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Lamine Sylla

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UNIVERSITE GASTON BERGER DE SAINT LOUIS

Ecole Doctorale des Sciences et des Technologies

UFR de Sciences Appliquées et de Technologie



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ÉQUATIONS DIFFÉRENTIELLES STOCHASTIQUES RÉTROGRADES ET APPLICATIONS

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par M. Lamine SYLLA

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CONTENTS

List of figures	1
Dédicace	2
Remerciements	5
Notations and Abbreviations	8
Résumé et Abstract	11
Synthèse Générale de la thèse en français	12
1 General information on SDE and BSDE	88
1.1 Stochastic differential equations	88
1.1.1 Some remarkable SDE	89
1.1.2 Quadratic variation of standard Brownian motion	91
1.1.3 Link SDE \leftrightarrow Parabolic PDEs (Feynman-Kac formula)	91
1.1.3.1 Parabolic PDE	94
1.2 Stochastic differential equations with jumps	95
1.2.1 Levy process and Poisson random measure	95
1.2.2 Integration with respect to the Poisson random measure μ	97
1.2.3 Integration with respect to the compensated Poisson random measure $\tilde{\mu}$	98
1.2.4 Ito formula for jump processes	98
1.2.5 Stochastic integral of Levy and semimartingale	100
1.2.6 Itô's formula for Levy's stochastic integrals	101
1.2.7 Itô formula for the semi-martingale in the multidimensional case	102

1.3	Existence and uniqueness of solution for stochastic differential equations with jumps	104
1.3.1	The nonlinear case	104
1.3.2	The linear case	105
1.4	General results on BSDE	106
1.4.1	Existence and uniqueness of BSDE solution	106
1.4.1.1	The Pardoux-Peng result	108
1.4.1.2	Picard iteration method	109
1.4.1.3	A priori estimate	109
1.4.2	Comparison theorem	110
1.4.2.1	Linear BSDE	110
1.4.3	BSDE with the non-Lipschitz generator in dimension 1	111
1.5	Links between BSDEs and PDE: viscosity solution	112
1.5.1	The Markovian framework	115
1.5.2	Feynman-Kac formula	117
1.5.3	viscosity solution	118
2	Reflected backward stochastic differential equation with jumps and viscosity solution of second order integro-partial differential equation without monotonicity condition:case with the measure of Lévy infinite	121
2.1	Introduction	121
2.2	Notations and Assumptions	122
2.3	Preliminaires	126
2.3.1	A class of diffusion processes with jumps	126
2.3.2	Existence and uniqueness for a RBSDE with jumps	126
2.3.3	Viscosity solutions of integro-differential partial equation with one obstacle	127
2.4	Estimates and Properties	127
2.5	Main Result	145
2.6	Extension	149
3	BSDEs driven by fractional Brownian motion with time delayed generators	152
3.1	Introduction	152
3.2	Fractional stochastic calculus	154
3.3	FBSDE with time delayed generators	156
3.3.1	Definitions and notations	156
3.3.2	Existence and uniqueness of solution	157
3.3.3	Example	162

4	Generalized Anticipated Backward Doubly Stochastic Differential Equations: with non-Lipschitz Condition	165
4.1	Introduction	165
4.2	Preliminaries	166
4.3	Generalized BDSDEs	169
4.3.1	Existence and uniqueness of solution	170
4.3.2	Comparison Theorem for GBDSDEs	174
4.4	Anticipated GBDSDEs	176
4.4.1	Existence and uniqueness of solution	177
4.4.2	Comparison Theorem for Anticipated GBDSDEs	182
5	Generalized Backward Stochastic Differential Equations driven by two mutually independent fractional Brownian motions	185
5.1	Preliminaries	186
5.1.1	Fractional Stochastic calculus	186
5.1.2	Definitions and notations	188
5.2	Fractional GBSDE	190
5.2.1	Existence and uniqueness of solution	190
5.2.2	Comparison Theorem	194
5.2.3	Connection with partial differential equations	196
	Conclusion	198
	A BSDE simulation	200
		200
	Bibliography	200
	B Research perspectives	203
B.1	Links between BSDE and Optimal Stochastic Control	203
B.2	Links between BSDE and Game Theory	209
	Zero-sum stochastic differential games	209
A	Proofs for some theorems and propositions in the first chapter	213
A.1	The proofs	213

LIST OF FIGURES

1.1	A trajectory of the process of the Black-Scholes-Samuelson equation $\mu = 0.75$ and $\sigma = 0.1$	89
1.2	According to the sign of volatility, the trajectory of the Black-Scholes-Samuelson process	90

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NOTATIONS AND ABBREVIATIONS

\mathbb{R} : the set of real numbers.

\mathbb{R}_+ : the set of positive real numbers.

\mathbb{N} : the set of integer numbers.

$E \times F$: Cartesian product of the sets E and F .

SDE: Stochastic Differential Equation.

BSDE : Backward Stochastic Differential Equation.

RBSDE : Reflected Backward Stochastic Differential Equation.

PDE : Partial Differential Equations.

dt : derivative with respect to t

$r.v$: random variable .

$Cov(X, Y)$: covariance of X and Y .

$\mathbb{E}[X]$: expected value of X .

$Var[X]$: variance of X .

B_t :standard Brownian motion.

W_t : Wiener process.

$\langle \cdot \rangle$: quadratic variation

$\lim_{x \rightarrow x_0}$: limit as x approaches x_0 .

$a.s$: almost surely.

$a.e$: almost everywhere.

\mathcal{F}_t : a filtration.

\mathcal{E} : tribe associated with the random measure of Poisson.

$\int_0^t f(s) ds < \infty$: f is integrable.

$\int_0^t f^2(s) ds < \infty$: f is square integrable.

f' : first derivative.

f'' : second derivative.

$\frac{\partial f}{\partial t}$: differential with respect to the variable t .

$x \cdot y$: scalar product.

H^* : the transpose of H .

∇f : gradient of f .

$C^{1,2}([0, T] \times \mathbb{R})$: C^1 with respect to the first variable and of class C^2 with respect to the second variable.

i.e: identically equivalent.

RCLL : right continuous with left limits .

LCRL : left continuous with right limits .

$(\Omega, \mathcal{F}, \mathbb{P})$: probability space.

$\#E$: The cardinal of the ensemble E .

\otimes : tensorial product.

$\mathbf{1}_A$: indicator on A .

Y^c : Y is continuous.

Y^d : Y is discontinuous.

\langle, \rangle : Hook between two processes.

$\| \cdot \|$: Euclidean norm.

s.a: such at.

iff: if and only if.

$y \in \mathbb{R}^{m \times n}$: will be considered as a matrix of size $m \times n$

$\|y\| = \sqrt{\text{trace}(yy^*)}$: Euclidean norm.

\mathcal{P}_n : the set of processes which are \mathcal{F}_t -progressively measurable from $\Omega \times [0, T]$ to value in \mathbb{R}^n .

$\mathcal{S}_n^2 = \left\{ \varphi \in \mathcal{P}_n \text{ and continuous, s.a. } \mathbb{E}[\sup_{t \leq T} |\varphi_t|^2] < \infty \right\}$.

$\mathbb{L}_T^{2,n} = \left\{ X \in \mathbb{R}^n; \quad X \in \mathcal{F}_T\text{-mesurable; } \|X\|^2 = \mathbb{E}(|X|^2) < \infty \right\}$.

$\mathbb{H}_T^{2,n} = \left\{ \varphi \in \mathcal{P}_n, \varphi_t \in \mathbb{R}^n, \quad \|\varphi\|^2 = \mathbb{E} \left(\int_0^T |\varphi_t|^2 dt \right) < \infty \right\}$.

For $\beta > 0$ and $\varphi \in \mathbb{H}_T^{2,n}$, $\|\varphi\|^2$: represents $\mathbb{E} \left(\int_0^T e^{\beta t} |\varphi_t|^2 dt \right)$

$\mathbb{H}_\beta^{2,n}$: represents the space $\mathbb{H}_T^{2,n}$ endowed with the norm $\|\cdot\|_\beta$.

$\mathbb{H}_T^{1,n} = \left\{ \varphi \in \mathcal{P}_n, \varphi_t \in \mathbb{R}^n, \quad \|\varphi\| = \mathbb{E} \left(\sqrt{\int_0^T |\varphi_t|^2 dt} \right) < \infty \right\}$.

Abstract

We are first interested on viscosity solutions of parabolic Partial Differential Equations through Reflected BSDEs where the existence and uniqueness of viscosity solutions for a class of system of differential integro-partial equations through the RBSDEs with jumps at one barrier.

Next we are interested in other classes of BSDEs, in particular by generalizing the work of Delong, L. and Imkeller, P. [22], by adopting the fractional Brownian motion with the Hurst parameter greater than $\frac{1}{2}$ to establish the existence and uniqueness of solutions for a sufficiently small time horizon or for a sufficiently small Lipschitz constant of a generator using the divergence operator of integral type.

Subsequently, we focus on generalized anticipated BSDEs, where under Mao's conditions, the existence and uniqueness of solutions of this family of BSDEs are proven, in extension of the work of Wu and all [76], who proved the existence and uniqueness of solutions under Lipschitz conditions.

The latest work is based on a class of generalized BSDEs driven by two mutually independent fractional Brownian motions where we show the existence and uniqueness of solutions for this family of BSDEs where in passing the link between the solution of this class of BSDEs and that of Partial Differential Equations (PDE) is established.

Keywords: Integro-partial Differential Equation; Reflected Stochastic Differential Equations with jumps; Viscosity solution; Non-local operator; Fractional Brownian Motion; Backward Stochastic Differential Equation; time delayed generators; Fubini's Theorem; Anticipated Generalized Backward Doubly Stochastic Differential Equation; existence and uniqueness; comparison theorem, non-Lipschitz conditions, Itô's representation formula and Gronwall lemma; Malliavin derivative.

Résumé

Ce travail s'intéresse d'abord aux solutions de viscosité des Équations aux Dérivées Partielles paraboliques à travers les EDSR Réfléchies où l'existence et l'unicité de solutions de viscosité pour une classe de système d'équations intégral-différentielles à travers les EDSRR à sauts à une barrière.

Ensuite l'on s'intéresse à d'autres classes d'EDSRs, notamment en généralisant les travaux Delong, L. et Imkeller, P.[22], en adoptant le mouvement brownien fractionnaire avec le paramètre de Hurst supérieur à $\frac{1}{2}$ pour établir l'existence et l'unicité des solutions pour un horizon temporel suffisamment petit ou pour une constante de Lipschitz suffisamment petite d'un générateur en utilisant l'opérateur de divergence de type intégrale.

Par la suite, l'on se consacre aux EDSRs anticipées généralisées, où sous les conditions de Mao, l'existence et l'unicité de solutions de cette famille des EDSRs sont prouvées, en extension des travaux de Wu and all [76], qui, eux, ont prouvé l'existence et l'unicité des solutions sous les conditions de Lipschitz.

Le dernier travail est basé sur une classe d'EDSRs généralisées dirigées par deux mouvements browniens fractionnaires mutuellement indépendants où l'on montre l'existence et l'unicité de solutions pour cette famille d'EDSRs où au passage le lien entre la solution de cette classe d'EDSRs et celle des Équations aux Dérivées Partielles (EDP) est établi.

SYNTHÈSE GÉNÉRALE DE LA THÈSE EN FRANÇAIS

Cette thèse de Doctorat de l'Université Gaston Berger de Saint-Louis du Sénégal est consacrée à deux grandes familles d'Équations Différentielles Stochastiques: celles dites Rétrogrades Réfléchies avec sauts à une barrière (EDSRR en abrégé) dirigées par un brownien simple et celles des Équations Différentielles Stochastiques Rétrogrades dirigées par un brownien fractionnaire. Il est à noter que nous nous sommes intéressés aux EDSRR à sauts avec une barrière,

$$\left\{ \begin{array}{l} \text{(i)} \quad dY_s^{i;t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i;t,x})_{i=1,m}, Z_s^{i;t,x}, U_s^{i;t,x})ds \\ \quad \quad \quad -dK_s^{i;t,x} + Z_s^{i;t,x}dB_s + \int_{\mathbb{E}} U_s^{i;t,x}(e)\tilde{\mu}(ds, de), \quad s \leq T; \\ \text{(ii)} \quad Y_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i;t,x} - \ell(s, X_s^{t,x}))dK_s^{i;t,x} = 0; \\ \text{(iii)} \quad Y_T^{i;t,x} = g^i(X_T^{t,x}). \end{array} \right. \quad (0.0.1)$$

pour montrer l'existence et l'unicité de la solution des équations aux dérivées partielles intégrales-différentielles suivantes,

$$\left\{ \begin{array}{l} \min \left\{ u^i(t, x) - \ell(t, x); -\partial_t u^i(t, x) - b(t, x)^\top D_x u^i(t, x) \right. \\ \left. - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 u^i(t, x)) - K_i u^i(t, x) - h^{(i)}(t, x, u^i(t, x)), \right. \\ \left. (\sigma^\top D_x u^i)(t, x), B_i u^i(t, x) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ u^i(T, x) = g^i(x); \end{array} \right. \quad (0.0.2)$$

où B_i et K_i sont définis comme suit:

$$\begin{aligned} B_i u^i(t, x) &= \int_E \gamma^i(t, x, e) (u^i(t, x + \beta(t, x, e)) - u^i(t, x)) \lambda(de); \\ K_i u^i(t, x) &= \int_E (u^i(t, x + \beta(t, x, e)) - u^i(t, x) - \beta(t, x, e)^\top D_x u^i(t, x)) \lambda(de). \end{aligned} \quad (0.0.3)$$

L'EDSRR avec saut (0.0.1) ne sera pas le seul outil d'aide à la résolution de l'équation (0.0.2), elle sera couplée aux Équations Différentielles Stochastiques (EDS en abrégé) avec saut,

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r + \int_t^s \int_E \beta(r, X_{r-}^{t,x}, e) \tilde{\mu}(dr, de), \quad (0.0.4)$$

pour $s \in [t, T]$ et $X_s^{t,x} = x$ si $s \leq t$.

L'un des objectifs de cette thèse est donc de résoudre (0.0.2) à travers (0.0.1) et (0.0.4) par abstraction des conditions habituelles (Lipschitz, monotonie sur le générateur f). D'autres travaux ont été abordés avec notamment l'utilisation du brownien fractionnaire pour généraliser le résultat de [22] en établissant l'existence et l'unicité de solution pour la famille d'équations différentielles stochastiques:

$$Y(t) = \xi + \int_t^T f(s, \eta_s, Y_s, Z_s) ds - \int_t^T Z(t) dB_s^H \quad 0 \leq t \leq T, \quad (0.0.5)$$

où B^H est un mouvement brownien fractionnaire avec Paramètre Hurst H supérieur à $1/2$, η est un processus stochastique donné par $\eta_t = \eta_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dB_s^H$, et $(Y_t, Z_t) = (Y(t+u), Z(t+u))_{-T \leq u \leq 0}$. L'intégrale stochastique utilisée dans l'équation ci-dessus est l'intégrale de type divergence. A l'aide du principe du point fixe, nous établissons notre résultat principal.

Partant de ce résultat, nous avons associé les travaux de Fan [28] et de T. Wang et J. Yu [73] pour explorer les équations différentielles doublement stochastiques rétrogrades généralisées anticipées avec condition non Lipschitz où il fallait montrer l'existence et l'unicité de solution pour l'équation différentielle stochastique

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) ds + \int_t^T g(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) dB_s \\ \quad + \int_t^T h(s, Y_s, Y_{s+\delta(s)}) d\Lambda_s - \int_t^T Z_s dW_s, \quad t \in [0, T] \\ Y_t = \xi_t, \quad Z_t = \eta_t, \quad t \in [T, T+K], \end{cases} \quad (0.0.6)$$

où K est une constante positive, $\delta(\cdot), \zeta(\cdot) : [0, T] \rightarrow \mathbb{R}_+$ sont des fonctions positives. En fin nous nous sommes intéressés à l'existence et à l'unicité de solution pour la famille d'équations différentielles stochastiques rétrogrades dirigées par deux browniens

fractionnaires mutuellement indépendants,

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds + \int_t^T g(s, \eta_s, Y_s) d\Lambda_s - \int_t^T Z_{1,s} dB_{1,s}^{H_1} - \int_t^T Z_{2,s} dB_{2,s}^{H_2},$$

$$t \in [0, T],$$

(0.0.7)

où $\{B_{1,t}^{H_1}, t \in [0, T]\}$ et $\{B_{2,t}^{H_2}, t \in [0, T]\}$ sont deux browniens fractionnaires mutuellement indépendants et $\{\Lambda_t, t \in [0, T]\}$ est un processus continu croissant à valeur réelle. La nouveauté dans ce type d'équations stochastiques réside dans le fait qu'il y a la présence d'un nouveau terme dans l'EDSR, à savoir une intégrale par rapport à un processus continu croissant.

Pour rappel les EDSR ont toujours fait l'objet de recherches depuis leur première utilisation en 1973 par Bismut [12] avec un générateur linéaire; après ces travaux, E. Pardoux et S. Peng [57] établissent le premier résultat d'existence et d'unicité de solution dans le cas non linéaire. Depuis, la littérature ne cesse de s'accroître avec notamment les travaux de Barles et al. [8] qui étendent les travaux de Pardoux et Peng aux équations aux dérivées partielles intégrées-différentielles.

En 1997, El Karoui et al. introduisent dans [23] les EDSRR avec une barrière, qui sont des EDSR dont la solution reste au dessus d'une fonction continue qui joue le rôle d'obstacle. Dans [23], les auteurs établissent l'existence et l'unicité de solution en utilisant la méthode de pénalisation ainsi que la méthode d'itération de Picard. Après ces travaux, Hamadène et Ouknine [38] font une généralisation sur les EDSRR à sauts avec une barrière, avec comme nouveauté un bruit composé du mouvement brownien et de la mesure de Poisson compensée [47]. Hamadène dans [37] en collaboration avec Morlais proposent une nouvelle définition de la notion de solution de viscosité différente de celle classique utilisée dans [8, 18] en s'affranchissant de la condition de monotonie habituelle en la première variable du générateur mais maintiennent le fait que la mesure de Lèvy reste finie. En 2016 Hamadène fait une extension de [37] en prenant la mesure de Lèvy infinie et il établit l'existence et l'unicité de solution en faisant une troncation de la mesure de Lèvy.

C'est donc partant de tous ces travaux antérieurs que nous nous sommes intéressés aux EDSR et aux EDSRR à sauts à une barrière.

Durant cette thèse, quatre résultats principaux ont été obtenus avec notamment un papier déjà publié dans lequel nous nous sommes inspirés de [36] en montrant l'existence et l'unicité de solution pour le cas d'EDSRR à sauts à une barrière dans le cas multidimensionnel pour la mesure de Lèvy infinie; il est consultable dans la revue *Acta Mathematica Scientia*, 2019, **39B**(3): pp. 819-844 (voir le lien suivant <https://link.springer.com/article/10.1007/s10473-019-0312-5>).

Notre second résultat est orienté vers le brownien fractionnaire en temps de retard (pub-

lié dans la revue: *Applicable Analysis*): consultable ici: <https://www.tandfonline.com/eprint/PCWXSOGHJKPGXBREVU7J/full?target=10.1080/00036811.2023.2207586>.

Le troisième papier est une orientation aux EDSRs doublement stochastique dites généralisées anticipées pour le cas non Lipschitz.

Le quatrième résultat nous plonge dans les EDSR généralisées dirigées par deux browniens fractionnaires mutuellement indépendants.

Ces deux derniers papiers sont soumis pour publication.

Nous ferons une synthèse de nos différents résultats obtenus durant cette thèse. Ainsi nous allons aborder le premier résultat obtenu.

Pour une meilleure compréhension de nos résultats, nous reviendrons sur les deux types de définition de solution de viscosité (voir [8] pour la définition classique et [36] pour la nouvelle définition); ce qui permettra au lecteur de comprendre le choix de nos hypothèses par rapport à ce qui était connu. De ces définitions, nous exposerons nos hypothèses de travail pour établir nos théorèmes d'existence et d'unicité de solution et nous adopterons les mêmes démarches(hypothèses, définition, théorème d'existence et d'unicité...) pour les autres résultats.

Barles and al.'s: solution de viscosité de l'équation (0.0.2)

Définition 1. *On dit qu'une famille de fonctions déterministes $u = (u^i)_{i=1,m}$ qui est continue $\forall i \in \{1, \dots, m\}$, est une sous-solution de viscosité (resp. super-solution de viscosité) de (0.0.2) si:*

- (i) $\forall x \in \mathbb{R}^k$, $u^i(x, T) \leq g^i(x)$ (resp. $u^i(x, T) \geq g^i(x)$);
- (ii) *Pour tout $(t, x) \in [0, T] \times \mathbb{R}^k$ et toute fonction ϕ de classe $C^{1,2}([0, T] \times \mathbb{R}^k)$ telle que (t, x) est un maximum global de $u^i - \phi$ (resp. minimum global de $u^i - \phi$) et $(u^i - \phi)(t, x) = 0$, on a*

$$\min \begin{cases} u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) \\ -h^i(t, x, (u^j(t, x))_{j=1,m}, \sigma^\top(t, x)) D_x \phi(t, x), B_i \phi(t, x) \end{cases} \leq 0$$

(resp.

$$\min \begin{cases} u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) \\ -h^i(t, x, (u^j(t, x))_{j=1,m}, \sigma^\top(t, x)) D_x \phi(t, x), B_i \phi(t, x)(t, x) \end{cases} \geq 0.$$

La famille $u = (u^i)_{i=1,m}$ est une solution de viscosité de (0.0.2) si elle est à la fois une sous-solution de viscosité et une sur-solution de viscosité.

Notons que $\mathcal{L}^X \phi(t, x) = b(t, x)^\top D_x \phi(t, x) + \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 \phi(t, x)) + K\phi(t, x)$;
où $K\phi(t, x) = \int_{\mathbb{E}} (\phi(t, x + \beta(t, x, e)) - \phi(t, x) - \beta(t, x, e)^\top D_x \phi(t, x)) \lambda(de)$.

Hamadène and al.'s: solution de viscosité de l'équation (0.0.2)

Définition 2. On dit qu'une famille de fonctions déterministes $u = (u^i)_{i=1,m}$, qui appartient à \mathcal{U} (voir [36] pour sa définition), $\forall i \in \{1, \dots, m\}$, est une sous-solution de viscosité (resp. super-solution de viscosité) de (0.0.2) si

- (i) Pour $\forall x \in \mathbb{R}^k$, $u^i(x, T) \leq g^i(x)$ (resp. $u^i(x, T) \geq g^i(x)$);
- (ii) Pour $\forall (t, x) \in [0, T] \times \mathbb{R}^k$ et toute fonction ϕ de classe $C^{1,2}([0, T] \times \mathbb{R}^k)$ tel que (t, x) est un maximum global de $u^i - \phi$ (resp. minimum global de $u^i - \phi$) et $(u^i - \phi)(t, x) = 0$, on a

$$\begin{aligned} & \min \left\{ u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) - h^i(t, x, (u^j(t, x))_{j=1,m}), \right. \\ & \left. \sigma^\top(t, x) D_x \phi(t, x), B_i u^i(t, x) \right\} \leq 0, \\ & \left(\text{resp. } \min \left\{ u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) - h^i(t, x, (u^j(t, x))_{j=1,m}), \right. \right. \\ & \left. \left. \sigma^\top(t, x) D_x \phi(t, x), B_i u^i(t, x) \right\} \geq 0 \right). \end{aligned}$$

La famille $u = (u^i)_{i=1,m}$ est une solution de viscosité de (0.0.2) s'il s'agit à la fois d'une sous-solution de viscosité et d'une super-solution de viscosité.

Notons que

$$\mathcal{L}^X \phi(t, x) = b(t, x)^\top D_x \phi(t, x) + \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 \phi(t, x)) + K\phi(t, x),$$

où

$$K\phi(t, x) = \int_{\mathbb{E}} (\phi(t, x + \beta(t, x, e)) - \phi(t, x) - \beta(t, x, e)^\top D_x \phi(t, x)) \lambda(de).$$

Remarque 1. Différence entre la définition 1 et la définition 2

Nous remarquons la présence $u = (u^i)_{i=1,m}$ pour 2 dans le terme B_i pour $i = 1, m$ donnant ainsi $B_i u^i$ qui n'est rien d'autre que (0.0.3) et ceci fera que nous serons amenés à montrer qu'il est bien défini vu que nous travaillons pour $\lambda = \infty$.

Pour montrer que (0.0.3) est bien défini, nous poserons un certain nombre d'hypothèses qui nous permettront également de résoudre l'équation (0.0.2) en montrant l'existence et l'unicité de solution selon la définition (2)

Résultats généraux du premier papier

Notations et hypothèses

Soit $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ une base stochastique telle que \mathcal{F}_0 contient tous les ensembles \mathbb{P} -null de \mathcal{F} , et $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$, $t \geq 0$, et nous supposons que la filtration est générée par les deux processus mutuellement indépendants:

- (i) $B := (B_t)_{t \geq 0}$ un mouvement brownien d -dimensionnel et
- (ii) une mesure aléatoire de Poisson μ sur $\mathbb{R}^+ \times E$, où $E := \mathbb{R}^\ell - \{0\}$ est muni de sa tribu borélienne \mathcal{E} ($\ell \geq 1$). Le compensateur $\nu(dt, de) = dt\lambda(de)$ est tel que $\{\tilde{\mu}([0, t] \times A) = (\mu - \lambda)([0, t] \times A)\}_{t \geq 0}$ est une martingale pour tout $A \in \mathcal{E}$ satisfaisant $\lambda(A) < \infty$. On suppose aussi que λ est une mesure σ -finie sur (E, \mathcal{E}) , et intègre la fonction $(1 \wedge |e|^2)$ et $\lambda(E) = \infty$.

Introduisons maintenant les espaces suivants:

- (iii) \mathcal{P} (resp. \mathbf{P}) le domaine sur $[0, T] \times \Omega$ de $\mathcal{F}_{t \leq T}$ -ensembles progressivement mesurables (resp. prévisibles).

(iv) Pour $\kappa \geq 1$, $\mathbb{L}_\kappa^2(\lambda)$ l'espace de fonctions mesurables boréliennes $\varphi := (\varphi(e))_{e \in E}$ définies de E dans \mathbb{R}^κ tel que $\|\varphi\|_{\mathbb{L}_\kappa^2(\lambda)}^2 = \int_E |\varphi(e)|_\kappa^2 \lambda(de) < \infty$; $\mathbb{L}_1^2(\lambda)$ sera simplement noté par $\mathbb{L}^2(\lambda)$;

(v) $\mathcal{S}^2(\mathbb{R}^\kappa)$ l'espace des fonctions càdlàg (continue à droite et limitée à gauche) \mathcal{P} -mesurable et à valeur dans \mathbb{R}^κ telles que $\mathbb{E}[\sup_{s \leq T} |Y_s|^2] < \infty$; \mathcal{A}_c^2 est son sous-espace continu non décroissant générant $(K_t)_{t \leq T}$ tel que $K_0 = 0$ et $\mathbb{E}[(K_T)^2] < \infty$;

(vi) $\mathbb{H}^2(\mathbb{R}^{\kappa \times d})$ l'espace des processus $Z := (Z_s)_{s \leq T}$ qui sont \mathcal{P} -mesurables, à valeur dans $\mathbb{R}^{\kappa \times d}$ et satisfaisant $\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty$;

(vii) $\mathbb{H}^2(\mathbb{L}_\kappa^2(\lambda))$ l'espace des processus $U := (U_s)_{s \leq T}$ qui sont \mathbf{P} -mesurables, à valeur dans $\mathbb{L}_\kappa^2(\lambda)$ et satisfaisant $\mathbb{E}\left[\int_0^T \|U_s(\omega)\|_{\mathbb{L}_\kappa^2(\lambda)}^2 ds\right] < \infty$;

(viii) Π_g l'ensemble des fonctions déterministes

$\varpi : (t, x) \in [0, T] \times \mathbb{R}^\kappa \mapsto \varpi(t, x) \in \mathbb{R}$ de croissance polynomiale, c'est-à-dire pour laquelle il existe deux constantes non négatives C et p telles que pour tout $(t, x) \in [0, T] \times \mathbb{R}^\kappa$,

$$|\varpi(t, x)| \leq C(1 + |x|^p).$$

Le sous-espace de Π_g des fonctions continues sera noté Π_g^c ;

(ix) \mathcal{U} la sous-classe de Π_g^c qui se compose des fonctions $\Phi : (t, x) \in [0, T] \times \mathbb{R}^\kappa \mapsto \mathbb{R}$ telles que pour deux constantes non négatives C et p , nous avons

$$|\Phi(t, x) - \Phi(t, x')| \leq C(1 + |x|^p + |x'|^p) |x - x'|, \text{ pour tout } t, x, x'.$$

(x) Pour tout processus $\theta := (\theta_s)_{s \leq T}$ et $t \in (0, T]$, $\theta_{t-} = \lim_{s \nearrow t} \theta_s$ et

$$\Delta_t \theta = \theta_t - \theta_{t-}.$$

Soit maintenant b et σ les fonctions suivantes :

$$b : (t, x) \in [0, T] \times \mathbb{R}^k \mapsto b(t, x) \in \mathbb{R}^k;$$

$$\sigma : (t, x) \in [0, T] \times \mathbb{R}^k \mapsto \sigma(t, x) \in \mathbb{R}^{k \times d}.$$

Nous supposons qu'elles sont conjointement continues en (t, x) et Lipschitz par rapport à x et uniformément en t , c'est-à-dire qu'il existe une constante C telle que,

$$\forall (t, x, x') \in [0, T] \times \mathbb{R}^{k+k}, \quad |b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C |x - x'|. \quad (0.0.8)$$

Notons que par (0.0.8) et par continuité, les fonctions b et σ sont à croissance linéaire, c'est-à-dire qu'il existe une constante C positive telles que

$$\forall (t, x, x') \in [0, T] \times \mathbb{R}^{k+k}, \quad |b(t, x)| + |\sigma(t, x)| \leq C |1 + x|. \quad (0.0.9)$$

Soit $\beta : (t, x, e) \in [0, T] \times \mathbb{R}^k \times E \mapsto \beta(t, x, e) \in \mathbb{R}^k$ soit une fonction mesurable, telle que pour une constante réelle positive C et pour tout $e \in E$, ce qui suit est vrai :

$$(i) \quad |\beta(t, x, e)| \leq C(1 \wedge |e|); \quad (0.0.10)$$

$$(ii) \quad |\beta(t, x, e) - \beta(t, x', e)| \leq C |x - x'| (1 \wedge |e|);$$

(iii) la correspondance $(t, x) \in [0, T] \times \mathbb{R}^k \mapsto \beta(t, x, e) \in \mathbb{R}^k$ est continue pour tout $e \in E$.

Nous allons maintenant présenter les objets qui sont spécifiquement connecté au EDSRR avec sauts, que nous traiterons. Soit ℓ la barrière de (0.0.1); $(g^i)_{i=1,m}$ et $(h^{(i)})_{i=1,m}$ soient deux fonctions définies comme suit : pour $i = 1, \dots, m$,

$$\begin{aligned} g^i : \mathbb{R}^k &\longrightarrow \mathbb{R}^m \\ x &\longmapsto g^i(x) \end{aligned}$$

et

$$\begin{aligned} h^{(i)} : [0, T] \times \mathbb{R}^{k+m+d+1} &\longrightarrow \mathbb{R} \\ (t, x, y, z, q) &\longmapsto h^{(i)}(t, x, y, z, q). \end{aligned}$$

De plus, nous supposons qu'ils satisfont,

(H1) La barrière réfléchissante ℓ est de valeur réelle et \mathcal{P} -processus mesurable satis-

faisant, $\ell \in \mathcal{U}$ c'est-à-dire qu'il est continu et qu'il existe deux constantes positive C et p telles que

$$|\ell(t, x) - \ell(t, x')| \leq C(1 + |x|^p + |x'|^p) |x - x'|, \quad \text{pour tout } t \geq 0, x, x'.$$

(H2) Pour tout $i \in \{1, \dots, m\}$, la fonction g^i appartient à \mathcal{U} .

(H3) Pour tout $i \in \{1, \dots, m\}$,

(i) la fonction $h^{(i)}$ est Lipschitz en (y, z, q) uniformément en (t, x) , c'est-à-dire qu'il existe une constante réelle positive C telle que pour tout $(t, x) \in [0, T] \times \mathbb{R}^k$, (y, z, q) et (y', z', q') éléments de \mathbb{R}^{m+d+1} ,

$$\left| h^{(i)}(t, x, y, z, q) - h^{(i)}(t, x, y', z', q') \right| \leq C(|y - y'| + |z - z'| + |q - q'|); \quad (0.0.11)$$

(ii) la correspondance $(t, x) \mapsto h^{(i)}(t, x, y, z, q)$, pour $(y, z, q) \in \mathbb{R}^{m+d+1}$, fixé appartient uniformément à \mathcal{U} , c'est-à-dire qu'il est continu et il existe des constantes C et p positives (qui ne dépendent de (y, z, q)) telles que

$$\left| h^{(i)}(t, x, y, z, q) - h^{(i)}(t, x', y, z, q) \right| \leq C(1 + |x|^p + |x'|^p) |x - x'|, \quad \text{pour tout } t \geq 0, x, x'. \quad (0.0.12)$$

Notons que d'après [8] de Barles et al, les fonctions $h^i(t, x, (y^i)_{i=1,m}, z, q)$ sont supposées croissantes par rapport à y^k lorsque les autres composants sont fixes. Mais cette hypothèse n'intervient pas dans cet article.

Ensuite, soit γ^i , $i = 1, \dots, m$ une famille Borélienne mesurable définie de $[0, T] \times \mathbb{R}^k \times E$ dans \mathbb{R} et satisfaisant aux conditions suivantes :

- (i) $|\gamma^i(t, x, e)| \leq C(1 \wedge |e|)$;
- (ii) $|\gamma^i(t, x, e) - \gamma^i(t, x', e)| \leq C(1 \wedge |e|) |x - x'| (1 + |x|^p + |x'|^p)$; (0.0.13)
- (iii) la correspondance $t \in [0, T] \mapsto \gamma^i(t, x, e) \in \mathbb{R}$ est continu pour tout $(x, e) \in \mathbb{R}^k \times E$.

Enfin, nous introduisons les fonctions suivantes $(f^{(i)})_{i=1,m}$ définies par

$$\forall (t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^{k+m+d} \times \mathbb{L}^2(\lambda),$$

$$f^{(i)}(t, x, y, z, \zeta) := h^{(i)} \left(t, x, y, z, \int_E \gamma^i(t, x, e) \zeta(e) \lambda(de) \right). \quad (0.0.14)$$

Les fonctions $(f^{(i)})_{i=1,m}$, bénéficient des deux propriétés suivantes :

(a) La fonction $f^{(i)}$ est Lipschitz en (y, z, ζ) uniformément en (t, x) , c'est-à-dire

qu'il existe une constante réelle positive C telle que

$$\left| f^{(i)}(t, x, y, z, \zeta) - f^{(i)}(t, x, y', z', \zeta') \right| \leq C(|y - y'| + |z - z'| + \|\zeta - \zeta'\|_{\mathbb{L}^2(\lambda)}), \quad (0.0.15)$$

car $h^{(i)}$ est uniformément Lipschitz en (y, z, q) et γ^i vérifie (0.0.11)-(i);

(b) La fonction $(t, x) \in [0, T] \times \mathbb{R}^k \mapsto f^{(i)}(t, x, 0, 0, 0)$ appartient à Π_g^c , et puis

$$\mathbb{E} \left[\int_0^T \left| f^{(i)}(r, X_r^{t,x}, 0, 0, 0) \right|^2 dr \right] < \infty.$$

Préliminaires

Une classe de processus de diffusion avec sauts

Soit $(t, x) \in [0, T] \times \mathbb{R}^d$ et $(X_s^{t,x})_{s \leq T}$ la solution de processus stochastique de (0.0.4). Sous hypothèses (0.0.8)–(0.0.10), la solution de l'équation (0.0.4) existe et est unique (voir [30] pour plus de détails). Nous énonçons quelques propriétés du processus $\{(X_s^{t,x}), s \in [0, T]\}$, qui se trouve dans [30].

Proposition 1. *Pour tout $t \geq 0$, il existe une version (voir [16] page 1 définition 3) de $\{(X_s^{t,x}), s \in [t, T]\}$ telle que $s \rightarrow X_s^t$ est un processus càdlàg (continu à droite et limité à gauche) et est $C^2(\mathbb{R}^d)$. De plus, il satisfait les estimations suivantes : $\forall p \geq 2, x, x' \in \mathbb{R}^d$ et $s \geq t$,*

$$\mathbb{E} \left[\sup_{t \leq r \leq s} \left| X_r^{t,x} - x \right|^p \right] \leq M_p(s - t)(1 + |x|^p), \quad (0.0.16)$$

$$\mathbb{E} \left[\sup_{t \leq r \leq s} \left| X_r^{t,x} - X_r^{t,x'} - (x - x') \right|^p \right] \leq M_p(s - t)(|x - x'|^p); \quad (0.0.17)$$

où M_p est une constante réelle.

