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Sonda Walha

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THÈSE EN CO-TUTELLE INTERNATIONALE

Pour obtenir le diplôme de doctorat

Spécialité Mathématiques

Préparée au sein de FSS de Sfax

Construction de solutions particulières de types ondes progressives pour le modèle de Frenkel-Kontorova et pour l'équation des ondes régularisée

Présentée et soutenue par
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*Construction de solutions particulières de type ondes
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l'équation des ondes régularisée*

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Chapitre 1

Introduction générale

Cette thèse porte sur la construction de solutions particulières de type ondes progressives ou ondes planes pour différentes équations aux dérivées partielles (EDP) et en particulier le modèle de Frenkel-Kontorova et une équation des ondes régularisée. Des résultats théoriques sur l'existence et l'unicité de ces solutions sont obtenus et un schéma numérique est proposé pour les calculer.

Ce mémoire, s'articule comme suit. Le chapitre 2 est destiné à l'étude d'existence et d'unicité des ondes progressives avec le terme d'accélération. Ce chapitre est un article publié à la revue *Journal of Dynamics and Differential Equations* : Existence and uniqueness of traveling wave for accelerated Frenkel-Kontorova model, *Journal of Dynamics and Differential Equations* : Volume 26, Issue 24 (2014), Page 1133-1169. Le chapitre 3 est réservé à l'homogénéisation numérique du modèle Frenkel-Kontorova dans le cas amorti. Le chapitre 4 est destiné à l'étude d'équation des ondes régularisée. Nous construisons une onde plane pour le problème approché et nous prouvons que cette solution satisfait certaines propriétés.

1 Motivation physique :

Les ondes progressives pour les équations de réaction-diffusion ont été introduites par Kolmogorov, Petrovsky et Piskunov [53] à la fin des années 30 et étudiées pour le modèle de la propagation des gènes dominants par Fisher [35]. Durant ces dernières années, un intérêt croissant a été porté à l'étude d'ondes progressives dans différents domaines. Nous citons à titre d'exemple la dynamique des populations [4, 5], la croissance des tumeurs [58], l'écologie [63], la propagation des flammes [27] ou encore la dynamique des dislocations [1] .

Nous nous intéressons dans ce mémoire à l'étude d'ondes progressives ϕ en particulier pour le modèle de Frenkel-Kontorova. Ces ondes sont des solutions particulières de l'EDP correspondante qui vont relier deux états stationnaires 0 et 1. Elles sont caractérisées par le fait que $u(x, t) = \phi(x - ct)$ soit solution de l'EDP avec $\phi(-\infty) = 0$ et $\phi(+\infty) = 1$ ([40], Chapitre 2). La fonction ϕ est appelée le profil de l'onde progressive et c est sa vitesse de déplacement. Il existe une littérature abondante sur l'existence, la stabilité et l'unicité de ce type d'ondes progressives (nous renvoyons le lecteur à [34, 51, 12, 67, 49, 14, 25, 15, 13]).

L'application qui a motivé notre étude est la théorie des dislocations. Une dislocation est un défaut linéaire qui se déplace dans un cristal. Cette théorie a été introduite par Orowan [59], Polanyi [61] et Taylor [64] dans les années 1930. Ils postulèrent que la déformation plastique des matériaux est provoquée par le mouvement de ces défauts linéaires dans les plans cristallins. Cette théorie a été confirmée dans les années 50 par les premières observations aux microscopes électroniques des dislocations. Un modèle très répandu pour décrire la dynamique des dislocations est celui proposé par Y. Frenkel et T. Kontorova [41] en 1938. Ils ont suggéré d'utiliser une chaîne d'atomes

pour décrire de manière simple la structure d'un cristal. Nous renvoyons le lecteur au livre [19] de Braun et Kivshar et à l'article [36] de Floria et Mazo pour une description détaillée du modèle de Frenkel-Kontorova et son utilisation dans d'autres applications.

2 Construction d'ondes progressives pour le modèle de Frenkel-Kontorova

Dans cette section, nous étudions l'interaction d'une chaîne d'atomes en présence d'une dislocation décrite par le modèle de Frenkel-Kontorova suivant :

$$m_0 \frac{d^2 u_i}{dt^2} + \frac{du_i}{dt} = u_{i+1} + u_{i-1} - 2u_i - \sin(2\pi(u_i - L)) - \sin(2\pi L) \quad (2.1)$$

où u_i est la position de la particule $i \in \mathbb{Z}$, $\frac{du_i}{dt}$ représente sa vitesse, $\frac{d^2 u_i}{dt^2}$ est le terme d'accélération de la i -ème particule, m_0 représente la masse d'une particule, $-\sin(2\pi L)$ est une force motrice qui provoque le déplacement de la chaîne d'atomes et $-\sin(2\pi(u_i - L))$ décrit la force créée par un potentiel périodique dont la période est 1.

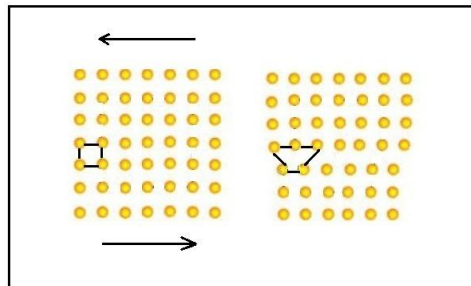


FIGURE 1.1 – Une dislocation dans un cristal

Nous prouvons un résultat d'existence d'ondes progressives pour le modèle (2.1). Comme dans [1], la construction repose sur l'existence de fonctions enveloppes (cf [39]) et utilise la notion de solution de viscosité [28]. Nous prouvons aussi l'unicité de la vitesse à l'aide d'un principe de comparaison et l'unicité du profil en utilisant un résultat de type principe de maximum fort. Ce résultat fait l'objet d'un article publié dans la revue *Journal of Dynamic Differential Equation* [40]. L'idée est de réécrire l'équation (2.1) sous la forme d'un système d'ordre 1. Pour ce faire, comme dans [39], nous introduisons le changement de variables suivant :

$$\xi(t) = u_i(t) + 2m_0 \frac{du_i}{dt}(t). \quad (2.2)$$

Nous obtenons alors le système

$$\begin{cases} \frac{du_i}{dt} = \frac{1}{2m_0}(\xi_i - u_i) \\ \frac{d\xi_i}{dt} = 2(u_{i+1} + u_{i-1} - 2u_i) + 2f_L(u_i) + \frac{1}{2m_0}(u_i - \xi_i) \end{cases} \quad (2.3)$$

avec

$$f_L(v) = -\sin(2\pi(v - L)) - \sin(2\pi L).$$

L'avantage de ce changement de variables est qu'il permet d'obtenir un système monotone si m_0 est assez petit (cf. Hypothèse (A)).

Les ondes progressives :

Nous cherchons une solution particulière, de type onde progressive, du système (2.3), *i.e.* une solution de la forme suivante :

$$\begin{cases} u_i(t) = \phi_1(i + ct) \\ \xi_i(t) = \phi_2(i + ct) \end{cases} \quad (2.4)$$

où (ϕ_1, ϕ_2) est le profil de l'onde progressive et c est la vitesse de la propagation. L'onde progressive recherchée doit satisfaire les propriétés suivantes :

$$\begin{cases} \phi_1(-\infty) = 0, \phi_1(+\infty) = 1 \\ \phi_2(-\infty) = 0, \phi_2(+\infty) = 1 \\ \phi_1' > 0, \phi_2' > 0. \end{cases} \quad (2.5)$$

La condition (2.5) exprime la présence d'une dislocation localisée par exemple en $\phi_1^{-1}\left(\frac{1}{2}\right)$. D'autre part, l'expression (2.4) signifie que le défaut se déplace à la vitesse c . Chaque onde ϕ_1 (resp. ϕ_2) est une transition de phase de 0 à 1. Cela permet donc d'étudier la vitesse de déplacement d'une dislocation en fonction de la force motrice exercée.

Nous injectons l'équation (2.4) dans l'équation (2.3). Le profil (ϕ_1, ϕ_2) satisfait donc

$$\begin{cases} c\phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c\phi_2'(z) = 2(\phi_1(z+1) + \phi_1(z-1) - 2\phi_1(z)) + 2f_L(\phi_1(z)) + \alpha_0(\phi_1(z) - \phi_2(z)) \end{cases} \quad (2.6)$$

avec $z = i + ct$ et $\alpha_0 = \frac{1}{2m_0}$.

Nous avons alors le résultat suivant.

Théorème 2.1 (Existence et unicité des ondes progressives pour le modèle de Frenkel-Kontorova et unicité de la vitesse). *Il existe une constante α^* , un réel c et deux fonctions $\phi_1 : \mathbb{R} \rightarrow \mathbb{R}$ et $\phi_2 : \mathbb{R} \rightarrow \mathbb{R}$ tel que pour $\alpha_0 \geq \alpha^*$*

$$\begin{cases} c\phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c\phi_2'(z) = 2(\phi_1(z+1) + \phi_1(z-1) - 2\phi_1(z)) + 2f_L(\phi_1(z)) + \alpha_0(\phi_1(z) - \phi_2(z)) \\ \phi_1, \phi_2 \text{ sont croissantes sur } \mathbb{R} \\ \phi_1(-\infty) = 0, \phi_1(+\infty) = 1 \\ \phi_2(-\infty) = 0, \phi_2(+\infty) = 1 \end{cases} \quad (2.7)$$

au sens classique si $c \neq 0$ et au sens presque partout si $c = 0$. De plus, si $c \neq 0$, les deux profils et la vitesse c sont uniques.

Le résultat présenté ci-dessus peut s'étendre à des modèles plus généraux.

Étant donnée la fonction $F : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$, on considère le système suivant (avec $N = 2m$) :

$$m_0 \frac{d^2 u_i}{dt^2} + \frac{du_i}{dt} = F(u_{i-m}, \dots, u_{i+m})$$

En utilisant le même changement de variables que précédemment (2.2), cela revient de manière plus générale à construire des fonctions ϕ_1, ϕ_2 vérifiant (2.5) telles que :

$$\begin{cases} c \phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c \phi_2'(z) = 2F((\phi_1(z + r_i))_{i=0, \dots, N}) + \alpha_0(\phi_1(z) - \phi_2(z)). \end{cases}$$

Afin de prouver l'existence d'une solution, nous avons besoin de mettre des hypothèses sur F (on note $f(v) = F(v, \dots, v)$).

Hypothèse (A) :

Régularité de F :

F est globalement Lipschitz sur $[0, 1]^N$;

Monotonie de F : $F(X_0, \dots, X_N)$ est croissante en X_i pour $i \neq 0$, et

$$2 \frac{\partial F}{\partial X_0} + \alpha_0 > 0. \tag{2.8}$$

Hypothèse (B) : Instabilité : $f(0) = 0 = f(1)$ et il existe $b \in [0, 1]$ tel que

$$f(b) = 0, \quad f_{|(0,b)} < 0, \quad f_{|(b,1)} > 0 \quad \text{et} \quad f'(b) > 0.$$

Régularité : F est C^1 dans un voisinage de $\{b\}^{N+1}$.

La preuve d'existence d'onde progressive est basée sur la construction de fonctions enveloppes.

En 1979, Aubry et André ont prédit que pour une certaine classe de potentiels quasi-périodique, une localisation de transition de phase peut se produire en une dimension

[7]. D'un point de vue mathématique, le problème est d'étudier les propriétés analytiques de la fonction enveloppe, solution d'une équation fonctionnelle. Cette fonction enveloppe est identifiée par une transition de phase.

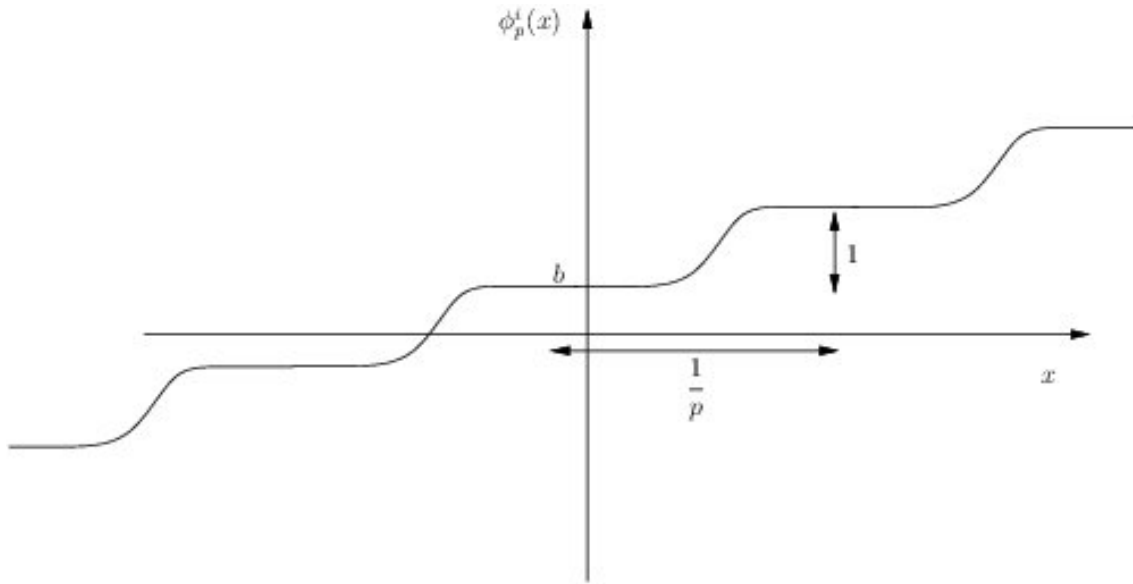
Pour notre modèle (2.3), nous considérons deux fonctions enveloppes h et g construites dans [39] et qui vérifient

$$\left\{ \begin{array}{l} \lambda_p h'_p(x) = \alpha_0(g_p(x) - h_p(x)) \\ \lambda_p g'_p(x) = 2F((h_p(x + p r_i))_{i=0, \dots, n}) + \alpha_0(h_p(x) - g_p(x)) \\ h_p(x + 1) = h_p(x) + 1 \\ g_p(x + 1) = g_p(x) + 1 \\ h'_p(x) \geq 0 \\ g'_p(x) \geq 0. \end{array} \right.$$

Nous considérons alors la remise à l'échelle suivante :

$$\left\{ \begin{array}{l} \phi_p^1(x) = h_p(px) \\ \phi_p^2(x) = g_p(px). \end{array} \right.$$

La représentation typique de ϕ_p^i est alors la suivante :



En passant à la limite $p \rightarrow 0$ pour ϕ_p^1 et ϕ_p^2 , on obtient le résultat suivant :

Théorème 2.2. (*Existence d'ondes progressives*) *Sous les conditions (A) et (B), il existe un réel c et deux fonctions $\phi_1 : \mathbb{R} \rightarrow \mathbb{R}$ et $\phi_2 : \mathbb{R} \rightarrow \mathbb{R}$ qui vérifient*

$$\left\{ \begin{array}{l} c \phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c \phi_2'(z) = 2 F((\phi_1(z + r_i))_{i=0, \dots, N}) + \alpha_0(\phi_1(z) - \phi_2(z)) \\ \phi_1, \phi_2 \text{ sont croissantes sur } \mathbb{R} \\ \phi_1(-\infty) = 0, \phi_1(+\infty) = 1 \\ \phi_2(-\infty) = 0, \phi_2(+\infty) = 1 \end{array} \right. \quad (2.9)$$

au sens classique $c \neq 0$ et au sens presque partout si $c = 0$.

La technique de preuve repose sur celle de [1] où les auteurs prouvent l'existence d'ondes progressives en utilisant des fonctions enveloppes pour le modèle de Frenkel-Kontorova dans le cas complètement amorti. Ceci permet d'obtenir ce résultat d'existence sous des hypothèses très faibles. Il a été prouvé dans [1] l'existence d'ondes progressives dans le cas amorti. Nous renvoyons également à Mallet-Paret [57] pour l'existence d'onde progressive dans le cas amorti mais avec des hypothèses plus fortes.

À notre connaissance, le théorème 2.2 est le premier résultat dans le cas accéléré.

Afin de prouver l'unicité de la vitesse, nous avons besoin d'introduire l'hypothèse suivante sur F .

Hypothèse (C) : Monotonie inverse près de $\{0\}^{N+1}$ et $E = \{1\}^{N+1}$. Il existe $\beta_0 > 0$ tel que pour $a > 0$

$$\begin{cases} F(X + (a, \dots, a)) < F(X) \text{ pour tout } X, X + (a, \dots, a) \in [0, \beta_0]^{N+1} \\ F(X + (a, \dots, a)) < F(X) \text{ pour tout } X, X + (a, \dots, a) \in [1 - \beta_0, 1]^{N+1} \end{cases}$$

Nous avons alors le résultat suivant sur l'unicité de la vitesse :

Théorème 2.3 (Unicité de la vitesse). *Sous la condition (A), nous considérons F définie sur $[0, 1]^{N+1}$. Soit $(c_1, (\phi_{11}, \phi_{12}))$ et $(c_2, (\phi_{21}, \phi_{22}))$ deux solutions de (2.6). Si de plus F satisfait (C), alors $c_1 = c_2$.*

Dans [1], les auteurs prouvent l'unicité de la vitesse en utilisant un principe de comparaison. De la même manière, nous prouvons l'unicité de la vitesse sous des hypothèses très faibles. Afin de prouver l'unicité du profil, nous avons besoin des hypothèses suivantes sur F .

Hypothèse (D+)

- i) Tous les r_i ont le même signe : nous supposons que $r_i \leq 0$, pour tout $i \in \{0, \dots, N\}$.

ou

- ii) Monotonie stricte : il existe i^+ tel que F est croissante en X_{i^+} avec $r_{i^+} > 0$.

Hypothèse (D-)

i) Tous les r_i ont le même signe : Nous supposons que $r_i \geq 0$, pour tout $i \in \{0, \dots, N\}$

ou

ii) Monotonie stricte : il existe i^- tel que F est croissante en X_{i^-} avec $r_{i^-} < 0$.

Le résultat suivant porte sur l'unicité du profil.

Théorème 2.4 (Unicité du profil). *Sous les conditions (A) et (B), il existe un réel c et deux fonctions $\phi_1 : \mathbb{R} \rightarrow \mathbb{R}$ et $\phi_2 : \mathbb{R} \rightarrow \mathbb{R}$ qui vérifient*

$$\left\{ \begin{array}{l} c \phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c \phi_2'(z) = 2 F((\phi_1(z + r_i))_{i=0, \dots, N}) + \alpha_0(\phi_1(z) - \phi_2(z)) \\ \phi_1, \phi_2 \text{ sont croissantes sur } \mathbb{R} \\ \phi_1(-\infty) = 0, \phi_1(+\infty) = 1 \\ \phi_2(-\infty) = 0, \phi_2(+\infty) = 1 \end{array} \right. \quad (2.10)$$

Si $c \neq 0$ alors sous les hypothèses (C) et (D+) i) ou ii) si $c > 0$ (resp. (D-) i) ou ii) si $c < 0$), (ϕ_1, ϕ_2) est unique et ϕ_1 et ϕ_2 sont strictement croissantes.

La preuve de l'unicité du profil dans le cas accéléré repose comme dans [1] sur des principes du maximum fort.

3 Homogénéisation numérique du modèle Frenkel-Kontorova

Nous étudions dans cette section un schéma numérique pour calculer l'hamiltonien effectif du modèle de Frenkel-Kontorova dans le cas amortie. Ce modèle est donné par la dynamique suivante :

$$\frac{du_i}{dt} = u_{i+1} + u_{i-1} - 2u_i - \sin(2\pi(u_i - L)) - \sin(2\pi L) \quad (3.11)$$

Nous étudions dans la section 3 du chapitre 3 l'homogénéisation numérique du modèle de Frenkel-Kontova dans le cas amorti. Nous rappelons que les solutions de (3.11) sont étudiées dans [37] en utilisant la notion de fonction enveloppe. Une fonction enveloppe h est telle que $u(y, \tau) = h(py + \lambda\tau, \tau)$ est une solution de (3.11) Pour une valeur de p fixé, la fonction enveloppe h vérifie

$$\lambda h'(z) = h(z - p) + h(z + p) - 2h(z) - \sin(2\pi(h(z) - L)) - \sin(2\pi L) \quad (3.12)$$

Un schéma numérique associé à (3.12) pourrait être écrit comme

$$\lambda \frac{h_{i+1} - h_i}{\Delta x} = h_{i-N} + h_{i+N} - 2h_i - \sin(2\pi(h_i - L)) - \sin(2\pi L) \quad (3.13)$$

où h_i est une approximation de $h(z)$, $N = \frac{p}{\Delta x}$ et Δx est la taille du maillage.

Nous nous intéressons au calcul numérique de l'hamiltonien effectif λ par deux méthodes : la méthode de calcul en temps long et la méthode de Newton. Ensuite, nous calculons la vitesse effective c

$$\lambda = c \times p.$$

Nous donnons la méthode de calcul en temps long pour calculer l'hamiltonien effectif λ dont on donne sa définition dans ce qui suit.

Définition 3.1. *l'hamiltonien effectif $\lambda(L, p)$ est définie par*

$$\lambda(L, p) = \lim_{t \rightarrow +\infty} \frac{u(y, t)}{t}$$

où u est la solution de l'équation (3.11) avec une condition initiale $u(y, 0) = py$.

Nous donnons maintenant une deuxième méthode, de type Newton, pour calculer λ pour le modèle de Frenkel-Kontorova dans le cas amorti. Cette méthode est inspirée

de [21].

Nous rappelons le schéma numérique associé à (3.12) :

$$\lambda \frac{h_{i+1} - h_i}{\Delta x} = h_{i-N} + h_{i+N} - 2h_i - \sin(2\pi(h_i - L)) - \sin(2\pi L) \quad (3.14)$$

où h_i est une approximation de $h(z)$, $N = \frac{p}{\Delta x}$ et Δx est la taille du maillage. Nous définissons une nouvelle fonction F telle que

$$F(h, \lambda) := \lambda \frac{h_{i+1} - h_i}{\Delta x} - (h_{i-N} + h_{i+N} - 2h_i - \sin(2\pi(h_i - L)) - \sin(2\pi L)).$$

Le problème discret est équivalent à

$$F(X) = 0 \quad (3.15)$$

où $X = (h, \lambda)$. Nous supposons que F est Fréchet différentiable et nous notons $J_F \in \mathbb{R}^{M \times N}$ sa matrice jacobienne. Nous cherchons les zéros de F en utilisant une méthode de type Newton. Nous utilisons la méthode itérative suivante

$$\begin{cases} X^{(k+1)} = X^{(k)} + \delta, & k \geq 0 \\ J_F(X^{(k)})\delta = -F(X^{(k)}) \end{cases} \quad (3.16)$$

Ce système peut être incompatible pour M et N arbitraires. Nous notons par J_F^\dagger la matrice pseudo-inverse de Moore-Penrose de la matrice jacobienne J_F de F . C'est une matrice d'ordre $N \times M$ telle que

$$J_F J_F^\dagger J_F = J_F, \quad J_F^\dagger J_F J_F^\dagger = J_F^\dagger, \quad (J_F J_F^\dagger)^t = J_F J_F^\dagger, \quad (J_F^\dagger J_F)^t = J_F^\dagger J_F.$$

Il est facile de prouver que

$$\delta^* := -J_F^\dagger(X^{(k)})F(X^{(k)})$$

est l'unique vecteur de la plus petite norme euclidienne qui minimise la norme euclidienne du résidu $J_F(X^{(k)})\delta + F(X^{(k)})$. La solution générale δ^* peut être efficacement

obtenu en évitant le calcul de la matrice pseudo-inverse de Moore-Penrose de la matrice jacobienne J_F . Nous appliquons la factorisation QR de J_F^t .

Pour les systèmes sous-déterminés ($N = M + 1$), par le minimum de la norme euclidienne

$$d^* := \arg \min_{\delta \in \mathbb{R}^n} \|\delta\|_2^2 \quad \text{subject to} \quad J_F(X)\delta + F(X) = 0 \quad (3.17)$$

à condition que le jacobien a un rang complet M . Nous appliquons la factorisation QR de J_F^t :

$$J_F^t = QR$$

où $Q = (Q_1 \quad Q_2) \in \mathbb{R}^{N \times M}$, $Q_1 \in \mathbb{R}^{N \times M}$ et $Q_2 \in \mathbb{R}^{N \times (N-M)}$, $R = (R_1 \quad 0)^t \in \mathbb{R}^{N \times M}$, $R_1 \in \mathbb{R}^{M \times M}$ est une matrice triangulaire supérieure et $0 \in \mathbb{R}^{(N-M) \times M}$.

Nous avons

$$J_F = R^t Q^t = R_1^t Q_1^t.$$

Soit

$$\delta = Qz = Q_1 z_1 + Q_2 z_2.$$

Nous obtenons

$$\begin{aligned} 0 &= J_F(X^{(k)})\delta + F(X^{(k)}) = R_1^t Q_1^t (Q_1 z_1 + Q_2 z_2) + F(X^{(k)}) \\ &= R_1^t z_1 + R_1^t Q_1^t Q_2 z_2 + F(X^{(k)}). \end{aligned}$$

Puisque $Q_1^t Q_2 = 0$ par l'orthogonalité de Q , nous obtenons

$$R_1^t z_1 + F(X^{(k)}) = 0. \quad (3.18)$$

Nous pouvons minimiser (voir (3.17))

$$\|Q^t \delta\|_2^2 = \|z_1\|_2^2 + \|z_2\|_2^2.$$

Nous prenons $z_2 = 0$, et nous concluons que

$$\delta^* = Q_1 z_1 = -Q_1 (R_1^{-1})^t F(X^{(k)})$$

où z_1 est calculé par (3.18).

Algorithme

Nous considérons l'algorithme suivant pour la solution de (3.15).

Étant donné une donnée initiale X et $\varepsilon > 0$,

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- Assembler $F(X)$ et $J_F(X)$
- Résoudre le système linéaire $J_F(X)\delta = -F(X)$ en utilisant la factorisation QR de $J_F(X)^t$
- Mettre à jour $X \leftarrow X + \delta$.

Jusqu'à $\|\delta\|_2^2 < \varepsilon$ ou $\|F(X)\|_2^2 < \varepsilon$.

Nous implémentons un code SCILAB. Dans ce code, nous employons plusieurs variantes bien connues et des modifications de la méthode de Newton classique, comme discuté dans les remarques suivantes.

Remark 3.2.

1. Parfois, les méthodes de type Newton ne convergent pas, en raison des oscillations autour d'un minimum de la fonction résiduelle $\|F(X)\|_2$. Dans ce cas, nous introduisons un paramètre dans l'étape de mise à jour : $X \leftarrow X + \mu\delta$ pour $0 < \mu < 1$.
2. Il peut arriver que $J_F(X)$ devienne une matrice singulière. Dans l'esprit de la méthode de Levenberg-Marquardt, nous pouvons régulariser $J_F(X)$ avec $\tau I + J_F(X)$, pour $\tau > 0$.

Dans la section 4, nous donnons les simulations numériques d'effective hamiltonian

et d'effective de la vitesse pour le modèle de Frenkel-Kontorova dans les cas amorti. Dans la section 5 ,Nous présentons des simulations de la méthode de type Newton. Nous comparons cette méthode avec la méthode de calcul en temps long.

4 Construction d'onde plane pour une équation des onde régularisée.

Nous nous intéressons maintenant à étudier une équation des ondes régularisée dans un domaine périodique. L'objectif de ce travail est de montrer l'existence d'onde plane pour l'équation des ondes. Comme il est difficile de résoudre ce problème, nous construisons une onde plane pour un problème approché et nous prouvons que cette solution satisfait certaines propriétés. Cette approche est inspirée de la thèse d'Arnad le Guicher [3] dans le cas $m = 0$.

Nous rappelons que l'équation des ondes est donnée par :

$$mu_{tt} + u_t = u_{xx} + f(u). \tag{4.19}$$

Afin d'obtenir un système d'ordre 1 et utiliser la technique déployée ci-dessus, nous posons

$$\xi := u + 2mu_t + 2\sqrt{m}u_x. \tag{4.20}$$

En multipliant la dernière équation (4.20) par $\frac{1}{2m}$, nous obtenons

$$u_t + \frac{1}{\sqrt{m}}u_x + \frac{1}{2m}u = \frac{1}{2m}\xi.$$

En dérivant l'équation (4.20) par rapport au temps, nous avons

$$\xi_t = u_t + 2mu_{tt} + 2\sqrt{m}u_{xt}.$$

En injectant l'équation (4.19) dans l'équation précédente, nous déduisons que

$$\xi_t = -u_t + 2u_{xx} + 2f(u) + 2\sqrt{m}u_{xt}.$$

En combinant l'équation précédente et (4.20), nous obtenons

$$\xi_t = \frac{1}{2m}u - \frac{1}{2m}\xi + \frac{1}{\sqrt{m}}u_x + 2u_{xx} + 2\sqrt{m}u_{xt} + 2f(u). \quad (4.21)$$

Or, en dérivant l'équation (4.20) par rapport à x , nous avons

$$\xi_x = u_x + 2mu_{xt} + 2\sqrt{m}u_{xx}. \quad (4.22)$$

Donc

$$\xi_t = \frac{1}{2m}u - \frac{1}{2m}\xi + \frac{1}{\sqrt{m}}\xi_x + 2f(u).$$

Nous avons finalement obtenu le système suivant

$$\begin{cases} u_t + \frac{1}{\sqrt{m}}u_x + \frac{1}{2m}u = \frac{1}{2m}\xi \\ \xi_t - \frac{1}{\sqrt{m}}\xi_x + \frac{1}{2m}\xi = \frac{1}{2m}u + 2f(u). \end{cases} \quad (4.23)$$

Afin d'obtenir un système monotone, nous ajoutons la condition suivante

$$\frac{1}{2m} + 2\frac{\partial f}{\partial u}(u) \geq 0.$$

Notre but est de construire une solution de type onde plane, c'est-à-dire

$$\begin{cases} u\left(x+1, t - \frac{p}{\lambda}\right) = u(x, t) \\ \xi\left(x+1, t - \frac{p}{\lambda}\right) = \xi(x, t) \end{cases}$$

avec $p^{-1} \in \mathbb{N}$ et $\lambda \in \mathbb{R}$.

