbb{Q}_\ell)$ et leurs H^0 sont donc des objets de $\text{Rep}(G_k, \mathbb{Q}_\ell)$. Le H^0 du membre de gauche est le dual de $H_{\text{ét},c}^n(X_{\bar{k}}, \mathbb{Q}_\ell)$ tandis que comme $p^!\mathbb{Q}_\ell \simeq \mathbb{Q}_{\ell,X}(d)[2d]$, le H^0 du membre de droite est $H_{\text{ét}}^{2d-n}(X_{\bar{k}}, \mathbb{Q}_\ell)$.

Dans le cas où le morphisme f n'est plus lisse mais simplement de type fini, la dualité de Poincaré ne fonctionne plus car on ne peut plus affirmer que $p^!\mathbb{Q}_\ell \simeq \mathbb{Q}_{\ell,X}(d)[2d]$. On peut interpréter cela comme une conséquence du fait que le foncteur de dualité ne respecte pas la sous-catégorie $\text{Sh}_c(X, \mathbb{Q}_\ell)$ de $D_c^b(X, \mathbb{Q}_\ell)$.

Beilinson, Bernstein, Deligne et Gabber ont construit dans [BBDG18] une sous-catégorie de $D_c^b(X, \mathbb{Q}_\ell)$ pour tout schéma X de type fini sur un corps qui est abélienne et préservée par le foncteur de dualité. Ils en ont déduit une nouvelle version de la dualité de Poincaré valable pour les schémas singuliers.

Au cours de cette étude, ils ont dégagé une notion centrale : la notion de *t-structure* sur une catégorie triangulée. Une t-structure sur une catégorie triangulée \mathcal{D} est une paire $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ de sous-catégories strictement pleines de \mathcal{D} telles que

- Pour tout objet M de $\mathcal{D}^{\leq 0}$ et tout objet N de $\mathcal{D}^{\geq 0}$, on a $\text{Hom}(M, N[-1]) = 0$.
- On a des inclusions $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}[-1]$ et $\mathcal{D}^{\geq 0}[-1] \subseteq \mathcal{D}^{\geq 0}$.
- Pour tout objet M de \mathcal{D} , il existe un triangle exact $M' \rightarrow M \rightarrow M''$ tel que M' est un objet de $\mathcal{D}^{\leq 0}$ et M'' est un objet $\mathcal{D}^{\geq 0}[-1]$.

Si n est un nombre entier, on notera de plus $\mathcal{D}^{\leq n}$ (resp. $\mathcal{D}^{\geq n}$) la sous-catégorie $\mathcal{D}^{\leq 0}[-n]$ (resp. $\mathcal{D}^{\geq 0}$) de \mathcal{D} et si M est un objet de \mathcal{D} , on notera $M \leq n$ (resp. $M \geq n$) si M appartient à $\mathcal{D}^{\leq n}$ (resp. $\mathcal{D}^{\geq n}$).

La notion de t-structure permet de décréter que des objets d'une catégorie triangulée sont placés en degrés positifs ou négatifs. La sous-catégorie pleine $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ formée des objets en degré 0 est appelé le *cœur* de la t-structure ; c'est une catégorie abélienne.

Par exemple, si \mathcal{A} est une catégorie abélienne, la catégorie triangulée $D(\mathcal{A})$ est munie d'une t-structure dite *ordinaire* : les objets t-positifs sont les objets dont les faisceaux de cohomologie en degré strictement négatif sont nuls et les objets t-négatifs sont les objets dont les faisceaux de cohomologie en degré strictement positif sont nuls. Le cœur de la t-structure ordinaire est équivalent à la catégorie abélienne \mathcal{A} .

Fixons à présent un schéma noethérien X , un nombre premier ℓ et Λ une extension d'un corps de nombres ℓ -adique ou l'anneau des entiers d'une telle extension. La catégorie triangulée $D_c^b(X, \Lambda)$ est alors munie d'une t-structure dite *ordinaire* dont le cœur est équivalent à la catégorie des faisceaux ℓ -adiques constructibles.

On donne à présent la construction des catégories de faisceaux pervers telle que Gabber l'a étendue à tous les schémas noethériens excellents dans [Gab08]. Si x est un point de X , on note $i_x : \{x\} \rightarrow X$ l'inclusion. On définit la *t-structure perverse* comme suit.

- ${}^{p \leq 0}D_c^b(X, \Lambda)$ est la sous-catégorie pleine de $D_c^b(X, \Lambda)$ formée des complexes K tels que pour tout point x de X , on a

$$i_x^*K \leq -\dim(\overline{\{x\}})$$

pour la t-structure ordinaire de $D_c^b(k(x), \Lambda)$.

- $p \geq 0$ $D_c^b(X, \Lambda)$ est la sous-catégorie pleine de $D_c^b(X, \Lambda)$ formée des complexes K tels que pour tout point x de X , on a

$$i_x^! K \geq -\dim(\overline{\{x\}})$$

pour la t-structure ordinaire de $D_c^b(k(x), \Lambda)$.

Le cœur de cette t-structure est la catégorie $\text{Perv}(X, \Lambda)$ des *faisceaux pervers*. Lorsque k est un corps fini, elle dispose de plusieurs propriétés qui la rapprochent de la catégorie des motifs mixtes sur un corps : elle admet une filtration par le poids et les gradués du faisceau $f_*\Lambda$ sont semi-simples pour tout morphisme $f: X \rightarrow \text{Spec}(k)$ de type fini. De plus, on peut décrire les objets simples de $\text{Perv}(X, \Lambda)$ en termes d'objets simples de catégories de systèmes locaux sur des sous-schémas. Enfin, dans le cas où Λ est un anneau de torsion première à la caractéristique de k , on peut formuler une définition analogue des faisceaux pervers et la catégorie $D_c^b(X, \Lambda)$ est alors équivalente à la catégorie dérivée bornée de $\text{Perv}(X, \Lambda)$. Tous ces éléments font de la catégorie des faisceaux pervers un bon analogue ℓ -adique de la catégorie conjecturale des motifs mixtes relatifs.

L'idée de Beilinson et Deligne pour construire une telle catégorie des motifs est alors de commencer par construire une catégorie triangulée qui aurait toutes les propriétés de la catégorie dérivée bornée de la catégorie des motifs mixtes puis de construire une t-structure sur cette catégorie afin de définir la catégorie abélienne des motifs mixtes. On s'attend à trouver deux t-structures : une *t-structure motivique ordinaire* dont le cœur $\text{MM}_{\text{ord}}(S)$ serait analogue à la catégorie des faisceaux ℓ -adique et une *t-structure motivique perverse* dont le cœur $\text{MM}_{\text{perv}}(S)$ serait analogue à la catégorie des faisceaux pervers.

C Catégories motiviques

La catégorie triangulée des motifs mixtes de Voevodsky

La première étape du programme de Beilinson et Deligne a été réalisée par Voevodsky dans [Voe92, Voe00]. Soit S un schéma noethérien de dimension finie, Voevodsky a construit une catégorie triangulée $\text{DM}_c(S)$ qui vérifie les propriétés que l'on attend de la catégorie dérivée bornée de la catégorie abélienne $\text{MM}_{\text{perv}}(S)$.

Il existe en fait de nombreux modèles de la catégorie $\text{DM}_c(S)$. Celui qui nous intéressera dans cette thèse, le modèle des *motifs étales*, a été proposé par F. Morel et étudié par Ayoub, Cisinski et Déglise [Ayo14c, CD16]. Une version de ce modèle est également présente dans la thèse de Voevodsky [Voe92] sous le nom de *h-motifs*.

En voici une construction : on part de la catégorie dérivée $D(\text{Sh}_{\text{ét}}(S, \mathbb{Q}))$ des faisceaux étales sur la catégorie Sm_S des S -schémas lisses à valeurs dans la catégorie des \mathbb{Q} -espaces vectoriels.

Un complexe de tels faisceaux étales K est dit \mathbb{A}^1 -local si pour tout S -schéma lisse X , et pour tout entier n , en notant p la projection $\mathbb{A}_X^1 \rightarrow X$, le morphisme $p^*: H_{\text{ét}}^n(X, F) \rightarrow H_{\text{ét}}^n(\mathbb{A}_X^1, F)$ est une équivalence. L'inclusion

$$\text{DA}_{\text{ét}}^{\text{eff}}(S) \rightarrow D(\text{Sh}_{\text{ét}}(S, \mathbb{Q}))$$

admet un adjoint à gauche noté $L_{\mathbb{A}^1}$. On note M_S^{eff} la composition de l'inclusion de Yoneda

$$\text{Sm}_S \rightarrow D(\text{Sh}_{\text{ét}}(S, \mathbb{Q}))$$

avec le foncteur L_{A^1} . On définit alors le motif de Tate T_S comme la cofibre de l'inclusion scindée

$$M_S^{\text{eff}}(S) \rightarrow M_S^{\text{eff}}(\mathbb{G}_{m,S}).$$

La dernière étape de la construction consiste à inverser formellement l'objet T_S ce qui peut être fait à l'aide par exemple de la notion de spectre de Tate [CD19, 5.3]. On obtient alors la catégorie $\text{DM}_{\text{ét}}(S)$ des *motifs étales* sur S à coefficients rationnels. On a de plus un foncteur

$$\Sigma^\infty : \text{DA}_{\text{ét}}^{\text{eff}}(S) \rightarrow \text{DM}_{\text{ét}}(S)$$

et on note M_S la composition de M_S^{eff} avec ce foncteur. Pour obtenir la catégorie $\text{DM}_{\text{ét},c}(S)$ évoquée plus haut, on considère la plus petite sous-catégorie de $\text{DM}_{\text{ét}}(S)$ stable par cône, décalage et rétractes et contenant les objets de la forme $M_S(X) \otimes T_S^n$ pour tout S -schéma lisse X et tout entier n .

On peut alors munir les catégories $\text{DM}_{\text{ét},c}(-)$ du formalisme des six foncteurs d'après les travaux d'Ayoub, Cisinski et Déglise [Ayo07, Ayo14c, CD19, CD16]. Si S est un schéma noethérien de dimension finie, on note $\mathbb{1}_S$ l'objet unité pour la structure monoïdale de la catégorie $\text{DM}_{\text{ét},c}(S)$. On définit alors le motif cohomologique $h_S(X) = f_* f^* \mathbb{1}_S$ de tout S -schéma de type fini.

Dans le but d'obtenir les catégories abéliennes des motifs mixtes $\text{MM}_{\text{ord}}(S)$ et $\text{MM}_{\text{perv}}(S)$, il faudrait alors équiper la catégorie $\text{DM}_{\text{ét}}(S)$ de deux t-structures : la t-structure motivique ordinaire et la t-structure motivique perverse. Bondarko a montré dans [Bon15] qu'il suffit en fait de construire la t-structure motivique ordinaire dans le cas des corps et de vérifier qu'elle satisfait une certaine propriété de compatibilité avec la structure de poids pour récupérer la t-structure motivique perverse sur une base arbitraire.

Pour caractériser la t-structure motivique sur $\text{MM}(k)$ avec k un corps, penchons-nous de nouveau sur la cohomologie singulière dans le cas où $S = \text{Spec}(\mathbb{C})$. Cette dernière peut être définie pour un \mathbb{C} -schéma X à partir du complexe $C^*(X^{\text{an}}, \mathbb{Q})$ des cochaînes singulières à coefficients rationnels sur l'analytifié X^{an} de X . Si pour tout schéma S , on note Sch_S^{ft} la catégorie des S -schémas de type fini, ceci fournit un foncteur contravariant

$$C^* : \text{Sch}_{\mathbb{C}}^{\text{ft}} \rightarrow \text{D}(\mathbb{Q}).$$

Ayoub a construit dans [Ayo10] un foncteur de *réalisation de Betti* ρ_B qui rend le diagramme suivant commutatif

$$\begin{array}{ccc} \text{Sch}_{\mathbb{C}}^{\text{ft}} & \xrightarrow{C^*} & \text{D}(\mathbb{Q}) \\ h_{\mathbb{C}} \downarrow & \nearrow \rho_B & \\ \text{DM}_{\text{ét}}(\mathbb{C}) & & \end{array} .$$

Les conjectures concernant les motifs sur le corps \mathbb{C} sont alors équivalentes à l'existence d'une t-structure sur $\text{DM}_{\text{ét},c}(\mathbb{C})$ telle que le foncteur ρ_B est t-exact et conservatif.

Dans le cas d'un schéma quelconque, on peut formuler une conjecture similaire en remplaçant la cohomologie singulière par la cohomologie ℓ -adique et en remplaçant la réalisation de Betti par la réalisation ℓ -adique.

Rigidité et réalisation ℓ -adique

Pour construire ce foncteur de réalisation ℓ -adique, il faut généraliser quelque peu la construction de la catégorie des motifs en remarquant que l'on peut définir de la même manière une catégorie $\mathrm{DM}_{\acute{e}t}(S, R)$ pour tout anneau commutatif R .

Soit $\rho^\# : \mathrm{D}(\mathrm{Sh}_{\acute{e}t}(S, R)) \rightarrow \mathrm{D}(S_{\acute{e}t}, R)$ la restriction au petit site étale. Le foncteur $\rho^\#$ admet un adjoint à gauche pleinement fidèle $\rho_\#$ qui envoie le faisceau représenté par un S -schéma étale X sur le faisceau étale sur Sm_S représenté par X . Si l'on pose $\rho! = \Sigma^\infty L_{\mathbb{A}^1} \rho_\#$, on obtient alors un foncteur

$$\rho! : \mathrm{D}(S_{\acute{e}t}, R) \rightarrow \mathrm{DM}_{\acute{e}t}(S, R).$$

Théorème. (*Suslin, Voevodsky, Ayoub, Cisinski, Déglise*) Soient S un schéma noethérien de dimension finie, ℓ un nombre premier inversible sur S , n un entier et $\Lambda = \mathbb{Z}/\ell^n \mathbb{Z}$. Alors, le foncteur

$$\rho! : \mathrm{D}(S_{\acute{e}t}, \Lambda) \rightarrow \mathrm{DM}_{\acute{e}t}(S, \Lambda).$$

est une équivalence de catégories triangulées monoïdales.

On renvoie le lecteur à [BH21, 3.2] pour la démonstration.

Pour définir le foncteur de réalisation ℓ -adique

$$\rho_\ell : \mathrm{DM}_{\acute{e}t,c}(S) \rightarrow \mathrm{D}_c^b(S, \mathbb{Q}_\ell),$$

on remarque d'abord que l'on dispose d'un foncteur

$$- \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^n \mathbb{Z} : \mathrm{DM}_{\acute{e}t,c}(S, \mathbb{Z}) \rightarrow \mathrm{DM}_{\acute{e}t,c}(S, \mathbb{Z}/\ell^n \mathbb{Z})$$

de changement des coefficients. On définit ensuite un foncteur

$$\rho_\ell : \mathrm{DM}_{\acute{e}t,c}(S, \mathbb{Z}) \rightarrow \mathrm{D}_c^b(S, \mathbb{Z}_\ell)$$

par passage à la limite et en utilisant le théorème ci-dessus. On renvoie le lecteur à [CD16, 7.2] pour le sens précis de ce passage à la limite.

Pour obtenir la version rationnelle de ce foncteur, on \mathbb{Q} -linéarise le foncteur ci-dessus au sens où on considère le foncteur induit sur les complétions idempotentes des catégories ayant les mêmes objets mais où l'on a tensorisé les ensembles de morphismes par \mathbb{Q} . Ce processus donne bien les bonnes catégories d'après [CD16, 7.2.23].

Vers la t -structure motivique : le cas des motifs d'Artin

Dans le cas d'un corps de base k , une partie du programme motivique est très bien connue et déjà réalisée. Notons Fini_k la catégorie des k -schémas finis. Celle-ci consiste à construire une catégorie $M^A(k)$ munie d'un foncteur contravariant

$$\mathrm{Fini}_k \rightarrow M^A(k)$$

tel que pour tout nombre premier ℓ inversible sur k , on ait un diagramme commutatif

$$\begin{array}{ccc} \mathrm{Fini}_k & \xrightarrow{H_{\acute{e}t}^0(-\bar{k}, \mathbb{Q}_\ell)} & \mathrm{Rep}(G_k, \mathbb{Q}_\ell) \\ h_k^0 \downarrow & \nearrow & \\ M^A(k) & & \end{array} .$$

Ici, on a inclus seulement le 0-ème groupe de cohomologie car les autres sont nuls. Remarquons par ailleurs que tout k -schéma fini X est muni d'un morphisme totalement inséparable vers un k -schéma fini étale X' . Par invariance de la cohomologie étale par morphisme totalement inséparable, on a

$$H_{\text{ét}}^0(X_{\bar{k}}, \mathbb{Q}_\ell) = H_{\text{ét}}^0(X'_k, \mathbb{Q}_\ell).$$

La correspondance de Galois-Grothendieck assure alors que

$$H_{\text{ét}}^0(X'_k, \mathbb{Q}_\ell) = \mathbb{Q}_\ell[X'(\bar{k})] = \mathbb{Q}_\ell[X(\bar{k})]$$

et par conséquent, la représentation $H_{\text{ét}}^0(X_{\bar{k}}, \mathbb{Q}_\ell)$ se factorise par un quotient fini. Une telle représentation est appelée *représentation d'Artin* de G_k . On peut alors définir $M^A(k)$ comme étant la catégorie des représentations d'Artin \mathbb{Q} -linéaires de G_k et définir le foncteur h_k^0 comme étant le foncteur

$$X \mapsto \mathbb{Q}[X(\bar{k})].$$

La théorie de Voevodsky permet également de retrouver ce résultat. On définit la catégorie $\text{DM}_{\text{ét},c}^A(k)$ des motifs d'Artin comme la plus petite sous-catégorie de $\text{DM}_{\text{ét},c}(k)$ stable par cône, décalage et rétractes et contenant les objets de la forme $h_k(X)$ avec X fini sur k . Voevodsky et Orgogozo ont montré dans [Voe00, Org04] (Orgogozo traite également le cas des 1-motifs) que cette catégorie est alors munie d'une t-structure, la t-structure *homotopique*, telle que les foncteurs de réalisation sont t-exacts. Le cœur de cette t-structure est équivalent à la catégorie des représentations d'Artin \mathbb{Q} -linéaires de G_k .

Ce résultat s'interprète également à l'aide du groupe de Galois motivique : le quotient du groupe de Galois motivique de k qui ne tient compte que des variétés de dimension 0 sur k est le groupe de Galois absolu de k .

Ces travaux ont été prolongés par Ayoub, Barbieri-Viale, Kahn, Pepin Lehalleur et Vaish dans [AB09, BVK16, Pep19b, Pep19a, Vai19] au cas des motifs d'Artin sur un schéma de base S quelconque. Ces auteurs construisent notamment la t-structure motivique ordinaire sur les motifs d'Artin au sens suivant.

Théorème. (Voevodsky, Orgogozo, Ayoub, Barbieri-Viale, Kahn, Pepin Lehalleur, Vaish) *Soit S un schéma noethérien excellent de dimension finie qui admet une résolution des singularités par altérations, soit ℓ un nombre premier inversible sur S et soit $n = 0, 1$.*

Alors, il existe une t-structure sur la catégorie triangulée $\text{DM}_{\text{ét}}^n(S, \mathbb{Q})$ des n -motifs étales qui induit une t-structure non dégénérée sur la sous-catégorie triangulée $\text{DM}_{\text{ét},c}^n(S, \mathbb{Q})$ des n -motifs étales constructibles telle que le foncteur

$$\text{DM}_{\text{ét},c}^n(S, \mathbb{Q}) \rightarrow D_c^b(S, \mathbb{Q}_\ell)$$

de réalisation ℓ -adique est t-exact quand on munit la catégorie triangulée $D_c^b(S, \mathbb{Q}_\ell)$ de sa t-structure ordinaire.

Le but de cette thèse est d'étendre le travail de ces auteurs dans deux directions ; on s'intéresse à la fois aux motifs à coefficients entiers et aux deux t-structures motiviques : l'ordinaire et la perverse.

D Présentation de la thèse

Rappels ∞ -catégoriques

Le premier chapitre de la thèse est à visée explicative, les différentes notions et théories abordées dans le reste du texte y sont présentées. On y rappelle d'abord la théorie des ∞ -catégories stables monoïdales. Celles-ci ont remplacé les catégories triangulées monoïdales en théorie de l'homotopie car elles ont l'avantage de conserver plus d'informations de nature homotopique sur les objets étudiés. Les triangles distingués sont par exemple décrits en termes de carrés cartésiens. Ils cessent donc d'être une *structure additionnelle* dont il faut équiper la catégorie considérée ; être un triangle distingué devient une *propriété*. Ceci permet par exemple d'obtenir la fonctorialité du cône d'un morphisme.

On commence par rappeler la théorie des ∞ -catégories et on explique comment les notions catégoriques usuelles sont définies dans ce cadre. Une des particularités de la théorie est que les morphismes entre deux objets d'une catégorie forment un *espace* ou ∞ -*groupoïde*, c'est à dire une ∞ -catégorie dont tous les morphismes sont inversibles. A partir de toute ∞ -catégorie, on peut construire une catégorie (usuelle) appelée sa *catégorie homotopique*. Par exemple, si X est un espace topologique, on peut lui associer un ∞ -groupoïde dont la catégorie homotopique est le groupoïde fondamental de X . Le groupoïde fondamental se « rappelle » de quels chemins sont homotopes mais ne se « rappelle » pas « comment » ; l' ∞ -groupoïde associé à X contient au contraire cette information et permet à partir d'elle de construire les groupes d'homotopie supérieurs.

On expose ensuite l'analogie de la *construction de Grothendieck* de [GR71, VI.8] dans le cadre ∞ -catégorique. Celle-ci permet de décrire les ∞ -foncteurs d'une ∞ -catégorie \mathcal{C} vers l' ∞ -catégorie des ∞ -catégories comme un type particulier de fibration au-dessus de \mathcal{C} . C'est de cette manière qu'on peut par exemple définir les adjonctions dans le cadre ∞ -catégorique. Puis, on rappelle la notion centrale de *présentabilité* : de manière informelle une ∞ -catégorie est présentable si tous ses objets s'obtiennent comme des ∞ -colimites filtrantes à partir d'un petit ensemble d'objets. Les ∞ -catégories de préfaisceaux en espaces sont l'exemple le plus fondamental d' ∞ -catégorie présentable : toute ∞ -catégorie peut s'obtenir par *localisation accessible*, c'est-à-dire en inversant formellement un petit ensemble de flèches, à partir d'une catégorie de préfaisceaux.

On définit ensuite la notion d' ∞ -catégorie stable ; la catégorie homotopique d'une ∞ -catégorie stable est triangulée. Une des notions qui joue un rôle central dans la thèse est la notion de *t-structure* sur une ∞ -catégorie stable. Celles-ci sont définies de la même façon que dans le cas triangulé et permettent encore d'obtenir des catégories abéliennes en considérant les objets concentrés en degré 0. Lurie a construit dans [Lur17, 1.3] les catégories dérivées des catégories abéliennes de Grothendieck qui sont des ∞ -catégories stables. Leurs catégories homotopiques sont bien les catégories dérivées au sens de Verdier.

La transposition dans le monde ∞ -catégorique des catégories monoïdales (symétriques) est l'un des points les plus délicats de la théorie développée par Lurie. Il est en effet nécessaire d'encoder de nombreuses cohérences supérieures de manière combinatoire. Comme dans le cas usuel des catégories monoïdales, on peut munir l' ∞ -catégorie des ∞ -catégories d'une structure monoïdale dont les algèbres commutatives sont les ∞ -catégories monoïdales. On présente également la *construction de Robalo* qui permet d'inverser formellement pour le produit tensoriel un objet d'une ∞ -catégorie monoïdale.

La dernière notion ∞ -catégorique dont nous avons besoin dans la suite de ce texte est la

notion d'*hyperfaisceaux*. Pour calculer la cohomologie étale, il est par exemple nécessaire de recourir à une notion généralisée de recouvrement (les *hyper-recouvrements*).

Rappels motiviques

On présente ensuite la théorie des catégories *motiviques* et *prémotiviques* développée dans [Ayo07, CD19]. Une catégorie prémotivique stable est un ∞ -foncteur contravariant de la catégorie des schémas dans l' ∞ -catégorie des ∞ -catégories monoïdales stables présentables tel que le tiré en arrière par tout morphisme lisse admet un adjoint à gauche et vérifiant la *formule de changement de base lisse* et la *formule de projection lisse*. Une telle catégorie est *motivique* si elle vérifie de plus les propriétés de *localisation*, de *Tate-stabilité* et d'*invariance par homotopie*.

Les catégories motiviques stables sont munies du formalisme des six foncteurs mais ne vérifient en général ni la pureté relative ni la pureté absolue. Elles ne vérifient pas non plus la dualité de Verdier si l'on ne se restreint pas à certains objets dits *constructibles*. On peut cependant donner une version plus générale de la pureté relative qui reste vraie dans toute catégorie motivique d'après le théorème de pureté de Morel et Voevodsky. On peut munir certaines catégories motiviques d'une structure supplémentaire appelée *orientation* et toute catégorie motivique orientée vérifie la propriété de pureté relative.

On donne enfin la construction de la catégorie motivique stable $\mathcal{DM}_{\text{ét}}$ des motifs étales qui est l'exact analogue ∞ -catégorique de la construction donnée plus haut de la catégorie triangulée des motifs étales. On rappelle de plus que c'est une catégorie motivique orientée qui vérifie la pureté absolue. De plus, on peut construire une sous-catégorie de motifs *constructibles* ou *géométriques* qui sont stables par les six opérations de Grothendieck et vérifient la dualité de Verdier. On termine le chapitre en revisitant la catégorie dérivée \mathcal{D}_c^b des faisceaux ℓ -adique de Bhatt-Scholze que l'on compare aux *motifs étales ℓ -complets* au moyen du théorème de rigidité de Suslin et Voevodsky. Cette comparaison permet de définir le foncteur de réalisation ℓ -adique.

Motifs d'Artin et complexes ℓ -adiques d'Artin

Si \mathcal{D}' est une sous- ∞ -catégorie pleine d'une ∞ -catégorie stable \mathcal{D} , on dit qu'elle est *épaisse* (resp. *localisante*) si elle est stable par ∞ -colimites finies, par ∞ -limites finies et par rétractes (resp. par ∞ -colimites et par ∞ -limites finies).

Définition. (3.1.2.2 (1) et (2)) Soit S un schéma noethérien de dimension finie et R un anneau commutatif.

1. La catégorie des motifs d'Artin (ou 0-motifs) notée $\mathcal{DM}_{\text{ét}}^A(S, R)$ est la plus petite sous- ∞ -catégorie localisante (au sens de la Définition 2.1.1.3) de $\mathcal{DM}_{\text{ét}}(S, R)$ contenant les motifs de la forme $f_*\mathbb{1}_X$ pour $f: X \rightarrow S$ fini.
2. La catégorie des motifs d'Artin constructibles notée $\mathcal{DM}_{\text{ét},c}^A(S, R)$ est la plus petite sous- ∞ -catégorie épaisse (au sens de la Définition 2.1.1.3) de $\mathcal{DM}_{\text{ét}}(S, R)$ contenant les mêmes générateurs.

Soit S un schéma noethérien de dimension finie, on s'intéresse dans cette thèse aux deux questions suivantes.

1. Existe-t-il une t-structure sur l' ∞ -catégorie stable $\mathcal{DM}_{\text{ét}}^A(S, \mathbb{Z})$ qui induise une t-structure non dégénérée sur la sous- ∞ -catégorie stable $\mathcal{DM}_{\text{ét},c}^A(S, \mathbb{Z})$ telle que pour tout nombre premier ℓ , le foncteur de réalisation ℓ -adique

$$\rho_\ell: \mathcal{DM}_{\text{ét},c}^A(S, \mathbb{Z}) \rightarrow \mathcal{D}_c^b(S[1/\ell], \mathbb{Z}_\ell)$$

est t-exact lorsqu'on munit l' ∞ -catégorie stable $\mathcal{D}_c^b(S[1/\ell], \mathbb{Z}_\ell)$ de sa t-structure **ordinaire** ?

2. Soit $R = \mathbb{Z}$ ou \mathbb{Q} . Existe-t-il une t-structure sur l' ∞ -catégorie stable $\mathcal{DM}_{\text{ét}}^A(S, R)$ qui induise une t-structure non dégénérée sur la sous- ∞ -catégorie stable $\mathcal{DM}_{\text{ét},c}^A(S, R)$ telle que pour tout nombre premier ℓ , le foncteur de réalisation ℓ -adique

$$\rho_\ell: \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}_c^b(S[1/\ell], R_\ell)$$

est t-exact lorsqu'on munit l' ∞ -catégorie stable $\mathcal{D}_c^b(S[1/\ell], R_\ell)$ de sa t-structure **perverse** ?

Dans le chapitre 2 on s'intéresse plus particulièrement à l'analogie ℓ -adique de ces questions. On définit en [2.2.1.3](#)(1) l' ∞ -catégorie des *complexes ℓ -adiques d'Artin* notée $\mathcal{D}^A(S, R)$ comme la plus petite sous- ∞ -catégorie épaisse de $\mathcal{D}_c^b(S, R)$ contenant les complexes de la forme $f_*\mathbb{Z}_{\ell,X}$ pour $f: X \rightarrow S$ fini. On se demande alors si les t-structures perverse et ordinaire de $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ induisent des t-structures sur la sous-catégorie formée des complexes d'Artin. L'avantage de ce cadre est qu'on dispose bien de t-structures, contrairement au cadre motivique où l'existence de celles-ci est conjecturale. Dans le chapitre 3, on résout les deux questions ci-dessus.

Identification des objets lisses

La première étape de notre étude dans le cadre ℓ -adique et dans le cadre motivique est l'identification de sous-catégories d'objets *lisses* qui jouent le rôle que jouent les faisceaux localement constants dans la théorie usuelle des complexes ℓ -adiques. On étend en particulier les résultats de [\[Org04\]](#) sur les motifs d'Artin en comparant les motifs d'Artin lisses avec les représentations d'Artin du groupe fondamental étale sur des schémas réguliers. En [2.2.1.3](#)(2) et en [3.1.2.2](#) (3) et (4), on définit ainsi pour tout schéma S

- l' ∞ -catégorie des *complexes ℓ -adiques d'Artin lisses* notée $\mathcal{D}^{smA}(S, R)$ comme la plus petite sous- ∞ -catégorie épaisse de $\mathcal{D}_c^b(S, R)$ contenant les motifs de la forme $f_*\mathbb{Z}_{\ell,X}$ pour $f: X \rightarrow S$ fini étale.
- l' ∞ -catégorie des *motifs d'Artin lisses* (resp. *motifs d'Artin lisses constructibles*) notée $\mathcal{DM}_{\text{ét},c}^{smA}(S, R)$ comme la plus petite sous- ∞ -catégorie localisante (resp. épaisse) de $\mathcal{DM}_{\text{ét}}(S, R)$ contenant les motifs de la forme $f_*\mathbb{1}_X$ pour $f: X \rightarrow S$ fini étale.

Dans l'esprit de [\[BS15, 7\]](#), si ξ est un point géométrique d'un schéma connexe S , la catégorie abélienne des représentations continues du groupe fondamental proétale $\pi_1^{\text{proét}}(S, \xi)$ est équivalente à la catégorie abélienne des *faisceaux ℓ -adiques lisses*. D'autre part, si π est un groupe topologique, on introduit la catégorie abélienne des *représentations d'origine Artin de π* comme la catégorie des extensions itérée de représentations d'Artin de π dans la catégorie des représentations continues de π . On dit qu'un faisceau ℓ -adique lisse est *d'Artin lisse* s'il correspond à une représentation d'origine Artin de $\pi_1^{\text{proét}}(S, \xi)$. On peut alors décrire la catégorie des complexes ℓ -adiques d'Artin comme suit.

Proposition. (2.2.2.1) Soit S un schéma noethérien de dimension finie et soit ℓ un nombre premier inversible sur S . Alors, l' ∞ -catégorie $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$ est la catégorie des complexes ℓ -adiques bornés dont les faisceaux de cohomologie sont d'Artin lisses.

Dans le cas motivique, on peut identifier les objets d'Artin lisses avec les catégories suivantes (voir la Définition 3.1.5.8). Soit S un schéma noethérien de dimension finie et R un anneau commutatif. Notons $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$ l' ∞ -catégorie stable des hyperfaisceaux sur le petit site étale de S à valeurs dans la catégorie dérivée $\mathcal{D}(R)$ des R -modules qui est équivalente à la catégorie dérivée de la catégorie $\text{Sh}(S_{\text{ét}}, R)$ des faisceaux étales de R -modules.

- L' ∞ -catégorie des faisceaux étales lisses $\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)$ est définie comme la sous-catégorie des objets dualisable de l' ∞ -catégorie $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$.
- L' ∞ -catégorie des faisceaux étales Ind-lisses $\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R)$ est définie comme la plus petite sous-catégorie localisante de l' ∞ -catégorie stable $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$ contenant $\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)$.

Théorème. (3.2.1.1, 3.1.5.9 et 3.1.5.10) Soit R une localisation de l'anneau des entiers d'un corps de nombres et soit S un schéma régulier.

1. On suppose que les exposants caractéristiques résiduels de S sont inversibles dans R . Alors, on a des équivalences d' ∞ -catégories stables monoïdales

$$\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R) \longrightarrow \mathcal{DM}_{\text{ét}}^{smA}(S, R)$$

$$\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R) \longrightarrow \mathcal{DM}_{\text{ét}, c}^{smA}(S, R).$$

2. La t -structure ordinaire de l' ∞ -catégorie $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$ induit des t -structures sur les sous- ∞ -catégories $\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)$ et $\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R)$.
 - (a) Le cœur de la t -structure induite sur $\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)$ est la catégorie $\text{Loc}_S(R)$ des faisceaux localement constants de R -modules de présentation finie.
 - (b) Le cœur de la t -structure induite sur $\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R)$ est la catégorie $\text{Ind Loc}_S(R)$ des colimites filtrantes dans la catégorie $\text{Sh}(S_{\text{ét}}, R)$ de faisceaux localement constants de R -modules de présentation finie.
3. Si le schéma S est connexe, et si ξ est un point géométrique de S , le foncteur fibre associé à ξ induit une équivalence de catégories abéliennes monoïdales

$$\xi^* : \text{Loc}_S(R) \rightarrow \text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R)$$

où $\text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R)$ est la catégorie abélienne des représentations d'Artin du groupe fondamental étale de S de point base ξ .

La t -structure ordinaire

Pour répondre à la première des deux questions ci-dessus dans le cadre ℓ -adique, on introduit les *faisceaux ℓ -adiques d'Artin* en mimant la définition usuelle des faisceaux constructibles : un faisceau ℓ -adique est d'Artin s'il existe une stratification du schéma de base par des sous-schémas localement fermés telle que la restriction de ce faisceau à chaque strate soit d'Artin lisse.

Proposition. (2.2.4.3) Soit S un schéma noethérien de dimension finie et soit ℓ un nombre premier inversible sur S . Alors, l' ∞ -catégorie $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ est la catégorie des complexes ℓ -adiques dont les faisceaux de cohomologie ordinaires sont d'Artin. En particulier, la t -structure ordinaire induit une t -structure sur $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ dont le cœur est la catégorie abélienne des faisceaux ℓ -adiques d'Artin.

Dans le cadre motivique, on étudie la t -structure homotopique ordinaire définie dans [Pep19b] et on étend le résultat principal de [Pep19a, Vai19] au cas des coefficients entiers.

Théorème. (3.4.2.7) Soit S un schéma noethérien de dimension finie admettant une résolution des singularités par altérations et soit R une localisation de l'anneau des entiers d'un corps de nombres K .

1. La t -structure homotopique ordinaire induit une t -structure sur $\mathcal{DM}_{\text{ét},c}^A(S, R)$.
2. Soit v une valuation non-archimédienne sur K et soit ℓ un nombre premier tel que v étend la valuation ℓ -adique. Alors, le foncteur de réalisation v -adique réduit

$$\bar{\rho}_v: \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}_c^b(S[1/\ell], R_v)$$

défini en 3.2.2.1 est t -exact lorsque le membre de gauche est muni de la t -structure homotopique ordinaire et le membre de droite est muni de la t -structure ordinaire.

Faisceaux pervers d'Artin

Le résultat principal du deuxième chapitre est le suivant.

Théorème. (2.3.3.4 et 2.3.4.1) Soient S un schéma noethérien excellent de dimension finie et ℓ un nombre premier inversible sur S . On suppose que S est de dimension au plus 2 ou que S est de dimension 3 et que les corps résiduels des points fermés de S sont séparablement clos ou réel clos. Alors, la t -structure perverse sur l' ∞ -catégorie stable $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ induit une t -structure sur la sous- ∞ -catégorie $\mathcal{D}^A(S, \mathbb{Z}_\ell)$.

Ce résultat permet de définir la catégorie des faisceaux pervers d'Artin $\text{Perv}^A(S, \mathbb{Z}_\ell)$ comme le cœur de la t -structure induite sur l' ∞ -catégorie stable $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ pour tout schéma S vérifiant les hypothèses énoncées ci-dessus (voir la Définition 2.3.2.2).

Ce résultat semble être optimal : si k est un corps fini et si ℓ est un nombre premier distinct de la caractéristique de k , on montre (voir la Remarque 2.3.5.2) que la t -structure perverse de $\mathcal{D}_c^b(\mathbb{A}_k^3, \mathbb{Z}_\ell)$ n'induit pas de t -structure sur $\mathcal{D}^A(\mathbb{A}_k^3, \mathbb{Z}_\ell)$.

Propriété universelle des t -structures homotopiques perverses

Pour contourner ce résultat négatif, on introduit une autre notion dite *homotopique* de faisceaux pervers d'Artin qui s'inspire de l'approche de [Pep19b, Vai19, Ayo07, 2.2.4] et de [BD17]. Si S est un schéma noethérien de dimension finie, on définit plus précisément la t -structure homotopique perverse sur la catégorie $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ des complexes ℓ -adiques *Ind-Artin* (qui est l'analogie ℓ -adique de celle des motifs d'Artin non nécessairement constructibles) en précisant comme dans le cas de la t -structure homotopique ordinaire une famille de générateurs. Elle a les propriétés suivantes d'après 2.4.5.1.

- Quand la t-structure homotopique perverse de $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ induit une t-structure sur $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, la t-structure induite est finale parmi celles qui rendent le foncteur d'inclusion

$$\mathcal{D}^A(S, \mathbb{Z}_\ell) \rightarrow \mathcal{D}_c^b(S, \mathbb{Z}_\ell)$$

t-exact à droite lorsqu'on munit le membre de droite de la t-structure perverse.

- Quand la t-structure perverse induit une t-structure sur $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, la t-structure homotopique perverse de $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ induit la même t-structure sur $\mathcal{D}^A(S, \mathbb{Z}_\ell)$.

Proposition. (2.4.7.1) *Soient S un schéma de type fini sur \mathbb{F}_p et $\ell \neq p$ un nombre premier. Alors, la t-structure homotopique perverse induit une t-structure sur $\mathcal{D}^A(S, \mathbb{Q}_\ell)$.*

Dans le cadre motivique, on peut également définir la t-structure homotopique perverse de la même façon. On a alors d'après 3.5.2.1(3) une propriété d'universalité de la t-structure similaire à celle énoncée ci-dessus : si S est un schéma noethérien excellent de dimension finie et si R est une localisation de l'anneau des entiers d'un corps de nombres, alors il existe une t-structure sur $\mathcal{DM}_{\text{ét},c}^A(S, R)$ dont les objets t-négatifs sont les objets qui sont envoyés sur des objets t-négatifs par toutes les réalisations v -adiques réduites si et seulement si la t-structure homotopique perverse induit une t-structure sur $\mathcal{DM}_{\text{ét},c}^A(S, R)$. Dans ce cas, les deux t-structures coïncident et la t-structure homotopique perverse est finale parmi celles qui rendent tous les foncteurs de réalisation v -adique t-exacts à droite.

Pour démontrer ce résultat, on prouve que la t-structure homotopique perverse admet une description locale similaire à celle de la t-structure perverse de [BBDG18].

Enfin, on a un résultat d'induction pour la t-structure homotopique perverse à coefficients rationnels et un résultat de t-exactitude des foncteurs de réalisation.

Proposition. (3.5.3.1) *Soit S un schéma noethérien excellent de dimension finie et soit K un corps de nombre. Alors, la t-structure homotopique perverse induit une t-structure sur $\mathcal{DM}_{\text{ét},c}^A(S, R)$.*

Si de plus S est de type fini sur \mathbb{F}_p , et si v est une valuation sur K qui n'étend pas la valuation ℓ -adique, le foncteur de réalisation v -adique

$$\mathcal{DM}_{\text{ét},c}^A(S, K) \rightarrow \mathcal{D}^A(S, K_v)$$

est t-exact quand les deux membres sont munis de la t-structure homotopique perverse.

Faisceaux pervers homotopiques d'Artin

Les t-structures homotopiques perverses sont étroitement liées aux foncteurs de *troncation d'Artin*. Dans le cas ℓ -adique, on définit pour tout schéma S noethérien de dimension finie le foncteur ω^0 de troncation d'Artin comme l'adjoint à droite de l'inclusion

$$\iota: \mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell) \rightarrow \mathcal{D}_{\text{Ind}}^{\text{coh}}(S, \mathbb{Z}_\ell)$$

des complexes ℓ -adiques Ind-Artin définis en 2.2.1.3(3) dans les complexes ℓ -adiques Ind-cohomologiques définis en 2.4.1.1.

Dans le cas où S est de type fini sur le corps \mathbb{F}_p et si l'on se restreint à des coefficients dans \mathbb{Q}_ℓ , la théorie des poids de Deligne développée dans [Del80] permet de donner une description

plus agréable du foncteur ω^0 . En effet, en utilisant les foncteurs de troncation définis par S. Morel dans [Mor05], Nair et Vaish ont défini dans [NV15] le foncteur de *troncation sans poids*

$$\omega^0: \mathcal{D}_m^b(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_m^b(S, \mathbb{Q}_\ell)$$

(noté $w_{\leq \text{Id}}$ dans [NV15]) où $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)$ désigne l' ∞ -catégorie stable des complexes ℓ -adiques mixtes de [BBDG18, 5].

D'après [2.4.6.2], les deux foncteurs notés ω^0 ci-dessus induisent des foncteurs équivalents

$$\mathcal{D}^{\text{coh}}(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}^A(S, \mathbb{Q}_\ell).$$

En particulier, ces deux foncteurs envoient les complexes ℓ -adiques cohomologiques vers des complexes ℓ -adiques d'Artin.

Dans le but d'étudier la catégorie abélienne $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ des *faisceaux pervers homotopiques d'Artin* définie comme le cœur de la t-structure homotopique perverse sur l' ∞ -catégorie stable $\mathcal{D}^A(S, \mathbb{Q}_\ell)$, on aimerait prouver des propriétés de t-exactitude du foncteur ω^0 . Pour que de telles propriétés aient du sens il est *a priori* nécessaire de disposer d'une t-structure sur les catégories considérées. Or, la question de la restriction de la t-structure perverse aux complexes cohomologiques n'est pas résolue et semble très délicate. Pour contourner ce problème, on définit l' ∞ -catégorie stable $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$ des *complexes ℓ -adiques sans poids* comme l'image essentielle du foncteur de troncation ω^0 (voir la Définition [2.4.8.1]). L'intuition qui conduit à une telle définition est la suivante : « Les complexes d'Artin sont les complexes cohomologiques sans poids ». Cette ∞ -catégorie stable $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$ est munie d'une t-structure, la *t-structure perverse sans poids*, telle que l'inclusion de $\mathcal{D}^A(S, \mathbb{Q}_\ell)$ munie de sa t-structure homotopique perverse soit t-exacte (on renvoie à [2.4.8.8](3)). On peut alors énoncer et démontrer un résultat de t-exactitude.

Théorème. [2.4.8.10] *Soit S un schéma de type fini sur \mathbb{F}_p , soit $\ell \neq p$ un nombre premier et soit*

$$\iota: \mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_m^b(S, \mathbb{Q}_\ell)$$

le foncteur d'inclusion. Lorsque le membre de gauche est muni de la t-structure perverse sans poids et le membre de droite de la t-structure perverse, l'adjonction (ι, ω^0) est une t-adjonction et le foncteur ω^0 est t-exact.

La conséquence principale de ce résultat est que la catégorie abélienne $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ est similaire à la catégorie abélienne des faisceaux pervers et contient le complexe d'intersection sans poids de [NV15].

Proposition. [2.4.9.4, 2.4.9.7 et 2.4.9.8] *Soit S un schéma de type fini sur \mathbb{F}_p et soit $\ell \neq p$ un nombre premier.*

1. *Tous les objets de $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ sont de longueur finie et les objets simples sont de la forme $\omega^0 j_{!*}(L[\dim(V)])$ où $j: V \hookrightarrow S$ parcourt l'ensemble des sous-schémas lisses de S et L parcourt la classe des faisceaux ℓ -adiques d'Artin simples sur V .*
2. *Soit IC_S le complexe d'intersection de S . Le complexe d'intersection sans poids $EC_S = \omega^0 IC_S$ de [NV15] est un objet simple de la catégorie abélienne $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$.*
3. *Si S est une surface, alors le complexe EC_S est également un faisceau pervers mais n'est pas nécessairement simple dans la catégorie des faisceaux pervers.*

Propriétés de Lefschetz faibles et motifs d'Artin pervers à coefficients rationnels

Dans le but de transposer les résultats précédents au cadre motivique et d'étudier la t-structure homotopique perverse sur les motifs d'Artin, on démontre des propriétés de type Lefschetz affine pour la t-structure homotopique perverse.

Théorème. ([3.5.3.5](#) et [3.5.3.10](#)) Soit S un schéma excellent, soit $f: X \rightarrow S$ un morphisme quasi-fini et soit K un corps de nombres. On suppose que le schéma X est nil-régulier

Alors, le foncteur

$$\omega^0 f_*: \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(X, K) \rightarrow \mathcal{DM}_{\text{ét},c}^A(S, K)$$

est t-exact vis-à-vis de la t-structure homotopique perverse.

Si l'on est de plus dans l'un des cas suivants

(a) Le schéma S est de dimension au plus 2.

(b) Il existe un nombre premier p tel que le schéma S est de type fini sur \mathbb{F}_p .

et si le morphisme f est affine, alors le foncteur

$$f_!: \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(X, K) \rightarrow \mathcal{DM}_{\text{ét},c}^A(S, K)$$

est t-exact vis-à-vis de la t-structure homotopique perverse.

La t-exactitude du foncteur $f_!$ avec f quasi-fini et affine s'étend à la catégorie des motifs M tels que $M \otimes_{\mathbb{Z}} \mathbb{Q}$ est constructible; Ce résultat constitue un des éléments principaux qui permet de comprendre la t-structure homotopique perverse.

La catégorie abélienne $M_{\text{perv}}^A(S, K)$ des motifs d'Artin pervers est définie comme le cœur de la t-structure homotopique perverse sur l' ∞ -catégorie stable $\mathcal{DM}_{\text{ét},c}(S, K)$. Elle est également similaire à la catégorie des faisceaux pervers et contient le complexe d'intersection motivique sans poids d'Ayoub et Zucker défini dans [\[AZ12\]](#); cette dernière affirmation est une conséquence de la t-exactitude du foncteur $\omega^0 j_*$. On peut de plus définir un foncteur $j_{!*}^A$ d'extension intermédiaire d'Artin.

Proposition. (Propositions [3.6.2.3](#) and [3.6.2.6](#)) Soient S un schéma excellent et K un corps de nombres.

1. Les objets de la catégorie abélienne $M_{\text{perv}}^A(S, K)$ sont de longueur finie.
2. Si $j: V \hookrightarrow S$ est l'inclusion d'un sous-schéma régulier connexe et si L est un objet simple de $\text{Loc}_V(K)$, alors le motif d'Artin pervers $j_{!*}^A(L[\delta(V)])$ est simple. Tout motif d'Artin pervers simple peut être obtenu de cette façon.
3. Soit $d = \delta(S)$ et soit \mathbb{E}_S le motif d'Ayoub-Zucker. Alors, le motif $\mathbb{E}_S[d]$ est un motif d'Artin pervers simple sur S .

Motifs d'Artin pervers à coefficients entiers

Quand l'anneau R des coefficients n'est plus un corps de nombre, la situation est très différente. On a tout d'abord le résultat suivant pour les schémas de dimension au plus 2.

Théorème. (3.5.4.1, 3.6.3.3 et 3.5.4.4) *Soit S un schéma noethérien excellent de dimension au plus 2 et soit R une localisation de l'anneau des entiers d'un corps de nombre K .*

1. *La t -structure homotopique perverse induit une t -structure sur $\mathcal{DM}_{\text{ét},c}^A(S, R)$.*
2. *Soit v une valuation non-archimédienne sur K et soit ℓ le nombre premier tel que v étend la valuation ℓ -adique. Alors, le foncteur de réalisation v -adique réduit*

$$\bar{\rho}_v: \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}_c^b(S[1/\ell], R_v)$$

défini en 3.2.2.1 est t -exact lorsqu'on munit le membre de gauche de la t -structure homotopique perverse et le membre de droite de la t -structure perverse.

3. *Soit $f: T \rightarrow S$ un morphisme de schémas quasi-fini affine. Alors, le foncteur*

$$f_!: \mathcal{DM}_{\text{ét},c}^A(T, R) \rightarrow \mathcal{DM}_{\text{ét},c}^A(S, R)$$

est t -exact vis-à-vis de la t -structure homotopique perverse.

On a cependant un résultat négatif pour les schémas de dimension supérieure. Si k est un corps, la t -structure homotopique perverse sur l' ∞ -catégorie stable $\mathcal{DM}_{\text{ét}}^A(\mathbb{A}_k^4, \mathbb{Z})$ n'induit pas de t -structure sur $\mathcal{DM}_{\text{ét},c}^A(\mathbb{A}_k^4, \mathbb{Z})$ (on renvoie à 3.5.4.6). Le cas des schémas de dimension 3 reste ouvert.

Description explicite des cœurs homotopiques pervers dans le cas des schémas de dimension 1

On décrit enfin les catégories $\text{Perv}^A(C, \mathbb{Z}_\ell)$ et $M_{\text{perv}}^A(C, R)$ quand C est un schéma excellent de dimension 1. Soit Γ l'ensemble des points génériques de C et soit $\nu: \tilde{C} \rightarrow C$ la normalisation de C . On introduit dans ce cadre une application *résidu* ∂_y pour tout point fermé y de \tilde{C} dans 2.5.1.5. La catégorie abélienne $\text{Perv}^A(C, \mathbb{Z}_\ell)$ est équivalente à la catégorie $\text{P}^A(C, \mathbb{Z}_\ell)$ définie en 2.5.1.7 comme suit (on renvoie à 2.5.1.1 pour les notations).

- Un objet de $\text{P}^A(C, \mathbb{Z}_\ell)$ est un quadruplet $(U, (M_\eta)_{\eta \in \Gamma}, (M_x)_{x \in C \setminus U}, (f_x)_{x \in C \setminus U})$ où
 1. U est un ouvert nil-régulier de C tel que l'immersion $U \rightarrow C$ est affine,
 2. pour tout $\eta \in \Gamma$, M_η est une représentation ℓ -adique d'origine Artin de $\pi_1^{\text{ét}}(U_\eta, \bar{\eta})$,
 3. pour tout $x \in C \setminus U$, M_x est un complexe dans $\mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((BG_{k(y)})_{\text{proét}}, \mathbb{Z}_\ell))$ (voir 2.1.5.8) placé en degrés $[0, 1]$,
 4. pour tout $x \in C \setminus U$, f_x est un élément de

$$\text{Hom}_{\mathcal{D}(\text{Sh}((BG_{k(x)})_{\text{proét}}, \mathbb{Z}_\ell))} \left(M_x, \bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left[\phi_y^*(M_{\eta(y)}) \right] \right) \right)$$

tel que $H^0(f_x)$ est injectif.

- Les éléments de $\text{Hom}_{\text{PA}(C, \mathbb{Z}_\ell)}((U, (M_\eta), (M_x), (f_x)), (V, (N_\eta), (N_x), (g_x)))$ sont les couples de la forme $((\Phi_\eta)_{\eta \in \Gamma}, (\Phi_x)_{x \in (C \setminus U) \cap (C \setminus V)})$ où $\Phi_\eta: M_\eta \rightarrow N_\eta$ est une application entre représentations de G_{K_η} , où $\Phi_x: M_x \rightarrow N_x$ est une application dans $\mathcal{D}(\text{Sh}((BG_{k(x)})_{\text{proét}}, \mathbb{Z}_\ell))$ et où le diagramme

$$\begin{array}{ccc}
M_x & \xrightarrow{f_x} & \bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left[\phi_y^*(M_{\eta(y)}) \right] \right) \\
\downarrow \Phi_x & & \downarrow \bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left(\phi_y^*(\Phi_{\eta(y)}) \right) \right) \\
N_x & \xrightarrow{g_x} & \bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left[\phi_y^*(N_{\eta(y)}) \right] \right)
\end{array}$$

est commutatif.

Dans le cas motivique, la catégorie abélienne $\text{M}_{\text{perv}}^A(S, R)$ est équivalente à la catégorie $\text{N}(S, R)$ (voir [3.6.4.9](#)) définie de manière similaire. On peut de plus donner une description explicite du foncteur de réalisation v -adique dans ce cadre.

Chapitre 1

Motifs étales

Ce chapitre est constitué de rappels sur la théorie des motifs étales. Il a pour but d'introduire les notions fondamentales de cette théorie dont on se servira dans la suite de la thèse.

Dans la première section, on rappelle la *théorie des ∞ -catégories* sur laquelle repose la version moderne de la théorie des motifs et notamment les notions centrales de présentabilité et de stabilité ainsi que les fondements de l'algèbre supérieure et de la théorie des topos supérieurs.

Dans la deuxième section on expose la théorie des motifs étales en elle-même. On rappelle d'abord le formalisme des *catégories prémotiviques* et *motiviques*. Celles-ci fournissent un cadre général dans lequel le formalisme des six foncteurs de Grothendieck peut être construit. On applique ensuite ce cadre au cas des motifs étales puis au cas des catégories ℓ -adiques. Enfin, on relie les motifs étales aux complexes ℓ -adiques au moyen du *théorème de rigidité* de Suslin et Voevodsky qui permet de construire le *foncteur de réalisation ℓ -adique*.

1.1 Rappels catégoriques

Dans le reste du texte, on appellera simplement *catégorie* une $(\infty, 1)$ -catégorie au sens de [Lur09].

1.1.1 Fondations

Catégories

On fixe trois univers $\mathbb{U} \in \mathbb{V} \in \mathbb{W}$. On appellera *petit* (resp. *gros*, resp. *très gros*) tout objet \mathbb{U} -petit (resp. \mathbb{V} -petit, resp. \mathbb{W} -petit). On note Δ la 1-catégorie simpliciale dont les objets sont les ensembles $[n] = \{0, \dots, n\}$, où n parcourt l'ensemble des entiers naturels, et dont les morphismes sont donnés par les applications croissantes.

Définition 1.1.1.1. *Un petit (resp. gros resp. très gros) ensemble simplicial est un 1-foncteur de la catégorie simpliciale Δ vers la 1-catégorie des petits (resp. gros, resp. très gros) ensembles.*

On note \mathbf{Set}_Δ (resp. $\mathbf{Set}_\Delta^{big}$) la 1-catégorie des petits (resp. gros) ensembles simpliciaux.

On a un 1-foncteur d'inclusion pleinement fidèle

$$\mathbf{Set}_\Delta \rightarrow \mathbf{Set}_\Delta^{big}$$

et si n est un entier et si $0 \leq k \leq n$, on note Δ^n le n -simplexe standard et $\Lambda_k^n \subseteq \Delta^n$ le k -ème cornet. On rappelle qu'une fibration intérieure est un morphisme $f: X \rightarrow S$ d'ensembles

simpliciaux tel que pour tout entier n , tout entier $0 < k < n$ et tout diagramme commutatif d'ensembles simpliciaux donné par les flèches pleines ci-dessous, il existe une flèche indiquée en pointillés qui rende le diagramme

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow \text{---} & \downarrow f \\ \Delta^n & \longrightarrow & S \end{array}$$

commutatif.

On rappelle que si n est un entier positif, les n -simplexes d'un ensemble simplicial K sont les morphismes d'ensembles simpliciaux $\Delta^n \rightarrow K$ et s'identifie aux éléments de $C([n])$ par le lemme de Yoneda. On notera d_i (resp. s_j) les faces (resp. les dégénérescences) d'un ensemble simplicial donné [Lur09, A.2.7].

Définition 1.1.1.2. 1. Une petite (resp. grosse, resp. très grosse) catégorie est un petit (resp. gros, resp. très gros) ensemble simplicial \mathcal{C} tel que l'unique application $\mathcal{C} \rightarrow \Delta^0$ est une fibration intérieure.

2. Un foncteur $\mathcal{C} \rightarrow \mathcal{D}$ entre deux catégories est un morphisme d'ensembles simpliciaux de \mathcal{C} vers \mathcal{D} .
3. Un objet d'une catégorie \mathcal{C} est un 0-simplexe de \mathcal{C} .
4. Un morphisme entre deux objets x et y d'une catégorie \mathcal{C} est un 1-simplexe $f: \Delta^1 \rightarrow \mathcal{C}$ tel que $d_0(f) = y$ et $d_1(f) = x$.
5. Si x est un objet de \mathcal{C} , l'identité de x notée Id_x est le morphisme $s_0(x)$ entre x et x .
6. Une composition de deux morphismes f et g dans une catégorie \mathcal{C} est 2-simplexe $\sigma: \Delta^2 \rightarrow \mathcal{C}$ tel que $d_0(\sigma) = g$ et $d_2(\sigma) = f$. On notera $f \circ g$ le morphisme $d_1(\sigma)$ que l'on appellera aussi une composition de f et g .
7. Un morphisme $f: x \rightarrow y$ dans une catégorie \mathcal{C} est une équivalence s'il existe un morphisme $g: y \rightarrow x$ tel que $g \circ f = \text{Id}_x$ et $f \circ g = \text{Id}_y$.

Les morphismes $f \circ g$ ci-dessus ne sont pas *a priori* définis correctement. Cependant, le théorème de Joyal [Lur09, 2.3.2.2] assure que la composition de deux morphismes est unique à choix contractile près.

On notera dans le reste du texte

- \mathcal{S} la grosse catégorie des petits complexes de Kan [Lur09, 1.2.16.1].
- \mathcal{S}^{big} la très grosse catégorie des gros complexes de Kan [Lur09, 1.2.16.1].
- Cat la grosse catégorie des petites catégories [Lur09, 3.0.0.1].
- CAT la très grosse catégorie des grosses catégories [Lur09, 3.0.0.1].

Si \mathcal{C} est une catégorie et si K est un ensemble simplicial, on note $\text{Fun}(K, \mathcal{C})$ la catégorie des foncteurs de K vers \mathcal{C} .

- Ses n -simplexes sont les morphismes d'ensembles simpliciaux $\Delta^n \times K \rightarrow \mathcal{C}$.

- Si $f: [n] \rightarrow [m]$ est une application croissante, elle induit un morphisme $f_{\#}: \Delta^n \rightarrow \Delta^m$ et la précomposition par $f_{\#}$ détermine une application des m -simplexes $\text{Fun}(K, \mathcal{C})$ vers ses n -simplexes.

Si $K = \mathcal{C}'$ est une catégorie, les 1-simplexes de la catégorie $\text{Fun}(\mathcal{C}', \mathcal{C})$ sont appelés 2-morphismes ou transformations naturelles et sont donc les foncteurs

$$\Delta^1 \times \mathcal{C}' \rightarrow \mathcal{C}.$$

Une *équivalence naturelle* est alors une transformation naturelle qui est une équivalence dans la catégorie $\text{Fun}(\mathcal{C}', \mathcal{C})$.

Soient \mathcal{C} est une catégorie et x et y deux objets de \mathcal{C} . Si $i \in \{0, 1\}$, notons ev_i l'application $\text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$ donnée par la précomposition avec l'inclusion $\{i\} \rightarrow \Delta^1$. On note $\text{Map}_{\mathcal{C}}(x, y)$ l'espace des applications de x vers y défini comme la limite dans la 1-catégorie \mathbf{Set}_{Δ} du diagramme

$$\begin{array}{ccc} & & \Delta^0 \\ & & \downarrow (x,y) \\ \text{Fun}(\Delta^1, \mathcal{C}) & \xrightarrow{(\text{ev}_0, \text{ev}_1)} & \mathcal{C} \times \mathcal{C}. \end{array}$$

\mathcal{C} est un complexe de Kan.

Tout foncteur $F: \mathcal{C} \rightarrow \mathcal{D}$ induit pour toute paire d'objets (x, y) de \mathcal{C} une application

$$\text{Map}_{\mathcal{C}}(x, y) \rightarrow \text{Map}_{\mathcal{D}}(F(x), F(y)).$$

On dit qu'un foncteur est *pleinement fidèle* si cette application est une équivalence faible (c'est-à-dire qu'elle induit une équivalence sur les groupes d'homotopie supérieurs) dans \mathbf{Set}_{Δ} .

Définition 1.1.1.3. Soit \mathcal{C} une catégorie. Un objet E de \mathcal{C} est final (resp. initial) si pour tout objet X , l'espace $\text{Map}_{\mathcal{C}}(X, E)$ (resp. $\text{Map}_{\mathcal{C}}(E, X)$) est contractile.

Enfin,

- Si \mathcal{C} est une 1-catégorie, on identifie \mathcal{C} à son nerf $N(\mathcal{C})$ [Lur09, 1.1.2]. Les n -simplexes de $N(\mathcal{C})$ sont donnés par les suites de n morphismes composables.
- Si \mathcal{C} est une catégorie, on note $\text{ho}(\mathcal{C})$ (le nerf de) sa catégorie homotopique. Cette dernière est la 1-catégorie dont les objets sont les objets de \mathcal{C} et dont les morphismes sont définis comme suit : si x et y sont des objets de \mathcal{C} , on définit

$$\text{Hom}_{\text{ho}(\mathcal{C})}(x, y) = \{f: \Delta^1 \rightarrow \mathcal{C} \mid d_1(f) = x \text{ et } d_0(f) = y\} / \sim$$

où \sim est la relation d'équivalence donnée sur l'ensemble des morphismes de x vers y par $f \sim g$ s'il existe un 2-simplexe $\sigma: \Delta^2 \rightarrow \mathcal{C}$ tel que $d_1(\sigma) = f$, $d_2(\sigma) = g$ et $d_0(\sigma) = \text{Id}_y$.

On a alors un foncteur

$$\mathcal{C} \rightarrow \text{ho}(\mathcal{C}).$$

- Si $F: \mathcal{C} \rightarrow \mathcal{D}$ est un foncteur, on dispose d'un 1-foncteur $\text{ho}(F): \text{ho}(\mathcal{C}) \rightarrow \text{ho}(\mathcal{D})$ induit par F . En effet, si x est un objet de \mathcal{C} , on pose $\text{ho}(F)(x) = F(x)$ et l'application

$$\{f: \Delta^1 \rightarrow \mathcal{C} \mid d_1(f) = x \text{ et } d_0(f) = y\} \rightarrow \{g: \Delta^1 \rightarrow \mathcal{D} \mid d_1(g) = F(x) \text{ et } d_0(g) = F(y)\}$$

induite par F passe au quotient pour la relation \sim .

- Si $\eta: F \rightarrow G$ est une transformation naturelle, alors, on a une transformation naturelle $\text{ho}(\eta): \text{ho}(F) \rightarrow \text{ho}(G)$. En effet, la transformation η est un foncteur $\Delta^1 \times \mathcal{C} \rightarrow \mathcal{D}$. Si x est un objet de \mathcal{C} , on définit alors $\text{ho}(\eta)(x): F(x) \rightarrow G(x)$ comme la classe dans $\text{Hom}_{\text{ho}(\mathcal{D})}(F(x), G(x))$ du morphisme obtenu par composition

$$\Delta^1 \times \{x\} \rightarrow \Delta^1 \times \mathcal{C} \rightarrow \mathcal{D}.$$

Un foncteur F est *essentiellement surjectif* si le foncteur $\text{ho}(F)$ induit entre les 1-catégories homotopiques est essentiellement surjectif. Une *équivalence de catégories* est alors un foncteur essentiellement surjectif et pleinement fidèle.

Dans le cas de catégories admettant des limites finies, on peut relier la notion d'équivalence à la notion usuelle entre 1-catégories.

Proposition 1.1.1.4. ([Cis19, 7.6.10]) *Soit $F: \mathcal{C} \rightarrow \mathcal{D}$ un foncteur compatible aux limites finies entre catégories ayant toutes les limites finies. Alors, le foncteur F est une équivalence de catégories si et seulement si le foncteur $\text{ho}(F)$ est une équivalence de catégories.*

On peut par ailleurs détecter les équivalences naturelles sur les catégories homotopiques : le résultat suivant est une conséquence de [Cis19, 3.5.12].

Proposition 1.1.1.5. *Soit $\eta: F \rightarrow G$ une transformation naturelle de foncteurs. Alors, la transformation η est une équivalence naturelle si et seulement si $\text{ho}(\eta)$ est une équivalence naturelle.*

Définition 1.1.1.6. ([Lur17, 1.2.11]) *Soit \mathcal{C} une grosse catégorie, soit \mathcal{E} un ensemble d'objets de \mathcal{C} et soit \mathcal{F} un ensemble de 1-morphismes entre les éléments de \mathcal{E} qui est stable par composition et contient les applications identités. Notons \mathcal{D} la 1-sous-catégorie de $\text{ho}(\mathcal{C})$ avec pour objets les éléments de \mathcal{E} et pour flèches les éléments de \mathcal{F} . Alors, la sous-catégorie de \mathcal{C} formée des objets contenus dans \mathcal{E} avec pour flèches les éléments de \mathcal{F} est la limite dans CAT du diagramme*

$$\begin{array}{ccc} & \mathcal{C} & \\ & \downarrow & \\ \mathcal{D} & \longrightarrow & \text{ho}(\mathcal{C}). \end{array}$$

Si \mathcal{F} est l'ensemble des flèches entre les objets de \mathcal{E} , on appelle cette catégorie la sous-catégorie pleine de \mathcal{C} formée des objets de \mathcal{E} .

Une sous-catégorie \mathcal{D} d'une catégorie \mathcal{C} est pleine si et seulement si le foncteur $\mathcal{D} \rightarrow \mathcal{C}$ est pleinement fidèle.

Limites et colimites

Si K et L sont des ensembles simpliciaux, on note $K * L$ la jointure de K et L [Lur09, 1.2.8.1]. On note K^\triangleright l'ensemble simplicial $K * \Delta^0$ et K^\triangleleft l'ensemble simplicial $\Delta^0 * K$. De manière informelle, l'ensemble simplicial K^\triangleright (resp. K^\triangleleft) peut être vu comme K auquel on a ajouté un objet final (resp. initial).

Définition 1.1.1.7. *Soit \mathcal{C} une catégorie et K un ensemble simplicial. Un diagramme à valeurs dans \mathcal{C} indexé par K est une application d'ensembles simpliciaux $K \rightarrow \mathcal{C}$.*

Si $p: K \rightarrow \mathcal{C}$ est un diagramme, on note $\mathcal{C}_{/p}$ (resp. $\mathcal{C}_{p/}$) la *catégorie des cones* (resp. *cocones*) sur p [Lur09, 1.2.9.2]. Les objets de la catégorie $\mathcal{C}_{/p}$ (resp. $\mathcal{C}_{p/}$) sont donnés par les morphismes $\bar{p}: K^\triangleleft \rightarrow \mathcal{C}$ (resp. $\bar{p}: K^\triangleright \rightarrow \mathcal{C}$) tels que $\bar{p}|_K = p$.

Définition 1.1.1.8. Soit $p: K \rightarrow \mathcal{C}$ un diagramme, une limite (resp. colimite) de p est un objet final (resp. initial) de la catégorie $\mathcal{C}_{/p}$ (resp. $\mathcal{C}_{p/}$). Si \bar{p} est une limite (resp. colimite) de K , on appelle aussi limite (resp. colimite) de K l'objet $\bar{p}|_{\Delta^0}$

Tout objet $\bar{p}: K^\triangleright \rightarrow \mathcal{C}$ de $\mathcal{C}_{p/}$ induit pour tout objet Y de \mathcal{C} un morphisme

$$\operatorname{colim}_{k \in K} \operatorname{Map}_{\mathcal{C}}(p(k), Y) \rightarrow \operatorname{Map}_{\mathcal{C}}(\bar{p}|_{\Delta^0}, Y).$$

Le diagramme \bar{p} est une colimite du diagramme $p: K \rightarrow \mathcal{C}$ si et seulement si le morphisme ci-dessus est une équivalence. On a une description similaire pour les limites.

Soit \mathcal{K} un ensemble de diagrammes d'une catégorie \mathcal{C} qui admettent des colimites. Un foncteur $F: \mathcal{C} \rightarrow \mathcal{D}$ est dit *compatible aux colimites de \mathcal{K}* si pour tout diagramme K dans \mathcal{K} , le diagramme $F \circ K$ admet une colimite et le morphisme naturel

$$\operatorname{colim}_{k \in K} F(k) \rightarrow F(\operatorname{colim}_{k \in K} k)$$

est une équivalence. On a une notion duale de compatibilité aux limites.

Définition 1.1.1.9. Soit S un petit ensemble simplicial et κ un petit cardinal régulier. On dit que S est κ -filtrant s'il existe une petite catégorie \mathcal{C} et une équivalence de Joyal $\square \mathcal{C} \rightarrow S$ dans \mathbf{Set}_{Δ} telle que tout diagramme $K \rightarrow \mathcal{C}$ dont la source K est un ensemble simplicial κ -petit admet un cocone $K^\triangleright \rightarrow \mathcal{C}$.

On appelle colimites κ -filtrantes les colimites indexées par des ensembles simpliciaux κ -filtrants. Si ω est le plus petit cardinal infini, on appelle filtrants les ensembles ω -filtrants et filtrantes les colimites ω -filtrantes.

Si $\kappa \leq \kappa'$ sont deux petits cardinaux réguliers et si S est κ' -filtrant alors il est κ -filtrant.

Si \mathcal{C} et \mathcal{D} sont des catégories qui admettent les limites ou colimites idoines, on note

- $\operatorname{LFun}(\mathcal{C}, \mathcal{D})$ (resp. $\operatorname{RFun}(\mathcal{C}, \mathcal{D})$) la sous-catégorie pleine de $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ formée des foncteurs compatibles aux colimites (resp. aux limites).
- $\operatorname{Fun}_{\kappa}(\mathcal{C}, \mathcal{D})$ la sous-catégorie pleine de $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ formée des foncteurs compatibles aux colimites κ -filtrantes pour un cardinal régulier κ fixé.

Construction de Grothendieck et adjonctions

Définition 1.1.1.10. Soit $p: \mathcal{C} \rightarrow \mathcal{D}$ un foncteur.

1. Soit $f: x \rightarrow y$ un morphisme dans \mathcal{C} .

¹au sens où le foncteur naturel $\operatorname{Fun}(S, \mathcal{D}) \rightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ induit une équivalence au niveau des classes d'isomorphismes.

(a) On dit que f est cartésien si le carré

$$\begin{array}{ccc} \mathcal{C}/f & \longrightarrow & \mathcal{C}/y \\ \downarrow & & \downarrow \\ \mathcal{D}/p(f) & \longrightarrow & \mathcal{D}/p(y) \end{array}$$

est cartésien.

(b) On dit que f est cocartésien si le carré

$$\begin{array}{ccc} \mathcal{C}/f & \longrightarrow & \mathcal{C}/x \\ \downarrow & & \downarrow \\ \mathcal{D}/p(f) & \longrightarrow & \mathcal{D}/p(x) \end{array}$$

est cartésien.

2. On dit que p est une fibration cartésienne (resp. cocartésienne) si p est une fibration intérieure et pour tout morphisme $f: x \rightarrow y$ dans \mathcal{D} et tout objet \tilde{y} (resp. \tilde{x}) de \mathcal{C} tel que $p(\tilde{y}) = y$ (resp. $p(\tilde{x}) = x$), il existe un morphisme cartésien (resp. cocartésien) $\tilde{f}: \tilde{x} \rightarrow \tilde{y}$ tel que $p(\tilde{f}) = f$.

On rappelle l'existence d'une fibration (co)cartésienne universelle $\mathcal{Z} \rightarrow \text{CAT}^{\text{op}}$ (resp. $\mathcal{Z}' \rightarrow \text{CAT}$) [Lur09, 3.3.2].

Toute fibration cartésienne $p: \mathcal{C}' \rightarrow \mathcal{C}$ entre grosses catégories admet d'après [Lur09, 3.3.2.3] une *application classifiante* au sens où il existe un foncteur $f: \mathcal{C} \rightarrow \text{CAT}^{\text{op}}$ muni d'une équivalence de fibrations cartésiennes $\mathcal{C}' \rightarrow \mathcal{Z} \times_{\text{CAT}^{\text{op}}} \mathcal{C}$ au-dessus de \mathcal{C} . On a un énoncé similaire pour les fibrations cocartésiennes. Ainsi, les fibrations (co)cartésiennes sont un moyen efficace d'encoder les foncteurs depuis une catégorie vers CAT ou CAT^{op} .

A titre d'exemple, on définit les adjonctions comme suit.

Définition 1.1.1.11. 1. Une adjonction est une application entre ensembles simpliciaux

$$\mathcal{M} \rightarrow \Delta^1$$

qui est une fibration cartésienne et cocartésienne.

2. Soient \mathcal{C} et \mathcal{D} deux catégories et $F: \mathcal{C} \rightarrow \mathcal{D}$ et $G: \mathcal{D} \rightarrow \mathcal{C}$ deux foncteurs. On dit que F est l'adjoint à gauche de G s'il existe une adjonction

$$\Phi: \mathcal{M} \rightarrow \Delta^1$$

telle que l'application $\Delta^1 \rightarrow \text{CAT}$ associée à Φ vue comme fibration cocartésienne est F et l'application $\Delta^1 \rightarrow \text{CAT}^{\text{op}}$ associée à Φ vue comme fibration cartésienne est G .

Tout foncteur adjoint à gauche est compatible aux colimites, tandis que tout foncteur adjoint à droite est compatible aux limites par [Lur09, 5.2.3.5].

D'après [Lur17, 5.2.2.8], une paire de foncteurs

$$F: \mathcal{C} \rightleftarrows \mathcal{D}: G$$

est une paire de foncteurs adjoints si et seulement si les conditions équivalentes suivantes sont vérifiées.

- a. Il existe une transformation naturelle $\varepsilon: \text{Id}_{\mathcal{D}} \rightarrow G \circ F$ telle que pour tout objet x de \mathcal{C} et tout objet y de \mathcal{D} , le morphisme induit

$$\text{Map}_{\mathcal{C}}(F(x), y) \xrightarrow{G} \text{Map}_{\mathcal{D}}(GF(x), G(y)) \xrightarrow{\varepsilon^*} \text{Map}_{\mathcal{D}}(x, G(y))$$

est une équivalence faible.

- b. Il existe une transformation naturelle $\eta: F \circ G \rightarrow \text{Id}_{\mathcal{C}}$ telle que pour tout objet x de \mathcal{C} et tout objet y de \mathcal{D} , le morphisme induit

$$\text{Map}_{\mathcal{C}}(x, G(y)) \xrightarrow{F} \text{Map}_{\mathcal{D}}(F(x), FG(y)) \xrightarrow{\eta^*} \text{Map}_{\mathcal{D}}(F(x), y)$$

est une équivalence faible.

1.1.2 Présentabilité

Catégories présentables

Définition 1.1.2.1. Soit \mathcal{C} une grosse catégorie, X un objet de \mathcal{C} et κ un petit cardinal régulier. On dit que X est κ -compact si le foncteur

$$\text{Map}_{\mathcal{C}}(X, -): \mathcal{C} \rightarrow \mathcal{S}^{\text{big}}$$

commute aux colimites κ -filtrantes. On note \mathcal{C}^{κ} la sous-catégorie pleine de \mathcal{C} formée des objets κ -compacts. Si ω est le plus petit cardinal infini, on appelle compact les objets ω -compacts.

Si $\kappa \leq \kappa'$ sont deux petits cardinaux réguliers, tout objet κ -compact est κ' -compact.

Si \mathcal{C} est une petite catégorie, notons $\widehat{\mathcal{C}}$ la catégorie $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$ des préfaisceaux à valeurs dans \mathcal{S} . On rappelle l'existence du plongement de Yoneda

$$h: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$$

construit dans [Lur09, 5.1.3] et on rappelle le fait que celui-ci est pleinement fidèle par [Lur09, 5.1.3.1]. La catégorie $\widehat{\mathcal{C}}$ admet toutes les petites colimites et toutes les petites limites. De plus, si \mathcal{D} est une catégorie admettant toutes les petites colimites, le foncteur de Yoneda induit une équivalence

$$\text{LFun}(\widehat{\mathcal{C}}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$$

d'après [Lur09, 5.1.5.6].

Si κ est un petit cardinal régulier, Lurie a de plus construit dans [Lur09, 5.3.5] la complétion $\text{Ind}_{\kappa}(\mathcal{C})$ de \mathcal{C} par rapport aux colimites κ -filtrantes. Celle-ci est une sous-catégorie pleine de $\widehat{\mathcal{C}}$ et le foncteur de Yoneda induit un foncteur

$$\mathcal{C} \rightarrow \text{Ind}_{\kappa}(\mathcal{C})$$

par [Lur09, 5.3.5.2]. Celui-ci est encore pleinement fidèle, la catégorie $\text{Ind}_{\kappa}(\mathcal{C})$ admet toutes les colimites κ -filtrantes et si \mathcal{D} est une catégorie admettant toutes les colimites κ -filtrantes, le foncteur de Yoneda induit une équivalence

$$\text{Fun}_{\kappa}(\text{Ind}_{\kappa}(\mathcal{C}), \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$$

d'après [Lur09, 5.3.5.10].

Définition 1.1.2.2. 1. Soit \mathcal{C} une grosse catégorie et κ un petit cardinal régulier. On dit que la catégorie \mathcal{C} est κ -accessible s'il existe une petite catégorie \mathcal{C}^0 et une équivalence de catégories

$$\mathrm{Ind}_\kappa(\mathcal{C}^0) \rightarrow \mathcal{C}.$$

2. Soit \mathcal{C} une grosse catégorie. On dit que \mathcal{C} est accessible s'il existe un petit cardinal régulier κ tel que \mathcal{C} est κ -accessible.
3. Soient κ un petit cardinal régulier et $F: \mathcal{C} \rightarrow \mathcal{D}$ un foncteur entre catégories κ -accessibles. On dit que le foncteur F est κ -accessible s'il préserve les colimites κ -filtrantes et envoie les objets κ -compacts sur des objets κ -compacts.

[Lur09, 5.3.5.10] assure que la restriction d'un foncteur κ -accessible aux objets κ -compact détermine essentiellement ce foncteur.

Proposition 1.1.2.3. ([Lur09, 5.4.2.2]) Soit \mathcal{C} une grosse catégorie et κ un petit cardinal régulier. Alors, la catégorie \mathcal{C} est κ -accessible si et seulement si les trois conditions suivantes sont vérifiées.

1. La catégorie \mathcal{C} est localement petite (i.e. ses espaces de morphismes sont faiblement équivalents à de petits complexes de Kan).
2. La catégorie \mathcal{C} admet les colimites κ -filtrantes.
3. La sous-catégorie \mathcal{C}^κ des objets κ -compacts de \mathcal{C} est essentiellement petite (i.e. elle est équivalente à une petite catégorie) et engendre \mathcal{C} par colimites κ -filtrantes.

Définition 1.1.2.4. Une grosse catégorie \mathcal{C} est présentable si elle est accessible et admet toutes les petites colimites.

On a d'après [Lur09, 5.5.1.1] la caractérisation suivante des catégories présentables.

Proposition 1.1.2.5. Soit \mathcal{C} une grosse catégorie. Alors, la catégorie \mathcal{C} est présentable si et seulement si les deux conditions suivantes sont vérifiées.

1. La catégorie \mathcal{C} est localement petite.
2. Il existe un petit cardinal κ et un petit ensemble d'objets κ -compacts \mathcal{E} de \mathcal{C} tels que tout objet de \mathcal{C} est la colimite d'un petit diagramme à valeurs dans la sous-catégorie pleine formée des objets de \mathcal{E} .

Un des intérêts principaux des catégories présentables est le résultat suivant.

Proposition 1.1.2.6. (Théorème du foncteur adjoint, [Lur09, 5.5.2.9]). Soit $F: \mathcal{C} \rightarrow \mathcal{D}$ un foncteur entre catégories présentables. Alors,

1. Le foncteur F admet un adjoint à droite si et seulement s'il est accessible et préserve les petites colimites.
2. Le foncteur F admet un adjoint à gauche si et seulement s'il est accessible et préserve les petites limites.

On note alors Pr^L (resp. Pr^R) la sous-catégorie de CAT formée des catégories présentables avec comme morphismes les foncteurs accessibles qui préservent les petites colimites (resp. les petites limites). Le résultat ci-dessus implique que Pr^R est la catégorie opposée de Pr^L .

Localisations

Un autre intérêt des catégories présentables est qu'on peut y inverser certaines classes de morphismes de manière contrôlée. On rappelle la notion de classe fortement saturée de morphismes d'une catégorie [Lur09, 5.5.4.5].

Définition 1.1.2.7. *Soit \mathcal{C} une catégorie présentable et S un petit ensemble de morphismes de \mathcal{C} .*

1. *Un objet X de \mathcal{C} est S -local si pour tout morphisme $s: T' \rightarrow T$, l'application*

$$\mathrm{Map}_{\mathcal{C}}(T, X) \rightarrow \mathrm{Map}_{\mathcal{C}}(T', X)$$

est une équivalence.

2. *Soit W la plus petite classe fortement saturée de morphismes de \mathcal{C} contenant S . C'est aussi la classe des morphismes f tels que pour tout objet S -local X , l'application $\mathrm{Map}(f, X)$ est une équivalence.*

On appelle localisation de \mathcal{C} par rapport à W et on note $\mathcal{C}[W^{-1}]$ la sous-catégorie pleine de \mathcal{C} formée des objets S -locaux.

La définition ci-dessus de la localisation ne dépend pas de l'ensemble d'objets S choisi mais seulement de la classe W pourvu que cette dernière soit engendrée par un petit ensemble de morphismes. Le résultat suivant découle de [Lur09, 5.5.4.15, 5.5.4.20].

Proposition 1.1.2.8. *Soient \mathcal{C} une catégorie présentable, S un petit ensemble de morphismes de \mathcal{C} et W la plus petite classe fortement saturée de morphismes de \mathcal{C} contenant S .*

1. *La catégorie $\mathcal{C}[W^{-1}]$ est présentable.*
2. *Le foncteur d'inclusion $\mathcal{C}[W^{-1}] \rightarrow \mathcal{C}$ admet un adjoint à gauche noté L_S .*
3. *Les éléments de W sont envoyés sur des équivalences dans $\mathcal{C}[W^{-1}]$.*
4. *Si pour toute grosse catégorie \mathcal{D} , on note $\mathrm{LFun}_W(\mathcal{C}, \mathcal{D})$ la sous-catégorie pleine de la catégorie $\mathrm{LFun}(\mathcal{C}, \mathcal{D})$ formée des foncteurs qui envoient les flèches qui sont dans W sur des équivalences dans \mathcal{D} , alors le foncteur*

$$\mathrm{LFun}(\mathcal{C}[W^{-1}], \mathcal{D}) \rightarrow \mathrm{LFun}_W(\mathcal{C}, \mathcal{D})$$

est une équivalence.

Enfin, on déduit de [Lur09, 5.5.1.1] le résultat de structure suivant sur les catégories présentables.

Proposition 1.1.2.9. *1. Si \mathcal{C} est une petite catégorie, alors la grosse catégorie $\widehat{\mathcal{C}}$ est présentable.*

2. *Toute catégorie présentable est équivalente à la localisation d'une catégorie de préfaisceaux sur une petite catégorie par rapport à une classe fortement saturée de morphismes engendrée comme dans la Définition 1.1.2.7 par un petit ensemble.*

1.1.3 Stabilité

Catégories stables

Définition 1.1.3.1. Soit \mathcal{C} une catégorie. On dit que \mathcal{C} est stable si elle vérifie les trois conditions suivantes.

1. La catégorie \mathcal{C} admet un objet nul, c'est-à-dire à la fois initial et final noté 0 .
2. La catégorie \mathcal{C} admet toutes les limites finies et toutes les colimites finies.
3. Un carré dans \mathcal{C} est cartésien si et seulement s'il est cocartésien.

Définition 1.1.3.2. Soit \mathcal{C} une catégorie stable.

1. Si X est un objet de \mathcal{C} , on appelle suspension de X et on note $X[1]$ la colimite du diagramme

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \\ 0 & & \end{array}$$

et on appelle désuspension de X et on note $X[-1]$ la limite du diagramme

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ 0 & \longrightarrow & X. \end{array}$$

2. Une suite de flèches

$$X \rightarrow Y \rightarrow Z$$

de \mathcal{C} est un triangle exact si le carré

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

est cartésien.

3. Si \mathcal{D} est une catégorie stable, un foncteur $F: \mathcal{C} \rightarrow \mathcal{D}$ est exact s'il commute aux limites finies.

La suspension et la désuspension sont des équivalences $\mathcal{C} \rightarrow \mathcal{C}$ qui sont inverses l'une de l'autre. De plus tout foncteur exact commute également aux colimites finies et tout foncteur qui commute aux colimites finies est exact. On note

- \mathcal{Sp} la catégorie des spectres (sur la catégorie \mathcal{S} des petits complexes de Kan) [Lur17, 1.4.3.1].
- CAT^{Ex} la sous-catégorie de la catégorie CAT formée des grosses catégories stables avec comme morphismes les foncteurs exacts.

- Pr_{Stb}^L (resp. Pr_{Stb}^R) la sous-catégorie pleine de la catégorie Pr^L (resp. Pr^R) formée des catégories stables présentables (tout morphisme dans Pr^L ou Pr^R entre catégories stables est exact).

La catégorie homotopique d'une catégorie stable \mathcal{C} est naturellement munie d'une structure de catégorie triangulée, les triangles exacts dans $\text{ho}(\mathcal{C})$ étant donnés par l'image des triangles exacts dans \mathcal{C} .

t-structures

Définition 1.1.3.3. Soit \mathcal{D} une catégorie stable. Une *t-structure* sur \mathcal{D} est une paire $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ de sous-catégories pleines de \mathcal{D} telles que

- Pour tout objet M de $\mathcal{D}^{\leq 0}$ et tout objet N de $\mathcal{D}^{\geq 0}$, on a $\text{Hom}(M, N[-1]) = 0$.
- On a des inclusions $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}[-1]$ et $\mathcal{D}^{\geq 0}[-1] \subseteq \mathcal{D}^{\geq 0}$.
- Pour tout objet M de \mathcal{D} , il existe un triangle exact $M' \rightarrow M \rightarrow M''$ tel que M' est un objet de $\mathcal{D}^{\leq 0}$ et M'' est un objet $\mathcal{D}^{\geq 0}[-1]$.

Si n est un entier, on pose $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ et $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$. On appelle *t-catégorie* une catégorie stable munie d'une *t-structure*.

Les conventions ci-dessus sont les conventions *cohomologiques*. Elles sont adoptées dans [BBDG18]. En revanche, les conventions adoptées dans [Lur17] sont les conventions *homologiques*. Si on veut transposer les énoncés de [Lur17] avec les conventions cohomologiques, il faut échanger n et $-n$ et inverser le sens des inégalités.

La catégorie $\mathcal{D}^{\leq 0}$ détermine la catégorie $\mathcal{D}^{\geq 0}$ et réciproquement. En effet, $\mathcal{D}^{\geq 0}$ est la catégorie pleine formée des objets Y tels que $\text{Hom}(X, Y) = 0$ pour tout objet X de $\mathcal{D}^{\leq -1}$.

D'après [Lur17, 1.2.1.6], pour tout entier n , la catégorie $\mathcal{D}^{\leq n}$ est une sous-catégorie de \mathcal{D} stable par colimites et la catégorie $\mathcal{D}^{\geq n}$ est une sous-catégorie de \mathcal{D} stable par limites. Par conséquent, l'inclusion $\mathcal{D}^{\leq n} \subseteq \mathcal{D}$ admet un adjoint à droite $\tau_{\leq n}$ et l'inclusion $\mathcal{D}^{\geq n} \subseteq \mathcal{D}$ admet un adjoint à gauche $\tau_{\geq n}$.

De plus, pour tout objet X de \mathcal{D} et tout entier n , on a d'après [Lur17, 1.2.1.8] un triangle exact

$$\tau_{\leq n} X \rightarrow X \rightarrow \tau_{\geq n+1} X$$

où les morphismes sont donnés par adjonction. Enfin, si m et n sont des entiers, la transformation naturelle

$$\tau_{\geq m} \circ \tau_{\leq n} \rightarrow \tau_{\leq n} \circ \tau_{\geq m},$$

obtenue par les unités et coïunités des adjonctions, est une équivalence de foncteurs $\mathcal{C} \rightarrow \mathcal{C}^{\geq m} \cap \mathcal{C}^{\leq n}$ par [Lur17, 1.2.1.10]. On note alors H^n le foncteur $\tau_{\leq n} \tau_{\geq n}(-)[n] \simeq \tau_{\geq n} \tau_{\leq n}(-)[n]$.

Définition 1.1.3.4. Soit \mathcal{D} une *t-catégorie*. Le cœur de \mathcal{D} est la catégorie $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. On le note \mathcal{D}^\heartsuit .

D'après [Lur17, 1.2.1.12], la catégorie \mathcal{D}^\heartsuit est une catégorie abélienne. De plus, le foncteur H^0 envoie les triangles exacts sur des suites exactes et on obtient de cette manière des suites exactes longues.

Définition 1.1.3.5. Soit $F: \mathcal{C} \rightarrow \mathcal{D}$ un foncteur exact entre t -catégories. On dit que

1. Le foncteur F est t -exact à droite (resp. à gauche) s'il envoie la catégorie $\mathcal{C}^{\leq 0}$ (resp. $\mathcal{C}^{\geq 0}$) dans la catégorie $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$).
2. Le foncteur F est t -exact s'il est t -exact à droite et à gauche.

Si $F: \mathcal{C} \rightarrow \mathcal{D}$ est un foncteur exact entre t -catégories, on note F^\heartsuit la composée

$$\mathcal{C}^\heartsuit \subseteq \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{H^0} \mathcal{D}^\heartsuit.$$

D'après [BBDG18, 1.3.17], on a alors le résultat suivant.

Proposition 1.1.3.6. 1. Soit $F: \mathcal{C} \rightarrow \mathcal{D}$ un foncteur t -exact à gauche (resp. à droite) entre t -catégories alors le foncteur F^\heartsuit est exact à gauche (resp. à droite).

2. Soit $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ une paire de foncteurs adjoints exacts entre t -catégories. Alors, le foncteur F est t -exact à droite si et seulement si le foncteur G est t -exact à gauche. Si c'est le cas, on dit que (F, G) est un t -adjonction et on a une adjonction

$$F^\heartsuit: \mathcal{C}^\heartsuit \rightleftarrows \mathcal{D}^\heartsuit: G^\heartsuit$$

telle que F^\heartsuit est exact à droite et G^\heartsuit est exact à gauche.

3. Soit $F: \mathcal{C} \rightarrow \mathcal{D}$ et $G: \mathcal{D} \rightarrow \mathcal{E}$ des foncteurs t -exact à gauche (resp. à droite) entre t -catégories. Alors, le foncteur $F \circ G$ est t -exact à gauche (resp. à droite) et on a $(F \circ G)^\heartsuit \simeq F^\heartsuit \circ G^\heartsuit$.

Enfin, la catégorie $\mathcal{S}\mathfrak{p}$ est munie d'une t -structure telle que $\mathcal{S}\mathfrak{p}^\heartsuit$ est équivalente à la catégorie \mathbf{Ab} des groupes abéliens d'après [Lur17, 1.4.3.6]. L'image d'un groupe abélien M par l'équivalence

$$H: \mathbf{Ab} \rightarrow \mathcal{S}\mathfrak{p}^\heartsuit$$

s'appelle le *spectre d'Eilenberg-MacLane* associé à M .

Catégories dérivées

On rappelle la définition suivante.

Définition 1.1.3.7. Soit A une catégorie abélienne. On dit que A est de Grothendieck si elle est présentable et si la collection de ses monomorphismes est stable par colimite filtrante.

Si A est une catégorie abélienne de Grothendieck, Lurie a construit dans [Lur17, 1.3.5] une catégorie stable $\mathcal{D}(A)$ appelée *catégorie dérivée de A* , obtenue à partir de la catégorie des complexes de chaînes sur A en inversant les quasi-isomorphismes, c'est-à-dire les applications f telles que pour tout entier n , le morphisme $H^n(f)$ soit un isomorphisme dans A . Par construction, la famille des foncteurs $(H^n)_{n \in \mathbb{Z}}$ est conservative.

De plus, si l'on note

$$\mathcal{D}(A)^{\leq n} = \{M \in \mathcal{D}(A) \mid \forall k > n, H^k(M) = 0\}$$

$$\mathcal{D}(A)^{\geq n} = \{M \in \mathcal{D}(A) \mid \forall k < n, H^k(M) = 0\},$$

alors, on a la proposition suivante d'après [Lur17, 1.3.5.15].

Proposition 1.1.3.8. 1. La catégorie $\mathcal{D}(A)$ est présentable.

2. La paire de sous-catégories pleines $(\mathcal{D}(A)^{\leq 0}, \mathcal{D}(A)^{\geq 0})$ est une t -structure sur $\mathcal{D}(A)$ et les foncteurs de cohomologie pour cette t -structure sont donnés par les foncteurs H^n pour tout entier n .

3. La catégorie $\mathcal{D}(A)^{\leq 0}$ est présentable et le foncteur $\tau_{\leq 0}$ est accessible.

4. Les foncteurs $\tau_{\leq n}$ et $\tau_{\geq n}$ sont compatibles aux colimites filtrantes pour tout entier n . En particulier, les foncteurs H^n sont aussi compatibles aux colimites filtrantes.

5. L'intersection

$$\bigcap_{n \in \mathbb{Z}} \mathcal{D}(A)^{\geq n}$$

est nulle.

Si A est la catégorie abélienne des petits R -modules, on notera $\mathcal{D}(R)$ la catégorie dérivée de A .

On peut aussi construire la catégorie dérivée bornée $\mathcal{D}^b(A)$ d'une catégorie abélienne A quelconque. Celle-ci est obtenue à partir de la catégorie des complexes de chaînes bornés sur A en inversant les quasi-isomorphismes. On renvoie à [BC20, 2.9] pour une construction précise. Dans le cas où A est de Grothendieck, la catégorie $\mathcal{D}^b(A)$ s'identifie à la sous-catégorie pleine de $\mathcal{D}(A)$ formée des complexes bornés.

Recollement

Définition 1.1.3.9. Soit \mathcal{X} une catégorie stable et soient $i_*: \mathcal{Z} \rightarrow \mathcal{X}$ et $j_*: \mathcal{U} \rightarrow \mathcal{X}$ deux foncteurs pleinement fidèles de sources stables. On dit que \mathcal{X} est un recollement de $(\mathcal{U}, \mathcal{Z})$ si les foncteurs i_* et j_* admettent des adjoints à gauche notés respectivement i^* et j^* tels que

1. On a $j^*i_* = 0$.
2. La famille (i^*, j^*) est conservative.

D'après [Lur17, A.8.7, A.8.11], on a le résultat suivant.

Proposition 1.1.3.10. 1. Soit $\Phi: (\Delta^1)^{\text{op}} \rightarrow \text{CAT}^{\text{Ex}}$ un diagramme, soit $F: \mathcal{M} \rightarrow \Delta^1$ la fibration cartésienne correspondante et \mathcal{X} la catégorie des sections de F . Notons \mathcal{U} la sous-catégorie de \mathcal{X} formée des sections cartésiennes de F et \mathcal{Z} la sous-catégorie de \mathcal{X} formée des sections σ de F telles que $\sigma(1) = 0$. Alors,

- (a) Les foncteurs d'évaluation $\text{ev}_0: \mathcal{Z} \rightarrow F^{-1}(0)$ et $\text{ev}_1: \mathcal{U} \rightarrow F^{-1}(1)$ sont des équivalences.
 - (b) Les foncteurs d'inclusion $\mathcal{Z} \rightarrow \mathcal{X}$ et $\mathcal{U} \rightarrow \mathcal{X}$ font de \mathcal{X} un recollement de $(\mathcal{U}, \mathcal{Z})$.
2. Soit \mathcal{X} une catégorie stable, soient $i_*: \mathcal{Z} \rightarrow \mathcal{X}$ et $j_*: \mathcal{U} \rightarrow \mathcal{X}$ deux foncteurs pleinement fidèles de sources stables qui font de \mathcal{X} un recollement de $(\mathcal{U}, \mathcal{Z})$ et soit $F: \mathcal{M} \rightarrow \Delta^1$ la fibration cartésienne associée au diagramme $(\Delta^1)^{\text{op}} \rightarrow \text{CAT}^{\text{Ex}}$ donné par le foncteur exact $i^*j_*: \mathcal{U} \rightarrow \mathcal{Z}$. Alors, la catégorie \mathcal{X} est équivalente à la catégorie des sections de F .

Remarque 1.1.3.11. Dans la situation du 1. de la proposition ci-dessus, on peut envisager la catégorie des sections de F comme la limite lax du foncteur Φ . Plus généralement, étant donné un diagramme $\Phi: I^{\text{op}} \rightarrow \text{CAT}^{\text{Ex}}$, on définit la limite lax de Φ comme la catégorie des sections de la fibration cartésienne correspondante. En utilisant le formalisme des $(\infty, 2)$ -catégories, on peut donner un sens à la notion de foncteur lax $I^{\text{op}} \rightarrow \text{CAT}^{\text{Ex}}$ puis à celle de limite lax d'un foncteur lax comme dans [AG18, 4.1]. Ceci permet par exemple de recoller des catégories le long d'une stratification d'un espace donné sans avoir à raisonner de manière inductive.

La proposition suivante est une conséquence de [Lur17, A.8.5, A.8.13].

Proposition 1.1.3.12. Soit \mathcal{X} une catégorie stable, soient $i_*: \mathcal{Z} \rightarrow \mathcal{X}$ et $j_*: \mathcal{U} \rightarrow \mathcal{X}$ deux foncteurs pleinement fidèles de sources stables qui font de \mathcal{X} un recollement de $(\mathcal{U}, \mathcal{Z})$. Notons i^* (resp. j^*) l'adjoint à gauche de i_* (resp. j_*).

1. Le foncteur i_* admet un adjoint à droite $i^!$.
2. Le foncteur j^* admet un adjoint à gauche $j_!$.
3. On a deux triangles exacts de transformations naturelles de la forme

$$\begin{aligned} j_!j^* &\rightarrow \text{Id} \rightarrow i_*i^* \\ i_*i^! &\rightarrow \text{Id} \rightarrow j_*j^* \end{aligned}$$

donnés par les unités et co-unités des adjonctions idoines.

4. On a $i^*j_! = 0$ et $i^!j_* = 0$.
5. Les foncteurs i_* , j_* et $j_!$ sont pleinement fidèles. Autrement dit, les morphismes d'adjonction $i^*i_* \rightarrow \text{Id} \rightarrow i^!i_*$ et $j^*j_* \rightarrow \text{Id} \rightarrow j^*j_!$ sont des équivalences.

Ainsi, lorsque \mathcal{X} est un recollement de $(\mathcal{U}, \mathcal{Z})$, la suite

$$\text{ho}(\mathcal{Z}) \xrightarrow{i_*} \text{ho}(\mathcal{X}) \xrightarrow{j_*} \text{ho}(\mathcal{U})$$

vérifie les hypothèses du formalisme du recollement de [BBDG18, 1.4.3]. Lorsque les catégories \mathcal{U} et \mathcal{Z} sont de plus munies de t-structures $(\mathcal{U}^{\leq 0}, \mathcal{U}^{\geq 0})$ et $(\mathcal{Z}^{\leq 0}, \mathcal{Z}^{\geq 0})$, on peut par conséquent définir une t-structure sur \mathcal{X} par recollement d'après [BBDG18, 1.4.10] en posant

$$\begin{aligned} \mathcal{X}^{\leq 0} &= \{M \in \mathcal{X} \mid j^*M \in \mathcal{U}^{\leq 0} \text{ et } i^*M \in \mathcal{Z}^{\leq 0}\} \\ \mathcal{X}^{\geq 0} &= \{M \in \mathcal{X} \mid j^*M \in \mathcal{U}^{\geq 0} \text{ et } i^!M \in \mathcal{Z}^{\geq 0}\}. \end{aligned}$$

D'après [BBDG18, 1.4.16], on a le résultat suivant.

Proposition 1.1.3.13. Soit \mathcal{X} une catégorie stable, soient $i_*: \mathcal{Z} \rightarrow \mathcal{X}$ et $j_*: \mathcal{U} \rightarrow \mathcal{X}$ deux foncteurs pleinement fidèles de sources stables qui font de \mathcal{X} un recollement de $(\mathcal{U}, \mathcal{Z})$. Alors, les adjonctions (i^*, i_*) , (j^*, j_*) , $(i_*, i^!)$ et $(j_!, j^*)$ sont des t-adjonctions. En particulier, les foncteurs j^* et i_* sont t-exacts.

1.1.4 Catégories monoïdales

Définition

On note Fin_* la 1-catégorie des entiers naturels vus comme ensembles finis pointés [Lur17, 2.0.0.2]. Ses objets sont les $\langle n \rangle = \{0, 1, \dots, n\}$ et les morphismes sont les applications qui envoient 0 sur 0. Si n est un entier et $1 \leq i \leq n$, on note $\rho_n^i: \langle n \rangle \rightarrow \langle 1 \rangle$ l'application qui envoie i sur 1 et tous les autres éléments de $\langle n \rangle$ sur 0.

Définition 1.1.4.1. Une catégorie monoïdale symétrique ou plus simplement catégorie monoïdale dans la suite de ce texte est une fibration cocartésienne

$$\mathcal{C}^\otimes \rightarrow \text{Fin}_*$$

telle que \mathcal{C}^\otimes est une grosse catégorie et telle que si $\underline{\mathcal{C}}: \text{Fin}_* \rightarrow \text{CAT}$ est le foncteur associé par la construction de Grothendieck, l'application

$$\prod_{i=1}^n \underline{\mathcal{C}}(\rho_n^i): \underline{\mathcal{C}}(\langle n \rangle) \rightarrow \underline{\mathcal{C}}(\langle 1 \rangle)^n$$

est une équivalence.

Si \mathcal{C}^\otimes est une catégorie monoïdale, on notera $\underline{\mathcal{C}}: \text{Fin}_* \rightarrow \text{CAT}$ le foncteur associé par la construction de Grothendieck et on notera \mathcal{C} la catégorie $\underline{\mathcal{C}}(\langle 1 \rangle)$. On dira que \mathcal{C}^\otimes est une *structure de catégorie monoïdale* sur \mathcal{C} . On voit en particulier que $\underline{\mathcal{C}}(\langle 0 \rangle)$ est contractile. On notera $\mathbb{1}$ l'image d'un objet de $\underline{\mathcal{C}}(\langle 0 \rangle)$ dans \mathcal{C} par le morphisme $\underline{\mathcal{C}}(\langle 0 \rangle) \rightarrow \underline{\mathcal{C}}(\langle 1 \rangle)$ induit par l'unique application $\langle 0 \rangle \rightarrow \langle 1 \rangle$.

Si (X_1, \dots, X_n) est un n -uplet d'objets de \mathcal{C} , on peut le voir comme un objet de $\underline{\mathcal{C}}(\langle n \rangle)$. On définit alors le produit tensoriel $X_1 \otimes \dots \otimes X_n$ comme étant l'image de cet objet par le morphisme $\underline{\mathcal{C}}(\langle n \rangle) \rightarrow \underline{\mathcal{C}}(\langle 1 \rangle)$ induit par l'application $\langle n \rangle \rightarrow \langle 1 \rangle$ qui envoie tous les entiers non nuls sur 1. Le caractère cocartésien de la fibration impose par exemple que l'on obtient un objet équivalent si l'on permute les X_i et fournit également des cohérences supérieures entre les équivalences. De plus, l'objet $\mathbb{1}$ peut être envisagé comme l'unité de ce produit tensoriel.

Un *foncteur monoïdal* $\mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ est la donnée d'une transformation naturelle $\underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$. La donnée d'un tel foncteur est équivalente à la donnée d'un foncteur $\mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ au-dessus de Fin_* qui envoie les morphismes cocartésiens sur des morphismes cocartésiens. On note $\text{Fun}^\otimes(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ la sous-catégorie pleine de $\text{Fun}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ formée des foncteurs monoïdaux.

Notons que la catégorie Fin_* est une catégorie monoïdale par le biais de l'application identité. La catégorie sous-jacente n'a qu'un seul objet. [Lur17, 2.2.1.2, 2.1.3.4, 2.2.1.1] donne les résultats suivants.

Proposition 1.1.4.2. Soit \mathcal{C}^\otimes une catégorie monoïdale.

1. Soit \mathcal{C}_0 une sous-catégorie pleine de la catégorie \mathcal{C} sous-jacente. On suppose que \mathcal{C}_0 est stable par équivalence et que le produit tensoriel

$$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

envoie $\mathcal{C}_0 \times \mathcal{C}_0$ dans \mathcal{C}_0 (on dit que \mathcal{C}_0 est une sous-catégorie monoïdale de \mathcal{C}). Alors, si on note \mathcal{C}_0^\otimes la sous-catégorie pleine de \mathcal{C}^\otimes formée des objets de la forme (X_1, \dots, X_n) avec les X_i dans \mathcal{C}_0 , la composition $\mathcal{C}_0^\otimes \subseteq \mathcal{C}^\otimes \rightarrow \text{Fin}_*$ fait de \mathcal{C}_0^\otimes une catégorie monoïdale.