Existence et unicité de solution pour une EDSRR avec sauts

Soit $(t, x) \in [0, T] \times \mathbb{R}^d$ et nous considérons l'EDSRR m -dimensionnel avec sauts suivant :

$$\left\{ \begin{array}{l} \text{(i)} \quad \vec{Y}^{t,x} := (Y^{i,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad Z^{t,x} := (Z^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad K^{t,x} := (K^{i,t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad U^{t,x} := (U^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} \quad dY_s^{i,t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i,t,x})_{i=1,m}, Z_s^{i,t,x}, U_s^{i,t,x}) ds - dK_s^{i,t,x} \\ \quad \quad \quad + Z_s^{i,t,x} dB_s + \int_{\mathbb{E}} U_s^{i,t,x}(e) \tilde{\mu}(ds, de), \quad s \leq T; \\ \text{(iii)} \quad Y_s^{i,t,x} \geq \ell(s, X_s^{t,x}) \text{ et } \int_0^T (Y_s^{i,t,x} - \ell(s, X_s^{t,x})) dK_s^{i,t,x} = 0, \end{array} \right. \quad (0.0.18)$$

où pour tout $i \in \{1, \dots, m\}$, $Z_s^{i;t,x}$ est la i ème ligne de $Z_s^{t,x}$, $K_s^{i;t,x}$ est le i ème composant de $K_s^{t,x}$, et $U_s^{i;t,x}$ est le i ème composant de $U_s^{t,x}$.

Le résultat suivant est lié à l'existence et à l'unicité de solution pour l'EDSRR avec sauts (0.0.18).

Sa preuve est donnée dans [38] en utilisant la méthode de pénalisation (voir p.5–12) et la méthode de l'enveloppe de Snell (voir p.14–16).

Proposition 2. *Supposons que les hypothèses (H1), (H2) et (H3) soient vérifiées. Pour tout $(t, x) \in [0, T] \times \mathbb{R}^d$, l'EDSRR (0.0.18) a une solution unique $(\vec{Y}^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x})$.*

Remarque 2. *La solution de cette EDSRR avec sauts existe et est unique en raison des éléments suivants : (i) $\mathbb{E}\left[|g(X_T^{t,x})|^2\right] < \infty$, par croissance polynomiale de g et l'estimation (0.0.17) sur $X^{t,x}$;*

(ii) pour tout $i = 1, \dots, m$, $f^{(i)}$ vérifie les propriétés (a)–(b) liées à l'uniforme Lipschitz (y, z, ζ) et $ds \otimes d\mathbb{P}$ -carré intégrable du processus $(f^{(i)}(s, X_s^{t,x}, 0, 0, 0))_{s \leq T}$.

Solutions de viscosité de l'équation aux dérivées partielles intégral-différentielle à un obstacle

Le résultat suivant sur un obstacle est prouvé avec deux obstacles dans Harraj et al (voir [42], Théorème 4.6. p.47 en utilisant (4.1) p.44), et il établit la relation entre la solution de (0.0.18) et celui du système (0.0.2).

Proposition 3. *Supposons que (H1), (H2) et (H3) soient remplies. Alors, il existe des fonctions continues déterministes $(u^i(t, x))_{i=1,m}$ qui appartiennent à Π_g telles que pour tout $(t, x) \in [0, T] \times \mathbb{R}^k$, la solution de l'EDSRR (0.0.18) vérifie que*

$$\forall i \in \{1, \dots, m\}, \forall s \in [t, T], Y_s^{i;t,x} = u^i(s, X_s^{t,x}). \quad (0.0.19)$$

De plus, si pour tout $i \in \{1, \dots, m\}$,

(i) $\gamma^i \geq 0$, et

(ii) pour tout (t, x, \vec{y}, z) fixé $\in [0, T] \times \mathbb{R}^{k+m+d}$, la correspondance $(q \in \mathbb{R}) \mapsto h^{(i)}(t, x, \vec{y}, z, q) \in \mathbb{R}$ est non décroissante, la fonction $(u^i)_{i=1,m}$ est une solution viscosité continue de (0.0.2) (au sens de Barles et al). Pour la preuve, voir [42] en appliquant la même méthode.

Enfin, la solution $(u^i)_{i=1,m}$ de (0.0.2) est unique dans la classe Π_g^c .

Remarque 3 (voir [42]). *Sous les hypothèses (H1), (H2) et (H3), il existe une solution de viscosité unique de (0.0.2) dans la classe de fonctions satisfaisant,*

$$\lim_{|x| \rightarrow +\infty} |u(t, x)| e^{-\tilde{A}[\log(|x|)]^2} = 0 \quad (0.0.20)$$

uniformément pour $t \in [0, T]$, pour $\tilde{A} > 0$.

Dans les sections à venir nous établissons nos premiers résultats de recherches.

Estimations et propriétés

Dans cette section, nous fournissons une estimation pour les fonctions $(u^i)_{i=1,m}$ définies dans (0.0.19).

Rappelons que $(\vec{Y}^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x}) := ((Y^{i;t,x})_{i=1,m}, (Z^{i;t,x})_{i=1,m}, (U^{i;t,x})_{i=1,m}, (K^{i;t,x})_{i=1,m})$ est l'unique solution de l'EDSRR avec sauts (0.0.18).

Lemme 1. *Sous les hypothèses (H1), (H2) et (H3), pour tout $p \geq 2$, il existe deux constantes non négatives C et ρ telles que*

$$\mathbb{E} \left[\left\{ \int_0^T ds \left(\int_E |U_s^{t,x}(e)|^2 \lambda(de) \right) \right\}^{\frac{p}{2}} \right] = \mathbb{E} \left[\left\{ \int_0^T ds \|U_s^{t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}} \right] \leq C (1 + |x|^\rho). \quad (0.0.21)$$

Ce lemme nous permettra de montrer que la famille $(B_i u^i)_{i=1,m}$ est bien définie. Sa preuve repose principalement sur la formule d'Itô en passant également par l'inégalité de BGD et le lemme de Gronwall. Nous proposons une sketch de la preuve.

Sketch de la preuve 1. *Rappelons d'abord que comme $X_s^{t,x} = x$ pour $s \in [0, t]$, alors l'unicité de la solution de l'EDSRR de (0.0.18) implique que*

$$Z_s^{t,x} = U_s^{t,x} = K_s^{t,x} = 0, ds \otimes d\mathbf{P}\text{-p.p. sur } [0, t] \times \Omega. \quad (0.0.22)$$

Ensuite, fixons $p \geq 2$. En utilisant la représentation (0.0.19), pour tout $i \in \{1, \dots, m\}$ et $s \in [t, T]$, nous avons

$$\begin{aligned} Y_s^{i;t,x} &= g^i(X_T^{t,x}) + \int_s^T f^{(i)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i;t,x}, U_r^{i;t,x}) dr + K_T^{i;t,x} - K_s^{i;t,x} \\ &\quad - \int_s^T Z_r^{i;t,x} dB_r - \int_s^T \int_E U_r^{i;t,x}(e) \tilde{\mu}(dr, de). \end{aligned} \quad (0.0.23)$$

Cela implique que le système d'EDSRR avec des sauts (0.0.18) se transforme en un système découplé car les équations dans (0.0.23) ne sont pas liées les unes aux autres.

Ensuite, pour tout $i = 1, \dots, m$, les fonctions u^i , g^i et $(t, x) \mapsto f^{(i)}(t, x, 0, 0, 0)$ sont à croissance polynomiale et finalement $y \mapsto f^{(i)}(t, x, y, 0, 0)$ est Lipschitz uniformément par rapport (t, x) . Alors, pour C et $\rho \geq 0$, nous avons

$$\mathbb{E} \left[\left| g^i(X_T^{t,x}) \right|^p + \left(\int_s^T \left| f^{(i)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, 0, 0) \right|^2 dr \right)^{\frac{p}{2}} \right] \leq C (1 + |x|^\rho). \quad (0.0.24)$$

Fixons maintenant $i_0 \in \{1, \dots, m\}$, puis pour $\forall s \in [t, T]$,

$$\left\{ \begin{array}{l} \text{(i)} \quad Y_s^{i_0, t, x} \in \mathcal{S}^2(\mathbb{R}), \quad Z_s^{i_0, t, x} \in \mathbb{H}^2(\mathbb{R}^d), \quad K_s^{i_0, t, x} \in \mathcal{A}_c^2, \quad U_s^{i_0, t, x} \in \mathbb{H}^2(\mathbb{L}^2(\lambda)); \\ \text{(ii)} \quad Y_s^{i_0, t, x} = g^{i_0}(X_T^{t, x}) + \int_s^T f^{(i_0)}(r, X_r^{t, x}, (u^j(X_r^{t, x}))_{j=1, m}, Z_r^{i_0, t, x}, U_r^{i_0, t, x}) dr \\ \quad \quad \quad + K_T^{i_0, t, x} - K_s^{i_0, t, x} - \int_s^T Z_r^{i_0, t, x} dB_r - \int_s^T \int_{\mathbb{E}} U_r^{i_0, t, x}(e) \tilde{\mu}(dr, de); \\ \text{(iii)} \quad Y_s^{i_0, t, x} \geq \ell(s, X_s^{t, x}) \text{ et } \int_0^T (Y_s^{i_0, t, x} - \ell(s, X_s^{t, x})) dK_s^{i_0, t, x} = 0. \end{array} \right.$$

En appliquant la formule d'Itô à $|Y_s^{i_0, t, x}|^2$ entre s et T , nous avons

$$|Y_s^{i_0, t, x}|^2 + \int_s^T |Z_r^{i_0, t, x}|^2 dr + \sum_{s \leq r \leq T} (\Delta Y_r^{i_0, t, x})^2 \quad (0.0.25)$$

$$\begin{aligned} &= |g^{i_0}(X_T^{t, x})|^2 + 2 \int_s^T Y_r^{i_0, t, x} f^{(i_0)}(r, X_r^{t, x}, (u^j(X_r^{t, x}))_{j=1, m}, Z_r^{i_0, t, x}, U_r^{i_0, t, x}) dr \quad (0.0.26) \\ &\quad + 2 \int_s^T Y_r^{i_0, t, x} dK_r^{i_0, t, x} - 2 \int_s^T \int_{\mathbb{E}} Y_r^{i_0, t, x} U_r^{i_0, t, x} \tilde{\mu}(dr, de) - 2 \int_s^T Y_r^{i_0, t, x} Z_r^{i_0, t, x} dB_r. \end{aligned}$$

Notons que $Y_r^{i_0, t, x} = u^{i_0}(r, X_r^{t, x})$ et nous avons $|u^{i_0}(r, X_r^{t, x})| \leq C(1 + |X_r^{t, x}|^q)$. Définissons $\Sigma = 1 + \sup_{s \leq T} |X_s^{t, x}|$. Par conséquent, $|Y_r^{i_0, t, x}| \leq C_q \Sigma^q$ et $|g^{i_0}(X_T^{t, x})| \leq C_q \Sigma^q$. En élevant à la puissance $\frac{p}{2}$ puis en prenant l'espérance, il résulte de (0.0.25) et du fait qu'il existe $C > 0$ tel que

$$\mathbb{E} \left[\left(\int_s^T \|U_r^{i_0, t, x}\|_{\mathbb{L}^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right] \leq C \mathbb{E} \left[\left(\sum_{s \leq r \leq T} (\Delta Y_r^{i_0, t, x})^2 \right)^{\frac{p}{2}} \right],$$

(voir [51] p.28-45),

$$\begin{aligned} &\mathbb{E} [|Y_s^{i_0, t, x}|^p] + \mathbb{E} \left[\left(\int_s^T |Z_r^{i_0, t, x}|^2 dr \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[\left(\int_s^T \|U_r^{i_0, t, x}\|_{\mathbb{L}^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right] \\ &\leq 5^{\frac{p}{2}-1} \mathbb{E} [|g^{i_0}(X_T^{t, x})|^p] \quad (0.0.27) \\ &\quad + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left(\int_s^T 2 |Y_r^{i_0, t, x} f^{(i_0)}(r, X_r^{t, x}, (u^j(X_r^{t, x}))_{j=1, m}, Z_r^{i_0, t, x}, U_r^{i_0, t, x})| dr \right)^{\frac{p}{2}} \right] \\ &\quad + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left| \int_s^T Y_r^{i_0, t, x} dK_r^{i_0, t, x} \right|^{\frac{p}{2}} \right] \\ &\quad + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left| \int_s^T \int_{\mathbb{E}} 2 Y_r^{i_0, t, x} U_r^{i_0, t, x} \tilde{\mu}(dr, de) \right|^{\frac{p}{2}} \right] + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left| \int_s^T 2 Y_r^{i_0, t, x} Z_r^{i_0, t, x} dB_r \right|^{\frac{p}{2}} \right]. \end{aligned}$$

Pour plus de compréhension, nous adoptons l'écriture suivante pour l'inégalité (0.0.27);

$$\begin{aligned} & \mathbb{E} \left[|Y_s^{i_0,t,x}|^p \right] + \mathbb{E} \left[\left(\int_s^T |Z_r^{i_0,t,x}|^2 dr \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[\left(\int_s^T \|U_r^{i_0,t,x}\|_{\mathbb{L}^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right] \\ & \leq 5^{\frac{p}{2}-1} \mathbb{E} \left[|g^{i_0}(X_T^{t,x})|^p \right] + 5^{\frac{p}{2}-1} T_1(s) + 5^{\frac{p}{2}-1} T_2(s) + 5^{\frac{p}{2}-1} T_3(s) + 5^{\frac{p}{2}-1} T_4(s) \end{aligned} \quad (0.0.28)$$

où

$$\begin{aligned} T_1(s) &= \mathbb{E} \left[\left(\int_s^T 2 \left| Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) \right| dr \right)^{\frac{p}{2}} \right]; \\ T_2(s) &= \mathbb{E} \left[\left| \int_s^T Y_r^{i_0,t,x} dK_r^{i_0,t,x} \right|^{\frac{p}{2}} \right]; \\ T_3(s) &= \mathbb{E} \left[\left| \int_s^T \int_{\mathbb{E}} 2Y_r^{i_0,t,x} U_r^{i_0,t,x} \tilde{\mu}(dr, de) \right|^{\frac{p}{2}} \right]; \\ T_4(s) &= \mathbb{E} \left[\left| \int_s^T 2Y_r^{i_0,t,x} Z_r^{i_0,t,x} dB_r \right|^{\frac{p}{2}} \right]. \end{aligned}$$

Nous allons estimer $T_1(s)$, $T_2(s)$, $T_3(s)$, et $T_4(s)$, $\forall s \in [t, T]$.

(a) Avant de commencer nos estimations, linéarisons f par rapport à $(u^j(X_r^{t,x}))_{j=1,m}$ et $Z_r^{i_0,t,x}$ c'est-à-dire,

$$\begin{aligned} & f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) \\ &= a_r^{t,x} Z_r^{i_0,t,x} + b_r^{t,x} (u^j(X_r^{t,x}))_{j=1,m} + f^{(i_0)}(r, X_r^{t,x}, 0, 0, U_r^{i_0,t,x}), \end{aligned}$$

où $a_r^{t,x}$ et $b_r^{t,x}$ sont des processus progressivement mesurables respectivement bornés par les constantes de Lipschitz de f en $Z_r^{t,x}$ et en $(u^j(X_r^{t,x}))_{j=1,m}$;

$$|a_r^{t,x}| \leq C_Z \text{ et } |b_r^{t,x}| \leq \lambda_1.$$

(b) Nous prenons aussi le fait que f est Lipschitz en $U_r^{i_0,t,x}$ c'est-à-dire qu'il existe une constante λ_2 telle que

$$|f^{(i_0)}(r, X_r^{t,x}, 0, 0, U_r^{i_0,t,x})| \leq |f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0)|$$

En combinant (a) et (b), nous avons

$$\begin{aligned} & |f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x})| \\ & \leq |a_r^{t,x} Z_r^{i_0,t,x}| + |b_r^{t,x} (u^j(X_r^{t,x}))_{j=1,m}| + |f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0)| + \lambda_2 \|U_r^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}. \end{aligned} \quad (0.0.29)$$

Commençons nos estimations.

Pour $T_1(s)$

En utilisant (0.0.29) et en élevant à la puissance $\frac{p}{2}$ puis en prenant l'espérance, il s'ensuit que

$$\begin{aligned}
T_1(s) &\leq 8^{\frac{p}{2}-1} C C_q^p \left\{ (C_Z T \epsilon_1^{-1})^{\frac{p}{2}} + (\lambda_1 T \epsilon_2^{-1})^{\frac{p}{2}} + C_q^p (\epsilon_2 \lambda_1 T)^{\frac{p}{2}} + (T \lambda_2 \epsilon_3^{-1})^{\frac{p}{2}} + (T \epsilon_4^{-1})^{\frac{p}{2}} \right. \\
&\quad \left. + (T \epsilon_4)^{\frac{p}{2}} \right\} |1 + |x|^{pq}| + 8^{\frac{p}{2}-1} (\epsilon_1 C_Z)^{\frac{p}{2}} \mathbb{E} \left[\left(\int_s^T |Z_r^{i_0, t, x}|^2 dr \right)^{\frac{p}{2}} \right] \\
&\quad + 8^{\frac{p}{2}-1} (\lambda_2 \epsilon_3)^{\frac{p}{2}} \mathbb{E} \left[\left(\int_s^T \|U_r^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right]. \tag{0.0.30}
\end{aligned}$$

Avant d'estimer $T_2(s)$, donnons d'abord une estimation de $\mathbb{E}[|K_T^{i_0, t, x} - K_s^{i_0, t, x}|^p]$, qui nous servira à estimer $T_2(s)$.

$$\begin{aligned}
K_T^{i_0, t, x} - K_s^{i_0, t, x} &= Y_s^{i_0, t, x} - g^{i_0}(X_T^{t, x}) - \int_s^T f^{(i_0)}(r, X_r^{t, x}, (u^j(X_r^{t, x}))_{j=1, m}, Z_r^{i_0, t, x}, U_r^{i_0, t, x}) dr \\
&\quad + \int_s^T Z_r^{i_0, t, x} dB_r + \int_s^T \int_{\mathbb{E}} U_r^{i_0, t, x}(e) \tilde{\mu}(dr, de).
\end{aligned}$$

Par (0.0.29) et l'inégalité de Cauchy-Schwartz, il s'ensuit que

$$\begin{aligned}
|K_T^{i_0, t, x} - K_s^{i_0, t, x}| &\leq 2C_q \Sigma^q + C_q \lambda_1 T \Sigma^q + C_q T \Sigma^q + C_Z \left(\int_s^T |Z_r^{i_0, t, x}|^2 dr \right)^{\frac{1}{2}} \\
&\quad + \lambda_2 \left(\int_s^T \|U_r^{i_0, t, x}\|^2 dr \right)^{\frac{1}{2}} + \int_s^T Z_r^{i_0, t, x} dB_r + \int_s^T \int_{\mathbb{E}} U_r^{i_0, t, x}(e) \tilde{\mu}(dr, de).
\end{aligned}$$

En élevant à la puissance p , et en prenant l'espérance et l'inégalité BDG, nous avons,

$$\begin{aligned}
\mathbb{E} \left[|K_T^{i_0, t, x} - K_s^{i_0, t, x}|^p \right] &\leq 5^{p-1} C C_q^p \left\{ 2^p + (T \lambda_1)^p + T^p \right\} |1 + |x|^{pq}| \\
&\quad + 5^{p-1} (C_Z^p + C_p) \mathbb{E} \left(\int_0^T |Z_s^{i_0, t, x}|^2 ds \right)^{\frac{p}{2}} \\
&\quad + 5^{p-1} (\lambda_2^p + C_p) \mathbb{E} \left\{ \int_0^T ds \|U_s^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}}. \tag{0.0.31}
\end{aligned}$$

Pour $T_2(s)$

$$\begin{aligned}
\int_s^T |Y_s^{i_0,t,x}| dK_s^{i_0,t,x} &\leq \int_s^T |(Y_s^{i_0,t,x} - \ell(s, X_s^{t,x}))| dK_s^{i_0,t,x} + \int_s^T |\ell(s, X_s^{t,x})| dK_s^{i_0,t,x} \\
&\leq \sup_{s \leq T} |\ell(s, X_s^{t,x})| K_T^{i_0,t,x} \\
&\leq \epsilon_5^{-1} \sup_{s \leq T} |\ell(s, X_s^{t,x})|^2 + \epsilon_5 (K_T^{i_0,t,x})^2 \\
&\leq \epsilon_5^{-1} C_q^2 \Sigma^{2q} + \epsilon_5 (K_T^{i_0,t,x})^2.
\end{aligned} \tag{0.0.32}$$

En utilisant (0.0.31) et (0.0.32), il s'ensuit que

$$\begin{aligned}
T_2(s) &\leq 2^{\frac{p}{2}-1} C C_q^p (\epsilon_5^{-1})^{\frac{p}{2}} |1 + |x|^{pq}| + 2^{\frac{p}{2}-1} (\epsilon_5)^{\frac{p}{2}} \mathbb{E} \left[(K_T^{i_0,t,x})^p \right] \\
&\leq 2^{\frac{p}{2}-1} C C_q^p (\epsilon_5^{-1})^{\frac{p}{2}} |1 + |x|^{pq}| + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} C C_q^p \{2^p + (T\lambda_1)^p + T^p\} |1 + |x|^{pq}| \\
&\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (C_Z^p + C_p) \mathbb{E} \left(\int_0^T |Z_s^{i_0,t,x}|^2 ds \right)^{\frac{p}{2}} \\
&\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (\lambda_2^p + C_p) \mathbb{E} \left\{ \int_0^T ds \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}} \\
&\leq \left\{ 2^{\frac{p}{2}-1} C C_q^p (\epsilon_5^{-1})^{\frac{p}{2}} + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} C C_q^p (2^p + (T\lambda_1)^p + T^p) \right\} (1 + |x|^{pq}) \\
&\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (C_Z^p + C_p) \mathbb{E} \left(\int_0^T |Z_s^{i_0,t,x}|^2 ds \right)^{\frac{p}{2}} \\
&\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (\lambda_2^p + C_p) \mathbb{E} \left\{ \int_0^T ds \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}}.
\end{aligned}$$

Pour $T_3(s)$

Par l'inégalité de BDG, nous avons,

$$\begin{aligned}
T_3(s) &\leq C_p \mathbb{E} \left(\int_0^T |Y_s^{i_0,t,x}|^2 |Z_s^{i_0,t,x}|^2 ds \right)^{\frac{p}{4}} \\
&\leq C_p \mathbb{E} \left(\sup_{s \leq T} |Y_s^{i_0,t,x}|^2 \int_0^T |Z_s^{i_0,t,x}|^2 ds \right)^{\frac{p}{4}} \\
&\leq C_p C_q^p \epsilon_6^{-1} (1 + |x|^{pq}) + C_p \epsilon_6 \mathbb{E} \left(\int_0^T |Z_s^{i_0,t,x}|^2 ds \right)^{\frac{p}{2}}.
\end{aligned} \tag{0.0.33}$$

Pour $T_4(s)$

Par l'inégalité de BDG, nous obtenons,

$$\begin{aligned}
T_4(s) &\leq C_p \mathbb{E} \left(\int_0^T |Y_s^{i_0,t,x}|^2 \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 ds \right)^{\frac{p}{4}} \\
&\leq C_p \mathbb{E} \left(\sup_{s \leq T} |Y_s^{i_0,t,x}|^2 \int_0^T \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 ds \right)^{\frac{p}{4}} \\
&\leq C_p C_q^p \epsilon_7^{-1} (1 + |x|^{pq}) + C_p \epsilon_7 \mathbb{E} \left(\int_0^T \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 ds \right)^{\frac{p}{2}}. \quad (0.0.34)
\end{aligned}$$

Finalemnt, en prenant l'estimation de $T_1(s)$, $T_2(s)$, $T_3(s)$, et $T_4(s)$, et en choisissant ϵ_1 , ϵ_2 , ϵ_3 , ϵ_4 , ϵ_5 , ϵ_6 , et ϵ_7 tels que $\{(\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (C_Z^p + C_p) + C_p \epsilon_6\} < 1$, $\{(\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (\lambda_2^p + C_p) + C_p \epsilon_7\} < 1$, et la somme de tous les coefficients de $(1 + |x|^{pq})$ est plus petite que 1.

Il s'ensuit que

$$\mathbb{E} \left[\left\{ \int_0^T ds \|U_s^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}} \right] \leq C (1 + |x|^\rho), \quad (0.0.35)$$

où $\rho = pq$.

Enfin, comme $i_0 \in \{1, \dots, m\}$ est arbitraire, nous obtenons alors l'estimation (0.0.21).

Proposition 4. Pour tout $i = 1, \dots, m$, u^i appartenant à \mathcal{U} .

Sketch de la preuve 2. Soient x et x' des éléments de \mathbb{R}^k . Soit $(\vec{Y}^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x})$ (resp. $(\vec{Y}^{t,x'}, Z^{t,x'}, U^{t,x'}, K^{t,x'})$) solution de l'EDSR à sauts (0.0.18) associée à $f(s, X_s^{t,x}, y, \eta, \zeta, g(X_T^{t,x}))$ (resp. $f(s, X_s^{t,x'}, y, \eta, \zeta, g(X_T^{t,x'}))$). En appliquant la formule d'Itô à $|\vec{Y}^{t,x} - \vec{Y}^{t,x'}|^2$ entre s et T et en prenant l'espérance, nous obtenons le résultat suivant pour $\forall s \in [t, T]$,

$$\begin{aligned}
&\mathbb{E} \left[\left| \vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'} \right|^2 + \int_s^T |\Delta Z_r|^2 dr + \int_s^T \|\Delta U_r\|_{\mathbb{L}^2(\lambda)}^2 dr \right] \quad (0.0.36) \\
&\leq \mathbb{E} \left[\left| g(X_T^{t,x}) - g(X_T^{t,x'}) \right|^2 + 2 \int_s^T \langle (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) , \Delta f(r) \rangle dr \right] \\
&\quad + \mathbb{E} \left[2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \right],
\end{aligned}$$

où les processus ΔX_r , ΔY_r , $\Delta f(r)$, ΔK_r , ΔZ_r , ΔU_r et $\Delta \ell_r$ sont définis comme suit: $\forall r \in [t, T]$,

$$\begin{aligned}
\Delta f(r) &:= ((\Delta f^{(i)}(r))_{i=1,m} = (f^{(i)}(r, X_r^{i;t,x}, \vec{Y}_r^{i;t,x}, Z_r^{i;t,x}, U_r^{i;t,x}) \\
&\quad - f^{(i)}(r, X_r^{i;t,x'}, \vec{Y}_r^{i;t,x'}, Z_r^{i;t,x'}, U_r^{i;t,x'}))_{i=1,m},
\end{aligned}$$

$$\begin{aligned}\Delta X_r &= X_r^{t,x} - X_r^{t,x'}, \Delta Y(r) = \vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'} = (Y_r^{j;t,x} - Y_r^{j;t,x'})_{j=1,m}, \\ \Delta K_r &= K_r^{t,x} - K_r^{t,x'}, \Delta Z_r = Z_r^{t,x} - Z_r^{t,x'}, \Delta U_r = U_r^{t,x} - U_r^{t,x'}\end{aligned}$$

et

$$\Delta \ell_r = \left(\ell(r, X_r^{t,x}) - \ell(r, X_r^{t,x'}) \right)$$

($\langle \cdot, \cdot \rangle$ est le produit scalaire usuel sur \mathbb{R}^m).

Maintenant, nous allons donner une estimation de chacun des trois termes du deuxième membre de l'inégalité (0.0.36). Par l'inégalité triangulaire et le fait que $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, l'inégalité de Cauchy-Schwartz, les relations (0.0.16) et (0.0.17) de la proposition 1

$$\bullet \quad \mathbb{E} \left[\left| g(X_T^{t,x}) - g(X_T^{t,x'}) \right|^2 \right] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}), \quad (0.0.37)$$

$$\bullet \quad \text{Par (iii) de (0.0.18) : } \mathbb{E} \left[2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \right] \quad (0.0.38)$$

peut être remplacée par

$$\mathbb{E} \left[2 \int_s^T (\ell(r, X_r^{t,x}) - \ell(r, X_r^{t,x'})) d(\Delta K_r) \right].$$

Par linéarité de l'intégrale et en utilisant (iii) de (0.0.18) nous obtenons notre résultat.

Maintenant, par (H1) et l'inégalité de Cauchy-Schwartz, nous obtenons,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} (\Delta \ell_t)^2 \right] \leq 2CC' |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}), \quad (0.0.39)$$

où $C' = \mathbb{E} [(\Delta K_T)^2]$.

• Pour compléter notre estimation de (0.0.36), nous devons traiter

$$\mathbb{E} \left[2 \int_s^T \langle (\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}), \Delta f(r) \rangle dr \right].$$

En tenant compte de l'expression de $f^{(i)}$ donnée par (0.0.14), nous décomposons alors

$\Delta f(r)$ de la manière suivante : pour $r \leq T$,

$$\begin{aligned}\Delta f(r) &= (\Delta f(r))_{i=1,m} = \Delta_1(r) + \Delta_2(r) + \Delta_3(r) + \Delta_4(r) \\ &= (\Delta_1^i(r) + \Delta_2^i(r) + \Delta_3^i(r) + \Delta_4^i(r))_{i=1,m},\end{aligned}$$

où pour tout $i = 1, \dots, m$,

$$\begin{aligned}\Delta_1^i(r) &= h^{(i)} \left(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \\ &\quad - h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right); \\ \Delta_2^i(r) &= h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \\ &\quad - h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right); \\ \Delta_3^i(r) &= h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \\ &\quad - h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right); \\ \Delta_4^i(r) &= h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de) \right) \\ &\quad - h^{(i)} \left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, \int_E \gamma^i(r, X_r^{t,x'}, e) U_r^{i;t,x'}(e) \lambda(de) \right).\end{aligned}$$

Par l'inégalité de Cauchy-Schwartz, l'inégalité $2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$, et l'estimation (0.0.17), nous avons,

$$\begin{aligned}& \mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_1(r) \rangle dr \right] \\ & \leq \mathbb{E} \left[\frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 dr + C^2 \epsilon \int_s^T |X_r^{t,x} - X_r^{t,x'}|^2 (1 + |X_r^{t,x}|^p + |X_r^{t,x'}|^p)^2 dr \right] \\ & \leq \mathbb{E} \left[\frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 dr \right] + C^2 \epsilon |x - x'|^2 (1 + |x|^p + |x'|^p)^2.\end{aligned} \quad (0.0.40)$$

De plus, comme $h^{(i)}$ est Lipschitz en (y, z, q) , alors

$$\mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_2(r) \rangle dr \right] \leq 2C \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right] \quad (0.0.41)$$

et

$$\mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_3(r) \rangle dr \right] \leq \mathbb{E} \left[\frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 dr + C^2 \epsilon \int_s^T |\Delta Z(r)|^2 dr \right]. \quad (0.0.42)$$

Il reste à obtenir un contrôle du dernier terme.

Pour tout $s \in [t, T]$, nous avons

$$\begin{aligned} & \mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_4(r) \rangle dr \right] \\ & \leq 2C \mathbb{E} \left[\int_s^T |\Delta Y(r)| dr \times \left| \int_E \left(\gamma(r, X_r^{t,x}, e) U_r^{t,x}(e) - \gamma(r, X_r^{t,x'}, e) U_r^{t,x'}(e) \right) \lambda(de) \right| \right]. \end{aligned} \quad (0.0.43)$$

Ensuite, en divisant les termes de croisement comme suit :

$$\begin{aligned} & \gamma(r, X_r^{t,x}, e) U_r^{t,x}(e) - \gamma(r, X_r^{t,x'}, e) U_r^{t,x'}(e) \\ & = \Delta U_s(e) \gamma(s, X_s^{t,x}, e) + U_s^{t,x'} \left(\gamma(s, X_s^{t,x}, e) - \gamma(s, X_s^{t,x'}, e) \right) \end{aligned}$$

et en définissant $\Delta \gamma_s(e) := \left(\gamma(s, X_s^{t,x}, e) - \gamma(s, X_s^{t,x'}, e) \right)$, nous obtenons

$$\begin{aligned} & \mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_4(r) \rangle dr \right] \\ & \leq 2C \mathbb{E} \left[\int_s^T |\Delta Y(r)| \times \left(\int_E (|U_r^{t,x'}(e) \Delta \gamma_r(e)| + |\Delta U_r(e) \gamma(r, X_r^{t,x}, e)|) \lambda(de) \right) dr \right] \\ & \leq \frac{2}{\epsilon} \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right] + C^2 \epsilon \mathbb{E} \left[\int_s^T \left(\int_E (|U_r^{t,x'}(e) \Delta \gamma_r(e)| \lambda(de))^2 dr \right) \right] \\ & \quad + C^2 \epsilon \mathbb{E} \left[\int_s^T \left(\int_E (|\Delta U_r(e) \gamma(r, X_r^{t,x}, e)| \lambda(de))^2 dr \right) \right]. \end{aligned} \quad (0.0.44)$$

Par l'inégalité de Cauchy-Schwartz, (0.0.13) et (0.0.17), et le résultat du lemme 1, il s'en suit que,

$$\begin{aligned} & \mathbb{E} \left[\int_s^T \left(\int_E (|U_r^{t,x'}(e) \Delta \gamma_r(e)| \lambda(de))^2 dr \right) \right] \\ & \leq \mathbb{E} \left[\int_s^T dr \left(\int_E |U_r^{t,x'}(e)|^2 \lambda(de) \right) \left(\int_E |\Delta \gamma_r(e)|^2 \lambda(de) \right) \right] \\ & \leq C \mathbb{E} \left[\left\{ \sup_{r \in [t, T]} |X_r^{t,x} - X_r^{t,x'}|^2 (1 + \sup_{r \in [t, T]} |X_r^{t,x}|^p + |X_r^{t,x'}|^p)^2 dr \right\} \right] \\ & \quad \times \mathbb{E} \left[\int_s^T dr \left(\int_E |U_r^{t,x'}(e)|^2 \lambda(de) \right) \right] \\ & \leq C \sqrt{\mathbb{E} \left[\left\{ \sup_{r \in [t, T]} |X_r^{t,x} - X_r^{t,x'}|^4 (1 + \sup_{r \in [t, T]} |X_r^{t,x}|^p + |X_r^{t,x'}|^p)^4 \right\} \right]} \\ & \quad \times \sqrt{\mathbb{E} \left[\left\{ \int_s^T dr \left(\int_E |U_r^{t,x'}(e)|^2 \lambda(de) \right) \right\}^2 \right]} \\ & \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}). \end{aligned} \quad (0.0.45)$$

Pour un exposant p et en utilisant encore une fois l'inégalité de Cauchy-Schwarz et (0.0.13)-(i), nous obtenons

$$\mathbb{E} \left[\int_s^T \left(\int_E (|\Delta U_r(e) \gamma(r, X_r^{t,x}, e)| \lambda(de)) \right)^2 dr \right] \quad (0.0.48)$$

$$\begin{aligned} &\leq \mathbb{E} \left[\int_s^T dr \left(\int_E (|\Delta U_r(e)|^2 \lambda(de)) \right) \left(\int_E |\gamma(r, X_r^{t,x}, e)|^2 \lambda(de) \right) \right] \\ &\leq C \mathbb{E} \left[\int_s^T dr \left(\int_E (|\Delta U_r(e)|^2 \lambda(de)) \right) \right]. \end{aligned} \quad (0.0.49)$$

En tenant compte maintenant des inégalités (0.0.40)–(0.0.48), nous obtenons

$$\begin{aligned} &\mathbb{E} \left[|\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}|^2 + \int_s^T |\Delta Z_r|^2 dr + \int_s^T \|\Delta U_r\|_{\mathbb{L}^2(\lambda)}^2 dr \right] \\ &\leq \mathbb{E} \left[|g(X_T^{t,x}) - g(X_T^{t,x'})|^2 + 2 \int_s^T \langle (\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}), \Delta f(r) \rangle dr \right] \\ &\quad + \mathbb{E} \left[2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \right] \\ &\leq |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}) (C + 2CC' + C^2\epsilon + C^3\epsilon) \\ &\quad + \left(\frac{3}{\epsilon} + 2C \right) \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right] \\ &\quad + C^2\epsilon \mathbb{E} \left[\int_s^T |\Delta Z(r)|^2 dr \right] + C^3\epsilon \mathbb{E} \left[\int_s^T dr \left(\int_E (|\Delta U_r(e)|^2 \lambda(de)) \right) \right]. \end{aligned}$$

En choisissant maintenant ϵ assez petit, nous en déduisons l'existence d'une constante $C \geq 0$ telle que pour tout $s \in [t, T]$,

$$\mathbb{E} \left[|\Delta Y(s)|^2 \right] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}) + \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right]$$

et par le lemme de Gronwall, cela implique que pour tout $s \in [t, T]$,

$$\mathbb{E} \left[|\Delta Y(s)|^2 \right] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}).$$

Enfin, en prenant $s = t$ et en considérant (0.0.19), nous obtenons le résultat souhaité.

Remarque 4. Ce résultat donne également une estimation de U , où nous utilisons la fonction $(h^{(i)})_{1 \leq i \leq m}$, contrairement à l'estimation (0.0.21).

Corollaire 1. Pour $u^i \in \mathcal{U}$ ($i = 1, \dots, m$), $B_i u^i$ définie dans (0.0.3) est bien posée car les fonctions β et $(\gamma_i)_{i=1, m}$ vérifient respectivement (0.0.10) et (0.0.13).