Nous considérons la condition initiale suivante pour l'équation (4.23)

$$u(0, x) = \xi(0, x) = px \quad \text{avec } p^{-1} \in \mathbb{N}. \quad (4.24)$$

Nous supposons que la fonction f est régulière et vérifie

$$\begin{cases} f \in Lip(\mathbb{R}) \\ f(v+l) = f(v) \quad \text{pour tout } l \in \mathbb{Z}, v \in \mathbb{R}. \end{cases} \quad (4.25)$$

Comme il est difficile de résoudre le problème (4.23), nous introduisons un problème approché avec un terme régularisant pour pouvoir contrôler les oscillations en espace et en temps. Pour $\delta > 0$, nous considérons

$$\begin{cases} u_t + \frac{1}{\sqrt{m}}u_x + \frac{1}{2m}u = \frac{1}{2m}\xi + \delta(a_0 + a_1[u(\cdot, t)](x))u_x & \text{dans } \mathbb{R} \times (0, +\infty) \\ \xi_t - \frac{1}{\sqrt{m}}\xi_x + \frac{1}{2m}\xi = \frac{1}{2m}u + 2f(u) + \delta(a_0 + a_1[\xi(\cdot, t)](x))\xi_x & \text{dans } \mathbb{R} \times (0, +\infty) \\ u_0(x) = \xi_0(x) = px & \text{on } \mathbb{R} \end{cases} \quad (4.26)$$

où $a_0 > \frac{1}{\sqrt{m}}$ et le terme non-local a_1 est définie pour une fonction $v : \mathbb{R} \rightarrow \mathbb{R}$

$$a_1[v](x) = \inf_{y \in \mathbb{R}} (v(y, t) - py) + px - v(x).$$

Le résultat qui suit porte sur l'existence de solution de type onde plane pour le problème approché (4.26).

Théorème 4.1. *Sous la condition (4.25), pour $p > 0$ avec $p^{-1} \in \mathbb{N}$, il existe*

$C_\delta > 0, \lambda \in \mathbb{R}$ et une solution (u, ξ) de (4.27) qui vérifient les propriétés suivantes :

$$\begin{cases} (i) |u(x, t) - px - \lambda t| \leq C_\delta, \quad |\xi(x, t) - px - \lambda t| \leq C_\delta \\ (ii) u\left(x + \frac{1}{p}, t\right) = 1 + u(x, t), \quad \xi\left(x + \frac{1}{p}, t\right) = 1 + \xi(x, t) \\ (iii) u_x \geq 0, \xi_x \geq 0 \\ (iv) |u(x + y, t) - u(x, t) - py| \leq 2, |\xi(x + y, t) - \xi(x, t) - py| \leq 2 \end{cases} \quad (4.27)$$

La propriété (i) permet de contrôler l'oscillation de la solution en temps. La propriété (ii) signifie que (u, ξ) est périodique plus linéaire en espace. La propriété (iii)

signifie que u et ξ sont croissantes en espace. La propriété (iv) permet de contrôler l'oscillation de la solution en espace.

Chapitre 2

Construction d'ondes progressives pour le modèle de Frenkel-Kontorova

Ce chapitre est un article publié à la revue *Journal of Dynamics and Differential Equations* : Existence and uniqueness of traveling wave for accelerated Frenkel-Kontorova model, *Journal of Dynamics and Differential Equations* : Volume 26, Issue 24 (2014), Page 1133-1169. Nous étudions l'existence et l'unicité des ondes progressives avec le terme d'accélération. Ce modèle consiste d'un système ODE qui décrit les mouvements des particules dans les interactions. Les applications les plus importantes que nous avons à l'esprit est le mouvement des défauts cristallins appelés dislocations. Pour ce modèle, nous montrons l'existence des ondes progressives sous des hypothèses très faible. L'unicité de la vitesse est étudié ainsi que l'unicité du profil en utilisant les différents types du principe de maximum fort. Comme ce que nous savons, c'est le premier résultat concernant les ondes progressives pour un système accéléré, spatialement discret.

Existence and uniqueness of traveling wave for accelerated Frenkel-Kontorova model

Abstract

In this paper, we study the existence and uniqueness of traveling wave solution for the accelerated Frenkel-Kontorova model. This model consists in a system of ODE that describes the motion particles in interaction. The most important applications we have in mind is the motion of crystal defects called dislocations. For this model, we prove the existence of traveling wave solutions under very weak assumptions. The uniqueness of the velocity is also studied as well as the uniqueness of the profile which used different types of strong maximum principle. As far as we know, this is the first result concerning traveling waves for accelerated, spatially discrete system.

AMS Classification : 35B27, 35F20, 45K05, 47G20, 49L25, 35B10.

Keywords : Frenkel-Kontorova models, Traveling waves, Viscosity solutions, Maximum principle, Hull function.

1 Introduction

In the present paper, we study the accelerated Frenkel-Kontorova model (F-K) which describes a chain of particles interacting by an harmonic potential. Besides its original aim of modeling crystal dislocations, the (F-K) model has many applications in physics such as the description of magnetic domain walls, atoms adsorbed on a crystalline surface or superionic conductors (see for instance the book of Braun and Kivshar [20] for an introduction to this model). The goal of this work is to prove the existence and the uniqueness of traveling wave as well as the uniqueness of the velocity. This work is a generalization of the one of Al Haj et al. [1] in which the authors study

the fully overdamped case.

The study of traveling waves in reaction-diffusion equations has been introduced in pioneering works of Fisher [35] and Kolmogorov, Petrovsky and Piskunov [52]. Existence of traveling waves solutions has been for instance obtained in [4, 17, 34, 51]. More generally, there is a huge literature about existence, uniqueness and stability of traveling waves with various non linearities with applications in particular in biology and combustion and we refer for instance to the references cited in [16, 24]. There are also several works on discrete or nonlocal versions of reaction-diffusion equations (see for instance [10, 11, 22, 23, 26, 29, 33, 46, 48, 56, 57, 66, 68, 69] and the references cited therein) and on damped hyperbolic equation (see [30, 44, 43, 42, 65]) but, as far as we know, there is no result concerning hyperbolic discrete in space equations.

1.1 The Frenkel-Kontorova model

The classical Frenkel-Kontorova (F-K) model describes the dynamics of crystal defects. If $u_i(t)$ is the position of the particle $i \in \mathbb{Z}$, then the classical (F-K) models is given by the following dynamics

$$m_0 \frac{d^2 u_i}{dt^2} + \frac{du_i}{dt} = u_{i+1} + u_{i-1} - 2u_i - \sin(2\pi(u_i - L)) - \sin(2\pi L)$$

where $\frac{d^2 u_i}{dt^2}$ denotes the acceleration of the i th particle, $\frac{du_i}{dt}$ is its velocity, m_0 denotes the mass of the particles, $-\sin(2\pi L)$ is a constant driving force which will cause the movement of the chain of atoms and $-\sin(2\pi(u_i - L))$ describes the force created by a periodic potential whose period is assumed to be 1. We set

$$f_L(v) = -\sin(2\pi(v - L)) - \sin(2\pi L)$$

and for all $i \in \mathbb{Z}$

$$\Xi_i(t) = u_i(t) + 2m_0 \frac{du_i}{dt}(t).$$

We replace it in equation (1.2) in order to obtain the following monotone system : for $i \in \mathbb{Z}$ and $t \in (0, +\infty)$,

$$\begin{cases} \frac{du_i}{dt} = \frac{1}{2m_0}(\Xi_i - u_i) \\ \frac{d\Xi_i}{dt} = 2(u_{i+1} + u_{i-1} - 2u_i) + f_L(u_i) + \frac{1}{2m_0}(u_i - \Xi_i). \end{cases} \quad (1.1)$$

We look for particular traveling wave solution of (1.1), which have the form

$$\begin{cases} u_i(t) = \phi_1(i + ct) \\ \Xi_i(t) = \phi_2(i + ct). \end{cases} \quad (1.2)$$

If we replace (1.2) in (1.1), then the profile (ϕ_1, ϕ_2) should satisfy

$$\begin{cases} c\phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c\phi_2'(z) = 2(\phi_1(z+1) + \phi_1(z-1) - 2\phi_1(z)) + 2f_L(\phi_1(z)) + \alpha_0(\phi_1(z) - \phi_2(z)) \end{cases} \quad (1.3)$$

with $z = i + ct$ and $\alpha_0 = \frac{1}{2m_0}$. We then have the following result.

Theorem 1.1 (Existence and uniqueness of traveling wave solution for Frenkel-Kontorova model). *There exists a constant α^* (which will be made precised later on assumption (A)) such that for all $\alpha_0 \geq \alpha^*$, there exist a unique real c and two functions $\phi_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $\phi_2 : \mathbb{R} \rightarrow \mathbb{R}$ that satisfy*

$$\begin{cases} c\phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c\phi_2'(z) = 2(\phi_1(z+1) + \phi_1(z-1) - 2\phi_1(z)) + 2f_L(\phi_1(z)) + \alpha_0(\phi_1(z) - \phi_2(z)) \\ \phi_1, \phi_2 \text{ are non-decreasing over } \mathbb{R} \\ \phi_1(-\infty) = 0, \phi_1(+\infty) = 1 \\ \phi_2(-\infty) = 0, \phi_2(+\infty) = 1 \end{cases} \quad (1.4)$$

in the classical sense if $c \neq 0$ and almost everywhere if $c = 0$. Moreover, if $c \neq 0$, then the two profiles are unique up to translation.

1.2 Main results for the general case

We now consider a generalization of system (1.3). Given a function $F : [0, 1]^{N+1} \rightarrow \mathbb{R}$, we consider the system

$$\begin{cases} c \phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c \phi_2'(z) = 2F((\phi_1(z + r_i))_{i=0, \dots, N}) + \alpha_0(\phi_1(z) - \phi_2(z)). \end{cases}$$

In order to provide our results, we introduce some assumptions on F .

Assumption (A)

- **Regularity of F** : F is globally Lipschitz continuous over $[0, 1]$;
- **Monotonicity of F** : $F(X_0, \dots, X_N)$ is non-decreasing in X_i for $i \neq 0$, and

$$2 \frac{\partial F}{\partial X_0} + \alpha_0 > 0. \tag{1.5}$$

We set $F(v, \dots, v) = f(v)$.

Assumption (B)

- **Instability** : $f(0) = 0 = f(1)$ and there exists $b \in [0, 1]$ such that

$$f(b) = 0, \quad f_{|(0,b)} < 0, \quad f_{|(b,1)} > 0 \quad \text{and} \quad f'(b) > 0.$$

- **Smoothness** : F is C^1 in a neighborhood of $\{b\}^{N+1}$.

We give the first main result concerning the existence of traveling wave.

Theorem 1.2 (Existence of a traveling wave). *Under assumptions (A) and (B), there*

exist a real c and two functions $\phi_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $\phi_2 : \mathbb{R} \rightarrow \mathbb{R}$ that solves

$$\left\{ \begin{array}{l} c \phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c \phi_2'(z) = 2 F((\phi_1(z + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_1(z) - \phi_2(z)) \\ \phi_1, \phi_2 \text{ are non-decreasing over } \mathbb{R} \\ \phi_1(-\infty) = 0, \phi_1(+\infty) = 1 \\ \phi_2(-\infty) = 0, \phi_2(+\infty) = 1 \end{array} \right. \quad (1.6)$$

in the classical sense if $c \neq 0$ and almost everywhere if $c = 0$.

In order to prove the uniqueness of the traveling waves, we require some additional assumptions :

Assumption (C) : Inverse monotonicity close to $\{0\}^{N+1}$ and $E = \{1\}^{N+1}$:

There exists $\beta_0 > 0$ such that for $a > 0$, we have

$$\left\{ \begin{array}{l} F(X + (a, \dots, a)) < F(X) \text{ for all } X, X + (a, \dots, a) \in [0, \beta_0]^{N+1} \\ F(X + (a, \dots, a)) < F(X) \text{ for all } X, X + (a, \dots, a) \in [1 - \beta_0, 1]^{N+1} \end{array} \right.$$

Assumption (D+)

- i) All the r_i have the same sign : We assume that $r_i \leq 0$, for all $i \in \{0, \dots, N\}$.
- ii) Strict monotonicity : F is increasing in X_{i+} with $r_{i+} > 0$.

Assumption (D-)

- i) All the r_i have the same sign : We assume that $r_i \geq 0$, for all $i \in \{0, \dots, N\}$.
- ii) Strict monotonicity : F is increasing in X_{i-} with $r_{i-} < 0$.

We give the second main results concerning the uniqueness of the velocity and of the profile.

Theorem 1.3 (Uniqueness of the velocity and of the profile). *We assume (A) and let $(c, (\phi_1, \phi_2))$, with $\phi_1, \phi_2 : \mathbb{R} \rightarrow [0, 1]$, be a solution of*

$$\begin{cases} c \phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c \phi_2'(z) = 2 F((\phi_1(z + r_i))_{i=0, \dots, N}) + \alpha_0(\phi_1(z) - \phi_2(z)) \\ \phi_1(-\infty) = 0, \phi_1(+\infty) = 1 \\ \phi_2(-\infty) = 0, \phi_2(+\infty) = 1. \end{cases} \quad (1.7)$$

Then, we have the following properties.

- (a) Uniqueness of the velocity : *Under the additional assumption (C), the velocity c is unique.*
- (b) Uniqueness of (ϕ_1, ϕ_2) : *If $c \neq 0$, then under the additional assumption (C) and (D+) i) or ii) if $c > 0$ (resp. (D-) i) or ii) if $c < 0$), (ϕ_1, ϕ_2) is unique (up to translation) and ϕ_1 and ϕ_2 are increasing.*

Remark 1.4. *We note that $F(X_0, X_1, X_2) = X_1 + X_2 - 2X_0 - \sin(2\pi(X_0 - L)) - \sin(2\pi L)$ satisfies assumptions (A), (B), (C), (D+)ii) and (D-)ii). Then Theorem 1.1 is a direct application of Theorems 1.2 and 1.3.*

For this paper, we define

$$r^* = \max_{i=0 \dots N} |r_i| \quad (1.8)$$

and we assume that $r^* > 0$ (otherwise, the system reduce to a single ODE).

1.3 Organization of the paper

In Section 2, we give the definition of viscosity solution and of Hull function. We also recall some basic results about monotone functions. Section 3 is devoted to the proof of existence of traveling waves, namely Theorem 2.2. In Section 4, we study the

question of uniqueness of the velocity by proving a comparison principal on the half line. Finally, in Section 5, we prove the uniqueness of the profile using different types of strong maximum principles.

2 Preliminary results

This section is divided into four subsections. The first one is devoted to the extension of the function F onto \mathbb{R}^{N+1} . In the second subsection, we give the definition of viscosity solution while the notion of hull functions is recalled in the third one. Finally, we present some results about monotone functions in the last subsection.

2.1 Extension of F

To construct the traveling waves, we will use the hull functions constructed in [38]. To do that, as in [1], we will need to extend the function F by \tilde{F} which is defined over \mathbb{R}^{N+1} and satisfied the following assumption :

Assumption (\tilde{A}) :

- a) Regularity : \tilde{F} is Lipschitz continuous over \mathbb{R}^{N+1} .
- b) Periodicity : $\tilde{F}(X_0 + 1, \dots, X_N + 1) = \tilde{F}(X_0, \dots, X_N)$ for every $X = (X_0, \dots, X_N) \in \mathbb{R}^{N+1}$.
- c) Monotonicity : $\tilde{F}(X_0, \dots, X_N)$ is non-decreasing in V_i for $i \neq 0$ and

$$2 \frac{\partial \tilde{F}}{\partial X_0} + \alpha_0 > 0. \tag{2.9}$$

Then, we have the following extension of the function F .

Lemma 2.1. *Given a function F defined over $Q = [0, 1]^{N+1}$ satisfying (A) and*

$F(1, \dots, 1) = F(0, \dots, 0)$, there exists an extension \tilde{F} defined over \mathbb{R}^{N+1} such that

$$\tilde{F}|_Q = F \text{ and } \tilde{F} \text{ satisfies } (\tilde{A}).$$

Démonstration. The construction is made in [1, Lemma 2.1]. The only thing to verify is that \tilde{F} satisfy (2.9) if F satisfy (1.5), but this is trivial by looking to the way the function \tilde{F} is constructed. \square

Remark 2.2.

We remark that, if (ϕ_1, ϕ_2) is a traveling wave for equation (2.6) with F replaced by \tilde{F} , then (ϕ_1, ϕ_2) is also a traveling wave of the same equation. This is a direct consequence of Lemma 2.1 and the fact that

$$\left\{ \begin{array}{l} (\phi_1, \phi_2) \text{ is non-decreasing over } \mathbb{R} \\ \phi_1(-\infty) = 0, \phi_1(+\infty) = 1 \\ \phi_2(-\infty) = 0, \phi_2(+\infty) = 1. \end{array} \right.$$

Theorem 2.2 is then a direct application of the following result.

Proposition 2.3 (Existence of traveling waves). *We assume that \tilde{F} satisfies (\tilde{A}) and (B). Then there exist a real c and two functions ϕ_1, ϕ_2 solutions of*

$$\left\{ \begin{array}{l} c \phi_1'(z) = \alpha_0(\phi_2(z) - \phi_1(z)) \\ c \phi_2'(z) = 2 \tilde{F}((\phi_1(z + r_i))_{i=0, \dots, N}) + \alpha_0(\phi_1(z) - \phi_2(z)) \\ \phi_1, \phi_2 \text{ are non-decreasing over } \mathbb{R} \\ \phi_1(-\infty) = 0, \phi_1(+\infty) = 1 \\ \phi_2(-\infty) = 0, \phi_2(+\infty) = 1 \end{array} \right. \quad (2.10)$$

in the classical sense if $c \neq 0$ and almost everywhere if $c = 0$.

Proof of Theorem 1.2. The proof of Theorem 1.2 is a direct consequence of Remark 2.2 and Proposition 2.3. \square

For simplicity of presentation, we call \tilde{F} as F in the rest of this section and in Section 3.

2.2 Viscosity solution

In this subsection, we give the definition of viscosity solution. We first recall the definition of the upper and the lower semi-continuous envelopes u^* and u_* :

$$u^*(y) = \limsup_{x \rightarrow y} u(x)$$

$$u_*(y) = \liminf_{x \rightarrow y} u(x).$$

Definition 2.4 (Viscosity solution). *Let $c \in \mathbb{R}$ and F be defined over \mathbb{R}^{N+1} . Let $u_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $u_2 : \mathbb{R} \rightarrow \mathbb{R}$ be two locally bounded and upper semi-continuous functions. (u_1, u_2) is called a sub-solution on an open set $\Omega \subset \mathbb{R}$ of*

$$\begin{cases} c u_1'(z) = \alpha_0(u_2(z) - u_1(z)) \\ c u_2'(z) = 2F((u_1(z + r_i))_{i=0, \dots, N}) + \alpha_0(u_1(z) - u_2(z)) \end{cases} \quad (2.11)$$

if for any test function $\psi \in C^1(\Omega)$ such that $(u_1 - \psi)$ (resp $(u_2 - \psi)$) reaches a local maximum at a point $z \in \Omega$ then we have

$$\begin{cases} c \psi'(z) \leq \alpha_0(u_2(z) - u_1(z)) \\ (\text{resp. } c \psi'(z) \leq 2F((u_1(z + r_i))_{i=0, \dots, N}) + \alpha_0(u_1(z) - u_2(z))). \end{cases}$$

Let $u_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $u_2 : \mathbb{R} \rightarrow \mathbb{R}$ be two locally bounded and lower semi-continuous functions. (u_1, u_2) is called a super-solution of (2.11) on Ω if for any test function $\psi \in C^1(\Omega)$ such that $(u_1 - \psi)$ (resp $(u_2 - \psi)$) reaches a local minimum at a point

$z \in \Omega$ then we have

$$\begin{cases} c \psi'(z) \geq \alpha_0(u_2(z) - u_1(z)) \\ (\text{resp. } c \psi'(z) \geq 2F((u_1(z + r_i))_{i=0,\dots,N}) + \alpha_0(u_1(z) - u_2(z))). \end{cases}$$

Finally, a locally bounded functions (u_1, u_2) is called a viscosity solution of (2.11) if $((u_1)^*, (u_2)^*)$ is a sub-solution and $((u_1)_*, (u_2)_*)$ is a super-solution.

2.3 Hull fonction

We present the notion of hull function for (1.3). This result has been proved in [38, Theorem 1.10].

Proposition 2.5 (Existence of hull functions). *Let F be a given function satisfying (\tilde{A}) and let $p > 0$. Then there exists a unique $\lambda_p \in \mathbb{R}$ such that there exists two locally bounded functions $h_p : \mathbb{R} \rightarrow \mathbb{R}$ and $g_p : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (in the viscosity sense) :*

$$\begin{cases} \lambda_p h'_p(x) = \alpha_0(g_p(x) - h_p(x)) \\ \lambda_p g'_p(x) = 2F((h_p(x + p r_i))_{i=0\dots n}) + \alpha_0(h_p(x) - g_p(x)) \\ h_p(x + 1) = h_p(x) + 1 \\ g_p(x + 1) = g_p(x) + 1 \\ h'_p(x) \geq 0 \\ g'_p(x) \geq 0. \end{cases} \quad (2.12)$$

We then define

$$\begin{cases} \phi_p^1(x) = h_p(px) \\ \phi_p^2(x) = g_p(px) \end{cases} \quad \text{and} \quad c_p = \frac{\lambda_p}{p}. \quad (2.13)$$

We now give some properties of the function (ϕ_1^p, ϕ_2^p) .

Lemma 2.6 (Properties of (ϕ_1^p, ϕ_2^p)). *We assume that F satisfies (\tilde{A}) . Then the function (ϕ_1^p, ϕ_2^p) defined in (2.13) satisfies in the viscosity sense*

$$\left\{ \begin{array}{l} c_p(\phi_p^1)' = \alpha_0(\phi_p^2 - \phi_p^1) \\ c_p(\phi_p^2)' = 2F((\phi_p^1(x + pr_i)_{i=0, \dots, N})) + \alpha_0(\phi_p^1 - \phi_p^2) \\ \phi_p^1\left(x + \frac{1}{p}\right) = \phi_p^1(x) + 1, \quad \phi_p^2\left(x + \frac{1}{p}\right) = \phi_p^2(x) + 1 \\ (\phi_p^1)'(x) \geq 0, \quad (\phi_p^2)'(x) \geq 0. \end{array} \right. \quad (2.14)$$

Moreover, if $c_p \neq 0$ then there exists $M > 0$ independent on p such that

$$|(\phi_i^p)'| \leq \frac{M}{|c_p|} \quad \text{for } 0 < p < \frac{1}{r^*} \quad \text{and } i = 1, 2. \quad (2.15)$$

Proof of Lemma 2.6. Equation (2.14) is obtained by the change of variables (2.13) in (2.12).

We now prove (2.15). We fix $p > 0$ such that

$$\frac{1}{p} \geq r^*.$$

We first remark that the function ψ defined by $\psi(x) = \alpha_0(\phi_p^2(x) - \phi_p^1(x))$ is bounded (because ψ is continuous and periodic) then there exists $M_1 > 0$ such that

$$|\psi(x)| \leq M_1.$$

This implies that

$$|(\phi_1^p)'| \leq \frac{M_1}{|c_p|}.$$

On the other side, since ϕ_p^1 is non-decreasing, we have

$$\begin{cases} |\phi_p^1(x + r_i) - \phi_p^1(x)| \leq \left| \phi_p^1\left(x + \frac{1}{p}\right) - \phi_p^1(x) \right| = 1 & \text{if } r_i \geq 0 \\ |\phi_p^1(x + r_i) - \phi_p^1(x)| \leq \left| \phi_p^1\left(x - \frac{1}{p}\right) - \phi_p^1(x) \right| = 1 & \text{if } r_i \leq 0. \end{cases}$$

Moreover, using that $F \in (\mathbb{R}^{N+1})$, we get

$$|F((\phi_p^1(x + r_i))_{i=0,\dots,N}) - F((\phi_p^1(x))_{i=0,\dots,N})| \leq L \begin{vmatrix} 1 \\ \vdots \\ 1 \end{vmatrix} =: L^1.$$

On the other hand, f is bounded (because f is Lipschitz continuous and periodic).

Therefore

$$|F((\phi_p^1(x + r_i))_{i=0,\dots,N})| \leq L^1 + |f|_{L^\infty(\mathbb{R})}$$

and

$$|2F((\phi_p^1(x))_{i=0,\dots,N}) + \alpha_0(\phi_p^2(x) - \phi_p^1(x))| \leq 2(L^1 + |f|_{L^\infty(\mathbb{R})}) + M_1 =: M_2.$$

This implies that

$$|(\phi_p^2)'| \leq \frac{M_2}{|c_p|}.$$

Taking $M = \max(M_1, M_2)$, we get the desired result. \square

2.4 Useful results for monotone functions

In this subsection, we recall some results about monotone function that will be used later for the proof of Proposition 2.3. We state Helly's Lemma and the equivalence between viscosity and almost everywhere solution.

First we recall Helly's Lemma which gives the convergence of subsequence in the almost everywhere sense.

Lemma 2.7 (Helly's Lemma). *Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of non-decreasing functions on $[a, b]$ verifying $|g_n| \leq C$. Then there exists a subsequence $(g_{n_j})_{j \in \mathbb{N}}$ such that*

$$g_{n_j} \rightarrow g \quad \text{a.e. on } [a, b]$$

where g is non-decreasing on $[a, b]$ and $|g| \leq C$.

Moreover, if $(g_n)_{n \in \mathbb{N}}$ is a sequence of non-decreasing functions on a bounded interval I and if

$$g_n \rightarrow g \quad \text{a.e. on } I$$

with g constant on $\overset{\circ}{I}$, then for every closed subset interval $I' \subset \overset{\circ}{I}$,

$$g_n \rightarrow g \quad \text{uniformly on } I'.$$

Démonstration. The first part of this lemma is the classical Helly's Lemma and a proof can be found in [2, Section 3.3, p. 70] while the second part is proved in [1, Lemma 2.10]. □

Finally, after the use of Lemma 3.3 we often need to apply the following lemma (which proof is very similar to the one of [1, Lemma 2.11]) in order to get a solution in the viscosity sense.

Lemma 2.8 (Equivalence between viscosity and a.e. solutions). *We assume that F satisfies (\tilde{A}) . Then ϕ_1 and ϕ_2 are viscosity solutions of*

$$\begin{cases} 0 = \alpha_0(\phi_2(x) - \phi_1(x)), \\ 0 = 2 F((\phi_1(x + r_i))_{i=0 \dots N}) + \alpha_0(\phi_1(x) - \phi_2(x)) \end{cases} \quad (2.16)$$

if and only if ϕ_1 and ϕ_2 are solutions in the almost everywhere sense of the same equation.

3 Construction of a traveling wave

This section is divided into two subsections. In the first one, we control the velocity of propagation and give some properties on the plateau of the profiles. The second subsection is devoted to the proof of Proposition 2.3.

3.1 Preliminary results

We begin to show that the velocity c_p is uniformly bounded in p .

Lemma 3.1 (Velocity c_p is bounded). *Under the assumption (\tilde{A}) and (B) , let c_p be the velocity given by (2.13). Then there exists $C > 0$ such that*

$$|c_p| \leq C \quad \text{for } 0 < p < \frac{1}{r^*} \quad \text{with } r^* = \max_{i=0, \dots, N} |r_i|.$$

Démonstration. We consider the functions ϕ_p^1 and ϕ_p^2 given by (2.13) and satisfying (2.14). Let c_p be the associated velocity given by (2.13). We assume by contradiction that when $p \rightarrow p_0 \in [0, \frac{1}{r^*}]$

$$\lim_{p \rightarrow p_0} c_p = +\infty$$

(the case $c_p \rightarrow -\infty$ being similar). Let $\bar{\phi}_p^1 = \phi_p^1(c_p x)$ and $\bar{\phi}_p^2 = \phi_p^2(c_p x)$ solution of

$$\begin{cases} (\bar{\phi}_p^1)' = \alpha_0(\bar{\phi}_p^2 - \bar{\phi}_p^1) \\ (\bar{\phi}_p^2)' = 2F((\bar{\phi}_p^1(x + \frac{r_i}{c_p}))_{i=0, \dots, N}) + \alpha_0(\bar{\phi}_p^1 - \bar{\phi}_p^2). \end{cases} \quad (3.17)$$

According to (2.15), we have

$$\begin{cases} |(\bar{\phi}_p^1)'| = |c_p(\phi_p^1)'| \leq M \\ |(\bar{\phi}_p^2)'| = |c_p(\phi_p^2)'| \leq M \end{cases}$$

for M independent of p . Since (3.17) is invariant by space translation, we assume that

$$\bar{\phi}_p^1(0) = b - \varepsilon.$$

for ε small enough. Using Ascoli theorem and diagonal extraction argument, we have, up to extract a subsequence, that

$$\begin{cases} \bar{\phi}_p^1 \rightarrow \bar{\phi}^1 \\ \bar{\phi}_p^2 \rightarrow \bar{\phi}^2. \end{cases}$$

Moreover, by stability of viscosity solutions, $\bar{\phi}^1$ and $\bar{\phi}^2$ satisfy

$$\begin{cases} (\bar{\phi}^1)'(x) = \alpha_0(\bar{\phi}^2(x) - \bar{\phi}^1(x)) \\ (\bar{\phi}^2)'(x) = 2F((\bar{\phi}^1(x))_{i=0,\dots,N}) + \alpha_0(\bar{\phi}^1(x) - \bar{\phi}^2(x)) \end{cases}$$

and

$$\bar{\phi}^1(0) = b - \varepsilon.$$

Since $(\bar{\phi}_p^1)' \geq 0$, $(\bar{\phi}_p^2)' \geq 0$, we have $(\bar{\phi}^1)' \geq 0$, $(\bar{\phi}^2)' \geq 0$. This implies that

$$\begin{cases} \alpha_0(\bar{\phi}^2(x) - \bar{\phi}^1(x)) \geq 0 \\ 2f(\bar{\phi}^1(x)) + \alpha_0(\bar{\phi}^1(x) - \bar{\phi}^2(x)) \geq 0. \end{cases}$$

Therefore

$$2f(\bar{\phi}^1(x)) \geq 0.$$

In particular, $2f(\bar{\phi}^1(0)) = 2f(b - \varepsilon) \geq 0$, which is a contradiction since $f(b - \varepsilon) < 0$ (see assumption (B)).

□

We continue with some properties on the plateau of the profiles. The following lemma shows that if one of the profile have a large enough plateau then the other profile has the same plateau.