2. Soit \mathcal{D} une catégorie, on peut munir $\text{Fun}(\mathcal{D}, \mathcal{C})$ d'une structure monoïdale en faisant le produit tensoriel coordonnées par coordonnées. Plus précisément, il existe une fibration cocartésienne $\text{Fun}(\mathcal{D}, \mathcal{C}^\otimes) \rightarrow \text{Fin}_*$ telle que pour tout objet D de \mathcal{D} , si on note ev_D le morphisme d'évaluation, on a un diagramme commutatif

$$\begin{array}{ccc} \text{Fun}(\mathcal{D}, \mathcal{C}^\otimes) & \xrightarrow{\text{ev}_D} & \mathcal{C}^\otimes \\ & \searrow & \swarrow \\ & \text{Fin}_* & \end{array} .$$

Définition 1.1.4.3. Soit \mathcal{C}^\otimes une catégorie monoïdale d'unité 1 et X un objet de \mathcal{C} . On dit que X est dualisable s'il existe un objet X' de \mathcal{C} et des morphismes

$$\alpha_X: 1 \rightarrow X' \otimes X \quad \text{et} \quad \beta_X: X \otimes X' \rightarrow 1$$

tels que les morphismes

$$X \simeq X \otimes 1 \xrightarrow{\text{Id}_X \otimes \alpha_X} X \otimes X' \otimes X \xrightarrow{\beta_X \otimes \text{Id}_X} 1 \otimes X$$

$$X' \simeq 1 \otimes X' \xrightarrow{\alpha_X \otimes \text{Id}_{X'}} X' \otimes X \otimes X' \xrightarrow{\text{Id}_{X'} \otimes \beta_X} X' \otimes 1 \simeq X'$$

sont homotopes aux applications identité dans \mathcal{C} . Ceci est équivalent à ce que pour toute paire d'objets (X, Y) de \mathcal{C} , l'application

$$\text{Map}_{\mathcal{C}}(X \otimes Y, Z) \xrightarrow{\alpha_{X'} \circ (X' \otimes -)} \text{Map}_{\mathcal{C}}(Y, X' \otimes Z)$$

induit une équivalence.

Catégories monoïdales présentables

On a également une notion de présentabilité pour les catégories monoïdales.

Définition 1.1.4.4. Une catégorie monoïdale présentable est une catégorie monoïdale \mathcal{C}^\otimes telle que la catégorie sous-jacente \mathcal{C} est présentable et telle que le produit tensoriel

$$\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

commute aux petites colimites en chaque variable.

D'après [Rob14, 3.6.1], on a le résultat suivant.

Proposition 1.1.4.5. Soit \mathcal{C}^\otimes une catégorie monoïdale présentable. Soit S un petit ensemble de morphismes de \mathcal{C} . Notons W la plus petite classe fortement saturée contenant S . On suppose que pour tout morphisme $f: X \rightarrow Y$ dans W , l'application $f \otimes \text{Id}: X \otimes Z \rightarrow Y \otimes Z$ est encore dans W pour tout objet Z de \mathcal{C} .

Alors, la sous-catégorie $\mathcal{C}[W^{-1}]$ est stable par le produit tensoriel, la catégorie monoïdale $\mathcal{C}[W^{-1}]$ est encore monoïdale présentable et le foncteur L_S est monoïdal.

Algèbres

Si $\langle n \rangle$ est un objet de Fin_* , on note $\langle n \rangle^o$ le sous-ensemble $\langle n \rangle \setminus \{0\}$ de $\langle n \rangle$. On dit qu'un morphisme $f: \langle n \rangle \rightarrow \langle m \rangle$ dans Fin_* est *inerte* si le morphisme

$$f^{-1}(\langle m \rangle^o) \rightarrow \langle m \rangle^o$$

induit par f est une bijection.

Définition 1.1.4.6. Soit \mathcal{C}^\otimes une catégorie monoïdale. Une algèbre commutative dans \mathcal{C}^\otimes est la donnée d'une section

$$s: \text{Fin}_* \rightarrow \mathcal{C}^\otimes$$

de la fibration cocartésienne

$$\mathcal{C}^\otimes \rightarrow \text{Fin}_*$$

qui envoie les morphismes inertes sur des flèches cocartésiennes.

Si s est une telle section, tous les $s(\langle n \rangle)$ sont de la forme (A, \dots, A) lorsqu'on les envisage comme des objet de \mathcal{C}^n . Le morphisme $\langle n \rangle \rightarrow \langle 1 \rangle$ fournit alors un morphisme $A \otimes A \rightarrow A$ et le morphisme $\langle 0 \rangle \rightarrow \langle 1 \rangle$ fournit un morphisme $\mathbb{1} \rightarrow A$. Ce sont ces applications qui font de A une algèbre commutative. Les autres applications encodent des cohérences supérieures.

Les algèbres commutatives d'une catégorie monoïdale forment une catégorie. Pour la définir, on définit d'abord la catégorie des foncteurs au-dessus d'une base donnée comme suit. Si $p: \mathcal{C} \rightarrow \mathcal{B}$ et $q: \mathcal{D} \rightarrow \mathcal{B}$ sont des foncteurs, on note $\text{Fun}_{\mathcal{B}}(\mathcal{C}, \mathcal{D})$ la catégorie des foncteurs de \mathcal{C} dans \mathcal{D} au-dessus de \mathcal{B} définie comme la limite dans la 1-catégorie \mathbf{Set}_Δ du diagramme

$$\begin{array}{ccc} & & \Delta^0 \\ & & \downarrow p \\ \text{Fun}(\mathcal{C}, \mathcal{D}) & \xrightarrow{q^o} & \text{Fun}(\mathcal{C}, \mathcal{B}). \end{array}$$

Si \mathcal{C}^\otimes est une catégorie monoïdale, la catégorie $\text{CAlg}(\mathcal{C})$ des algèbres commutatives dans \mathcal{C} est alors définie comme la sous-catégorie pleine de $\text{Fun}_{\text{Fin}_*}(\text{Fin}_*, \mathcal{C}^\otimes)$ formée des sections qui envoient les morphismes inertes sur des flèches cocartésiennes. Le foncteur

$$\text{CAlg}(\mathcal{C}) \rightarrow \mathcal{C}$$

donné par $F \mapsto F(\langle 1 \rangle)$ est alors fidèle.

Modules sur une algèbre

On définit maintenant un module sur une algèbre commutative. La construction que nous donnons est due à Iwasa et Annala [\[AI23, A\]](#).

On définit une 1-catégorie \mathbb{M} comme suit.

- Les objets de \mathbb{M} sont les paires (I, S) avec I un objet de Fin_* et S un sous-ensemble pointé de I .
- Si (I, S) et (J, T) sont des objets de \mathbb{M} , un morphisme de (I, S) vers (J, T) est un morphisme $f: I \rightarrow J$ dans Fin_* tel que $S \subseteq f^{-1}(T)$ et qui se restreint en une bijection $f^{-1}(T^o) \rightarrow T^o$.

La catégorie \mathbb{M} n'est pas monoïdale mais vérifie la propriété plus faible d'être une *opétrade* au sens de [Lur17, 2.1]. En particulier, elle est munie d'une fibration intérieure

$$\pi: \mathbb{M} \rightarrow \text{Fin}_*$$

donnée par l'oubli de la deuxième coordonnée. De plus, tout morphisme inerte $f: \langle n \rangle \rightarrow \langle m \rangle$ admet un relèvement cocartésien de source fixée $(\langle n \rangle, S)$ donné par f et dont le but est $(\langle n \rangle, f(S))$. On dit encore que ces morphismes sont *inertes*. Dans le cas des morphismes ρ_n^i , cela donne une bijection

$$\pi^{-1}(\langle n \rangle) \rightarrow \pi^{-1}(\langle 1 \rangle)^n.$$

Remarquons que la catégorie $\pi^{-1}(\langle 1 \rangle)$ a deux objets $\mathfrak{a} := (\langle 1 \rangle, \{0\})$ et $\mathfrak{m} := (\langle 1 \rangle, \{0, 1\})$. On peut alors penser à \mathfrak{a} comme à une algèbre commutative dans \mathbb{M} : on a une section

$$\mathfrak{a}: \text{Fin}_* \rightarrow \mathbb{M}$$

de π donnée par $\langle n \rangle \mapsto (\langle n \rangle, \{0\})$ qui vérifie des conditions similaires à celles d'être un foncteur monoïdal. On peut alors envisager \mathfrak{m} comme un module sur \mathfrak{a} .

Définition 1.1.4.7. Soit \mathcal{C}^\otimes une catégorie monoïdale.

1. Un module dans \mathcal{C} est la donnée d'un foncteur $m: \mathbb{M} \rightarrow \mathcal{C}^\otimes$ tel que le diagramme

$$\begin{array}{ccc} \mathbb{M} & \xrightarrow{m} & \mathcal{C}^\otimes \\ & \searrow \pi & \swarrow p \\ & \text{Fin}_* & \end{array}$$

est commutatif si p désigne le morphisme structural qui fait de \mathcal{C}^\otimes une catégorie monoïdale.

2. Soit

$$A: \text{Fin}_* \rightarrow \mathcal{C}^\otimes$$

une algèbre commutative. Un A -module est la donnée d'un module $m: \mathbb{M} \rightarrow \mathcal{C}^\otimes$ dans \mathcal{C} tel que le diagramme

$$\begin{array}{ccc} \text{Fin}_* & \xrightarrow{a} & \mathbb{M} \\ & \searrow A & \downarrow m \\ & & \mathcal{C}^\otimes \end{array}$$

est commutatif.

On note alors $\text{Mod}(\mathcal{C}^\otimes)$ la catégorie des modules dans \mathcal{C} et $\text{Mod}_A(\mathcal{C}^\otimes)$ la catégorie des A -modules (vues comme des sous-catégories pleines de $\text{Fun}_{\text{Fin}_*}(\mathbb{M}, \mathcal{C}^\otimes)$).

On a un foncteur

$$\mathfrak{A}: \text{Mod}(\mathcal{C}^\otimes) \rightarrow \text{CAlg}(\mathcal{C}^\otimes)$$

donné par la précomposition par a qui est une fibration cocartésienne d'après [Lur17, 4.5.3.1] et cartésienne d'après [Lur17, 3.4.3.4] et on a un foncteur

$$\mathfrak{M}: \text{Mod}(\mathcal{C}^\otimes) \rightarrow \mathcal{C}$$

donné par $F \mapsto F(\mathfrak{m})$. Si A une algèbre commutative dans \mathcal{C}^\otimes , lorsqu'on restreint le foncteur \mathfrak{M} à la sous-catégorie pleine $\text{Mod}_A(\mathcal{C}^\otimes)$ de $\text{Mod}(\mathcal{C}^\otimes)$, on obtient un foncteur fidèle $\text{Mod}_A(\mathcal{C}^\otimes) \rightarrow \mathcal{C}$.

Enfin, en utilisant le fait que \mathfrak{A} est une fibration cartésienne et cocartésienne, on obtient deux foncteurs

$$\text{Mod}_-(\mathcal{C}^\otimes): \text{CAlg}(\mathcal{C}^\otimes) \rightarrow \text{CAT}$$

et

$$\text{CAlg}(\mathcal{C}^\otimes) \rightarrow \text{CAT}^{\text{op}}$$

et par conséquent, on a pour tout morphisme d'algèbres $\sigma: A \rightarrow B$ une adjonction

$$\sigma^*: \text{Mod}_A(\mathcal{C}^\otimes) \rightleftarrows \text{Mod}_B(\mathcal{C}^\otimes): \sigma_*.$$

Le foncteur σ_* est par définition le foncteur d'oubli et le foncteur σ^* est d'après [Lur17, 4.6.2.17] le foncteur $- \otimes_A B$.

Structure monoïdale sur Pr^L

Il suit de [Lur17, 4.8.1.15] le résultat suivant.

Proposition 1.1.4.8. *Il existe une structure monoïdale sur Pr^L telle que le produit tensoriel $\mathcal{C} \otimes \mathcal{C}'$ de deux catégories présentables \mathcal{C} et \mathcal{C}' est muni d'une application $\mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{C} \otimes \mathcal{C}'$ telle que $\text{LFun}(\mathcal{C} \otimes \mathcal{C}', \mathcal{D})$ s'identifie au moyen de cette application à la sous-catégorie pleine de $\text{Fun}(\mathcal{C} \times \mathcal{C}', \mathcal{D})$ formée des foncteurs qui préservent les colimites séparément en chaque variable.*

De plus d'après [Lur17, 4.8.1.17], le produit tensoriel de deux catégories \mathcal{C} et \mathcal{D} est donné par la formule $\text{RFun}(\mathcal{C}^{\text{op}}, \mathcal{D})$. En utilisant [Lur09, 5.5.2.2], l'unité monoïdale de la catégorie Pr^L est donc la catégorie \mathcal{S} des espaces. Enfin, la catégorie $\text{CAlg}(\text{Pr}^L)$ s'identifie à la catégorie des catégories monoïdales présentables d'après [Lur17, 4.8.1.9].

On a également une version stable de ce résultat [Lur17, 4.8.2.18].

Proposition 1.1.4.9. *La structure monoïdale sur Pr^L induit une structure monoïdale sur la sous-catégorie pleine Pr_{Stb}^L dont l'unité pour le produit tensoriel est la catégorie \mathcal{Sp} des spectres.*

On en déduit notamment que \mathcal{Sp} est une catégorie monoïdale.

En utilisant la construction d'Eilenberg-MacLane, on obtient un foncteur

$$\text{CAlg}(\mathbf{Ab}) \xrightarrow{H} \text{CAlg}(\mathcal{Sp}) \xrightarrow{\text{Mod}_-(\mathcal{Sp})} \text{CAlg}(\text{Pr}_{\text{Stb}}^L).$$

Si R est un anneau commutatif, on note Mod_R la catégorie monoïdale des HR -modules dans \mathcal{Sp} , c'est-à-dire l'image de R par le foncteur ci-dessus. D'après le théorème de Schwede et Shipley [Lur17, 7.1.2.1], on a une équivalence entre Mod_R et la catégorie dérivée $\mathcal{D}(R)$. De plus, si $\sigma: R \rightarrow R'$ est un morphisme d'anneaux, alors on a une adjonction

$$\sigma^* = - \otimes_R R': \text{Mod}_R \rightleftarrows \text{Mod}_{R'}: \mathcal{O} = \sigma_*$$

où \mathcal{O} désigne le foncteur d'oubli.

Définition 1.1.4.10. *Soit R un anneau commutatif. On dit qu'une catégorie stable présentable est R -linéaire si c'est un Mod_R -module dans la catégorie Pr^L .*

On note $\text{CAlg}_R(\text{Pr}^L)$ la catégorie des catégories stables monoïdales R -linéaires (c'est par définition la catégorie des Mod_R -algèbres dans Pr_{Stb}^L).

Enrichissement des catégories stables et R -linéaires présentables

Soit \mathcal{C} une catégorie stable présentable R -linéaire. On dispose d'un foncteur

$$\mathrm{Mod}_R \otimes \mathcal{C} \rightarrow \mathcal{C}$$

qui définit l'action de Mod_R sur \mathcal{C} . On en déduit un foncteur

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \rightarrow \mathrm{RFun}(\mathrm{Mod}_R^{\mathrm{op}}, \mathcal{S})$$

donné par la formule $(x, y) \mapsto \mathrm{Map}_{\mathcal{C}}(- \otimes x, y)$. Comme d'après [Lur09, 5.5.2.2], on a

$$\mathrm{RFun}(\mathrm{Mod}_R^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Mod}_R,$$

on en déduit que toute catégorie stable présentable R -linéaire est enrichie sur Mod_R . De même, toute catégorie stable est enrichie sur $\mathcal{S}\mathcal{P}$. On notera encore $\mathrm{Map}_{\mathcal{C}}(x, y)$ le R -module des applications de x dans y lorsqu'il n'y a pas d'ambiguïté. Celui-ci est par construction envoyé sur le spectre des applications de x vers y par le foncteur d'oubli $\mathrm{Mod}_R \rightarrow \mathcal{S}\mathcal{P}$ et ce dernier est envoyé sur l'espace des applications de x vers y par le foncteur de désuspension infinie $\Omega^\infty : \mathcal{S}\mathcal{P} \rightarrow \mathcal{S}$.

Tenseur-inversion d'un élément dans une catégorie monoïdale

Définition 1.1.4.11. Soient \mathcal{C}^\otimes une catégorie monoïdale d'unité 1 et X un objet de \mathcal{C} . On dit que X est \otimes -inversible s'il existe un objet X^* de \mathcal{C} tel que $X \otimes X^*$ est équivalent à l'objet 1.

De manière équivalente, l'objet X est \otimes -inversible si le foncteur

$$X \otimes - : \mathcal{C} \rightarrow \mathcal{C}$$

est une équivalence de catégories.

Le résultat suivant est dû à Voevodsky et Robalo (voir [Voe98, 4.3]).

Proposition 1.1.4.12. ([Rob14, 4.2.5, 4.2.13]) Soient \mathcal{C}^\otimes une catégorie monoïdale présentable et X un objet de \mathcal{C} . Soit $\mathrm{Stab}_X(\mathcal{C})$ la colimite dans Pr^L de la suite

$$\dots \xrightarrow{X \otimes -} \mathcal{C} \xrightarrow{X \otimes -} \mathcal{C} \xrightarrow{X \otimes -} \mathcal{C} \xrightarrow{X \otimes -} \dots$$

On suppose que l'action de la permutation cyclique sur $X \otimes X \otimes X$ devient une application identité après tensorisation par X un nombre fini de fois.

1. La catégorie $\mathrm{Stab}_X(\mathcal{C})$ est admet une structure monoïdale canonique.
2. Le foncteur canonique $\Sigma_X^\infty : \mathcal{C} \rightarrow \mathrm{Stab}_X(\mathcal{C})$ est monoïdal, envoie X sur un objet \otimes -inversible et est universel pour cette propriété. Il est de plus compatible aux petites colimites.

1.1.5 Hyperfaisceaux

Faisceaux en espaces

Définition 1.1.5.1. Soit \mathcal{C} une petite catégorie.

1. Soit X un objet de \mathcal{C} . Un crible sur X est la donnée d'un sous-objet de $h(X)$ dans $\widehat{\mathcal{C}}$.
2. Une topologie de Grothendieck sur \mathcal{C} est la donnée pour tout objet X de \mathcal{C} d'un ensemble de crible de X appelés cribles couvrants tel que les conditions suivantes sont vérifiées.
 - (a) Pour tout objet X de \mathcal{C} , le crible $h(X)$ est couvrant.
 - (b) Pour tout objet X de \mathcal{C} , tout crible couvrant R et tout morphisme $f: Y \rightarrow X$, le crible $R \times_X Y$ sur Y est couvrant.
 - (c) Si X est un objet de \mathcal{C} , si R est un crible sur X et si R' est un crible couvrant tel que pour tout $h(Y) \rightarrow R'$ avec Y dans \mathcal{C} , le crible $R \times_X Y$ sur Y est couvrant, alors, le crible R est couvrant.

On appelle *site* la donnée d'une petite catégorie munie d'une topologie de Grothendieck.

Définition 1.1.5.2. Soit (\mathcal{C}, τ) un site. Notons S l'ensemble des cribles couvrants $R \rightarrow h(X)$ d'objets X de \mathcal{C} . Un τ -faisceau sur \mathcal{C} est un objet S -local de $\widehat{\mathcal{C}}$ au sens de la Définition [1.1.2.7](#). On note $\mathcal{S}\mathcal{H}_\tau^{\text{usuel}}(\mathcal{C})$ la catégorie des τ -faisceaux sur \mathcal{C} et $L_\tau^0: \widehat{\mathcal{C}} \rightarrow \mathcal{S}\mathcal{H}_\tau^{\text{usuel}}(\mathcal{C})$ le foncteur de S -localisation.

Les résultats précédents impliquent que $\mathcal{S}\mathcal{H}_\tau^{\text{usuel}}(\mathcal{C})$ est une catégorie monoïdale présentable.

Hyperfaisceaux en espaces

On note Δ_+ la catégorie des ensembles finis (éventuellement vides) totalement ordonnés. On a $\Delta_+ = \Delta^\triangleleft$. Notons $\iota: \Delta^1 \rightarrow \Delta_+$ l'inclusion donnée par l'application vide $\emptyset \rightarrow [0]$. Si n est un entier, on note $\Delta^{\leq n}$ la sous-catégorie pleine de Δ formée des objets de cardinal au plus $n + 1$. On rappelle que les foncteurs de restriction

$$\iota^*: \text{Fun}(\Delta_+^{\text{op}}, \mathcal{C}) \rightarrow \text{Fun}((\Delta^1)^{\text{op}}, \mathcal{C})$$

et

$$\text{sk}_n: \text{Fun}(\Delta^{\text{op}}, \mathcal{C}) \rightarrow \text{Fun}((\Delta^{\leq n})^{\text{op}}, \mathcal{C})$$

ont des adjoints à droite appelés *extension de Kan à droite le long de ι* et de *l'inclusion $(\Delta^{\leq n})^{\text{op}} \subseteq \Delta^{\text{op}}$* . Cette dernière application est aussi appelée *n -cosquelette*. Remarquons également que l'on a $\Delta_+^{\text{op}} = (\Delta^{\text{op}})^\triangleright$.

Définition 1.1.5.3. Soit \mathcal{C} une catégorie et soit X un objet de \mathcal{C} .

1. Une résolution simpliciale de X est un diagramme colimite $p: \Delta_+^{\text{op}} = (\Delta^{\text{op}})^\triangleright \rightarrow \mathcal{C}$ tel que $p|_{\Delta^0} = X$.
2. Le nerf de Čech d'un morphisme $f: Y \rightarrow X$ est le diagramme $\check{C}(f): \Delta_+ \rightarrow X$ donné par l'extension de Kan à droite de f le long de ι .

3. Un épimorphisme effectif est un morphisme $f: Y \rightarrow X$ tel que le nerf de Čech de f est un diagramme colimite.

Si \mathcal{U} est un objet simplicial d'une catégorie \mathcal{C} , et si n est un entier, on note $\text{cosk}_n(\mathcal{U})$ l'image par l'extension de Kan à droite le long de l'inclusion $(\Delta^{\leq n})^{\text{op}} \subseteq \Delta^{\text{op}}$ de $\text{sk}_n(\mathcal{U})$. C'est encore un objet simplicial de \mathcal{C} .

Définition 1.1.5.4. Soit (\mathcal{C}, τ) un site et soit X un objet de \mathcal{C} . Un hyper-recouvrement de X est un objet simplicial \mathcal{U} de $\mathcal{C}_{/X}$ tel que pour tout entier n , le morphisme canonique

$$L_\tau^0 h(\mathcal{U}_n) \rightarrow L_\tau^0 h(\text{cosk}_{n-1}(\mathcal{U})_n),$$

où le foncteur cosk est calculé dans la catégorie $\mathcal{C}_{/X}$, est un épimorphisme effectif.

Définition 1.1.5.5. Soit (\mathcal{C}, τ) un site. Notons S l'ensemble des flèches de la forme

$$\text{colim}_{[n] \in \Delta} h(\mathcal{U}_n) \rightarrow h(X)$$

pour \mathcal{U} un hyper-recouvrement d'un objet X . Un τ -hyperfaisceau sur \mathcal{C} est un préfaisceau S -local. On note $L_\tau: \widehat{\mathcal{C}} \rightarrow \mathcal{S}\mathfrak{h}_\tau(\mathcal{C})$ le foncteur de S -localisation.

On peut à présent définir la catégorie des hyperfaisceaux

Hyperfaisceaux à valeurs dans une catégorie

Définition 1.1.5.6. Soit (\mathcal{C}, τ) un site et \mathcal{D} une catégorie qui admet toutes les petites limites. Un τ -hyperfaisceau à valeurs dans \mathcal{D} est un foncteur $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ tel que si $\overline{F}: \widehat{\mathcal{C}}^{\text{op}} \rightarrow \mathcal{D}$ est l'extension de F qui préserve les petites limites, le foncteur \overline{F} se factorise par L_τ^{op} . On note $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \mathcal{D})$ la catégorie des τ -hyperfaisceaux à valeurs dans \mathcal{D} .

De manière équivalente, un foncteur $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ est un hyperfaisceau si pour tout objet X de \mathcal{C} et tout hyper-recouvrement \mathcal{U} de X , l'application canonique

$$F(X) \rightarrow \lim_{n \in \Delta} F(\mathcal{U}_n)$$

est une équivalence.

Si la catégorie \mathcal{D} est de plus présentable, en combinant la Proposition [1.1.2.8](#) et [Lur17, 4.8.1.17](#), on obtient

$$\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \mathcal{D}) = \mathcal{S}\mathfrak{h}_\tau(\mathcal{C}) \otimes \mathcal{D}.$$

Enfin, si \mathcal{D}^\otimes est une catégorie monoïdale présentable, la sous-catégorie $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \mathcal{D})$ de $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{D})$ est stable par le produit tensoriel coordonnée par coordonnée. On obtient ainsi une catégorie monoïdale $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \mathcal{D})^\otimes$ de catégorie sous-jacente $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \mathcal{D})$.

Si R est un anneau, on note $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, R)$ la catégorie $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \text{Mod}_R)$. Le foncteur de Yoneda induit alors un foncteur

$$R(-): \mathcal{C} \rightarrow \mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, R).$$

Un des intérêts principaux des hyperfaisceaux est le résultat suivant dont l'intuition première remonte à Verdier.

Proposition 1.1.5.7. ([Lur18](#), 2.1.2.3) *Soit \mathcal{C} une 1-catégorie et τ une topologie de Grothendieck sur \mathcal{C} . Si R est un anneau commutatif, on note $\mathrm{Sh}_\tau(\mathcal{C}, R)$ la catégorie abélienne des faisceaux de petits R -modules (au sens usuel). Alors,*

1. *La catégorie stable $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, R)$ est munie d'une t -structure dont le cœur est équivalent à la catégorie $\mathrm{Sh}_\tau(\mathcal{C}, R)$.*
2. *L'inclusion $\mathrm{Sh}_\tau(\mathcal{C}, R) \rightarrow \mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, R)$ s'étend en un foncteur*

$$\mathcal{D}(\mathrm{Sh}_\tau(\mathcal{C}, R)) \rightarrow \mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, R)$$

qui est une équivalence de catégories.

1.2 La catégorie stable des motifs étales

Tous les schémas qui apparaissent dans le texte ci-dessous sont supposés noethériens et de dimension finie. De plus, tous les morphismes lisses (et étales) sont supposés implicitement être séparés et de type fini. On note Sm la classe des morphismes lisses. Si S est un schéma, on note $S_{\mathrm{ét}}$ le petit site étale sur S . Enfin, on note Sch la catégorie des schémas (noethériens de dimension finie).

Tous les anneaux sont par convention commutatifs. On note

$$R_S: \mathrm{Sm}_S \rightarrow \mathcal{S}\mathfrak{h}_\tau(\mathrm{Sm}_S, R)$$

le foncteur induit par le foncteur de Yoneda.

1.2.1 Formalisme général des catégories motiviques

Catégories prémotiviques stables

On rappelle la notion de catégorie prémotivique de [\[CD19, 1\]](#).

Définition 1.2.1.1. *Soit*

$$\mathcal{D}: \mathrm{Sch}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{Stb}}^L)$$

un foncteur. Si S est un schéma, on note $\mathcal{D}(S)^\otimes$ l'image de S et $\mathcal{D}(S)$ la catégorie sous-jacente. Si f est un morphisme de schémas, on note f^ le foncteur $\mathcal{D}(f)$ et f_* son adjoint à droite. On note \otimes le produit tensoriel. On dit que \mathcal{D} est une catégorie prémotivique stable si les conditions suivantes sont vérifiées.*

1. *Pour tout morphisme lisse f , le foncteur f^* admet un adjoint à gauche $f_\#$.*
2. *Pour tout carré cartésien*

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \downarrow q & & \downarrow p \\ T & \xrightarrow{f} & S \end{array}$$

tel que le morphisme f est lisse, la transformation naturelle $g_\#p^ \rightarrow q^*f_\#$ donnée par adjonction est une équivalence.*

3. Pour tout morphisme lisse $f: T \rightarrow S$, tout objet M de $\mathcal{D}(T)$ et tout objet N de $\mathcal{D}(S)$, l'application naturelle

$$f_{\#}(M \otimes f^*N) \rightarrow f_{\#}M \otimes N$$

est une équivalence dans $\mathcal{D}(S)$.

Un morphisme ϕ entre deux catégories prémotiviques stables vues comme objets de la catégorie $\text{Fun}(\text{Sch}^{\text{op}}, \text{CAlg}(\text{Pr}_{\text{Stb}}^L))$ est prémotivique si chaque foncteur ϕ_S admet un adjoint à droite et si pour tout morphisme lisse $f: T \rightarrow S$, la transformation naturelle $f_{\#}\phi_T \rightarrow \phi_S f_{\#}$ est une équivalence.

On appellera *adjonction prémotivique* la paire formée par un morphisme prémotivique et son adjoint. La proposition suivante est immédiate.

Proposition 1.2.1.2. *Soit \mathcal{D} une catégorie prémotivique stable.*

1. Pour tout carré cartésien

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \downarrow q & & \downarrow p \\ T & \xrightarrow{f} & S \end{array}$$

tel que le morphisme f est lisse, la transformation naturelle $q_* f^* \rightarrow g^* p_*$ donnée par adjonction est une équivalence.

2. Si $j: U \rightarrow S$ une immersion ouverte, alors les foncteurs $j_{\#}$ et j_* sont pleinement fidèles.

Image directe exceptionnelle

On commence par rappeler la propriété du support. On considère une catégorie stable prémotivique \mathcal{D} . Si

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & X_1 \\ \downarrow q & & \downarrow p \\ X_2 & \xrightarrow{j} & S \end{array} \quad (*)$$

est un diagramme commutatif de schémas tel que p et q sont propres et j et γ sont des immersions ouvertes, on peut construire un morphisme de foncteurs

$$j_{\#} q_* \rightarrow p_* \gamma_{\#}.$$

Définition 1.2.1.3. *Soit \mathcal{D} une catégorie prémotivique stable. On dit que \mathcal{D} vérifie la propriété du support si le morphisme ci-dessus est une équivalence pour tout diagramme commutatif de la forme (*).*

Définition 1.2.1.4. Soit \mathcal{D} une catégorie prémotivique stable qui vérifie la propriété du support. Si $f: X \rightarrow S$ est un morphisme séparé de type fini, on choisit au moyen du théorème de Nagata [Sta23, 0F41] une compactification de f , c'est-à-dire une factorisation

$$\begin{array}{ccc} X & \xrightarrow{j} & \overline{X} \\ & \searrow f & \downarrow p \\ & & S \end{array}$$

de f telle que j est une immersion ouverte et p est un morphisme propre. On définit alors le foncteur

$$f_! : \mathcal{D}(X) \rightarrow \mathcal{D}(S)$$

comme la composition $p_* \circ j_\#$.

On peut suivre la méthode de [AGV22, 4.3.5] pour montrer que le foncteur $f_!$ ne dépend pas de la compactification choisie : il suffit de remplacer dans la démonstration la référence à [AGV22, 4.1.4] par la propriété du support.

Fixons à présent un schéma S . Notons Sep_S la catégorie des S -schémas séparés de type fini. Si \mathcal{D} est une catégorie prémotivique stable qui vérifie la propriété du support, on peut adapter les arguments de [AGV22, 4.3.9] (ceux-ci généralisent au cadre des catégories stable la construction de Deligne donnée dans [AGV73, XVII.3.3.2] et dont on peut trouver un résumé dans le cas triangulé dans [CD16, 2.2.7]) pour construire un foncteur

$$\mathcal{D}_! : \text{Sep}_S \rightarrow \text{CAT}^{\text{Ex}}$$

qui envoie un S -schéma séparé de type fini X sur la catégorie $\mathcal{D}(X)$ et envoie tout morphisme f vers le foncteur $f_!$. On verra qu'on peut étendre le foncteur $\mathcal{D}_!$ de manière unique à tous les schémas de type fini sur S sous certaines hypothèses supplémentaires.

Si l'on note \mathcal{D}_* le foncteur

$$\text{Sep}_S \rightarrow \text{CAT}^{\text{Ex}}$$

obtenu en restreignant le foncteur obtenu par adjonction à partir de \mathcal{D} , on obtient un morphisme $\alpha: \mathcal{D}_! \rightarrow \mathcal{D}_*$ qui induit donc pour tout morphisme f séparé une application

$$\alpha_f: f_! \rightarrow f_*.$$

Cette application est de plus un isomorphisme si f est propre.

Définition 1.2.1.5. Soit \mathcal{D} une catégorie prémotivique stable. On dit que \mathcal{D} vérifie la propriété d'adjointabilité si pour tout morphisme propre f , le foncteur f_* est compatible aux petites colimites.

Si \mathcal{D} vérifie la propriété du support et la propriété d'adjointabilité, le foncteur $\mathcal{D}_!$ se factorise par la catégorie Pr_{Stb}^L et on obtient donc un foncteur

$$\mathcal{D}^! : \text{Sep}_S^{\text{op}} \rightarrow \text{Pr}_{\text{Stb}}^R$$

qui envoie un morphisme f sur l'adjoint à droite $f^!$ de $f_!$.

On note Sch_S^{tf} la catégorie des S -schémas de type fini. Si \mathcal{D} est un faisceau Zariski, le foncteur $\mathcal{D}^!$ est aussi un faisceau Zariski. Suivant la méthode de [AGV22, 4.3.18], on obtient le résultat suivant.

Proposition 1.2.1.6. *Soit \mathcal{D} une catégorie prémotivique stable vérifiant les propriétés du support et d'adjointabilité et qui est un faisceau Zariski. Alors, le foncteur $\mathcal{D}_! : \text{Sep}_S \rightarrow \text{Pr}_{\text{Stb}}^L$ se prolonge de manière unique en un foncteur*

$$\mathcal{D}_! : \text{Sch}_S^{\text{tf}} \rightarrow \text{Pr}_{\text{Stb}}^L$$

dont le foncteur adjoint $\mathcal{D}^!$ est un faisceau Zariski.

Morphismes d'échange

Proposition 1.2.1.7. *Soit \mathcal{D} une catégorie prémotivique stable.*

1. *On dit que \mathcal{D} vérifie la formule de changement de base propre si pour tout carré cartésien*

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \downarrow q & & \downarrow p \\ T & \xrightarrow{f} & S \end{array}$$

tel que le morphisme p est propre, la transformation naturelle $q_* f^* \rightarrow g^* p_*$ donnée par adjonction est une équivalence.

2. *On dit que \mathcal{D} vérifie la formule de projection si pour tout morphisme propre $p: X \rightarrow Y$, tout objet M de $\mathcal{D}(X)$ et tout objet N de $\mathcal{D}(Y)$, la transformation naturelle donnée par adjonctions*

$$p_* M \otimes N \rightarrow p_*(M \otimes p^* N)$$

est une équivalence.

Le foncteur

$$\text{Sch}^{\text{op}} \rightarrow \text{CAT}$$

donné par $S \mapsto \text{Sch}_S^{\text{tf}}$ donne par la construction de Grothendieck une fibration cocartésienne

$$\pi : \mathcal{E} \rightarrow \text{Sch}^{\text{op}}.$$

- Les objets de \mathcal{E} s'identifient alors aux paires (S, X) avec S un schéma et X un S -schéma de type fini.
- Les morphismes entre une paire (S, X) et une paire (T, Y) sont donnés par les paires (g, f) avec $g: T \rightarrow S$ et $f: T \times_S Y \rightarrow X$ des morphismes de schémas.

En appliquant les constructions de [AGV22, 4.4, 4.5], on peut montrer le résultat suivant.

Proposition 1.2.1.8. *Soit \mathcal{D} une catégorie prémotivique stable qui vérifie les propriétés du support et d'adjointabilité, la formule de changement de base propre et la formule de projection. On suppose que \mathcal{D} est un faisceau Zariski. Alors, il existe des foncteurs*

$$\mathcal{D}_!^{\otimes} : \mathcal{E} \rightarrow \text{Mod}(\text{Pr}_{\text{Stb}}^L)$$

et

$$\mathcal{D}_*^! : \mathcal{E}^{\text{op}} \rightarrow \text{Pr}_{\text{Stb}}^R$$

qui vérifient les propriétés suivantes.

1. La composition de $\mathcal{D}_!^\otimes$ avec le foncteur $\mathfrak{A}: \text{Mod}(\text{Pr}_{\text{Stb}}^L) \rightarrow \text{CAlg}(\text{Pr}_{\text{Stb}}^L)$ est équivalente au foncteur

$$\mathcal{E} \xrightarrow{\pi} \text{Sch}^{\text{op}} \xrightarrow{\mathcal{D}} \text{CAlg}(\text{Pr}_{\text{Stb}}^L).$$

2. Notons $\mathcal{D}_!^*: \mathcal{E} \rightarrow \text{Pr}_{\text{Stb}}^L$ la composition de $\mathcal{D}_!^\otimes$ avec le foncteur $\mathfrak{M}: \text{Mod}(\text{Pr}_{\text{Stb}}^L) \rightarrow \text{Pr}_{\text{Stb}}^L$. Les foncteurs $\mathcal{D}_!^*$ et $\mathcal{D}_*^!$ sont échangés par l'équivalence $(\text{Pr}_{\text{Stb}}^R)^{\text{op}} \simeq \text{Pr}_{\text{Stb}}^L$.
3. Le foncteur \mathcal{D} (resp. \mathcal{D}_*) est obtenu à partir du foncteur $\mathcal{D}_!^\otimes$ (resp. $\mathcal{D}_*^!$) par précomposition avec le foncteur $\text{Sch}^{\text{op}} \rightarrow \mathcal{E}$ (resp. $\text{Sch} \rightarrow \mathcal{E}^{\text{op}}$) donné par l'application diagonale $S \mapsto (S, S)$.
4. Le foncteur $\mathcal{D}_!$ (resp. $\mathcal{D}^!$) est obtenu à partir du foncteur $\mathcal{D}_!^*$ (resp. $\mathcal{D}_*^!$) par restriction à Sch_S^{tf} .

On a la description informelle suivante de ces foncteurs.

- Le foncteur $\mathcal{D}_!^\otimes$ envoie une paire (S, X) vers un foncteur $F: \mathbb{M} \rightarrow \text{Pr}_{\text{Stb}}^L$ qui fait de la catégorie $\mathcal{D}(X)$ un $\mathcal{D}(S)^\otimes$ -module. L'action de $\mathcal{D}(S)$ sur $\mathcal{D}(X)$ est donnée par $(M, N) \mapsto f^*M \otimes N$ où $f: X \rightarrow S$ est le morphisme structurel.
- Le foncteur $\mathcal{D}_!^\otimes$ envoie un morphisme (g, f) entre une paire (S, X) et une paire (T, Y) sur le morphisme de $\mathcal{D}(S)^\otimes$ -module donné par

$$f_!(g')^*: \mathcal{D}(Y) \rightarrow \mathcal{D}(X)$$

où $g': T \times_S Y \rightarrow Y$ est le tiré en arrière de g .

- Le foncteur $\mathcal{D}_*^!$ envoie une paire (S, X) vers la catégorie $\mathcal{D}(X)$.
- Le foncteur $\mathcal{D}_*^!$ envoie un morphisme (g, f) entre une paire (S, X) et une paire (T, Y) sur le foncteur

$$g'_*f^!: \mathcal{D}(X) \rightarrow \mathcal{D}(Y).$$

Soit $f: X \rightarrow S$ un morphisme de type fini. Comme le foncteur $f_!$ est un morphisme de $\mathcal{D}(S)^\otimes$ -modules $\mathcal{D}(X) \rightarrow \mathcal{D}(S)$, on obtient une équivalence

$$f_!M \otimes N \rightarrow f_!(M \otimes f^*N)$$

puis par adjonction, en déduit des isomorphismes naturels

$$\underline{\text{Hom}}(f_!M, N) \rightarrow f_*\underline{\text{Hom}}(M, f^!N)$$

$$f^!\underline{\text{Hom}}(M, N) \rightarrow \underline{\text{Hom}}(f^*M, f^!N).$$

Catégories motiviques

Définition 1.2.1.9. Une catégorie prémotivique stable vérifie la propriété de localisation si $\mathcal{D}(\emptyset) = 0$ et si pour toute immersion fermée $i: F \rightarrow S$, le foncteur i_* est pleinement fidèle et en notant $j: U \rightarrow S$ l'immersion ouverte complémentaire, la famille (i^*, j^*) est conservative.

Dans la situation ci-dessus, les foncteurs $i_*: \mathcal{D}(Z) \rightarrow \mathcal{D}(X)$ et $j_*: \mathcal{D}(U) \rightarrow \mathcal{D}(X)$ font de $\mathcal{D}(X)$ un recollement de $(\mathcal{D}(U), \mathcal{D}(Z))$ d'après [CD19, 2.3.1].

Si \mathcal{D} est une catégorie prémotivique et si S est un schéma, on note $\mathbb{1}_S$ l'objet unité de $\mathcal{D}(S)$. Si X est un S -schéma lisse, on note $M_S(X)$ et on appelle *prémotif de X* l'objet $f_{\#}f^*\mathbb{1}_S$. On a une inclusion scindée

$$\mathbb{1}_S \rightarrow M_S(\mathbb{G}_{m,S})$$

donnée par la section constante égale à 1. Le *Twist de Tate* est l'opération produit tensoriel par le décalé de -1 de la cofibre de cette inclusion. On la note (1).

Définition 1.2.1.10. 1. On dit qu'une catégorie prémotivique stable est Tate stable si l'opération (1) admet un inverse noté (-1) .

2. On dit qu'une catégorie prémotivique stable vérifie l'invariance par homotopie si le morphisme canonique $M_S(\mathbb{A}_S^1) \rightarrow \mathbb{1}_S$ est une équivalence.

3. Une catégorie motivique stable est une catégorie prémotivique stable qui est Tate stable et vérifie l'invariance par homotopie et la propriété de localisation.

Le théorème suivant est du aux efforts combinés de Voevodsky, Röndings, Ayoub, Cisinski, Déglise, Drew et Gallauer. Il se déduit de [CD19, 2.4.26, 2.4.28] et de [DG22, 7.13] (cette dernière référence concerne la deuxième partie du théorème).

Théorème 1.2.1.11. Soit \mathcal{D} une catégorie motivique stable. Alors, la catégorie \mathcal{D} vérifie les propriétés du support et d'adjointabilité, la formule de changement de base propre et la formule de projection. De plus, le foncteur $\text{Sch}^{\text{op}} \rightarrow \text{Pr}_{\text{Stb}}^L$ sous-jacent à \mathcal{D} est un faisceau pour la topologie de Nisnevich.

On en déduit le résultat suivant.

Proposition 1.2.1.12. Soit \mathcal{D} une catégorie motivique stable.

1. On a trois paires de foncteurs adjoints exacts comme ci-dessous.

- $f^*: \mathcal{D}(Y) \rightleftarrows \mathcal{D}(X): f_*$ pour tout morphisme de schémas $f: X \rightarrow Y$.
- $f_!: \mathcal{D}(Y) \rightleftarrows \mathcal{D}(X): f^!$ pour tout morphisme de type fini de schémas $f: X \rightarrow Y$.
- $(\otimes, \underline{\text{Hom}})$ une structure monoïdale symétrique fermée sur chaque $\mathcal{D}(X)$.

De plus les foncteurs de type f^* sont monoïdaux.

2. Les foncteurs f_* et $f_!$ (resp. f^* et $f^!$) fournissent des foncteurs covariants (resp. contravariants) de la catégorie des schémas (de type fini sur une base fixée pour les foncteurs exceptionnels) vers la catégorie CAT.

3. Il existe un morphisme de foncteurs

$$\alpha: f_! \rightarrow f_*$$

qui est une équivalence si f est propre.

4. Pour tout carré cartésien de schémas

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow g' & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

tel que f est de type fini, on a des isomorphismes naturels

$$g^* f! \rightarrow f'_!(g')^*$$

$$g'_*(f')^! \rightarrow f^! g_*.$$

5. Pour tout morphisme de type fini $f: X \rightarrow Y$, on a des isomorphismes naturels

$$(f_! M) \otimes N \rightarrow f_!(M \otimes f^* N)$$

$$\underline{\mathrm{Hom}}(f_! M, N) \rightarrow f_* \underline{\mathrm{Hom}}(M, f^! N)$$

$$f^! \underline{\mathrm{Hom}}(M, N) \rightarrow \underline{\mathrm{Hom}}(f^* M, f^! N).$$

6. Pour toute immersion fermée $i: F \rightarrow X$ de complémentaire $j: X \setminus F \rightarrow X$, on a deux triangles exacts de transformations naturelles de la forme

$$j_! j^! \rightarrow \mathrm{Id} \rightarrow i_* i^*$$

$$i_! i^! \rightarrow \mathrm{Id} \rightarrow j_* j^*$$

donnés par les unités et co-unités des adjonctions idoines.

Pureté relative

Si S est un schéma et si \mathcal{D} est une catégorie motivique stable, la descente Nisnevich permet de montrer que

$$M_S(\mathbb{P}_S^1) \simeq \mathbb{1}_S \oplus \mathbb{1}_S(1)[2].$$

Définition 1.2.1.13. Soient \mathcal{D} une catégorie motivique stable.

1. Soit $p: \mathcal{E} \rightarrow X$ un fibré vectoriel et s la section nulle de p . Le motif de Thom de \mathcal{E} est le motif $\mathrm{MTh}_X(\mathcal{E}) = p_{\#} s_*(\mathbb{1}_X)$.
2. Une orientation de \mathcal{D} est une transformation naturelle de foncteurs $\mathrm{Sch} \rightarrow \mathbf{Set}$ notée

$$c_1: \mathrm{Pic} \rightarrow \mathrm{Hom}_{\mathcal{D}(-)}(\mathbb{1}_-, \mathbb{1}_-(1)[2])$$

telle que pour tout schéma S , l'image de la classe $\mathcal{O}(-1)$ par l'application

$$\mathrm{Pic}(\mathbb{P}_S^1) \rightarrow \mathrm{Hom}_{\mathcal{D}(\mathbb{P}_S^1)}(\mathbb{1}_{\mathbb{P}_S^1}, \mathbb{1}_{\mathbb{P}_S^1}(1)[2]) \simeq \mathrm{Hom}_{\mathcal{D}(S)}(M_S(\mathbb{P}_S^1), \mathbb{1}_S(1)[2])$$

est la projection canonique $M_S(\mathbb{P}_S^1) \rightarrow \mathbb{1}_S(1)[2]$.

Le motif de Thom d'un fibré \mathcal{E} est toujours \otimes -inversible d'après [CD19, 2.4.14] et on note $\mathrm{MTh}_X(-\mathcal{E})$ l'inverse de $\mathrm{MTh}_X(\mathcal{E})$ pour le produit tensoriel.

On déduit de la combinaison de [CD19, 2.4.40] et de [DG22, 7.14] qu'une orientation fournit pour tout fibré vectoriel $\mathcal{E} \rightarrow X$ de rang n un isomorphisme

$$\mathfrak{t}_{\mathcal{E}}: \mathrm{MTh}_X(\mathcal{E}) \rightarrow \mathbb{1}_X(n)[2n]$$

et que ces isomorphismes vérifient les propriétés de cohérence listées en [CD19, 2.4.38].²

Le théorème de pureté sous sa forme énoncée dans [CD19, 2.4.37] (on renvoie à [MV99, 3.2.24] pour l'énoncé original de Morel et Voevodsky qui porte sur le cas où \mathcal{D} est la catégorie homotopique stable des schémas) implique le résultat suivant.

Proposition 1.2.1.14. *Soit \mathcal{D} une catégorie motivique.*

1. *Pour tout morphisme lisse f , si on note T_f le fibré tangent de f , on a des isomorphismes naturels*

$$\mathfrak{p}'_f: f_{\#} \rightarrow f_!(\mathrm{MTh}_X(T_f) \otimes -)$$

$$\mathfrak{p}_f: f^* \rightarrow \mathrm{MTh}_X(-T_f) \otimes f^!$$

2. *Si \mathcal{D} est munie d'une orientation, et si f est un morphisme lisse de dimension relative d , cette orientation induit des isomorphismes*

$$\mathfrak{p}'_f: f_{\#} \rightarrow f_!(d)[2d]$$

$$\mathfrak{p}_f: f^* \rightarrow f^!(-d)[-2d].$$

1.2.2 Catégories stables de motifs étales

Construction

Rappelons d'abord que si $f: T \rightarrow S$ est un morphisme de schémas, on a un foncteur

$$f^*: \mathrm{Sm}_S \rightarrow \mathrm{Sm}_T$$

donné par $X \mapsto X \times_S T$. Ce foncteur s'étend de manière unique en un foncteur

$$f^{-1}: \widehat{\mathrm{Sm}}_S \rightarrow \widehat{\mathrm{Sm}}_T$$

qui préserve les colimites.

Les foncteurs de type $f^* = L_{\mathrm{ét}} f^{-1}$ donnent ensuite une structure de catégorie pré motivique stable aux foncteurs $\mathcal{S}\mathfrak{h}_{\mathrm{ét}}(\mathrm{Sm}_-, R)^{\otimes}$ et $\mathcal{S}\mathfrak{h}_{\mathrm{ét}}(\mathrm{Sm}_-, \mathcal{S}\mathfrak{p})^{\otimes}$.

Définition 1.2.2.1. *Soit S un schéma et R un anneau. On note $\mathcal{D}\mathcal{A}_{\mathrm{ét}}^{\mathrm{eff}}(S, R)$ (resp. $\mathcal{S}\mathcal{H}_{\mathrm{ét}}^{\mathrm{eff}}(S)$) la sous-catégorie pleine de $\mathcal{S}\mathfrak{h}_{\mathrm{ét}}(\mathrm{Sm}_S, R)$ (resp. $\mathcal{S}\mathfrak{h}_{\mathrm{ét}}(\mathrm{Sm}_S, \mathcal{S}\mathfrak{p})$) formée des faisceaux \mathbb{A}^1 -locaux ou \mathbb{A}^1 -invariants, c'est-à-dire tels que pour tout S -schéma lisse X , si on note $p: \mathbb{A}_X^1 \rightarrow X$ la projection canonique, le morphisme $p^*: F(X) \rightarrow F(\mathbb{A}_X^1)$ est une équivalence.*

²Il manque en fait un axiome dans les propriétés énoncées dans [CD19] : le triangle de localisation et l'invariance par homotopie fournissent pour tout schéma X un isomorphisme $\mathrm{MTh}_X(\mathbb{A}_X^1) \rightarrow \mathbb{1}_X(1)[2]$; on demande que $\mathfrak{t}_{\mathbb{A}_X^1}$ soit donné par cette isomorphisme.

Les faisceaux \mathbb{A}^1 -invariants sont les objets locaux par rapport aux morphismes $p^* : R_S(X) \rightarrow R_S(\mathbb{A}_X^1)$. On note alors

$$L_{\mathbb{A}^1} : \mathcal{S}\mathcal{H}_{\text{ét}}(\text{Sm}_S, R) \rightarrow \mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(S, R)$$

$$L_{\mathbb{A}^1} : \mathcal{S}\mathcal{H}_{\text{ét}}(\text{Sm}_S, \mathcal{S}\mathcal{P}) \rightarrow \mathcal{S}\mathcal{H}_{\text{ét}}^{\text{eff}}(S)$$

les foncteurs d' \mathbb{A}^1 -localisation. Par ce qui précède, la catégorie $\mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(S, R)$ (resp. $\mathcal{S}\mathcal{H}_{\text{ét}}^{\text{eff}}(S)$) est sous-jacente à une unique catégorie monoïdale présentable $\mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(S, R)^{\otimes}$ (resp. $\mathcal{S}\mathcal{H}_{\text{ét}}^{\text{eff}}(S)^{\otimes}$) telle que le foncteur d' \mathbb{A}^1 -localisation est monoïdal. De plus, en combinant la Proposition [1.1.2.8](#) et [Lur17](#), 4.8.1.17], on a

$$\mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(S, R) = \mathcal{S}\mathcal{H}_{\text{ét}}^{\text{eff}}(S) \otimes \text{Mod}_R.$$

Si $f : T \rightarrow S$ est un morphisme de schémas, on notera encore f^* le foncteur $L_{\mathbb{A}^1}f^*$ lorsqu'il n'y a pas d'ambiguïté. Les foncteurs de type f^* font des foncteurs $\mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(-, R)^{\otimes}$ et $\mathcal{S}\mathcal{H}_{\text{ét}}^{\text{eff}}(-)^{\otimes}$ des catégories prémotiviques stables.

On note M_S^{eff} la composée de $L_{\mathbb{A}^1}$ et du foncteur R_S .

Définition 1.2.2.2. Soit S un schéma et R un anneau. On note T_S l'image par $L_{\mathbb{A}^1}$ de la cofibre de l'inclusion scindée $R_S(S) \rightarrow R_S(\mathbb{G}_{m,S})$ (resp. $S \rightarrow \mathbb{G}_{m,S}$) induite par la section constante égale à 1 du morphisme canonique $\mathbb{G}_{m,S} \rightarrow S$. On note

$$\mathcal{D}\mathcal{M}_{\text{ét}}(S, R)^{\otimes} := \text{Stab}_{T_S}(\mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(S, R)^{\otimes})$$

la catégorie stable monoïdale des motifs étales (resp.

$$\mathcal{S}\mathcal{H}_{\text{ét}}(S)^{\otimes} := \text{Stab}_{T_S}(\mathcal{S}\mathcal{H}_{\text{ét}}^{\text{eff}}(S)^{\otimes})$$

la catégorie homotopique stable des S -schémas).

Les catégories ci-dessus sont bien définies grâce à la construction de Robalo donnée dans la Proposition [1.1.4.12](#) et [Voe98](#), 4.4]. Par construction de la stabilisation, on obtient de plus

$$\mathcal{D}\mathcal{M}_{\text{ét}}(S, R) = \mathcal{S}\mathcal{H}_{\text{ét}}(S) \otimes \text{Mod}_R.$$

Cela permet de définir une adjonction

$$\Sigma^{\infty} : \mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(S, R) \rightleftarrows \mathcal{D}\mathcal{M}_{\text{ét}}(S, R) : \Omega^{\infty}$$

telle que le foncteur Σ^{∞} soit monoïdal. On note M_S la composée de Σ^{∞} et de M_S^{eff} . Si l'on a besoin de spécifier l'anneau R des coefficients, on notera $M_S(X, R)$ le motif d'un S -schéma lisse X . On note $\mathbb{1}_S = M_S(S)$ l'unité pour le produit tensoriel et par $\mathbb{1}_S(1)$ le motif $\Sigma^{\infty}T_S[-1]$. Si n est un entier, on note $\mathbb{1}_S(n) = \mathbb{1}_S(1)^{\otimes n}$. Si M est un motif étale, on note $M(n)$ le motif $M \otimes \mathbb{1}_S(n)$.

On peut montrer la proposition suivante dans l'esprit de [Hoy17](#), 4.8] et [AGV22](#), 2.3.5].

Proposition 1.2.2.3. La construction $S \mapsto \mathcal{D}\mathcal{M}_{\text{ét}}(S, R)^{\otimes}$ s'étend en un foncteur

$$\mathcal{D}\mathcal{M}_{\text{ét}}(-, R)^{\otimes} : \text{Sch}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{Stb}}^L)$$

qui envoie un morphisme f de schémas vers le foncteur $\Sigma^{\infty}L_{\mathbb{A}^1}f^*$ (ce dernier sera encore noté f^* s'il n'y a pas d'ambiguïté). Le foncteur $\mathcal{D}\mathcal{M}_{\text{ét}}(-, R)$ est alors un hyperfaisceau étale et est de plus une catégorie prémotivique stable.

On a alors une suite d'adjonctions prémotiviques

$$\mathcal{S}\mathcal{H}_{\text{ét}}(-, R) \xleftarrow[\iota]{L_{\mathbb{A}^1}} \mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(-, R) \xleftarrow[\Omega^\infty]{\Sigma^\infty} \mathcal{D}\mathcal{M}_{\text{ét}}(-, R) .$$

On peut rajouter à gauche de ce diagramme une paire de foncteurs adjoints en prenant garde au fait qu'elle ne sera pas prémotivique. L'inclusion de sites $\rho: S_{\text{ét}} \rightarrow \text{Sm}_S$ fournit en effet pour tout schéma S une paire de foncteurs adjoints

$$\rho_{\#}: \mathcal{S}\mathcal{H}(S_{\text{ét}}, R) \rightleftarrows \mathcal{S}\mathcal{H}_{\text{ét}}(\text{Sm}_S, R): \rho^* .$$

Celle-ci s'étend en un morphisme entre objets de $\text{Fun}(\text{Sch}^{\text{op}}, \text{CAlg}(\text{Pr}_{\text{Stb}}^L))$.

Le formalisme des six foncteurs

Par définition, la catégorie prémotivique stable $\mathcal{D}\mathcal{M}_{\text{ét}}(-, R)^{\otimes}$ vérifie l'invariance par homotopie et est Tate-stable. Il suffit donc de prouver la propriété de localisation pour montrer que c'est une catégorie motivique stable. Ceci découle alors de [CD16, 5.6.2]³. De plus, cette catégorie est orientée d'après [CD16, 5.6.1]. Enfin, elle vérifie la pureté absolue (voir [CD16, A.2.9]). Ainsi, on a le résultat suivant.

Théorème 1.2.2.4. (*Ayoub, Cisinski, Déglise*) *Soit R un anneau commutatif.*

1. *On a trois paires de foncteurs adjoints exacts comme ci-dessous.*

- $f^*: \mathcal{D}\mathcal{M}_{\text{ét}}(Y, R) \rightleftarrows \mathcal{D}\mathcal{M}_{\text{ét}}(X, R): f_*$ pour tout morphisme de schémas $f: X \rightarrow Y$.
- $f_!: \mathcal{D}\mathcal{M}_{\text{ét}}(Y, R) \rightleftarrows \mathcal{D}\mathcal{M}_{\text{ét}}(X, R): f^!$ pour tout morphisme de type fini de schémas $f: X \rightarrow Y$.
- (\otimes, Hom) structure monoïdale symétrique fermée.

De plus les foncteurs de type f^ sont monoïdaux.*

2. *Les foncteurs f_* et $f_!$ (resp. f^* et $f^!$) fournissent des foncteurs covariants (resp. contravariants) de la catégorie des schémas (de type fini sur une base fixée pour les foncteurs exceptionnels) vers la catégorie CAT.*

3. *Il existe un morphisme de foncteurs*

$$\alpha: f_! \rightarrow f_*$$

qui est une équivalence si f est propre.

4. *Pour tout carré cartésien de schémas*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow g' & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

³Le catégorie prémotivique sur laquelle ce résultat est démontré dans [CD16] est celle des h-motifs. Elle est cependant équivalente à celle des motifs étales : [CD16, 5.5.7] s'applique dans la généralité voulue en remplaçant dans la preuve [Ayo14c, 4.1] par [BH21, 3.2].

tel que f est de type fini, on a des isomorphismes naturels

$$\begin{aligned} g^* f_! &\rightarrow f'_!(g')^* \\ g'_!(f')^! &\rightarrow f^! g_*. \end{aligned}$$

5. Pour tout morphisme de type fini $f: X \rightarrow Y$, on a des isomorphismes naturels

$$\begin{aligned} (f_! M) \otimes N &\rightarrow f_!(M \otimes f^* N) \\ \underline{\mathrm{Hom}}(f_! M, N) &\rightarrow f_* \underline{\mathrm{Hom}}(M, f^! N) \\ f^! \underline{\mathrm{Hom}}(M, N) &\rightarrow \underline{\mathrm{Hom}}(f^* M, f^! N). \end{aligned}$$

6. Pour toute immersion fermée $i: F \rightarrow X$ de complémentaire $j: X \setminus F \rightarrow X$, on a deux triangles exacts de transformations naturelles de la forme

$$\begin{aligned} j_! j^! &\rightarrow \mathrm{Id} \rightarrow i_* i^* \\ i_! i^! &\rightarrow \mathrm{Id} \rightarrow j_* j^* \end{aligned}$$

donnés par les unités et coïunités des adjonctions idoïnes.

7. Pour tout morphisme lisse $f: X \rightarrow S$ de dimension relative d , on a un isomorphisme

$$\mathbf{p}_f: f^* \rightarrow f^!(-d)[-2d].$$

8. Si $i: F \rightarrow X$ est une immersion fermée purement de codimension c entre schémas réguliers, on a un isomorphisme canonique

$$\mathbb{1}_F(-c)[-2c] \rightarrow i^! \mathbb{1}_X.$$

Théorème de rigidité

Le résultat suivant fait le lien entre les motifs étales et les faisceaux étales dans le cas de torsion. La première version de ce résultat est due à Suslin et Voevodsky [SV96, 4.3]. La version ci-dessous est [BH21, 3.2].

Proposition 1.2.2.5. (Théorème de rigidité) Soient n un entier et R un anneau de n -torsion. Alors, le foncteur $\mathcal{S}\mathfrak{h}(-_{\text{ét}}, R)$ est une catégorie prémotivique lorsqu'on le restreint à la catégorie des $\mathbb{Z}[1/n]$ -schémas et l'adjonction

$$\rho_! = \Sigma^\infty L_{\mathbb{A}^1} \rho_{\#}: \mathcal{S}\mathfrak{h}(-_{\text{ét}}, R) \rightleftarrows \mathcal{DM}_{\text{ét}}(-, R): \rho^* \iota \Omega^\infty = \rho^!$$

est une équivalence de catégories prémotiviques.

Dans le cas d'un schéma général, on a de plus le résultat suivant d'après [CD16, A.3.4], lequel permet de se ramener au cas d'un $\mathbb{Z}[1/n]$ -schéma pour décrire les motifs à coefficients dans un anneau de n -torsion.

Proposition 1.2.2.6. Soient n un entier, R un anneau de n -torsion et S est un schéma. Notons $j_n: S[1/n] := S \times_{\mathbb{Z}} \mathbb{Z}[1/n] \rightarrow S$ l'immersion ouverte canonique. Alors le morphisme

$$j_n^*: \mathcal{DM}_{\text{ét}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}(S[1/n], R)$$

est une équivalence de catégories.

Motifs constructibles

Définition 1.2.2.7. *Soit S un schéma et R un anneau commutatif. La catégorie $\mathcal{DM}_{\text{ét},c}(S, R)$ des motifs étales constructibles est la plus petite sous-catégorie de $\mathcal{DM}_{\text{ét}}(S, R)$ stable par limites finies, colimites finies et rétractes et contenant les motifs de la forme $M_S(X)(n)$ avec X lisse sur S et n un entier.*

Dans le cas où on se restreint aux schémas de base excellents vérifiant une certaine hypothèse sur les dimensions cohomologiques ponctuelles (voir [Ayo14c, 7.3]) et aux morphismes entre schémas quasi-projectifs sur le schéma de base, les six opérations préservent les motifs constructibles d'après [Ayo14c, 8.10, 8.12]. On peut en fait relâcher ces hypothèses [CD16] aux motifs étales. Alternativement, on peut aussi utiliser [CD16, 6.2.14] ainsi que l'équivalence pré-motivique entre les motifs étales et les h-motifs. On obtient alors le résultat suivant.

Proposition 1.2.2.8. *Soit R un anneau commutatif. Alors les six foncteurs préservent la catégorie $\mathcal{DM}_{\text{ét},c}(-, R)$ des motifs étales constructibles si on se restreint aux schémas (noethériens de dimension finie) quasi-excellents et aux morphismes de type fini entre ces schémas. Plus précisément, les propriétés suivantes sont vérifiées.*

1. *Soit S un schéma noethérien quasi-excellent de dimension finie et soient M et N des motifs étales constructibles sur S , alors les motifs étales $M \otimes N$ et $\underline{\text{Hom}}(M, N)$ sont constructibles.*
2. *Soient $f: X \rightarrow S$ un morphisme de schémas et M un motif étale constructible sur S , alors, le motif f^*M est constructible.*
3. *Soit $f: X \rightarrow S$ un morphisme de schémas de type fini, soit M un motif étale constructible sur S et N un motif étale constructible sur X , alors, les motifs $f^!M$, $f_!N$ et f_*N sont constructibles.*

Les points 1 à 8 du théorème ci-dessus sont par conséquent valables pour les motifs constructibles (pourvu qu'on se limite aux morphismes de type fini dans le cas des foncteur de type f_*). On a également la dualité de Verdier d'après [CD16, 6.2.17] et [CD19, 4.4.24].

Proposition 1.2.2.9. *Soit S un schéma régulier, R un anneau commutatif et K_S un objet \otimes -invertible de $\mathcal{DM}_{\text{ét},c}(S, R)$. Si $f: X \rightarrow S$ est un morphisme séparé de type fini, on pose*

$$D_{X/S} := \underline{\text{Hom}}(-, f^!K_S).$$

On a alors les propriétés suivantes.

1. *Pour tout S -schéma séparé de type fini X , la transformation naturelle*

$$\text{Id} \rightarrow D_{X/S} \circ D_{X/S}$$

est une équivalence.

2. *On suppose de plus que R est une \mathbb{Q} -algèbre. Alors, pour tout S -schéma séparé de type fini X et toute paire (M, N) de motifs sur X , on a un isomorphisme canonique*

$$D_{X/S}(M \otimes D_{X/S}(N)) \simeq \underline{\text{Hom}}(M, N)$$

et pour tout morphisme $f: Y \rightarrow X$ entre S -schémas séparés de type fini, on a des isomorphismes canoniques

$$D_{Y/S} \circ f^* \simeq f^! \circ D_{X/S}$$

$$f^* \circ D_{X/S} \simeq D_{Y/S} \circ f^!$$

$$D_{X/S} \circ f_! \simeq f_* \circ D_{Y/S}$$

$$f_! \circ D_{Y/S} \simeq D_{X/S} \circ f_*.$$

Enfin, on a la propriété de continuité d'après [CD16, 6.3.7, 6.3.9].

Proposition 1.2.2.10. *Soit R une localisation de l'anneau des entiers d'un corps de nombres. On considère un schéma X qui est la limite d'un système projectif de schémas $(X_i)_{i \in I}$ avec applications de transitions affines.*

Alors, on a

$$\operatorname{colim} \mathcal{DM}_{\text{ét},c}(X_i, R)^\otimes = \mathcal{DM}_{\text{ét},c}(X, R)^\otimes$$

dans la catégorie des catégories monoïdales.

En particulier, si $(M_i)_{i \in I}$ et $(N_i)_{i \in I}$ sont des sections cartésiennes de la catégorie fibrée $\mathcal{DM}_{\text{ét},c}(-, R)$ au-dessus du diagramme de schémas $(X_i)_{i \in I}$ et si on note M et N les tirés en arrière respectifs de M_i et N_i le long des applications $X \rightarrow X_i$, alors les applications canoniques

$$\operatorname{colim}_{i \in I} \operatorname{Hom}(M_i, N_i) \rightarrow \operatorname{Hom}(M, N)$$

$$\operatorname{colim}_{i \in I} \operatorname{Map}(M_i, N_i) \rightarrow \operatorname{Map}(M, N)$$

sont des équivalences.

1.2.3 La réalisation ℓ -adique et le formalisme de Bhatt-Scholze

Changement de coefficients

Soit $\sigma: R \rightarrow R'$ un morphisme d'anneaux. L'adjonction

$$\sigma^*: \operatorname{Mod}_R \rightleftarrows \operatorname{Mod}'_R: \sigma_*$$

fournit une adjonction pré motivique

$$\sigma^*: \mathcal{DM}_{\text{ét}}(-, R) \rightleftarrows \mathcal{DM}_{\text{ét}}(-, R'): \sigma_*.$$

Si S est un schéma, on peut de plus identifier $\mathcal{DM}_{\text{ét}}(S, R)$ avec la catégorie des $R_S(S)$ -modules dans $\mathcal{SH}_{\text{ét}}(S)$. Le foncteur σ_* est alors le foncteur d'oubli et le foncteur σ^* est le foncteur $- \otimes_R R' := - \otimes_{R_S(S)} R'_S(S)$.

Enfin, l'application $\sigma \mapsto \sigma^*$ provient du foncteur

$$\operatorname{CAlg}(\mathbf{Ab}) \xrightarrow{H} \operatorname{CAlg}(\mathcal{S}\mathcal{P}) \xrightarrow{h(S) \otimes \operatorname{Id}} \operatorname{CAlg}(\widehat{\operatorname{Sm}}_S \otimes \mathcal{S}\mathcal{P}) \xrightarrow{L_{\mathbb{A}^1} L_{\text{ét}}} \operatorname{CAlg}(\mathcal{SH}(S)) \xrightarrow{\operatorname{Mod}_-} \operatorname{CAlg}(\operatorname{Pr}_{\text{Stb}}^L).$$

Motifs ℓ -complets

Dans ce paragraphe, on donne la construction de la réalisation ℓ -adique. On fixe pour cela un anneau de valuation discrète R de paramètre local ℓ tel que le corps $R/\ell R$ est fini. On note \mathfrak{l} la caractéristique de ce dernier.

Définition 1.2.3.1. *Soit \mathcal{C} une catégorie stable R -linéaire. Un objet M de \mathcal{C} est ℓ -complet si l'application naturelle*

$$M \rightarrow \lim M \otimes_R (R/\ell^n R)$$

est une équivalence. On note \mathcal{C}_ℓ la sous-catégorie pleine de \mathcal{C} formée des objets ℓ -complets.

On a un foncteur de ℓ -complétion $\rho_\ell^0: \mathcal{C} \rightarrow \mathcal{C}_\ell$ qui envoie un objet M vers l'objet $\lim M \otimes_R (R/\ell^n R)$. Ce foncteur est adjoint à gauche de l'inclusion $\mathcal{C}_\ell \subseteq \mathcal{C}$ et est d'après [CD16, 7.2.8] la localisation par rapport aux applications $0 \rightarrow N$ où N parcourt l'ensemble des objets tels que la multiplication par ℓ est inversible sur N .

Si \mathcal{D} est une catégorie prémotivique, on pose

$$\mathcal{D}_\ell(S) = \mathcal{D}(S)_\ell.$$

Par ce qui précède, on a une unique structure monoïdale sur $\mathcal{D}(S)_\ell$ telle que le foncteur ρ_ℓ^0 est monoïdal. Les foncteurs du type $\rho_\ell^0 f^*$ pour $f: T \rightarrow S$ un morphisme de schémas font alors de \mathcal{D}_ℓ une catégorie prémotivique stable et on a une adjonction prémotivique

$$\rho_\ell^0: \mathcal{D} \rightleftarrows \mathcal{D}_\ell.$$

Comme les foncteurs de type f_* et $\underline{\text{Hom}}$ sont compatibles aux limites, ils commutent avec le foncteur ρ_ℓ^0 .

Si la catégorie \mathcal{D} est motivique, la catégorie \mathcal{D}_ℓ est également motivique. De plus, si elle est orientée, la catégorie \mathcal{D}_ℓ est également orientée. Enfin, si \mathcal{D} vérifie la pureté absolue, la catégorie \mathcal{D}_ℓ aussi. On déduit de [CD16, A.1.17] que le foncteur ρ_ℓ^0 commute avec les six opérations. En notant $\mathcal{DM}_{\text{ét}}(-, R_\ell)$ la catégorie motivique $\mathcal{DM}_{\text{ét}}(-, R)_\ell$, on a par conséquent le résultat suivant (voir [CD16, 7.2.11]).

Proposition 1.2.3.2. *La catégorie prémotivique $\mathcal{DM}_{\text{ét}}(-, R_\ell)$ satisfait le formalisme des six foncteurs et la pureté absolue. De plus, le foncteur ρ_ℓ^0 commute aux six foncteurs.*

On a la variante suivante du théorème de rigidité d'après [AGV22, 2.10.4].

Proposition 1.2.3.3. *Le foncteur $\mathcal{S}\mathfrak{h}(-_{\text{ét}}, R)_\ell$ est une catégorie prémotivique lorsqu'on le restreint à la catégorie des $\mathbb{Z}[1/\mathfrak{l}]$ -schémas et l'adjonction*

$$\rho_! : \mathcal{S}\mathfrak{h}(-_{\text{ét}}, R)_\ell \rightleftarrows \mathcal{DM}_{\text{ét}}(S, R_\ell) : \rho^!$$

est une équivalence de catégories prémotiviques.

Enfin, on a une version constructible de cette variante du théorème de rigidité. On peut alors définir la catégorie des objets ℓ -complets constructibles comme suit.

Définition 1.2.3.4. *Soit S un schéma.*

1. La catégorie $\mathcal{DM}_{\text{ét},c}(S, R_\ell)$ des motifs étales ℓ -complets constructibles est la plus petite sous-catégorie de $\mathcal{DM}_{\text{ét}}(S, R)$ stable par limites finies, colimites finies et rétractes et contenant les motifs ℓ -complets de la forme $\rho_\ell^0(M_S(X))(n)$ avec X lisse sur S .
2. Un faisceau étale ℓ -complet est constructible s'il existe une stratification de X telle que la restriction de ce faisceau à chaque strate est dualisable. On note $\mathcal{S}\mathfrak{h}_{\ell,c}(S_{\text{ét}}, R)$ la sous-catégorie pleine de $\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)_\ell$ formée des faisceaux constructibles.

On déduit de la proposition précédente le lien suivant entre ces deux catégories (voir [CD16, 7.2.21])

Proposition 1.2.3.5. *Soit S un $\mathbb{Z}[1/\ell]$ -schéma. Le foncteur ρ_ℓ induit une équivalence de catégories monoïdales*

$$\mathcal{S}\mathfrak{h}_{\ell,c}(S_{\text{ét}}, R) \rightarrow \mathcal{DM}_{\text{ét},c}(S, R_\ell).$$

Enfin, on déduit de la Proposition [1.2.3.2] le résultat suivant.

Proposition 1.2.3.6. ([CD16, 7.2.16]) *Les six foncteurs préservent la catégorie $\mathcal{DM}_{\text{ét},c}(-, R_\ell)$ des motifs étales ℓ -complets constructibles si on se restreint aux schémas (noethériens de dimension finie) quasi-excellents et aux morphismes de type fini entre ces schémas.*

Le formalisme de Bhatt-Scholze revisité

On fixe comme dans le paragraphe précédent un anneau de valuation discrète R et un paramètre local ℓ de R et on suppose de plus que le corps $R/\ell R$ est fini. On note R_ℓ le complété de R pour la valuation ℓ -adique et ℓ la caractéristique de $R/\ell R$. On expose ici le formalisme ℓ -adique de Bhatt-Scholze dans la version proposée par [HRS20].

L'anneau topologique R_ℓ définit pour tout schéma X un faisceau $R_{\ell,X}$ sur le site $X_{\text{proét}}$ à valeurs dans la catégorie $\mathcal{S}\mathfrak{p}$. Ce faisceau est alors une algèbre dans la catégorie $\mathcal{S}\mathfrak{h}(X_{\text{proét}}, \mathcal{S}\mathfrak{p})$ des hyperfaisceaux proétales sur $X_{\text{proét}}$.

Définition 1.2.3.7. 1. *On définit la catégorie stable monoïdale présentable $\mathcal{D}(X, R_\ell)^\otimes$ des faisceaux ℓ -adiques comme la catégorie des $R_{\ell,X}$ -modules dans $\mathcal{S}\mathfrak{h}(X_{\text{proét}}, \mathcal{S}\mathfrak{p})$.*

2. *Un faisceau ℓ -adique est constructible s'il existe une stratification de X telle que la restriction de ce faisceau à chaque strate est dualisable. La sous-catégorie pleine $\mathcal{D}_c^b(X, R_\ell)$ de $\mathcal{D}(X, R_\ell)$ formée des objets constructibles est une sous-catégorie monoïdale.*

On peut définir de manière analogue la catégorie des faisceaux étales constructibles que l'on note $\mathcal{S}\mathfrak{h}_c(X_{\text{ét}}, R)$.

Les foncteurs de type f^* font de $\mathcal{S}\mathfrak{h}(-, R_\ell)$ une catégorie prémotivique. On peut en fait montrer qu'elle est motivique si l'on se restreint aux $\mathbb{Z}[1/\ell]$ -schémas en combinant des résultats classiques de [AGV73] avec le résultat ci-dessous dû à Bhatt et Scholze.

Pour formuler ce dernier, rappelons qu'on a un morphisme de sites

$$\nu: S_{\text{proét}} \rightarrow S_{\text{ét}},$$

lequel induit pour tout entier n une adjonction prémotivique

$$\nu^*: \mathcal{S}\mathfrak{h}(-_{\text{ét}}, R/\ell^n R) \rightleftarrows \mathcal{S}\mathfrak{h}(-_{\text{proét}}, R/\ell^n R): \nu_*$$

D'après [HRS20, 3.40], on a alors le résultat suivant.

Proposition 1.2.3.8. *Soit X un schéma. Le foncteur ν^* induit pour tout entier n une équivalence de catégories*

$$\nu^*: \mathcal{S}\mathcal{H}_c(X_{\text{ét}}, R/\ell^n R) \rightarrow \mathcal{S}\mathcal{H}_c(X_{\text{proét}}, R/\ell^n R).$$

De plus, la formule $M \mapsto \lim \nu^*(M)/\ell^n$ induit un foncteur

$$\Psi: \mathcal{S}\mathcal{H}_{\ell,c}(X_{\text{ét}}, R) \rightarrow \mathcal{D}_c^b(X, R_\ell)$$

qui est une équivalence.

Définition 1.2.3.9. *La réalisation ℓ -adique est le foncteur*

$$\rho_\ell = \Psi \rho^! \rho_\ell^0: \mathcal{DM}_{\text{ét},c}(S, R) \rightarrow \mathcal{D}_c^b(S, R_\ell).$$

On déduit le résultat suivant de [CD16, 7.2.16].

Proposition 1.2.3.10. *Le foncteur*

$$\rho_\ell: \mathcal{DM}_{\text{ét},c}(-, R) \rightarrow \mathcal{D}_c^b(-, R_\ell)$$

commute aux six foncteurs si l'on se restreint aux schémas quasi-excellents et aux morphismes de type fini entre ces schémas.

La réalisation ℓ -adique rationnelle

On fixe comme dans le paragraphe précédent un anneau de valuation discrète R et un paramètre local ℓ de R . On note R_ℓ le complété de R pour la valuation ℓ -adique. Soit K le corps des fractions de R . On note K_ℓ le complété de K pour la valuation ℓ -adique. On définit comme avant la catégorie $\mathcal{D}_c^b(X, K_\ell)$.

Définition 1.2.3.11. *Soit \mathcal{C} une catégorie stable R -linéaire. On définit la K -linéarisation $\mathcal{C} \otimes_R K$ de \mathcal{C} comme la complétion idempotente de la catégorie stable K -linéaire ayant les mêmes objets que \mathcal{C} et telle que*

$$\text{Map}_{\mathcal{C} \otimes_R K}(M, N) = \text{Map}_{\mathcal{C}}(M, N) \otimes_R K.$$

D'après [CD16, 7.2.23], on a

$$\mathcal{D}_c^b(S, K_\ell) = \mathcal{D}_c^b(S, R) \otimes_R K$$

et

$$\mathcal{DM}_{\text{ét},c}(S, K) = \mathcal{DM}_{\text{ét},c}(S, R) \otimes_R K.$$

On définit alors comme suit le foncteur de réalisation ℓ -adique.

Définition 1.2.3.12. *La réalisation ℓ -adique est le foncteur*

$$\rho_\ell \otimes_R K: \mathcal{DM}_{\text{ét},c}(S, K) \rightarrow \mathcal{D}_c^b(S, K_\ell)$$

que l'on note encore ρ_ℓ .

Comme on dispose d'une équivalence pré-motivique entre les motifs étales et les h-motifs, [CD16, 7.2.24] implique le résultat suivant (on renvoie aussi le lecteur à [Ayo14c, 6.6]).

Proposition 1.2.3.13. *Le foncteur*

$$\rho_\ell: \mathcal{DM}_{\text{ét},c}(-, K) \rightarrow \mathcal{D}_c^b(-, K_\ell)$$

commute aux six foncteurs si l'on se restreint aux schémas quasi-excellents et aux morphismes de type fini entre ces schémas.

Chapter 2

Artin Perverse Sheaves

We show that the perverse t-structure induces a t-structure on the category $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ of Artin ℓ -adic complexes when S is an excellent scheme of dimension less than 2 and provide a counterexample in dimension 3. The heart $\text{Perv}^A(S, \mathbb{Z}_\ell)$ of this t-structure can be described explicitly in terms of representations in the case of 1-dimensional schemes.

When S is of finite type over a finite field, we also construct a perverse homotopy t-structure over $\mathcal{D}^A(S, \mathbb{Q}_\ell)$ and show that it is the best possible approximation of the perverse t-structure. We describe the simple objects of its heart $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ and show that the weightless truncation functor ω^0 is t-exact. We also show that the weightless intersection complex $EC_S = \omega^0 IC_S$ is a simple Artin homotopy perverse sheaf. If S is a surface, it is also a perverse sheaf but it need not be simple in the category of perverse sheaves.

Introduction

In this chapter, we aim to define abelian categories of Artin perverse sheaves. Those categories should be categories of perverse sheaves which come from finite schemes over the base scheme. Letting S be a noetherian finite dimensional scheme, we study to this end the stable ∞ -category of *Artin ℓ -adic complexes* $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ which is defined as the thick stable ∞ -subcategory of the stable ∞ -category $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ of ℓ -adic complexes generated by the complexes of the form $f_* \mathbb{Z}_{\ell, X}$ for $f: X \rightarrow S$ finite, where $\mathbb{Z}_{\ell, X}$ denotes the constant sheaf with value \mathbb{Z}_ℓ over X . To define categories of perverse sheaves, we will define perverse t-structures on those categories.

The motivation for this problem comes from the conjectural motivic t-structure. We have at our disposal several stable ∞ -categories of constructible étale motives $\mathcal{DM}_{\text{ét}, c}(-)$ endowed with Grothendieck's six functors formalism and with ℓ -adic realization functors [Voe92, VSF11, Ayo07, Ayo14c, CD19, CD16]. Let S be a noetherian finite-dimensional scheme. One of the major open problems regarding the stable ∞ -category $\mathcal{DM}_{\text{ét}, c}(S)$ is the existence of a motivic t-structure, whose heart would be the abelian category of mixed motivic sheaves that satisfies Beilinson's conjectures [Jan94].

More precisely, working by analogy with the stable ∞ -category $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$, there are two possible versions of the motivic t-structure: the *perverse motivic t-structure* and the *ordinary motivic t-structure*. As Beilinson proved in [Bei87b] that the category $\mathcal{D}_c^b(S, \mathbb{Q}_\ell)$ is the bounded derived category of the abelian category of perverse sheaves over S , the category $\mathcal{DM}_{\text{ét}, c}(S)$ is by analogy conjectured to be the derived category of the conjectural abelian category of *perverse motives* which would be the heart of the conjectural motivic t-structure. In [Bon15],

Bondarko has shown that the perverse motivic t-structure on the stable ∞ -category $\mathcal{DM}_{\acute{e}t,c}(S)$ can be recovered from the ordinary motivic t-structures on the stable ∞ -categories $\mathcal{DM}_{\acute{e}t,c}(K)$ (assuming they exist) when K runs through the set of residue fields of S .

Proving that the motivic t-structure exists seems completely out of reach at the moment. However, if $n = 0, 1$, the ordinary motivic t-structure has been constructed in [Org04, Ayo11, AB09, BVK16, Pep19b, Pep19a, Vai19] on the subcategory $\mathcal{DM}_{\acute{e}t,c}^n(S, \mathbb{Q})$ of *constructible n -motives*, which is the thick stable ∞ -subcategory generated by the cohomological motives of proper S -schemes of dimension less than n . We state their main result:

Theorem. (Voevodsky, Orgogozo, Ayoub, Barbieri-Viale, Kahn, Pepin Lehalleur, Vaish) *Let S be a noetherian excellent finite dimensional scheme admitting resolution of singularities by alterations. Let ℓ be a prime number invertible on S . Let $n = 0, 1$.*

Then, there exists a non-degenerate t-structure on the stable ∞ -category $\mathcal{DM}_{\acute{e}t,c}^n(S, \mathbb{Q})$ such that the ℓ -adic realization functor:

$$\mathcal{DM}_{\acute{e}t,c}^n(S, \mathbb{Q}) \rightarrow \mathcal{D}_c^b(S, \mathbb{Q}_\ell)$$

is t-exact when the stable ∞ -category $\mathcal{D}_c^b(S, \mathbb{Q}_\ell)$ is endowed with its ordinary t-structure.

In the case of the dimensional subcategories $\mathcal{DM}_{\acute{e}t,c}^n(S, \mathbb{Q})$, Bondarko's approach, which defines the perverse motivic t-structure from the ordinary motivic t-structure, does not apply. It indeed requires to use Grothendieck's six functors formalism and some of the six functors do not behave well with respect to the dimensional subcategories. Therefore, the existence of the perverse motivic t-structure on the stable ∞ -category $\mathcal{DM}_{\acute{e}t,c}^n(S, \mathbb{Q})$ does not follow formally from the existence of the ordinary motivic t-structure on the stable ∞ -categories $\mathcal{DM}_{\acute{e}t,c}^n(K, \mathbb{Q})$ when K runs through the set of residue fields of S .

Another natural problem is to determine whether those results can be extended to the case of étale motives with integral coefficients: Voevodsky has shown in [VSF11, 4.3.8] that it is not possible to construct a reasonable motivic t-structure on the stable ∞ -category of *Nisnevich motives* $\mathcal{DM}_{\text{Nis},c}(S, \mathbb{Z})$; however, nothing is known about the t-structure on the stable ∞ -category of étale motives.

Hence, the above theorem leaves two open questions:

1. Is there a t-structure on the stable ∞ -category $\mathcal{DM}_{\acute{e}t,c}^n(S, \mathbb{Z})$ such that the ℓ -adic realization functor:

$$\mathcal{DM}_{\acute{e}t,c}^n(S, \mathbb{Z}) \rightarrow \mathcal{D}_c^b(S, \mathbb{Z}_\ell)$$

is t-exact when $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ is endowed with its *ordinary* t-structure ?

2. Let $R = \mathbb{Z}$ or \mathbb{Q} . Is there a t-structure on the stable ∞ -category $\mathcal{DM}_{\acute{e}t,c}^n(S, R)$ such that the ℓ -adic realization functor:

$$\mathcal{DM}_{\acute{e}t,c}^n(S, R) \rightarrow \mathcal{D}_c^b(S, R_\ell)$$

is t-exact when $\mathcal{D}_c^b(S, R_\ell)$ is endowed with its *perverse* t-structure ?

One of the goals of this chapter is to determine what answer to this question could reasonably be expected in the case when $n = 0$, looking at the analogous situation in the setting of ℓ -adic complexes. Namely, the stable ∞ -category $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ is the ℓ -adic analog of the stable ∞ -category of 0-motives (sometimes called Artin motives). The analog of the above problem is

then to determine whether the ordinary and the perverse t-structures on the stable ∞ -category of constructible ℓ -adic complexes induce a t-structure on the stable ∞ -category of Artin ℓ -adic complex. The advantage of this setting is that the ordinary and the perverse t-structure exist while they are still conjectural in the motivic world.

To investigate this problem and to state some of our results, we need to introduce the stable ∞ -subcategory $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$ of $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ of *smooth Artin ℓ -adic complexes* which is the stable ∞ -subcategories of $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ generated by the $f_*\mathbb{Z}_{\ell, X}$, for $f: X \rightarrow S$ finite and étale. This category plays in this chapter the role that the category of lisse ℓ -adic complexes plays in the theory of perverse sheaves. It is closely linked to a subclass of representations of the pro-étale fundamental group of S .

To explain this link, we use two ingredients. On the first hand, we need Bhatt and Scholze's construction of the category of constructible ℓ -adic complex: they defined it in [BS15] as the category of constructible objects of the derived category $\mathcal{D}(\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell))$ of sheaves of $\mathbb{Z}_{\ell, S}$ -modules on the pro-étale site of S . In the spirit of [BS15, 7], if ξ is a geometric point of S and if S is connected, the abelian category of continuous representations of the pro-étale fundamental group $\pi_1^{\mathrm{pro\acute{e}t}}(S, \xi)$ is equivalent to the abelian category of *lisse ℓ -adic sheaves*.

On the second hand, letting π be a topological group, we introduce the abelian category of *representations of Artin origin* of π as the category of iterated extensions of Artin representations of π in the category of continuous representations. We then say that a lisse ℓ -adic sheaf is *smooth Artin* if it corresponds to a representation of Artin origin of $\pi_1^{\mathrm{pro\acute{e}t}}(S, \xi)$. We can extend this definition to non-connected schemes by asking the sheaf to be smooth Artin when restricted to every connected component. We can now state the link between smooth Artin ℓ -adic sheaves and representations more precisely:

Proposition. (Proposition 2.2.2.1) *Let S be a noetherian finite dimensional scheme and let ℓ be a prime number which is invertible on S . Then, the category $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$ is the category of those ℓ -adic complexes whose ordinary cohomology sheaves are smooth Artin. In particular, the ordinary t-structure induces a t-structure on $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$ whose heart is the category of smooth Artin ℓ -adic sheaves.*

To answer our question regarding the ordinary t-structure, we introduce the notion of Artin ℓ -adic sheaves by mimicking the definition of constructible sheaves: an ℓ -adic sheaf is *Artin* if there is a stratification of S such that its pullback to any stratum is smooth Artin.

Proposition. (Corollary 2.2.4.3) *Let S be a noetherian finite dimensional scheme and let ℓ be a prime number which is invertible on S . Then, the category $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ is the category of those ℓ -adic complexes whose ordinary cohomology sheaves are Artin. In particular, the ordinary t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ whose heart is the category of Artin ℓ -adic sheaves.*

Our main result concerns the perverse t-structure:

Theorem. (Theorem 2.3.3.4 and Proposition 2.3.4.1) *Let S be a noetherian finite dimensional excellent scheme and let ℓ be a prime number which is invertible on S . Assume that S is of dimension at most 2, or that S is of dimension 3 and that the residue fields of the closed points of S are separably closed or real closed. Then, the perverse t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$.*

We let $\mathrm{Perv}^A(S, \mathbb{Z}_\ell)$ be the heart of the induced perverse t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ when it is defined.

The above result seems rather optimal: if k is a finite field and ℓ is a prime number invertible on k , we prove (see Example [2.3.5.2](#)) that the perverse t-structure of $\mathcal{D}_c^b(\mathbb{A}_k^3, \mathbb{Z}_\ell)$ does not induce a t-structure on $\mathcal{D}^A(\mathbb{A}_k^3, \mathbb{Z}_\ell)$.

To circumvent this negative result, we introduce another notion of Artin perverse sheaves inspired by the approach of [\[Pep19b\]](#), [\[Vai19\]](#), [\[Ayo07\]](#), 2.2.4 and [\[BD17\]](#). More precisely, we define a *perverse homotopy t-structure* on the category $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ of *Ind-Artin ℓ -adic complexes* formally by specifying the set of t-negative objects. It has the following properties (see Proposition [2.4.5.1](#)).

- When the perverse homotopy t-structure on the stable ∞ -category $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ induces a t-structure on the subcategory $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, the induced t-structure is final among those such that the inclusion functor

$$\mathcal{D}^A(S, \mathbb{Z}_\ell) \rightarrow \mathcal{D}_c^b(S, \mathbb{Z}_\ell)$$

is right t-exact when the right hand side is endowed with its perverse t-structure.

- When the perverse t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, then so does the perverse homotopy t-structure and both induced t-structures coincide.

The properties of this t-structure are closely related to the properties of the right adjoint ω^0 of the inclusion functor

$$\iota: \mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell) \rightarrow \mathcal{D}_{\text{Ind}}^{\text{coh}}(S, \mathbb{Z}_\ell)$$

of Ind-Artin ℓ -adic complexes into Ind-cohomological ℓ -adic complexes. The motivic analog of this functor was first introduced in [\[AZ12\]](#) to study the reductive Borel-Serre compactification of a symmetric variety.

When S is of finite type over \mathbb{F}_p and working with coefficients in \mathbb{Q}_ℓ , the functor ω^0 is more tractable. We have at our disposal the stable ∞ -category of mixed ℓ -adic sheaves $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)$ of [\[BBDG18\]](#), 5 and Deligne's theory of weights (see [\[Del80\]](#)). Using the truncation functors defined by Morel in [\[Mor05\]](#), Nair and Vaish defined in [\[NV15\]](#) a *weightless truncation functor*

$$\omega^0: \mathcal{D}_m^b(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_m^b(S, \mathbb{Q}_\ell)$$

(denoted by $w_{\leq \text{Id}}$ in [\[NV15\]](#)) which they claim to be the ℓ -adic analog of Ayoub and Zucker's functor.

We show the following results:

Proposition. (*Propositions [2.4.6.2](#) and [2.4.7.1](#) and Remarks [2.4.6.3](#) and [2.4.7.3](#)*) *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number.*

1. *The two functors denoted above by ω^0 induce equivalent functors*

$$\mathcal{D}^{\text{coh}}(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}^A(S, \mathbb{Q}_\ell).$$

In particular, both functors send cohomological ℓ -adic complexes to Artin ℓ -adic complexes.

2. *The perverse homotopy t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Q}_\ell)$. We denote its heart by $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ and call it the abelian category of Artin homotopy perverse sheaves.*

3. The abelian categories $\text{Perv}^A(S, \mathbb{Q}_\ell)$ and $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ coincide when they are both well defined.

To study the abelian category $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$, we would like to show that the functor ω^0 is t-exact. However, this t-exactness statement does not make sense as it is: it is unknown whether the perverse truncation functors preserve cohomological complexes. To get around this issue, we define the stable ∞ -category $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$ of *weightless ℓ -adic complexes* as the essential image of the weightless truncation functor. The main insight leading to this definition is that Artin ℓ -adic complexes are the cohomological complexes which are also weightless. This stable ∞ -category $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$ is then endowed with a t-structure such that the inclusion of $\mathcal{D}^A(S, \mathbb{Q}_\ell)$ equipped with its perverse homotopy t-structure is t-exact (see Proposition 2.4.8.8(3)). We can now provide a meaningful t-exactness statement:

Theorem. (Theorem 2.4.8.10) *Let S be a scheme of finite type over \mathbb{F}_p and let $\ell \neq p$ be a prime number. Let*

$$\iota: \mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_m^b(S, \mathbb{Q}_\ell)$$

be the inclusion. When the left hand side is endowed with the weightless perverse t-structure and the right hand side is endowed with the perverse t-structure, the adjunction (ι, ω^0) is a t-adjunction and the functor ω^0 is t-exact.

A consequence of this result is that the abelian category $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ is similar to the abelian category of perverse sheaves and contains the weightless intersection complex of [NV15]:

Proposition. (Propositions 2.4.9.4 and 2.4.9.7 and Example 2.4.9.8) *Let S be a scheme of finite type over \mathbb{F}_p and let $\ell \neq p$ be a prime number.*

1. *All the objects of $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ are of finite length. Its simple objects can be described as the $\omega^0 j_{!*}(L[\dim(V)])$ where $j: V \hookrightarrow S$ runs through the set of inclusions of smooth subschemes of S and L runs through the set of simple smooth Artin ℓ -adic sheaves over V .*
2. *Let IC_S be the intersection complex of S . The weightless intersection complex $EC_S = \omega^0 IC_S$ of [NV15] is a simple object of the abelian category $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$.*
3. *If S is a surface, then, the complex EC_S is also a perverse sheaf, but it need not be simple in the category of perverse sheaves.*

Finally, we describe the category $\text{Perv}^A(C, \mathbb{Z}_\ell)$ when C is an excellent scheme of dimension 1. Let Γ be the set of generic points of C . Let $\nu: \tilde{C} \rightarrow C$ be the normalization of C . In this setting, we introduce a residue map ∂_y for any closed point y of \tilde{C} in Lemma 2.5.1.5. The category $\text{Perv}^A(C, \mathbb{Z}_\ell)$ is equivalent to the category $\text{P}^A(C, \mathbb{Z}_\ell)$ (see Definition 2.5.2.1) which is defined as follows:

- An object of $\text{P}^A(C, \mathbb{Z}_\ell)$ is a quadruple $(U, (M_\eta)_{\eta \in \Gamma}, (M_x)_{x \in C \setminus U}, (f_x)_{x \in C \setminus U})$ where
 1. U is a nil-regular open subset of C such that the immersion $U \rightarrow C$ is affine,
 2. for all $\eta \in \Gamma$, M_η is an ℓ -adic representation of Artin origin of $\pi_1^{\text{ét}}(U_\eta, \bar{\eta})$,
 3. for all $x \in C \setminus U$, M_x is a complex in $\mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((BG_{k(x)})_{\text{proét}}, \mathbb{Z}_\ell))$ (see Remark 2.1.5.8) placed in degrees $[0, 1]$,

4. for all $x \in C \setminus U$, f_x is an element of

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Rep}(G_{k(x)}, \mathbb{Z}_\ell))} \left(M_x, \bigoplus_{\nu(y)=x} \mathrm{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left[\phi_y^*(M_{\eta(y)}) \right] \right) \right)$$

such that $H^0(f_x)$ is injective.

- An element of $\mathrm{Hom}_{\mathcal{P}^A(C, \mathbb{Z}_\ell)}((U, (M_\eta), (M_x), (f_x)), (V, (N_\eta), (N_x), (g_x)))$ is a couple of the form $((\Phi_\eta)_{\eta \in \Gamma}, (\Phi_x)_{x \in (C \setminus U) \cap (C \setminus V)})$ where $\Phi_\eta: M_\eta \rightarrow N_\eta$ is a map of representations of G_{K_η} , where $\Phi_x: M_x \rightarrow N_x$ is a map in $\mathcal{D}(\mathrm{Sh}((BG_{k(x)})_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell))$ and where the diagram

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & \bigoplus_{\nu(y)=x} \mathrm{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left[\phi_y^*(M_{\eta(y)}) \right] \right) \\ \downarrow \Phi_x & & \downarrow \bigoplus_{\nu(y)=x} \mathrm{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left(\phi_y^*(\Phi_{\eta(y)}) \right) \right) \\ N_x & \xrightarrow{g_x} & \bigoplus_{\nu(y)=x} \mathrm{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left[\phi_y^*(N_{\eta(y)}) \right] \right) \end{array}$$

is commutative.

In the next chapter, we study the motivic case and prove that we have similar results. One of the main interests of the results of this chapter is that the motivic constructions will realize into our constructions.

2.1 Preliminaries

All schemes are assumed to be **noetherian** and of **finite dimension**; furthermore all smooth (and étale) morphisms and all quasi-finite morphisms are also implicitly assumed to be separated and of finite type.

In this text we will freely use the language of $(\infty, 1)$ -categories of [Lur09, Lur17]. The term *category* will by default mean $(\infty, 1)$ -category, and the term *2-category* will mean $(\infty, 2)$ -category. When we refer to derived categories, we refer to the ∞ -categorical version.

There are several definitions of the derived category of constructible ℓ -adic sheaves. The first definition was given in [BBDG18] over schemes over a field. This definition was then extended to all noetherian finite dimensional schemes in [Eke07 [CD16, 7.2], [BS15] and [LYF14, XIII.4]. In this chapter, we will use the model of [BS15]. We will use the theorems of [BBDG18, 4] over excellent schemes; in particular, to use the affine Lefschetz theorem [BBDG18, 4.1.1] and its consequences, one has to replace [AGV73, XIV.3.1] in the proof of [BBDG18, 4.1.1] with [LYF14, XV.1.1.2].

We adopt the cohomological convention for t-structures (*i.e* the convention of [BBDG18, 1.3.1] and the opposite of [Lur17, 1.2.1.1]): a t-structure on a stable category \mathcal{D} is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of strictly full subcategories of \mathcal{D} having the following properties:

- For any object M of $\mathcal{D}^{\leq 0}$ and any object N of $\mathcal{D}^{\geq 0}$, $\pi_0 \mathrm{Map}(M, N[-1]) = 0$.
- We have inclusions $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}[-1]$ and $\mathcal{D}^{\geq 0}[-1] \subseteq \mathcal{D}^{\geq 0}$.

- For any object M of \mathcal{D} , there exists an exact triangle $M' \rightarrow M \rightarrow M''$ where M' is an object of $\mathcal{D}^{\leq 0}$ and M'' is an object of $\mathcal{D}^{\geq 0}[-1]$.

A stable category endowed with a t-structure is called a t-category. If \mathcal{D} is a t-category, we denote by $\mathcal{D}^\heartsuit = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ the heart of the t-structure which is an abelian category and by \mathcal{D}^b the full subcategory of bounded objects which is a stable t-category.

Let $i: Z \rightarrow X$ be a closed immersion and $j: U \rightarrow X$ be the complementary open immersion. We call localization triangles the exact triangles of functors on ℓ -adic sheaves over X :

$$j_!j^* \rightarrow \text{Id} \rightarrow i_*i^*. \quad (2.1.0.1)$$

$$i_!i^! \rightarrow \text{Id} \rightarrow j_*j^*. \quad (2.1.0.2)$$

Let S be a scheme. A *stratification* of S is a partition \mathcal{S} of S into non-empty equidimensional locally closed subschemes called *strata* such that the topological closure of any stratum is a union strata.

If S is a scheme and ξ is a geometric point of S , we denote by $\pi_1^{\text{ét}}(S, \xi)$ the étale fundamental group of S with base point ξ defined in [GR71] and $\pi_1^{\text{proét}}(S, \xi)$ the pro-étale fundamental group of S with base point ξ defined in [BS15, 7].

If k is a field, we denote by G_k its absolute Galois group.

If p is a prime number, we denote by \mathbb{F}_p the field with p elements, by \mathbb{Z}_p the ring of p -adic integers, by \mathbb{Q}_p the field of p -adic numbers and by $\overline{\mathbb{Q}_p}$ its algebraic closure.

If $\phi: H \rightarrow G$ is a map of groups, we denote by ϕ^* the forgetful functor which maps representations of G to representations of H .

If H is a finite index subgroup of a group G , we denote by Ind_H^G the induced representation functor which maps representations of H to representations of G .

2.1.1 Notions of abelian and stable subcategories generated by a set of objects

Recall the following definitions (see [Sta23, 02MN]).

Definition 2.1.1.1. *Let \mathcal{A} be an abelian category.*

1. A Serre subcategory of \mathcal{A} is a nonempty full subcategory \mathcal{B} of \mathcal{A} such that given an exact sequence

$$M' \rightarrow M \rightarrow M''$$

such that M' and M'' are objects of \mathcal{B} , then M is an object of \mathcal{B} .

2. A weak Serre subcategory of \mathcal{A} is a nonempty full subcategory \mathcal{B} of \mathcal{A} such that given an exact sequence

$$M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow M_5$$

such that for all $i \in \{1, 2, 4, 5\}$, M_i is an object of \mathcal{B} , then M_3 is an object of \mathcal{B} .

3. Let \mathcal{E} be a set of objects of \mathcal{A} , the Serre (resp. weak Serre) subcategory of \mathcal{A} generated by \mathcal{E} is the smallest Serre (resp. weak Serre) subcategory of \mathcal{A} whose set of objects contains \mathcal{E} .

Note that a Serre subcategory of an abelian category A is also a weak Serre subcategory of A and that if B is a weak Serre subcategory of A , then B is abelian and the inclusion functor $B \rightarrow A$ is exact and fully faithful.

Recall that a strictly full subcategory is a full subcategory whose set of objects is closed under isomorphisms.

Proposition 2.1.1.2. *Let A be a strictly full abelian subcategory of an abelian category B . Assume that A is closed under subobject and quotient in B .*

Let A^ be the full subcategory of B spanned by the objects obtained as successive extensions of the objects A . Then*

1. *The category A^* is the smallest weak Serre subcategory of B containing A .*
2. *The category A^* is a Serre subcategory of B .*

Proof. Any weak Serre subcategory containing A is closed under extensions and therefore contains A^* . Hence it suffices to show that A^* is a Serre subcategory of B .

The subcategory A^* is strictly full, contains the zero object and any extension of objects of A^* is an object of A^* . Therefore using [Sta23, 02MP], it suffices to show that A^* is closed under subobject and quotient.

We first define subcategories of B by induction. Let $A_0 = A$ and if $n \geq 0$ is an integer, let A_{n+1} be the full subcategory of B whose objects are the objects M of B such that there exists an exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

where M' and M'' are objects of A_n .

By definition, A^* is the full subcategory of B whose objects are objects of A_n for some non-negative integer n . Therefore it suffices to show that for all $n \geq 0$, the subcategory A_n is closed under subobject and quotient.

The case where $n = 0$ is the hypothesis on A . Assume that any subobject and any quotient of any object of A_n lies in A_n and let M be an object of A_{n+1} . Let N be a subobject of M , we show that N and M/N are objects of A_{n+1} .

We have an exact sequence

$$0 \rightarrow M' \rightarrow M \xrightarrow{\pi} M'' \rightarrow 0$$

where M' and M'' are objects of A_n . This yields exact sequences:

$$\begin{aligned} 0 \rightarrow \ker(\pi|_N) \rightarrow N \xrightarrow{\pi|_N} \pi(N) \rightarrow 0, \\ 0 \rightarrow M'/\ker(\pi|_N) \rightarrow M/N \rightarrow M''/\pi(N) \rightarrow 0. \end{aligned}$$

Since $\pi(N)$ is a subobject of M'' , it is an object of A_n and since $\ker(\pi|_N)$ is a subobject of M' , it is also an object of A_n . Thus N is an object of A_{n+1} . Similarly, M/N is an object of A_{n+1} . \square

Definition 2.1.1.3. *Let \mathcal{C} be a stable category*

1. *A thick subcategory of \mathcal{C} is a full subcategory \mathcal{D} of \mathcal{C} which is closed under finite limits, finite colimits and retracts.*

2. A localizing subcategory of \mathcal{C} is a full subcategory \mathcal{D} of \mathcal{C} which is closed under finite limits and arbitrary colimits.
3. Let \mathcal{E} be a set of objects. We call thick (resp. localizing) subcategory generated by \mathcal{E} the smallest thick (resp. localizing) subcategory of \mathcal{C} whose set of objects contains \mathcal{E} .

2.1.2 Induced t-structures

Recall the following definition of [BBDG18, 1.3.19]:

Definition 2.1.2.1. Let \mathcal{D} be a t-category and \mathcal{D}' be a full stable subcategory of \mathcal{D} . If $(\mathcal{D}^{\leq 0} \cap \mathcal{D}', \mathcal{D}^{\geq 0} \cap \mathcal{D}')$ defines a t-structure on \mathcal{D}' , we say that \mathcal{D}' is a sub-t-category of \mathcal{D} , that the t-structure of \mathcal{D} induces a t-structure on \mathcal{D}' and call the latter the induced t-structure.