Sketch de la preuve 3. Le principal point à noter est que λ intègre $(1 \wedge |e|^p)$, $\forall p \geq 2$.

Nous avons

$$\begin{aligned}
|B_i u^i(t, x)| &\leq \int_{\mathbb{E}} |\gamma^i(t, x, e)| \times |(u^i(t, x + \beta(t, x, e)) - u^i(t, x))| \lambda(\mathrm{d}e) \\
&\leq \int_{\mathbb{E}} C(1 \wedge |e|) |\beta(t, x, e)| (1 + |x + \beta(t, x, e)|^p + |x|^p) \lambda(\mathrm{d}e) \\
&\leq C^2(1 + |x|^p(1 + 2^{p-1})) \int_{\mathbb{E}} C(1 \wedge |e|^2) \lambda(\mathrm{d}e) \\
&\quad + (2^{\frac{p-1}{p}} C^{\frac{p+2}{p}})^p \int_{\mathbb{E}} C(1 \wedge |e|^p) \lambda(\mathrm{d}e).
\end{aligned}$$

Ce qui termine la preuve.

Maintenant, par la remarque 4, la dernière estimation de U confirme le résultat suivant.

Proposition 5. *Pour tout $i = 1, \dots, m$, $(t, x) \in [0, T] \times \mathbb{R}^k$, nous avons*

$$U_s^{i;t,x}(e) = u^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u^i(s, X_{s-}^{t,x}), \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}s \otimes \mathrm{d}\lambda\text{-p.p. sur } \Omega \times [t, T] \times E. \quad (0.0.50)$$

Sketch de la preuve 4. Notons tout d'abord que la mesure λ n'étant pas finie, alors nous ne pouvons pas utiliser la même technique que dans [37], où les auteurs utilisent les sauts de processus et (0.0.19). Dans notre cas, $U^{i;t,x}$ n'est que carré intégrable et pas nécessairement intégrable par rapport à $\mathrm{d}\mathbb{P} \otimes \mathrm{d}s \otimes \mathrm{d}\lambda$. Par conséquent, nous commençons par tronquer la mesure de Lévy de la même manière que dans [36].

Étape 1 Troncature de la mesure Lévy

Pour tout $k^* \geq 1$, introduisons d'abord une nouvelle mesure aléatoire de Poisson μ_{k^*} (obtenue à partir de la troncature de μ) et son compensateur associé ν_{k^*} comme suit :

$$\mu_{k^*}(\mathrm{d}s, \mathrm{d}e) = 1_{\{|e| \geq \frac{1}{k^*}\}} \mu(\mathrm{d}s, \mathrm{d}e) \quad \text{et} \quad \nu_{k^*}(\mathrm{d}s, \mathrm{d}e) = 1_{\{|e| \geq \frac{1}{k^*}\}} \nu(\mathrm{d}s, \mathrm{d}e).$$

Ce qui veut dire que, $\tilde{\mu}_{k^*}(\mathrm{d}s, \mathrm{d}e) := (\mu_{k^*} - \nu_{k^*})(\mathrm{d}s, \mathrm{d}e)$ est la mesure martingale aléatoire associée. Le principal point à remarquer est que

$$\begin{aligned}
\lambda_{k^*}(E) &= \int_E \lambda_{k^*}(\mathrm{d}e) = \int_E 1_{\{|e| \geq \frac{1}{k^*}\}} \lambda(\mathrm{d}e) \\
&= \int_{\{|e| \geq \frac{1}{k^*}\}} \lambda(\mathrm{d}e) = \lambda(\{|e| \geq \frac{1}{k^*}\}) < \infty.
\end{aligned} \quad (0.0.51)$$

Comme dans [36], introduisons le processus ${}^{k^*}X^{t,x}$ résolvant l'EDS standard de type diffusion de sauts suivante:

$$\begin{aligned}
{}^{k^*}X_s^{t,x} &= x + \int_t^s b(r, {}^{k^*}X_r^{t,x}) \mathrm{d}r + \int_t^s \sigma(r, {}^{k^*}X_r^{t,x}) \mathrm{d}B_r \\
&\quad + \int_t^s \int_{\mathbb{E}} \beta(r, {}^{k^*}X_{r-}^{t,x}, e) \tilde{\mu}_{k^*}(\mathrm{d}r, \mathrm{d}e), \quad t \leq s \leq T; \quad {}^{k^*}X_r^{t,x} = x \quad \text{si } s \leq t.
\end{aligned} \quad (0.0.52)$$

Notons qu'en raison des hypothèses sur b , σ , β , le processus ${}^{k^*}X^{t,x}$ existe et est unique. De plus, il satisfait aux mêmes estimations que dans (0.0.17) car λ_{k^*} n'est qu'une troncature à l'origine de λ , qui intègre $(1 \wedge |e|^2)_{e \in E}$.

D'autre part, considérons l'EDSRR markovienne avec sauts suivante

$$\left\{ \begin{array}{l} \text{(i)} \quad \mathbb{E} \left[\sup_{s \leq T} |{}^{k^*}Y_s^{t,x}|^2 + \int_s^T |{}^{k^*}Z_r^{t,x}|^2 dr + \int_s^T \|{}^{k^*}U_r^{t,x}\|_{\mathbb{L}^2(\lambda_{k^*})}^2 dr \right] < \infty \\ \text{(ii)} \quad {}^{k^*}Y^{t,x} := ({}^{k^*}Y^{i,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad {}^{k^*}Z^{t,x} := ({}^{k^*}Z^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad {}^{k^*}K^{t,x} := ({}^{k^*}K^{i,t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad {}^{k^*}U^{t,x} := ({}^{k^*}U^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda_{k^*})); \\ \text{(iii)} \quad {}^{k^*}Y_s^{t,x} = g({}^{k^*}X_T^{t,x}) + \int_s^T f_{\mu_{k^*}}(r, {}^{k^*}X_r^{t,x}, {}^{k^*}Y_r^{t,x}, {}^{k^*}Z_r^{t,x}, {}^{k^*}U_r^{t,x}) dr + {}^{k^*}K_T^{t,x} - {}^{k^*}K_s^{t,x} \\ \quad - \int_s^T \left\{ {}^{k^*}Z_r^{t,x} dB_r + \int_E U_r^{t,x}(e) \tilde{\mu}_{k^*}(dr, de) \right\}, \quad s \leq T; \\ \text{(iv)} \quad {}^{k^*}Y_s^{i,t,x} \geq \ell(s, {}^{k^*}X_s^{t,x}) \text{ and } \int_0^T ({}^{k^*}Y_s^{i,t,x} - \ell(s, {}^{k^*}X_s^{t,x})) d({}^{k^*}K_s^{i,t,x}) = 0. \end{array} \right. \quad (0.0.53)$$

Enfin, introduisons les fonctions suivantes $(f^{(i)})_{i=1,m}$ définies comme suit : pour tout $(t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{L}_m^2(\lambda_{k^*})$,

$$\begin{aligned} f_{\mu_{k^*}}(t, x, y, z, \zeta) &= (f_{\mu_{k^*}}^{(i)}(t, x, y, z_i, \zeta_i))_{i=1,m} \\ &:= \left(h^{(i)} \left(t, x, y, z, \int_E \gamma^i(t, x, e) \zeta_i(e) \lambda_{k^*}(de) \right) \right)_{i=1,m}. \end{aligned}$$

Tout d'abord, soulignons que cette dernière EDSRR est liée à la filtration $(\mathcal{F}_s^{k^*})_{s \leq T}$, engendrée par le mouvement brownien et la mesure aléatoire indépendante μ_{k^*} . Cependant, ce point ne soulève pas de problèmes majeurs car pour tout $s \leq T$, $\mathcal{F}_s^{k^*} \subset \mathcal{F}_s$ et par la relation entre μ et μ_{k^*} . Ensuite, par les propriétés des fonctions b , σ , β et par les mêmes opinions des propositions 2 et 3, il existe un unique quadriplet $({}^{k^*}Y^{t,x}, {}^{k^*}K^{t,x}, {}^{k^*}Z^{t,x}, {}^{k^*}U^{t,x})$ résolvant (0.0.53) et il existe aussi une fonction u^{k^*} de $[0, T] \times \mathbb{R}^k$ dans \mathbb{R}^m de Π_g^c tel que

$$\forall s \in [t, T], \quad {}^{k^*}Y^{t,x} := u^{k^*}(s, {}^{k^*}X^{t,x}), \quad \mathbb{P} - p.s. \quad (0.0.54)$$

De plus, comme dans la proposition 4, il existe des constantes positives C et p qui ne dépendent pas de k^* telles que pour $\forall t, x, x'$,

$$|u^{k^*}(t, x) - u^{k^*}(t, x')| \leq C |x - x'| (1 + |x|^p + |x'|^p). \quad (0.0.55)$$

Enfin, comme λ_{k^*} est fini, alors nous avons ce qui suit; relation entre le processus ${}^{k^*}U^{t,x} := ({}^{k^*}U^{i,t,x})_{i=1,m}$ et les fonctions déterministes $u^{k^*} := (u_i^{k^*})_{i=1,m}$ (voir [37]):

$\forall i = 1, \dots, m$, nous avons

$$\begin{aligned} {}^{k^*}U_s^{i;t,x}(e) &= u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x} + \beta(s, {}^{k^*}X_{s-}^{t,x}, e)) - u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x}), \\ &\quad d\mathbb{P} \otimes ds \otimes d\lambda_{k^*}\text{-p.p. sur } \Omega \times [t, T] \times E. \end{aligned}$$

Ceci est principalement dû au fait que ${}^{k^*}U^{t,x}$ appartient à $\mathbb{L}^1 \cap \mathbb{L}^2(d\lambda_s \otimes d\mathbb{P} \otimes d\lambda_{k^*})$ car $\lambda_{k^*}(E) < \infty$ et ensuite nous pouvons diviser l'intégrale stochastique $\tilde{\mu}_{k^*}$ dans (0.0.53). Par conséquent, pour tout $i = 1, \dots, m$, nous avons

$$\begin{aligned} {}^{k^*}U_s^{i;t,x}(e)1_{\{|e| \geq \frac{1}{k^*}\}} &= (u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x} + \beta(s, {}^{k^*}X_{s-}^{t,x}, e)) - u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x}))1_{\{|e| \geq \frac{1}{k^*}\}}, \\ &\quad d\mathbb{P} \otimes ds \otimes d\lambda_{k^*}\text{-p.p. sur } \Omega \times [t, T] \times E. \end{aligned} \quad (0.0.56)$$

Étape 2 Convergence des processus auxiliaires

Démontrons maintenant le résultat de convergence suivant :

$$\begin{aligned} &\mathbb{E} \left[\sup_{s \leq T} |Y_s^{t,x} - {}^{k^*}Y_s^{t,x}|^2 + (K_T^{t,x} - {}^{k^*}K_T^{t,x})^2 + \int_0^T |Z_s^{t,x} - {}^{k^*}Z_s^{t,x}|^2 ds \right. \\ &\quad \left. + \int_0^T ds \int_E \lambda(de) |U_s^{t,x}(e) - {}^{k^*}U_s^{t,x}(e)1_{\{|e| \geq \frac{1}{k^*}\}}|^2 \right] \xrightarrow{k^* \rightarrow +\infty} 0, \end{aligned} \quad (0.0.57)$$

où $(Y^{t,x}, K^{t,x}, Z^{t,x}, U^{t,x})$ est la solution de l'EDSRR avec des sauts (0.0.18). Tout d'abord, notons que le résultat de convergence suivant a été établi dans [36]

$$\mathbb{E} \left[\sup_{s \leq T} |X_s^{t,x} - {}^{k^*}X_s^{t,x}|^2 \right] \xrightarrow{k^* \rightarrow +\infty} 0. \quad (0.0.58)$$

Concentrons-nous maintenant sur (0.0.57). Notons que, nous pouvons appliquer la formule d'Itô et prendre l'espérance, même si les EDSSR sont liées à des filtrages et des mesures aléatoires de Poisson qui ne sont pas les mêmes, à cause des éléments suivants :

(i) $\mathcal{F}_s^{k^*} \subset \mathcal{F}_s, \forall s \leq T$;

(ii) Pour tout $s \leq T$,

$$\int_0^s \int_E U^{i;t,x}(e) \tilde{\mu}_{k^*}(dr, de) = \int_0^s \int_E U^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k^*}\}} \tilde{\mu}(dr, de)$$

puis la première $(\mathcal{F}_s^{k^*})_{s \leq T}$ - martingale est aussi une $(\mathcal{F}_s)_{s \leq T}$ - martingale.

Pour $\forall s \in [0, T]$, nous avons

$$\begin{aligned}
& \mathbb{E} \left[\left| \vec{Y}_s^{t,x} - {}^{k^*} Y_s^{t,x} \right|^2 + \left| {}^{k^*} \Delta K_T \right|^2 \right. \\
& \quad \left. + \int_0^T \left\{ \left| Z_s^{t,x} - {}^{k^*} Z_s^{t,x} \right|^2 + \int_E \left| U_s^{t,x} - {}^{k^*} U_s^{t,x} 1_{\{|e| \geq \frac{1}{k^*}\}} \right|^2 \lambda(de) \right\} ds \right] \\
& \leq \mathbb{E} \left[\left| g(X_T^{t,x}) - g({}^{k^*} X_T^{t,x}) \right|^2 + 2 \int_s^T \left(\vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} \right) \times {}^{k^*} \Delta f(r) dr \right] \\
& \quad + \mathbb{E} \left[\sup_{s \leq T} \left| {}^{k^*} \Delta \ell_s \right|^2 \right], \tag{0.0.59}
\end{aligned}$$

où les processus ${}^{k^*} \Delta X_r$, ${}^{k^*} \Delta Y_r$, ${}^{k^*} \Delta f(r)$, ${}^{k^*} \Delta K_r$, ${}^{k^*} \Delta Z_r$, ${}^{k^*} \Delta U_r$, et ${}^{k^*} \Delta \ell_r$ sont définis comme suit, pour $\forall r \in [0, T]$,

$$\begin{aligned}
{}^{k^*} \Delta f(r) & := \left(({}^{k^*} \Delta f^{(i)}(r))_{i=1,m} = (f^{(i)}(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, U_r^{i;t,x}) \right. \\
& \quad \left. - f_{k^*}^{(i)}(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{t,x}, {}^{k^*} U_r^{t,x}) \right)_{i=1,m}, \\
{}^{k^*} \Delta X_r & = X_r^{t,x} - {}^{k^*} X_r^{t,x}, \quad {}^{k^*} \Delta Y(r) = \vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} = (Y_r^{j;t,x} - {}^{k^*} Y_r^{j;t,x})_{j=1,m}, \\
{}^{k^*} \Delta K_r & = K_r^{t,x} - {}^{k^*} K_r^{t,x}, \quad {}^{k^*} \Delta Z_r = Z_r^{t,x} - {}^{k^*} Z_r^{t,x}, \quad {}^{k^*} \Delta U_r = U_r^{t,x} - {}^{k^*} U_r^{t,x} 1_{\{|e| \geq \frac{1}{k^*}\}}
\end{aligned}$$

et

$${}^{k^*} \Delta \ell_r = \left(\ell(r, X_r^{t,x}) - \ell(r, {}^{k^*} X_r^{t,x}) \right).$$

Posons, pour $r \leq T$,

$$\begin{aligned}
{}^{k^*} \Delta f(r) & = (f(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{t,x}, U_r^{t,x}) - f_{k^*}(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{t,x}, {}^{k^*} U_r^{t,x})) \\
& = A(r) + B(r) + C(r) + D(r),
\end{aligned}$$

où pour tout $i = 1, \dots, m$,

$$\begin{aligned}
A(r) &= \left(h^{(i)} \left(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right. \\
&\quad \left. - h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right)_{i=1,m}, \\
B(r) &= \left(h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right. \\
&\quad \left. - h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right)_{i=1,m}, \\
C(r) &= \left(h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right. \\
&\quad \left. - h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right)_{i=1,m}, \\
D(r) &= \left(h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right. \\
&\quad \left. - h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{i;t,x}, \int_E \gamma^i(r, {}^{k^*} X_r^{t,x}, e) {}^{k^*} U_r^{i;t,x}(e) \lambda_{k^*}(\mathrm{d}e) \right) \right)_{i=1,m}.
\end{aligned}$$

Par (0.058) et le fait que $g \in \mathcal{U}$ et $\ell \in \mathcal{U}$, nous avons

$$\mathbb{E} \left[\left| g(X_T^{t,x}) - g({}^{k^*} X_T^{t,x}) \right|^2 \right]_{k^* \rightarrow +\infty} \rightarrow 0 \quad (0.060)$$

et

$$\mathbb{E} \left[\sup_{s \leq T} \left| \ell(X_s^{t,x}) - \ell({}^{k^*} X_s^{t,x}) \right|^2 \right]_{k^* \rightarrow +\infty} \rightarrow 0. \quad (0.061)$$

Maintenant, nous allons nous intéresser à $\mathbb{E} \left[\int_s^T \left(\vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} \right) \times {}^{k^*} \Delta f(r) \mathrm{d}r \right]$ pour obtenir (0.057).

Par (H1) et (0.011), pour $\forall r \in [0, T]$, nous obtenons

$$\begin{aligned}
|A(r)| &\leq C \left| X_r^{t,x} - {}^{k^*} X_r^{t,x} \right| \left(1 + \left| X_r^{t,x} \right|^p + \left| {}^{k^*} X_r^{t,x} \right|^p \right), \\
|B(r)| &\leq C \left| \vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} \right| \text{ and } |C(r)| \leq \left| Z_r^{t,x} - {}^{k^*} Z_r^{t,x} \right|;
\end{aligned} \quad (0.062)$$

où C est une constante réelle positive.

Considérons enfin $D(r)$. Tout d'abord, notons que $D(r) = (D_i(r))_{i=1,m}$, où

$$\begin{aligned}
D_i(r) &= h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \\
&\quad - h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{i;t,x}, \int_E \gamma^i(r, {}^{k^*} X_r^{t,x}, e) {}^{k^*} U_r^{i;t,x}(e) \lambda_{k^*}(\mathrm{d}e) \right).
\end{aligned}$$

Mais comme $h^{(i)}$ est Lipschitz par rapport au dernier composant q , alors,

$$\begin{aligned}
|D(r)|^2 &\leq C \left\{ \int_E \left| \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) - \gamma^i(r, X_r^{t,x, k^*}, e) U_r^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k^*}\}} \right|^2 \lambda(de) \right\} \\
&\leq C \left\{ \left\{ \int_E \left| \gamma^i(r, X_r^{t,x}, e) - \gamma^i(r, X_r^{t,x, k^*}, e) \right| \left| U_r^{i;t,x}(e) \right| \lambda(de) \right\}^2 \right. \\
&\quad \left. + \left\{ \int_E \left| \gamma^i(r, X_r^{t,x}, e) \right| \left| U_r^{i;t,x}(e) - U_r^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k^*}\}} \right| \lambda(de) \right\}^2 \right\} \\
&\leq C \left\{ \left| X_r^{t,x} - X_r^{t,x, k^*} \right| \left(1 + |X_r^{t,x}|^p + |X_r^{t,x, k^*}|^p \right) \left| \int_E U_r^{i;t,x}(e) \lambda(de) \right|^2 \right. \\
&\quad \left. + C \int_E (1 \wedge |e|) \left| U_r^{i;t,x}(e) - U_r^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k^*}\}} \right|^2 \lambda(de) \right\}, \tag{0.0.63}
\end{aligned}$$

et (0.0.59) devient, en utilisant les majorations obtenues en (0.0.62) et (0.0.63),

$$\begin{aligned}
&\mathbb{E} \left[\left| \vec{Y}_s^{t,x} - Y_s^{t,x, k^*} \right|^2 + \left| {}^{k^*} \Delta K_T \right|^2 \right] \tag{0.0.64} \\
&+ \int_0^T \left\{ \left| Z_s^{t,x} - Z_s^{t,x, k^*} \right|^2 + \int_E \left| U_s^{t,x} - U_s^{t,x, k^*} 1_{\{|e| \geq \frac{1}{k^*}\}} \right|^2 \lambda(de) \right\} ds \\
&\leq \mathbb{E} \left[\left| g(X_T^{t,x}) - g(X_T^{t,x, k^*}) \right|^2 \right] + \mathbb{E} \left[\sup_{s \leq T} \left| \ell(X_s^{t,x}) - \ell(X_s^{t,x, k^*}) \right|^2 \right] \\
&+ C \mathbb{E} \left[\int_s^T \left| \vec{Y}_s^{t,x} - Y_s^{t,x, k^*} \right|^2 \right] + C \mathbb{E} \left[\int_0^T \left| X_r^{t,x} - X_r^{t,x, k^*} \right|^2 \left(1 + |X_r^{t,x}|^p + |X_r^{t,x, k^*}|^p \right)^2 dr \right] \\
&+ C \mathbb{E} \left[\int_0^T dr \left\{ \left| X_r^{t,x} - X_r^{t,x, k^*} \right| \left(1 + |X_r^{t,x}|^p + |X_r^{t,x, k^*}|^p \right) \int_E U_r^{i;t,x}(e) \lambda(de) \right\}^2 \right]. \tag{0.0.65}
\end{aligned}$$

Les deux premiers termes convergent vers 0 par (0.0.60) et (0.0.61). Pour le quatrième terme, nous avons,

$$\begin{aligned}
&\mathbb{E} \left[\int_0^T \left| X_r^{t,x} - X_r^{t,x, k^*} \right|^2 \left(1 + |X_r^{t,x}|^p + |X_r^{t,x, k^*}|^p \right)^2 dr \right] \\
&\leq \mathbb{E} \left[\sup_{r \leq T} \left| X_r^{t,x} - X_r^{t,x, k^*} \right|^2 \int_0^T \left(1 + |X_r^{t,x}|^p + |X_r^{t,x, k^*}|^p \right)^2 dr \right] \\
&\leq \left\{ \mathbb{E} \left[\sup_{r \leq T} \left| X_r^{t,x} - X_r^{t,x, k^*} \right|^2 \right] \right\}^{\frac{1}{2}} \\
&\quad \times \left\{ \mathbb{E} \left[\left(\int_0^T \left(1 + |X_r^{t,x}|^p + |X_r^{t,x, k^*}|^p \right)^2 \left| X_r^{t,x} - X_r^{t,x, k^*} \right| \right)^2 dr \right] \right\}^{\frac{1}{2}}.
\end{aligned}$$

Le premier facteur du membre de droite de cette inégalité tend vers 0 quand $k^* \rightarrow \infty$ à cause de (0.0.58) et le deuxième facteur est uniformément borné par les estimations

uniformes (0.0.17) de $X^{t,x}$ et ${}^{k^*}X^{t,x}$. Notons aussi que le dernier terme converge vers 0 lorsque $k^* \rightarrow \infty$, c'est une conséquence de (0.0.58), le fait que ${}^{k^*}X^{t,x}$ vérifie les estimations (0.0.17) uniformément, l'inégalité de Cauchy-Schwartz (utilisée deux fois) et enfin (0.0.21) du lemme 1. Ensuite, par le lemme de Gronwall, on en déduit d'abord que pour tout $s \leq T$,

$$\mathbb{E} \left[\left| \bar{Y}_s^{t,x} - {}^{k^*}Y_s^{t,x} \right|^2 \right] \xrightarrow{k^* \rightarrow +\infty} 0, \quad (0.0.66)$$

et prenant $s = t$, nous obtenons, $u^{k^*}(t, x) \xrightarrow{k^* \rightarrow +\infty} u(t, x)$. Comme $(t, x) \in [0, T] \times \mathbb{R}^k$ est arbitraire, alors $u^{k^*} \xrightarrow{k^* \rightarrow +\infty} u$ point par point.

Ensuite, en revenant à (0.0.64), en prenant la limite par rapport à k^* et en utilisant la croissance polynomiale uniforme de u^{k^*} et le théorème de convergence dominé de Lebesgue, nous obtenons,

$$\mathbb{E} \left[\int_t^T \int_E \left| U_s^{t,x} - {}^{k^*}U_s^{t,x} 1_{\{|e| \geq \frac{1}{k^*}\}} \right|^2 \lambda(de) ds \right] \xrightarrow{k^* \rightarrow +\infty} 0. \quad (0.0.67)$$

Étape 3 Conclusion

Tout d'abord, notons que par (0.0.55) et la convergence ponctuelle de $(u^{k^*})_{k^*}$ vers u , si $(x_{k^*})_{k^*}$ est une suite de \mathbb{R}^k , qui converge vers x , alors $((u^{k^*}(t, x_{k^*}))_{k^*})$ converge vers $u(t, x)$.

Considérons maintenant une sous-suite que nous notons encore $\{{}^{k^*}X_s^{t,x}\}$, telle que

$$\sup_{s \leq T} \left| X_s^{t,x} - {}^{k^*}X_s^{t,x} \right|^2 \xrightarrow{k^* \rightarrow +\infty} 0, \mathbb{P} - \text{p.s.}$$

(et alors $\left| X_{s-}^{t,x} - {}^{k^*}X_{s-}^{t,x} \right| \xrightarrow{k^* \rightarrow +\infty} 0$ de plus $\left| X_{s-}^{t,x} - {}^{k^*}X_{s-}^{t,x} \right| \leq \sup_{s \leq T} \left| X_s^{t,x} - {}^{k^*}X_s^{t,x} \right|^2$). Par (0.0.58), cette sous-séquence existe. Comme la correspondance $x \mapsto \beta(t, x, e)$ est Lipschitz, alors la séquence;

$$\begin{aligned} \left({}^{k^*}U_s^{t,x}(e) 1_{\{|e| \geq \frac{1}{k^*}\}} \right)_{k^*} = & \left(\left\{ u_i^{k^*} \left(s, {}^{k^*}X_{s-}^{t,x} + \beta \left(s, {}^{k^*}X_{s-}^{t,x}, e \right) \right. \right. \right. \\ & \left. \left. \left. - u_i^{k^*} \left(s, {}^{k^*}X_{s-}^{t,x} \right) \right\} 1_{\{|e| \geq \frac{1}{k^*}\}} \right)_{k^* \geq 1} \end{aligned} \quad (0.0.68)$$

$$\begin{aligned}
& \left(\{u_i^{k^*}(s, X_{s^-}^{t,x} + \beta(s, X_{s^-}^{t,x}, e)) - u_i^{k^*}(s, X_{s^-}^{t,x})\} 1_{\{|e| \geq \frac{1}{k^*}\}} \right)_{k^* \geq 1} \xrightarrow{k^* \rightarrow +\infty} \\
& (u_i(s, X_{s^-}^{t,x} + \beta(s, X_{s^-}^{t,x}, e)) - u_i(s, X_{s^-}^{t,x})) \\
& d\mathbb{P} \otimes ds \otimes d\lambda - a.e. \quad \text{on } \Omega \times [t, T] \times E,
\end{aligned} \tag{0.0.69}$$

pour tout $i = 1, \dots, m$. Enfin, de (0.0.67), nous en déduisons que

$$U_s^{t,x}(e) = (u_i(s, X_{s^-}^{t,x} + \beta(s, X_{s^-}^{t,x}, e)) - u_i(s, X_{s^-}^{t,x})), \quad \text{sur } \Omega \times [t, T] \times E, \tag{0.0.70}$$

qui est le résultat souhaité.

Résultat principal

Théorème 1. *Sous les hypothèses (H1), (H2) et (H3), (0.0.2) a une solution unique, qui est le m -uplet des fonctions $(u^i)_{i=1,m}$ défini dans la proposition 3 par (0.0.19).*

Sketch de la preuve 5.

Étape 1 : Existence

Supposons que les hypothèses (H1), (H2) et (H3) soient remplies, alors l'EDSRR multidimensionnel à sauts suivant

$$\left\{ \begin{array}{l}
\text{(i)} \quad \underline{Y}^{\vec{t},x} := (\underline{Y}^{i;t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \underline{Z}^{t,x} := (\underline{Z}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\
\quad \underline{K}^{t,x} := (\underline{K}^{i;t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad \underline{U}^{t,x} := (\underline{U}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\
\text{(ii)} \quad \underline{Y}_s^{i;t,x} = g^i(X_T^{t,x}) + \underline{K}_T^{i;t,x} - \underline{K}_s^{i;t,x} - \int_s^T \underline{Z}^{i;t,x} dB_r - \int_s^T \int_E \underline{U}_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\
\quad + \int_s^T h^{(i)} \left(r, X_r^{t,x}, \underline{Y}_r^{i;t,x}, \underline{Z}_r^{i;t,x}, \right. \\
\quad \left. \int_E \gamma^i(t, X_r^{t,x}, e) \{ (u^i(t, X_{r^-}^{t,x} + \beta(t, X_{r^-}^{t,x}, e)) - u^i(t, X_{r^-}^{t,x})) \} \lambda(de) \right) dr; \\
\text{(iii)} \quad \underline{Y}_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ et } \int_0^T (\underline{Y}_s^{i;t,x} - \ell(s, X_s^{t,x})) d\underline{K}_s^{i;t,x} = 0
\end{array} \right. \tag{0.0.71}$$

a une solution unique $(\underline{Y}, \underline{Z}, \underline{K}, \underline{U})$. Ensuite, comme pour tout $i = 1, \dots, m$, u^i appartient à \mathcal{U} , alors par (0.0.19) Proposition 3, il existe une famille de fonctions continues déterministes à croissance polynomiale $(\underline{u}^i)_{i=1,m}$, ainsi pour tout $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\forall s \in [t, T], \underline{Y}_s^{i;t,x} = \underline{u}^i(s, X_s^{t,x}).$$

Telle que par la même proposition, la famille $(\underline{u}^i)_{i=1,m}$ est une solution de viscosité du système suivant :

$$\begin{cases} \min \left\{ \underline{u}^i(t, x) - \ell(t, x); -\partial_t \underline{u}^i(t, x) - b(t, x)^\top D_x \underline{u}^i(t, x) \right. \\ \left. - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 \underline{u}^i(t, x)) - K_i \underline{u}^i(t, x) - h^{(i)}(t, x, (\underline{u}^j(t, x))_{j=1,m}), \right. \\ \left. (\sigma^\top D_x \underline{u}^i)(t, x), B_i u^i(t, x) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ u^i(T, x) = g^i(x). \end{cases} \quad (0.0.72)$$

Maintenant, sachons que la famille $(\underline{u}^i)_{i=1,m}$ est une solution de viscosité; notre objectif principal est de trouver la relation entre $(\underline{u}^i)_{i=1,m}$ et $(u^i)_{i=1,m}$ qui est définie dans (0.0.19). Pour cela, considérons le système d'EDSRR à sauts

$$\begin{cases} \text{(i)} & \vec{Y}^{t,x} := (Y^{i;t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad Z^{t,x} := (Z^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ & K^{t,x} := (K^{i;t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad U^{t,x} := (U^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} & Y_s^{i;t,x} = g^i(X_T^{t,x}) + K_T^{i;t,x} - K_s^{i;t,x} - \int_s^T Z^{i;t,x} dB_r - \int_s^T \int_E U_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\ & \quad + \int_s^T h^{(i)}(r, X_r^{t,x}, Y_r^{i;t,x}, Z_r^{i;t,x}, \int_E \gamma^i(t, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de)) dr; \\ \text{(iii)} & Y_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i;t,x} - \ell(s, X_s^{t,x})) dK_s^{i;t,x} = 0. \end{cases} \quad (0.0.73)$$

Par l'unicité de la solution de l'EDSRR à sauts (0.0.71), puis pour $\forall s \in [t, T]$ et $\forall i \in \{1, \dots, m\}$, $\underline{Y}_s^{i;t,x} = Y_s^{i;t,x}$. Donc, $\underline{u}^i = u^i$, telle que par (0.0.67), nous obtenons

$$U_s^{t,x}(e) = (u_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u_i(s, X_{s-}^{t,x})), \quad \text{sur } \Omega \times [t, T] \times E,$$

qui donne la solution de viscosité au sens de la définition 2 (voir [36]) en remplaçant (0.0.68) dans $h^{(i)}$ de (0.0.72).

Étape 2 : Unicité

Soit $(\bar{u}^i)_{i=1,m}$ une autre famille de \mathcal{U} qui est la solution de viscosité du système (0.0.2) au sens de la définition 2 et nous considérons l'EDSRR à sauts définie avec \bar{u}^i .

$$\left\{ \begin{array}{l}
\text{(i)} \quad \bar{Y}^{\rightarrow t,x} := (\bar{Y}^{i;t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \bar{Z}^{t,x} := (\bar{Z}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\
\quad \bar{K}^{t,x} := (\bar{K}^{i;t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad \bar{U}^{t,x} := (\bar{U}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\
\text{(ii)} \quad \bar{Y}_s^{i;t,x} = g^i(X_T^{t,x}) + \bar{K}_T^{i;t,x} - \bar{K}_s^{i;t,x} - \int_s^T \bar{Z}^{i;t,x} dB_r - \int_s^T \int_E \bar{U}_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\
\quad + \int_s^T h^{(i)} \left(r, X_r^{t,x}, \bar{Y}_r^{i;t,x}, \bar{Z}_r^{i;t,x}, \right. \\
\quad \left. \int_E \gamma^i(t, X_r^{t,x}, e) (\bar{u}_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - \bar{u}_i(s, X_{s-}^{t,x})) \lambda(de) \right) dr; \\
\text{(iii)} \quad \bar{Y}_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ et } \int_0^T (\bar{Y}_s^{i;t,x} - \ell(s, X_s^{t,x})) d\bar{K}_s^{i;t,x} = 0.
\end{array} \right. \tag{0.0.74}$$

Par la formule de Feynman Kac, $\bar{u}^i(s, X_s^{t,x}) = Y_s^{i;t,x}$, où $Y_s^{i;t,x}$ satisfait l'EDSRR à sauts (0.0.2) associée à (0.0.1). L'EDRSS à sauts (0.0.74) a une solution et elle est unique en supposant que (H1), (H2) et (H3) sont vérifiées. D'après (0.0.19) de la proposition 3, il existe une famille de fonctions continues déterministes à croissance polynomiale $(v^i)_{i=1,m}$, qui pour tout $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\forall s \in [t, T], \quad \bar{Y}_s^{i;t,x} = v^i(s, X_s^{t,x}).$$

Telle que par la même proposition, la famille $(v^i)_{i=1,m}$ est une solution de viscosité du système suivant :

$$\left\{ \begin{array}{l}
\min \left\{ v^i(t, x) - \ell(t, x); -\partial_t v^i(t, x) - b(t, x)^\top D_x v^i(t, x) \right. \\
\quad \left. - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 v^i(t, x)) - K_i v^i(t, x) - h^{(i)}(t, x, (v^j(t, x))_{j=1,m}), \right. \\
\quad \left. (\sigma^\top D_x v^i)(t, x), B_i \bar{u}^i(t, x) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\
u^i(T, x) = g^i(x).
\end{array} \right. \tag{0.0.75}$$

Par l'unicité de la solution de (0.0.73), \bar{u}^i est la solution de viscosité de (0.0.72), et par la proposition 3 $v^i = \bar{u}^i$ pour $\forall i \in \{1, \dots, m\}$. Maintenant, pour terminer notre preuve, montrons que sur $\Omega \times [t, T] \times E$, $ds \otimes d\mathbb{P} \otimes d\lambda$ -p.p., pour $\forall i \in \{1, \dots, m\}$, nous avons

$$\begin{aligned}
\bar{U}_s^{i;t,x}(e) &= (v^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - v^i(s, X_{s-}^{t,x})) \\
&= (\bar{u}_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - \bar{u}_i(s, X_{s-}^{t,x})).
\end{aligned} \tag{0.0.76}$$

Par la Remarque 3.4 dans [36], considérons $(x_{k^*})_{k^* \geq 1}$, une séquence de \mathbb{R}^k , qui converge vers $x \in \mathbb{R}^k$ et les deux EDSRR suivants avec sauts (l'adaptation est par

rapport \mathcal{F}^{k^*}):

$$\left\{ \begin{array}{l} \text{(i)} \quad \bar{Y}^{\bar{i};k^*,t,x} := (\bar{Y}^{\bar{i};k^*,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \bar{Z}^{k^*,t,x} := (\bar{Z}^{\bar{i};k^*,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \bar{K}^{k^*,t,x} := (\bar{K}^{\bar{i};k^*,t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad \bar{U}^{k^*,t,x} := (\bar{U}^{\bar{i};k^*,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} \quad \bar{Y}_s^{\bar{i};k^*,t,x} = g^i(X_T^{k^*,t,x}) + \bar{K}_T^{\bar{i};k^*,t,x} - \bar{K}_s^{\bar{i};k^*,t,x} - \int_s^T \bar{Z}^{\bar{i};k^*,t,x} dB_r \\ - \int_s^T \int_E \bar{U}_r^{\bar{i};k^*,t,x}(e) \tilde{\mu}(dr, de) + \int_s^T h^{(i)}\left(r, X_r^{k^*,t,x}, \bar{Y}_r^{\bar{i};k^*,t,x}, \bar{Z}_r^{\bar{i};k^*,t,x}, \right. \\ \left. \int_E \gamma^i(t, X_r^{k^*,t,x}, e)(\bar{u}_i(s, X_{s-}^{k^*,t,x} + \beta(s, X_{s-}^{k^*,t,x}, e)) \right. \\ \left. - \bar{u}_i(s, X_{s-}^{k^*,t,x})) \lambda(de) \right) dr; \\ \text{(iii)} \quad \bar{Y}_s^{\bar{i};k^*,t,x} \geq \ell(s, X_s^{k^*,t,x}) \text{ et } \int_0^T (\bar{Y}_s^{\bar{i};k^*,t,x} - \ell(s, X_s^{k^*,t,x})) d\bar{K}_s^{\bar{i};k^*,t,x} = 0; \end{array} \right. \quad (0.0.77)$$

et

$$\left\{ \begin{array}{l} \text{(i)} \quad \bar{Y}^{\bar{i};k^*,t,x_{k^*}} := (\bar{Y}^{\bar{i};k^*,t,x_{k^*}})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \bar{Z}^{k^*,t,x_{k^*}} := (\bar{Z}^{\bar{i};k^*,t,x_{k^*}})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \bar{K}^{k^*,t,x_{k^*}} := (\bar{K}^{\bar{i};k^*,t,x_{k^*}})_{i=1,m} \in \mathcal{A}_c^2, \quad \bar{U}^{k^*,t,x_{k^*}} := (\bar{U}^{\bar{i};k^*,t,x_{k^*}})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} \quad \bar{Y}_s^{\bar{i};k^*,t,x_{k^*}} = g^i(X_T^{k^*,t,x_{k^*}}) + \bar{K}_T^{\bar{i};k^*,t,x_{k^*}} - \bar{K}_s^{\bar{i};k^*,t,x_{k^*}} - \int_s^T \bar{Z}^{\bar{i};k^*,t,x_{k^*}} dB_r \\ - \int_s^T \int_E \bar{U}_r^{\bar{i};k^*,t,x_{k^*}}(e) \tilde{\mu}(dr, de) + \int_s^T h^{(i)}\left(r, X_r^{k^*,t,x_{k^*}}, \bar{Y}_r^{\bar{i};k^*,t,x_{k^*}}, \right. \\ \left. \bar{Z}_r^{\bar{i};k^*,t,x_{k^*}}, \int_E \gamma^i(t, X_r^{k^*,t,x_{k^*}}, e)(\bar{u}_i(s, X_{s-}^{k^*,t,x_{k^*}} \right. \\ \left. + \beta(s, X_{s-}^{k^*,t,x_{k^*}}, e)) - \bar{u}_i(s, X_{s-}^{k^*,t,x_{k^*}})) \lambda(de) \right) dr; \\ \text{(iii)} \quad \bar{Y}_s^{\bar{i};k^*,t,x_{k^*}} \geq \ell(s, X_s^{k^*,t,x_{k^*}}) \text{ et } \int_0^T (\bar{Y}_s^{\bar{i};k^*,t,x_{k^*}} - \ell(s, X_s^{k^*,t,x_{k^*}})) d\bar{K}_s^{\bar{i};k^*,t,x_{k^*}} = 0. \end{array} \right. \quad (0.0.78)$$

Par la preuve de l'étape 2 de la proposition 5, $(\bar{Y}^{\bar{i};k^*,t,x}, \bar{K}^{\bar{i};k^*,t,x}, \bar{Z}^{\bar{i};k^*,t,x}, \bar{U}^{\bar{i};k^*,t,x} 1_{\{|e| \geq \frac{1}{k^*}\}})_{k^*}$ converge vers $(\bar{Y}^{\bar{i};t,x}, \bar{K}^{\bar{i};t,x}, \bar{Z}^{\bar{i};t,x}, \bar{U}^{\bar{i};t,x})$ à valeurs dans $\mathcal{S}^2(\mathbb{R}) \times \mathcal{A}_c^2 \times \mathbb{H}^2(\mathbb{R}^{\kappa \times d}) \times \mathbb{H}^2(\mathbb{L}^2(\lambda))$.