Lemma 3.2 (Properties on the plateau of the profiles). *Let (ϕ_1, ϕ_2) be solution of*

$$\begin{cases} c\phi_1'(x) = \alpha_0(\phi_2(x) - \phi_1(x)) \\ c\phi_2'(x) = 2 F((\phi_1(x + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_1(x) - \phi_2(x)) \\ \phi_1' \geq 0, \phi_2' \geq 0. \end{cases} \quad (3.18)$$

We assume that there exists a constant C , a point $x_0 \in \mathbb{R}$ and $a > r^*$ such that

$$\phi_1(x) = C \quad \forall x \in (x_0 - a, x_0 + a) \quad \text{or} \quad \phi_2(x) = C \quad \forall x \in (x_0 - a, x_0 + a).$$

Then

$$\phi_1(x) = \phi_2(x) = C \quad \forall x \in (x_0 - a, x_0 + a).$$

Démonstration. If $\phi_1(x) = C \quad \forall x \in (x_0 - a, x_0 + a)$, then $\phi_1'(x) = 0 \quad \forall x \in (x_0 - a, x_0 + a)$ and the first equation of (3.18) implies the result.

Let us then assume that

$$\phi_2(x) = C \quad \forall x \in (x_0 - a, x_0 + a).$$

We set

$$\psi_1(x) = (\phi_1)_*(x + a) - (\phi_1)^*(x - a) \quad \text{and} \quad \psi_2(x) = (\phi_2)_*(x + a) - (\phi_2)^*(x - a).$$

Then (ψ_1, ψ_2) is solution of

$$\begin{cases} c\psi_1'(x) \geq \alpha_0(\psi_2(x) - \psi_1(x)) \\ c\psi_2'(x) \geq 2 [F(((\phi_1)_*(x + a + r_i))_{i=0,\dots,N}) - F(((\phi_1)^*(x - a + r_i))_{i=0,\dots,N})] \\ \quad + \alpha_0(\psi_1(x) - \psi_2(x)). \end{cases} \quad (3.19)$$

Since ϕ_1 and ϕ_2 are non-decreasing, we have $\psi_1 \geq 0$ and $\psi_2 \geq 0$. Moreover, $\psi_2(x_0) = 0$.

Hence, x_0 is a point of minimum of ψ_2 and the second equation of (3.19) implies that

$$\begin{aligned} 0 &\geq 2 [F(((\phi_1)_*(x_0 + a + r_i))_{i=0,\dots,N}) - F(((\phi_1)^*(x_0 - a + r_i))_{i=0,\dots,N})] + \alpha_0 \psi_1(x_0) \\ &\geq 2 [F(((\phi_1)^*(x_0 - a) + \psi_1(x_0), (\phi_1)^*(x_0 - a + r_i))_{i=1,\dots,N}) - F(((\phi_1)^*(x_0 - a + r_i))_{i=0,\dots,N})] \\ &\quad + \alpha_0 \psi_1(x_0) \end{aligned}$$

where we have used the monotony of F for the second inequality. We set

$$G(x) = 2F(((\phi_1)^*(x_0 - a) + x, (\phi_1)^*(x_0 - a + r_i))_{i=1,\dots,N}) + \alpha_0 x.$$

Then, by assumption (1.5), G is strictly increasing. Using that

$$0 \geq G(\psi_1(x_0)) - G(0),$$

we deduce that $\psi_1(x_0) = 0$ (recall that $\psi_1 \geq 0$). This implies that ϕ_1 is constant over $(x_0 - a, x_0 + a)$ and by the first equation of (3.18), this constant is also equal to C .

□

In the proof of Proposition 2.3, we will need to pass to the limit for (ϕ_p^1, ϕ_p^2) . This is the goal of the following lemma.

Lemma 3.3 (Passing to the limit for (ϕ_p^1, ϕ_p^2)). *For every $n \in \mathbb{N}$, let $(c^n, \phi_1^n, \phi_2^n)$ be a solution of*

$$\begin{cases} c^n(\phi_1^n)'(x) = \alpha_0(\phi_2^n(x) - \phi_1^n(x)) \\ c^n(\phi_2^n)'(x) = 2 F((\phi_1^n(x + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_1^n(x) - \phi_2^n(x)) \\ (\phi_1^n)' \geq 0, (\phi_2^n)' \geq 0 \end{cases} \quad (3.20)$$

satisfying

$$\begin{cases} \phi_1^n(x+1) \leq \phi_1^n(x) + 1 \\ \phi_2^n(x+1) \leq \phi_2^n(x) + 1 \\ |c^n| \leq M_0 \\ |c^n(\phi_1^n)'| \leq M_1, |c^n(\phi_2^n)'| \leq M_2 \end{cases}$$

where M_0, M_1 and M_2 are positive constant. We also assume that there exists $M_3 > 0$ and $x^* \in \mathbb{R}$ such that $|\phi_1^n(x^*)| \leq M_3$.

Then there exists (c, ϕ_1, ϕ_2) such that, up to extract a subsequence, $c^n \rightarrow c$, $\phi_1^n \rightarrow \phi_1$ and $\phi_2^n \rightarrow \phi_2$ a.e. and (c, ϕ_1, ϕ_2) is a viscosity solution of

$$\begin{cases} c\phi_1'(x) = \alpha_0(\phi_2(x) - \phi_1(x)) \\ c\phi_2'(x) = 2 F((\phi_1(x + r_i))_{i=0, \dots, N}) + \alpha_0(\phi_1(x) - \phi_2(x)) \\ \phi_1' \geq 0, \phi_2' \geq 0 \end{cases} \quad (3.21)$$

Démonstration. Up to translate ϕ_1 , we assume that $x^* = 0$. Since $|c^n| \leq M_0$, up to extract a subsequence, we can assume that

$$c^n \rightarrow c \quad \text{as } n \rightarrow +\infty.$$

We study two cases for c .

Case 1 : $c \neq 0$. For n large enough, we have $|c^n| \geq \frac{|c|}{2} \neq 0$. Hence for n large enough, we have

$$|(\phi_1^n)'| \leq \frac{2M_1}{|c|} \quad \text{and} \quad |(\phi_2^n)'| \leq \frac{2M_2}{|c|}.$$

Using Ascoli's Theorem and the diagonal extraction argument, we can assume, up to a subsequence, that $(\phi_1^n)_n$ and $(\phi_2^n)_n$ converge locally uniformly on \mathbb{R} respectively to

ϕ_1 and ϕ_2 . By stability, ϕ_1 and ϕ_2 satisfy in the viscosity sense

$$\begin{cases} c\phi_1'(x) = \alpha_0(\phi_2(x) - \phi_1(x)) \\ c\phi_2'(x) = 2 F((\phi_1(x + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_1(x) - \phi_2(x)) \\ \phi_1' \geq 0, \phi_2' \geq 0. \end{cases} \quad (3.22)$$

Case 2 : $c = 0$. We have $\phi_1^n(1+x) \leq \phi_1^n(x) + 1$. This implies, using the fact that $\phi_1^n(0) \leq M_3$, that

$$\begin{cases} \phi_1^n(x) \leq [x] + M_3 & \text{for } x \geq 0 \\ \phi_1^n(x) \geq -[|x|] - M_3 & \text{for } x \leq 0 \end{cases} \quad (3.23)$$

Using also the fact that $0 \leq |c^n(\phi_1^n)'| \leq M_1$, we get

$$\begin{cases} \phi_2^n(x) \leq [x] + \frac{M_1}{\alpha_0} + M_3 & \text{for } x \geq 0 \\ \phi_2^n(x) \geq -[|x|] - M_3 & \text{for } x \leq 0 \end{cases} \quad (3.24)$$

Using Helly's Lemma (Lemma 2.7), up to extract a subsequence, we have $\phi_1^n \rightarrow \phi_1$ a.e. and $\phi_2^n \rightarrow \phi_2$ a.e.. This implies that

$$\begin{cases} c^n \int_{b_1}^{b_2} (\phi_1^n)'(x) dx = \alpha_0 \int_{b_1}^{b_2} (\phi_2^n(x) - \phi_1^n(x)) dx \\ c^n \int_{b_1}^{b_2} (\phi_2^n)'(x) dx = 2 \int_{b_1}^{b_2} F((\phi_1^n(x + r_i))_{i=0,\dots,N}) dx + \alpha_0 \int_{b_1}^{b_2} (\phi_1^n(x) - \phi_2^n(x)) dx. \end{cases}$$

for every $b_1 \leq b_2$. That is

$$\begin{cases} c^n(\phi_1^n(b_2) - \phi_1^n(b_1)) = \alpha_0 \int_{b_1}^{b_2} (\phi_2^n(x) - \phi_1^n(x)) dx \\ c^n(\phi_2^n(b_2) - \phi_2^n(b_1)) = 2 \int_{b_1}^{b_2} F((\phi_1^n(x + r_i))_{i=0,\dots,N}) dx + \alpha_0 \int_{b_1}^{b_2} (\phi_1^n(x) - \phi_2^n(x)) dx. \end{cases}$$

We have

$$\begin{cases} \alpha_0(\phi_2^n(x) - \phi_1^n(x)) \rightarrow \alpha_0(\phi_2(x) - \phi_1(x)) \text{ a.e.} \\ 2 F((\phi_1^n(x + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_1^n(x) - \phi_2^n(x)) \rightarrow 2 F((\phi_1(x + r_i))_{i=0,\dots,N}) + \\ \alpha_0(\phi_1(x) - \phi_2(x)) \text{ a.e.} \end{cases}$$

and (because of (3.23) and F is Lipschitz continuous and f is bounded)

$$|\alpha_0(\phi_2^n(x) - \phi_1^n(x))| \leq M_1$$

$$|F((\phi_1^n(x + r_i))_{i=0,\dots,N})| \leq C(1 + |x|),$$

i.e.

$$|F((\phi_1^n(x + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_1^n(x) - \phi_2^n(x))| \leq C(1 + |x|) + M_1.$$

Thus, using Lebesgue's dominated convergence Theorem, we pass to the limit as $n \rightarrow +\infty$ and we get

$$\begin{cases} 0 = \alpha_0 \int_{b_1}^{b_2} (\phi_2(x) - \phi_1(x)) dx \\ 0 = 2 \int_{b_1}^{b_2} F((\phi_1(x + r_i))_{i=0,\dots,N}) dx + \alpha_0 \int_{b_1}^{b_2} (\phi_1(x) - \phi_2(x)) dx. \end{cases}$$

which implies that

$$\begin{cases} 0 = \alpha_0(\phi_2(x) - \phi_1(x)) \text{ a.e.} \\ 0 = 2F((\phi_1(x + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_1(x) - \phi_2(x)) \text{ a.e.} \end{cases}$$

Since $(\phi_1^n)' \geq 0$, $(\phi_2^n)' \geq 0$, we get $\phi_1' \geq 0$, $\phi_2' \geq 0$ and so Lemma 2.8 implies

$$\begin{cases} 0 = \alpha_0(\phi_2(x) - \phi_1(x)) \\ 0 = 2F((\phi_1(x + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_1(x) - \phi_2(x)) \\ \phi_1' \geq 0 \\ \phi_2' \geq 0 \end{cases} \quad (3.25)$$

in the viscosity sense.

□

We finish this subsection with the following proposition which help us to identify the value of the plateau of the profiles.

Proposition 3.4 (The value of the plateau of the profile are close to the zero of f).

We assume that F satisfies (\tilde{A}) and let $a > r^*$. For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$

such that for all function (c, ϕ_1, ϕ_2) solution of

$$\left\{ \begin{array}{l} c\phi_1'(x) = \alpha_0(\phi_2(x) - \phi_1(x)) \\ c\phi_2'(x) = 2 F((\phi_1(x + r_i))_{i=0, \dots, N}) + \alpha_0(\phi_1(x) - \phi_2(x)) \\ \phi_1' \geq 0, \phi_2' \geq 0 \\ \phi_1(x + 1) \leq \phi_1(x) + 1, \phi_2(x + 1) \leq \phi_2(x) + 1 \\ |c| \leq M_0 \\ |c\phi_1'| \leq M_1, |c\phi_2'| \leq M_2 \end{array} \right.$$

and for all $x_0 \in \mathbb{R}$ satisfying

$$(\phi_1)_*(x_0 + a) - (\phi_1)^*(x_0 - a) \leq \delta(\varepsilon) \quad \text{or} \quad (\phi_2)_*(x_0 + a) - (\phi_2)^*(x_0 - a) \leq \delta(\varepsilon),$$

we have

$$\text{dist}(\alpha_1, \{0, b\} + \mathbb{Z}) < \varepsilon \quad \text{for all} \quad \alpha_1 \in [(\phi_1)_*(x_0), (\phi_1)^*(x_0)]$$

and

$$\text{dist}(\alpha_2, \{0, b\} + \mathbb{Z}) < \varepsilon \quad \text{for all} \quad \alpha_2 \in [(\phi_2)_*(x_0), (\phi_2)^*(x_0)].$$

Démonstration. The proof is decomposed into three steps.

Step 1 : Construction of a sequence. We assume by contradiction that there exists $\varepsilon > 0$ such that for all $\delta_n \rightarrow 0$ and $(c^n, \phi_1^n, \phi_2^n)$ solution of

$$\left\{ \begin{array}{l} c^n(\phi_1^n)'(x) = \alpha_0(\phi_2^n(x) - \phi_1^n(x)) \\ c^n(\phi_2^n)'(x) = 2 F((\phi_1^n(x + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_1^n(x) - \phi_2^n(x)) \\ (\phi_1^n)' \geq 0, (\phi_2^n)' \geq 0 \\ \phi_1^n(x + 1) \leq \phi_1^n(x) + 1, \phi_2^n(x + 1) \leq \phi_2^n(x) + 1 \\ |c^n| \leq M_0 \\ |c^n(\phi_1^n)'| \leq M_1, |c^n(\phi_2^n)'| \leq M_2 \end{array} \right. \quad (3.26)$$

and there exists $(x_n)_n \subset \mathbb{R}$ satisfying

$$(\phi_1^n)_*(x_n + a) - (\phi_1^n)^*(x_n - a) \leq \delta_n \rightarrow 0 \quad \text{or} \quad (\phi_2^n)_*(x_n + a) - (\phi_2^n)^*(x_n - a) \leq \delta_n \rightarrow 0 \quad (3.27)$$

and $\alpha_1^n \in [(\phi_1^n)_*(x_n), (\phi_1^n)^*(x_n)]$ such that

$$\text{dist}(\alpha_1^n, \{0, b\} + \mathbb{Z}) \geq \varepsilon > 0 \quad (3.28)$$

or $\alpha_2^n \in [(\phi_2^n)_*(x_n), (\phi_2^n)^*(x_n)]$ such that

$$\text{dist}(\alpha_2^n, \{0, b\} + \mathbb{Z}) \geq \varepsilon > 0. \quad (3.29)$$

Up to translate the profile, we assume that

$$\left\{ \begin{array}{l} x_n \equiv 0 \\ \phi_1^n(0) \in [0, 1) \quad \text{for all } n. \end{array} \right. \quad (3.30)$$

Step 2 : Passing to limit $n \rightarrow +\infty$. Using Lemma 3.3, we deduce that there exists (c, ϕ_1, ϕ_2) such that, up to extract a subsequence, $c^n \rightarrow c$, $\phi_1^n \rightarrow \phi_1$ and $\phi_2^n \rightarrow \phi_2$ a.e.

and (c, ϕ_1, ϕ_2) is solution of

$$\begin{cases} c\phi_1'(x) = \alpha_0(\phi_2(x) - \phi_1(x)) \\ c\phi_2'(x) = 2 F((\phi_1(x + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_1(x) - \phi_2(x)) \\ \phi_1' \geq 0, \phi_2' \geq 0 \end{cases} \quad (3.31)$$

Step 3 : Getting a contradiction. We pass to the limit in (3.27) with $x_n = 0$.

This implies that

$$(\phi_1)_*(a) \leq (\phi_1)^*(-a) \quad \text{or} \quad (\phi_2)_*(a) \leq (\phi_2)^*(-a). \quad (3.32)$$

Since ϕ_1 and ϕ_2 are non-decreasing, we have $\phi_1 = K_1$ on $[-a, a]$ or $\phi_2 = K_1$ on $[-a, a]$.

Using Lemma 3.2 we then get that

$$\phi_2 = \phi_1 = K_1 \quad \text{on} \quad [-a, a].$$

Using (3.31), we deduce that for $x = 0$

$$0 = F((\phi_1(x + r_i))_{i=0,\dots,N}) = F((K_1)_{i=0,\dots,N}) = f(K_1).$$

Hence $K_1 \in \{0, b\} + \mathbb{Z}$. Although, since $\alpha_1^n \in [(\phi_1^n)_*(0), (\phi_1^n)^*(0)]$ then $\alpha_1^n \rightarrow \alpha_1 \in \{K_1\}$.

But, passing to the limit in (3.28), yields

$$\text{dist}(\alpha_1, \{0, b\} + \mathbb{Z}) \geq \varepsilon > 0,$$

which is a contradiction. Similarly, if we pass to the limit in (3.29), we then get

$$\text{dist}(\alpha_2, \{0, b\} + \mathbb{Z}) \geq \varepsilon > 0,$$

which is also a contradiction.

□

3.2 Proof of Proposition 2.3

We are now able to give the proof of Proposition 2.3.

Proof of Proposition 2.3.

Step 0 : Introduction. Let $p > 0$ and (ϕ_p^1, ϕ_p^2) (given by (2.13)) be two non-decreasing functions solution of

$$\begin{cases} c_p (\phi_p^1)'(x) = \alpha_0(\phi_p^2(x) - \phi_p^1(x)) \\ c_p (\phi_p^2)'(x) = 2 F((\phi_p^1(x + r_i))_{i=0, \dots, N}) + \alpha_0(\phi_p^1(x) - \phi_p^2(x)). \end{cases} \quad (3.33)$$

with

$$\phi_p^1\left(x + \frac{1}{p}\right) = 1 + \phi_p^1(x) \quad \text{and} \quad \phi_p^2\left(x + \frac{1}{p}\right) = 1 + \phi_p^2(x). \quad (3.34)$$

Up to translate ϕ_p^1 , we assume that

$$\begin{cases} (\phi_p^1)_*(0) \leq b \\ (\phi_p^1)^*(0) \geq b. \end{cases} \quad (3.35)$$

Our goal is to pass to the limit as p tends to zero.

Step 1 : Passing to the limit $p \rightarrow 0$. We want to apply Lemma 3.3. The only thing we have to show is that $\phi_p^1(0)$ is bounded. From (3.34) and (3.35), we deduce that

$$b - 1 \leq \phi_p^1\left(-\frac{1}{2p}\right) \leq (\phi_p^1)_*(0) \leq b \leq (\phi_p^1)^*(0) \leq \phi_p^1\left(\frac{1}{2p}\right) \leq b + 1.$$

Thus

$$b - 1 \leq \phi_p^1(0) \leq b + 1.$$

Using Lemma 3.3, we then deduce that there exists (c, ϕ_1, ϕ_2) such that, up to extract a subsequence, $c_p \rightarrow c$, $\phi_p^1 \rightarrow \phi_1$ and $\phi_p^2 \rightarrow \phi_2$ a.e. and (c, ϕ_1, ϕ_2) is solution of

$$\begin{cases} c\phi_1'(x) = \alpha_0(\phi_2(x) - \phi_1(x)) \\ c\phi_2'(x) = 2 F((\phi_1(x + r_i))_{i=0, \dots, N}) + \alpha_0(\phi_1(x) - \phi_2(x)) \\ \phi_1' \geq 0, \phi_2' \geq 0. \end{cases} \quad (3.36)$$

We also note that $(\phi_1)_*(0) \leq b$ and $(\phi_1)^*(0) \geq b$.

Step 2 : Properties of the limit (ϕ^1, ϕ^2) .

Step 2.1 : The oscillation of (ϕ^1, ϕ^2) is bounded. Let $R > 0$. For every p such that $R \leq \frac{1}{2p}$ we have

$$\phi_p^1(R) - \phi_p^1(-R) \leq \phi_p^1\left(\frac{1}{2p}\right) - \phi_p^1\left(\frac{-1}{2p}\right) = 1.$$

Passing to the limit as $p \rightarrow 0$, we get

$$\phi^1(R) - \phi^1(-R) \leq 1.$$

Sending $R \rightarrow +\infty$, we deduce that

$$\phi^1(+\infty) - \phi^1(-\infty) \leq 1.$$

We get in the same way that

$$\phi^2(+\infty) - \phi^2(-\infty) \leq 1.$$

Step 2.2 : $\phi^1(\pm\infty) \in \mathbb{Z} \cup (b \pm \mathbb{Z})$ and $\phi^2(\pm\infty) = \phi^1(\pm\infty)$. We define

$$\phi_n^1(x) = \phi^1(x - n), \quad \text{and} \quad \phi_n^2(x) = \phi^2(x - n).$$

Since (3.36) is invariant by translation, we get that (ϕ_n^1, ϕ_n^2) is still solution of (3.36). Moreover, since (ϕ^1, ϕ^2) is non-decreasing and bounded, (ϕ_n^1, ϕ_n^2) is also non-decreasing and bounded. Thus (ϕ_n^1, ϕ_n^2) converges as $n \rightarrow +\infty$ and we denote by $(\phi^1(-\infty), \phi^2(-\infty))$ its limit. By stability of viscosity solution, we then get that

$$\begin{cases} 0 = \alpha_0(\phi^2(-\infty) - \phi^1(-\infty)) \\ 0 = 2F(((\phi^1(-\infty))_{i=0,\dots,N}) + \alpha_0(\phi^1(-\infty) - \phi^2(-\infty))). \end{cases}$$

The first equation implies that $\phi^1(-\infty) = \phi^2(-\infty)$ while the second implies that $f(\phi_1(-\infty)) = 0$ and so $\phi_1(-\infty) \in \mathbb{Z} \cup (b + \mathbb{Z})$.

In the same way, we get $\phi_1(+\infty) \in \mathbb{Z} \cup (b + \mathbb{Z})$ and $\phi^1(+\infty) = \phi^2(+\infty)$.

Step 3 : $\phi^1(\pm\infty) \notin \{b\} + \mathbb{Z}$. Since

$$\phi^1(+\infty) - \phi^1(-\infty) \leq 1 \quad \text{and} \quad \begin{cases} (\phi^1)_*(0) \leq b \\ (\phi^1)^*(0) \geq b, \end{cases}$$

we obtain that $\phi^1(-\infty) \in \{b-1, 0, b\}$ and $\phi^1(+\infty) \in \{b, 1, b+1\}$. If $\phi^1(+\infty) = b+1$ then $\phi^1(-\infty) = b$ and if $\phi^1(-\infty) = b-1$ then $\phi^1(+\infty) = b$. Thus, it is sufficient to exclude the cases $\phi^1(\pm\infty) = b$. At the end, this will prove that that $\phi^1(+\infty) = 1$ and $\phi^1(-\infty) = 0$ (and so by step 2.2, $\phi^2(+\infty) = 1$ and $\phi^2(-\infty) = 0$).

By contradiction, we assume that

$$\phi^1(+\infty) = b.$$

(the case $\phi^1(-\infty) = b$ being similar). Let $x_0 = 2r^*$, where $r^* = \max_{i=0,\dots,N} |r_i|$. Since

$$b = \phi^1(+\infty) \geq (\phi^1)^*(0) \geq b$$

then $\phi^1(x) = b$ for all $x > 0$. Hence

$$\phi^1(x_0) = \phi^1(x_0 \pm a) = b$$

for $r^* < a < 2r^*$.

Step 3.1 : Introduce z_p and y_p . For any $\varepsilon > 0$ small enough ($\varepsilon < \min(b, 1 - b)/2$),

let z_p and $y_p \in \mathbb{R}$ such that

$$\left\{ \begin{array}{l} (\phi_p^1)_*(z_p) \leq b + \varepsilon \\ (\phi_p^1)^*(z_p) \geq b + \varepsilon \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} (\phi_p^1)_*(y_p) \leq b - \varepsilon \\ (\phi_p^1)^*(y_p) \geq b - \varepsilon. \end{array} \right. \quad (3.37)$$

Let

$$\psi_p^1(x) = (\phi_p^1)_*(x + a) - (\phi_p^1)^*(x - a)$$

$$\psi_p^2(x) = (\phi_p^2)_*(x + a) - (\phi_p^2)^*(x - a).$$

Note that (ψ_p^1, ψ_p^2) is lower semi-continuous and solution of

$$\left\{ \begin{array}{l} c_p(\psi_p^1)'(x) \geq \alpha_0(\psi_p^2(x) - \psi_p^1(x)) \\ c_p(\psi_p^2)'(x) \geq 2(F(((\phi_p^1)_*(x + a + r_i))_{i=0,\dots,N}) - F(((\phi_p^1)^*(x - a + r_i))_{i=0,\dots,N})) \\ \quad + \alpha_0(\psi_p^1(x) - \psi_p^2(x)). \end{array} \right. \quad (3.38)$$

Moreover, we have

$$b + \varepsilon \in [(\phi_p^1)_*(z_p), (\phi_p^1)^*(z_p)] \quad \text{such that} \quad \text{dist}(b + \varepsilon, \{0, b\} + \mathbb{Z}) \geq \varepsilon.$$

Then there exists $\delta(\varepsilon)$ (given by Proposition 3.1) independent of p such that (for $a > r^*$)

$$\psi_1(z_p) \geq \delta(\varepsilon) > 0 \quad \text{and} \quad \psi_2(z_p) \geq \delta(\varepsilon) > 0. \quad (3.39)$$

Similarly, we get that

$$\psi_1(y_p) \geq \delta(\varepsilon) > 0 \quad \text{and} \quad \psi_2(y_p) \geq \delta(\varepsilon) > 0. \quad (3.40)$$

Using the uniform convergence of ϕ_p^1 to ϕ^1 (see the second part of Lemma 3.3 if $c = 0$), we also get that

$$\phi_p^1(x_0) \rightarrow b$$

and

$$\psi_p^1(x_0) = (\phi_p^1)_*(x_0 + a) - (\phi_p^1)^*(x_0 - a) \rightarrow 0 \quad \text{as } p \rightarrow 0.$$

Step 3.2 : Equation satisfied by (ψ^1, ψ^2) at its point of minimum. Since

$$\begin{cases} z_p \rightarrow +\infty \text{ as } p \rightarrow +\infty \\ y_p \leq 0, \end{cases}$$

we have $x_0 \in [y_p, z_p]$ for p small enough. We define

$$\begin{aligned} m_p^1 &= \min_{x \in [y_p, z_p]} \psi_p^1(x) = \psi_p^1(x_p^1) \geq 0 \quad \text{with } x_p^1 \in [y_p, z_p] \\ m_p^2 &= \min_{x \in [y_p, z_p]} \psi_p^2(x) = \psi_p^2(x_p^2) \geq 0 \quad \text{with } x_p^2 \in [y_p, z_p]. \end{aligned}$$

Note that

$$m_p^1 = \psi_p^1(x_p^1) \leq \psi_p^1(x_0) \rightarrow 0 \quad \text{as } p \rightarrow 0. \quad (3.41)$$

Since, by (3.39) and (3.40),

$$\psi_p^1(y_p) \geq \delta(\varepsilon) > 0, \quad \psi_p^1(z_p) \geq \delta(\varepsilon) > 0,$$

we have

$$x_p^1 \in (y_p, z_p)$$

and from (3.38), we get that

$$0 = c_p(\psi_p^1)'(x_p^1) \geq \alpha_0(\psi_p^2(x_p^1) - \psi_p^1(x_p^1)).$$

This gives that

$$m_p^1 = \psi_p^1(x_p^1) \geq \psi_p^2(x_p^1) \geq m_p^2 \geq 0.$$

Thus

$$m_p^2 \rightarrow 0 \quad \text{as } p \rightarrow 0.$$

Using that

$$\psi_p^2(y_p) \geq \delta(\varepsilon) > 0 \quad \text{and} \quad \psi_p^2(z_p) \geq \delta(\varepsilon) > 0,$$

we then get

$$x_p^2 \in (y_p, z_p). \tag{3.42}$$

Then by (3.38), we have

$$\begin{cases} 0 = c_p(\psi_p^1)'(x_p^1) & \geq \alpha_0(\psi_p^2(x_p^1) - \psi_p^1(x_p^1)) \\ 0 = c_p(\psi_p^2)'(x_p^2) & \geq 2(F(((\phi_p^1)^*(x_p^2 + a + r_i))_{i=0,\dots,N}) - F(((\phi_p^1)^*(x_p^2 - a + r_i))_{i=0,\dots,N})) \\ & + \alpha_0(\psi_p^1(x_p^2) - \psi_p^2(x_p^2)). \end{cases} \tag{3.43}$$

Using that

$$\alpha_0(\psi_p^1(x_p^2) - \psi_p^2(x_p^2)) \geq \alpha_0(\psi_p^1(x_p^1) - \psi_p^2(x_p^1)) \geq 0,$$

we deduce that

$$0 \geq F(((\phi_p^1)^*(x_p^2 + a + r_i))_{i=0,\dots,N}) - F(((\phi_p^1)^*(x_p^2 - a + r_i))_{i=0,\dots,N}). \tag{3.44}$$

Step 3.3 : $\psi_p^1(x_p^2 + r_i) \geq \psi_p^1(x_p^1) = m_p^1$ for all i . Using (3.42) we get

$$b - \varepsilon \leq (\phi_p^1)^*(y_p) \leq \phi_p^1(x_p^2) \leq (\phi_p^1)_*(z_p) \leq b + \varepsilon. \tag{3.45}$$

Using Lemma 3.3, we deduce that there exists ϕ_0^1 and ϕ_0^2 such that

$$\phi_p^1(x_p^2 + \cdot) \rightarrow \phi_0^1 \quad \text{and} \quad \phi_p^2(x_p^2 + \cdot) \rightarrow \phi_0^2 \quad \text{a.e. on } \mathbb{R}$$

and (ϕ_0^1, ϕ_0^2) is solution of

$$\begin{cases} c(\phi_0^1)'(z) = \alpha_0(\phi_0^2(z) - \phi_0^1(z)) \\ c(\phi_0^2)'(z) = 2F((\phi_0^1(z + r_i))_{i=0,\dots,N}) + \alpha_0(\phi_0^1(z) - \phi_0^2(z)). \end{cases} \tag{3.46}$$

Using that

$$0 \leq (\phi_p^2)_*(x_p^2 + a) - (\phi_p^2)^*(x_p^2 - a) = \psi_p^2(x_p^2) \leq \psi_p^2(x_p^1) \leq \psi_p^1(x_p^1) = m_p^1 \rightarrow 0 \quad \text{as } p \rightarrow 0 \quad (3.47)$$

we deduce that $\phi_0^2 = K_1$ on $(-a, a)$. Using Lemma 3.2, we deduce that $\phi_0^1 = K_1$ on $(-a, a)$ with $K_1 \in (b - \varepsilon, b + \varepsilon)$ (by (3.45)).