First notice the following properties on sub-t-categories:

Proposition 2.1.2.2. Let \mathcal{D} be a t-category and \mathcal{D}_0 be a strictly full sub-t-category of \mathcal{D}^b . Then,

1. The subcategory \mathcal{D}_0 is thick.
2. The category \mathcal{D}_0^\heartsuit is a weak Serre subcategory of \mathcal{D}^\heartsuit .

Proof. We first prove (1). Since \mathcal{D}_0 is a strictly full stable subcategory of \mathcal{D} , it is closed under finite limits and colimits.

Let M be an object of \mathcal{D}_0 . Assume that $M = M' \oplus M''$ in \mathcal{D} . We prove that M' is an object of \mathcal{D}_0 .

Since M is bounded, so is M' . Hence, by dévissage, it suffices to show that $H^k(M')$ lies in \mathcal{D}_0 for any integer k . Therefore, we can assume that M lies in \mathcal{D}^\heartsuit .

Let $p: M \rightarrow M$ be the projector corresponding to M' . Let C be the cone of p . Then, $M' = \ker(p) = H^0(C)$ lies in \mathcal{D}_0^\heartsuit which proves (1).

We now prove (2). Since the inclusion functor $\mathcal{D}_0^\heartsuit \rightarrow \mathcal{D}^\heartsuit$ is fully faithful and exact, by [Sta23, 0754], it suffices to show that \mathcal{D}_0^\heartsuit is closed under extensions. But if

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is an exact sequence in \mathcal{D}^\heartsuit with M' and M'' objects of \mathcal{D}_0^\heartsuit , the triangle

$$M' \rightarrow M \rightarrow M''$$

is exact in \mathcal{D} . Since \mathcal{D}_0 is a stable subcategory of \mathcal{D} , this proves that M is an object of \mathcal{D}_0 and thus of \mathcal{D}_0^\heartsuit . This finishes the proof. \square

The following lemma will allow us to restrict t-structures to thick subcategories.

Lemma 2.1.2.3. Let \mathcal{D} be a t-category. Let \mathcal{D}' be a thick subcategory of \mathcal{D}^b . Assume that $\mathcal{D}' \cap \mathcal{D}^\heartsuit$ contains a set of generators of \mathcal{D}' as a thick subcategory of \mathcal{D} . Then, the following conditions are equivalent.

1. The t-structure of \mathcal{D} induces a t-structure on \mathcal{D}' .

2. If $f: M \rightarrow N$ is a map in $\mathcal{D}' \cap \mathcal{D}^\heartsuit$, $\ker(f)$ is an object of \mathcal{D}' .

3. If $f: M \rightarrow N$ is a map in $\mathcal{D}' \cap \mathcal{D}^\heartsuit$, $\operatorname{coker}(f)$ is an object of \mathcal{D}' .

Proof. Condition (1) implies conditions (2) and (3). If $f: M \rightarrow N$ is a map in $\mathcal{D}' \cap \mathcal{D}^\heartsuit$ and C is the cone of f , we have an exact triangle:

$$\ker(f) \rightarrow C \rightarrow \operatorname{coker}(f)[-1].$$

Thus, conditions (2) and (3) are equivalent.

Assume now that conditions (2) and (3) hold. First notice that $\mathcal{D}' \cap \mathcal{D}^\heartsuit$ is closed under extensions since \mathcal{D}' is. Therefore, the subcategory $\mathcal{D}' \cap \mathcal{D}^\heartsuit$ of \mathcal{D}^\heartsuit is weak Serre using [Sta23, 0754].

We consider the full subcategory \mathcal{C} of \mathcal{D}' made of those objects M such that for all integer n , the object $H^n(M)$ lies in \mathcal{D}' . The subcategory \mathcal{C} is closed under suspensions and direct factors. We now show that it is closed under extensions.

Let

$$M' \rightarrow M \rightarrow M''$$

be an exact triangle in \mathcal{D}' such that M' and M'' are objects of \mathcal{C} . Then, if n is an integer, we get an exact sequence in \mathcal{D}^\heartsuit :

$$H^{n-1}(M'') \rightarrow H^n(M') \rightarrow H^n(M) \rightarrow H^n(M'') \rightarrow H^{n+1}(M').$$

Since $\mathcal{D}' \cap \mathcal{D}^\heartsuit$ is a weak Serre subcategory of \mathcal{D}^\heartsuit and since M' and M'' are objects of \mathcal{C} , the object $H^n(M)$ lies in \mathcal{D}' .

Therefore, the subcategory \mathcal{C} is thick and contains a set of generators of \mathcal{D}' , whence, we have $\mathcal{C} = \mathcal{D}'$. This shows that the subcategory \mathcal{D}' of \mathcal{D}^b is stable under the truncation functor $\tau_{\geq 0}$, thus by [BBDG18, 1.3.19], the t-structure of \mathcal{D} induces a t-structure on \mathcal{D}' . \square

In fact, Lemma 2.1.2.3 implies that weak Serre subcategories of the heart of a t-category on the one hand and strictly full bounded sub-t-category of the same t-category on the other hand are in one-to-one correspondence. More precisely, we define the following particular sub-t-categories:

Definition 2.1.2.4. Let \mathcal{D} be a t-category. Let A be a weak Serre subcategory of \mathcal{D}^\heartsuit . We will denote by \mathcal{D}_A^b the full subcategory of \mathcal{D} spanned by those bounded objects C of \mathcal{D} such that for all integer n , the object $H^n(C)$ is in A .

Remark 2.1.2.5. Be careful that in general, the category \mathcal{D}_A^b need not coincide with the category $\mathcal{D}^b(A)$. However, by [BCKW19, 7.60], there is a canonical t-exact functor:

$$\mathcal{D}^b(A) \rightarrow \mathcal{D}_A^b.$$

We then have the following proposition:

Proposition 2.1.2.6. Let \mathcal{D} be a t-category. Then,

1. Let A be a weak Serre subcategory of \mathcal{D}^\heartsuit . Then, the category \mathcal{D}_A^b is a strictly full sub-t-category of \mathcal{D}^b . Moreover, we have $(\mathcal{D}_A^b)^\heartsuit = A$ as subcategories of \mathcal{D}^\heartsuit .

2. Let \mathcal{D}_0 be a strictly full sub-t-category of \mathcal{D}^b . Then, the category \mathcal{D}_0^\heartsuit is a weak Serre subcategory of \mathcal{D}^\heartsuit and $\mathcal{D}_{\mathcal{D}_0^\heartsuit}^b = \mathcal{D}_0$ as subcategories of \mathcal{D} .

Proof. The first assertion follows from Lemma 2.1.2.3 and from the fact that A generates \mathcal{D}_A^b as a thick subcategory of \mathcal{D}^b .

The first part of the second assertion follows from Proposition 2.1.2.2.

Finally, the cohomology objects of any object of \mathcal{D}_0 lie in \mathcal{D}_0^\heartsuit . Thus, any object of \mathcal{D}_0 lies in $\mathcal{D}_{\mathcal{D}_0^\heartsuit}^b$. Moreover, if M is an object of $\mathcal{D}_{\mathcal{D}_0^\heartsuit}^b$, to show that it lies in \mathcal{D}_0 , it suffices by dévissage to show that for all n , the object $H^n(M)$ is an object of \mathcal{D}_0 . This is true by definition. \square

2.1.3 Finite permutation resolutions

In this paragraph, we recall a result of Balmer and Gallauer (see [BG23]). If G is a finite group, R is a ring and H is a subgroup of G , the R -module $R[G/H]$ is endowed with a left $R[G]$ -module structure letting G act on G/H by left multiplication.

Proposition 2.1.3.1. *Let G be a finite group. Let R be a regular ring and let M be a representation of G with coefficients in R . Then, the complex $M \oplus M[1]$ of $R[G]$ -modules is quasi-isomorphic as a complex of $R[G]$ -modules to a complex*

$$\cdots \rightarrow 0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0 \rightarrow \cdots$$

where each P_i is placed in degree $-i$ and is isomorphic to $R[G/H_1] \oplus \cdots \oplus R[G/H_m]$ for some subgroups H_1, \dots, H_m of G .

Proof. We say that a bounded complex P of $R[G]$ -modules is a *permutation complex* if the P_i are isomorphic to $R[G/H_1] \oplus \cdots \oplus R[G/H_m]$ for some subgroups H_1, \dots, H_m of G .

Let $\mathcal{P}(R, G)$ be the subcategory of $\mathcal{D}^b(R[G])$ made of those complexes N which admit a quasi-isomorphism $P \rightarrow N$, where P is permutation complex. Then, [BG23, 4.6] states that the idempotent completion of $\mathcal{P}(R, G)$ is the category $\mathcal{D}^b(R[G])$ itself.

Moreover, using [BG23, 2.13], the idempotent completion of $\mathcal{P}(R, G)$ is the subcategory made of those complexes N such that $N \oplus N[1]$ lies in $\mathcal{P}(R, G)$. Thus, the complex $M \oplus M[1]$ admits a quasi-isomorphism $P \rightarrow M \oplus M[1]$, where P is permutation complex.

Finally, noting that $M \oplus M[1]$ lies in nonpositive degree, we can modify P so that the P_i vanish for $i > 0$ by [BG23, 2.4] which yields the result. \square

2.1.4 Artin representations and representations of Artin origin

Let ℓ be a prime number. The topology on \mathbb{Z}_ℓ induces a topology on every \mathbb{Z}_ℓ -module of finite type.

In this paragraph and in the following paragraphs, we only consider the case of \mathbb{Z}_ℓ -coefficients for simplicity; however, all the definitions can be formulated with coefficients in \mathbb{Q}_ℓ (or even a finite extension of \mathbb{Q}_ℓ , its ring of integers, $\overline{\mathbb{Q}}_\ell$ or the subring of elements of $\overline{\mathbb{Q}}_\ell$ which are integral over \mathbb{Z}_ℓ) and all the propositions still hold.

Recall the following definitions:

Definition 2.1.4.1. *Let π be a topological group and let ℓ be a prime number.*

1. A continuous representation of π with coefficients in \mathbb{Z}_ℓ is a \mathbb{Z}_ℓ -module of finite type endowed with a continuous action of π . We will denote by $\text{Rep}(\pi, \mathbb{Z}_\ell)$ the abelian category of continuous representation of π with coefficients in \mathbb{Z}_ℓ .
2. An Artin representation of π is a continuous representation of π which factors through a finite quotient of π . We will denote by $\text{Rep}^A(\pi, \mathbb{Z}_\ell)$ the abelian category of Artin representation of π with coefficients in \mathbb{Z}_ℓ .

The category $\text{Rep}^A(\pi, \mathbb{Z}_\ell)$ is a strictly full abelian subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$; in general, it is not a weak Serre subcategory. Indeed, consider an additive character $\chi: \pi \rightarrow \mathbb{Z}_\ell$ that does not factor through a finite quotient (for example if $\pi = \hat{\mathbb{Z}}$, write $\hat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$ and take the projection to \mathbb{Z}_ℓ). Then, we have a morphism $\pi \rightarrow GL_2(\mathbb{Z}_\ell)$ given by

$$\begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}$$

which defines a continuous representation M of π on \mathbb{Z}_ℓ^2 . The exact sequence:

$$0 \rightarrow \mathbb{Z}_\ell \rightarrow M \rightarrow \mathbb{Z}_\ell \rightarrow 0$$

is an exact sequence of representations if π acts trivially on both copies of \mathbb{Z}_ℓ . Hence, this gives an extension of Artin representations which is not Artin.

This leads us to the following definition:

Definition 2.1.4.2. *Let π be a topological group and let ℓ be a prime number. A continuous representation is of Artin origin if it is obtained as a successive extension of Artin representations in $\text{Rep}(\pi, \mathbb{Z}_\ell)$. We denote by $\text{Rep}^A(\pi, \mathbb{Z}_\ell)^*$ the category of representation of Artin origin.*

Proposition [2.1.1.2](#) implies the following proposition.

Proposition 2.1.4.3. *Let π be a topological group and let ℓ be a prime number. Then,*

1. *The category $\text{Rep}^A(\pi, \mathbb{Z}_\ell)^*$ is the smallest weak Serre subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$ containing $\text{Rep}^A(\pi, \mathbb{Z}_\ell)$.*
2. *The category $\text{Rep}^A(\pi, \mathbb{Z}_\ell)^*$ is a Serre subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$.*

The following proposition shows that the Artin origin condition can be checked with coefficients in \mathbb{Q}_ℓ .

Proposition 2.1.4.4. *Let π be a topological group. Let M be a continuous representation with coefficients in \mathbb{Z}_ℓ . Assume that $M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a representation of Artin origin with coefficients in \mathbb{Q}_ℓ . Then, M itself is of Artin origin.*

Proof. Let $A_0 = \text{Rep}^A(\pi, \mathbb{Q}_\ell)$ and if n is a non-negative integer, let A_{n+1} be the full subcategory of $\text{Rep}(\pi, \mathbb{Q}_\ell)$ whose objects are the extensions of the objects of A_n in $\text{Rep}(\pi, \mathbb{Q}_\ell)$. There is a non-negative integer n such that $M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is an object of A_n .

Let n be a non-negative integer. We prove by induction on n that if M is a continuous representation with coefficients in \mathbb{Z}_ℓ such that $M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ lies in A_n , then M is of Artin origin.

Notice first that we can assume M to be torsion free. Indeed, let M_{tors} be the sub- \mathbb{Z}_ℓ -module of torsion elements of M . Then, the representation on M induces a sub-representation on M_{tors} . Moreover, we have an exact sequence of representations:

$$0 \rightarrow M_{\text{tors}} \rightarrow M \rightarrow M/M_{\text{tors}} \rightarrow 0.$$

Now, the abelian group M_{tors} is finite. Therefore, the representation on M_{tors} is an Artin representation. Moreover, we have

$$M/M_{\text{tors}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Thus, we can assume that M is torsion free.

Now, if, $n = 0$, the representation $M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is an Artin representation. Since M is torsion free, the group homomorphism

$$\text{Aut}_{\mathbb{Z}_\ell}(M) \rightarrow \text{GL}_{\mathbb{Q}_\ell}(M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$$

is injective. Therefore, the morphism $\pi \rightarrow \text{Aut}_{\mathbb{Z}_\ell}(M)$ which gives rise to the $\mathbb{Z}_\ell[\pi]$ -module structure of M factors through a finite quotient and thus M is an Artin representation.

Assume now that $n > 0$. We get an exact sequence:

$$0 \rightarrow N' \xrightarrow{i} M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{p} N'' \rightarrow 0$$

such that N' and N'' are objects of \mathbf{A}_{n-1} .

Let M' (resp. M'') be the inverse image (resp. the image) of M along the map i (resp. through the map p).

There is a \mathbb{Q}_ℓ -basis of N' whose image lies in M . Therefore,

$$N' = M' \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Furthermore, there is a \mathbb{Q}_ℓ -basis of N'' who lies in the image of M . Thus,

$$N'' = M'' \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Hence, by induction, M' and M'' are of Artin origin.

The map i and p induce maps $i_0: M' \rightarrow M$ and $p_0: M \rightarrow M''$ such that $p_0 \circ i_0 = 0$. Letting K be the kernel of p_0 , the map i_0 factors through K .

Moreover, the rank of K as a \mathbb{Z}_ℓ -module is $\dim_{\mathbb{Q}_\ell}(M) - \dim_{\mathbb{Q}_\ell}(M'')$ which is also the rank of M' . This yields two exact sequences:

$$0 \rightarrow K \rightarrow M \rightarrow M'' \rightarrow 0$$

$$0 \rightarrow M' \rightarrow K \rightarrow F \rightarrow 0$$

where F is of rank 0.

Since F is finite as an abelian group, it is an Artin representation. Thus, K is of Artin origin and finally M is of Artin origin. \square

Artin representations are contained in a more explicit Serre subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$ which we will call *strongly of weight 0*. To prove that a representation is not of Artin origin, it is often convenient to prove that it is not strongly of weight 0.

Definition 2.1.4.5. Let π be a topological group and ℓ be a prime number. Let M be a continuous representation of π with coefficients in \mathbb{Z}_ℓ . We say that M is strongly of weight 0 if for all $g \in \pi$, the eigenvalues of the matrix defined by the action of g on $M \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$ are roots of unity. We will denote by $\text{Rep}^{w=0}(\pi, \mathbb{Z}_\ell)$ the abelian category of representations that are strongly of weight 0.

Remark 2.1.4.6. In the theory of weights developed by Deligne in [Del80], letting k be a finite field and letting $\pi = G_k = \hat{\mathbb{Z}}$ be its absolute Galois group, an ℓ -adic representation of G_k is of weight 0 if the action of the topological generator F of G_k are algebraic numbers such that all their conjugates are of absolute value 1 as complex numbers (i.e. are roots of unity by a theorem of Kronecker). Hence, representations that are strongly of weight 0 of G_k are contained in representations of weight 0 of G_k .

Proposition 2.1.4.7. Let π be a topological group and ℓ be a prime number. The abelian category $\text{Rep}^{w=0}(\pi, \mathbb{Z}_\ell)$ is a Serre subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$ and contains the category representations of Artin origin.

Proof. First, any sub-representation and any quotient of a representation that is strongly of weight 0 is strongly of weight 0. Now, if

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is an exact sequence of continuous representations where M' and M'' are strongly of weight 0, the sequence

$$0 \rightarrow M' \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell} \rightarrow M \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell} \rightarrow M'' \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell} \rightarrow 0$$

is still exact and if g is an element of π , its action on M is the action of a matrix of the form $A = \begin{bmatrix} A' & * \\ 0 & A'' \end{bmatrix}$ where A' (resp. A'') is the matrix of the action of g on M' (resp. M''). The set of eigenvalues of A is the union of the set of eigenvalues of A' and A'' and those are roots of unity. Hence, the category $\text{Rep}^{w=0}(\pi, \mathbb{Z}_\ell)$ is a Serre subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$.

Now, it suffices to show that Artin representations are strongly of weight 0. Let M be an Artin representation of π . Then, the representation $M \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$ is an Artin representation of π with coefficients in $\overline{\mathbb{Q}_\ell}$. Since it factors through a finite quotient G of π , if $g \in \pi$, the element $g^{|G|}$ of π acts trivially on $M \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$. \square

2.1.5 Categories of ℓ -adic sheaves and complexes

In this paragraph, we recall the definitions of [BS15, HRS20] of various ℓ -adic categories. If S is a scheme, denote by $S_{\text{proét}}$ the pro-étale site of S introduced in [BS15, 4].

Any totally disconnected topological space M induces a sheaf on $S_{\text{proét}}$:

$$U \mapsto \mathcal{C}^0(U, M) = \mathcal{C}^0(\pi_0(U), M).$$

we denote this sheaf by M_S and call it the constant sheaf of value M . If $M = \Lambda$ is a totally disconnected topological ring, then the sheaf Λ_S is a sheaf of rings on $S_{\text{proét}}$.

Definition 2.1.5.1. Let S be a scheme (noetherian by conventions) and ℓ be a prime number invertible on S . We define:

1. The abelian category of ℓ -adic sheaves $\mathrm{Sh}(S, \mathbb{Z}_\ell)$ as the abelian category of sheaves of $\mathbb{Z}_{\ell,S}$ -modules on $S_{\mathrm{pro\acute{e}t}}$.
2. The abelian category of lisse ℓ -adic sheaves $\mathrm{Loc}_S(\mathbb{Z}_\ell)$ as the category of sheaves of $\mathbb{Z}_{\ell,S}$ -modules L such that there exists a pro-étale cover $\{S_i \rightarrow S\}$ such that for all i , there exists a \mathbb{Z}_ℓ -module of finite type M such that $L|_{S_i}$ is isomorphic to M_{S_i} .
3. The abelian category of constructible ℓ -adic sheaves $\mathrm{Sh}_c(S, \mathbb{Z}_\ell)$ as the category of sheaves of $\mathbb{Z}_{\ell,S}$ -modules L such that there exists a stratification of S such that for any stratum T , the sheaf $L|_T$ is lisse.
4. The derived category of ℓ -adic sheaves $\mathcal{D}(S, \mathbb{Z}_\ell)$ as the derived category of $\mathrm{Sh}(S, \mathbb{Z}_\ell)$.
5. The stable category of lisse ℓ -adic sheaves $\mathcal{D}_{\mathrm{lisse}}(S, \mathbb{Z}_\ell)$ as category of dualizable objects of $\mathcal{D}(S, \mathbb{Z}_\ell)$.
6. The derived category of constructible ℓ -adic sheaves $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ as the category of complexes of sheaves of $\mathbb{Z}_{\ell,S}$ -modules C such that there exists a stratification of S such that for any stratum T , the sheaf $C|_T$ is lisse.

Remark 2.1.5.2. Beware that in [BS15], they denote by $\mathrm{Loc}_S(\mathbb{Z}_\ell)$ the category of locally free sheaves of $\mathbb{Z}_{\ell,S}$ -modules of finite rank. We want the category of lisse sheaves to be abelian, hence we chose to include all sheaves that are locally of finite type. This is also the convention of [Del77].

These categories are related to each other by the following proposition.

Proposition 2.1.5.3. ([HRS20, 3.28]) Let S be a scheme and ℓ be a prime number invertible on S . The canonical t -structure of $\mathcal{D}(S, \mathbb{Z}_\ell)$ induces a t -structure on $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ (resp. $\mathcal{D}_{\mathrm{lisse}}(S, \mathbb{Z}_\ell)$) whose heart is $\mathrm{Sh}_c(S, \mathbb{Z}_\ell)$ (resp. $\mathrm{Loc}_S(\mathbb{Z}_\ell)$).

The above proposition shows in particular that $\mathrm{Sh}_c(S, \mathbb{Z}_\ell)$ and $\mathrm{Loc}_S(\mathbb{Z}_\ell)$ are weak Serre subcategories of $\mathrm{Sh}(S, \mathbb{Z}_\ell)$. Furthermore, we have the following characterization of lisse sheaves:

Proposition 2.1.5.4. Let S be a connected scheme and ℓ be a prime number invertible on S . Let ξ be a geometric point of S . The fiber functor associated to ξ induces an equivalence of abelian monoidal categories

$$\xi^* : \mathrm{Loc}_S(\mathbb{Z}_\ell) \rightarrow \mathrm{Rep}(\pi_1^{\mathrm{pro\acute{e}t}}(S, \xi), \mathbb{Z}_\ell)$$

where $\pi_1^{\mathrm{pro\acute{e}t}}(S, \xi)$ is the pro-étale fundamental group of S with base point ξ of [BS15, 7.4.2]

Remark 2.1.5.5. The proposition above with coefficients in \mathbb{Z}_ℓ still holds with the étale fundamental group by [Del77, Chapitre 2, 2.4]; we need the pro-étale fundamental group only in the case when the ring of coefficients is an algebraic extension of \mathbb{Q}_ℓ . In this setting, the proposition does not follow from the proof below. However, it is a consequence of [BS15, 7.4.7, 7.4.10].

Proof. Let Loc_S be the infinite Galois category (see [BS15, 7.2.1]) of locally constant sheaves of sets on $S_{\mathrm{pro\acute{e}t}}$. By definition of the pro-étale fundamental group, the fiber functor

$$\xi^* : \mathrm{Loc}_S \rightarrow \pi_1^{\mathrm{pro\acute{e}t}}(S, \xi) - \mathrm{Set}$$

is an equivalence.

The constant sheaf $(\mathbb{Z}/\ell^n\mathbb{Z})_S$ is sent to the set $\mathbb{Z}/\ell^n\mathbb{Z}$ endowed with the trivial action of $\pi_1^{\text{proét}}(S, \xi)$. Thus, ξ^* induces an equivalence

$$\text{Loc}_S(\mathbb{Z}/\ell^n\mathbb{Z}) \rightarrow \text{Rep}(\pi_1^{\text{proét}}(S, \xi), \mathbb{Z}/\ell^n\mathbb{Z}).$$

The proposition follows by passing to the limit in the 2-category of categories. \square

Recall that Bhatt and Scholze defined using the theory of infinite Galois categories of [BS15, 7.3] a map

$$\pi_1^{\text{proét}}(S, \xi) \rightarrow \pi_1^{\text{ét}}(S, \xi)$$

that exhibits $\pi_1^{\text{ét}}(S, \xi)$ as the profinite completion of $\pi_1^{\text{proét}}(S, \xi)$ by [BS15, 7.4.3], and which is a group isomorphism when S is geometrically unibranch. We now claim that the formation of $\pi_1^{\text{proét}}(S, \xi)$ can be upgraded into a functor with respect to (S, ξ) which is compatible with the map above.

Lemma 2.1.5.6. *Let $f: T \rightarrow S$ be a map of schemes. Let $\xi': \text{Spec}(\Omega) \rightarrow T$ be a geometric point of T and let $\xi = f \circ \xi'$. Then,*

1. *The map f induces a continuous map of groups*

$$\pi_1^{\text{proét}}(f): \pi_1^{\text{proét}}(T, \xi') \rightarrow \pi_1^{\text{proét}}(S, \xi)$$

such that $\pi_1^{\text{proét}}$ is a functor and the map $\pi_1^{\text{proét}} \rightarrow \pi_1^{\text{ét}}$ is a natural transformation.

2. *Assume that f is finite and étale, then, the map $\pi_1^{\text{proét}}(f)$ is injective and its image is a finite index subgroup.*

3. *Let ℓ be a prime number which is invertible on S . Then, the square*

$$\begin{array}{ccc} \text{Loc}_S(\mathbb{Z}_\ell) & \xrightarrow{f^*} & \text{Loc}_T(\mathbb{Z}_\ell) \\ \downarrow \xi^* & & \downarrow (\xi')^* \\ \text{Rep}(\pi_1^{\text{proét}}(S, \xi), \mathbb{Z}_\ell) & \xrightarrow{\pi_1^{\text{proét}}(f)^*} & \text{Rep}(\pi_1^{\text{proét}}(T, \xi'), \mathbb{Z}_\ell) \end{array}$$

is commutative.

Proof. If X is a scheme, let Loc_X (resp. $\text{Loc}_X^{\text{ét}}$) be the infinite Galois category (see [BS15, 7.2.1]) of locally constant sheaves of sets on $X_{\text{proét}}$ (resp. $X_{\text{ét}}$).

Now, if $f: T \rightarrow S$ is a map of schemes, let Loc_f (resp. $\text{Loc}_f^{\text{ét}}$) be the map $f^*: \text{Loc}_S \rightarrow \text{Loc}_T$. This defines contravariant functors Loc and $\text{Loc}^{\text{ét}}$ from the category of schemes to the category of infinite Galois category. Moreover, the map of Bhatt and Scholze between the pro-étale and the étale fundamental group is induced by the natural map $\text{Loc}^{\text{ét}} \rightarrow \text{Loc}$. Thus, the first assertion follows from the structure theorem of infinite Galois categories [BS15, 7.2.5].

Now, by definition of the map $\pi_1^{\text{proét}}(f)$, the square:

$$\begin{array}{ccc} \text{Loc}_S & \xrightarrow{f^*} & \text{Loc}_T \\ \downarrow \xi^* & & \downarrow (\xi')^* \\ \pi_1^{\text{proét}}(S, \xi) - \text{Set} & \xrightarrow{\pi_1^{\text{proét}}(f)^*} & \pi_1^{\text{proét}}(T, \xi') - \text{Set} \end{array}$$

is commutative which implies the third assertion.

We now prove the second assertion. We have a functor $f_*: \text{Loc}_T \rightarrow \text{Loc}_S$ which is both left and right adjoint to f^* and sends a locally constant sheaf L to the sheaf $X \mapsto L(X \times_S T)$. Moreover, if L is a locally constant sheaf on S , the map $L \rightarrow f_* f^* L$ is injective. Therefore, using [Lar21, 2.37], the map $\pi_1^{\text{proét}}(f)$ is a topological embedding.

Now, the profinite completion of a quotient of topological groups is the quotient of the profinite completions of those groups. Thus, the profinite completion of $\pi_1^{\text{proét}}(S, \xi)/\pi_1^{\text{proét}}(T, \xi')$ is $\pi_1^{\text{ét}}(S, \xi)/\pi_1^{\text{ét}}(T, \xi')$ which is finite by [GR71, V.6.3]. \square

Corollary 2.1.5.7. *Let $f: T \rightarrow S$ be a finite étale map and let ℓ be a prime number which is invertible on S . Let ξ' be a geometric point of T and let $\xi = f \circ \xi'$. Then, the functor $f_*: \text{Sh}(T, \mathbb{Z}_\ell) \rightarrow \text{Sh}(S, \mathbb{Z}_\ell)$ induces a map $\text{Loc}_T(\mathbb{Z}_\ell) \rightarrow \text{Loc}_S(\mathbb{Z}_\ell)$ and the square:*

$$\begin{array}{ccc} \text{Loc}_T(\mathbb{Z}_\ell) & \xrightarrow{f_*} & \text{Loc}_S(\mathbb{Z}_\ell) \\ \downarrow (\xi')^* & & \downarrow \xi^* \\ \text{Rep}(\pi_1^{\text{proét}}(T, \xi'), \mathbb{Z}_\ell) & \xrightarrow{\text{Ind}_{\pi'}^\pi} & \text{Rep}(\pi_1^{\text{proét}}(S, \xi), \mathbb{Z}_\ell) \end{array}$$

where $\pi' = \pi_1^{\text{proét}}(T, \xi')$ and $\pi = \pi_1^{\text{proét}}(S, \xi)$ is commutative.

Proof. The functor f_* is right adjoint to the functor f^* . On the other hand, the functor $\text{Ind}_{\pi'}^\pi$ is right adjoint to the functor $\pi_1^{\text{proét}}(f)^*$ which yields the result. \square

Finally, recall that the fibered category $\mathcal{D}_c^b(-, \mathbb{Z}_\ell)$ over the category of noetherian excellent schemes is endowed with the 6 functors formalism (see [BS15, 6.7]).

Remark 2.1.5.8. *If k is a field, the topos $\tilde{k}_{\text{proét}}$ is equivalent to the topos $(\widetilde{BG}_k)_{\text{proét}}$ where $(BG_k)_{\text{proét}}$ is the site of profinite continuous G_k -sets with covers given by continuous surjections (see [BS15, 4.1.10]). Therefore, the abelian category $\text{Sh}(k_{\text{proét}}, \mathbb{Z}_\ell)$ identifies with the abelian category $\text{Sh}((BG_k)_{\text{proét}}, \mathbb{Z}_\ell)$ of modules over the profinite G_k -set \mathbb{Z}_ℓ endowed with trivial action (in the case of coefficients in \mathbb{Q}_ℓ , the topological ring \mathbb{Q}_ℓ with its trivial action still defines a sheaf of rings on $(BG_k)_{\text{proét}}$ using [BS15, 4.3.2]).*

Finally, the stable category $\mathcal{D}_c^b(k, \mathbb{Z}_\ell)$ identifies with the stable subcategory of \mathbb{Z}_ℓ -perfect complexes $\mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((BG_k)_{\text{proét}}, \mathbb{Z}_\ell))$ of $\mathcal{D}^b(\text{Sh}((BG_k)_{\text{proét}}, \mathbb{Z}_\ell))$ which is the full subcategory made of those complexes whose restriction to $\mathcal{D}^b(\mathbb{Z}_\ell)$ is perfect (i.e. whose cohomology sheaves are finite when seen as \mathbb{Z}_ℓ -modules).

The heart of $\mathcal{D}_c^b(k, \mathbb{Z}_\ell)$ then identifies with $\text{Rep}(G_k, \mathbb{Z}_\ell)$.

2.1.6 Reminders on cdh-descent and Mayer-Vietoris for closed immersions

Definition 2.1.6.1. *We say following [Voe10, 2] and [CD19, 2.1.11] that a cartesian square of schemes:*

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow i \\ Y & \xrightarrow{f} & X \end{array}$$

is cdh-distinguished if f is proper and i is a closed immersion such that, letting U be its open complement in X , the map $f^{-1}(U) \rightarrow U$ induced by f is an isomorphism.

The definition of [CD16, 7.2] of ℓ -adic complexes (which is equivalent to the definition of [BS15] by [CD16, 7.2.21]) yields the following:

Proposition 2.1.6.2. (*cdh-descent*) *Let ℓ be a prime number. A cdh-distinguished square of $\mathbb{Z}[1/\ell]$ -schemes:*

$$\begin{array}{ccc} E & \xrightarrow{p} & F \\ \downarrow i_E & & \downarrow i \\ Y & \xrightarrow{f} & X \end{array}$$

induces an exact triangle:

$$\mathbb{Z}_{\ell, X} \rightarrow f_*\mathbb{Z}_{\ell, Y} \oplus i_*\mathbb{Z}_{\ell, F} \rightarrow i_*p_*\mathbb{Z}_{\ell, E}.$$

The following formulas are consequences of [DDØ21, 5.1.7]. They are closely linked to the Rapoport-Zink complex [RZ82, 2.5] and are generalizations of the Mayer-Vietoris exact triangle for closed immersions [Ayo07, 2.2.31].

Proposition 2.1.6.3. *Let $i: E \rightarrow Y$ be a closed immersion and let ℓ be a prime number invertible on Y . Assume that $E = \bigcup_{i \in I} E_i$ with I a finite set and $(E_i)_{i \in I}$ a family of closed subschemes of E . If $J \subseteq I$, write $E_J = \bigcap_{i \in J} E_i$ and write $i_J: E_J \rightarrow Y$ the canonical closed immersion. Then, the canonical maps $(i_J)_*i_J^!\mathbb{Z}_{\ell, Y} \rightarrow i_*i^!\mathbb{Z}_{\ell, Y}$ exhibit $i_*i^!\mathbb{Z}_{\ell, Y}$ as the colimit*

$$\operatorname{colim}_{p \in (\Delta^{\text{inj}})^{\text{op}}} \left(\bigoplus_{\substack{J \subseteq I \\ |J|=p+1}} (i_J)_*i_J^!\mathbb{Z}_{\ell, Y} \right).$$

where Δ^{inj} is the category of finite ordered sets with morphisms the injective maps.

Proposition 2.1.6.4. *Let $i: E \rightarrow Y$ be a closed immersion and let ℓ be a prime number invertible on Y . Assume that $E = \bigcup_{i \in I} E_i$ with I a finite set and $(E_i)_{i \in I}$ a family of closed subschemes of E . If $J \subseteq I$, write $E_J = \bigcap_{i \in J} E_i$ and write $i_J: E_J \rightarrow Y$ the canonical closed immersion. Then, the canonical maps $i_*i^*\mathbb{Z}_{\ell, Y} \rightarrow (i_J)_*i_J^*\mathbb{Z}_{\ell, Y}$ exhibit $i_*i^*\mathbb{Z}_{\ell, Y}$ as the limit*

$$\lim_{p \in (\Delta^{\text{inj}})^{\text{op}}} \left(\bigoplus_{\substack{J \subseteq I \\ |J|=p+1}} (i_J)_*i_J^*\mathbb{Z}_{\ell, Y} \right).$$

where Δ^{inj} is the category of finite ordered sets with morphisms the injective maps.

2.1.7 Perverse sheaves

In this paragraph, we recall the definition of the perverse t-structure over base scheme endowed with a dimension function following [Gab08]. First recall the definition of dimension functions (see for instance [BD17, 1.1.1]):

Definition 2.1.7.1. *Let S be a scheme. A dimension function on S is a function $\delta: S \rightarrow \mathbb{Z}$ such that for any immediate specialization y of a point x (i.e a specialization such that $\operatorname{codim}_{\overline{\{x\}}}(y) = 1$), we have $\delta(x) = \delta(y) + 1$.*

Two dimension functions on a scheme differ by a Zariski-locally constant function. Moreover, if S is universally catenary and integral, $\delta(x) = \dim(X) - \text{codim}_X(x)$ is a dimension function on S . Finally, any S -scheme of finite type $f: X \rightarrow S$ inherits a canonical dimension function by setting

$$\delta(x) = \delta(f(x)) + \text{tr.deg } k(x)/k(f(x)).$$

The auto-dual perversity $p_{1/2}: Z \mapsto -\delta(Z)$ induces two t-structures $[p_{1/2}]$ and $[p_{1/2}^+]$ on $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$. In the thesis, we only consider the t-structure $[p_{1/2}]$. We now recall its definition over a general base scheme.

Definition 2.1.7.2. (Compare with [Gab08, 2]) *Let S be a scheme endowed with a dimension function and ℓ be a prime number invertible on S . For every point $x \in S$, let $i_x: \{x\} \rightarrow S$ be the inclusion. We define*

- *the full subcategory ${}^p\mathcal{D}^{\leq 0}(S, \mathbb{Z}_\ell)$ as the subcategory of those complexes K such that for any point $x \in S$, we have $i_x^* K \leq p_{1/2}(\{x\}) = -\delta(x)$ with respect to the ordinary t-structure of $\mathcal{D}(k(x), \mathbb{Z}_\ell)$.*
- *the full subcategory ${}^p\mathcal{D}^{\geq 0}(S, \mathbb{Z}_\ell)$ as the subcategory of those complexes K such that for any point $x \in S$, we have $i_x^! K \geq p_{1/2}(\{x\}) = -\delta(x)$ with respect to the ordinary t-structure of $\mathcal{D}(k(x), \mathbb{Z}_\ell)$.*

In the case when the ring of coefficients is $\mathbb{Z}/\ell\mathbb{Z}$, [Gab08, 6] implies that this defines a t-structure on $\mathcal{D}(S, \mathbb{Z}/\ell\mathbb{Z})$ and [Gab08, 8.2] implies that $\mathcal{D}_c^b(S, \mathbb{Z}/\ell\mathbb{Z})$ is a sub-t-category of $\mathcal{D}(S, \mathbb{Z}/\ell\mathbb{Z})$. The proofs of [Gab08] actually apply with coefficients in \mathbb{Z}_ℓ which defines a t-structure on $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$.

Alternatively, when the scheme S is assumed to be excellent, this is a t-structure for the same reasons as in [BBDG18, 2.2.9-2.2.17] (see [Mor18, 2.2] for a more precise explanation; the proof uses the absolute purity property [LYF14, XVI.3.1]). The following result is an important step of the latter proof and will be used in the rest of the chapter.

Proposition 2.1.7.3. *Let S be a regular connected scheme endowed with a dimension function δ and ℓ be a prime number invertible on S . The perverse t-structure of $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ induces a t-structure on $\mathcal{D}_{\text{lisce}}(S, \mathbb{Z}_\ell)$ which coincides with the ordinary t-structure shifted by $\delta(S)$. In particular, the heart of this t-structure is $\text{Loc}_S(\mathbb{Z}_\ell)[\delta(S)]$.*

Proof. Since the category $\mathcal{D}_{\text{lisce}}(S, \mathbb{Z}_\ell)$ is bounded with respect to the shifted ordinary t-structure, by dévissage it suffices to show that the objects of $\text{Loc}_S(\mathbb{Z}_\ell)$ are in degree $\delta(S)$ with respect to the perverse t-structure.

Let L be a locally constant sheaf of \mathbb{Z}_ℓ -modules on S and let x be a point of S . The sheaf $i_x^* L$ is in degree $0 \leq \delta(S) - \delta(x)$. Furthermore, since the scheme S is regular and connected, by the absolute purity property [LYF14, XVI.3.1], we have

$$i_x^! L = i_x^* L(-c)[-2c]$$

where $c = \text{codim}_S(x) = \delta(S) - \delta(x)$. Therefore, the sheaf $i_x^! L$ is in degree $2c \geq \delta(S) - \delta(x)$. Hence, the sheaf L is in degree $\delta(S)$ with respect to the perverse t-structure which finishes the proof. \square

If n is an integer, we will denote by ${}^p H^n$ the n -th cohomology functor with respect to the perverse t-structure.

Definition 2.1.7.4. *Let S be a scheme endowed with a dimension function and ℓ be a prime number that is invertible on S . The category $\text{Perv}(S, \mathbb{Z}_\ell)$ of perverse sheaves is the heart of the perverse t -structure on $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$.*

Finally, note that Propositions [2.1.2.2](#) and [2.1.7.3](#) imply that $\text{Loc}_S(\mathbb{Z}_\ell)[\delta(S)]$ is a weak Serre subcategory of $\text{Perv}(S, \mathbb{Z}_\ell)$ when the scheme S is regular and connected.

2.2 Artin ℓ -adic complexes

In this section, we define abelian categories of Artin ℓ -adic sheaves and stable categories of Artin ℓ -adic complexes. We show that the ordinary t -structure on the derived category of ℓ -adic sheaves induces a t -structure on the category of Artin ℓ -adic complexes such that the heart of this t -structure is the category of Artin ℓ -adic sheaves.

2.2.1 Artin ℓ -adic sheaves and complexes

Definition 2.2.1.1. *Let S be a scheme and ℓ a prime number that is invertible on S .*

1. *We define the abelian category $\text{Sh}^{\text{smA}}(S, \mathbb{Z}_\ell)$ of smooth Artin ℓ -adic sheaves over S as the Serre subcategory of $\text{Loc}_S(\mathbb{Z}_\ell)$ made of those objects M such that for any geometric point ξ of S , the representation ξ^*M of $\pi_1^{\text{proét}}(S, \xi)$ is of Artin origin.*
2. *We define the abelian category $\text{Sh}^A(S, \mathbb{Z}_\ell)$ of Artin ℓ -adic sheaves over S as the category of constructible ℓ -adic sheaves such that there exists a stratification of S such that for any stratum T , the ℓ -adic sheaf $L|_T$ is smooth Artin.*

Remark 2.2.1.2. *Let S be a connected scheme and ℓ a prime number invertible on S . Let ξ be a geometric point of S . Then, the functor ξ^* induces an equivalence from $\text{Sh}^{\text{smA}}(S, \mathbb{Z}_\ell)$ to $\text{Rep}^A(\pi_1^{\text{proét}}(S, \xi), \mathbb{Z}_\ell)^*$ by Proposition [2.1.5.4](#).*

Definition 2.2.1.3. *Let S be a scheme and ℓ a prime number which is invertible on S .*

1. *The stable category $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ of Artin ℓ -adic complexes is the thick stable subcategory of the stable category $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ generated by the complexes of the form $f_*\mathbb{Z}_{\ell, X}$ with $f: X \rightarrow S$ finite.*
2. *The stable category $\mathcal{D}^{\text{smA}}(S, \mathbb{Z}_\ell)$ of smooth Artin ℓ -adic complexes over S with coefficients in \mathbb{Z}_ℓ is the thick stable subcategory of the stable category $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ generated by the complexes of the form $f_*\mathbb{Z}_{\ell, X}$ with $f: X \rightarrow S$ étale and finite.*
3. *The stable category $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ (resp. $\mathcal{D}_{\text{Ind}}^{\text{smA}}(S, \mathbb{Z}_\ell)$) is the localizing subcategory of the stable category $\mathcal{D}(S, \mathbb{Z}_\ell)$ with the generators as in (1) (resp. (2)).*

Proposition 2.2.1.4. *Let S be a scheme and ℓ a prime number which is invertible on S . Then, the stable category $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ is presentable.*

Proof. This follows from [[Lur17](#), 1.4.4.11(2)]. □

2.2.2 Description of smooth Artin ℓ -adic complexes

Proposition 2.2.2.1. *Let S be a scheme and ℓ a prime number which is invertible on S . The ordinary t -structure of $\mathcal{D}(S, \mathbb{Z}_\ell)$ induces a t -structure on $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$. The heart of this t -structure is equivalent to $\mathrm{Sh}^{smA}(S, \mathbb{Z}_\ell)$. In particular, the subcategory $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$ is the subcategory made of those complexes C of ℓ -adic sheaves such that for all integer n , the sheaf $H^n(C)$ is smooth Artin.*

Proof. We can assume that S is connected. By Lemma 2.1.2.3 and Proposition 2.1.2.6, it suffices to show that

$$\mathcal{D}^{smA}(S, \mathbb{Z}_\ell) \cap \mathrm{Sh}(S, \mathbb{Z}_\ell) = \mathrm{Sh}^{smA}(S, \mathbb{Z}_\ell).$$

Let $f: X \rightarrow S$ be étale and finite. Recall that f_* is t -exact with respect to the ordinary t -structure of $\mathcal{D}(S, \mathbb{Z}_\ell)$. Therefore, using Corollary 2.1.5.7, the object $f_*\mathbb{Z}_{\ell, X}$ lies in $\mathrm{Sh}^{smA}(S, \mathbb{Z}_\ell)$.

Thus, by Propositions 2.1.2.2, 2.1.2.6 and 2.1.4.3, the category $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$ is included in $\mathcal{D}^b(S, \mathbb{Z}_\ell)_{\mathrm{Sh}^{smA}(S, \mathbb{Z}_\ell)}$ (see Definition 2.1.2.4). Thus, we have

$$\mathcal{D}^{smA}(S, \mathbb{Z}_\ell) \cap \mathrm{Sh}(S, \mathbb{Z}_\ell) \subseteq \mathrm{Sh}^{smA}(S, \mathbb{Z}_\ell).$$

Take now M a smooth Artin ℓ -adic sheaf. It corresponds to a representation of Artin origin of $\pi_1^{\mathrm{proét}}(S, \xi)$ that we still denote by M .

Assume first that M is an Artin representation. Let G be a finite quotient of $\pi_1^{\mathrm{proét}}(S, \xi)$ through which M factors. Then, by Proposition 2.1.3.1 the complex $M \oplus M[1]$ of $\mathbb{Z}_\ell[G]$ -modules is quasi-isomorphic as a complex of $\mathbb{Z}_\ell[G]$ -modules to a complex

$$P = \cdots \rightarrow 0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0 \rightarrow \cdots$$

where each P_i is placed in degree $-i$ and is isomorphic to $\mathbb{Z}_\ell[G/H_1] \oplus \cdots \oplus \mathbb{Z}_\ell[G/H_m]$ for some subgroups H_1, \dots, H_m of G .

The exact functor

$$\mathrm{Rep}(G, \mathbb{Z}_\ell) \rightarrow \mathrm{Rep}(\pi_1^{\mathrm{proét}}(S, \xi), \mathbb{Z}_\ell) \rightarrow \mathrm{Sh}(S, \mathbb{Z}_\ell)$$

induces an exact functor

$$\mathcal{D}^b(\mathrm{Rep}(G, \mathbb{Z}_\ell)) \rightarrow \mathcal{D}(S, \mathbb{Z}_\ell).$$

If H is a subgroup of G , the finite set G/H is endowed with an action of $\pi_1^{\mathrm{proét}}(S, \xi)$ and therefore corresponds to a finite locally constant sheaf on $S_{\mathrm{ét}}$, and therefore to a finite étale S -scheme $f: X \rightarrow S$. The image of $\mathbb{Z}_\ell[G/H]$ through the functor above is then $f_*\mathbb{Z}_{\ell, X}$.

Thus, the object $M \oplus M[1]$, and therefore the object M itself, is in the thick category generated by the $f_*\mathbb{Z}_{\ell, X}$ with $f: X \rightarrow S$ finite and étale, which is $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$.

Now, since $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$ is closed under extensions, any representation of Artin origin of $\pi_1^{\mathrm{proét}}(S, \xi)$ lies in $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$ which finishes the proof. \square

Corollary 2.2.2.2. *If k is a field and ℓ is a prime number that is distinct from the characteristic of k , then, the category $\mathcal{D}^A(k, \mathbb{Z}_\ell)$ is equivalent to the category of the complexes C in $\mathcal{D}^b(\mathrm{Sh}((BG_k)_{\mathrm{proét}}, \mathbb{Z}_\ell))$ (see Remark 2.1.5.8) such that the sheaf $H^n(C)$ identifies with a representation of Artin origin for all n .*

2.2.3 The trace of the six functors on Artin ℓ -adic complexes

Artin ℓ -adic complexes are stable under some of the six functors of constructible ℓ -adic complexes. The results of this paragraph could also have been formulated with the categories $\mathcal{D}_{\text{Ind}}^A(-, \mathbb{Z}_\ell)$, replacing thick subcategories with localizing subcategories when needed. We will first show the following lemma which was proved by Ayoub and Zucker in the motivic setting and in the case of schemes of finite type over a field of characteristic 0 and was generalized to arbitrary schemes by Pepin Lehalleur.

Proposition 2.2.3.1. *Let S be a scheme and ℓ a prime number which is invertible on S . The category $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ is the thick subcategory of $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ generated by any of the following families of objects:*

1. $f_*\mathbb{Z}_{\ell, X}$ for $f: X \rightarrow S$ finite.
2. $f_!\mathbb{Z}_{\ell, X}$ for $f: X \rightarrow S$ étale.
3. $f_!\mathbb{Z}_{\ell, X}$ for $f: X \rightarrow S$ quasi-finite.

Proof. We follow the proof of [Pep19b, 1.28].

Let $j: U \rightarrow S$ be an open immersion. Let $i: F \rightarrow S$ be the reduced complementary closed immersion. By localization (2.1.0.1), we have an exact triangle

$$j_!\mathbb{Z}_{\ell, U} \rightarrow \mathbb{Z}_{\ell, S} \rightarrow i_*\mathbb{Z}_{\ell, F}.$$

Thus, $j_!\mathbb{Z}_{\ell, U}$ is an Artin ℓ -adic complex.

Now, let $f: X \rightarrow S$ be quasi-finite. Using Zariski's Main Theorem [Gro67, 18.12.13] and the case of open immersions, we get that $f_!\mathbb{Z}_{\ell, X}$ is an Artin ℓ -adic complex.

Let $\mathcal{C}(S)$ be the thick subcategory of $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ generated by the $f_!\mathbb{Z}_{\ell, X}$ with $f: X \rightarrow S$ étale. To prove the proposition, it suffices to show that if $g: Y \rightarrow S$ is finite, $g_*\mathbb{Z}_{\ell, Y}$ is in $\mathcal{C}(S)$.

Notice that the fibered category $\mathcal{C}(-)$ is closed under the functor p^* for any morphism p and the functor $q_!$ for any quasi-finite morphism q .

Furthermore, when g is a universal homeomorphism, we have $g_*\mathbb{Z}_{\ell, Y} = \mathbb{Z}_{\ell, S}$ by topological invariance of the small étale site (see [Sta23, 04DY]).

We now prove this claim by noetherian induction on S . We can assume that Y is reduced using the topological invariance of the small étale site once again, we have

$$g_*\mathbb{Z}_{\ell, Y} = g'_*\mathbb{Z}_{\ell, Y_{\text{red}}}$$

where Y_{red} is the reduction of Y and $g': Y_{\text{red}} \rightarrow S$ is the canonical morphism.

Moreover, there is an open immersion $j: U \rightarrow S$ such that the pullback map $g_U: Y \times_S U \rightarrow U$ is the composition of a finite étale map and a purely inseparable morphism. Thus, $j^*g_*\mathbb{Z}_{\ell, Y} = (g_U)_*\mathbb{Z}_{\ell, Y \times_S U}$ lies in $\mathcal{C}(U)$.

Furthermore, if $i: Z \rightarrow S$ is the reduced complementary immersion, the complex $i^*g_*\mathbb{Z}_{\ell, Y}$ is in $\mathcal{C}(Z)$ by noetherian induction and proper base change. Finally, by localization, it suffices to prove the lemma below. \square

Lemma 2.2.3.2. *Keep the same notations, then, the functor i_* sends $\mathcal{C}(Z)$ to $\mathcal{C}(S)$.*

Proof. We follow the proof of [Pep19b, 1.18(ii), 1.14(iii)]. It suffices to show that if $f: X \rightarrow Z$ is étale, $i_* f_! \mathbb{Z}_{\ell, X}$ lies in $\mathcal{C}(S)$. As in the proof of [Pep19b, 1.14] we use Mayer-Vietoris exact triangles to reduce to the case where $S = \text{Spec}(A)$, $Z = \text{Spec}(A/I)$ and $X = \text{Spec}(B)$ where A is a ring, I is an ideal of A and B is a standard étale A/I -algebra, *i.e.* B is of the form

$$B = (A/I)[x_1, \dots, x_n]/(P_1, \dots, P_n)$$

and the determinant of the matrix $(\frac{\partial P_i}{\partial x_j})_{1 \leq i, j \leq n}$ is invertible in A/I .

We can then lift the equations P_i to $\tilde{P}_i \in A[x_1, \dots, x_n]$. The open subset V of S over which the map

$$\tilde{f}: X' = \text{Spec}(A[x_1, \dots, x_n]/(\tilde{P}_1, \dots, \tilde{P}_n)) \rightarrow \text{Spec}(A) = S$$

is étale contains Z and \tilde{f} extends f . Since the functor $(V \rightarrow S)_!$ sends $\mathcal{C}(V)$ to $\mathcal{C}(S)$, we can replace S with V and therefore assume that \tilde{f} is étale.

We then have a localization triangle:

$$j_! j^* \tilde{f}_! \mathbb{Z}_{\ell, X'} \rightarrow \tilde{f}_! \mathbb{Z}_{\ell, X'} \rightarrow i_* f_! \mathbb{Z}_{\ell, X}.$$

Therefore, $i_* f_! \mathbb{Z}_{\ell, X}$ lies in $\mathcal{C}(S)$ as wanted. \square

Corollary 2.2.3.3. *Let ℓ be a prime number. The six functors from $\mathcal{D}(-, \mathbb{Z}_{\ell})$ induce by restriction the following functors on $\mathcal{D}^A(-, \mathbb{Z}_{\ell})$:*

1. f^* for any morphism f ,
2. $f_!$ for any quasi-finite morphism f ,
3. A monoidal structure \otimes .

Remark 2.2.3.4. *Notice that on the categories $\mathcal{D}_{\text{Ind}}^A(-, \mathbb{Z}_{\ell})$, we have formal right adjoints to the three functors that we have described.*

In the case of smooth Artin ℓ -adic complexes, we also have functors \otimes , f^* for any morphism f and $g_* = g_!$ for any finite étale morphism g . Since f^* and g_* are t-exact, they induce functors on the hearts. Furthermore, the monoidal structure on $\mathcal{D}^{\text{sm}A}(S, \mathbb{Z}_{\ell})$ induces a monoidal structure on its heart: map two objects M and N to $H^0(M \otimes N)$. Using Lemma [2.1.5.6], corollary [2.1.5.7], and proposition [2.2.2.1], we identify those functors with classical constructions on categories of representations:

Proposition 2.2.3.5. *Let $f: T \rightarrow S$ be a map between connected schemes and let ℓ be a prime number which is invertible on S . Let ξ' be a geometric point of T and let $\xi = f \circ \xi'$. Then,*

1. *The functor $f^*: \mathcal{D}^{\text{sm}A}(S, \mathbb{Z}_{\ell}) \rightarrow \mathcal{D}^{\text{sm}A}(T, \mathbb{Z}_{\ell})$ induces the forgetful functor*

$$\pi_1^{\text{proét}}(f)^*: \text{Rep}^A(\pi_1^{\text{proét}}(S, \xi), \mathbb{Z}_{\ell})^* \rightarrow \text{Rep}^A(\pi_1^{\text{proét}}(T, \xi'), \mathbb{Z}_{\ell})^*$$

by restriction.

2. *Furthermore, if f is finite and étale, the map induced by f_* by restriction is the induced representation functor $\text{Ind}_{\pi'}^{\pi}$, where $\pi = \pi_1^{\text{proét}}(S, \xi)$ and $\pi' = \pi_1^{\text{proét}}(T, \xi')$.*
3. *The monoidal structure of $\mathcal{D}^{\text{sm}A}(S, \mathbb{Z}_{\ell})$ induces by restriction a monoidal structure on $\text{Rep}^A(\pi_1^{\text{proét}}(S), \mathbb{Z}_{\ell})^*$ which is the usual monoidal structure.*

2.2.4 Artin ℓ -adic complexes as stratified complexes of representations

In this paragraph, we show that for any Artin ℓ -adic complex, there is a stratification of the base scheme such that the restriction of the complex to any stratum is smooth Artin. We then deduce that the ordinary t-structure of $\mathcal{D}(S, \mathbb{Z}_\ell)$ induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ whose heart is $\text{Sh}^A(S, \mathbb{Z}_\ell)$.

Proposition 2.2.4.1. *Let S be a scheme, ℓ be a prime number that is invertible on S and M be an ℓ -adic complex. Then, the complex M is an Artin ℓ -adic complex if and only if there is a stratification of S such that for any stratum T , the complex $M|_T$ is a smooth Artin ℓ -adic complex.*

Proof. To prove the "only if" part of the proposition, proceed by induction on the number of strata and use Corollary 2.2.3.3. We now prove the converse starting with a lemma:

Lemma 2.2.4.2. *Let S be a scheme, ℓ be a prime number that is invertible on S and M be an Artin ℓ -adic complex. Then, there is a dense open subscheme U of S such that $M|_U$ is smooth Artin.*

Proof. Let \mathcal{C} be the full subcategory of $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ made of those objects M such that there is a dense open subscheme U of S such that $M|_U$ is smooth Artin. Then, the subcategory \mathcal{C} is closed under finite (co)limits and retracts and is therefore thick.

Furthermore, if $f: W \rightarrow S$ is étale, there is a dense open immersion $j: U \rightarrow S$ such that the pull-back morphism $g: V \rightarrow U$ is finite étale.

Therefore, the complex

$$(f! \mathbb{Z}_{\ell, W})|_U = j^* f! \mathbb{Z}_{\ell, W} = g! \mathbb{Z}_{\ell, V}$$

is smooth Artin. Thus, the complex $f! \mathbb{Z}_{\ell, W}$ lies in \mathcal{C} . Hence, using the description of generators of Proposition 2.2.3.1, the category \mathcal{C} is equivalent to $\mathcal{D}^A(S, \mathbb{Z}_\ell)$. \square

To finish the proof, we proceed by noetherian induction on S . Let M be an Artin ℓ -adic sheaf over S .

The lemma gives a dense open subscheme U of S such that $M|_U$ is smooth Artin. Using the induction hypothesis, there is a stratification \mathcal{S}_F of $F = S \setminus U$ such that the restriction of $M|_F$ to any stratum of \mathcal{S}_F is smooth Artin. \square

Corollary 2.2.4.3. *Let S be a scheme, ℓ be a prime number that is invertible on S . The canonical t-structure of $\mathcal{D}(S, \mathbb{Z}_\ell)$ induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$. The heart of this t-structure is equivalent to $\text{Sh}^A(S, \mathbb{Z}_\ell)$. In particular, the subcategory $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ is the subcategory made of those complexes C of ℓ -adic sheaves such that for all integer n , the sheaf $H^n(C)$ is Artin.*

Proof. Let M be an Artin ℓ -adic complex. Let $\tau_{\geq 0}$ be the truncation functor with respect to the ordinary t-structure on $\mathcal{D}(S, \mathbb{Z}_\ell)$. To show that the canonical t-structure of $\mathcal{D}(S, \mathbb{Z}_\ell)$ induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, it suffices to show that $\tau_{\geq 0} M$ is Artin.

Let \mathcal{S} be a stratification of S such that for all $T \in \mathcal{S}$, the complex $M|_T$ is smooth Artin. Since the pullback to T functor is t-exact, we have $(\tau_{\geq 0} M)|_T = \tau_{\geq 0}(M|_T)$. The latter is smooth Artin by Proposition 2.2.2.1. Therefore, the complex $\tau_{\geq 0} M$ is Artin.

Furthermore, using Propositions [2.2.2.1](#) and [2.2.4.1](#), we get that any Artin ℓ -adic sheaf is an Artin ℓ -adic complex, in other words, we have

$$\mathrm{Sh}^A(S, \mathbb{Z}_\ell) \subseteq \mathcal{D}^A(S, \mathbb{Z}_\ell).$$

Finally, if M is an object of $\mathcal{D}^A(S, \mathbb{Z}_\ell)^\heartsuit$, there exists a stratification \mathcal{S} of S such that for all $T \in \mathcal{S}$, the complex $M|_T$ is smooth Artin and thus, it is a smooth Artin ℓ -adic sheaf. \square

Corollary 2.2.4.4. *Let S be a scheme, ℓ be a prime number that is invertible on S . Let M be a constructible ℓ -adic complex with coefficients in \mathbb{Z}_ℓ . Then, the ℓ -adic complex M is Artin if and only if $M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is Artin as an ℓ -adic complex with coefficients in \mathbb{Q}_ℓ .*

Proof. Since M is constructible, there is a stratification \mathcal{S} of S such that for all $T \in \mathcal{S}$, the complex $M|_T$ is lisse. Thus, for any integer n , the sheaf $H^n(M|_T)$ is lisse.

Refining the stratification if needed, we can assume the strata to be connected. Let ξ_T be a geometric point of T . Then, for any integer n , the sheaf $H^n(M|_T)$ corresponds through the equivalence of Proposition [2.1.5.4](#) to an object of $\mathrm{Rep}(\pi_1^{\mathrm{pro\acute{e}t}}(T, \xi_T), \mathbb{Z}_\ell)$ that we denote by $E^n(M, T)$.

Since the functor $- \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is t-exact, we have

$$H^n(M|_T) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = H^n((M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)|_T)$$

for any integer n .

Since $M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is Artin, refining the stratification further, we can assume the ℓ -adic sheaves $H^n((M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)|_T)$ to be smooth Artin for any integer n Corollary [2.2.4.3](#).

Therefore, the representation $E^n(M, T) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ belongs to $\mathrm{Rep}^A(\pi_1^{\mathrm{pro\acute{e}t}}(T, \xi_T), \mathbb{Q}_\ell)^*$ for any integer n . Thus, by Proposition [2.1.4.4](#), the representation $E^n(M, T)$ belongs for any integer n to the abelian category $\mathrm{Rep}^A(\pi_1^{\mathrm{pro\acute{e}t}}(T, \xi_T), \mathbb{Z}_\ell)^*$. Therefore, by Corollary [2.2.4.3](#), the complex M is Artin. \square

2.3 The perverse t-structure

From now on, all schemes are assumed to be excellent and endowed with a dimension function δ (see for instance [\[BD17, 1.1.1\]](#)). One of the main goals of this chapter is to understand when the perverse t-structure can be restricted to Artin ℓ -adic complexes and to study Artin perverse sheaves when that is the case.

2.3.1 The smooth Artin case

Using Propositions [2.1.7.3](#) and [2.2.2.1](#), we have:

Proposition 2.3.1.1. *Let S be a regular connected scheme and ℓ be a prime number which is invertible on S . Then, the perverse t-structure of $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ induces a t-structure on $\mathcal{D}^{\mathrm{sm}A}(S, \mathbb{Z}_\ell)$. This t-structure coincides with the ordinary t-structure of $\mathcal{D}^{\mathrm{sm}A}(S, \mathbb{Z}_\ell)$ (see Proposition [2.2.2.1](#)) shifted by $\delta(S)$. In particular, the heart of this t-structure is $\mathrm{Sh}^{\mathrm{sm}A}(S, \mathbb{Z}_\ell)[\delta(S)]$ which is equivalent to $\mathrm{Rep}^A(\pi_1^{\mathrm{pro\acute{e}t}}(S, \xi), \mathbb{Z}_\ell)^*$ if ξ is a geometric point of S .*

In particular the case of fields translates to:

Corollary 2.3.1.2. *Let k be a field and ℓ a prime number which is distinct from the characteristic of k . Then, the perverse t -structure of $\mathcal{D}_c^b(k, \mathbb{Z}_\ell)$ induces a t -structure on $\mathcal{D}^A(k, \mathbb{Z}_\ell)$. This t -structure coincides with the ordinary t -structure of Corollary 2.2.2.2 shifted by $\delta(k)$. In particular, the heart of this t -structure is equivalent to $\text{Rep}^A(G_k, \mathbb{Z}_\ell)^*$.*

In the case where the base scheme is not connected but is still regular, we have:

Corollary 2.3.1.3. *Let S be a regular scheme and ℓ be a prime number which is invertible on S . Then, the perverse t -structure of $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ induces a t -structure on $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$. If T is a connected component of S , denote by $i_T: T \rightarrow S$ the clopen immersion. The heart of this t -structure is then the full subcategory made of those objects of the form*

$$\bigoplus_{T \in \pi_0(S)} (i_T)_* L_T[\delta(T)]$$

where L_T is an object of $\text{Sh}^{smA}(T, \mathbb{Z}_\ell)$.

Remark 2.3.1.4. *Keeping the notations of Corollary 2.3.1.3, the heart of the perverse t -structure on $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$ is equivalent (as an abelian category) to $\text{Sh}^{smA}(S, \mathbb{Z}_\ell)$.*

Definition 2.3.1.5. *Let S be a regular scheme. We define the category $\text{Perv}^{smA}(S, \mathbb{Z}_\ell)$ of smooth Artin perverse sheaves to be the heart of the perverse t -structure on $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$.*

2.3.2 General properties

We have the following restriction property:

Proposition 2.3.2.1. *Let S be a scheme and ℓ be a prime number which is invertible on S . Let T be a locally closed subscheme of S . Then, if the perverse t -structure on $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ induces a t -structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, then, the perverse t -structure on $\mathcal{D}_c^b(T, \mathbb{Z}_\ell)$ induces a t -structure on $\mathcal{D}^A(T, \mathbb{Z}_\ell)$.*

Proof. We can assume that T is either an open subset or a closed subset of S .

Assume first that T is open and denote by $j: T \rightarrow S$ the canonical immersion. Let M be an Artin ℓ -adic complex over T . It suffices to show that ${}^p\tau_{\geq 0}M$ is an Artin ℓ -adic complex. But since the functor j^* is t -exact with respect to the perverse t -structure, we have

$${}^p\tau_{\geq 0}M = j^*({}^p\tau_{\geq 0}j_!M).$$

Therefore, using the stability properties of Artin ℓ -adic complexes of Corollary 2.2.3.3, the ℓ -adic complex ${}^p\tau_{\geq 0}M$ is Artin.

Assume now that T is closed and denote by $i: T \rightarrow S$ the canonical immersion. Let M be an Artin ℓ -adic complex over T . Since the functor i_* is t -exact with respect to the perverse t -structure, we have

$${}^p\tau_{\geq 0}M = i^*({}^p\tau_{\geq 0}i_*M).$$

Therefore, using the stability properties of Artin ℓ -adic complexes of Corollary 2.2.3.3, the ℓ -adic complex ${}^p\tau_{\geq 0}M$ is Artin. \square

Definition 2.3.2.2. *Let S be a scheme and ℓ be a prime number which is invertible on S . Assume that the perverse t -structure on $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ induces a t -structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$. Then, we define the category $\text{Perv}^A(S, \mathbb{Z}_\ell)$ of Artin perverse sheaves to be the heart of this t -structure.*

The following properties are analogous to the classical properties of perverse sheaves:

Proposition 2.3.2.3. *Let ℓ be a prime number. Assume that all schemes below are defined over $\mathbb{Z}[1/\ell]$ and that the perverse t -structures on their categories of constructible ℓ -adic sheaves induce t -structures on their categories of Artin ℓ -adic sheaves.*

1. *Let $f: T \rightarrow S$ be a quasi-finite morphism of schemes. Then, the functor*

$${}^p H^0 f_! : \text{Perv}^A(T, \mathbb{Z}_\ell) \rightarrow \text{Perv}^A(S, \mathbb{Z}_\ell)$$

is right exact.

Furthermore, if f is affine, ${}^p H^0 f_! = f_!$ and this functor is exact.

2. *Let $g: T \rightarrow S$ be a morphism of schemes. Assume that $\dim(g) \leq d$. Then, the functor*

$${}^p H^d g^* : \text{Perv}^A(S, \mathbb{Z}_\ell) \rightarrow \text{Perv}^A(T, \mathbb{Z}_\ell)$$

is right exact.

Furthermore, if f is étale, ${}^p H^0 f^ = f^*$ and this functor is exact.*

3. *Consider a cartesian square of schemes*

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \downarrow q & & \downarrow p \\ T & \xrightarrow{f} & S \end{array}$$

such that p is quasi-finite and $\dim(f) \leq d$. Then we have a canonical equivalence:

$${}^p H^d f^*({}^p H^0 p_!) \rightarrow {}^p H^0 q_!({}^p H^d g^*).$$

4. *Let S be a scheme. Let $i: F \rightarrow C$ be a closed immersion and $j: U \rightarrow C$ be the open complement. Let M be a perverse ℓ -adic sheaf on S . Then, we have an exact sequence of perverse Artin ℓ -adic sheaves:*

$$0 \rightarrow i_* {}^p H^{-1} i^* M \rightarrow {}^p H^0 j_! j^* M \rightarrow M \rightarrow i_* {}^p H^0 i^* M \rightarrow 0.$$

When j is affine, we have an exact sequence of perverse Artin ℓ -adic sheaves:

$$0 \rightarrow i_* {}^p H^{-1} i^* M \rightarrow j_! j^* M \rightarrow M \rightarrow i_* {}^p H^0 i^* M \rightarrow 0.$$

Proof. (1) and (2) follow from the analogous properties of perverse sheaves [BBDG18, 4.1.2, 4.2.4], from the stability properties of Artin ℓ -adic complexes of Corollary 2.2.3.3 and from general properties of t -structures described in [BBDG18, 1.3.17]. Note that [BBDG18, 4.1.2] still applies in this general setting (and not only when the base scheme is of finite type over a field) since it is a consequence of the absolute purity property of [AGV73, XIV.3.1] which has been generalized by Gabber in [LYF14, XV.1.1.2] in the case of quasi-excellent base schemes.

Property (3) follows from base change, from properties (1) and (2) and from [BBDG18, 1.3.17]. Finally, property (4) follows from [BBDG18, 1.4.19] which is the analogous property on perverse sheaves and from (1) and (2). \square

2.3.3 Construction of Artin perverse sheaves over schemes of dimension 2 or less

To prove the main theorem of this chapter, we first need a few lemmas.

Lemma 2.3.3.1. *Let k be a field, let ℓ be a prime number distinct from the characteristic of k and let $p: E \rightarrow \text{Spec}(k)$ be a proper morphism. Let*

$$E \rightarrow \pi_0(E/k) \xrightarrow{q} \text{Spec}(k)$$

be the Stein factorization of p .

Then, $R^0 p_* \mathbb{Z}_{\ell, X} = q_* \mathbb{Z}_{\ell, \pi_0(E/k)}$.

Proof. For any scheme X and any ring of coefficients in R ,

$$H_{\text{ét}}^0(X, R) = R^{\pi_0(X)}.$$

The lemma follows from this result applied to $R = \mathbb{Z}/\ell^n \mathbb{Z}$ for all n . \square

One of our main tools will be the following lemma, which is closely linked to the Rapoport-Zink spectral sequence [RZ82, 2.8].

Lemma 2.3.3.2. *Let ℓ be a prime number. Consider a diagram of $\mathbb{Z}[1/\ell]$ -schemes:*

$$\begin{array}{ccc} E & \xrightarrow{p} & F \\ \downarrow i & & \\ Y & & \end{array}$$

where p is a proper map and i is a closed immersion of a simple normal crossing divisor E into a regular scheme Y .

Write $E = \bigcup_{i \in I} E_i$ where I is a finite set and for all $J \subseteq I$, the closed subscheme $E_J := \bigcap_{i \in J} E_i$ of Y is a regular subscheme of codimension $|J|$.

For all $J \subseteq I$, let $p_J: E_J \rightarrow F$ be the natural map. Then, there is a spectral sequence such that:

$$E_1^{p,q} = \begin{cases} \bigoplus_{\substack{J \subseteq I \\ |J|=p+1}} {}^p H^{q-2}((p_J)_* \mathbb{Z}_{\ell, E_J})(-p-1) & \text{if } p \geq 0 \\ 0 & \text{if } p < 0 \end{cases} \implies {}^p H^{p+q}(p_* i^! \mathbb{Z}_{\ell, Y}).$$

Proof. Denote $i_J: E_J \rightarrow Y$ the inclusion. Proposition 2.1.6.3 asserts that $i_* i^! \mathbb{Z}_{\ell, Y}$ is the colimit:

$$\text{colim}_{p \in \Delta^{inj}} \left(\bigoplus_{\substack{J \subseteq I \\ |J|=p+1}} (i_J)_* i_J^! \mathbb{Z}_{\ell, Y} \right).$$

where Δ^{inj} is the category of finite ordered sets with morphisms the injective maps.

We can truncate (naively) this diagram in each degree to write $i_*i^!\mathbb{Z}_{\ell,Y}$ as the colimit of a sequential diagram $(M_p)_{p \geq 0}$ such that the cofiber of the map $M_{p-1} \rightarrow M_p$ is

$$M(p-1, p) := \bigoplus_{\substack{J \subseteq I \\ |J|=p+1}} (i_J)_* i_J^! \mathbb{Z}_{\ell,Y}[p].$$

Pulling back to E , pushing forward to F and using the absolute purity [LYF14, XV.1.1.2], we can write $p_*i^!\mathbb{Z}_{\ell,Y}$ as the colimit of a sequential diagram $(N_p)_{p \geq 0}$ such that the cofiber of the map $N_{p-1} \rightarrow N_p$ is

$$N(p-1, p) := \bigoplus_{\substack{J \subseteq I \\ |J|=p+1}} (p_J)_* \mathbb{Z}_{\ell,E_J}(-p-1)[-p-2].$$

But by [Lur17, 1.2.2.14] such a sequential diagram gives rise to a spectral sequence such that

$$E_1^{p,q} = \bigoplus_{\substack{J \subseteq I \\ |J|=p+1}} {}^p H^{p+q}(N(p-1, p)) \implies {}^p H^{p+q}(p_*i^!\mathbb{Z}_{\ell,Y}).$$

Finally,

$${}^p H^{p+q}(N(p-1, p)) = {}^p H^{q-2}((p_J)_* \mathbb{Z}_{\ell,E_J})(-p-1)$$

which finishes the proof. □

The following consequence of this lemma will be useful:

Corollary 2.3.3.3. *Keep the notations of the lemma. Take the convention that $\delta(F) = \dim(F)$. Then,*

1. ${}^p H^k(p_*i^!\mathbb{Z}_{\ell,Y})$ vanishes for $k = 0, 1$.
2. Write $I_0 = \{i \in I \mid \delta(p_i(E_i)) = 0\}$. If $i \in I_0$, let

$$E_i \rightarrow Z_i \xrightarrow{q_i} F.$$

be the Stein factorization of p_i , so that the map q_i is finite. Then,

- (a) The image of q_i is a finite subset of closed points of codimension $\delta(F)$ of F if $i \in I_0$.
- (b) We have ${}^p H^2(p_*i^!\mathbb{Z}_{\ell,Y}) = \bigoplus_{i \in I_0} (q_i)_* \mathbb{Z}_{\ell,Z_i}(-1)$.

Proof. If i is an element of I , we denote by W_i the image of E_i through the map p and by $x_i \in F$ the generic point of W_i .

Notice first that if $x \in F$, we have

$$\delta(x) \geq \dim(F) - \text{codim}_F(x) \geq \dim(\overline{\{x\}}).$$

Applying this to the point of x_i for $i \in I_0$ proves (2)(a). We now prove the remaining statements.

Letting y be a closed point of E_J of maximal codimension, We have

$$\delta(E_J) - \dim(E_J) = \delta(E_J) - \text{codim}_{E_J}(y) = \delta(y) \geq \delta(p(y)) \geq 0.$$

Since the relative dimension of p_J is at most $\dim(E_J)$, using [BBDG18, 4.2.4] and the fact that \mathbb{Z}_{ℓ, E_J} lies in degree $\delta(E_J)$ with respect to the perverse t-structure, we get that ${}^p H^q((p_J)_* \mathbb{Z}_{\ell, E_J})$ vanishes if q is a negative integer.

Therefore, the only non-vanishing term of the spectral sequence of Lemma 2.3.3.2 such that $p + q \leq 2$ is the term

$$E_1^{0,2} = \bigoplus_{i \in I} {}^p H^0((p_i)_* \mathbb{Z}_{\ell, E_i})(-1).$$

Now, if $i \notin I_0$, we claim that ${}^p H^0((p_i)_* \mathbb{Z}_{\ell, E_i})$ vanishes.

Assume first that W_i is of positive dimension. Then, no fibers of p_i can be of dimension $\dim(E_i)$ because the map $E_i \rightarrow W_i$ is surjective and because E_i is irreducible. Therefore, the relative dimension of p_i is at most $\dim(E_i) - 1$. Since the sheaf \mathbb{Z}_{ℓ, E_i} lies in degree $\delta(E_i) \geq \dim(E_i)$, [BBDG18, 4.2.4] ensures that the sheaf ${}^p H^0((p_i)_* \mathbb{Z}_{\ell, E_i})$ vanishes.

In the case where $W_i = \{x_i\}$, we get as before that

$$\delta(E_i) - \dim(E_i) \geq \delta(x_i) = \delta(W_i) > 0$$

and therefore, the sheaf ${}^p H^0((p_i)_* \mathbb{Z}_{\ell, E_i})$ vanishes by [BBDG18, 4.2.4] as before.

The corollary then follows from Lemma 2.3.3.1 □

The following statement is the main result of this chapter.

Theorem 2.3.3.4. *Assume that S is an excellent scheme of dimension 2 or less and ℓ be a prime number which is invertible on S , then, the perverse t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_{\ell})$.*

Proof. We can assume that S is reduced and connected. We proceed by noetherian induction on S . If $\dim(S) = 0$, the result follows from Corollary 2.3.1.2.

Assume now that $\dim(S) > 0$. If $j: U \rightarrow S$ is an open immersion, denote by $\mathcal{D}_U^A(S, \mathbb{Z}_{\ell})$ the subcategory of $\mathcal{D}^A(S, \mathbb{Z}_{\ell})$ made of those complexes M such that the ℓ -adic complex $j^* M$ is smooth Artin. Using Lemma 2.2.4.2, every object of $\mathcal{D}^A(S, \mathbb{Z}_{\ell})$ lies in some $\mathcal{D}_U^A(S, \mathbb{Z}_{\ell})$ for some dense open immersion $j: U \rightarrow S$. Shrinking U if needed, we can assume that U is regular since S is excellent. Using Lemma 2.3.3.5 below, we may also assume that the morphism j is affine.

Therefore, it suffices to show that for any dense affine open immersion $j: U \rightarrow S$ with U regular, the perverse t-structure induces a t-structure on the subcategory $\mathcal{D}_U^A(S, \mathbb{Z}_{\ell})$. Let now $j: U \rightarrow S$ be such an immersion, let $i: F \rightarrow S$ be the reduced closed complementary immersion. We let

- $\mathcal{S} = \mathcal{D}_c^b(S, \mathbb{Z}_{\ell})$,
- $\mathcal{U} = \mathcal{D}_c^b(U, \mathbb{Z}_{\ell})$ and $\mathcal{U}_0 = \mathcal{D}^{smA}(U, \mathbb{Z}_{\ell})$,
- $\mathcal{F} = \mathcal{D}_c^b(F, \mathbb{Z}_{\ell})$ and $\mathcal{F}_0 = \mathcal{D}^A(F, \mathbb{Z}_{\ell})$.

The category \mathcal{S} is a gluing of the pair $(\mathcal{U}, \mathcal{F})$ along the fully faithful functors j_* and i_* in the sense of [Lur17, A.8.1] and the perverse t-structure on \mathcal{S} is obtained by gluing the perverse t-structures of \mathcal{U} and \mathcal{F} in the sense of [BBDG18, 1.4.10]. In addition, the perverse t-structure induces a t-structure on \mathcal{U}_0 by Corollary 2.3.1.3 and on \mathcal{F}_0 by induction. Furthermore, since the morphism j is affine, the functor $j_! : \mathcal{U} \rightarrow \mathcal{S}$ is t-exact by the Affine Lefschetz Theorem.

Therefore, using Lemma 2.3.3.6 below, it suffices to show that for any object M of $\text{Perv}^A(F, \mathbb{Z}_\ell)$, any object N of $\text{Perv}^{smA}(U, \mathbb{Z}_\ell)$ and any map

$$f: M \rightarrow {}^p H^{-1}(i^* j_* N),$$

the kernel K of f is an Artin ℓ -adic complex on F .

If $\dim(S) = 1$, the scheme F is 0-dimensional. Thus, Corollary 2.3.1.2 implies that the subcategory $\text{Perv}^A(F, \mathbb{Z}_\ell)$ of $\text{Perv}(F, \mathbb{Z}_\ell)$ is Serre. Since K is a subobject of M , it is an object of $\text{Perv}^A(F, \mathbb{Z}_\ell)$ and thus it is an Artin ℓ -adic complex.

If $\dim(S) = 2$, the scheme F is of dimension at most 1.

Let $Z \rightarrow F$ be a closed immersion such that the open complementary immersion is dense and affine. Then, the scheme Z is of dimension at most 0 and therefore, by Corollary 2.3.1.2, the subcategory $\text{Perv}^A(Z, \mathbb{Z}_\ell)$ of $\text{Perv}(Z, \mathbb{Z}_\ell)$ is Serre. Furthermore, Lemma 2.3.3.10 below implies that the perverse sheaf ${}^p H^{-1}(i^* j_* N)$ is Artin.

Finally, using Lemma 2.3.3.7 below, those two facts imply together that K is an Artin perverse sheaf over F which finishes the proof. \square

Lemma 2.3.3.5. *Let S be a (noetherian) scheme. Let $U \subseteq S$ be an open subscheme. Then, there is a dense open subscheme V of U such that the map $V \rightarrow S$ is affine.*

Proof. The proof of the existence of affine stratifications on noetherian schemes of [Sta23, 0F2X] provides a dense open affine subscheme W of S such that the immersion $W \rightarrow S$ is affine. Since a composition of affine morphisms is affine, we may replace S with W . Therefore we can assume that S is an affine scheme. By the prime avoidance lemma [Sta23, 00DS], there is an affine open V contained in U which contains the generic points of S . The map $V \rightarrow S$ is affine and thus, the lemma is proved. \square

Lemma 2.3.3.6. *Let \mathcal{U} and \mathcal{F} be t-categories, let $j_* : \mathcal{U} \rightarrow \mathcal{S}$ and $i_* : \mathcal{F} \rightarrow \mathcal{S}$ be fully faithful functors such that \mathcal{S} is a gluing of the pair $(\mathcal{U}, \mathcal{F})$ in the sense of [Lur17, A.8.1] (and therefore the axioms of [BBDG18, 1.4.3] are satisfied) and let \mathcal{U}_0 (resp. \mathcal{F}_0) be a sub-t-category of \mathcal{U} (resp. \mathcal{F}). Endow \mathcal{S} with the glued t-structure of [BBDG18, 1.4.10] and denote by i^* (resp. j^*) the left adjoint functor to i_* (resp. j_*). Assume that the following properties hold.*

(i) *The stable subcategory \mathcal{S}_0 of \mathcal{S} made of those objects M such that $i^* M$ belongs to \mathcal{F}_0 and $j^* M$ belongs to \mathcal{U}_0 is bounded.*

(ii) *The left adjoint functor $j_!$ to j^* is t-exact.*

(iii) *For any object M of \mathcal{F}_0^\heartsuit , any object N of \mathcal{U}_0^\heartsuit and any map*

$$f: M \rightarrow H^{-1}(i^* j_* N),$$

the kernel of f lies in \mathcal{F}_0 .

Then, the t-structure of \mathcal{S} induces a t-structure on the stable subcategory \mathcal{S}_0 .

Proof. Step 1: Let M be an object of \mathcal{F}_0^\heartsuit , let N be an object of \mathcal{U}_0^\heartsuit . By [BBDG18, 1.4.16], the object i_*M belongs to \mathcal{S}^\heartsuit . Furthermore, using our second hypothesis, the object $j_!N$ also belongs to \mathcal{S}^\heartsuit . Let $f: i_*M \rightarrow j_!N$ be an arbitrary map. In this step we prove that the objects $\ker(f)$ and $\operatorname{coker}(f)$ belong to \mathcal{S}_0 .

By [Lur17, A.8.13], we have an exact triangle:

$$j_!N \rightarrow j_*N \rightarrow i_*i^*j_*N$$

which by [BBDG18, 1.4.16] yields an exact sequence:

$$0 \rightarrow i_*^p H^{-1}(i^*j_*N) \rightarrow j_!N \rightarrow H^0(j_*N).$$

Notice that by [BBDG18, 1.4.16], we have

$$H^0(j_*M) = \tau_{\leq 0}(j_*M).$$

Since $j^*i_* = 0$, we get

$$\operatorname{Hom}(i_*M, \tau_{\leq 0}(j_*M)) = \operatorname{Hom}(i_*M, j_*N) = 0.$$

Thus, the map f factors through the object $i_*H^{-1}(i^*j_*N)$. Since the functor i_* is t-exact, we get

$$\ker(f) = i_* \ker(M \rightarrow H^{-1}(i^*j_*N)).$$

Therefore, our last assumption implies that object $\ker(f)$ belongs to \mathcal{S}_0 .

To finish the first step, let C be the kernel of the map $j_!N \rightarrow \operatorname{coker}(f)$. We have exact triangles:

$$\ker(f) \rightarrow i_*M \rightarrow C$$

$$C \rightarrow j_!N \rightarrow \operatorname{coker}(f).$$

The first exact triangle shows that C belongs to \mathcal{S}_0 . Thus, using the second exact triangle, the object $\operatorname{coker}(f)$ also lies in \mathcal{S}_0 .

Step 2: We now prove the lemma. First notice that by dévissage, it suffices to show that if P belongs to \mathcal{S}_0 and n is an integer, then, the object $H^n(P)$ of \mathcal{S} lies in \mathcal{S}_0 .

The localization triangle [BBDG18, 1.4.3.4] yields an exact sequence:

$$i_*H^{n-1}(i^*P) \xrightarrow{\alpha} j_!H^n(j^*P) \rightarrow H^n(P) \rightarrow i_*H^n(i^*P) \xrightarrow{\beta} j_!H^{n+1}(j^*P)$$

which yields an exact triangle:

$$\operatorname{coker}(\alpha) \rightarrow H^n(P) \rightarrow \ker(\beta).$$

For any integer m , the object $H^m(i^*P)$ belongs to \mathcal{F}_0^\heartsuit and the object $H^m(j^*P)$ lies in \mathcal{U}_0^\heartsuit . Thus, using our first step, the objects $\operatorname{coker}(\alpha)$ and $\ker(\beta)$ belong to \mathcal{S}_0 and so does the object $H^n(P)$ which finishes the proof. \square

Lemma 2.3.3.7. *Let F be a reduced excellent scheme and ℓ be a prime number which is invertible on F . Assume that the perverse t-structure on $\mathcal{D}_c^b(F, \mathbb{Z}_\ell)$ induces a t-structure on $\mathcal{D}^A(F, \mathbb{Z}_\ell)$. Let M be an Artin perverse sheaf over F , let P be a perverse sheaf over F and let $f: M \rightarrow P$ be an arbitrary map of perverse sheaves.*

Assume that for any reduced closed immersion $\iota: Z \rightarrow F$ such that the open complementary immersion is dense and affine,

(i) The subcategory $\text{Perv}^A(Z, \mathbb{Z}_\ell)$ of $\text{Perv}(Z, \mathbb{Z}_\ell)$ is Serre.

(ii) The perverse sheaf ${}^p H^{-1}(\iota^* P)$ over Z is an Artin perverse sheaf.

Then, the kernel of f is an Artin perverse sheaf over F .

Remark 2.3.3.8. The notation $\text{Perv}^A(Z, \mathbb{Z}_\ell)$ in (i) makes sense by Proposition [2.3.2.1](#).

Proof. Using Lemma [2.2.4.2](#) we have $\gamma: V \rightarrow F$ a dense open immersion such that $\gamma^* M$ is a smooth Artin ℓ -adic complex. Shrinking V if needed, we can assume that V is regular and that $\gamma^* P$ is a lisse perverse sheaf. Using Lemma [2.3.3.5](#), we may assume that γ is affine so that the functor $\gamma_!$ is t-exact by the Affine Lefschetz Theorem. Let $\iota: Z \rightarrow F$ be the complementary closed immersion of γ . We let

- $\mathcal{F} = \mathcal{D}_c^b(F, \mathbb{Z}_\ell)$,
- $\mathcal{V} = \mathcal{D}_{\text{lisse}}(V, \mathbb{Z}_\ell)$ and $\mathcal{V}_0 = \mathcal{D}^{smA}(U, \mathbb{Z}_\ell)$,
- $\mathcal{Z} = \mathcal{D}_c^b(Z, \mathbb{Z}_\ell)$ and $\mathcal{Z}_0 = \mathcal{D}^A(Z, \mathbb{Z}_\ell)$.

By Corollary [2.3.1.3](#), the subcategory $\mathcal{V}_0^\heartsuit = \text{Perv}^{smA}(V, \mathbb{Z}_\ell)$ of \mathcal{V}^\heartsuit is Serre while the subcategory \mathcal{Z}_0^\heartsuit of \mathcal{Z}^\heartsuit is Serre by assumption.

Furthermore, using [\[BBDG18, 1.4.17.1\]](#), the exact and fully faithful functor

$$\iota_*: \text{Perv}(Z, \mathbb{Z}_\ell) \rightarrow \text{Perv}(F, \mathbb{Z}_\ell)$$

maps $\text{Perv}(Z, \mathbb{Z}_\ell)$ into a Serre subcategory.

Finally, by Corollary [2.3.1.3](#), the perverse t-structure on $\mathcal{D}^A(F, \mathbb{Z}_\ell)$ induces a t-structure on the subcategory $\mathcal{D}_V^A(F, \mathbb{Z}_\ell)$ made of those complexes M such that the ℓ -adic complex $\gamma^* M$ is smooth Artin. Therefore, the result follows from Lemma [2.3.3.9](#) below. \square

Lemma 2.3.3.9. *Let \mathcal{V} and \mathcal{Z} be t-categories, let $\gamma_*: \mathcal{V} \rightarrow \mathcal{F}$ and $f_{\iota_*}: \mathcal{Z} \rightarrow \mathcal{F}$ be fully faithful functors such that \mathcal{F} is a gluing of the pair $(\mathcal{U}, \mathcal{F})$ in the sense of [\[Lur17, A.8.1\]](#) (and therefore the axioms of [\[BBDG18, 1.4.3\]](#) are satisfied) and let \mathcal{V}_0 (resp. \mathcal{Z}_0) be a sub-t-category of \mathcal{V} (resp. \mathcal{Z}). Endow \mathcal{F} with the glued t-structure of [\[BBDG18, 1.4.10\]](#) and denote by ι^* (resp. γ^*) the left adjoint functor to ι_* (resp. γ_*). Assume that the following properties hold.*

(i) *The t-structure on \mathcal{F} induces a bounded t-structure on the stable subcategory \mathcal{F}_0 of \mathcal{F} made of those objects M such that $\iota^* M$ belongs to \mathcal{F}_0 and $\gamma^* M$ belongs to \mathcal{U}_0 .*

(ii) *The left adjoint functor $\gamma_!$ to γ^* is t-exact.*

(iii) *The exact and fully faithful functor $\iota_*: \mathcal{Z}^\heartsuit \rightarrow \mathcal{F}^\heartsuit$ maps \mathcal{Z}^\heartsuit into a Serre subcategory.*

(iv) *The subcategory \mathcal{V}_0^\heartsuit (resp. \mathcal{Z}_0^\heartsuit) of \mathcal{V}^\heartsuit (resp. \mathcal{Z}^\heartsuit) is Serre.*

Then, if M is an object of \mathcal{F}_0^\heartsuit , if P is an object of \mathcal{F}^\heartsuit such that $H^{-1}(\iota^ P)$ belongs to \mathcal{Z}_0^\heartsuit and if $f: M \rightarrow N$ is a map in \mathcal{F}^\heartsuit , the kernel of f belongs to \mathcal{F}_0 .*

Proof. Let K be the kernel of f . Using the t-exactness assumption on $\gamma_!$ and [BBDG18, 1.4.16], we have commutative diagrams with exact rows in \mathcal{F}^\heartsuit .

$$\begin{array}{ccccccc}
0 & \longrightarrow & \iota_* H^{-1}(\iota^* M) & \longrightarrow & \gamma_! \gamma^* M & \longrightarrow & Q \longrightarrow 0 \\
& & \downarrow \iota_* u & & \downarrow \gamma_! v & & \downarrow g \\
0 & \longrightarrow & \iota_* H^{-1}(\iota^* P) & \longrightarrow & \gamma_! \gamma^* P & \longrightarrow & R \longrightarrow 0 \\
& & & & & & \\
0 & \longrightarrow & Q & \longrightarrow & M & \longrightarrow & \iota_* H^0(\iota^* M) \longrightarrow 0 \\
& & \downarrow g & & \downarrow f & & \downarrow \iota_* w \\
0 & \longrightarrow & R & \longrightarrow & P & \longrightarrow & \iota_* H^0(\iota^* P) \longrightarrow 0
\end{array}$$

which, by the snake lemma, yield exact sequences:

$$\begin{aligned}
0 &\rightarrow \iota_* \ker(u) \xrightarrow{a} \gamma_! \ker(v) \rightarrow \ker(g) \xrightarrow{b} \iota_* \operatorname{coker}(u) \\
&\quad \quad \quad 0 \rightarrow \ker(g) \rightarrow K \xrightarrow{c} \iota_* \ker(w).
\end{aligned}$$

Since by assumption, the subcategory \mathcal{V}_0^\heartsuit of \mathcal{V}^\heartsuit is Serre, the object $\ker(v)$ belongs to \mathcal{V}_0^\heartsuit . Moreover, the object $H^0(\iota^* M)$, $H^{-1}(\iota^* M)$ and $H^{-1}(\iota^* P)$ belong to \mathcal{Z}_0^\heartsuit . Since the subcategory \mathcal{Z}_0^\heartsuit of \mathcal{Z}^\heartsuit is Serre, the objects $\ker(w)$, $\ker(u)$ and $\operatorname{coker}(u)$ then belong to \mathcal{Z}_0^\heartsuit .

Since we have an exact triangle

$$\iota_* \ker(u) \xrightarrow{a} \gamma_! \ker(v) \rightarrow \operatorname{coker}(a),$$

the object $\operatorname{coker}(a)$ therefore belongs to \mathcal{F}_0^\heartsuit .

As the functor $\iota_*: \mathcal{Z}^\heartsuit \rightarrow \mathcal{F}^\heartsuit$ maps \mathcal{Z}^\heartsuit into a Serre subcategory, the image of $\ker(g)$ through b is of the form $\iota_* C$, where C is a subobject of $\operatorname{coker}(u)$ and therefore belongs to \mathcal{Z}_0^\heartsuit .

Likewise, the image of K through c is of the form $\iota_* K'$ where K' is a subobject of $\ker(w)$ and is therefore belongs to \mathcal{Z}_0^\heartsuit .

Thus, we get exact triangles:

$$\operatorname{coker}(a) \rightarrow \ker(g) \rightarrow \iota_* C$$

$$\ker(g) \rightarrow K \rightarrow \iota_* K'$$

Hence, the object $\ker(g)$ belongs to \mathcal{F}_0^\heartsuit and so does K which finishes the proof. \square

Lemma 2.3.3.10. *Let S be a reduced excellent connected scheme of dimension 2 and ℓ be a prime number which is invertible on S . Let $j: U \rightarrow S$ be a dense affine open immersion with U regular. Let $i: F \rightarrow S$ be the reduced closed complementary immersion. Let N be an object of $\operatorname{Perv}^{smA}(U, \mathbb{Z}_\ell)$. Let $\iota: Z \rightarrow F$ be a reduced closed immersion of positive codimension. Then, the perverse sheaf ${}^p H^{-1}(\iota^* {}^p H^{-1}(i^* j_* N))$ is Artin.*

Proof. We may assume that $\delta(S) = 2$. Using Corollary [2.2.4.4], it suffices to show the same result with coefficients in \mathbb{Q}_ℓ . Furthermore, replacing S with the closure of a connected component of U , we may assume that U is connected. Thus, since U is regular, Proposition [2.3.1.1] ensures that

$$\operatorname{Perv}^{smA}(U, \mathbb{Z}_\ell) = \operatorname{Sh}^{smA}(U, \mathbb{Z}_\ell)[2].$$

Step 1: We show that we can assume that the image of N through the equivalence of Proposition 2.1.5.4 is an Artin representation of $\pi_1^{\text{proét}}(U)$ (throughout this proof, we omit the choice of a base point to lighten the notations).

Assume the result for the objects whose image through the equivalence of Proposition 2.1.5.4 is an Artin representation. Let $A_0 = \text{Rep}^A(\pi_1^{\text{proét}}(U), \mathbb{Q}_\ell)$ and if $n \geq 0$ is an integer, let A_{n+1} be the full subcategory of $\text{Rep}(\pi_1^{\text{proét}}(U), \mathbb{Q}_\ell)$ whose objects are the extensions of objects of A_n in $\text{Rep}(\pi_1^{\text{proét}}(U), \mathbb{Q}_\ell)$. There is a non-negative integer n such that the image of N through the equivalence of Proposition 2.1.5.4 is an object of A_n .

We now prove the result by induction on n . If $n = 0$, this is our assumption. Suppose that the result holds for an integer n . Let N be an object of A_{n+1} . We have an exact sequence

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

where N' and N'' are objects of A_n .

This exact sequence yields an exact triangle:

$$i^*j_*N' \rightarrow i^*j_*N \rightarrow i^*j_*N''.$$

Since i^*j_* is of cohomological amplitude $[-1, 0]$, this yields an exact sequence:

$$0 \rightarrow {}^pH^{-1}(i^*j_*N') \rightarrow {}^pH^{-1}(i^*j_*N) \rightarrow {}^pH^{-1}(i^*j_*N'').$$

Letting C be the image of the map ${}^pH^{-1}(i^*j_*N) \rightarrow {}^pH^{-1}(i^*j_*N'')$ and C' be the cokernel of the map $C \rightarrow {}^pH^{-1}(i^*j_*N'')$, we get two exact triangles:

$$\begin{aligned} {}^pH^{-1}(i^*j_*N') &\rightarrow {}^pH^{-1}(i^*j_*N) \rightarrow C \\ C &\rightarrow {}^pH^{-1}(i^*j_*N'') \rightarrow C'. \end{aligned}$$

Applying ι^* to the first exact triangle yields an exact sequence:

$${}^pH^{-1}(\iota^*{}^pH^{-1}(i^*j_*N')) \rightarrow {}^pH^{-1}(\iota^*{}^pH^{-1}(i^*j_*N)) \rightarrow {}^pH^{-1}(\iota^*C)$$

Furthermore, since ι^* has cohomological amplitude $[-1, 0]$, the second exact triangle yields an exact sequence:

$$0 \rightarrow {}^pH^{-1}(\iota^*C) \rightarrow {}^pH^{-1}(\iota^*{}^pH^{-1}(i^*j_*N''))$$

Since Z is at most 0-dimensional, Corollary 2.3.1.2 implies that $\text{Perv}^A(Z, \mathbb{Q}_\ell)$ is a Serre subcategory of $\text{Perv}(Z, \mathbb{Q}_\ell)$. Furthermore, by our induction hypothesis, the perverse sheaves ${}^pH^{-1}(\iota^*{}^pH^{-1}(i^*j_*N''))$ and ${}^pH^{-1}(\iota^*{}^pH^{-1}(i^*j_*N'))$ are objects of $\text{Perv}^A(Z, \mathbb{Q}_\ell)$. Thus, so is ${}^pH^{-1}(\iota^*C)$ and therefore, so is ${}^pH^{-1}(\iota^*{}^pH^{-1}(i^*j_*N))$.

Step 2: In this step we prove that we can assume S to be normal and that we can assume that $N = \mathbb{Q}_{\ell, U}[2]$.

Since the category $\text{Rep}^A(\pi_1^{\text{proét}}(U), \mathbb{Q}_\ell)$ is semisimple by Maschke's lemma, the the image of N through the equivalence of Proposition 2.1.5.4 is a direct factor of an object whose image is of the form

$$\mathbb{Q}_\ell[G_1] \oplus \cdots \oplus \mathbb{Q}_\ell[G_n]$$

where G_1, \dots, G_n are finite quotients of $\pi_1^{\text{proét}}(U)$.

Thus, N is a direct factor of an object of the form $f_*\mathbb{Q}_{\ell, V}$ where $f: V \rightarrow U$ is a finite étale map. Therefore, we can assume that $N = f_*\mathbb{Q}_{\ell, V}$.

Let \bar{V} be the relative normalization of S in V . Since S is excellent and noetherian, it is Nagata by [Sta23, 033Z]. Thus, [Sta23, 0AVK] ensures that the structural map $\nu: \bar{V} \rightarrow S$ is finite. Furthermore, the scheme \bar{V} is normal by [Sta23, 035L].

Consider the following commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\gamma} & \bar{V} & \xleftarrow{\psi} & F' \\ \downarrow f & & \downarrow \nu & & \downarrow \nu_F \\ U & \xrightarrow{j} & S & \xleftarrow{i} & F \end{array}$$

made of cartesian squares (the left-hand square is cartesian by construction of the relative normalization). In particular, the map ψ is a closed immersion and the map γ is the open complementary immersion.

Now, we have

$$i^*j_*N = i^*j_*f_*\mathbb{Q}_{\ell,V}[2] = i^*\nu_*\gamma_*\mathbb{Q}_{\ell,V}[2] = (\nu_F)_*\psi^*\gamma_*\mathbb{Q}_{\ell,V}[2].$$

Since ν_F is finite, it is perverse t-exact. Thus,

$${}^pH^{-1}(\iota^*{}^pH^{-1}(i^*j_*N)) = {}^pH^{-1}(\iota^*(\nu_F)_*{}^pH^{-1}(\psi^*\gamma_*\mathbb{Q}_{\ell,V}[2])).$$

Let $\nu_Z: Z' \rightarrow Z$ be the pullback of ν_F along the map ι and $\varepsilon: Z' \rightarrow F'$ be the obvious map. Since ν_Z is finite, we get

$${}^pH^{-1}(\iota^*(\nu_F)_*{}^pH^{-1}(\psi^*\gamma_*\mathbb{Q}_{\ell,V}[2])) = (\nu_Z)_*{}^pH^{-1}(\varepsilon^*{}^pH^{-1}(\psi^*\gamma_*\mathbb{Q}_{\ell,V}[2])).$$

Therefore, we can replace S with \bar{V} , U with V , Z with Z' in order to assume that S is normal and that $N = \mathbb{Q}_{\ell,U}[2]$.

Step 3: In this step, we study the singularities of S . We want to prove that the perverse sheaf ${}^pH^{-1}(\iota^*{}^pH^{-1}(i^*j_*\mathbb{Q}_{\ell,U}))$ is Artin.

First, notice that if $Y \rightarrow S$ is a closed immersion and $Z \cap Y = \emptyset$, letting $\xi: F \setminus (Y \cap F) \rightarrow F$ be the canonical open immersion, and letting $u: Z \rightarrow F \setminus (Y \cap F)$ be the canonical closed immersion, we have $\iota = \xi \circ u$. Since ξ is an open immersion, we get

$${}^pH^{-1}(\iota^*{}^pH^{-1}(i^*j_*\mathbb{Q}_{\ell,U})) = {}^pH^{-1}(u^*{}^pH^{-1}(\xi^*i^*j_*\mathbb{Q}_{\ell,U})).$$

Notice that the immersion $i \circ \xi$ induces a closed immersion $F \setminus (Y \cap F) \rightarrow S \setminus Y$. We can therefore remove any closed subset which does not intersect Z from S without changing ${}^pH^{-1}(\iota^*{}^pH^{-1}(i^*j_*\mathbb{Q}_{\ell,U}))$.

Since S is normal, we can therefore assume that its singular locus is contained in Z . Moreover, we can assume that the irreducible components of F only cross at points of Z .

Since S is excellent, Lipman's Theorem on embedded resolution of singularities applies (see [Sta23, 0BGP,0BIC,0ADX] for precise statements). We get a cdh-distinguished square:

$$\begin{array}{ccc} E & \xrightarrow{i_E} & \tilde{S} \\ \downarrow p & & \downarrow f \\ F & \xrightarrow{i} & S \end{array}$$

such that the map f induces an isomorphism over $S \setminus Z$, the scheme \tilde{S} is regular and the subscheme E of \tilde{S} is a simple normal crossing divisor. By removing a closed subset from S which does not intersect Z , we can assume that crossings only occur over Z .

Let $\gamma: U \rightarrow \tilde{S}$ be the complementary open immersion of i_E . We have $j = f \circ \gamma$. Therefore, we get

$$i^* j_* \mathbb{Q}_{\ell,U} = i^* f_* \gamma_* \mathbb{Q}_{\ell,U} = p_* i_E^* \gamma_* \mathbb{Q}_{\ell,U}.$$

We have a localization exact triangle:

$$(i_E)_* i_E^! \mathbb{Q}_{\ell,\tilde{S}} \rightarrow \mathbb{Q}_{\ell,\tilde{S}} \rightarrow \gamma_* \mathbb{Q}_{\ell,U}.$$

Applying $p_* i_E^*$, we get an exact triangle:

$$p_* i_E^! \mathbb{Q}_{\ell,\tilde{S}} \rightarrow p_* \mathbb{Q}_{\ell,E} \rightarrow i^* j_* \mathbb{Q}_{\ell,U}.$$

Since E is a simple normal crossing divisor, Corollary [2.3.3.3](#) ensures that the perverse sheaf ${}^p H^1(p_* i_E^! \mathbb{Q}_{\ell,\tilde{S}})$ is trivial. Furthermore, since p induces an isomorphism over $F \setminus Z$, the only irreducible components of E whose image through p is 0-dimensional have their image contained in Z . Therefore, Corollary [2.3.3.3](#) also ensures that

$${}^p H^2(p_* i_E^! \mathbb{Q}_{\ell,\tilde{S}}) = \iota_* M$$

where M is a perverse sheaf over Z .

Hence, we have an exact sequence:

$$0 \rightarrow {}^p H^1(p_* \mathbb{Q}_{\ell,E}) \rightarrow {}^p H^1(i^* j_* \mathbb{Q}_{\ell,U}) \xrightarrow{\alpha} \iota_* M.$$

Using [\[BBDG18, 1.4.17.1\]](#), the exact and fully faithful functor

$$\iota_*: \text{Perv}(Z, \mathbb{Q}_{\ell}) \rightarrow \text{Perv}(F, \mathbb{Q}_{\ell})$$

maps $\text{Perv}(Z, \mathbb{Q}_{\ell})$ to a Serre subcategory. Thus, the image of ${}^p H^1(i^* j_* \mathbb{Q}_{\ell,U})$ through α is of the form $\iota_* M'$ where M' is an object of $\text{Perv}(Z, \mathbb{Q}_{\ell})$.