Soit $((v_{i=1,m}^{k^*}))_{k^* \geq 1}$ la suite de fonctions déterministes continues, telles que pour tout $t \leq T$ et $s \in [t, T]$,

$$\bar{Y}_s^{\bar{i};k^*,t,x} = \bar{v}_i^{k^*}(s, {}^{k^*}X_s^{t,x}) \text{ and } \bar{Y}_s^{\bar{i};k^*,t,x_{k^*}} = \bar{v}_i^{k^*}(s, {}^{k^*}X_s^{t,x_{k^*}}) \quad \forall i = 1, \dots, m.$$

Telle que par la preuve de la proposition 5 à l'étape 1 et à l'étape 2, nous avons respectivement :

(i) $\bar{U}_s^{\bar{i};k^*,t,x}(e) = (v^i(s, {}^{k^*}X_{s-}^{t,x} + \beta(s, {}^{k^*}X_{s-}^{t,x}, e)) - v^i(s, {}^{k^*}X_{s-}^{t,x}))$, $ds \otimes d\mathbb{P} \otimes d\lambda_{k^*}$ - $p.p$ sur $[t, T] \times \Omega \times E$;

(ii) la suite $((v_{i=1,m}^{k^*}))_{k^* \geq 1}$ converge vers $v^i(t, x)$ en utilisant (0.0.66). Soit $x_{k^*} \rightarrow_{k^*}$

x , où nous prenons l'estimation suivante qui est obtenue par la formule d'Itô et les propriétés de $h^{(i)}$:

$$\mathbb{E} \left[\left| \vec{Y}_s^{k^*, t, x_{k^*}} - Y_s^{k^*, t, x} \right|^2 + \left| K_T^{k^*, t, x_{k^*}} - K_T^{k^*, t, x} \right|^2 \right] \quad (0.0.79)$$

$$+ \int_0^T \left\{ \left| Z_s^{k^*, t, x_{k^*}} - Z_s^{k^*, t, x} \right|^2 + \int_E \left| U_s^{k^*, t, x_{k^*}} - U_s^{k^*, t, x} \right|^2 \lambda_{k^*}(de) \right\} ds \Bigg] \\ \leq \mathbb{E} \left[\left| g(k^* X_T^{t, x_{k^*}}) - g(k^* X_T^{t, x}) \right|^2 \right] + \mathbb{E} \left[\sup_{s \leq T} \left| \ell(k^* X_s^{t, x_{k^*}}) - \ell(k^* X_s^{t, x}) \right|^2 \right] \quad (0.0.80)$$

$$+ C \mathbb{E} \left[\int_s^T \left| \vec{Y}_r^{k^*, t, x_{k^*}} - \vec{Y}_r^{k^*, t, x} \right|^2 dr \right] \\ + C \mathbb{E} \left[\int_0^T \left| X_r^{k^*, t, x_{k^*}} - X_r^{k^*, t, x} \right|^2 (1 + |X_r^{k^*, t, x_{k^*}}|^p + |X_r^{k^*, t, x}|^p)^2 dr \right] \\ + C \sum_{i=1, m} \mathbb{E} \left[\int_s^T \left| B_i \bar{u}^i(r, X_r^{k^*, t, x_{k^*}}) - B_i \bar{u}^i(r, X_r^{k^*, t, x}) \right|^2 dr \right]. \quad (0.0.81)$$

Ensuite, en utilisant (0.0.60) et (0.0.61), la continuité de la fonction $(t, x) \mapsto B_i \bar{u}^i(t, x)$, et le fait qu'il appartienne à Π_g , et d'autre part, la majoration du quatrième terme de (0.0.64) ; nous pouvons utiliser le lemme de Gronwall, pour $s = t$, $\forall i = 1, \dots, m$,

$$v_i^{k^*}(t, x_{k^*}) \longrightarrow_{k^*} v_i^{k^*}(t, x).$$

Donc, par (i)–(ii), nous avons, pour tout $i = 1, \dots, m$,

$$\bar{U}_s^{i; t, x}(e) = (v^i(s, X_{s-}^{t, x} + \beta(s, X_{s-}^{t, x}, e)) - v^i(s, X_{s-}^{t, x})) \quad (0.0.82)$$

$$ds \otimes d\mathbb{P} \otimes d\lambda\text{-a.e. in } [t, T] \times \Omega \times E, \quad \forall i \in \{1, \dots, m\}. \quad (0.0.83)$$

Par ce résultat, nous pouvons remplacer $(\bar{u}_i(s, X_{s-}^{t, x} + \beta(s, X_{s-}^{t, x}, e)) - \bar{u}_i(s, X_{s-}^{t, x}))$ par $\bar{U}_s^{i; t, x}(e)$ dans (0.0.76), nous en déduisons que le quadruplet $(\bar{Y}^{t, x}, \bar{K}^{t, x}, \bar{Z}^{t, x}, \bar{U}^{t, x})$ vérifie que pour $\forall i \in \{1, \dots, m\}$,

$$\left\{ \begin{array}{l} \text{(i)} \quad \bar{Y}^{\vec{t}, x} := (\bar{Y}^{i; t, x})_{i=1, m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \bar{Z}^{t, x} := (\bar{Z}^{i; t, x})_{i=1, m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad \bar{K}^{t, x} := (\bar{K}^{i; t, x})_{i=1, m} \in \mathcal{A}_c^2, \quad \bar{U}^{t, x} := (\bar{U}^{i; t, x})_{i=1, m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} \quad \bar{Y}_s^{i; t, x} = g^i(X_T^{t, x}) + \bar{K}_T^{i; t, x} - \bar{K}_s^{i; t, x} - \int_s^T \bar{Z}_r^{i; t, x} dB_r - \int_s^T \int_E \bar{U}_r^{i; t, x}(e) \tilde{\mu}(dr, de). \\ \quad \quad \quad + \int_s^T h^{(i)}(r, X_r^{t, x}, \bar{Y}_r^{i; t, x}, \bar{Z}_r^{i; t, x}, \int_E \gamma^i(r, X_r^{t, x}, e) \bar{U}_r^{i; t, x} \lambda(de)) dr; \\ \text{(iii)} \quad \bar{Y}_s^{i; t, x} \geq \ell(s, X_s^{t, x}) \text{ and } \int_0^T (\bar{Y}_s^{i; t, x} - \ell(s, X_s^{t, x})) d\bar{K}_s^{i; t, x} = 0. \end{array} \right. \quad (0.0.84)$$

Il s'ensuit que

$$\bar{Y}^{i;t,x} = Y^{i;t,x}, \text{ for } \forall i \in \{1, \dots, m\}.$$

Avec l'unicité de la solution de (0.0.74), nous avons $u^i = \bar{u}^i = v^i$, ce qui signifie que la solution de (0.0.2) au sens de la définition 2 est unique à l'intérieur de la classe \mathcal{U} .

Extension

Dans cette section, nous allons redéfinir la fonction $h^{(i)}$ comme une fonction de $\|U^{i,t,x}\|_{\mathbb{L}^2(\lambda)}$ ($\forall i \in \{1, \dots, m\}$). Et pour montrer que les résultats de la section précédente restent valables, considérons les fonctions $f^{(i)}$, pour tout $i \in \{1, \dots, m\}$, définies par

$$\forall (t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{m+d} \times \mathbb{L}^2(\lambda)$$

et

$$f^{(i)}(t, x, y, z, \zeta) = h^{(i)}(t, x, y, z, \|\zeta\|_{\mathbb{L}^2(\lambda)}),$$

où les fonctions $(h^{(i)})_{i=1,m}$ sont les mêmes que celles définies plus haut. On rappelle que le résultat du théorème 1 est obtenu en ayant principalement $U_s^{t,x}(e) = (u^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u^i(s, X_{s-}^{t,x}))$; cela permet d'avoir la définition 2 en passant par une modification de l'expression de $B_i u^i$ ($\forall i \in \{1, \dots, m\}$).

Nous montrons que $\|U_s^{i,t,x}(e)\|_{\mathbb{L}^2(\lambda)}^2 = \|(u^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u^i(s, X_{s-}^{t,x}))\|_{\mathbb{L}^2(\lambda)}^2$ et que dans ce cas, $B_i u^i$ ($\forall i \in \{1, \dots, m\}$) est bien définie. Maintenant, soit $(t, x) \in [0, T] \times \mathbb{R}^d$ et considérons l'EDSRR m -dimensionnel avec sauts suivant :

$$\left\{ \begin{array}{l} \text{(i)} \quad \vec{Y}^{t,x} := (Y^{i,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad Z^{t,x} := (Z^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad K^{t,x} := (K^{i,t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad U^{t,x} := (U^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \quad \forall i \in \{1, \dots, m\} \quad Y_T^{i;t,x} = g^i(X_T^{t,x}) \text{ et}; \\ \text{(ii)} \quad dY_s^{i;t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i;t,x})_{i=1,m}, Z_s^{i;t,x}, \|U_s^{t,x}(e)\|_{\mathbb{L}^2(\lambda)}) ds \\ \quad \quad - dK_s^{i;t,x} + Z_s^{i;t,x} dB_s + \int_E U_s^{i;t,x}(e) \tilde{\mu}(ds, de), \quad s \leq T; \\ \text{(iii)} \quad Y_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ et } \int_0^T (Y_s^{i;t,x} - \ell(s, X_s^{t,x})) dK_s^{i;t,x} = 0. \end{array} \right. \quad (0.0.85)$$

En supposant que (H1), (H2) et (H3) sont vérifiées et par (0.0.19) dans la proposition 3, il existe une famille de fonctions continues déterministes à croissance polynomiale $(w^i)_{i=1,m}$ et ainsi pour tout $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\forall s \in [t, T], \quad Y_s^{i;t,x} = w^i(s, X_s^{t,x}),$$

telle que par la même proposition, la famille $(w^i)_{i=1,m}$ est une solution de viscosité du

système suivant :

$$\left\{ \begin{array}{l} \min \left\{ w^i(t, x) - \ell(t, x); -\partial_t w^i(t, x) - b(t, x)^\top D_x w^i(t, x) \right. \\ \left. - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 w^i(t, x)) - K_i w^i(t, x) - h^{(i)}(t, x, (w^j(t, x))_{j=1, m}), \right. \\ \left. (\sigma^\top D_x w^i)(t, x), B_i w^i(t, x) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ w^i(T, x) = g^i(x). \end{array} \right. \quad (0.0.86)$$

En effet, en utilisant le lemme 1 et le fait que $U^{i;t, x_k}$ converge vers $U^{i;t, x}$ ($\forall i \in \{1, \dots, m\}$) lorsque $x_k \xrightarrow{k} x$, nous en déduisons que $\|U_s^{i;t, x_k}(e)\|_{\mathbb{L}^2(\lambda)} \xrightarrow{k} \|U_s^{i;t, x}(e)\|_{\mathbb{L}^2(\lambda)}$.

De plus, à partir de la propriété de $h^{(i)}$ et de la preuve de l'étape 2 dans le théorème 1 (unicité de la solution de viscosité),

$$\|U_s^{i;t, x}(e)\|_{\mathbb{L}^2(\lambda)}^2 = \|(w^i(s, X_{s-}^{t, x} + \beta(s, X_{s-}^{t, x}, e)) - w^i(s, X_{s-}^{t, x}))\|_{\mathbb{L}^2(\lambda)}^2,$$

D'où

$$B_i w^i = \left\{ \int_E |(w^i(s, X_{s-}^{t, x} + \beta(s, X_{s-}^{t, x}, e)) - w^i(s, X_{s-}^{t, x}))|^2 \lambda(\text{de}) \right\}^{\frac{1}{2}}.$$

Du fait du corollaire 1, nous en déduisons que $B_i w^i$ ($\forall i \in \{1, \dots, m\}$) est bien défini.

Résultats généraux du second papier

Cette partie traite d'une classe d'équations différentielles stochastiques rétrogrades conduites par un mouvement brownien fractionnaire (avec un paramètre de Hurst H supérieur à $1/2$) avec des générateurs temporisés. Dans ce type d'équation, un générateur à l'instant t peut dépendre des valeurs d'une solution dans le passé, pondérées par une fonction de retard temporel, par exemple de type moyenne mobile. Nous établissons un résultat d'existence et d'unicité des solutions pour un horizon de temps suffisamment petit ou pour une constante de Lipschitz suffisamment petite d'un générateur. L'intégrale stochastique utilisée tout au long de cette partie est l'intégral de type opérateur de divergence.

Dans cette partie, notre objectif est de généraliser le résultat établi dans [22] à ce EDSR fractionnaire:

$$Y(t) = \xi + \int_t^T f(s, \eta_s, Y_s, Z_s) ds - \int_t^T Z(t) dB_s^H \quad 0 \leq t \leq T, \quad (0.0.87)$$

où B^H est un mouvement Brownien fractionnaire avec paramètre de Hurst supérieur à $1/2$,

η est un processus stochastique donné par $\eta_t = \eta_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dB_s^H$, et

$(Y_t, Z_t) = (Y(t+u), Z(t+u))_{-T \leq u \leq 0}$. A l'aide du principe du point fixe, nous établirons notre résultat principal.

Préliminaires

Soient Ω un ensemble non vide, \mathcal{F} une σ -algèbre d'ensembles de Ω et \mathbb{P} une mesure de probabilité définie sur \mathcal{F} . Le triplet $(\Omega, \mathcal{F}, \mathbb{P})$ définit un espace de probabilité et \mathbb{E} l'espérance mathématique par rapport à la mesure de probabilité \mathbb{P} .

Rappelons que, pour $H \in (0, 1)$, un mouvement brownien fractionnaire $(B^H(t))_{t \geq 0}$ de paramètre de Hurst H est continu et processus gaussien centré avec covariance

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H} \text{ droite}), \quad t, s \geq 0.$$

Notons $\phi(x) = H(2H-1)|x|^{2H-2}$ $x \in \mathbb{R}$. Soient ξ et η des fonctions mesurables sur $[0, T]$. Définissons $\langle \eta, \xi \rangle_t = \int_0^t \int_0^t \phi(u-v)\xi(u)\eta(v)dudv$ et $\|\xi\|_t^2 = \langle \xi, \xi \rangle_t$. Notons que, pour tout $t \in [0, T]$, $\langle \eta, \xi \rangle_t$ est le produit scalaire au sens de Hilbert. Soit \mathcal{H} la complétion de l'ensemble des fonctions mesurables telles que $\|\xi\|_t^2 < +\infty$.

Pour une fonction polynomiale de k variables f , soit \mathcal{P}_T l'ensemble de tous les polynômes de mouvement brownien fractionnaire de la forme

$$F(\omega) = f \left(\int_0^T \xi_1(t)dB_t^H(\omega), \int_0^T \xi_2(t)dB_t^H(\omega), \dots, \int_0^T \xi_k(t)dB_t^H(\omega) \right)$$

où $(\xi_n)_{n \in \mathbb{N}}$ est une suite dans \mathcal{H} telle que $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$. La dérivée de Malliavin de $F \in \mathcal{P}_T$ est donnée par

$$D_s^H F = \sum_{i=1}^k \frac{\partial f}{\partial x_i} \left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \dots, \int_0^T \xi_k(t)dB_t^H \right) \xi_i(s), \quad 0 \leq s \leq T.$$

Il est bien connu que l'opérateur de divergence $D^H : L^2(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\Omega, \mathcal{F}, \mathcal{H})$ est fermable. On peut donc considérer que l'espace $\mathbb{D}_{1,2}$ est le complément de \mathcal{P}_T de norme

$$\|F\|_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}\|D_s^H F\|_T^2.$$

On définit $\mathbb{D}_s^H F = \int_0^T \phi(s-r)D_r^H F dr$ et on note $\mathbb{L}_H^{1,2}$ l'espace de tous les processus stochastiques $F : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathcal{H}$ tel que

$$\mathbb{E} \left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 ds dt \right) < +\infty.$$

On a le théorème suivant (voir [43, Proposition 6.25])

Théorème 2. Si $F \in \mathbb{L}_H^{1,2}$, alors l'intégrale stochastique de type Itô-Skorohod $\int_0^T F_s dB_s^H$

existe dans $L^2(\Omega, \mathcal{F}, \mathbb{P})$ et satisfait

$$\mathbb{E} \left(\int_0^T F_s dB_s^H \right) = 0 \quad \text{et} \quad \mathbb{E} \left(\int_0^T F_s dB_s^H \right)^2 = \mathbb{E} \left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_t^H F_s \mathbb{D}_s^H F_t ds dt \right).$$

L'intégration par parties suivante donnée dans [43, Théorème 11.1] sera utile dans la suite

Theorem 0.0.1. *Soit $f_i(s), g_i(s), i = 1, 2$, sont dans $\mathbb{D}_{1,2}$ et $\mathbb{E} \int_0^T (|f_i(s)| + |g_i(s)|) ds < +\infty$. Supposons que $\mathbb{D}_t^H f_1(s)$ et $\mathbb{D}_t^H f_2(s)$ sont continûment différentiables par rapport à $(s, t) \in [0, T] \times [0, T]$ pour tout $\omega \in \Omega$. Supposons que $\mathbb{E} \int_0^T \int_0^T |\mathbb{D}_t^H f_i(s)|^2 ds dt < +\infty$ et notons*

$$X_i(t) = \int_0^t g_i(s) ds + \int_0^t f_i(s) dB_s^H, \quad i = 1, 2.$$

Alors nous avons pour $0 \leq t \leq T$,

$$\begin{aligned} X_1(t)X_2(t) &= \int_0^t X_1(s)g_2(s)ds + \int_0^t X_1(s)f_2(s)dB_s^H + \int_0^t X_2(s)g_1(s)ds \\ &\quad + \int_0^t X_2(s)f_1(s)dB_s^H + \int_0^t \mathbb{D}_s^H X_1(s)f_2(s)ds + \int_0^t \mathbb{D}_s^H X_2(s)f_1(s)ds. \end{aligned}$$

Terminons cette section en donnant une formule d'Itô pour l'intégrale de type divergence (voir [43, Theorem 10.3]).

Theorem 0.0.2. *(voir [26]) Soient μ et σ des fonctions différentielles stochastiques. Si*

$$X(t) = X(0) + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dB_s^H, \quad 0 \leq t \leq T,$$

nous avons

$$\begin{aligned} \varphi(t, X(t)) &= \varphi(0, X(0)) + \int_0^t \frac{\partial \varphi}{\partial s}(s, X(s))ds + \int_0^t \frac{\partial \varphi}{\partial x}(s, X(s))\mu(X(s)) ds \quad (0.0.88) \\ &\quad + \int_0^t \frac{\partial \varphi}{\partial x}(s, X(s))\sigma(X(s))dB_s^H + \int_0^t \frac{\partial^2 \varphi}{\partial x^2}(s, X_s)\sigma(X(s))\mathbb{D}_s^H X_s ds. \end{aligned}$$

où

$$\begin{aligned} \mathbb{D}_s^H X_t &= \int_0^t \mu'(X(r))\mathbb{D}_s^H X_r + \int_0^t \sigma'(X(r))\mathbb{D}_s^H X_r \\ &\quad + H(2H - 1) \int_0^t \sigma(X(s))|s - r|^{2H-2} dr. \end{aligned}$$

Grâce à ces résultats préliminaires, nous sommes maintenant en mesure d'étudier notre sujet principal.

Equations différentielles stochastiques progressives rétrogrades avec générateurs temporisés

Définitions et notations

Considérons le processus suivant

$$\eta_t = \eta_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s^H \quad 0 \leq t \leq T,$$

où η_0 , b et σ satisfont:

- $\eta_0 \in \mathbb{R}$ est une constante;
- $b, \sigma : [0, T] \rightarrow \mathbf{R}$ sont des fonctions déterministes continues, σ est différentiable et $\sigma(t) \neq 0, t \in [0, T]$.

Notons que,

$$\|\sigma\|_t^2 = H(2H-1) \int_0^t \int_0^t |u-v|^{2H-2} \sigma(u)\sigma(v)du dv,$$

nous avons

$$\frac{d}{dt} (\|\sigma\|_t^2) = \sigma(t)\hat{\sigma}(t) > 0 \quad \text{where } \hat{\sigma}(t) = \int_0^t \phi(t-v)\sigma(v)dv, \quad 0 \leq t \leq T.$$

Nous introduisons les ensembles suivants (où \mathbb{E} désigne l'espérance mathématique par rapport à la mesure de probabilité \mathbb{P}) :

- $\mathcal{C}_{\text{pol}}^{1,2}([0, T] \times \mathbb{R})$ est l'espace de toutes les fonctions de classe $\mathcal{C}^{1,2}$ sur $[0, T] \times \mathbb{R}$, qui, avec leurs dérivées, sont à croissance polynomiale,
- $\mathcal{V}_{[0,T]} = \left\{ Y(\cdot) = \psi(\cdot, \eta_\cdot); \quad \psi \in \mathcal{C}_{\text{pol}}^{1,2}([0, T] \times \mathbb{R}), \quad \frac{\partial \psi}{\partial t} \text{ est borné, } t \in [0, T] \right\}$,
- $\tilde{\mathcal{V}}_T^{H,\beta}$ la complétion de $\mathcal{V}_{[0,T]}$ sous la norme suivante (où $\beta > 0$)

$$\left\| Y \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}} = \left(\int_0^T t^{2H-1} \mathbb{E}[e^{\beta t} |Y(t)|^2] dt \right)^{1/2} = \left(\int_0^T t^{2H-1} \mathbb{E}[e^{\beta t} |\psi(t, \eta_t)|^2] dt \right)^{1/2}.$$

- $\mathcal{B}^2([0, T], \mathbb{R}) = \tilde{\mathcal{V}}_{[0,T]}^{1/2,\beta} \times \tilde{\mathcal{V}}_{[0,T]}^{H,\beta}$ est un espace de Banach avec la norme

$$\left\| (Y, Z) \right\|_{\mathcal{B}^2}^2 = \left\| Y \right\|_{\tilde{\mathcal{V}}_T^{1/2,\beta}}^2 + \left\| Z \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2$$

- Notons enfin $\mathbf{L}_{-T}^2(\mathbb{R})$ l'espace des fonctions mesurables $\varphi : [-T, 0] \rightarrow \mathbb{R}$ satisfaisant

$$\int_{-T}^0 |\varphi(t)|^2 dt < +\infty.$$

Comme rappeler plus haut, le but de cette partie est d'étudier les équations différentielles stochastiques rétrogrades dirigées par le mouvement brownien fractionnaire avec des générateurs temporisés dont la dynamique est décrite par

$$Y(t) = \xi + \int_t^T f(r, \eta_s, Y_s, Z_s) ds - \int_t^T Z(s) dB_s^H, \quad 0 \leq t \leq T, \quad (0.0.89)$$

où le générateur f au temps $s \in [0, T]$ dépend des valeurs passées de la solution notées $Y_s := Y(s+u)_{-T \leq u \leq 0}$ et $Z_s := Z(s+u)_{-T \leq u \leq 0}$. On fixe toujours $Z(t) = 0$ et $Y(t) = Y(0)$ pour $t < 0$.

Plus précisément, nous établissons un résultat d'existence et d'unicité pour (0.0.127) sous les hypothèses suivantes :

- **(H1)**: $\xi = h(\eta_T)$ pour une fonction h avec dérivée bornée, $\mathbb{E} [e^{\beta T} |\xi|^2] < +\infty$;
- **(H2)**: le générateur $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbf{L}_{-T}^2(\mathbb{R}) \times \mathbf{L}_{-T}^2(\mathbb{R}) \rightarrow \mathbb{R}$ est un produit mesurable, \mathbb{F} -adapté et Lipschitz continu en ce sens que pour une mesure de probabilité α sur $([-T, 0] \times \mathcal{B}([-T, 0]))$

$$|f(t, x, y_t, z_t) - f(t, x, y'_t, z'_t)|^2 \leq K \int_{-T}^0 (|y(t+u) - y'(t+u)|^2 + |z(t+u) - z'(t+u)|^2) \alpha(du),$$

pour $\mathbb{P} \otimes \lambda$ -p.p. $(\omega, t) \in \Omega \times [0, T]$ et pour tout $x \in \mathbb{R}$; $y, z, y', z' \in \mathbf{L}_{-T}^2(\mathbb{R})$.

- **(H3)** : $f(s, \cdot, \cdot, \cdot) = 0$ pour $t < 0$.

Definition 0.0.3. *Un couple de processus (Y, Z) est appelé une solution d'EDSR fractionnaire (0.0.127) avec des paramètres (ξ, f) , s'il satisfait eq.(0.0.127) avec $(Y, Z) \in \mathcal{B}^2([0, T], \mathbb{R})$.*

Existence et unicité de solution

Grâce à ce résultat, nous avons la représentation donnée dans la [5, Proposition 3.6].

Proposition 6. *Soit $(Y_t, Z_t)_{0 \leq t \leq T}$ une solution de l'EDSR fractionnaire (0.0.127). Alors:*

(i) *Nous avons la représentation stochastique*

$$\mathbb{D}_t^H Y_t = \frac{\hat{\sigma}(t)}{\sigma(t)} Z_t, \quad 0 \leq t \leq T,$$

(ii) *Il existe une constante $M > 0$ telle que*

$$\frac{t^{2H-1}}{M} \leq \frac{\hat{\sigma}(t)}{\sigma(t)} \leq M t^{2H-1}, \quad 0 \leq t \leq T.$$

Commentaire de la proposition 6:

► Le (i): $\mathbb{D}_t^H Y_t = \frac{\hat{\sigma}(t)}{\sigma(t)} Z_t$, $0 \leq t \leq T$, est prouvé pour $t \in [0, T]$. Mais pour le cas "temporisé", nous avons $t + u < 0 \Rightarrow Z(t + u) = 0$; aussi pour $t + u \in [0, T]$, $\mathbb{D}_t^H Y_t = \frac{\hat{\sigma}(t)}{\sigma(t)} Z_t$,

► Le (ii): $\frac{t^{2H-1}}{M} \leq \frac{\hat{\sigma}(t)}{\sigma(t)} \leq Mt^{2H-1}$, $0 \leq t \leq T$. est obtenu pour $t \in [0, T]$ et son utilisation est combinée avec $\mathbb{D}_t^H Y_t = \frac{\hat{\sigma}(t)}{\sigma(t)} Z_t$, (voir [54]) qui reste vrai pour $t + u \in [0, T]$ et le (ii) reste valable pour $t + u \in [0, T]$

Nous sommes en mesure de prouver notre résultat principal.

Résultat principal

Théorème 3. *Sous les hypothèses (H1), (H2) et (H3). Pour un horizon temporel T suffisamment petit ou pour une constante de Lipschitz K suffisamment petite, l'équation différentielle stochastique rétrograde (0.0.127) a une solution unique $(Y, Z) \in \mathcal{B}^2([0, T], \mathbb{R})$.*

Sketch de la preuve 6. *Pour prouver l'existence et l'unicité d'une solution, nous suivons l'idée classique en construisant un schéma de Picard et montrons sa convergence. Pour cela, considérons maintenant la suite $(Y^n, Z^n)_{n \geq 0}$ donnée par*

$$\begin{cases} Y^0(t) = Z^0(t) = 0; \\ Y^{n+1}(t) = \xi + \int_t^T f(s, \eta_s, Y_s^n, Z_s^n) ds - \int_t^T Z^{n+1}(t) dB_s^H, \quad 0 \leq t \leq T. \end{cases} \quad (0.0.90)$$

Soit $(Y^n, Z^n)_{n \geq 0} \in \mathcal{B}^2([0, T], \mathbb{R})$, l'équation (0.0.90) à une unique solution $(Y^{n+1}, Z^{n+1})_{n \geq 0} \in \mathcal{B}^2([0, T], \mathbb{R})$ (see, [22] p.1517 et [46] p.57-58).

Définissons le processus $\delta \in \{Y, Z\}$, $n, m \geq 1$, $\bar{\delta}^{n+1} = \delta^{n+1} - \delta^n$. On voit facilement que la paire de processus $(\bar{Y}^{n+1}(t), \bar{Z}^{n+1}(t))_{0 \leq t \leq T}$ résout l'équation

$$\bar{Y}^{n+1}(t) = \int_t^T \left(f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right) ds - \int_t^T \bar{Z}^{n+1}(s) dB_s^H, \quad 0 \leq t \leq T.$$

Par la formule d'Itô, nous avons

$$\begin{aligned} d|\bar{Y}_t^{n+1}|^2 &= -2\bar{Y}_t^{n+1}(t) \left(f(t, \eta_t, Y_t^n, Z_t^n) - f(t, \eta_t, Y_t^{n-1}, Z_t^{n-1}) \right) dt \\ &\quad + 2\bar{Y}_t^{n+1}(t) Z_t^{n+1}(t) dB_t^H + 2\mathbb{D}_t^H \bar{Y}_t^{n+1}(t) \bar{Z}_t^{n+1}(t) dt. \end{aligned}$$

En appliquant la formule d'Itô's à nouveau $e^{\beta t} |\bar{Y}^{n+1}(t)|^2$, (où β sera choisi ultérieure-

ment), nous obtenons pour $n \geq 1$ et $0 \leq t \leq T$,

$$\begin{aligned} e^{\beta t} |\bar{Y}^{n+1}(t)|^2 &= 2 \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \left(f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right) ds \\ &\quad - 2 \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s) dB_s^H - 2 \int_t^T e^{\beta s} \mathbb{D}_s^H \bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s) ds \\ &\quad - \beta \int_t^T e^{\beta s} |\bar{Y}^{n+1}(s)|^2 ds. \end{aligned}$$

En utilisant le fait que $(\bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s))_{t \leq s \leq T} \in \tilde{\mathcal{V}}_{[0,T]}^{1/2, \beta} \times \tilde{\mathcal{V}}_{[0,T]}^{H, \beta}$ et $\mathcal{V}_{[0,T]} \subset \mathbb{L}_H^{1,2}$ (voir lemme 8 dans [54]) ce qui implique en particulier $F_s = e^{\beta s} \bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s) \in \mathbb{L}_H^{1,2}$.

Alors par le théorème 2, nous avons $\mathbb{E} \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s) dB_s^H = 0$.

Prenons l'espérance mathématique des deux côtés, alors nous avons

$$\begin{aligned} \mathbb{E} \left(e^{\beta t} |\bar{Y}^{n+1}(t)|^2 \right) &= 2 \mathbb{E} \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \left(f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right) ds \\ &\quad - 2 \mathbb{E} \int_t^T e^{\beta s} \mathbb{D}_s^H \bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s) ds - \beta \mathbb{E} \int_t^T e^{\beta s} |\bar{Y}^{n+1}(s)|^2 ds. \quad (0.0.91) \end{aligned}$$

Il est connu que $\mathbb{D}_s^H \bar{Y}^{n+1}(s) = \frac{\hat{\sigma}(s)}{\sigma(s)} \bar{Z}^{n+1}(s)$ et il existe $M > 0$ telle que pour tout $t \in [0, T]$, $\frac{t^{2H-1}}{M} \leq \frac{\hat{\sigma}(t)}{\sigma(t)} \leq M t^{2H-1}$.

Ainsi, nous déduisons de (0.0.91)

$$\begin{aligned} \mathbb{E} \left(e^{\beta t} |\bar{Y}^{n+1}(t)|^2 + \beta \int_t^T e^{\beta s} |\bar{Y}^{n+1}(s)|^2 ds + \frac{2}{M} \int_t^T e^{\beta s} s^{2H-1} |\bar{Z}^{n+1}(s)|^2 ds \right) \\ \leq 2 \mathbb{E} \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \left(f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right) ds. \quad (0.0.92) \end{aligned}$$

Par $2ab \leq \lambda a^2 + b^2/\lambda$ pour $\lambda > 0$, nous déduisons que

$$\begin{aligned} 2 \mathbb{E} \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \left(f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right) ds &\leq \lambda \mathbb{E} \int_t^T e^{\beta s} |\bar{Y}^{n+1}(s)|^2 ds \\ &\quad + \frac{1}{\lambda} \mathbb{E} \int_t^T e^{\beta s} \left| f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right|^2 ds, \end{aligned}$$

ce qui implique

$$\begin{aligned} (\beta - \lambda) \mathbb{E} \int_0^T e^{\beta s} |\bar{Y}^{n+1}(s)|^2 ds + \frac{2}{M} \mathbb{E} \int_0^T e^{\beta s} s^{2H-1} |\bar{Z}^{n+1}(s)|^2 ds \\ \leq \frac{1}{\lambda} \mathbb{E} \int_0^T e^{\beta s} \left| f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right|^2 ds. \quad (0.0.93) \end{aligned}$$

De l'hypothèse **(H2)**, nous obtenons

$$\begin{aligned}
& \frac{1}{\lambda} \mathbb{E} \int_0^T e^{\beta s} \left| f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right|^2 ds \\
& \leq \frac{K}{\lambda} \mathbb{E} \int_0^T e^{\beta s} \left[\int_{-T}^0 \left(|\bar{Y}^n(s+u)|^2 + |\bar{Z}^n(s+u)|^2 \right) \alpha(du) \right] ds \\
& \leq \frac{K}{\lambda} (I_1 + I_2), \tag{0.0.94}
\end{aligned}$$

où $I_1 := \mathbb{E} \int_0^T e^{\beta s} \int_{-T}^0 |\bar{Y}^n(s+u)|^2 \alpha(du) ds$ et $I_2 := \mathbb{E} \int_0^T e^{\beta s} \int_{-T}^0 |\bar{Z}^n(s+u)|^2 \alpha(du) ds$.