The second equation of (3.46) implies that $f(K_1) = 0$, which gives $K_1 = b$. We then deduce (using the uniform Lipschitz continuity or Helly's Lemma in the case $c = 0$), that

$$\sup_{(x_p^2 - a + \delta, x_p^2 + a - \delta)} |\phi_p^1(x) - b| \rightarrow 0 \quad \text{for all } \delta > 0,$$

which implies

$$(\phi_p^1)_*(x_p^2 + a - \delta), (\phi_p^1)^*(x_p^2 - a + \delta) \rightarrow b \quad \text{as } p \rightarrow 0 \quad (3.48)$$

Using (3.37), we deduce, taking δ small enough ($0 < \delta \leq a - r^*$), that

$$y_p \leq x_p^2 + r_i \leq z_p \quad \text{for all } i$$

which gives that

$$\psi_p^1(x_p^2 + r_i) \geq \psi_p^1(x_p^1) = m_p^1. \quad (3.49)$$

Step 3.4 : Getting a contradiction. In this step, we assume that $m_p^1 > 0$ and we want to get a contradiction. Set

$$k_i = \begin{cases} (\phi_p^1)_*(x_p^2 + r_i + a) & \text{if } r_i \leq 0 \\ (\phi_p^1)^*(x_p^2 + r_i - a) & \text{if } r_i > 0 \end{cases}$$

and note that

$$k_i \in [(\phi_p^1)^*(x_p^2 - a), (\phi_p^1)_*(x_p^2 + a)]$$

which implies that $k_i \rightarrow b$ as $p \rightarrow 0$ (by (3.48)).

Hence from (3.44),(3.49) and using the monotonicity of F , we get

$$0 \geq F((a_i)_{i=0,\dots,N}) - F((c_i)_{i=0,\dots,N})$$

where

$$a_i = \begin{cases} k_i & \text{if } r_i \leq 0 \\ k_i + m_p^1 & \text{if } r_i > 0 \end{cases}$$

and

$$c_i = \begin{cases} k_i - m_p^1 & \text{if } r_i \leq 0 \\ k_i & \text{if } r_i > 0. \end{cases}$$

Therefore from the fact that $k_i \rightarrow b$ and $m_p^1 \rightarrow 0$, we deduce that

$$a_i \rightarrow b \quad \text{and} \quad c_i \rightarrow b \quad \text{as } p \rightarrow 0.$$

Since F is C^1 near $\{b\}^{n+1}$ and $c_i + t(a_i - c_i) = c_i + t m_p^1$, we have

$$\begin{aligned} 0 &\geq \int_0^1 dt \sum_{i=0}^N (a_i - c_i) \left(\frac{\partial F}{\partial X_i} (c_j + t(a_j - c_j)_{j=0,\dots,N}) \right) \\ &= \int_0^1 dt \sum_{i=0}^N m_p^1 \left(\frac{\partial F}{\partial X_i} ((c_j + t m_p^1)_{j=0,\dots,N}) \right). \end{aligned}$$

Using that $m_p^1 > 0$, we get

$$\begin{aligned} 0 &\geq \int_0^1 dt \sum_{i=0}^N \frac{\partial F}{\partial X_i} ((c_j + t m_p^1)_{j=0,\dots,N}) \\ &= f'(b) + \int_0^1 dt \left(\sum_{i=0}^N \frac{\partial F}{\partial X_i} ((c_j + t m_p^1)_{j=0,\dots,N}) - \sum_{i=0}^N \frac{\partial F}{\partial X_i} (b, \dots, b) \right). \end{aligned}$$

But F is C^1 near $\{b\}^{N+1}$ and $c_i + t m_p^1 \rightarrow b$ for all i , thus

$$\int_0^1 dt \left(\sum_{i=0}^N \frac{\partial F}{\partial X_i} ((c_j + t m_p^1)_{j=0,\dots,N}) - \sum_{i=0}^N \frac{\partial F}{\partial X_i} (b, \dots, b) \right) \rightarrow 0 \quad \text{as } p \rightarrow 0.$$

This implies that

$$0 \geq f'(b) > 0$$

which is a contradiction with assumption (B).

Step 5 : $m_p^1 > 0$. We split this step into two cases :

Case 1 : F is strongly increasing in some direction. We assume that F satisfies

$$\frac{\partial F}{\partial X_{i_1}} \geq \delta_0 > 0. \quad (3.50)$$

We assume by contradiction that $m_p^1 = 0$. Thus

$$\psi_p^1(x_p^1) = (\phi_p^1)_*(x_p^1 + a) - (\phi_p^1)^*(x_p^1 - a) = 0.$$

Since ϕ_p^1 is non-decreasing, we get

$$\phi_{p|(x_p^1 - a, x_p^1 + a)}^1 = \phi_p^1(x_p^1) = b.$$

The first equation of (3.33) implies that

$$\phi_{p|(x_p^1 - a, x_p^1 + a)}^2 = b.$$

Let $d_1 \geq x_p^1 + a$ be the first real number such that

$$\phi_p^1(d_1 + \eta_1) > b \text{ for every } \eta_1 > 0.$$

We choose $0 < \eta_1 < r_{i_1}$ and set

$$x_1 = d_1 + \eta_1 - r_{i_1}.$$

From the definition of d_1 , we deduce that

$$\phi_p^1 = \phi_p^2 = b \text{ on a neighborhood of } x_1.$$

hence $(\phi_p^2)'(x_1) = 0$. Moreover, we have

$$\begin{cases} \phi_p^1(x_1 + r_i) \geq b \text{ for all } i \neq i_1 \\ \phi_p^1(x_1 + r_{i_1}) = \phi_p^1(d_1 + \eta_1) > b \text{ for } i = i_1. \end{cases}$$

Thus, the second equation of (3.33) implies that

$$\begin{aligned}
 0 &= c(\phi_p^2)'(x_1) = 2F((\phi_p^1(x_1 + r_i))_{i=0, \dots, N}) \\
 &\geq 2F(b, \dots, \phi_p^1(x_1 + r_{i_1}), \dots, b) \\
 &\geq f(b) + \delta_0(\phi_p^1(d_1 + \eta_1) - b) \\
 &= \delta_0(\phi_p^1(d_1 + \eta_1) - b) > 0.
 \end{aligned}$$

This is a contradiction.

Case 2 : Create the monotonicity. In fact, we can always assume (3.50) for a modification F_p of F , where

$$F_p(X_0, X_1, \dots, X_N) = F(X_0, X_1, \dots, X_N) + p(X_{i_1} - X_0).$$

Then the whole construction works for F replaced by F_p with the additional monotonicity property (3.50) with $\delta_0 = p$. Once we pass to the limit $p \rightarrow 0$, we still get the same contradiction as in Step 3.4 and we recuperate the construction of traveling wave (ϕ_1, ϕ_2) of (1.6) for the function F .

□

4 Uniqueness of the velocity c

In the first subsection, we prove a comparison principle on $(-\infty, r^*]$ and then another one on $[-r^*, +\infty)$. These two comparison principles will be used in the second subsection to prove the uniqueness of the velocity.

4.1 Comparison principle on the half-line

Theorem 4.1 (Comparison principle on $(-\infty, r^*]$). *Let $F : [0, 1]^{N+1} \rightarrow \mathbb{R}$ satisfying*

(A) *and assume that*

$$\left\{ \begin{array}{l} \text{there exists } \beta_0 > 0 \text{ such that if:} \\ Y = (Y_0, \dots, Y_N), Y + (a, \dots, a) \in [0, \beta_0^{N+1}] \\ \text{then } F(Y + (a, \dots, a)) < F(Y) \text{ if } a > 0. \end{array} \right. \quad (4.51)$$

Let (u_1, u_2) and $(v_1, v_2) : (-\infty, r^*]^2 \rightarrow [0, 1]^2$ be respectively a sub and a super-solution of

$$\left\{ \begin{array}{ll} c \ u'_1(z) = \alpha_0(u_2(z) - u_1(z)) & \text{on } (-\infty, 0) \\ c \ u'_2(z) = 2F((u_1(z + r_i))_{i=0, \dots, N}) + \alpha_0(u_1(z) - u_2(z)) & \text{on } (-\infty, 0) \end{array} \right. \quad (4.52)$$

We suppose that

$$\left\{ \begin{array}{l} u_1 \leq \beta_0 \text{ on } (-\infty, 2r^*] \\ u_2 \leq \beta_0 \text{ on } (-\infty, 2r^*] \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} u_1 \leq v_1 \text{ on } [0, r^*] \\ u_2 \leq v_2 \text{ on } [0, r^*]. \end{array} \right.$$

Then

$$\left\{ \begin{array}{l} u_1 \leq v_1 \text{ on } (-\infty, r^*] \\ u_2 \leq v_2 \text{ on } (-\infty, r^*]. \end{array} \right.$$

Corollary 4.2 (Comparison principle on $[-r^*, +\infty)$). *Let $F : [0, 1]^{N+1} \rightarrow \mathbb{R}$ satisfying*

(A) *and assume that*

$$\left\{ \begin{array}{l} \text{there exists } \beta_0 > 0 \text{ such that if:} \\ Y = (Y_0, \dots, Y_N), Y + (a, \dots, a) \in [1 - \beta_0, 1]^{N+1} \\ \text{then } F(Y + (a, \dots, a)) < F(Y) \text{ if } a > 0. \end{array} \right. \quad (4.53)$$

Let (u_1, u_2) and $(v_1, v_2) : [-r^*, +\infty)^2 \rightarrow [0, 1]^2$ be respectively a sub and a super-solution of (4.52) on $(0, +\infty)$. We assume that

$$\left\{ \begin{array}{l} v_1 \geq 1 - \beta_0 \text{ on } [-2r^*, +\infty) \\ v_2 \geq 1 - \beta_0 \text{ on } [-2r^*, +\infty) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} u_1 \leq v_1 \text{ on } [-r^*, 0] \\ u_2 \leq v_2 \text{ on } [-r^*, 0]. \end{array} \right.$$

Then

$$\left\{ \begin{array}{l} u_1 \leq v_1 \text{ on } [-r^*, +\infty] \\ u_2 \leq v_2 \text{ on } [-r^*, +\infty]. \end{array} \right.$$

Lemma 4.3 (Transformation of a solution of (4.52)). Let $(u_1, u_2), (v_1, v_2) : (-\infty, r^*]^2 \rightarrow [0, 1]^2$ be respectively a sub and a super-solution of (4.52). Then

$$\left\{ \begin{array}{l} \hat{u}_1(x) = 1 - u_1(-x) \\ \hat{u}_2(x) = 1 - u_2(-x) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \hat{v}_1(x) = 1 - v_1(-x) \\ \hat{v}_2(x) = 1 - v_2(-x) \end{array} \right.$$

are respectively a super and a sub-solution of (4.52) on $(0 + \infty)$ with F, c, r_i (for all $i \in \{0, \dots, N\}$) replaced by \hat{F}, \hat{c} and \hat{r}_i , given by

$$\left\{ \begin{array}{l} \hat{F}(X_0, \dots, X_N) = -F(1 - X_0, \dots, 1 - X_N) \\ \hat{c} = -c \\ \hat{r}_i = -r_i \end{array} \right. \quad (4.54)$$

with $\hat{F} : [0, 1]^{N+1} \rightarrow \mathbb{R}$ satisfying (A), (B) and (C), where b and f are replaced by

$$\left\{ \begin{array}{l} \hat{b} = 1 - b \\ \hat{f}(v) = -f(1 - v). \end{array} \right.$$

Démonstration. Let $(u_1, u_2) : (-\infty, r^*]^2 \rightarrow [0, 1]^2$ be a sub-solution of (4.52) and set

$$\hat{u}_1(x) = 1 - u_1(-x), \quad \hat{u}_2(x) = 1 - u_2(-x).$$

We have

$$\begin{cases} c \hat{u}'_1(x) = c u'_1(-x) \leq \alpha_0(u_2(-x) - u_1(-x)) \\ c \hat{u}'_2(x) = c u'_2(-x) \leq -2F((u_1(-x + r_i))_{i=0, \dots, N}) + \alpha_0(u_1(-x) - u_2(-x)) \end{cases}$$

thus

$$\begin{cases} \hat{c} \hat{u}'_1(x) \geq \alpha_0(\hat{u}_2(x) - \hat{u}_1(x)) \\ \hat{c} \hat{u}'_2(x) \geq 2F((1 - \hat{u}_1(x - r_i))_{i=0, \dots, N}) + \alpha_0(\hat{u}_2(x) - \hat{u}_1(x)). \end{cases}$$

Hence (\hat{u}_1, \hat{u}_2) is a super-solution of (4.52) on $(0, +\infty)$. Similarly, we prove that (\hat{v}_1, \hat{v}_2) is a sub-solution of the same equation on $(0, +\infty)$.

□

The proof of Corollary 4.2 is now very easy.

Proof of Corollary 4.2. Let $(u_1, u_2), (v_1, v_2) : [-r^*, +\infty)^2 \rightarrow [0, 1]^2$ be a sub and a super-solution of (4.52) on $(0, +\infty)$ such that $v_1 \geq 1 - \beta_0$ and $v_2 \geq 1 - \beta_0$ on $[-2r^*, +\infty)$. We set

$$\hat{u}_1(x) = 1 - u_1(-x), \quad \hat{u}_2(x) = 1 - u_2(-x), \quad \hat{v}_1(x) = 1 - v_1(-x) \quad \text{and} \quad \hat{v}_2(x) = 1 - v_2(-x).$$

Thus $\hat{v}_1, \hat{v}_2 \leq \beta_0$ on $(-\infty, 2r^*]$ and by Lemma 4.3 (\hat{u}_1, \hat{u}_2) and (\hat{v}_1, \hat{v}_2) are respectively a super and a sub-solution of (4.52). Since F satisfies (4.53), we deduce that \hat{F} satisfies (4.51). By Theorem 4.1, we then get that

$$\begin{cases} \hat{v}_1 \leq \hat{u}_1 & \text{on } (-\infty, r^*] \\ \hat{v}_2 \leq \hat{u}_2 & \text{on } (-\infty, r^*]. \end{cases}$$

This implies that

$$\begin{cases} u_1 \leq v_1 & \text{on } [-r^*, +\infty) \\ u_2 \leq v_2 & \text{on } [-r^*, +\infty). \end{cases}$$

□

We now go back to the proof of Theorem 4.1

Proof of Theorem 4.1. Let (u_1, u_2) and $(v_1, v_2) : (-\infty, r^*]^2 \rightarrow [0, 1]^2$ be respectively a sub and a super-solution of (4.52) such that

$$\left\{ \begin{array}{l} u_1 \leq \beta_0 \quad \text{on} \quad (-\infty, 2r^*] \\ u_2 \leq \beta_0 \quad \text{on} \quad (-\infty, 2r^*] \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} u_1 \leq v_1 \quad \text{on} \quad [0, r^*] \\ u_2 \leq v_2 \quad \text{on} \quad [0, r^*] \end{array} \right.$$

Step 0 : Introduction. Let

$$\left\{ \begin{array}{l} \bar{v}_1 = \min(v_1, \beta_0) \\ \bar{v}_2 = \min(v_2, \beta_0) \end{array} \right.$$

According to (4.51) we have

$$F(\beta_0, \dots, \beta_0) \leq 0.$$

Therefore the constant β_0 is a super-solution of (4.52) and then (\bar{v}_1, \bar{v}_2) is a super-solution of (4.52) on $(-\infty, 0)$ with $u_1 \leq \bar{v}_1, u_2 \leq \bar{v}_2$ on $[0, r^*]$. Moreover, since $\bar{v}_1 \leq v_1$ and $\bar{v}_2 \leq v_2$, it is sufficient to prove the comparison principle (Theorem 4.1) between (u_1, u_2) and (\bar{v}_1, \bar{v}_2) with $u_1, \bar{v}_1, u_2, \bar{v}_2 \in [0, \beta_0]$. For simplicity, we note (\bar{v}_1, \bar{v}_2) as (v_1, v_2) .

Step 1 : Doubling the variables. We assume by contradiction that

$$M = \sup_{x \in (-\infty, r^*]} \max(u_1(x) - v_1(x), u_2(x) - v_2(x)) > 0$$

Let $0 < \varepsilon, \alpha < 1$ and we define

$$\begin{aligned} \psi_1(x, y) &= u_1(x) - v_1(y) - \frac{|x - y|^2}{2\varepsilon} + \alpha x \\ \psi_2(x, y) &= u_2(x) - v_2(y) - \frac{|x - y|^2}{2\varepsilon} + \alpha x \end{aligned}$$

and

$$M_{\varepsilon, \alpha} = \sup_{x, y \in (-\infty, r^*]} \max(\psi_1(x, y), \psi_2(x, y))$$

Since the function ψ_1 and ψ_2 are upper semi-continuous and satisfy $\psi_1(x, y), \psi_2(x, y) \rightarrow -\infty$ as $|(x, y)| \rightarrow +\infty$, we deduce that ψ_1 and ψ_2 reach their maximum respectively at $(x_\varepsilon^1, y_\varepsilon^1)$ and $(x_\varepsilon^2, y_\varepsilon^2) \in (-\infty, r^*]^2$. We also denote by $(x_\varepsilon, y_\varepsilon, i_\varepsilon) \in (-\infty, r^*]^2 \times \{1, 2\}$ such that

$$M_{\varepsilon, \alpha} = \psi_{i_\varepsilon}(x_\varepsilon, y_\varepsilon).$$

Moreover, for α small enough, we get that

$$M_{\varepsilon, \alpha} \geq \frac{M}{2} > 0.$$

Using also the fact that $u_1(x_\varepsilon^1) - v_1(x_\varepsilon^1) \leq \beta_0$ and $u_2(x_\varepsilon^2) - v_2(x_\varepsilon^2) \leq \beta_0$, we get then

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} - \alpha x_\varepsilon \leq \beta_0. \quad (4.55)$$

Step 2 : for all α and ε small enough, we have $x_\varepsilon, y_\varepsilon \in (-\infty, 0)$. By contradiction, assume that there exists α small enough and $\varepsilon \rightarrow 0$ such that $x_\varepsilon \in [0, r^*]$ or $y_\varepsilon \in [0, r^*]$ and $i_\varepsilon = i$. We suppose that $x_\varepsilon \in [0, r^*]$ (the case $y_\varepsilon \in [0, r^*]$ being similar). Using (4.55), we deduce that $\bar{y}_\varepsilon \in [-\sqrt{2(\beta_0 + r^*)\varepsilon}, r^*]$. Then x_ε and y_ε converge to $x_0 \in [0, r^*]$ as $\varepsilon \rightarrow 0$. We deduce that

$$0 < \frac{M}{2} \leq \limsup_{\varepsilon \rightarrow 0} (u_i(x_\varepsilon) - v_i(y_\varepsilon)) \leq u_i(x_0) - v_i(y_0) \leq 0$$

which is a contradiction.

Step 3 : Viscosity inequalities. Using that $x \mapsto \psi_1(x, y_\varepsilon^1)$ reaches a maximum at point x_ε^1 , we get that

$$c \left(\frac{x_\varepsilon^1 - y_\varepsilon^1}{\varepsilon} + \alpha \right) \leq \alpha_0 (u_2(x_\varepsilon^1) - u_1(x_\varepsilon^1)).$$

In the same way, we have

$$c \left(\frac{x_\varepsilon^1 - y_\varepsilon^1}{\varepsilon} \right) \geq \alpha_0 (v_2(y_\varepsilon^1) - v_1(y_\varepsilon^1)).$$

Subtracting the two inequalities, we deduce that

$$c\alpha \leq \alpha_0 \left((u_2(x_\varepsilon^1) - v_2(y_\varepsilon^1)) - (u_1(x_\varepsilon^1) - v_1(y_\varepsilon^1)) \right) \quad (4.56)$$

In the same way (using ψ_2) we get that

$$\begin{aligned} c\alpha \leq & 2 \left[F((u_1(x_\varepsilon^2 + r_i))_{i=0,\dots,N}) - F((v_1(y_\varepsilon^2 + r_i))_{i=0,\dots,N}) \right] \\ & + \alpha_0 \left((u_1(x_\varepsilon^2) - v_1(y_\varepsilon^2)) - (u_2(x_\varepsilon^2) - v_2(y_\varepsilon^2)) \right) \end{aligned} \quad (4.57)$$

Step 4 : Passing to the limit $\varepsilon, \alpha \rightarrow 0$. We set

$$u_{j,i}^{\varepsilon,k} = u_j(x_\varepsilon^k + r_i), \quad \text{and} \quad v_{j,i}^{\varepsilon,k} = v_j(x_\varepsilon^k + r_i).$$

The proof is split into two cases :

Case 1 : $\exists \varepsilon, \alpha \rightarrow 0$ such that $i_\varepsilon = 1$. In that case, equation (4.56) implies that

$$\psi_2(x_\varepsilon^2, y_\varepsilon^2) \geq \psi_2(x_\varepsilon^1, y_\varepsilon^1) \geq \psi_1(x_\varepsilon^1, y_\varepsilon^1) + \frac{c\alpha}{\alpha_0} \geq \frac{M}{2} + \frac{c\alpha}{\alpha_0} \geq \frac{M}{4}$$

for α small enough. Hence, using classical arguments, we deduce that

$$\frac{|x_\varepsilon^2 - y_\varepsilon^2|^2}{2\varepsilon}, \alpha x_\varepsilon^2 \rightarrow 0 \quad \text{as} \quad \varepsilon, \alpha \rightarrow 0. \quad (4.58)$$

Moreover, using that $\psi_1(x_\varepsilon^2 + r_i, y_\varepsilon^2 + r_i) \leq \psi_1(x_\varepsilon^1, y_\varepsilon^1)$, we have

$$u_{1,i}^{\varepsilon,2} \leq v_{1,i}^{\varepsilon,2} + m_\varepsilon + \delta_i^\varepsilon$$

where $m_\varepsilon = u_1(x_\varepsilon^1) - v_1(y_\varepsilon^1)$, and $\delta_i^\varepsilon = \frac{|x_\varepsilon^2 - y_\varepsilon^2|^2}{2\varepsilon} - \alpha(x_\varepsilon^2 + r_i)$ (note that $\alpha x_\varepsilon^1 \leq 0$). Since $u_{j,i}^{\varepsilon,k}, v_{j,i}^{\varepsilon,k} \in [0, \beta_0]$ and $\frac{M^1}{2} \leq m_\varepsilon \leq \beta_0$, we deduce that as $\varepsilon, \alpha \rightarrow 0$

$$\begin{cases} u_{j,i}^{\varepsilon,k} \rightarrow u_{1,i}^k \\ v_{j,i}^{\varepsilon,k} \rightarrow v_{1,i}^k \\ m_\varepsilon \rightarrow m_0 = u_{1,0}^1 - v_{1,0}^1 \\ \delta_i^\varepsilon \rightarrow 0 \end{cases}$$

with $u_{j,i}^k, v_{j,i}^k \in [0, \beta_0]$, $0 < \frac{M^1}{2} \leq m_0 \leq \beta_0$ and

$$u_{1,i}^2 \leq v_{1,i}^2 + m_0.$$

Passing to the limit in (4.57) implies that

$$0 \leq 2(F((u_{1,i}^2)_{i=0,\dots,N}) - F((v_{1,i}^2)_{i=0,\dots,N})) + \alpha_0((u_{1,0}^2 - v_{1,0}^2) - (u_{2,0}^2 - v_{2,0}^2)).$$

We define $\bar{m} = m_0 - (u_{1,0}^2 - v_{1,0}^2) \geq 0$. Thus, using the monotony (1.5), we get

$$0 \leq 2(F(u_{1,0}^2 + \bar{m}, (u_{1,i}^2)_{i=1,\dots,N}) - F((v_{1,i}^2)_{i=0,\dots,N})) + \alpha_0((u_{1,0}^1 - v_{1,0}^1) - (u_{2,0}^2 - v_{2,0}^2)).$$

Passing to the limit in (4.56) and using the fact that $u_{2,0}^1 - v_{2,0}^1 \leq u_{2,0}^2 - v_{2,0}^2$, we get that

$$\alpha_0((u_{1,0}^1 - v_{1,0}^1) - (u_{2,0}^2 - v_{2,0}^2)) \leq 0.$$

Thus, up to redefine $u_{1,0}^2$ by $u_{1,0}^2 + \bar{m} \in [0, \beta_0]$, we get

$$0 \leq F(u_{1,0}^2, (u_{1,i}^2)_{i=1,\dots,N}) - F((v_{1,i}^2)_{i=0,\dots,N}). \quad (4.59)$$

with $u_{j,i}^k, v_{j,i}^k \in [0, \beta_0]$, $0 < \frac{M^1}{2} \leq m_0 \leq \beta_0$,

$$u_{1,i}^2 \leq v_{1,i}^2 + m_0 \quad \text{and} \quad u_{1,0}^2 = v_{1,0}^2 + m_0.$$

Case 2 : $\exists \varepsilon, \alpha \rightarrow 0$ such that $i_\varepsilon = 2$. Using that $\psi_1(x_\varepsilon^2 + r_i, y_\varepsilon^2 + r_i) \leq \psi_2(x_\varepsilon^2, y_\varepsilon^2)$, we get that

$$u_{1,i}^{\varepsilon,2} \leq v_{1,i}^{\varepsilon,2} + m_\varepsilon + \delta_i^\varepsilon$$

where $m_\varepsilon = u_2(x_\varepsilon^2) - v_2(y_\varepsilon^2)$, and $\delta_i^\varepsilon = -\alpha r_i$. Since $u_{j,i}^{\varepsilon,k}, v_{j,i}^{\varepsilon,k} \in [0, \beta_0]$ and $\frac{M^1}{2} \leq m_\varepsilon \leq \beta_0$, we deduce that as $\varepsilon, \alpha \rightarrow 0$

$$\left\{ \begin{array}{l} u_{j,i}^{\varepsilon,k} \rightarrow u_{1,i}^k \\ v_{j,i}^{\varepsilon,k} \rightarrow v_{1,i}^k \\ m_\varepsilon \rightarrow m_0 = u_{2,0}^2 - v_{2,0}^2 \\ \delta_i^\varepsilon \rightarrow 0 \end{array} \right.$$

with $u_{j,i}^k, v_{j,i}^k \in [0, \beta_0]$, $0 < \frac{M^1}{2} \leq m_0 \leq \beta_0$ and

$$u_{1,i}^2 \leq v_{1,i}^2 + m_0.$$

Passing to the limit in (4.57) implies that

$$0 \leq 2(F((u_{1,i}^2)_{i=0,\dots,N}) - F((v_{1,i}^2)_{i=0,\dots,N})) + \alpha_0((u_{1,0}^2 - v_{1,0}^2) - (u_{2,0}^2 - v_{2,0}^2)).$$

We define $\bar{m} = m_0 - (u_{1,0}^2 - v_{1,0}^2) \geq 0$. Thus, using the monotony (1.5), we get

$$\begin{aligned} 0 &\leq 2(F(u_{1,0}^2 + \bar{m}, (u_{1,i}^2)_{i=1,\dots,N}) - F((v_{1,i}^2)_{i=0,\dots,N})) + \alpha_0((u_{1,0}^2 - v_{1,0}^2) - (u_{2,0}^2 - v_{2,0}^2) + \bar{m}) \\ &= 2(F(u_{1,0}^2 + \bar{m}, (u_{1,i}^2)_{i=1,\dots,N}) - F((v_{1,i}^2)_{i=0,\dots,N})) \end{aligned}$$

Thus, up to redefine $u_{1,0}^2$ by $u_{1,0}^2 + \bar{m} \in [0, \beta_0]$, we get

$$0 \leq F((u_{1,i}^2)_{i=0,\dots,N}) - F((v_{1,i}^2)_{i=0,\dots,N}). \quad (4.60)$$

with $u_{j,i}^k, v_{j,i}^k \in [0, \beta_0]$, $0 < \frac{M^1}{2} \leq m_0 \leq \beta_0$,

$$u_{1,i}^2 \leq v_{1,i}^2 + m_0 \quad \text{and} \quad u_{1,0}^2 = v_{1,0}^2 + m_0.$$

Step 5 : Getting a contradiction. We claim that for all i , there exists $l_i, l'_i \geq 0$ such that

$$u_{1,i}^2 + l_i = v_{1,i}^2 - l'_i + m_0, \quad (4.61)$$

and

$$\begin{cases} \bar{u}_{1,i}^2 := u_{1,i}^2 + l_i \leq \beta_0 \\ \bar{v}_{1,i}^2 := v_{1,i}^2 - l'_i \geq 0. \end{cases}$$

Recall that for all $i \in \{0, \dots, N\}$, we have

$$\begin{cases} u_{1,i}^2, v_{1,i}^2 \in [0, \beta_0] \\ u_{1,i}^2 \leq v_{1,i}^2 + m_0 \\ u_{1,0}^2 - v_{1,0}^2 = m_0 \leq \beta_0. \end{cases}$$

If for some i , $u_{1,i}^2 = v_{1,i}^2 + m_0$, then it suffices to take $l_i = l'_i = 0$. Assume then that $u_{1,i}^2 < v_{1,i}^2 + m_0$.

Case 1 : $u_{1,i}^2, v_{1,i}^2 \in (v_{1,0}^2, u_{1,0}^2)$. Set $l_i = u_{1,0}^2 - u_{1,i}^2$ and $l'_i = v_{1,i}^2 - v_{1,0}^2$. Then

$$\begin{cases} \bar{u}_{1,i}^2 = u_{1,i}^2 + l_i = u_{1,0}^2 \leq \beta_0 \\ \bar{v}_{1,i}^2 = v_{1,i}^2 - l'_i = v_{1,0}^2 \geq 0, \end{cases}$$

and $\bar{u}_{1,i}^2 = \bar{v}_{1,i}^2 + m_0$.

Case 2 : $u_{1,i}^2 > u_{1,0}^2$ and $v_{1,i}^2 > v_{1,0}^2$. Since $u_{1,i}^2 - v_{1,0}^2 > m_0$, then there exists $l'_i < v_{1,i}^2 - v_{1,0}^2$ such that

$$u_{1,i}^2 = v_{1,i}^2 - l'_i + m_0$$

and $\bar{v}_{1,i}^2 = v_{1,i}^2 - l'_i > v_{1,0}^2 \geq 0$. Thus, it is sufficient to take $l_i = 0$.

Case 3 : $u_{1,i}^2 < u_{1,0}^2$ and $v_{1,i}^2 < v_{1,0}^2$. This case can be treated as Case 2 by taking $l'_i = 0$ and $l_i < u_{1,0}^2 - u_{1,i}^2$.