Thus, we have an exact triangle:

$${}^p H^1(p_* \mathbb{Q}_{\ell,E}) \rightarrow {}^p H^1(i^* j_* \mathbb{Q}_{\ell,U}) \rightarrow \iota_* M'.$$

Applying ι^* , we get an exact triangle:

$$\iota^{*p} H^1(p_* \mathbb{Q}_{\ell,E}) \rightarrow \iota^{*p} H^1(i^* j_* \mathbb{Q}_{\ell,U}) \rightarrow M'.$$

Since M' is placed in degree 0, we get:

$${}^p H^{-1}(\iota^{*p} H^1(i^* j_* \mathbb{Q}_{\ell,U})) = {}^p H^{-1}(\iota^{*p} H^1(p_* \mathbb{Q}_{\ell,E})).$$

Step 4: In this step we compute ${}^p H^1(p_* \mathbb{Q}_{\ell,E})$ and we finish the proof.

There is a finite set I and regular 1-dimensional closed subschemes E_i of \tilde{X} for $i \in I$, such that $E = \bigcup_{i \in I} E_i$. Choose a total order on the finite set I . If $i < j$, write $E_{ij} = E_i \cap E_j$. Consider for $J \subseteq I$ of cardinality at most 2 the obvious diagram:

$$\begin{array}{ccc} E_J & \xrightarrow{u_{E_J}} & E \\ & \searrow p_J & \downarrow p \\ & & F \end{array}$$

Proposition [2.1.6.4](#) then yields an exact triangle:

$$\mathbb{Q}_{\ell,E} \rightarrow \bigoplus_{i \in I} (u_{E_i})_* (u_{E_i})^* \mathbb{Q}_{\ell,E} \rightarrow \bigoplus_{i < j} (u_{E_{ij}})_* (u_{E_{ij}})^* \mathbb{Q}_{\ell,E}.$$

Applying p_* , we get an exact triangle:

$$p_* \mathbb{Q}_{\ell,E} \rightarrow \bigoplus_{i \in I} (p_i)_* \mathbb{Q}_{\ell,E_i} \rightarrow \bigoplus_{i < j} (p_{ij})_* \mathbb{Q}_{\ell,E_{ij}}.$$

Since crossings only occur over Z , the map p_{ij} factors through Z for any $i < j$. Write $p_{ij} = \iota \circ \pi_{ij}$. The morphism π_{ij} is then finite.

Let

$$I_0 = \{i \in I \mid \delta(p(E_i)) = 0\}.$$

If $i \in I \setminus I_0$, we have $\delta(E_i) = 1$ and p_i is finite. If $i \in I_0$, the map p_i factors through Z and we can write $p_i = \iota \circ \pi_i$. Finally, notice that $\delta(Z) = 0$ and thus, for any $i < j$, we have $\delta(E_{ij}) = 0$. Hence, we get an exact sequence:

$$0 \rightarrow \iota_* P \rightarrow {}^p H^1(p_* \mathbb{Q}_{\ell,E}) \rightarrow R \oplus \iota_* Q \rightarrow 0$$

where

$$P = \operatorname{coker} \left(\bigoplus_{i \in I_0} {}^p H^0((\pi_i)_* \mathbb{Q}_{\ell,E_i}) \rightarrow \bigoplus_{i < j} (\pi_{ij})_* \mathbb{Q}_{\ell,E_{ij}} \right),$$

$$Q = \bigoplus_{i \in I} {}^p H^1((\pi_i)_* \mathbb{Q}_{\ell,E_i}) \text{ and } R = \bigoplus_{i \in I \setminus I_0} (p_i)_* \mathbb{Q}_{\ell,E_i}[1].$$

Therefore, we get an exact triangle:

$$\iota_* P \rightarrow {}^p H^1(p_* \mathbb{Q}_{\ell,E}) \rightarrow R \oplus \iota_* Q$$

which yields an exact triangle:

$$P \rightarrow \iota^* {}^p H^1(p_* \mathbb{Q}_{\ell,E}) \rightarrow \iota^* R \oplus Q.$$

Thus, the perverse sheaf ${}^p H^{-1}(\iota^* {}^p H^1(p_* \mathbb{Q}_{\ell,E}))$ over Z is a subobject of ${}^p H^{-1}(\iota^* R)$. Now, we have

$${}^p H^{-1}(\iota^* R) = \bigoplus_{i \in I \setminus I_0} {}^p H^0((q_i)_* \mathbb{Q}_{\ell,E_i^Z})$$

where for all $i \in I \setminus J$, the map $q_i: E_i^Z \rightarrow Z$ is the pullback of p_i along ι . Since p_i is finite, so is q_i and thus, we have

$${}^p H^0((q_i)_* \mathbb{Q}_{\ell,E_i^Z}) = (q_i)_* \mathbb{Q}_{\ell,E_i^Z}.$$

Thus, the perverse sheaf ${}^p H^{-1}(\iota^* R)$ is Artin.

Hence, the perverse sheaf ${}^p H^{-1}(\iota^* {}^p H^1(i^* j_* \mathbb{Q}_{\ell,U}))$ is also Artin. \square

2.3.4 The case of 3-folds with real closed or separably closed closed points.

In the case of 3-folds, we will show that the perverse t-structure can be restricted to Artin ℓ -adic complexes in some cases, and that it is impossible in other cases.

Proposition 2.3.4.1. *Let S be an excellent 3-fold such that all the closed points of S have a separably closed or real closed residue field and ℓ be a prime number which is invertible on S . Then, the perverse t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$.*

Remark 2.3.4.2. *In the proof, we only use that the residue fields of the closed points of S have finite absolute Galois groups. Recall that a field has a finite absolute Galois group if and only if it is separably closed or real closed.*

Proof. The proof is almost the same as the proof of Theorem 2.3.3.4 but we need to use Lemma 2.3.4.3 below.

Using Lemma 2.3.3.6, it suffices to show that if $j: U \rightarrow S$ is a dense affine open immersion with U regular, letting $i: F \rightarrow S$ be the reduced closed complementary immersion, for any object M of $\text{Perv}^A(F, \mathbb{Z}_\ell)$, any object N of $\text{Perv}^{smA}(U, \mathbb{Z}_\ell)$ and any map

$$f: M \rightarrow {}^p H^{-1}(i^* j_* N),$$

the kernel K of f is an Artin ℓ -adic complex on F .

The scheme F is of dimension at most 2. Let $\iota: Z \rightarrow F$ be a closed immersion such that the open complementary immersion is dense and affine. Then, Z is of dimension at most 1 and therefore, Lemma 2.3.4.3(2) below ensures that the subcategory $\text{Perv}^A(Z, \mathbb{Z}_\ell)$ of $\text{Perv}(Z, \mathbb{Z}_\ell)$ is Serre.

Lemma 2.3.3.7 shows that to finish the proof, it suffices to prove that the perverse sheaf ${}^p H^{-1}(\iota^* {}^p H^{-1}(i^* j_* N))$ is Artin.

Using Lemma 2.3.4.3(1) below, we only need to prove that if η is a generic point of Z , and $i_\eta: \{\eta\} \rightarrow Z$ is the canonical immersion, the perverse sheaf $i_\eta^* {}^p H^{-1}(\iota^* {}^p H^{-1}(i^* j_* N))$ over $k(\eta)$ is Artin.

We have a commutative diagram:

$$\begin{array}{ccccccc} \{\eta\} & \xrightarrow{u_\eta} & \text{Spec}(\mathcal{O}_{F,\eta}) & \xrightarrow{\psi} & \text{Spec}(\mathcal{O}_{S,\eta}) & \xleftarrow{\gamma} & U \times_S \text{Spec}(\mathcal{O}_{S,\eta}) \\ \downarrow i_\eta & & \downarrow \xi & & \downarrow \alpha & & \downarrow \beta \\ Z & \xrightarrow{\iota} & F & \xrightarrow{i} & S & \xleftarrow{j} & U \end{array}$$

made of cartesian squares and such that the vertical maps are filtering limits of open immersions with affine transition maps. If f is such a filtering limit, the functor f^* is perverse t-exact by definition of the perverse t-structure. Thus, we have

$$i_\eta^* {}^p H^{-1}(\iota^* {}^p H^{-1}(i^* j_* N)) = {}^p H^{-1}(u_\eta^* \xi^* {}^p H^{-1}(i^* j_* N)) = {}^p H^{-1}(u_\eta^* {}^p H^{-1}(\psi^* \alpha^* j_* N)).$$

Now, since j is of finite type and since α is filtering limits of open immersions with affine transition maps, Lemma 2.3.4.4 below yields

$$i_\eta^* {}^p H^{-1}(\iota^* {}^p H^{-1}(i^* j_* N)) = {}^p H^{-1}(u_\eta^* {}^p H^{-1}(\psi^* \gamma_*(\beta^* N))).$$

We can now apply Lemma 2.3.3.10: the scheme $\mathrm{Spec}(\mathcal{O}_{S,\eta})$ is of dimension 2, the perverse sheaf β^*N is a smooth Artin over $U \times_S \mathrm{Spec}(\mathcal{O}_{S,\eta})$, the open immersion γ is affine and dense and the complementary open immersion of u_η is also affine and dense.

Thus, the perverse sheaf $i_\eta^{*p}H^{-1}(l^{*p}H^{-1}(i^*j_*N))$ over $k(\eta)$ is Artin. This finishes the proof. \square

Lemma 2.3.4.3. *Let C be an excellent 1-dimensional connected scheme and ℓ be a prime number which is invertible on C . Assume that the closed points of C have finite absolute Galois groups. Then,*

1. *Let M be a constructible ℓ -adic complex over C . Then, the complex M is Artin if and only if for any generic point η of C , the complex η^*M is over $k(\eta)$ is Artin.*
2. *The subcategory $\mathrm{Perv}^A(C, \mathbb{Z}_\ell)$ of $\mathrm{Perv}(C, \mathbb{Z}_\ell)$ is Serre.*

Proof. We first claim that if F is a 0-dimensional scheme such that the residue fields of F have finite absolute Galois groups, then

$$\mathcal{D}_c^b(F, \mathbb{Z}_\ell) = \mathcal{D}^A(F, \mathbb{Z}_\ell).$$

To prove this claim, we can assume that F is reduced and connected and therefore that $F = \mathrm{Spec}(k)$ where k is a field with finite Galois group. Since every continuous representation of G_k is in fact an Artin representation, Corollary 2.2.2.2 then implies our claim.

We now prove the first assertion. The "only if" part follows from the usual stability properties of Artin complexes of Corollary 2.2.3.3. We now prove the "if" part.

Let M be a constructible ℓ -adic complex over C . Assume that for any generic point η of C , the ℓ -adic complex η^*M is Artin. The above claim implies that the ℓ -adic complex $M|_F$ is Artin for any closed subset F of C of positive codimension. Thus, using Proposition 2.2.4.1, it suffices to find a dense open subset U such that the ℓ -adic complex $M|_U$ is Artin. By Corollary 2.2.4.3, we can assume that M is concentrated in degree 0 with respect to the ordinary t-structure of $\mathcal{D}_c^b(C, \mathbb{Z}_\ell)$ i.e. that it lies in $\mathrm{Sh}_c(C, \mathbb{Z}_\ell)$.

Let U be a dense open subset such that $M|_U$ is lisse i.e. lies in $\mathrm{Loc}_U(\mathbb{Z}_\ell)$. Since C is excellent, we can assume that U is regular. We now claim that $M|_U$ lies in $\mathrm{Sh}^{smA}(U, \mathbb{Z}_\ell)$. To show this, we can assume that U is connected and therefore only has one generic point η . Since U is regular, by [BS15, 7.4.10], we have

$$\pi_1^{\mathrm{pro\acute{e}t}}(U, \bar{\eta}) = \pi_1^{\acute{e}t}(U, \bar{\eta}).$$

Let ϕ be the canonical group homomorphism $G_{k(\eta)} \rightarrow \pi_1^{\acute{e}t}(U, \bar{\eta})$. The fiber functor associated to the separable closure $\bar{\eta}$ of η induces by Proposition 2.1.5.4 a commutative square

$$\begin{array}{ccc} \mathrm{Loc}_U(\mathbb{Z}_\ell) & \longrightarrow & \mathrm{Rep}(\pi_1^{\acute{e}t}(U, \bar{\eta}), \mathbb{Z}_\ell) \\ \downarrow \eta^* & & \downarrow \phi^* \\ \mathrm{Loc}_\eta(\mathbb{Z}_\ell) & \longrightarrow & \mathrm{Rep}(G_{k(\eta)}, \mathbb{Z}_\ell) \end{array}$$

where the horizontal arrows are equivalences. By [GR71, V.8.2], the homomorphism ϕ is surjective. Thus, a continuous representation N of $\pi_1^{\acute{e}t}(U, \bar{\eta})$ is of Artin origin if and only if ϕ^*N

is of Artin origin as a representation of $G_{k(\eta)}$. Hence, since η^*M lies in $\text{Sh}^{smA}(\eta, \mathbb{Z}_\ell)$, the lisse sheaf $M|_U$ is smooth Artin which proves the first assertion.

Finally, the first assertion implies the second one using that

$$\text{Perv}^A(C, \mathbb{Z}_\ell) = \mathcal{D}^A(C, \mathbb{Z}_\ell) \cap \text{Perv}(C, \mathbb{Z}_\ell),$$

that the functor η^* is perverse t-exact and that $\text{Perv}^A(k(\eta), \mathbb{Z}_\ell)$ is a Serre subcategory of $\text{Perv}(k(\eta), \mathbb{Z}_\ell)$. \square

Lemma 2.3.4.4. *Let ℓ be a prime number. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow p & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

be a cartesian square of $\mathbb{Z}[1/\ell]$ -schemes such that f is a filtering limit of étale morphisms with affine transition maps and q is of finite type. Then, the exchange transformation

$$f^*q_* \rightarrow p_*p^*f^*q_* \simeq p_*g^*q^*q_* \rightarrow p_*g^*$$

is an equivalence of functors from $\mathcal{D}_c^b(Y', \mathbb{Z}_\ell)$ to $\mathcal{D}_c^b(X, \mathbb{Z}_\ell)$.

Proof. If S is a scheme and ℓ is invertible on S , the functor

$$- \otimes_{\mathbb{Z}_\ell} \mathbb{Z}/\ell\mathbb{Z}: \mathcal{D}_c^b(S, \mathbb{Z}_\ell) \rightarrow \mathcal{D}(S_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z})$$

is conservative. Thus, the result follows from [CD16, 1.1.14]. \square

2.3.5 A counter-example

Proposition 2.3.5.1. *Let X be a normal scheme of dimension 3 with singular locus a closed point x of codimension 3. Let ℓ be a prime number which is invertible on X . Assume that the field $k(x)$ is finite. Let $f: \widetilde{X} \rightarrow X$ be a resolution of singularities of X and assume that the exceptional divisor E is smooth over $k(x)$ and has a non-zero first ℓ -adic betti number.*

Then, taking the convention that $\delta(X) = 3$, the ℓ -adic complex ${}^pH^2(\mathbb{Z}_{\ell, X})$ is not an Artin ℓ -adic complex over X .

Proof. We have a cdh-distinguished square:

$$\begin{array}{ccc} E & \xrightarrow{i_E} & \widetilde{X} \\ \downarrow p & & \downarrow f \\ \{x\} & \xrightarrow{i} & X \end{array}$$

which by Proposition [2.1.6.2] yields an exact triangle:

$$\mathbb{Z}_{\ell, X} \rightarrow f_*\mathbb{Z}_{\ell, \widetilde{X}} \oplus i_*\mathbb{Z}_{\ell, x} \rightarrow i_*p_*\mathbb{Z}_{\ell, E}.$$

Letting $j: U \rightarrow X$ be the complementary open immersion of i , the localization triangle [2.1.0.2] associated to the immersions $U \rightarrow \widetilde{X}$ and i_E yields after applying f_* an exact triangle:

$$i_*p_*i_E^!\mathbb{Z}_{\ell, \widetilde{X}} \rightarrow f_*\mathbb{Z}_{\ell, \widetilde{X}} \rightarrow j_*\mathbb{Z}_{\ell, U}.$$

Now, since j is of relative dimension 0, the functor j_* is perverse left t-exact. Since U is regular and $\delta(U) = 3$, the complex $j_*\mathbb{Z}_{\ell,U}$ is in perverse degree at least 3. Hence, applying Corollary [2.3.3.3](#), the perverse sheaf ${}^pH^1(f_*\mathbb{Z}_{\ell,\tilde{X}})$ is trivial and

$${}^pH^2(f_*\mathbb{Z}_{\ell,\tilde{X}}) = i_*{}^pH^2(p_*i_E^!\mathbb{Z}_{\ell,\tilde{X}}).$$

Using [\[BBDG18, 1.4.17.1\]](#), the fully faithful exact functor i_* maps $\text{Perv}(k(x), \mathbb{Z}_\ell)$ to a Serre subcategory of $\text{Perv}(X, \mathbb{Z}_\ell)$. Therefore, we get an exact sequence:

$$0 \rightarrow i_*{}^pH^1(p_*\mathbb{Z}_{\ell,X}) \rightarrow {}^pH^2(\mathbb{Z}_{\ell,X}) \rightarrow i_*M \rightarrow 0$$

where M is a perverse sheaf over $k(x)$.

Hence, once again using [\[BBDG18, 1.4.17.1\]](#), there is a perverse sheaf P over $k(x)$ such that ${}^pH^2(\mathbb{Z}_{\ell,X}) = i_*P$. The perverse sheaf ${}^pH^1(p_*\mathbb{Z}_{\ell,X})$ is a subobject of P .

It now suffices to show that ${}^pH^1(p_*\mathbb{Z}_{\ell,X})$ does not correspond to representation of Artin origin of $G_{k(x)}$. By Proposition [2.1.4.7](#) and Remark [2.1.4.6](#), it suffices to show that it is not of weight 0 as a perverse sheaf over $k(x)$.

But by [\[BBDG18, 5.1.14, 5.4.4\]](#), the perverse sheaf ${}^pH^1(p_*\mathbb{Z}_{\ell,X})$ is pure of weight 1. Since the first ℓ -adic Betti number $b_1(E)$ is non-zero, the perverse sheaf ${}^pH^1(p_*\mathbb{Z}_{\ell,X})$ is non-zero and thus, it is not of Artin origin. \square

Example 2.3.5.2. *Let E be an elliptic curve over a finite field k and X be the affine cone over $E \times \mathbb{P}_k^1$. Let ℓ be a prime number distinct from the characteristic of k . The scheme X has a single singular point x and the blow-up \tilde{X} of X at x is a resolution of singularities of X with exceptional divisor $E \times \mathbb{P}_k^1$.*

As $b_1(E \times \mathbb{P}_k^1) = 1$, the proposition applies and therefore, the perverse t-structure does not induce a t-structure on $\mathcal{D}^A(X, \mathbb{Z}_\ell)$.

Finally, notice that by Noether's normalization lemma, there is a finite map $X \rightarrow \mathbb{A}_k^3$. Thus, the perverse t-structure does not induce a t-structure on $\mathcal{D}^A(\mathbb{A}_k^3, \mathbb{Z}_\ell)$. As \mathbb{A}_k^3 is a closed subscheme of \mathbb{A}_k^n for $n \geq 3$, the perverse t-structure does not induce a t-structure on $\mathcal{D}^A(\mathbb{A}_k^n, \mathbb{Z}_\ell)$ if $n \geq 3$.

Remark 2.3.5.3. *The condition that $\dim(S) \leq 2$ or $\dim(S) \leq 3$ and the residue field of closed points of S are separably closed or real closed seems rather optimal: loosely speaking, if X becomes more singular, the perverse cohomology sheaves of $\mathbb{Z}_{\ell,X}$ should become more complicated. Here, the simplest possible singularity on a scheme of dimension 3 over a finite field already renders the cohomology sheaf not Artin.*

2.4 The perverse homotopy t-structure

Recall (see Proposition [2.4.2.1](#) below) that we can formally define t-structures on a presentable stable category by setting a certain set of objects to be the t-negative objects of our t-structure. Thus, an other attempt to define a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ is to define it by generators on $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ and then to see when this t-structure can be restricted to $\mathcal{D}^A(S, \mathbb{Z}_\ell)$.

2.4.1 Cohomological ℓ -adic complexes and the six functors for them

Definition 2.4.1.1. *Let S be a scheme and ℓ be a prime number invertible on S . The category of cohomological (resp. ind-cohomological) ℓ -adic complexes over S is the thick (resp. localizing) subcategory $\mathcal{D}^{\text{coh}}(S, \mathbb{Z}_\ell)$ (resp. $\mathcal{D}_{\text{Ind}}^{\text{coh}}(S, \mathbb{Z}_\ell)$) of $\mathcal{D}(S, \mathbb{Z}_\ell)$ generated by the objects $f_*\mathbb{Z}_{\ell, X}$ for $f: X \rightarrow S$ proper.*

In [Pep19b, 1.12], Pepin Lehalleur proved that (constructible) cohomological motives are stable under some of the six operations when the base schemes have resolution of singularities by alterations. In [CD16, 6.2], using Gabber's method, Cisinski and Déglise showed that constructible étale motives are endowed with the six functors formalism when the base schemes are quasi-excellent. One can mimic their proof in the case of cohomological ℓ -adic complexes. We outline how to do this:

1. The fibred subcategory $\mathcal{D}^{\text{coh}}(-, \mathbb{Z}_\ell)$ of $\mathcal{D}(-, \mathbb{Z}_\ell)$ is closed under the tensor product operation, negative Tate twists, the functors f^* for any morphism f and the functors $f_!$ for any separated morphism of finite type f .
2. From the absolute purity property and the stability of \mathcal{D}^{coh} under negative Tate twists and Proposition 2.1.6.3, we deduce that if i is the immersion of a simple normal crossing divisor with regular target X , the ℓ -adic complex $i^!\mathbb{Z}_{\ell, X}$ is cohomological.
3. We want to prove Gabber's lemma for cohomological complexes: if X is quasi-excellent, then for any dense open immersion $U \rightarrow X$, the ℓ -adic complex $j_*\mathbb{Z}_{\ell, U}$ is cohomological. Notice that the fibred stable category $\mathcal{D}^{\text{coh}}(-, \mathbb{Z}_\ell)$ has the properties [CD16, 6.2.9(b)(c)] replacing $\mathcal{DM}_h(-, R)$ with $\mathcal{D}(-, \mathbb{Z}_\ell)$. It does not necessarily have property [CD16, 6.2.9(a)]; however, in the proof of Gabber's lemma of [CD16, 6.2.7], we only need the following weaker version: for any scheme X , the subcategory $\mathcal{D}^{\text{coh}}(X, \mathbb{Z}_\ell)$ of $\mathcal{D}(X, \mathbb{Z}_\ell)$ is thick and contains the object $\mathbb{Z}_{\ell, X}$.
4. Follow the proofs of [CD16, 6.2.13, 6.2.14] to prove the following result.

Proposition 2.4.1.2. *The fibered category $\mathcal{D}^{\text{coh}}(-, \mathbb{Z}_\ell)$ over the category of quasi-excellent (noetherian) $\mathbb{Z}[1/\ell]$ -schemes is stable under the following operations:*

- tensor product and negative Tate twists.
- f^* for any morphism f .
- f_* and $f_!$ for any separated and of finite type morphism f .
- $f^!$ for any quasi-finite morphism f .

2.4.2 Definition for Ind-Artin ℓ -adic complexes

Proposition 2.4.2.1. *Let \mathcal{C} be a presentable stable category which admits small coproducts and push-outs. Given a family \mathcal{E} of objects, the smallest subcategory \mathcal{E}_- closed under extensions, small coproducts and positive shifts is the set of non-positive objects of a t -structure.*

Proof. [Lur17, 1.2.1.16]. □

In the setting of Proposition [2.4.2.1](#) we will call this t-structure the *t-structure generated by* \mathcal{E} . In the motivic setting, t-structures given by a set of generators have been used to define the ordinary homotopy t-structure in [\[Ayo07, 2.2.3\]](#), the perverse homotopy t-structure in [\[Ayo07, 2.2.3\]](#) or [\[BD17\]](#) and the ordinary motivic t-structure on 0-motives and 1-motives in [\[Pep19b\]](#). Our perverse homotopy t-structure is similar to a "shifted" version of Pepin Lehalleur's ordinary motivic t-structure.

Definition 2.4.2.2. *Let S be a scheme and ℓ be a prime number that is invertible on S . Then, the perverse homotopy t-structure on $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ is the t-structure generated by the complexes $f_*\mathbb{Z}_{\ell, X}[\delta(X)]$ for $f: X \rightarrow S$ finite.*

We use the notation hp to denote any object related to the perverse homotopy t-structure. For instance, letting n be an integer, we denote by ${}^{\text{hp}}H^n$ the n -th cohomology functor with respect to this t-structure.

Using the same method as in Proposition [2.2.3.1](#), the perverse homotopy t-structure is also generated by the $f_*\mathbb{Z}_{\ell, X}[\delta(X)]$ for $f: X \rightarrow S$ quasi-finite. By definition, an object M of $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ lies in ${}^{\text{hp}}\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)^{\geq n}$ if and only if for any finite map $f: X \rightarrow S$ and any integer $p > -n$, we have

$$\text{Hom}_{\mathcal{D}(S, \mathbb{Z}_\ell)}(f_*\mathbb{Z}_{\ell, X}[\delta(X) + p], M) = 0.$$

In other words, the object M lies in ${}^{\text{hp}}\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)^{\geq n}$ if and only if the chain complex $\text{Map}(f_*\mathbb{Z}_{\ell, X}, M)$ is $(n - \delta(X) - 1)$ -connected.

To study this t-structure, we need to introduce the Artin truncation functor:

Definition 2.4.2.3. *Let S be a scheme and ℓ be a prime number that is invertible on S . Since the category $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ is presentable by Proposition [2.2.1.4](#), the adjunction theorem [\[Lur09, 5.5.1.9\]](#) gives rise to a right adjoint functor to the inclusion of Ind-Artin ℓ -adic complexes into Ind-cohomological complexes of ℓ -adic sheaves:*

$$\omega^0: \mathcal{D}_{\text{Ind}}^{\text{coh}}(S, \mathbb{Z}_\ell) \rightarrow \mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell).$$

We call this functor the Artin truncation functor.

Remark 2.4.2.4. *In the motivic setting, this functor was first introduced in [\[AZ12, 2.2\]](#). Ayoub and Zucker predicted that the ℓ -adic realization of their functor was a punctual weightless truncation of the ℓ -adic realization functor.*

Such a punctual weightless truncation functor $w_{\leq \text{Id}}$ was then introduced when S is of finite type over a finite field in [\[NV15\]](#) by Vaish and Nair over the category $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)$ of mixed ℓ -adic complexes of [\[BBDG18, 5\]](#). Finally, by giving an explicit description of Ayoub and Zucker's functor which mimics the construction of [\[NV15\]](#), Vaish showed in [\[Vai16\]](#) that the ℓ -adic realization of Ayoub and Zucker's functor is the weightless truncation of its ℓ -adic realization.

We will show that when restricted to cohomological ℓ -adic complexes, our functor coincides with Vaish and Nair's (see Section [2.4.6](#) below).

Proposition 2.4.2.5. *Let ℓ be a prime number. Let f be a quasi-finite morphism of quasi-excellent $\mathbb{Z}[1/\ell]$ -schemes and g be morphism of $\mathbb{Z}[1/\ell]$ -schemes which is essentially of finite type. Then, with respect to the perverse homotopy t-structure,*

1. The adjunction $(f_!, \omega^0 f^!)$ is a t-adjunction.

2. If M is t -non-positive, the functor $- \otimes_S M$ is right t -exact.
3. If $\dim(g) \leq d$, the adjunction $(g^*[d], \omega^0 g_*[-d])$ is a t -adjunction.

Proof. The full subcategory of those Ind-Artin ℓ -adic complexes N such that $f_! N$, $g^* N[d]$ and $N \otimes M$ are perverse homotopy t -non-positive is closed under extensions, small coproducts and negative shifts and contains the set of generators of the t -structure. Thus, by definition of the perverse homotopy t -structure, this subcategory contains all perverse homotopy t -non-positive objects. Hence, the functors $f_!$, $g^*[d]$ and $- \otimes M$ are right t -exact with respect to the perverse homotopy t -structure and the result follows. \square

Corollary 2.4.2.6. *Let ℓ be a prime number. Let f be a morphism of quasi-excellent $\mathbb{Z}[1/\ell]$ -schemes. Then, with respect to the perverse homotopy t -structure,*

1. If f is étale, the functor $f^* = f^! = \omega^0 f^!$ is t -exact.
2. If f is finite, the functor $f_! = f_* = \omega^0 f_*$ is t -exact.

2.4.3 The smooth Artin case

Compare the following with Proposition [2.3.1.1](#).

Proposition 2.4.3.1. *Let S be a regular scheme and ℓ be a prime number which is invertible on S . Then, the perverse homotopy t -structure induces a t -structure on $\mathcal{D}^{smA}(S, \mathbb{Z}_\ell)$ which coincides with the t -structure induced by the perverse t -structure. When S is connected, it coincides with the ordinary t -structure shifted by $\delta(S)$.*

Proof. It suffices to show that the inclusion functor

$$\mathcal{D}^{smA}(S, \mathbb{Z}_\ell) \rightarrow \mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$$

is t -exact when the left hand side is endowed with the perverse t -structure given by Proposition [2.3.1.1](#) and the right hand side is endowed with the perverse homotopy t -structure.

We can assume that S is connected. By dévissage, it suffices to show that

$$\text{Sh}^{smA}(S, \mathbb{Z}_\ell)[\delta(S)] \subseteq \mathcal{D}^{smA}(S, \mathbb{Z}_\ell) \cap {}^{\text{hp}}\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)^\heartsuit$$

Let M be an object of $\text{Sh}^{smA}(S, \mathbb{Z}_\ell)[\delta(S)]$. The object M is in perverse degree 0 by Proposition [2.3.1.1](#). If $f: X \rightarrow S$ is finite, the complex $f_* \mathbb{Z}_{\ell, X}$ is in perverse degree at most $\delta(X)$. Thus, the complex $\text{Map}(f_* \mathbb{Z}_{\ell, X}, M)$ is $(-\delta(X) - 1)$ -connected. Hence M is non-negative with respect to the perverse homotopy t -structure.

We now show that M is non-negative with respect to the perverse homotopy t -structure. It suffices to show that M can be obtained from the set of generators of the t -structure by successively taking negative shifts, coproducts and extensions.

Let ξ be a geometric point of S . The sheaf $M[-\delta(S)]$ corresponds to a representation of Artin origin of $\pi_1^{\text{proét}}(S, \xi)$ that we denote by N . We can assume that N is an Artin representation, as representations of Artin origin are obtained as a successive extensions of Artin representations.

Let G be a finite quotient of $\pi_1^{\text{proét}}(S, \xi)$ through which N factors. Then, by Proposition [2.1.3.1](#), the complex $N \oplus N[1]$ of $\mathbb{Z}_\ell[G]$ -modules is quasi-isomorphic to a complex

$$P = \cdots \rightarrow 0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0 \rightarrow \cdots$$

where each P_i is placed in degree $-i$ and is isomorphic to $\mathbb{Z}_\ell[G/H_1] \oplus \cdots \oplus \mathbb{Z}_\ell[G/H_m]$ for some subgroups H_1, \dots, H_m of G .

If H is a subgroup of G , there is a finite étale S -scheme $f: X \rightarrow S$ such that the image of $\mathbb{Z}_\ell[G/H]$ through the exact functor

$$\Psi: \mathcal{D}^b(\text{Rep}(G, \mathbb{Z}_\ell)) \rightarrow \mathcal{D}(S, \mathbb{Z}_\ell)$$

is $f_*\mathbb{Z}_{\ell, X}$. Hence, the image Q_i of $P_i[\delta(S)]$ is t-non-positive with respect to the perverse homotopy t-structure.

Furthermore, we have exact triangles

$$Q_n[n] \rightarrow M \oplus M[1] \rightarrow Q_{<n}$$

$$Q_k[k] \rightarrow Q_{<k+1} \rightarrow Q_{<k} \text{ if } 0 < k < n$$

where $Q_{<k}[-\delta(S)]$ is the image of the truncated complex

$$\cdots \rightarrow 0 \rightarrow P_{k-1} \rightarrow P_{k-2} \rightarrow \cdots \rightarrow P_0 \rightarrow 0 \rightarrow \cdots$$

through Ψ .

Since $Q_{<1} = Q_0$, a quick induction shows that $Q_{<k}$ is t-non-positive with respect to the perverse homotopy t-structure. Thus, the complex $M \oplus M[1]$ is t-non-positive and therefore, the complex M itself is. □

2.4.4 Gluing formalism

Proposition 2.4.4.1. *Let S be a quasi-excellent scheme. Let ℓ be a prime number that is invertible on S . Let $i: F \rightarrow S$ be a closed immersion and $j: U \rightarrow S$ be the open complement. Then, the category $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ is a gluing of the pair $(\mathcal{D}_{\text{Ind}}^A(U, \mathbb{Z}_\ell), \mathcal{D}_{\text{Ind}}^A(Z, \mathbb{Z}_\ell))$ along the fully faithful functors i_* and $\omega^0 j_*$ in the sense of [Lur17, A.8.1], i.e. the functors i_* and $\omega^0 j_*$ have left adjoint functors i^* and j^* such that*

1. We have $j^*i_* = 0$.
2. The family (i^*, j^*) is conservative.

In particular, by [Lur17, A.8.5, A.8.13], the sequence

$$\mathcal{D}_{\text{Ind}}^A(F, \mathbb{Z}_\ell) \xrightarrow{i_*} \mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell) \xrightarrow{j^*} \mathcal{D}_{\text{Ind}}^A(U, \mathbb{Z}_\ell)$$

satisfies the gluing formalism of [BBDG18, 1.4.3].

Moreover, the perverse homotopy t-structure on $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ is obtained by gluing the t-structures of $\mathcal{D}_{\text{Ind}}^A(U, \mathbb{Z}_\ell)$ and $\mathcal{D}_{\text{Ind}}^A(F, \mathbb{Z}_\ell)$, (see [BBDG18, 1.4.9]) i.e. for all object M of $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, we have

1. $M \geq_{\text{hp}} 0$ if and only if $j^*M \geq_{\text{hp}} 0$ and $\omega^0 i^!M \geq_{\text{hp}} 0$.
2. $M \leq_{\text{hp}} 0$ if and only if $j^*M \leq_{\text{hp}} 0$ and $i^*M \leq_{\text{hp}} 0$.

Proof. This follows from Proposition [2.4.2.5](#), Corollary [2.4.2.6](#) and the usual properties of the six functors. \square

Corollary 2.4.4.2. *Let S be a quasi-excellent scheme. Let ℓ be a prime number that is invertible on S . Let M be an object of $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$. Then, we have*

1. $M \geq_{\text{hp}} 0$ if and only if there is a stratification of S such that for any stratum $i: T \hookrightarrow S$, we have $\omega^0 i^! M \geq_{\text{hp}} 0$.
2. $M \leq_{\text{hp}} 0$ if and only if there is a stratification of S such that for any stratum $i: T \hookrightarrow S$, we have $i^* M \leq_{\text{hp}} 0$.

In particular if S is excellent and M is an object of $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, then, $M \leq_{\text{hp}} 0$ if and only if there is a stratification of S with nil-regular strata, such that for any stratum $i: T \hookrightarrow S$, the ℓ -adic complex $i^ M$ is smooth Artin and $i^* M \leq_p 0$.*

Proof. Recall that an excellent scheme admits a stratification with nil-regular strata. Therefore, the last point follows from Proposition [2.2.4.1](#) and Proposition [2.4.3.1](#). \square

2.4.5 The perverse homotopy and the perverse t-structures

Proposition 2.4.5.1. *Let S be an excellent scheme and ℓ be a prime number that is invertible on S . Then,*

1. *Let M be an object of $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, then M is perverse homotopy t-negative if and only if it is perverse t-negative.*
2. *Let M be an object of $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, then if M is perverse t-positive, then it is perverse homotopy t-positive.*
3. *If the perverse t-structure on $\mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, then the perverse homotopy t-structure on $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$. When this is the case, both induced t-structures coincide.*

Remark 2.4.5.2. *We can reformulate the first point: when the perverse homotopy t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, it is final among those t-structures such that the inclusion $\mathcal{D}^A(S, \mathbb{Z}_\ell) \rightarrow \mathcal{D}_c^b(S, \mathbb{Z}_\ell)$ is right t-exact when the right hand side is endowed with the perverse t-structure.*

Proof. The first point follows from Corollary [2.4.4.2](#)(2). To prove the second assertion, let be M a perverse t-positive constructible ℓ -adic complex. Then, it is in the right orthogonal of the set of non-positive objects of the perverse t-structure. Thus, it is in the right orthogonal of the set of non-positive objects of the perverse homotopy t-structure that are Artin ℓ -adic complexes; in particular for any finite map $f: X \rightarrow S$ and any $p > -n$, the complex M is right orthogonal to $f_* \mathbb{Z}_{\ell, X}[\delta(X) + p]$ and hence M is perverse homotopy t-positive.

If the perverse t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$, the objects of the subcategory

$$\mathcal{D} = \mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)^{\leq 0} \cap \mathcal{D}^A(S, \mathbb{Z}_\ell)$$

of $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ are the t-non-positive objects of a t-structure. Furthermore, the subcategory \mathcal{D}^+ of positive objects of this t-structure is contained in the subcategory of perverse homotopy t-positive objects. But \mathcal{D}^+ contains the perverse homotopy t-positive objects since they are orthogonal to the elements of \mathcal{D} . This finishes the proof. \square

2.4.6 Comparison of the Artin truncation functor with the weightless truncation functor for schemes of finite type over a finite field and with coefficients \mathbb{Q}_ℓ

Let S be a scheme of finite type over a finite field. In this paragraph and in the following ones, we set $\delta(S) = \dim(S)$ which is a consistent choice for any such S . Recall that Vaish and Nair introduced in [NV15, 3.1.1, 3.1.5] an exact functor

$$w_{\leq \text{Id}}: \mathcal{D}_m^b(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_m^b(S, \mathbb{Q}_\ell)$$

where $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)$ is the category of mixed ℓ -adic complexes of [BBDG18, 5]. We now recall its construction and some of its fundamental properties.

First recall Morel's weight truncation of [Mor05]. If T is a scheme of finite type over a finite field and if a is an integer, Morel defined

1. The subcategory ${}^w\mathcal{D}_m^b(T, \mathbb{Q}_\ell)^{\leq a}$ to be the full subcategory of $\mathcal{D}_m^b(T, \mathbb{Q}_\ell)$ made of those objects whose perverse cohomology sheaves are of weight a or less.
2. The subcategory ${}^w\mathcal{D}_m^b(T, \mathbb{Q}_\ell)^{> a}$ to be the full subcategory of $\mathcal{D}_m^b(T, \mathbb{Q}_\ell)$ made of those objects whose perverse cohomology sheaves are of weight a or less.

According to [Mor05, 3.1.1], the pair $({}^w\mathcal{D}_m^b(T, \mathbb{Q}_\ell)^{\leq a}, {}^w\mathcal{D}_m^b(T, \mathbb{Q}_\ell)^{> a})$ is a t-structure on $\mathcal{D}_m^b(T, \mathbb{Q}_\ell)$ and both subcategories ${}^w\mathcal{D}_m^b(T, \mathbb{Q}_\ell)^{\leq a}$ and ${}^w\mathcal{D}_m^b(T, \mathbb{Q}_\ell)^{> a}$ of $\mathcal{D}_m^b(T, \mathbb{Q}_\ell)$ are stable. By definition, they are also closed under retract and therefore thick.

Given a stratification \mathcal{S} of S , every stratum T is therefore endowed with a t-structure

$$({}^w\mathcal{D}_m^b(T, \mathbb{Q}_\ell)^{\leq \dim(T)}, {}^w\mathcal{D}_m^b(T, \mathbb{Q}_\ell)^{\dim(T)}).$$

Nair and Vaish then define a t-structure $({}^{\mathcal{S}}\mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{\leq \text{Id}}, {}^{\mathcal{S}}\mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{> \text{Id}})$ on $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)$ by gluing those t-structures. Letting ${}^{\mathcal{S}}w_{\text{Id}}$ be the non-negative truncation functor, by [NV15, 3.1.3], the following properties hold.

1. The functor

$$w_{\leq \text{Id}} = \lim_{\mathcal{S}} {}^{\mathcal{S}}w_{\leq \text{Id}}$$

is well defined and we have $w_{\leq \text{Id}} = {}^{\mathcal{S}}w_{\leq \text{Id}}$ for any sufficiently fine stratification \mathcal{S} .

2. The functor

$$w_{> \text{Id}} = \text{colim}_{\mathcal{S}} {}^{\mathcal{S}}w_{> \text{Id}}$$

is well defined and we have $w_{> \text{Id}} = {}^{\mathcal{S}}w_{> \text{Id}}$ for any sufficiently fine stratification \mathcal{S} .

Finally, by [NV15, 3.1.7], the functors $(w_{\leq \text{Id}}, w_{> \text{Id}})$ are the truncation functors with respect to t-structure $(\mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{\leq \text{Id}}, \mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{> \text{Id}})$ and the subcategories $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{\leq \text{Id}}$ and $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{> \text{Id}}$ are thick stable subcategories of $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)$. Therefore, the induced functor

$$w_{\leq \text{Id}}: \mathcal{D}_m^b(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{\leq \text{Id}}$$

is exact and is right adjoint to the inclusion functor $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{\leq \text{Id}} \rightarrow \mathcal{D}_m^b(S, \mathbb{Q}_\ell)$.

Example 2.4.6.1. (Compare with [AZ12, 3.11]). Let k be a finite field. Then, the functor $w_{\leq \text{Id}}$ on $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)$ is Morel's functor $w_{\leq 0}$. Therefore, if $f: X \rightarrow \text{Spec}(k)$ is a proper smooth morphism, letting

$$X \rightarrow Z \xrightarrow{g} \text{Spec}(k)$$

be the Stein factorization of f , we have

$$w_{\leq \text{Id}} f_* \mathbb{Q}_{\ell, X} = g_* \mathbb{Q}_{\ell, Z}.$$

Proposition 2.4.6.2. Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Then, the functors ω^0 and $w_{\leq \text{Id}}$ induce equivalent functors

$$\mathcal{D}^{\text{coh}}(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}^A(S, \mathbb{Q}_\ell)$$

on the subcategory $\mathcal{D}^{\text{coh}}(S, \mathbb{Q}_\ell)$ of $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)$.

Remark 2.4.6.3. A priori, the functor ω^0 sends cohomological ℓ -adic complexes to Ind-Artin ℓ -adic complexes. This proposition proves that in fact, it sends cohomological ℓ -adic complexes to Artin ℓ -adic complexes.

Proof. Recall that in [CD16, 5.1, 7.2.10], Cisinski and Déglise constructed a triangulated category $\text{DM}(S, \mathbb{Q})$ endowed with a realization functor

$$\rho_\ell: \text{DM}(S, \mathbb{Q}) \rightarrow \text{D}(S, \mathbb{Q}_\ell)$$

that commutes with the six operations by [CD16, 7.2.24]. We can promote their construction to a stable ∞ -categorical one following [Kha16] and [Rob14].

Now, recall that Vaish constructed a functor (see [Vai16, 3.1]) $w_{\leq \text{Id}}^{\text{mot}}$ from the category of cohomological motives to the category of Artin motives over S .

Lemma 2.4.6.4. Keep the same notations, then $\rho_\ell w_{\leq \text{Id}}^{\text{mot}} = w_{\leq \text{Id}} \rho_\ell$

Proof. We follow the ideas of [Vai16, 4.2.4]. The proof of [Vai16, 4.2.4] tells us that

$$\rho_\ell \mathcal{DM}^{\text{coh}}(S, \mathbb{Q})^{\leq \text{Id}} \subseteq \mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{\leq \text{Id}}$$

and that $\rho_\ell \mathcal{DM}^{\text{coh}}(S, \mathbb{Q})^{> \text{Id}} \subseteq \mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{> \text{Id}}$.

Applying ρ_ℓ to the canonical 2-morphism $\varepsilon^{\text{mot}}: w_{\leq \text{Id}}^{\text{mot}} \rightarrow \text{Id}$ yields a 2-morphism

$$\rho_\ell \varepsilon^{\text{mot}}: \rho_\ell w_{\leq \text{Id}}^{\text{mot}} \rightarrow \rho_\ell.$$

Since $\rho_\ell w_{\leq \text{Id}}^{\text{mot}}$ factors through $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{\leq \text{Id}}$, the functor $\rho_\ell \varepsilon^{\text{mot}}$ factors as

$$\rho_\ell w_{\leq \text{Id}}^{\text{mot}} \xrightarrow{\alpha} w_{\leq \text{Id}} \rho_\ell \xrightarrow{\varepsilon \rho_\ell} \rho_\ell$$

where $\varepsilon: w_{\leq \text{Id}} \rightarrow \text{Id}$ is the canonical 2-morphism. Therefore, we have a commutative square:

$$\begin{array}{ccc} \rho_\ell w_{\leq \text{Id}}^{\text{mot}} & \xrightarrow{\rho_\ell \varepsilon^{\text{mot}}} & \rho_\ell \\ \downarrow \alpha & & \parallel \\ w_{\leq \text{Id}} \rho_\ell & \xrightarrow{\varepsilon \rho_\ell} & \rho_\ell \end{array} .$$

This yields a morphism of exact triangles:

$$\begin{array}{ccccc} \rho_\ell w_{\leq \text{Id}}^{\text{mot}} & \longrightarrow & \rho_\ell & \longrightarrow & \rho_\ell w_{> \text{Id}}^{\text{mot}} \\ \downarrow \alpha & & \parallel & & \downarrow \\ w_{\leq \text{Id}} \rho_\ell & \longrightarrow & \rho_\ell & \longrightarrow & w_{> \text{Id}} \rho_\ell \end{array} .$$

If M is a cohomological morive, as $\rho_\ell w_{\leq \text{Id}}^{\text{mot}} M$ lies in $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{\leq \text{Id}}$ and $\rho_\ell w_{> \text{Id}}^{\text{mot}} M$ lies in $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{> \text{Id}}$, by uniqueness of the truncation, the map $\alpha(M)$ is an isomorphism. Therefore, the map α is an isomorphism. \square

Using the lemma, we see that if $f: X \rightarrow S$ is proper, the ℓ -adic complex $w_{\leq \text{Id}} f_* \mathbb{Q}_{\ell, X}$ is Artin. Hence, the functor $w_{\leq \text{Id}}$ has its essential image included in $\mathcal{D}^A(S, \mathbb{Q}_\ell)$.

Recall that the functor $w_{\leq \text{Id}}: \mathcal{D}_m^b(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_m^b(S, \mathbb{Q}_\ell)^{\leq \text{Id}}$ is exact and is right adjoint to the inclusion functor. Now, if $g: Y \rightarrow S$ is finite, the motive $g_* \mathbb{Q}_{\ell, Y}$ lies in $\mathcal{D}^{\text{coh}}(S, \mathbb{Q}_\ell)^{\leq \text{Id}}$ by [NV15, 3.1.8], hence, if M is a cohomological ℓ -adic complex, the adjunction map

$$\text{Map}(g_* \mathbb{Q}_{\ell, Y}, w_{\leq \text{Id}} M) \rightarrow \text{Map}(g_* \mathbb{Q}_{\ell, Y}, M)$$

is an equivalence. Therefore, if N is an ind-Artin ℓ -adic complex and M is a cohomological ℓ -adic complex, the adjunction map

$$\text{Map}(N, w_{\leq \text{Id}} M) \rightarrow \text{Map}(N, M)$$

is an equivalence. Proposition 2.4.6.2 follows. \square

Using the Artin truncation functor ω^0 , we prove that there is a trace of the six functors formalism on Artin ℓ -adic sheaves with coefficients in \mathbb{Q}_ℓ over a finite field.

Corollary 2.4.6.5. *Let $\ell \neq p$ be prime numbers. Then, on schemes of finite type over \mathbb{F}_p , the six functors from $\mathcal{D}^{\text{coh}}(-, \mathbb{Q}_\ell)$ (see Section 2.4.1) induce the following adjunctions on $\mathcal{D}^A(-, \mathbb{Q}_\ell)$:*

1. $(f^*, \omega^0 f_*)$ for any separated morphism of finite type f ,
2. $(f_!, \omega^0 f^!)$ for any quasi-finite morphism f ,
3. $(\otimes, \omega^0 \text{Hom})$.

Remark 2.4.6.6. *These functors have quite remarkable features. Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Let X be a scheme of finite type over k .*

First, Vaish and Nair used them to define the weightless intersection complex $EC_X = \omega^0 j_ \mathbb{Q}_{\ell, U}$ with $j: U \rightarrow X$ the open immersion of a regular subscheme of X (see [NV15, 3.1.15]). This is an analog of Ayoub and Zucker's motive \mathbb{E}_X (see [AZ12, 2.20]).*

Furthermore, if $f: \tilde{X} \rightarrow X$ is a resolution of singularities, then we have $EC_X = \omega^0 f_ \mathbb{Q}_{\ell, \tilde{X}}$ (see [NV15, 4.1.2]). The proof amounts to the following fact: if $i: F \rightarrow S$ is a closed immersion of dense complement and S is regular, the functor $\omega^0 i^!$ vanishes on $\mathcal{D}^{\text{smA}}(S, \mathbb{Q}_\ell)$ (see the proof of [NV15, 4.1.1]).*

2.4.7 The perverse homotopy t-structure for schemes of finite type over a finite field and with coefficients \mathbb{Q}_ℓ

We will now show that the perverse homotopy t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Q}_\ell)$ when S is of finite type over a finite field. To show this, we will construct a t-structure on $\mathcal{D}^A(S, \mathbb{Q}_\ell)$ by gluing the perverse t-structure on smooth Artin objects along stratifications and show that it coincides with the perverse homotopy t-structure.

Proposition 2.4.7.1. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Then,*

1. *The perverse homotopy t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Q}_\ell)$.*
2. *Let $i: F \rightarrow S$ be a closed immersion and $j: U \rightarrow S$ be the open complement. The sequence*

$$\mathcal{D}^A(F, \mathbb{Q}_\ell) \xrightarrow{i_*} \mathcal{D}^A(S, \mathbb{Q}_\ell) \xrightarrow{j^*} \mathcal{D}^A(U, \mathbb{Q}_\ell)$$

satisfies the axioms of the gluing formalism of [BBDG18, 1.4.3] and the perverse homotopy t-structure on $\mathcal{D}^A(S, \mathbb{Q}_\ell)$ is obtained by gluing the t-structures of $\mathcal{D}^A(U, \mathbb{Q}_\ell)$ and $\mathcal{D}^A(F, \mathbb{Q}_\ell)$, (see [BBDG18, 1.4.9]).

Proof. We can assume that S is reduced. The first part of (2) is a direct consequence of Proposition 2.4.4.1 and Corollary 2.4.6.5. To prove the rest of the proposition, we need to adapt the ideas of [BBDG18, 2.2.10] to our setting.

We first need a few definitions:

- If \mathcal{S} is a stratification of S , we say that a locally closed subscheme X of S is an \mathcal{S} -subscheme of S if X is a union of strata of \mathcal{S} .
- A pair $(\mathcal{S}, \mathcal{L})$ is *admissible* if
 - \mathcal{S} is a stratification of S with smooth strata everywhere of the same dimension.
 - for every stratum T of \mathcal{S} , $\mathcal{L}(T)$ is a finite set of isomorphism classes of smooth Artin perverse sheaves (see Definition 2.3.1.5).
- If $(\mathcal{S}, \mathcal{L})$ is an admissible pair and X is an \mathcal{S} -subscheme of S , the category $\mathcal{D}_{\mathcal{S}, \mathcal{L}}(X, \mathbb{Q}_\ell)$ of $(\mathcal{S}, \mathcal{L})$ -constructible Artin ℓ -adic complexes over X is the full subcategory of $\mathcal{D}^A(X, \mathbb{Q}_\ell)$ made of those objects M such that for any stratum T of \mathcal{S} contained in X , the restriction of M to T is a smooth Artin ℓ -adic complex and its perverse cohomology sheaves are successive extensions of objects whose isomorphism classes lie in $\mathcal{L}(T)$.
- A pair $(\mathcal{S}', \mathcal{L}')$ refines a pair $(\mathcal{S}, \mathcal{L})$ if every stratum S of \mathcal{S} is a union of strata of \mathcal{S}' and any perverse sheaf M over a stratum T of \mathcal{S} whose isomorphism class lies in $\mathcal{L}(T)$ is $(\mathcal{S}', \mathcal{L}')$ -constructible.

If $(\mathcal{S}, \mathcal{L})$ is an admissible pair and X is an \mathcal{S} -subscheme of S , the category $\mathcal{D}_{\mathcal{S}, \mathcal{L}}(X, \mathbb{Q}_\ell)$ is a stable subcategory of $\mathcal{D}^A(S, \mathbb{Q}_\ell)$ (i.e. it is closed under finite (co)limits).

If $(\mathcal{S}, \mathcal{L})$ is an admissible pair and $i: U \rightarrow V$ is an immersion between \mathcal{S} -subschemes of S , the functors $i_!$ and i^* preserve $(\mathcal{S}, \mathcal{L})$ -constructible objects.

Now, we say that an admissible pair $(\mathcal{S}, \mathcal{L})$ is *superadmissible* if letting $i: T \rightarrow S$ be the immersion of a stratum T of \mathcal{S} into S and letting M be a perverse sheaf over T whose isomorphism class lies in $\mathcal{L}(T)$, the complex $\omega^0 i_* M$ is $(\mathcal{S}, \mathcal{L})$ -constructible.

The proof of [AZ12, 3.16] yields equivalences $\omega^0 i_* \omega^0 \simeq \omega^0 i_*$ and $\omega^0 i^! \omega^0 \simeq \omega^0 i^!$ for any quasi-finite morphism i .

We now claim that an admissible pair $(\mathcal{S}, \mathcal{L})$ can always be refined into a superadmissible one.

Assume that this is true when we replace S with the union of the strata of dimension n or more. Let $i: T \rightarrow S$ and $i': T' \rightarrow S$ be immersions of a strata of \mathcal{S} . Let M be a perverse sheaf over T whose isomorphism class lies in $\mathcal{L}(T)$. If T and T' are of dimension at least n , then, the perverse cohomology sheaves of $(i')^* \omega^0 i_* M$ are obtained as successive extensions of objects whose isomorphism classes lie in $\mathcal{L}(T')$ by induction.

If $T = T'$, then

$$(i')^* \omega^0 i_* M = \omega^0 i^* \omega^0 i_* M = \omega^0 i^* i_* M = M$$

whose isomorphism class lies in $\mathcal{L}(T)$.

Otherwise, if T is of dimension $n - 1$, we can always replace T with an open subset, so that the closure of T is disjoint from T' . Then,

$$(i')^* \omega^0 i_* M = \omega^0 (i')^* i_* M = 0.$$

Assume now that T is of dimension at least n and that T' is of dimension $n - 1$. We can replace T' with an open subset so that for any M in $\mathcal{L}(T)$, the complex $(i')^* \omega^0 i_* M$ is smooth Artin by Corollary 2.4.6.5 and Proposition 2.2.4.1. We can also add all the isomorphism classes of the ${}^p H^n((i')^* \omega^0 i_* M)$ to $\mathcal{L}(T')$ because there are finitely many of them. This proves our claim when we replace S with the union of the strata of dimension $n - 1$ or more.

Hence, any admissible pair can be refined into a superadmissible one.

If $(\mathcal{S}, \mathcal{L})$ is a superadmissible pair and $i: U \rightarrow V$ is an immersion between \mathcal{S} -subschemes of S , we claim that the functors $\omega^0 i_*$ and $\omega^0 i^!$ preserve $(\mathcal{S}, \mathcal{L})$ -constructibility. The proof is the same as [BBDG18, 2.1.13] using as above the commutation of ω^0 with the six functors.

Hence, if $(\mathcal{S}, \mathcal{L})$ is a superadmissible pair, we can define a t-structure on $(\mathcal{S}, \mathcal{L})$ -constructible objects by gluing the perverse t-structure on the strata. By Corollary 2.4.4.2, the positive (resp. negative) objects of this t-structure are the positive (resp. negative) objects of the perverse homotopy t-structure that are $(\mathcal{S}, \mathcal{L})$ -constructible.

Thus, the perverse homotopy t-structure induces a t-structure over $(\mathcal{S}, \mathcal{L})$ -constructible objects. Therefore, $(\mathcal{S}, \mathcal{L})$ -constructible objects are stable under the truncation functors of the perverse homotopy t-structure.

Any object M of $\mathcal{D}^A(S, \mathbb{Q}_\ell)$ is $(\mathcal{S}, \mathcal{L})$ -constructible for some superadmissible pair $(\mathcal{S}, \mathcal{L})$. Indeed, by Proposition 2.2.4.1, there is an admissible pair $(\mathcal{S}, \mathcal{L})$ such that M is $(\mathcal{S}, \mathcal{L})$ -constructible. Such a pair is can always be refined into one that is superadmissible.

Hence, the subcategory $\mathcal{D}^A(S, \mathbb{Q}_\ell)$ of the category $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Q}_\ell)$ is stable under the truncation functors of the perverse homotopy t-structure. Therefore, the perverse homotopy t-structure induces a t-structure on $\mathcal{D}^A(S, \mathbb{Q}_\ell)$.

Now, the second part (2) follows from Proposition 2.4.4.1. □

Definition 2.4.7.2. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. We denote by $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ and call abelian category of Artin homotopy perverse sheaves over S the heart of the perverse homotopy t-structure on $\mathcal{D}^A(S, \mathbb{Q}_\ell)$.*

Remark 2.4.7.3. *When both are defined, the abelian category $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ coincides with the abelian category $\text{Perv}^A(S, \mathbb{Q}_\ell)$ (see Definition 2.3.2.2). Recall that this is the case when S is of finite type over the field \mathbb{F}_p and $\dim(S) \leq 2$ according to our main theorem.*

With the assumptions of the definition, there is a right exact functor

$$\iota: \text{Perv}^A(S, \mathbb{Q}_\ell)^\# \rightarrow \text{Perv}(S, \mathbb{Q}_\ell).$$

It is exact if and only if the perverse t-structure induces a t-structure on Artin ℓ -adic complexes.

2.4.8 Weightless perverse t-structure

In the next section, we will study a functor akin to the intermediate extension functor of [BBDG18]. To show that this functor behaves nicely, one would like to restrict the perverse t-structure over $\mathcal{D}^{\text{coh}}(S, \mathbb{Q}_\ell)$ and to show that the functor ω^0 is t-exact.

However, it is not known to the author of this thesis whether the perverse t-structure preserves cohomological motives. To circumvent this issue, notice however that, using [BBDG18, 5.1.7], the perverse t-structure of $\mathcal{D}_c^b(S, \mathbb{Q}_\ell)$ induces a t-structure on $\mathcal{D}_m^b(S, \mathbb{Q}_\ell)$ and that the heart $\text{Perv}_m(S, \mathbb{Q}_\ell)$ of this t-structure is a Serre subcategory of $\text{Perv}(S, \mathbb{Q}_\ell)$. Furthermore, notice that using [BBDG18, 5.1.6], the fibered category $\mathcal{D}_m^b(-, \mathbb{Q}_\ell)$ is endowed with the six operations. We will now define a t-structure on a certain subcategory of *weightless complexes* such that the inclusion of Artin complexes into this category is t-exact and the right adjoint functor of the inclusion of weightless complexes into mixed complexes is t-exact.

Definition 2.4.8.1. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. We denote by $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$ and call category of weightless ℓ -adic complexes the subcategory of the non-positive objects of the t-structure $(w_{\leq \text{Id}}, w_{> \text{Id}})$ of Vaish and Nair.*

Remark 2.4.8.2. *We chose the notation $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$ rather than the notation $\mathcal{D}^{w \leq \text{Id}}(S, \mathbb{Q}_\ell)$, because as stated in Proposition 2.4.8.5 below, the punctual weight of the standard cohomology sheaves of lisse weightless complexes are non-positive.*

Since the functor $w_{\leq \text{Id}}$ of Vaish and Nair was proved in Section 2.4.6 to coincide with ω^0 over cohomological motives and is right adjoint to the inclusion of weightless complexes into the category of mixed complexes, we will from this point on denote the functor $w_{\leq \text{Id}}$ by ω^0 .

We have an analog of Corollary 2.4.6.5:

Proposition 2.4.8.3. *Let $\ell \neq p$ be prime numbers. Then, on schemes of finite type over \mathbb{F}_p , the six functors from $\mathcal{D}_m^b(-, \mathbb{Q}_\ell)$ induce the following adjunctions on $\mathcal{D}^{w \leq 0}(-, \mathbb{Q}_\ell)$:*

1. $(f^*, \omega^0 f_*)$ for any separated morphism of finite type f ,
2. $(f_!, \omega^0 f^!)$ for any quasi-finite morphism f ,
3. $(\otimes, \omega^0 \underline{\text{Hom}})$.

Proof. The first stability assertion follows from [NV15, 3.1.10] while the third follows from [NV15, 3.2.3].

In the case when f is finite, the second stability assertion follows from [NV15, 3.1.8]. Using Zariski's Main Theorem [Gro67, 18.12.13], it suffices to prove that if $j: U \rightarrow S$ is an open immersion and M is weightless over U , then $j_!M$ is still weightless over S .

Recall that a mixed complex N defined over a scheme X is weightless if and only if there is an adapted stratification \mathcal{X} of X in the sense of [NV15, 3.1.6] such that for each stratum Y of \mathcal{X} , the perverse cohomology sheaves of the restriction of N to Y are of weight at most $\dim(Y)$.

Take an adapted stratification of S such that U is a union of strata. Then, perverse cohomology sheaves of the restriction of $j_!M$ to any stratum T contained in U is of weight at most $\dim(T)$ since M is weightless, while the restriction of $j_!M$ to any stratum disjoint from U vanishes. Therefore, the complex $j_!M$ is weightless, which yields the result. \square

Definition 2.4.8.4. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number.*

1. *We denote by $\text{Loc}_S^{w \leq 0}(\mathbb{Q}_\ell)$ and call category of weightless lisse ℓ -adic sheaves the full subcategory of lisse mixed ℓ -adic sheaves over S that are of non-positive weight in the sense of [BBDG18, 5].*
2. *We denote by $\mathcal{D}_{\text{lisse}}^{w \leq 0}(S, \mathbb{Q}_\ell)$ and call category of weightless lisse ℓ -adic complexes the subcategory of $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$ made of those objects which are lisse as ℓ -adic complexes.*

Since representations of weight at most 0 of $\hat{\mathbb{Z}}$ form a Serre subcategory of $\text{Rep}(\hat{\mathbb{Z}}, \mathbb{Q}_\ell)$, the category $\text{Loc}_S^{w \leq 0}(\mathbb{Q}_\ell)$ is a Serre subcategory of $\text{Loc}_S(\mathbb{Q}_\ell)$.

Proposition 2.4.8.5. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Then, the ordinary t -structure of $\mathcal{D}_{\text{lisse}}(S, \mathbb{Q}_\ell)$ induces a t -structure on $\mathcal{D}_{\text{lisse}}^{w \leq 0}(S, \mathbb{Q}_\ell)$. The heart of this t -structure is $\text{Loc}_S^{w \leq 0}(\mathbb{Q}_\ell)$. In particular, $\mathcal{D}_{\text{lisse}}^{w \leq 0}(S, \mathbb{Q}_\ell)$ is the subcategory of those complexes C of ℓ -adic sheaves such that for all integer n , the sheaf $H^n(C)$ is weightless and lisse.*

Proof. Using Lemma [2.1.2.3], it suffices to show that

$$\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell) \cap \text{Loc}_S(\mathbb{Q}_\ell) = \text{Loc}_S^{w \leq 0}(\mathbb{Q}_\ell).$$

Let M be an object of $\text{Loc}_S^{w \leq 0}(\mathbb{Q}_\ell)$. It lies in $\text{Loc}_S(\mathbb{Q}_\ell)$ by definition. Furthermore, if a stratification \mathcal{S} is adapted to M in the sense of [NV15, 3.1.6] and T is a stratum of \mathcal{S} , since $M|_T$ is lisse and T is smooth, for any integer n , we have

$${}^p H^n(M|_T) = \begin{cases} M|_T[\dim(T)] & \text{if } n = \dim(T) \\ 0 & \text{otherwise} \end{cases}.$$

Since the ℓ -adic complex $M|_T$ is of weight at most 0, the ℓ -adic complex $M|_T[\dim(T)]$ is of weight at most $\dim(T)$. Therefore, for any integer n , the ℓ -adic complex ${}^p H^n(M|_T)$ is of weight at most $\dim(T)$. Hence, the complex M lies in $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$.

Conversely, if M lies in $\mathcal{D}_{\text{lisse}}^{w \leq 0}(S, \mathbb{Q}_\ell) \cap \text{Loc}_S(\mathbb{Q}_\ell)$, letting \mathcal{S} be adapted to M and T be a stratum of \mathcal{S} , the complex ${}^p H^{\dim(T)}(M|_T) = M|_T[\dim(T)]$ is of weight at most $\dim(T)$. Thus, the sheaf $M|_T$ is of weight at most 0 and therefore, the sheaf M lies in $\text{Loc}_S^{w \leq 0}(\mathbb{Q}_\ell)$. \square

Corollary 2.4.8.6. *Let S be a smooth scheme over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Then, the perverse t -structure of $\mathcal{D}_{\text{lisse}}(S, \mathbb{Q}_\ell)$ induces a t -structure on $\mathcal{D}_{\text{lisse}}^{w \leq 0}(S, \mathbb{Q}_\ell)$. If S is connected, the heart of this t -structure is $\text{Loc}_S^{w \leq 0}(\mathbb{Q}_\ell)[\dim(S)]$.*

If S is a scheme of finite type over \mathbb{F}_p and $\ell \neq p$ is prime, we now define a t-structure on $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$ by gluing the perverse t-structures over strata. Proceeding as in Proposition [2.4.7.1](#), we define the following:

- A pair $(\mathcal{S}, \mathcal{L})$ is *admissible* if
 - \mathcal{S} is a stratification of S with smooth strata everywhere of the same dimension.
 - for every stratum T of \mathcal{S} , $\mathcal{L}(T)$ is a finite set of isomorphism classes of weightless lisse perverse sheaves (see Corollary [2.4.8.6](#)).
- If $(\mathcal{S}, \mathcal{L})$ is an admissible pair and X is an \mathcal{S} -subscheme of S , the category $\mathcal{D}_{\mathcal{S}, \mathcal{L}}(X, \mathbb{Q}_\ell)$ of $(\mathcal{S}, \mathcal{L})$ -constructible weightless ℓ -adic complexes over X is the full subcategory of the stable category $\mathcal{D}^{w \leq 0}(X, \mathbb{Q}_\ell)$ made of those objects M such that for any stratum T of \mathcal{S} contained in X , the restriction of M to T is a weightless lisse ℓ -adic complex and its perverse cohomology sheaves are successive extensions of objects whose isomorphism classes lie in $\mathcal{L}(T)$.
- An admissible pair $(\mathcal{S}, \mathcal{L})$ is *superadmissible* if letting $i: T \rightarrow S$ be the immersion of a stratum T of \mathcal{S} into S and letting M be a perverse sheaf over T whose isomorphism class lies in $\mathcal{L}(T)$, the complex $\omega^0 i_* M$ is $(\mathcal{S}, \mathcal{L})$ -constructible.

As before, an admissible pair $(\mathcal{S}, \mathcal{L})$ can always be refined into a superadmissible one, replacing Proposition [2.2.4.1](#) with the fact that weightless complexes admit a stratification over which they are weightless and lisse.

If $(\mathcal{S}, \mathcal{L})$ is a superadmissible pair and $i: U \rightarrow V$ is an immersion between \mathcal{S} -subschemes of S , the functors $i_!$, i^* , $\omega^0 i_*$ and $\omega^0 i^!$ preserve $(\mathcal{S}, \mathcal{L})$ -constructibility.

Hence, if $(\mathcal{S}, \mathcal{L})$ is a superadmissible pair, we can define a t-structure on $(\mathcal{S}, \mathcal{L})$ -constructible objects by gluing the perverse t-structures on the strata.

Any object M of $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$ is $(\mathcal{S}, \mathcal{L})$ -constructible for some superadmissible pair $(\mathcal{S}, \mathcal{L})$. If $(\mathcal{S}', \mathcal{L}')$ refines $(\mathcal{S}, \mathcal{L})$, the t-structure defined on $\mathcal{D}_{\mathcal{S}', \mathcal{L}'}(X, \mathbb{Q}_\ell)$ induces the t-structure defined on $\mathcal{D}_{\mathcal{S}, \mathcal{L}}(X, \mathbb{Q}_\ell)$. Hence, we can define non-ambiguously a t-structure on $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$.

Notice that if M is Artin, we can chose $(\mathcal{S}, \mathcal{L})$ above to be such that the $\mathcal{L}(T)$ are smooth Artin for any stratum T of \mathcal{S} . Therefore, our t-structure induces the perverse homotopy t-structure on $\mathcal{D}^A(S, \mathbb{Q}_\ell)$.

Definition 2.4.8.7. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number.*

1. *We call weightless perverse t-structure the t-structure defined above on $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$.*
2. *We denote by $\text{Perv}^{w \leq 0}(S, \mathbb{Q}_\ell)$ its heart and call it the abelian category of weightless perverse sheaves.*

We use the notation wp to denote the objects related to the weightless perverse t-structure. By construction, we have:

Proposition 2.4.8.8. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number.*

1. Let $i: F \rightarrow S$ be a closed immersion and $j: U \rightarrow S$ be the open complement. The sequence

$$\mathcal{D}^{w \leq 0}(F, \mathbb{Q}_\ell) \xrightarrow{i_*} \mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell) \xrightarrow{j^*} \mathcal{D}^{w \leq 0}(U, \mathbb{Q}_\ell)$$

satisfies the axioms of the gluing formalism (see [BBDG18, 1.4.3] and Proposition [2.4.4.1]) and the weightless perverse t -structure on $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$ is obtained by gluing the weightless perverse t -structures of $\mathcal{D}^{w \leq 0}(U, \mathbb{Q}_\ell)$ and $\mathcal{D}^{w \leq 0}(F, \mathbb{Q}_\ell)$ (see [BBDG18, 1.4.9]).

2. Let M be an object of $\mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$. Then,

- (a) $M \geq_{\text{wp}} 0$ if and only if there is a stratification of S such that for any stratum $i: T \hookrightarrow S$, we have $\omega^0 i^! M \geq_{\text{wp}} 0$.
- (b) $M \leq_{\text{wp}} 0$ if and only if there is a stratification of S such that for any stratum $i: T \hookrightarrow S$, we have $i^* M \leq_{\text{wp}} 0$.

3. The inclusion functor

$$\mathcal{D}^A(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$$

is t -exact when the left hand side is endowed with the perverse homotopy t -structure and the right hand side is endowed with the weightless perverse t -structure.

Proceeding as in Proposition [2.4.5.1], this implies:

Corollary 2.4.8.9. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Let M be a weightless ℓ -adic complex. Then,*

- 1. *The complex M is weightless perverse t -negative if and only if it is perverse t -negative.*
- 2. *The complex M be a weightless ℓ -adic complex. If the complex M is perverse t -positive, then it is weightless perverse t -positive.*

Theorem 2.4.8.10. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Let*

$$\iota: \mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_m^b(S, \mathbb{Q}_\ell)$$

be the inclusion. When the left hand side is endowed with the weightless perverse t -structure and the right hand side is endowed with the perverse t -structure, the adjunction (ι, ω^0) is a t -adjunction and the functor ω^0 is t -exact.

Proof. Corollary [2.4.8.9] implies that the functor ι is right t -exact. Thus, the functor ω^0 is left t -exact. But by [NV15, 3.2.6], the functor ω^0 is perverse right exact, *i.e.* it sends perverse t -negative objects to perverse t -negative objects. But any such object is t -negative with respect to the weightless perverse t -structure using Corollary [2.4.8.9]. Thus, the functor ω^0 is right t -exact and therefore, it is t -exact. \square

We can now define the weightless truncation functor over mixed perverse sheaves.

Definition 2.4.8.11. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. We call weightless truncation the exact functor induced by ω^0 :*

$$\omega^0: \text{Perv}_m(S, \mathbb{Q}_\ell) \rightarrow \text{Perv}^{w \leq 0}(S, \mathbb{Q}_\ell).$$

Finally, Theorem [2.4.8.10](#) implies an analog of the affine Lefschetz theorem for the perverse homotopy t-structure.

Corollary 2.4.8.12. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Let $j: U \rightarrow S$ be an affine open immersion. Then, the functors*

$$j_!, \omega^0 j_*: \mathcal{D}^{smA}(U, \mathbb{Q}_\ell) \rightarrow \mathcal{D}^A(S, \mathbb{Q}_\ell)$$

are t-exact with respect to the perverse homotopy t-structure on both sides.

Proof. The functor

$$\omega^0 j_*: \mathcal{D}^{smA}(U, \mathbb{Q}_\ell) \rightarrow \mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$$

identifies with the composition of t-exact functors:

$$\mathcal{D}^{smA}(U, \mathbb{Q}_\ell) \xrightarrow{\iota} \mathcal{D}_m^b(U, \mathbb{Q}_\ell) \xrightarrow{j_*} \mathcal{D}_m^b(S, \mathbb{Q}_\ell) \xrightarrow{\omega^0} \mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$$

and is therefore t-exact using Proposition [2.4.3.1](#) and Theorem [2.4.8.10](#). Since the inclusion functor

$$\mathcal{D}^A(S, \mathbb{Q}_\ell) \rightarrow \mathcal{D}^{w \leq 0}(S, \mathbb{Q}_\ell)$$

is t-exact by Proposition [2.4.8.8](#), the result follows for $\omega^0 j_*$. The case of the functor $j_!$ is similar. \square

2.4.9 Artin intermediate extension and simple Artin homotopy perverse sheaves

We now show that the abelian category of Artin homotopy perverse sheaves shares some features with the category of perverse sheaves.

Let $j: U \rightarrow S$ be an open immersion of schemes of finite type over \mathbb{F}_p . Recall (see [\[BBDG18, 1.4.22\]](#)) that the intermediate extension functor $j_{!*}$ is defined as the image of the map ${}^p H^0 j_! \rightarrow {}^p H^0 j_*$. In fact, whenever the gluing formalism of [\[BBDG18, 1.4.3\]](#) is satisfied, [\[BBDG18, 1.4.22\]](#) provides an intermediate extension functor. Hence, Proposition [2.4.7.1](#) allows us to define an *Artin intermediate extension functor*:

Definition 2.4.9.1. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Let $j: U \rightarrow S$ be an open immersion. The Artin intermediate extension functor $j_{!*}^A$ is the functor which associates to an Artin homotopy perverse sheaf M over U the image of the map*

$${}^{hp} H^0 j_! M \rightarrow {}^{hp} H^0 \omega^0 j_* M.$$

Since the weightless truncation functor and the inclusion of Artin homotopy perverse sheaves into weightless perverse sheaves are exact, the following proposition holds.

Proposition 2.4.9.2. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Let $j: U \rightarrow S$ be an open immersion. Then, we have $j_{!*}^A = \omega^0 j_{!*}$.*

Using [\[BBDG18, 1.4.23\]](#), we have the following description of the intermediate extension functor (compare with [\[BBDG18, 2.1.9\]](#)):

Proposition 2.4.9.3. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Let $i: F \rightarrow S$ be a closed immersion and $j: U \rightarrow S$ be the open complementary immersion. Let M be an Artin homotopy perverse sheaf over U . Then, the complex $j_{!*}^A(M)$ over S is the only extension P of M to S such that $i^*P < 0$ and $\omega^0 i^! P > 0$.*

The following result is similar to [BBDG18, 4.3.1].

Proposition 2.4.9.4. *Let S be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number.*

1. *The abelian category of Artin homotopy perverse sheaves over S with coefficients in \mathbb{Q}_ℓ is artinian and noetherian: every object is of finite length.*
2. *If $j: V \hookrightarrow S$ is the inclusion of a smooth subscheme and if L is an irreducible object of $\mathrm{Sh}^{smA}(V, \mathbb{Q}_\ell)$, then the Artin homotopy perverse sheaf $j_{!*}^A(L[\dim(V)])$ is simple. Every simple Artin perverse sheaf over S is obtained in this way.*

Proof. The proof is the same as in [BBDG18, 4.3.1] replacing [BBDG18, 4.3.2, 4.3.3] with the following lemmas. \square

Lemma 2.4.9.5. *Let V be a smooth connected \mathbb{F}_p -scheme. Let $\ell \neq p$ be a prime number. Let L be an irreducible object of $\mathrm{Sh}^{smA}(V, \mathbb{Q}_\ell)$. Then, if $F = L[\dim(V)]$, for any open immersion $j: U \hookrightarrow V$, we have $F = j_{!*}^A j^* F$.*

Proof. We use Proposition [2.4.9.3]. Let $i: F \rightarrow S$ be the reduced complementary closed immersion of j . Then, the ℓ -adic sheaf i^*L over F is smooth Artin. It is therefore in degree at most $\dim(F)$ with respect to the perverse homotopy t-structure. Thus, the ℓ -adic complex i^*F is in degree at most $\dim(F) - \dim(V) < 0$.

Finally, the functor $\omega^0 i^!$ vanishes over $\mathcal{D}^{smA}(V, \mathbb{Q}_\ell)$ since V is smooth (see the proof of [NV15, 4.1.1]). \square

Lemma 2.4.9.6. *Let V be a smooth connected \mathbb{F}_p -scheme. Let $\ell \neq p$ be a prime number. Let L be a simple object of $\mathrm{Sh}^{smA}(V, \mathbb{Q}_\ell)$. Then, the Artin homotopy perverse sheaf $L[\dim(V)]$ is simple.*

Proof. The proof is exactly the same as [BBDG18, 4.3.3]. \square

Proposition 2.4.9.7. *Let X be a scheme of finite type over \mathbb{F}_p . Let $\ell \neq p$ be a prime number. Recall the construction of the weightless intersection complex EC_X of Vaish and Nair (see Remark [2.4.6.6]). Let $j: U \rightarrow X$ be an open immersion with U regular. Then, we have*

1. $EC_X = j_{!*}^A \mathbb{Q}_{\ell,U}[\dim(U)]$.
2. EC_X is a simple Artin homotopy perverse sheaf.
3. If X is a surface, EC_X is an Artin perverse sheaf; in particular, it is a perverse sheaf.

Proof. Let $IC_X = j_{!*} \mathbb{Q}_{\ell,U}[\dim(U)]$ be the intersection complex, then

$$EC_X = \omega^0 IC_X$$

by [NV15, 3.1.13]. This proves (1) by Proposition [2.4.9.2]; (2) follows by the description of simple objects given in Proposition [2.4.9.4] and (3) follows from the comparison between Artin perverse sheaves and Artin homotopy perverse sheaves of Remark [2.4.7.3]. \square

However, when X is a surface, the weightless intersection complex EC_X need not be simple in the category of perverse sheaves:

Example 2.4.9.8. *Let X be a normal surface with a single singular point $i: \{x\} \rightarrow X$. Let $f: Y \rightarrow X$ be a resolution of singularities such that the fiber of f above x is a simple normal crossing divisor with components of positive genus. Then, EC_X is not simple as a perverse sheaf.*

Indeed, let $i_E: E \rightarrow X$ be the pull-back of i along f and $p: E \rightarrow \{x\}$ be the pull-back of f along i .

We have exact triangles:

$$EC_X \rightarrow f_*\mathbb{Q}_{\ell,Y}[2] \rightarrow w_{>\text{Id}}f_*\mathbb{Q}_{\ell,Y}[2]$$

and

$$w_{>\text{Id}}j_!\mathbb{Q}_{\ell,U} \rightarrow w_{>\text{Id}}f_*\mathbb{Q}_{\ell,Y} \rightarrow w_{>\text{Id}}i_*p_*\mathbb{Q}_{\ell,E}.$$

But as $j_!\mathbb{Q}_{\ell,U}$ is Artin, we have $w_{>\text{Id}}j_!\mathbb{Q}_{\ell,U} = 0$.

In addition, by [NV15, 3.1.8, 3.1.10], the functor i_* is t -exact with respect to the t -structure $(\mathcal{D}^{\leq \text{Id}}(-, \mathbb{Q}_{\ell}), \mathcal{D}^{> \text{Id}}(-, \mathbb{Q}_{\ell}))$. Thus, i_* commutes with $w_{>\text{Id}}$. But over $\mathcal{D}_c^b(k(x), \mathbb{Q}_{\ell})$, the functor $w_{>\text{Id}}$ is defined as Morel's weight truncation functor $w_{\geq 1}$ (see [Mor05, 4.1]).

Now, write $E = \bigcup_{i \in I} E_i$ where E_i is regular. If $J \subseteq I$, write $E_J = \bigcap_{i \in J} E_i$ and $p_J: E_J \rightarrow \{x\}$ the structural morphism. Proposition 2.1.6.4 gives an exact triangle

$$p_*\mathbb{Q}_{\ell,E} \rightarrow \bigoplus_{i \in I} (p_i)_*\mathbb{Q}_{\ell,E_i} \rightarrow \bigoplus_{J \subseteq I, |J|=2} (p_J)_*\mathbb{Q}_{\ell,E_J}.$$

If $|J| = 2$, the ℓ -adic complex $(p_J)_*\mathbb{Q}_{\ell,E_J}$ is pure of weight 0. Therefore, applying Morel's truncation functor $w_{\geq 1}$, we get an isomorphism:

$$\begin{aligned} w_{\geq 1}p_*\mathbb{Q}_{\ell,E} &= \bigoplus_{i \in I} w_{\geq 1}(p_i)_*\mathbb{Q}_{\ell,E_i} \\ &= \bigoplus_{i \in I} R^1(p_i)_*\mathbb{Q}_{\ell,E_i}[-1] \oplus \bigoplus_{i \in I} R^2(p_i)_*\mathbb{Q}_{\ell,E_i}[-2]. \end{aligned}$$

Hence, we have

$$w_{>\text{Id}}i_*p_*\mathbb{Q}_{\ell,E} = i_* \left(\bigoplus_{i \in I} R^1(p_i)_*\mathbb{Q}_{\ell,E_i}[-1] \oplus \bigoplus_{i \in I} R^2(p_i)_*\mathbb{Q}_{\ell,E_i}[-2] \right).$$

Now, using [DCM09, 1.8.1], we have

$$f_*\mathbb{Q}_{\ell,Y} = IC_X \oplus i_* \left(\bigoplus_{i \in I} R^2(p_i)_*\mathbb{Q}_{\ell,E_i} \right) [-2].$$

Hence, we have an exact sequence of perverse sheaves:

$$0 \rightarrow i_* \left(\bigoplus_{i \in I} R^1(p_i)_*\mathbb{Q}_{\ell,E_i} \right) \rightarrow EC_X \rightarrow IC_X \rightarrow 0.$$

Therefore, since one of the E_i has positive genus, $i_* \left(\bigoplus_{i \in I} R^1(p_i)_*\mathbb{Q}_{\ell,E_i} \right)$ is non-zero and thus EC_X is not simple.

2.5 Explicit description of Artin perverse sheaves over schemes of dimension 1

2.5.1 Perverse sheaves as gluing of representations over schemes of dimension 1

The material in this section is a variation on Beilinson's method described in [Bei87a]. Let M be a perverse sheaf over an excellent base scheme S . Recall that there is a stratification \mathcal{S} of S such that for all $T \in \mathcal{S}$, $M|_T$ is lisse. We want to recover M from the $M|_T$ and additional data. This is possible using the nearby and vanishing cycle functors in general. However, in the case when S is 1-dimensional, we have another way of doing this.

We take the convention that $\delta(S) = 1$. Assume for simplicity that S is reduced. The scheme S is then the disjoint union of two regular strata U and F over which M is lisse and such that U is open and dense. Shrinking U , we can also assume that the open immersion $j: U \rightarrow S$ is affine by Lemma [2.3.3.5]. Write $i: F \rightarrow S$ the closed immersion. The ℓ -adic complex $M|_U$ is a lisse perverse sheaf and the ℓ -adic complex $M|_F$ lies in perverse degrees $[-1, 0]$ by [BBDG18, 4.1.10].

By localization, we have an exact triangle:

$$j_!M|_U \rightarrow M \rightarrow i_*M|_F$$

which gives rise to an exact sequence:

$$0 \rightarrow i_*^p H^{-1}M|_F \rightarrow j_!M|_U \rightarrow M \rightarrow i_*^p H^0M|_F \rightarrow 0.$$

Hence, we can recover M as the cone of a map $i_*M|_F[-1] \rightarrow j_!M|_U$ such that the induced map $i_*^p H^{-1}M|_F \rightarrow j_!M|_U$ is a monomorphism.

Applying the localization triangle ([2.1.0.1]) to the functor j_* , we have an exact triangle:

$$j_! \rightarrow j_* \rightarrow i_* i^* j_*.$$

Moreover, we have $j^* i_* = 0$. Therefore, we have

$$\mathrm{Hom}(i_*M|_F[-1], j_!M|_U) = \mathrm{Hom}(M|_F, i^* j_*M|_U)$$

and we have an exact sequence:

$$0 \rightarrow i_*^p H^{-1}(i^* j_*M|_U) \rightarrow j_!M|_U \rightarrow j_*M|_U \rightarrow i_*^p H^0(i^* j_*M|_U) \rightarrow 0.$$

Therefore, a map $i_*^p H^{-1}M|_F \rightarrow j_!M|_U$ is a monomorphism if and only if the ${}^p H^{-1}$ of the corresponding map $M|_F \rightarrow i^* j_*M|_U$ is a monomorphism.

Conversely, a lisse perverse sheaf M_U over U , a (lisse) ℓ -adic complex M_F over F which lies in perverse degrees $[-1, 0]$ and a map $M_F \rightarrow i^* j_*M_U$ such that the induced map ${}^p H^{-1}(M_F) \rightarrow {}^p H^{-1}(i^* j_*M_U)$ is a monomorphism, give rise to a unique perverse sheaf M over S .

Since $\delta(S) = 1$, an ℓ -adic complex N is a lisse perverse sheaf over U if and only if $N[-1]$ is a lisse sheaf. Moreover, since $\delta(F) = 0$, the perverse and ordinary t-structures coincide over $\mathcal{D}_c^b(F, \mathbb{Z}_\ell)$. Hence, the following data determine a unique perverse sheaf over S :

1. A dense affine open immersion $j: U \rightarrow S$ with U regular. Denote by $i: F \rightarrow S$ the complementary reduced closed immersion.

2. A locally constant ℓ -adic sheaf M_U over U .
3. A (lisse) ℓ -adic complex M_F over F placed in (ordinary) degrees $[-1, 0]$.
4. A map $\phi: M_F \rightarrow i^*j_*M_U$ such that ${}^pH^{-1}(\phi)$ is a monomorphism.

We can compute $i^*j_*M_U$ more explicitly. The first case to consider is the case of a regular 1-dimensional scheme. We first need some notations:

Notations 2.5.1.1. *Let C be a regular connected 1-dimensional scheme and K be its field of functions. Then, if U is an open subset of C and x is a closed point of C .*

- We let K_x be the completion of K with respect to the valuation defined by x on K .
- We let $\phi_x: \text{Spec}(K_x) \rightarrow \text{Spec}(K)$ be the induced map.
- We let $G_0(x)$ be inertia subgroup of G_{K_x} .

The following lemma is classical (use [GR71, V.8.2, X.2.1]):

Lemma 2.5.1.2. *Let C be a regular connected 1-dimensional scheme of generic point η , U an open subset of C and x a closed point. Let $K = k(\eta)$ be the field of regular functions on C . Then,*

1. We have an exact sequence of groups:

$$1 \rightarrow G_0(x) \rightarrow G_{K_x} \rightarrow G_{k(x)} \rightarrow 1$$

2. If $x \notin U$, the map

$$G_{K_x} \rightarrow G_K \rightarrow \pi_1^{\text{ét}}(U, \bar{\eta})$$

is injective.

In this setting, we have a morphism of sites $(BG_{K_x})_{\text{proét}} \rightarrow (BG_{k(x)})_{\text{proét}}$ which sends the continuous G_{K_x} -set M to the continuous $G_{k(x)}$ -set $M^{G_0(x)}$. This morphism then induces a functor

$$\mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((BG_{K_x})_{\text{proét}}, \mathbb{Z}_\ell)) \rightarrow \mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((BG_{k(x)})_{\text{proét}}, \mathbb{Z}_\ell))$$

that we will denote by ∂_x .

Proposition 2.5.1.3. *Let C be a regular connected 1-dimensional scheme of generic point η and field of regular functions K . Let $i: F \rightarrow C$ be a reduced closed immersion and let $j: U \rightarrow C$ be the open complement. If $x \in F$, let $i_x: \{x\} \rightarrow F$ be the closed immersion. Finally, let ℓ be a prime number which is invertible on C .*

Then, the diagram

$$\begin{array}{ccc}
 \text{Loc}_U(\mathbb{Z}_\ell) & \xrightarrow{i^*j_*} & \mathcal{D}_c^b(F, \mathbb{Z}_\ell) \\
 \bar{\eta}^* \downarrow & & \uparrow \bigoplus_{x \in F} (i_x)_* \\
 \text{Rep}(\pi_1^{\text{ét}}(U, \bar{\eta}), \mathbb{Z}_\ell) & & \\
 \downarrow & \prod_{x \in F} \phi_x^* & \prod_{x \in F} \partial_x \\
 \text{Rep}(G_K, \mathbb{Z}_\ell) & \longrightarrow \prod_{x \in F} \text{Rep}(G_{K_x}, \mathbb{Z}_\ell) & \longrightarrow \prod_{x \in F} \mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((BG_{k(x)})_{\text{proét}}, \mathbb{Z}_\ell))
 \end{array}$$

is commutative.

Proof. First, note that $i^*j_* = \bigoplus_{x \in F} (i_x)_* i_x^* i^* j_*$. We can therefore assume that F has a single point x .

Then, by [AGV73, VIII.5] and [BS15, 5.5.4], if n is an integer and M lies in $\text{Loc}_U(\mathbb{Z}_\ell)$, the pro-étale sheaf $H^n(i^*j_*M)$ over $\{x\}$ identifies with the $G_{k(x)}$ -module:

$$H_{\text{ét}}^n(\text{Spec}(L), \phi_x^* \bar{\eta}^* M) = H^n(G_0(x), \phi_x^* \bar{\eta}^* M)$$

where L is the maximal unramified extension of K_x . This yields the result. \square

We will now tackle the case of a general 1-dimensional scheme.

Notations 2.5.1.4. *Let C be an excellent 1-dimensional scheme. Let $\nu: \tilde{C} \rightarrow C$ be the normalization of C . Let Γ be the set of generic points of \tilde{C} . Let U is an open subset of C , let y be a closed point of \tilde{C} and let $\eta \in \Gamma$.*

- We let $U_\eta = \overline{\{\eta\}} \cap \nu^{-1}(U) \subseteq \tilde{C}$.
- We let K_η be the field of regular functions on C_η .
- We let $\eta(y)$ be the unique element of Γ such that $y \in C_{\eta(y)}$.
- We let K_y be the completion of $K_{\eta(y)}$ with respect to the valuation defined by y on $K_{\eta(y)}$.
- We let $\phi_y: \text{Spec}(K_y) \rightarrow \text{Spec}(K_{\eta(y)})$ be the induced map.
- We let $G_0(y)$ be the inertia subgroup of G_{K_y} .

The following lemma follows from Lemma 2.5.1.2.

Lemma 2.5.1.5. *Let C be a 1-dimensional excellent scheme. Let $\nu: \tilde{C} \rightarrow C$ be the normalization of C . Let Γ be the set of generic points of \tilde{C} . Let U be a nil-regular open subset of C , let y be a closed point of \tilde{C} and let $\eta \in \Gamma$. Then,*

1. *We have an exact sequence of groups:*

$$1 \rightarrow G_0(y) \rightarrow G_{K_y} \rightarrow G_{k(y)} \rightarrow 1$$

2. *If $y \notin U_{\eta(y)}$, the map*

$$G_{K_y} \rightarrow G_{K_{\eta(y)}} \rightarrow \pi_1^{\text{ét}}(U_{\eta(y)}, \overline{\eta(y)})$$

is injective.

In this setting, we have a morphism of sites $(BG_{K_y})_{\text{proét}} \rightarrow (BG_{k(y)})_{\text{proét}}$ given by $M \mapsto M^{G_0(y)}$. This morphism then induces a functor

$$\mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((BG_{K_y})_{\text{proét}}, \mathbb{Z}_\ell)) \rightarrow \mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((BG_{k(y)})_{\text{proét}}, \mathbb{Z}_\ell))$$

that we will denote by ∂_y .

Corollary 2.5.1.6. *Let C be a 1-dimensional excellent scheme and let ℓ be a prime number which is invertible on C . Let $\nu: \tilde{C} \rightarrow C$ be the normalization of C . Let Γ be the set of generic points of C . Let $i: F \rightarrow C$ be a closed immersion and $j: U \rightarrow C$ be the open complement. Assume that U is nil-regular.*

Then, the diagram

$$\begin{array}{ccc}
 \mathrm{Loc}_U(\mathbb{Z}_\ell) & \xrightarrow{i^* j_*} & \mathcal{D}_C^b(F, \mathbb{Z}_\ell) \\
 \downarrow \prod_{\eta \in \Gamma} \bar{\eta}^* & & \uparrow \bigoplus_{x \in F} (i_x)_* \\
 \prod_{\eta \in \Gamma} \mathrm{Rep}(\pi_1^{\acute{e}t}(U_\eta, \bar{\eta}), \mathbb{Z}_\ell) & & \prod_{x \in F} \mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\mathrm{Sh}((BG_{k(x)})_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell)) \\
 \downarrow & & \uparrow \prod_{x \in F} \bigoplus_{\nu(y)=x} \mathrm{Ind}_{G_{k(y)}}^{G_{k(x)}} \\
 \prod_{\eta \in \Gamma} \mathrm{Rep}(G_{K_\eta}, \mathbb{Z}_\ell) & & \prod_{x \in F} \prod_{\nu(y)=x} \mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\mathrm{Sh}((BG_{k(y)})_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell)) \\
 \prod_{\eta \in \Gamma} \prod_{y \in \{\eta\} \cap \nu^{-1}(F)} \phi_y^* \downarrow & & \\
 \prod_{\eta \in \Gamma} \prod_{y \in \{\eta\} \cap \nu^{-1}(F)} \mathrm{Rep}(G_{K_y}, \mathbb{Z}_\ell) & \xrightarrow{\prod_{y \in \nu^{-1}(F)} \partial_y} & \prod_{y \in \nu^{-1}(F)} \mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\mathrm{Sh}((BG_{k(y)})_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell)) = \prod_{x \in F} \prod_{\nu(y)=x} \mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\mathrm{Sh}((BG_{k(y)})_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell)) \\
 \parallel & & \\
 \prod_{y \in \nu^{-1}(F)} \mathrm{Rep}(G_{K_y}, \mathbb{Z}_\ell) & &
 \end{array}$$

is commutative.

Proof. Let $\nu_U: U_{\mathrm{red}} \rightarrow U$ (resp. $\nu_F: \tilde{F} \rightarrow F$) be the inverse image of ν along j (resp. i). Let \tilde{j} (resp. \tilde{i}) be the inclusion of U_{red} (resp. \tilde{F}) in \tilde{C} .

Recall that ν_U^* is an equivalence of categories. Thus,

$$i^* j_* = i^* j_* (\nu_U)_* \nu_U^* = i^* \nu_* \tilde{j}_* \nu_U^* = (\nu_F)_* \tilde{i}^* \tilde{j}_* \nu_U^*.$$

The result then follows from Proposition 2.5.1.3 and Proposition 2.2.3.5. \square

This discussion leads us to the following definition:

Definition 2.5.1.7. *Let C be a 1-dimensional excellent scheme and let ℓ be a prime number invertible on C . Let Γ be the set of generic points of C . Let $\nu: \tilde{C} \rightarrow C$ be the normalization of C . We define an abelian category $\mathbf{P}(C, \mathbb{Z}_\ell)$ as follows:*

- *An object of $\mathbf{P}(C, \mathbb{Z}_\ell)$ is a quadruple $(U, (M_\eta)_{\eta \in \Gamma}, (M_x)_{x \in C \setminus U}, (f_x)_{x \in C \setminus U})$ where*
 1. *U is a nil-regular open subset of C such that the immersion $U \rightarrow C$ is affine,*
 2. *for all $\eta \in \Gamma$, M_η is an ℓ -adic representation of $\pi_1^{\acute{e}t}(U_\eta, \bar{\eta})$,*
 3. *for all $x \in C \setminus U$, M_x a complex in $\mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\mathrm{Sh}((BG_{k(x)})_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell))$ placed in degrees $[0, 1]$,*
 4. *for all $x \in C \setminus U$, f_x is an element of*

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{Rep}(G_{k(x)}, \mathbb{Z}_\ell))} \left(M_x, \bigoplus_{\nu(y)=x} \mathrm{Ind}_{G_{k(y)}}^{G_{k(x)}} (\partial_y [\phi_y^*(M_\eta(y))]) \right)$$

such that $H^0(f_x)$ is injective.

- An element of $\mathrm{Hom}_{\mathbf{P}(C, \mathbb{Z}_\ell)}((U, (M_\eta), (M_x), (f_x)), (V, (N_\eta), (N_x), (g_x)))$ is a couple of the form $((\Phi_\eta)_{\eta \in \Gamma}, (\Phi_x)_{x \in (C \setminus U) \cap (C \setminus V)})$ where $\Phi_\eta: M_\eta \rightarrow N_\eta$ is a map of representations of G_{K_η} , where $\Phi_x: M_x \rightarrow N_x$ is a map in $\mathcal{D}(\mathrm{Sh}((BG_{k(x)})_{\mathrm{pro\acute{e}t}}, \mathbb{Z}_\ell))$ and such that the diagram:

$$\begin{array}{ccc}
 M_x & \xrightarrow{f_x} & \bigoplus_{\nu(y)=x} \mathrm{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left[\phi_y^*(M_{\eta(y)}) \right] \right) \\
 \downarrow \Phi_x & & \downarrow \bigoplus_{\nu(y)=x} \mathrm{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left(\phi_y^*(\Phi_{\eta(y)}) \right) \right) \\
 N_x & \xrightarrow{g_x} & \bigoplus_{\nu(y)=x} \mathrm{Ind}_{G_{k(y)}}^{G_{k(x)}} \left(\partial_y \left[\phi_y^*(N_{\eta(y)}) \right] \right)
 \end{array}$$

is commutative.

Our discussion then yields the following proposition.

Proposition 2.5.1.8. *Let C be an excellent 1-dimensional scheme and ℓ be a prime number invertible on C . Then, the category $\mathrm{Perv}(C, \mathbb{Z}_\ell)$ is equivalent to $\mathbf{P}(C, \mathbb{Z}_\ell)$.*

2.5.2 Artin perverse sheaves over schemes of dimension 1

Let M be an Artin perverse sheaf over an excellent 1-dimensional base scheme S . Recall that there is a stratification \mathcal{S} of S such that for all $T \in \mathcal{S}$, $M|_T$ is smooth Artin. We want to recover M from the $M|_T$ and additional data. This is possible in the case when S is 1-dimensional using the method of the previous paragraph.

We again take the convention that $\delta(S) = 1$.

Definition 2.5.2.1. *Let S be a 1-dimensional excellent scheme. Let ℓ be a prime number invertible on S . Let Γ be the set of generic points of S . We define the full subcategory $\mathbf{P}^A(S, \mathbb{Z}_\ell)$ of $\mathbf{P}(S, \mathbb{Z}_\ell)$ as the subcategory made of those quadruple $(U, (M_\eta)_{\eta \in \Gamma}, (M_x)_{x \in C \setminus U}, (f_x)_{x \in C \setminus U})$ such that for all $\eta \in \Gamma$, the representation M_η is of Artin origin and for all $x \in S \setminus U$, the representations $H^0(M_x)$ and $H^1(M_x)$ are of Artin origin.*

Proposition 2.5.2.2. *Let S be an excellent 1-dimensional scheme and ℓ be a prime number invertible on S . Then, $\mathrm{Perv}^A(S, \mathbb{Z}_\ell)$ is equivalent to $\mathbf{P}^A(S, \mathbb{Z}_\ell)$.*

Remark 2.5.2.3. *It is possible that the nearby and vanishing cycle preserve the Artin condition over 1-dimensional schemes which would give a description of Artin perverse sheaves which is more in the spirit of [Bei87a]. In the case of Artin homotopy perverse sheaves, it may even be possible to have a similar description for every scheme of finite type over a finite field by using the Artin truncation of nearby and vanishing cycles.*

Example 2.5.2.4. *Let k be an algebraically closed field of characteristic 0. A perverse Artin ℓ -adic sheaf over \mathbb{P}_k^1 is a quadruple $(\mathbb{P}_k^1 \setminus F, M, (M_x)_{x \in F}, (f_x)_{x \in F})$ where F is a finite set of points of \mathbb{P}_k^1 , where M is a representation of Artin origin of $\pi_1^{\acute{e}t}(\mathbb{P}_k^1 \setminus F)$, where M_x is a perfect complex of \mathbb{Z}_ℓ -modules placed in degrees $[0, 1]$ and where*

$$f_x: M_x \rightarrow \partial_x[\phi_x^*(M)]$$

is a map in the category $\mathcal{D}(\mathbb{Z}_\ell)$ such that $H^0(f_x)$ is injective.

We can always assume that the point at infinity is in F . Write $F = \{\infty\} \sqcup F'$. Let $m = |F'|$. Let $x \in F'$, then, the field $k((X-x))$ is the completion of the field $k(X)$ of regular functions on \mathbb{P}_k^1 with respect to the valuation defined by x .

On the other hand, the field $k((1/X))$ is the completion of the field $k(X)$ with respect to the valuation of the point at infinity.

Furthermore, recall that the absolute Galois group of the field $k((X))$ is $\hat{\mathbb{Z}}$ since k is algebraically closed and of characteristic 0.

The étale fundamental group of the scheme $\mathbb{P}_k^1 \setminus F$ is the profinite completion of the free group over F' ; we denote by g_x for $x \in F'$ its topological generators. The map

$$G_{k((X-x))} \rightarrow \pi_1^{\text{ét}}(\mathbb{P}_k^1 \setminus F)$$

is the only continuous map that sends the topological generator of $\hat{\mathbb{Z}}$ to g_x . Moreover, the map

$$G_{k((1/X))} \rightarrow \pi_1^{\text{ét}}(\mathbb{P}_k^1 \setminus F)$$

is the map that sends the topological generator to a certain product $(g_{x_1} \cdots g_{x_m})^{-1}$ where $F' = \{x_1, \dots, x_m\}$.

Now, the action of $\pi_1^{\text{ét}}(\mathbb{P}_k^1 \setminus F)$ on M factors through a finite quotient G and is therefore uniquely determined by the representation of a finite group with m marked generators which are the images of the g_x for $x \in F'$.

Finally, let

$$\partial: \text{Rep}(\hat{\mathbb{Z}}, \mathbb{Z}_\ell) \rightarrow \mathcal{D}^b(\mathbb{Z}_\ell)$$

be the functor that sends a continuous $\mathbb{Z}_\ell[\hat{\mathbb{Z}}]$ module M to the derived module of fixed points under the action of the topological generator.

A perverse Artin ℓ -adic sheaf over \mathbb{P}_k^1 is equivalent to the following data:

- A finite number of distinct points x_1, \dots, x_m of \mathbb{A}_k^1 (i.e. of elements of k).
- A finite group G generated by m elements g_1, \dots, g_m ; let $\phi_i: \hat{\mathbb{Z}} \rightarrow G$ be the map given by g_i and $\phi_\infty: \hat{\mathbb{Z}} \rightarrow G$ be the map given by $g_\infty = (g_1 \cdots g_m)^{-1}$.
- A \mathbb{Z}_ℓ -linear representation M of G which is finite as a \mathbb{Z}_ℓ -module.
- Perfect complexes of \mathbb{Z}_ℓ -modules $M_1, \dots, M_m, M_\infty$ placed in degrees $[0, 1]$.
- Maps $f_i: M_i \rightarrow \partial(\phi_i^* M)$ for $i = 1, \dots, m, \infty$, such that the maps

$$H^0(f_i): H^0(M_i) \rightarrow M^{g_i}$$

are injective.

Example 2.5.2.5. Let $S = \text{Spec}(\mathbb{Z}_p)$. A perverse Artin ℓ -adic sheaf over S is given by a triple (M, N, f) where M is a representation of Artin origin of $G_{\mathbb{Q}_p}$, where N is a complex in $\mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((B\hat{\mathbb{Z}})_{\text{proét}}, \mathbb{Z}_\ell))$ placed in degrees $[0, 1]$ with cohomology of Artin origin and where $f: N \rightarrow \partial(M)$, where ∂ is the map

$$\mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((BG_{\mathbb{Q}_p})_{\text{proét}}, \mathbb{Z}_\ell)) \rightarrow \mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((B\hat{\mathbb{Z}})_{\text{proét}}, \mathbb{Z}_\ell))$$

induced by the map $M \mapsto M^{G_0}$ and $H^0(f)$ is injective.

Example 2.5.2.6. Let k be an algebraically closed field of characteristic 0. Let A be the localization of the ring $k[X, Y]/(XY)$ at the prime ideal (X, Y) . It is the local ring at the intersection point of two lines in \mathbb{P}_k^2 .

Let $S = \text{Spec}(A)$. The scheme S has two generic points of residue field $k(X)$ and the residue field at the only closed point is k . the normalization of S is given by the scheme

$$\text{Spec}(k[X]_{(X)} \times k[Y]_{(Y)}).$$

Let $\partial: \text{Rep}(\hat{\mathbb{Z}}, \mathbb{Z}_\ell) \rightarrow \mathcal{D}^b(\mathbb{Z}_\ell)$ be the derived functor of fixed points and let

$$\phi: \hat{\mathbb{Z}} = G_{k((X))} \rightarrow G_{k(X)}$$

be the inclusion.