L'utilisation du théorème de Fubini conduit à

$$\begin{aligned}
I_1 & := \mathbb{E} \int_0^T e^{\beta s} \int_{-T}^0 |\bar{Y}^n(s+u)|^2 \alpha(du) ds \leq \mathbb{E} \int_{-T}^0 e^{-\beta u} \left(\int_0^T e^{\beta(s+u)} |\bar{Y}^n(s+u)|^2 ds \right) \alpha(du) \\
& \leq \int_{-T}^0 e^{-\beta u} \mathbb{E} \left(\int_u^{T+u} e^{\beta(v)} |\bar{Y}^n(v)|^2 dv \right) \alpha(du) \\
& \leq \frac{e^{\beta T}}{\beta} \left\| \bar{Y}^n \right\|_{\tilde{\mathcal{V}}_T^{\frac{1}{2}, \beta}}^2; \tag{0.0.95}
\end{aligned}$$

et

$$\begin{aligned}
I_2 & := \mathbb{E} \int_0^T e^{\beta s} \int_{-T}^0 |\bar{Z}^n(s+u)|^2 \alpha(du) ds \leq \mathbb{E} \int_{-T}^0 e^{-\beta u} \left(\int_0^T e^{\beta(s+u)} |\bar{Z}^n(s+u)|^2 ds \right) \alpha(du) \\
& \leq \int_{-T}^0 e^{-\beta u} \mathbb{E} \left(\int_u^{T+u} \frac{1}{v^{2H-1}} e^{\beta(v)} v^{2H-1} |\bar{Z}^n(v)|^2 dv \right) \alpha(du) \\
& \leq \int_{-T}^0 e^{-\beta u} \sup_{u \leq v \leq T+u} \left(\frac{1}{v^{2H-1}} \right) \alpha(du) \mathbb{E} \left(\int_u^{T+u} e^{\beta(v)} v^{2H-1} |\bar{Z}^n(v)|^2 dv \right) \\
& \leq \int_{-T}^0 \frac{e^{-\beta u}}{u^{2H-1}} \alpha(du) \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H, \beta}}^2; \tag{0.0.96}
\end{aligned}$$

où $\sup_{u \leq v \leq T+u} \left(\frac{1}{v^{2H-1}} \right) \leq \frac{1}{u^{2H-1}}$ (rappelons que $1/2 \leq H \leq 1$).

En utilisant l'inégalité de Hölder, nous déduisons que

$$\begin{aligned}
I_2 &\leq \left(\int_{-T}^0 e^{-2\beta u} \alpha(du) \right)^{\frac{1}{2}} \left(\int_{-T}^0 u^{2(1-2H)} \alpha(du) \right)^{\frac{1}{2}} \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 \\
&\leq \left(\frac{e^{2\beta T}}{2\beta} \right)^{\frac{1}{2}} \sqrt{T^{2(1-2H)}} \times \sqrt{T} \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 \\
&\leq \frac{e^{\beta T} T^{1-2H} \times \sqrt{T}}{\sqrt{2\beta}} \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 ; \text{ where } \sup_{-T \leq u \leq 0} (u^{2(1-2H)}) \leq T^{2(1-2H)}
\end{aligned}$$

En rassemblant les inégalités obtenues plus haut, nous déduisons de (0.093) que

$$\begin{aligned}
(\beta - \lambda) \left\| \bar{Y}^{n+1} \right\|_{\tilde{\mathcal{V}}_T^{\frac{1}{2},\beta}}^2 + \frac{2}{M} \left\| \bar{Z}^{n+1} \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 &\leq \frac{K e^{\beta T}}{\lambda \beta} \left\| \bar{Y}^n \right\|_{\tilde{\mathcal{V}}_T^{\frac{1}{2},\beta}}^2 + \frac{K e^{\beta T} T^{1-2H} \times \sqrt{T}}{\lambda \sqrt{2\beta}} \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 \\
&\leq \frac{K C e^2}{\beta \lambda} \left(\left\| \bar{Y}^n \right\|_{\tilde{\mathcal{V}}_T^{\frac{1}{2},\beta}}^2 + \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 \right), \quad (0.097)
\end{aligned}$$

avec $T = \frac{2}{\beta}$ et $C = \max \{1, T^{1-2H}\}$.

Par conséquent, si nous choisissons $\lambda = \lambda_0 = \frac{1}{T}$ satisfaisant $\beta - \lambda_0 = \frac{2}{M}$, alors nous obtenons

$$\left\| (\bar{Y}^{n+1}, \bar{Z}^{n+1}) \right\|_{\mathcal{B}^2}^2 \leq \frac{T^3 K C e^2}{2} \left\| (\bar{Y}^n, \bar{Z}^n) \right\|_{\mathcal{B}^2}^2. \quad (0.098)$$

Pour un T suffisamment petit ou pour un K suffisamment petit, l'inégalité (0.098) est une contraction, et il existe une unique limite $(Y, Z) \in \mathcal{B}^2([0, T], \mathbb{R})$ d'une suite convergente $(Y^n, Z^n)_{n \in \mathbb{N}}$ qui satisfait l'équation du point fixe

$$Y(t) = \xi + \int_t^T f(r, \eta_s, Y_s, Z_s) ds - \int_t^T Z(s) dB_s^H, \quad 0 \leq t \leq T.$$

Avec les mêmes arguments, en répétant la technique ci-dessus, nous obtenons une unicité de la solution des Équations différentielles stochastiques progressives rétrogrades avec générateurs temporisés sur $[0, T]$.

Résultats généraux du troisième papier

Cette partie étudie les équations différentielles doublement stochastiques rétrogrades généralisées anticipées (EDDSRGA) dont les coefficients satisfont une condition non-lipschitzienne. Nous avons eu à établir l'existence et l'unicité est formulé et prouvé pour ces types d'équations. Un théorème de comparaison a été également établi.

Préliminaires

Soient Ω un ensemble non vide, \mathcal{F} une σ -algèbre d'ensembles de Ω et \mathbb{P} une mesure de probabilité définie sur \mathcal{F} . Le triplet $(\Omega, \mathcal{F}, \mathbb{P})$ définit un espace de probabilité, supposé complet. Pour un fixe réel $0 < T \leq \infty$, nous supposons donnés deux processus mutuellement indépendants

- un mouvement brownien ℓ -dimensionnel $(B_t)_{0 \leq t \leq T}$,
- un mouvement brownien d -dimensionnel $(W_t)_{0 \leq t \leq T}$.

Pour tout $t \in [0, T + K]$, on définit :

$$\mathcal{F}_t = \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,T+K}^B, \quad 0 \leq t \leq T + K,$$

où pour tout processus $\{\phi_t\}_{t \geq 0}$, $\mathcal{F}_{s,t}^\phi = \sigma\{\phi_r - \phi_s, ; s \leq r \leq t\} \vee \mathcal{N}$, $;$ $\mathcal{F}_t^\phi = \mathcal{F}_{0,t}^\phi$. \mathcal{N} dénote la classe de \mathbb{P} -ensembles nuls de \mathcal{F} . Notez que $(\mathcal{F}_t)_{0 \leq t \leq T}$ ne constitue pas une filtration classique.

Soit Λ_t un processus continu, croissant et \mathcal{F}_t -adapté sur $[0, T + K]$ avec $\Lambda_0 = 0$. Pour $k \in \mathbb{N}^*$ $\mu > 0$ et $\beta > 0$, on considère les ensembles suivants (où \mathbb{E} dénote l'espérance mathématique par rapport à la mesure de probabilité \mathbb{P}) :

- $L^2(\mathcal{F}_T, \mathbb{R}^k)$ l'espace de \mathcal{F}_T -variable aléatoire mesurable telle que

$$\mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 \right] < +\infty;$$

- $\mathcal{S}_{[0,T]}^2(\mathcal{F}, \mathbb{R}^k)$ l'espace des processus \mathcal{F}_t -adaptés càdlàg

$$\Psi : [0, T] \times \Omega \longrightarrow \mathbb{R}^k, \quad \|\Psi\|_{\mathcal{S}^2}^2 = \mathbb{E} \left(\sup_{0 \leq t \leq T} e^{\mu t + \beta \Lambda_t} |\Psi_t|^2 \right) < +\infty;$$

- $\mathcal{M}_{[0,T]}^2(\mathcal{F}, \mathbb{R}^{k \times d})$ l'espace de \mathcal{F}_t -processus progressivement mesurables

$$\Psi : [0, T] \times \Omega \longrightarrow \mathbb{R}^k, \quad \|\Psi\|_{\mathcal{M}^2(\mathbb{R}^{k \times d})}^2 = \mathbb{E} \int_0^T e^{\mu t + \beta \Lambda_t} |\Psi_t|^2 dt < +\infty;$$

- $\mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k) = \mathcal{S}_{[0,T]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[0,T]}^2(\mathcal{F}, \mathbb{R}^{k \times d})$ doté de la norme

$$\|(Y, Z)\|_{\mathcal{B}_{\mathcal{F}}^2(T, \mathbb{R}^k)}^2 = \|Y\|_{\mathcal{S}^2(\beta, \mathbb{R}^k)}^2 + \|Z\|_{\mathcal{M}^2(\beta, \mathbb{R}^{k \times d})}^2;$$

- **S** soit l'ensemble de toutes les fonctions non décroissantes, continues et concaves $\rho(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfaisant $\rho(0) = 0$; on peut trouver une paire de constantes

positives a et b telles que
 $\rho(v) \leq a + bv$ et $\int_{0^+} \frac{du}{\rho(u)} = +\infty$.

Étant donné ξ une variable aléatoire à valeur réelle mesurable et les fonctions

$$F, G : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow L^2(\mathcal{F}_t, \mathbb{R}^k) \quad \text{et} \quad H : \Omega \times [0, T] \times \mathbb{R}^k \rightarrow L^2(\mathcal{F}_t, \mathbb{R}^k).$$

Alors, d'après le théorème 2.1 de [14], l'EDSR généralisée suivante de paramètres (ξ, F, G, H, Λ) ,

$$Y_t = \xi_T + \int_t^T F(s, Y_s, Z_s) ds + \int_t^T G(s, Y_s, Z_s) dB_s + \int_t^T H(s, Y_s) d\Lambda_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (0.0.99)$$

admet une solution unique $(Y, Z) \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$ sous les conditions de Lipschitz.

Pour chaque $t \in [0, T]$, nous supposons trois fonctions :

$$\begin{aligned} f &: \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^{k \times d}) \rightarrow L^2(\mathcal{F}_t, \mathbb{R}^k), \\ g &: \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^{k \times d}) \rightarrow L^2(\mathcal{F}_t, \mathbb{R}^k), \\ h &: \Omega \times [0, T] \times \mathbb{R}^k \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \rightarrow L^2(\mathcal{F}_t, \mathbb{R}^k). \end{aligned}$$

Nous nous intéressons à l'EDDSRGA

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) ds + \int_t^T g(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) dB_s \\ \quad + \int_t^T h(s, Y_s, Y_{s+\delta(s)}) d\Lambda_s - \int_t^T Z_s dW_s, \quad t \in [0, T] \\ Y_t = \xi_t, \quad Z_t = \eta_t, \quad t \in [T, T+K], \end{cases} \quad (0.0.100)$$

où K est une constante positive, $\delta, \zeta : [0, T] \rightarrow \mathbb{R}_+$ sont des fonctions continues et pour $\sigma(\cdot) = \delta(\cdot)$, $z(\cdot) : \text{tel que}$:

(A1) : il existe une constante $K \geq 0$ telle que pour tout $t \in [0, T]$, $t + \sigma(t) \leq T + K$;

(A2) : Il existe $M \geq 0$ tel que pour $0 \leq t \leq T$ et une fonction positive, intégrable Φ ,

$$\int_t^T \Phi(s + \sigma(s)) ds \leq M \int_t^{T+K} \Phi(s) ds \quad \text{et} \quad \int_t^T \Phi(s + \sigma(s)) d\Lambda_s \leq M \int_t^{T+K} \Phi(s) d\Lambda_s.$$

Remarque 5. Si l'hypothèse **(A2)** est vérifiée, alors pour tout $t \in [0, T]$, $\beta > 0$, $\mu > 0$ et une fonction positive, intégrable $\Phi(\cdot)$, nous avons :

$$\int_t^T e^{\mu s + \beta \Lambda_s} \Phi(s + \sigma(s)) ds \leq M \int_t^{T+K} e^{\mu s + \beta \Lambda_s} \Phi(s) ds, \quad ; \sigma \in \{\delta, \zeta\}.$$

En fait,

$$\begin{aligned} \int_t^T e^{\mu s + \beta \Lambda_s} \Phi(s + \sigma(s)) ds &= \int_t^T e^{-\mu \sigma(s)} e^{\mu(s + \sigma(s)) + \beta \Lambda_s} \Phi(s + \sigma(s)) ds \\ &\leq \int_t^T e^{-\mu \sigma(s)} e^{\mu(s + \sigma(s)) + \beta \Lambda_{(s + \sigma(s))}} \Phi(s + \sigma(s)) ds \\ &\leq M \int_t^{T+K} e^{\mu s + \beta \Lambda_s} \Phi(s) ds. \end{aligned}$$

Similairement, nous avons

$$\int_t^T e^{\mu s + \beta \Lambda_s} \Phi(s + \sigma(s)) d\Lambda_s \leq M \int_t^{T+K} e^{\mu s + \beta \Lambda_s} \Phi(s) d\Lambda_s.$$

Rappelons le résultat suivant donné dans [[73], Theorem 1]:

Proposition 7. *Supposons qu'il existe $K > 0$ tel que f et g soient des fonctions K -lipschitz et que h satisfasse la condition de monotonie. Alors (0.0.100) a une solution unique $(Y_t, Z_t) \in \mathcal{B}_{\mathcal{F}}^2([0, T + K], \mathbb{R}^k)$.*

Avant d'étudier la solution de l'EDDSRGA (0.0.100) et d'établir son théorème de comparaison, nous donnons quelques résultats de l'EDDSRG (0.0.99), qui joueront un rôle clé dans ce qui suit.

EDDSRs Généralisés

Dans cette partie, nous nous intéressons aux équations différentielles doublement stochastiques généralisées suivantes

$$\begin{aligned} Y_t &= \xi_T + \int_t^T F(s, Y_s, Z_s) ds + \int_t^T G(s, Y_s, Z_s) dB_s \\ &\quad + \int_t^T H(s, Y_s) d\Lambda_s - \int_t^T Z_t dW_s, \quad 0 \leq t \leq T. \end{aligned} \quad (0.0.101)$$

On suppose qu'il existe $\rho \in \mathbf{S}$ tel que ξ, F, G et H vérifient les hypothèses **(H1)** :

(H1.1) il existe $C > 0$, pour tout $t \in [0, T]$, $(t, y, z), (t, y', z') \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$, nous avons

$$\left| f(t, y, z) - f(t, y', z') \right|^2 \vee \left| g(t, y, z) - g(t, y', z') \right|^2 \leq C \left(\rho (|y - y'|^2) + |z - z'|^2 \right),$$

$$\left\langle y - y', h(t, y) - h(t, y') \right\rangle \leq C |y - y'|^2.$$

(H1.2) La condition d'intégrabilité est vérifiée

$$\mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + \int_0^T e^{\mu s + \beta \Lambda_s} |F(s, 0, 0)|^2 ds + \int_0^T e^{\mu s + \beta \Lambda_s} |G(s, 0, 0)|^2 ds \right] < +\infty.$$

Définition 3. Une paire de processus (Y, Z) est appelée une solution de l'EDDSRGA (0.0.99) si $(Y_t, Z_t) \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$ et satisfait (0.0.99).

Existence et unicité de solution

Pour prouver l'existence d'une solution, considérons la séquence de processus $(Y_t^n, Z_t^n)_{n \geq 0}$ donnée par

$$\begin{cases} Y_t^0 = 0; \forall 0 \leq t \leq T, \\ Y_t^n = \xi_T + \int_t^T F(s, Y_s^{n-1}, Z_s^n) ds + \int_t^T G(s, Y_s^{n-1}, Z_s^n) dB_s + \int_t^T H(s, Y_s^n) d\Lambda_s \\ \quad - \int_t^T Z_s^n dW_s. \end{cases}$$

Afin d'obtenir le résultat d'existence et d'unicité, nous avons besoin du résultat suivant. Supposons que les hypothèses (H1.1) et (H1.2) soient satisfaites et que $\xi_T \in L^2(\mathcal{F}_T, \mathbb{R}^k)$. Il existe une constante positive C telle que

$$\sup_{n \geq 0} \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 \right] \leq C(1 + \mathbb{E}[X_T]), \quad t \in [0, T], \quad (0.0.102)$$

où

$$X_T = e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + \int_0^T e^{\mu s + \beta \Lambda_s} |F(s, 0, 0)|^2 ds. \quad (0.0.103)$$

Sketch de la preuve 7. La formule d'Itô appliquée à $e^{\mu t + \beta \Lambda_t} |Y_t^n|^2$, donne

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 (\mu ds + \beta \Lambda_s) \right] \\ &= \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, F(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) \rangle ds \right] \\ &+ 2 \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, H(s, Y_s^n, Y_{s+\delta(s)}^n) \rangle d\Lambda_s \right] \\ &+ \int_t^T e^{\mu s + \beta \Lambda_s} |G(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n)|^2 ds. \end{aligned} \quad (0.0.104)$$

En utilisant l'inégalité $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ (où $\varepsilon > 0$ sera choisi plus tard), nous

déduisons des hypothèses **(H1.1)** et **(H1.2)** :

- $2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, F(s, Y_s^{n-1}, Z_s^n) \rangle ds \leq 2\lambda \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 ds$
 $+ \frac{1}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |F(s, 0, 0)|^2 ds + \frac{C}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\rho(|Y_s^{n-1}|^2) + |Z_s^n|^2) ds;$
- $\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |G(s, Y_s^{n-1}, Z_s^n)|^2 ds \leq C \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\rho(|Y_s^{n-1}|^2) + |Z_s^n|^2) ds;$
- $2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, H(s, Y_s^n) \rangle d\Lambda_s \leq 2C \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 d\Lambda_s.$

En combinant les inégalités obtenues plus haut, nous déduisons de **(0.0.104)** que

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 \left((\mu - 2\lambda) ds + (\beta - 2C) d\Lambda_s \right) \right] \\ & \leq \frac{C(1 + \lambda)}{\lambda} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} \left(\rho(|Y_s^{n-1}|^2) + |Z_s^n|^2 \right) ds \right] + C_0 \mathbb{E}[X_T], \quad t \in [0, T]; \end{aligned}$$

où X_T est donné par **(0.0.103)** et C_0 est une constante positive.

Choisissez $\lambda = \lambda_0$ tel que $\lambda_0 = \frac{2C}{1-2C} \leq \frac{2\mu-1}{4}$ et β tel que $\beta \geq 2C + \frac{1}{2}$. Il existe donc une constante positive $C' > 0$ (dont la valeur peut varier d'une ligne à l'autre) telle que

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \frac{1}{2} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 (ds + d\Lambda_s) + \frac{1}{2} \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds \right] \\ & \leq C' \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \rho(|Y_s^{n-1}|^2) ds + C' \mathbb{E}[X_T], \quad t \in [0, T]; \end{aligned}$$

En particulier, en mettant $q_n(t) = \sup_{n \in \mathbb{N}} \mathbb{E}[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2]$, nous obtenons

$$q_n(t) \leq C'(1 + \mathbb{E}[X_T]) + C' \int_t^T q_n(s) ds, \quad t \in [0, T].$$

L'inégalité de Gronwall donne

$$\sup_{n \in \mathbb{N}} \mathbb{E}[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2] \leq C(1 + \mathbb{E}[X_T]), \quad t \in [0, T].$$

Cela donne immédiatement **(0.0.102)**.

Le théorème suivant est le résultat principal de cette section. Dans ce théorème, à l'aide du lemme, nous pouvons maintenant prouver notre théorème d'existence et d'unicité.

Résultat principal sur l'existence et l'unicité de solution

Théorème 4. *Supposons que les hypothèses (H1.1) et (H1.2) soient satisfaites et que $\xi_T \in L^2(\mathcal{F}_T, \mathbb{R}^k)$. Alors l'EDSRG (0.0.99) a une unique solution $(Y_t, Z_t)_{0 \leq t \leq T} \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$.*

Sketch de la preuve 8. (i) Existence. *Nous considérons la séquence définie dans (0.0.102). Pour un processus $\varrho \in \{Y, Z\}$ et $n \in \mathbb{N}, m \in \mathbb{N}$, nous définissons $\bar{\varrho}_t^{n,m} = \varrho_t^n - \varrho_t^m$, et pour $\Psi \in \{F, G\}$*

$$\begin{aligned}\Delta \Psi^{(n,m)}(s) &= \Psi(s, Y_s^{n-1}, Z_s^n); -\Psi(s, Y_s^{m-1}, Z_s^m) \\ \Delta H^{(n,m)}(s) &= H(s, Y_s^n) - H(s, Y_s^m).\end{aligned}$$

Alors, il est évident que $(\bar{Y}_t^{n,m}, \bar{Z}_t^{n,m})_{0 \leq t \leq T}$ vérifie l'EDSR généralisée suivante

$$\bar{Y}_t^{n,m} = \int_t^T \Delta f^{(n,m)}(s) ds + \int_t^T \Delta g^{(n,m)}(s) dB_s + \int_t^T \Delta h^{(n,m)}(s) d\Lambda_s - \int_t^T \bar{Z}_s^{n,m} dW_s, \quad t \in [0, T]. \quad (0.0.105)$$

Par les mêmes calculs que dans le lemme, nous avons

$$\begin{aligned}\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 \right] + \frac{1}{2} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s^{n,m}|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s^{n,m}|^2 ds \right] \\ \leq C' \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \rho(|\bar{Y}_s^{n-1, m-1}|^2) ds, \quad t \in [0, T] \quad (0.0.106)\end{aligned}$$

Définissons

$$q(t) = \lim_{n,m \rightarrow \infty} \sup \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 \right], \quad t \in [0, T].$$

Alors par (0.0.106), le lemme de Fatou et le fait que ρ est une fonction concave continue croissante, nous obtenons

$$q(t) \leq C' \int_t^{T+K} \rho(q(s)) ds, \quad t \in [0, T]T.$$

Par conséquent, nous pouvons utiliser l'inégalité de Bihari pour obtenir $q(t) = 0$, soit

$$\lim_{n,m \rightarrow \infty} \sup \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 \right] = 0 \quad t \in [0, T]$$

Ainsi, à partir de l'inégalité (0.0.106), nous obtenons pour $t \in [0, T]$

$$\lim_{n,m \rightarrow \infty} \mathbb{E} \left(e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s^{n,m}|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s^{n,m}|^2 ds \right) = 0,$$

Alors il existe $(Y_t, Z_t)_{0 \leq t \leq T} \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$ tel que

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(e^{\mu t + \beta \Lambda_t} |Y_t^n - Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n - Y_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n - Z_s|^2 ds \right) = 0.$$

Soit $n \rightarrow +\infty$ dans (0.0.102), nous obtenons $\forall t \in [0, T]$,

$$Y_t = \xi_T + \int_t^T F(s, Y_s, Z_s) ds + \int_t^T G(s, Y_s, Z_s) dB_s + \int_t^T H(s, Y_s) d\Lambda_s - \int_t^T Z_s dW_s$$

Cela montre que $(Y_t, Z_t)_{0 \leq t \leq T} \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$ vérifie l'EDSR généralisée (0.0.99). La preuve de l'existence est complète.

(ii) Unicité. Soient $(Y^i, Z^i) \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$, ; $i = 1, 2$ deux solutions quelconques de (0.0.99).

Définissons $\bar{Y}_t = Y_t^1 - Y_t^2$, $\bar{Z}_t = Z_t^1 - Z_t^2$ et pour une fonction $\Psi \in \{F, G\}$,

$$\Delta \Psi(s) = \Psi(s, Y_s^1, Z_s^1) - \Psi(s, Y_s^2, Z_s^2) \quad \text{and} \quad \Delta H(s) = H(s, Y_s^1) - H(s, Y_s^2).$$

Nous considérons l'équation suivante

$$\bar{Y}_t = \int_t^T \Delta F(s) ds + \int_t^T \Delta G(s) dB_s + \int_t^T \Delta H(s) d\Lambda_s - \int_t^T \bar{Z}_s dW_s, \quad t \in [0, T]. \quad (0.0.107)$$

Par les mêmes calculs que dans le lemme , nous obtenons

$$\begin{aligned} \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 \right] + \frac{1}{2} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s|^2 ds \right] & (0.0.108) \\ & \leq C' \mathbb{E} \int_t^T \rho(e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2) ds. \end{aligned}$$

ce qui donne

$$\mathbb{E} [e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2] \leq C' \mathbb{E} \int_t^T \rho(e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2) ds, \quad t \in [0, T].$$

L'utilisation du théorème de Fubini et de l'inégalité de Jensen donnent

$$\mathbb{E} [e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2] \leq C' \int_t^T \rho \left(\mathbb{E} [e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2] \right) ds, \quad t \in [0, T].$$

Nous pouvons alors utiliser l'inégalité de Bihari pour obtenir $\mathbb{E} [e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2] = 0$, pour tout $t \in [0, T]$. Cela implique $\bar{Z}_t = 0$, pour tout $t \in [0, T]$.

Théorème de comparaison pour les EDDSRs Généralisées

Dans cette partie, notre objectif est d'établir le théorème de comparaison pour les EDDSRs généralisées 1-dimensionnels suivants sous des coefficients non Lipschitz :

$$Y_t^{(i)} = \xi_T^i + \int_t^T F^i(s, Y_s^{(i)}, Z_s^{(i)}) ds + \int_t^T G^i(s, Y_s^{(i)}, Z_s^{(i)}) dB_s + \int_t^T H^i(s, Y_s^{(i)}) d\Lambda_s - \int_t^T Z_s^{(i)} dW_s, \quad t \in [0, T]. \quad (0.0.109)$$

Alors d'après le théorème 4, (0.0.131) a une unique solution $(Y_t^{(i)}, Z_t^{(i)})_{0 \leq t \leq T} \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R})$.

Donnons un théorème de comparaison des EDDSRs généralisées sous des hypothèses non Lipschitz, qui jouera un rôle clé dans ce qui suit.

Théorème 5. Soient $(Y^{(1)}, Z^{(1)})$ et $(Y^{(2)}, Z^{(2)})$ des solutions de l'EDDSR généralisée 1-dimensionnelle (0.0.131) respectivement. Supposons que F^1, F^2, G^1, G^2, H^1 et H^2 satisfassent **(H1)** et $\xi_T^{(1)}, \xi_T^{(2)} \in L^2(\mathcal{F}_T, \mathbb{R})$. Nous supposons que :

- (i) $\xi_T^{(1)} \geq \xi_T^{(2)}$, p.s.;
- (ii) Pour tout $t \in [0, T]$, nous obtenons
 - $F^1(t, Y^{(1)}, Z^{(1)}) \geq F^2(t, Y^{(1)}, Z^{(1)})$ ou $F^1(t, Y^{(2)}, Z^{(2)}) \geq F^2(t, Y^{(2)}, Z^{(2)})$, p.s.;
 - $G^1(t, Y^{(1)}, Z^{(1)}) \geq G^2(t, Y^{(1)}, Z^{(1)})$ ou $G^1(t, Y^{(2)}, Z^{(2)}) \geq G^2(t, Y^{(2)}, Z^{(2)})$, p.s.;
 - $H^1(t, Y^{(1)}) \geq H^2(t, Y^{(1)})$ ou $H^1(t, Y^{(2)}) \geq H^2(t, Y^{(2)})$, p.s.;

Alors

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad p.s., \quad t \in [0, T].$$

Sketch de la preuve 9. Pour tout $t \in [0, T]$, nous notons

$$\begin{aligned} \widehat{Y}_t &= Y_t^{(2)} - Y_t^{(1)}, \quad \widehat{Z}_t = Z_t^{(2)} - Z_t^{(1)}, \quad \widehat{\xi}_t = \xi_t^{(2)} - \xi_t^{(1)}; \\ \widehat{F}(t) &= F^2(t, Y_t^{(2)}, Z_t^{(2)}) - F^1(t, Y_t^{(1)}, Z_t^{(1)}), \quad \widehat{G}(t) = G^2(t, Y_t^{(2)}, Z_t^{(2)}) - G^1(t, Y_t^{(1)}, Z_t^{(1)}) \\ \text{et } \widehat{H}(t) &= H^2(t, Y_t^{(2)}) - H^1(t, Y_t^{(1)}). \end{aligned}$$

Alors $(\widehat{Y}, \widehat{Z})$ satisfait

$$\widehat{Y}_t = \widehat{\xi}_T + \int_t^T \widehat{F}(s) ds + \int_t^T \widehat{G}(s) dB_s + \int_t^T \widehat{H}(s) d\Lambda_s - \int_t^T \widehat{Z}_s dW_s, \quad t \in [0, T]. \quad (0.0.110)$$

Par application de l'extension de la formule d'Itô à $e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2$, nous avons

$$\begin{aligned}
& \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 (\mu ds + \beta d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s^+ > 0\}} |\widehat{Z}_s|^2 ds \right] \\
&= 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \widehat{F}(s) ds + 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \widehat{H}(s) d\Lambda_s \\
&+ \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s^+ > 0\}} |\widehat{G}(s)|^2 ds. \tag{0.0.111}
\end{aligned}$$

Compte tenu de **(H1)**, de l'inégalité de Young et de l'inégalité de Jensen, pour tout $\lambda > 0$, nous avons :

- $$\begin{aligned}
2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \widehat{F}(s) ds &\leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \left(F^1(t, Y_t^{(2)}, Z_t^{(2)}) \right. \\
&\quad \left. - F^1(t, Y_t^{(1)}, Z_t^{(1)}) \right) ds \\
&\leq \frac{C}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 ds + \lambda \int_t^T \rho \left(\mathbb{E} \left[e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \right] \right) ds \\
&\quad + \lambda \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s^+ > 0\}} |\widehat{Z}_s|^2 ds;
\end{aligned}$$
- $$\begin{aligned}
\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s^+ > 0\}} |\widehat{G}(s)|^2 ds &\leq C \int_t^T \rho \left(\mathbb{E} \left[e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \right] \right) ds \\
&\quad + C \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s^+ > 0\}} |\widehat{Z}_s|^2 ds;
\end{aligned}$$
- $$\begin{aligned}
2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \widehat{H}(s) d\Lambda_s &\leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \left(H^1(t, Y_t^{(2)}) - H^1(t, Y_t^{(1)}) \right) d\Lambda_s \\
&\leq C \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 d\Lambda_s.
\end{aligned}$$

Alors, grâce aux inégalités ci-dessus, nous obtenons,

$$\begin{aligned}
& \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \left(\left(\mu - \frac{C}{\lambda} \right) ds + (\beta - C) d\Lambda_s \right) \right] \\
&+ (1 - c - \lambda) \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s^+ > 0\}} |\widehat{Z}_s|^2 ds \leq (C + \lambda) \int_t^T \rho \left(\mathbb{E} \left[e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \right] \right) ds \tag{0.0.112}
\end{aligned}$$

En choisissant $\beta \geq C$ et $\lambda = \lambda_0$ tels que $\lambda_0 + C \leq 1$ et $\frac{C}{\lambda_0} \leq \mu$, nous obtenons

$$\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 \right] \leq (C + \lambda) \int_t^T \rho \left(\mathbb{E} \left[e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \right] \right) ds, \quad t \in [0, T]. \tag{0.0.113}$$

Par le même argument que dans la preuve d'unicité du théorème 4, nous avons

$$\mathbb{E} \left[e^{\mu t + \beta \Lambda t} |\widehat{Y}_t^+|^2 \right] = 0, \text{ pour tout } t \in [0, T]. \text{ Ce qui implique}$$

$$Y_t^{(1)} \geq Y_t^{(2)}, \text{ a.s., } t \in [0, T].$$

Grâce à ces résultats préliminaires, nous sommes maintenant en mesure d'étudier notre sujet principal.

EDSR Généralisées Anticipées

Notre étude porte principalement sur l'équation suivante

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) ds + \int_t^T g(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) dB_s \\ \quad + \int_t^T h(s, Y_s, Y_{s+\delta(s)}) d\Lambda_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \\ Y_t = \xi_t, \quad Z_t = \eta_t, \quad t \in [T, T+K]. \end{cases} \quad (0.0.114)$$

Définition 4. Une paire de processus (Y, Z) est appelée une solution à l'EDSR généralisée anticipée (0.0.114) si $(Y_t, Z_t) \in \mathcal{B}_{\mathcal{F}}^2([0, T+K], \mathbb{R}^k)$ et satisfait (0.0.114).

Nous supposons qu'il existe $\rho \in \mathbf{S}$ tel que ξ, f, g et h satisfassent les hypothèses **(H2)** :

(H2.1) pour tout $t \in [0, T]$, $r, \bar{r} \in [t, T+K]$, $(t, y, z, \theta(r), \varphi(\bar{r}))$, $(t, y', z', \theta'(r), \varphi'(\bar{r})) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^{k \times d})$,

$$\begin{aligned} \left| f(t, y, z, \theta(r), \varphi(\bar{r})) - f(t, y', z', \theta'(r), \varphi'(\bar{r})) \right|^2 &\leq C\rho (|y - y'|^2) \\ &\quad + \mathbb{E}^{\mathcal{F}_t}[\rho (|\theta(r) - \theta'(r)|^2)] \\ &\quad + C|z - z'|^2 + \mathbb{E}^{\mathcal{F}_t}[|\varphi(\bar{r}) - \varphi'(\bar{r})|^2], \end{aligned}$$

$$\begin{aligned} \left| g(t, y, z, \theta(r), \varphi(\bar{r})) - g(t, y', z', \theta'(r), \varphi'(\bar{r})) \right|^2 &\leq C\rho (|y - y'|^2) \\ &\quad + \mathbb{E}^{\mathcal{F}_t}[\rho (|\theta(r) - \theta'(r)|^2)] \\ &\quad + \alpha_1|z - z'|^2 + \alpha_2\mathbb{E}^{\mathcal{F}_t}[|\varphi(\bar{r}) - \varphi'(\bar{r})|^2]; \end{aligned}$$

$$\left\langle y - y', h(t, y, \theta(r)) - h(t, y', \theta'(r)) \right\rangle \leq C|y - y'|^2 + \mathbb{E}^{\mathcal{F}_t}[|\theta(r) - \theta'(r)|^2],$$

où $C > 0$, $\alpha_1 \in (0, \frac{1}{2})$ et $\alpha_2 \in (0, \frac{1}{M})$ satisfaisant $\alpha_1 + \alpha_2 M \in (0, \frac{1}{2})$.