Finally, going back to (4.59) or (4.60), since F is non-decreasing, we deduce that

$$\begin{aligned}
 0 &\leq F((u_{1,i}^2)_{i=0,\dots,N}) - F((v_{1,i}^2)_{i=0,\dots,N}) \\
 &\leq F((\bar{u}_{1,i}^2)_{i=0,\dots,N}) - F((\bar{v}_{1,i}^2)_{i=0,\dots,N}) \\
 &= F((\bar{u}_{1,i}^2)_{i=0,\dots,N}) - F((\bar{u}_{1,i}^2 - m_0)_{i=0,\dots,N}) \\
 &< 0.
 \end{aligned}$$

Last inequality takes place since F verifies ((1.5)) for $\bar{u}_{1,i}^2, \bar{u}_{1,i}^2 - m_0 \in [0, \beta_0]$ and $m_0 > 0$. Therefore, we get a contradiction. □

4.2 Uniqueness of the velocity

In this subsection, we use Theorem 4.1 and Corollary 4.2 in order to prove the uniqueness of the velocity c .

Proposition 4.4 (Uniqueness of the velocity). *Under assumptions (A), we consider the function F defined on $[0, 1]^{N+1}$. Let $(c_1, (\phi_{11}, \phi_{12}))$ and $(c_2, (\phi_{21}, \phi_{22}))$ be two solutions of (1.7), with $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} : \mathbb{R} \rightarrow [0, 1]$. If F satisfies in addition (C), then $c_1 = c_2$.*

Démonstration. Assume that $(c_1, (\phi_{11}, \phi_{12}))$ and $(c_2, (\phi_{21}, \phi_{22}))$ are solutions of (1.7) and assume by contradiction that $c_1 < c_2$. We have

$$\left\{ \begin{array}{l} \phi_{11}(-\infty) = 0, \phi_{11}(+\infty) = 1 \\ \phi_{12}(-\infty) = 0, \phi_{12}(+\infty) = 1. \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \phi_{21}(-\infty) = 0, \phi_{21}(+\infty) = 1 \\ \phi_{22}(-\infty) = 0, \phi_{22}(+\infty) = 1. \end{array} \right.$$

We set $\delta = \min(\beta_0, \frac{1}{4})$ where β_0 is given in assumption (C) and up to translate (ϕ_{11}, ϕ_{12}) and (ϕ_{21}, ϕ_{22}) , we assume that

$$\begin{cases} \phi_{11}(x) \geq 1 - \delta & \forall x \geq -2r^* \\ \phi_{12}(x) \geq 1 - \delta & \forall x \geq -2r^* \end{cases}$$

and

$$\begin{cases} \phi_{21}(x) \leq \delta & \forall x \leq 2r^* \\ \phi_{22}(x) \leq \delta & \forall x \leq 2r^*. \end{cases}$$

This implies that

$$\begin{cases} \phi_{21}(x) \leq \phi_{11}(x) & \text{over } [-r^*, r^*] \\ \phi_{22}(x) \leq \phi_{12}(x) & \text{over } [-r^*, r^*]. \end{cases}$$

Moreover, since $c_1 < c_2$, we have

$$\begin{cases} c_1 (\phi_{21})'(x) \leq c_2 (\phi_{21})'(x) = \alpha_0(\phi_{22}(x) - \phi_{21}(x)) \\ c_1 (\phi_{22})'(x) \leq c_2 (\phi_{22})'(x) = 2F((\phi_{21}(x - r_i))_{i=0, \dots, N}) + \alpha_0(\phi_{21}(x) - \phi_{22}(x)). \end{cases}$$

Thus $(c_1, (\phi_{21}, \phi_{22}))$ is a sub-solution of (1.7). Using Corollary 4.2, we deduce that

$$\begin{cases} \phi_{21} \leq \phi_{11} & \text{over } [-r^*, +\infty) \\ \phi_{22} \leq \phi_{12} & \text{over } [-r^*, +\infty). \end{cases}$$

Similarly, using Theorem 4.1, we get that

$$\begin{cases} \phi_{21} \leq \phi_{11} & \text{over } (-\infty, r^*] \\ \phi_{22} \leq \phi_{12} & \text{over } (-\infty, r^*]. \end{cases}$$

Therefore

$$\begin{cases} \phi_{21} \leq \phi_{11} & \text{over } \mathbb{R} \\ \phi_{22} \leq \phi_{12} & \text{over } \mathbb{R}. \end{cases}$$

We set

$$\begin{cases} u_1(t, x) = \phi_{11}(x + c_1 t) \\ u_2(t, x) = \phi_{12}(x + c_1 t) \\ u_3(t, x) = \phi_{21}(x + c_2 t) \\ u_4(t, x) = \phi_{22}(x + c_2 t). \end{cases}$$

Then for $i = 1, j = 2$ and $i = 3, j = 4$, we have

$$\begin{cases} \partial_t u_i(t, x) = \alpha_0(u_j(t, x) - u_i(t, x)) \\ \partial_t u_j(t, x) = 2F((u_i(t, x + ri))_{i=0, \dots, N}) + \alpha_0(u_i(t, x) - u_j(t, x)). \end{cases} \quad (4.62)$$

Moreover, at time $t = 0$,

$$\begin{cases} u_1(0, x) = \phi_{11}(x) \geq \phi_{21}(x) = u_3(0, x) \text{ over } \mathbb{R} \\ u_2(0, x) = \phi_{12}(x) \geq \phi_{22}(x) = u_4(0, x) \text{ over } \mathbb{R}. \end{cases} \quad (4.63)$$

Then, applying the comparison principle for equation (4.62), we get

$$\begin{cases} u_1 \geq u_3 \quad \forall t \geq 0 \quad \forall x \in \mathbb{R} \\ u_2 \geq u_4 \quad \forall t \geq 0 \quad \forall x \in \mathbb{R}. \end{cases}$$

Taking $x = y - c_1 t$, yields

$$\begin{cases} \phi_{11}(y) \geq \phi_{21}(y + (c_2 - c_1)t) \quad \forall t \geq 0, \forall y \in \mathbb{R} \\ \phi_{12}(y) \geq \phi_{22}(y + (c_2 - c_1)t) \quad \forall t \geq 0, \forall y \in \mathbb{R}. \end{cases}$$

Using that $c_1 < c_2$ and passing to the limit $t \rightarrow +\infty$, we get

$$\begin{cases} \phi_{11}(y) \geq \phi_{21}(+\infty) = 1 \quad \forall y \in \mathbb{R} \\ \phi_{12}(y) \geq \phi_{22}(+\infty) = 1 \quad \forall y \in \mathbb{R}. \end{cases}$$

But $\phi_{11}(-\infty) = 0$ and $\phi_{12}(-\infty) = 0$, hence a contradiction. Therefore $c_1 \geq c_2$.

Similarly, we prove that $c_2 \geq c_1$. Thus $c_1 = c_2$.

□

5 Uniqueness of the profile

This section is devoted to the proof of the uniqueness of the profiles (under assumption $(D\pm)$) using two different types of strong maximum principle. **Different types of strong maximum principle**

Lemma 5.1 (Half Strong Maximum Principle). *Let $F : [0, 1]^{N+1} \rightarrow \mathbb{R}$ satisfying assumption (A) and let (ϕ_{11}, ϕ_{12}) and (ϕ_{21}, ϕ_{22}) be respectively a viscosity sub and super-solution of (1.7), with $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} : \mathbb{R} \rightarrow [0, 1]$. We assume that*

$$\left\{ \begin{array}{l} \phi_{21} \geq \phi_{11} \quad \text{on } \mathbb{R} \\ \phi_{22} \geq \phi_{12} \quad \text{on } \mathbb{R} \\ \phi_{21}(0) = \phi_{11}(0) \\ \phi_{22}(0) = \phi_{12}(0). \end{array} \right.$$

If $c > 0$ (resp. $c < 0$), then

$$\left\{ \begin{array}{l} \phi_{11} = \phi_{21} \quad \text{for all } x \leq 0 \text{ (resp. } x \geq 0) \\ \phi_{12} = \phi_{22} \quad \text{for all } x \leq 0 \text{ (resp. } x \geq 0). \end{array} \right.$$

Démonstration. We do the proof in the case where $c > 0$. By contradiction, assume that there exists $x_0 < 0$ such that

$$\phi_{21}(x_0) > \phi_{11}(x_0) \quad \text{or} \quad \phi_{22}(x_0) > \phi_{12}(x_0).$$

Let $w_1(x) = \phi_{21}(x) - \phi_{11}(x)$ and $w_2(x) = \phi_{22}(x) - \phi_{12}(x)$. A simple computation gives that

$$\left\{ \begin{array}{l} c w_1'(x) \geq \alpha_0 (w_2(x) - w_1(x)) \\ c w_2'(x) \geq 2(F((\phi_{21}(x + r_i))_{i=0, \dots, N}) - F((\phi_{11}(x + r_i))_{i=0, \dots, N})) + \alpha_0 (w_1(x) - w_2(x)). \end{array} \right. \quad (5.64)$$

Using that F is non-decreasing w.r.t. X_i for all $i \neq 0$, we get

$$\begin{cases} c w_1'(x) \geq \alpha_0 (w_2(x) - w_1(x)), \\ c w_2'(x) \geq 2 (F(\phi_{11}(x) + w_1(x), (\phi_{11}(x + r_i))_{i=1, \dots, N}) - F(\phi_{11}(x), (\phi_{11}(x + r_i))_{i=1, \dots, N})) \\ \quad + \alpha_0 (w_1(x) - w_2(x)) \end{cases}$$

Let $w(x) = w_1(x) + w_2(x)$. Then

$$c w'(x) \geq 2 (F(\phi_{11}(x) + w_1(x), (\phi_{11}(x + r_i))_{i=1, \dots, N}) - F(\phi_{11}(x), (\phi_{11}(x + r_i))_{i=1, \dots, N}))$$

Since F is globally Lipschitz continuous (we denote by L its Lipschitz constant), we have

$$w'(x) \geq -\frac{2L}{c} w_1(x) \geq -\frac{2L}{c} w(x). \quad (5.65)$$

We note that $y(x) := w(x_0) \exp\left(-\frac{2L(x-x_0)}{c}\right)$ satisfied (5.65) for all $x_0 \in \mathbb{R}$.

Using the comparison principle, we deduce that

$$w(x) \geq w(x_0) \exp\left(-\frac{2L(x-x_0)}{c}\right) \text{ for all } x \geq x_0. \quad (5.66)$$

Since $w_1(x_0) > 0$ or $w_2(x_0) > 0$, we have

$$w(x_0) > 0.$$

This implies that

$$w(x) > 0 \text{ for all } x \geq x_0.$$

In particular, for $x = 0$, we get

$$w_1(0) > 0 \quad \text{or} \quad w_2(0) > 0,$$

i.e.

$$\phi_{21}(0) > \phi_{11}(0) \quad \text{or} \quad \phi_{22}(0) > \phi_{12}(0),$$

which is a contradiction.

□

We now use Lemma 5.1 in order to get a Strong Maximum Principle under assumption $(D\pm)$ ii).

Lemma 5.2 (Strong Maximum Principle under $(D\pm)$ ii). *Let $F : [0, 1]^{N+1} \rightarrow \mathbb{R}$ satisfying (A). Let (ϕ_{11}, ϕ_{12}) and (ϕ_{21}, ϕ_{22}) , with $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} : \mathbb{R} \rightarrow [0, 1]$, be respectively a viscosity sub and super-solution of (2.9) such that*

$$\left\{ \begin{array}{l} \phi_{21} \geq \phi_{11} \quad \text{on } \mathbb{R} \\ \phi_{22} \geq \phi_{12} \quad \text{on } \mathbb{R} \\ \phi_{21}(0) = \phi_{11}(0) \\ \phi_{22}(0) = \phi_{12}(0). \end{array} \right.$$

a) *If F is increasing w.r.t. X_{i_0} for a certain $i_0 \neq 0$ then*

$$\left\{ \begin{array}{l} \phi_{21}(k r_{i_0}) = \phi_{11}(k r_{i_0}) \quad \text{for all } k \in \mathbb{N} \\ \phi_{22}(k r_{i_0}) = \phi_{12}(k r_{i_0}) \quad \text{for all } k \in \mathbb{N}. \end{array} \right.$$

b) *If we suppose moreover that F satisfies $(D+)$ ii) if $c > 0$ or $(D-)$ ii) if $c < 0$ then*

$$\left\{ \begin{array}{l} \phi_{21}(x) = \phi_{11}(x) \quad \text{for all } x \in \mathbb{R} \\ \phi_{22}(x) = \phi_{12}(x) \quad \text{for all } x \in \mathbb{R}. \end{array} \right.$$

Démonstration. a) We assume for simplicity of notation that $i_0 = 1$. As in the proof of Lemma 5.1, we define $w_1(x) = \phi_{21}(x) - \phi_{11}(x)$ and $w_2(x) = \phi_{22}(x) - \phi_{12}(x)$ which satisfy

$$\left\{ \begin{array}{l} c w_1'(x) \geq \alpha_0(w_2(x) - w_1(x)) \\ c w_2'(x) \geq 2(F((\phi_{21}(x + r_i))_{i=0, \dots, N}) - F((\phi_{11}(x + r_i))_{i=0, \dots, N})) + \alpha_0(w_1(x) - w_2(x)). \end{array} \right. \quad (5.67)$$

Using that $w_1(0) = 0, w_2(0) = 0$ and $w_1, w_2 \geq 0$ on \mathbb{R} (hence 0 is a point of minimum of w_1 and w_2), we deduce that

$$\begin{cases} 0 \geq \alpha_0(w_2(0) - w_1(0)) \\ 0 \geq 2(F((\phi_{21}(r_i))_{i=0,\dots,N}) - F((\phi_{11}(r_i))_{i=0,\dots,N})) + \alpha_0(w_1(0) - w_2(0)). \end{cases}$$

Thus

$$0 \geq 2(F((\phi_{21}(r_i))_{i=0,\dots,N}) - F((\phi_{11}(r_i))_{i=0,\dots,N})).$$

Using the fact that $\phi_{21}(0) = \phi_{11}(0)$ and that F is monotone w.r.t. X_i for all $i \neq 0$, we get

$$F((\phi_{21}(r_i))_{i=0,\dots,N}) = F((\phi_{11}(r_i))_{i=0,\dots,N}).$$

Since F is increasing w.r.t. X_1 , we deduce that

$$\phi_{21}(r_1) = \phi_{11}(r_1),$$

i.e. $w_1(r_1) = 0$. Hence r_1 is a point of minimum of w_1 . The first equation of (5.67) then implies

$$0 \geq w_2(r_1) - w_1(r_1) = w_2(r_1).$$

Since $w_2 \geq 0$, we deduce that $w_2(r_1) = 0$, i.e.

$$\phi_{22}(r_1) = \phi_{12}(r_1).$$

Repeating the above argument replacing 0 by r_1 , we get that

$$\phi_{21}(k r_1) = \phi_{11}(k r_1) \quad \text{and} \quad \phi_{22}(k r_1) = \phi_{12}(k r_1) \quad \text{for all } k \in \mathbb{N}.$$

b) We assume that $c > 0$ and that F satisfies $(D+)$ ii) (the other case where $c < 0$ being similar). By contradiction, we suppose that there exists $x \in \mathbb{R}$, such that

$$\phi_{21}(x) > \phi_{11}(x) \quad \text{or} \quad \phi_{22}(x) > \phi_{12}(x).$$

Let $k \in \mathbb{N}$ big enough such that $k r_{i_+} > x$. Using Lemma 5.1, and the fact that

$$\begin{cases} \phi_{11}(k r_{i_+}) = \phi_{21}(k r_{i_+}) \\ \phi_{12}(k r_{i_+}) = \phi_{22}(k r_{i_+}). \end{cases}$$

We get that

$$\begin{cases} \phi_{21}(x) = \phi_{11}(x) \\ \phi_{22}(x) = \phi_{12}(x). \end{cases}$$

which is a contradiction. □

Lemma 5.3 (Comparison Principle under $(D\pm) i$). *We assume that $c > 0$ (resp. $c < 0$) and let F satisfying (A) and $(D+) i$ (resp. $(D-) i$). Let (ϕ_{11}, ϕ_{12}) and (ϕ_{21}, ϕ_{22}) be two solutions of (2.9), with $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} : \mathbb{R} \rightarrow [0, 1]$. We assume that $\phi_{11}, \phi_{21} \in C^2$ and $\phi_{12}, \phi_{22} \in C^1$ and that*

$$\phi_{11}(0) = \phi_{21}(0) \quad \text{and} \quad \phi_{12}(0) = \phi_{22}(0).$$

Suppose moreover that

$$\begin{cases} \phi_{21}(x) \geq \phi_{11}(x) \quad \text{on} \quad [-r^*, 0] \quad (\text{resp. on } [0, r^*]) \\ \phi_{22}(x) \geq \phi_{12}(x) \quad \text{on} \quad [-r^*, 0] \quad (\text{resp. on } [0, r^*]) \end{cases}$$

then

$$\begin{cases} \phi_{21}(x) \geq \phi_{11}(x) \quad \text{for all } x \geq -r^* \quad (\text{resp. } x \leq r^*) \\ \phi_{22}(x) \geq \phi_{12}(x) \quad \text{for all } x \geq -r^* \quad (\text{resp. } x \leq r^*). \end{cases}$$

Démonstration. We assume that $c > 0$ (the case $c < 0$ being similar). We define the

functions $w_1(x) = \phi_{11}(x) - \phi_{21}(x)$ and $w_2(x) = \phi_{12}(x) - \phi_{22}(x)$ which satisfy

$$\begin{cases} c w_1'(x) = \alpha_0(w_2(x) - w_1(x)) \\ c w_2'(x) = 2(F((\phi_{11}(x + r_i))_{i=0,\dots,N}) - F((\phi_{21}(x + r_i))_{i=0,\dots,N})) + \alpha_0(w_1(x) - w_2(x)). \end{cases} \quad (5.68)$$

By the first equation, we then deduce that

$$w_2' = w_1' + \frac{c}{\alpha_0} w_1''.$$

The second equation then implies that

$$\frac{c^2}{\alpha_0} w_1'' + 2c w_1' = 2(F((\phi_{11}(x + r_i))_{i=0,\dots,N}) - F((\phi_{21}(x + r_i))_{i=0,\dots,N})).$$

Since $\phi_{11} \leq \phi_{21}$ on $[-r^*, 0]$ and $r_i \leq 0$ for all $i \neq 0$, then for all $x \in [0, \min_{i \neq 0}(-r_i)]$, we

have $\phi_{11}(x + r_i) \leq \phi_{21}(x + r_i)$ for $i \neq 0$. This implies that

$$\begin{aligned} \frac{c^2}{\alpha_0} w_1'' + 2c w_1' &\leq 2(F(\phi_{21}(x) + w_1(x), (\phi_{21}(x + r_i))_{i=1,\dots,N}) - F(\phi_{21}(x), (\phi_{21}(x + r_i))_{i=1,\dots,N})) \\ &\leq 2L |w_1(x)| \end{aligned}$$

where L is the Lipschitz constant of F . Moreover, $w_1(0) = 0$, $w_1'(0) = \frac{\alpha_0}{c}(w_2(0) - w_1(0)) = 0$ and $y = 0$ is a solution of $\frac{c^2}{\alpha_0} y'' + 2c y' = 2L y$, then using the comparison principle, we deduce that

$$w_1 \leq 0 \quad \text{for all } x \in [0, \min_{i \neq 0}(-r_i)]$$

i.e.

$$\phi_{11} \leq \phi_{21} \quad \text{for all } x \in [0, \min_{i \neq 0}(-r_i)].$$

Using the second equation of (5.68) and the fact that $\phi_{11}(x + r_i) \leq \phi_{21}(x + r_i)$ for

$i \neq 0$ for all $x \in [0, \min_{i \neq 0}(-r_i)]$, we deduce that

$$\begin{aligned} c w_2'(x) &\leq 2(F(\phi_{21}(x) + w_1(x), (\phi_{21}(x + r_i))_{i=1, \dots, N}) - F(\phi_{21}(x), (\phi_{21}(x + r_i))_{i=1, \dots, N})) \\ &\quad + \alpha_0(w_1(x) - w_2(x)) \\ &= G(w_1(x)) - G(0) - \alpha_0 w_2(x) \end{aligned}$$

where $G(t) = 2F(\phi_{21}(x) + t, (\phi_{21}(x + r_i))_{i=1, \dots, N}) + \alpha_0 t$. Using that G is non-decreasing (see (1.5)) and $w_1(x) \leq 0$, we deduce that

$$c w_2'(x) \leq -\alpha_0 w_2(x).$$

Using again that $w_2(0) = 0$ and $y = 0$ is a solution of $w'(x) = \frac{-\alpha_0}{c} w_2(x)$, we deduce by the comparison principle that

$$w_2 \leq 0 \quad \text{for all } x \in [0, \min_{i \neq 0}(-r_i)].$$

We repeat the above argument several times, each on the new extended interval. We deduce that

$$\begin{cases} \phi_{11} \leq \phi_{21} & \text{for all } x \geq -r^* \\ \phi_{12} \leq \phi_{22} & \text{for all } x \geq -r^*. \end{cases}$$

□

We use Lemma 5.1 and Lemma 5.3 in order to prove the Strong Maximum principle under $(D\pm) i$.

Lemma 5.4 (Strong Maximum Principle under $(D\pm) i$). *We assume that $c > 0$ (resp. $c < 0$) and F satisfies (A) and $(D+) i$ (resp. $(D-) i$). Let (ϕ_{11}, ϕ_{12}) and (ϕ_{21}, ϕ_{22}) be respectively a viscosity sub and a super-solution of (2.9), with $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} : \mathbb{R} \rightarrow$*

$[0, 1]$. We assume that $\phi_{11}, \phi_{21} \in C^2$ and $\phi_{12}, \phi_{22} \in C^1$ and that

$$\left\{ \begin{array}{l} \phi_{21} \geq \phi_{11} \quad \text{on } \mathbb{R} \\ \phi_{22} \geq \phi_{12} \quad \text{on } \mathbb{R} \\ \phi_{21}(0) = \phi_{11}(0) \\ \phi_{22}(0) = \phi_{12}(0). \end{array} \right.$$

Then

$$\left\{ \begin{array}{l} \phi_{11}(x) = \phi_{21}(x) \quad \text{for all } x \text{ in } \mathbb{R} \\ \phi_{12}(x) = \phi_{22}(x) \quad \text{for all } x \text{ in } \mathbb{R}. \end{array} \right.$$

Démonstration. Let $c > 0$. Using Lemma 5.1, we deduce that

$$\left\{ \begin{array}{l} \phi_{11} = \phi_{21} \quad \text{for all } x \leq 0 \\ \phi_{21} = \phi_{22} \quad \text{for all } x \leq 0. \end{array} \right.$$

By Lemma 5.3, we then deduce that

$$\left\{ \begin{array}{l} \phi_{11} \geq \phi_{21} \quad \text{for all } x \geq -r^* \\ \phi_{12} \geq \phi_{22} \quad \text{for all } x \geq -r^*. \end{array} \right.$$

which gives the result. □

Lemma 5.5 (Ordering two solutions of (4.20) up to translation). *We assume that $c \neq 0$ and let $F : [0, 1]^{N+1} \rightarrow \mathbb{R}$ satisfyng (A) and (C). Let (ϕ_{11}, ϕ_{12}) and (ϕ_{21}, ϕ_{22}) be respectively a viscosity sub and super-solution of (1.7), with $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} : \mathbb{R} \rightarrow$*

$[0, 1]$. There exists a shift $a^* \in \mathbb{R}$ and some $x_0 \in [-r^*, r^*]$ such that

$$\begin{cases} \phi_{21}^{a^*} \geq \phi_{11} & \text{on } \mathbb{R} \\ \phi_{22}^{a^*} \geq \phi_{21} & \text{on } \mathbb{R} \\ \phi_{21}^{a^*}(x_0) = \phi_{11}(x_0) \\ \phi_{22}^{a^*}(x_0) = \phi_{12}(x_0), \end{cases}$$

where

$$\begin{cases} \phi_{21}^{a^*}(x) = \phi_{21}(x + a^*) \\ \phi_{22}^{a^*}(x) = \phi_{22}(x + a^*). \end{cases}$$

Démonstration. The idea of the proof is to translate (ϕ_{21}, ϕ_{22}) and then to compare it with (ϕ_{11}, ϕ_{12}) .

Step 1 : Family of solutions above (ϕ_{11}, ϕ_{12}) . For $a \in \mathbb{R}$, we define

$$\begin{cases} \phi_{21}^a(x) = \phi_{21}(x + a) \\ \phi_{22}^a(x) = \phi_{22}(x + a). \end{cases}$$

For some $a > 0$ large enough, we have

$$\begin{cases} \phi_{21}^{\bar{a}} \geq \phi_{11} & \text{on } [-2r^*, 2r^*] \text{ for all } \bar{a} > a \\ \phi_{22}^{\bar{a}} \geq \phi_{12} & \text{on } [-2r^*, 2r^*] \text{ for all } \bar{a} > a. \end{cases}$$

Using the comparison principle (Theorem 4.1 and Corollary 4.2), we deduce that for all $\bar{a} \geq a$, we have

$$\begin{cases} \phi_{21}^{\bar{a}} \geq \phi_{11} & \text{on } \mathbb{R} \\ \phi_{22}^{\bar{a}} \geq \phi_{12} & \text{on } \mathbb{R}. \end{cases}$$

Step 2 : There exists a^* such that $\phi_{21}^{a^*}$ and ϕ_{11} touch at $x_0 \in [-r^*, r^*]$, $\phi_{22}^{a^*}$ and ϕ_{12} touch at $x_0 \in [-r^*, r^*]$. Let

$$\begin{cases} a_1^* = \inf\{a \in \mathbb{R}, \phi_{21}^{\bar{a}} \geq \phi_{11} \text{ on } \mathbb{R} \text{ for all } \bar{a} \geq a\} \\ a_2^* = \inf\{a \in \mathbb{R}, \phi_{22}^{\bar{a}} \geq \phi_{12} \text{ on } \mathbb{R} \text{ for all } \bar{a} \geq a\}. \end{cases}$$

We set $a^* = \max(a_1^*, a_2^*)$. We define $k_1(x) = \phi_{21}^{a^*}(x) - \phi_{11}(x)$ and $k_2(x) = \phi_{22}^{a^*}(x) - \phi_{12}(x)$ which satisfy

$$\begin{cases} c k_1'(x) \geq \alpha_0(k_2(x) - k_1(x)) \\ c k_2'(x) \geq 2(F((\phi_{21}^{a^*}(x + r_i))_{i=0, \dots, N}) - F((\phi_{11}(x + r_i))_{i=0, \dots, N})) + \alpha_0(k_1(x) - k_2(x)). \end{cases} \quad (5.69)$$

We now prove that $a_2^* = a_1^*$. By contradiction, assume that $a_2^* \neq a_1^*$. If $a_2^* > a_1^*$, we have

$$\begin{cases} \phi_{21}^{a_2^*}(x) > \phi_{11}(x) \\ \phi_{22}^{a_2^*}(x) \geq \phi_{12}(x) \\ \phi_{22}^{a_2^*}(x_0) = \phi_{12}(x_0). \end{cases}$$

Since F is non-decreasing in X_i for $i \neq 0$, we have

$$\begin{aligned} F((\phi_{21}^{a_2^*}(x + r_i))_{i=0, \dots, N}) &\geq F(\phi_{21}^{a_2^*}(x), (\phi_{11}(x + r_i))_{i=1, \dots, N}) \\ &\geq F(\phi_{11}(x) + k_1(x), (\phi_{11}(x + r_i))_{i=1, \dots, N}) \end{aligned}$$

Thus (since x_0 is a point of minimum of k_2 and $k_2(x_0) = 0$)

$$0 \geq 2(F(\phi_{11}(x_0) + k_1(x_0), (\phi_{11}(x_0 + r_i))_{i=1, \dots, N}) - F((\phi_{11}(x_0 + r_i))_{i=0, \dots, N})) + \alpha_0 k_1(x_0) \quad (5.70)$$

We define $G : \mathbb{R} \rightarrow \mathbb{R}$ by

$$G(y) = 2F(\phi_{11}(x_0) + y, (\phi_{11}(x_0 + r_i))_{i=1 \dots N}) + \alpha_0 y. \quad (5.71)$$

Then

$$0 \geq (G(k_1(x_0)) - G(0)).$$

But $k_1(x_0) > 0$ and, by assumption (1.5), $G'(y) > 0$, so we get a contradiction.

If $a_1^* > a_2^*$, then

$$\left\{ \begin{array}{l} \phi_{22}^{a^*}(x) > \phi_{12}(x) \\ \phi_{21}^{a^*}(x) \geq \phi_{11}(x) \\ \phi_{21}^{a^*}(x_0) = \phi_{11}(x_0). \end{array} \right.$$

Since $k_1(x_0) = 0$ and $k_1(x) \geq 0$, we deduce that x_0 is a point of minimum of k_1 and so by the first equation of (5.69), we get

$$0 \geq \alpha_0 k_2(x_0) > 0$$

wich is a contradiction.

Thus $a_1^* = a_2^* = a^*$ and so

$$\left\{ \begin{array}{l} \phi_{21}^{a^*}(x) \geq \phi_{11}(x) \\ \phi_{22}^{a^*}(x) \geq \phi_{12}(x) \\ \phi_{21}^{a^*}(x_0) = \phi_{11}(x_0) \\ \phi_{22}^{a^*}(x_0) = \phi_{12}(x_0). \end{array} \right.$$

□

Lemma 5.6 (Monotonicity of the profiles). *Assume that $c > 0$ (resp. $c < 0$) and let $F : [0, 1]^{N+1} \rightarrow \mathbb{R}$ satisfying (A), (C) and (D+) i) or ii). Let $\phi_1, \phi_2 : \mathbb{R} \rightarrow [0, 1]$ be a solution of (1.7). Then ϕ_1 and ϕ_2 are increasing on \mathbb{R} .*

Démonstration. Assume that $c > 0$ (the proof when $c < 0$ being similar) and let (ϕ_1, ϕ_2) be a solution of (2.9).