Now, an Artin perverse sheaf over S is given by a quadruple (M_1, M_2, N, f) where M_1 and M_2 are representations of Artin origin of $G_{k(X)}$, where N is a complex of \mathbb{Z}_ℓ -modules placed in degrees $[0, 1]$ and where $f: N \rightarrow \partial(\phi^*(M_1 \oplus M_2))$ is such that $H^0(f)$ is injective.

Example 2.5.2.7. Let $S = \text{Spec}(\mathbb{Z}[\sqrt{5}]_{(2)})$. This scheme has two points: its generic point has residue field $\mathbb{Q}(\sqrt{5})$ and its closed point has residue field \mathbb{F}_2 . The normalization of S is $\text{Spec}\left(\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]_{(2)}\right)$ which is a discrete valuation ring. The residue field of its closed point is the field \mathbb{F}_4 with 4 elements. The completion of $\mathbb{Q}(\sqrt{5})$ with respect to the valuation defined by 2 is the field $\mathbb{Q}_2(\sqrt{5})$.

Let ∂ be the map

$$\mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((BG_{\mathbb{Q}(\sqrt{5})})_{\text{proét}}, \mathbb{Z}_\ell)) \rightarrow \mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((B\hat{\mathbb{Z}})_{\text{proét}}, \mathbb{Z}_\ell))$$

induced by the map $M \mapsto M^{G_0}$.

Let $\phi: \hat{\mathbb{Z}} \rightarrow \hat{\mathbb{Z}}$ be the multiplication by 2 map. The map ϕ^* has right adjoint functor

$$\text{Ind}_\phi: \text{Sh}((B\hat{\mathbb{Z}})_{\text{proét}}, \mathbb{Z}_\ell) \rightarrow \text{Sh}((B\hat{\mathbb{Z}})_{\text{proét}}, \mathbb{Z}_\ell).$$

Thus an Artin perverse sheaf over S is given by a triple (M, N, f) where M is a representation of Artin origin of $G_{\mathbb{Q}(\sqrt{5})}$, where N is a complex in $\mathcal{D}_{\mathbb{Z}_\ell\text{-perf}}^b(\text{Sh}((B\hat{\mathbb{Z}})_{\text{proét}}, \mathbb{Z}_\ell))$ placed in degrees $[0, 1]$ with cohomology of Artin origin and where

$$f: N \rightarrow \text{Ind}(\partial(M))$$

is such that $H^0(f)$ is injective.

Chapter 3

Abelian categories of Artin motives

We study motivic t-structures on the stable category $\mathcal{DM}_{\acute{e}t,c}^A(S, \mathbb{Z})$ of constructible Artin étale motives over an excellent base scheme S .

When the scheme S is regular, we exhibit a subcategory of smooth Artin motives which can be related to local systems and to representations of the étale fundamental group.

When the scheme S allows resolution of singularities, we construct the ordinary motivic t-structure.

When the scheme S is of dimension 2 or less we construct the perverse motivic t-structure and prove an analog of the Affine Lefschetz Theorem. We show that the perverse motivic t-structure cannot exist on $\mathcal{DM}_{\acute{e}t,c}^A(\mathbb{A}_k^4, \mathbb{Z})$ with k a field.

We construct the perverse homotopy t-structure on $\mathcal{DM}_{\acute{e}t,c}^A(S, \mathbb{Q})$ which is the best possible approximation of the perverse motivic t-structure. When S is of finite type over \mathbb{F}_p , the ℓ -adic realization functor is t-exact when we restrict its target to the category of Artin ℓ -adic sheaves endowed with its perverse homotopy t-structure.

When S is 1-dimensional, we describe the category $M_{\text{perv}}^A(S, \mathbb{Z})$ of perverse Artin étale motives in terms of representations of étale fundamental groups of regular locally closed subschemes of S . We also describe the ℓ -adic realization in this case.

Finally, the category $M_{\text{perv}}^A(S, \mathbb{Q})$ is similar to the category of perverse sheaves: its objects are of finite length and we can describe its simple objects. For instance, the motivic weightless intersection complex \mathbb{E}_S of Ayoub and Zucker is a simple Artin perverse sheaf.

Introduction

Thanks to [Voe92, Ayo07, CD19, Ayo14c, CD16, Rob14, Kha16, Ayo21], we now have at our disposal stable ∞ -categories of mixed étale motives $\mathcal{DM}_{\acute{e}t}$ endowed with Grothendieck's six functors formalism and with ℓ -adic realization functors. However, to connect those categories to Grothendieck's conjectural theory of motives, an essential structure is still missing: the motivic t-structure. Its heart would be the abelian category of mixed motivic sheaves which would satisfy Beilinson's conjectures [Jan94].

This problem seems completely out of reach at the moment. However, when k is a field, Beilinson introduced in [Bei02] the dimensional filtration on the stable ∞ -category $\mathcal{DM}_{\acute{e}t,c}(k, \mathbb{Q})$ of constructible étale motives over k with rational coefficients and showed that the existence of the motivic t-structure implies the existence of t-structures on the subcategory $\mathcal{DM}_{\acute{e}t,c}^n(k, \mathbb{Q})$ of n -motives which is the subcategory generated by the cohomological motives of proper k -schemes

of dimension less than n . Defining the motivic t-structure on those dimensional subcategories is hopefully easier.

This question also generalizes to an arbitrary base scheme if we define the subcategory $\mathcal{DM}_{\acute{e}t,c}^n(S, \mathbb{Q})$ of n -motives in the same fashion. Working by analogy with the derived category of ℓ -adic complexes, there are two possible versions of the motivic t-structure: the *perverse motivic t-structure* and the *ordinary motivic t-structure*. In [Bon15], Bondarko has shown that the perverse motivic t-structure on $\mathcal{DM}_{\acute{e}t,c}(S, \mathbb{Q})$ can be recovered from the ordinary motivic t-structures on every $\mathcal{DM}_{\acute{e}t,c}(k, \mathbb{Q})$ (assuming they exist) when k runs through the set of residue fields of S . However, his approach does not apply to the dimensional subcategories $\mathcal{DM}_c^n(S, \mathbb{Q})$.

The strongest result on this problem concerns the ordinary motivic t-structure and stems from the work of Voevodsky, Orgogozo, Ayoub, Barbieri-Viale, Kahn, Pépin Lehalleur and Vaish.

Theorem. ([Org04, Ayo11, AB09, BVK16, Pep19b, Pep19a, Vai19]) *Let S be a noetherian excellent finite dimensional scheme allowing resolution of singularities by alterations, let ℓ be a prime number invertible on S and let $n = 0, 1$.*

Then, there is a t-structure on the stable ∞ -category $\mathcal{DM}_{\acute{e}t}^n(S, \mathbb{Q})$ which induces a non-degenerate t-structure on the subcategory $\mathcal{DM}_{\acute{e}t,c}^n(S, \mathbb{Q})$ of constructible n -motives and such that the ℓ -adic realization functor

$$\rho_\ell: \mathcal{DM}_c^n(S, \mathbb{Q}) \rightarrow \mathcal{D}_c^b(S, \mathbb{Q}_\ell)$$

is t-exact when the derived category $\mathcal{D}_c^b(S, \mathbb{Q}_\ell)$ of constructible ℓ -adic complexes is endowed with its ordinary t-structure.

There are two main directions in which this result can be generalized. More precisely, the above theorem leaves two open questions.

1. Is there a t-structure on the stable ∞ -category $\mathcal{DM}_{\acute{e}t}^n(S, \mathbb{Z})$ inducing a non-degenerate t-structure on the subcategory $\mathcal{DM}_{\acute{e}t,c}^n(S, \mathbb{Z})$ of constructible étale n -motives and such that for any prime number ℓ , the ℓ -adic realization functor

$$\rho_\ell: \mathcal{DM}_c^n(S, \mathbb{Z}) \rightarrow \mathcal{D}_c^b(S[1/\ell], \mathbb{Z}_\ell)$$

is t-exact when the stable ∞ -category $\mathcal{D}_c^b(S[1/\ell], \mathbb{Z}_\ell)$ is endowed with its **ordinary** t-structure ?

2. Let $R = \mathbb{Z}$ or \mathbb{Q} . Is there a t-structure on the stable ∞ -category $\mathcal{DM}_{\acute{e}t}^n(S, R)$ which induces a non-degenerate t-structure on the subcategory $\mathcal{DM}_{\acute{e}t,c}^n(S, R)$ of constructible étale n -motives and such that for any prime number ℓ , the ℓ -adic realization functor

$$\rho_\ell: \mathcal{DM}_c^n(S, R) \rightarrow \mathcal{D}_c^b(S[1/\ell], R_\ell)$$

is t-exact when the stable ∞ -category $\mathcal{D}_c^b(S[1/\ell], \mathbb{Z}_\ell)$ is endowed with its **perverse** t-structure ?

Voevodsky showed in [VSF11, 4.3.8] that when working with with integral coefficients, it is not possible to construct a reasonable motivic t-structure on the stable ∞ -category $\mathcal{DM}_{Nis}(k, \mathbb{Z})$ of Nisnevich motives for a general base field k . However, nothing is known about motivic t-structures on $\mathcal{DM}_{\acute{e}t}^n(S, \mathbb{Z})$.

To goal of this chapter is to answer both of the above questions in the case of étale motives. Let R be a ring of coefficients. We introduce the following categories of motives.

- The category $\mathcal{DM}_{\acute{e}t,c}^A(S, R)$ of constructible Artin étale motives (or 0-motives with the above terminology) defined as the thick subcategory of the stable ∞ -category $\mathcal{DM}_{\acute{e}t}(S, R)$ of étale motives of [Ayo14c, CD16] generated by the cohomological motives of finite S -schemes.
- The category $\mathcal{DM}_{\acute{e}t}^A(S, R)$ of Artin étale motives defined as the localizing subcategory of $\mathcal{DM}_{\acute{e}t}(S, R)$ with the same generators.

Rigidity for torsion motives

One of the main contrasts between Artin étale motives with rational coefficients and Artin étale motives with integral coefficients is that the category of integral motives will contain many motives which should come from higher dimensional motives with torsion coefficients. This is in fact an instance of the rigidity theorem. Let $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$ be the stable ∞ -category of hypersheaves on the small étale site of S with value in the derived category of R -modules. We have by Proposition 3.1.4.3 a functor

$$\rho!: \mathcal{S}\mathcal{h}(S_{\acute{e}t}, R) \rightarrow \mathcal{DM}_{\acute{e}t}^A(S, R)$$

which sends a sheaf to the infinite Tate suspension of the \mathbb{A}^1 -localization of its extension to the big étale site. This functor is in fact very close to being an equivalence when restricted to torsion objects. More precisely, we define

- The stable category of *torsion étale motives* (resp. *p^∞ -torsion étale motives*) denoted by $\mathcal{DM}_{\acute{e}t,\text{tors}}(S, R)$ (resp. $\mathcal{DM}_{\acute{e}t,p^\infty-\text{tors}}(S, R)$) as the subcategory of the stable category $\mathcal{DM}_{\acute{e}t}(S, R)$ made of those objects M such that the motive $M \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $M \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$) vanishes.
- The stable category of *torsion étale sheaves* (resp. *p^∞ -torsion étale sheaves*) $\mathcal{S}\mathcal{h}_{\text{tors}}(S_{\acute{e}t}, R)$ (resp. $\mathcal{S}\mathcal{h}_{p^\infty-\text{tors}}(S_{\acute{e}t}, R)$) as the subcategory of the stable category $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$ made of those sheaves M such that the sheaf $M \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $M \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$) vanishes.

The following result is then a consequence of the Rigidity Theorem [BH21, 3.2].

Proposition. (Corollary 3.1.3.4 and Proposition 3.1.4.10) *Let S be a scheme and let R be a good ring (see Notations and Conventions below). Then,*

1. *Torsion motives are Artin motives.*
2. *If p is a prime number, let $j_p: S[1/p] \rightarrow S$ be the open immersion. Then, the functor*

$$\prod_{p \text{ prime}} \mathcal{S}\mathcal{h}_{p^\infty-\text{tors}}(S_{\acute{e}t}, R) \rightarrow \mathcal{DM}_{\acute{e}t,\text{tors}}(S, R)$$

which sends an object $(M_p)_{p \text{ prime}}$ to the object $\bigoplus_{p \text{ prime}} j_p^ \rho! M_p$ is an equivalence.*

This result will allow us to construct an ordinary t-structure and a perverse t-structure on torsion motives. One of the main ideas of this chapter is to deduce information on the t-structures on Artin motives with integral coefficients from the t-structures on Artin motives with rational coefficients and the t-structures on torsion motives.

Smooth Artin motives

The subcategory of dualizable (or lisse) objects of the derived category of ℓ -adic sheaves plays a key role in the study of ℓ -adic sheaves and perverse sheaves. They can be described in terms of local systems and are therefore much more tractable than general perverse sheaves. In this chapter, the subcategory that will play the role of lisse ℓ -adic sheaves is the category of smooth Artin motives.

- The category $\mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R)$ of constructible smooth Artin étale motives defined as the thick subcategory of $\mathcal{DM}_{\text{ét}}(S, R)$ generated by the cohomological motives of finite étale S -schemes.
- The category $\mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$ of smooth Artin étale motives defined as the localizing subcategory of $\mathcal{DM}_{\text{ét}}(S, R)$ with the same generators.

An important step towards the answer of the questions above is to describe the categories $\mathcal{DM}_{\text{ét},(c)}^{\text{sm}A}(S, R)$ more precisely. Extending a result of [Org04], we link them together the more concrete and tractable category of Artin representations of the étale fundamental group of S .

The categories of smooth Artin motives can be identified to the following categories.

- The category of lisse étale sheaves $\mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R)$ as the subcategory of dualizable objects of the stable ∞ -category $\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$.
- The category of Ind-lisse étale sheaves $\mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(S, R)$ as the localizing subcategory of the stable ∞ -category $\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$ generated by $\mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R)$.

Theorem. (Theorem 3.2.1.1, Propositions 3.1.5.9 and 3.1.5.10 and Corollary 3.2.1.7) *Let R be a regular good ring and let S be a regular scheme.*

1. Assume that the residue characteristic exponents of S are invertible in R . Then, the functor $\rho_!$ of Proposition 3.1.4.3 induces monoidal equivalences

$$\mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(S, R) \longrightarrow \mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$$

$$\mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R) \longrightarrow \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R).$$

2. The ordinary t -structure on the stable ∞ -category $\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$ induces t -structures on the subcategories $\mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R)$ and $\mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(S, R)$.

- (a) The heart of the induced t -structure on $\mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R)$ is the category $\text{Loc}_S(R)$ of locally constant sheaves of R -modules with finitely presented fibers.
- (b) The heart of the induced t -structure on $\mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(S, R)$ is the category $\text{Ind Loc}_S(R)$ of filtered colimits of locally constant sheaves of R -modules with finitely presented fibers.

3. If S is connected, letting ξ be a geometric point of S , the fiber functor associated to ξ induces an equivalence of abelian monoidal categories

$$\xi^* : \text{Loc}_S(R) \rightarrow \text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R)$$

where $\text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R)$ is the category of Artin representations of the étale fundamental group of S with base point ξ .

4. If S is the spectrum of a field k of characteristic exponent p . Then, the functor $\rho_!$ induces monoidal equivalences

$$\mathcal{D}(\mathrm{Mod}(G_k, R[1/p])) \simeq \mathcal{S}\mathcal{h}(k_{\acute{\mathrm{e}}\mathrm{t}}, R[1/p]) \longrightarrow \mathcal{DM}_{\acute{\mathrm{e}}\mathrm{t}}^A(k, R),$$

$$\mathcal{D}^b(\mathrm{Rep}^A(G_k, R[1/p])) \simeq \mathcal{S}\mathcal{h}_{\mathrm{lisse}}(k, R[1/p]) \longrightarrow \mathcal{DM}_{\acute{\mathrm{e}}\mathrm{t},c}^A(k, R),$$

where $\mathrm{Rep}^A(G_k, R[1/p])$ (resp. $\mathrm{Mod}(G_k, R[1/p])$) is the category of Artin representation (resp. discrete representation) of the absolute Galois group G_k of k with coefficients in $R[1/p]$.

The above theorem shows in particular that the stable ∞ -category $\mathcal{DM}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{sm}A}(S, R)$ (resp. $\mathcal{DM}_{\acute{\mathrm{e}}\mathrm{t},c}^{\mathrm{sm}A}(S, R)$) is endowed with a t-structure and identifies the heart with an abelian category of étale sheaves. Furthermore, this theorem shows that we can describe the objects of $\mathcal{DM}_{\acute{\mathrm{e}}\mathrm{t},c}^A(S, R)$ as a gluing in the sense of [BBDG18] of Artin representations.

The last consequence of the above theorem is a version with integral coefficients of the conservativity conjecture for Artin étale motives which is slightly different from the version with rational coefficients proved in [Pep19b]. Indeed, we need to include all the v -adic realization where v runs through the set of valuations on the fraction field of R which are non-negative on R (i.e. such that $\{x \in R \mid v(x) > 0\}$ defines a prime ideal of R). Otherwise any motive of ℓ -torsion would provide a counterexample to the result.

The Artin truncation functor

The existence of motivic t-structures and more generally the properties of étale motives are closely related to the Artin truncation functor ω^0 which is the right adjoint functor of the inclusion functor $\iota: \mathcal{DM}_{\acute{\mathrm{e}}\mathrm{t}}^A(S, R) \rightarrow \mathcal{DM}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{coh}}(S, R)$ of Artin étale motives into the category of cohomological étale motives. This functor was first introduced in [AZ12] to study the relative Borel-Serre compactification of a locally symmetric variety.

One of the main features of the functor ω^0 is that when the ring R is a number field, it preserves constructible objects. Unfortunately, when R is a localization of the ring of integers of a number field which is not a number field, we show that this results fails completely: if $f: X \rightarrow S$ is a morphism of finite type which is not quasi-finite, then, the Artin motive $\omega^0 f_! \mathbb{1}_X$ is not constructible under some mild conditions on S and R (see Proposition 3.3.1.1). However, this functor is still tractable, even with integral coefficients when the base scheme is the spectrum of a field and we can generalize some of the computations of [AZ12].

Theorem. (Theorem 3.3.1.4 and Corollary 3.3.1.8) *Let R be a localization of the ring of integers of a number field, let k be a field of characteristic exponent p and let $f: X \rightarrow \mathrm{Spec}(k)$ be a morphism of finite type. Assume that the scheme X is regular.*

1. *The diagram*

$$\begin{array}{ccc} \mathcal{S}\mathcal{h}_{\mathrm{Ind\ lisse}}(X, R) & \xrightarrow{\rho_!} & \mathcal{DM}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{sm}A}(X, R) \\ \downarrow f_* & & \downarrow \omega^0 f_* \\ \mathcal{S}\mathcal{h}(k_{\acute{\mathrm{e}}\mathrm{t}}, R) & \xrightarrow{\rho_!} & \mathcal{DM}_{\acute{\mathrm{e}}\mathrm{t}}^A(k, R) \end{array}$$

is commutative.

2. We have

$$H^0(\omega^0 f_* \mathbb{1}_X) = h_k(\pi_0(X/k)).$$

3. Using Notations [3.3.1.7](#), we have

$$H^n(\omega^0 f_* \mathbb{1}_X) = \alpha_! [H_{\text{ét}}^n(X_{\bar{k}}, R[1/p])]$$

and

$$H_{\text{ét}}^n(X_{\bar{k}}, R[1/p]) = \begin{cases} R[1/p][\pi_0(X_{\bar{k}})] & \text{if } n = 0 \\ \mu^{n-1}(X, R) & \text{if } n \geq 2 \\ 0 & \text{otherwise} \end{cases}.$$

Hence, in the above computation of $\omega^0(f_* \mathbb{1}_X)$, we have a contribution of weight zero which is $H^0(\omega^0(f_* \mathbb{1}_X))$ and a non-constructible torsion part which does not appear when R is a \mathbb{Q} -algebra and comes from motives with torsion coefficients of weight at least one.

The ordinary motivic t-structure

We introduce two *homotopy t-structures* whose definitions are inspired by the approach of [Pep19b](#), [Vai19](#), [Ayo07](#), [BD17](#). More precisely, we define an *ordinary homotopy t-structure* studied in Section [3.4](#) and a *perverse homotopy t-structure* studied in Section [3.5](#) on the category $\mathcal{DM}_{\text{ét}}^A(S, R)$ by giving a family of t-non-positive generators. We then determine when those t-structures induce t-structures on the category of constructible objects and when the realization functor is t-exact.

We extend the main results of [Pep19a](#), [Vai19](#) about the ordinary motivic t-structure to the case of integral coefficients.

Theorem. (Theorem [3.4.2.7](#)) *Let S be a scheme allowing resolution of singularities by alterations and let R be a regular good ring. Then,*

1. *The ordinary homotopy t-structure induces a non-degenerate t-structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)$. We still call this t-structure the ordinary homotopy t-structure.*
2. *Assume that the ring R is a localization of the ring of integers of a number field K . Let v be a non-archimedean valuation on K and let ℓ be the prime number such that v extends the ℓ -adic valuation. Then, the reduced v -adic realization functor*

$$\bar{\rho}_v: \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}_c^b(S[1/\ell], R_v)$$

of Definition [3.2.2.1](#) is t-exact when the left hand side is endowed with the ordinary homotopy t-structure and the right hand side is endowed with the ordinary t-structure.

To prove this result, we first study the case of smooth Artin motives and of torsion motives. We show that the inclusion functors

$$\mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

and

$$\mathcal{DM}_{\text{tors}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

are t-exact when the left hand sides are endowed with their ordinary t-structures and the right hand side is endowed with the ordinary homotopy t-structure (see Propositions [3.4.1.8](#) and [3.4.2.1](#)). We then show a t-exactness result about the pullback functor (see Proposition [3.4.2.5](#)) and finally, we prove the theorem by stratifying any constructible motive by smooth Artin motives.

The perverse motivic t-structure and the weak Lefschetz Properties

Let t_0 be a t-structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)$. We say that

- the t-structure t_0 is the *perverse motivic t-structure* if for any constructible Artin étale motive M , the motive M is t_0 -non-positive if and only if for all non-archimedean valuation v on K which is non-negative on R , the complex $\bar{\rho}_v(M)$ is perverse t-non-positive (see Section [3.1.6](#)).
- the t-structure t_0 is the *perverse motivic t-structure in the strong sense* if for any non-archimedean valuation v on K which is non-negative on R , the functor

$$\bar{\rho}_v: \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}_c^b(S, R_v)$$

is t-exact when the left hand side is endowed with the t-structure t_0 and the right hand side is endowed with the perverse t-structure.

A perverse motivic t-structure in the strong sense is perverse motivic and if a perverse motivic t-structure exists, it is unique. Therefore, the perverse motivic t-structure is, if it exists, the best possible approximation of the perverse motivic t-structure in the strong sense. Hence, the answer to question (2) above can be divided into two step: first to determine when the perverse motivic t-structure exists and second to determine when the ℓ -adic realization functor is t-exact.

To achieve the first step of this program, we can in fact reduce the statement to a statement about the perverse homotopy t-structure. Indeed, the only suitable candidate to be the perverse motivic t-structure is the perverse homotopy t-structure. More precisely (see Proposition [3.5.2.1\(3\)](#)), if S is excellent and R is a localization of \mathbb{Z} , then the perverse motivic t-structure exists if and only if the perverse homotopy t-structure induces a t-structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)$. In this case, the perverse homotopy t-structure is the perverse motivic t-structure.

To prove this statement, we show that the perverse homotopy t-structure can be described locally in way which is similar to the local description of the perverse t-structure of [\[BBDG18\]](#).

Proposition. (*Proposition [3.5.2.1\(1\)](#)*) *Let S be a scheme and R be a regular good ring. If x is a point of S , denote by $i_x: \{x\} \rightarrow S$ the inclusion. Let M be an Artin motive over S with coefficients in R . Then,*

1. *The Artin motive M is perverse homotopy t-non-negative if and only if it is bounded below with respect to the ordinary homotopy t-structure and for any point x of S , we have*

$$\omega^0 i_x^! M \geq_{\text{ord}} -\delta(x).$$

2. *Assume that the Artin motive M is constructible. Then, it is perverse homotopy t-non-positive if and only if for any point x of S , we have*

$$i_x^* M \leq_{\text{ord}} -\delta(x).$$

When the ring of coefficients is a number field, following the ideas of [Vai19, Rui22], we show that the perverse homotopy t-structure induces a t-structure on the stable ∞ -category $\mathcal{DM}_{\text{ét},c}^A(S, R)$. This t-structure is therefore the perverse motivic t-structure. Furthermore, if p is a prime number, if the scheme S is of finite type over \mathbb{F}_p and if Λ is an ℓ -adic field, we defined in the second chapter a perverse homotopy t-structure on the category $\mathcal{D}^A(S, \Lambda)$ of Artin ℓ -adic sheaves. We show in this chapter that if K is a number field and if v is a valuation on K which does not extend the ℓ -adic valuation, the v -adic realization functor

$$\mathcal{DM}_{\text{ét},c}^A(S, K) \rightarrow \mathcal{D}^A(S, K_v)$$

is t-exact when both sides are endowed with the perverse homotopy t-structure.

We then prove weak Lefschetz properties which will play a key role in the rest of the chapter.

Theorem. (Proposition 3.5.3.5 and Corollary 3.5.3.10) *Let S be an excellent scheme, let $f: X \rightarrow S$ be a quasi-finite morphism and let K be a number field. Assume that the scheme X is nil-regular.*

Then, the functor

$$\omega^0 f_*: \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(X, K) \rightarrow \mathcal{DM}_{\text{ét},c}^A(S, K)$$

is perverse homotopy t-exact.

Assume furthermore that we are in one of the following cases

(a) *We have $\dim(S) \leq 2$.*

(b) *There is a prime number p such that the scheme S is of finite type over \mathbb{F}_p .*

and that the morphism f is affine, then, the functor

$$f_!: \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(X, K) \rightarrow \mathcal{DM}_{\text{ét},c}^A(S, K)$$

is perverse homotopy t-exact.

The t-exactness of the functor $f_!$ with f quasi-finite and affine can in fact be extended to the category of those motives M such that $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is constructible. This result is one of the main ingredients to understand the perverse homotopy t-structure on Artin motives with integral coefficients.

When R is not a number field, the situation is indeed different. First, we have a positive result for schemes of dimension 2 or less which may be seen as the main result of this chapter. The proof of this theorem involves computations around the Artin truncation functor.

Theorem. (Theorems 3.5.4.1 and 3.6.3.3 and Proposition 3.5.4.4) *Let S be an excellent scheme of dimension 2 or less, let R be a localization of the ring of integers of a number field K . Then,*

1. *The perverse homotopy t-structure induces a t-structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)$.*
2. *Let v be a non-archimedean valuation on K and let ℓ be the prime number such that v extends the ℓ -adic valuation. Then, the reduced v -adic realization functor*

$$\bar{\rho}_v: \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}_c^b(S[1/\ell], R_v)$$

of Definition 3.2.2.1 is t-exact when the left hand side is endowed with the perverse homotopy t-structure and the right hand side is endowed with the perverse t-structure.

3. Let $f: T \rightarrow S$ be a quasi-finite and affine morphism of schemes. Then, the functor

$$f_!: \mathcal{DM}_{\text{ét},c}^A(T, R) \rightarrow \mathcal{DM}_{\text{ét},c}^A(S, R)$$

is perverse homotopy t-exact.

However, unlike in the case of Artin motives with coefficients in a number field, we have a negative result for higher dimensional schemes. If k is a field, the perverse homotopy t-structure of the stable ∞ -category $\mathcal{DM}_{\text{ét}}^A(\mathbb{A}_k^4, \mathbb{Z})$ does not induce a t-structure on the subcategory $\mathcal{DM}_{\text{ét},c}^A(\mathbb{A}_k^4, \mathbb{Z})$ (see Example 3.5.4.6). The case of 3-dimensional schemes remains open.

Perverse Artin motives

The weak Affine Lefschetz properties also allow us to describe the object of the heart of the perverse homotopy t-structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)$ which we call the category of perverse Artin étale motive over S and denote by $M_{\text{perv}}^A(S, R)$.

If S is an excellent scheme of dimension 1, the category $M_{\text{perv}}^A(S, R)$ is equivalent to the category $N(S, R)$ (see Definition 3.6.4.9) which is defined completely in terms of representations and maps of representations. In addition, we give an explicit description of the v -adic realization functor in terms of the category $N(S, R)$ if S is of dimension 1.

Finally, if the ring of coefficient is a number field K , the category $M_{\text{perv}}^A(S, K)$ is similar to the category of perverse sheaves with coefficients in an ℓ -adic number field and we can define an analog of the intermediate extension functor denoted $j_{!*}^A$. Furthermore, the category of perverse Artin motives contains the motivic weightless complex of [AZ12]; this last assertion is a consequence of the perverse homotopy t-exactness of the functor $\omega^0 j_{!*}$.

Proposition. (Propositions 3.6.2.3 and 3.6.2.6) *Let S be an excellent scheme.*

1. *The abelian category of perverse Artin motives with rational coefficients on S is artinian and noetherian: every object is of finite length.*
2. *If $j: V \hookrightarrow S$ is the inclusion of a regular connected subscheme and if L is a simple object of $\text{Loc}_V(K)$, then the perverse Artin motive $j_{!*}^A(\rho_! L[\delta(V)])$ is simple. Every simple perverse Artin motive is obtained this way.*
3. *Let $d = \delta(S)$. Recall the motive \mathbb{E}_S of Ayoub and Zucker. Then, the motive $\mathbb{E}_S[d]$ is a simple perverse Artin motive over S .*

Notations and conventions

All schemes are assumed to be **noetherian** and of **finite dimension**; furthermore all smooth (and étale) morphisms and all quasi-finite morphisms are implicitly assumed to be separated and of finite type.

We let Sm be the class of smooth morphisms of schemes. For a scheme S , we let $S_{\text{ét}}$ (resp. Sm_S) be the category of étale (resp. smooth) S -schemes.

In this text we will freely use the language of $(\infty, 1)$ -categories of [Lur09, Lur17]. The term *category* will mean $(\infty, 1)$ -category, and the term *2-category* will mean $(\infty, 2)$ -category. When we refer to derived categories, we refer to the ∞ -categorical version.

Recall that a strictly full subcategory is a full subcategory whose set of objects is closed under isomorphisms.

We adopt the cohomological convention for t-structures (*i.e* the convention of [BBDG18, 1.3.1] which is the opposite of the convention of [Lur17, 1.2.1.1]): a t-structure on a stable category \mathcal{D} is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of strictly full subcategories of \mathcal{D} having the following properties:

- For any object M of $\mathcal{D}^{\leq 0}$ and any object N of $\mathcal{D}^{\geq 0}$, the abelian group $\pi_0 \text{Map}(M, N[-1])$ vanishes.
- We have inclusions $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}[-1]$ and $\mathcal{D}^{\geq 0}[-1] \subseteq \mathcal{D}^{\geq 0}$.
- For any object M of \mathcal{D} , there exists an exact triangle $M' \rightarrow M \rightarrow M''$ where M' is an object of $\mathcal{D}^{\leq 0}$ and M'' is an object of $\mathcal{D}^{\geq 0}[-1]$.

A stable category endowed with a t-structure is called a t-category. If \mathcal{D} is a t-category, we denote by $\mathcal{D}^\heartsuit = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ the heart of the t-structure which is an abelian category and by \mathcal{D}^b the full subcategory of bounded objects which is a stable t-category.

We will extensively use the results and notations of [CD16]. We also freely use the language of [CD19]. Their definitions and results are formulated in the setup of triangulated categories but can readily be adapted to our framework following the ideas of [Ayo21, Kha16, Rob14, AGV22].

If \mathcal{D} is a premotivic category, for any scheme S , we denote by $\mathbb{1}_S$ the unit object of the monoidal category $\mathcal{D}(S)$.

If S is a scheme and R is a ring, we denote by $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$ the category of étale hypersheaves on the small étale site $S_{\text{ét}}$ with value in the derived category of R -modules and by $\text{Sh}(S, R)$ its heart which is the category of sheaves of R -modules over $S_{\text{ét}}$. Furthermore, we denote by $\mathcal{S}\mathcal{h}_{\text{ét}}(S, R)$ the category of étale hypersheaves over Sm_S with value in the derived category of R -modules and by $\text{Sh}_{\text{ét}}(S, R)$ its heart which is the category of étale sheaves of R -modules over Sm_S .

There are several models for the category of étale motives. We will use Ayoub's model developed in [Ayo07]. If R is a ring, we denote by $\mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(S, R)$ the stable subcategory of $\mathcal{S}\mathcal{h}_{\text{ét}}(S, R)$ made of its \mathbb{A}^1 -local objects. Then, the stable category $\mathcal{D}\mathcal{M}_{\text{ét}}(S, R)$ of *étale motives* over S with coefficients in R is the \mathbb{P}^1 -stabilization of the stable category $\mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(S, R)$.

Let S be a scheme. Recall that the h-topology is the topology on the category of schemes of finite type over S whose covers are universal topological epimorphisms. The premotivic category of h-motives defined in [CD16, 5.1.1] is equivalent over noetherian schemes of finite dimension to the premotivic category of étale motives ([CD16, 5.5.7] applies in this generality, replacing [Ayo14c, 4.1] by [BH21, 3.2] in the proof). Therefore, étale motives satisfy h-descent and we can use the results proved in [CD16] in the setting of h-motives to prove statements about étale motives.

Let $i: Z \rightarrow X$ be a closed immersion and $j: U \rightarrow X$ be the complementary open immersion. We call localization triangles the exact triangles of functors:

$$j_!j^* \rightarrow \text{Id} \rightarrow i_*i^*. \quad (3.0.0.1)$$

$$i_!i^! \rightarrow \text{Id} \rightarrow j_*j^*. \quad (3.0.0.2)$$

All rings are commutative and noetherian. A ring is said to be *good* if it is either a localization of the ring of integers of a number field or noetherian and of positive characteristic.

We denote by Δ^{inj} the category of finite ordered sets with morphisms the injective maps.

If S is a scheme. A *stratification* of S is a partition of S into non-empty equidimensional locally closed subschemes called *strata* such that the topological closure of any stratum is a union strata.

If S is a scheme and ξ is a geometric point of S , we denote by $\pi_1^{\text{ét}}(S, \xi)$ the étale fundamental group of S with base point ξ defined in [GR71] and $\pi_1^{\text{proét}}(S, \xi)$ the pro-étale fundamental group of S with base point ξ defined in [BS15, 7].

If x is a point of a scheme S , we denote by $k(x)$ the residue field of S at the point x .

If k is a field, we denote by G_k its absolute Galois group.

We denote by \mathbb{Z} the ring of rational integers, by \mathbb{Q} the ring of rational numbers and by \mathbb{C} the ring of complex numbers. If p is a prime number, we denote by \mathbb{F}_p the field with p elements, by \mathbb{Z}_p the ring of p -adic integers, by \mathbb{Q}_p the field of p -adic numbers and by $\overline{\mathbb{Q}_p}$ the latter's algebraic closure. Finally, if \mathfrak{p} is a prime ideal of the ring \mathbb{Z} we denote by $\mathbb{Z}_{\mathfrak{p}}$ the localization of the ring \mathbb{Z} with respect to \mathfrak{p} .

If $\phi: H \rightarrow G$ is a map of group, we denote by ϕ^* the forgetful functor which maps representations of G to representations of H .

If H is a finite index subgroup of a group G , we denote by Ind_H^G the induced representation functor which maps representations of H to representations of G .

If R is a localization of the ring of integers of a number field K and if v is a valuation on K , we denote by R_v the completion of R with respect to v . If the valuation v is non-negative on R , we denote by $R_{(v)}$ the localization of R with respect to the prime ideal $\{x \in R \mid v(x) > 0\}$.

Remark. *The assumption that all schemes are noetherian and of finite dimension is absolutely essential in the proof of the continuity property and of the absolute purity property (among others) for étale motives. In the case of Nisnevich motives, one can have the continuity over quasi-compact quasi-separated schemes (see [Hoy15, C]). In fact, as over noetherian schemes of finite dimension, étale and Nisnevich motives with rational coefficients coincide, some of our results can be extended to this setting.*

3.1 Preliminaries

3.1.1 Categorical notions and lemmas

Notions of abelian and stable subcategories generated by a set of objects

Recall the following definitions (see [Sta23, 02MN]).

Definition 3.1.1.1. *Let A be an abelian category.*

1. A Serre subcategory of A is a nonempty full subcategory B of A such that given an exact sequence

$$M' \rightarrow M \rightarrow M''$$

in A such that M' and M'' are objects of B , then, the object M belongs to B .

2. A weak Serre subcategory of A is a nonempty full subcategory B of A such that given an exact sequence

$$M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow M_5$$

in \mathcal{A} such that for all $i \in \{1, 2, 4, 5\}$, the object M_i belongs to \mathcal{B} , then, the object M_3 belongs to \mathcal{B} .

3. Let \mathcal{E} be a set of objects of \mathcal{A} , the Serre (resp. weak Serre) subcategory of \mathcal{A} generated by \mathcal{E} is the smallest Serre (resp. weak Serre) subcategory of \mathcal{A} whose set of objects contains \mathcal{E} .

Note that a Serre subcategory of an abelian category \mathcal{A} is also a weak Serre subcategory of \mathcal{A} and that if \mathcal{B} is a weak Serre subcategory of \mathcal{A} , then \mathcal{B} is abelian and the inclusion functor $\mathcal{B} \rightarrow \mathcal{A}$ is exact and fully faithful.

Definition 3.1.1.2. Let \mathcal{C} be a stable category

1. A thick subcategory of \mathcal{C} is a full subcategory \mathcal{D} of \mathcal{C} which is closed under finite limits, finite colimits and retracts.
2. A localizing subcategory of \mathcal{C} is a full subcategory \mathcal{D} of \mathcal{C} which is closed under finite limits and arbitrary colimits.
3. Let \mathcal{E} be a set of objects. We call thick (resp. localizing) subcategory generated by \mathcal{E} the smallest thick (resp. localizing) subcategory of \mathcal{C} whose set of objects contains \mathcal{E} .

Induced t-structure

Recall the following definition from [BBDG18, 1.3.19]:

Definition 3.1.1.3. Let \mathcal{D} be a t-category and \mathcal{D}' be a full stable subcategory of \mathcal{D} . If $(\mathcal{D}^{\leq 0} \cap \mathcal{D}', \mathcal{D}^{\geq 0} \cap \mathcal{D}')$ defines a t-structure on \mathcal{D}' , we say that \mathcal{D}' is a sub-t-category of \mathcal{D} , that the t-structure of \mathcal{D} induces a t-structure on \mathcal{D}' and call the latter the induced t-structure.

Recall the following properties of sub-t-categories.

Proposition 3.1.1.4. (Proposition [2.1.2.2]) Let \mathcal{D} be a t-category and \mathcal{D}_0 be a strictly full sub-t-category of \mathcal{D}^b . Then,

1. The subcategory \mathcal{D}_0 is thick.
2. The category \mathcal{D}_0^\heartsuit is a weak Serre subcategory of \mathcal{D}^\heartsuit .

Lemma 3.1.1.5. (Lemma [2.1.2.3]) Let \mathcal{D} be a t-category. Let \mathcal{D}' be a thick subcategory of \mathcal{D}^b . Assume that $\mathcal{D}' \cap \mathcal{D}^\heartsuit$ contains a set of generators of \mathcal{D}' as a thick subcategory of \mathcal{D} . Then, the following conditions are equivalent.

1. The t-structure of \mathcal{D} induces a t-structure on \mathcal{D}' .
2. If $f: M \rightarrow N$ is a map in $\mathcal{D}' \cap \mathcal{D}^\heartsuit$, the object $\ker(f)$ of \mathcal{D}^\heartsuit belongs to \mathcal{D}' .
3. If $f: M \rightarrow N$ is a map in $\mathcal{D}' \cap \mathcal{D}^\heartsuit$, the object $\operatorname{coker}(f)$ of \mathcal{D}^\heartsuit belongs to \mathcal{D}' .

t-structure generated by a set of objects

This lemma will allow us to define the t-structure which we will study in this chapter.

Proposition 3.1.1.6. ([Lur17, 1.4.4.11]) *Let \mathcal{C} be a presentable stable category. Given a small family \mathcal{E} of objects, the smallest subcategory \mathcal{E}_- closed under small colimits and extensions is the set of non-positive objects of a t-structure.*

In this setting, we will call this t-structure the *t-structure generated by \mathcal{E}* . When a set \mathcal{E} of objects has a small set of isomorphism classes, we still call *t-structure generated by \mathcal{E}* the t-structure generated by a small family of representatives of \mathcal{E} .

Proposition 3.1.1.7. *Let \mathcal{C} be a category endowed with a Grothendieck topology τ and let Λ be a ring. Denote by $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \Lambda)$ the t-category of Λ -modules in the category of τ -hypersheaves of spectra and denote by*

$$\Lambda(-): \mathcal{C} \rightarrow \mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \Lambda)$$

the τ -hypersheafification composed with the Yoneda functor. Then,

1. *The category $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \Lambda)$ is generated as a localizing subcategory of itself by the sheaves of the form $\Lambda(X)$ where X runs through the set of objects of \mathcal{C} .*
2. *Assume that \mathcal{C} has a small set of isomorphism classes. Then, the t-structure on $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \Lambda)$ is generated by the sheaves of the form $\Lambda(X)$ where X runs through the set of objects of \mathcal{C} .*

Proof. Since $\mathcal{S}\mathfrak{h}_\tau(\mathcal{C}, \Lambda)$ is a localization of the category $\mathcal{P}\mathcal{S}\mathfrak{h}(\mathcal{C}, \Lambda)$ of presheaves of Λ -modules and since every t-non-positive sheaf is the sheafification of a t-non-positive presheaf it suffices to prove the analogous result for the category of presheaves.

The category $\mathcal{P}\mathcal{S}\mathfrak{h}(\mathcal{C}, \Lambda)$ is compactly generated by the sheaves of the form $\Lambda(X)$ where X runs through the set of objects of \mathcal{C} . The first assertion then follows from [Nee01, 8.4.1].

We now prove the second assertion. A presheaf M is t-positive with respect to the ordinary t-structure on $\mathcal{P}\mathcal{S}\mathfrak{h}(\mathcal{C}, \Lambda)$ if and only if for object X of \mathcal{C} , the complex $M(X)$ of Λ -modules is 0-connected. But since for any object X of \mathcal{C} , we have

$$M(X) = \text{Map}_{\mathcal{P}\mathcal{S}\mathfrak{h}(\mathcal{C}, \Lambda)}(\Lambda(X), M),$$

this is equivalent to the condition that M is t-positive with respect to the t-structure generated by the sheaves of the form $\Lambda(X)$ with X an object of \mathcal{C} . Hence, both t-structures coincide. \square

The following lemma shows how t-structures generated by a set of objects behave with respect to change of coefficients.

Lemma 3.1.1.8. *Let \mathcal{C} be \mathbb{Z} -linear presentable stable category, let \mathcal{E} be a small family of objects of \mathcal{C} , let n be an integer and let A be a localization of \mathbb{Z} . Denote by $\sigma_n: \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ and by $\sigma_A: \mathbb{Z} \rightarrow A$ the natural ring morphisms.*

If $R = \mathbb{Z}/n\mathbb{Z}$ or $R = A$ and if $\sigma: \mathbb{Z} \rightarrow R$ is the natural ring morphism, we denote by

$$\sigma^*: \mathcal{C} \rightleftarrows \mathcal{C} \otimes_{\text{Mod}_{\mathbb{Z}}} \text{Mod}_R: \sigma_*$$

the adjunction given by [Lur17, 4.6.2.17]. Denote by \mathcal{E}_R the family of objects of $\mathcal{C} \otimes_{\text{Mod}_{\mathbb{Z}}} \text{Mod}_R$ made of the σ^*M .

Endow \mathcal{C} (resp. $\mathcal{C} \otimes_{\text{Mod}_{\mathbb{Z}}} \text{Mod}_{\mathbb{Z}/n\mathbb{Z}}$, resp. $\mathcal{C} \otimes_{\text{Mod}_{\mathbb{Z}}} \text{Mod}_A$) with the t-structure generated by \mathcal{E} (resp. $\mathcal{E}_{\mathbb{Z}/n\mathbb{Z}}$, resp. \mathcal{E}_A) in the sense of Proposition [2.4.2.1].

1. The functors of the form σ_* are t-exact.
2. Assume furthermore that if M belongs to \mathcal{E} and N is an object of \mathcal{C} , the map

$$\sigma_A^* \text{Map}_{\mathcal{C}}(M, N) \rightarrow \text{Map}_{\mathcal{C}}(M, \sigma_A^* N)$$

is an equivalence. Then, the functor σ_A^* is t-exact.

Proof. Let $R = \mathbb{Z}/n\mathbb{Z}$ or $R = A$ and let $\sigma: \mathbb{Z} \rightarrow R$ be the natural ring morphism. The colimit preserving exact functor σ^* sends the generators of the t-structure to t-non-positive objects and is therefore right t-exact. By adjunction, the functor σ_* is therefore left t-exact.

Notice then that if M lies in \mathcal{E} , the object $\sigma_n^* M$ is the cofiber of the map $M \xrightarrow{\times n} M$. Therefore the object $\sigma_n^* M$ is a colimit of t-non-positive objects of \mathcal{C} and is therefore t-non-positive.

Since the subcategory of t-non-positive objects of the stable category $\mathcal{C} \otimes_{\text{Mod}_{\mathbb{Z}}} \text{Mod}_{\mathbb{Z}/n\mathbb{Z}}$ is the smallest subcategory closed under small colimits and extensions containing the $M \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ when M runs through \mathcal{E} , and since the exact functor $(\sigma_n)_*$ preserves small colimits, the functor $(\sigma_n)_*$ is therefore right t-exact.

On the other hand, write $A = \Sigma^{-1}\mathbb{Z}$ with Σ a multiplicative system. Then, the object $\sigma_A^* M$ is the colimit of the diagram of value M with transition maps the multiplications by s for $s \in \Sigma$. Hence, it is t-non-positive. Since the functor $(\sigma_A)_*$ is also colimit preserving, this yields its t-exactness.

To prove the second assertion, it suffices to show that the exact functor σ_A^* is left-t-exact under our assumption. Take N a t-non-negative object of \mathcal{C} . Let M be an object of \mathcal{E} . Then, we have

$$\text{Map}_{\mathcal{C} \otimes_{\text{Mod}_{\mathbb{Z}}} \text{Mod}_A}(\sigma_A^* M, \sigma_A^* N) = \text{Map}(M, \sigma_A^* N)$$

But, by assumption, we have

$$\text{Map}(M, \sigma_A^* N) = \sigma_A^* \text{Map}(M, N).$$

Thus, the chain complex $\text{Map}(\sigma_A^* M, \sigma_A^* N)$ is (-1) -connected and the object $\sigma_A^*(N)$ is t-non-negative. Therefore, the functor σ_A^* is t-exact. \square

A t-exactness lemma

In our study, we will often need the following:

Lemma 3.1.1.9. *Let I be a small set, let \mathcal{D} be a t-category. For any index i in I , let \mathcal{D}_i be a t-category and let $T_i: \mathcal{D} \rightarrow \mathcal{D}_i$ be a t-exact functor. Assume that the family $(T_i)_{i \in I}$ is conservative.*

Then, an object M of \mathcal{D} is t-positive (resp. t-negative) if and only if for any index i in I , the object $T_i(M)$ is t-positive (resp. t-negative) in \mathcal{D}_i .

Proof. The t-exactness of the T_i yields the "only if" part of the lemma.

We now prove the "if" part. Let M be an object of \mathcal{D} such that for any $i \in I$, the object $T_i(M)$ of \mathcal{D}_i is t-positive. Then, for any $i \in I$, the functor T_i sends the map $M \rightarrow \tau_{>0} M$ to an equivalence since t-exact functors commute with the positive truncation. Therefore, the map $M \rightarrow \tau_{>0} M$ is an equivalence and M is t-positive. The t-negative case is similar. \square

Idempotent complete \mathfrak{p} -localization

We recall some notions from [CD16, B].

Definition 3.1.1.10. *Let \mathcal{C} be a \mathbb{Z} -linear stable category and let \mathfrak{p} be a prime ideal of \mathbb{Z} .*

1. *The naive \mathfrak{p} -localization $\mathcal{C}_{\mathfrak{p}}^{\text{na}}$ of \mathcal{C} is the category with the same objects as \mathcal{C} and such that if M and N are objects of \mathcal{C} , we have*

$$\text{Map}_{\mathcal{C}_{\mathfrak{p}}^{\text{na}}}(M, N) = \text{Map}_{\mathcal{C}}(M, N) \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}}.$$

2. *The \mathfrak{p} -localization $\mathcal{C}_{\mathfrak{p}}$ of \mathcal{C} is the idempotent completion of the naive \mathfrak{p} -localization $\mathcal{C}_{\mathfrak{p}}^{\text{na}}$ of \mathcal{C} .*

If \mathcal{C} is a \mathbb{Z} -linear stable category and if \mathfrak{p} is a prime ideal of \mathbb{Z} , we have a canonical colimit-preserving exact functor

$$\mathcal{C} \rightarrow \mathcal{C}_{\mathfrak{p}}.$$

Proposition 3.1.1.11. *Let \mathcal{C} be a \mathbb{Z} -linear stable category and let \mathfrak{p} be a prime ideal of \mathbb{Z} .*

1. *The category $\mathcal{C}_{\mathfrak{p}}$ is universal among the $\mathbb{Z}_{\mathfrak{p}}$ -linear idempotent complete stable categories under \mathcal{C} .*
2. *Assume that the stable category \mathcal{C} is endowed with a t -structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$.*
 - (a) *Let $\mathcal{C}_{\mathfrak{p}}^{\leq 0, \text{na}}$ (resp. $\mathcal{C}_{\mathfrak{p}}^{\geq 0, \text{na}}$) be the full subcategory of $\mathcal{C}_{\mathfrak{p}}^{\text{na}}$ made of those objects lying in the essential image of the functor $\mathcal{C}^{\leq 0} \rightarrow \mathcal{C}_{\mathfrak{p}}^{\text{na}}$ (resp. of the functor $\mathcal{C}^{\geq 0} \rightarrow \mathcal{C}_{\mathfrak{p}}^{\text{na}}$). Then, the pair $(\mathcal{C}_{\mathfrak{p}}^{\leq 0, \text{na}}, \mathcal{C}_{\mathfrak{p}}^{\geq 0, \text{na}})$ defines a t -structure on the stable category $\mathcal{C}_{\mathfrak{p}}^{\text{na}}$ which is the unique t -structure such that the canonical functor $\mathcal{C} \rightarrow \mathcal{C}_{\mathfrak{p}}^{\text{na}}$ is t -exact.*
 - (b) *Let $\mathcal{C}_{\mathfrak{p}}^{\leq 0}$ (resp. $\mathcal{C}_{\mathfrak{p}}^{\geq 0}$) be the full subcategory of $\mathcal{C}_{\mathfrak{p}}$ made of those objects which are retract of objects of $\mathcal{C}^{\leq 0, \text{na}}$ (resp. $\mathcal{C}^{\geq 0, \text{na}}$) (resp. $\mathcal{C}_{\mathfrak{p}}^{\leq 0, \text{na}}$ (resp. $\mathcal{C}_{\mathfrak{p}}^{\geq 0, \text{na}}$)). Then, the pair $(\mathcal{C}_{\mathfrak{p}}^{\leq 0}, \mathcal{C}_{\mathfrak{p}}^{\geq 0})$ defines a t -structure on the stable category $\mathcal{C}_{\mathfrak{p}}$ which is the unique t -structure such that the canonical functor $\mathcal{C} \rightarrow \mathcal{C}_{\mathfrak{p}}$ is t -exact.*

Proof. The first assertion follows from the proof of [CD16, B.1.6]. The second assertion follows from the proofs of [CD16, B.2.1] and [CD16, B.2.2]. \square

If \mathcal{C} and \mathcal{D} are \mathbb{Z} -linear stable categories, if $F: \mathcal{C} \rightarrow \mathcal{D}$ is an exact functor and if \mathfrak{p} is a prime ideal of \mathbb{Z} , we have canonical exact functors

$$\begin{aligned} F_{\mathfrak{p}}^{\text{na}}: \mathcal{C}_{\mathfrak{p}}^{\text{na}} &\rightarrow \mathcal{D}_{\mathfrak{p}}^{\text{na}}. \\ F_{\mathfrak{p}}: \mathcal{C}_{\mathfrak{p}} &\rightarrow \mathcal{D}_{\mathfrak{p}}. \end{aligned}$$

Proposition 3.1.1.12. *Let \mathcal{C} and \mathcal{D} be \mathbb{Z} -linear stable categories, let $F: \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor and let \mathfrak{p} be a prime ideal of \mathbb{Z} .*

1. *If the functor F is fully faithful, so are the functors $F_{\mathfrak{p}}^{\text{na}}$ and $F_{\mathfrak{p}}$.*
2. *If the stable categories \mathcal{C} and \mathcal{D} are endowed with t -structures and if the functor F is right (or left) t -exact, so are the functors $F_{\mathfrak{p}}^{\text{na}}$ and $F_{\mathfrak{p}}$ when the stable categories $\mathcal{C}_{\mathfrak{p}}^{\text{na}}$, $\mathcal{D}_{\mathfrak{p}}^{\text{na}}$, $\mathcal{C}_{\mathfrak{p}}$ and $\mathcal{D}_{\mathfrak{p}}$ are endowed with the t -structures defined in Proposition 3.1.1.11.*

Proposition 3.1.1.13. ([CD16, B.1.7]) *Let \mathcal{C} be a \mathbb{Z} -linear stable category and let \mathcal{D} be a thick stable subcategory of \mathcal{C} . An object M of \mathcal{C} belongs to \mathcal{D} if and only if the image of M in $\mathcal{C}_{\mathfrak{p}}$ belongs to $\mathcal{D}_{\mathfrak{p}}$ for any maximal ideal \mathfrak{p} of \mathbb{Z} .*

3.1.2 Reminders and complements on étale motives

Let R be a (commutative) ring.

Yoneda embedding

If S is a scheme, let $R_S: \text{Sm}_S \rightarrow \mathcal{S}\mathfrak{h}_{\text{ét}}(S, R)$ be the Yoneda embedding. Let $\mathcal{D}(-, R)$ be an Sm-premotivic category endowed with an Sm-premotivic adjunction (see for instance [CD16, A])

$$\chi_!: \mathcal{S}\mathfrak{h}_{\text{ét}}(-, R) \rightleftarrows \mathcal{D}(-, R) : \chi^!$$

We will call geometric sections of $\mathcal{D}(-, R)$ the 2-functor

$$\chi_! R_- : \text{Sm}_- \rightarrow \mathcal{D}(-, R).$$

For a scheme S we let $M_S(-, R)$ (resp. $M_S^{\text{eff}}(-, R)$) be the geometric sections of $\mathcal{D}\mathcal{M}_{\text{ét}}(S, R)$ (resp. $\mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(S, R)$). When the ring R is non-ambiguous, we will simply denote this functor by M_S (resp. M_S^{eff}).

Finally, we let $\underline{R} = R_S(S)$ when the base scheme S is unambiguous.

Chain of adjunctions

We have a chain of Sm-premotivic adjunctions of stable Sm-premotivic categories (*i.e.* all the functors are exact, the left adjoints are monoidal and the functors are compatible with the premotivic structure):

$$\mathcal{S}\mathfrak{h}_{\text{ét}}(-, R) \xrightleftharpoons[\iota]{L_{\mathbb{A}^1}} \mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}}(-, R) \xrightleftharpoons[\Omega^\infty]{\Sigma^\infty} \mathcal{D}\mathcal{M}_{\text{ét}}(-, R) .$$

Here,

- All the above categories are Sm-premotivic. This can be proved in the same fashion as in the case of stable \mathbb{A}^1 -homotopy theory (see [Kha16, Chapter 1]).
- $(L_{\mathbb{A}^1}, \iota)$ is the \mathbb{A}^1 -localization adjunction.
- $(\Sigma^\infty, \Omega^\infty)$ is the \mathbb{P}^1 -stabilization adjunction (see [Rob14, Section 4]).

Furthermore, if R is an n -torsion ring with n an integer, the functor Σ^∞ is an equivalence of premotivic categories by [BH21, 3.2].

Grothendieck's six functors formalism

Since we have a premotivic equivalence between h-motives and étale motives, [CD16, 5.6.2] implies that étale motives are endowed with the six functors formalism and have the absolute purity property in the triangulated case. Since étale motives are premotivic in the ∞ -categorical setting, using [Kha16, Chapter 2: 3.6.6 and 4.2.3] and the triangulated case [CD16], we have the full six functors formalism and the absolute purity property.

Let S be a scheme. In this formalism, if $f: X \rightarrow S$ is a smooth morphism, there is an equivalence of functors

$$M_S(X) = f_! f^! \mathbb{1}_S.$$

If X is a scheme of finite type over S , we denote by $M_S^{\text{BM}}(X, R)$ the object $f_! \mathbb{1}_X$ and by $h_S(X, R)$ the object $f_* \mathbb{1}_X$. Again, when the ring R is non-ambiguous, those functors will simply be denoted by M_S^{BM} and h_S .

Small étale site

We can add the category $\mathcal{S}\mathfrak{h}(-_{\text{ét}}, R)$ at the left of the above diagram using the adjunction:

$$\rho_{\#}: \mathcal{S}\mathfrak{h}(-_{\text{ét}}, R) \rightleftarrows \mathcal{S}\mathfrak{h}_{\text{ét}}(-, R): \rho^*$$

given by the inclusion of the small étale site into the big étale site. The functor $\rho_{\#}$ is fully faithful. We still denote by

$$R_S: S_{\text{ét}} \rightarrow \mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$$

the Yoneda embedding. This will be harmless as the sheaf on $S_{\text{ét}}$ representing an étale S -scheme X is sent by the functor $\rho_{\#}$ to the sheaf on Sm_S which represents X .

If R is an n -torsion ring, where the integer n is invertible on S , then the functor $L_{\mathbb{A}^1} \rho_{\#}$ and therefore, the functor $\rho_! = \Sigma^{\infty} L_{\mathbb{A}^1} \rho_{\#}$ is a monoidal equivalence (it is the rigidity theorem [BH21, 3.2]).

Proposition 3.1.2.1. *Let R be a ring and let*

$$\rho_! = \Sigma^{\infty} L_{\mathbb{A}^1} \rho_{\#}: \mathcal{S}\mathfrak{h}(-_{\text{ét}}, R) \rightarrow \mathcal{DM}_{\text{ét}}(-, R).$$

Then,

1. The functor $\rho_!$ is monoidal.
2. For any morphism f , there is a canonical isomorphism

$$\rho_! f^* \rightarrow f^* \rho_!.$$

3. For any quasi-finite morphism f , there is a canonical isomorphism

$$\rho_! f_! \rightarrow f_! \rho_!.$$

Proof. [CD16, 4.4.2] yields the first two assertions and the third assertion in the case of étale morphisms. Therefore, using Zariski’s Main Theorem [Gro67, 18.12.13], it suffices to prove that for any finite morphism f , there is a canonical isomorphism

$$\rho_! f_* \rightarrow f_* \rho_!.$$

Assume first that $f: X \rightarrow S$ is a closed immersion and let $j: U \rightarrow S$ be the complementary open immersion. The localization triangles (3.0.0.1) in the fibered category of étale sheaves and in the fibered category of étale motives yield a morphism of exact triangles

$$\begin{array}{ccccc} \rho_! j_! j^* & \longrightarrow & \rho_! & \longrightarrow & \rho_! f_* f^* \\ \downarrow & & \parallel & & \downarrow \\ j_! j^* \rho_! & \longrightarrow & \rho_! & \longrightarrow & f_* f^* \rho_! \end{array} .$$

The left vertical arrows is an equivalence using the first assertion and the third assertion in the case of the map j . Therefore, we get an isomorphism

$$\rho_! f_* f^* \rightarrow f_* f^* \rho_!$$

and therefore, using the first assertion, we get an equivalence

$$\rho_! f_* f^* \rightarrow f_* \rho_! f^*.$$

Since the functor f^* is essentially surjective, this yields the result for the map f .

If the map f is purely inseparable the functor f_* is an equivalence with inverse f^* using [CD16, 6.3.16] in the motivic setting and using the topological invariance of the small étale site [Sta23, 04DY] in the case of étale sheaves. Since the functors $\rho_!$ and f^* commute, so do the functors $\rho_!$ and f_* .

Assume now that $f: X \rightarrow S$ is a general finite morphism. We prove the claim by noetherian induction on S . We can assume that the scheme X is reduced using the case of closed immersions.

Let Γ be the subscheme of S made of its generic points. The pullback map $X \times_S \Gamma \rightarrow \Gamma$ is finite over a finite disjoint union of spectra of fields and can therefore be written as a composition

$$X \times_S \Gamma \xrightarrow{g} \Gamma' \xrightarrow{h} \Gamma$$

where g is finite étale and h is purely inseparable. Hence, by [Gro67, 8.8.2], if $j: U \rightarrow S$ is a small enough dense open immersion, the pullback map $f_U: X \times_S U \rightarrow U$ can be written as a composition

$$X \times_S U \xrightarrow{g_U} U' \xrightarrow{h_U} U$$

such that the pullback of g_U (resp. h_U) to Γ is g (resp. h).

Shrinking U further, we can assume that the map g_U is finite étale by [Gro67, 8.10.5(x)] and [Ayo15, 1.A.5] and that the map h_U is purely inseparable by [Gro67, 8.10.5(vii)]. Hence, there is a dense open subscheme U such that the map $f_U: X \times_S U \rightarrow U$ is the composition of a finite étale map and a purely inseparable morphism. Therefore, the transformation

$$\rho_!(f_U)_* \rightarrow (f_U)_* \rho_!$$

is an equivalence.

Let $i: Z \rightarrow S$ be the reduced closed immersion which is complementary to j and let $f_Z: X \times_S Z \rightarrow Z$ be the pullback map. By noetherian induction, the transformation

$$\rho_!(f_Z)_* \rightarrow (f_Z)_* \rho_!$$

is an equivalence.

The localization triangles (3.0.0.1) in the fibered category of étale sheaves and in the fibered category of étale motives yield a morphism of exact triangles

$$\begin{array}{ccccc} \rho_! j_! j^* f_* & \longrightarrow & \rho_! f_* & \longrightarrow & \rho_! i_* i^* f_* \\ \downarrow & & \downarrow & & \downarrow \\ j_! j^* f_* \rho_! & \longrightarrow & f_* \rho_! & \longrightarrow & i_* i^* f_* \rho_! \end{array}.$$

The discussion above and proper base change ensure that the leftmost and rightmost vertical arrows are equivalences. Therefore, the middle vertical arrow is an equivalence which finishes the proof. \square

Constructible objects

Let S be a scheme and let R be a ring. We define subcategories of constructible objects of the above categories.

- The category $\mathcal{S}\mathcal{h}_c(S_{\acute{e}t}, R)$ of *constructible étale sheaves over S* is the thick subcategory of $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$ generated by the sheaves of the form $R_S(X)$ where X runs through the set of étale S -schemes.
- The category $\mathcal{S}\mathcal{h}_{\acute{e}t,c}(S, R)$ of *constructible étale sheaves over Sm_S* is the thick subcategory of $\mathcal{S}\mathcal{h}_{\acute{e}t}(S, R)$ generated by the sheaves of the form $R_S(X)$ where X runs through the set of smooth S -schemes.
- The category $\mathcal{D}\mathcal{A}_{\acute{e}t,c}^{\text{eff}}(S, R)$ of *constructible effective étale motives over S* is the thick subcategory of $\mathcal{D}\mathcal{A}_{\acute{e}t}^{\text{eff}}(S, R)$ generated by the effective motives of the form $M_S^{\text{eff}}(X)$ where X runs through the set of smooth S -schemes.
- The category $\mathcal{D}\mathcal{M}_{\acute{e}t,c}(S, R)$ of *constructible étale motives over S* is the thick subcategory of $\mathcal{D}\mathcal{M}_{\acute{e}t}(S, R)$ generated by the motives of the form $M_S(X)(n)$ where X runs through the set of smooth S -schemes and n runs through the set of integers.

Those categories generate their non-constructible counterparts in the sense that the family $\text{Hom}(M, -)$ is conservative when M runs through the subcategory of constructible objects.

Subcategories of dimensional motives

Let S be a scheme and R be a ring. We can define subcategories of the stable category $\mathcal{D}\mathcal{M}_{\acute{e}t}(S, R)$ as follows:

- The category $\mathcal{D}\mathcal{M}_{\acute{e}t}^{\text{coh}}(S, R)$ of *cohomological étale motives over S* is the localizing subcategory generated by the motives of the form

$$h_S(X) = M_S^{\text{BM}}(X)$$

where X runs through the set of proper S -schemes.

- The category $\mathcal{D}\mathcal{M}_{\acute{e}t,c}^{\text{coh}}(S, R)$ *constructible cohomological étale motives over S* is the thick subcategory generated by the motives of the form

$$h_S(X) = M_S^{\text{BM}}(X)$$

where X runs through the set of proper S -schemes.

- The category $\mathcal{D}\mathcal{M}_{\acute{e}t}^n(S, R)$ of *n -étale motives over S* is the localizing subcategory generated by the motives of the form

$$h_S(X) = M_S^{\text{BM}}(X)$$

where X runs through the set of proper S -schemes of relative dimension at most n .

- The category $\mathcal{DM}_{\acute{e}t,c}^n(S, R)$ of *constructible n -étale motives* over S is the thick subcategory generated by the motives of the form

$$h_S(X) = M_S^{\text{BM}}(X)$$

where X runs through the set of proper S -schemes of relative dimension at most n .

The main focus of this text will be the following categories:

Definition 3.1.2.2. *Let S be a scheme and R be a ring. We define*

1. *The category $\mathcal{DM}_{\acute{e}t}^A(S, R)$ of Artin étale motives to be the category of 0-étale motives.*
2. *The category $\mathcal{DM}_{\acute{e}t,c}^A(S, R)$ of constructible Artin étale motives to be the category of constructible 0-motives.*
3. *The category $\mathcal{DM}_{\acute{e}t}^{\text{sm}A}(S, R)$ of smooth Artin étale motives over S to be the localizing subcategory of $\mathcal{DM}_{\acute{e}t}(S, R)$ generated by the motives of the form*

$$h_S(X) = M_S^{\text{BM}}(X) = M_S(X)$$

where X runs through the set of étale covers over S .

4. *The category $\mathcal{DM}_{\acute{e}t,c}^{\text{sm}A}(S, R)$ constructible smooth Artin étale motives over S to be the thick subcategory of $\mathcal{DM}_{\acute{e}t}(S, R)$ generated by the motives of the form*

$$h_S(X) = M_S^{\text{BM}}(X) = M_S(X)$$

where X runs through the set of étale covers over S .

Using Zariski's Main Theorem [Gro67, 18.12.13], (constructible) Artin étale motives are generated by the motives of the form $h_S(X)$ where X runs through the set of finite S -schemes.

We can define subcategories of the stable category $\mathcal{DA}_{\acute{e}t}^{\text{eff}}(S, R)$ in a similar fashion and we will use similar notations.

Beware that in general (in the case of integral coefficients), all Artin motives that belong to the category $\mathcal{DM}_{\acute{e}t,c}(S, R)$ of constructible étale motives are not constructible Artin motives.

The functors \otimes and f^* (where f is any morphism) induce functors over the categories $\mathcal{DM}_{\acute{e}t,c}^{(\text{sm})A}(-, R)$.

Change of coefficients

Let \mathcal{D} be one of the fibered categories $\mathcal{DM}_{\acute{e}t}$, $\mathcal{DA}_{\acute{e}t}^{\text{eff}}$, $\mathcal{S}\mathcal{H}_{\acute{e}t}$ and $\mathcal{S}\mathcal{H}(-_{\acute{e}t})$. The fibered category $\mathcal{D}(-, R)$ is the category of R -module objects in $\mathcal{D}(-, \mathbb{Z})$. Therefore, if $\sigma: R \rightarrow R'$ is a ring morphism, we can construct an adjunction

$$\sigma^*: \mathcal{D}(-, R) \rightleftarrows \mathcal{D}(-, R'): \sigma_*$$

which is Sm-premotivic except in the case of $\mathcal{S}\mathcal{H}(-_{\acute{e}t})$ (see [CD16, 5.4]). The functor σ_* is the forgetful functor and the functor σ^* is the tensor product with R' .

Proposition 3.1.2.3. *Let \mathcal{D} be one of the fibered categories $\mathcal{DM}_{\acute{e}t}$, $\mathcal{DA}_{\acute{e}t}^{\text{eff}}$, $\mathcal{S}\mathcal{h}_{\acute{e}t}$ and $\mathcal{S}\mathcal{h}(-_{\acute{e}t})$. Let $\sigma: R \rightarrow R'$ be a ring morphism of the form $\sigma_A: R \rightarrow R \otimes_{\mathbb{Z}} A$ or $\sigma_n: R \rightarrow R/nR$, where A is a localization of \mathbb{Z} and n is an integer.*

1. *The functor σ^* commutes with the functors $\rho_{\#}$, ρ^* , $L_{\mathbb{A}^1}$, ι , Σ^{∞} and Ω^{∞} defined in Section [3.1.2](#).*
2. *Let n be an integer, let S be a scheme and let M and N be objects of $\mathcal{D}(S, R)$. Then, the canonical maps*

$$\begin{aligned}\sigma_n^* \text{Hom}_{\mathcal{D}(S,R)}(M, N) &\rightarrow \text{Hom}_{\mathcal{D}(S,R)}(M, \sigma_n^* N) \\ \sigma_n^* \text{Map}_{\mathcal{D}(S,R)}(M, N) &\rightarrow \text{Map}_{\mathcal{D}(S,R)}(M, \sigma_n^* N)\end{aligned}$$

are equivalences.

3. *Let A be a localization of \mathbb{Z} , let S be a scheme and let M and N be objects of $\mathcal{D}(S, R)$. Assume that M is constructible in the sense of Section [3.1.2](#). Then, the canonical maps*

$$\begin{aligned}\sigma_A^* \text{Hom}_{\mathcal{D}(S,R)}(M, N) &\rightarrow \text{Hom}_{\mathcal{D}(S,R)}(M, \sigma_A^* N) \\ \sigma_A^* \text{Map}_{\mathcal{D}(S,R)}(M, N) &\rightarrow \text{Map}_{\mathcal{D}(S,R)}(M, \sigma_A^* N)\end{aligned}$$

are equivalences.

4. *If \mathcal{D} is one of the categories $\mathcal{DM}_{\acute{e}t}$, $\mathcal{DA}_{\acute{e}t}^{\text{eff}}$ and $\mathcal{S}\mathcal{h}_{\acute{e}t}$, the functor σ^* commutes with the six functors formalism.*

Proof. The functor ρ^* is the restriction to the small étale site and therefore, it commutes with the forgetful functor σ_* and with the functor σ^* . Hence, its left adjoint $\rho_{\#}$ also commutes with the functor σ^* .

The functor ι is the inclusion of \mathbb{A}^1 -local sheaves into the fibered category $\mathcal{S}\mathcal{h}_{\acute{e}t}$. Since the \mathbb{A}^1 -locality property does not depend on the ring of coefficients, the functor ι commutes with the forgetful functor σ_* . Therefore the functors $L_{\mathbb{A}^1}$ and σ^* commute. Furthermore, [\[CD16, 5.4.5, 5.4.11\]](#) ensure that if M is \mathbb{A}^1 -local, then so is $\sigma^*(M)$ which implies that ι and σ^* commute.

Following [\[CD16, 5.4\]](#) the functor σ^* commutes with geometric sections. Since the functors σ^* and Σ^{∞} are colimit-preserving and since by [Proposition 3.1.1.7](#), the category $\mathcal{DA}_{\acute{e}t}^{\text{eff}}(S, R)$ is generated as a localizing subcategory of itself by geometric sections, those functors commute. The proof of [\[CD16, 5.5.5\]](#) shows that the functor Ω^{∞} commutes with the functor σ_A^* . Finally, we have exact triangle of functors

$$\text{Id} \xrightarrow{\times n} \text{Id} \rightarrow \sigma_n^*$$

in $\mathcal{DM}_{\acute{e}t}$ and $\mathcal{DA}_{\acute{e}t}^{\text{eff}}$ which induce an exact triangles of functors when applying Ω^{∞} to the left or to the right

$$\begin{aligned}\Omega^{\infty} \xrightarrow{\times n} \Omega^{\infty} &\rightarrow \Omega^{\infty} \sigma_n^* \\ \Omega^{\infty} \xrightarrow{\times n} \Omega^{\infty} &\rightarrow \sigma_n^* \Omega^{\infty}\end{aligned}$$

Hence, the functors Ω^{∞} and σ_n^* commute, which finishes the proof of the first assertion. The second assertion follows from the exactness of the triangle

$$\text{Id} \xrightarrow{\times n} \text{Id} \rightarrow \sigma_n^*$$

in the categories $\mathcal{D}(S, R)$ and $\mathcal{D}(R)$.

In the case where $\mathcal{D} = \mathcal{DM}_{\acute{e}t}$, the third assertion follows from [CD16, 5.4.11]. Tensoring with A respects \mathbb{A}^1 -local objects by [CD16, 1.1.11]. Therefore, the third assertion in the case where $\mathcal{D} = \mathcal{DA}_{\acute{e}t}^{\text{eff}}$ follows from the same assertion when $\mathcal{D} = \mathcal{S}\mathcal{h}_{\acute{e}t}$. The assertion in both of the cases when $\mathcal{D} = \mathcal{S}\mathcal{h}_{\acute{e}t}$ and when $\mathcal{D} = \mathcal{S}\mathcal{h}(-_{\acute{e}t})$ follows from [CD16, 1.1.11].

Using [CD16, 5.4.5, 5.4.11, A.1.1.16], we see that the six functors commute with σ^* in the case of $\mathcal{DM}_{\acute{e}t}$. We can mimic the proofs of [CD16, 5.4.5, 5.4.11] in the other cases to prove the last assertion. \square

Proposition 3.1.2.4. *Let \mathcal{D}_c be on of the fibered categories $\mathcal{DM}_{\acute{e}t,c}$, $\mathcal{DA}_{\acute{e}t,c}^{\text{eff}}$, $\mathcal{S}\mathcal{h}_{\acute{e}t,c}$ or $\mathcal{S}\mathcal{h}_c(-_{\acute{e}t})$, let \mathfrak{p} be a prime ideal of \mathbb{Z} and let S be a scheme. Then, with the notations of Section 3.1.1 the functor*

$$\mathcal{D}_c(S, R)_{\mathfrak{p}} \rightarrow \mathcal{D}_c(S, R \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})$$

is an equivalence.

Proof. With the notations of Section 3.1.1, the functor

$$\mathcal{D}_c(S, R)_{\mathfrak{p}}^{\text{na}} \rightarrow \mathcal{D}_c(S, R \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})$$

is fully faithful by Proposition 3.1.2.3. Since the category $\mathcal{D}_c(S, R \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})$ is idempotent complete, the functor

$$\mathcal{D}_c(S, R)_{\mathfrak{p}} \rightarrow \mathcal{D}_c(S, R \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})$$

is therefore fully faithful. Its essential image is therefore a thick subcategory which contains the generators of $\mathcal{D}_c(S, R \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})$. Therefore, it is an equivalence. \square

Lemma 3.1.2.5. ([CD16, 5.4.12]) *Let \mathcal{D} be one of the fibered categories $\mathcal{DM}_{\acute{e}t}$, $\mathcal{DA}_{\acute{e}t}^{\text{eff}}$, $\mathcal{S}\mathcal{h}_{\acute{e}t}$ and $\mathcal{S}\mathcal{h}(-_{\acute{e}t})$. Let S be a scheme and let R be a good ring, then with the above notations, the family $(\sigma_{\mathbb{Q}}^*, \sigma_p^* \mid p \text{ prime})$ is conservative.*

We have another similar conservativity lemma. If p is a prime number, denote by $\mathbb{Z}_{(p)}$ the localization of \mathbb{Z} with respect to the prime ideal generated by p . Denote by r_p the ring morphism $\sigma_{\mathbb{Z}_{(p)}}$. Then, the morphisms $\sigma_{\mathbb{Q}}$ and σ_p factor through r_p . This yields the following corollary.

Corollary 3.1.2.6. *Let S be a scheme and let R be a good ring, then with the above notations, the family $(r_p^* \mid p \text{ prime})$ is conservative.*

We finish this section with an analog of Proposition 3.1.2.3(2) for infinite torsion. If p is a prime number denote by $\mathbb{Z}(p^{\infty}) = \mathbb{Z}[1/p]/\mathbb{Z}$ the Prüfer p -group.

Proposition 3.1.2.7. *Keep the same notations as in Proposition 3.1.2.3. Let S be a scheme and let M and P be objects of $\mathcal{D}(S, R)$. Assume that M is constructible in the sense of Section 3.1.2.*

1. *The canonical functor*

$$\text{Map}_{\mathcal{D}(S,R)}(M, P) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \rightarrow \text{Map}_{\mathcal{D}(S,R)}(M, P \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$$

is an equivalence.

2. Let p be a prime number. The canonical functor

$$\mathrm{Map}_{\mathcal{D}(S,R)}(M, P) \otimes_{\mathbb{Z}} \mathbb{Z}(p^\infty) \rightarrow \mathrm{Map}_{\mathcal{D}(S,R)}(M, P \otimes_{\mathbb{Z}} \mathbb{Z}(p^\infty))$$

is an equivalence.

3. The canonical functor

$$\bigoplus_{p \text{ prime}} \mathrm{Map}_{\mathcal{D}(S,R)}(M, P \otimes_{\mathbb{Z}} \mathbb{Z}(p^\infty)) \rightarrow \mathrm{Map}_{\mathcal{D}(S,R)}(M, P \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$$

is an equivalence.

4. The canonical functor

$$\mathrm{colim}_n \mathrm{Map}(M, \sigma_n^*(P)) \rightarrow \mathrm{Map}(M, \mathrm{colim}_n \sigma_n^*(P))$$

is an equivalence.

5. Let p be a prime number. The canonical functor

$$\mathrm{colim}_n \mathrm{Map}(M, \sigma_{p^n}^*(P)) \rightarrow \mathrm{Map}(M, \mathrm{colim}_n \sigma_{p^n}^*(P))$$

is an equivalence.

Proof. We have an exact triangle

$$P \rightarrow P \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow P \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}.$$

Thus, we get a morphism of exact triangles

$$\begin{array}{ccccc} \mathrm{Map}(M, P) & \longrightarrow & \mathrm{Map}(M, P) \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathrm{Map}(M, P) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}(M, P) & \longrightarrow & \mathrm{Map}(M, P \otimes_{\mathbb{Z}} \mathbb{Q}) & \longrightarrow & \mathrm{Map}(M, P \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \end{array}.$$

Since the left and the middle vertical arrows are equivalences by Proposition [3.1.2.3](#), the right vertical arrow is an equivalence which yields Assertion (1).

Similarly, we have an exact triangle

$$P \rightarrow P \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \rightarrow P \otimes_{\mathbb{Z}} \mathbb{Z}(p^\infty).$$

Thus, we get a morphism of exact triangles:

$$\begin{array}{ccccc} \mathrm{Map}(M, P) & \longrightarrow & \mathrm{Map}(M, P) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] & \longrightarrow & \mathrm{Map}(M, P) \otimes_{\mathbb{Z}} \mathbb{Z}(p^\infty) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}(M, P) & \longrightarrow & \mathrm{Map}(M, P \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]) & \longrightarrow & \mathrm{Map}(M, P \otimes_{\mathbb{Z}} \mathbb{Z}(p^\infty)) \end{array}.$$

Since the left and the middle vertical arrows are equivalences by Proposition [3.1.2.3](#), the right vertical arrow is an equivalence which yields Assertion (2).

Since

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty),$$

Assertion (3) follows from Assertions (1) and (2).

To prove Assertion (4), observe we have a commutative diagram

$$\begin{array}{ccc} \operatorname{colim}_n \operatorname{Map}(M, \sigma_n^*(P)) & \longrightarrow & \operatorname{Map}(M, \operatorname{colim}_n \sigma_n^*(P)) \\ \downarrow & & \downarrow \\ \operatorname{colim}_n \sigma_n^* \operatorname{Map}(M, P) & & \\ \downarrow & & \\ \operatorname{Map}(M, P) \otimes_R (R \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \operatorname{Map}(M, P \otimes_R (R \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})) \end{array}$$

where the right vertical arrow and the bottom left vertical arrow are equivalences because tensor products and colimits commute, the upper left vertical arrow is an equivalence by [CD16, 5.4.5] and the bottom horizontal arrow is an equivalence by Assertion (1). This yields Assertion (4).

We can derive Assertion (5) from Assertion (2) in a similar fashion. \square

Continuity

Let \mathcal{D}_c be on of the fibered categories $\mathcal{DM}_{\acute{e}t,c}$, $\mathcal{DM}_{\acute{e}t,c}^{\operatorname{coh}}$, $\mathcal{DM}_{\acute{e}t,c}^n$ or $\mathcal{DM}_{\acute{e}t,c}^{smA}$.

Proposition 3.1.2.8. *Let R be a good ring. Consider a scheme X which is the limit of a projective system of schemes with affine transition maps $(X_i)_{i \in I}$.*

Then, in the 2-category of stable monoidal categories, we have

$$\operatorname{colim} \mathcal{D}_c(X_i, R) = \mathcal{D}_c(X, R).$$

Proof. The proof is the same as [CD16, 6.3.9], replacing [Gro66, 8.10.5, 17.7.8] with [Pep19b, 1.24]. \square

3.1.3 Torsion étale motives and torsion étale sheaves

The Six Functors and the Rigidity Theorem

Definition 3.1.3.1. *Let S be a scheme, let R be a ring and let p be a prime number. We define stable categories as follows.*

1. *The stable category of torsion étale motives $\mathcal{DM}_{\acute{e}t,\operatorname{tors}}(S, R)$ as the subcategory of the stable category $\mathcal{DM}_{\acute{e}t}(S, R)$ made of those objects M such that the motive $M \otimes_{\mathbb{Z}} \mathbb{Q}$ vanishes.*
2. *The stable category of p^∞ -torsion étale motives $\mathcal{DM}_{\acute{e}t,p^\infty-\operatorname{tors}}(S, R)$ as the subcategory of the stable category $\mathcal{DM}_{\acute{e}t}(S, R)$ made of those objects M such that the motive $M \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ vanishes.*
3. *The stable category of torsion étale sheaves $\mathcal{S}\mathfrak{h}_{\operatorname{tors}}(S_{\acute{e}t}, R)$ as the subcategory of the stable category $\mathcal{S}\mathfrak{h}(S_{\acute{e}t}, R)$ made of those sheaves M such that the sheaf $M \otimes_{\mathbb{Z}} \mathbb{Q}$ vanishes.*

4. The stable category of p^∞ -torsion étale sheaves $\mathcal{S}\mathfrak{h}_{p^\infty\text{-tors}}(S_{\text{ét}}, R)$ as the subcategory of the stable category $\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$ made of those sheaves M such that the sheaf $M \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ vanishes.

Proposition 3.1.3.2. *Let R be a ring and let p be a prime number. Then, the six functors of $\mathcal{DM}_{\text{ét}}(-, R)$ induce functors over $\mathcal{DM}_{\text{ét,tors}}(-, R)$ and $\mathcal{DM}_{\text{ét},p^\infty\text{-tors}}(-, R)$.*

Proof. The six functors commute with tensor product with \mathbb{Q} and with $\mathbb{Z}[1/p]$ by Proposition 3.1.2.3 which implies the result. \square

Scholze has shown in [Sch22, 7.15] that the category $\mathcal{S}\mathfrak{h}_{\text{tors}}(-_{\text{ét}}, \mathbb{Z})$ is endowed with the six functors formalism. The proof applies to an arbitrary noetherian ring R . We deduce in the same fashion that they induce functors over $\mathcal{S}\mathfrak{h}_{p^\infty\text{-tors}}(-_{\text{ét}}, R)$.

The following result is an analog of the rigidity theorem [BH21, 3.2] for torsion motives.

Proposition 3.1.3.3. *Let S be a scheme and let R be a good ring. Then,*

1. The functor

$$\prod_{p \text{ prime}} \mathcal{DM}_{\text{ét},p^\infty\text{-tors}}(S, R) \rightarrow \mathcal{DM}_{\text{ét,tors}}(S, R)$$

which sends an object $(M_p)_{p \text{ prime}}$ to the object $\bigoplus_{p \text{ prime}} M_p$ is an equivalence.

2. Let p be a prime number. Let $j_p: S[1/p] \rightarrow S$ be the open immersion. Then, the functor

$$j_p^*: \mathcal{DM}_{\text{ét},p^\infty\text{-tors}}(S, R) \rightarrow \mathcal{DM}_{\text{ét},p^\infty\text{-tors}}(S[1/p], R)$$

is an equivalence.

3. Let p be a prime number. Let S be a scheme over which p is invertible. Then, the map $\rho_!$ of Section 3.1.2 induces an equivalence

$$\mathcal{S}\mathfrak{h}_{p^\infty\text{-tors}}(S_{\text{ét}}, R) \rightarrow \mathcal{DM}_{\text{ét},p^\infty\text{-tors}}(S, R)$$

4. All those equivalences commute with the six functors formalism.

Proof. Proof of Assertion (1): For any prime number p , let $\mathbb{Z}(p^\infty)$ be Prüfer p -group. If A is an abelian group, denote by \underline{A} the image through the functor

$$\rho!: \mathcal{S}\mathfrak{h}(S_{\text{ét}}, R) \rightarrow \mathcal{DM}_{\text{ét}}(S, R)$$

of the constant sheaf with value A . We have an exact triangle

$$\mathbb{1}_S \rightarrow \mathbb{1}_S \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \rightarrow \underline{\mathbb{Z}(p^\infty)},$$

therefore, if M is a p^∞ -torsion étale motive, we have

$$M = M \otimes_{\mathbb{Z}} \underline{\mathbb{Z}(p^\infty)}[-1].$$

Furthermore, if p and q are distinct prime numbers, the canonical map

$$\mathbb{Z}(q^\infty) \rightarrow \mathbb{Z}(q^\infty) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

is an isomorphism.

For any prime number p , let M_p and N_p be p^∞ -torsion étale motives. Notice that if p is a prime number, we have

$$\begin{aligned} \bigoplus_{\substack{q \text{ prime} \\ q \neq p}} N_q &= \bigoplus_{\substack{q \text{ prime} \\ q \neq p}} N_q \otimes_{\mathbb{Z}} \mathbb{Z}(q^\infty)[-1] \\ &= \bigoplus_{\substack{q \text{ prime} \\ q \neq p}} N_q \otimes_{\mathbb{Z}} (\mathbb{Z}(q^\infty) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p])[-1] \\ &= \left(\bigoplus_{\substack{q \text{ prime} \\ q \neq p}} N_q \right) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \end{aligned}$$

Therefore, we get

$$\text{Map} \left(M_p, \bigoplus_{\substack{q \text{ prime} \\ q \neq p}} N_q \right) = 0.$$

Hence, the map

$$\prod_{p \text{ prime}} \text{Map}(M_p, N_p) \rightarrow \text{Map} \left(\bigoplus_{p \text{ prime}} M_p, \bigoplus_{p \text{ prime}} N_p \right)$$

is an equivalence. Therefore, the functor

$$\prod_{p \text{ prime}} \mathcal{DM}_{\text{ét}, p^\infty\text{-tors}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}, \text{tors}}(S, R)$$

is fully faithful.

We now prove that this functor is essentially surjective. We have an exact triangle

$$\mathbb{1}_S \rightarrow \mathbb{1}_S \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \underline{\mathbb{Q}/\mathbb{Z}}.$$

Let now M be a torsion étale motive. We get that

$$M = M \otimes_{\mathbb{Z}} \underline{\mathbb{Q}/\mathbb{Z}}[-1].$$

Furthermore, we have

$$\underline{\mathbb{Q}/\mathbb{Z}} = \bigoplus_{p \text{ prime}} \underline{\mathbb{Z}(p^\infty)}.$$

and since the functor $\rho_!$ commutes with colimits, we therefore get that

$$M = \bigoplus_{p \text{ prime}} \left(M \otimes_{\mathbb{Z}} \underline{\mathbb{Z}(p^\infty)}[-1] \right).$$

As

$$\left(M \otimes_{\mathbb{Z}} \underline{\mathbb{Z}(p^\infty)}[-1] \right) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0,$$

the motive $M \otimes_{\mathbb{Z}} \underline{\mathbb{Z}(p^\infty)}[-1]$ is of p^∞ -torsion and therefore, the functor

$$\prod_{p \text{ prime}} \mathcal{DM}_{\text{ét}, p^\infty\text{-tors}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}, \text{tors}}(S, R)$$

is essentially surjective which finishes the proof of the first assertion.

Proof of Assertion (2):

The 2-morphism

$$j_p^*(j_p)_* \rightarrow \text{Id}$$

is an equivalence. Since by Proposition [3.1.3.2](#), both functors j_p^* and $(j_p)_*$ induce functors on p^∞ -torsion motives, the functor

$$j_p^*: \mathcal{DM}_{\text{ét}, p^\infty\text{-tors}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}, p^\infty\text{-tors}}(S[1/p], R)$$

is essentially surjective.

To show that the functor j_p^* is fully faithful when restricted to the category of p^∞ -torsion étale motives, it suffices to show that the natural map $\text{Id} \rightarrow (j_p)_*j_p^*$ is an equivalence when restricted to the category of p^∞ -torsion étale motives. Notice that the functor:

$$- \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}: \mathcal{DM}_{\text{ét}, p^\infty\text{-tors}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}(S, R/pR)$$

is conservative by Lemma [3.1.2.5](#) and commutes with the functors $(j_p)_*$ and j_p^* by [\[CD16, 5.4.5\]](#). Therefore the full faithfulness of j_p^* follows from the same result with coefficients in R/pR .

Since R/pR is a $\mathbb{Z}/p\mathbb{Z}$ -algebra, we have by [\[CD16, 5.4\]](#) a premotivic adjunction

$$\mathcal{DM}_{\text{ét}}(S, \mathbb{Z}/p\mathbb{Z}) \simeq \mathcal{DM}_{\text{ét}}(S, R/pR).$$

Therefore, the result follows from [\[CD16, A.3.4\]](#)

Proof of Assertion (3):

Let N be a p^∞ -torsion étale motive over S . Then, as before, we have

$$N = N \otimes_{\mathbb{Z}} \underline{\mathbb{Z}(p^\infty)}[-1] = \text{colim } N \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}[-1].$$

If n is a non-negative integer, let

$$N_n = N \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}[-1].$$

The étale motive N_n lies in $\mathcal{DM}_{\text{ét}}(S, R/p^nR)$ and lies therefore in the essential image of the functor $\rho_!$ by the rigidity theorem [\[BH21, 3.2\]](#). Let M_n be a p^∞ -torsion étale sheaf such that

$$\rho_!M_n = N_n.$$

The rigidity theorem also ensures that the map

$$\text{Id} \otimes_{\mathbb{Z}} (\times p): N_n \rightarrow N_{n+1}$$

is the image through $\rho_!$ of a map

$$\phi_n: M_n \rightarrow M_{n+1}.$$

Letting M be the colimit of the M_n with transition maps the ϕ_n , we have

$$\rho_!M = N$$

since the functor $\rho_!$ preserves colimits.

Hence, the functor $\rho_!$ is essentially surjective.

Recall now that the map $\rho_!$ has a right adjoint functor $\rho^!$ defined in Section 3.1.2. To finish the proof it suffices to show that the natural map

$$\mathrm{Id} \rightarrow \rho^! \rho_!$$

is an equivalence. Since the functor $-\otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is conservative and commutes with the functors $\rho^!$ and $\rho_!$ by Proposition 3.1.2.3, it suffices to show the same result with coefficients $\mathbb{Z}/p\mathbb{Z}$ which follows from the rigidity theorem [BH21, 3.2].

Proof of Assertion (4):

The fourth assertion follows from the fact that all those functors can be promoted into Sm-premotivic functors (as defined in [CD16, A.1.7]) and from [CD16, A.1.17]. \square

Putting those results together, we get the following result.

Corollary 3.1.3.4. *Keep the notations of Proposition 3.1.3.3. The functor*

$$\prod_{p \text{ prime}} \mathcal{S}\mathfrak{h}_{p^\infty\text{-tors}}(S[1/p]_{\acute{\text{e}}\text{t}}, R) \rightarrow \mathcal{D}\mathcal{M}_{\acute{\text{e}}\text{t}, \text{tors}}(S, R)$$

which sends an object $(M_p)_{p \text{ prime}}$ to the object $\bigoplus_{p \text{ prime}} j_p^(\rho_! M_p)$ is an equivalence. This equivalence commutes with the six functors.*

The ordinary t-structure

Proposition 3.1.3.5. *Let S be a scheme and let R be a ring. The ordinary t-structure on $\mathcal{S}\mathfrak{h}(S_{\acute{\text{e}}\text{t}}, R)$ is generated in the sense of Proposition 3.1.1.6 by the étale sheaves of the form $R_S(X)$ with X étale over S .*

Proof. The proposition follows from Proposition 3.1.1.7. \square

Proposition 3.1.3.6. *Let S be a scheme and R be a ring. Then, the ordinary t-structure of the stable category $\mathcal{S}\mathfrak{h}(S_{\acute{\text{e}}\text{t}}, R)$ induces a t-structure on the stable subcategory $\mathcal{S}\mathfrak{h}_{p^\infty\text{-tors}}(S_{\acute{\text{e}}\text{t}}, R)$.*

Proof. Let $\tau_{\geq 0}$ denote the truncation with respect to the ordinary t-structure of $\mathcal{S}\mathfrak{h}(S_{\acute{\text{e}}\text{t}}, R)$. Let M be an étale sheaf of p^∞ -torsion. The functor $\otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ is t-exact with respect to the ordinary t-structure as $\mathbb{Z}[1/p]$ is flat over \mathbb{Z} . Therefore, we have

$$\tau_{\geq 0}(M) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = \tau_{\geq 0}(M \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]).$$

Furthermore, the sheaf $\tau_{\geq 0}(M \otimes_{\mathbb{Z}} \mathbb{Z}[1/p])$ vanishes and therefore, the sheaf $\tau_{\geq 0}(M)$ is of p^∞ -torsion. We conclude by using [BBDG18, 1.3.19]. \square

A product of t-categories is endowed with a t-structure by taking the product of the subcategories of t-non-positive objects as the subcategory of t-non-positive objects and the product of the subcategories of t-non-negative objects as the subcategory of t-non-negative objects.

This allows us to define an ordinary t-structure on p^∞ -torsion étale motives.

Definition 3.1.3.7. *Let S be a scheme and let R be a good ring. The ordinary t-structure over the stable category $\mathcal{D}\mathcal{M}_{\acute{\text{e}}\text{t}, \text{tors}}(S, R)$ is the only t-structure such that the functor*

$$\prod_{p \text{ prime}} \mathcal{S}\mathfrak{h}_{p^\infty\text{-tors}}(S[1/p]_{\acute{\text{e}}\text{t}}, R) \rightarrow \mathcal{D}\mathcal{M}_{\acute{\text{e}}\text{t}, \text{tors}}(S, R)$$

of Corollary 3.1.3.4 is a t-exact equivalence when the left hand side is endowed with the product of the ordinary t-structures.

Remark 3.1.3.8. *Let S be a scheme and let R be a good ring. If all the residue characteristics of S are invertible in R , then, the functor $\rho_!$ of Section [3.1.2](#) induces a t -exact equivalence*

$$\mathcal{S}\mathfrak{h}_{\text{tors}}(S_{\text{ét}}, R) \rightarrow \mathcal{D}\mathcal{M}_{\text{ét,tors}}(S, R)$$

when both sides are endowed with their ordinary t -structures.

Proposition 3.1.3.9. *Let $f: T \rightarrow S$ be a morphism of schemes and let R be a good ring. The functor f^* induces a t -exact functor*

$$\mathcal{D}\mathcal{M}_{\text{ét,tors}}(S, R) \rightarrow \mathcal{D}\mathcal{M}_{\text{ét,tors}}(T, R)$$

when both sides are endowed with their ordinary t -structures.

Proof. Since the functor of Corollary [3.1.3.4](#) commutes with the six functors, it suffices to show that if p is a prime number and is invertible on S , the functor

$$f^*: \mathcal{S}\mathfrak{h}_{p^\infty\text{-tors}}(S_{\text{ét}}, R) \rightarrow \mathcal{S}\mathfrak{h}_{p^\infty\text{-tors}}(T_{\text{ét}}, R)$$

is t -exact when both sides are endowed with their ordinary t -structures which follows from the t -exactness of the functor

$$f^*: \mathcal{S}\mathfrak{h}(S_{\text{ét}}, R) \rightarrow \mathcal{S}\mathfrak{h}(T_{\text{ét}}, R).$$

□

Strengthening of the continuity property

The ordinary t -structure gives us a setting in which we can generalize the continuity property of [\[CD16, 6.3.7\]](#).

Proposition 3.1.3.10. *Let \mathcal{D} be one of the fibered categories $\mathcal{D}\mathcal{M}_{\text{ét}}$ or $\mathcal{S}\mathfrak{h}(-_{\text{ét}})$ and let R be a good ring. Consider a scheme X which is the limit of a projective system of schemes with affine transition maps $(X_i)_{i \in I}$.*

Let $(M_i)_{i \in I}$ and $(N_i)_{i \in I}$ be two cartesian sections of the fibered category $\mathcal{D}(-, R)$ over the diagram of schemes $(X_i)_{i \in I}$ and denote by M and N the respective pullback of M_i and N_i along the map $X \rightarrow X_i$. If each M_i is constructible and each $N_i \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ is bounded below with respect to the ordinary t -structure on $\mathcal{D}_{\text{tors}}(X_i, R)$, then the canonical maps

$$\text{colim}_{i \in I} \text{Hom}_{\mathcal{D}(X_i, R)}(M_i, N_i) \rightarrow \text{Hom}_{\mathcal{D}(X, R)}(M, N)$$

$$\text{colim}_{i \in I} \text{Map}_{\mathcal{D}(X_i, R)}(M_i, N_i) \rightarrow \text{Map}_{\mathcal{D}(X, R)}(M, N)$$

are equivalences.

Proof. Following the proof of [\[CD16, 6.3.7\]](#), we can assume that the ring R is a \mathbb{Q} -algebra or a $\mathbb{Z}/n\mathbb{Z}$ -algebra with n a power of some prime number p .

If R is a \mathbb{Q} -algebra, the result is [\[CD16, 5.2.5\]](#) when $\mathcal{D} = \mathcal{D}\mathcal{M}_{\text{ét}}$ and the result follows from [\[AGV73, VI.8.5.7\]](#) when $\mathcal{D} = \mathcal{S}\mathfrak{h}(-_{\text{ét}})$.

If R is a $\mathbb{Z}/n\mathbb{Z}$ -algebra, we can assume that p (and therefore n) is invertible on S by virtue of [\[CD16, A.3.4\]](#). Applying the Rigidity Theorem [\[BH21, 3.2\]](#), we can assume that $\mathcal{D} = \mathcal{S}\mathfrak{h}(-_{\text{ét}})$. In this case, the result follows from [\[AGV73, IX.2.7.3\]](#). □

The perverse t-structure

We now define a perverse t-structure on the category of torsion étale motives over base schemes endowed with a dimension function following [Gab08]. First recall the definition of dimension functions (see for instance [BD17, 1.1.1]).

Definition 3.1.3.11. *Let S be a scheme. A dimension function on S is a function*

$$\delta: S \rightarrow \mathbb{Z}$$

such that for any immediate specialization y of a point x , i.e a specialization such that

$$\text{codim}_{\overline{\{x\}}}(y) = 1,$$

we have $\delta(x) = \delta(y) + 1$.

Two dimension functions on a scheme differ by a Zariski-locally constant function. Moreover, if a scheme S is universally catenary and integral, the formula

$$\delta(x) = \dim(S) - \text{codim}_S(x)$$

defines a dimension function on S . Finally, any S -scheme of finite type $f: X \rightarrow S$ inherits a canonical dimension function by setting

$$\delta(x) = \delta(f(x)) + \text{tr.deg } k(x)/k(f(x)).$$

Let R be a ring. The auto-dual perversity $p_{1/2}: Z \mapsto -\delta(Z)$ induces two t-structures $[p_{1/2}]$ and $[p_{1/2}^+]$ on $\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$. In this chapter, we only consider the t-structure $[p_{1/2}]$.

Definition 3.1.3.12. ([Gab08, 2]) *Let S be a scheme endowed with a dimension function δ and let R be a ring. If x is a point of S , denote by $i_x: \{x\} \rightarrow S$ the inclusion. We define*

- *the full subcategory ${}^p\mathcal{S}\mathfrak{h}^{\leq 0}(S_{\text{ét}}, R)$ as the subcategory made of those étale sheaves M such that for any point x of S , we have*

$$i_x^* M \leq p_{1/2}(\overline{\{x\}}) = -\delta(x)$$

with respect to the ordinary t-structure of $\mathcal{S}\mathfrak{h}(k(x)_{\text{ét}}, R)$.

- *the full subcategory ${}^p\mathcal{S}\mathfrak{h}^{\geq 0}(S_{\text{ét}}, R)$ as the subcategory made of those bounded below étale sheaves M such that for any point x of S , we have*

$$i_x^! M \geq p_{1/2}(\overline{\{x\}}) = -\delta(x)$$

with respect to the ordinary t-structure of $\mathcal{S}\mathfrak{h}(k(x)_{\text{ét}}, R)$.

[Gab08, 6] implies that this defines a t-structure on $\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$. Which we call the perverse t-structure.

Proposition 3.1.3.13. *Let S be a scheme endowed with a dimension function δ and let R be a ring. The perverse t-structure on $\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$ is generated in the sense of Proposition 3.1.1.6 by the étale sheaves of the form $f_! R[\delta(X)]$ with $f: X \rightarrow S$ quasi-finite.*

Proof. When S is the spectrum of a field, the result follows from Proposition [3.1.3.5](#).

Denote by t_0 the t-structure generated by the étale sheaves of the form $f_! \underline{R}[\delta(X)]$ with $f: X \rightarrow S$ quasi-finite. It suffices to show that a sheaf M is perverse t-non-negative if and only if it is t_0 -non-negative.

Hence, it suffices to show that a sheaf M is t_0 -non-negative if and only if it is bounded below and for any point x of S , the étale sheaf $i_x^! M$ is t_0 -non-negative.

To prove the "only if" part, notice that if f is a quasi-finite morphism, the adjunction $(f_!, f^!)$ is a t_0 -adjunction. Indeed, the full subcategory of those étale sheaves N such that $f_! N$ is t_0 -non-positive is closed under extensions and small colimits and contains the set of generators of t_0 . Thus, this subcategory contains all t_0 -non-positive objects. Hence, the functors $f_!$ is right t_0 -exact and therefore, the functor $f^!$ is left- t_0 -exact. Hence, if M is t_0 -non-negative and if x is a point of S , the étale sheaf $i_x^! M$ is t_0 -non-negative which yields

$$i_x^! M \geq -\delta(x)$$

with respect to the ordinary t-structure on $\mathcal{S}\mathfrak{h}(k(x)_{\text{ét}}, R)$.

Furthermore, if M is t_0 -non-negative, if c is a lower bound for the function δ on S , the complex $\text{Map}(R_S(X), M)$ is $(c - 1)$ -connected for any étale S -scheme X . Hence, the étale sheaf sheaf M is bounded below and it is therefore perverse t-non-negative.

We now prove the converse. Let M be a bounded below étale sheaf such that for all point x of S , the étale sheaf $i_x^! M$ is t_0 -non-negative. Let $f: X \rightarrow S$ be quasi-finite. For any point y of X , let $Z(y)$ be the set of open neighborhoods of y in the closure $\overline{\{y\}}$ of $\{y\}$. If U is an element of $Z(y)$, denote by $f_U: U \rightarrow S$ the inclusion. Then, by [\[BD17, 3.1.5\]](#), we have the δ -niveau spectral sequence

$$E_{p,q}^1 \Rightarrow \pi_{-p-q} \text{Map}_{\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)}(f_! \underline{R}, M)$$

with

$$E_{p,q}^1 = \bigoplus_{y \in X, \delta(y)=p} \text{colim}_{U \in Z(y)} \pi_{-p-q} \text{Map}_{\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)}((f_U)_! \underline{R}, M)$$

If y is a point of X and if $x = f(y)$, let $\iota_y: \{y\} \rightarrow X$ be the inclusion and let $f_y: \{y\} \rightarrow \{x\}$ be the morphism induced by f . Then, since the sheaf M is bounded below, Proposition [3.1.3.10](#) yields

$$\begin{aligned} \text{colim}_{U \in Z(y)} \text{Map}_{\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)}((f_U)_! \underline{R}, M) &= \text{Map}_{\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)}((f \circ \iota_y)_! \underline{R}, M) \\ &= \text{Map}_{\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)}((i_x)_! (f_y)_! \underline{R}, M) \\ &= \text{Map}_{\mathcal{S}\mathfrak{h}(k(x)_{\text{ét}}, R)}((f_y)_! \underline{R}, i_x^! M). \end{aligned}$$

Thus, we get

$$E_{p,q}^1 = \bigoplus_{y \in X, \delta(y)=p} \pi_{-p-q} \text{Map}_{\mathcal{S}\mathfrak{h}(k(f(y))_{\text{ét}}, R)}((f_y)_! \underline{R}, i_{f(y)}^! M).$$

Since for any point x of S , the sheaf $i_x^! M$ is t_0 -non-negative, for any point y of X , the complex $\text{Map}_{\mathcal{S}\mathfrak{h}(k(f(y))_{\text{ét}}, R)}((f_y)_! \underline{R}, i_{f(y)}^! M)$ is $(-\delta(y) - 1)$ -connected and thus it is $(-\delta(X) - 1)$ -connected.