(H2.2) pour tout $s \in [0, T]$, $f(s, 0) = f(s, 0, 0, 0, 0)$ et la condition d'intégrabilité est vérifiée

$$\mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + \int_0^T e^{\mu s + \beta \Lambda_s} |f(s, 0)|^2 ds + \int_T^{T+K} e^{\mu s + \beta \Lambda_s} \left(\rho(|\xi_s|^2)(ds + d\Lambda_s) + |\eta_s|^2 \right) ds \right] < +\infty.$$

Résultat principal

Existence et unicité de la solution

Pour prouver l'existence d'une solution, considérons la séquence de processus $(Y_t^n, Z_t^n)_{n \geq 0}$ donnée par

$$\begin{cases} Y_t^0 = 0; \quad \forall 0 \leq t \leq T, \\ Y_t^n = \xi_T + \int_t^T f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) ds + \int_t^T g(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) dB_s \\ \quad + \int_t^T h(s, Y_s^n, Y_{s+\delta(s)}^n) d\Lambda_s - \int_t^T Z_s^n dW_s \quad t \in [0, T], \\ Y_t^n = \xi_t, \quad Z_t^n = \eta_t, \quad \forall t \in [T, T+K]. \end{cases} \quad (0.0.115)$$

Grâce à [[73], Théorème 1], cette suite est bien définie. Dans ce qui suit, nous établissons deux résultats qui seront utiles dans la suite. Supposons que les hypothèses **(A1)**, **(A2)** et **(H2.1)** et **(H2.2)** soient vérifiées et $\xi_T \in L^2(\mathcal{F}_T, \mathbb{R}^k)$. Alors pour tout $(\xi, \eta) \in \mathcal{S}_{[T, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[T, T+K]}^2(\mathcal{F}, \mathbb{R}^{k \times d})$. Il existe une constante positive C telle que

$$\sup_{n \geq 0} \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 \right] \leq C(1 + \mathbb{E}[\Gamma_T]), \quad t \in [0, T+K], \quad (0.0.116)$$

où

$$\begin{aligned} \Gamma_T = e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + \int_T^{T+K} e^{\mu s + \beta \Lambda_s} (\rho(|\xi_s|^2) + |\eta_s|^2) ds + \int_T^{T+K} e^{\mu s + \beta \Lambda_s} |\xi_s|^2 d\Lambda_s \\ + \int_0^T e^{\mu s + \beta \Lambda_s} |f(s, 0)|^2 ds. \end{aligned} \quad (0.0.117)$$

Sketch de la preuve 10. La formule d'Itô appliquée à $e^{\mu t + \beta \Lambda_t} |Y_t^n|^2$, donne

$$\begin{aligned}
& \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 (\mu ds + \beta \Lambda_s) \right] \\
&= \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} \left\langle Y_s^n, f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) \right\rangle ds \right] \\
&+ 2 \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} \left\langle Y_s^n, h(s, Y_s^n Y_{s+\delta(s)}^n) \right\rangle d\Lambda_s \right] \\
&+ \int_t^T e^{\mu s + \beta \Lambda_s} |g(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n)|^2 ds \Big]. \tag{0.0.118}
\end{aligned}$$

En utilisant l'inégalité $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ (où $\varepsilon > 0$ sera choisi plus tard), nous déduisons des hypothèses **(H2.1)**, **(H2.2)**, **(A1)** et **(A2)** :

- $$\begin{aligned}
2 \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \left\langle Y_s^n, f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) \right\rangle ds &\leq 2\lambda \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 ds \\
&+ \frac{1}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |f(s, 0)|^2 ds + \frac{C}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\rho(|Y_s^{n-1}|^2) + |Z_s^n|^2) ds \\
&+ \frac{C}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\mathbb{E}^{\mathcal{F}_s} [\rho(|Y_{s+\delta(s)}^{n-1}|^2)] + \mathbb{E}^{\mathcal{F}_s} [|Z_{s+\zeta(s)}^n|^2]) ds \\
&\leq 2\lambda \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 ds \\
&+ \frac{C+M}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\rho(|Y_s^{n-1}|^2) + |Z_s^n|^2) ds \\
&+ \mathbb{E} \int_T^{T+K} e^{\mu s + \beta \Lambda_s} (\rho(|\xi_s|^2) + |\eta_s|^2) ds;
\end{aligned}$$
- $$\begin{aligned}
\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |g(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n)|^2 ds &\leq \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (C\rho(|Y_s^{n-1}|^2) \\
&+ \alpha_1 |Z_s^n|^2) ds \\
&+ \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\mathbb{E}^{\mathcal{F}_s} [\rho(|Y_{s+\delta(s)}^{n-1}|^2)] + \alpha_2 \mathbb{E}^{\mathcal{F}_s} [|Z_{s+\zeta(s)}^n|^2]) ds \\
&\leq \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} ((C+M)\rho(|Y_s^{n-1}|^2) + (\alpha_1 + \alpha_2 M)|Z_s^n|^2) ds \\
&+ \mathbb{E} \int_T^{T+K} e^{\mu s + \beta \Lambda_s} (\rho(|\xi_s|^2) + \alpha_2 |\eta_s|^2) ds;
\end{aligned}$$

$$\begin{aligned}
& \bullet \quad 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \left\langle Y_s^n, h(s, Y_s^n, Y_{s+\delta(s)}^n) \right\rangle d\Lambda_s \\
& \leq 2C\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 d\Lambda_s + 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbb{E}^{\mathcal{F}_s} \left[|Y_{s+\delta(s)}^n|^2 \right] d\Lambda_s \\
& \leq 2(C+M)\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 d\Lambda_s + \mathbb{E} \int_T^{T+K} e^{\mu s + \beta \Lambda_s} |\xi_s|^2 d\Lambda_s
\end{aligned}$$

En rassemblant les inégalités obtenues plus haut, nous déduisons de (0.0.118) que

$$\begin{aligned}
& \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 \left((\mu - 2\lambda) ds \right. \right. \\
& \qquad \qquad \qquad \left. \left. + [\beta - 2(C+M)] d\Lambda_s \right) \right] \\
& \leq \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} \left(\frac{(C+M)(1+\lambda)}{\lambda} \rho(|Y_s^{n-1}|^2) + \frac{C+M+\lambda(\alpha_1+\alpha_2 M)}{\lambda} |Z_s^n|^2 \right) ds \right] \\
& \quad + C_0 \mathbb{E} [\Gamma_T];
\end{aligned}$$

où Γ_T est donné par (0.0.117) et C_0 est une constante positive.

Choisissez $\lambda = \lambda_0$ tel que $\lambda_0 = \frac{2(1+c)}{1-2(\alpha_1+\alpha_2 M)} \leq \frac{2\mu-1}{4}$ et β tel que $\beta - 2(M+c) \geq \frac{1}{2}$. Il existe donc une constante positive $C' > 0$ (dont la valeur peut varier d'une ligne à l'autre) telle que

$$\begin{aligned}
& \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \frac{1}{2} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 (ds + d\Lambda_s) + \frac{1}{2} \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds \right] \\
& \leq C' \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \rho(|Y_s^{n-1}|^2) ds + C' \mathbb{E} [\Gamma_T], \quad t \in [0, T+K].
\end{aligned}$$

De manière similaire à la preuve du lemme, on pouvons montrer que

$$\sup_{n \in \mathbb{N}} \mathbb{E} [e^{\mu t + \beta \Lambda_t} |Y_t^n|^2] \leq C(1 + \mathbb{E}[\Gamma_T]), \quad t \in [0, T+K].$$

A l'aide du lemme, nous pouvons établir le théorème d'existence et d'unicité suivant dans cette partie.

Théorème 6. *Sous les hypothèses (A1), (A2), (H2.1) et (H2.2) et $\xi_T \in L^2(\mathcal{F}_T, \mathbb{R}^k)$. Alors pour tout $(\xi, \eta) \in \mathcal{S}_{[T, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[T, T+K]}^2(\mathcal{F}, \mathbb{R}^{k \times d})$ l'EDSR généralisée anticipée (0.0.114) a une solution unique $(Y_t, Z_t)_{0 \leq t \leq T+K} \in \mathcal{B}_{\mathcal{F}}^2([0, T+K], \mathbb{R}^k)$.*

Sketch de la preuve 11. (i) Existence. *Nous considérons la séquence définie dans (0.0.115). Pour un processus $\varrho \in \{Y, Z\}$ et $n \in \mathbb{N}, m \in \mathbb{N}$, nous définissons $\bar{\varrho}_t^{n,m} = \varrho_t^n - \varrho_t^m$, et pour $\Psi \in \{f, g\}$*

$$\begin{aligned}\Delta\Psi^{(n,m)}(s) &= \Psi\left(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n\right) - \Psi\left(s, Y_s^{m-1}, Z_s^m, Y_{s+\delta(s)}^{m-1}, Z_{s+\zeta(s)}^m\right) \\ \Delta h^{(n,m)}(s) &= h\left(s, Y_s^n, Y_{s+\delta(s)}^n\right) - h\left(s, Y_s^m, Y_{s+\delta(s)}^m\right).\end{aligned}$$

Alors, il est évident que $(\bar{Y}_t^{n,m}, \bar{Z}_t^{n,m})_{0 \leq t \leq T+K}$ vérifie l'EDSR généralisée anticipée suivante

$$\begin{cases} \text{Pour tout } t \in [0, T] \\ \bar{Y}_t^{n,m} = \int_t^T \Delta f^{(n,m)}(s) ds + \int_t^T \Delta g^{(n,m)}(s) dB_s + \int_t^T \Delta h^{(n,m)}(s) d\Lambda_s - \int_t^T \bar{Z}_s^{n,m} dW_s, \\ \bar{Y}_t^{n,m} = 0, \quad \bar{Z}_t^{n,m} = 0, \quad t \in [T, T+K]. \end{cases} \quad (0.0.119)$$

Par les mêmes calculs que dans le lemme , nous obtenons

$$\begin{aligned}\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 \right] + \frac{1}{2} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s^{n,m}|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s^{n,m}|^2 ds \right] \\ \leq C' \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \rho(|\bar{Y}_r^{n-1, m-1}|^2) dr, \quad t \in [0, T+K]. \end{aligned} \quad (0.0.120)$$

Définissons

$$q(t) = \lim_{n,m \rightarrow \infty} \sup \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 \right], \quad t \in [0, T+K].$$

Alors par (0.0.120), le lemme de Fatou et le fait que ρ est une fonction concave continue croissante, nous obtenons

$$q(t) \leq C' \int_t^{T+K} \rho(q(s)) ds, \quad t \in [0, T+K].$$

Par conséquent, nous pouvons utiliser l'inégalité de Bihari pour obtenir $q(t) = 0$, soit

$$\lim_{n,m \rightarrow \infty} \sup \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 \right] = 0, \quad t \in [0, T+K].$$

Ainsi, à partir de l'inégalité (0.0.120), nous obtenons

$$\begin{aligned}\lim_{n,m \rightarrow \infty} \mathbb{E} \left(e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s^{n,m}|^2 (ds + d\Lambda_s) \right. \\ \left. + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s^{n,m}|^2 ds \right) = 0, \quad t \in [0, T+K].\end{aligned}$$

Alors il existe $(Y_t, Z_t)_{0 \leq t \leq T+K} \in \mathcal{B}_{\mathcal{F}}^2([0, T+K], \mathbb{R}^k)$ tel que

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(e^{\mu t + \beta \Lambda_t} |Y_t^n - Y_t|^2 + \int_t^{T+K} e^{\mu s + \beta \Lambda_s} |Y_s^n - Y_s|^2 (ds + d\Lambda_s) + \int_t^{T+K} e^{\mu s + \beta \Lambda_s} |Z_s^n - Z_s|^2 ds \right) = 0.$$

Soit $n \rightarrow +\infty$ dans (0.0.115), nous obtenons

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) ds + \int_t^T g(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) dB_s \\ \quad + \int_t^T h(s, Y_s, Y_{s+\delta(s)}) d\Lambda_s - \int_t^T Z_s dW_s \quad \forall t \in [0, T], \\ Y_t = \xi_t, \quad Z_t = \eta_t, \quad \forall t \in [T, T+K]. \end{cases}$$

Cela montre que $(Y_t, Z_t)_{0 \leq t \leq T+K} \in \mathcal{B}_{\mathcal{F}}^2([0, T+K], \mathbb{R}^k)$ satisfait l'EDSR généralisée anticipée (0.0.114). La preuve de l'existence est complète.

(ii) **Unicité.** Soit $(Y^i, Z^i) \in \mathcal{B}_{\mathcal{F}}^2([0, T+K], \mathbb{R}^k)$, $i = 1, 2$ soit deux solutions quelconques.

Définissons $\bar{Y}_t = Y_t^1 - Y_t^2$, $\bar{Z}_t = Z_t^1 - Z_t^2$ et pour une fonction $\Psi \in \{f, g\}$,

$$\begin{aligned} \Delta \Psi(s) &= \Psi(s, Y_s^1, Z_s^1, Y_{s+\delta(s)}^1, Z_{s+\zeta(s)}^1) - \Psi(s, Y_s^2, Z_s^2, Y_{s+\delta(s)}^2, Z_{s+\zeta(s)}^2), \quad \text{and} \\ \Delta h(s) &= h(s, Y_s^1, Y_{s+\delta(s)}^1) - h(s, Y_s^2, Y_{s+\delta(s)}^2) \end{aligned}$$

Nous considérons l'équation suivante

$$\begin{cases} \bar{Y}_t = \int_t^T \Delta f(s) ds + \int_t^T \Delta g(s) dB_s + \int_t^T \Delta h(s) d\Lambda_s - \int_t^T \bar{Z}_s dW_s, \quad t \in [0, T], \\ \bar{Y}_t = 0, \quad \bar{Z}_t = 0, \quad t \in [T, T+K]. \end{cases} \quad (0.0.121)$$

Par les mêmes calculs que dans le lemme, nous obtenons

$$\begin{aligned} \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 \right] + \frac{1}{2} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s|^2 ds \right] & (0.0.122) \\ & \leq C' \mathbb{E} \int_t^T \rho(e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2) ds. \end{aligned}$$

Ce qui donne

$$\mathbb{E} [e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2] \leq C' \mathbb{E} \int_t^{T+K} \rho(e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2) ds.$$

L'utilisation du théorème de Fubini et de l'inégalité de Jensen donnent

$$\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 \right] \leq C' \int_t^{T+K} \rho \left(\mathbb{E} \left[e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 \right] \right) ds, \quad t \in [0, T + K].$$

Nous pouvons alors utiliser l'inégalité de Bihari pour obtenir $\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 \right] = 0$, pour tout $t \in [0, T + K]$. Cela implique $\bar{Z}_t = 0$.

Théorème de comparaison pour les EDDSRs généralisées anticipées

Dans cette partie, nous donnerons le théorème de comparaison de l'EDSR généralisée anticipée 1-dimensionnelle suivante sous des coefficients non Lipschitz :

$$\left\{ \begin{array}{l} \forall 0 \leq t \leq T \\ Y_t^{(i)} = \xi_T^i + \int_t^T f^i(s, Y_s^{(i)}, Z_s^{(i)}, Y_{s+\delta(s)}^{(i)}, Z_{s+\zeta^{(i)}(s)}^{(i)}) ds \\ \quad + \int_t^T g^i(s, Y_s^{(i)}, Z_s^{(i)}, Y_{s+\delta(s)}^{(i)}, Z_{s+\zeta^{(i)}(s)}^{(i)}) dB_s + \int_t^T h^i(s, Y_s^{(i)}, Y_{s+\delta(s)}^{(i)}) d\Lambda_s \\ \quad - \int_t^T Z_s^{(i)} dW_s, \\ Y_t^{(i)} = \xi_t^{(i)}, \quad Z_t^{(i)} = \eta_t^{(i)}, \quad T \leq t \leq T + K \end{array} \right.$$

où $i = 1, 2$, et **(A1)**, **(A2)** et **(H)** sont satisfaites. Alors d'après le théorème 6, (0.0.123) admet une solution unique.

Théorème 7. Soient $(Y^{(1)}, Z^{(1)})$ et $(Y^{(2)}, Z^{(2)})$ deux solutions de l'EDSR généralisée anticipée. Sous l'hypothèse **(H1)**, supposons que:

(i) $\xi_t^{(1)} \geq \xi_t^{(2)}$, p.s., $t \in [T, T + K]$;

(ii) pour tout $t \in [0, T]$, nous obtenons

- $f^1(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(1)}(t)}^{(1)}) \geq f^2(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(2)}(t)}^{(2)})$, p.s.,
- $G^1(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(1)}(t)}^{(1)}) \geq G^2(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(2)}, Z_{t+\zeta^{(2)}(t)}^{(2)})$, p.s.,
- $h^1(t, Y_t^{(1)}, Y_{t+\delta(t)}^{(1)}) \geq h^2(t, Y_t^{(1)}, Y_{t+\delta(t)}^{(2)})$, p.s.

(iii) pour tout $t \in [0, T]$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ et $\varphi(\bar{r}) \in \mathcal{M}_{[0, T]}^2(\mathcal{F}, \mathbb{R}^d)$, $\bar{r} \in [t, T + K]$, $f^2(t, y, z, \cdot, \varphi(\bar{r}))$ est croissante.

Alors

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad p.s., \quad t \in [T, T + K].$$

Sketch de la preuve 12. Pour $i = 1, 2$ et pour tout $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, nous notons:

- $F^i(t, y, z) = f^i(t, y, z, Y_{t+\delta(t)}^{(i)}, Z_{t+\zeta^{(i)}(t)}^{(i)})$,
- $G^i(t, y, z) = g^i(t, y, z, Y_{t+\delta(t)}^{(i)}, Z_{r+\zeta^{(i)}(r)}^{(i)})$,
- $H^i(t, y) = h^i(t, y, z, Y_{t+\delta(t)}^{(i)})$.

Alors $(Y^{(i)}, Z^{(i)})$ est aussi l'unique solution de l'équation suivante

$$\begin{aligned} Y_t^{(i)} &= \xi_T^{(i)} + \int_t^T F^i(s, Y_s^{(i)}, Z_s^{(i)}) ds + \int_t^T G^i(s, Y_s^{(i)}, Z_s^{(i)}) dB_s + \int_t^T H^i(s, Y_s^{(i)}) d\Lambda_s \\ &\quad - \int_t^T Z_s^{(i)} dW_s, \quad t \in [0, T]. \end{aligned} \quad (0.0.123)$$

Introduisons l'équation suivante

$$\left\{ \begin{aligned} Y_t^{(3)} &= \xi_T^{(2)} + \int_t^T f^2(s, Y_s^{(3)}, Z_s^{(3)}, Y_{s+\delta(s)}^{(1)}, Z_{s+\zeta^{(2)}(s)}^{(2)}) ds + \int_t^T G^2(s, Y_s^{(3)}, Z_s^{(3)}) dB_s \\ &\quad + \int_t^T H^2(s, Y_s^{(3)}) d\Lambda_s - \int_t^T Z_s^{(3)} dW_s, \quad t \in [0, T], \\ Y_t^{(3)} &= \xi_t^{(2)}, \quad Z_t^{(3)} = \eta_t^{(2)}, \quad t \in [T, T+K]. \end{aligned} \right.$$

Si nous définissons $\tilde{F}^2(s, y, z) = f^2(s, y, z, Y_{s+\delta(s)}^{(1)}, Z_{s+\zeta^{(2)}(s)}^{(2)})$, alors cette équation est équivalente à

$$\left\{ \begin{aligned} Y_t^{(3)} &= \xi_T^{(2)} + \int_t^T \tilde{F}^2(s, Y_s^{(3)}, Z_s^{(3)}) ds + \int_t^T G^2(s, Y_s^{(3)}, Z_s^{(3)}) dB_s + \int_t^T H^2(s, Y_s^{(3)}) d\Lambda_s \\ &\quad - \int_t^T Z_s^{(3)} dW_s, \quad t \in [0, T], \\ Y_t^{(3)} &= \xi_t^{(2)}, \quad Z_t^{(3)} = \eta_t^{(2)}, \quad t \in [T, T+K], \end{aligned} \right.$$

qui admet une unique solution $(Y^{(3)}, Z^{(3)}) \in \mathcal{S}_{[0, T+K]}^2(\mathcal{F}, \mathbb{R}) \times \mathcal{M}_{[0, T+K]}^2(\mathcal{F}, \mathbb{R}^d)$, par le théorème 6. De plus $(Y^{(3)}, Z^{(3)})$ résout également de manière unique.

$$\begin{aligned} Y_t^{(3)} &= \xi_T^{(2)} + \int_t^T \tilde{F}^2(s, Y_s^{(3)}, Z_s^{(3)}) ds + \int_t^T G^2(s, Y_s^{(3)}, Z_s^{(3)}) dB_s + \int_t^T H^2(s, Y_s^{(3)}) d\Lambda_s \\ &\quad - \int_t^T Z_s^{(3)} dW_s, \quad t \in [0, T]. \end{aligned}$$

D'après les hypothèses **(i)** et **(ii)**, pour $t \in [0, T]$, il s'ensuit que $\xi_t^{(1)} \geq \xi_t^{(2)}$, *p.s.*, $F^1(t, Y_t^{(1)}, Z_t^{(1)}) \geq \tilde{F}^2(t, Y_t^{(1)}, Z_t^{(1)})$, $G^1(t, Y_t^{(1)}, Z_t^{(1)}) \geq G^2(t, Y_t^{(1)}, Z_t^{(1)})$ et $F^1(t, Y_t^{(1)}) \geq H^2(t, Y_t^{(1)})$. Donc le théorème **11** peut conduire à $Y_t^{(1)} \geq Y_t^{(3)}$, *p.s.*, pour tout $t \in [0, T]$, ce qui implique

$$Y_t^{(1)} \geq Y_t^{(3)}, \text{ p.s.}, \quad t \in [0, T + K].$$

De même, nous obtenons l'équation suivante

$$\begin{cases} Y_t^{(4)} = \xi_T^{(2)} + \int_t^T f^2(s, Y_s^{(4)}, Z_s^{(4)}, Y_{s+\delta(s)}^{(3)}, Z_{s+\zeta^{(2)}(s)}^{(2)})ds + \int_t^T G^2(s, Y_s^{(4)}, Z_s^{(4)})dB_s \\ \quad + \int_t^T H^2(s, Y_s^{(4)})d\Lambda_s - \int_t^T Z_s^{(4)}dW_s, \quad t \in [0, T], \\ Y_t^{(4)} = \xi_t^{(2)}, \quad Z_t^{(4)} = \eta_t^{(2)}, \quad t \in [T, T + K]. \end{cases}$$

Avec un argument similaire et l'hypothèse **iii)** sur la fonction f^2 , nous avons

$$Y_t^{(3)} \geq Y_t^{(4)}, \text{ p.s.}, \quad t \in [0, T + K].$$

Maintenant, pour $m = 5, 6, \dots$, nous introduisons l'équation suivante

$$\begin{cases} Y_t^{(m)} = \xi_T^{(2)} + \int_t^T f^2(s, Y_s^{(m)}, Z_s^{(m)}, Y_{s+\delta(s)}^{(m-1)}, Z_{s+\zeta^{(2)}(s)}^{(2)})ds + \int_t^T G^2(s, Y_s^{(m)}, Z_s^{(m)})dB_s \\ \quad + \int_t^T H^2(s, Y_s^{(m)})d\Lambda_s - \int_t^T Z_s^{(m)}dW_s, \quad t \in [0, T], \\ Y_t^{(m)} = \xi_t^{(2)}, \quad Z_t^{(m)} = \eta_t^{(2)}, \quad t \in [T, T + K]. \end{cases}$$

Grâce au théorème **11**, par induction nous avons

$$Y_t^{(4)} \geq Y_t^{(5)} \geq Y_t^{(6)} \geq \dots \geq Y_t^{(m-1)} \geq Y_t^{(m)}, \text{ p.s.}, \quad t \in [0, T + K].$$

De manière similaire à la preuve du théorème **6**, nous pouvons prouver que $(Y_t^{(m)}, Z_t^{(m)})$ converge dans $\mathcal{B}_{\mathcal{F}}^2([0, T + K], \mathbb{R})$ à $(Y_t^{(2)}, Z_t^{(2)})$. Puisque pour tout $t \in [0, T + K]$,

$$Y_t^{(4)} \geq Y_t^{(5)} \geq Y_t^{(6)} \geq \dots \geq Y_t^{(2)}, \text{ p.s.},$$

il tient immédiatement que

$$Y_t^{(1)} \geq Y_t^{(2)}, \text{ p.s.}, \quad t \in [0, T + K].$$

Résultats généraux du quatrième papier

Cette partie traite d'une classe d'équations différentielles stochastiques rétrogrades généralisées pilotées par deux mouvements browniens fractionnaires mutuellement indépendants (EDSRGF en abrégé). L'existence et l'unicité de solution pour l'EDSRGF ainsi qu'un théorème de comparaison sont obtenus. De plus, nous montrons le lien entre cette solution et la solution de l'équation aux dérivées partielles.

Notre étude est consacré à l'équation suivante appelée EDSR généralisée fractionnaire:

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds + \int_t^T g(s, \eta_s, Y_s) d\Lambda_s - \int_t^T Z_{1,s} dB_{1,s}^{H_1} - \int_t^T Z_{2,s} dB_{2,s}^{H_2}, \quad t \in [0, T], \quad (0.0.124)$$

où $\{B_{1,t}^{H_1}, ; t \in [0, T]\}$ et $\{B_{2,t}^{H_2}, ; t \in [0, T]\}$ sont deux mouvements browniens fractionnaires mutuellement indépendants et $\{\Lambda_t, t \in [0, T]\}$ est un processus continu croissant à valeur réelle. La nouveauté dans ce type d'équations stochastiques réside dans le fait de la présence d'un nouveau terme dans l'EDSR, à savoir une intégrale par rapport à un processus continu croissant.

Préliminaires

Calcul stochastique fractionnaire

Nous supposons qu'il existe deux mouvements browniens fractionnaires mutuellement indépendants

$B^H \in \{B_1^{H_1}, B_2^{H_2}\}$ avec le paramètre de Hurst $H \geq \frac{1}{2}$.

Théorème 8. *Soit $\sigma_1, \sigma_2 \in \mathcal{H}$ des fonctions continues déterministes. Notons*

$$X_t = X_0 + \int_0^t \alpha(s) ds + \int_0^t \sigma_1(s) dB_{1,s}^{H_1} + \int_0^t \sigma_2(s) dB_{2,s}^{H_2},$$

où X_0 est une constante, $\alpha(t)$ est une fonction déterministe avec $\int_0^t |\alpha(s)| ds < +\infty$.

Soit $F(t, x)$ continûment dérivable par rapport à t et deux fois continûment dérivable

par rapport à x . Alors

$$F(t, X_t) = F(0, X_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) dX_s \\ + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s) \left[\frac{d}{ds} \|\sigma_1\|_s^2 + \frac{d}{ds} \|\sigma_2\|_s^2 \right] ds, \quad 0 \leq t \leq T.$$

Terminons cette section en donnant une règle de dérivation par chaîne d'Itô fractionnaire (voir[[29], Theorem 3.2]).

Théorème 9. *Supposons que pour $j = 1, 2$, les processus μ_j , α_j et ϑ_j satisfont*

$$\mathbb{E} \left[\int_0^T \mu_j^2(s) ds + \int_0^T \alpha_j^2(s) ds + \int_0^T \vartheta_j^2(s) ds \right] < +\infty.$$

Supposons que $\mathbb{D}_t^{H_1} \alpha_j(s)$ et $\mathbb{D}_t^{H_2} \vartheta_j(s)$ sont continûment dérivables par rapport à (s, t) dans $[0, T]^2$ pour tout $\omega \in \Omega$. Soient X_t et Y_t deux processus satisfaisant

$$X_t = X_0 + \int_0^t \mu_1(s) ds + \int_0^t \alpha_1(s) dB_{1,s}^{H_1} + \int_0^t \vartheta_1(s) dB_{2,s}^{H_2}, \quad 0 \leq t \leq T, \\ Y_t = Y_0 + \int_0^t \mu_2(s) ds + \int_0^t \alpha_2(s) dB_{1,s}^{H_1} + \int_0^t \vartheta_2(s) dB_{2,s}^{H_2}, \quad 0 \leq t \leq T.$$

Si les conditions suivantes sont satisfaites:

$$\mathbb{E} \left[\int_0^T |\mathbb{D}_t^{H_1} \alpha_i(s)|^2 ds dt \right] < +\infty \quad \text{et} \quad \mathbb{E} \left[\int_0^T |\mathbb{D}_t^{H_2} \vartheta_i(s)|^2 ds dt \right] < +\infty$$

Alors

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s \\ + \int_0^t \left[\alpha_1(s) \mathbb{D}_s^{H_1} Y_s + \alpha_2(s) \mathbb{D}_s^{H_1} X_s + \vartheta_1(s) \mathbb{D}_s^{H_2} Y_s + \vartheta_2(s) \mathbb{D}_s^{H_2} X_s \right] ds,$$

qui peut s'écrire formellement comme

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \left[\alpha_1(t) \mathbb{D}_t^{H_1} Y_t + \alpha_2(t) \mathbb{D}_t^{H_1} X_t + \vartheta_1(t) \mathbb{D}_t^{H_2} Y_t + \vartheta_2(t) \mathbb{D}_t^{H_2} X_t \right] dt.$$

Définitions et notations

Soit G un sous-ensemble connexe ouvert de \mathbb{R}^d tel que pour $\ell \in \mathcal{C}^2(\mathbb{R}^d)$, $G = \{x ; \ell(x) > 0\}$ et $\partial G = \{x ; \ell(x) = 0\}$ et $|\nabla \ell(x)| = 1$ pour $x \in \partial G$.

Soit $\eta_0 \in G$ et (η_t, Λ_t) une solution de l'EDS réfléchie suivante par rapport aux mouve-

ments browniens fractionnaires

$$\eta_t = \eta_0 + \int_0^t b(s)ds + \int_0^t \nabla \ell(s) d\Lambda_s + \int_0^t \sigma_1(s) dB_{1,s}^{H_1} + \int_0^t \sigma_2(s) dB_{2,s}^{H_2}, \quad t \in [0, T], \quad (0.0.125)$$

où les coefficients η_0 , b , σ_1 et σ_2 satisfont:

- η_0 est constante donnée et $b : [0, T] \rightarrow \mathbb{R}$ est une fonction déterministe continue;
- $\sigma_1, \sigma_2 : [0, T] \rightarrow \mathbb{R}$ sont des fonctions déterministes différentiable $\sigma_1(t) \neq 0$, $\sigma_2(t) \neq 0$ telles que

$$|\sigma|_t^2 = \|\sigma_1\|_t^2 + \|\sigma_2\|_t^2, \quad t \in [0, T], \quad (0.0.126)$$

$$\text{où} \quad \|\sigma_i\|_t^2 = H_i(2H_i-1) \int_0^t \int_0^t |u-v|^{2H_i-2} \sigma_i(u) \sigma_i(v) dudv, \quad i = 1, 2.$$

- Λ est un processus croissant, $\Lambda_0 = 0$, et $\int_0^T (\eta_t - a) d\Lambda_t \leq 0$ pour tout $a \in G$.

La prochaine Remarque sera utile dans la suite.

Remarque 6. Pour $i = 1, 2$, la fonction $|\sigma|_t^2$ définie par (0.0.126) est continûment dérivable par rapport à t sur $[0, T]$, et pour une constante appropriée $C_0 > 0$,

$$\inf_{t \in [0, T]} \frac{\widehat{\sigma}_i(t)}{\sigma_i(t)} \geq C_0 \quad \text{où} \quad \widehat{\sigma}_i(t) = \int_0^t \phi(t, v) \sigma_i(v) dv.$$

Étant donné ξ une variable aléatoire à valeur réelle mesurable et les fonctions

$$f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad ; g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R},$$

nous considérons l'EDSR généralisée fractionnaire suivante avec les paramètres (ξ, f, g, Λ)

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds + \int_t^T g(s, \eta_s, Y_s) d\Lambda_s - \int_t^T Z_{1,s} dB_{1,s}^{H_1} \\ &\quad - \int_t^T Z_{2,s} dB_{2,s}^{H_2}, \quad t \in [0, T]. \end{aligned} \quad (0.0.127)$$

Avant de donner la définition de la solution de l'équation ci-dessus, nous introduisons les ensembles suivants (où $\mu, \beta > 0$) :

- $\mathcal{C}_{\text{pol}}^{1,2}([0, T] \times \mathbb{R})$ est l'espace de tous les $\mathcal{C}^{1,2}$ -fonctions sur $[0, T] \times \mathbb{R}$, qui, avec leurs dérivées, sont à croissance polynomiale,
- $\mathcal{V}_{[0, T]} = \left\{ Y = \psi(\cdot, \eta); \quad \psi \in \mathcal{C}_{\text{pol}}^{1,2}([0, T] \times \mathbb{R}), \quad \frac{\partial \psi}{\partial t} \text{ is bounded, } t \in [0, T] \right\}$,
- $\widetilde{\mathcal{V}}_{[0, T]}^2$ et $\widetilde{\mathcal{V}}_{([0, T], \Lambda)}^2$ la complétion de $\mathcal{V}_{[0, T]}$ sous la norme suivante respectivement

$$\|Y\|_{\widetilde{\mathcal{V}}_{[0, T]}^2}^2 = \mathbb{E} \int_0^T e^{\mu t + \beta \Lambda_t} |Y_t|^2 dt \quad \text{and} \quad \|Y\|_{\widetilde{\mathcal{V}}_{([0, T], \Lambda)}^2}^2 = \mathbb{E} \int_0^T e^{\mu t + \beta \Lambda_t} |Y_t|^2 d\Lambda_t.$$

- $\mathcal{B}^2([0, T], \Lambda) = \left(\tilde{\mathcal{V}}_{[0, T]}^2 \cap \tilde{\mathcal{V}}_{([0, T], \Lambda)}^2 \right) \times \tilde{\mathcal{V}}_{[0, T]}^2 \times \tilde{\mathcal{V}}_{[0, T]}^2$ muni de la norme

$$\begin{aligned} \left\| (Y, Z_1, Z_2) \right\|_{\mathcal{B}^2([0, T], \Lambda)}^2 &= \mathbb{E} \left[\int_0^T e^{\mu t + \beta \Lambda_t} |Y_t|^2 (dt + d\Lambda_t) \right. \\ &\quad \left. + \int_0^T e^{\mu t + \beta \Lambda_t} (|Z_{1,t}|^2 + |Z_{2,t}|^2) dt \right] \end{aligned}$$

Définition 5. Un triplet de processus $(Y_t, Z_{1,t}, Z_{2,t})_{0 \leq t \leq T}$ est appelé solution de l'EDSR généralisée fractionnaire (0.0.127), si $(Y_t, Z_{1,t}, Z_{2,t})_{0 \leq t \leq T} \in \mathcal{B}^2([0, T], \Lambda)$ et satisfait (0.0.127).

La proposition suivante sera utile dans la suite.

Proposition 8. Soit $(Y_t, Z_{1,t}, Z_{2,t})_{0 \leq t \leq T} \in \mathcal{B}^2([0, T], \Lambda)$ une solution de l'EDSR généralisée fractionnaire (0.0.127). Alors pour presque $t \in [0, T]$,

$$\mathbb{D}_t^{H_1} Y_t = \frac{\hat{\sigma}_1(t)}{\sigma_1(t)} Z_{1,t} \quad \text{et} \quad \mathbb{D}_t^{H_2} Y_t = \frac{\hat{\sigma}_2(t)}{\sigma_2(t)} Z_{2,t}.$$

EDSR généralisée fractionnaire

Existence et unicité de la solution

Nous supposons que les coefficients f et g de l'EDSR généralisée fractionnaire et ξ vérifient les hypothèses suivantes:

(H1) $\xi_T = h(\eta_T)$, où $h : \mathbb{R} \rightarrow \mathbb{R}$ est une fonction différentiable et $\mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 \right] < +\infty$.

(H2) Il existe une constante, $x \in \mathbb{R}$, $(y, y') \in \mathbb{R}^2$, $(z_1, z'_1) \in \mathbb{R}^2$, $(z_2, z'_2) \in \mathbb{R}^2$, nous avons

- $|f(t, x, y, z_1, z_2) - f(t, x, y', z'_1, z'_2)|^2 \leq K(|y - y'|^2 + |z_1 - z'_1|^2 + |z_2 - z'_2|^2);$
- $|g(t, x, y) - g(t, x, y')|^2 \leq K|y - y'|^2$

Nous établissons d'abord une estimation a priori sur la solution.

Proposition 9. Sous les conditions **(H1)** et **(H2)**, si $(Y, Z_1, Z_2) \in \mathcal{B}^2([0, T], \Lambda)$ est

une solution de (0.0.127), alors il existe une constante C telle que, pour tout $t \in [0, T]$

$$\begin{aligned} & \mathbb{E} \left(e^{\mu t + \beta \Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_{1,s}|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_{2,s}|^2 ds \right) \\ & \leq C \mathbb{E} \left(e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |f(s, \eta, 0, 0, 0)|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |g(s, \eta, 0)|^2 d\Lambda_s \right). \end{aligned}$$

Sketch de la preuve 13. Par C nous désignerons une constante qui peut varier d'une ligne à l'autre.