Step 1 : (ϕ_1, ϕ_2) are non-decreasing. The goal is to show that $\phi_1(x+a) \geq \phi_1(x)$ and $\phi_2(x+a) \geq \phi_2(x)$ for all $a \geq 0$. As in the proof of Lemma 5.5, we deduce that for $a \geq 0$ large enough and for all $\bar{a} \geq a$, we have

$$\phi_1^{\bar{a}}(x) := \phi_1(x + \bar{a}) \geq \phi_1(x) \quad \text{and} \quad \phi_2^{\bar{a}}(x) := \phi_2(x + \bar{a}) \geq \phi_2(x) \quad \text{on} \quad [-2r^*, 2r^*].$$

Thus using the comparison principle (Theorem 4.1 and Corollary 4.2), we deduce that for all $\bar{a} \geq a$, we have

$$\phi_1^{\bar{a}}(x) \geq \phi_1(x) \quad \text{and} \quad \phi_2^{\bar{a}}(x) \geq \phi_2(x) \quad \text{on} \quad \mathbb{R}.$$

Let

$$\begin{cases} a_1^* = \inf\{a \in \mathbb{R}, \phi_2^{\bar{a}} \geq \phi_{11} \quad \text{on} \quad \mathbb{R} \quad \text{for all} \quad \bar{a} \geq a\} \\ a_2^* = \inf\{a \in \mathbb{R}, \phi_2^{\bar{a}} \geq \phi_{12} \quad \text{on} \quad \mathbb{R} \quad \text{for all} \quad \bar{a} \geq a\}. \end{cases}$$

As in the proof of Lemma 5.5, we can prove that $a_1^* = a_2^* = a^*$. We want to prove that $a^* = 0$. By definition of a^* , there exists some x_0 such that

$$\begin{cases} \phi_1^{a^*} \geq \phi_1 \quad \text{on} \quad \mathbb{R} \\ \phi_2^{a^*} \geq \phi_2 \quad \text{on} \quad \mathbb{R} \\ \phi_1^{a^*}(x_0) = \phi_1(x_0) \\ \phi_2^{a^*}(x_0) = \phi_2(x_0). \end{cases} \tag{5.72}$$

Then, using the Strong Maximum Principle Lemma 5.2 or Lemma 5.4 (note that, since $c \neq 0$, $\phi_1, \phi_2 \in C^1$ and so the first equation of (2.9) gives that $\phi_1 \in C^2$, then we can apply Lemma 5.4), we get that $\phi_1^{a^*} = \phi_1$, i.e., ϕ_1 is periodic of period a^* . But $\phi_1(-\infty) = 0$ and $\phi_1(+\infty) = 1$, thus $a^* = 0$.

Step 2 : (ϕ_1, ϕ_2) are increasing. Let $a > 0$, we want to show that $\phi_1(x+a) > \phi_1(x)$ and $\phi_2(x+a) > \phi_2(x)$. From Step 1, we have $\phi_1(x+a) \geq \phi_1(x)$ and $\phi_2(x+a) \geq \phi_2(x)$.

Assume that there exists x_0 such that

$$\phi_1(x_0 + a) = \phi_1(x_0) \quad \text{or} \quad \phi_2(x_0 + a) = \phi_2(x_0).$$

As in the proof of Lemma 5.5, we can prove that

$$\phi_1(x_0 + a) = \phi_1(x_0) \quad \text{and} \quad \phi_2(x_0 + a) = \phi_2(x_0).$$

Using the Strong Maximum Principle (Lemma 5.2 or Lemma 5.4), we get that $a = 0$, which is a contradiction. Thus

$$\phi_1(x + a) > \phi_1(x) \quad \text{and} \quad \phi_2(x + a) > \phi_2(x) \quad \text{on} \quad \mathbb{R} \quad \text{for any} \quad a > 0.$$

□

Proof of Theorem 1.3. The proof of the uniqueness of the velocity is a direct consequence of Proposition 4.4. For the uniqueness of the profiles, it suffices to use Lemma 5.5 and the Strong Maximum Principle (Lemma 5.2 or Lemma 5.4). Note that, since $c \neq 0$, $\phi_1, \phi_2 \in C^1$ and so the first equation of (2.9) gives that $\phi_1 \in C^2$, then we can apply Lemma 5.4. Finally the strict monotonicity of ϕ_1 and ϕ_2 follows from Lemma 5.6. □

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Chapitre 3

L'étude numérique de l'homogénéisation du modèle Frenkel-Kontorova

Dans cet article, nous nous intéressons à l'homogénéisation du modèle de Frenkel-Kontorova. Nous présentons deux méthodes pour calculer l'hamiltonien effectif : la méthode grand temps et la méthode de Newton. Quelques simulations d'effective hamiltonien et d'effective de la vitesse sont fournies.

A numerical study of the homogenization of Frenkel-Kontorova models

Abstract

In this paper we are interested in the numerical homogenization of Frenkel-Kontorova model. We present two methods for computing the effective hamiltonian : large time method and Newton-like method. Some simulations of the effective hamiltonian and the effective of velocity are provided.

AMS Classification : 35B27, 35F20, 35F25, 35K55, 49L25, 65N06, 65N12, 74N05.

Keywords : continuous viscosity solution, dislocations dynamics, effective hamiltonian, finite difference scheme, Frenkel-Kontorova models, Hamilton-Jacobi equation, numerical homogenization, traveling wave.

1 Introduction

The goal of this paper is to compute the effective hamiltonian for the Frenkel-Kontorova model. The Frenkel-Kontorova model is a physical model used in various fields : biology, mechanics, chemistry etc.. For a good overview on the Frenkel-Kontorova model, we refer the reader to the book [19] of Braun and Kivshar.

2 Organization of the article

In section 3, we study the numerical homogenization of overdamped F-K model and we give two methods to calculate the effective hamiltonian : large time method and Newton-like method. In section 4, we propose the numerical simulations of the effective hamiltonian and the effective velocity using large time method . In section 5, we present some simulations of the Newton-like method and we compare this method with the large-t method.

3 Numerical homogenization of fully overdamped Frenkel-Kontorova models

The Frenkel-Kontorova model is given by the following dynamics

$$\frac{du_i}{dt} = u_{i+1} + u_{i-1} - 2u_i - \sin(2\pi(u_i - L)) - \sin(2\pi L) \quad (3.1)$$

where $\frac{du_i}{dt}$ is its velocity, m_0 denotes the mass of the particles, $-\sin(2\pi L)$ is a constant driving force which will cause the movement of the chain of atoms and $-\sin(2\pi(u_i - L))$ describes the force created by a periodic potential whose period is assumed to be 1.

We study in this section the numerical homogenization of fully overdamped Frenkel-Kontorova model. Homogenization results are obtained for fully overdamped Frenkel-Kontorova model in [37] and for accelerated Frenkel-Kontorova model with n types of particles in [38].

We recall that the solutions of (3.1) are studied in [37] using the notion of hull function. A hull function h is such that $u(y, \tau) = h(py + \lambda\tau, \tau)$ is a solution of (3.1).

For fixed p , the hull function h satisfies

$$\lambda h'(z) = h(z - p) + h(z + p) - 2h(z) - \sin(2\pi(h(z) - L)) - \sin(2\pi L) \quad (3.2)$$

A numerical scheme associated to (3.2) could be written as

$$\lambda \frac{h_{i+1} - h_i}{\Delta x} = h_{i-N} + h_{i+N} - 2h_i - \sin(2\pi(h_i - L)) - \sin(2\pi L) \quad (3.3)$$

where h_i is an approximation of $h(z)$, $N = \frac{p}{\Delta x}$ and Δx is the mesh size. we are interested in the numerical computation of λ by two methods : large time method and Newton-like method. Then we calculate the effective velocity c

$$\lambda = c \times p.$$

3.1 Large time method

We give the large time method in order to calculate the effective Hamiltonian λ which is given its definition in the following.

Definition 3.1. *The effective hamiltonian $\lambda(L, p)$ is defined by*

$$\lambda(L, p) = \lim_{t \rightarrow +\infty} \frac{u(y, t)}{t}$$

where u is the solution of the equation (3.1) with the initial condition $u(y, 0) = py$

3.2 Newton-like method for the fully overdamped Frenkel-Kontorova model

In this subsection, we present the Newton like method introduced in[21].

We recall the numerical scheme associated to (3.2)

$$\lambda \frac{h_{i+1} - h_i}{\Delta x} = h_{i-N} + h_{i+N} - 2h_i - \sin(2\pi(h_i - L)) - \sin(2\pi L) \quad (3.4)$$

We set a new function F such that

$$F(h, \lambda) := \lambda \frac{h_{i+1} - h_i}{\Delta x} - (h_{i-N} + h_{i+N} - 2h_i - \sin(2\pi(h_i - L)) - \sin(2\pi L)).$$

The discrete problem is equivalent to solve

$$F(X) = 0 \tag{3.5}$$

where $X = (h, \lambda)$. Assuming that F is Fréchet differentiable with Jacobian $J_F \in \mathbb{R}^{M \times N}$, we would like to approximate the zeros of F using the Newton Method

$$J_F(X^{(k)})\delta = -F(X^{(k)}), \quad X^{(k+1)} = X^{(k)} + \delta, \quad k \geq 0$$

but this system can be inconsistent for arbitrary M and N underdetermined if $M < N$ and overdetermined if $M > N$. We denote by J_F^\dagger the Moore-Penrose pseudoinverse of the Jacobian J_F , namely the unique $N \times M$ matrix such that

$$J_F J_F^\dagger J_F = J_F, \quad J_F^\dagger J_F J_F^\dagger = J_F^\dagger, \quad (J_F J_F^\dagger)^t = J_F J_F^\dagger, \quad (J_F^\dagger J_F)^t = J_F^\dagger J_F.$$

It can be easily proved that

$$\delta^* := -J_F^\dagger(X^{(k)})F(X^{(k)})$$

is the unique vector of the smallest Euclidean norm which minimizes the Euclidean norm of the residual $J_F(X^{(k)})\delta + F(X^{(k)})$.

The generalized solution δ^* can be efficiently obtained avoiding the computation of the Moore-Penrose pseudoinverse of the Jacobian J_F . We use the QR factorization of J_F^t . For underdetermined systems ($N = M + 1$) by the min Euclidean norm least-squares solution

$$d^* := \arg \min_{\delta \in \mathbb{R}^n} \|\delta\|_2^2 \quad \text{subject to} \quad J_F(X)\delta + F(X) = 0 \tag{3.6}$$

provided that the Jacobian has full row rank M . Therefore, factoring $J_F^t = QR$ with $Q = (Q_1 \ Q_2) \in \mathbb{R}^{N \times M}$ orthogonal, $Q_1 \in \mathbb{R}^{N \times M}$ and $Q_2 \in \mathbb{R}^{N \times (N-M)}$, $R = (R_1 \ 0)^t \in \mathbb{R}^{N \times M}$, $R_1 \in \mathbb{R}^{M \times M}$ upper triangular and $0 \in \mathbb{R}^{(N-M) \times M}$.

We have

$$J_F = R^t Q^t = R_1^t Q_1^t.$$

Moreover, We set

$$\delta = Qz = Q_1 z_1 + Q_2 z_2.$$

We get

$$\begin{aligned} 0 &= J_F(X^{(k)})\delta + F(X^{(k)}) = R_1^t Q_1^t (Q_1 z_1 + Q_2 z_2) + F(X^{(k)}) \\ &= R_1^t z_1 + R_1^t Q_1^t Q_2 z_2 + F(X^{(k)}). \end{aligned}$$

Since $Q_1^t Q_2 = 0$ by the orthogonality of Q , we obtain

$$R_1^t z_1 + F(X^{(k)}) = 0. \tag{3.7}$$

It follows that we can minimize (see (3.6))

$$\| Q^t \delta \|_2^2 = \| z_1 \|_2^2 + \| z_2 \|_2^2.$$

We just we take $z_2 = 0$, and we conclude that

$$\delta^* = Q_1 z_1 = -Q_1 (R_1^{-1})^t F(X^{(k)}).$$

where z_1 is computed by (3.7).

Algorithm

We consider the following algorithm for the solution of (3.5).

Given an initial guess X and a tolerance $\varepsilon > 0$,

REPEAT

- Assemble $F(X)$ and $J_F(X)$
- Solve the linear system $J_F(X)\delta = -F(X)$ in the least-squares sense, using the QR factorization of $J_F(X)^t$
- Update $X \leftarrow X + \delta$

UNTIL $\|\delta\|_2^2 < \varepsilon$ OR $\|F(X)\|_2^2 < \varepsilon$

We implement the algorithm in SCILAB and we use the QR factorization. In the code implementation of the algorithm above, we employ several well known variants and modifications of the classical Newton method, as discussed in the following remarks.

Remark 3.2.

1. *Sometimes Newton-like methods do not converge, due to oscillations around a minimum of the residual function $\|F(X)\|_2$. In this case we introduce a dumping parameter in the update step : $X \leftarrow X + \mu\delta$ for some $0 < \mu < 1$.*
2. *It may happen that $J_F(X)$ is nearly singular or rank deficient, so that the least-squares cannot be computed. In the spirit of the Levenberg-Marquardt method, we can regularize $J_F(X)$ with $\tau I + J_F(X)$, for some $\tau > 0$.*

4 Numerical simulation by the Large time method

4.1 Numerical simulation of effective hamiltonian

For the simulations we have the following particular choices.

Figure	3.1	3.2	3.4	3.5
p	0.02	$k * 0.02, k \in \{1, 2, 3, 4, 5\}$	$(-0.1, 0.1)$	$(-0.1, 0.1)$
L	$(-0.24, 0.24)$	$(0, 0.24)$	$(-0.24, 0.24)$	$(-0.24, 0.24)$
Δx	0.005	0.005	0.005	0.005
Δt	0.01	0.01	0.01	0.01
$t_1 = 100 * \Delta t$	1	1	1	1

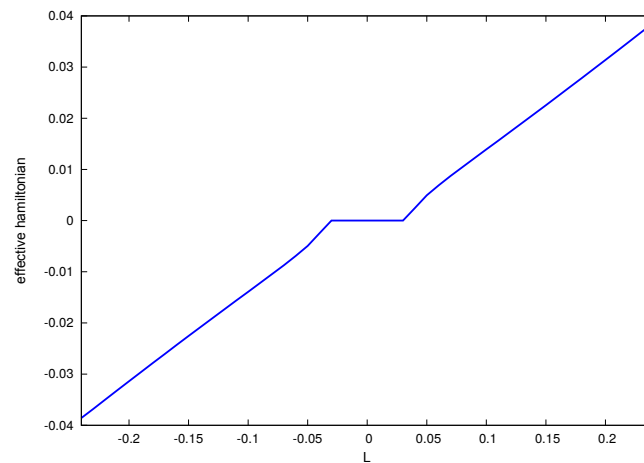


FIGURE 3.1 – $\bar{H}^{num}(L)$ for $p = 0.02$

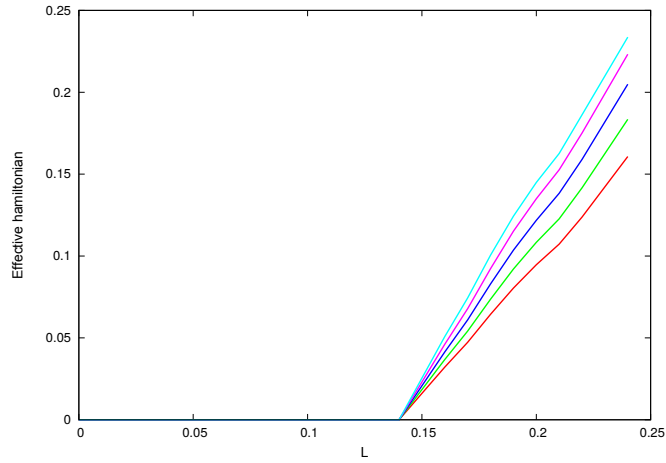


FIGURE 3.2 – $\bar{H}^{num}(L, p)$ as a function of L

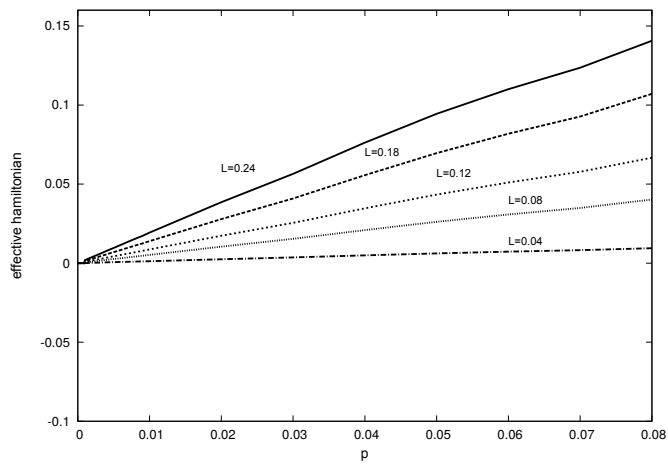


FIGURE 3.3 – $\bar{H}^{num}(L, p)$ as a function of p

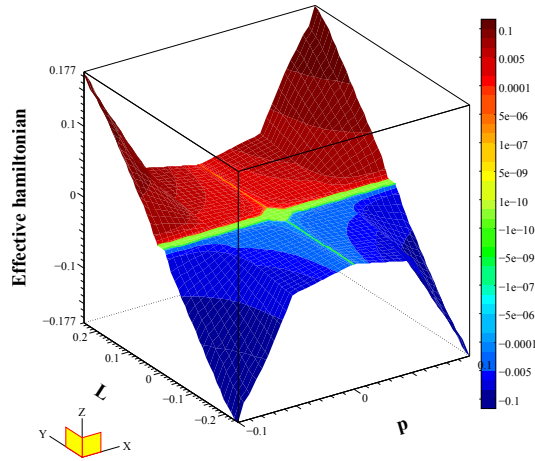


FIGURE 3.4 – Graph of $\bar{H}^{num}(L, p)$

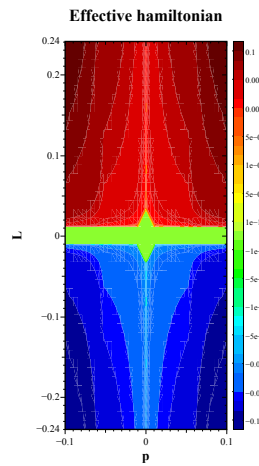


FIGURE 3.5 – Level sets of the effective hamiltonian $\bar{H}^{num}(L, p)$

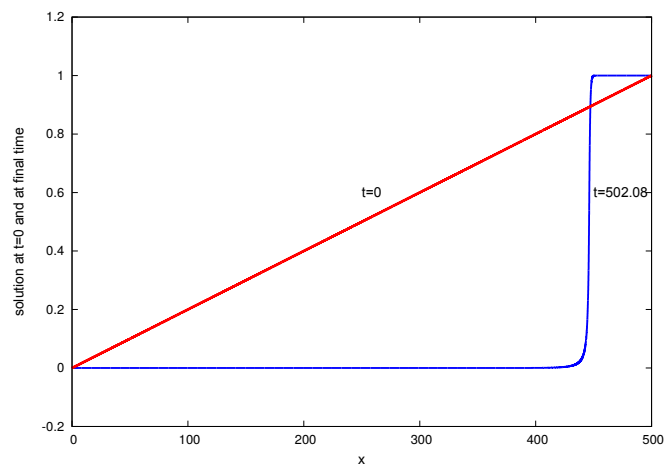


FIGURE 3.6 – Solution at initial and final times

4.2 Numerical simulation of effective velocity

For the simulations we have the following particular choices.

Figure	3.7	3.8	3.10	3.11
p	0.02	$k*0.02, k \in \{1, 2, 3, 4, 5\}$	$(-0.1, 0.1)$	$(-0.1, 0.1)$
L	$(-0.24, 0.24)$	$(0, 0.24)$	$(-0.24, 0.24)$	$(-0.24, 0.24)$
Δx	0.005	0.005	0.005	0.005
Δt	0.01	0.01	0.01	0.01
$t_1 = 100 * \Delta t$	1	1	1	1

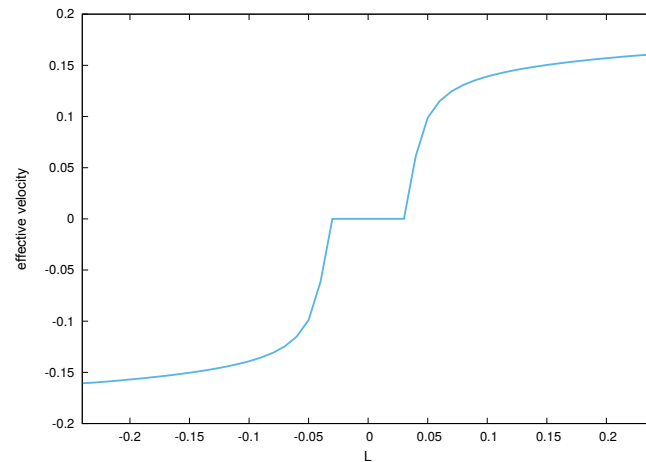


FIGURE 3.7 – $c^{num}(L)$ for $p = 0.02$

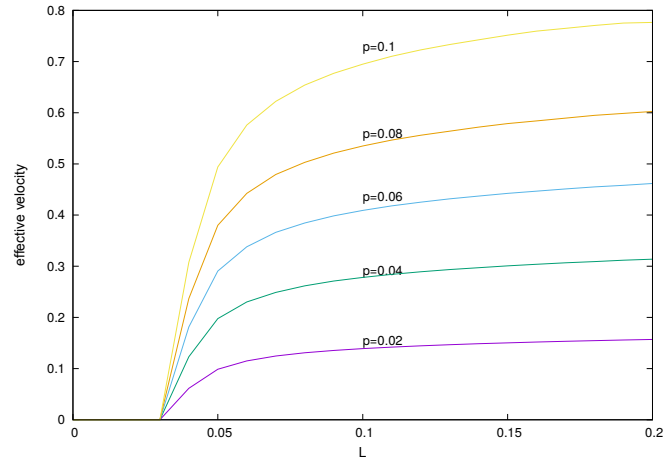


FIGURE 3.8 – $c^{num}(L, p)$ as a function of L

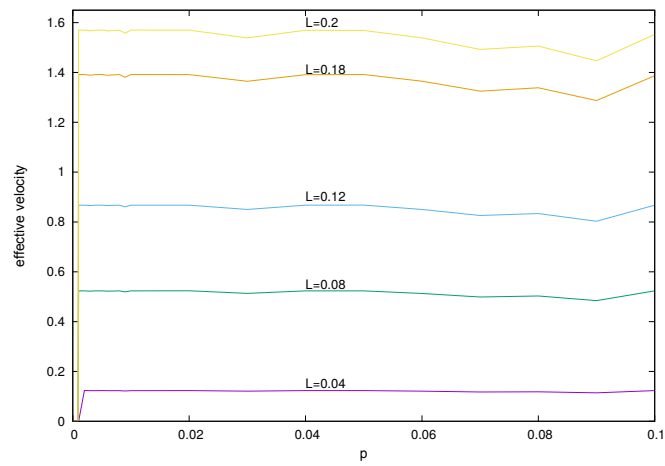


FIGURE 3.9 – $c^{num}(L, p)$ as a function of p

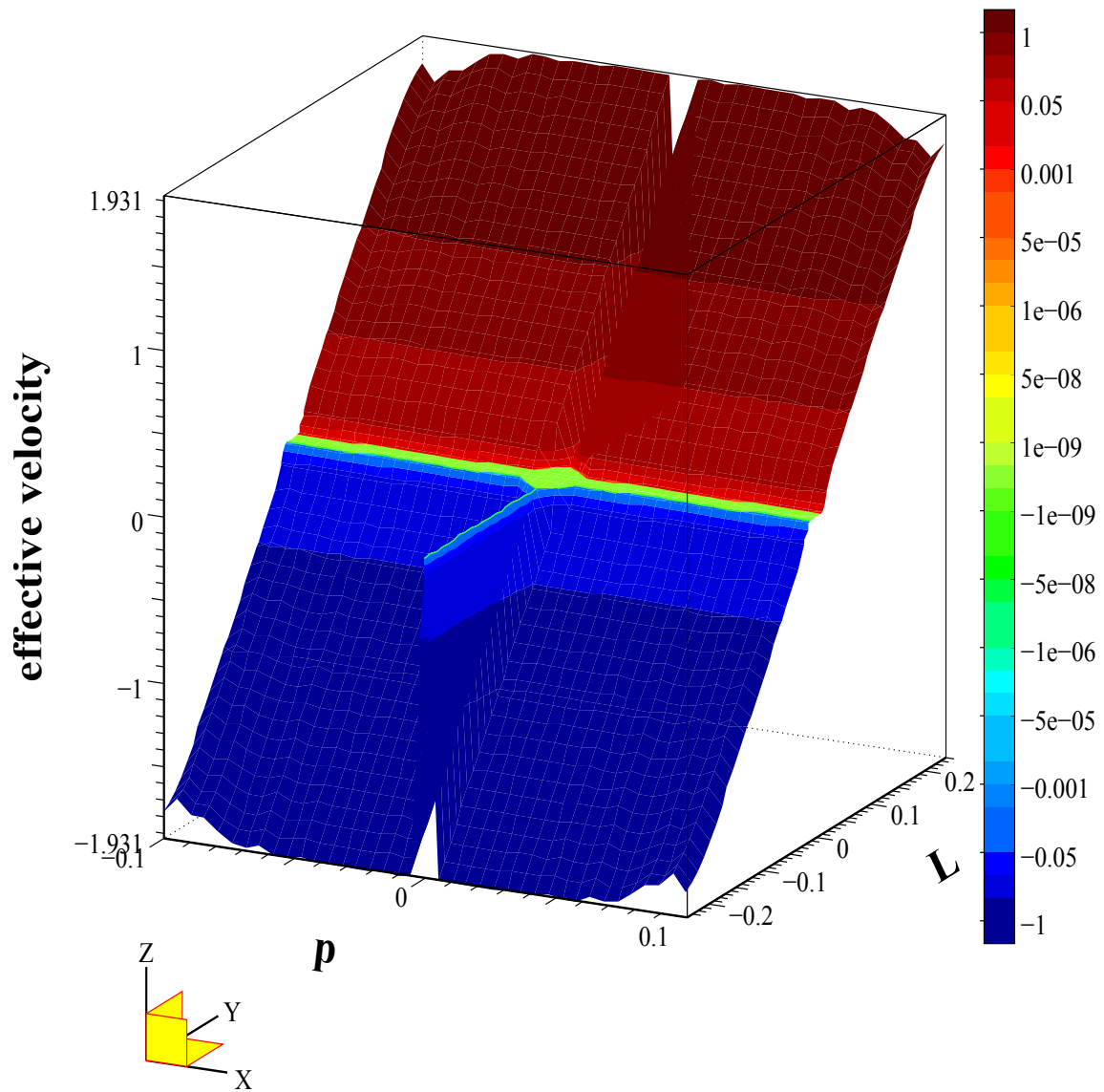


FIGURE 3.10 – Graph of $c^{num}(L, p)$

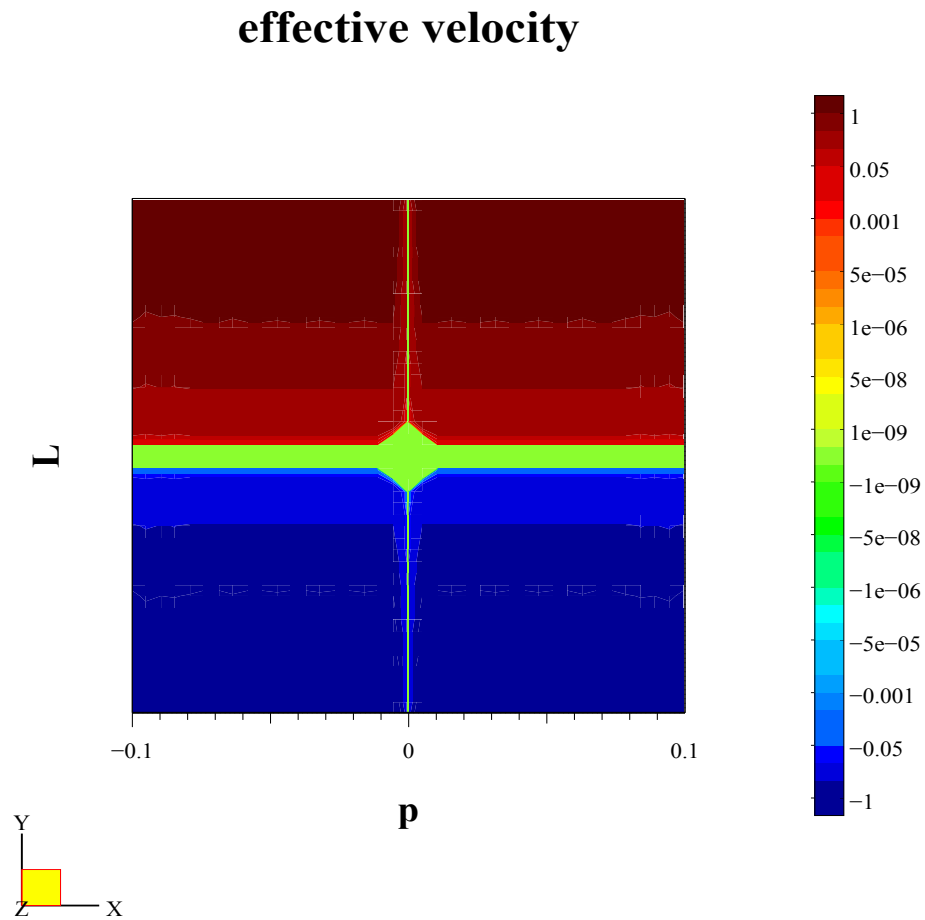


FIGURE 3.11 – Level sets of the effective velocity $c^{num}(L, p)$

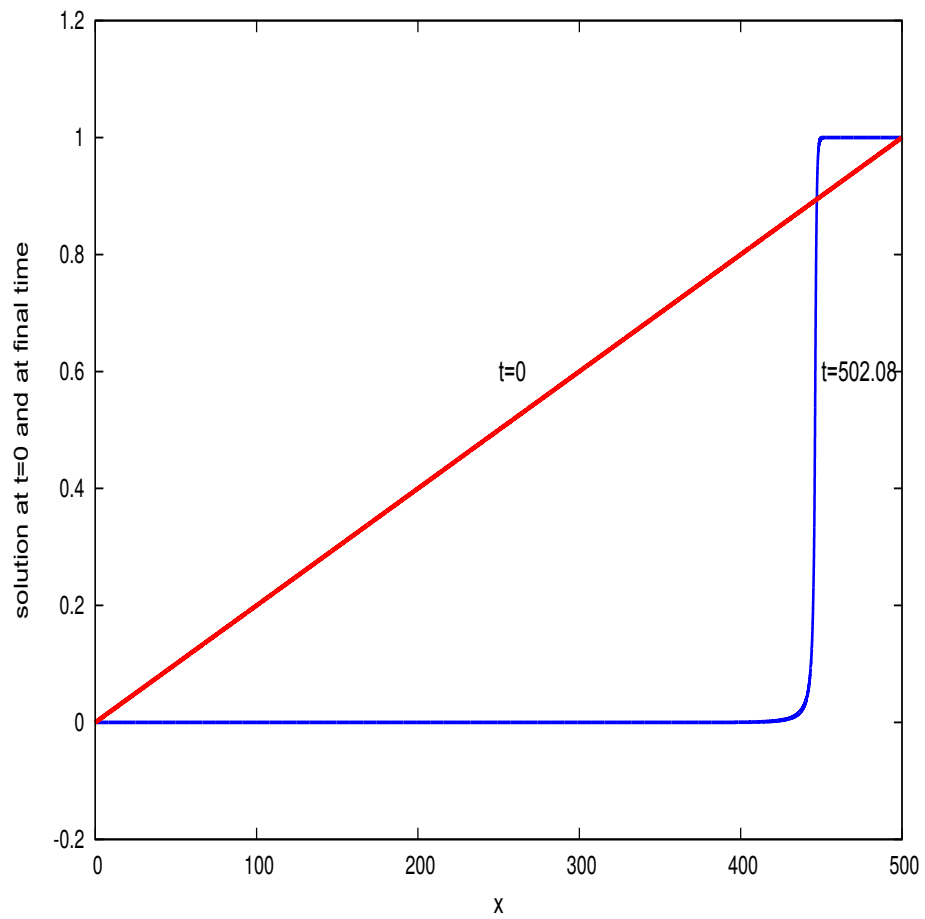


FIGURE 3.12 – Solution at initial and final times

5 Numerical results

We present in this subsection some simulations of the Newton-like method. We compare this method with the large-t method. Newton-like method is more efficient than the large-t method. The number of iterations of the Newton-like method is less than 100 iterations and the number of iterations of the large-t method is more than 1000 iterations. Therefore, the Newton-like method is accurate.