Thus, if $n > \delta(X)$, the R -module $\pi_{-n} \text{Map}_{\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)}(f_! \underline{R}, M)$ vanishes. Hence, the étale sheaf M is t_0 -non-negative. \square

Proposition 3.1.3.14. *Let S be a scheme endowed with a dimension function and let R be a ring. Then, the perverse t-structure of the stable category $\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$ induces a t-structure on the stable subcategory $\mathcal{S}\mathfrak{h}_{p^\infty\text{-tors}}(S_{\text{ét}}, R)$.*

Proof. To apply the same proof as in Proposition 3.1.3.6, we only need to show that the functor $-\otimes_{\mathbb{Z}}\mathbb{Z}[1/p]$ is perverse t-exact. This is true because if x is a point of S , denoting by $i_x: \{x\} \rightarrow S$ the inclusion, the functors i_x^* and $i_x^!$ commute with the functor $-\otimes_{\mathbb{Z}}\mathbb{Z}[1/p]$. \square

Definition 3.1.3.15. Let S be a scheme endowed with a dimension function and let R be a ring. The perverse t-structure over the stable category $\mathcal{DM}_{\text{ét,tors}}(S, R)$ is the only t-structure such that the functor

$$\prod_{p \text{ prime}} \mathcal{S}\mathfrak{h}_{p^\infty\text{-tors}}(S[1/p]_{\text{ét}}, R) \rightarrow \mathcal{DM}_{\text{ét,tors}}(S, R)$$

of Corollary 3.1.3.4 is a t-exact equivalence when the left hand side is endowed with the product of the perverse t-structures.

Remark 3.1.3.16. Let S be a scheme endowed with a dimension function and let R be a ring. If all the residue characteristics of S are invertible in R , then, the functor $\rho_!$ of Section 3.1.2 induces a t-exact equivalence

$$\mathcal{S}\mathfrak{h}_{\text{tors}}(S_{\text{ét}}, R) \rightarrow \mathcal{DM}_{\text{ét,tors}}(S, R)$$

when both sides are endowed with their perverse t-structures.

The definition of the perverse t-structure and the fact that the equivalence $\rho_!$ of Section 3.1.2 commutes with the six functors yield the following characterization of the perverse t-structure.

Proposition 3.1.3.17. Let S be a scheme endowed with a dimension function and let R be a ring. If x is a point of S , denote by $i_x: \{x\} \rightarrow S$ the inclusion. Then,

- The subcategory ${}^p\mathcal{DM}_{\text{ét,tors}}^{\leq 0}(S, R)$ is made of those torsion motives M such that for any point x of S , we have

$$i_x^*M \leq -\delta(x)$$

with respect to the ordinary t-structure of $\mathcal{DM}_{\text{ét,tors}}(k(x), R)$.

- The subcategory ${}^p\mathcal{DM}_{\text{ét,tors}}^{\geq 0}(S, R)$ is made of those bounded below torsion motives M such that for any point x of S , we have

$$i_x^!M \geq -\delta(x)$$

with respect to the ordinary t-structure of $\mathcal{DM}_{\text{ét,tors}}(k(x), R)$.

Proposition 3.1.3.18. (Affine Lefschetz Theorem for torsion motives) Let $f: T \rightarrow S$ be an affine morphism of schemes and let R be a ring. Assume that the scheme S is quasi-excellent and endowed with a dimension function. Then, the functor $f_!$ induces a left t-exact functor

$$\mathcal{DM}_{\text{ét,tors}}(T, R) \rightarrow \mathcal{DM}_{\text{ét,tors}}(S, R)$$

when both sides are endowed with their perverse t-structures.

Proof. Since the functor of Corollary 3.1.3.4 commutes with the six functors, it suffices to show that if p is a prime number and invertible on S , the functor

$$f_! : \mathcal{S}h_{p^\infty\text{-tors}}(T_{\acute{e}t}, R) \rightarrow \mathcal{S}h_{p^\infty\text{-tors}}(S_{\acute{e}t}, R)$$

is left t-exact when both sides are endowed with their perverse t-structures.

Let M be a p^∞ -torsion étale sheaf which is perverse t-non-negative. We have

$$M = \operatorname{colim}_n M \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}[-1].$$

Let n be a positive integer. Since we have an exact triangle

$$M \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}[-1] \rightarrow M \xrightarrow{\times p^n} M,$$

the sheaf $M \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}[-1]$ is perverse t-non-negative.

Therefore, by Lemma 3.1.3.19, the sheaf $f_!(M \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}[-1])$ is perverse t-non-negative. Since the functor $f_!$ is a left adjoint, it is compatible with colimits and therefore, using Proposition 3.1.2.7, the sheaf $f_!(M)$ is perverse t-non-negative. \square

Lemma 3.1.3.19. *Let $f: T \rightarrow S$ be an affine morphism of schemes, let p be a prime number which is invertible on S and let n be a positive integer. Then, the functor*

$$f_! : \mathcal{S}h(T_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z}) \rightarrow \mathcal{S}h(S_{\acute{e}t}, \mathbb{Z}/p^n\mathbb{Z})$$

is left t-exact.

Proof. By definition of the perverse t-structure, we can assume that the scheme S is the spectrum of a strictly henselian ring. In that case, since the étale sheaves of the form $g_!\mathbb{Z}/p^n\mathbb{Z}[\delta(X)]$ with $g: X \rightarrow S$ quasi-finite are compact, the perverse t-structure is compatible with filtered colimits.

Let M be an étale sheaf over $T_{\acute{e}t}$ which is t-non-negative we want to prove that the sheaf $f_!M$ is . The sheaf M is a filtering colimit of constructible sheaves. Therefore, we can assume that the sheaf M is constructible. The result is then exactly the Affine Lefschetz Theorem [LYF14, XV.1.1.2]. \square

3.1.4 Reminders and complements on Artin motives

Generators and stability

We now state and generalize some results from [AZ12] and [Pep19b] that will be useful in our discussion. The following two results were stated in [AZ12] for schemes over a perfect field and extended over any schemes in [Pep19b] where they assume that $R = \mathbb{Q}$. However, their proofs apply if R is any commutative ring.

Proposition 3.1.4.1. ([Pep19b, 1.28]) *Let S be a scheme and let R be a ring. The category $\mathcal{DM}_{\acute{e}t}^A(S, R)$ (resp. $\mathcal{DM}_{\acute{e}t,c}^A(S, R)$) is the localizing (resp. thick) subcategory of $\mathcal{DM}_{\acute{e}t}(S, R)$ generated by any of the following families of objects.*

1. The family of the $M_S^{\text{BM}}(X)$, with X quasi-finite over S .

2. The family of the $M_S^{\text{BM}}(X) = h_S(X)$, with X finite over S .
3. The family of the $M_S^{\text{BM}}(X) = M_S(X)$, with X étale over S .

Proposition 3.1.4.2. ([Pep19b, 1.17]) Let R be a ring. The categories $\mathcal{DM}_{\text{ét},(c)}^A(-, R)$ are closed under the following operations.

1. The functor f^* , where f is any morphism.
2. The functor $f_!$, where f is a quasi-finite morphism.
3. Tensor product.

Furthermore, the fibered category $\mathcal{DM}_{\text{ét},(c)}(-, R)$ satisfies the localization property (3.0.0.1).

If $j: U \rightarrow S$ is an open immersion, the motive $j_*\mathbb{1}_U$ need not be Artin, the fibered category $\mathcal{DM}_{\text{ét}}(-, R)$ does not satisfy the localization property (3.0.0.2).

On the other hand, the description of generators yields the following statement.

Proposition 3.1.4.3. Let S be a scheme and let R be a ring. Then, the functor $\rho_!$ of Section 3.1.2 has its essential image contained in $\mathcal{DM}_{\text{ét}}^A(S, R)$. We still denote by

$$\rho_!: \mathcal{S}\mathfrak{h}(S_{\text{ét}}, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

the induced functor.

Proof. The functor $\rho_!$ of Section 3.1.2 is colimit preserving. Therefore, the result follows from Proposition 3.1.1.7 and from the fact that if X is an étale S -scheme, we have

$$\rho_!R_S(X) = M_S(X).$$

□

Finally, we can identify the \mathfrak{p} -localization of the category of constructible Artin motives

Proposition 3.1.4.4. Let S be a scheme, let R be a ring and let \mathfrak{p} be a prime ideal of \mathbb{Z} . Then, with the notations of Section 3.1.1, the functor

$$\mathcal{DM}_{\text{ét},c}^A(S, R)_{\mathfrak{p}} \rightarrow \mathcal{DM}_{\text{ét},c}^A(S, R \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})$$

is an equivalence.

Proof. The proof is the same as the proof of Proposition 3.1.2.4. □

The Artin truncation functor

Let S be a scheme and let R be a ring. The adjunction theorem [Lur09, 5.5.2.9] ensures the existence of a right adjoint exact functor to the inclusion of Artin motives into cohomological motives:

$$\omega^0: \mathcal{DM}_{\text{ét}}^{\text{coh}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R).$$

This functor was first introduced in [AZ12, 2.2] and was further studied in [Pep19b] and [Vai16]. We have a 2-morphism $\delta: \omega^0 \rightarrow \text{id}$ given by the counit of the adjunction. We recall the basic properties of ω^0 and its compatibilities with the six functors formalism from [AZ12, 2.15] and [Pep19b, 3.3, 3.7]:

Proposition 3.1.4.5. *Let R be a ring.*

1. *The 2-morphism $\delta: \omega^0 \rightarrow \text{id}$ induces an equivalence when restricted to Artin motives. Moreover, we have $\delta(\omega^0) = \omega^0(\delta)$ and this 2-morphism is an equivalence $\omega^0 \circ \omega^0 \rightarrow \omega^0$.*
2. *Let f be a morphism of schemes.*
 - (a) *The 2-morphism $\delta(f^*\omega^0): \omega^0 f^*\omega^0 \rightarrow f^*\omega^0$ is invertible.*
 - (b) *We have a 2-morphism $\alpha_f: f^*\omega^0 \rightarrow \omega^0 f^*$ such that*
 - i. *we have $\delta(f^*) \circ \alpha_f = \boxed{1} f^*(\delta)$ and $\alpha_f \circ \delta(f^*\omega^0) = \omega^0 f^*(\delta)$.*
 - ii. *the 2-morphism α_f is invertible when f is étale.*
 - iii. *letting $i: F \rightarrow S$ be a closed immersion, letting $j: U \rightarrow S$ be the complementary open immersion and letting M be an object of $\mathcal{DM}_{\text{ét}}(S, R)$ such that $j^*(M)$ belongs to $\mathcal{DM}_{\text{ét}}^A(U, R)$, the map $\alpha_i(M)$ is invertible.*
3. *Let f be a morphism of finite type.*
 - (a) *The 2-morphism $\omega^0 f_*(\delta): \omega^0 f_*\omega^0 \rightarrow \omega^0 f_*$ is invertible.*
 - (b) *We have a 2-morphism $\beta_f: \omega^0 f_* \rightarrow f_*\omega^0$ such that:*
 - i. *we have $f_*(\delta) \circ \beta_f = \delta(f_*)$ and $\beta_f \circ \omega^0 f_*(\delta) = \delta(f_*\omega^0)$.*
 - ii. *the 2-morphism β_f is invertible when the morphism f is finite.*
4. *Let f be a quasi-finite morphism.*
 - (a) *The maps $\delta(f_!\omega^0): \omega^0 f_!\omega^0 \rightarrow f_!\omega^0$ and $\omega^0 f^!(\delta): \omega^0 f^!\omega^0 \rightarrow \omega^0 f^!$ are invertible.*
 - (b) *We have 2-morphisms $\eta_f: f_!\omega^0 \rightarrow \omega^0 f_!$ and $\gamma_f: \omega^0 f^! \rightarrow f^!\omega^0$ such that*
 - i. *we have $\delta(f_!) \circ \eta_f = f_!(\delta)$, $\eta_f \circ \delta(f_!\omega^0) = \omega^0 f_!(\delta)$, $f^!(\delta) \circ \gamma_f = \delta(f^!)$ and $\gamma_f \circ \omega^0 f^!(\delta) = \delta(f^!\omega^0)$.*
 - ii. *the 2-morphism η_f is invertible when f is finite and coincides with β_f^{-1} modulo the natural isomorphism $f_! \simeq f_*$.*
 - iii. *the 2-morphism γ_f is invertible when f is étale and coincides with α_f^{-1} modulo the natural isomorphism $f^* \simeq f^!$.*
5. *Assume that the ring R is a \mathbb{Q} -algebra.*
 - (a) *The map α_f is an equivalence when f is smooth.*
 - (b) *Assume that S is regular. Let X be a smooth and proper S -scheme and $\pi_0(X/S)$ be its Stein factorization over S . Then, the canonical map $h_S(\pi_0(X/S)) \rightarrow \omega^0 h_S(X)$ is an equivalence.*

Proof. This is [\[Pep19b, 3.3,3.7\]](#). □

As a consequence, we get an analog of the localization triangle [\(3.0.0.2\)](#) for Artin motives:

¹A more rigorous statement would be that $f^*(\delta)$ is a composition of $\delta(f^*)$ and α_f (in the ∞ -categorical sense).

Corollary 3.1.4.6. *Let $i: Z \rightarrow S$ be a closed immersion, let $j: U \rightarrow S$ be the complementary open immersion and let R be a ring. Then, we have an exact triangle*

$$i_! \omega^0 i^! \rightarrow \text{id} \rightarrow \omega^0 j_* j^*$$

in $\mathcal{DM}_{\text{ét}}^A(S, R)$.

Proof. Apply the functor ω^0 to the exact localization triangle (3.0.0.2) and use the fact that the functor η_i of Proposition 3.1.4.5 above an equivalence to get the result. \square

We will see later that this triangle need not induce a triangle in $\mathcal{DM}_{\text{ét},c}^A(S, R)$ in general.

In [AZ12], the authors proved that if S is a quasi-projective scheme over a field of characteristic 0 and if $R = \mathbb{Q}$, then, the functor ω_S^0 preserves constructible objects (see [AZ12, 2.15 (vii)]). In [Pep19b, 3.7], Pepin Lehalleur extended the result to schemes allowing resolutions of singularities by alterations. In the proof, Pepin Lehalleur uses this hypothesis only to ensure that the localization triangle (3.0.0.2) belongs to $\mathcal{DM}_{\text{ét},c}^{\text{coh}}(-, R)$. Using the appendix, we see that this assumption is in fact unnecessary and we deduce the following statement.

Proposition 3.1.4.7. *Let S be a quasi-excellent scheme and R a \mathbb{Q} -algebra. Then, the functor ω_S^0 preserves constructible objects.*

Using the Artin truncation functor ω^0 , we prove that there is a trace of the six functors formalism on constructible Artin motives with rational coefficients ; this is not true with integral coefficient. Namely, if k is a field, and i is the inclusion of a point in \mathbb{P}_k^1 , the motive $\omega^0 i^! \mathbb{1}_{\mathbb{P}_k^1}$ is not constructible (see Corollary 3.3.1.8 below).

Proposition 3.1.4.8. *Let R be a \mathbb{Q} -algebra. Then, the six functors on $\mathcal{DM}_{\text{ét}}(-, R)$ induce*

1. a functor f^* for any morphism f ,
2. a functor $\omega^0 f_*$ for any separated morphism of finite type f whose target is quasi-excellent,
3. a functor $f_!$ for any quasi-finite morphism f ,
4. a functor $\omega^0 f^!$ for any quasi-finite morphism f with a quasi-excellent source,
5. functors \otimes and $\omega^0 \underline{\text{Hom}}$

on $\mathcal{DM}_{\text{ét},c}^A(-, R)$. Moreover, the fibered category $\mathcal{DM}_{\text{ét},c}^A(S, R)$ satisfies the localization property (3.0.0.1) and if $i: Z \rightarrow S$ is a closed immersion and if $j: U \rightarrow S$ is the complementary open immersion, we have an exact triangle

$$i_! \omega^0 i^! \rightarrow \text{id} \rightarrow \omega^0 j_* j^*$$

in $\mathcal{DM}_{\text{ét},c}^A(S, R)$.

Proof. We already proved the proposition in the case of left adjoint functors in Proposition 3.1.4.2. The case of right adjoint functors follows from the fact that $\mathcal{DM}_{\text{ét},c}^{\text{coh}}(-, R)$ is endowed with the six functors formalism and from Proposition 3.1.4.7. \square

Finally, the functor ω^0 is compatible with change of coefficients.

Proposition 3.1.4.9. *Let $\sigma: R \rightarrow R'$ be a ring morphism of the form $\sigma_A: R \rightarrow R \otimes_{\mathbb{Z}} A$ or $\sigma_n: R \rightarrow R/nR$ where A is a localization of \mathbb{Z} and n is an integer. Assume that the ring R is flat over \mathbb{Z} . Then, we have invertible 2-morphisms*

$$u: \sigma_*\omega^0 \rightarrow \omega^0\sigma_* \text{ and } v: \sigma^*\omega^0 \rightarrow \omega^0\sigma^*$$

such that the diagrams

$$\begin{array}{ccc} \sigma_*\omega^0 & \xrightarrow{u} & \omega^0\sigma_* \\ & \searrow \sigma_*(\delta) & \downarrow \delta(\sigma_*) \\ & & \sigma_* \end{array} \quad \text{and} \quad \begin{array}{ccc} \sigma^*\omega^0 & \xrightarrow{u} & \omega^0\sigma^* \\ & \searrow \sigma^*(\delta) & \downarrow \delta(\sigma^*) \\ & & \sigma^* \end{array}$$

are commutative.

Proof. Let us first show that any object of $\mathcal{DM}_{\text{ét}}^A(S, R')$ lies in $\mathcal{DM}_{\text{ét}}^A(S, R)$. Let X be an étale S -scheme.

In the case where $R' = R \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$, since R is flat over \mathbb{Z} , we have an exact sequence

$$0 \rightarrow R \xrightarrow{\times n} R \rightarrow R \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \rightarrow 0.$$

Since the six functors formalism commute with change of coefficients by Proposition [3.1.2.3](#), this implies that $M_S(X, R')$ is the cone of the map

$$M_S(X, R) \xrightarrow{\times n} M_S(X, R).$$

Therefore, the set of generators of $\mathcal{DM}_{\text{ét}}^A(S, R')$ belongs to $\mathcal{DM}_{\text{ét}}^A(S, R)$.

In the case where $R' = R \otimes_{\mathbb{Z}} A$, write $A = \Sigma^{-1}\mathbb{Z}$ where Σ is a multiplicative subset of \mathbb{Z} . Then, the motive $M_S(X, R')$ is the colimit of the inductive system given by the multiplication by s maps $M_S(X, R) \rightarrow M_S(X, R)$ for $s \in \Sigma$. Therefore, the set of generators of $\mathcal{DM}_{\text{ét}}^A(S, R')$ lies in $\mathcal{DM}^A(S, R)$.

Therefore, the functor σ_* preserves Artin motives. The existence of a 2-morphisms u such that $\sigma_*(\delta) = \delta(\sigma_*) \circ u$ is therefore a consequence of the definition of ω^0 . Likewise, since the functor σ^* preserves Artin motives, we get a 2-morphism v such that $\sigma^*(\delta) = \delta(\sigma^*) \circ v$. We now show that u and v are invertible.

To show that the 2-morphism u is invertible, it suffices to show that for any object M of $\mathcal{DM}_{\text{ét}}^A(S, R)$ and any object N of $\mathcal{DM}_{\text{ét}}^A(S, R')$, the map

$$\text{Map}(M, \sigma_*\omega^0 N) \rightarrow \text{Map}(M, \sigma_* N)$$

induced by $\sigma_*(\delta)$ is invertible. This amounts to the fact that the map

$$\text{Map}(\sigma^* M, \omega^0 N) \rightarrow \text{Map}(\sigma^* M, N)$$

induced by δ is invertible which is true by definition of the functor ω^0 .

To show that the 2-morphism v is invertible, it suffices to show that for any object M of $\mathcal{DM}_{\text{ét},c}^A(S, R')$ and any object N of $\mathcal{DM}_{\text{ét}}^A(S, R)$, the map

$$\text{Map}(M, \sigma^*\omega^0 N) \rightarrow \text{Map}(M, \sigma^* N)$$

induced by $\sigma^*(\delta)$ is invertible. We can assume that $M = M_S(X, R')$ for X an étale S -scheme. We have $M = \sigma^*M_S(X, R)$. But the diagram

$$\begin{array}{ccc} \mathrm{Map}(\sigma^*M_S(X, R), \sigma^*\omega^0N) & \xrightarrow{\sigma^*(\delta)\circ-} & \mathrm{Map}(\sigma^*M_S(X, R), \sigma^*N) \\ \uparrow & & \uparrow \\ \sigma^*\mathrm{Map}(M_S(X, R), \omega^0N) & \xrightarrow{\sigma^*(\delta)\circ-} & \sigma^*\mathrm{Map}(M_S(X, R), N) \end{array}$$

is commutative, its vertical arrows are invertible by [CD16, 5.4.5, 5.4.11] and its lower horizontal arrow is invertible by definition of ω^0 . This finishes the proof. \square

Torsion motives are Artin motives

Proposition 3.1.4.10. *Let S be a scheme and let R be a ring. The stable category of torsion étale motives $\mathcal{DM}_{\text{ét,tors}}(S, R)$ is the subcategory of the stable category $\mathcal{DM}_{\text{ét}}^A(S, R)$ made of those Artin motives M such that the motive $M \otimes_{\mathbb{Z}} \mathbb{Q}$ vanishes.*

Proof. If an Artin motive M is such that the motive $M \otimes_{\mathbb{Z}} \mathbb{Q}$ vanishes, it is by definition a torsion motive. Therefore, it suffices to show that torsion motives are Artin motives.

Let M be a torsion motive. Using Proposition 3.1.3.3, we can decompose M as

$$M = \bigoplus_{p \text{ prime}} M_p$$

where M_p is of p^∞ -torsion. Therefore, we can assume that M is of p^∞ -torsion for some prime number p . Proposition 3.1.3.3 gives a p^∞ -torsion étale sheaf N over the open subscheme $S[1/p]$ of S such that

$$M = (j_p)_! \rho_! N$$

where $j_p: S[1/p] \rightarrow S$ is the open immersion.

By Proposition 3.1.4.3, the motive $\rho_! N$ is an Artin motive. Therefore, by Proposition 3.1.4.2, the motive M is an Artin motive. \square

Corollary 3.1.4.11. *Let S be a scheme and let R be a ring. The functor $\delta: \omega^0 \rightarrow \mathrm{Id}$ induces an equivalence over $\mathcal{DM}_{\text{ét,tors}}(S, R)$.*

3.1.5 Artin representations and lisse étale sheaves

Finite permutation resolutions

In this paragraph, we recall a result of Balmer and Gallauer (see [BG23]). If G is a finite group, R is a ring and H is a subgroup of G , the R -module $R[G/H]$ is endowed with a left $R[G]$ -module structure letting G act on G/H by left multiplication.

Proposition 3.1.5.1. *Let G be a finite group, let R be a regular ring and let M be a representation of G with coefficients in R . Then, the complex $M \oplus M[1]$ of $R[G]$ -modules is quasi-isomorphic as a complex of $R[G]$ -modules to a complex of the form*

$$\cdots \rightarrow 0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0 \rightarrow \cdots$$

where each P_i is placed in degree $-i$ and is isomorphic to $R[G/H_1] \oplus \cdots \oplus R[G/H_m]$ for some subgroups H_1, \dots, H_m of G .

Proof. This is Proposition 2.1.3.1. \square

Discrete representations, Artin representations

Recall the following definitions:

Definition 3.1.5.2. *Let π be a topological group and let R be a ring. An $R[\pi]$ -module M is discrete if the action of π on M is continuous when M is endowed with the discrete topology. We will denote by $\text{Mod}(\pi, R)$ the Grothendieck abelian category of discrete $R[\pi]$ -modules.*

Remark 3.1.5.3. *Keep the same notations. An $R[\pi]$ -module is discrete if and only if the stabilizer of every point is open. Equivalently, this means that M is the union of all the M^U for U an open subgroup of π .*

If the topological group π is profinite, this also means that M is the union of all the M^H for H a normal subgroup of π of finite index.

Example 3.1.5.4. *Let k be a field of characteristic exponent p . If n is a positive integer coprime to p , the absolute Galois group G_k of k acts on the abelian group μ_n of roots of unity in the algebraic closure of k . Let (T_n) , where n runs through the set of positive integers which are coprime to p , be an inductive system of discrete representations of G_k such that T_n is of n -torsion. Set $T = \text{colim } T_n$ and let m be an integer, we can define a discrete representation*

$$T(m) := \text{colim } T_n \otimes \mu_n^{\otimes m}$$

of G_k . We call this representation the m -th discrete Tate twist of T (beware that it depends on the inductive system and not only on T itself in general). Notice that by definition, T is isomorphic to every $T(m)$ as an abelian group but not as a G_k -module in general.

For example, let $X \rightarrow \text{Spec}(k)$ be a morphism of finite type, let \bar{k} be the algebraic closure of k , let $X_{\bar{k}}$ be the scalar extension of X to \bar{k} and let n be a non-negative integer. Then, the m -th discrete Tate twist of the discrete Galois representation $H_{\text{ét}}^n(X_{\bar{k}}, \mathbb{Q}/\mathbb{Z}[1/p])$ will by convention be the discrete representation

$$\text{colim}_{(N,p)=1} H_{\text{ét}}^n(X_{\bar{k}}, \mathbb{Z}/N\mathbb{Z}) \otimes \mu_N^{\otimes m}.$$

Remark 3.1.5.5. *With the notation of the example above, we have in fact $T(m) \simeq T \otimes_{\widehat{\mathbb{Z}}} \widehat{\mathbb{Z}}'(m)$ where $\widehat{\mathbb{Z}}' = \prod_{\ell \neq p} \mathbb{Z}_{\ell}$.*

Definition 3.1.5.6. *Let π be a profinite group and let R be a ring. An Artin representation of π is a discrete $R[\pi]$ -module which is of finite presentation as an R -module. We will denote by $\text{Rep}^A(\pi, R)$ the abelian category of Artin representations.*

Remark 3.1.5.7. *Keep the same notations. An $R[\pi]$ -module is an Artin representation if and only if it is of finite presentation as an R -module and the action of π factors through a finite quotient. Indeed, if M is an Artin representation, the stabilizer of a finite generating family is open and thus contains a normal subgroup N of finite index. Therefore, the action of π factors through π/N .*

Locally constant étale sheaves

Definition 3.1.5.8. *Let S be a scheme and let R be a regular ring. We define:*

1. The abelian category $\mathrm{Sh}(S_{\acute{e}t}, R)$ of étale sheaves of R -modules as the heart of the ordinary t -structure on the stable category $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$.
2. The abelian category $\mathrm{Loc}_S(R)$ of locally constant sheaves of R -modules as the subcategory of the abelian category $\mathrm{Sh}(S_{\acute{e}t}, R)$ made of those sheaves which are étale-locally constant with finitely presented fibers.
3. The abelian category $\mathrm{Ind}\mathrm{Loc}(S, R)$ of Ind-locally constant sheaves of R -modules as the subcategory of the abelian category $\mathrm{Sh}(S_{\acute{e}t}, R)$ made of those sheaves which are filtered colimits of locally constant sheaves of R -modules.
4. The category of lisse étale sheaves $\mathcal{S}\mathcal{h}_{\mathrm{lisse}}(S, R)$ as the subcategory of dualizable objects of the stable category $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$.
5. The category of Ind-lisse étale sheaves $\mathcal{S}\mathcal{h}_{\mathrm{Ind}\mathrm{lisse}}(S, R)$ as the localizing subcategory of the stable category $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$ generated by $\mathcal{S}\mathcal{h}_{\mathrm{lisse}}(S, R)$.

These categories are related to each other by the following proposition.

Proposition 3.1.5.9. *Let S be a scheme and let R be a regular ring.*

1. An étale sheaf is lisse if and only if it is locally constant with perfect fibers.
2. The ordinary t -structure on the stable category $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$ induces a t -structure on the subcategory $\mathcal{S}\mathcal{h}_{\mathrm{lisse}}(S, R)$ whose heart is the abelian category $\mathrm{Loc}_S(R)$.
3. The ordinary t -structure on the stable category $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$ induces a t -structure on the subcategory $\mathcal{S}\mathcal{h}_{\mathrm{Ind}\mathrm{lisse}}(S, R)$ whose heart is the abelian category $\mathrm{Ind}\mathrm{Loc}(S, R)$.

Proof. The first assertion follows from [Sta23, 0FPP]. To prove the second assertion, notice that, as a consequence of the first assertion, lisse sheaves are bounded with respect to the ordinary t -structure of $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$. Therefore, Lemma 3.1.1.5 ensures that it suffices to show that

$$\mathrm{Loc}_S(R) = \mathcal{S}\mathcal{h}_{\mathrm{lisse}}(S, R) \cap \mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)^{\heartsuit}.$$

The fact that locally constant sheaves with finitely presented fibers have perfect fibers follows from the hypothesis that the ring R is regular, since in this case, an R -module is perfect if and only if it is finitely presented by [Sta23, 066Z]. Moreover, an abelian sheaf with perfect fibers is finitely presented. This finishes the proof of the second assertion.

Any Ind-lisse étale sheaf is a filtered colimit of lisse étale sheaves. Furthermore, by [Lur17, 1.3.5.21], the ordinary t -structure of $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$ is compatible with filtered colimits. Therefore, if $M = \mathrm{colim}_{i \in I} M_i$ is an Ind-lisse étale sheaf, where I is a filtered simplicial set and each M_i is lisse, we have

$$\tau_{\geq 0}(M) = \tau_{\geq 0}(\mathrm{colim} M_i) = \mathrm{colim} \tau_{\geq 0}(M_i).$$

Therefore, the sheaf $\tau_{\geq 0}(M)$ is Ind-lisse. Hence, by [BBDG18, 1.3.19], the ordinary t -structure of $\mathcal{S}\mathcal{h}(S_{\acute{e}t}, R)$ induces a t -structure on the category $\mathcal{S}\mathcal{h}_{\mathrm{Ind}\mathrm{lisse}}(S, R)$. An object L of the heart of this t -structure is a filtered colimit of lisse sheaves M_i . But then, we have,

$$L = \mathrm{colim} M_i = H^0(\mathrm{colim} M_i) = \mathrm{colim} H^0(M_i).$$

Therefore, the sheaf L is Ind-locally constant. Conversely, Ind-locally constant sheaves lie in the heart of the ordinary t -structure and are Ind-lisse. This finishes the proof of the third assertion. \square

The above proposition shows in particular that $\text{Loc}_S(R)$ is a weak Serre subcategory of $\text{Sh}(S_{\text{ét}}, R)$. We have the following characterization of lisse sheaves:

Proposition 3.1.5.10. *Let S be a connected scheme and let R be a regular ring. Let ξ be a geometric point of S . The fiber functor associated to ξ induces an equivalence of abelian monoidal categories*

$$\xi^* : \text{Loc}_S(R) \rightarrow \text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R)$$

where $\pi_1^{\text{ét}}(S, \xi)$ is the étale fundamental group of S with base point ξ of [GR71, V].

Proof. Let Loc_S be the Galois category of locally constant sheaves of sets on $S_{\text{ét}}$. By definition of the étale fundamental group, the fiber functor

$$\xi^* : \text{Loc}_S \rightarrow \pi_1^{\text{ét}}(S, \xi) - \text{Set}$$

is an equivalence.

The constant sheaf $R_S(S)$ is sent to the set R endowed with the trivial action of $\pi_1^{\text{ét}}(S, \xi)$. Thus, the functor ξ^* induces an equivalence

$$\text{Loc}_S(R) \rightarrow \text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R).$$

□

Proposition 3.1.5.11. *Let $f: T \rightarrow S$ be a morphism of schemes, let $\xi': \text{Spec}(\Omega) \rightarrow T$ be a geometric point of T , let $\xi = f \circ \xi'$ and let R be a regular ring. Denote by $\pi_1^{\text{ét}}(f): \pi_1^{\text{ét}}(T, \xi') \rightarrow \pi_1^{\text{ét}}(S, \xi)$ the group homomorphism induced by f .*

1. *The square*

$$\begin{array}{ccc} \text{Loc}_S(R) & \xrightarrow{f^*} & \text{Loc}_T(R) \\ \downarrow \xi^* & & \downarrow (\xi')^* \\ \text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R) & \xrightarrow{\pi_1^{\text{ét}}(f)^*} & \text{Rep}^A(\pi_1^{\text{ét}}(T, \xi'), R) \end{array}$$

is commutative.

2. *Assume that f is finite and étale, then, the map $\pi_1^{\text{ét}}(f)$ is injective and its image is a finite index subgroup. Furthermore, the functor*

$$f_* : \mathcal{S}\mathcal{h}(T, R) \rightarrow \mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$$

induces a functor $\text{Loc}_T(R) \rightarrow \text{Loc}_S(R)$ and the square

$$\begin{array}{ccc} \text{Loc}_T(R) & \xrightarrow{f_*} & \text{Loc}_S(R) \\ \downarrow (\xi')^* & & \downarrow \xi^* \\ \text{Rep}^A(\pi_1^{\text{ét}}(T, \xi'), R) & \xrightarrow{\text{Ind}_{\pi'}^{\pi}} & \text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R) \end{array}$$

where $\pi' = \pi_1^{\text{ét}}(T, \xi')$ and $\pi = \pi_1^{\text{ét}}(S, \xi)$, *is commutative.*

Proof. If X is a scheme, let Loc_X be the Galois category of locally constant sheaves of sets on $X_{\text{ét}}$. By definition of the map $\pi_1^{\text{ét}}(f)$, the square:

$$\begin{array}{ccc} \text{Loc}_S & \xrightarrow{f^*} & \text{Loc}_T \\ \downarrow \xi^* & & \downarrow (\xi')^* \\ \pi_1^{\text{ét}}(S, \xi) - \text{Set} & \xrightarrow{\pi_1^{\text{ét}}(f)^*} & \pi_1^{\text{ét}}(T, \xi') - \text{Set} \end{array}$$

is commutative which implies the first assertion.

The first part of the second assertion is [GR71, V.6.3]. To finish the proof, notice that on the one hand, the functor f_* is left adjoint to the functor f^* and that on the other hand, the functor $\text{Ind}_{\pi'}^{\pi}$ is left adjoint to the functor $\pi_1^{\text{ét}}(f)^*$ which yields the result. \square

Since the functor f^* (for any morphism of schemes f) and the functor f_* (for any finite étale morphism f) are compatible with colimits, this yields the following result.

Corollary 3.1.5.12. *Let $f: T \rightarrow S$ be a morphism of schemes, let ξ' be a geometric point of T , let $\xi = f \circ \xi'$ and let R be a regular ring. Denote by $\pi_1^{\text{ét}}(f): \pi_1^{\text{ét}}(T, \xi') \rightarrow \pi_1^{\text{ét}}(S, \xi)$ the group homomorphism induced by f .*

1. *The functor $f^*: \mathcal{S}\mathfrak{h}(S_{\text{ét}}, R) \rightarrow \mathcal{S}\mathfrak{h}(T, R)$ induces a functor*

$$\text{Ind Loc}_T(R) \rightarrow \text{Ind Loc}_S(R)$$

which sends an Ind-locally constant sheaf $\text{colim } M_i$ to the Ind-locally constant sheaf

$$\text{colim}[(\xi')^*]^{-1} \left(\pi_1^{\text{ét}}(f)^*(\xi^* M_i) \right).$$

2. *Assume that f is finite and étale, then, the functor $f_*: \mathcal{S}\mathfrak{h}(T, R) \rightarrow \mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$ induces a functor*

$$\text{Ind Loc}_T(R) \rightarrow \text{Ind Loc}_S(R)$$

which sends an Ind-locally constant sheaf $\text{colim } M_i$ to the Ind-locally constant sheaf

$$\text{colim}[(\xi')^*]^{-1} (\text{Ind}_{\pi'}^{\pi}(\xi^* M_i))$$

where $\pi' = \pi_1^{\text{ét}}(T, \xi')$ and $\pi = \pi_1^{\text{ét}}(S, \xi)$.

In the case when S is the spectrum of a field k , the results of this paragraph can be formulated in a more satisfying way.

Proposition 3.1.5.13. *Let k be a field and let R be a regular ring.*

1. *If $\xi: \text{Spec}(\bar{k}) \rightarrow \text{Spec}(k)$ is the separable closure of k , the fiber functor associated to ξ induces an equivalence of abelian monoidal categories*

$$\xi^*: \text{Sh}(k_{\text{ét}}, R) \rightarrow \text{Mod}(G_k, R)$$

and an equivalence of monoidal stable categories

$$\xi^*: \mathcal{S}\mathfrak{h}(k_{\text{ét}}, R) \rightarrow \mathcal{D}(\text{Mod}(G_k, R)).$$

2. The functor ξ^* induces an equivalence

$$\xi^*: \mathcal{S}\mathfrak{h}_{\text{lisse}}(k_{\acute{e}t}, R) \rightarrow \mathcal{D}_{\text{perf}}(\text{Mod}(G_k, R))$$

where $\mathcal{D}_{\text{perf}}(\text{Mod}(G_k, R))$ denotes the subcategory of the stable category $\mathcal{D}(\text{Mod}(G_k, R))$ made of those complexes which are perfect as complexes of R -modules.

3. We have

$$\mathcal{S}\mathfrak{h}(k_{\acute{e}t}, R) = \mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(k_{\acute{e}t}, R)$$

$$\text{Sh}(k_{\acute{e}t}, R) = \text{Ind Loc}(k, R).$$

4. The canonical functor

$$\mathcal{D}^b(\text{Rep}^A(G_k, R)) \rightarrow \mathcal{D}_{\text{perf}}(\text{Mod}(G_k, R))$$

is an equivalence.

Proof. The first assertion follows from [AGV73, VIII.2.1].

The second assertion follows from the first assertion and the description of lisse sheaves Proposition 3.1.5.9(1).

Furthermore, any complex of discrete R -linear G_k -module is a (filtered) colimit of complexes with perfect underlying complex of R -modules. Therefore, we get

$$\mathcal{S}\mathfrak{h}(k_{\acute{e}t}, R) = \mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(k_{\acute{e}t}, R)$$

and since the ordinary t-structure on the stable category $\mathcal{S}\mathfrak{h}(k_{\acute{e}t}, R)$ is compatible with filtered colimits, we also get

$$\text{Sh}(k_{\acute{e}t}, R) = \text{Ind Loc}(k, R).$$

The last assertion is a consequence of [BGI71, I.4.12]. We give an alternative proof for the reader's convenience. Denote by

$$\psi: \mathcal{D}^b(\text{Rep}^A(G_k, R)) \rightarrow \mathcal{D}_{\text{perf}}(\text{Mod}(G_k, R))$$

the canonical functor. It suffices to show that the canonical map

$$\text{Map}_{\mathcal{D}^b(\text{Rep}^A(G_k, R))}(M, N) \rightarrow \text{Map}_{\mathcal{D}(\text{Mod}(G_k, R))}(\psi(M), \psi(N))$$

is an equivalence for any bounded complexes M and N of Artin representations of G_k .

Notice that the objects of $\mathcal{D}^b(\text{Rep}^A(G_k, R))$ are dualizable. Therefore, we can assume that M is the R -module R endowed with the trivial G_k -action placed in degree 0. Furthermore, by dévissage, we can assume that N is placed in degree 0.

Therefore, it suffices to show that for all n , the map

$$\text{Ext}_{\text{Rep}^A(G_k, R)}^n(R, N) \rightarrow H^n(G_k, N)$$

is an equivalence.

Now, we have

$$H^n(G_k, N) = \text{colim } H^n(G_k/H, N^H)$$

where H runs through the set of normal subgroups of finite index of G_k .

Since N is an Artin representation, any normal subgroup H of G_k contains a normal subgroup H' such that the action of H' on N is trivial. Hence, the set of normal subgroups H of finite index of G_k such that $N^H = N$ is final in the set of all normal subgroups of finite index of G_k .

Therefore, we have $H^n(G_k, N) = \operatorname{colim} H^n(G_k/H, N)$ where H runs through the set of normal subgroups of finite index of G_k such that the action of H on N is trivial.

In addition, if H is a normal subgroup of G_k which acts trivially on N , the abelian group $H^n(G_k/H, N)$ coincides with the $\operatorname{Ext}_{\operatorname{Rep}^A(G_k/H, R)}^n(R, N)$ and can by [Sta23, 06XU] be computed as the abelian group of classes of sequences

$$0 \rightarrow R \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \rightarrow N \rightarrow 0$$

where each M_i is an R -linear representation of G_k/H and two sequences M and M' are equivalent if there exists a third sequence M'' and morphisms of sequences $M \leftarrow M'' \rightarrow M'$.

Furthermore, using [Sta23, 06XU] again, the abelian group $\operatorname{Ext}_{\operatorname{Rep}^A(G_k, R)}^n(R, N)$ can be computed as the abelian group of classes of sequences

$$0 \rightarrow R \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \rightarrow N \rightarrow 0$$

where each M_i is an Artin representation of G_k and two sequences M and M' are equivalent if there exists a third sequence M'' and morphisms of sequences $M \leftarrow M'' \rightarrow M'$.

Any such sequence in fact is a sequence of R -linear representation of G_k/H for some normal subgroup H and any equivalence $M \leftarrow M'' \rightarrow M'$ between two sequences M and M' comes from an equivalence of sequences of R -linear representation of G_k/H for some normal subgroup H . Therefore, the map

$$\operatorname{colim} \operatorname{Ext}_{\operatorname{Rep}^A(G_k/H, R)}^n(R, N) \rightarrow \operatorname{Ext}_{\operatorname{Rep}^A(G_k, R)}^n(R, N),$$

where H runs through the set of normal subgroups of finite index of G_k such that the action of H on N is trivial, is an equivalence. This finishes the proof. \square

Proposition 3.1.5.13 and corollary 3.1.5.12 yield the following proposition.

Proposition 3.1.5.14. *Let $f: \operatorname{Spec}(L) \rightarrow \operatorname{Spec}(K)$ be a field extension, let $\xi': \operatorname{Spec}(\bar{L}) \rightarrow \operatorname{Spec}(L)$ be the separable closure of L and let $\xi = f \circ \xi'$. Denote by $\phi: G_L \rightarrow G_K$ the group homomorphism induced by f .*

1. *The square*

$$\begin{array}{ccc} \operatorname{Sh}(K_{\text{ét}}, R) & \xrightarrow{f^*} & \operatorname{Sh}(L_{\text{ét}}, R) \\ \downarrow \xi^* & & \downarrow (\xi')^* \\ \operatorname{Mod}(G_K, R) & \xrightarrow{\phi^*} & \operatorname{Mod}(G_L, R) \end{array}$$

is commutative.

2. *Assume that L is a finite extension of K . Then, the square*

$$\begin{array}{ccc} \operatorname{Sh}(L_{\text{ét}}, R) & \xrightarrow{f^*} & \operatorname{Sh}(K_{\text{ét}}, R) \\ \downarrow (\xi')^* & & \downarrow \xi^* \\ \operatorname{Mod}(G_L, R) & \xrightarrow{\operatorname{Ind}_{G_L}^{G_K}} & \operatorname{Mod}(G_K, R) \end{array},$$

where $\pi' = \pi_1^{\text{ét}}(T, \xi')$ and $\pi = \pi_1^{\text{ét}}(S, \xi)$, is commutative.

Generators of lisse and Ind-lisse sheaves

Proposition 3.1.5.15. *Let S be a scheme and let R be a regular ring.*

1. *The thick stable subcategory of $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$ generated by the sheaves of the form $R_S(X)$ with X finite and étale over S is $\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)$.*
2. *The localizing stable subcategory of $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$ generated by the sheaves of the form $R_S(X)$ with X finite and étale is $\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R)$.*

Proof. By definition of Ind-lisse sheaves the second assertion follows from the first.

To prove the first assertion, first note that $\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)$ is a thick stable subcategory of $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$. Furthermore, if X is a finite étale S -scheme, the sheaf $R_S(X)$ is dualizable with dual itself and is therefore lisse. Hence, the thick stable subcategory of $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$ generated by the sheaves of the form $R_S(X)$ with X finite and étale over S is contained in $\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)$.

By Proposition 3.1.5.9, lisse sheaves are bounded with respect to the ordinary t-structure. Therefore, it suffices by dévissage to prove that any locally constant sheaf lies in the thick subcategory generated by the sheaves of the form $R_S(X)$ with X finite and étale over S which follows from the stronger Lemma 3.1.5.16 below. \square

Lemma 3.1.5.16. *Let S be a scheme and let R be a regular ring and let $\mathcal{C}(S, R)$ be the smallest subcategory of $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$ which contains the sheaves of the form $R_S(X)$ with X finite and étale over S and is closed under finite colimits, extensions and retracts. Then, the category $\text{Loc}_S(R)$ is contained in $\mathcal{C}(S, R)$.*

Proof. We can assume that the scheme S is connected. Let ξ be a geometric point of S and let M be a locally constant étale sheaf. By Proposition 3.1.5.10, the sheaf M corresponds to an Artin representation N of $\pi_1^{\text{ét}}(S, \xi)$ through the fiber functor associated to ξ . Let G be a finite quotient of $\pi_1^{\text{ét}}(S, \xi)$ through which N factors. Then, by Proposition 3.1.5.1 the complex $M \oplus M[1]$ of finite $R[G]$ -modules is quasi-isomorphic to a complex of the form

$$P = \cdots \rightarrow 0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0 \rightarrow \cdots$$

where each P_i is placed in degree $-i$ and is isomorphic to $R[G/H_1] \oplus \cdots \oplus R[G/H_m]$ for some subgroups H_1, \dots, H_m of G .

Furthermore, we have an exact functor

$$\mathcal{D}^b(\text{Rep}(G, R)) \rightarrow \mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$$

and if H is a subgroup of G , the finite set G/H is endowed with an action of $\pi_1^{\text{ét}}(S, \xi)$ and therefore corresponds to a finite locally constant sheaf on $S_{\text{ét}}$, and therefore to a finite étale S -scheme X . The image of $R[G/H]$ through the functor above is then $R_S(X)$.

Thus, the object $M \oplus M[1]$, and therefore the object M itself, belong to the smallest subcategory which contains the sheaves of the form $R_S(X)$ with X finite and étale over S and is closed under finite colimits, extensions and retracts. \square

Another consequence of Lemma 3.1.5.16 is the following statement.

Proposition 3.1.5.17. *Let S be a scheme and let R be a regular ring.*

1. The smallest subcategory of $\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)$ which contains the sheaves of the form $R_S(X)$ with X finite and étale over S and is closed under finite colimits, extensions and retracts is the subcategory $\mathcal{S}\mathcal{h}_{\text{lisse}}^{\leq 0}(S, R)$ of t -non-positive objects of the ordinary t -structure on $\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)$.
2. The ordinary t -structure on the stable category $\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R)$ is generated by the sheaves of the form $R_S(X)$ with X finite and étale over S .

Proof. By dévissage, the first assertion follows from Lemma [3.1.5.16](#).

Let M be a t -non-positive Ind-lisse étale sheaf. Write M as a filtered colimit of lisse étale sheaves M_i . Since the ordinary t -structure is compatible with filtered colimits, we have

$$M = \tau_{\leq 0}(M) = \tau_{\leq 0}(\text{colim } M_i) = \text{colim } \tau_{\leq 0}(M_i).$$

Using the first assertion, each sheaf $\tau_{\leq 0}(M_i)$ belongs to the smallest subcategory of the stable category $\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R)$ which contains the sheaves of the form $R_S(X)$ with X finite and étale over S and is closed under small colimits and extensions. Therefore, so does the sheaf M . This finishes the proof. \square

Proposition 3.1.5.18. *Let S be a scheme and let R be a regular ring. Then, with the notations of Section [3.1.1](#), the canonical functor*

$$\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)_{\mathfrak{p}} \rightarrow \mathcal{S}\mathcal{h}_{\text{lisse}}(S, R \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})$$

is an equivalence.

Proof. The proof is the same as the proof of Proposition [3.1.2.4](#). \square

3.1.6 Reminders on the ℓ -adic theory

In this section, we recall the ℓ -adic formalism. We also recall the main definitions and the main results of the second chapter.

Let S be a scheme and ℓ be a prime number invertible on S . Let Λ be a finite extension of \mathbb{Q}_{ℓ} or the ring of integers of a finite extension of \mathbb{Q}_{ℓ} . There are several definitions of the *derived category of constructible ℓ -adic sheaves* $\mathcal{D}_c^b(S, \Lambda)$. The first definition was given in [\[BBDG18\]](#) for S of finite type over a field. In [\[Eke07\]](#), [\[CD16, 7.2\]](#), [\[BS15\]](#) and [\[LYF14, XIII.4\]](#), this definition was extended to all quasi-compact quasi-separated schemes.

The definition that was chosen in the second chapter is the definition of [\[BS15\]](#). If S is a scheme, denote by $S_{\text{proét}}$ the proétale site of S introduced in [\[BS15, 4\]](#). Any totally disconnected topological space M gives rise to a constant sheaf on $S_{\text{proét}}$ with value M denoted by M_S . The derived category of ℓ -adic sheaves $\mathcal{D}(S, \Lambda)$ is defined as the derived category of the category $\text{Sh}(S, \Lambda)$ of sheaves of Λ_S modules on the site $S_{\text{proét}}$.

ℓ -adic realization

Let S be a scheme, let R be a localization of the ring of integers of a number field K , let v be a non-archimedean valuation on the field K and let ℓ be the prime number such that v extends the ℓ -adic valuation. Assuming that the prime number ℓ is invertible on S , Cisinski and Déglise

defined an ℓ -adic realization functor, or in this context more accurately a v -adic realization functor

$$\rho_v : \mathcal{DM}_{\acute{e}t,c}(S, R) \rightarrow \mathcal{D}_c^b(S, R_v).$$

This functor preserves constructible objects and respects the six functors formalism.

To describe this v -adic realization functor, we must use the description of ℓ -adic complexes given in [CD16, 7.2]. Assume first R is a discrete valuation ring with valuation v and let λ be a uniformizer of R .

Recall that an étale motive M is λ -complete if the canonical map

$$M \rightarrow \lim M/\lambda^n$$

is an equivalence. Cisinski and Déglise defined the derived category of λ -adic sheaves as the full subcategory $\mathcal{DM}_{\acute{e}t}(S, \hat{R}_\lambda)$ of the stable category $\mathcal{DM}_{\acute{e}t}(S, R)$ made of those motives which are λ -complete.

We can define the subcategory $\mathcal{Sh}(S_{\acute{e}t}, R)_\lambda$ of λ -complete objects in the stable category $\mathcal{Sh}(S_{\acute{e}t}, R)$ in a similar fashion. The proof of [CD16, 7.2.21] implies that the map ρ_l of Section 3.1.2 induces an equivalence

$$\mathcal{Sh}_\lambda(S, R) \rightarrow \mathcal{DM}_{\acute{e}t}(S, \hat{R}_\lambda)$$

with inverse the functor $\rho^!$.

Cisinski and Déglise then define the λ -adic realization functor

$$\rho_\lambda^{\text{CD}} : \mathcal{DM}_{\acute{e}t}(S, R) \rightarrow \mathcal{DM}_{\acute{e}t}(S, \hat{R}_\lambda)$$

as the functor that sends a motive M to the motive $\lim M/\lambda^n$. Finally, they defined the subcategory $\mathcal{DM}_{\acute{e}t,c}(S, \hat{R}_\lambda)$ of *constructible λ -complete étale motives* as the thick subcategory of the stable category $\mathcal{DM}_{\acute{e}t}(S, \hat{R}_\lambda)$ generated by the essential image through the functor ρ_λ^{CD} of the subcategory of constructible motives.

We now explain the link between the models of [CD16, 7.2] and [BS15] for ℓ -adic complexes. On the one hand, we can define the subcategory $\mathcal{Sh}_{\lambda,c}(S, R)$ of *constructible λ -complete étale sheaves* as the essential image of the stable subcategory $\mathcal{DM}_{\acute{e}t,c}(S, \hat{R}_\lambda)$ of the stable category $\mathcal{DM}_{\acute{e}t}(S, \hat{R}_\lambda)$ through the functor $\rho^!$.

On the other hand, let

$$\nu : S_{\text{proét}} \rightarrow S_{\acute{e}t}$$

be the canonical morphism of sites. Then, unwinding the definitions of the maps described in the proofs of [CD16, 7.2.21] and [BS15, 5.5], the map

$$\Psi : \mathcal{Sh}(S_{\acute{e}t}, R)_\lambda \rightarrow \mathcal{D}(S, R_v)$$

which sends the étale sheaf M to the proétale sheaf $\lim \nu^*(M)/\lambda^n$ induces an equivalence

$$\mathcal{Sh}_{\lambda,c}(S, R) \rightarrow \mathcal{D}_c^b(S, R_v).$$

We can therefore define a v -adic realization functor

$$\rho_v = \Psi \circ \rho^! \circ \rho_\lambda^{\text{CD}} : \mathcal{DM}_{\acute{e}t}(S, R) \rightarrow \mathcal{D}(S, R_v).$$

In the case when v is non-negative on R , we can define the v -adic realization functor by pre-composing the v -adic realization functor

$$\rho_v: \mathcal{DM}_{\text{ét}}(S, R_{(v)}) \rightarrow \mathcal{D}(S, R_v)$$

with the change of coefficients from R to $R_{(v)}$.

In the case when v is not non-negative R , we can only define the realization functor on the subcategory of constructible motives. Notice that R_v is the completion of K with respect to the valuation v . Let

$$A = \{x \in K \mid v(x) \geq 0\}.$$

The ring A is a discrete valuation ring. Let λ be a uniformizer of A . We define the v -adic realization functor ρ_v as the composition

$$\mathcal{DM}_{\text{ét},c}(S, R) \rightarrow \mathcal{DM}_{\text{ét},c}(S, K) \xrightarrow{\rho_{\lambda \otimes_A K}} \mathcal{D}_c^b(S, R_v),$$

where, with the notations of Section 3.1.1, Proposition 3.1.2.4 ensures that the functors

$$\mathcal{DM}_{\text{ét},c}(S, A)_{(0)} \rightarrow \mathcal{DM}_{\text{ét},c}(S, K)$$

and where [CD16, 7.2.23] ensures that the functor

$$\mathcal{D}_c^b(S, A_v)_{(0)} \rightarrow \mathcal{D}_c^b(S, K_v)$$

is an equivalence.

Ordinary t-structure

In the rest of this section, we let S be a scheme, we let ℓ be a prime number invertible on S and we let Λ be a finite extension of \mathbb{Q}_ℓ or the ring of integers of a finite extension of \mathbb{Q}_ℓ .

The ordinary t-structure of the stable category $\mathcal{D}(S, \Lambda)$ induces a t-structure on the subcategory $\mathcal{D}_c^b(S, \Lambda)$. The heart of this t-structure is the abelian category $\text{Sh}_c(S, \Lambda)$ *constructible ℓ -adic sheaves*.

Following [HRS20], the ordinary t-structure of the stable category $\mathcal{D}(S, \Lambda)$ also induces a t-structure on the subcategory $\mathcal{D}_{\text{lisse}}(S, \Lambda)$ of dualizable objects of the stable category $\mathcal{D}(S, \Lambda)$. The heart of this t-structure is the abelian category $\text{Loc}_S(\Lambda)$ of locally (with respect to the proétale topology) constant sheaves with finitely presented fibers.

We can give a more concrete description of the abelian category $\text{Loc}_S(\Lambda)$. The topology on Λ induces a topology on every Λ -module of finite presentation. Letting π be a topological group, a *continuous representation* of π with coefficients in Λ is a Λ -module of finite presentation endowed with a continuous action of π . We denote by $\text{Rep}(\pi, \Lambda)$ the abelian category of continuous representation of π with coefficients in Λ . Then, if the scheme S is connected and if ξ is a geometric point of S , Proposition 2.1.5.4 implies that the fiber functor ξ^* associated to ξ induces an equivalence

$$\text{Loc}_S(\Lambda) \rightarrow \text{Rep}(\pi_1^{\text{proét}}(S, \xi), \Lambda)$$

where $\pi_1^{\text{proét}}(S, \xi)$ is the proétale fundamental group of Bhatt and Scholze defined in [BS15, 7].

Perverse t-structure

Assume that the scheme S is endowed with a dimension function δ . If x is a point of S , denote by $i_x: \{x\} \rightarrow S$ the inclusion. We define

- the full subcategory ${}^{p \leq 0} \mathcal{D}_c^b(S, \Lambda)$ as the subcategory made of those ℓ -adic complexes M such that for any point x of S , we have

$$i_x^* M \leq -\delta(x)$$

with respect to the ordinary t-structure of $\mathcal{D}_c^b(k(x), \Lambda)$.

- the full subcategory ${}^{p \geq 0} \mathcal{D}_c^b(S, \Lambda)$ as the subcategory made of those bounded below ℓ -adic complexes M such that for any point x of S , we have

$$i_x^* M \geq -\delta(x)$$

with respect to the ordinary t-structure of $\mathcal{D}_c^b(k(x), \Lambda)$.

Gabber's proof in [Gab08] can be used to prove that δ gives rise to a perverse t-structure on $\mathcal{D}_c^b(S, \Lambda)$ which we call the *perverse t-structure*. The heart of this t-structure is the abelian category of *perverse sheaves* denoted by $\text{Perv}(S, \Lambda)$.

We denote with the small letter p the notions related to the perverse t-structure on $\mathcal{D}_c^b(S, \Lambda)$. For instance, we denote by ${}^p \tau_{\leq 0}$ is the left truncation with respect to the perverse t-structure.

Finally, if the scheme S is regular, the perverse t-structure induces a t-structure on the stable subcategory $\mathcal{D}_{\text{lisse}}(S, \Lambda)$ which coincides with the ordinary t-structure shifted by $\delta(S)$.

Artin ℓ -adic complexes

If X is a scheme, recall that Λ_X denotes the constant sheaf with value Λ on X . The following ℓ -adic analogs of Artin motives and smooth Artin motives are defined in Definition 2.2.1.3.

- The stable category $\mathcal{D}^A(S, \Lambda)$ of *Artin ℓ -adic complexes* is the thick stable subcategory of $\mathcal{D}_c^b(S, \Lambda)$ generated by the complexes of the form $f_* \mathbb{Z}_{\ell, X}$ with $f: X \rightarrow S$ finite.
- The stable category $\mathcal{D}^{smA}(S, \Lambda)$ of *smooth Artin ℓ -adic complexes* over S with coefficients Λ is the thick stable subcategory of $\mathcal{D}_c^b(S, \Lambda)$ generated by the complexes of the form $f_* \Lambda_X$ with $f: X \rightarrow S$ étale and finite.
- The stable category $\mathcal{D}_{\text{Ind}}^A(S, \Lambda)$ is the localizing subcategory of $\mathcal{D}(S, \Lambda)$ generated by the complexes of the form $f_* \mathbb{Z}_{\ell, X}$ with $f: X \rightarrow S$ finite.

Let R be a localization of the ring of integers of a number field, and let v be a valuation on the fraction field of R such that $v(\ell)$ is positive. Notice that the ℓ -adic realization functor induces functors

$$\mathcal{DM}_{\text{ét}}^A(S, R) \rightarrow \mathcal{D}_{\text{Ind}}^A(S, R_v)$$

$$\mathcal{DM}_{\text{ét}, c}^A(S, R) \rightarrow \mathcal{D}^A(S, R_v)$$

$$\mathcal{DM}_{\text{ét}, c}^{smA}(S, R) \rightarrow \mathcal{D}^{smA}(S, R_v).$$

Artin ℓ -adic sheaves

The ordinary t-structure of the stable category $\mathcal{D}_c^b(S, \Lambda)$ induces a t-structure on the stable subcategory $\mathcal{D}^{smA}(S, \Lambda)$ and on the stable subcategory $\mathcal{D}^A(S, \Lambda)$. We can describe the heart of those t-structures which are weak Serre subcategories of $\text{Sh}_c(S, \Lambda)$.

Letting π be a topological group, the category of representation of *Artin origin* of π , denoted by $\text{Rep}^A(\pi, \Lambda)^*$, is the smallest Serre subcategory of $\text{Rep}(\pi, \Lambda)$ which contains Artin representations. It is made of those objects of $\text{Rep}(\pi, \Lambda)$ obtained as successive extensions of Artin representations. The abelian category $\text{Sh}^{smA}(S, \Lambda)$ of *smooth Artin ℓ -adic sheaves* is then defined as the subcategory of the abelian category $\text{Loc}_S(\Lambda)$ made of those sheaves whose image through the fiber functor associated to any geometric point ξ of S in a representation of Artin origin of the topological group $\pi_1^{\text{proét}}(S, \xi)$. When the scheme S is connected, the fiber functor associated to any geometric point ξ of S induces an equivalence

$$\text{Sh}^{smA}(S, \Lambda) \rightarrow \text{Rep}^A(\pi_1^{\text{proét}}(S, \xi), \Lambda)^*.$$

With those definitions, Proposition [2.2.2.1](#) ensures that the stable category $\mathcal{D}^{smA}(S, \Lambda)$ is the subcategory of the stable category $\mathcal{D}_c^b(S, \Lambda)$ made of those objects whose (ordinary) cohomology sheaves are smooth Artin and that therefore, the ordinary t-structure on the stable category $\mathcal{D}_c^b(S, \Lambda)$ induces a t-structure on the stable subcategory $\mathcal{D}^{smA}(S, \Lambda)$ whose heart is the abelian category $\text{Sh}^{smA}(S, \Lambda)$.

Finally, the abelian category $\text{Sh}^A(S, \Lambda)$ of *Artin ℓ -adic sheaves* is defined as the subcategory of the abelian category $\text{Sh}(S, \Lambda)$ made of those objects which are locally constant on every stratum of some stratification of S . Corollary [2.2.4.3](#) implies that the stable category $\mathcal{D}^A(S, \Lambda)$ is then the subcategory of the stable category $\mathcal{D}_c^b(S, \Lambda)$ made of those objects whose cohomology sheaves are Artin. Therefore, the ordinary t-structure on the stable category $\mathcal{D}_c^b(S, \Lambda)$ induces a t-structure on the stable subcategory $\mathcal{D}^A(S, \Lambda)$ whose heart is the abelian category $\text{Sh}^A(S, \Lambda)$.

Artin perverse sheaves

First, if S is connected and regular, then, by Proposition [2.3.1.1](#), the perverse t-structure on the stable category $\mathcal{D}_c^b(S, \Lambda)$ induces t-structures on the stable subcategory $\mathcal{D}^{smA}(S, \Lambda)$ and, letting δ be a dimension function on S , this t-structure coincides with the ordinary t-structure shifted by $\delta(S)$.

If S is an excellent scheme of dimension 0, 1 or 2, or a scheme of dimension 3 such that all the residue fields of its closed points are real closed or separably closed, then, by Theorem [2.3.3.4](#) and Proposition [2.3.4.1](#), the perverse t-structure on $\mathcal{D}_c^b(S, \Lambda)$ induces a t-structure on $\mathcal{D}^A(S, \Lambda)$. We denote its heart by $\text{Perv}^A(S, \Lambda)$ and we call it the abelian category of *Artin perverse sheaves*.

Perverse homotopy t-structure

If S excellent, letting δ be a dimension function on S , we define the *perverse homotopy t-structure* on $\mathcal{D}_{\text{Ind}}^A(S, \Lambda)$ as the t-structure generated by the complexes of the form $f_*\Lambda_X[\delta(X)]$ with $f: X \rightarrow S$ finite.

If S is connected and regular. Then, by Proposition [2.4.3.1](#), the perverse homotopy t-structure on the stable category $\mathcal{D}_{\text{Ind}}^A(S, \Lambda)$ both induce t-structures on the stable subcategory $\mathcal{D}^{smA}(S, \Lambda)$. In addition, this t-structure coincides with the perverse t-structure. Moreover, Proposition [2.4.5.1](#) yields the following statements.

- Let M be an object of $\mathcal{D}^A(S, \Lambda)$, then M is perverse homotopy t-negative if and only if it is perverse t-negative.
- If the perverse t-structure on $\mathcal{D}_c^b(S, \Lambda)$ induces a t-structure on $\mathcal{D}^A(S, \Lambda)$, then the perverse homotopy t-structure on $\mathcal{D}_{\text{ind}}^A(S, \Lambda)$ induces a t-structure on $\mathcal{D}^A(S, \Lambda)$. When this is the case, those two t-structures coincide.

Therefore, when the perverse homotopy t-structure induces a t-structure on $\mathcal{D}^A(S, \Lambda)$, the induced t-structure is the best possible approximation of the perverse t-structure.

Finally, if S is a scheme of finite type over \mathbb{F}_p and if Λ is a finite extension of \mathbb{Q}_ℓ , then Proposition 2.4.7.1 ensures that the perverse homotopy t-structure on $\mathcal{D}_{\text{ind}}^A(S, \Lambda)$ induces a t-structure on $\mathcal{D}^A(S, \Lambda)$. We denote its heart by $\text{Perv}^A(S, \Lambda)^\#$ and we call it the abelian category of *Artin homotopy perverse sheaves*.

3.1.7 Some computations of étale cohomology and étale motivic cohomology

Recall that the étale motivic cohomology groups of a scheme X are the groups

$$H_{\mathcal{M}, \text{ét}}^{n,m}(X, R) = \text{Hom}_{\text{DM}_{\text{ét}}(X, R)}(\mathbb{1}_X, \mathbb{1}_X[n](m)).$$

We will need the following computations. The first one is a slight generalization of a [Ayo14c, 11.1].

Lemma 3.1.7.1. *Let S be a scheme, let R be a good ring and let $n < 0$ and $m \leq 0$ be integers. Then,*

$$H_{\mathcal{M}, \text{ét}}^{n,m}(S, R) = 0.$$

Proof. By Lemma 3.1.2.5, we are reduced to the case where R is a $\mathbb{Z}/p\mathbb{Z}$ -algebra or a \mathbb{Q} -algebra.

In the first case, by the rigidity theorem [BH21, 3.2], the category $\mathcal{DM}_{\text{ét}}(S, R)$ is equivalent to the category $\mathcal{D}(S[1/p]_{\text{ét}}, R)$ and we have

$$H_{\mathcal{M}, \text{ét}}^{n,m}(X, R) = H_{\text{ét}}^n(X, R(m))$$

which vanishes since the integer n is negative. In the case when the ring R is a \mathbb{Q} -algebra, Ayoub’s result [Ayo14c, 11.1] applies. \square

We will also need to compute étale cohomology with coefficients invertible on the base scheme and with constant coefficients. First recall:

Proposition 3.1.7.2. *(Deninger) Let X be a geometrically unibranch scheme which is the disjoint union of its irreducible components. Then, we have*

$$H_{\text{ét}}^i(X, \mathbb{Q}) = \begin{cases} \mathbb{Q}^{\pi_0(X)} & \text{if } i = 0. \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By [Den88, 2.1], this result is true when the scheme X is normal. If X is a geometrically unibranch scheme which is the disjoint union of its irreducible components, the normalization map $\nu: X^\nu \rightarrow X$ is a universal homeomorphism by [Sta23, 0GIQ] and therefore, the lemma follows from the topological invariance of the small étale site [Gro67, 18.1.2]. \square

If X is a scheme and ℓ is a prime number which is invertible on X , write $b_n(X, \ell)$ the dimension (over \mathbb{Q}_ℓ) of the n -th ℓ -adic cohomology group of X . Let $T_n(X, \ell)$ be the torsion subgroup of the group $H_{\text{ét}}^n(X, \mathbb{Z}_\ell)$. Finally denote by $\mathbb{Z}(\ell^\infty)$ the Prüfer ℓ -group. Then we have the following result.

Proposition 3.1.7.3. *Let X be a geometrically unibranch scheme which is the disjoint union of its irreducible components, let R be a ring which is flat over \mathbb{Z} and in which all the residue characteristics of X are invertible and let $e(X)$ be the set of residue characteristic exponents of X . Then,*

$$H_{\text{ét}}^n(X, R) = \begin{cases} R^{\pi_0(X)} & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ R \otimes_{\mathbb{Z}} \left[\bigoplus_{\ell \notin e(X)} \left(\mathbb{Z}(\ell^\infty)^{b_{n-1}(X, \ell)} \oplus T_{n-1}(X, \ell) \right) \right] & \text{if } n \geq 2 \end{cases}$$

Proof. As R is flat over \mathbb{Z} , we may suppose that R is the subring

$$\mathbb{Z}[1/e(X)] = \left\{ \frac{n}{p_1^{m_1} \cdots p_r^{m_r}} \mid p_i \in e(X) \right\}$$

of \mathbb{Q} . We have an exact sequence

$$0 \rightarrow \mathbb{Z}[1/e(X)] \rightarrow \mathbb{Q} \rightarrow \bigoplus_{\ell \notin e(X)} \mathbb{Z}(\ell^\infty) \rightarrow 0.$$

Hence, by Proposition [3.1.7.2](#), we get

$$H_{\text{ét}}^n(X, \mathbb{Z}[1/e(X)]) = \begin{cases} \mathbb{Z}[1/e(X)]^{\pi_0(X)} & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ \bigoplus_{\ell \notin e(X)} \text{colim}_N H_{\text{ét}}^{n-1}(X, \mathbb{Z}/\ell^N \mathbb{Z}) & \text{if } n \geq 2 \end{cases}$$

This yields the result. □

Remark 3.1.7.4. *If a complete theory of abelian motives was available, all the $b_i(X, \ell)$ would coincide. Thus, for $n \geq 2$, we would have*

$$H_{\text{ét}}^n(X, R) = R \otimes_{\mathbb{Z}} \left[\mathbb{Q}/\mathbb{Z}[1/e(X)]^{b_i(X)} \oplus \bigoplus_{\ell \notin e(X)} T_{n-1}(X, \ell) \right]$$

Notice that, using Artin's Theorem [\[AGV73, XI.4.4\]](#) or Deligne's work on Weil's conjectures (see [\[Del80\]](#)) all the $b_i(X, \ell)$ coincide if X is defined over an algebraically closed field of characteristic 0 or over $\overline{\mathbb{F}}_p$.

Finally, if X is a curve over any separably closed field k , using [\[AGV73, IX.4.7\]](#), we have

$$H_{\text{ét}}^2(X, R) = (R \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[1/p])^{2g}$$

$$H_{\text{ét}}^3(X, R) = R \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[1/p]$$

and $H_{\text{ét}}^n(X, R) = 0$ for $n \geq 4$.

3.2 Description of smooth Artin motives and applications

3.2.1 Equivalence between smooth Artin motives and lisse étale sheaves over a regular basis

In this paragraph, we give an explicit description of smooth Artin étale motives over a regular basis. This description will not work over a non-regular basis, even when it is normal (see Remark [3.2.1.2](#) below).

Theorem 3.2.1.1. *Let R be a regular good ring and let S be a regular scheme. Assume that the residue characteristic exponents of S are invertible in R .*

The functor $\rho_!$ of Proposition [3.1.4.3](#) induces monoidal equivalences

$$\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R) \longrightarrow \mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R),$$

$$\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R) \longrightarrow \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R).$$

Proof. We can assume the scheme S to be connected. If V is an étale cover of S , the functor $\rho_!$ sends the sheaf $R_S(V)$ to the motive $M_S(V)$. Using Proposition [3.1.5.15](#), the fact that the ring R is regular and the fact that $\rho_!$ commutes with colimits and finite limits, it induces exact functors

$$\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R) \longrightarrow \mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$$

$$\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R) \longrightarrow \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R).$$

Assuming that the second functor is an equivalence, the first functor is also an equivalence since $\mathcal{S}\mathcal{h}_{\text{lisse}}(S, R)$ is the thick stable subcategory of $\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R)$ generated by the sheaves of the form $R_S(V)$ when V runs through the set of étale covers of S and $\mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R)$ is the thick stable subcategory of $\mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$ generated by the motives of the form $M_S(V)$ when V runs through the set of étale covers of S .

We now prove that the functor

$$\mathcal{S}\mathcal{h}_{\text{Ind lisse}}(S, R) \longrightarrow \mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$$

induced by $\rho_!$ is an equivalence. Assuming that the functor $\rho_!$ is fully faithful, its essential surjectivity follows from the fact that $\mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$ is generated as a localizing subcategory of itself by the essential image of the functor $\rho_!$. Hence, it suffices to prove that the functor above is fully faithful.

Let $\rho^!$ be the right adjoint of $\rho_!$. The full faithfulness of $\rho_!$ holds for any good ring. It is equivalent to the fact that the ordinary transformation $\alpha: Id \rightarrow \rho^!\rho_!$ is an equivalence. Using Lemma [3.1.2.5](#) and proposition [3.1.2.3](#), we may assume that the ring R is either of ℓ -torsion where ℓ is invertible in S , or a \mathbb{Q} -algebra. The case where R is of ℓ -torsion follows from [\[BH21, 3.2\]](#). Therefore, we can assume that the ring R is a \mathbb{Q} -algebra.

The family of functors $\text{Map}(R_S(W), -)$ when W runs through the set of étale S -schemes is conservative. Therefore, it suffices to show that for any Ind-lisse étale sheaf M and any étale S -scheme W , the canonical map

$$\text{Map}(R_S(W), M) \rightarrow \text{Map}(R_S(W), \rho^!\rho_!M)$$

is an equivalence. This is equivalent to the fact that the canonical map

$$\mathrm{Map}(R_S(W), M) \rightarrow \mathrm{Map}(M_S(W), \rho_! M)$$

is an equivalence.

Since R is a \mathbb{Q} -algebra, by [CD16, 1.1.4, 1.1.9] the sheaf $R_S(W)$ is a compact object of $\mathcal{S}\mathcal{H}(S_{\text{ét}}, R)$ and the motive $M_S(W)$ is a compact object of $\mathcal{D}\mathcal{M}_{\text{ét}}(S, R)$. As the stable category $\mathcal{S}\mathcal{H}_{\text{Ind lisse}}(S, R)$ generated as a localizing subcategory of itself by objects of the form $R_S(V)$ where V is finite étale over S , it therefore suffices to show that for any étale finite scheme V over S and any étale scheme W over S , the map

$$\rho_{W,V}: \mathrm{Map}(R_S(W), R_S(V)) \rightarrow \mathrm{Map}(M_S(W), M_S(V))$$

is an equivalence.

Recall now, that the étale sheaf $R_S(V)$ is dualizable with itself as a dual. As the functor $\rho_!$ is monoidal, this implies that the motive $M_S(V)$ is also dualizable with itself as a dual. Hence, the map $\rho_{W,V}$ is an equivalence if and only if the map $\rho_{W \times_S V, S}$ is an equivalence. As

$$\rho_{X,S} = \bigoplus_{T \in \pi_0(X)} \rho_{T,S},$$

it suffices to show that the maps $\rho_{T,S}$ are equivalences for T étale over S and connected. But if T is étale over S , the map $\rho_{T,S}$ is an equivalence if and only if the canonical map

$$\psi_T: \mathrm{Map}(R_T(T), R_T(T)) \rightarrow \mathrm{Map}(\mathbb{1}_T, \mathbb{1}_T)$$

is an equivalence.

Such a T is irreducible and geometrically unibranch. Therefore, using Proposition [3.1.7.2], we have

$$\begin{aligned} \pi_n(\mathrm{Map}(R_T(T), R_T(T))) &= H_{\text{ét}}^n(T, R) \\ &= \begin{cases} R & \text{if } n = 0. \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Moreover, as such a T is also regular (and connected), using [Pep19b, B.5],

$$\begin{aligned} \pi_n(\mathrm{Map}(\mathbb{1}_T, \mathbb{1}_T)) &= H_{\mathcal{M}, \text{ét}}^{n,0}(T, R) \\ &= \begin{cases} R & \text{if } n = 0. \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, it is sufficient to show that the map $\pi_0(\psi_T)$ is non-zero in order to show that it is an equivalence. But $\pi_0(\psi_T)$ is the natural map

$$\mathrm{Hom}(R_T(T), R_T(T)) \rightarrow \mathrm{Hom}(\mathbb{1}_T, \mathbb{1}_T)$$

and this map sends the identity of $R_T(T)$ to the identity of $\mathbb{1}_T$. □

Remark 3.2.1.2. *Let R be a regular good ring and let S be a geometrically unibranch scheme which is the disjoint union of its irreducible components. Assume that the residue characteristic exponents of S are invertible in R .*

Then, the functor $\theta_! = L_{\mathbb{A}^1} \rho_{\#}$ (see Section 3.1.2) induces monoidal equivalences

$$\begin{aligned} \mathcal{S}h_{\text{Ind lisse}}(S, R) &\longrightarrow \mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}, \text{sm}A}(S, R) \\ \mathcal{S}h_{\text{lisse}}(S, R) &\longrightarrow \mathcal{D}\mathcal{A}_{\text{ét}, c}^{\text{eff}, \text{sm}A}(S, R) \end{aligned}$$

Indeed, for the same reasons as in the proof of Theorem 3.2.1.1, we reduce to showing that the natural maps

$$\text{Map}(R_S(T), R_S(S)) \rightarrow \text{Map}(L_{\mathbb{A}^1} R_S(T), L_{\mathbb{A}^1} R_S(S))$$

are equivalences for T connected and étale over S and R a \mathbb{Q} -algebra (beware that on the right hand side, $R_S(T)$ and $R_S(S)$ are seen as sheaves on Sm_S). This follows from Lemma 3.2.1.3 below.

Hence, if S is regular, the functor

$$\Sigma^\infty : \mathcal{D}\mathcal{A}_{\text{ét}}^{\text{eff}, \text{sm}A}(S, R) \rightarrow \mathcal{D}\mathcal{M}_{\text{ét}}^{\text{sm}A}(S, R)$$

is an equivalence. This is not true in general if S is normal but non-regular since [Pep19b, B.7] provides an example where étale cohomology in degree n differs from motivic cohomology in degree $(n, 0)$ for some integer n . In particular Theorem 3.2.1.1 is false when the scheme S is not regular.

Lemma 3.2.1.3. *Let S be a scheme, let X be a geometrically unibranch smooth S -scheme which is the disjoint union of its irreducible components and let R be a regular good ring, where all residue characteristics of X are invertible. Then, the étale sheaf $R_S(X)$ on Sm_S is \mathbb{A}^1 -local.*

Proof. The functor $L_{\mathbb{A}^1}$ commutes with the functors $- \otimes_{\mathbb{Z}} \mathbb{Q}$ and $- \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ by Proposition 3.1.2.3. Using [AGV73, XV.2.2] and Proposition 3.1.7.2, The lemma follows Lemma 3.1.2.5 applied to the canonical map $R_S(X) \rightarrow L_{\mathbb{A}^1} R_S(X)$. \square

We now state some consequences of Theorem 3.2.1.1.

Corollary 3.2.1.4. *Let R be a regular good ring and let S be a connected regular scheme. Assume that any residue characteristic exponent of S is invertible in R . Then, the canonical map*

$$H_{\text{ét}}^n(S, R) \rightarrow H_{\mathcal{M}, \text{ét}}^{n, 0}(S, R)$$

is an isomorphism.

Combining Theorem 3.2.1.1 with Propositions 3.1.5.9 and 3.1.5.10 yields the following statement.

Corollary 3.2.1.5. *Let R be a regular good ring and let S be a regular scheme. Assume that the residue characteristic exponents of S are invertible in R .*

1. *The category $\mathcal{D}\mathcal{M}_{\text{ét}}^{\text{sm}A}(S, R)$ is endowed with a non-degenerate t -structure such that the functor*

$$\mathcal{S}h_{\text{Ind lisse}}(S, R) \longrightarrow \mathcal{D}\mathcal{M}_{\text{ét}}^{\text{sm}A}(S, R)$$

induced by the functor $\rho_!$ of Proposition 3.1.4.3 is t -exact when the left hand side is endowed with the ordinary t -structure. We still call this t -structure the ordinary t -structure. Its heart is equivalent to the abelian category $\text{Ind Loc}(S, R)$.

2. The category $\mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R)$ is endowed with a non-degenerate t -structure such that the functor

$$\mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R) \longrightarrow \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R)$$

induced by the functor $\rho_!$ of Proposition [3.1.4.3](#) is t -exact when the left hand side is endowed with the ordinary t -structure. We still call this t -structure the ordinary t -structure. Its heart is equivalent to the abelian category $\text{Loc}_S(R)$. If S is connected and ξ is a geometric point of S , it is also equivalent to the abelian category $\text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R)$.

If n is an integer, we denote by H^n the n -th cohomology functor with respect to the ordinary t -structure on smooth Artin motives.

Corollary 3.2.1.6. *Let R be a regular good ring and let S be a connected regular scheme of characteristic exponent p . Then, the functor $\rho_!$ of Proposition [3.1.4.3](#) induces monoidal equivalences*

$$\begin{aligned} \mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(S, R[1/p]) &\longrightarrow \mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R) \\ \mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R[1/p]) &\longrightarrow \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R). \end{aligned}$$

Corollary 3.2.1.7. *Let k be a field of characteristic exponent p . Let R be a regular good ring. Then, the functor $\rho_!$ induces monoidal equivalences*

$$\begin{aligned} \mathcal{D}(\text{Mod}(G_k, R[1/p])) &\simeq \mathcal{S}\mathfrak{h}(k_{\text{ét}}, R[1/p]) \longrightarrow \mathcal{DM}_{\text{ét}}^A(k, R) \\ \mathcal{D}^b(\text{Rep}^A(G_k, R[1/p])) &\simeq \mathcal{S}\mathfrak{h}_{\text{lisse}}(k, R[1/p]) \longrightarrow \mathcal{DM}_{\text{ét},c}^A(k, R). \end{aligned}$$

Proof. Combine Corollary [3.2.1.6](#) with Proposition [3.1.5.13](#). □

Remark 3.2.1.8. *It is not true in general that the canonical functor*

$$\mathcal{D}^b(\text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R)) \rightarrow \mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R)$$

is an equivalence. Indeed, letting k be a separably closed field of characteristic 0, the étale fundamental group of \mathbb{P}_k^1 is trivial. Therefore, the abelian category of its \mathbb{Z} -linear Artin representations is the category of finitely generated abelian groups, which for any geometric point ξ yields

$$\text{Ext}_{\text{Rep}^A(\pi_1^{\text{ét}}(\mathbb{P}_k^1, \xi), \mathbb{Z})}^3(\mathbb{Z}, \mathbb{Z}) = 0.$$

On the other hand, we have

$$H_{\text{ét}}^3(\mathbb{P}_k^1, \mathbb{Z}) = \text{Ext}_{\text{Sh}_{\text{ét}}(\mathbb{P}_k^1, \mathbb{Z})}^3(\mathbb{Z}, \mathbb{Z})$$

and this group is isomorphic to \mathbb{Q}/\mathbb{Z} by Remark [3.1.7.4](#). Thus, the above functor cannot be an equivalence.

Finally, when R is a \mathbb{Q} -algebra, it was however shown in [[CDN22](#), 2.3.4] that the above functor is an equivalence.

We end this section with a description of the ℓ -adic realization functor on smooth Artin motives over a regular scheme.

Proposition 3.2.1.9. *Let S be a regular scheme and let R be a localization of the ring of integers of a number field K , let v be a non-archemidian valuation on the field K and let ℓ be the prime number such that v extends the ℓ -adic valuation. Assume that ℓ is invertible on S and that all the residue characteristic exponents of S are invertible in R .*

1. *If v is non-negative on R , denote by λ a uniformizer of $R_{(v)}$. Let Ξ be the functor*

$$\mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(S, R) \rightarrow \mathcal{S}\mathfrak{h}_{\lambda}(S, R_{(v)})$$

that sends an étale sheaf M to the λ -complete sheaf $\lim(M \otimes_R R_{(v)})/\lambda^n$.

Then, with the notations of Section 3.1.6, the square

$$\begin{array}{ccc} \mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(S, R) & \xrightarrow{\rho^!} & \mathcal{D}\mathcal{M}_{\text{ét}}^{\text{sm}A}(S, R) \\ \downarrow \Xi & & \downarrow \rho_v \\ \mathcal{S}\mathfrak{h}_{\lambda}(S, R_{(v)}) & \xrightarrow{\Psi} & \mathcal{D}(S, R_v) \end{array}$$

is commutative.

Furthermore, when restricted to the subcategory of constructible smooth Artin étale motives, the v -adic realization functor is t -exact with respect to the ordinary t -structures and induces the exact functor

$$- \otimes_R R_v : \text{Loc}_S(R) \rightarrow \text{Sh}^{\text{sm}A}(S, R_v)$$

between the hearts.

2. *If v is not non-negative on R , Let A be the subring of K on which the valuation v is non-negative and let λ be a uniformizer of A . Then, with the notations of Section 3.1.6, the square*

$$\begin{array}{ccc} \mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R) & \xrightarrow{\rho^!} & \mathcal{D}\mathcal{M}_{\text{ét},c}^{\text{sm}A}(S, R) \\ \downarrow \Xi' & & \downarrow \rho_v \\ \mathcal{S}\mathfrak{h}_{\lambda,c}(S, A)_{(0)} & \xrightarrow{\Psi'} & \mathcal{D}_c^b(S, R_v) \end{array}$$

is commutative using the notations of Section 3.1.1, if Ψ' denotes the functor $\Psi_{(0)}$ and Ξ' denotes the functor $\Xi_{(0)} \circ (- \otimes_R K)$.

Furthermore, the v -adic realization functor is t -exact with respect to the ordinary t -structures and induces the exact functor

$$- \otimes_R R_v : \text{Loc}_S(R) \rightarrow \text{Sh}^{\text{sm}A}(S, R_v)$$

between the hearts.

Proof. The second assertion follows from the first assertion. We now prove the first assertion. Recall that with the notations of Section 3.1.6, we have

$$\rho_v = \Psi \circ \rho^! \circ \rho_{\lambda}^{\text{CD}}.$$

But as $\rho^!$ is a right adjoint functor, it commutes with limits which yields

$$\rho^! \circ \rho_\lambda^{\text{CD}} \circ \rho_! = \Xi.$$

To show that the ℓ -adic realization functor is t-exact when restricted to the subcategory of constructible smooth Artin étale motives, it suffices to show that the functor

$$\Xi \circ \Psi: \mathcal{S}\mathcal{H}_{\text{lisse}}(S, R) \rightarrow \mathcal{D}_c^b(S, R_v)$$

is t-exact. Let M be a locally constant étale sheaf with coefficients in R . We have

$$\Xi(\Psi(M)) = \lim \nu^*(M)/\lambda^n.$$

It suffices to show that $\lim \nu^*(M)/\lambda^n$ lies in the heart of the ordinary t-structure on $\mathcal{D}_c^b(S, R_v)$. Now, by definition of the ordinary t-structure (see [BBDG18, 2.2.13]) if i is an integer, we have

$$H^i(\lim \nu^*(M)/\lambda^n) = \lim H^i(\nu^*(M/\lambda^n)) = \lim \nu^* \text{Tor}_i^R(M, R/\lambda^n R).$$

Now, the exact sequence

$$0 \rightarrow R \xrightarrow{\times \lambda^n} R \rightarrow R/\lambda^n R \rightarrow 0$$

yields

$$\text{Tor}_i^R(M, R/\lambda^n R) = \begin{cases} M/\lambda^n M & \text{if } i = 0 \\ M[\lambda^n] & \text{if } i = -1 \\ 0 & \text{otherwise} \end{cases}$$

where $M[\lambda^n]$ denotes the λ^n -torsion of M . As the transition map $M[\lambda^{n+1}] \rightarrow M[\lambda^n]$ is the multiplication by λ , we get $\lim M[\lambda^n] = 0$. The result follows. \square

3.2.2 The conservativity conjecture for Artin motives

The conservativity conjecture, proved for Artin motives with rational coefficients in [Pep19b], asserts that the ℓ -adic realization functor [CD16, 7.2.11] is conservative. With integral coefficients, we need to include realization functors with respect to every ℓ . Indeed, if M is an Artin motive of p -torsion and $\ell \neq p$, the ℓ -adic realization of M vanishes.

Furthermore, the stable category of constructible ℓ -adic complexes has a non-pathological behavior only when ℓ is invertible on the base scheme. Therefore, we introduce the following functor.

Definition 3.2.2.1. *Let S be a scheme, let R be a localization of the ring of integers of a number field K , let v be a valuation on K and let ℓ be the prime number such that v extends the ℓ -adic valuation.*

The reduced v -adic realization functor $\bar{\rho}_v$ is the composition

$$\mathcal{D}\mathcal{M}_{\text{ét}}(S, R) \xrightarrow{j_\ell^*} \mathcal{D}\mathcal{M}_{\text{ét}}(S[1/\ell], R) \xrightarrow{\rho_v} \mathcal{D}(S[1/\ell], R_v),$$

where j_ℓ denotes the open immersion $S[1/\ell] \rightarrow S$ and where the functor ρ_v is the v -adic realization functor defined in Section [3.1.6].

We still denote by $\bar{\rho}_v$ the restriction of this functor to the stable subcategory of constructible Artin motives.

A consequence of Theorem [3.2.1.1](#) is the conservativity conjecture in the case of Artin motives.

Proposition 3.2.2.2. *Let S be a scheme, let R be a localization of the ring of integers of a number field K . If v is a valuation on K , denote by R_v the completion of R with respect to v and by K_v the completion of K with respect to v .*

1. *If there is a prime number ℓ which is invertible on S , let v be a valuation which extends the ℓ -adic valuation then, the functor*

$$\bar{\rho}_v : \mathcal{DM}_{\text{ét},c}^A(S, K) \rightarrow \mathcal{D}(S, K_v)$$

is conservative.

2. *If ℓ and ℓ' are two distinct prime numbers, let v be a valuation which extends the ℓ -adic valuation and let v' be a valuation which extends the ℓ' -adic valuation. Then, when working with coefficients in K , the family $(\bar{\rho}_v, \bar{\rho}_{v'})$ is conservative.*

3. *The family*

$$(\bar{\rho}_v : \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}_c^b(S[1/\ell(v)], R_v))$$

where v runs through the set of non-archimedean valuations on K which are non-negative on R and $\ell(v)$ denotes the prime number such that v extends the ℓ -adic valuation is conservative.

Proof. If x is a point of S , denote by $i_x : \{x\} \rightarrow S$ the inclusion. The family of functors $(i_x^*)_{x \in S}$ is conservative by [\[CD19, 4.3.17\]](#). In addition, the v -adic realization commutes with the six functors by [\[CD16, 7.2.16\]](#). Hence, to prove the proposition, we may assume that S is the spectrum of a field k . Let p be the characteristic exponent of k .

We prove the first assertion. Since $\ell \neq p$, we have

$$\rho_v = \bar{\rho}_v.$$

Denote by $(BG_k)_{\text{proét}}$ the site of profinite continuous G_k -sets with covers given by continuous surjections (see [\[BS15, 4.1.10\]](#)). Using Proposition [3.2.1.9](#), the diagram

$$\begin{array}{ccc} \mathcal{D}^b(\text{Rep}^A(G_k, K)) & \simeq & \mathcal{S}\mathcal{H}_{\text{lisse}}(k, K) \xrightarrow{\rho_l} \mathcal{DM}_{\text{ét}}^A(k, K) \\ \downarrow -\otimes_K K_v & & \downarrow \rho_v \\ \mathcal{D}^b(\text{Rep}(G_k, K_v)) & \xrightarrow{\alpha} & \mathcal{D}(\text{Sh}((BG_k)_{\text{proét}}, K_v)) \simeq \mathcal{D}(k, K_v) \end{array},$$

where α is the derived functor of the fully faithful exact functor

$$\text{Rep}(G_k, K_v) \rightarrow \text{Sh}((BG_k)_{\text{proét}}, K_v),$$

is commutative.

Since the functors α and $-\otimes_K K_v$ are conservative and since the functor ρ_l is an equivalence, the functor ρ_v is conservative.

With the assumptions of the second assertion, one of the prime numbers ℓ and ℓ' is distinct from p and the proof is the same.

We now prove the third assertion. Let v be a valuation on K such that $v(p) = 0$ (i.e. v does not extend the p -adic valuation) Using Proposition [3.2.1.9](#), the diagram

$$\begin{array}{ccc} \mathcal{D}^b(\mathrm{Rep}^A(G_k, R[1/p])) & \simeq & \mathcal{S}\mathcal{H}_{\mathrm{lisse}}(k, R[1/p]) \xrightarrow{\rho^!} \mathcal{DM}_{\acute{\mathrm{e}}\mathrm{t}}^A(k, R) \\ \downarrow -\otimes_R R_v & & \downarrow \rho_v \\ \mathcal{D}^b(\mathrm{Rep}(G_k, R_v)) & \xrightarrow{\alpha} & \mathcal{D}(\mathrm{Sh}((BG_k)_{\mathrm{pro\acute{e}t}}, R_v)) \simeq \mathcal{D}(k, R_v) \end{array},$$

where α is the derived functor of the fully faithful exact functor

$$\mathrm{Rep}(G_k, R_v) \rightarrow \mathrm{Sh}((BG_k)_{\mathrm{pro\acute{e}t}}, R_v),$$

is commutative.

Now, if M is a complex in $\mathcal{D}^b(\mathrm{Rep}^A(G_k, R[1/p]))$ such that for any valuation v such that $v(p) = 0$, we have

$$M \otimes_R R_v = 0,$$

then, the complex M vanishes. Hence, the family $(\bar{\rho}_\ell)$ is conservative. \square

3.2.3 Constructible Artin motives as stratified representations

Let S be a scheme. Recall that a *stratification* of S is a finite partition of S into equidimensional locally closed connected subsets of S (called *strata*) such that the topological closure of any stratum is a union of strata.

Proposition 3.2.3.1. *Let S be a scheme, let M be a constructible Artin étale motive over S . Then, there is a stratification of S such that, for any stratum T , the Artin motive $M|_T$ is smooth. Moreover if the scheme S is excellent, we can assume the strata to be regular.*

Proof. We can assume that S is reduced and connected. We proceed by noetherian induction on S . The local rings at the generic points of S are fields. Let M be a constructible Artin étale motive over S . As Artin étale motives over a field are smooth, by Proposition [3.1.2.8](#), there is an open dense subscheme U of S such that $M|_U$ is smooth. If the scheme S is excellent, we can furthermore assume that U is regular. Using the induction hypothesis, there is a stratification of $S \setminus U$ such that for any stratum T , the motive $M|_T$ is a constructible smooth Artin motive and we are done. \square

Corollary 3.2.3.2. *Let S be an excellent scheme with finitely many residue characteristics and let R be a regular good ring. Write $S_0 = S \times_{\mathbb{Z}} \mathbb{Q}$ and $S_p = S \times_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ for any prime number p . Then, we have prime numbers p_1, \dots, p_r such that*

$$S = S_0 \sqcup S_{p_1} \sqcup \dots \sqcup S_{p_r}.$$

Furthermore, the subscheme S_0 is open in S and the subschemes S_{p_1}, \dots, S_{p_r} are closed in S .

Let M be a constructible Artin étale motive over S . We have a stratification \mathcal{S}_i of every S_i when i runs through the set $\{0, p_1, \dots, p_r\}$ (and therefore a stratification of S), lisse étale sheaves $M_T \in \mathcal{S}\mathcal{H}_{\mathrm{lisse}}(T, R)$ for T in \mathcal{S}_0 and lisse étale sheaves $M_T \in \mathcal{S}\mathcal{H}_{\mathrm{lisse}}(T, R[1/p_i])$ for T in \mathcal{S}_{p_i} , such that for any stratum T , we have

$$M|_T = \rho_!(M_T).$$

In particular, the motive M can be constructed from the M_T by gluing using the localization triangle (3.0.0.1).

Corollary 3.2.3.3. *Let S be an excellent scheme and let R be a regular good ring. Assume that all residue characteristics of S are invertible in R .*

Let M be a constructible Artin étale motive over S . We have a stratification of S and lisse étale sheaves $M_T \in \mathcal{S}\mathfrak{h}_{\text{lisse}}(T, R)$ for every stratum T , such that we have

$$M|_T = \rho_!(M_T).$$

In particular, the motive M can be constructed from the M_T by gluing using the localization triangle (3.0.0.1).

Corollary 3.2.3.4. *Let S be an excellent scheme, let p be a prime number and let R be a $\mathbb{Z}_{(p)}$ -algebra.*

Write $S_p = S \times_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. Then, we have

$$S = S[1/p] \sqcup S_p.$$

Furthermore the subscheme $S[1/p]$ is open in S and the subscheme S_p is closed in S .

Let M be a constructible Artin étale motive over S with coefficients in R . We have a stratification $\mathcal{S}[1/p]$ of $S[1/p]$ with regular strata, a stratification \mathcal{S}_p of S_p with regular strata, lisse étale sheaves $M_T \in \mathcal{S}\mathfrak{h}_{\text{lisse}}(T, R[1/p])$ for T in \mathcal{S}_p and lisse étale sheaves $M_T \in \mathcal{S}\mathfrak{h}_{\text{lisse}}(T, R)$ for $T \in \mathcal{S}[1/p]$, such that for any stratum T , we have

$$M|_T = \rho_!(M_T).$$

In particular, the motive M can be constructed from the M_T by gluing using the localization triangle (3.0.0.1).

3.3 Computations around the Artin truncation functor

3.3.1 The Artin truncation over a field

With integral coefficients, Proposition 3.1.4.7 is (very) false, namely we have:

Proposition 3.3.1.1. *Let R be a localization of the ring of integers of a number field K and let $f: X \rightarrow S$ be a separated morphism of finite type. If the object $\omega^0 f_! \mathbb{1}_X$ of $\mathcal{D}\mathcal{M}_{\text{ét}}^A(S, R)$ belongs to $\mathcal{D}\mathcal{M}_{\text{ét},c}^A(S, R)$, then we are in one of the following cases:*

1. *we have $R = K$.*
2. *the map f is quasi-finite.*
3. *there is a prime number p such that $R[1/p]$ is a \mathbb{Q} -algebra and all the points x of S such that the set $f^{-1}(x)$ is infinite have the same residue characteristic p .*

Proposition 3.3.1.1 will follow from Theorem 3.3.1.4 below. To state this result, we need to introduce the following notion:

Definition 3.3.1.2. Let R be a localization of the ring of integers of a number field and let k be a field of characteristic exponent p . If n is a positive integer, we denote by μ_n the sheaf of n -th roots of unity over the small étale site of k .

Let $f: X \rightarrow \text{Spec}(k)$ be a morphism of finite type. Let m be an integer, we define the m -th Tate twist of the étale sheaf $f_* \left[R[1/p] \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \right]$ as the étale sheaf

$$f_* \left[R[1/p] \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \right] (m) := \text{colim}_{(n,p)=1} f_* \left[R[1/p] \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \right] \otimes_{\mathbb{Z}} \mu_n^{\otimes m}.$$

Remark 3.3.1.3. Notice that through the equivalence of Proposition 3.1.5.13, the i -th cohomology group of $f_* \left[R[1/p] \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \right] (m)$ seen as a complex of discrete G_k -modules is the discrete G_k -module $H_{\text{ét}}^i(X_{\bar{k}}, R[1/p] \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})(m)$ defined in Example 3.1.5.4.

We can now state the main result of this section:

Theorem 3.3.1.4. Let R be a localization of the ring of integers of a number field, let k be a field of characteristic exponent p , let $\sigma: R \rightarrow R[1/p]$ be the localization morphism and let $f: X \rightarrow \text{Spec}(k)$ be a morphism of finite type. Assume that the scheme X is regular.

Then,

1. The diagram

$$\begin{array}{ccc} \mathcal{S}\mathcal{h}_{\text{Ind lisse}}(X, R[1/p]) & \xrightarrow{\sigma_* \rho_!} & \mathcal{D}\mathcal{M}_{\text{ét}}^{\text{sm}A}(X, R) \\ \downarrow f_* & & \downarrow \omega^0 f_* \\ \mathcal{S}\mathcal{h}(k_{\text{ét}}, R[1/p]) & \xrightarrow{\sigma_* \rho_!} & \mathcal{D}\mathcal{M}_{\text{ét}}^A(k, R) \end{array}$$

is commutative.

2. We have

$$\omega^0 f_* \mathbb{1}_X = \rho_! f_* R[1/p].$$

3. If m is a positive integer, we have

$$\omega^0(f_* \mathbb{1}_X(-m)) = \rho_! \left[f_* \left(R[1/p] \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \right) [-1](-m) \right].$$

Proof. We can assume by Corollary 3.2.1.7 that p is invertible in R . We have pairs of adjoint functors

$$f^*: \mathcal{D}\mathcal{M}_{\text{ét}}^A(k, R) \rightleftarrows \mathcal{D}\mathcal{M}_{\text{ét}}^{\text{sm}A}(X, R): \omega^0 f_*$$

and

$$f^*: \mathcal{S}\mathcal{h}_{\text{Ind lisse}}(k, R) \rightleftarrows \mathcal{S}\mathcal{h}_{\text{Ind lisse}}(X, R): f_*$$

Since the functor $\rho_!$ is an equivalence by Theorem 3.2.1.1 and commutes with the functor f^* by Proposition 3.1.2.1, this yields the first assertion.

The second assertion follows from the first one, since we have

$$\mathbb{1}_X = \rho_! R.$$

We now prove the third assertion. Let $\pi: \mathbb{P}_X^m \rightarrow \text{Spec}(k)$ be the canonical projection. We have

$$\pi_* \mathbb{1}_{\mathbb{P}_X^m} = \bigoplus_{i=0}^m f_* \mathbb{1}_X(-i)[-2i]$$

by the projective bundle formula.

On the other hand, we have an exact triangle étale sheaves

$$\pi_* (\underline{R}) \rightarrow \pi_* (R \otimes_{\mathbb{Z}} \mathbb{Q}) \rightarrow \pi_* (R \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}).$$

Proposition [3.1.7.2](#) yields

$$\pi_* (R \otimes_{\mathbb{Z}} \mathbb{Q}) = f_* (R \otimes_{\mathbb{Z}} \mathbb{Q})$$

and the projective bundle formula in étale cohomology yields

$$\pi_* (R \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) = \bigoplus_{i=0}^m f_* (R \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) (-i)[-2i].$$

The result follows by induction on m . □

Definition 3.3.1.5. *Let $f : X \rightarrow \text{Spec}(k)$ be a morphism of finite type, let \bar{k} be the algebraic closure of k and let $X_{\bar{k}}$ be the pull-back of X to \bar{k} . The set $\pi_0(X_{\bar{k}})$ of connected components of $X_{\bar{k}}$ is a finite G_k -set and therefore corresponds to a finite étale k -scheme through Galois-Grothendieck correspondence [[GR71](#), VI]. We let $\pi_0(X/k)$ be this scheme and call it the scheme of connected components of X .*

Remark 3.3.1.6. *When f is proper, this scheme coincides with the scheme $\pi_0(X/k)$ defined in [[AZ12](#)] as the Stein factorization of f . In general, the scheme $\pi_0(X/k)$ coincides with the Stein factorization of any compactification \bar{X} of X which is normal in X in the sense that X is its own relative normalization in \bar{X} .*

Notations 3.3.1.7. *Let k be a field of characteristic exponent p and let $\xi : \text{Spec}(\bar{k}) \rightarrow \text{Spec}(k)$ be the algebraic closure of k .*

1. *Let R be a regular ring. We let*

$$\alpha_1 : \mathcal{D}(\text{Mod}(G_k, R[1/p])) \rightarrow \mathcal{DM}^A(k, R)$$

be the equivalence of Corollary [3.2.1.7](#) (which is the composition of ρ_1 and of the inverse of the fiber functor ξ^ associated to ξ) and we let $\alpha^!$ be its inverse.*

2. *Let R be a localization of the ring of integers of a number field, let $f : X \rightarrow S$ be a morphism of finite type and let $X_{\bar{k}}$ be the pull-back of X to \bar{k} . We denote by $\mu^n(X, R)$ the G_k -module $H_{\text{ét}}^n(X_{\bar{k}}, R[1/p] \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$.*

Corollary 3.3.1.8. *Let R be a localization of the ring of integers of a number field, let k be a field of characteristic exponent p and let $f : X \rightarrow S$ be a morphism of finite type. Assume that the scheme X is regular.*

Then,

1. *We have*

$$H^0(\omega^0 f_* \mathbb{1}_X) = h_k(\pi_0(X/k)).$$

2. We have

$$H^n(\omega^0 f_* \mathbb{1}_X) = \alpha_! [H_{\text{ét}}^n(X_{\bar{k}}, R[1/p])]$$

and

$$H_{\text{ét}}^n(X_{\bar{k}}, R[1/p]) = \begin{cases} R[1/p][\pi_0(X_{\bar{k}})] & \text{if } n = 0 \\ \mu^{n-1}(X, R) & \text{if } n \geq 2 \\ 0 & \text{otherwise} \end{cases}.$$

3. If m is a positive integer and if n is an integer, we have

$$H^n(\omega^0 f_* \mathbb{1}_X(-m)) = \alpha_! [\mu^{n-1}(X, R)(-m)].$$

We can now prove Proposition [3.3.1.1](#). Notice that it follows from the following lemma.

Lemma 3.3.1.9. *Let R be a localization of the ring of integers of a number field, let k be a field of characteristic exponent p and let $f: X \rightarrow \text{Spec}(k)$ be a morphism of finite type. Assume that the ring $R[1/p]$ is not a \mathbb{Q} -algebra and that the morphism f is not finite. Then, the Artin motive $\omega^0 f_! \mathbb{1}_X$ is not constructible.*

Using Theorem [3.2.1.1](#), the lemma follows from the following more precise statement.

Lemma 3.3.1.10. *Let R be a localization of the ring of integers of a number field, let k be a field of characteristic exponent p , let $f: X \rightarrow \text{Spec}(k)$ be a separated morphism of finite type and let $d = \dim(X)$. Assume that the ring $R[1/p]$ is not a \mathbb{Q} -algebra and that $d > 0$. Then, the motive $H^{2d+1}(\omega^0 f_! \mathbb{1}_X)$ is not the image of an Artin representation through the map $\alpha_!$. Furthermore, if $n > 2d + 1$, the motive $H^n(\omega^0 f_! \mathbb{1}_X)$ vanishes.*

Example 3.3.1.11. *With the notations of the lemma, assume that $R = \mathbb{Z}$ and that $X = \mathbb{P}_k^1$. Then, by Corollary [3.3.1.8](#), the motive $H^3(\omega^0 f_! \mathbb{1}_{\mathbb{P}_k^1})$ is equivalent to $\alpha_! [\mathbb{Q}/\mathbb{Z}[1/p](-1)]$. The idea of this lemma is to prove that for a general k -scheme of finite type X with n connected components, it still remains true that the top cohomology object of $\omega^0 f_! \mathbb{1}_X$ is placed in degree $\dim(X) + 1$ and corresponds through the equivalence $\alpha_!$ to the direct sum of n copies of $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} R$ (which is not of finite presentation as an R -module) endowed with some action of G_k .*

Proof. Since change of coefficients preserves constructible objects, we may assume that there is a prime number $\ell \neq p$ such that the ring R is a flat $\mathbb{Z}_{(\ell)}$ -algebra. We may also assume that the scheme X is connected.