De la formule d'Itô, nous obtenons

$$\begin{aligned} e^{\mu t + \beta \Lambda_t} |Y_t|^2 &= e^{\mu T + \beta \Lambda_T} |\xi|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds \\ &+ 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s - 2 \int_t^T e^{\mu s + \beta \Lambda_s} Z_{1,s} \mathbb{D}_s^{H_1} Y_s ds \\ &- 2 \int_t^T e^{\mu s + \beta \Lambda_s} Z_{2,s} \mathbb{D}_s^{H_2} Y_s ds - 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s Z_{1,s} dB_{1,s}^{H_1} \\ &- 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s Z_{2,s} dB_{2,s}^{H_2} - \mu \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 ds - \beta \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 d\Lambda_s \end{aligned}$$

Prenons l'espérance mathématique des deux côtés, alors nous avons

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 (\mu ds + \beta d\Lambda_s) \right. \\ & \quad \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} (Z_{1,s} \mathbb{D}_s^{H_1} Y_s + Z_{2,s} \mathbb{D}_s^{H_2} Y_s) ds \right] \\ &= \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds \right. \\ & \quad \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s \right] \\ & - 2 \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} Y_s Z_{1,s} dB_{1,s}^{H_1} + \int_t^T e^{\mu s + \beta \Lambda_s} Y_s Z_{2,s} dB_{2,s}^{H_2} \right] \end{aligned}$$

En utilisant le fait que $\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} Y_s Z_{i,s} dB_{i,s}^{H_i} = 0$. Nous en déduisons donc

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 (\mu ds + \beta d\Lambda_s) \right. \\ & \quad \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} (Z_{1,s} \mathbb{D}_s^{H_1} Y_s + Z_{2,s} \mathbb{D}_s^{H_2} Y_s) ds \right] \\ & = \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds \right. \\ & \quad \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s \right]. \end{aligned}$$

Nous savons que, par la proposition 8, $\mathbb{D}_s^{H_1} Y_s = \frac{\widehat{\sigma}_1(s)}{\sigma_1(s)} Z_{1,s}$ et $\mathbb{D}_s^{H_2} Y_s = \frac{\widehat{\sigma}_2(s)}{\sigma_2(s)} Z_{2,s}$. Par la remarque 6, nous obtenons

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 (\mu ds + \beta d\Lambda_s) + 2C_0 \int_t^T e^{\mu s + \beta \Lambda_s} (|Z_{1,s}|^2 + |Z_{2,s}|^2) ds \right] \\ & \leq \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds \right. \\ & \quad \left. + \int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s \right]. \end{aligned} \tag{0.0.128}$$

En utilisant les estimations standard $2ab \leq \lambda a^2 + \frac{1}{\lambda} b^2$ (où $\lambda > 0$) et l'hypothèse **(H2)**, nous obtenons

- $2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds$

$$\begin{aligned} & \leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s| \left(|f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) - f(s, \eta_s, 0, 0, 0)| + |f(s, \eta_s, 0, 0, 0)| \right) ds \\ & \leq \frac{K + 2\lambda^2}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 ds + \frac{K}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (|Z_{1,s}|^2 + |Z_{2,s}|^2) ds \\ & \quad + \frac{1}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |f(s, \eta_s, 0, 0, 0)|^2 ds \end{aligned}$$
- $2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s$

$$\begin{aligned} & \leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s| \left(|g(s, \eta_s, Y_s) - g(s, \eta_s, 0)| + |g(s, \eta_s, 0)| \right) d\Lambda_s \\ & \leq \frac{K + 2\lambda^2}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 d\Lambda_s \\ & \quad + \frac{1}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |g(s, 0)|^2 d\Lambda_s. \end{aligned}$$

Choisisant $\lambda = \lambda_0$ tel que $\mu, \beta \geq \frac{K+2\lambda_0^2+\lambda_0}{\lambda_0}$ et $C_0 \geq \frac{K+\lambda_0}{2\lambda_0}$, nous déduisons de (0.0.128)

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} (|Z_{1,s}|^2 + |Z_{2,s}|^2) ds \right] \\ & \leq C \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |f(s, \eta_s, 0, 0, 0)|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |g(s, 0)|^2 d\Lambda_s \right]. \end{aligned}$$

Le résultat principal de cette partie est le théorème suivant :

Théorème 10. *Supposons que les hypothèses (H1) et (H2) soient. Alors l'EDSR généralisée fractionnaire (0.0.127) a une unique solution $(Y_t, Z_{1,t}, Z_{2,t})_{0 \leq t \leq T} \in \mathcal{B}^2([0, T], \Lambda)$.*

Sketch de la preuve 14. *Considérons la correspondance $\Gamma : \mathcal{B}^2([0, T], \Lambda) \rightarrow \mathcal{B}^2([0, T], \Lambda)$ définie par $(U, V_1, V_2) \mapsto \Gamma(U, V_1, V_2) = (Y, Z_1, Z_2)$. Nous allons montrer que l'application Γ est une contraction, où (Y, Z_1, Z_2) est une solution de l'EDSR généralisée fractionnaire suivante:*

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, \eta_s, U_s, V_{1,s}, V_{2,s}) ds + \int_t^T g(s, \eta_s, U_s) d\Lambda_s - \int_t^T Z_{1,s} dB_{1,s}^{H_1} \\ &\quad - \int_t^T Z_{2,s} dB_{2,s}^{H_2}, \quad t \in [0, T]. \end{aligned} \tag{0.0.129}$$

Définissons pour un processus $\delta \in \{U, V_1, V_2, Y, Z_1, Z_2\}$, $\bar{\delta} = \delta - \delta'$ et les fonctions

$$\Delta f(s) = f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) - f(s, \eta_s, Y'_s, Z'_{1,s}, Z'_{2,s}), \quad \Delta g(s) = g(s, \eta_s, Y_s) - g(s, \eta_s, Y'_s).$$

Ensuite, le triplet $(\bar{Y}, \bar{Z}_1, \bar{Z}_2)$ est solution de l'EDSR généralisée fractionnaire

$$\bar{Y}_t = \int_t^T \Delta f(s) ds + \int_t^T \Delta g(s) d\Lambda_s - \int_t^T \bar{Z}_{1,s} dB_{1,s}^{H_1} - \int_t^T \bar{Z}_{2,s} dB_{2,s}^{H_2}, \quad t \in [0, T].$$

En appliquant la formule d'Itô à $e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2$, nous obtenons que

$$\begin{aligned} e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 &= 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \Delta f(s) ds + 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \Delta g(s) d\Lambda_s \\ &\quad - 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Z}_{1,s} \mathbb{D}_s^{H_1} \bar{Y}_s ds - 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Z}_{2,s} \mathbb{D}_s^{H_2} \bar{Y}_s ds \\ &\quad - 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \bar{Z}_{1,s} dB_{1,s}^{H_1} - 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \bar{Z}_{2,s} dB_{2,s}^{H_2} \\ &\quad - \mu \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 ds - \beta \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 d\Lambda_s \end{aligned}$$

Prenons l'espérance mathématique des deux côtés, alors nous avons

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 (\mu ds + \beta d\Lambda_s) \right. \\ & \quad \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} (\bar{Z}_{1,s} \mathbb{D}_s^{H_1} \bar{Y}_s + \bar{Z}_{2,s} \mathbb{D}_s^{H_2} \bar{Y}_s) ds \right] \\ & = 2\mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \Delta f(s) ds + \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \Delta g(s) d\Lambda_s \right]. \end{aligned}$$

Par les mêmes calculs que dans la proposition 9, nous en déduisons que

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 ((\mu - \lambda) ds + (\beta - \lambda) d\Lambda_s) \right. \\ & \quad \left. + 2C_0 \int_t^T e^{\mu s + \beta \Lambda_s} (|\bar{Z}_{1,s}|^2 + |\bar{Z}_{2,s}|^2) ds \right] \\ & \leq \frac{K}{\lambda} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{U}_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} (|\bar{V}_{1,s}|^2 + |\bar{V}_{2,s}|^2) ds \right], \quad t \in [0, T]. \end{aligned} \tag{0.0.130}$$

Prenons $\lambda = \lambda_0$ telle que $\frac{1}{2} \min \{\mu - \lambda_0, \beta - \lambda_0\} \geq 2C_0 = \frac{4K}{3\lambda_0}$, nous obtenons

$$\begin{aligned} & \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_{1,s}|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_{2,s}|^2 ds \right] \\ & \leq \frac{3}{4} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{U}_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{V}_{1,s}|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{V}_{2,s}|^2 ds \right], \quad t \in [0, T]. \end{aligned}$$

Sous les conditions **(H1)** et **(H2)**, si $(Y, Z_1, Z_2) \in \mathcal{B}^2([0, T], \Lambda)$ est une solution de (0.0.127), alors il existe une constante C telle que, pour tout $t \in [0, T]$

Théorème de comparaison

Dans cette section, notre objectif est d'établir le théorème de comparaison pour les EDSRs généralisées fractionnaires suivantes, pour tout $t \in [0, T]$,

$$Y_t^{(i)} = h^i(\eta_T) + \int_t^T f^i(s, \eta_s, Y_s^{(i)}, Z_{1,s}^{(i)}, Z_{2,s}^{(i)}) ds + \int_t^T g^i(s, \eta_s, Y_s^{(i)}) d\Lambda_s - \int_t^T Z_{1,s}^{(i)} dB_s^{H_1} - \int_t^T Z_{2,s}^{(i)} dB_s^{H_2}, \tag{0.0.131}$$

où $i = 1, 2$. Alors d'après le théorème 10, (0.0.131) admet une unique solution $(Y^{(i)}, Z_1^{(i)}, Z_2^{(i)}) \in \mathcal{B}^2([0, T], \Lambda)$.

Donnons un théorème de comparaison des EDSR sous les hypothèses de Lipschitz.

Théorème 11. *Pour $i = 1, 2$, supposons que f^i et g^i satisfont **(H1)**. Soit h^i continûment dérivable. Soit $(Y^{(i)}, Z_1^{(i)}, Z_2^{(i)})$ deux solutions de l'EDSR généralisée fractionnaire (0.0.131). Nous supposons que :*

(i) $h^1(\eta_T) \geq h^2(\eta_T)$, *p.s.;*

(ii) *pour tout $t \in [0, T]$, $f^1(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}) \geq f^2(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)})$ ou $f^1(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) \geq f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)})$, p.s.;*

(iii) *pour tout $t \in [0, T]$, $g^1(t, \eta_t, Y_t^{(1)}) \geq g^2(t, \eta_t, Y_t^{(1)})$ ou $g^1(t, \eta_t, Y_t^{(2)}) \geq g^2(t, \eta_t, Y_t^{(2)})$, p.s.;*

Alors

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad p.s., \quad t \in [0, T].$$

Sketch de la preuve 15. *Pour tout $t \in [0, T]$, nous notons*

$$\begin{aligned} \widehat{Y}_t &= Y_t^{(2)} - Y_t^{(1)}, \quad \widehat{Z}_{1,t} = Z_{1,t}^{(2)} - Z_{1,t}^{(1)}, \quad \widehat{Z}_{2,t} = Z_{2,t}^{(2)} - Z_{2,t}^{(1)}, \\ \widehat{h}(\eta_T) &= h^2(\eta_T) - h^1(\eta_T); \\ \Delta f(t) &= f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) - f^1(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}), \\ \Delta g(t) &= g^2(t, \eta_t, Y_t^{(2)}) - g^1(t, \eta_t, Y_t^{(1)}) \end{aligned}$$

Alors $(\widehat{Y}, \widehat{Z}_1, \widehat{Z}_2)$ satisfait

$$\widehat{Y}_t = \widehat{h}(\eta_T) + \int_t^T \Delta f(s) ds + \int_t^T \Delta g(s) d\Lambda_s + \int_t^T \widehat{Z}_{1,s} dB_s^{H_1} - \int_t^T \widehat{Z}_{2,s} dB_s^{H_2}, \quad t \in [0, T]. \quad (0.0.132)$$

En appliquant l'extension de la formule Itô à $e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2$, nous avons

$$\begin{aligned} &\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 (\mu ds + \beta d\Lambda_s) \right. \\ &\quad \left. + 2C_0 \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} (|\widehat{Z}_{1,s}|^2 + |\widehat{Z}_{2,s}|^2) ds \right] \\ &\leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \Delta f(s) ds + 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \Delta g(s) d\Lambda_s \end{aligned} \quad (0.0.133)$$

Compte tenu de **(H1)** et de **ii)**, de l'inégalité de Young et de l'inégalité de Jensen,

pour tout $\lambda > 0$, nous avons:

- $$\begin{aligned}
& 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \Delta f(s) ds \\
&= 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \left(f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) - f^1(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}) \right) ds \\
&\leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \left(f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) - f^2(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}) \right) ds \\
&\leq \frac{\lambda^2 + K}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 ds + \frac{K}{\lambda} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} \left(|\widehat{Z}_{1,s}|^2 + |\widehat{Z}_{2,s}|^2 \right) ds
\end{aligned}$$
- $$\begin{aligned}
& 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \Delta s(s) d\Lambda_s = 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \left(g^2(t, \eta_t, Y_t^{(2)}) - g^1(t, \eta_t, Y_t^{(1)}) \right) d\Lambda_s \\
&\leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \left(g^2(t, \eta_t, Y_t^{(2)}) - g^2(t, \eta_t, Y_t^{(1)}) \right) d\Lambda_s \\
&\leq \frac{\lambda^2 + K}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 d\Lambda_s
\end{aligned}$$

Alors, grâce aux inégalités ci-dessus, nous obtenons

$$\begin{aligned}
& \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 (\mu ds + \beta d\Lambda_s) \right. \\
& \left. + 2C_0 \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} \left(|\widehat{Z}_{1,s}|^2 + |\widehat{Z}_{2,s}|^2 \right) ds \right] \\
& \leq \frac{\lambda^2 + K}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 (ds + d\Lambda_s) + \frac{K}{\lambda} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} \left(|\widehat{Z}_{1,s}|^2 + |\widehat{Z}_{2,s}|^2 \right) ds
\end{aligned} \tag{0.0.134}$$

En choisissant $\lambda = \lambda_0$ tel que $\min\{\mu, \beta\} \geq \frac{\lambda_0^2 + K}{\lambda_0}$ et $\lambda_0 \geq \frac{K}{2C_0}$, nous avons

$$\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 \right] \leq 0, \quad t \in [0, T].$$

Ce qui implique

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad a.s., \quad t \in [0, T].$$

cela termine la preuve du théorème.

Connexion avec les équations aux dérivées partielles

Nous considérons les EDSRs généralisées fractionnaires suivantes

$$\begin{cases} dY_t = -f(t, \eta_t, Y_t, Z_{1,t}, Z_{2,t}) dt - g(t, \eta_t, Y_t) d\Lambda_t + Z_{1,t} dB_t^{H_1} + Z_{2,t} dB_t^{H_2} \\ Y_T = \xi_T. \end{cases} \tag{0.0.135}$$

Notons

$$\tilde{\sigma}(t) = \left[\frac{d}{ds} \|\sigma_1\|_s^2 + \frac{d}{ds} \|\sigma_2\|_s^2 \right]$$

La fonction $u(t, x)$ satisfait l'équation aux dérivées partielles suivante (EDP en abrégé)

$$\begin{cases} u'_t(t, x) + \frac{1}{2}\tilde{\sigma}(t)u''_{xx}(t, x) + b(t)u'_x(t, x) + f(t, x, u(t, x), \sigma_1(t)u'_x(t, x), \sigma_2(t)u'_x(t, x)) = 0; \\ \nabla\ell(t)u'_x(t, x) + g(t, x, u(t, x)) = 0; \\ u(T, x) = h(x) \end{cases} \quad (0.0.136)$$

Donnons maintenant le théorème suivant

Théorème 12. *Si l'EDP (0.0.136) admet une solution $u(t, x)$ continûment dérivable en t et continûment dérivable deux fois en x , alors*

$$(Y_t, Z_{1,t}, Z_{2,t}) = (u(t, \eta_t), \sigma_1(t)u'_x(t, \eta_t), \sigma_2(t)u'_x(t, \eta_t))$$

satisfait (0.0.135).

Sketch de la preuve 16. *A partir de la formule d'Itô fractionnaire (Théorème 8), nous obtenons*

$$\begin{aligned} du(t, \eta_t) &= u'_t(t, \eta_t)dt + u'_x(t, \eta_t) \left[b(t)dt + \nabla\ell(t)d\Lambda_t + \sigma_1(t)dB_{1,t}^{H_1} + \sigma_2(t)dB_{2,t}^{H_2} \right] \\ &\quad + \frac{1}{2}\tilde{\sigma}(t)u''_{xx}(t, \eta_t)dt \\ &= \left[u'_t(t, \eta_t) + b(t)u'_x(t, \eta_t) + \frac{1}{2}\tilde{\sigma}(t)u''_{xx}(t, \eta_t) \right] dt + u'_x(t, \eta_t)\nabla\ell(t)d\Lambda_t \\ &\quad + \sigma_1(t)u'_x(t, \eta_t)dB_{1,t}^{H_1} + \sigma_2(t)u'_x(t, \eta_t)dB_{2,t}^{H_2}. \end{aligned}$$

Comme $u(t, x)$ satisfait l'EDP (0.0.136), nous avons,

$$\begin{aligned} du(t, \eta_t) &= -f(t, \eta_t, u(t, \eta_t), \sigma_1(t)u'_x(t, \eta_t), \sigma_2(t)u'_x(t, \eta_t))dt - g(t, \eta_t, u(t, \eta_t))d\Lambda_t \\ &\quad + \sigma_1(t)u'_x(t, \eta_t)dB_{1,t}^{H_1} + \sigma_2(t)u'_x(t, \eta_t)dB_{2,t}^{H_2}. \end{aligned}$$

Remarque 7. *De la preuve ci-dessus, nous voyons aussi que si le l'équation aux dérivées partielles non linéaires (0.0.136) a une solution unique, alors l'équation différentielle stochastique rétrograde fractionnaire généralisée (0.0.135) a également une solution unique.*

INTRODUCTION

This thesis from Gaston Berger University in Saint-Louis, Senegal is devoted to Backward Stochastic Differential Equations (BSDE for short) and in Reflected BSDE (RBSDE for short) with jump and one obstacle,

$$\left\{ \begin{array}{l} \text{(i)} \quad dY_s^{i;t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i;t,x})_{i=1,m}, Z_s^{i;t,x}, U_s^{i;t,x})ds \\ \quad \quad \quad -dK_s^{i;t,x} + Z_s^{i;t,x}dB_s + \int_{\mathbf{E}} U_s^{i;t,x}(e)\tilde{\mu}(ds, de), \quad s \leq T; \\ \text{(ii)} \quad Y_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i;t,x} - \ell(s, X_s^{t,x}))dK_s^{i;t,x} = 0; \\ \text{(iii)} \quad Y_T^{i;t,x} = g^i(X_T^{t,x}). \end{array} \right. \quad (0.0.137)$$

this equation was used for show existence and uniqueness solution for integro-differential partial equation

$$\left\{ \begin{array}{l} \min \left\{ u^i(t, x) - \ell(t, x); -\partial_t u^i(t, x) - b(t, x)^\top D_x u^i(t, x) \right. \\ \left. - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 u^i(t, x)) - K_i u^i(t, x) - h^{(i)}(t, x, u^i(t, x)), \right. \\ \left. (\sigma^\top D_x u^i)(t, x), B_i u^i(t, x) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ u^i(T, x) = g^i(x); \end{array} \right. \quad (0.0.138)$$

where B_i and K_i are defined as follows:

$$\begin{aligned} B_i u^i(t, x) &= \int_{\mathbf{E}} \gamma^i(t, x, e)(u^i(t, x + \beta(t, x, e)) - u^i(t, x))\lambda(de); \\ K_i u^i(t, x) &= \int_{\mathbf{E}} (u^i(t, x + \beta(t, x, e)) - u^i(t, x) - \beta(t, x, e)^\top D_x u^i(t, x))\lambda(de). \end{aligned} \quad (0.0.139)$$

The RBSDE with jump (0.0.137) will not be the only tool to help solve equation (0.0.138), it will be coupled with the Stochastic Differential Equations (SDE for short) with jump,

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r + \int_t^s \int_E \beta(r, X_{r-}^{t,x}, e) \tilde{\mu}(dr, de), \quad (0.0.140)$$

for $s \in [t, T]$ and $X_s^{t,x} = x$ if $s \leq t$.

One of the objectives of this thesis is therefore to solve (0.0.138) through (0.0.137) and (0.0.140) by abstraction of the usual conditions (Lipschitz, monotony on the generator f).

Other works have been approached with in particular the use of the fractional Brownian to generalize the result of [22] by establishing the existence and the uniqueness of solution for the family of stochastic differential equation:

$$Y(t) = \xi + \int_t^T f(s, \eta_s, Y_s, Z_s) ds - \int_t^T Z(t) dB_s^H \quad 0 \leq t \leq T, \quad (0.0.141)$$

where B^H is a fractional Brownian motion with Hurst Parameter H greater than $1/2$, η is a stochastic process given by $\eta_t = \eta_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dB_s^H$, and $(Y_t, Z_t) = (Y(t+u), Z(t+u))_{-T \leq u \leq 0}$. The stochastic integral used in the above equation is the divergence-type integral. Using the fixed point principle, we establish our main result.

Starting from this result, we associated the work of Fan [28] and of T. Wang and J. Yu [73] to explore the anticipated generalized retrograde doubly stochastic differential equations with non-Lipschitz condition where it was necessary to show the existence and uniqueness of solution for the stochastic differential equation

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) ds + \int_t^T g(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) dB_s \\ \quad + \int_t^T h(s, Y_s, Y_{s+\delta(s)}) d\Lambda_s - \int_t^T Z_s dW_s, \quad t \in [0, T] \\ Y_t = \xi_t, \quad Z_t = \eta_t, \quad t \in [T, T+K], \end{cases} \quad (0.0.142)$$

where K is a positive constant, $\delta(\cdot), \zeta(\cdot) : [0, T] \rightarrow \mathbb{R}_+$ are positive functions.

Finally we are interested in the existence and uniqueness of solution for the family of backward stochastic differential equation governed by two mutually independent

fractional Brownians,

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds + \int_t^T g(s, \eta_s, Y_s) d\Lambda_s - \int_t^T Z_{1,s} dB_{1,s}^{H_1} - \int_t^T Z_{2,s} dB_{2,s}^{H_2}, \quad t \in [0, T],$$

where $\{B_{1,t}^{H_1}, t \in [0, T]\}$ and $\{B_{2,t}^{H_2}, t \in [0, T]\}$ two mutually independent fractional Brownians and $\{\Lambda_t, t \in [0, T]\}$ is a real-valued continuous increasing process. The novelty in this type of stochastic equations resides in the fact of the presence of a new term in the EDSR, namely an integral with respect to an increasing continuous process.

It should be remembered that these types of equations (BSDE) have always been the subject of research since their first use in 1973 by Bismut [12] with a linear generator; after this work, E. Pardoux and S. Peng [57] establish the first result of existence and uniqueness of solution in the nonlinear case. Since then, literature has continued to increase, in particular with the work of Barles et al. [8] which will extend the work of Pardoux and Peng to integro-differential partial equations. In 1997, El Karoui et al. introduce in [23] the RBSDEs with one obstacle, which are BSDEs whose solution remains above a continuous function which plays the obstacle role. In [23], the authors establish the existence and uniqueness of solution using the penalization method as well as the Picard iteration method. After these works, Hamadene and Ouknine [38] make a generalization on the RBSDE with jumps and one barrier(obstacle), with the novelty of a noise made up of Brownian motion and the compensated Poisson measure [47]. Hamadene in [37] in collaboration with Morlais propose a new definition of the concept of solution of different viscosity from the classic used in [8, 18] by overcoming the condition of monotony usual in the first variable of the generator but maintain the fact that the measure of Levy finished. In [36], 2016 Hamadene made an extension of [37] by taking the measure of Levy infinite and he establishes the existence and the uniqueness of the solution by making a truncation of the Levy measure.

It is therefore on the basis of all these previous works that we are interested in RBSDE with jumps at a barrier in the multidimensional case with Levy measure infinite as in [36].

As mentioned above, EDSRs have undergone significant mutations since their appearance ranging from simple Brownian motion to fractional case; without forgetting the case not Lipschitz.

Recently in [22], Delong and Imkeller introduced BSDEs with time delayed generators defined by

$$Y(t) = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z(s) dW_s \quad 0 \leq t \leq T, \quad (0.0.143)$$

where for all $t \in [0, T]$,

$$(Y_t, Z_t) = (Y(t+u), Z(t+u))_{-T \leq u \leq 0} \quad (0.0.144)$$

represents all the past of the process solution until t . An existence and uniqueness result under a small global Lipschitz K or a small horizon time T has been derived. This restrictive is inevitable, since authors with two examples, proved that this result can not be extended in the general case. However, for some special class of generators, existence and uniqueness result may still hold for arbitrary both time horizon and Lipschitz constant.

On the other hand, several authors investigate BSDEs with respect to fractional Brownian motion (fractional BSDEs in short). In [11], Bender gave one of the earliest result on fractional BSDEs. The author established an explicit solution of a class of linear fractional BSDEs with arbitrary Hurst parameter H . This is done essentially by means of solution of a specific linear parabolic PDE. There are two major obstacles depending on the properties of fractional Brownian motion: Firstly, the fractional Brownian motion is not a semimartingale except for the case of Brownian motion ($H = 1/2$), hence the classical Itô calculus based on semimartingales cannot be transposed directly to the fractional case. Secondly, there is no martingale representation theorem with respect to the fractional Brownian motion. Studying nonlinear fractional BSDEs, Hu and Peng [45] overcame successfully the second obstacle in the case $H > 1/2$ by means of the quasi-conditional expectation. The authors prove existence and uniqueness of the solution but with some restrictive assumptions on the generator. In this same spirit, Maticiuc and Nie [54] interesting in backward stochastic variational inequalities, improved this first result by weakening the required condition on the drift of the stochastic equation.

Other authors like Fan[28] have had to work without the so-called Lipschitz condition where they use a concave function satisfying Mao's condition [53] to be able to establish an a priori approximation.

It is therefore on the basis of all these results that we have tried to make our contribution to the study of EDSRs.

During this thesis, four main results were obtained with in particular a paper already published in which we were inspired by [36] by showing the existence and the uniqueness of solution for the case of EDSRR with jumps at a barrier in the multidimensional case for the infinite Lévy measure; it can be consulted in the journal *Acta Mathematica Scientia*, 2019, **39B**(3): pp. 819-844 (see the following link <https://link.springer.com/article/10.1007/s10473-019-0312-5>).

Our second result is oriented towards the fractional Brownian in delay time (it can be consulted in the journal: *Applicable Analysis*): see the following link <https://www.>

tandfonline.com/eprint/PCWXSGGHJKPGXBREVU7J/full?target=10.1080/00036811.2023.2207586.

The third paper is an orientation to the Generalized Anticipated Backward Doubly Stochastic Differential Equations: with non-Lipschitz Condition.

The fourth result immerses us in doubly stochastic BSDE driven by two mutually independent fractional Brownians.

These last two papers are submitted for publication .

For a better understanding of our results and to give a boost to readers who would like to get involved in the field, we will start with reminders of certain concepts related to BSDE and RBSDE, which will undoubtedly help the reader to become familiar with the subject and everything that revolves around.

CHAPTER 1

GENERAL INFORMATION ON SDE AND BSDE

The objective of this chapter is to come back to the basic notions of the theory SDE and BSDE with some applications.

1.1 Stochastic differential equations

Theorem-Definition 1.1.1. *Let $(B_t, t \in \mathbb{R}_+)$ a standard Brownian motion (with respect to filtration $(\mathcal{F}_t, t \in \mathbb{R}_+)$), $x_0 \in \mathbb{R}$ and f, g Lipschitzian. So there is a unique process $(X_t, t \in \mathbb{R}_+)$ continuous and adapted to $(\mathcal{F}_t, t \in \mathbb{R}_+)$ such as*

$$X_t = x_0 + \int_0^t f(s, X_s) ds + \int_0^t g(s, X_s) dB_s, \quad a.s. \quad \forall t \in \mathbb{R}_+. \quad (1.1.1)$$

Moreover, $\mathbb{E}(\sup_{0 \leq t \leq T} X_t^2) < \infty$ for $T > 0$.

- The solution (X_t) of the equation (1.1.1) is also called a strong solution of the equation.
- The functions f and g are called respectively the term of drift and the term of diffusion of the equation.

The equation (1.1.1) is called Stochastic Differential Equations for short SDE. Its differential form is given by $dX_t = f(t, X_t)dt + g(t, X_t)dB_t$

The fields of application of SDE are vast: physics, biology, population dynamics, ecology, financial mathematics, signal processing, control theory...

1.1.1 Some remarkable SDE

Example 1.1.1. (*Black-Scholes-Samuelson equation*)

$$dX_t = \mu X_t dt + \sigma X_t dB_t \text{ et } X_0 = x_0,$$

with $\mu \in \mathbb{R}$ et $\sigma, x_0 > 0$ fixed, is an example of an equation admitting a unique solution (X_t) , because $f(x) = \mu x$ and $g(x) = \sigma x$ are linear, therefore Lipschitzian.

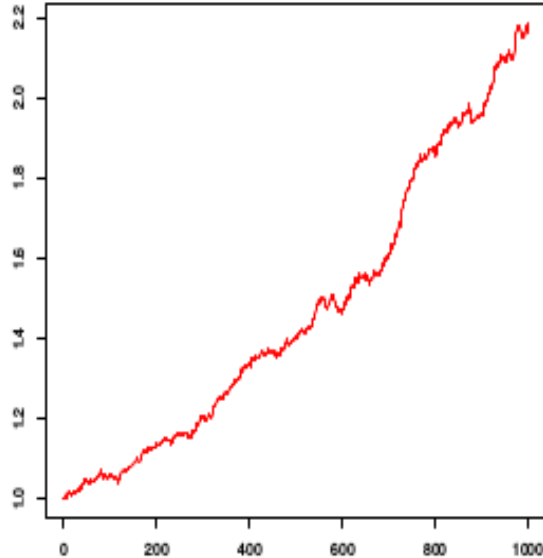


Figure 1.1: A trajectory of the process of the Black-Scholes-Samuelson equation $\mu = 0.75$ and $\sigma = 0.1$.

This equation finds its applications in finance in particular for writing the price of an asset and in this case we will call the coefficient in front of the Brownian its volatility, and the coefficient in front of dt expected return.[68]

Example 1.1.2. (*Ornstein-Uhlenbeck process*)[71]

The Ornstein-Uhlenbeck process, named after Leonard Ornstein and George Uhlenbeck and also known as the mean-reverting process, is a stochastic process described by the SDE

$$dX_t = -\theta(X_t - \mu) dt + \sigma dB_t \text{ et } X_0 = x_0,$$

where θ, μ et σ are deterministic parameters.

The Vasicek model of interest rates is an example of an Ornstein-Uhlenbeck process where the coefficients are positive and constant.

The CIR Process, the model of Cox, Ingersoll and Ross (1985) is an extension of the Vasicek model and the Ornstein-Uhlenbeck process which introduces the square root of the instantaneous interest rate into the coefficient of the stochastic term. [20]

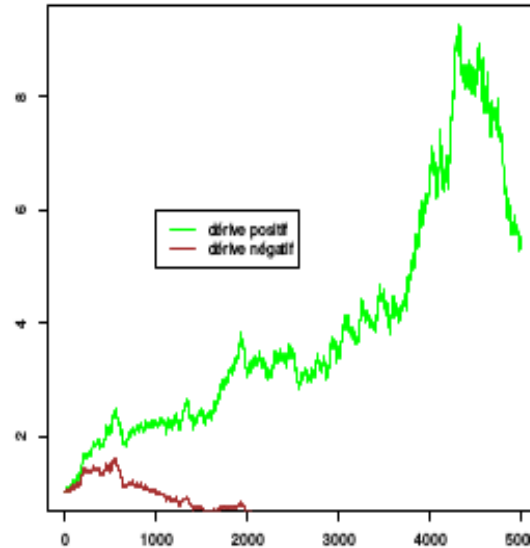


Figure 1.2: According to the sign of volatility, the trajectory of the Black-Scholes-Samuelson process

Example 1.1.3. (*The SIR model based on SDE*)[72, 80]

The model « S.I.R. » is a classic model in epidemiology for the spread of a contagious disease.

The model « S.I.R. » was first presented by Kermack and Mckendrick in London and Cambridge in 1927 to explain a posteriori the evolution of the plague epidemic in Bombay in 1905-1906.

At each moment we decide to divide the population into three categories (called "compartments" in the language of epidemiology):

- the individuals « **Susceptibles** » (**S**): those who have never had the disease, and can contract it;
- the individuals « **Infected** » (**I**) : the sick are also the contagious (this is a hypothesis of this model);
- the individuals « **Recovered** » (**R**): those who have had the disease in the past and are now immune to it (this is an assumption of this model). This group includes people who have died (since they can no longer contract the disease, and because it is convenient).

The model is described by the following SDE system:

$$\begin{cases} dS(t) = -\left[\frac{\beta S(t)I(t)}{N}\right] - \sqrt{\frac{\beta S(t)I(t)}{N}} dW_1(t) \\ dI(t) = -\left[\frac{\beta S(t)I(t)}{N} - \gamma I(t)\right] dt + \sqrt{\frac{\beta S(t)I(t)}{N}} dW_1(t) - \sqrt{\gamma I(t)} dW_2(t) \end{cases} \quad (1.1.2)$$

where $N = S(t) + I(t) + R(t)$ is a constant representing the total population. The absence of patients is obtained when $S(t) = N$ and $I(t) = R(t) = 0$. The parameters β and γ represent respectively the rate of infected and the rate of cured.

$W_1(t)$ and $W_2(t)$ are two independent Wiener processes.

The notation $B(t) = W(t)$ is also used when $B(t)$ denotes a Brownian motion.

1.1.2 Quadratic variation of standard Brownian motion

Let $(B_t, t \in \mathbb{R}_+)$ a standard Brownian motion (for convenience, we will sometimes note $B_t = B(t)$). For $t > 0$, we define

$$\langle B \rangle_t^{(n)} = \sum_{i=1}^n \left(B\left(\frac{it}{2^n}\right) - B\left(\frac{(i-1)t}{2^n}\right) \right)^2.$$

Proposition-Definition 1.1.1.

$$\text{For } t > 0, \lim_{n \rightarrow \infty} \langle B \rangle_t^{(n)} = t \quad \text{a.s.}$$

and we define the quadratic variation of the standard Brownian motion $\langle B \rangle_t$ as being given by this limit (by convention we set $\langle B \rangle_0 = 0$).

Consequence of 1.1.1

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{2n} \left| B\left(\frac{it}{2^n}\right) - B\left(\frac{(i-1)t}{2^n}\right) \right| = \infty \quad \text{a.s. .}$$

So the process (B_t) is not bounded variation.

Remark 1.1.1. *The fact that the process (B_t) is not with bounded variation implies that its trajectories are not derivable.*

Remark 1.1.2. *The process $(B_t^2 - \langle B \rangle_t, t \in \mathbb{R}_+)$ is a martingale. This property will even be used to define the quadratic variation of a continuous martingale of integral square. For more details see [52].*

1.1.3 Link SDE \leftrightarrow Parabolic PDEs (Feynman-Kac formula)

Definition 1.1.1. *A simple predictable process (with respect to filtration (\mathcal{F}_t)) is a process $(H_t, t \in [0, T])$ such as*

$$H_t = \sum_{i=1}^n X_i \mathbf{1}_{]t_{i-1}, t_i]}(t), \quad t \in [0, T],$$

where $0 = t_0 < t_1 < \dots < t_n = T$ form a partition of $[0, T]$ and X_i is a random variable $\mathcal{F}_{t_{i-1}}$ -measurable and bounded $\forall i \in \{1, \dots, n\}$.

We can see that over the interval $]t_{i-1}, t_i]$, the value of the process (H_t) is determined by information $\mathcal{F}_{t_{i-1}}$, hence the name "predictable" (the name "simple" comes from the fact that the process takes only a finite number of values (random!)).

Remark 1.1.3. *An adapted and continuous process on the left is the continuous-time equivalent of a predictable discrete-time process: the value of such a process at time t cannot differ too much from its value at the moment $t - \epsilon$, when it is $\mathcal{F}_{t-\epsilon}$ -measurable because adapted.*

Property 1.1.1. (Properties of the stochastic integral)

- Linearity: $\int_0^t (g(s, X_s) + ch(s, X_s)) dB_s = \int_0^t g(s, X_s) dB_s + c \int_0^t h(s, X_s) dB_s.$
- Zero expectation: $\mathbb{E} \left[\int_0^t g(s, X_s) dB_s \right] = 0.$
- Isometry: $\text{Cov} \left(\int_0^t g(s, X_s) dB_s, \int_0^r h(u, X_u) dB_u \right) = \mathbb{E} \left[\int_0^{t \wedge r} g(s, X_s) h(s, X_s) ds \right]$
- $\int_0^t g(s, X_s) dB_s$ is a continuous martingale of square integrable such that
$$\mathbb{E} \left[\sup_{t \in [0, T]} \left(\int_0^T g(s, X_s) dB_s \right)^2 \right] \leq 4 \mathbb{E} \left[\int_0^T g(s, X_s)^2 ds \right]$$

The details of these properties are exposed in [52] with an extension on stochastic integrals with respect to a martingales.

Definition 1.1.2. (Itô process)

We call Itô process, a process X to real values such as

$$\mathbb{P} - a.s. \quad \forall 0 \leq t \leq T, \quad X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dB_s ,$$

where X_0 is \mathcal{F}_0 -measurable, K and H are two progressively measurable processes verifying the conditions, \mathbb{P} -a.e.

$$\int_0^T |K_s| ds < \infty \quad \text{et} \quad \int_0^T H_s^2 ds < \infty.$$

Theorem 1.1.1. (Basic Itô Formula)

Let $(B_t, t \in \mathbb{R}^+)$ a standard Brownian motion with respect to $(\mathcal{F}_t, t \in \mathbb{R}^+)$ and $f \in C^2(\mathbb{R})$ (i.e f, f', f'') are continuous. It is further assumed that

$$\mathbb{E} \left[\int_0^t (f'(B_s))^2 ds \right] < \infty. \tag{1.1.3}$$

Then for everything $t > 0$,

$$f(B_t) - f(B_0) = \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds \quad a.s. \quad (1.1.4)$$

Remark 1.1.4. *Seen that $(f'(B_t), t \geq 0)$ is a continuous process adapted to $(\mathcal{F}_t, t \in \mathbb{R}^+)$ and the condition (1.1.3) is verified, $\int_0^t f'(B_s) dB_s$ is well defined and $\int_0^t f''(B_s) ds$ is also because the application $s \mapsto f''(B_s)$ is continuous.*

Theorem 1.1.2. *Let $(B_t, t \in \mathbb{R}^+)$ a standard Brownian motion with respect to $(\mathcal{F}_t, t \in \mathbb{R}^+)$ and $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ (i.e. $f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ are continuous). It is further assumed that*

$$\mathbb{E} \left[\int_0^t \left(\frac{\partial f}{\partial x}(s, B_s) \right)^2 ds \right] < \infty . \quad (1.1.5)$$

Then for all $t > 0$,

$$f(t, B_t) - f(0, B_0) = \int_0^t \frac{\partial f}{\partial s}(s, B_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s) ds \quad a.e.. \quad (1.1.6)$$

We can extend the theorem 1.1.2 to the case of a m -dimensional Brownian motion and an Itô n -dimensional process. The hypotheses on the coefficients are those of the Definition 1.1.2.

Let us first give some rules that we will use in the following:

1. if $X_t = \int_0^t H_s dM_s$, then

$$dX_t = H_t dM_t \quad et \quad d\langle X \rangle_t = H_t^2 d\langle M \rangle_t , \quad (*)$$

2. note that if $M_t = \int_0^t K_s dB_s$, then

$$X_t = \int_0^t H_s K_s dB_s \quad et \quad d\langle X \rangle_t = \int_0^t H_s^2 K_s^2 ds . \quad (**)$$

Now let's move on to the extension theorem.