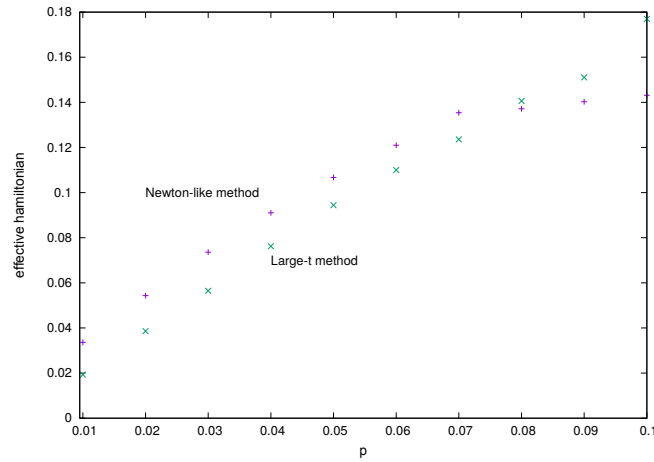


FIGURE 3.13 – $\bar{H}^{num}(p)$ for $L = 0.24$, comparison Newton method and large-t method

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Chapitre 4

Construction d'une solution de type onde plane pour une équation des ondes régularisée

Dans ce travail, nous étudions une équation d'onde dans un support périodique. Selon certaines hypothèses, nous construisons une solution d'onde plane pour le problème approché et nous montrons que cette solution satisfait certaines propriétés. Nous définissons un opérateur non local et un correcteur à terme afin de contrôler les oscillations de la solution dans l'espace et dans le temps. Nous prouvons la construction d'une solution d'onde plane pour problème approché en utilisant la notion de solution de viscosité.

Construction of a plane wave like solution for a wave equation

Abstract

In this paper, we study a wave equation in a periodic medium. Under certain assumptions, we construct a plane wave like solution, and show that this solution satisfy some properties. We define a non-local operator and a term corrector in order to control the oscillations of the solution in space and in time. We prove the construction of a plane wave like solution for the approached problem using the notion of viscosity solution.

1 Introduction

The theory of plane gravitational waves in general relativity has its origin in the papers of Einstein published in [31, 32]. The plane gravitational waves have been examined mathematically by Takeno [65] and it was shown that these waves are transformable into each other using a suitable transformation. Khapekar, Patil, Deshmukh, Pawar [54] have obtained the plane wave like solutions of the Einstein-Maxwell equations in presence of null fluids and null currents with particular cases.

We study the wave equation

$$mu_{tt} + u_t = u_{xx} + f(u). \tag{1.1}$$

In order to get a system of order 1, we set

$$\xi := u + 2mu_t + 2\sqrt{m}u_x.$$

Using this new function, the equation (1.1) can be rewritten in the following form

$$\begin{cases} u_t + \frac{1}{\sqrt{m}}u_x + \frac{1}{2m}u = \frac{1}{2m}\xi \\ \xi_t - \frac{1}{\sqrt{m}}\xi_x + \frac{1}{2m}\xi = \frac{1}{2m}u + 2f(u). \end{cases} \tag{1.2}$$

We consider an initial condition

$$u(0, x) = \xi(0, x) = px \quad \text{with } p > 0. \quad (1.3)$$

In order to get a monotone system, we add a condition

$$\frac{1}{2m} + 2f'(u) \geq 0. \quad (1.4)$$

Our goal is to construct a plane wave like solution of (1.2) satisfying

$$\begin{cases} u\left(x+1, t - \frac{p}{\lambda}\right) = u(x, t) \\ \xi\left(x+1, t - \frac{p}{\lambda}\right) = \xi(x, t) \end{cases}$$

with $p > 0$ and $\lambda \in \mathbb{R}$. We assume that the function f satisfies the following conditions.

Regularity and periodicity of f :

$$\begin{cases} f \in Lip(\mathbb{R}, \mathbb{R}) \\ f(v+l) = f(v) \quad \text{for all } l \in \mathbb{Z}, v \in \mathbb{R}. \end{cases} \quad (1.5)$$

It is difficult to prove this problem. So in this work, we prove the existence of a plane like solution for the Cauchy problem.

1.1 A approximated problem

We have to regularize the equation and to consider the following problem for $\delta > 0$

$$\begin{cases} u_t + \frac{1}{\sqrt{m}}u_x + \frac{1}{2m}u = \frac{1}{2m}\xi + \delta(a_0 + a_1[u(\cdot, t)](x))u_x \text{ in } \mathbb{R} \times \mathbb{R} \\ \xi_t - \frac{1}{\sqrt{m}}\xi_x + \frac{1}{2m}\xi = \frac{1}{2m}u + 2f(u) + \delta(a_0 + a_1[\xi(\cdot, t)](x))\xi_x \text{ in } \mathbb{R} \times \mathbb{R} \end{cases} \quad (1.6)$$

where $a_0 > \frac{1}{\sqrt{m}}$ is a constant. The non-local term is defined for a function $v : \mathbb{R} \rightarrow \mathbb{R}$

by

$$a_1[v](x) = \inf_{y \in \mathbb{R}} (v(y, t) - py) + px - v(x).$$

The regularization term will allow us to control the oscillation in space and time of the solution. The main result concerning (2.8) is the existence of plane wave like solution.

Theorem 1.1 (Plane like solution for (1.6)). *Let $\delta > 0$. Under assumption (1.4)-(1.5), for a given $p > 0$, there exists $C_\delta > 0, \lambda \in \mathbb{R}$ and a solution (u, ξ) of (1.6) which satisfies the following properties :*

$$\left\{ \begin{array}{l} (i) |u(x, t) - px - \lambda t| \leq C_\delta, \quad |\xi(x, t) - px - \lambda t| \leq C_\delta \\ (ii) u\left(x + \frac{1}{p}, t\right) = 1 + u(x, t), \quad \xi\left(x + \frac{1}{p}, t\right) = 1 + \xi(x, t) \\ (iii) u_x \geq 0, \xi_x \geq 0 \\ (iv) u\left(x + 1, t - \frac{p}{\lambda}\right) = u(x, t), \xi\left(x + 1, t - \frac{p}{\lambda}\right) = \xi(x, t) \end{array} \right. \quad (1.7)$$

1.2 Organization of the paper

In Section 2, we recall the definition of viscosity solution for the Cauchy problem associated to (1.6) and we give a comparison principle and a gradient estimate for the solution. In Section 3, we prove Theorem 1.1.

2 Preliminary results

2.1 Definition of viscosity solution

In this subsection, we give the definition of viscosity solution. We first recall the definition of upper and lower semi-continuous envelopes u^* and u_* :

$$u^*(y) = \limsup_{x \rightarrow y} u(x) \quad \text{and} \quad u_*(y) = \liminf_{x \rightarrow y} u(x).$$

In order to prove Theorem 1.1, we will consider the following Cauchy problem

$$\begin{cases} u_t + \frac{1}{\sqrt{m}}u_x + \frac{1}{2m}u = \frac{1}{2m}\xi + \delta(a_0 + a_1[u(\cdot, t)](x))u_x \text{ in } \mathbb{R} \times (0, +\infty) \\ \xi_t - \frac{1}{\sqrt{m}}\xi_x + \frac{1}{2m}\xi = \frac{1}{2m}u + 2f(u) + \delta(a_0 + a_1[\xi(\cdot, t)](x))\xi_x \text{ in } \mathbb{R} \times (0, +\infty) \\ u_0(x) = \xi_0(x) = px \text{ on } \mathbb{R} \end{cases} \quad (2.8)$$

In order to well define the equation, the infimum of $u(y, t) - py$ and of $\xi(y, t) - py$ should be finite. Hence we will only consider solution u, ξ satisfying for some $C_0(T) > 0$: for all $t \in [0, T)$ and all $x, y \in \mathbb{R}$

$$|u(x + y, t) - u(x, t) - py| \leq C_0, \quad |\xi(x + y, t) - \xi(x, t) - py| \leq C_0. \quad (2.9)$$

Definition 2.1 (Viscosity solution). *Let Ω be an open subset of \mathbb{R} . Let $u : \mathbb{R} \times (0, +\infty) \rightarrow \mathbb{R}$ and $\xi : \mathbb{R} \times (0, +\infty) \rightarrow \mathbb{R}$ be two locally bounded and upper semi-continuous functions. Then (u, ξ) is called a sub-solution on $\Omega \times (0, +\infty)$ of (2.8) if for any test function $\psi \in C^1(\Omega \times (0, +\infty))$ such that $(u - \psi)$ (resp $(\xi - \psi)$) reaches a local maximum at a point $(\bar{x}, \bar{t}) \in \Omega \times (0, +\infty)$, we have*

$$\psi_t(\bar{x}, \bar{t}) + \frac{1}{\sqrt{m}}\psi_x(\bar{x}, \bar{t}) + \frac{1}{2m}u(\bar{x}, \bar{t}) \leq \frac{1}{2m}\xi(\bar{x}, \bar{t}) + \delta(a_0 + a_1[u(\cdot, \bar{t})](\bar{x}))\psi_x(\bar{x}, \bar{t})$$

(resp.

$$\psi_t(\bar{x}, \bar{t}) - \frac{1}{\sqrt{m}}\psi_x(\bar{x}, \bar{t}) + \frac{1}{2m}\xi(\bar{x}, \bar{t}) \leq \frac{1}{2m}u(\bar{x}, \bar{t}) + 2f(u(\bar{x}, \bar{t})) + \delta(a_0 + a_1[\xi(\cdot, \bar{t})](\bar{x}))\psi_x(\bar{x}, \bar{t})).$$

Let $u : \mathbb{R} \times (0, +\infty) \rightarrow \mathbb{R}$ and $\xi : \mathbb{R} \times (0, +\infty) \rightarrow \mathbb{R}$ be two locally bounded and lower semi-continuous functions. Then (u, ξ) is called a super-solution of (2.8) on $\Omega \times (0, +\infty)$ if for any test function $\psi \in C^1(\Omega \times (0, +\infty))$ such that $(u - \psi)$ (resp.

$(\xi - \psi)$ reaches a local minimum at a point $(\bar{x}, \bar{t}) \in \Omega \times (0, +\infty)$, we have

$$\psi_t(\bar{x}, \bar{t}) + \frac{1}{\sqrt{m}}\psi_x(\bar{x}, \bar{t}) + \frac{1}{2m}u(\bar{x}, \bar{t}) \geq \frac{1}{2m}\xi(\bar{x}, \bar{t}) + \delta(a_0 + a_1[u(\cdot, \bar{t})](\bar{x}))\psi_x(\bar{x}, \bar{t})$$

(resp.

$$\psi_t(\bar{x}, \bar{t}) - \frac{1}{\sqrt{m}}\psi_x(\bar{x}, \bar{t}) + \frac{1}{2m}\xi(\bar{x}, \bar{t}) \geq \frac{1}{2m}u(\bar{x}, \bar{t}) + 2f(u(\bar{x}, \bar{t})) + \delta(a_0 + a_1[\xi(\cdot, \bar{t})](\bar{x}))\psi_x(\bar{x}, \bar{t})).$$

Finally, a locally bounded function (u, ξ) is called a viscosity solution of (2.8) if (u^*, ξ^*)

is a sub-solution and (u_*, ξ_*) is a super-solution of (2.10).

2.2 Comparison principle and existence

We begin this subsection by introducing a change of variables on the unknown functions that will be used thorough the paper.

Lemma 2.2. *Let $p > 0$ and (u, ξ) be a sub-solution (resp. a super-solution) of (2.8) such that*

$$|u(x, t) - px| \leq K_1 t, \quad |\xi(x, t) - px| \leq K_1 t. \quad (2.10)$$

We set

$$\begin{cases} \bar{u}(x, t) := e^{-\mu t} u(x, t) \\ \bar{\xi}(x, t) := e^{-\mu t} \xi(x, t). \end{cases} \quad (2.11)$$

Then

$$|\bar{u}(x, t) - e^{-\mu t} px| \leq e^{-\mu t} K_1 t, \quad |\bar{\xi}(x, t) - e^{-\mu t} px| \leq e^{-\mu t} K_1 t \quad (2.12)$$

and $(\bar{u}, \bar{\xi})$ is a sub-solution (resp. a super-solution) of

$$\begin{cases} \bar{u}_t + \mu \bar{u} + \frac{1}{\sqrt{m}}\bar{u}_x + \frac{1}{2m}\bar{u} = \frac{1}{2m}\bar{\xi} + \delta(a_0 + a_1[e^{\mu t}\bar{u}(\cdot, t)](x))\bar{u}_x \\ \bar{\xi}_t + \mu \bar{\xi} - \frac{1}{\sqrt{m}}\bar{\xi}_x + \frac{1}{2m}\bar{\xi} = \frac{1}{2m}\bar{u} + 2\bar{f}(\bar{u}) + \delta(a_0 + a_1[e^{\mu t}\bar{\xi}(\cdot, t)](x))\bar{\xi}_x \end{cases} \quad (2.13)$$

with

$$\bar{f}(\bar{u}) = e^{-\mu t} f(e^{\mu t} \bar{u}) \quad (2.14)$$

Démonstration. This is a straightforward computation. \square

We now give a comparison principle for equation (2.8).

Proposition 2.3 (Comparison principle for (2.8)). *Let $\delta \geq 0$ and $p > 0$. Under assumptions (1.4)-(1.5), let (u_1, ξ_1) be a viscosity sub-solution and (u_2, ξ_2) be a viscosity super-solution of (2.8) on $Q = \mathbb{R} \times (0, +\infty)$.*

We assume that

$$u_1(x, t) \leq px + K_1 t, \quad u_2(x, t) \geq px - K_1 t$$

and

$$\xi_1(x, t) \leq px + K_1 t, \quad \xi_2(x, t) \geq px - K_1 t.$$

If

$$\begin{cases} u_1 \leq u_2 & \text{in } \mathbb{R} \times \{0\} \\ \xi_1 \leq \xi_2 & \text{in } \mathbb{R} \times \{0\} \end{cases}$$

then

$$\begin{cases} u_1 \leq u_2 & \text{in } Q \\ \xi_1 \leq \xi_2 & \text{in } Q. \end{cases}$$

Proof of Proposition 2.3

Let $T > 0$. We will prove that

$$M = \sup_{(x,t) \in \mathbb{R} \times (0,T)} \max(u_1(x, t) - u_2(x, t), \xi_1(x, t) - \xi_2(x, t)) \leq 0.$$

We set

$$\begin{cases} \bar{u}(x, t) := e^{-\mu t} u(x, t) \\ \bar{\xi}(x, t) := e^{-\mu t} \xi(x, t) \end{cases} \quad (2.15)$$

with $\mu > 0$. Using lemma 2.2, we get that $(\bar{u}_1, \bar{\xi}_1), (\bar{u}_2, \bar{\xi}_2)$ are a sub- and a super-solution of (2.13) with (2.14). We then have to prove that

$$\bar{M} = \sup_{(x,t) \in \mathbb{R} \times (0,T)} \max(\bar{u}_1(x,t) - \bar{u}_2(x,t), \bar{\xi}_1(x,t) - \bar{\xi}_2(x,t)) \leq 0.$$

We assume by contradiction that $\bar{M} > 0$. Then

$$M_1 = \sup_{(x,y,t) \in \mathbb{R} \times \mathbb{R} \times (0,T)} \max(\psi_1(x,y,t), \psi_2(x,y,t)) > 0$$

where

$$\begin{cases} \psi_1(x,y,t) = \bar{u}_1(x,t) - \bar{u}_2(y,t) - \frac{|x-y|^2}{2\varepsilon} - \frac{\eta}{T-t} - \alpha(x^2 + y^2) \\ \psi_2(x,y,t) = \bar{\xi}_1(x,t) - \bar{\xi}_2(y,t) - \frac{|x-y|^2}{2\varepsilon} - \frac{\eta}{T-t} - \alpha(x^2 + y^2) \end{cases}$$

for $\varepsilon, \eta, \alpha$ small enough. We have

$$\begin{aligned} \psi_1(x,y,t) &\leq px \cdot e^{-\mu t} - py \cdot e^{-\mu t} + 2K_1 t e^{-\mu t} - \frac{|x-y|^2}{2\varepsilon} - \frac{\eta}{T-t} - \alpha(x^2 + y^2) \\ &\leq \left(p - \frac{|x-y|}{2\varepsilon} \right) |x-y| + 2K_1 t - \frac{\eta}{T-t} - \alpha(x^2 + y^2) \\ &\leq \frac{p^2}{2} + 2K_1 T - \frac{\eta}{T-t} - \alpha(x^2 + y^2) \end{aligned}$$

where we have used that the function $\phi : r \rightarrow pr - \frac{r^2}{2\varepsilon}$ is bounded by $\frac{p^2}{2}$. We then get that $\psi_1(x,y,t) \rightarrow -\infty$ as $|(x,y)| \rightarrow +\infty$. Since ψ_1 is upper semi-continuous, we deduce that ψ_1 reaches its maximum. We prove similarly that ψ_2 reaches its maximum. We denote by (x,y,t) a point of maximum in \bar{M} . We recall that we classically have

$$\lim_{\alpha \rightarrow 0} \alpha x, \alpha y = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \frac{(x-y)^2}{\varepsilon} = 0.$$

Case 1 We assume that the maximum is reached for ψ_1 . By doubling the time variable and then passing to the limit, we get the existence of $a, b \in \mathbb{R}$ with $a - b = \frac{\eta}{(T-t)^2}$ such

that

$$\begin{aligned} & a + \mu \bar{u}_1(x, t) + \frac{1}{\sqrt{m}} \left(\frac{x-y}{\varepsilon} + 2\alpha x \right) + \frac{1}{2m} \bar{u}_1(x, t) \\ & \leq \frac{1}{2m} \bar{\xi}_1(x, t) + \delta \left(a_0 + \inf_{y' \in \mathbb{R}} (e^{\mu t} \bar{u}_1(y', t) - py') + px - e^{\mu t} \bar{u}_1(x, t) \right) \left(\frac{x-y}{\varepsilon} + 2\alpha x \right) \end{aligned}$$

and

$$\begin{aligned} & b + \mu \bar{u}_2(y, t) + \frac{1}{\sqrt{m}} \left(\frac{x-y}{\varepsilon} - 2\alpha y \right) + \frac{1}{2m} \bar{u}_2(y, t) \\ & \geq \frac{1}{2m} \bar{\xi}_2(y, t) + \delta \left(a_0 + \inf_{y' \in \mathbb{R}} (e^{\mu t} \bar{u}_2(y', t) - py') + py - e^{\mu t} \bar{u}_2(y, t) \right) \left(\frac{x-y}{\varepsilon} - 2\alpha y \right). \end{aligned}$$

Subtracting these two inequalities, we get

$$\begin{aligned} & \frac{\eta}{T^2} + \mu(\bar{u}_1(x, t) - \bar{u}_2(y, t)) + \frac{2\alpha}{\sqrt{m}}(x+y) + \frac{1}{2m}(\bar{u}_1(x, t) - \bar{u}_2(y, t)) \\ & \leq \frac{1}{2m}(\bar{\xi}_1(x, t) - \bar{\xi}_2(y, t)) + \delta \left(a_0 + \inf_{y' \in \mathbb{R}} (e^{\mu t} \bar{u}_1(y', t) - py') + px - e^{\mu t} \bar{u}_1(x, t) \right) \\ & \quad \left(\frac{x-y}{\varepsilon} + 2\alpha x \right) - \delta \left(a_0 + \inf_{y' \in \mathbb{R}} (e^{\mu t} \bar{u}_2(y', t) - py') + py - e^{\mu t} \bar{u}_2(y, t) \right) \\ & \quad \left(\frac{x-y}{\varepsilon} - 2\alpha y \right). \end{aligned} \tag{2.16}$$

Using that for all $y' \in \mathbb{R}$

$$e^{\mu t} \bar{u}_1(y', t) - e^{\mu t} \bar{u}_2(y', t) \leq e^{\mu t} \bar{u}_1(x, t) - e^{\mu t} \bar{u}_2(y, t) + 2\alpha y' e^{\mu t},$$

we deduce that

$$\lim_{\alpha \rightarrow 0} \delta \left(\inf_{y' \in \mathbb{R}} e^{\mu t} (\bar{u}_1(y', t) - py') - \inf_{y' \in \mathbb{R}} (e^{\mu t} \bar{u}_2(y', t) - py') - e^{\mu t} \bar{u}_1(x, t) + e^{\mu t} \bar{u}_2(y, t) \right) = 0.$$

Passing to the limit $\alpha \rightarrow 0$ in (2.16) and using that the maximum is reached for the function ψ_1 and that $\psi_1 > 0$, we then get

$$\frac{\eta}{T^2} \leq \delta p \lim_{\alpha \rightarrow 0} \frac{(x-y)^2}{\varepsilon}.$$

Taking ε small enough, we get a contradiction.

Case 2 We assume that the maximum is reached for ψ_2 . By doubling the time variable and then passing to the limit, we get the existence of $a, b \in \mathbb{R}$ with $a - b = \frac{\eta}{(T-t)^2}$ such that

$$\begin{aligned} & a + \mu \bar{\xi}_1(x, t) - \frac{1}{\sqrt{m}} \left(\frac{x-y}{\varepsilon} + 2\alpha x \right) + \frac{1}{2m} \bar{\xi}_1(x, t) \\ & \leq \frac{1}{2m} \bar{u}_1(x, t) + 2\bar{f}(\bar{u}_1(x, t)) + \delta \left(a_0 + \inf_{y' \in \mathbb{R}} (e^{\mu t} \bar{\xi}_1(y', t) - py') + px - e^{\mu t} \bar{\xi}_1(x, t) \right) \\ & \quad \left(\frac{x-y}{\varepsilon} + 2\alpha x \right) \end{aligned}$$

and

$$\begin{aligned} & b + \mu \bar{\xi}_2(y, t) - \frac{1}{\sqrt{m}} \left(\frac{x-y}{\varepsilon} - 2\alpha y \right) + \frac{1}{2m} \bar{\xi}_2(y, t) \\ & \geq \frac{1}{2m} \bar{u}_2(y, t) + 2\bar{f}(\bar{u}_2(y, t)) + \delta \left(a_0 + \inf_{y' \in \mathbb{R}} (e^{\mu t} \bar{\xi}_2(y', t) - py') + py - e^{\mu t} \bar{\xi}_2(y, t) \right) \\ & \quad \left(\frac{x-y}{\varepsilon} - 2\alpha y \right). \end{aligned}$$

Subtracting these two inequalities, we get

$$\begin{aligned} & \frac{\eta}{T^2} + \mu(\bar{\xi}_1(x, t) - \bar{\xi}_2(y, t)) - \frac{2\alpha}{\sqrt{m}}(x + y) \tag{2.17} \\ & \leq \frac{1}{2m}(\bar{u}_1(x, t) + \bar{\xi}_2(y, t) - \bar{\xi}_1(x, t)) + 2\bar{f}(\bar{u}_1(x, t)) - \frac{1}{2m}\bar{u}_2(y, t) - 2\bar{f}(\bar{u}_2(y, t)) \\ & \quad + \delta \left(a_0 + \inf_{y' \in \mathbb{R}} (e^{\mu t} \bar{\xi}_1(y', t) - py') + px - e^{\mu t} \bar{\xi}_1(x, t) \right) \left(\frac{x-y}{\varepsilon} + 2\alpha x \right) \\ & \quad - \delta \left(a_0 + \inf_{y' \in \mathbb{R}} (e^{\mu t} \bar{\xi}_2(y', t) - py') + py - e^{\mu t} \bar{\xi}_2(y, t) \right) \left(\frac{x-y}{\varepsilon} - 2\alpha y \right) \end{aligned}$$

Using that $\bar{u}_1(x, t) + \bar{\xi}_2(y, t) - \bar{\xi}_1(x, t) \leq \bar{u}_2(y, t)$ and the fact that the function $u \mapsto$

$\frac{1}{2m} + 2\bar{f}(u)$ is non-decreasing (see (1.4)), we deduce that

$$\begin{aligned}
 & \frac{1}{2m}(\bar{u}_1(x, t) + \bar{\xi}_2(y, t) - \bar{\xi}_1(x, t)) + 2\bar{f}(\bar{u}_1(x, t)) - \frac{1}{2m}\bar{u}_2(y, t) - 2\bar{f}(\bar{u}_2(y, t)) \\
 & - \mu(\bar{\xi}_1(x, t) - \bar{\xi}_2(y, t)) \\
 & \leq 2\bar{f}(\bar{u}_1(x, t)) - 2\bar{f}(\bar{u}_1(x, t) + \bar{\xi}_2(y, t) - \bar{\xi}_1(x, t)) - \mu(\bar{\xi}_1(x, t) - \bar{\xi}_2(y, t)) \\
 & \leq (2\|f'\|_\infty - \mu)(\bar{\xi}_1(x, t) - \bar{\xi}_2(y, t)) \\
 & \leq 0
 \end{aligned}$$

for $\mu \geq 2\|f'\|_\infty$. Using the same arguments as in case 1, we also get

$$\lim_{\alpha \rightarrow 0} \delta \left(\inf_{y' \in \mathbb{R}} e^{\mu t} (\bar{\xi}_1(y', t) - py') - \inf_{y' \in \mathbb{R}} (e^{\mu t} \bar{\xi}_2(y', t) - py') - e^{\mu t} \bar{\xi}_1(x, t) + e^{\mu t} \bar{\xi}_2(y, t) \right) = 0.$$

Passing to the limit $\alpha \rightarrow 0$ in (2.17) and using the previous estimates, we finally deduce

$$\frac{\eta}{T^2} \leq \delta p \lim_{\alpha \rightarrow 0} \frac{(x - y)^2}{\varepsilon}.$$

Taking ε small enough, we get a contradiction.

Proposition 2.4 (Existence and uniqueness for (2.8)). *Let $\delta \geq 0$ and $p > 0$. Then there exists a unique solution (u, ξ) of (2.8) such that*

$$|u(x, t) - px| \leq K_1 t, \quad |\xi(x, t) - px| \leq K_1 t$$

with $K_1 = 2\|f\|_\infty + \frac{1}{\sqrt{m}}p$.

Démonstration. We consider the barriers functions

$$\begin{cases} u^\pm(x, t) = px \pm K_1 t \\ \xi^\pm(x, t) = px \pm K_1 t. \end{cases} \quad (2.18)$$

We have that (u^+, ξ^+) (resp. (u^-, ξ^-)) is a viscosity super-solution (resp. a viscosity sub-solution) of (2.8) with initial data $u_0(x) = \xi_0(x) = px$. Using Perron's method

and Proposition 2.3, we deduce that there exists a unique continuous viscosity solution (u, ξ) such that

$$u^- \leq u \leq u^+ \quad \text{and} \quad \xi^- \leq \xi \leq \xi^+.$$

□

2.3 Bound on the gradient

We now give gradient estimates for the solution of (2.8). We begin by an estimate from below.

Proposition 2.5 (Bound on the gradient from below). *Let $\delta \geq 0$, $p > 0$ and (u, ξ) be the solution of (2.8). Under assumptions (1.4)-1.5, (u, ξ) satisfy*

$$u_x \geq 0 \quad \text{and} \quad \xi_x \geq 0.$$

Démonstration. Let $T > 0$. By contradiction, we assume that

$$M = \sup_{\substack{(x,y,t) \in \mathbb{R} \times \mathbb{R} \times (0,T) \\ y \geq x}} \max(u(x,t) - u(y,t), \xi(x,t) - \xi(y,t)) > 0.$$

We set

$$\begin{cases} \bar{u}(x,t) := e^{-\mu t} u(x,t) \\ \bar{\xi}(x,t) := e^{-\mu t} \xi(x,t). \end{cases} \quad (2.19)$$

Then

$$\bar{M} = \sup_{\substack{(x,y,t) \in \mathbb{R} \times \mathbb{R} \times (0,T) \\ y \geq x}} \max(\bar{u}(x,t) - \bar{u}(y,t), \bar{\xi}(x,t) - \bar{\xi}(y,t)) > 0.$$

For $\alpha, \eta > 0$, we set

$$\begin{cases} \psi_1(x,y,t) = \bar{u}(x,t) - \bar{u}(y,t) - \frac{\eta}{T-t} - \alpha(x^2 + y^2) \\ \psi_2(x,y,t) = \bar{\xi}(x,t) - \bar{\xi}(y,t) - \frac{\eta}{T-t} - \alpha(x^2 + y^2) \end{cases}$$

then, for η, α small enough,

$$M_1 = \sup_{\substack{(x,y,t) \in \mathbb{R} \times \mathbb{R} \times (0,T) \\ y \geq x}} \max(\psi_1(x, y, t), \psi_2(x, y, t)) > 0.$$

We have

$$\begin{aligned} \psi_1(x, y, t) &\leq e^{-\mu t} p(x - y) + 2K_1 t e^{-\mu t} - \frac{\eta}{T-t} - \alpha(x^2 + y^2) \\ &\leq 2K_1 T - \alpha(x^2 + y^2). \end{aligned}$$

This implies that $\psi_1(x, y, t) \rightarrow -\infty$ as $\|x\|, \|y\| \rightarrow +\infty$. Since \bar{u} is continuous, we deduce that ψ_1 reaches its maximum. We prove similarly that ψ_2 reaches its maximum. We denote by (x, y, t) (with $y \geq x$) a point of maximum of $\max(\psi_1(x, y, t), \psi_2(x, y, t))$. Since $M_1 > 0$, we deduce that $y > x$. Moreover, we classically have $\alpha x, \alpha y \rightarrow 0$ as $\alpha \rightarrow 0$.