We proceed by induction on the dimension of X . By compactifying X using Nagata's theorem [\[Sta23, 0F41\]](#), and by using the localization triangle [\(3.0.0.1\)](#) and the induction hypothesis, we may assume that the morphism f is proper. By normalizing X , by using cdh-descent and by using the induction hypothesis again, we may also assume that the scheme X is normal.

Using Gabber's ℓ -alterations [\[LYF14, IX.1.1\]](#), there is a proper and surjective map $p: Y \rightarrow X$ such that Y is regular and integral and p is generically the composition of an étale Galois cover of group of order prime to ℓ and of a finite surjective purely inseparable morphism.

Therefore, there is a closed subscheme F of X of positive codimension such that the induced map

$$p: Y \setminus p^{-1}(F) \rightarrow X \setminus F$$

is the composition of an étale Galois cover of group G , where the order of G is prime to ℓ , and of a finite surjective purely inseparable morphism.

Let X' be the relative normalization of X in $Y \setminus p^{-1}(F)$. Then, the map $X' \rightarrow X$ is finite and the scheme X' is endowed with a G -action. Furthermore, the map $X'/G \rightarrow X$ is purely inseparable since X is normal.

Using [Ayo07, 2.1.165], for any integer n we have a canonical isomorphism

$$H^n(\omega^0 h_k(X)) \rightarrow H^n(\omega^0 h_k(X'))^G.$$

Furthermore, by cdh-descent, and by using the induction hypothesis, the canonical map

$$H^n(\omega^0 h_k(X')) \rightarrow H^n(\omega^0 h_k(Y))$$

is an isomorphism if $n > 2d$.

Now, by Poincaré Duality in étale cohomology [Del77] and by Theorem 3.3.1.4, since d is positive, we have

$$H^{2d+1}(\omega^0 h_k(Y)) = \alpha_! [\mathbb{Q}/\mathbb{Z}(-d) \otimes_{\mathbb{Z}} R[\pi_0(Y_{\bar{k}})]]$$

and the G -action on $H^{2d+1}(\omega^0 h_k(Y))$ is induced by the action of G on the connected components of $Y_{\bar{k}}$.

Since the map $Y \rightarrow X'$ is an isomorphism in codimension 1, the map

$$\pi_0(Y_{\bar{k}}) \rightarrow \pi_0(X'_{\bar{k}})$$

is a bijection. Thus, we get

$$H^{2d+1}(\omega^0 h_k(Y)) = \alpha_! [\mathbb{Q}/\mathbb{Z}(-d) \otimes_{\mathbb{Z}} R[\pi_0(X'_{\bar{k}})]]^G$$

If $x \in \mathbb{Q}/\mathbb{Z}(-d)$ and if S is a connected component of the scheme $X'_{\bar{k}}$, denote by $[x]_S$ the element $x \otimes e_S$ of $\mathbb{Q}/\mathbb{Z}(-d) \otimes_{\mathbb{Z}} R[\pi_0(X'_{\bar{k}})]$ where e_S is the element of the basis of $R[\pi_0(X'_{\bar{k}})]$ attached to S . Then, the G -fixed points are exactly the elements of the form:

$$\sum_{\substack{T \in \pi_0(X_{\bar{k}}), S \in \pi_0(X'_{\bar{k}}) \\ p(S)=T}} [x_T]_S$$

where $x_T \in \mathbb{Q}/\mathbb{Z}(-d)$. Therefore, this set of G -fixed points is isomorphic as a G_k -module to

$$\mathbb{Q}/\mathbb{Z}(-d) \otimes_{\mathbb{Z}} R[\pi_0(X_{\bar{k}})]$$

which is not an Artin representation. □

We now explore more consequences of Theorem 3.3.1.4.

Corollary 3.3.1.12. *Let R be a localization of the ring of integers of a number field, let k be a field of characteristic exponent p and let $f: X \rightarrow S$ be a proper morphism. Assume that*

$$X = \bigcup_{i \in I} X_i$$

where I is a finite set and X_i is a closed subscheme of X .

For $J \subseteq I$, let

$$X_J = \bigcap_{j \in J} X_j$$

and let $f_J: X_J \rightarrow \text{Spec}(k)$ be the structural morphism. Assume that for any such J , the scheme X_J is nil-regular.

1. The Artin motive $\omega^0 f_* \mathbb{1}_X$ is the image through the functor $\alpha_!$ of the following limit

$$\lim_{[n] \in \Delta^{\text{inj}}} \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} (f_J)_* \underline{R[1/p]}$$

where the degeneracy maps are given by the appropriate closed immersions (see [DDØ21, 5.1.3]).

2. Let Y be a regular scheme and $i: X \rightarrow Y$ be a closed immersion. Let r be a non-negative integer. Assume that for all $J \subseteq I$, the subscheme X_J is of codimension $|J|$ in Y . Then, the Artin motive $\omega^0 f_* i^! \mathbb{1}_Y(-r)$ is the image through the functor $\alpha_!$ of the following colimit

$$\text{colim}_{[n] \in \Delta^{\text{inj}}} \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} (f_J)_* \left(\underline{R[1/p]} \otimes \underline{\mathbb{Q}/\mathbb{Z}} \right) (-n-1-r)[-2n-3]$$

Proof. We begin with the first assertion. First, [DDØ21, 5.7] asserts that the canonical map

$$f_* \mathbb{1}_X \rightarrow \lim_{[n] \in \Delta^{\text{inj}}} \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} (f_J)_* \mathbb{1}_{X_J}$$

is an equivalence.

Since ω^0 is an exact functor, it is compatible with finite limits. Therefore, the canonical map

$$\omega^0 f_* \mathbb{1}_X \rightarrow \lim_{[n] \in \Delta^{\text{inj}}} \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} \omega^0 (f_J)_* \mathbb{1}_{X_J}$$

is an equivalence. The first assertion then follows from Theorem 3.3.1.4.

We now prove the second assertion. If $J \subseteq I$, let $i_J: X_J \rightarrow X$ be the inclusion. Again, [DDØ21, 5.7] asserts that the canonical map

$$i^! \mathbb{1}_Y \rightarrow \text{colim}_{[n] \in \Delta^{\text{inj}}} \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} (i_J)_* (i \circ i_J)^! \mathbb{1}_X$$

is an equivalence.

Since ω^0 is an exact functor, it is compatible with finite colimits, and therefore the canonical map

$$\text{colim}_{[n] \in \Delta^{\text{inj}}} \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} \omega^0 f_* (i_J)_* (i \circ i_J)^! \mathbb{1}_X(-r) \rightarrow \omega^0 f_* i^! \mathbb{1}_Y(-r)$$

is an equivalence.

Using absolute purity property [CD16, 5.6.2], we get that $\omega^0 f_* i^! \mathbb{1}_Y(-r)$ is equivalent to

$$\text{colim}_{[n] \in \Delta^{\text{inj}}} \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} \omega^0 f_* (i_J)_* \mathbb{1}_{X_J}(-n-1-r)[-2n-2].$$

The result then follows from Theorem 3.3.1.4. □

With the notations of Corollary 3.3.1.12, we now want to describe the cohomology of $\omega^0 f_* \mathbb{1}_X$ seen as the image of a discrete G_k -module through the functor $\alpha_!$. Using [Lur17, 1.2.2.7], Corollary 3.3.1.12 provides:

Corollary 3.3.1.13. *Keep the notations of Corollary 3.3.1.12.*

1. *The limit of (1) of Corollary 3.3.1.12 induces a spectral sequence:*

$$\alpha_! E_1^{n,m} \Rightarrow H^{n+m}(\omega^0 f_* \mathbb{1}_X)$$

where

$$E_1^{n,m} = \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} H_{\text{ét}}^m((X_J)_{\bar{k}}, R[1/p]) = \begin{cases} \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} R[1/p] [\pi_0((X_J)_{\bar{k}})] & \text{if } m = 0 \\ \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} \mu^{m-1}(X_J, R) & \text{if } m \geq 2 \\ 0 & \text{otherwise} \end{cases} .$$

2. *the colimit of (2) of Corollary 3.3.1.12 induces a spectral sequence:*

$$\alpha_! E_1^{n,m} = \alpha_! \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} \mu^{m-n-3}(X_J)(-n-1-r) \implies H^{n+m}(\omega^0 f_* i^! \mathbb{1}_Y(-r)).$$

Proof. The Artin motive $\omega^0 f_* \mathbb{1}_X$ is the image through the functor $\alpha_!$ of the limit

$$\lim_{n \in \Delta^{\text{inj}}} \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} (f_J)_* R[1/p].$$

We can truncate this diagram in each degree to write $\omega^0 f_* \mathbb{1}_X$ as the limit of a sequential diagram $(M^n)_{n \geq 0}$ such that the fiber of the map $M^n \rightarrow M^{n-1}$ is $\alpha_! \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} (f_J)_* R[1/p]$.

By [Lur17, 1.2.2.14], this sequential diagram gives rise to a spectral sequence such that

$$\alpha_! E_1^{n,m} = \alpha_! \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} H_{\text{ét}}^m((X_J)_{\bar{k}}, R[1/p]) \implies H^{n+m}(\omega^0 f_* \mathbb{1}_X).$$

This proves the first assertion. We now prove the second assertion.

As in the proof of the first assertion, we can compute the motive $\omega^0 f_* i^! \mathbb{1}_Y$ using Corollary 3.3.1.12 as the colimit of a sequential diagram $(N_p)_{p \geq 0}$ such that the cofiber of the map $N_{p-1} \rightarrow N_p$ is $\rho_! \bigoplus_{\substack{J \subseteq I \\ |J|=p+1}} (f_J)_* (R[1/p] \otimes \mathbb{Q}/\mathbb{Z})(-n-1-r)[-2n-3]$.

Now, by [Lur17, 1.2.2.14] this sequential diagram gives rise to a spectral sequence such that

$$\alpha_! E_1^{n,m} = \alpha_! \bigoplus_{\substack{J \subseteq I \\ |J|=n+1}} \mu^{m-n-3}(X_J)(-n-1-r) \implies H^{n+m}(\omega^0 f_* i^! \mathbb{1}_Y)$$

which is exactly the second assertion. □

3.3.2 Artin residues over 1-dimensional schemes

In this section, we study an analog of the nearby cycle functor (or more accurately residue functor) in our setting. Namely, let R be a localization of the ring of integers of a number field, let C be an excellent 1-dimensional scheme, let $i: F \rightarrow C$ be a closed immersion of positive codimension and let $j: U \rightarrow C$ be the complementary open immersion; we study the functor

$$\omega^0 i^* j_* : \mathcal{DM}_{\acute{e}t,c}^{smA}(U, R) \rightarrow \mathcal{DM}_{\acute{e}t}^A(F, R).$$

First, notice that the scheme F is 0-dimensional. Therefore, the scheme F is the disjoint union of its points. Therefore, denoting by

$$i_x: \{x\} \rightarrow F$$

the inclusion for $x \in F$, we get

$$\omega^0 i^* j_* = \bigoplus_{x \in F} (i_x)_* i_x^* \omega^0 i^* j_*.$$

Now, since the map i_x is an open immersion, by Proposition [3.1.4.5](#), the 2-morphism $i_x^* \omega^0 \rightarrow \omega^0 i_x^*$ is an equivalence. Therefore, the functor $\omega^0 i^* j_*$ is equivalent to the functor

$$\bigoplus_{x \in F} (i_x)_* \omega^0 (i \circ i_x)^* j_*.$$

Hence, studying the case where F is a point will yield a description in the general case. Assume from now on that F is reduced to a single point. Let x be the only point of F and let p be the characteristic exponent of $k(x)$.

Now, recall that Theorem [3.2.1.1](#) states that if S is a regular scheme, and if all the residue characteristic exponents of S are invertible in R , the functor

$$\rho_! : \mathcal{S}\mathcal{h}_{\text{lis}}(S, R) \rightarrow \mathcal{DM}_{\acute{e}t}^{smA}(S, R)$$

is an equivalence.

If the scheme U is regular and if all the residue characteristic exponents of U are invertible in R , the functor $\omega^0 i^* j_*$ above can be seen as a functor

$$\mathcal{S}\mathcal{h}_{\text{lis}}(U, R) \rightarrow \mathcal{S}\mathcal{h}(k(x)_{\acute{e}t}, R[1/p]).$$

Since the functor $\rho_!$ is the left adjoint functor of a premotivic adjunction, we have an equivalence $\rho_! i^* \xrightarrow{\sim} i^* \rho_!$ and an exchange transformation $\rho_! j_* \rightarrow j_* \rho_!$.

This gives a transformation

$$\rho_! i^* j_* \rightarrow i^* j_* \rho_!$$

of functors

$$\mathcal{S}\mathcal{h}_{\text{lis}}(U, R) \rightarrow \mathcal{DM}_{\acute{e}t}(F, R).$$

Furthermore, if M is a lisse étale sheaf over U , then, the motive $\rho_! i^* j_* M$ is Artin by Proposition [3.1.4.3](#). Therefore, we have a transformation

$$\bar{\Theta}: \rho_! i^* j_* \rightarrow \omega^0 i^* j_* \rho_!.$$

of functors

$$\mathcal{S}\mathcal{h}_{\text{lisse}}(U, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(F, R).$$

Finally, let

$$\Theta = \bar{\Theta} \otimes_R R[1/p].$$

Since tensoring with $R[1/p]$ induces an equivalence

$$\mathcal{DM}_{\text{ét}}^A(F, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(F, R[1/p]),$$

the functor Θ can be seen as a transformation

$$\rho_! i^* j_* \otimes_R R[1/p] \rightarrow \omega^0 i^* j_* \rho_!.$$

Proposition 3.3.2.1. *Let R be a localization of the ring of integers of a number field, let C be an excellent 1-dimensional scheme, let x be a closed point of C , let $i: \{x\} \rightarrow C$ be the closed immersion, let $j: U \rightarrow C$ be the complementary open immersion and let p be the characteristic exponent of $k(x)$.*

Assume that the scheme U is nil-regular and the residue characteristics of U are invertible in R .

Then, the transformation

$$\Theta: \rho_! i^* j_* \otimes_R R[1/p] \rightarrow \omega^0 i^* j_* \rho_!$$

is an equivalence. In particular, the square

$$\begin{array}{ccc} \mathcal{S}\mathcal{h}_{\text{lisse}}(U, R) & \xrightarrow{\rho_!} & \mathcal{DM}_{\text{ét},c}^A(U, R) \\ \downarrow i^* j_* \otimes_R R[1/p] & & \downarrow \omega^0 i^* j_* \\ \mathcal{S}\mathcal{h}_{\text{lisse}}(\{x\}, R[1/p]) & \xrightarrow{\rho_!} & \mathcal{DM}_{\text{ét},c}^A(\{x\}, R) \end{array}$$

is commutative.

Proof. Tensoring with $\mathbb{Z}/n\mathbb{Z}$ or \mathbb{Q} commutes with the six functors formalism and with $\rho_!$ by Proposition 3.1.2.7, it also commutes with the functor ω^0 by Proposition 3.1.4.9. Using the conservativity property (Lemma 3.1.2.5), we can therefore assume that the ring R is either a \mathbb{Q} -algebra or an n -torsion algebra with n an integer prime to p .

In the n -torsion case, the proposition follows from the fact that ω^0 is an equivalence by Corollary 3.1.4.11 and from the rigidity theorem and the fact that the six functors commute with the functor $\rho_!$ in the torsion case by Proposition 3.1.3.3.

Assume now that the ring R is a \mathbb{Q} -algebra.

Let $\nu: \tilde{C} \rightarrow C$ be the normalization of C . Write

$$\begin{array}{ccccc} U_{\text{red}} & \xrightarrow{\gamma} & \tilde{C} & \xleftarrow{\iota} & F \\ \nu_U \downarrow & & \downarrow \nu & & \downarrow \nu_F \\ U & \xrightarrow{j} & C & \xleftarrow{i} & \{x\} \end{array}$$

the commutative diagram such that both squares are cartesian. Notice that U_{red} is the reduced subscheme associated to U . Therefore, the functor $(\nu_U)_*$ is an equivalence and commutes with the functor $\rho_!$. Hence, it suffices to show that

$$\Theta(\nu_U)_* : \rho_! i^* j_* (\nu_U)_* \rightarrow \omega^0 i^* j_* (\nu_U)_* \rho_!$$

is an equivalence.

Therefore, it suffices to show that the map $\rho_! i^* \nu_* \gamma_* \rightarrow \omega^0 i^* \nu_* \gamma_* \rho_!$ is an equivalence.

Using proper base change, this is equivalent to the fact that the map

$$\rho_! (\nu_F)_* \iota^* \gamma_* \rightarrow \omega^0 (\nu_F)_* \iota^* \gamma_* \rho_!$$

is an equivalence. Since the map induced by ν_F over the reduced schemes is finite, the functor $(\nu_F)_*$ commutes with the functor $\rho_!$ by Proposition [3.1.2.1](#) and with the functor ω^0 by Proposition [3.1.4.5](#). Thus, it suffices to show that the map

$$\rho_! \iota^* \gamma_* \rightarrow \omega^0 \iota^* \gamma_* \rho_!$$

is an equivalence.

Moreover, since the scheme F is 0-dimensional, we can assume as before that F is a point. Therefore, we can assume that the 1-dimensional scheme C is normal and therefore regular.

Since by Proposition [3.1.5.10](#) the stable category $\mathcal{S}\mathcal{H}_{\text{lisse}}(U, R)$ is generated as a thick subcategory of itself by the sheaves of the form $R_U(V) = f_* \underline{R}$ for $f: V \rightarrow U$ finite and étale and since the functors $\rho_! i^* j_*$ and $\omega^0 i^* j_*$ are exact and therefore compatible with finite limits, finite colimits and retracts, it suffices to show that the map

$$\Theta(R_U(V)) : \rho_! i^* j_* f_* \underline{R} \rightarrow \omega^0 i^* j_* f_* \mathbb{1}_V$$

is an equivalence.

To show that this map is an equivalence, we introduce $\bar{f}: C_V \rightarrow C$ the normalization of C in V . Write

$$\begin{array}{ccccc} V & \xrightarrow{\gamma} & C_V & \xleftarrow{\iota} & Z \\ \downarrow f & & \downarrow \bar{f} & & \downarrow p \\ U & \xrightarrow{j} & C & \xleftarrow{i} & \{x\} \end{array}$$

the commutative diagram such that both squares are cartesian. The map $\Theta(R_U(V))$ is an equivalence if and only if the map

$$\rho_! i^* \bar{f}_* \gamma_* \underline{R} \rightarrow \omega^0 i^* \bar{f}_* \gamma_* \mathbb{1}_V$$

is an equivalence. The morphism p induces a finite and étale map over the reduced schemes and therefore the functor p_* commutes with the functor $\rho_!$ by Proposition [3.1.2.1](#) and with the functor ω^0 by Proposition [3.1.4.5](#). Hence, using proper base change, it suffices to show that the map

$$\rho_! \iota^* \gamma_* \underline{R} \rightarrow \omega^0 \iota^* \gamma_* \mathbb{1}_V$$

is an equivalence.

Hence, we can assume that $V = U$. Therefore, we need to prove that the map

$$\Theta(\underline{R}) : \rho_! i^* j_* \underline{R} \rightarrow \omega^0 i^* j_* \mathbb{1}_U$$

is an equivalence. Using Proposition [3.1.4.5](#), the exchange transformation

$$i^* \omega^0 j_* \rightarrow \omega^0 i^* j_*$$

is an equivalence. Furthermore, by Proposition [3.1.2.1](#), the exchange transformation $\rho_! i^* \rightarrow i^* \rho_!$ is an equivalence. Therefore, it suffices to show that the exchange map

$$\rho_! j_* \underline{R} \rightarrow \omega^0 j_* \mathbb{1}_U$$

is an equivalence which is a direct consequence of the two lemmas below. □

Lemma 3.3.2.2. *Let R be a \mathbb{Q} algebra. Let S be a regular scheme, and $j: U \rightarrow S$ an open immersion with complement a simple normal crossing divisor. Then, the canonical map*

$$\mathbb{1}_S \rightarrow \omega^0 j_* \mathbb{1}_U$$

is an equivalence.

Proof. This result generalizes [\[AZ12, 2.11\]](#). Let $i: D \rightarrow S$ be the closed immersion which is complementary to j . Using Proposition [3.1.4.8](#), we have an exact triangle

$$i_* \omega^0 i^! \mathbb{1}_S \rightarrow \mathbb{1}_S \rightarrow \omega^0 j_* \mathbb{1}_U.$$

It therefore suffices to show that

$$\omega^0 i^! \mathbb{1}_S = 0.$$

Using [\[DDØ21, 5.7\]](#), we can reduce to the case when D is a regular subscheme of codimension $c > 0$. In this setting, we have

$$\omega^0 i^! \mathbb{1}_S = \omega^0(\mathbb{1}_D(-c))[-2c]$$

and the latter vanishes by [\[Pep19b, 3.9\]](#). □

Lemma 3.3.2.3. *Let R be a \mathbb{Q} algebra, let S be a normal scheme, and let $j: U \rightarrow S$ be an open immersion. Then, the canonical map*

$$\underline{R} \rightarrow j_* \underline{R}$$

is an equivalence.

Proof. The canonical map $\underline{R} \rightarrow j_* \underline{R}$ is an equivalence if and only if for any geometric point ξ of S , the canonical map

$$R \rightarrow \xi^* j_* \underline{R}$$

is an equivalence. If n is an integer, [\[AGV73, VIII.5\]](#) yields

$$H^n(\xi^* j_* \underline{R}) = H^n_{\text{ét}}(\text{Spec}(\mathcal{O}_{S,x}^{sh}) \times_S U, R)$$

where $\mathcal{O}_{S,x}^{sh}$ is the strict henselization of the local ring $\mathcal{O}_{S,x}$ of S at x .

Since the scheme S is normal, so is the scheme $\text{Spec}(\mathcal{O}_{S,x}^{sh}) \times_S U$. Moreover, this scheme is also connected. Therefore Proposition [3.1.7.2](#) yields the result. □

Remark 3.3.2.4. Proposition [3.3.2.1](#) is specific to curves. Assume that $R = \mathbb{Q}$. A reasonable equivalent to Proposition [3.3.2.1](#) would then be that if S is an excellent scheme, $i: F \rightarrow S$ a closed immersion and $j: U \rightarrow S$ is the complementary open immersion, with F a Cartier divisor (so that the morphism j is affine) and U regular, then, the 2-morphism

$$\Theta: \rho_! i^* j_* \rightarrow \omega^0 i^* j_* \rho_!$$

is an equivalence.

However, this statement is not true. We now give a counterexample. Let k be a field. Take $S = \text{Spec}(k[x, y, z]/(x^2 + y^2 - z^2))$ to be the affine cone over the rational curve of degree 2 in \mathbb{P}^2 given by the homogeneous equation $x^2 + y^2 = z^2$. The scheme S has a single singular point $P = (0, 0, 0)$.

Let F be the Cartier divisor of S given by the equation $x = 0$, so that F is the union of the two lines

$$L_1: y - z = 0 \text{ and } L_2: y + z = 0$$

which contain P and therefore the complement U of F is regular.

The blow up $B_P(S)$ of S at P is a resolution of singularities of S and the exceptional divisor E above F is the reunion of \mathbb{P}_k^1 and of the strict transform L'_i of the line L_i for $i \in \{1, 2\}$. Each L'_i crosses \mathbb{P}_k^1 at a single point and L'_1 and L'_2 do not cross each other.

We therefore have a commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{\gamma} & \tilde{S} = B_P(S) & \xleftarrow{\iota} & E = L'_1 \cup \mathbb{P}_k^1 \cup L'_2 \\ \parallel & & \downarrow f & & \downarrow p \\ U & \xrightarrow{j} & S & \xleftarrow{i} & F = L_1 \cup L_2 \end{array}$$

where both squares are cartesian.

Now, Lemma [3.3.2.3](#) yields

$$\rho_! i^* j_* \underline{R} = \mathbb{1}_F.$$

On the other hand, we have

$$\omega^0 i^* j_* \mathbb{1}_U = \omega^0 i^* f_* \gamma_* \mathbb{1}_U = \omega^0 p_* \iota^* \omega^0 \gamma_* \mathbb{1}_U$$

by Proposition [3.1.4.5](#). But Lemma [3.3.2.2](#) ensures that

$$\omega^0 \gamma_* \mathbb{1}_U = \mathbb{1}_{\tilde{S}}$$

and thus, we get

$$\omega^0 i^* j_* \mathbb{1}_U = \omega^0 p_* \mathbb{1}_E.$$

Finally, notice that $\omega^0 p_* \mathbb{1}_E$ does not coincide with $\mathbb{1}_F$ since by Theorem [3.3.1.4](#) $\omega^0 i_x^! \mathbb{1}_F$ is trivial while $\omega^0 i_x^! \omega^0 p_* \mathbb{1}_E$ is equivalent to $\mathbb{1}_x$.

3.4 The ordinary homotopy t -structure on Artin motives

3.4.1 Definition for non-constructible Artin motives

Definition 3.4.1.1. *Let S be a scheme and let R be a ring. The ordinary homotopy t -structure on $\mathcal{DM}_{\text{ét}}^A(S, R)$ is the t -structure generated in the sense of Proposition 3.1.1.6 by the family of the motives $M_S^{\text{BM}}(X)$ with X quasi-finite over S .*

We denote with the small letters ord the notions related to the ordinary homotopy t -structure on $\mathcal{DM}_{\text{ét}}^A(S, R)$. For instance, we denote by $\text{ord}\tau_{\leq 0}$ the left truncation with respect to the ordinary homotopy t -structure and by t_{ord} the ordinary homotopy t -structure itself.

Using the same method as Proposition 3.1.4.1, this t -structure is also generated by family of the motives $M_S^{\text{BM}}(X)$ with X finite (resp. étale) over S . Thus, an object M of $\mathcal{DM}_{\text{ét}}^A(S, R)$ is in $\mathcal{DM}_{\text{ét}}^A(S, R)^{\geq n}$ if and only if for any étale S -scheme X , the complex $\text{Map}_{\mathcal{DM}_{\text{ét}}(S, R)}(M_S(X), M)$ is $(n - 1)$ -connected.

Proposition 3.4.1.2. *Let R be a ring. Let f be a quasi-finite morphism of schemes and let g be a morphism of schemes. Then,*

1. *The adjunction $(f_!, \omega^0 f^!)$ is a t_{ord} -adjunction.*
2. *If M is a t_{ord} -non-positive Artin motive, the functor $- \otimes_S M$ is right t_{ord} -exact.*
3. *The adjunction $(g^*, \omega^0 g_*)$ is a t_{ord} -adjunction.*

Proof. The full subcategory of those Artin motives N such that $f_! N$, $g^* N[d]$ and $N \otimes M$ are t_{ord} -non-positive is closed under extensions, small coproducts and negative shifts and contains the set of generators of the t -structure. Thus, by definition of t_{ord} , this subcategory contains any t_{ord} -non-positive object. Hence, the functors $f_!$, $g^*[d]$ and $- \otimes M$ are right t_{ord} -exact. \square

Corollary 3.4.1.3. *Let R be a ring and let f be a morphism of schemes. Then,*

1. *If f is étale, the functor $f^* = f^! = \omega^0 f^!$ is t_{ord} -exact.*
2. *If f is finite, the functor $f_! = f_* = \omega^0 f_*$ is t_{ord} -exact.*

Proposition 3.4.1.4. *Let S be a scheme, let R be a ring, let $i: F \rightarrow S$ be a closed immersion and $j: U \rightarrow S$ be the open complement. Then, the category $\mathcal{DM}_{\text{ét}}^A(S, R)$ is a gluing of the pair $(\mathcal{DM}_{\text{ét}}^A(U, R), \mathcal{DM}_{\text{ét}}^A(Z, R))$ along the fully faithful functors i_* and $\omega^0 j_*$ in the sense of [Lur17, A.8.1], i.e. the functors i_* and $\omega^0 j_*$ have left adjoint functors i^* and j^* such that*

1. *We have $j^* i_* = 0$.*
2. *The family (i^*, j^*) is conservative.*

In particular, by [Lur17, A.8.5, A.8.13], the sequence

$$\mathcal{DM}_{\text{ét}}^A(F, R) \xrightarrow{i_*} \mathcal{DM}_{\text{ét}}^A(S, R) \xrightarrow{j^*} \mathcal{DM}_{\text{ét}}^A(U, R)$$

satisfies the axioms of the gluing formalism of [BBDG18, 1.4.3].

Moreover, the ordinary homotopy t -structure on $\mathcal{DM}_{\text{ét}}^A(S, R)$ is obtained by gluing the t -structures of $\mathcal{DM}_{\text{ét}}^A(U, \mathbb{Z}_\ell)$ and $\mathcal{DM}_{\text{ét}}^A(F, R)$ (see [BBDG18, 1.4.9]) i.e. for any object M of $\mathcal{DM}_{\text{ét}}^A(S, R)$, we have

1. $M \geq_{\text{ord}} 0$ if and only if $j^*M \geq_{\text{ord}} 0$ and $\omega^0 i^! M \geq_{\text{ord}} 0$.
2. $M \leq_{\text{ord}} 0$ if and only if $j^*M \leq_{\text{ord}} 0$ and $i^*M \leq_{\text{ord}} 0$.

Proof. This follows from Proposition [3.4.1.2](#), Corollary [3.4.1.3](#) and from the usual properties of the six functors. \square

Corollary 3.4.1.5. *Let S be a scheme, let R be a ring and let M be an object of $\mathcal{DM}_{\text{ét}}^A(S, R)$. Then, we have*

1. $M \geq_{\text{ord}} 0$ if and only if there is a stratification of S such that for any stratum $i: T \hookrightarrow S$, we have $\omega^0 i^! M \geq_{\text{ord}} 0$.
2. $M \leq_{\text{ord}} 0$ if and only if there is a stratification of S such that for any stratum $i: T \hookrightarrow S$, we have $i^* M \leq_{\text{ord}} 0$.

Lemma 3.4.1.6. *Let S be a scheme and let R be a ring. Let X be finite over S . Then, the motive $h_S(X)$ lies in the heart of the ordinary homotopy t -structure.*

Proof. If f is finite, the functor f_* is t_{ord} -exact by Corollary [3.4.1.3](#) and therefore, we may assume that $X = S$. By definition, we have

$$\mathbb{1}_S \leq_{\text{ord}} 0.$$

Let U be étale over S , and let n be an integer, by definition, we have

$$\pi_n \text{Map}_{\mathcal{DM}_{\text{ét}}(S, R)}(M_S(U), \mathbb{1}_S) = H_{\mathcal{M}, \text{ét}}^{n, 0}(U, R).$$

But if the integer n is negative, this group vanishes by Lemma [3.1.7.1](#) and the result follows. \square

We now explore the compatibility of the ordinary homotopy t -structure with respect to change of coefficients.

Proposition 3.4.1.7. *Let S be a scheme, let R be a ring, let n be an integer and let A be a localization of \mathbb{Z} . Denote by $\sigma_n: R \rightarrow R/nR$ and by $\sigma_A: R \rightarrow R \otimes_{\mathbb{Z}} A$ the natural ring morphisms. Recall the notations of Section [3.1.2](#). Then,*

1. The functors $(\sigma_A)_*$ and $(\sigma_n)_*$ are t_{ord} -exact.
2. the functor σ_A^* is t_{ord} -exact.

Proof. This follows from Lemma [3.1.1.8](#) and from Proposition [3.1.2.3](#). \square

Proposition 3.4.1.8. *Let S be a scheme and let R be a ring. Recall Proposition [3.1.4.10](#). Then, the inclusion functor*

$$\mathcal{DM}_{\text{ét}, \text{tors}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

is t -exact when the left hand side is endowed with the ordinary t -structure (see Definition [3.1.3.7](#)) and when the right hand side is endowed with the ordinary homotopy t -structure.

Proof. If X is étale over S , the functor $\rho_!$ maps the étale sheaf $R_S(X)$ to the motive $M_S(X)$ which is t_{ord} -non-positive. Recall that Proposition 3.1.3.5 asserts that the sheaves of the form $R_S(X)$ with X étale over S generate the ordinary t -structure on the stable category $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$. Since the exact functor

$$\rho_! : \mathcal{S}\mathcal{h}(S_{\text{ét}}, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

of Proposition 3.1.4.3 is compatible with small colimits, it is therefore right t -exact.

If p is a prime number, we denote by $j_p : S[1/p] \rightarrow S$ the open immersion. Recall that by Proposition 3.1.3.3, the functor j_p^* induces an equivalence

$$\mathcal{DM}_{\text{ét}, p^\infty\text{-tors}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}, p^\infty\text{-tors}}(S[1/p], R)$$

with inverse $(j_p)_! = (j_p)_*$.

Now, let M be a torsion étale motive. Assume first that M is t -non-positive with respect to the ordinary t -structure. By definition, this means that

$$M = \bigoplus_{p \text{ prime}} (j_p)_! \rho_! N_p,$$

where for any prime number p , N_p is a p^∞ -torsion étale sheaf which is t -non-positive with respect to the ordinary t -structure.

By right t -exactness of the functor $\rho_!$, for any prime number p , the motive $\rho_! N_p$ is t_{ord} -non-positive. Hence, using Proposition 3.4.1.2, so is the motive $(j_p)_! \rho_! N_p$. Therefore, the motive M is t_{ord} -non-positive.

Assume now that M is t -non-negative with respect to the ordinary t -structure. Let X be an étale S -scheme. By Proposition 3.1.2.7, we get

$$\text{Map}_{\mathcal{DM}_{\text{ét}}(S, R)}(M_S(X), M) = \bigoplus_{p \text{ prime}} \text{colim}_n \text{Map}_{\mathcal{DM}_{\text{ét}}(S, R)}(M_S(X), M \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z}[-1]).$$

If p is a prime number and if n is an integer, denote by M_{p^n} the motive $M \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z}[-1]$. We have

$$\text{Map}_{\mathcal{DM}_{\text{ét}}(S, R)}(M_S(X), M_{p^n}) = \text{Map}_{\mathcal{DM}_{\text{ét}}(S, R/nR)}(M_S(X, R/nR), M_{p^n}).$$

Since by [CD16, A.3.4], we have

$$M_{p^n} = (j_p)_* j_p^* M_{p^n},$$

we get

$$\text{Map}_{\mathcal{DM}_{\text{ét}}(S, R/nR)}(M_S(X, R/nR), M_{p^n}) = \text{Map}_{\mathcal{DM}_{\text{ét}}(S[1/p]_{\text{ét}}, R/nR)}(M_{S[1/p]}(X[1/p]), j_p^*(M_{p^n})).$$

And therefore, we get

$$\text{Map}_{\mathcal{DM}_{\text{ét}}(S, R)}(M_S(X), M_{p^n}) = \text{Map}_{\mathcal{S}\mathcal{h}(S[1/p]_{\text{ét}}, R)}(R_{S[1/p]}(X[1/p]), \rho^! j_p^*(M_{p^n})).$$

If p is a prime number and if n is an integer, the p^∞ -torsion étale motive $M \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z}[-1]$ is t -non-negative with respect to the ordinary t -structure. Therefore, by definition of the ordinary t -structure, the p^∞ -torsion étale sheaf $\rho^! j_p^*(M_{p^n})$ is t -non-negative with respect to the ordinary t -structure. Therefore, the complex $\text{Map}_{\mathcal{S}\mathcal{h}(S[1/p]_{\text{ét}}, R)}(R_{S[1/p]}(X[1/p]), \rho^! j_p^*(M_{p^n}))$ is (-1) -connected. Hence, the complex $\text{Map}_{\mathcal{DM}_{\text{ét}}(S, R)}(M_S(X), M)$ is also (-1) -connected. Hence, the motive M is t_{ord} -non-negative. \square

3.4.2 The ordinary homotopy t-structure on constructible Artin motives and the t-exactness of the ℓ -adic realization

We begin this section with the case of smooth Artin motives when the residue characteristic exponents of the base scheme are invertible in the ring of coefficients.

Proposition 3.4.2.1. *Let R be a regular good ring and let S be a regular scheme. Assume that the residue characteristic exponents of S are invertible in R .*

Then, the functor $\rho_!$ of Proposition 3.1.4.3 induces a fully faithful t-exact functor

$$\rho_! : \mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

when the left hand side is endowed with the ordinary t-structure and the right hand side is endowed with the ordinary homotopy t-structure.

In particular, the ordinary t-structure on the stable category $\mathcal{DM}_{\text{ét}}^A(S, R)$ induces a t-structure on the stable subcategories $\mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$ and $\mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R)$.

Proof. In the proof of Proposition 3.4.1.8, we showed that the exact functor

$$\rho_! : \mathcal{S}\mathfrak{h}(S_{\text{ét}}, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

is right t-exact. Therefore, it suffices to show that the exact functor

$$\rho_! : \mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

is left t-exact. Let N be an Ind-lisse étale sheaf and let X be an étale S -scheme. By Theorem 3.2.1.1, we have

$$\text{Map}_{\mathcal{DM}_{\text{ét}}(S;R)}(M_S(X), \rho_!N) = \text{Map}_{\mathcal{S}\mathfrak{h}(S_{\text{ét}},R)}(R_S(X), N).$$

Hence, if the sheaf N is t-non-negative, by Proposition 3.1.1.7, so is the motive $\rho_!N$. Therefore, the functor

$$\rho_! : \mathcal{S}\mathfrak{h}_{\text{Ind lisse}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

is t-exact.

By Theorem 3.2.1.1, the functor $\rho_!$ is fully faithful with essential image the stable subcategory $\mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$. Therefore, the ordinary homotopy t-structure on the stable category $\mathcal{DM}_{\text{ét}}^A(S, R)$ induces a t-structure on the stable subcategory $\mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$.

Finally, since the ordinary t-structure on the stable category $\mathcal{S}\mathfrak{h}(S_{\text{ét}}, R)$ induces by Proposition 3.1.5.9 a t-structure on the stable subcategory $\mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R)$, the ordinary homotopy t-structure on the stable category $\mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$ also induces a t-structure on the stable subcategory $\mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R)$ by Theorem 3.2.1.1. \square

We want to show that the ordinary homotopy t-structure induces a t-structure on constructible Artin motives and to study the properties of the induced t-structure. In the case of Artin motives with rational coefficients, this was done in [Pep19a]. We will use Proposition 3.4.1.8 to extend Pepin Lehalleur's result to arbitrary regular good rings. The first step is to extend Pepin Lehalleur's result on the t-exactness of the functor f^* for any morphism of schemes f to a result with integral coefficients. To that end, we need to introduce the following auxiliary notion.

Definition 3.4.2.2. Let S be a scheme and let R be a ring. An Artin étale motive M over S with coefficients in R is \mathbb{Q} -constructible if the Artin motive $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is constructible. We denote by $\mathcal{DM}_{\text{ét}, \mathbb{Q}-c}^A(S, R)$ the thick stable subcategory of the stable category $\mathcal{DM}_{\text{ét}, c}^A(S, R)$ made of those objects which are \mathbb{Q} -constructible.

Recall the following definition from [Pep19a].

Definition 3.4.2.3. We say that a scheme S allows resolution of singularities by alterations if for any separated S -scheme X of finite type and any nowhere dense closed subset Z of X , there is a proper alteration $g: X' \rightarrow X$ with X' regular and such that $g^{-1}(Z)$ is a strict normal crossing divisor.

Proposition 3.4.2.4. Let S be an excellent scheme allowing resolution of singularities by alterations and let R be a regular good ring. The ordinary homotopy t -structure on the stable category $\mathcal{DM}_{\text{ét}}^A(S, R)$ induces a t -structure on the stable subcategory $\mathcal{DM}_{\text{ét}, \mathbb{Q}-c}^A(S, R)$.

Proof. Assume first that the ring R is a \mathbb{Q} -algebra. The proposition precisely asserts that the ordinary homotopy t -structure induces a t -structure on the stable category $\mathcal{DM}_{\text{ét}, c}^A(S, R)$.

But by [Pep19b, 4.17], the inclusion functor

$$\mathcal{DM}_{\text{ét}}^A(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^1(S, R)$$

is t -exact when the left hand side is endowed with the ordinary homotopy t -structure and the right hand side is endowed with the standard motivic t -structure of Pepin Lehalleur defined in [Pep19b, 4.4]. By [Pep19a, 4.1], the truncation ${}^{\text{std}}\tau_{\leq 0}$ with respect to the standard t -structure preserves compact objects. Therefore, the functor ${}^{\text{std}}\tau_{\leq 0} = {}^{\text{ord}}\tau_{\leq 0}$ preserves $\mathcal{DM}_{\text{ét}, c}^A(S, R)$. Hence, using [BBDG18, 1.3.19], the ordinary homotopy t -structure induces a t -structure on the stable category $\mathcal{DM}_{\text{ét}, c}^A(S, R)$.

Take now a general ring R . Let M be a \mathbb{Q} -constructible Artin motive. Proposition [3.4.1.7] implies that

$${}^{\text{ord}}\tau_{\leq 0}(M) \otimes_{\mathbb{Z}} \mathbb{Q} = {}^{\text{ord}}\tau_{\leq 0}(M \otimes_{\mathbb{Z}} \mathbb{Q}).$$

But the Artin motive ${}^{\text{ord}}\tau_{\leq 0}(M \otimes_{\mathbb{Z}} \mathbb{Q})$ is constructible using the result with coefficients in $R \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore, the Artin motive ${}^{\text{ord}}\tau_{\leq 0}(M)$ is \mathbb{Q} -constructible. The result then follows from [BBDG18, 1.3.19]. \square

We can now formulate and prove an analog of Pepin Lehalleur’s result on the t -exactness of the functor f^* to for any morphism of schemes f .

Proposition 3.4.2.5. Let S be an excellent scheme allowing resolution of singularities by alterations, let $f: T \rightarrow S$ be a morphism of schemes and let R be a ring.

Then, the functor

$$f^*: \mathcal{DM}_{\text{ét}, \mathbb{Q}-c}^A(S, R) \rightarrow \mathcal{DM}_{\text{ét}, \mathbb{Q}-c}^A(T, R)$$

is t -exact when both sides are endowed with the ordinary homotopy t -structure.

Proof. Proposition [3.4.1.2] implies that the functor f^* is right t_{ord} -exact. Hence, it suffices to show that it is left t_{ord} -exact.

Let M be a \mathbb{Q} -constructible motive over S which is t_{ord} -non-negative. We have an exact triangle

$$M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1] \rightarrow M \rightarrow M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By Proposition 3.4.1.7, the motive $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is t_{ord} -non-negative. Since the subcategory of t_{ord} -non-negative objects is closed under limits, the motive $M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1]$ is also t_{ord} -non-negative.

Furthermore, we have an exact triangle

$$f^*(M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1]) \rightarrow f^*(M) \rightarrow f^*(M \otimes_{\mathbb{Z}} \mathbb{Q}).$$

By [Pep19a, 4.1], the motive $f^*(M \otimes_{\mathbb{Z}} \mathbb{Q})$ is t_{std} -non-negative and therefore t_{ord} -non-negative by [Pep19b, 4.17]. Furthermore, by Proposition 3.1.3.9, the motive $f^*(M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1])$ is t_{ord} -non-negative. Since the subcategory of t -non-negative objects is closed under extensions, the motive $f^*(M)$ is also t_{ord} -non-negative. \square

Using, the above result, the t -exactness statement of Proposition 3.4.2.1 for lisse sheaves and without the assumption that the residue characteristics are invertible in the ring of coefficients.

Proposition 3.4.2.6. *Let R be a regular good ring and let S be a regular scheme.*

Then, the functor $\rho_!$ of Proposition 3.1.4.3 induces a t -exact functor

$$\rho_! : \mathcal{S}\mathfrak{h}_{\text{lisse}}(S, R) \rightarrow \mathcal{D}\mathcal{M}_{\text{ét}}^A(S, R)$$

when the left hand side is endowed with the ordinary t -structure and the right hand side is endowed with the ordinary homotopy t -structure.

Proof. The family $(i_x^*)_{x \in S}$ is conservative by [CD19, 4.3.17] and the functors i_x^* are t -exact when x is a point of S by Proposition 3.4.2.5. Therefore, by Lemma 3.1.1.9 and proposition 3.1.2.1, we can assume that S is the spectrum of a field k . In that case, by Corollary 3.2.1.7, letting $\sigma : R \rightarrow R[1/p]$ be the canonical morphism, the down horizontal and right vertical arrows of the diagram

$$\begin{array}{ccc} \mathcal{S}\mathfrak{h}_{\text{lisse}}(k, R) & \xrightarrow{\rho_!} & \mathcal{D}\mathcal{M}_{\text{ét}}^A(k, R) \\ \downarrow \sigma^* & & \downarrow \sigma^* \\ \mathcal{S}\mathfrak{h}_{\text{lisse}}(k, R[1/p]) & \xrightarrow{\rho_!} & \mathcal{D}\mathcal{M}_{\text{ét}}^A(k, R[1/p]) \end{array}$$

are equivalences; those equivalences are furthermore t -exact by Propositions 3.4.1.7 and 3.4.2.1. Since the left vertical arrow is also t -exact, this finishes the proof. \square

We can now prove the main result of this section which extends [Pep19a, 4.1].

Theorem 3.4.2.7. *Let S be a scheme allowing resolution of singularities by alterations and let R be a regular good ring. Then,*

1. *The ordinary homotopy t -structure induces a t -structure on $\mathcal{D}\mathcal{M}_{\text{ét},c}^A(S, R)$. We still call this t -structure the ordinary homotopy t -structure.*
2. *The objects of $\mathcal{D}\mathcal{M}_{\text{ét},c}^A(S, R)$ are bounded with respect to the ordinary homotopy t -structure which is therefore non-degenerate.*
3. *Let $f : T \rightarrow S$ be a morphism of schemes. Then, the functor*

$$f^* : \mathcal{D}\mathcal{M}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}\mathcal{M}_{\text{ét},c}^A(T, R)$$

is t_{ord} -exact.

4. If x is a point of S , denote by $i_x: \{x\} \rightarrow S$ the inclusion. The family

$$(i_x^*: \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{DM}_{\text{ét},c}^A(k(x), R) \mid x \in S)$$

is a conservative family of t_{ord} -exact functors.

5. Assume that the ring R is a localization of the ring of integers of a number field K . Let v be a non-archimedean valuation on K and let ℓ be the prime number such that v extends the ℓ -adic valuation. Then, the reduced v -adic realization functor

$$\bar{\rho}_v: \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}_c^b(S[1/\ell], R_v)$$

of Definition 3.2.2.1 is t -exact when the left hand side is endowed with the ordinary homotopy t -structure and the right hand side is endowed with the ordinary t -structure.

Proof. First notice that if the first assertion is true, the third assertion makes sense and follows from Proposition 3.4.2.5. Furthermore, the third assertion implies the fourth assertion since the family $(i_x^*)_{x \in S}$ is conservative by [CD19, 4.3.17]. Using Lemma 3.1.1.9, the fourth assertion implies the fifth assertion. Finally the second assertion follows from Lemma 3.4.1.6. Hence, it suffices to show the first assertion.

If \mathfrak{p} is a prime ideal of \mathbb{Z} and if X is a scheme, we say that an Artin motive is \mathfrak{p} -constructible when, with the notations of Section 3.1.1, its image in the category $\mathcal{DM}_{\text{ét}}^A(X, R)_{\mathfrak{p}}$ belongs to $\mathcal{DM}_{\text{ét},c}^A(X, R)_{\mathfrak{p}}$. Notice that

- if f is a morphism of schemes, the functor f^* preserve \mathfrak{p} -constructible Artin motives.
- if f is a quasi-finite morphism of schemes, the functor $f_!$ preserve \mathfrak{p} -constructible Artin motives.
- the category of \mathfrak{p} -constructible motives over X is a thick subcategory of $\mathcal{DM}_{\text{ét}}^A(X, R)$ for any scheme X .

Let M be a constructible Artin motive over S . We want to show that the Artin motive ${}^{\text{ord}}\tau_{\leq 0}(M)$ is constructible. By Proposition 3.1.1.13, it suffices to show that it is \mathfrak{p} -constructible for any maximal ideal \mathfrak{p} of \mathbb{Z} .

Fix a maximal ideal \mathfrak{p} of \mathbb{Z} . Let p be a generator of \mathfrak{p} , let $S_p = S \times_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$, let $i_p: S_p \rightarrow S$ be the immersion and let $j_p: S[1/p] \rightarrow S$ be the complementary open immersion.

We have an exact triangle

$$(j_p)_! j_p^* {}^{\text{ord}}\tau_{\leq 0}(M) \rightarrow {}^{\text{ord}}\tau_{\leq 0}(M) \rightarrow (i_p)_* i_p^* {}^{\text{ord}}\tau_{\leq 0}(M).$$

The functors $(i_p)_*$ and $(j_p)_!$ preserve \mathfrak{p} -constructible Artin motives. Therefore, it suffices to show that the Artin motives $j_p^* {}^{\text{ord}}\tau_{\leq 0}(M)$ and $i_p^* {}^{\text{ord}}\tau_{\leq 0}(M)$ are \mathfrak{p} -constructible.

Using Proposition 3.4.2.5, we have

$$j_p^* {}^{\text{ord}}\tau_{\leq 0}(M) = {}^{\text{ord}}\tau_{\leq 0}(j_p^* M)$$

and

$$i_p^* {}^{\text{ord}}\tau_{\leq 0}(M) = {}^{\text{ord}}\tau_{\leq 0}(i_p^* M).$$

Hence, we can assume that either p is invertible on S or that S is of characteristic p .

Proposition [3.2.3.1](#) then gives a stratification of S with regular strata such that for any stratum $i: T \rightarrow S$, the motive i^*M is a constructible smooth Artin motive. Assume that for any stratum $i: T \rightarrow S$, the motive $\text{ord}\tau_{\leq 0}(i^*M)$ is \mathfrak{p} -constructible. Using the localization triangle and the stability of \mathfrak{p} -constructible objects under the functors f^* and $f_!$ for f quasi-finite, we can prove by induction on the number of strata that the motive $\text{ord}\tau_{\leq 0}(M)$ is constructible. Hence, we can assume that the scheme S is regular and that the motive M is a constructible smooth Artin motive.

If p is invertible on S , the result follows from Lemma [3.4.2.8](#) below.

Assume that S is of characteristic p . It suffices to show that the Artin motive $\text{ord}\tau_{\leq 0}(M)$ is a constructible smooth Artin motive. Using [[CD16](#), A.3.4], we can assume that p is invertible in R . The result then follows from Proposition [3.4.2.1](#). \square

Lemma 3.4.2.8. *Let R be a regular good ring, let S be a regular scheme, let \mathfrak{p} be a maximal ideal of \mathbb{Z} and let p be a generator of \mathfrak{p} . Assume that p is invertible in S .*

1. The functor $\rho_!$ of Proposition [3.1.4.3](#) induces a monoidal equivalence

$$\mathcal{S}\mathcal{H}_{\text{lisse}}(S, R)_{\mathfrak{p}} \longrightarrow \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R)_{\mathfrak{p}}.$$

2. The functor $\rho_!$ of Proposition [3.1.4.3](#) induces a t -exact fully faithful functor

$$(\rho_!)_{\mathfrak{p}}: \mathcal{S}\mathcal{H}_{\text{lisse}}(S, R)_{\mathfrak{p}} \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)_{\mathfrak{p}}$$

when the left hand side is endowed with the t -structure obtained by applying Proposition [3.1.1.11](#) to the ordinary t -structure and the right hand side is endowed with the t -structure obtained by applying Proposition [3.1.1.11](#) to the ordinary homotopy t -structure.

In particular, the ordinary t -structure on the stable category $\mathcal{DM}_{\text{ét}}^A(S, R)_{\mathfrak{p}}$ induces a t -structure on the stable subcategory $\mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)_{\mathfrak{p}}$.

Proof. The first assertion follows from Propositions [3.1.4.4](#) and [3.1.5.18](#) and Theorem [3.2.1.1](#). The second assertion then follows from the first assertion and from Propositions [3.1.1.12](#) and [3.4.2.6](#). \square

Definition 3.4.2.9. *Let S be a scheme allowing resolution of singularities by alterations and let R be a regular good ring. The abelian category of ordinary Artin motives $M_{\text{ord}}^A(S, R)$ is the heart of the ordinary homotopy t -structure of $\mathcal{DM}_{\text{ét},c}^A(S, R)$.*

Corollary 3.4.2.10. *Let S be a scheme allowing resolution of singularities by alterations and let R be a regular good ring. The following properties hold.*

1. Let $f: T \rightarrow S$ be a morphism of schemes. Then, the functor f^* induces an exact functor

$$f^*: M_{\text{ord}}^A(S, R) \rightarrow M_{\text{ord}}^A(T, R).$$

2. If x is a point of S , denote by $i_x: \{x\} \rightarrow S$ the inclusion and by $p(x)$ the characteristic exponent of $k(x)$. The family

$$\left(\alpha^! i_x^* : M_{\text{ord}}^A(S, R) \rightarrow \text{Rep}^A(G_{k(x)}, R[1/p(x)]) \mid x \in S \right)$$

is a conservative family of exact functors.

3. Assume that the ring R is a localization of the ring of integers of a number field K . Let v be a non-archimedean valuation on K and let ℓ be the prime number such that v extends the ℓ -adic valuation. Then, the reduced v -adic realization functor induces an exact and conservative functor

$$\bar{\rho}_v : M_{\text{ord}}^A(S, R) \rightarrow \text{Sh}^A(S[1/\ell], R_v).$$

3.5 The perverse homotopy t -structure on Artin motives

From now on, all schemes are assumed to be endowed with a dimension function δ (see Definition 3.1.3.11). If S is a scheme and if Λ is an ℓ -adic field or its ring of integers, we denote with the small letters p the notions related to the perverse t -structure on $\mathcal{D}_c^b(S, \Lambda)$. For instance, we denote by ${}^p\tau_{\leq 0}$ the left truncation with respect to the perverse t -structure and t_p denotes the perverse t -structure itself.

One of the main goals of this chapter is to construct the perverse motivic t -structure on constructible Artin étale motives in the following sense

Definition 3.5.0.1. *Let S be a scheme and R be a localization of the ring of integers of a number field K . Let t_0 be a t -structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)$. Recall the reduced v -adic realization functor (see Definition 3.2.2.1). We say that*

1. the t -structure t_0 is the perverse motivic t -structure if for any constructible Artin étale motive M , the motive M is t_0 -non-positive if and only if for all non-archimedean valuation v on K which is non-negative on R , the complex $\bar{\rho}_v(M)$ is t_p -non-positive (see Section 3.1.6).
2. the t -structure t_0 is the perverse motivic t -structure in the strong sense if for any non-archimedean valuation v on K which is non-negative on R , the functor

$$\bar{\rho}_v : \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}_c^b(S, R_v)$$

is t -exact when the left hand side is endowed with the t -structure t_0 and the right hand side is endowed with the perverse t -structure.

Remark 3.5.0.2. 1. If the perverse motivic t -structure exists, then it is unique.

2. If t is the perverse motivic t -structure in the strong sense, Lemma 3.1.1.9 and Proposition 3.2.2.2 ensure that it is the perverse motivic t -structure.

The most suitable candidate to be the perverse motivic t -structure will be the *perverse homotopy t -structure* (see Proposition 3.5.2.1 below) which we will construct and study in this section.

3.5.1 Definition for non-constructible Artin motives

Definition 3.5.1.1. *Let S be a scheme and let R be a ring. The perverse homotopy t -structure on $\mathcal{DM}_{\text{ét}}^A(S, R)$ is the t -structure generated in the sense of Proposition 3.1.1.6 by the family of the motives $M_S^{\text{BM}}(X)[\delta(X)]$ with X quasi-finite over S .*

We denote with the small letters hp the notions related to the perverse t-structure on $\mathcal{DM}_{\text{ét}}^A(S, R)$. For instance, we denote by ${}^{hp}\tau_{\leq 0}$ the left truncation with respect to the ordinary homotopy t-structure and t_{hp} denotes the perverse t-structure itself.

Using the same method as Proposition [3.1.4.1](#), this t-structure is also generated by family of the motives $h_S(X)[\delta(X)]$ with X finite over S . Thus, an object M of $\mathcal{DM}_{\text{ét}}^A(S, R)$ belongs to $\mathcal{DM}_{\text{ét}}^A(S, R)^{hp \geq n}$ if and only if for any finite S -scheme X the complex $\text{Map}_{\mathcal{DM}_{\text{ét}}^A(S, R)}(h_S(X), M)$ is $(n - \delta(X) - 1)$ -connected.

Proposition 3.5.1.2. *Let R be a ring. Let f be a quasi-finite morphism of schemes and let g be a morphism of relative dimension d .*

1. *The adjunction $(f_!, \omega^0 f^!)$ is a t_{hp} -adjunction.*
2. *If M is a t_{hp} -non-positive Artin motive, then, the functor $- \otimes_S M$ is right t_{hp} -exact.*
3. *If $\dim(g) \geq d$, then, the adjunction $(g^*[d], \omega^0 g_*[d])$ is a t_{hp} -adjunction.*

Proof. The proof is the same as the proof of Proposition [3.4.1.2](#) □

Corollary 3.5.1.3. *Let R be a ring and let f be a morphism of schemes. Then,*

1. *If f is étale, the functor $f^* = f^! = \omega^0 f^!$ is t_{hp} -exact.*
2. *If f is finite, the functor $f_! = f_* = \omega^0 f_*$ is t_{hp} -exact.*

Proposition 3.5.1.4. *Let S be a scheme, let R be a ring, let $i: F \rightarrow S$ be a closed immersion and let $j: U \rightarrow S$ be the open complement. Then, the perverse homotopy t-structure on $\mathcal{DM}_{\text{ét}}^A(S, R)$ is obtained by gluing the t-structures of $\mathcal{DM}_{\text{ét}}^A(U, \mathbb{Z}_\ell)$ and $\mathcal{DM}_{\text{ét}}^A(F, R)$ (see [\[BBDG18, 1.4.9\]](#)) i.e. for any object M of $\mathcal{DM}_{\text{ét}}^A(S, R)$, we have*

1. *$M \geq_{hp} 0$ if and only if $j^* M \geq_{hp} 0$ and $\omega^0 i^! M \geq_{hp} 0$.*
2. *$M \leq_{hp} 0$ if and only if $j^* M \leq_{hp} 0$ and $i^* M \leq_{hp} 0$.*

Proof. This follows from Propositions [3.4.1.4](#) and [3.5.1.2](#) and corollary [3.5.1.3](#). □

Corollary 3.5.1.5. *Let S be a scheme, let R be a ring and let M be an object of $\mathcal{DM}_{\text{ét}}^A(S, R)$. Then, we have*

1. *$M \geq_{hp} 0$ if and only if there is a stratification of S such that for any stratum $i: T \hookrightarrow S$, we have $\omega^0 i^! M \geq_{hp} 0$.*
2. *$M \leq_{hp} 0$ if and only if there is a stratification of S such that for any stratum $i: T \hookrightarrow S$, we have $i^* M \leq_{hp} 0$.*

Proposition 3.5.1.6. *Let S be a scheme and let R be a ring. Then, the constructible objects are bounded with respect to the perverse homotopy t-structure on $\mathcal{DM}_{\text{ét}}^A(S, R)$.*

Proof. The subcategory of bounded objects is thick. Therefore, it suffices to show that the motives of the form $f_* \mathbb{1}_X$ with $f: X \rightarrow S$ finite are bounded. By Corollary [3.5.1.3](#), the functor f_* is t_{hp} -exact when f is finite. Therefore, it suffices to show that the object $\mathbb{1}_S$ is bounded

with respect to t_{hp} for any scheme S . We already know that it lies in degree at most $\delta(S)$ by definition of t_{hp} .

Let Y be a finite S -scheme. The objects $h_S(Y)$ and $\mathbb{1}_S$ belong to the heart of the ordinary homotopy t -structure by Lemma 3.4.1.6. Therefore, the complex $\text{Map}(h_S(Y), \mathbb{1}_S)$ is (-1) -connected. Therefore, if c is a lower bound for δ , the complex $\text{Map}(h_S(X), \mathbb{1}_S[c])$ is $(\delta(X) - 1)$ -connected and therefore, the object $\mathbb{1}_S$ is bounded below by c . \square

Proposition 3.5.1.7. *Let S be a scheme, let R be a ring, let n be an integer and let A be a localization of \mathbb{Z} . Denote by $\sigma_n: R \rightarrow R/nR$ and by $\sigma_A: R \rightarrow R \otimes_{\mathbb{Z}} A$ the natural ring morphisms. Recall the notations of Section 3.1.2. Then,*

1. *The functors $(\sigma_A)_*$ and $(\sigma_n)_*$ are t_{hp} -exact.*
2. *the functor σ_A^* is t_{hp} -exact.*

Proof. This follows from Lemma 3.1.1.8 and from Proposition 3.1.2.3. \square

We now describe the perverse homotopy t -structure in the case when the base scheme is the spectrum of a field.

Proposition 3.5.1.8. *Let k be a field of characteristic exponent p and let R be a regular good ring. Then, the following properties hold.*

1. *The ordinary homotopy and the perverse homotopy t -structure on the stable category $\mathcal{DM}_{\text{ét}}^A(k, R)$ coincide up to a shift of $\delta(k)$.*
2. *The perverse homotopy t -structure induces a t -structure on $\mathcal{DM}_{\text{ét},c}^A(k, R)$.*
3. *If R is good, then, the functors*

$$\begin{aligned} \rho! : \mathcal{S}\mathfrak{h}_{\text{lisse}}(k, R[1/p]) &\rightarrow \mathcal{DM}_{\text{ét},c}^A(k, R) \\ \rho! : \mathcal{S}\mathfrak{h}(k_{\text{ét}}, R[1/p]) &\rightarrow \mathcal{DM}_{\text{ét}}^A(k, R) \end{aligned}$$

of Corollary 3.2.1.7 are t -exact equivalences when the right hand sides are endowed with the perverse homotopy t -structure and the left hand sides are endowed with the ordinary t -structure shifted by $\delta(k)$.

4. *If R is a localization of the ring of integers of a number field K and v is a non-archimedean valuation on K , the v -adic realization functor*

$$\mathcal{DM}_{\text{ét},c}^A(k, R) \rightarrow \mathcal{D}_c^b(k, R_v)$$

is t -exact when the left hand side is endowed with the perverse homotopy t -structure and the right hand side is endowed with the perverse t -structure.

Proof. Notice that t_{ord} and t_{hp} have the same set of generators up to a shift of $\delta(k)$. This proves the first point. The other points follow from the first point and from the properties of t_{ord} . \square

Proposition 3.5.1.9. *Let S be a scheme and let R be a ring. The inclusion functor*

$$\mathcal{DM}_{\text{ét},\text{tors}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

is t -exact when the left hand side is endowed with the perverse t -structure of Definition 3.1.3.15 and the right hand side is endowed with the perverse homotopy t -structure.

Proof. If $f: X \rightarrow S$ is quasi-finite, Proposition 3.1.2.1 implies that the functor $\rho_!$ maps the étale sheaf $f_! \underline{R}$ to the motive $M_S^{BM}(X)$. Recall that Proposition 3.1.3.13 asserts that the sheaves of the form $f_! \underline{R}[\delta(X)]$ with $f: X \rightarrow S$ quasi-finite generate the perverse t-structure on the stable category $\mathcal{S}\mathcal{h}(S_{\text{ét}}, R)$. Since the exact functor

$$\rho_!: \mathcal{S}\mathcal{h}(S_{\text{ét}}, R) \rightarrow \mathcal{DM}_{\text{ét}}(S, R)$$

is compatible with small colimits, it is therefore right t-exact.

We can prove the left t-exactness using the same proof as the proof of the left t-exactness in Proposition 3.4.1.8. We also give a shorter proof which uses Proposition 3.5.2.1(1)(a) below.

Let M be a torsion étale motive which is perverse t-non-negative. Proposition 3.1.3.17 ensures that M is bounded below (with respect to the ordinary t-structure) and that for any point x of S , we have

$$i_x^! M \geq -\delta(x)$$

with respect to the ordinary t-structure. Corollary 3.1.4.11 then ensures that

$$i_x^! M = \omega^0 i_x^! M.$$

Therefore, using Proposition 3.4.1.8, we have

$$\omega^0 i_x^! M \geq_{\text{ord}} -\delta(x).$$

Proposition 3.5.2.1(1)(a) below ensures that the Artin motive M is therefore t_{hp} -non-negative. \square

3.5.2 Locality and consequences

The following proposition summarizes the main features of the perverse homotopy t-structure. They are analogous to the properties of the perverse t-structure on ℓ -adic sheaves.

Proposition 3.5.2.1. *Let S be a scheme and R be a regular good ring. If x is a point of S , denote by $i_x: \{x\} \rightarrow S$ the inclusion. Then, the following properties hold.*

1. *Let M be an Artin motive over S with coefficients in R . Then,*

(a) *The Artin motive M is t_{hp} -non-negative if and only if it is bounded below with respect to t_{ord} and for any point x of S , we have*

$$\omega^0 i_x^! M \geq_{hp} 0,$$

i.e. $\omega^0 i_x^! M \geq_{\text{ord}} -\delta(x)$.

(b) *Assume that the Artin motive M is constructible. Then, it is t_{hp} -non-positive if and only if for any point x of S , we have*

$$i_x^* M \leq_{hp} 0,$$

i.e. $i_x^ M \leq_{\text{ord}} -\delta(x)$.*

2. *Assume that the scheme S is regular. Then,*

- (a) Assume that the regular scheme S is connected. Then, the functor $\rho_!$ of Proposition [3.1.4.3](#) induces a t -exact functor

$$\rho_! : \mathcal{S}\mathcal{h}_{\text{lisse}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

when the left hand side is endowed with the ordinary t -structure shifted by $\delta(S)$ and the right hand side is endowed with the perverse homotopy t -structure.

In particular, when the residue characteristic exponents of S are invertible in R , the perverse homotopy t -structure on the stable category $\mathcal{DM}_{\text{ét}}^A(S, R)$ induces a t -structure on the stable subcategory $\mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R)$.

- (b) If R is a localization of the ring of integers of a number field K , if the residue characteristic exponents of S are invertible in R , and if v is a non-archimedean valuation on K , the v -adic realization functor

$$\mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R) \rightarrow \mathcal{D}_c^b(S, R_v)$$

is t -exact when the left hand side is endowed with the perverse homotopy t -structure and the right hand side is endowed with the perverse t -structure.

3. Assume that S is excellent and that the ring R a localization of the ring of integers of a number field K . Then, the perverse motivic t -structure exists if and only if the perverse homotopy t -structure induces a t -structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)$. In that case, the perverse homotopy t -structure is the perverse motivic t -structure.

Proof. Notice that Assertion (2)(b) follows from Assertion (2)(a) and Theorem [3.4.2.7](#). We now prove the other assertions.

Proof of Assertion (1)(a):

If $M \geq_{hp} 0$, we already know from Proposition [3.5.1.2](#) that for any point x of S , we have $\omega^0 i_x^! M \geq_{hp} 0$. Furthermore, if c is a lower bound for the function δ on S , the complex $\text{Map}(M_S(X), M)$ is $(c - 1)$ -connected for any étale S -scheme X . Therefore, the Artin motive M is bounded below with respect to the ordinary homotopy t -structure.

Conversely, assume that M is t_{ord} -bounded below and that for any point x of S , the Artin motive $\omega^0 i_x^! M$ is t_{hp} -non-negative. Let $f : X \rightarrow S$ be quasi-finite. As in the proof of Proposition [3.1.3.13](#), the δ -niveau spectral sequence of [\[BD17, 3.1.5\]](#) and the strong continuity property of Proposition [3.1.3.10](#) yields

$$E_{p,q}^1 \Rightarrow \pi_{-p-q} \text{Map}_{\mathcal{DM}_{\text{ét}}(S,R)}(M_S^{\text{BM}}(X), M)$$

with

$$E_{p,q}^1 = \bigoplus_{y \in X, \delta(y)=p} \pi_{-p-q} \text{Map}_{\mathcal{DM}_{\text{ét}}(k(f(y)),R)}(M_{f(y)}^{\text{BM}}(y), \omega^0 i_{f(y)}^! M).$$

Since for any point x of S , we have $\omega^0 i_x^! M \geq_{hp} 0$, the complex

$$\text{Map}(M_{f(y)}^{\text{BM}}(y), \omega^0 i_{f(y)}^! M)$$

is $(-\delta(X) - 1)$ -connected for any point y of X .

Hence, if $n > \delta(X)$, the R -module $\pi_{-n} \text{Map}_{\mathcal{DM}_{\text{ét}}(S,R)}(M_S^{\text{BM}}(X), M)$ vanishes. Therefore, the Artin motive M is t_{hp} -non-negative.

Proof of Assertion (2)(a):

Proposition [3.1.5.17](#) implies that the functor

$$\rho_! : \mathcal{S}\mathcal{H}_{\text{Ind lisse}}(S, R) \rightarrow \mathcal{DM}_{\text{ét}}^A(S, R)$$

is right t-exact when the left hand side is endowed with the ordinary t-structure shifted by $-\delta(S)$ and the right hand side is endowed with the perverse homotopy t-structure. Since the scheme S is regular and connected $\delta(x) = -\text{codim}_S(x)$ is a dimension function on S . In particular, we can assume that $\delta(S) = 0$.

By dévissage, it therefore suffices to show that any étale sheaf which belongs to $\text{Loc}_S(R)$ is sent through the functor $\rho_!$ to a t_{hp} -non-negative object. Let M be such an étale sheaf. We have an exact triangle

$$M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1] \rightarrow M \rightarrow M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The sheaf $M \otimes_{\mathbb{Z}} \mathbb{Q}$ lies in $\text{Loc}_S(R \otimes_{\mathbb{Z}} \mathbb{Q})$.

Furthermore, if n is a positive integer, the sheaf $M \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}[-1]$ is t-non-negative with respect to the ordinary t-structure. Let x be a point of S . Proposition [3.1.2.7](#) yields

$$i_x^!(M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1]) = \text{colim}_n i_x^!(M \otimes \mathbb{Z}/n\mathbb{Z}[-1]).$$

Therefore, since the scheme S is regular, and since the sheaf $M \otimes \mathbb{Z}/n\mathbb{Z}[-1]$ is lisse, we get by absolute purity [[LYF14](#), XVI.3.1.1] that

$$i_x^!(M \otimes \mathbb{Z}/n\mathbb{Z}[-1]) = i_x^*(M \otimes \mathbb{Z}/n\mathbb{Z}[-1])(\delta(x))[2\delta(x)].$$

Therefore, we get

$$i_x^!(M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1]) = i_x^*(M \otimes \mathbb{Q}/\mathbb{Z}[-1])(\delta(x))[2\delta(x)].$$

Finally, we get

$$\omega^0 i_x^! \rho_!(M \otimes \mathbb{Q}/\mathbb{Z}[-1]) = \rho_! [i_x^*(M \otimes \mathbb{Q}/\mathbb{Z}[-1])(\delta(x))] [2\delta(x)]$$

where the left hand side in degree $-2\delta(x)$ or more. It is therefore in degree $-\delta(x)$ or more and therefore, the motive $\rho_!(M \otimes \mathbb{Q}/\mathbb{Z}[-1])$ is t-non-negative.

Hence, the torsion étale sheaf $M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1]$ is perverse t-non-negative and therefore it is perverse homotopy t-non-negative by Proposition [3.5.1.9](#).

We can therefore assume that the ring R is a \mathbb{Q} -algebra. In that case, Corollary [3.3.1.8](#) ensures that $\omega^0 i_x^!(M)$ vanishes when x is not the generic point of S . When x is the generic point of S , we get

$$\omega^0 i_x^!(M) = i_x^*(M).$$

Therefore, the sheaf $\omega^0 i_x^!(M)$ is in degree $0 = -\delta(x)$. Hence, Assertion (1)(a) ensures that M is perverse homotopy t-non-negative

Proof of Assertion (1)(b):

If $M \leq_{hp} 0$, we already know from Proposition [3.5.1.2](#) that for any point x of S , we have $i_x^* M \leq_{hp} 0$.

We now prove the converse. Corollary [3.1.2.6](#) and Proposition [3.5.1.7](#) ensure that the family $(- \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})_{p \text{ prime}}$ is conservative family which is made of t_{hp} -exact functors. Using

Lemma [3.1.1.9](#), we can therefore assume that the ring R is a $\mathbb{Z}_{(p)}$ -algebra where p is a prime number.

Write

$$S = S[1/p] \sqcup S_p$$

where $S_p = S \times_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. The subscheme $S[1/p]$ of S is open while the subscheme S_p is closed.

Let M be a constructible Artin motive and such that for all point x of S , we have

$$i_x^* M \leq_{hp} 0.$$

Corollary [3.2.3.4](#) ensures that we have a stratification $\mathcal{S}[1/p]$ of $S[1/p]$ with regular strata and a stratification \mathcal{S}_p of S_p with regular strata, lisse étale sheaves $M_T \in \mathcal{S}\mathcal{H}_{\text{lisse}}(T, R[1/p])$ for $T \in \mathcal{S}_p$ and lisse étale sheaves $M_T \in \mathcal{S}\mathcal{H}_{\text{lisse}}(T, R)$ for $T \in \mathcal{S}[1/p]$ such that for all any stratum T , we have

$$M|_T = \rho! M_T.$$

Lemma [3.5.2.2](#) below then ensures that $M|_T \leq_{hp} 0$ for each $T \in \mathcal{S}$. Therefore, using Corollary [3.5.1.5](#), we get

$$M \leq_{hp} 0.$$

Proof of Assertion (3):

Let M be in $\mathcal{DM}_{\text{ét},c}^A(S, R)$. Then,

$$\begin{aligned} M \leq_{hp} 0 &\iff \forall x \in S, i_x^* M \leq_{hp} 0 \\ &\iff \forall x \in S, \forall v \text{ valuation } i_x^* \bar{\rho}_v(M) \leq_p 0 \\ &\iff \forall v \text{ valuation } \bar{\rho}_v(M) \leq_p 0. \end{aligned}$$

Indeed the first equivalence follows from Assertion (1)(b); the second equivalence follows from Proposition [3.5.1.8](#) and the fact that the v -adic realization respects the six functors formalism; finally the third equivalence is a characterization of the perverse t -structure (see [\[BBDG18, 2.2.12\]](#)).

Hence, if the perverse homotopy t -structure induces a t -structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)$, it is the perverse motivic t -structure.

Conversely, assume that the perverse motivic t -structure t_{pm} exists. Then, its t -non-positive objects are the t_{hp} -non-negative objects which are constructible. Therefore, the t_{pm} -positive objects are those constructible objects N such that for all constructible object M with $M \leq_{hp} 0$, we have

$$\text{Hom}_{\mathcal{DM}_{\text{ét}}^A(S,R)}(M, N) = 0.$$

But the generators of the t_{hp} are constructible and thus, those objects are exactly the t_{hp} -positive objects which are constructible. □

Lemma 3.5.2.2. *Let S be a regular connected scheme and R be a regular ring and let*

$$\eta: \text{Spec}(K) \rightarrow S$$

be the generic point of S . Then, the functor

$$\eta^*: \mathcal{S}\mathcal{H}_{\text{lisse}}(S, R) \rightarrow \mathcal{S}\mathcal{H}_{\text{lisse}}(K, R)$$

is conservative and t -exact with respect to the ordinary t -structure.

Proof. The functor η^* is t-exact.

Assume that $f: C \rightarrow D$ is a map in $\mathcal{S}\mathcal{H}_{\text{lisse}}(S, R)$ such that $\eta^*(f)$ is invertible. Then, for any integer n , the map $H^n(\eta^*(f))$ is invertible. Since the functor η^* is t-exact, the map $\eta^*H^n(f)$ is invertible.

Let ξ be a geometric point with image η . By [GR71, V.8.2], the canonical map $G_K \rightarrow \pi_1^{\text{ét}}(S, \xi)$ is surjective. Thus, the map

$$\text{Rep}^A(\pi_1^{\text{ét}}(S, \xi), R) \rightarrow \text{Rep}^A(G_K, R)$$

induced by η^* is conservative. Therefore, for any integer n , the map $H^n(f)$ is invertible. Thus, the map f is invertible. \square

Remark 3.5.2.3. *Let S be a regular scheme and let R be a regular good ring. Assume that the residue characteristic exponents of S are invertible in R . Then, the perverse homotopy t-structure of $\mathcal{DM}_{\text{ét}}^A(S, R)$ induces a t-structure on $\mathcal{DM}_{\text{ét},c}^{\text{sm}A}(S, R)$. If T is a connected component of S , denote by $i_T: T \rightarrow S$ the clopen immersion. The heart of this t-structure is then the full subcategory made of those objects of the form*

$$\bigoplus_{T \in \pi_0(S)} (i_T)_* \rho_! L_T[\delta(T)]$$

where L_T is an object of $\text{Loc}_T(R)$ for any connected component T of S .

Proposition 3.5.2.4. *Let S be a regular scheme and let R be a regular good ring. Then, the Artin motive $\mathbb{1}_S$ is in degree $\delta(S)$ with respect to the perverse homotopy t-structure.*

Proof. Since S is regular, the function $\delta = -\text{codim}_S$ is a dimension function on S (for which $\delta(S) = 0$). Without loss of generality, we may work with this dimension function. We already know that $\mathbb{1}_S \leq_{hp} 0$ since it is a generator of the t-structure.

Let x be a point of S . Then, by the absolute purity property [CD16, 5.2.7], we have

$$\omega^0 i_x^! \mathbb{1}_S = \omega^0(\mathbb{1}_x(\delta(x)))[2\delta(x)].$$

If $\delta(x) = 0$, the point x is a generic point of S . Therefore, we have

$$\omega^0 i_x^! \mathbb{1}_S = \mathbb{1}_x$$

and therefore, the Artin motive $\omega^0 i_x^! \mathbb{1}_S$ in degree $\delta(x) = 0$.

Assume that $\delta(x) < 0$. Using Theorem 3.3.1.4, the Artin motive $\omega^0(\mathbb{1}_x(\delta(x)))$ is in degree 1 for the ordinary t-structure. Thus, the Artin motive $\omega^0 i_x^! \mathbb{1}_S$ is in degree $1 - 2\delta(x) \geq -\delta(x)$. \square

Definition 3.5.2.5. *Let S be a regular scheme and let R be a regular good ring. Assume that the residue characteristic exponents of S are invertible in R . The abelian category of perverse smooth Artin motives $\text{M}_{\text{perv}}^{\text{sm}A}(S, R)$ is the heart of the perverse homotopy t-structure on the stable category $\mathcal{DM}_{\text{ét}}^{\text{sm}A}(S, R)$.*

3.5.3 Induced t -structure on constructible motives with rational coefficients and weak Lefschetz properties

Proposition 3.5.3.1. *Let S be an excellent scheme and let K be a number field.*

1. *Then, the perverse homotopy t -structure on the stable category $\mathcal{DM}_{\text{ét}}^A(S, K)$ induces a t -structure on the stable subcategory $\mathcal{DM}_{\text{ét},c}^A(S, K)$.*
2. *Let $i: F \rightarrow S$ be a closed immersion and let $j: U \rightarrow S$ be the open complement. The sequence*

$$\mathcal{DM}_{\text{ét},c}^A(F, K) \xrightarrow{i_*} \mathcal{DM}_{\text{ét},c}^A(S, K) \xrightarrow{j^*} \mathcal{DM}_{\text{ét},c}^A(U, K)$$

satisfies the axioms of the gluing formalism of [BBDG18, 1.4.3] and the perverse homotopy t -structure on $\mathcal{DM}_{\text{ét},c}^A(S, K)$ is obtained by gluing the perverse homotopy t -structures of $\mathcal{DM}_{\text{ét},c}^A(U, K)$ and $\mathcal{DM}_{\text{ét},c}^A(F, K)$ (see [BBDG18, 1.4.9]).

3. *Let p be a prime number and let v be a non-archimedean valuation on K which does not extend the p -adic valuation. Assume that the scheme S is of finite type over \mathbb{F}_p . Then, the v -adic realization functor*

$$\rho_v: \mathcal{DM}_{\text{ét},c}^A(S, K) \rightarrow \mathcal{D}^A(S, K_v)$$

is t -exact and conservative when the right hand side is endowed with the perverse homotopy t -structure.

Proof. We can assume that S is reduced. The first part of (2) is a direct consequence of Proposition 3.5.1.4 and Proposition 3.1.4.8. To prove the rest of the proposition, we need to adapt the ideas of [BBDG18, 2.2.10] to our setting. This technique is very similar to Proposition 2.4.7.1.

We first need a few definitions:

- If \mathcal{S} is a stratification of S , we say that a locally closed subscheme X of S is an \mathcal{S} -subscheme of S if X is a union of strata of \mathcal{S} .
- A pair $(\mathcal{S}, \mathcal{L})$ is *admissible* if
 - \mathcal{S} is a stratification of S with regular strata everywhere of the same dimension.
 - for every stratum T of \mathcal{S} , $\mathcal{L}(T)$ is a finite set of isomorphism classes of objects of $M_{\text{perv}}^{\text{sm}A}(S, K)$ (see Definition 3.5.2.5).
- If $(\mathcal{S}, \mathcal{L})$ is an admissible pair and X is an \mathcal{S} -subscheme of S , the category $\mathcal{DM}_{\mathcal{S},\mathcal{L}}(X, K)$ of $(\mathcal{S}, \mathcal{L})$ -constructible Artin motives over X is the full subcategory of $\mathcal{DM}_{\text{ét}}^A(X, K)$ made of those objects M such that for any stratum T of \mathcal{S} contained in X , the restriction of M to T is a smooth Artin motive and its perverse cohomology sheaves are successive extensions of objects whose isomorphism classes lie in $\mathcal{L}(T)$.
- A pair $(\mathcal{S}', \mathcal{L}')$ refines a pair $(\mathcal{S}, \mathcal{L})$ if every stratum S of \mathcal{S} is a union of strata of \mathcal{S}' and any perverse smooth Artin motive M over a stratum T of \mathcal{S} whose isomorphism class lies in $\mathcal{L}(T)$ is $(\mathcal{S}', \mathcal{L}')$ -constructible.

If $(\mathcal{S}, \mathcal{L})$ is an admissible pair and if X is an \mathcal{S} -subscheme of S , the category $\mathcal{DM}_{\mathcal{S}, \mathcal{L}}(X, K)$ is a stable subcategory of $\mathcal{DM}_{\text{ét}}^A(S, K)$ (i.e. it is closed under finite (co)limits).

If $(\mathcal{S}, \mathcal{L})$ is an admissible pair and $i: U \rightarrow V$ is an immersion between \mathcal{S} -subschemes of S , the functors $i_!$ and i^* preserve $(\mathcal{S}, \mathcal{L})$ -constructible objects.

Now, we say that an admissible pair $(\mathcal{S}, \mathcal{L})$ is *superadmissible* if letting $i: T \rightarrow S$ be the immersion of a stratum T of \mathcal{S} into S and letting M be a perverse smooth Artin motive over T whose isomorphism class lies in $\mathcal{L}(T)$, the complex $\omega^0 i_* M$ is $(\mathcal{S}, \mathcal{L})$ -constructible.

We now claim that an admissible pair $(\mathcal{S}, \mathcal{L})$ can always be refined into a superadmissible one.

Assume that this is true when we replace S with the union of the strata of dimension n or more. Let $i: T \rightarrow S$ and $i': T' \rightarrow S$ be immersions of strata of \mathcal{S} . Let M be a perverse smooth Artin motive on T whose isomorphism class lies in $\mathcal{L}(T)$. If T and T' are of dimension at least n , then, the perverse cohomology sheaves of $(i')^* \omega^0 i_* M$ are obtained as successive extensions of objects of whose isomorphism classes lie in $\mathcal{L}(T')$ by induction.

If $T = T'$, then, by Proposition [3.1.4.5](#), we have

$$(i')^* \omega^0 i_* M = \omega^0 i'^* \omega^0 i_* M = \omega^0 i'^* i_* M = M$$

whose isomorphism class lies in $\mathcal{L}(T)$.

Otherwise, if T is of dimension $n - 1$, we can always replace T with an open subset, so that the closure of T is disjoint from T' . Then, by Proposition [3.1.4.5](#), we have

$$(i')^* \omega^0 i_* M = \omega^0 (i')^* i_* M = 0.$$

Assume now that T is of dimension n or more and that T' is of dimension $n - 1$. We can replace T' with an open subset so that for any M in $\mathcal{L}(T)$, the motive $(i')^* \omega^0 i_* M$ is smooth Artin by Propositions [3.1.4.8](#) and [3.2.3.1](#). We can also add all the isomorphism classes of the ${}^{hp}H^n((i')^* \omega^0 i_* M)$ to $\mathcal{L}(T')$ as there are finitely many of them.

Hence, any admissible pair can be refined into a superadmissible one.

If $(\mathcal{S}, \mathcal{L})$ is a superadmissible pair and if $i: U \rightarrow V$ is an immersion between \mathcal{S} -subschemes of S , we claim that the functors $\omega^0 i_*$ and $\omega^0 i^!$ preserve $(\mathcal{S}, \mathcal{L})$ -constructibility. The proof is the same as [\[BBDG18, 2.1.13\]](#) using as above the commutation of ω^0 with the six functors described in Proposition [3.1.4.5](#).

Hence, if $(\mathcal{S}, \mathcal{L})$ is a superadmissible pair, we can define a t-structure on $(\mathcal{S}, \mathcal{L})$ -constructible objects by gluing the perverse homotopy t-structures on the strata. By Proposition [3.5.2.1](#), the positive (resp. negative) objects of this t-structure are the positive (resp. negative) objects of the perverse homotopy t-structure that are $(\mathcal{S}, \mathcal{L})$ -constructible.

Thus, the perverse homotopy t-structure induces a t-structure on $(\mathcal{S}, \mathcal{L})$ -constructible objects. Therefore, the subcategory of $(\mathcal{S}, \mathcal{L})$ -constructible objects is stable under the truncation functors of the perverse homotopy t-structure.

Any object M of $\mathcal{DM}_{\text{ét}}^A(S, K)$ is $(\mathcal{S}, \mathcal{L})$ -constructible for some superadmissible pair $(\mathcal{S}, \mathcal{L})$. Indeed, by Proposition [3.2.3.1](#), there is an admissible pair $(\mathcal{S}, \mathcal{L})$ such that M is $(\mathcal{S}, \mathcal{L})$ -constructible. Such a pair is can always be refined into one that is superadmissible.

Hence, $\mathcal{DM}_{\text{ét}}^A(S, K)$ is stable under the truncation functors of the perverse homotopy t-structure. Therefore, the perverse homotopy t-structure induces a t-structure on $\mathcal{DM}_{\text{ét}}^A(S, K)$.

Now, the second part (2) follows from Proposition [3.5.1.4](#). Finally the third assertion follows from the construction of the perverse homotopy t-structure on constructible objects

that we used to prove the first assertion, from the same construction in the ℓ -adic setting of Proposition 2.4.7.1 and from the fact that the functor ω^0 commutes with the ℓ -adic realization functor by Lemma 2.4.6.4. \square

Remark 3.5.3.2. *Another possible proof of the first assertion is to show that the perverse homotopy t -structure can be constructed on constructible Artin motives by using Vaish’s formalism of punctual gluing [Vai19, 3].*

Using the same proof as Proposition 3.4.2.4, replacing Proposition 3.4.1.7 with Proposition 3.5.1.7 yields

Corollary 3.5.3.3. *Let S be an excellent scheme and let R be a ring. Then, the perverse homotopy t -structure on the stable category $\mathcal{DM}_{\text{ét}}^A(S, R)$ induces a t -structure on the stable subcategory $\mathcal{DM}_{\text{ét}, \mathbb{Q}-c}^A(S, R)$.*

We now prove weak motivic versions of the Affine Lefschetz Theorem (also known as the Artin vanishing theorem) of [AGV73, XIV.3.1], [LYF14, XV.1.1.2] and [BBDG18, 4.1.1] in the setting of Artin motives. We begin with the case of schemes of dimension at most 1.

Proposition 3.5.3.4. *Let S be an excellent scheme, let $f: U \rightarrow S$ be a quasi-finite morphism and let K be a number field. Assume that $\dim(S) \leq 1$, then, the functor*

$$f_!: \mathcal{DM}_{\text{ét}, c}^A(U, K) \rightarrow \mathcal{DM}_{\text{ét}, c}^A(S, K)$$

is t_{hp} -exact.

Proof. Corollary 3.5.1.3 implies that when g is a finite morphism, the functor $g_!$ is perverse homotopy t -exact. When the scheme S is 0-dimensional, the morphism f is finite and the result holds.

Assume that the scheme S is 1-dimensional. Zariski’s Main Theorem [Gro67, 18.12.13] provides a finite morphism g and an open immersion $j: U \rightarrow X$ such that $f = g \circ j$. Therefore it suffices to show that the functor

$$j_!: \mathcal{DM}_{\text{ét}, c}^A(U, K) \rightarrow \mathcal{DM}_{\text{ét}, c}^A(X, K)$$

is t_{hp} -exact.

We already know by Proposition 3.5.1.2 that $j_!$ is right t -exact.

Let M be a constructible Artin motive over U . Assume that M t_{hp} -non-negative. We want to show that the Artin motive $j_!(M)$ is t_{hp} -non-negative.

Let x be a point of X . Using Proposition 3.5.2.1, we can replace X with a neighborhood of x . Since the scheme X is 1-dimensional, we can therefore assume that the complement of U is the closed subscheme $\{x\}$ and replace U with any neighborhood of the generic points. In particular, we can assume that U is nil-regular and that M is smooth Artin using the continuity property of Proposition 3.1.2.8.

Let $i: \{x\} \rightarrow X$ be the canonical immersion. Applying the localization triangle (3.0.0.1) to the Artin motive $\omega^0 j_*(M)$ yields an exact triangle

$$j_!(M) \rightarrow \omega^0 j_*(M) \rightarrow i_* i^* \omega^0 j_*(M).$$

By Proposition [3.5.1.2](#), the motive $\omega^0 j_*(M)$ is t_{hp} -non-negative. Furthermore, Proposition [3.1.4.5](#) ensures that

$$i^* \omega^0 j_*(M) = \omega^0 i^* j_*(M).$$

Using Proposition [3.3.2.1](#), the Artin motive $\omega^0 i^* j_*(M)$ is in degree at least $\delta(X)$ with respect to the ordinary homotopy t-structure. Therefore, we get

$$\omega^0 i^* j_*(M) \geq_{hp} -1.$$

This ensures that the Artin motive $j_!(M)$ is perverse homotopy t-non-negative. □

Proposition 3.5.3.5. (*Affine Lefschetz Theorem for Artin motives*) *Let S be an excellent scheme, let $f: X \rightarrow S$ be a quasi-finite affine morphism and let K be a number field. Assume that X is nil-regular and that we are in one of the following cases*

- (a) *We have $\dim(S) \leq 2$.*
- (b) *There is a prime number p such that the scheme S is of finite type over \mathbb{F}_p .*

Then, the functor

$$f_!: \mathcal{DM}_{\text{ét},c}^{smA}(X, K) \rightarrow \mathcal{DM}_{\text{ét},c}^A(S, K)$$

is t_{hp} -exact.

Proof. We can assume that the scheme S is connected. As in the proof of Proposition [3.5.3.4](#), it suffices to show that if $j: U \rightarrow S$ is an affine open immersion such that the scheme U is nil-regular and if M is a t_{hp} -non-negative smooth Artin motive over U , the Artin motive $j_!M$ is t_{hp} -non-negative.

Suppose that Assumption (b) holds, *i.e.* there is a prime number p such that the scheme S is of finite type over \mathbb{F}_p . Let v be a non-archimedean valuation on K which does not extend the p -adic valuation. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{DM}_{\text{ét},c}^{smA}(U, K) & \xrightarrow{j_!} & \mathcal{DM}_{\text{ét},c}^A(S, K) \\ \downarrow \rho_v & & \downarrow \rho_v \\ \mathcal{D}^{smA}(U, K_v) & \xrightarrow{j_!} & \mathcal{D}^A(S, K_v) \end{array}$$

such that the vertical arrows are conservative and t-exact when the categories which appear in the bottom row are endowed with the perverse homotopy t-structure. Using Lemma [3.1.1.9](#), it suffices to show that the functor $j_!$ is t-exact which follows from Corollary [2.4.8.12](#).

We now suppose that Assumption (a) holds *i.e.* that $\dim(S) \leq 2$. By Proposition [3.5.3.4](#), we can assume that $\dim(S) = 2$. Take the convention that $\delta(S) = 2$. Let $i: F \rightarrow S$ be the reduced complementary closed immersion of j . As in the proof of Proposition [3.5.3.4](#), it suffices to show that the functor $\omega^0 i^* j_*$ has cohomological amplitude bounded below by -1 .

Step 1: In this step, we show that we can assume that the scheme S is normal.

Let $\nu: \hat{S} \rightarrow S$ be the normalization of S . Write

$$\begin{array}{ccccc} U_{\text{red}} & \xrightarrow{\gamma} & \hat{S} & \xleftarrow{\iota} & \hat{F} \\ \nu_U \downarrow & & \downarrow \nu & & \downarrow \nu_F \\ U & \xrightarrow{j} & S & \xleftarrow{i} & F \end{array}$$

the commutative diagram such that both squares are cartesian. Notice that U_{red} is the reduced scheme associated to U and therefore, the functor $(\nu_U)_*$ is a t -exact equivalence. Hence, it suffices to show that the functor $\omega^0 i^* j_* (\nu_U)_*$ has cohomological amplitude bounded below by -1 . This functor is equivalent to $\omega^0 (\nu_F)_* \iota^* \gamma_*$. Since ν_F is a finite morphism, the functor $(\nu_F)_*$ is t_{hp} -exact by Corollary 3.5.1.3 and commutes with ω^0 by Proposition 3.1.4.5. Therefore, suffices to show that the functor $\omega^0 \iota^* \gamma_*$ has cohomological amplitude bounded below by -1 . Therefore, we can assume that the scheme S is normal.

Step 2: In this step, we show that it suffices to show that $\omega^0 i^* j_* \mathbb{1}_U \geq_{hp} 1$.

Using Proposition 3.5.1.6, the objects of $\mathcal{DM}_{\text{ét},c}^{smA}(U, K)$ are bounded with respect to the perverse homotopy t -structure. Therefore, it suffices to show that the objects of the heart of the t -category $\mathcal{DM}_{\text{ét},c}^{smA}(U, R)$ are sent to objects of degree at least -1 with respect to the perverse homotopy t -structure. Letting ξ be a geometric point of U , the heart of $\mathcal{DM}_{\text{ét},c}^{smA}(U, R)$ is equivalent to $\text{Rep}^A(\pi_1^{\text{ét}}(U, \xi), R)$ by Theorem 3.2.1.1 and Proposition 3.1.5.10. Since K is a \mathbb{Q} -algebra, the latter is a semi-simple category by Maschke's Theorem. Thus, every object of the heart is a retract of an object of the form $f_* \mathbb{1}_V[2]$ with $f: V \rightarrow U$ finite and étale. Therefore, it suffices to show that $\omega^0 i^* j_* f_* \mathbb{1}_V \geq_{hp} 1$.

Let $\bar{f}: S_V \rightarrow S$ be the normalization of S in V . Write

$$\begin{array}{ccccc} V & \xrightarrow{\gamma} & S_V & \xleftarrow{\iota} & F_V \\ f \downarrow & & \downarrow \bar{f} & & \downarrow p \\ U & \xrightarrow{j} & S & \xleftarrow{i} & F \end{array}$$

the commutative diagram such that both squares are cartesian. By proper base change, we get

$$\omega^0 i^* j_* f_* \mathbb{1}_V = \omega^0 p_* \iota^* \gamma_* \mathbb{1}_V.$$

Since the map p is finite, the functor p_* is t_{hp} -exact by Corollary 3.5.1.3 and commutes with the functor ω^0 by Proposition 3.1.4.5. Therefore, it suffices to show that

$$\omega^0 \iota^* \gamma_* \mathbb{1}_V \geq_{hp} 1.$$

Hence, we can assume that $V = U$.

Step 3: Since the scheme S is normal its singular points are in codimension 2. Let x be a point of S . Using Proposition 3.5.2.1, we can replace S with a neighborhood of x . We can therefore assume that the singular locus of S is either empty or the single point x . Let $f: \tilde{S} \rightarrow S$ be a resolution of singularities of S such that E is a simple normal crossing divisor. Since the scheme S is excellent, such a resolution exists by Lipman's Theorem on embedded resolution of singularities (see [Sta23, 0BGP,0BIC,0ADX]). Write

$$\begin{array}{ccccc} U & \xrightarrow{\gamma} & \tilde{S} & \xleftarrow{\iota} & E \\ \parallel & & \downarrow f & & \downarrow p \\ U & \xrightarrow{j} & S & \xleftarrow{i} & F \end{array}$$

the commutative diagram such that both squares are cartesian.

Then, by proper base change, we get

$$\omega^0 i^* j_* \mathbb{1}_U = \omega^0 p_* \iota^* \gamma_* \mathbb{1}_U.$$

The latter is by Proposition [3.1.4.5](#) equivalent to $\omega^0 p_* \mathbb{L}_E$ which by Lemma [3.3.2.2](#) is equivalent to $\omega^0 p_* \mathbb{L}_E$. Therefore, it suffices to show that $\omega^0 p_* \mathbb{L}_E \geq_{hp} 1$.

Write $E = \bigcup_{i \in J} E_i$ with J finite, where the E_i are regular and of codimension 1, the $E_{ij} = E_i \cap E_j$ are of codimension 2 and regular if $i \neq j$ and the intersections of 3 distinct E_i are empty. By cdh-descent, we have an exact triangle

$$\omega^0 p_* \mathbb{L}_E \rightarrow \bigoplus_{i \in J} \omega^0 (p_i)_* \mathbb{L}_{E_i} \rightarrow \bigoplus_{\{i,j\} \subseteq J} (p_{ij})_* \mathbb{L}_{E_{ij}}.$$

Since we have $\delta(E_i) = 1$ for any i and since $\delta(E_{ij}) = 0$ if $i \neq j$, Propositions [3.5.1.2](#) and [3.5.2.4](#) imply that the Artin motives $\omega^0 (p_i)_* \mathbb{L}_{E_i}$ and $(p_{ij})_* \mathbb{L}_{E_{ij}}$ are t_{hp} -non-negative. Hence, we get $\omega^0 p_* \mathbb{L}_E \geq_{hp} 0$ and that ${}^{hp}H^0(\omega^0 p_* \mathbb{L}_E)$ is the kernel of the map

$$\bigoplus_{i \in J} {}^{hp}H^0(\omega^0 (p_i)_* \mathbb{L}_{E_i}) \rightarrow \bigoplus_{\{i,j\} \subseteq J} (p_{ij})_* \mathbb{L}_{E_{ij}}.$$

Hence, to finish the proof, it suffices to show that the kernel of the above map vanishes.

Let

$$F_0 = \{y \in F \mid \dim(f^{-1}(y)) > 0\}.$$

The scheme F_0 is 0-dimensional. Since we can work locally around x , we may assume that F_0 is either empty or the single point x and that the image of any E_{ij} through the map p is $\{x\}$.

Assume that the scheme F_0 is empty, then, the morphism p is finite. Furthermore, for any index i in J , the Artin motive \mathbb{L}_{E_i} is in degree 1 with respect to the perverse homotopy t-structure by Proposition [3.5.2.4](#), therefore, the Artin motive ${}^{hp}H^0(\omega^0 (p_i)_* \mathbb{L}_{E_i})$ vanishes. Thus, the motive ${}^{hp}H^0(\omega^0 p_* \mathbb{L}_E)$ vanishes.

Assume now that $F_0 = \{x\}$. Let

$$I = \{i \in J \mid p(E_i) = \{x\}\}$$

and let $k: \{x\} \rightarrow F$ be the inclusion. If an index i belongs to $J \setminus I$, the morphism p_i is finite and therefore, the motive ${}^{hp}H^0(\omega^0 (p_i)_* \mathbb{L}_{E_i})$ vanishes.

If an index i belongs to I , let

$$E_i \rightarrow G_i \xrightarrow{g_i} \{x\}$$

be the Stein factorization of p_i . Corollary [3.3.1.8](#) yields

$${}^{hp}H^0(\omega^0 (p_i)_* \mathbb{L}_{E_i}) = k_*(g_i)_* \mathbb{L}_{G_i}$$

and therefore

$$\bigoplus_{i \in I} {}^{hp}H^0(\omega^0 (p_i)_* \mathbb{L}_{E_i}) = k_* \bigoplus_{i \in I} (g_i)_* \mathbb{L}_{G_i}.$$

Now, if $i \neq j$, the morphism p_{ij} factors through $\{x\}$, so that we have a morphism $g_{ij}: E_{ij} \rightarrow \{x\}$ with $p_{ij} = k \circ g_{ij}$.

We have

$$\bigoplus_{\{i,j\} \subseteq J} (p_{ij})_* \mathbb{L}_{E_{ij}} = k_* \bigoplus_{\{i,j\} \subseteq J} (g_{ij})_* \mathbb{L}_{E_{ij}}.$$

Hence, the Artin motive ${}^{hp}H^0(\omega^0 p_* \mathbb{L}_E)$ is the kernel of the map

$$k_* \bigoplus_{i \in I} (g_i)_* \mathbb{L}_{G_i} \rightarrow k_* \bigoplus_{\{i,j\} \subseteq J} (g_{ij})_* \mathbb{L}_{E_{ij}}.$$

Since the functor k_* is t -exact, we get ${}^{hp}H^0(\omega^0 p_* \mathbb{1}_E) = k_* P$ with P the kernel of the map

$$\bigoplus_{i \in I} (g_i)_* \mathbb{1}_{G_i} \rightarrow \bigoplus_{\{i,j\} \subseteq J} (g_{ij})_* \mathbb{1}_{E_{ij}}.$$

Let $\kappa = k(x)$ and let $\Gamma = G_\kappa$. Recall that if $g: X \rightarrow \text{Spec}(\kappa)$ is a finite morphism, the Artin motive $g_* \mathbb{1}_X$ corresponds to the Artin representation $K[X_{\bar{\kappa}}]$ of Γ through the equivalence $\alpha_!$ (see Notations [3.3.1.7](#)). Hence, we have

$$P = \alpha_! P'$$

where P' is the kernel of the morphism

$$\bigoplus_{i \in I} K[(G_i)_{\bar{\kappa}}] \rightarrow \bigoplus_{\{i,j\} \subseteq J} K[(E_{ij})_{\bar{\kappa}}]$$

of Artin representations of Γ with coefficients in K .

Now, since K is flat over \mathbb{Z} , we have

$$P' = P_0 \otimes_{\mathbb{Z}} K$$

where P_0 is the kernel of the map

$$\bigoplus_{i \in I} \mathbb{Z}[(G_i)_{\bar{\kappa}}] \rightarrow \bigoplus_{\{i,j\} \subseteq J} \mathbb{Z}[(E_{ij})_{\bar{\kappa}}].$$

Since the underlying \mathbb{Z} -module of P_0 is of finite type, and since K is a \mathbb{Q} -algebra, it suffices to show that P_0 is of rank 0.

Denote by N the $\mathbb{Z}[\Gamma]$ -module $\bigoplus_{i \in I} \mathbb{Z}[(G_i)_{\bar{\kappa}}]$, by Q the $\mathbb{Z}[\Gamma]$ -module $\bigoplus_{\{i,j\} \subseteq J} \mathbb{Z}[(E_{ij})_{\bar{\kappa}}]$ and by R the image of N through the map $N \rightarrow Q$.

We have an exact sequence

$$0 \rightarrow P_0 \rightarrow N \rightarrow R \rightarrow 0.$$

Let ℓ be a prime number, let n be a positive integer and denote by Λ the ring $\mathbb{Z}/\ell^n \mathbb{Z}$. We get an exact sequence

$$\text{Tor}_{\mathbb{Z}}^1(R, \Lambda) \rightarrow P_0 \otimes_{\mathbb{Z}} \Lambda \rightarrow N \otimes_{\mathbb{Z}} \Lambda \rightarrow R \otimes_{\mathbb{Z}} \Lambda \rightarrow 0.$$

Assume that the map $N \otimes_{\mathbb{Z}} \Lambda \rightarrow R \otimes_{\mathbb{Z}} \Lambda$ is injective. Then, the induced map $N \otimes_{\mathbb{Z}} \Lambda \rightarrow R \otimes_{\mathbb{Z}} \Lambda$ is also injective and we get a surjection

$$\text{Tor}_{\mathbb{Z}}^1(R, \Lambda) \rightarrow P_0 \otimes_{\mathbb{Z}} \Lambda.$$

Since $\text{Tor}_{\mathbb{Z}}^1(R, \Lambda)$ is the ℓ^n -torsion subgroup of R and since the latter is of finite type, there is an integer n_0 such that if $n \geq n_0$, the group $\text{Tor}_{\mathbb{Z}}^1(R, \Lambda)$ is of ℓ^{n_0} -torsion. If P_0 is not of rank 0, the group $P_0 \otimes_{\mathbb{Z}} \Lambda$ is not of ℓ^{n_0} -torsion for $n > n_0$.

Hence, to show that P_0 is of rank 0 and to finish the proof, it suffices to show that there is a prime number ℓ such that for any positive integer n , letting $\Lambda = \mathbb{Z}/\ell^n \mathbb{Z}$, the map

$$\bigoplus_{i \in I} \Lambda[(G_i)_{\bar{\kappa}}] \rightarrow \bigoplus_{\{i,j\} \subseteq J} \Lambda[(E_{ij})_{\bar{\kappa}}]$$

has a trivial kernel.

Step 4: Let ℓ be a prime number which is invertible in κ . Replacing S with a neighborhood of x if needed, we can assume that ℓ is invertible on S . Let n be a positive integer. Denote by Λ the ring $\mathbb{Z}/\ell^n\mathbb{Z}$. In this step, we show that the map

$$\bigoplus_{i \in I} \Lambda[(G_i)_{\bar{\kappa}}] \rightarrow \bigoplus_{\{i,j\} \subseteq J} \Lambda[(E_{ij})_{\bar{\kappa}}]$$

has trivial kernel.