Case 1 We assume that the maximum is reached for ψ_1 . Using Lemma 2.2, doubling the variable in time and then passing to the limit, we get the existence of $a, b \in \mathbb{R}$ with $a - b = \frac{\eta}{(T-t)^2}$ such that

$$\begin{aligned} &a + \mu \bar{u}(x, t) + \frac{2}{\sqrt{m}} \alpha x + \frac{1}{2m} \bar{u}(x, t) \\ &\leq \frac{1}{2m} \bar{\xi}(x, t) + 2\alpha x \delta(a_0 + a_1[e^{\mu t} \bar{u}(\cdot, t)](y)) \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &b + \mu \bar{u}(y, t) - \frac{2}{\sqrt{m}} \alpha y + \frac{1}{2m} \bar{u}(y, t) \\ &\geq \frac{1}{2m} \bar{\xi}(y, t) - 2\alpha y \delta(a_0 + a_1[e^{\mu t} \bar{u}(\cdot, t)](y)). \end{aligned} \tag{2.21}$$

We subtract (2.21) to (2.20), we use that $\bar{u}(x, t) \geq \bar{u}(y, t)$ and that $\psi_1(x, y, t) \geq \psi_2(x, y, t)$ and we take the limit $\alpha \rightarrow 0$ to get

$$\frac{\eta}{T^2} \leq 0.$$

This is a contradiction.

Case 2 We assume the maximum is reached for ψ_2 . Using Lemma 2.2, doubling the variable in time and then passing to the limit, we get the existence of $a, b \in \mathbb{R}$ with $a - b = \frac{\eta}{(T-t)^2}$ such that

$$\begin{aligned} & a + \mu \bar{\xi}(x, t) - \frac{2}{\sqrt{m}} \alpha x + \frac{1}{2m} \bar{\xi}(x, t) \\ & \leq \frac{1}{2m} \bar{u}(x, t) + 2\bar{f}(\bar{u}(x, t)) + 2\alpha x \delta(a_0 + a_1[e^{\mu t} \bar{\xi}(\cdot, t)](x)) \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} & b + \mu \bar{\xi}(y, t) + \frac{2}{\sqrt{m}} \alpha y + \frac{1}{2m} \bar{\xi}(y, t) \\ & \geq \frac{1}{2m} \bar{u}(y, t) + 2\bar{f}(\bar{u}(y, t)) - 2\alpha y \delta(a_0 + a_1[e^{\mu t} \bar{\xi}(\cdot, t)](y)). \end{aligned} \quad (2.23)$$

Subtracting (2.23) to (2.22), we get

$$\begin{aligned} \frac{\eta}{T^2} \leq & \mu(\bar{\xi}(y, t) - \bar{\xi}(x, t)) + \frac{1}{2m}(\bar{u}(x, t) + \bar{\xi}(y, t) - \bar{\xi}(x, t)) + 2\bar{f}(\bar{u}(x, t) + \bar{\xi}(y, t) \\ & - \bar{\xi}(x, t)) - 2\bar{f}(\bar{u}(y, t)) - \frac{1}{2m}\bar{u}(y, t) + 2\bar{f}(\bar{u}(x, t)) - 2\bar{f}(\bar{u}(x, t) + \bar{\xi}(y, t) \\ & - \bar{\xi}(x, t)) + o(\alpha) \end{aligned}$$

Using that $u \mapsto \frac{1}{2m}u + 2\bar{f}(u)$ is non-decreasing and that $\bar{u}(x, t) + \bar{\xi}(y, t) - \bar{\xi}(x, t) \leq \bar{u}(y, t)$, we get

$$\frac{\eta}{T^2} \leq (2\|f'\|_\infty - \mu)(\bar{\xi}(x, t) - \bar{\xi}(y, t)) + o(\alpha).$$

For $\mu \geq 2\|f'\|_\infty$ and α small enough, we get a contradiction. \square

We now turn to the gradient estimate from above

Proposition 2.6 (Bound on the gradient from above). *Let $\delta, p > 0$ and (u, ξ) be the solution of (2.8). Under assumptions (1.4)-(1.5), (u, ξ) satisfy*

$$u_x \leq L \quad \text{and} \quad \xi_x \leq L$$

with

$$L = p + \frac{2}{\delta}\|f'\|_\infty. \quad (2.24)$$

Démonstration. We set

$$\begin{cases} \bar{u}(x, t) := e^{-\mu t} u(x, t) \\ \bar{\xi}(x, t) := e^{-\mu t} \xi(x, t). \end{cases} \quad (2.25)$$

Let $T > 0$. By contradiction, we assume that

$$\bar{M} = \sup_{\substack{(x, y, t) \in \mathbb{R} \times \mathbb{R} \times (0, T) \\ x \geq y}} \max(\bar{u}(x, t) - \bar{u}(y, t) - Le^{-\mu t}(x - y), \bar{\xi}(x, t) - \bar{\xi}(y, t) - Le^{-\mu t}|x - y|) > 0.$$

Then

$$M_1 = \sup_{\substack{(x, y, t) \in \mathbb{R} \times \mathbb{R} \times (0, T) \\ x \geq y}} \max(\psi_1(x, y, t), \psi_2(x, y, t)) > 0$$

with

$$\begin{aligned} \psi_1(x, y, t) &= \bar{u}(x, t) - \bar{u}(y, t) - L(x - y)e^{-\mu t} - \frac{\eta}{T-t} - \alpha(x^2 + y^2) \\ \psi_2(x, y, t) &= \bar{\xi}(x, t) - \bar{\xi}(y, t) - L(x - y)e^{-\mu t} - \frac{\eta}{T-t} - \alpha(x^2 + y^2). \end{aligned}$$

We have

$$\begin{aligned} \psi_1(x, y, t) &\leq p(x - y)e^{-\mu t} + 2K_1te^{-\mu t} - L(x - y)e^{-\mu t} - \frac{\eta}{T-t} - \alpha(x^2 + y^2) \\ &\leq 2K_1T - \frac{\eta}{T-t} - \alpha(x^2 + y^2). \end{aligned}$$

where we have used that $p \leq L$. This implies that $\psi_1(x, y, t) \rightarrow -\infty$ as $\|x\|, \|y\| \rightarrow +\infty$. Then we deduce that ψ_1 reaches its maximum (and similarly for ψ_2). We denote by (x, y, t) with $(x \geq y)$ a point of maximum of $\max(\psi_1(x, y, t), \psi_2(x, y, t))$. Since $M_1 > 0$, we deduce that $x > y$. Moreover, we have $\alpha x, \alpha y \rightarrow 0$ as $\alpha \rightarrow 0$.

Case 1 We assume that the maximum is reached for ψ_1 . Using Lemma 2.2, doubling the variable in time and then passing to the limit, we get the existence of $a, b \in \mathbb{R}$ with

$$a - b = \frac{\eta}{(T - t)^2} - \mu L(x - y)e^{-\mu t}$$

such that

$$\begin{aligned} &a + \mu \bar{u}(x, t) + \frac{1}{\sqrt{m}}(2\alpha x + Le^{-\mu t}) + \frac{1}{2m} \bar{u}(x, t) \\ &\leq \frac{1}{2m} \bar{\xi}(x, t) + \delta(a_0 + a_1[e^{\mu t} \bar{u}(\cdot, t)](x))(2\alpha x + Le^{-\mu t}) \end{aligned} \quad (2.26)$$

and

$$\begin{aligned} & b + \mu\bar{u}(y, t) + \frac{1}{\sqrt{m}}(-2\alpha y + Le^{-\mu t}) + \frac{1}{2m}\bar{u}(y, t) \\ & \geq \frac{1}{2m}\bar{\xi}(y, t) + \delta(a_0 + a_1[e^{\mu t}\bar{u}(\cdot, t)](y))(-2\alpha y + Le^{-\mu t}) \end{aligned} \quad (2.27)$$

Using that

$$a_1[e^{\mu t}\bar{u}(\cdot, t)](x) - a_1[e^{\mu t}\bar{u}(\cdot, t)](y) = e^{\mu t}(\bar{u}(y, t) - \bar{u}(x, t)) + p(x - y),$$

and subtracting (2.27) to (2.26), we get

$$\begin{aligned} \frac{\eta}{T^2} - \mu L(x - y)e^{-\mu t} & \leq \mu(\bar{u}(y, t) - \bar{u}(x, t)) + \frac{1}{2m}(\bar{u}(y, t) - \bar{u}(x, t) - \bar{\xi}(y, t) + \bar{\xi}(x, t)) \\ & \quad + \delta L(\bar{u}(y, t) - \bar{u}(x, t)) + \delta Le^{-\mu t}p(x - y) + o(\alpha) \\ & \leq (\mu + \delta L)(\bar{u}(y, t) - \bar{u}(x, t)) + \delta Le^{-\mu t}p(x - y) + o(\alpha) \end{aligned}$$

where we have used that $\psi_1(x, y, t) \geq \psi_2(x, y, t)$. Using that $\psi_1(x, y, t) > 0$, we deduce that

$$\bar{u}(y, t) - \bar{u}(x, t) \leq -L(x - y)e^{-\mu t}.$$

This implies that

$$\frac{\eta}{T^2} \leq \delta L(x - y)e^{-\mu t}(p - L) + o(\alpha)$$

where we have used the fact that $L \geq p$. This is a contradiction for α small enough.

Case 2 We assume that the maximum is reached for ψ_2 . Using Lemma 2.2, doubling the variable in time and then passing to the limit, we get the existence of $a, b \in \mathbb{R}$ with

$$a - b = \frac{\eta}{(T - t)^2} - \mu L(x - y)e^{-\mu t}$$

such that

$$\begin{aligned} & a + \mu\bar{\xi}(x, t) - \frac{1}{\sqrt{m}}(Le^{-\mu t} + 2\alpha x) + \frac{1}{2m}\bar{\xi}(x, t) \\ & \leq \frac{1}{2m}\bar{u}(x, t) + 2\bar{f}(\bar{u}(x, t)) + \delta(a_0 + a_1[e^{\mu t}\bar{\xi}(\cdot, t)](x))(Le^{-\mu t} + 2\alpha x) \end{aligned} \quad (2.28)$$

$$\begin{aligned}
 & b + \mu \bar{\xi}(y, t) - \frac{1}{\sqrt{m}}(Le^{-\mu t} - 2\alpha y) + \frac{1}{2m} \bar{\xi}(y, t) \\
 & \geq \frac{1}{2m} \bar{u}(y, t) + 2\bar{f}(\bar{u}(y, t)) + \delta(a_0 + a_1[e^{\mu t} \bar{\xi}(\cdot, t)](y))(Le^{-\mu t} - 2\alpha y)
 \end{aligned} \tag{2.29}$$

Using that

$$a_1[e^{\mu t} \bar{\xi}(\cdot, t)](x) - a_1[e^{\mu t} \bar{\xi}(\cdot, t)](y) = e^{\mu t}(\bar{\xi}(y, t) - \bar{\xi}(x, t)) + p(x - y),$$

and subtracting (2.29) to (2.28), we get

$$\begin{aligned}
 \frac{\eta}{T^2} - \mu L(x - y)e^{-\mu t} & \leq \mu(\bar{\xi}(y, t) - \bar{\xi}(x, t)) + \frac{1}{2m}(\bar{\xi}(y, t) - \bar{\xi}(x, t) - \bar{u}(y, t) + \bar{u}(x, t)) \\
 & \quad + \delta L(\bar{\xi}(y, t) - \bar{\xi}(x, t)) + \delta Le^{-\mu t} p(x - y) + 2\bar{f}(\bar{u}(x, t)) \\
 & \quad - 2\bar{f}(\bar{u}(y, t)) + o(\alpha) \\
 & \leq (\mu + \delta L)(\bar{\xi}(y, t) - \bar{\xi}(x, t)) + \delta Le^{-\mu t} p(x - y) + o(\alpha) \\
 & \quad + \frac{1}{2m}(\bar{\xi}(y, t) - \bar{\xi}(x, t) + \bar{u}(x, t)) + 2\bar{f}(\bar{\xi}(y, t) - \bar{\xi}(x, t) + \bar{u}(x, t)) \\
 & \quad - \frac{1}{2m}\bar{u}(y, t) - 2\bar{f}(\bar{u}(y, t)) + 2\bar{f}(\bar{u}(x, t)) - 2\bar{f}(\bar{\xi}(y, t) - \bar{\xi}(x, t) \\
 & \quad + \bar{u}(x, t)).
 \end{aligned}$$

Using that $\frac{1}{2m} \cdot + 2\bar{f}(\cdot)$ is non-decreasing and that $\bar{u}(x, t) + \bar{\xi}(y, t) - \bar{\xi}(x, t) \leq \bar{u}(y, t)$,

we get

$$\frac{\eta}{T^2} \leq (\mu + \delta L)(\bar{\xi}(y, t) - \bar{\xi}(x, t)) + \delta Le^{-\mu t} p(x - y) + \mu L(x - y)e^{-\mu t} + 2\|f'\|_{\infty}(\bar{\xi}(x, t) - \bar{\xi}(y, t)) + o(\alpha).$$

Using that $\psi_2(x, y, t) > 0$, we deduce that

$$\bar{\xi}(y, t) - \bar{\xi}(x, t) \leq -L(x - y)e^{-\mu t}.$$

This implies that

$$\begin{aligned}
 \frac{\eta}{T^2} &\leq (\mu - 2\|f'\|_\infty)(\bar{\xi}(y, t) - \bar{\xi}(x, t)) + \delta L(x - y)e^{-\mu t}(p - L) + \mu L(x - y)e^{-\mu t} + o(\alpha) \\
 &\leq -L(x - y)e^{-\mu t}(\mu - 2\|f'\|_\infty) + \delta L(x - y)e^{-\mu t}(p - L) + \mu L|x - y|e^{-\mu t} + o(\alpha) \\
 &\leq L(x - y)e^{-\mu t}(p\delta + 2\|f'\|_\infty - L\delta) + o(\alpha)
 \end{aligned}$$

where we have used the fact that $L = p + \frac{2}{\delta}\|f'\|_\infty$. This is a contradiction for α small enough. \square

3 Proof of Theorem 1.1

Some properties of the solution of (2.8) We now give in this subsection a result for the control of the solution of the Cauchy problem (2.8).

Proposition 3.1 (Control of the solution of the Cauchy problem). *Let (u, ξ) be be solution of (2.8) given by Proposition 2.4 with initial data $u_0(x) = \xi_0(x) = px$ where $p > 0$. Then, there exists C_δ such that (u, ξ) satisfies*

$$\left\{ \begin{array}{l}
 (i) |u(x, t) - px - \lambda t| \leq C_\delta, \quad |\xi(x, t) - px - \lambda t| \leq C_\delta \\
 (ii) u\left(x + \frac{1}{p}, t\right) = 1 + u(x, t), \quad \xi\left(x + \frac{1}{p}, t\right) = 1 + \xi(x, t) \\
 (iii) u_x \geq 0, \xi_x \geq 0 \\
 (iv) |u(x + y, t) - u(x, t) - py| \leq 2, |\xi(x + y, t) - \xi(x, t) - py| \leq 2
 \end{array} \right. \quad (3.30)$$

Proof of Proposition 3.1.

Step 1 : Proof of (ii) and (iii)

Using that

$$u_0\left(x + \frac{1}{p}\right) = \xi_0\left(x + \frac{1}{p}\right) = u_0(x) + 1 = \xi_0(x) + 1$$

and the fact that $(u + 1, \xi + 1)$ and $\left(u\left(\cdot + \frac{1}{p}\right), \xi\left(\cdot + \frac{1}{p}\right)\right)$ are solutions of (2.8) with the same initial data, we deduce by the comparison principle that

$$\begin{aligned} u\left(x + \frac{1}{p}, t\right) &= 1 + u(x, t) \\ \xi\left(x + \frac{1}{p}, t\right) &= 1 + \xi(x, t). \end{aligned}$$

Moreover, (iii) is a direct consequence of Proposition 2.5

Step 2 : Control of the space oscillations

Lemma 3.2. *For all $t > 0$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, we have*

$$\begin{cases} |u(x + y, t) - u(x, t) - py| \leq 2 \\ |\xi(x + y, t) - \xi(x, t) - py| \leq 2 \end{cases} \quad (3.31)$$

Proof of Lemma 3.2. In order to prove this lemma, we set

$$w(x, t) = u(x, t) - px. \quad (3.32)$$

We have

$$\begin{aligned} w\left(x + \frac{1}{p}, t\right) &= u\left(x + \frac{1}{p}, t\right) - p\left(x + \frac{1}{p}\right) \\ &= 1 + u(x, t) - px - 1 \\ &= u(x, t) - px \\ &= w(x, t) \end{aligned}$$

then w is $\frac{1}{p}$ periodic. Let $0 < y < z < \frac{1}{p}$. Since u is increasing, then we have

$$\begin{aligned} 0 &\leq u(y, t) \leq u(z, t) \leq u\left(y + \frac{1}{p}, t\right) \\ 0 &\leq w(y, t) + py \leq w(z, t) + pz \leq w\left(y + \frac{1}{p}, t\right) + p\left(y + \frac{1}{p}\right) \\ -1 &\leq py - pz \leq w(z, t) - w(y, t) \leq py - pz + 1 \\ -1 &\leq w(z, t) - w(y, t) \leq 2 \end{aligned}$$

$$|w(z, t) - w(y, t)| \leq 2.$$

We take $z = x + y$ and $y = x$, we get

$$|u(x + y, t) - u(x, t) - py| \leq 2.$$

□

To control the oscillation in time, we have to bound $|u - \xi|$

Step 3 : estimate of the solution $|u - \xi|_{L^\infty(\mathbb{R} \times (0, +\infty))}$

Lemma 3.3. *We have*

$$|u - \xi|_{L^\infty(\mathbb{R} \times (0, +\infty))} \leq c_1$$

where

$$c_1 = m(2 \|f\|_\infty + \delta(a_0 + 4)L). \quad (3.33)$$

Démonstration. We set $w = u - \xi$. Using that (u, ξ) is a solution of (2.8) joint to Propositions 2.5 and 2.6 and estimate (3.31), we get that

$$\begin{aligned} u_t &\leq \frac{1}{2m}(\xi(x, t) - u(x, t)) + \delta(a_0 + 2)L, \\ \xi_t &\geq \frac{1}{2m}(u(x, t) - \xi(x, t)) - 2\|f\|_\infty - \delta 2L \end{aligned}$$

Let $\phi \in C^1$ be such that $w - \phi$ reaches a local maximum at (\bar{x}, \bar{t}) . We define

$$M_\varepsilon = \sup_{(x, y, t) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty)} \left(u(x, t) - \xi(y, s) - \phi(x, t) - \frac{|x - y|^2}{2\varepsilon} - \frac{|t - s|^2}{2\varepsilon} - |t - \bar{t}|^2 - |x - \bar{x}|^2 \right).$$

It is then easy to show that this supremum is reached at some point $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$ such that $x_\varepsilon, y_\varepsilon \rightarrow \bar{x}$ and $s_\varepsilon, t_\varepsilon \rightarrow \bar{t}$ as $\varepsilon \rightarrow 0$. We then get

$$\begin{aligned} &\phi_t(x_\varepsilon, t_\varepsilon) + (t_\varepsilon - \bar{t}) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} \\ &\leq -\frac{1}{2m}(u(x_\varepsilon, t_\varepsilon) - \xi(x_\varepsilon, t_\varepsilon)) + \delta(a_0 + 2)L \end{aligned}$$

and

$$\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} \geq \frac{1}{2m}(u(x_\varepsilon, s_\varepsilon) - \xi(x_\varepsilon, s_\varepsilon)) - 2\|f\|_\infty - \delta 2L.$$

Subtracting the two inequalities and passing to the limit as $\varepsilon \rightarrow 0$, we get

$$\phi_t(\bar{x}, \bar{t}) \leq -\frac{1}{m}w(\bar{x}, \bar{t}) + 2\|f\|_\infty + \delta(a_0 + 4)L.$$

This implies, that w satisfies in the viscosity sense

$$w_t \leq -\frac{1}{m}w + 2\|f\|_\infty + \delta(a_0 + 4)L.$$

Applying Gronwall's lemma, we get

$$w(x, t) \leq m(2\|f\|_\infty + \delta(a_0 + 4)L)$$

□

Step 4 : Control of the time oscillation of the solution.

In the following, we need to control the time oscillation of the solution of (2.8). We have the following lemma.

Lemma 3.4. *There exists $C_\delta > 0$ and $\lambda \in \mathbb{R}$ such that for all $t > 0$ and for all $x \in \mathbb{R}$,*

we have

$$\begin{cases} |u(x, t) - px - \lambda t| \leq C_\delta \\ |\xi(x, t) - px - \lambda t| \leq C_\delta. \end{cases} \quad (3.34)$$

Proof of Lemma 3.4. We set

$$\lambda_+^u(T) := \sup_{t \geq 0} \frac{u(0, t+T) - u(0, t)}{T}, \quad \lambda_-^u(T) := \inf_{t \geq 0} \frac{u(0, t+T) - u(0, t)}{T},$$

$$\lambda_+^\xi(T) = \sup_{t \geq 0} \frac{\xi(0, t+T) - \xi(0, t)}{T}, \quad \lambda_-^\xi(T) = \inf_{t \geq 0} \frac{\xi(0, t+T) - \xi(0, t)}{T}$$

$$\lambda_+(T) = \sup(\lambda_+^u(T), \lambda_+^\xi(T)) \quad \text{and} \quad \lambda_-(T) = \inf(\lambda_-^u(T), \lambda_-^\xi(T)).$$

Our goal is to prove that $\lambda_+(T)$ and $\lambda_-(T)$ have a common limit as $T \rightarrow \infty$. We will apply the following classical lemma from ergodic theory (see for instance [60]).

Lemma 3.5. *Let $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a continuous function which is sub-additive, i.e. for all $t, s \geq 0$*

$$V(t + s) \leq V(t) + V(s).$$

Then, $\frac{V(t)}{t}$ has a limit l as $t \rightarrow +\infty$, and

$$l = \inf_{t \geq 0} \frac{V(t)}{t}.$$

From the definition of λ_+^u and λ_+^ξ , we deduce that $T \mapsto T\lambda_+^u(T)$, $T \mapsto T\lambda_+^\xi(T)$, $T \mapsto -T\lambda_-^u(T)$ and $T \mapsto -T\lambda_-^\xi(T)$ are sub-additive. If we can prove that these quantities $\lambda_\pm^u(T), \lambda_\pm^\xi(T)$ are finite then we will know that they converge. It will just remain to prove that $\lambda_+(T)$ and $\lambda_-(T)$ have the same limit.

Step 4.1 : first control of the oscillation in time.

We prove that λ_+ and λ_- are finite.

Lemma 3.6. *For all $T > 0$, we have*

$$-K_1 - \frac{c_2}{T} \leq \lambda_-(T) \leq \lambda_+(T) \leq K_1 + \frac{c_2}{T}. \tag{3.35}$$

where $c_2 = c_1 + 3$

Démonstration. Using (3.31) we get

$$u(y, t) \geq \Delta + py - 2$$

$$\xi(y, t) \geq \Delta + py - 2$$

where $\Delta = \inf(u(0, t), \xi(0, t))$.

Recalling (see Proposition 2.4) that $\lfloor \Delta - 2 \rfloor + py - K_1\tau$ is a sub-solution of (2.8) and using the comparison principle on the interval $[t, t + \tau)$, we deduce that

$$\begin{cases} u(y, t + \tau) \geq \lfloor \Delta - 2 \rfloor + py - K_1\tau \\ \xi(y, t + \tau) \geq \lfloor \Delta - 2 \rfloor + py - K_1\tau. \end{cases} \quad (3.36)$$

We now want to estimate Δ from below. We have

$$\lfloor \Delta - 2 \rfloor \geq \Delta - 3 \geq u(0, t) - 3 - c_1,$$

where c_1 is defined in Lemma 3.3. Similarly for ξ , we have

$$\lfloor \Delta - 2 \rfloor \geq \xi(0, t) - c_1 - 3.$$

Injecting this two estimates in (3.36), we get

$$\begin{cases} u(y, t + \tau) \geq u(0, t) - c_2 + py - K_1\tau \\ \xi(y, t + \tau) \geq \xi(0, t) - c_2 + py - K_1\tau. \end{cases} \quad (3.37)$$

Similarly, we also get

$$\begin{cases} u(y, t + \tau) \leq u(0, t) + c_2 + py + K_1\tau \\ \xi(y, t + \tau) \leq \xi(0, t) + c_2 + py + K_1\tau. \end{cases} \quad (3.38)$$

Taking $y = 0$ we get (3.35). □

Step 4.2 : Refined control of the oscillation in time.

We estimate $\lambda_+ - \lambda_-$ in order to prove that they have a common limit.

Lemma 3.7. *For all $T > 0$ we have*

$$|\lambda_+(T) - \lambda_-(T)| \leq \frac{c_3}{T}$$

with

$$c_3 := 5 + 4c_1 + 2c_2 + 2K_1.$$

Proof of Lemma 3.7. By definition of $\lambda_+(T)$ and $\lambda_-(T)$, for all $\varepsilon > 0$, there exists $t^\pm > 0$ and $v^\pm \in \{u, \xi\}$ such that

$$\left| \lambda_+ - \frac{v^+(0, t^+ + T) - v^+(0, t^+)}{T} \right| \leq \varepsilon, \quad \left| \lambda_- - \frac{v^-(0, t^- + T) - v^-(0, t^-)}{T} \right| \leq \varepsilon$$

We choose $\beta \in [0, 1]$ such that $t^+ - t^- - \beta = k \in \mathbb{Z}$. We set

$$\Delta^u := u(0, t^+) - u(0, t^- + \beta), \quad \Delta^\xi := \xi(0, t^+) - \xi(0, t^- + \beta)$$

and

$$\Delta := \sup(\Delta^u, \Delta^\xi).$$

Using (3.31) we have

$$u(y, t^+) \leq u(y, t^- + \beta) + 4 + \lceil \Delta \rceil$$

and

$$\xi(y, t^+) \leq \xi(y, t^- + \beta) + 4 + \lceil \Delta \rceil.$$

Using the comparison principle for (2.10), we deduce that

$$\begin{cases} u(y, t^+ + T) \leq u(y, t^- + \beta + T) + 4 + \lceil \Delta \rceil \\ \xi(y, t^+ + T) \leq \xi(y, t^- + \beta + T) + 4 + \lceil \Delta \rceil \end{cases} \quad (3.39)$$

We now estimate $\lceil \Delta \rceil$ from above. We have

$$\begin{aligned} \lceil \Delta \rceil &\leq \Delta + 1 \\ &\leq u(0, t^+ + T) - u(0, t^- + \beta + T) + 2c_1 + 1 \end{aligned}$$

where c_1 is defined in Lemma 3.3. Similarly for ξ , we have

$$\lceil \Delta \rceil \leq \xi(0, t^+ + T) - \xi(0, t^- + \beta + T) + 2c_1 + 1. \quad (3.40)$$

We inject these two estimates in (3.39) and get

$$u(y, t^+ + T) \leq u(y, t^- + \beta + T) + 4 + \Delta^u + 2c_1 + 1.$$

Taking $y = 0$ and using (3.37) (with $t = t^-$ and $\tau = \beta$) and (3.38) ($t = t^- + T$ and $\tau = \beta$), we obtain

$$u(0, t^+ + T) - u(0, t^+) \leq u(0, t^- + T) - u(0, t^-) + 4 + 2c_1 + 1 + 2c_2 + 2K_1\beta.$$

Similarly for ξ , we also have

$$\xi(0, t^+ + T) - \xi(0, t^+) \leq \xi(0, t^- + T) - \xi(0, t^-) + 4 + 2c_1 + 1 + 2c_2 + 2K_1\beta.$$

We finally get

$$v^+(0, t^+ + T) - v^+(0, t^+) \leq v^-(0, t^- + T) - v^-(0, t^-) + c_3$$

where

$$c_3 = 5 + 4c_1 + 2c_2 + 2K_1.$$

This implies

$$\lambda_+(T) \leq \lambda_-(T) + \frac{c_3}{T} + 2\varepsilon.$$

□

Step 4.3 : Conclusion. We now conclude that $\lim_{T \rightarrow +\infty} \lambda_{\pm}(T)$ are equal. If λ denotes this common limit, we also have, by Lemma 3.5, that for every $T > 0$

$$\lambda_-(T) \geq \lambda \geq \lambda_+(T).$$

Using Lemma 3.7, we also have

$$\lambda_+(T) \leq \lambda_-(T) + \frac{c_3}{T}$$

and so

$$\lambda_-(T) \leq \lambda \leq \lambda_-(T) + \frac{c_3}{T}.$$

We finally deduce (using a similar argument for λ_+) that

$$|\lambda_{\pm}(T) - \lambda| \leq \frac{c_3}{T}. \quad (3.41)$$

Combining estimates (3.41) and (3.31), we get for $T = t$

$$|u(y, t) - u(0, 0) - py - \lambda t| \leq c_3 + 2$$

and we finally obtain (3.34), with $C_{\delta} = c_3 + 2$.

□

□

Chapitre 5

Conclusion et Perspectives

Le travail de cette thèse a permis de répondre à plusieurs questions en analyse mathématique concernant l'existence et l'unicité d'ondes progressives pour le modèle de Frenkel-Kontorova dans le cas accéléré et en analyse numérique pour le calcul de l'hamiltonien effectif du modèle de Frenkel-Kontorova dans le cas amorti. On a montré dans le chapitre 4 l'existence d'une onde plane pour le problème approché.

Afin de compléter cette étude, il serait intéressant d'étudier l'homogénéisation numérique du modèle de Frenkel-Kontorova dans le cas accéléré à la suite du Chapitre 3. Il serait aussi intéressant de montrer l'existence d'onde plane pour l'équation d'onde à la suite du Chapitre 4 et d'étudier le problème d'homogénéisation associé.

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