The classical Affine Lefschetz Theorem [LYF14, XV.1.1.2] on $\mathcal{D}_c^b(S, \Lambda)$ endowed with its perverse t-structure ensures that

$${}^p H^0(i^* j_* \underline{\Lambda}) = 0.$$

As before, the sheaf $i^* j_* \underline{\Lambda}$ is equivalent to the sheaf $p_* \iota^* \gamma_* \underline{\Lambda}$. Now, the localization triangle in $\mathcal{D}_c^b(S, \Lambda)$ yields an exact triangle

$$\iota_* \iota^! \underline{\Lambda} \rightarrow \underline{\Lambda} \rightarrow \gamma_* \underline{\Lambda}.$$

Applying $p_* \iota^*$, we get an exact triangle

$$p_* \iota^! \underline{\Lambda} \rightarrow p_* \underline{\Lambda} \rightarrow p_* \iota^* \gamma_* \underline{\Lambda}.$$

Furthermore, by cdh descent, we have an exact triangle

$$\bigoplus_{\{i,j\} \subseteq J} (p_{ij})_* \underline{\Lambda}(-2)[-4] \rightarrow \bigoplus_{i \in J} (p_i)_* \underline{\Lambda}(-1)[-2] \rightarrow p_* \iota^! \underline{\Lambda}$$

Thus, the perverse sheaf ${}^p H^k(p_* \iota^! \underline{\Lambda})$ vanishes if $k < 2$. Therefore, the map

$${}^p H^0(p_* \underline{\Lambda}) \rightarrow {}^p H^0(p_* \iota^* \gamma_* \underline{\Lambda})$$

is an equivalence and therefore, the perverse sheaf ${}^p H^0(p_* \underline{\Lambda})$ vanishes.

Finally, the same method as in the third step of our lemma shows that

$${}^p H^0(p_* \underline{\Lambda}) = k_* P_\Lambda$$

where P_Λ is the étale sheaf of $\kappa_{\text{ét}}$ which corresponds through Galois-Grothendieck theory to the kernel P'_Λ of the map

$$\bigoplus_{i \in I} \Lambda[(G_i)_{\bar{\kappa}}] \rightarrow \bigoplus_{\{i,j\} \subseteq J} \Lambda[(E_{ij})_{\bar{\kappa}}]$$

of Γ -modules. Therefore, the Γ -module P'_Λ vanishes which finishes the proof. \square

Remark 3.5.3.6. Let k be a sub-field of \mathbb{C} , let S be a k -scheme of finite type, let $f: X \rightarrow S$ be a quasi-finite affine morphism and let K be a number field. By using the Betti realization instead of the v -adic realization and by adapting the proof of Corollary 2.4.8.12 to the setting of Mixed Hodge Modules, it should be possible to prove that the functor

$$f!: \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(X, K) \rightarrow \mathcal{DM}_{\text{ét},c}^A(S, K)$$

is perverse homotopy t -exact.

We can now prove the main result of this section.

Theorem 3.5.3.7. *Let S be an excellent scheme, let $f: X \rightarrow S$ be a quasi-finite morphism of schemes and let R be a regular good ring. Assume that X is nil-regular and that we are in one of the following cases*

- (a) *We have $\dim(S) \leq 2$.*
- (b) *There is a prime number p such that the scheme S is of finite type over \mathbb{F}_p .*

Then, the functor

$$f_!: \mathcal{DM}_{\acute{e}t, \mathbb{Q}-c}^{smA}(X, R) \rightarrow \mathcal{DM}_{\acute{e}t, \mathbb{Q}-c}^A(X, R)$$

is t_{hp} -exact.

Proof. Proposition 3.4.1.2 implies that the functor $f_!$ is right t_{hp} -exact. Hence, it suffices to show that it is left t_{hp} -exact.

Let M be a \mathbb{Q} -constructible motive over X which is t_{hp} -non-negative. We have an exact triangle

$$M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1] \rightarrow M \rightarrow M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By Proposition 3.4.1.7, the motive $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is t_{hp} -non-negative. Since the subcategory of t_{hp} -non-negative is closed under limits, the motive $M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1]$ is also t_{hp} -non-negative.

Furthermore, we have an exact triangle

$$f_!(M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1]) \rightarrow f_!(M) \rightarrow f_!(M \otimes_{\mathbb{Z}} \mathbb{Q}).$$

By Proposition 3.5.3.5, the motive $f_!(M \otimes_{\mathbb{Z}} \mathbb{Q})$ is t_{hp} -non-negative and by Proposition 3.1.3.18, the motive $f_!(M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}[-1])$ is t_{hp} -non-negative. Since the subcategory of t_{hp} -non-negative Artin motives is closed under extensions, the motive $f_!(M)$ is also t_{hp} -non-negative. \square

We also have a t_{hp} -exactness result about the functor $\omega^0 f_*$ when f is a proper morphism.

Proposition 3.5.3.8. *Let S be a scheme allowing resolution of singularities by alterations, let $f: X \rightarrow S$ be a proper morphism and let K be a number field. Then, the functor*

$$\omega^0 f_*: \mathcal{DM}_{\acute{e}t, c}^A(X, K) \rightarrow \mathcal{DM}_{\acute{e}t, c}^A(S, K)$$

is right t_{hp} -exact.

Proof. Recall that since f is proper, we have

$$\omega^0 f_* = \omega^0 f_!$$

Since the functor $f_!$ is a left adjoint functor, it respects small colimits. Furthermore, with rational coefficients, the functor ω^0 also respects small colimits by [Pep19b, 3.5] and the proof applies in the setting of motives with coefficients in K . Hence, it suffices to show that the Artin motives of the form $\omega^0 f_* h_X(Y)[\delta(Y)]$ with Y finite over S are perverse homotopy t -non-positive. Hence, replacing X with Y , it suffices to show that the Artin motive $\omega^0 f_* \mathbb{1}_X[\delta(X)]$ is perverse homotopy t -non-positive.

This follows from Lemma 3.5.3.9. \square

Lemma 3.5.3.9. *Let K be a number field and let $f: X \rightarrow S$ be a morphism of finite type. Then, the Artin motive $\omega^0 f_* \mathbb{1}_X[\delta(X)]$ belongs to the smallest subcategory of $\mathcal{DM}_{\text{ét},c}^A(S, K)$ which is closed under finite colimits, extensions and retracts and contains the motives of the form $g_* \mathbb{1}_Y[\delta(Y)]$ with $g: Y \rightarrow S$ finite.*

Proof. We now proceed by noetherian induction on X . If $\dim(X) = 0$, the morphism f is finite and the result follows from Corollary 3.5.1.3.

Step 1: Assume first that the map f is proper.

Using the induction hypothesis and cdh-descent, we can assume the scheme X to be normal. Let $q: \widetilde{X} \rightarrow X$ be a resolution of singularities by alterations and let $q': X' \rightarrow X$ be the relative normalization of X in $\widetilde{X} \setminus B$ where B is the inverse image of the singular locus of X . Using [Ayo07, 2.1.165], the motive $\omega^0 f_* q'_* \mathbb{1}_{X'}$ is a retract of the motive $\omega^0 f_* q_* \mathbb{1}_{\widetilde{X}}$. Hence, using the induction hypothesis and cdh-descent, we can assume the scheme X to be regular and connected.

Let $\eta = \text{Spec}(\mathbb{K})$ be a generic point of S . Let $f_{\mathbb{K}}: X_{\mathbb{K}} \rightarrow \mathbb{K}$ be the pullback of f to \mathbb{K} . De Jong's resolution of singularities yield a proper alteration $X''_{\mathbb{K}} \rightarrow X_{\mathbb{K}} \rightarrow X_{\mathbb{K}}$ such that the map $X_{\mathbb{K}} \rightarrow X_{\mathbb{K}}$ is proper and birational, the map $X''_{\mathbb{K}} \rightarrow X_{\mathbb{K}}$ is a finite morphism which is generically the composition of an étale cover and of a purely inseparable morphism and such that the structural map $X''_{\mathbb{K}} \rightarrow \mathbb{K}$ is the composition of a purely inseparable morphism and of a smooth morphism. Therefore by [Gro66, 8], we have a dense open subset U and a commutative diagram

$$\begin{array}{ccccc} X''_U & \xrightarrow{q} & X'_U & \xrightarrow{p} & X_U \\ & \searrow f''_U & \downarrow f'_U & \swarrow f_U & \\ & & U & & \end{array}$$

such that the map f_U is the pullback of the map f to U , the map q is proper and birational, the map q is finite and is generically the composition of an étale Galois cover and of a purely inseparable morphism and the map f''_U is the composition of a purely inseparable morphism and of a smooth morphism.

Shrinking U if needed, we can assume that it is regular. Denote by $j: U \rightarrow S$ the immersion, by $i: F \rightarrow S$ its reduced closed complement and by $f_F: X_F \rightarrow F$ the pullback of f along the map i . By cdh-descent and induction, we can assume that the scheme X_F is simple normal crossing in X by replacing X with some abstract blow-up.

By localization, we have an exact triangle

$$i_* i^! f_* \mathbb{1}_X \rightarrow f_* \mathbb{1}_X \rightarrow j_*(f_U)_* \mathbb{1}_{X_U}.$$

By Corollary 3.3.1.13, the motive $\omega^0 i^! f_* \mathbb{1}_X$ vanishes since the scheme X is regular and since X_F is a simple normal crossing divisor in X . This yields an equivalence

$$\omega^0 f_* \mathbb{1}_X = \omega^0 j_*(f_U)_* \mathbb{1}_{X_U}$$

By Proposition 3.1.4.5, we therefore get that

$$\omega^0 f_* \mathbb{1}_X = \omega^0 j_* \omega^0 (f_U)_* \mathbb{1}_{X_U}.$$

Let Z be the complement of the open subset of X_U over which the map p is an isomorphism and let Z' be its pullback along the map p . By cdh-descent, and by using the induction hypothesis on the structural maps $Z \rightarrow U \rightarrow S$ and $Z' \rightarrow U \rightarrow S$, it suffices to show that

$$\omega^0 j_* \omega^0 (f'_U)_* \mathbb{1}_{X'_U}[\delta(X)] \ll_{hp} 0.$$

Furthermore, [Ayo07, 2.1.165] implies that the motive $\omega^0(f'_U)_*\mathbb{1}_{X'_U}$ is a retract of the motive $\omega^0(f''_U)_*\mathbb{1}_{X''_U}$ and therefore, it suffices to show that

$$\omega^0 j_* \omega^0(f''_U)_*\mathbb{1}_{X''_U}[\delta(X)] \leq_{hp} 0.$$

Since the map f''_U is the composition of a smooth and proper morphism and of a purely inseparable morphism, letting

$$X''_U \rightarrow Y \xrightarrow{g} U$$

be the Stein factorization of f''_U yields by Proposition 3.1.4.5(5) that

$$\omega^0(f''_U)_*\mathbb{1}_{X''_U} = g_*\mathbb{1}_Y.$$

On the other hand, the localization property also gives an exact triangle

$$j!g_*\mathbb{1}_Y \rightarrow \omega^0 j_*g_*\mathbb{1}_Y \rightarrow i_*\omega^0 i^* j_*g_*\mathbb{1}_Y$$

and therefore, using Proposition 3.5.1.2, it suffices to show that

$$\omega^0 i^* j_*g_*\mathbb{1}_Y[\delta(X) - 1] \leq_{hp} 0.$$

Let Y_1 be a relative normalization of S in Y and let \bar{Y} be a resolution of singularities by alterations of Y_1 . Let $k: \bar{Y} \rightarrow S$ be the structural map. The scheme \bar{Y} is regular and there is a map $e: Y' \rightarrow Y$ which the composition of a finite étale cover and of a purely inseparable morphism such that we have a comutative diagram

$$\begin{array}{ccccc} Y' & \xrightarrow{\gamma} & \bar{Y} & \xleftarrow{\iota} & Y_F \\ g \circ e \downarrow & & \downarrow k & & \downarrow g_F \\ U & \xrightarrow{j} & S & \xleftarrow{i} & F \end{array}$$

which is made of cartesian square.

The Artin motive $\omega^0 i^* j_*g_*\mathbb{1}_Y$ is by [Ayo07, 2.1.165] a retract of the motive $\omega^0 i^* j_*(g \circ e)_*\mathbb{1}_{Y'}$ and by proper base change, we get

$$\omega^0 i^* j_*(g \circ e)_*\mathbb{1}_{Y'} = \omega^0(g_F)_*\omega^0 \iota^* \gamma_*\mathbb{1}_Y.$$

But using Proposition 3.1.4.5 and lemma 3.3.2.2, we have

$$\omega^0 \iota^* \gamma_*\mathbb{1}_Y = \iota^* \omega^0 \gamma_*\mathbb{1}_Y = \mathbb{1}_{Y_F}.$$

Therefore, it suffices to show that

$$\omega^0(g_F)_*\mathbb{1}_{Y_F}[\delta(X) - 1] \leq_{hp} 0.$$

Since

$$\dim(Y_F) \leq \dim(Y) - 1 \leq \dim(X) - 1,$$

the result follows by induction.

Step 2: Assume now that the map f is a general morphism.

Using Nagata’s Theorem [Sta23, 0F41], we can write

$$f = j \circ \bar{f}$$

where \bar{f} is proper and j is an open immersion. Therefore, we can assume that $f = j: U \rightarrow S$ is an open immersion.

We want to show that

$$\omega^0 j_* \mathbb{1}_U[\delta(U)] \geq_{hp} 0.$$

Let now $p: Y \rightarrow S$ be a resolution of singularities by alterations. We can proceed as before to show that the motive $\omega^0 j_* \mathbb{1}_U$ is a retract of the motive $\omega^0 p_* \mathbb{1}_Y$. The result then follows from the case of proper morphisms. \square

Corollary 3.5.3.10. *Let S be a scheme allowing resolution of singularities by alteration, let $f: X \rightarrow S$ be a quasi-finite morphism of schemes and let K be a number field. Assume that X is nil-regular.*

Then, the functor

$$\omega^0 f_*: \mathcal{DM}_{\acute{e}t,c}^{smA}(X, K) \rightarrow \mathcal{DM}_{\acute{e}t,c}^A(S, K)$$

is t-exact when both sides are endowed with the perverse homotopy t-structure.

Proof. Using Proposition 3.5.1.6, the objects of $\mathcal{DM}_{\acute{e}t,c}^{smA}(X, K)$ are bounded with respect to the perverse homotopy t-structure. Therefore, it suffices to show that the objects of the heart of the t-category $\mathcal{DM}_{\acute{e}t,c}^{smA}(X, R)$ are sent to objects of degree 0 with respect to the perverse homotopy t-structure. Since the latter is a semi-simple category by Maschke’s Theorem, every object of the heart is a retract of an object of the form $g_* \mathbb{1}_V[\delta(X)]$ with $g: V \rightarrow X$ finite and étale. Therefore, it suffices to show that

$$\omega^0 f_* g_* \mathbb{1}_V[\delta(V)] \geq_{hp} 0$$

which follows from Lemma 3.5.3.9. \square

3.5.4 Induced t-structure on constructible Artin motives over a base scheme of dimension at most 2

Theorem 3.5.4.1. *Let S be an excellent scheme of dimension 2 or less and let R be a regular good ring, then, the perverse homotopy t-structure on the stable category $\mathcal{DM}_{\acute{e}t}^A(S, R)$ induces a t-structure on the stable subcategory $\mathcal{DM}_{\acute{e}t,c}^A(S, R)$.*

Proof. We can assume that S is reduced and connected. We proceed by noetherian induction on S . If $\dim(S) = 0$, the result follows from Proposition 3.5.1.8.

Assume now that $\dim(S) > 0$. If \mathfrak{p} be a prime ideal of \mathbb{Z} , if X is a scheme and if M is an Artin motive over X , we denote by $M_{\mathfrak{p}}$ the image of M in $\mathcal{DM}_{\acute{e}t}^A(S, R)_{\mathfrak{p}}$ (with the notations of Section 3.1.1) and we say that an object N of $\mathcal{DM}_{\acute{e}t,c}^A(S, R)_{\mathfrak{p}}$ is constructible when it belongs to $\mathcal{DM}_{\acute{e}t,c}^A(X, R)_{\mathfrak{p}}$. Recall that an Artin motive M is \mathfrak{p} -constructible when $M_{\mathfrak{p}}$ is constructible. By Proposition 3.1.1.13, it suffices to show that for any constructible Artin motive M , the Artin motive ${}^{hp}\tau_{\leq 0}(M)$ is \mathfrak{p} -constructible for any maximal ideal \mathfrak{p} of \mathbb{Z} .

Let \mathfrak{p} be a maximal ideal of \mathbb{Z} . By Proposition 3.1.1.11, there is a unique t-structure on the stable category $\mathcal{DM}_{\acute{e}t}^A(S, R)_{\mathfrak{p}}$ such that the canonical functor

$$\mathcal{DM}_{\acute{e}t}^A(S, R) \rightarrow \mathcal{DM}_{\acute{e}t}^A(S, R)_{\mathfrak{p}}$$

is t -exact. We still call this t -structure the *perverse homotopy t -structure*. If M is an Artin motive, we get by t -exactness of the functor $M \mapsto M_{\mathfrak{p}}$ that

$${}^{hp}\tau_{\leq 0}(M)_{\mathfrak{p}} = {}^{hp}\tau_{\leq 0}(M_{\mathfrak{p}}).$$

Hence, if the perverse homotopy t -structure on the stable category $\mathcal{DM}_{\text{ét}}^A(S, R)_{\mathfrak{p}}$ induces a t -structure on the subcategory $\mathcal{DM}_{\text{ét},c}^A(S, R)_{\mathfrak{p}}$, then the Artin motive ${}^{hp}\tau_{\leq 0}(M)$ is \mathfrak{p} -constructible for any Artin motive M .

Therefore, it suffices to show that the perverse homotopy t -structure induces a t -structure on the subcategory $\mathcal{DM}_{\text{ét},c}^A(S, R)_{\mathfrak{p}}$ for any maximal ideal \mathfrak{p} of \mathbb{Z} .

The properties of $\mathcal{DM}_{\text{ét},(c)}^A(-, R)$ transfer to $\mathcal{DM}_{\text{ét},(c)}^A(-, R)_{\mathfrak{p}}$: we get functors of the form f^* for any morphism f , functors of the form $\omega^0 f_*$ for any morphism of finite type f , functors of the form $f_!$ and $\omega^0 f^!$ for any quasi-finite morphism f and localization triangles. The t -exactness properties of Proposition 3.5.1.2 and corollary 3.5.1.3 and the Affine Lefschetz Theorem (Theorem 3.5.3.7) remain true in this setting.

Let p be a generator of \mathfrak{p} . If p is invertible on a regular connected scheme X , the functor $\rho_!$ of Proposition 3.1.4.3 induces a t -exact fully faithful functor

$$\rho_! : \mathcal{S}\mathcal{h}_{\text{lisse}}(X, R)_{\mathfrak{p}} \rightarrow \mathcal{DM}_{\text{ét}}^A(X, R)_{\mathfrak{p}}$$

when the left hand side is endowed with the ordinary t -structure shifted by $\delta(X)$ and the right hand side is endowed with the perverse homotopy t -structure. Indeed, the full faithfulness follows from Lemma 3.4.2.8 while the t -exactness follows from Propositions 3.1.1.12 and 3.5.2.1.

On the other hand, if a regular scheme X is of characteristic p , the functor $\rho_!$ induces a t -exact fully faithful functor

$$\rho_! : \mathcal{S}\mathcal{h}_{\text{lisse}}(X, R[1/p])_{\mathfrak{p}} \rightarrow \mathcal{DM}_{\text{ét}}^A(X, R)_{\mathfrak{p}}$$

when the left hand side is endowed with the ordinary t -structure shifted by $\delta(X)$ and the right hand side is endowed with the perverse homotopy t -structure. Indeed, the full faithfulness follows from Corollary 3.2.1.6 while the t -exactness follows from Proposition 3.5.2.1.

We say that a connected scheme X has the property \mathcal{P} if one of the two following properties hold:

- The scheme X is of characteristic p .
- The prime number p is invertible on X .

We say that a scheme X has the property \mathcal{P} if all its connected components have the property \mathcal{P} . The above discussion implies that in that case, the perverse homotopy t -structure induces a t -structure on $\mathcal{DM}_{\text{ét},c}^{smA}(X, R)_{\mathfrak{p}}$ and we let $M_{\text{perv}}^{smA}(X, R)_{\mathfrak{p}}$ be the heart of this t -structure.

Notice furthermore that if F is any closed subscheme of S which is of positive codimension, the induction hypothesis and Proposition 3.1.1.11 imply that the perverse homotopy t -structure induces a t -structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)_{\mathfrak{p}}$. We let $M_{\text{perv}}^A(F, R)_{\mathfrak{p}}$ be the heart of this induced t -structure.

Using Lemma 3.5.4.2 below, it suffices to show that if $j: U \rightarrow S$ is a dense affine open immersion with U regular and having the property \mathcal{P} , letting $i: F \rightarrow S$ be the reduced closed complementary immersion, for any object M of $M_{\text{perv}}^A(F, R)_{\mathfrak{p}}$, any object N of $M_{\text{perv}}^{smA}(U, R)_{\mathfrak{p}}$ and any map

$$f: M \rightarrow {}^{hp}H^{-1}(\omega^0 i^* j_* N),$$

the kernel K of f is a constructible Artin motive.

If $\dim(S) = 1$, the scheme F is 0-dimensional. For any profinite group G and any noetherian ring Λ , the subcategory $\text{Rep}^A(G, \Lambda)$ of $\text{Mod}(G, \Lambda)$ is Serre. Therefore, Proposition 3.5.1.8 and corollary 3.2.1.7 imply that the subcategory $\mathcal{M}_{\text{perv}}^A(F, R)_{\mathfrak{p}}$ of $\mathcal{DM}_{\text{ét}}^A(F, R)_{\mathfrak{p}}^{\heartsuit}$ is Serre. Since K is a subobject of M , it is an object of $\mathcal{M}_{\text{perv}}^A(F, R)_{\mathfrak{p}}$ and thus it constructible.

If $\dim(S) = 2$, the scheme F is of dimension at most 1.

By Lemma 3.5.4.3 below, we have an exact sequence

$$0 \xrightarrow{u} M_1 \rightarrow {}^{hp}H^{-1}(\omega^0 i^* j_* N) \xrightarrow{v} M_2$$

in $\mathcal{DM}_{\text{ét}}^A(F, R)_{\mathfrak{p}}^{\heartsuit}$ such that M_1 and M_2 are perverse Artin motives. Let K_1 be the kernel of the map

$$M \xrightarrow{f} {}^{hp}H^{-1}(\omega^0 i^* j_* N) \rightarrow M_2,$$

it is a perverse Artin motive. Furthermore, the map $f_0: K_1 \rightarrow {}^{hp}H^{-1}(\omega^0 i^* j_* N)$ induced by f factors through M_1 since

$$u \circ f_0 = 0$$

and the kernel of the induced map

$$K_1 \rightarrow M_1$$

is K which is therefore a perverse Artin motive. \square

Lemma 3.5.4.2. *Let S be a reduced excellent scheme, let R be a regular good ring, let \mathfrak{p} be a maximal ideal of \mathbb{Z} and let p be a generator of \mathfrak{p} .*

Assume that:

- (i) *The perverse homotopy t-structure induces a t-structure on the subcategory $\mathcal{DM}_{\text{ét},c}^A(F, R)_{\mathfrak{p}}$ of $\mathcal{DM}_{\text{ét}}^A(F, R)_{\mathfrak{p}}$, for any closed subscheme F of S which is of positive codimension.*
- (ii) *If $j: U \rightarrow S$ is a dense affine open immersion with U regular and having the property \mathcal{P} , letting $i: F \rightarrow S$ be the reduced closed complementary immersion, for any object M of $\mathcal{M}_{\text{perv}}^A(F, R)_{\mathfrak{p}}$, any object N of $\mathcal{M}_{\text{perv}}^{\text{sm}A}(U, R)_{\mathfrak{p}}$ and any map*

$$f: M \rightarrow {}^{hp}H^{-1}(\omega^0 i^* j_* N),$$

the kernel of f is a perverse Artin motive on F .

Then, the perverse homotopy t-structure induces a t-structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)_{\mathfrak{p}}$.

Proof. If $j: U \rightarrow S$ is an open immersion, denote by $\mathcal{DM}_{\text{ét},c,U}^A(S, R)_{\mathfrak{p}}$ the subcategory of $\mathcal{DM}_{\text{ét},c}^A(S, R)_{\mathfrak{p}}$ made of those objects M such that j^*M belongs to $\mathcal{DM}_{\text{ét},c}^{\text{sm}A}(U, R)_{\mathfrak{p}}$. Using Lemma 2.2.4.2, every object of $\mathcal{DM}_{\text{ét},c}^A(S, R)_{\mathfrak{p}}$ lies in some $\mathcal{DM}_{\text{ét},c,U}^A(S, R)_{\mathfrak{p}}$ for some dense open immersion $j: U \rightarrow S$. Shrinking U if needed, we can assume that U is regular since S is excellent. Shrinking U further, we can assume that it has the property \mathcal{P} . Finally, using Lemma 2.3.3.5, we may assume that the morphism j is affine.

Therefore, it suffices to show that for any dense affine open immersion $j: U \rightarrow S$ with U regular and having the property \mathcal{P} , the perverse homotopy t-structure induces a t-structure on $\mathcal{DM}_{\text{ét},c,U}^A(S, R)_{\mathfrak{p}}$. Let now $j: U \rightarrow S$ be such an immersion, let $i: F \rightarrow S$ be the reduced closed complementary immersion. We let

- $\mathcal{S} = \mathcal{DM}_{\text{ét}}^A(S, R)_{\mathfrak{p}}$,
- $\mathcal{U} = \mathcal{DM}_{\text{ét}}^A(U, R)_{\mathfrak{p}}$ and $\mathcal{U}_0 = \mathcal{DM}_{\text{ét},c}^{smA}(U, R)_{\mathfrak{p}}$,
- $\mathcal{F} = \mathcal{DM}_{\text{ét}}^A(F, R)_{\mathfrak{p}}$ and $\mathcal{F}_0 = \mathcal{DM}_{\text{ét},c}^A(F, R)_{\mathfrak{p}}$.

The category \mathcal{S} is a gluing of the pair $(\mathcal{U}, \mathcal{F})$ along the fully faithful functors $\omega^0 j_*$ and i_* in the sense of [Lur17, A.8.1] and the perverse homotopy t -structure on \mathcal{S} is obtained by gluing the perverse t -structures of \mathcal{U} and \mathcal{F} in the sense of [BBDG18, 1.4.10] by Proposition 2.4.4.1. In addition, the perverse homotopy t -structure induces a t -structure on \mathcal{U}_0 and on \mathcal{F}_0 by assumption. Furthermore, since the morphism j is affine, the functor $j_! : \mathcal{U} \rightarrow \mathcal{S}$ is t -exact.

Finally Proposition 3.5.1.6 ensures that the objects of $\mathcal{DM}_{\text{ét},c}^A(S, R)_{\mathfrak{p}}$ are bounded with respect to the perverse homotopy t -structure. We conclude by using Lemma 2.3.3.6. \square

Lemma 3.5.4.3. *Let S be a reduced excellent connected scheme of dimension 2, let R be a regular good ring, let $j : U \rightarrow S$ be a dense affine open immersion with U regular and having property \mathcal{P} , let $i : F \rightarrow S$ be the reduced closed complementary immersion and let N be an object of $M_{\text{perv}}^{smA}(U, R)_{\mathfrak{p}}$.*

Then, the object ${}^{hp}H^{-1}(\omega^0 i^ j_* N)$ fits in $\mathcal{DM}_{\text{ét}}^A(F, R)_{\mathfrak{p}}^{\heartsuit}$ into an exact sequence of the form*

$$0 \rightarrow M_1 \rightarrow {}^{hp}H^{-1}(\omega^0 i^* j_* N) \rightarrow M_2$$

where M_1 and M_2 belong to $M_{\text{perv}}^A(F, R)_{\mathfrak{p}}$.

Proof. We use the convention that $\delta(S) = 2$. Replacing S with the closure of a connected component of U , we may assume that U is connected. Thus, since U is regular and has property \mathcal{P} , we have

$$M_{\text{perv}}^{smA}(U, R)_{\mathfrak{p}} = \rho_! \text{Loc}_U(\Lambda_0)_{\mathfrak{p}}[2]$$

where $\Lambda_0 = R$ if p is invertible on U and $\Lambda_0 = R[1/p]$ if U is of characteristic p .

Notice that the result holds when the ring R is torsion by Corollary 3.1.4.11 and therefore, we can assume that the ring R is the localization of the ring of integers of a number field K .

Denote by Λ the ring $\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}}$. By Propositions 3.1.1.11 and 3.1.5.18, the canonical functor

$$\text{Loc}_U(\Lambda_0)_{\mathfrak{p}} \rightarrow \text{Loc}_U(\Lambda)$$

is an equivalence.

Step 1: Let ξ be a geometric point of U and let

$$\varepsilon_! : \text{Rep}^A(\pi_1^{\text{ét}}(U, \xi), \Lambda) \rightarrow \text{Loc}_U(\Lambda)$$

be the inverse of the equivalence of Proposition 3.1.5.10, let $\alpha_! = \varepsilon_! \rho_!$ and let $\alpha^!$ be the inverse of $\alpha_!$.

Let $P = \alpha^! N[-2]$ and let P_{tors} be the sub-representation of P made of the torsion elements. There is a positive integer n such that $p^n P_{\text{tors}}$ vanishes; moreover, the $\Lambda[\pi_1^{\text{ét}}(U, \xi)]$ -module P_{tors} is an Artin representation. Therefore, the $\Lambda[\pi_1^{\text{ét}}(U, \xi)]$ -module P/P_{tors} is also an Artin representation and it is torsion free.

Using the Affine Lefschetz Theorem (Theorem 3.5.3.7), the functor $\omega^0 i^* j_*$ is of perverse homotopy cohomological amplitude bounded below by -1 . Therefore, the exact triangle

$$\omega^0 i^* j_* \alpha_! P_{\text{tors}}[2] \rightarrow \omega^0 i^* j_* N \rightarrow \omega^0 i^* j_* \alpha_!(P/P_{\text{tors}})[2]$$

induces an exact sequence

$$0 \rightarrow {}^{hp}H^1(\omega^0 i^* j_* \alpha_! P_{\text{tors}}) \rightarrow {}^{hp}H^{-1}(\omega^0 i^* j_* N) \rightarrow {}^{hp}H^1(\omega^0 i^* j_* \alpha_!(P/P_{\text{tors}})).$$

Using Corollary [3.1.4.11](#), the object ${}^{hp}H^1(\omega^0 i^* j_* \alpha_! P_{\text{tors}})$ lies in $M_{\text{perv}}^A(F, R)_{\mathfrak{p}}$ and therefore, it suffices to prove that the object ${}^{hp}H^1(\omega^0 i^* j_* \alpha_!(P/P_{\text{tors}}))$ of $\mathcal{DM}_{\text{ét}}^A(F, R)_{\mathfrak{p}}^{\heartsuit}$ is a subobject of an object of $M_{\text{perv}}^A(F, R)_{\mathfrak{p}}$.

Since the Artin representation P/P_{tors} is torsion free, the canonical map

$$P/P_{\text{tors}} \rightarrow P/P_{\text{tors}} \otimes_{\Lambda} K$$

is injective. The category $\text{Rep}^A(\pi_1^{\text{ét}}(U, \xi), K)$ is semi-simple by Maschke's lemma. Therefore, the representation $P/P_{\text{tors}} \otimes_{\Lambda} K$ is a direct factor of a representation

$$Q_K := K[G_1] \oplus \cdots \oplus K[G_n]$$

where G_1, \dots, G_n are finite quotients of $\pi_1^{\text{ét}}(U, \xi)$.

Denote by λ_K the map $P/P_{\text{tors}} \otimes_{\Lambda} K \rightarrow Q_K$. There is an integer m such that the image of P/P_{tors} in Q_K is contained in the lattice

$$Q_0 := \frac{1}{p^m} (\Lambda[G_1] \oplus \cdots \oplus \Lambda[G_n])$$

in Q_K . The representation Q_0 is isomorphic to $\Lambda[G_1] \oplus \cdots \oplus \Lambda[G_n]$. Thus, we get an embedding

$$P/P_{\text{tors}} \rightarrow \Lambda[G_1] \oplus \cdots \oplus \Lambda[G_n].$$

Since the functor $\omega^0 i^* j_*$ is of perverse homotopy cohomological amplitude bounded below by -1 , we get an embedding

$${}^{hp}H^1(\omega^0 i^* j_* \alpha_!(P/P_{\text{tors}})) \rightarrow \bigoplus_{i=1}^n {}^{hp}H^1(\omega^0 i^* j_* \alpha_! \Lambda[G_i]).$$

Furthermore, if $1 \leq i \leq n$, the motive $\alpha_! \Lambda[G_i]$ is of the form $f_* \mathbb{1}_V$ where $f: V \rightarrow U$ is a finite étale map.

Therefore, it suffices to show that the following claim: "If $f: V \rightarrow U$ is a finite étale map, and if $N = f_* \mathbb{1}_V$, the object ${}^{hp}H^1(\omega^0 i^* j_* N)$ belongs to $M_{\text{perv}}^A(F, R)_{\mathfrak{p}}$."

Step 2: In this step we show that to prove the above claim, we can assume S to be normal and that $N = \mathbb{1}_U$.

Let \bar{V} be the relative normalization of S in V . Since S is excellent and noetherian, it is Nagata by [\[Sta23, 033Z\]](#). Thus, [\[Sta23, 0AVK\]](#) ensures that the structural map $\nu: \bar{V} \rightarrow S$ is finite. Furthermore, the scheme \bar{V} is normal by [\[Sta23, 035L\]](#).

Consider the following commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\gamma} & \bar{V} & \xleftarrow{\iota} & F' \\ \downarrow f & & \downarrow \nu & & \downarrow \nu_F \\ U & \xrightarrow{j} & S & \xleftarrow{i} & F \end{array}$$

made of cartesian squares.

Now, we have

$$\omega^0 i^* j_* N = \omega^0 i^* j_* f_* \mathbb{1}_V = \omega^0 i^* \nu_* \gamma_* \mathbb{1}_V = (\nu_F)_* \omega^0 \iota^* \gamma_* \mathbb{1}_V.$$

Since ν_F is finite, it is perverse homotopy t -exact. Thus, we have

$${}^{hp}H^1(\omega^0 i^* j_* N) = (\nu_F)_* {}^{hp}H^1(\omega^0 \psi^* \gamma_* \mathbb{1}_V).$$

Therefore, we can replace S with \bar{V} and U with V in order to assume that S is normal and that $N = \mathbb{1}_U$.

Step 3: In this step, we study the singularities of S .

First, notice that if $Y \rightarrow S$ is a closed immersion and if $F \cap Y = \emptyset$, we can factor i into a closed immersion $u: F \rightarrow S \setminus Y$ and an open immersion $\xi: S \setminus Y \rightarrow S$. Letting $j': U \setminus Y \rightarrow S \setminus Y$ be the canonical immersion, smooth base change implies that

$$\omega^0 i^* j_* \mathbb{1}_U = \omega^0 u^* \xi^* j_* \mathbb{1}_U = \omega^0 u^* j'_* \mathbb{1}_{U \setminus Y}.$$

We can therefore remove any closed subset which does not intersect F from S without changing ${}^{hp}H^1(\omega^0 i^* j_* \mathbb{1}_U)$. Since S is normal, its singular locus lies in codimension 2 and we can therefore assume that it is contained in F .

Since S is excellent, Lipman's Theorem on embedded resolution of singularities applies (see [\[Sta23, 0BGP,0BIC,0ADX\]](#) for precise statements). We get a cdh-distinguished square

$$\begin{array}{ccc} E & \xrightarrow{i_E} & \tilde{S} \\ \downarrow p & & \downarrow f \\ F & \xrightarrow{i} & S \end{array}$$

such that the map f induces an isomorphism over U , the scheme \tilde{S} is regular and the subscheme E of \tilde{S} is a simple normal crossing divisor.

Let $\gamma: U \rightarrow \tilde{S}$ be the complementary open immersion of i_E . We have $j = f \circ \gamma$. Therefore, we get

$$\omega^0 i^* j_* \mathbb{1}_U = \omega^0 i^* f_* \gamma_* \mathbb{1}_U = \omega^0 p_* i_E^* \gamma_* \mathbb{1}_U.$$

We have a localization exact triangle

$$(i_E)_* i_E^! \mathbb{1}_{\tilde{S}} \rightarrow \mathbb{1}_{\tilde{S}} \rightarrow \gamma_* \mathbb{1}_U,$$

applying $\omega^0 p_* i_E^*$, we get an exact triangle

$$\omega^0 p_* i_E^! \mathbb{1}_{\tilde{S}} \rightarrow \omega^0 p_* \mathbb{1}_E \rightarrow \omega^0 i^* j_* \mathbb{1}_U$$

and therefore, we get an exact sequence

$${}^{hp}H^1(\omega^0 p_* i_E^! \mathbb{1}_{\tilde{S}}) \rightarrow {}^{hp}H^1(\omega^0 p_* \mathbb{1}_E) \rightarrow {}^{hp}H^1(\omega^0 i^* j_* \mathbb{1}_U) \rightarrow {}^{hp}H^2(\omega^0 p_* i_E^! \mathbb{1}_{\tilde{S}})$$

and an equivalence

$${}^{hp}H^0(\omega^0 p_* i_E^! \mathbb{1}_{\tilde{S}}) = {}^{hp}H^0(\omega^0 p_* \mathbb{1}_E)$$

(to get this last equivalence, we use the Affine Lefschetz Theorem).

Since E is a simple normal crossing divisor, there is a finite set I and regular 1-dimensional closed subschemes E_i of \widetilde{X} for $i \in I$, such that $E = \bigcup_{i \in I} E_i$. Choose a total order on the finite set I . If $i < j$, write $E_{ij} = E_i \cap E_j$. Consider for $J \subseteq I$ of cardinality at most 2 the obvious diagram:

$$\begin{array}{ccc} E_J & \xrightarrow{u_{E_J}} & E \\ & \searrow p_J & \downarrow p \\ & & F \end{array}$$

By cdh-descent and absolute purity, we have an exact triangle

$$\bigoplus_{i < j} (u_{E_{ij}})_* \mathbb{1}_{E_{ij}}(-2)[-4] \rightarrow \bigoplus_{i \in I} (u_{E_i})_* \mathbb{1}_{E_i}(-1)[-2] \rightarrow i_E^! \mathbb{1}_{\widetilde{S}}$$

and applying $\omega^0 p_*$ yields an exact triangle

$$\bigoplus_{i < j} \omega^0 \left((p_{ij})_* \mathbb{1}_{E_{ij}}(-2) \right) [-4] \rightarrow \bigoplus_{i \in I} \omega^0 \left((p_i)_* \mathbb{1}_{E_i}(-1) \right) [-2] \rightarrow \omega^0 p_* i_E^! \mathbb{1}_{\widetilde{S}}.$$

Let $J \subseteq I$ be of cardinality at most 2 and let c be the cardinality of J . We have an exact triangle

$$\omega^0 \left((p_J)_* (\mathbb{1}_{E_J} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})(-c) \right) [-1] \rightarrow \omega^0 \left((p_J)_* \mathbb{1}_{E_J}(-c) \right) \rightarrow \omega^0 \left((p_J)_* (\mathbb{1}_{E_J} \otimes_{\mathbb{Z}} \mathbb{Q})(-c) \right)$$

[Pep19b, 3.9] ensures that the motive $\omega^0 \left((p_J)_* (\mathbb{1}_{E_J} \otimes_{\mathbb{Z}} \mathbb{Q})(-c) \right)$ vanishes. Therefore, using Corollary [3.1.4.11], we get an equivalence

$$\omega^0 \left((p_J)_* \mathbb{1}_{E_J}(-c) \right) = (p_J)_* (\mathbb{1}_{E_J} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})(-c)[-1]$$

But then, using Proposition [3.5.1.9] and the usual properties of the perverse t-structure described in [BBDG18, 4.2.4], we get

$$(p_J)_* (\mathbb{1}_{E_J} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})(-c) \geq_{hp} 0.$$

Therefore, we have

$$\omega^0 \left((p_J)_* \mathbb{1}_{E_J}(-c) \right) \geq_{hp} 1$$

and thus, we have

$$\omega^0 \left((p_J)_* \mathbb{1}_{E_J}(-c) \right) [-2c] \geq_{hp} 1 + 2c$$

which yields

$$\omega^0 p_* i_E^! \mathbb{1}_{\widetilde{S}} \geq_{hp} 3.$$

Hence, we get an equivalence

$${}^{hp}H^1(\omega^0 p_* \mathbb{1}_E) = {}^{hp}H^1(\omega^0 i^* j_* \mathbb{1}_U)$$

and the motive ${}^{hp}H^0(\omega^0 p_* \mathbb{1}_E)$ vanishes.

Step 4: In this step we compute the motive ${}^{hp}H^1(\omega^0 p_* \mathbb{1}_E)$ and we finish the proof. By cdh-descent, we have an exact triangle

$$\mathbb{1}_E \rightarrow \bigoplus_{i \in I} (u_{E_i})_* (u_{E_i})^* \mathbb{1}_E \rightarrow \bigoplus_{i < j} (u_{E_{ij}})_* (u_{E_{ij}})^* \mathbb{1}_E.$$

and applying $\omega^0 p_*$, we get an exact triangle

$$\omega^0 p_* \mathbb{1}_E \rightarrow \bigoplus_{i \in I} \omega^0 (p_i)_* \mathbb{1}_{E_i} \rightarrow \bigoplus_{i < j} (p_{ij})_* \mathbb{1}_{E_{ij}}.$$

Let

$$I_0 = \{i \in I \mid \delta(p(E_i)) = 0\}.$$

If $i \in I \setminus I_0$, we have $\delta(E_i) = 1$ and the morphism p_i is finite. If $i \in I_0$, the map p_i factors through the singular locus Z of S which is 0-dimensional and we can write $p_i = \iota \circ \pi_i$ with $\iota: Z \rightarrow S$ the closed immersion. Finally, notice that $\delta(Z) = 0$ and thus, for any $i < j$, we have $\delta(E_{ij}) = 0$. Hence, since the motive ${}^{hp}H^0(\omega^0 p_* \mathbb{1}_E)$ vanishes, we get an exact sequence

$$0 \rightarrow P \rightarrow \bigoplus_{i < j} (p_{ij})_* \mathbb{1}_{E_{ij}} \rightarrow {}^{hp}H^1(\omega^0 p_* \mathbb{1}_E) \rightarrow \bigoplus_{i \in I \setminus I_0} (p_i)_* \mathbb{1}_{E_i} \oplus Q \rightarrow 0$$

where $P = \iota_* \bigoplus_{i \in I_0} {}^{hp}H^0(\omega^0(\pi_i)_* \mathbb{1}_{E_i})$ and $Q = \iota_* \bigoplus_{i \in I_0} {}^{hp}H^1(\omega^0(\pi_i)_* \mathbb{1}_{E_i})$.

If $i \in I_0$, let

$$E_i \rightarrow F_i \xrightarrow{q_i} Z$$

be the Stein factorization of π_i . Corollary [3.3.1.8](#) yields

$$P = \iota_* \bigoplus_{i \in I_0} (q_i)_* \mathbb{1}_{F_i}$$

and

$$Q = 0.$$

Hence, the objects P and Q belong to $M_{\text{perv}}^A(F, R)_p$ and therefore, so does the object ${}^{hp}H^1(\omega^0 p_* \mathbb{1}_E)$ which finishes the proof. \square

Proposition 3.5.4.4. *Let S be an excellent scheme of dimension 2 or less, let $f: X \rightarrow S$ be a quasi-finite affine morphism of schemes and let R be a regular good ring. Assume that X is nil-regular.*

Then, the functor

$$f_!: \mathcal{DM}_{\text{ét},c}^{\text{sm}A}(X, R) \rightarrow \mathcal{DM}_{\text{ét},c}^A(X, R)$$

is t -exact when both sides are endowed with the perverse homotopy t -structure.

Proof. This follows from Theorem [3.5.3.7](#). \square

Example 3.5.4.5. *Let X be a normal scheme of dimension 4 with a single singular point x of codimension 4, let k be the residue field at x , let p be the characteristic exponent of k and let $f: \widetilde{X} \rightarrow X$ be a resolution of singularities of X .*

Assume that the exceptional divisor above x is isomorphic to a smooth k -scheme E such that $b_1(E, \ell) \neq 0$ for some prime number ℓ distinct from the characteristic exponent of E and that the ring of coefficients is \mathbb{Z} . Take the convention that $\delta(X) = 4$.

We have a cdh-distinguished square

$$\begin{array}{ccc} E & \xrightarrow{i_E} & \widetilde{X} \\ \downarrow \pi & & \downarrow f \\ \text{Spec}(k) & \xrightarrow{i} & X \end{array}$$

which yields an exact triangle

$$\mathbb{1}_X \rightarrow \omega^0 f_* \mathbb{1}_{\tilde{X}} \oplus i_* \mathbb{1}_k \rightarrow i_* \omega^0 \pi_* \mathbb{1}_E.$$

Furthermore, letting $U = X \setminus \{x\} = \tilde{X} \setminus E$ and letting $j: X \setminus \{x\} \rightarrow X$ be the open immersion, the localization property (3.0.0.2) yields an exact triangle

$$i_* \omega^0 \pi_*(i_E)^! \mathbb{1}_{\tilde{X}} \rightarrow \omega^0 f_* \mathbb{1}_{\tilde{X}} \rightarrow \omega^0 j_* \mathbb{1}_U.$$

Since by Propositions 3.5.1.2 and 3.5.2.4, the Artin motive $\omega^0 j_* \mathbb{1}_U$ lies in degree at least 4, we get that

$${}^{hp}H^i(\omega^0 f_* \mathbb{1}_{\tilde{X}}) = i_* {}^{hp}H^i(\omega^0 \pi_*(i_E)^! \mathbb{1}_{\tilde{X}})$$

for $i < 4$.

Using the absolute purity property, we have

$$\omega^0 \pi_*(i_E)^! \mathbb{1}_{\tilde{X}} = \omega^0 (\pi_* \mathbb{1}_E(-1))[-2]$$

and therefore, by Corollary 3.3.1.8, we get

$${}^{hp}H^i(\omega^0 f_* \mathbb{1}_{\tilde{X}}) = \begin{cases} 0 & \text{if } i < 3 \\ i_*(\alpha! \mathbb{Q}/\mathbb{Z}[1/p](-1)) & \text{if } i = 3, \end{cases}$$

using Notations 3.3.1.7.

On the other hand, we have

$${}^{hp}H^2(\omega^0 \pi_* \mathbb{1}_E) = i_* \alpha! \mu_1(E, \mathbb{Z})$$

by Corollary 3.3.1.8. Moreover, the Artin motive $i_* \mathbb{1}_k$ is in the heart of the perverse homotopy t -structure. Hence, we get an exact sequence

$$0 \rightarrow i_* \alpha! \mu_1(E, \mathbb{Z}) \rightarrow {}^{hp}H^3(\mathbb{1}_X) \rightarrow i_* M \rightarrow 0,$$

where M is the kernel of the map ${}^{hp}H^3(\omega^0 \pi_*(i_E)^! \mathbb{1}_{\tilde{X}}) \rightarrow {}^{hp}H^3(\omega^0 \pi_* \mathbb{1}_E)$. Thus, the motive ${}^{hp}H^3(\mathbb{1}_X)$ is in the image of the functor i_* . Since the representation $\mu_1(E, \mathbb{Z})$ of G_k is not of finite type as a \mathbb{Z} -module (it has $\mathbb{Z}(\ell^\infty)^{b_1(E, \ell)}$ as a sub-module), the Artin motive ${}^{hp}H^3(\mathbb{1}_X)$ cannot be constructible.

Example 3.5.4.6. Let X be the affine cone over an abelian variety E embedded in some \mathbb{P}^N . The scheme X has a single singular point x and the blow up of X at x is a resolution of singularities of X with exceptional divisor E .

By Noether's normalization lemma, there is a finite map $X \rightarrow \mathbb{A}_k^4$. Thus, the perverse homotopy t -structure does not induce a t -structure on $\mathcal{DM}_{\text{ét},c}^A(\mathbb{A}_k^4, \mathbb{Z})$. As \mathbb{A}_k^4 is a closed subscheme of \mathbb{A}_k^n for $n \geq 4$, the perverse t -structure does not induce a t -structure on $\mathcal{DM}_{\text{ét},c}^A(\mathbb{A}_k^n, \mathbb{Z})$ if $n \geq 4$.

Remark 3.5.4.7. Loosely speaking, if the scheme X becomes more singular, the perverse homotopy cohomology sheaves of $\mathbb{1}_X$ should become more complicated. Here, the simplest possible singularity on a scheme of dimension 4 over a finite field already renders the cohomology sheaves not constructible. Therefore, Theorem 3.5.4.1 should not hold for schemes of dimension 4 or more. The case of 3-folds, however, remains open.

3.6 The abelian category of perverse Artin étale motives

3.6.1 Definition and first properties

Definition 3.6.1.1. *Let S be an excellent scheme and let R be a good ring. Assume that the perverse homotopy t -structure on $\mathcal{DM}_{\text{ét}}^A(S, R)$ induces a t -structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)$. We define the abelian category of perverse Artin étale motives over S with coefficients in R , as the heart of the perverse homotopy t -structure on $\mathcal{DM}_{\text{ét},c}^A(S, R)$. We denote by $M_{\text{perv}}^A(S, R)$ this category.*

Proposition 3.6.1.2. *Let R be a good ring. Assume that the perverse homotopy t -structure induces a t -structure on the category of constructible Artin étale motives with R coefficients over all schemes appearing below.*

1. *Let $f: T \rightarrow S$ be a quasi-finite morphism of schemes. Then, the functor $f_!$ induces a right exact functor*

$${}^{hp}H^0 f_!: M_{\text{perv}}^A(T, R) \rightarrow M_{\text{perv}}^A(S, R).$$

Furthermore, if the functor f is finite, we have ${}^{hp}H^0 f_! = f_!$ and this functor is exact.

2. *Let $g: T \rightarrow S$ be a morphism of schemes. Assume that $\dim(g) \leq d$. Then, the functor g^* induces a right exact functor*

$${}^{hp}H^d g^*: M_{\text{perv}}^A(S, R) \rightarrow M_{\text{perv}}^A(T, R).$$

Furthermore, if the functor f is étale, we have $H^0 f^ = f^*$ and this functor is exact.*

3. *Consider a cartesian square of schemes*

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \downarrow q & & \downarrow p \\ T & \xrightarrow{f} & S \end{array}$$

such that p is quasi-finite and $\dim(f) \leq d$. Then we have a canonical equivalence

$${}^{hp}H^d f^* {}^{hp}H^0 p_! \rightarrow {}^{hp}H^0 q_! {}^{hp}H^d g^*.$$

4. *Let S be a scheme. Assume that $\dim(S) \leq 2$. Let $f: T \rightarrow S$ be a quasi-finite and affine morphism. Then, the functor*

$${}^{hp}H^0 f_!: M_{\text{perv}}^A(T, R) \rightarrow M_{\text{perv}}^A(S, R).$$

is exact and ${}^{hp}H^0 f_! = f_!$.

5. *Let $i: F \rightarrow S$ be a closed immersion and $j: U \rightarrow S$ be the open complement. Assume that $\dim(S) \leq 2$ and that j is affine. Then, we have an exact sequence:*

$$0 \rightarrow i_* {}^{hp}H^{-1} i^* M \rightarrow j_! j^* M \rightarrow M \rightarrow i_* {}^{hp}H^0 i^* M \rightarrow 0.$$

6. Let p be a prime number. Assume that the ring $R = K$ is a number field and that the scheme S is of finite type over \mathbb{F}_p . Let v be a non-archimedean valuation on K which does not extend the p -adic valuation. Then, the v -adic realization functor ρ_v induces an exact and conservative functor

$$M_{\text{perv}}^A(S, K) \rightarrow \text{Perv}^A(S, K_v)^\#.$$

7. Let R be a $\mathbb{Z}/n\mathbb{Z}$ -algebra where n is invertible on S . Then, the functor $\rho_!$ induces an equivalence of categories

$$\rho_!: \text{Perv}(S, R) \rightarrow M_{\text{perv}}^A(S, R).$$

Proof. This follows from Propositions [3.5.1.2](#), [3.5.1.9](#), [3.5.3.1](#) and [3.5.4.4](#). \square

3.6.2 Intermediate extension, simple objects and the Ayoub-Zucker weightless intersection complex (rational coefficients)

We now show that the abelian category of perverse Artin motives with coefficients in a number field K has similar features as the category of perverse sheaves.

First, we recall the definition of the intermediate extension functor given by the gluing formalism (see [\[BBDG18, 1.4.22\]](#)):

Definition 3.6.2.1. Let S be a scheme, let K be a number field and let $j: U \rightarrow S$ be an open immersion. The intermediate extension functor $j_{!*}^A$ is the functor which to an object M in $M_{\text{perv}}^A(S, K)$ associates the image of the morphism

$${}^{hp}H^0 j_! M \rightarrow {}^{hp}H^0 \omega^0 j_* M.$$

Using [\[BBDG18, 1.4.23\]](#), we have the following description of this functor (compare with [\[BBDG18, 2.1.9\]](#)):

Proposition 3.6.2.2. Let S be a scheme, let $i: F \rightarrow S$ be a closed immersion, let $j: U \rightarrow S$ be the open complementary immersion and let M be in $M_{\text{perv}}^A(U, K)$. Then, the Artin motive $j_{!*}^A(M)$ is the only extension P of M to S such that $i^* P <_{{}^{hp}} 0$ and $\omega^0 i^! P >_{{}^{hp}} 0$.

The following result is similar to [\[BBDG18, 4.3.1\]](#).

Proposition 3.6.2.3. Let S be an excellent scheme.

1. The abelian category of perverse Artin motives with rational coefficients on S is artinian and noetherian: every object is of finite length.
2. If $j: V \hookrightarrow S$ is the inclusion of a regular connected subscheme and if L is a simple object of $\text{Loc}_V(K)$, then the perverse Artin motive $j_{!*}^A(\rho_! L[\delta(V)])$ is simple. Every simple perverse Artin motive is obtained this way.

Proof. The proof is the same as in [\[BBDG18, 4.3.1\]](#) replacing [\[BBDG18, 4.3.2, 4.3.3\]](#) with the following lemmas. \square

Lemma 3.6.2.4. Let V be a regular connected scheme. Let L be a simple object of $\text{Loc}_V(K)$. Then, letting $F = \rho_! L[\delta(V)]$, for any open immersion $j: U \hookrightarrow V$, we have

$$F = j_{!*}^A j^* F.$$

Proof. Let $i: F \rightarrow V$ be the reduced complementary closed immersion to j . Then, the motive i^*F belongs to $\rho_! \text{Loc}_F(R)[\delta(X)]$ at is therefore in degree at most $\delta(F) - \delta(X) < 0$ with respect to the perverse homotopy t-structure.

Now, if x is a point of F , we have

$$\omega^0 i_x^! F = 0$$

by absolute purity and using Corollary 3.3.1.8. Therefore, by Proposition 3.5.2.1, we get

$$\omega^0 i^! F >_{hp} 0.$$

Therefore, the motive F is an extension of j^*F such that $i^*F <_{hp} 0$ and $\omega^0 i^! F >_{hp} 0$ and the proposition follows from Proposition 3.6.2.2. \square

Lemma 3.6.2.5. *Let V be a regular connected scheme. If L is a simple object of $\text{Loc}_V(K)$, then $\rho_! L[\delta(V)]$ is simple in $M_{\text{perv}}^A(V, K)$.*

Proof. The proof is exactly the same as [BBDG18, 4.3.3]. \square

Proposition 3.6.2.6. *(compare with Proposition 2.4.9.7) Let X be a scheme allowing resolution of singularities by alterations, let $j: U \rightarrow X$ be an open immersion with U nil-regular and let $d = \delta(X)$. Ayoub and Zucker defined (see [AZ12, 2.20]) a motive*

$$\mathbb{E}_X = \omega^0 j_* \mathbb{1}_U.$$

Then,

1. $\mathbb{E}_X[d] = j_{!*}^A(\mathbb{1}_U[d])$.
2. $\mathbb{E}_X[d]$ is a simple perverse Artin motive over X . In particular, $\omega^0 j_* \mathbb{1}_U$ is concentrated in degree d with respect to the perverse homotopy t-structure.

Proof. First, notice that Assertion (2) follows from Assertion (1) and Proposition 3.6.2.3. Corollary 3.5.3.10 ensures that $\omega^0 j_* \mathbb{1}_U$ is in degree d with respect to the perverse homotopy t-structure. Moreover, we have a triangle:

$$j_! \mathbb{1}_U \rightarrow \omega^0 j_* \mathbb{1}_U \rightarrow i_* \omega^0 i^* j_* \mathbb{1}_U.$$

thus, it suffices to show that ${}^{hp}H^d(\omega^0 i^* j_* \mathbb{1}_U) = 0$. To prove this, note that we can assume X to be normal and connected as in the first step of the proof of Proposition 3.5.3.5. By assumption, there is a proper and surjective map $p: Y \rightarrow X$ such that Y is regular and integral and p is generically the composition of an étale Galois cover and of a finite surjective purely inseparable morphism.

Therefore, there is a closed subscheme S of X of positive codimension such that $p: Y \setminus p^{-1}(S) \rightarrow X \setminus S$ is the composition of an étale Galois cover of group G and of a finite surjective purely inseparable morphism.

Let X' be the relative normalization of X in $Y \setminus p^{-1}(S)$. Then, the scheme X' is finite over X and endowed with a G -action and the morphism $X'/G \rightarrow X$ is purely inseparable since X is normal. We have a commutative diagram

$$\begin{array}{ccccc} U' & \xrightarrow{j'} & X' & \xleftarrow{i'} & F' \\ p_U \downarrow & & \downarrow p & & \downarrow p_F \\ U & \xrightarrow{j} & X & \xleftarrow{i} & F \end{array}$$

made of cartesian squares. Using [Ayo07, 2.1.165], the motive $\mathbb{1}_U$ is a retract of the motive $(p_U)_*\mathbb{1}_{U'}$. Moreover, we have

$$\omega^0 i^* j_* (p_U)_* \mathbb{1}_{U'} = (p_F)_* \omega^0 (i')^* j'_* \mathbb{1}_{U'}.$$

Thus, we can replace X with X' and assume that the group G is trivial. In this case, the morphism p is birational. Let $\pi: E \rightarrow F$ be the pullback of p along i . Using the same argument as in the first step of the proof of Proposition 3.5.3.5, we get that

$$\omega^0 i^* j_* \mathbb{1}_U = \omega^0 \pi_* \mathbb{1}_E.$$

The proof then follows from Proposition 3.5.3.8. \square

3.6.3 Description of perverse Artin motives and t-exactness of the realization

The following description of Artin perverse motives is a variation on Beilinson's gluing method developed in [Bei87a]. Let M be an Artin perverse motive over an excellent base scheme S of dimension at most 2. Proposition 3.2.3.1 gives a stratification of S with regular strata and such that for any stratum T , the motive $M|_T$ is a constructible smooth Artin motive. We can recover the motive M from the motives $M|_T$ and additional data.

We take the convention that $\delta(S) = 2$ and assume for simplicity that the scheme S is reduced. Let U be the (disjoint) union of the strata containing the generic points of S . Without loss of generality, we can assume that U is a single stratum over which the motive M a constructible smooth Artin motive. Shrinking the open subset U , we can also assume that the immersion $j: U \rightarrow S$ is affine. Write $i: F \rightarrow S$ the reduced complementary closed immersion. The motive $M|_U$ is a perverse Artin motive and the constructible Artin motive $M|_F$ is in perverse degree $[-1, 0]$.

By localization, we have an exact triangle

$$i_* M|_F[-1] \rightarrow j_! M|_U \rightarrow M.$$

Furthermore, by Proposition 3.6.1.2, we have an exact sequence

$$0 \rightarrow i_* {}^{hp}H^{-1}M|_F \rightarrow j_! M|_U \rightarrow M \rightarrow i_* {}^{hp}H^0M|_F \rightarrow 0$$

Hence, we can recover M as the cofiber of a map $i_* M|_F[-1] \rightarrow j_! M|_U$ such that the induced map $i_* {}^{hp}H^{-1}M|_F \rightarrow j_! M|_U$ is a monomorphism.

Conversely, a smooth Artin perverse sheaf M_U over U , a constructible Artin motive M_F in perverse degrees $[-1, 0]$ over F and a connecting map $i_* M_F[-1] \rightarrow j_! M_U$ such that the induced map $i_* {}^{hp}H^{-1}M_F \rightarrow j_* M_U$ is a monomorphism, give rise to a unique perverse Artin motive M over S .

Now, recall that we have an exact triangle

$$j_! \rightarrow \omega^0 j_* \rightarrow i_* \omega^0 i^* j_*$$

and that $j^* i_* = 0$. Therefore, we have an exact sequence

$$0 \rightarrow i_* {}^{hp}H^{-1}(\omega^0 i^* j_* M_U) \rightarrow j_! M_U \rightarrow {}^{hp}H^0(\omega^0 j_* M_U) \rightarrow i_* {}^{hp}H^0(\omega^0 i^* j_* M_U) \rightarrow 0.$$

and we have

$$\mathrm{Hom}(i_*M_F[-1], j_!M_U) = \mathrm{Hom}(M_F, \omega^0 i^* j_* M_U).$$

Therefore, if $\phi: i_*M_F[-1] \rightarrow j_!M_U$ is any map, the map ${}^{hp}H^0(\phi)$ is a monomorphism if and only if the induced map

$${}^{hp}H^{-1}(M_F) \rightarrow {}^{hp}H^{-1}(\omega^0 i^* j_* M_U)$$

is a monomorphism.

Similarly, since the stratification of S induces a stratification of F , the Artin motive M_F is uniquely encoded by the data of a smooth constructible Artin motive M_1 concentrated in perverse degrees $[-1, 0]$ over an affine open subset U_1 of F , a constructible Artin motive N_1 in perverse degrees $[-2, 0]$ on $F_1 = F \setminus U_1$ and a connecting map with a similar condition.

This discussion provides the following proposition:

Proposition 3.6.3.1. *Let S be an excellent scheme of dimension 2 or less and let R be a regular good ring. Then, perverse Artin étale motive over S are uniquely encoded by the following data.*

- A triple (U_0, U_1, U_2) of locally closed subschemes of S such that each U_k is nil-regular, purely of codimension k , is open in the scheme $F_{k-1} := S \setminus (U_0 \cup \dots \cup U_{k-1})$ and such that the open immersion

$$j_k: U_k \rightarrow F_{k-1}$$

is affine (take the conventions that $F_{-1} = S$). Denote by

$$i_k: F_k \rightarrow F_{k-1}$$

the closed immersion.

- A triple (M_0, M_1, M_2) of constructible smooth Artin motives on U_i such that each M_i is placed in perverse degree $[-i, 0]$.
- Two connecting maps

$$\begin{cases} \phi_2: M_2 \rightarrow \omega^0 i_1^*(j_1)_* M_1 \\ \phi_1: M_{12} \rightarrow \omega^0 i_0^*(j_0)_* M_0 \end{cases}$$

such that ${}^{hp}H^{-k}(\phi_k)$ is a monomorphism and where M_{12} is the cofiber of the map

$$(i_1)_* M_2[-1] \rightarrow (j_1)_! M_1$$

induced by ϕ_2 .

Remark 3.6.3.2. *If $\dim(S) \leq 1$, $U_2 = \emptyset$, $M_2 = 0$ and the data of ϕ_1 is trivial so that $M_{12} = M_1$.*

This description yields the t-exactness if the ℓ -adic realization functor.

Theorem 3.6.3.3. *Let S be an excellent scheme of dimension 2 or less, let R be a localization of the ring of integers of a number field K , let v be a non-archimedean valuation on K and let ℓ be the prime number such that v extends the ℓ -adic valuation. Then, the reduced v -adic realization functor*

$$\bar{\rho}_v: \mathcal{DM}_{\text{ét},c}^A(S, R) \rightarrow \mathcal{D}_c^b(S[1/\ell], R_v)$$

of Definition [3.2.2.1](#) is t-exact when the left hand side is endowed with the perverse homotopy t-structure and the right hand side is endowed with the perverse t-structure.

Proof. First, notice that ρ_v is right t-exact because the generators of the perverse homotopy t-structure are sent to perverse t-non-positive objects. Using same trick as in the proof of Theorem 3.4.2.7, we can assume that $R = K$. We may also assume that ℓ is invertible on S . Now, it suffices to show that any perverse Artin étale motive on S is sent to a complex which is non-negative for the perverse t-structure. Let M be such a motive on S .

By Proposition 3.6.3.1, the motive M is given by

- A triple (U_0, U_1, U_2) of locally closed subschemes of S such that each U_k is nil-regular, purely of codimension k , is open in the scheme $F_{k-1} := S \setminus (U_0 \cup \dots \cup U_{k-1})$ and such that the open immersion

$$j_k: U_k \rightarrow F_{k-1}$$

is affine (take the conventions that $F_{-1} = S$). Denote by

$$i_k: F_k \rightarrow F_{k-1}$$

the closed immersion.

- A triple (M_0, M_1, M_2) of constructible smooth Artin motives on U_i such that each M_i is placed in perverse degree $[-i, 0]$.
- Two connecting maps

$$\begin{cases} \phi_2: M_2 \rightarrow \omega^0 i_1^*(j_1)_* M_1 \\ \phi_1: M_{12} \rightarrow \omega^0 i_0^*(j_0)_* M_0 \end{cases}$$

such that ${}^h p H^{-k}(\phi_k)$ is a monomorphism and where M_{12} is the cofiber of the map $(i_1)_* M_2[-1] \rightarrow (j_1)_! M_1$ induced by ϕ_2 .

By Proposition 3.5.2.1, the ℓ -adic complex $\rho_v(M)|_{U_i} = \rho_v(M_i)$ is in perverse degree $[-i, 0]$. To show that $\rho_v(M)$ is a perverse sheaf, it suffices to show that the natural maps

$$\begin{cases} \psi_2: \rho_v(M_2) \rightarrow \rho_v(i_1^*(j_1)_* M_1) \\ \psi_1: \rho_v(M_{12}) \rightarrow \rho_v(i_0^*(j_0)_* M_0) \end{cases}$$

are such that ${}^p H^{-k}(\psi_k)$ is a monomorphism.

Notice now that if $k \in \{1, 2\}$ and $M_{\geq k} = \begin{cases} M_{12} & \text{if } k = 1 \\ M_2 & \text{if } k = 2 \end{cases}$, we have a commutative diagram

$$\begin{array}{ccc} \rho_v(M_{\geq k}) & \xrightarrow{\rho_v(\phi_k)} & \rho_v(\omega^0 i_{k-1}^*(j_{k-1})_* M_{k-1}) \\ & \searrow \psi_k & \downarrow \rho_v(\delta(i_{k-1}^*(j_{k-1})_* M_{k-1})) \\ & & \rho_v(i_{k-1}^*(j_{k-1})_* M_{k-1}) \end{array}$$

Working by induction on $\dim(S)$, we may assume that the maps

$$\mathcal{DM}_{\text{ét},c}^A(F_{k-1}, K) \rightarrow \mathcal{D}_c^b(F_{k-1}, K_v)$$

are t-exact. Thus, the map ${}^p H^{-k}(\rho_v(\phi_k))$ is a monomorphism. To prove the theorem, it therefore suffices to show that the map ${}^p H^{-k}[\rho_v(\delta(i_{k-1}^*(j_{k-1})_* M_{k-1}))]$ is a monomorphism.

Write

$$M_1 = A[1] \oplus B[0],$$

which is possible since the category of smooth Artin motives with coefficients in K is semi-simple by Maschke's theorem. Then, using Theorem 3.5.3.7 and the induction hypothesis, the ℓ -adic complex $\rho_v(\omega^0 i_1^*(j_1)_* B)$ is in degree at least -1 with respect to the perverse t-structure. By the usual affine Lefschetz theorem, so is $\rho_v(i_1^*(j_1)_* B)$, since by Proposition 3.5.2.1, the ℓ -adic complex $\rho_v(B)$ is a perverse sheaf. Hence, the map ${}^p H^{-2}[\rho_v(\delta(i_1^*(j_1)_* M_1))]$ is a monomorphism if and only if the map

$${}^p H^{-1}[\rho_v(\delta)] : {}^p H^{-1}[\rho_v(\omega^0 i_1^*(j_1)_* A)] \longrightarrow {}^p H^{-1}[\rho_v(i_1^*(j_1)_* A)]$$

is a monomorphism. Hence, to prove the theorem, it suffices to show that the maps

$${}^p H^{-1}[\rho_v(\delta)] : {}^p H^{-1}[\rho_v(\omega^0 i_1^*(j_1)_* A)] \longrightarrow {}^p H^{-1}[\rho_v(i_1^*(j_1)_* A)]$$

and

$${}^p H^{-1}[\rho_v(\delta)] : {}^p H^{-1}[\rho_v(\omega^0 i_0^*(j_0)_* M_0)] \longrightarrow {}^p H^{-1}[\rho_v(i_0^*(j_0)_* M_0)]$$

are monomorphisms which follows from lemma 3.6.3.4 below. □

Lemma 3.6.3.4. *Let S be an excellent scheme of dimension 2 or less, let K be a number field, let v be a non-archimedian valuation on K , let ℓ be the prime number such that v extends the ℓ -adic valuation, let $i: F \rightarrow S$ be a closed immersion and let $j: U \rightarrow S$ be the open complementary immersion. Assume that the prime number ℓ is invertible on S , that the morphism j is affine and that the scheme U is nil-regular. Let M be an object of $M^{smA}(U, K)$. Then, the map*

$${}^p H^{-1} \rho_v(\delta) : {}^p H^{-1}[\rho_v(\omega^0 i^* j_* M)] \longrightarrow {}^p H^{-1}[\rho_v(i^* j_* M)]$$

is a monomorphism.

Proof. Let $d = \dim(S)$. Take the convention that $\delta(S) = d$. As in the proof of Proposition 3.5.3.5, we can assume that $M = \mathbb{1}_U[d]$, that the scheme S is normal and that the scheme U is regular.

Case 1: Assume that $\dim(S) = 1$. Since the closed subscheme F is 0-dimensional, we can assume it to be reduced to a point $\{x\}$. Recall that the 2-morphism

$$\Theta: \rho_! i^* j_* \rightarrow \omega^0 i^* j_* \rho_!$$

of Proposition 3.3.2.1 is defined so that we have a commutative diagram

$$\begin{array}{ccc} \rho_! i^* j_* & \xrightarrow{\Theta} & \omega^0 i^* j_* \\ & \searrow Ex & \downarrow \delta \\ & & i^* j_* \rho_! \end{array}$$

where Ex is the exchange map.

Moreover, the map $\Theta(\underline{K})$ is an equivalence by Proposition 3.3.2.1. Therefore, it suffices to show that the map

$${}^p H^0 \rho_v(Ex) : {}^p H^0 \rho_v(\rho_! i^* j_* \underline{K}) \longrightarrow {}^p H^0(i^* j_* \underline{K}_v)$$

is a monomorphism. Now, this map can be by [AGV73, VIII.5] identified with the canonical map

$$K_v \rightarrow H_{\text{ét}}^0(\mathcal{O}_{S,x}^{sh} \times_S U, K_v)$$

where $\mathcal{O}_{S,x}^{sh}$ is the strict henselization of the local ring of S at x . Since S is normal, this map is an isomorphism by Proposition 3.1.7.2.

Case 2: Assume that $\dim(S) = 2$. Since S is normal its singular point are of codimension 2. Let $f: \tilde{S} \rightarrow S$ be a resolution of singularities of S such that the inverse image of F is a simple normal crossing divisor (which exists by Lipman's theorem). We have a commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{\gamma} & \tilde{S} & \xleftarrow{\iota} & E \\ \parallel & & \downarrow f & & \downarrow p \\ U & \xrightarrow{j} & S & \xleftarrow{i} & F \end{array}$$

made of cartesian squares which yields a commutative diagram

$$\begin{array}{ccc} \rho_v(\omega^0 i^* j_* \mathbb{1}_U) & \longrightarrow & \rho_v(\omega^0 p_* \iota^* \gamma_* \mathbb{1}_U) \\ \downarrow \rho_v(\delta) & & \downarrow \rho_v(\delta) \\ \rho_v(i^* j_* \mathbb{1}_U) & \longrightarrow & \rho_v(p_* \iota^* \gamma_* \mathbb{1}_U) \end{array}$$

where the horizontal maps are equivalences. Thus, it suffices to show that the map

$${}^p H^1 \rho_v(\delta): {}^p H^1 \rho_v(\omega^0 p_* \iota^* \gamma_* \mathbb{1}_U) \longrightarrow {}^p H^1 \rho_v(p_* \iota^* \gamma_* \mathbb{1}_U)$$

is a monomorphism.

The localization triangle (3.0.0.2) induces a morphism of exact triangles:

$$\begin{array}{ccccc} \rho_v(\omega^0 p_* \iota^! \mathbb{1}_{\tilde{S}}) & \longrightarrow & \rho_v(\omega^0 p_* \mathbb{1}_E) & \longrightarrow & \rho_v(\omega^0 p_* \iota^* \gamma_* \mathbb{1}_U) \\ \rho_v(\delta) \downarrow & & \rho_v(\delta) \downarrow & & \downarrow \rho_v(\delta) \\ \rho_v(p_* \iota^! \mathbb{1}_{\tilde{S}}) & \longrightarrow & \rho_v(p_* \mathbb{1}_E) & \longrightarrow & \rho_v(p_* \iota^* \gamma_* \mathbb{1}_U) \end{array}$$

By Lemma 3.3.2.2, we have

$$\omega^0 p_* \iota^! \mathbb{1}_{\tilde{S}} = 0.$$

Moreover, using the cdh-descent argument which appears the last step of the proof of Proposition 3.5.3.5 we can show that

$${}^p H^1(\rho_v(p_* \iota^! \mathbb{1}_{\tilde{S}})) = 0.$$

Hence, we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}^p H^1 \rho_v(\omega^0 p_* \mathbb{1}_E) & \longrightarrow & {}^p H^1 \rho_v(\omega^0 p_* \iota^* \gamma_* \mathbb{1}_U) & \longrightarrow & 0 \\ & & \downarrow {}^p H^1 \rho_v(\delta) & & \downarrow {}^p H^1 \rho_v(\delta) & & \\ 0 & \longrightarrow & {}^p H^1 \rho_v(p_* \mathbb{1}_E) & \longrightarrow & {}^p H^1 \rho_v(p_* \iota^* \gamma_* \mathbb{1}_U) & & \end{array}$$

Therefore, it suffices to show that the map

$${}^p H^1 \rho_v(\delta): {}^p H^1 \rho_v(\omega^0 p_* \mathbb{1}_E) \rightarrow {}^p H^1 \rho_v(p_* \mathbb{1}_E)$$

is a monomorphism.

Write $E = \bigcup_{i \in J} E_i$ with J finite, where for any index i , the scheme E_i is regular and of codimension 1, where for any distinct indices i and j the scheme $E_{ij} = E_i \cap E_j$ is of codimension 2 and regular and where the intersections of 3 distinct E_i are empty.

By cdh-descent, we have a morphism exact triangles

$$\begin{array}{ccccc}
 \rho_v \omega^0 p_* \mathbb{1}_E & \longrightarrow & \bigoplus_{i \in J} \rho_v \omega^0 (p_i)_* \mathbb{1}_{E_i} & \longrightarrow & \bigoplus_{\{i,j\} \subseteq J} \rho_v (p_{ij})_* \mathbb{1}_{E_{ij}} \\
 \downarrow \rho_v(\delta) & & \downarrow \rho_v(\delta) & & \parallel \\
 \rho_v p_* \mathbb{1}_E & \longrightarrow & \bigoplus_{i \in J} \rho_v (p_i)_* \mathbb{1}_{E_i} & \longrightarrow & \bigoplus_{\{i,j\} \subseteq J} \rho_v (p_{ij})_* \mathbb{1}_{E_{ij}}
 \end{array}$$

which yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 \bigoplus_{i \in J} {}^p H^0 \rho_v \omega^0 (p_i)_* \mathbb{1}_{E_i} & \longrightarrow & \bigoplus_{\{i,j\} \subseteq J} \rho_v (p_{ij})_* \mathbb{1}_{E_{ij}} & \longrightarrow & {}^p H^1 \rho_v \omega^0 p_* \mathbb{1}_E & \longrightarrow & \bigoplus_{i \in J} {}^p H^1 \rho_v \omega^0 (p_i)_* \mathbb{1}_{E_i} \longrightarrow 0 \\
 \downarrow {}^p H^0 \rho_v(\delta) & & \parallel & & \downarrow {}^p H^1 \rho_v(\delta) & & \downarrow {}^p H^1 \rho_v(\delta) \\
 \bigoplus_{i \in J} {}^p H^0 \rho_v (p_i)_* \mathbb{1}_{E_i} & \longrightarrow & \bigoplus_{\{i,j\} \subseteq J} \rho_v (p_{ij})_* \mathbb{1}_{E_{ij}} & \longrightarrow & {}^p H^1 \rho_v p_* \mathbb{1}_E & \longrightarrow & \bigoplus_{i \in J} {}^p H^1 \rho_v \omega^0 (p_i)_* \mathbb{1}_{E_i} \longrightarrow 0
 \end{array}$$

The proof then follows from the Five Lemma and from Lemma [3.6.3.5](#) below. □

Lemma 3.6.3.5. *Let S be an excellent 1-dimensional scheme and let $f: X \rightarrow S$ be a proper map, let K be a number field, let v be a non-archimedean valuation on K , let ℓ be the prime number such that v extends the ℓ -adic valuation. Assume that the scheme X is regular and connected. Then,*

1. *The map*

$${}^p H^0(\rho_v \omega^0 f_* \mathbb{1}_X) \rightarrow {}^p H^0(\rho_v f_* \mathbb{1}_X)$$

is an isomorphism.

2. *The map*

$${}^p H^1(\rho_v \omega^0 f_* \mathbb{1}_X) \rightarrow {}^p H^1(\rho_v f_* \mathbb{1}_X)$$

is a monomorphism.

Proof. If f is finite, then, $\omega^0 f_* \mathbb{1}_X = f_* \mathbb{1}_X$ and both statement hold. Otherwise, the image of X is a point and the lemma follows from Corollary [3.3.1.8](#). □

3.6.4 Explicit description of perverse Artin motives on schemes of dimension 1

This description can be made more explicit in the case of 1-dimensional excellent schemes. We begin with some notations:

Notations 3.6.4.1. *Let C be a regular connected 1-dimensional scheme and K be its field of functions. Then, if U is an open subset of C and if x is a closed point of C .*

- *We let K_x be the completion of K with respect to the valuation defined by x on K .*

- We let $\phi_x: \text{Spec}(K_x) \rightarrow \text{Spec}(K)$ be the induced map.
- We let $G_0(x)$ be inertia subgroup of G_{K_x} .
- We let $p(x)$ be the characteristic exponent of the residue field $k(x)$ of C at x .

The following lemma is classical (see for example Lemma 2.5.1.2).

Lemma 3.6.4.2. *Let C be a regular connected 1-dimensional scheme of generic point η , U an open subset of C and x a closed point. Let $K = k(\eta)$ be the field of regular functions on C . Then,*

1. We have an exact sequence of groups:

$$1 \rightarrow G_0(x) \rightarrow G_{K_x} \rightarrow G_{k(x)} \rightarrow 1$$

2. If $x \notin U$, the map

$$G_{K_x} \rightarrow G_K \rightarrow \pi_1^{\text{ét}}(U, \bar{\eta})$$

is injective.

In this setting, we denote by

$$\partial_x: \text{Rep}^A(G_{K_x}, R) \rightarrow \text{Rep}^A(G_{k(x)}, R[1/p(x)])$$

the functor given by $M \mapsto M^{G_0(x)} \otimes_R R[1/p(x)]$.

Proposition 3.6.4.3. *Let C be a regular connected 1-dimensional scheme of generic point η and field of regular functions $k(C)$, let R be a localization of the ring of integers of a number field, let $i: F \rightarrow C$ be a reduced closed immersion and let $j: U \rightarrow C$ be the open complement. If x is a point of F , we denote by $i_x: \{x\} \rightarrow F$ be the closed immersion. Assume that all the residue characteristics of U are invertible in R .*

Then, the diagram

$$\begin{array}{ccc}
 M^{\text{sm}A}(U, R) & \xrightarrow{h^p \tau_{\leq 0} \omega^0 i^* j_*} & \mathcal{DM}_{\text{ét}}^A(F, R) \\
 \rho_{! [+1]} \uparrow & & \uparrow \bigoplus_{x \in F} (i_x)_* \\
 \text{Loc}_U(R) & & \prod_{x \in F} \mathcal{DM}_{\text{ét}}^A(k(x), R) \\
 \bar{\eta}^* \downarrow & & \uparrow \prod_{x \in F} \rho_{! [+1]} \\
 \text{Rep}^A(\pi_1^{\text{ét}}(U, \bar{\eta}), R) & & \prod_{x \in F} \text{Loc}_x(R[1/p(x)]) \\
 \downarrow & & \downarrow \prod_{x \in F} \bar{x}^* \\
 \text{Rep}^A(G_K, R) & \xrightarrow{\prod_{x \in F} \phi_x^*} \text{Rep}^A(G_{K_x}, R) \xrightarrow{\prod_{x \in F} \partial_x} & \text{Rep}^A(G_{k(x)}, R[1/p(x)])
 \end{array}$$

is commutative.

Remark 3.6.4.4. *Less formally, this diagram shows that, under the identification of Theorem 3.2.1.1, the functor*

$${}^{hp}\tau_{\leq 0}\omega^0 i^* j_* : M^{smA}(U, R) \rightarrow \mathcal{DM}_{\acute{e}t}^A(F, R)$$

can be identified with the functor

$$\mathrm{Rep}^A(\pi_1^{\acute{e}t}(U, \bar{\eta}), R)[+1] \rightarrow \prod_{x \in F} \mathcal{D}^b(\mathrm{Rep}^A(G_{k(x)}, R[1/p(x)]))$$

that sends the object $M[1]$ where M is an Artin representation of $\pi_1^{\acute{e}t}(U, \bar{\eta})$ to the complex

$$\bigoplus_{x \in F} \left([\phi_x^*(M)]^{G_0(x)} \otimes_R R[1/p(x)][+1] \right).$$

Proof. We can assume that the closed subscheme F is reduced to a point x . By Proposition 3.3.2.1, it suffices to show that the diagram

$$\begin{array}{ccccc} \mathrm{Loc}_U(R) & \xrightarrow{\tau_{\leq 1} i^* j_*} & \mathcal{S}\mathcal{h}(k(x), R[1/p(x)]) & & \\ \bar{\eta}^* \downarrow & & \uparrow & & \\ \mathrm{Rep}^A(\pi_1^{\acute{e}t}(U, \bar{\eta}), R) & & \mathrm{Loc}_x(R[1/p(x)]) & & \\ \downarrow & & \downarrow \bar{x}^* & & \\ \mathrm{Rep}^A(G_K, R) & \xrightarrow{\phi_x^*} \mathrm{Rep}^A(G_{K_x}, R) & \xrightarrow{\partial_x} \mathrm{Rep}^A(G_{k(x)}, R[1/p(x)]) & & \end{array}$$

is commutative.

Let M be a locally constant étale sheaf over U with finitely presented fibers. Identify M with the corresponding Artin representation of $\pi_1^{\acute{e}t}(U, \bar{\eta})$. By [AGV73, VIII.5], if n is an integer, the étale sheaf $H^n(i^* j_* M)$ identifies with the $G_{k(x)}$ -module

$$H_{\acute{e}t}^n(\mathrm{Spec}(L), \phi_x^* M) = H^n(G_0(x), \phi_x^* M)$$

where L is the maximal unramified extension of K_x . Therefore, the sheaf $H^1(i^* j_* M)$ vanishes and we have

$$H^0(i^* j_* M) = [\phi_x^*(M)]^{G_0(x)} \otimes_R R[1/p(x)]$$

which yields the result. □

We now give an analog to this statement in the general case.

Notations 3.6.4.5. *Let C be an excellent 1-dimensional scheme, let $\nu: \tilde{C} \rightarrow C$ be the normalization of C , let Γ be the set of generic points of \tilde{C} , let U is an open subset of C , let y be a closed point of \tilde{C} and let $\eta \in \Gamma$.*

- We let $U_\eta = \overline{\{\eta\}} \cap \nu^{-1}(U) \subseteq \tilde{C}$.
- We let K_η be the field of regular functions on C_η .
- We let $\eta(y)$ be the unique element of Γ such that $y \in C_{\eta(y)}$.
- We let K_y be the completion of $K_{\eta(y)}$ with respect to the valuation defined by y on $K_{\eta(y)}$.

- We let $\phi_y : \text{Spec}(K_y) \rightarrow \text{Spec}(K_{\eta(y)})$ be the induced map.
- We let $G_0(y)$ be the inertia subgroup of G_{K_y} .

Lemma 3.6.4.6. *Let C be a 1-dimensional excellent scheme, let $\nu: \tilde{C} \rightarrow C$ be the normalization of C , let Γ be the set of generic points of \tilde{C} , let U be a nil-regular open subset of C , let y be a closed point of \tilde{C} and let $\eta \in \Gamma$. Then,*

1. *We have an exact sequence of groups:*

$$1 \rightarrow G_0(y) \rightarrow G_{K_y} \rightarrow G_{k(y)} \rightarrow 1$$

2. *If $y \notin U_{\eta(y)}$, the map*

$$G_{K_y} \rightarrow G_{K_{\eta(y)}} \rightarrow \pi_1^{\text{ét}}(U_{\eta(y)}, \overline{\eta(y)})$$

is injective.

In this setting, we denote by

$$\partial_y: \text{Rep}^A(G_{K_y}, R) \rightarrow \text{Rep}^A(G_{k(y)}, R[1/p(y)])$$

the functor given by $M \mapsto M^{G_0(y)} \otimes_R R[1/p(y)]$.

Corollary 3.6.4.7. *Let C be a 1-dimensional excellent scheme, let $\nu: \tilde{C} \rightarrow C$ be the normalization of C , let Γ be the set of generic points of C , let $i: F \rightarrow C$ be a closed immersion, let $j: U \rightarrow C$ be the open complement and let R be the localization of a number field. Assume that the scheme U is nil-regular and that the residue characteristic exponents of U are invertible in R .*

Then, the diagram

$$\begin{array}{ccc}
\text{M}_{\text{perv}}^{\text{sm}A}(U, R) & \xrightarrow{h^p \tau_{\leq 0} \omega^0 i^* j_*} & \mathcal{DM}_{\text{ét}}^A(F, R) \\
\uparrow \rho_{[+1]} & & \uparrow \bigoplus_{x \in F} (i_x)_* \\
\text{Loc}_U(R) & & \prod_{x \in F} \mathcal{DM}_{\text{ét}}^A(k(x), R) \\
\downarrow \prod_{\eta \in \Gamma} \bar{\eta}^* & & \uparrow \rho_{[+1]} \\
\prod_{\eta \in \Gamma} \text{Rep}^A(\pi_1^{\text{ét}}(U_{\eta}, \bar{\eta}^*), R) & & \prod_{x \in F} \text{Loc}_x(R[1/p(x)]) \\
\downarrow & & \downarrow \prod_{x \in F} \bar{x}^* \\
\prod_{\eta \in \Gamma} \text{Rep}^A(G_{K_{\eta}}, R) & & \prod_{x \in F} \text{Rep}^A(G_{k(x)}, R[1/p(x)]) \\
\downarrow \prod_{\eta \in \Gamma} \left(\prod_{y \in \{\eta\} \cap \nu^{-1}(F)} \phi_y^* \right) & & \downarrow \prod_{x \in F} \bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(x)}}^{G_{k(y)}} \\
\prod_{\eta \in \Gamma} \left(\prod_{y \in \{\eta\} \cap \nu^{-1}(F)} \text{Rep}^A(G_{K_y}, R) \right) & & \prod_{x \in F} \bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(x)}}^{G_{k(y)}} \\
\parallel & & \uparrow \\
\prod_{y \in \nu^{-1}(F)} \text{Rep}^A(G_{K_y}, R) & \xrightarrow{\prod_{y \in \nu^{-1}(F)} \partial_y} & \prod_{y \in \nu^{-1}(F)} \text{Rep}^A(G_{k(y)}, R[1/p(y)]) = \prod_{x \in F} \prod_{\nu(y)=x} \text{Rep}^A(G_{k(y)}, R[1/p(x)])
\end{array}$$

is commutative.

Remark 3.6.4.8. *Less formally, the commutativity of this diagram shows that, under the identification of Theorem [3.2.1.7](#), the functor the functor*

$${}^{hp}\tau_{\leq 0}\omega^0 i^* j_* : M^{smA}(U, R) \rightarrow \mathcal{DM}_{\acute{e}t}^A(F, R)$$

can be identified with the functor that sends the object $(M_\eta[+1])_{\eta \in \Gamma}$ where each M_η is an Artin representation of G_{K_η} to

$$\left(\bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(y)}}^{G_{k(x)}} \left([\phi_y^*(M_{\eta(y)})]^{G_0(y)} \otimes_R R[1/p(x)] \right) [+1] \right)_{x \in F}.$$

Proof. Let $\nu_U : U_{\text{red}} \rightarrow U$ (resp. $\nu_F : \tilde{F} \rightarrow F$) be the pullback of the map ν along j (resp. i).

Let \tilde{j} (resp. \tilde{i}) be the inclusion of U_{red} (resp. \tilde{F}) in \tilde{C} . Recall that the functor ν_U^* is an equivalence. Thus, we have

$$\omega^0 i^* j_* = \omega^0 i^* j_* (\nu_U)_* \nu_U^* = \omega^0 i^* \nu_* \tilde{j}_* \nu_U^* = (\nu_F)_* \omega^0 \tilde{i}_* \tilde{j}_* \nu_U^*.$$

The result then follows from Proposition [3.6.4.3](#) and from Proposition [3.1.5.11](#). □

We can now describe the category of Artin motives over a 1-dimensional excellent scheme.

Definition 3.6.4.9. *Let C be a 1-dimensional excellent scheme, let Γ be the set of generic points of C , let $\nu : \tilde{C} \rightarrow C$ be a normalization of C and let R be a localization of a number field. We define an abelian category $\mathbb{N}(C, R)$ as follows.*

- *An object of $\mathbb{N}(C, R)$ is a quadruple $(U, (M_\eta)_{\eta \in \Gamma}, (M_x)_{x \in C \setminus U}, (f_x)_{x \in C \setminus U})$ where*
 1. *U is a nil-regular open subset of C such that the immersion $U \rightarrow C$ is affine and such that the residue characteristic exponents of U are invertible in R ,*
 2. *for all $\eta \in \Gamma$, M_η is a representation of $\pi_1^{\acute{e}t}(U_\eta, \bar{\eta})$,*
 3. *for all $x \in C \setminus U$, M_x is a complex of representations of $G_{k(x)}$ placed in degrees $[0, 1]$*
 4. *for all $x \in C \setminus U$, f_x is an element of*

$$\text{Hom}_{\mathbb{D}(\text{Rep}^A(G_{k(x)}, R))} \left(M_x, \bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(y)}}^{G_{k(x)}} (\partial_y \phi_y^*(M_{\eta(y)})) \right)$$

such that $H^0(f_x)$ is a monomorphism.

- *An element of $\text{Hom}_{\mathbb{N}(C, R)}((U, (M_\eta), (M_x), (f_x)), (V, (N_\eta), (N_x), (g_x)))$ is a couple of the form $((\Phi_\eta)_{\eta \in \Gamma}, (\Phi_x)_{x \in (C \setminus U) \cap (C \setminus V)})$ where $\Phi_\eta : M_\eta \rightarrow N_\eta$ is a map of representations of G_{K_η} , where $\Phi_x : M_x \rightarrow N_x$ is a map of representations of $G_{k(x)}$ and the diagram*

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & \bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(y)}}^{G_{k(x)}} (\partial_y \phi_y^*(M_{\eta(y)})) \\ \downarrow \Phi_x & & \downarrow \bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(y)}}^{G_{k(x)}} (\partial_y \phi_y^*(\Phi_{\eta(y)})) \\ N_x & \xrightarrow{g_x} & \bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(y)}}^{G_{k(x)}} (\partial_y \phi_y^*(N_{\eta(y)})) \end{array}$$

is commutative.

Proposition 3.6.3.1 and Corollary 3.6.4.7 give us the following proposition:

Proposition 3.6.4.10. *Let C be an excellent 1-dimensional scheme, and let R be a localization of the ring of integers of a number field. Assume that the residue characteristic exponents of a dense open subset of C are invertible in R . Then, the category $M_{\text{perv}}^A(C, R)$ is equivalent to $N(C, R)$.*

Remark 3.6.4.11. *As in Corollary 3.2.3.2 we can also give a description in the case where S has finitely many residue characteristics.*

Example 3.6.4.12. *Let k be an algebraically closed field of characteristic 0. A perverse Artin étale motive over \mathbb{P}_k^1 is given by a quadruple $(\mathbb{P}_k^1 \setminus F, M, (M_x)_{x \in F}, (f_x)_{x \in F})$ where F is a finite set of points of \mathbb{P}_k^1 , where M is an Artin representation of $\pi_1^{\text{ét}}(\mathbb{P}_k^1 \setminus F)$, where M_x is a complex of R -modules of finite type placed in degrees $[0, 1]$ and where*

$$f_x: M_x \rightarrow [\phi_x^*(M)]^{G_0(x)}$$

is such that the morphism $H^0(f_x)$ of R -modules is injective.

We can always assume that the point at infinity belongs to F . Write $F = \{\infty\} \sqcup F'$ and let $m = |F'|$.

Let $x \in F'$, then, the field $k((X-x))$ is the completion of the field $k(X)$ of regular functions on \mathbb{P}_k^1 with respect to the valuation defined by x .

On the other hand, the field $k((1/X))$ is the completion of $k(X)$ with respect to the valuation given by the point at infinity.

Furthermore, recall that the absolute Galois group of the field $k((X))$ is $\hat{\mathbb{Z}}$.

The étale fundamental group of the scheme $\mathbb{P}_k^1 \setminus F$ where F is a finite set of closed points is the profinite completion of the free group over F' ; we denote by g_x for $x \in F'$ its topological generators. The map

$$G_{k((x-x))} \rightarrow \pi_1^{\text{ét}}(\mathbb{P}_k^1 \setminus F)$$

is the only continuous map that sends the topological generator of $\hat{\mathbb{Z}}$ to g_x .

The map $G_{k((1/X))} \rightarrow \pi_1^{\text{ét}}(\mathbb{P}_k^1 \setminus F)$ is the only continuous map which that sends the topological generator to a certain product $(g_{x_1} \cdots g_{x_m})^{-1}$ where $F' = \{x_1, \dots, x_m\}$.

Now, the action of $\pi_1^{\text{ét}}(\mathbb{P}_k^1 \setminus F)$ on M factors through a finite quotient G and is therefore the same as a representation of a finite group with m marked generators which are the images of the g_x for $x \in F'$.

Thus, a perverse Artin étale motive over \mathbb{P}_k^1 is equivalent to the following data:

- A finite number of distinct points x_1, \dots, x_m of \mathbb{A}_k^1 (i.e. of elements of k).
- A finite group G generated by m elements g_1, \dots, g_m ; let $\phi_i: \hat{\mathbb{Z}} \rightarrow G$ be the map given by g_i and $\phi_\infty: \hat{\mathbb{Z}} \rightarrow G$ be the map given by $g_\infty = (g_1 \cdots g_m)^{-1}$.
- An R -linear representation M of G .
- Complexes $M_1, \dots, M_m, M_\infty$ of R -modules of finite type placed in degrees $[0, 1]$.
- Maps $f_i: M_i \rightarrow (\phi_i^* M)^{g_i}$ for $i = 1, \dots, m, \infty$ such that

$$H^0(f_i): H^0(M_i) \rightarrow (\phi_i^* M)^{g_i}$$

is injective.

Example 3.6.4.13. Let $S = \text{Spec}(\mathbb{Z}_p)$. A perverse Artin étale motives over S is given by a triple (M, N, f) where M is an Artin representation of $G_{\mathbb{Q}_p}$, where N is a complex of Artin representation of $\hat{\mathbb{Z}}$ with coefficients in $R[1/p]$ placed in degrees $[0, 1]$ and $f: N \rightarrow M^{G_0} \otimes_R R[1/p]$ (where G_0 is the inertia group) is a map such that the morphism $H^0(f)$ of representations of $\hat{\mathbb{Z}}$ with coefficients in $R[1/p]$ is injective.

Example 3.6.4.14. Let k be an algebraically closed field of characteristic 0. Let A be the localization of the ring $k[X, Y]/(XY)$ at the prime ideal (X, Y) . It is the local ring at the intersection point of two lines in \mathbb{P}_k^2 .

Let $S = \text{Spec}(A)$. The scheme S has two generic points of residue field $k(X)$ and the residue field at the closed point is k . The normalization of S is given by the scheme

$$\text{Spec}(k[X]_{(X)} \times k[Y]_{(Y)}).$$

Let ε be the topological generator of $\hat{\mathbb{Z}}$ and let

$$\phi: \hat{\mathbb{Z}} = G_{k((X))} \rightarrow G_{k(X)}$$

be the inclusion.

A perverse Artin étale motive on S is given by a quadruple (M_1, M_2, N, f) where M_1 and M_2 are Artin representations of $G_{k(X)}$, where N is a complex of R -modules placed in degrees $[0, 1]$ and where

$$f: N \rightarrow (\phi^*(M_1 \oplus M_2))^\varepsilon$$

is such that $H^0(f)$ is injective.

Example 3.6.4.15. Let $S = \text{Spec}(\mathbb{Z}[\sqrt{5}]_{(2)})$. This scheme has two points: its generic point has residue field $\mathbb{Q}(\sqrt{5})$ and its closed point has residue field \mathbb{F}_2 (the field with two elements). The normalization of S is $\text{Spec}\left(\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]_{(2)}\right)$ which is a discrete valuation ring. The residue field of its closed point is the field \mathbb{F}_4 with 4 elements. The completion of the field $\mathbb{Q}(\sqrt{5})$ with respect to the valuation defined by 2 is the field $\mathbb{Q}_2(\sqrt{5})$. Let G_0 be the inertia subgroup of $G_{\mathbb{Q}(\sqrt{5})}$.

Let $\phi: \hat{\mathbb{Z}} \rightarrow \hat{\mathbb{Z}}$ be the multiplication by 2. The map ϕ^* has a right adjoint functor

$$\text{Ind}_\phi: \text{Rep}(\hat{\mathbb{Z}}, \mathbb{Z}_\ell) \rightarrow \text{Rep}(\hat{\mathbb{Z}}, \mathbb{Z}_\ell).$$

Thus a perverse Artin motive on S is given by a triple (M, N, f) where M is an Artin representation of $G_{\mathbb{Q}(\sqrt{5})}$, where N is a complex Artin representations of $\hat{\mathbb{Z}}$ placed in degrees $[0, 1]$ and where

$$f: N \rightarrow \text{Ind}(M^{G_0})$$

is such that $H^0(f)$ is injective.

Recall that there is an analogous description in the ℓ -adic setting (see Definition [2.5.1.7](#)). If G is a profinite group and if Λ is an ℓ -adic field or its ring of integers, we denote by $BG_{\text{proét}}$ the site of profinite continuous G -sets with covers given by continuous surjections (see [BS15](#), 4.1.10]). If k is a field, the stable category $\mathcal{D}(k, \Lambda)$ identifies with the stable category $\mathcal{S}\mathfrak{h}((BG_k)_{\text{proét}}, \Lambda)$ of hypersheaves of modules over the hypersheaf of rings on $(BG_k)_{\text{proét}}$ defined by the topological ring Λ endowed with its trivial action using [BS15](#), 4.3.2]). The stable category $\mathcal{D}_c^b(k, \Lambda)$ then identifies with the stable subcategory $\mathcal{S}\mathfrak{h}_{\text{lisse}}((BG_k)_{\text{proét}}, \Lambda)$ of $\mathcal{S}\mathfrak{h}((BG_k)_{\text{proét}}, \Lambda)$ made of those

objects which are dualizable. The heart of $\mathcal{D}_c^b(k, \Lambda)$ then identifies with the category $\text{Rep}(G_k, \Lambda)$ of continuous representations of G_k with coefficients in Λ .

Let C be an excellent scheme of dimension 1, let $\nu: \tilde{C} \rightarrow C$ be a normalization of C , let Γ be the set of generic points of \tilde{C} , let U be a nil-regular open subset of C , let y be closed point of \tilde{C} , let η be an element of Γ , let R be a localization of a number field L and let v be a non-archemidian valuation on L . Assume that the prime number ℓ such that the valuation v extends the ℓ -adic valuation is invertible on C . We denote by

$$\partial_y^v: \mathcal{S}\mathcal{H}_{\text{lisse}}((BG_{K_y})_{\text{proét}}, R_v) \rightarrow \mathcal{S}\mathcal{H}_{\text{lisse}}((BG_{k(y)})_{\text{proét}}, R_v)$$

the derived functor of the map given by $M \mapsto M^{G_0(y)}$.

We still denote by ∂_y^v the induced map

$$\partial_y^v: \text{Rep}^A(G_{K_y}, R_v) \rightarrow \mathcal{S}\mathcal{H}_{\text{lisse}}((BG_{k(y)})_{\text{proét}}, R_v)$$

We have a 2-morphism

$$e_y: \partial_y \otimes_R R_v \rightarrow \partial_y^v.$$

We can describe the ℓ -adic valuation in terms of our description and of the description of Definition 2.5.1.7.

Proposition 3.6.4.16. *Let C be an excellent 1-dimensional scheme, let R be a localization of the ring of integers of a number field L , let v be a non-archimedean valuation on L and let ℓ be the prime number such that v extends the ℓ -adic valuation. We have a commutative diagram*

$$\begin{array}{ccc} M_{\text{perv}}^A(C, R_{(v)}) & \xrightarrow{\bar{\rho}_v} & \text{Perv}^A(C[1/\ell], R_v) \\ \downarrow & & \downarrow \\ N(C, R_{(v)}) & \xrightarrow{\theta_v} & P(C[1/\ell], R_v) \end{array}$$

where

$$\begin{aligned} \theta_v(U, (M_\eta)_{\eta \in \Gamma}, (M_x)_{x \in C \setminus U}, (f_x)_{x \in C \setminus U}) = \\ (U, (M_\eta \otimes_{R_{(v)}} R_v)_{\eta \in \Gamma}, (M_x \otimes_{R_{(v)}} R_v)_{x \in C \setminus U}, (\psi_x^v(f_x))_{x \in C \setminus U}) \end{aligned}$$

setting

$$\psi_x^v(f_x) = \left(\bigoplus_{\nu(y)=x} \text{Ind}_{G_{k(y)}}^{G_{k(x)}} (e_y [\phi_y^*(M_{\eta(y)} \otimes_{R_{(v)}} R_v)]) \right) \circ (f_x \otimes_{R_{(v)}} R_v)$$

for any closed point x of C .

Appendix: The six functors formalism for cohomological étale motives without resolution of singularities (following Gabber)

In [Pep19b, 1.12], Pépin Lehalleur proved that (constructible) cohomological motives are stable under the six functors. However, one has to assume that $R = \mathbb{Q}$ and to have the functors $f^!$ and f_* one also has to assume to have resolutions of singularities by alterations. This is not

necessary: one can mimic the method of Gabber [CD16, 6.2] in the cohomological case. Thus, we can prove that if X (resp. Y) is a quasi-excellent scheme, and $f : Y \rightarrow X$, f_* (resp. $f^!$) preserves cohomological constructible motives. We will outline how to do this.

1. Let R be a ring. The fibred subcategory $\mathcal{DM}_{\acute{e}t,c}^{\text{coh}}(-, R)$ of $\mathcal{DM}_{\acute{e}t}(-, R)$ is stable under tensor product, negative Tate twists, f^* if f is any morphism and $f_!$ if f is separated and of finite type.
2. From the absolute purity property and the stability of $\mathcal{DM}_{\acute{e}t,c}^{\text{coh}}$ under negative Tate twists and using [Ayo07, 2.2.31], we deduce that if i is the immersion of a simple normal crossing divisor with regular target X , $i^! \mathbb{1}_X$ is a cohomological constructible étale motive.
3. For any prime ideal \mathfrak{p} of \mathbb{Z} , we define \mathfrak{p} -constructible cohomological étale motives as in [CD16, 6.2.2] and show as in [CD16, 6.2.3] that an étale motive is a cohomological constructible étale motive if and only if it is \mathfrak{p} -cohomological constructible for all prime ideal \mathfrak{p} . Moreover, if p is a prime number then as in [CD16, 6.2.4], an étale motive over a scheme of characteristic p is (p) -constructible cohomological if and only if it is (0) -cohomological constructible.
4. Let X be a scheme and \mathfrak{p} a prime ideal of \mathbb{Z} . Denote by $\mathcal{C}_0(X)$ the category of \mathfrak{p} -cohomological constructible étale motives. It has the properties [CD16, 6.2.9(b)(c)]. It does not necessarily have property [CD16, 6.2.9(a)] however, in the proof of [CD16, 6.2.7] we only need the following weaker version: $\mathcal{C}_0(X)$ is a thick subcategory of $\mathcal{DM}_{\acute{e}t}(X, R)$ which contains the object $\mathbb{1}_X$. From this discussion, we deduce Gabber's lemma for cohomological étale motives: if X is quasi-excellent, and if either $\mathfrak{p} = (0)$ or $\mathfrak{p} = (p)$ and for any point x of X , $\text{char}(k(x)) \neq p$, then for any dense immersion $U \rightarrow X$, the étale motive $j_*(\mathbb{1}_U)$ is a \mathfrak{p} -cohomological constructible étale motive.
5. Follow the proofs of [CD16, 6.2.13, 6.2.14] to prove that $\mathcal{DM}_{\acute{e}t,c}^{\text{coh}}(-, R)$ is endowed with the six functors formalism over quasi-excellent schemes.

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