Theorem 1.1.3. *Let X a process of Itô to values in \mathbb{R}^n : for $i = 1, \dots, n$,*

$$X_t^i = X_0^i + \int_0^t K_s^i ds + \sum_{k=1}^m \int_0^t H_s^{i,k} dB_s^k .$$

If f is twice differentiable in x and once in t , then we have

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_j \partial x_i}(s, X_s) d\langle X^i, X^j \rangle_s, \end{aligned}$$

with $dX_s^i = K_s^i ds + \sum_{k=1}^m H_s^{i,k} dB_s^k$ and $d\langle X^i, X^j \rangle_s = \sum_{k=1}^m H_s^{i,k} H_s^{j,k} ds$.

This theorem can be understood in matrix form by taking X as the column vector of \mathbb{R}^n with coordinates X^i , K the vector of \mathbb{R}^n of coordinates K^i and B the vector of \mathbb{R}^m of coordinates B^j . We then introduce the size matrix $n \times m$, $H = (H^{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$. With these notations, we have:

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dB_s,$$

where $H_s dB_s$ is a matrix-column vector product. Itô's formula is written as, denoting H^* the transpose of H

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \int_0^t \nabla f(s, X_s) \cdot dX_s \\ &\quad + \frac{1}{2} \int_0^t \text{trace}(D^2 f(s, X_s) H_s H_s^*) ds, \end{aligned}$$

or by using the calculation rules established in (*) et (**)

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \int_0^t \left(\frac{\partial f}{\partial s}(s, X_s) + \nabla f(s, X_s) \cdot K_s \right) ds + \int_0^t \nabla f(s, X_s) \cdot H_s dB_s \\ &\quad + \frac{1}{2} \int_0^t \text{trace}(D^2 f(s, X_s) H_s H_s^*) ds. \end{aligned}$$

1.1.3.1 Parabolic PDE

Let $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}$, $(B, t \in \mathbb{R}_+)$ a standard Brownian motion and $f, g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ two functions jointly continuous in (t, x) and Lipschitzian in x . We pose $X_t^{t_0, x_0} = X_t - X_{t_0} + x_0$, $t \in [t_0, +\infty[$ the SDE solution $dX_t = f(t, X_t) dt + g(t, X_t) dB_t$ and $X_{t_0} = x_0$. We further assume that there exists a constant $K > 0$ such that $|g(t, x)| \geq K$, $\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

This last hypothesis is the hypothesis of "non-degenerate diffusion"

Theorem 1.1.4. Let $h \in C(\mathbb{R})$ and $T > 0$. Given the assumptions made above on f

and g , there is a “unique” function $u \in C^{1,2}([0, T] \times \mathbb{R})$ which verify

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + f(t, x) \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} g^2(t, x) \frac{\partial^2 u}{\partial x^2}(t, x) = 0, & \forall (t, x) \in [0, T] \times \mathbb{R} \\ u(T, x) = h(x) & \forall x \in \mathbb{R} \end{cases} \quad (1.1.7)$$

Lemma 1.1.1. *Let u the solution of equation (1.1.7). For all $(t_0, x_0) \in [0, T] \times \mathbb{R}$ fixed, le process $(u(t, X_t^{t_0, x_0}), t \in [t_0, T])$ is a martingale.*

Property 1.1.2. (Feynman-Kac formula)

Let u the solution of equation (1.1.7). For $\forall (t_0, x_0) \in [0, T] \times \mathbb{R}$, we have $u(t_0, x_0) = \mathbb{E} [h(X_T^{t_0, x_0})]$

Proof 1.1.1. *By the lemma 1.1.1 and the terminal condition (retrograde) in (1.1.7), we found that*

$$u(t_0, x_0) = u(t_0, X_{t_0}^{t_0, x_0}) = \mathbb{E} [u(T, X_T^{t_0, x_0})] = \mathbb{E} [h(X_T^{t_0, x_0})]$$

1.2 Stochastic differential equations with jumps

1.2.1 Levy process and Poisson random measure

Definition 1.2.1 (Function RCLL « right continuous with left limits », function LCRL « left continuous with right limits »).

1) A function $f : [0, T] \rightarrow \mathbb{R}^n$ is said right continuous with left limits (“RCLL”), if for $\forall t \in [0, T]$

$$f(t^+) = \lim_{s \rightarrow t^+} f(s) = f(t) \text{ and } f(t^-) = \lim_{s \rightarrow t^-} f(s) \text{ exist}$$

2) A function $f : [0, T] \rightarrow \mathbb{R}^n$ is said left continuous with right limits (“LCRL”), if for $\forall t \in [0, T]$

$$f(t^-) = \lim_{s \rightarrow t^-} f(s) = f(t) \text{ and } f(t^+) = \lim_{s \rightarrow t^+} f(s) \text{ exist}$$

Remark 1.2.1.

- If f is “RCLL” for $\forall t \in [0, T]$, then $\Delta f(t) = f(t) - f(t^-)$ is called the jump of f in t .

- If f is "LCRL" for $\forall t \in [0, T]$, then $\Delta f(t) = f(t) - f(t^+)$ is called the jump of f in t .

Definition 1.2.2 (Levy process).

Let $X = (X_t; t \geq 0)$ a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- We say that X has **independent increments** if, for $\forall n \in \mathbb{N}$ and for $\forall 0 \leq t_1 \leq t_2 \dots \leq t_n < +\infty$, random variables $X_{t_{j+1}} - X_{t_j}; 1 \leq j \leq n - 1$ are independents.
- We say that X is in **stationary increments** if, for all $t, s > 0$, the r.v $X_{t+s} - X_s$ has the same law as $X_t - X_0$.
- We say that X is a **Levy process** (resulting from 0) if:
 - 1) $X_0 = 0$ almost surely,
 - 2) X has independent and stationary increases,
 - 3) X is stochastically continuous : for $\forall \epsilon > 0$ and $t \geq 0$:

$$\lim_{h \rightarrow 0} \mathbb{P}(|X_{t+h} - X_t| > \epsilon) = 0$$

- If X is a Levy process, we define the associated **jump process** $\Delta X = (\Delta X_t; t \geq 0)$ with $\Delta X_t = X_t - X_{t^-}$ under the remark. [1.2.1](#)

Definition 1.2.3 (Random Poisson measure).

Let (E, \mathcal{E}) a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probabilized space.

Let λ a σ -finite measure on (E, \mathcal{E}) . A random measure of Poisson μ on (E, \mathcal{E}) is a collection of random variables $(\mu(B); B \in \mathcal{E})$ such that:

- 1) for $\forall B \in \mathcal{E}$ such that $\lambda(B) < +\infty$, $\mu(B)$ follows a Poisson distribution with parameter $\lambda(B)$,
- 2) if A_1, \dots, A_m are disjoint sets of \mathcal{E} ,
the random variables $\mu(A_1), \dots, \mu(A_m)$ are independents,
- 3) For $\forall \omega \in \Omega$, the application $A \rightarrow \mu(A, \omega)$ is a count measure on (E, \mathcal{E})

Note that if X is a Levy process then the quantity $\mu([0, t] \times A) = \#\{0 \leq s \leq t; \Delta X_s \in A\}$ is a random Poisson measure on $\mathbb{R}_+ \times \mathbb{R}^d \setminus \{0\}$ of intensity $dt \otimes d\lambda$ with $t\lambda(A) = \mathbb{E}[\mu([0, t] \times A)]$.

Just as in the case of the Brownian which has finite quadratic variation which plays a major role in the a priori estimates that we will see in the next parts, we define the compensated Poisson random measure $\tilde{\mu}$ which will allow us to define a space of type \mathbb{L}^2 under the intensity λ .

For $t \geq 0$ and A bounded below, we define the compensated Poisson random measure $\tilde{\mu}$ by:

$$\tilde{\mu}([0, t] \times A) = \mu([0, t] \times A) - t\lambda(A)$$

Remark 1.2.2. Note that in the literature the notation $\mu([0, t] \times A)$ is equivalent to the notation $\mu(t, A)$

1.2.2 Integration with respect to the Poisson random measure μ

Let μ a random Poisson measure of intensity $dt \otimes d\lambda$ on $\mathbb{R}_+ \times \mathbb{R}^l \setminus \{0\}$. If $f : \mathbb{R}^l \setminus \{0\} \rightarrow \mathbb{R}^n$ is a Borel-measurable function and if $A \in \mathbb{B}(\mathbb{R}^l \setminus \{0\})$ verify $\mu(A) < +\infty$, we define for $\forall t \geq 0$ the integral with respect to the Poisson random measure of f by

$$\int_A f(e) \mu(t, de) = \sum_{e \in A} f(e) \mu(t, \{e\}).$$

Note that the definition above does not pose a problem because the sum is random finite. If μ is associated with a Levy process, this integral coincides with

$$\int_A f(e) \mu(t, de) = \sum_{0 \leq s \leq t} f(\Delta X_s) \mathbf{1}_A(\Delta X_s).$$

because $\mu(t, A) = \#\{0 \leq s \leq t; \Delta X_s \in A\}$. Moreover, the process $t \rightarrow \int_A f(e) \mu(t, de)$ is an adapted stochastic process and RCLL. We also notice that if λ is a Levy measure then $0 \notin \bar{A} \Rightarrow \lambda(A) < +\infty$ so that $\int_A f(e) \mu(t, de)$ is well defined as long as A is lower bound.

We will now assume that μ is associated with a Levy process.

Theorem 1.2.1. Let A be a lower bound Borel set. Then :

1) If $f \in \mathbb{L}^1(A, \lambda)$ we have:

$$\mathbb{E} \left[\int_A f(e) \mu(t, de) \right] = t \int_A f(e) \lambda(de)$$

2) If $f \in \mathbb{L}^2(A, \lambda)$ we have:

$$\text{Var} \left[\int_A f(e) \mu(t, de) \right] = t \int_A |f(e)|^2 \lambda(de)$$

3) If $f \in \mathbb{L}^2(A, \lambda)$ and $g \in \mathbb{L}^2(B, \lambda)$ we have:

$$\text{Cov} \left[\int_A f(e) \mu(t, de), \int_B g(e) \mu(t, de) \right] = t \int_{A \cap B} f(e)g(e) \lambda(de)$$

Remark 1.2.3. The construction of the integral with respect to the random Poisson measure μ is done following the same steps as in the case of the construction of the Lebesgue integral with in particular the elementary functions, the staged or simple functions, the positive functions and arbitrary functions. We invite the reader who wishes

to refer to [47, 15] to see the method of construction of such integrals and also to have an outline on the proof of the theorem 1.2.1

1.2.3 Integration with respect to the compensated Poisson random measure

$\tilde{\mu}$

The stochastic integral with respect to $\tilde{\mu}$ is an object defined by a limit \mathbb{L}^2 . Indeed, its construction is analogous to that of the stochastic integral with respect to the motion Brownian and relies like it on an isometric property.

Definition 1.2.4. For $\forall f \in \mathbb{L}^1(A, \lambda)$ we define the integral with respect to the compensated random Poisson measure $\tilde{\mu}$ by

$$\int_A f(e) \tilde{\mu}(t, de) = \int_A f(e) \mu(t, de) - t \int_A f(e) \lambda(de)$$

Property 1.2.1. Let A be a lower bound Borel set.

- 1) If $f \in \mathbb{L}^1(A, \lambda)$ we have:
the process $t \rightarrow \int_A f(e) \tilde{\mu}(t, de)$ is a martingale RCLL
- 2) If $f \in \mathbb{L}^1(A, \lambda)$ we have:

$$\mathbb{E} \left[\int_A f(e) \tilde{\mu}(t, de) \right] = 0$$

- 3) If $f \in \mathbb{L}^2(A, \lambda)$ and $g \in \mathbb{L}^2(B, \lambda)$ we have:

$$\text{Cov} \left[\int_A f(e) \tilde{\mu}(t, de), \int_B g(e) \tilde{\mu}(t, de) \right] = t \int_{A \cap B} f(e)g(e) \lambda(de)$$

1.2.4 Ito formula for jump processes

The following proposition gives the simplest instance of the Itô formula with jumps, in the case of a standard Poisson process of measure μ and intensity λ .

Proposition 1.2.1. Itô's formula for a measurement process μ

If $(X_t)_{t \geq 0}$ is a stochastic process defined by $X_t = \int_0^t \int_A K(r, e) \mu(dr, de)$ where K is predictable then, for any function $f \in C(\mathbb{R})$ and for $\forall t \geq 0$, it almost surely :

$$f(X_t) = f(X_0) + \int_0^t \int_A f(X_{r-} + K(r, e)) - f(X_{r-}) \mu(dr, de) \quad (1.2.1)$$

Remark 1.2.4. The relation (1.2.1) is equivalent to the following two relations:

1)

$$f(X_t) = f(X_0) + \sum_{0 \leq s \leq t, \Delta X_s \neq 0} f(X_{s-} + K(s, \Delta X_s)) - f(X_{s-}) \quad (1.2.2)$$

2)

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{s-}) dX_s \\ &+ \sum_{0 \leq s \leq t, \Delta X_s \neq 0} f(X_{s-} + K(s, \Delta X_s)) - f(X_{s-}) - K(s, \Delta X_s) f'(X_{s-}). \end{aligned}$$

3) The relation 2) is obtained by adding and subtracting $K(s, \Delta X_s) f'(X_{s-})$ from the sum of the relation (1.2.2).

This technique will be of great use to us in the next chapter, notably with the linearization of the generator.

We will now see the Itô formula in the case of a Poisson process for the compensated random measure $\tilde{\mu}$

Proposition 1.2.2. Itô formula for a measurement process $\tilde{\mu}$

If $(X_t)_{t \geq 0}$ is a stochastic process defined by $X_t = \int_0^t \int_A K(r, e) \tilde{\mu}(dr, de)$ where K is predictable such that $\mathbb{P} \left(\int_0^t \int_A |K(s, e)|^2 \lambda(de) ds < +\infty \right) = 1$ then, for all function $f \in C(\mathbb{R})$ and for all $t \geq 0$, we almost surely have :

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t \int_A f'(X_{s-}) K(s, e) \tilde{\mu}(ds, de) \\ &+ \int_0^t \int_A \left(f(X_{s-} + K(s, e)) - f(X_{s-}) - K(s, e) f'(X_{s-}) \right) \lambda(de) ds \end{aligned} \quad (1.2.3)$$

Remark 1.2.5.

a)

$$\begin{aligned} &\int_0^t \int_A \left(f(X_{s-} + K(s, e)) - f(X_{s-}) - K(s, e) f'(X_{s-}) \right) \lambda(de) ds = \\ &\sum_{0 \leq s \leq t, \Delta X_s \neq 0} f(X_{s-} + K(s, \Delta X_s)) - f(X_{s-}) - K(s, \Delta X_s) f'(X_{s-}) \end{aligned}$$

b) The differential form of (1.2.3) is given by:

$$df(X_t) = f'(X_{t-}) dX_t + \left[f(X_t) - f(X_{t-}) - f'(X_{t-}) \Delta X_t \right] \quad (1.2.4)$$

c) In propositions 1.2.1 and 1.2.2, it should be noted that $f(X_0)$ could be replaced by $f(0)$ because a Levy process is zero at the origin.

1.2.5 Stochastic integral of Levy and semimartingale

Definition 1.2.5. *Stochastic integral of Levy*

We call Levy's stochastic integral a process $(Y_t)_{t \geq 0}$ such that $t \geq 0$

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \psi(r) dr + \int_0^t \varphi(r) dB_r + \int_0^t \int_{\mathbb{R}} H(r, e) \mu(dr, de) \\ &+ \int_0^t \int_{\mathbb{R}} K(r, e) \tilde{\mu}(dr, de), \end{aligned} \quad (1.2.5)$$

where

1. X_0 is F_0 -measurable,
2. φ is a process adapted such as $\forall t \geq 0, \int_0^t \varphi^2(r) dr < +\infty$
3. ψ is a process adapted such as $\forall t \geq 0, \int_0^t |\psi(r)| dr < +\infty$
4. H is a predictable process such as $\forall t \geq 0, \int_0^t \int_{\mathbb{R}} |H(r, e)| \mu(dr, de) < +\infty$
5. K is a predictable process such as $\forall t \geq 0, \int_0^t \int_{\mathbb{R}} K^2(r, e) \lambda(de) dr < +\infty$

Example 1.2.1. *Le processus de Lévy*

$$\begin{aligned} Y_t &= Y_0 + bt + \sqrt{c}B_t + \int_0^t \int_{\mathbb{R}} x \mathbf{1}_{\{|x|>1\}} \mu(dr, de) \\ &+ \int_0^t \int_{\mathbb{R}} x \mathbf{1}_{\{|x| \leq 1\}} \tilde{\mu}(dr, de), \end{aligned} \quad (1.2.6)$$

is obtained by choosing $\psi(t) = b$, $\varphi(t) = \sqrt{c}$, $H(t, x) = x \mathbf{1}_{\{|x|>1\}}$, $K(t, x) = x \mathbf{1}_{\{|x| \leq 1\}}$

Definition 1.2.6. *semi-martingale*

A semi-martingale is process $(Y_t)_{t \geq 0}$ $(F_t)_{t \geq 0}$ -adapted RCLL which admits decomposition $\forall t \geq 0, Y_t = Y_0 + A_t + M_t$ where

- $(A_t)_{t \geq 0}$ is a finite variation process (i.e. being written as the difference of two increasing functions) RCLL zero at 0,
- $(M_t)_{t \geq 0}$ is a local martingale RCLL zero at 0.

Remark 1.2.6. *The decomposition of the semi-martingale $(Y_t)_{t \geq 0}$ is usually not unique. Nevertheless, when it is continuous, it admits a unique decomposition $Y_t = Y_0 + A_t + M_t$, with $(M_t)_{t \geq 0}$ local martingale continues, $(A_t)_{t \geq 0}$ is a finite variation process continuous, both currently zero $t = 0$.*

In the general case, there exists a unique continuous local martingale and zero in 0 denoted by $(Y_t^c)_{t \geq 0}$ such that $\forall t \geq 0$, $Y_t = Y_0 + Y_t^c + A_t + M_t^d$ with $(A_t)_{t \geq 0}$ finite variation process RCLL zero in 0 and $(M_t^d)_{t \geq 0}$ local martingale RCLL purely discontinuous in the sense where $(M_t^d N_t)_{t \geq 0}$ is a local martingale for any continuous local martingale $(N_t)_{t \geq 0}$. The process $(Y_t^c)_{t \geq 0}$ is called the continuous martingale part of the semi-martingale $(Y_t)_{t \geq 0}$.

Example 1.2.2. Levy's stochastic integrals are a special class of semi-martingales such as $Y_t^c = \int_0^t \varphi(r) dB_r$ and for which all the following decompositions indexed by $\alpha \geq 0$ are among others possible

$$\begin{aligned} M_t &= \int_0^t \varphi(r) dB_r + \int_0^t \int_{\mathbb{R}} \left(K(r, e) + H(r, e) \mathbf{1}_{\{|H(r, e)| \leq \alpha(1 \wedge e^2)\}} \right) \tilde{\mu}(dr, de) \\ A_t &= \int_0^t \psi(r) dr + \int_0^t \int_{\mathbb{R}} H(r, e) \mathbf{1}_{\{|H(r, e)| > \alpha(1 \wedge e^2)\}} \mu(dr, de) \\ &+ \int_0^t \int_{\mathbb{R}} H(r, e) \mathbf{1}_{\{|H(r, e)| \leq \alpha(1 \wedge e^2)\}} \lambda(de) dr \end{aligned} \quad (1.2.7)$$

1.2.6 Itô's formula for Levy's stochastic integrals

By combining Itô's formula (1.1.6) of theorem 1.1.2 for Itô processes with those stated in the propositions 1.2.1 and 1.2.2 we obtain the proposition following giving the Itô formula for the stochastic integrals of Levy.

Proposition 1.2.3. Let the stochastic integral of Levy $(Y_t)_{t \geq 0}$ defined by $\forall t \geq 0$

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \psi(s) ds + \int_0^t \varphi(s) dB_s + \int_0^t \int_{\mathbb{R}} H(s, e) \mu(ds, de) \\ &+ \int_0^t \int_{\mathbb{R}} K(s, e) \tilde{\mu}(ds, de), \end{aligned} \quad (1.2.8)$$

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is class C^2 then

$$\begin{aligned} f(t, Y_t) - f(0, Y_0) &= \int_0^t \frac{\partial f}{\partial s}(s, Y_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, Y_s) dY_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, Y_s) \varphi^2(s) ds \\ &+ \int_0^t \int_{\mathbb{R}} f(Y_{s-} + H(s, e)) - f(Y_{s-}) \mu(ds, de) \\ &+ \int_0^t \int_{\mathbb{R}} f'(Y_{s-}) K(s, e) \tilde{\mu}(ds, de) \\ &+ \int_0^t \int_{\mathbb{R}} \left(f(Y_{s-} + K(s, e)) - f(Y_{s-}) - K(s, e) f'(Y_{s-}) \right) \lambda(de) ds \end{aligned} \quad (1.2.9)$$

Remark 1.2.7. The relation (1.2.9) can be written as follows

$$\begin{aligned} f(t, Y_t) - f(0, Y_0) &= \int_0^t \frac{\partial f}{\partial s}(s, Y_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, Y_{s-}) dY_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, Y_s) \varphi^2(s) ds \\ &+ \sum_{0 \leq s \leq t} f(Y_s) - f(Y_{s-}) - f'(Y_{s-}) \Delta Y_s \end{aligned}$$

Corollary 1.2.1. If Y is a Levy-type stochastic integral like (1.2.5), then almost surely:

$$\sum_{0 \leq s \leq T} |\Delta Y_s|^2 < +\infty. \quad (1.2.10)$$

1.2.7 Itô formula for the semi-martingale in the multidimensional case

Itô's formula of proposition 1.2.3 for one-dimensional semi-martingales with jumps is easily generalized to n dimensional case.

Proposition 1.2.4. Consider a dimensional d - semi-martingales with jumps $Y_t = (Y_t^1, Y_t^2, \dots, Y_t^d)$ where for $i = 1, 2, \dots, d$

$$\begin{aligned} Y_t^i &= Y_0^i + \int_0^t \psi^i(s) ds + \int_0^t \varphi^i(s) dB_s^i + \int_0^t \int_{\mathbb{R}} H^i(s, e) \mu(ds, de) \\ &+ \int_0^t \int_{\mathbb{R}} K^i(s, e) \tilde{\mu}(ds, de), \end{aligned} \quad (1.2.11)$$

If $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is of class C^2 then

$$\begin{aligned} f(t, Y_t) - f(0, Y_0) &= \int_0^t \frac{\partial f}{\partial s}(s, Y_s) ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x^i}(s, Y_s) dY_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x^j \partial x^i}(s, Y_s) \varphi^i(s) \varphi^j(s) ds \\ &+ \int_0^t \int_{\mathbb{R}} f(Y_{s-} + H(s, e)) - f(Y_{s-}) \mu(ds, de) \\ &+ \sum_{i=1}^d \int_0^t \int_{\mathbb{R}} \frac{\partial f}{\partial x^i}(s, Y_s) K^i(s, e) \tilde{\mu}(ds, de) \quad (1.2.12) \\ &+ \int_0^t \int_{\mathbb{R}} (f(Y_{s-} + K(s, e)) \\ &- f(Y_{s-}) - \sum_{i=1}^d K^i(s, e) \frac{\partial f}{\partial x^i}(s, Y_s)) \lambda(de) ds \end{aligned}$$

Remark 1.2.8. Considering the decomposition $Y_t^i = Y_0^i + Y_t^{c,i} + A_t^i + M_t^{d,i}$ made in the remark 1.2.6 for $i = 1, 2, \dots, d$; Itô's formula in the d - dimensional case can be rewritten as follows:

$$\begin{aligned} f(t, Y_t) - f(0, Y_0) &= \int_0^t \frac{\partial f}{\partial s}(s, Y_s) ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x^i}(s, Y_{s-}) dY_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(s, Y_s) d \langle Y^{c,i}, Y^{c,j} \rangle_s \\ &\quad + \sum_{0 \leq s \leq t} \left(f(Y_s) - f(Y_{s-}) - \sum_{i=1}^d \Delta Y_s^i \frac{\partial f}{\partial x^i}(s, Y_s) \right) \end{aligned} \quad (1.2.13)$$

Remark 1.2.9. (Application of Itô's formula: Integration formula by parts)

Let $(Y_t^1)_{t \geq 0}$ and $(Y_t^2)_{t \geq 0}$ two semi-martingales with real values. Then $\forall t \geq 0$

$$Y_t^1 Y_t^2 = Y_0^1 Y_0^2 + \int_0^t Y_{s-}^1 dY_s^2 + \int_0^t Y_{s-}^2 dY_s^1 + [Y^1, Y^2]_t \quad (1.2.14)$$

with $[Y^1, Y^2]_t = \langle Y^{c,1}, Y^{c,2} \rangle + \sum_{0 \leq s \leq t} \Delta Y_s^1 \Delta Y_s^2$.

Indeed, considering the function

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (y_1, y_2) &\longmapsto f(y_1, y_2) = y_1 y_2 \end{aligned}$$

We have:

$$\frac{\partial f(y_1, y_2)}{\partial y_1} = y_2, \quad \frac{\partial f(y_1, y_2)}{\partial y_2} = y_1, \quad \frac{\partial^2 f(y_1, y_2)}{\partial y_1^2} = 0, \quad \frac{\partial^2 f(y_1, y_2)}{\partial y_2^2} = 0, \quad \frac{\partial^2 f(y_1, y_2)}{\partial y_2 \partial y_1} = 1, \quad \frac{\partial^2 f(y_1, y_2)}{\partial y_1 \partial y_2} = 1.$$

By Itô's formula (1.2.13) of the remark 1.2.8,

$$\begin{aligned} Y_t^1 Y_t^2 &= Y_0^1 Y_0^2 + \int_0^t Y_{s-}^1 dY_s^2 + \int_0^t Y_{s-}^2 dY_s^1 + \int_0^t d \langle Y^{c,1}, Y^{c,2} \rangle_s \\ &\quad + \sum_{0 \leq s \leq t} Y_s^1 Y_s^2 - Y_{s-}^1 Y_{s-}^2 - Y_{s-}^1 (Y_s^2 - Y_{s-}^2) - Y_{s-}^2 (Y_s^1 - Y_{s-}^1) \\ &= Y_0^1 Y_0^2 + \int_0^t Y_{s-}^1 dY_s^2 + \int_0^t Y_{s-}^2 dY_s^1 + \langle Y^{c,1}, Y^{c,2} \rangle_t \\ &\quad + \sum_{0 \leq s \leq t} Y_s^1 Y_s^2 + Y_{s-}^1 Y_{s-}^2 - Y_{s-}^1 Y_s^2 - Y_s^1 Y_{s-}^2 \\ &= Y_0^1 Y_0^2 + \int_0^t Y_{s-}^1 dY_s^2 + \int_0^t Y_{s-}^2 dY_s^1 + \langle Y^{c,1}, Y^{c,2} \rangle_t \\ &\quad + \sum_{0 \leq s \leq t} \Delta Y_s^1 \Delta Y_s^2 \end{aligned} \quad (1.2.15)$$

1.3 Existence and uniqueness of solution for stochastic differential equations with jumps

Definition 1.3.1. $\{Y_t\}_{t \geq 0}$ is called the solution of the stochastic differential equation with jumps (1.2.8), iff $\{Y\}_{t \geq 0}$ satisfies the equation (1.2.8). In the case where $Y_t \in \mathcal{F}_t$, $\forall t \geq 0$, where \mathcal{F}_t is the σ -algebra generated (and completed) by the Wiener process and the Poisson process, then Y_t is called a strong solution.

Definition 1.3.2. There is uniqueness of solution, if for two solutions $\{Y_t^i\}_{t \geq 0}$, $i = 1, 2$ satisfying (1.2.8) on the same probability space, the same Wiener process $\{B_t\}_{t \geq 0}$ and the same compensated Poisson measure $\tilde{\mu}(dt, de)$, $\forall t \geq 0$, $\mathbb{P}(\sup_{t \geq 0} |Y_t^1 - Y_t^2| = 0) = 1$

1.3.1 The nonlinear case

We will recall two important results which intervene in the properties and estimates serving to show the uniqueness of the solution. These are the inequality due to Burkholder, Davis, Gundy, a result better known as BDG inequalities, and Gronwall's lemma.

Theorem 1.3.1. (BDG inequalities)

Let $p > 0$ a real. There are two constants c_p and C_p such that, for any continuous local martingale X , zero at zero,

$$c_p \mathbb{E} \left[\langle X, X \rangle_{\infty}^{\frac{p}{2}} \right] \leq \mathbb{E} \left[\sup_{t \geq 0} |X_t|^p \right] \leq C_p \mathbb{E} \left[\langle X, X \rangle_{\infty}^{\frac{p}{2}} \right].$$

In particular, if $T > 0$,

$$c_p \mathbb{E} \left[\langle X, X \rangle_T^{\frac{p}{2}} \right] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t|^p \right] \leq C_p \mathbb{E} \left[\langle X, X \rangle_T^{\frac{p}{2}} \right].$$

For Gronwall's lemma, it was first established by the Swedish mathematician Thomas Hacon Gronwall [34] in 1911 as differential. The so-called integral form of the lemma was proposed by Bellmen [10] in 1943.

Lemma 1.3.1. (Gronwall)

Let φ , ψ and y be three continuous functions on a segment $[a, b]$ with positive values and satisfying the inequality

$$\forall t \in [a, b] \quad y(t) \leq \varphi(t) + \int_a^t y(s) \psi(s) \, ds$$

Then

$$\forall t \in [a, b] \quad y(t) \leq \varphi(t) + \int_a^t \varphi(s)\psi(s) \exp\left(\int_s^t \psi(u) du\right) ds$$

Corollary 1.3.1. *Let $T \geq 0$ and g be a positive measurable function bounded on $[0, T]$. We assume that there are constants $p \geq 0, q \geq 0$ such that $\forall t \in [0, T]$ we have:*

$$g(t) \leq p + q \int_0^t g(s) ds \text{ alors on a } g(t) \leq pe^{qt}.$$

It corresponds to the case where $\varphi = p$ et $\psi = q$

Corollary 1.3.2. *Let $y : [a, b] \rightarrow \mathbb{R}^n$ a function of class C^1 verifying*

$$\exists \alpha > 0, \exists \beta \geq 0, \forall t \in [a, b], \quad \|y'(t)\| \leq \beta + \alpha \|y(t)\|$$

Then

$$\forall t \in [a, b], \quad \|y(t)\| \leq \|y(a)\|e^{\alpha(t-a)} + \frac{\beta}{\alpha}(e^{\alpha(t-a)} - 1)$$

Theorem 1.3.2. *Under the assumptions of Theorem 117 of [65] using Lemmas 114 and 115 in which Rong Suti uses Gronwall's lemma, the equation (1.2.8) admits a unique solution*

1.3.2 The linear case

Example 1.3.1. *(Merton's model for jumping diffusion processes)*

Merton's model assumes that the price of the underlying asset moves according to

$$dS_t = bS_{t-}dt + \sigma S_{t-}dB_t + S_{t-}(J_t + 1)dN_t \quad (1.3.1)$$

where S_{t-} represents the value of S just before a jump to t , if there is one and $(N_t)_{t \geq 0}$ is a Poisson process. We assume that the J_t jump sizes are distributed identically and independently. It is also assumed that the three processes $(B_t)_{t \geq 0}$, $(N_t)_{t \geq 0}$ and $(J_t)_{t \geq 0}$ are independent.

The equation ref 1.3.1 has the exact solution:

$$S_t = S_0 \exp\left\{bt + \sigma B_t - \frac{\sigma^2 t}{2} + J_t\right\}$$

Merton [55] considers the case where the jump sizes Y_i are normally distributed, with $J_t = \sum_{i=1}^{N_t} Y_i$

1.4 General results on BSDE

1.4.1 Existence and uniqueness of BSDE solution

Consider $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a filtered probability space and the random variable ξ measurable with respect to \mathcal{F}_T . We would like to solve the following differential equation

$$\frac{-dY_t}{dt} = f(Y_t), \quad t \in [0, T], \quad \text{with } Y_T = \xi. \quad (1.4.1)$$

By imposing that, for any instant t , Y_t does not depend on the future after t that is to say that the process Y is adapted for filtration $(\mathcal{F}_t)_{t \geq 0}$.

Theorem 1.4.1. (Representation of Brownian martingales, BRIAND, [16])

Let M be a martingale (RCLL) of square integral for filtration $(\mathcal{F}_t^B)_{t \in [0, T]}$. So there is a unique process $(H_t)_{t \in [0, T]}$, belonging to $\mathcal{H}_T^2(\mathbb{R}^k)$, such that

$$\mathbb{P} - a.s. \quad \forall 0 \leq t \leq T, \quad M_t = M_0 + \int_0^t H_s dB_s.$$

It is important to note that this result implies that, in Brownian filtration, the martingales are continuous.

Corollary 1.4.1. If ξ is an integrable square random variable, (\mathcal{F}_T^B) -measurable, there is a unique process $(H_t)_{t \in [0, T]} \in \mathcal{H}_T^2(\mathbb{R}^n)$ such that

$$\xi = \mathbb{E}[\xi] + \int_0^T H_s dB_s.$$

Let us not forget that the position of our problem is to solve the differential equation (1.4.1). Let us take the simplest example, namely $f \equiv 0$. The natural candidate is $Y_t = \xi$ which is not adapted if ξ is not a determinist (depending only on t). The best approximation say in L^2 adapted is the martingale $Y_t = \mathbb{E}(\xi/\mathcal{F}_t)$. If we work with the natural filtration of a Brownian motion, the theory of representation of Brownian martingales allows us to construct a process Z of integral square and adapted such as:

$$Y_t = \mathbb{E}(\xi/\mathcal{F}_t) = \mathbb{E}[\xi] + \int_0^t Z_s dB_s.$$

Which also gives

$$Y_t = \xi - \int_t^T Z_s dB_s. \quad (1.4.2)$$

Indeed

$$\begin{aligned}
 Y_T &= \mathbb{E}[\xi] + \int_0^T Z_s dB_s \\
 \xi &= \mathbb{E}[\xi] + \int_0^t Z_s dB_s + \int_t^T Z_s dB_s \\
 &= Y_t - \mathbb{E}[\xi] + \mathbb{E}[\xi] + \int_t^T Z_s dB_s \\
 Y_t &= \xi - \int_t^T Z_s dB_s.
 \end{aligned}$$

Remark 1.4.1. The relation (1.4.2) can be understood in differential form as follows: $-dY_t = -Z_t dB_t$ with $Y_T = \xi$.

We therefore see appearing on the simplest example an unknown second which is the process Z whose role is to make the process Y adapted. Therefore, as a second variable appears, to get the larger generality, we allow f to depend on the process Z ; the equation therefore becomes:

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dB_t \quad \text{avec} \quad Y_T = \xi.$$

Consider the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s^* dB_s, \quad 0 \leq t \leq T, \quad (1.4.3)$$

where:

- Z^* is the transpose of Z (i.e $z_{ij}^* = z_{ji}$),
- B is a m -dimensional Brownian motion,
- $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^n$ is $\mathcal{P} \otimes \mathcal{B}^n \otimes \mathcal{B}^{m \times n}$ -measurable,
- Y and Z are progressively measurable with value respectively in \mathbb{R}^n and $\mathbb{R}^{m \times n}$.

$$(\mathbf{H1}) \begin{cases} i) \xi \in \mathbb{L}_T^{2,n} \text{ and } f(t, 0, 0)_{t \leq T} \in \mathbb{H}_T^{2,n} \\ ii) f \text{ is uniformly lipschitz in } (y, z), \text{ i.e., there is a constant } C > 0 \text{ a.s.} \\ \forall (y, y', z, z') \quad |f(t, y, z) - f(t, y', z')| \leq C (|y - y'| + |z - z'|), \quad dt \otimes d\mathbb{P} \text{ a.s. .} \end{cases}$$

Definition 1.4.1. A solution of the BSDE (1.4.3) is a couple of processes $\{(Y_t, Z_t)\}_{0 \leq t \leq T}$ verifying:

1. Y and Z are progressively measurable with value respectively in \mathbb{R}^n and $\mathbb{R}^{m \times n}$.
2. \mathbb{P} -a.s. $\int_0^T \{|f(r, Y_r, Z_r)| + \|Z_r\|^2\} dr < \infty$.

3. \mathbb{P} -a.s. we have:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s^* dB_s, \quad 0 \leq t \leq T.$$

Proposition 1.4.1. *Suppose there is a process $\{f_t\}_{0 \leq t \leq T}$, positive, belonging to $\mathbb{H}_T^{2,1}$ and a positive constant λ such that*

$$\forall (t, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{m \times n}, \quad |f(t, y, z)| \leq f_t + \lambda(|y| + \|z\|). \quad (1.4.4)$$

If $\{(Y_t, Z_t)\}_{0 \leq t \leq T}$ is a solution of the BSDE (1.4.3) such that $Z \in \mathbb{H}_T^{2,1}(\mathbb{R}^{m \times n})$ then Y belongs to \mathcal{S}_n^2 .

1.4.1.1 The Pardoux-Peng result

The first result of existence and uniqueness for the solution of the BSDE in case of the generator is non-linear. This result is due to E. Pardoux and S. Peng in "Adapted Solutions of Backward Stochastic Differential Equation, Systems and Control Letters".

Theorem 1.4.2. *(Pardoux and Peng)*

*Under the assumption **(H1)** the BSDE (1.4.3) admits a unique solution (Y, Z) with the parameters (f, ξ) .*

Remark 1.4.2.

- (1) *In the hypothesis **(H1)** instead of considering the ii), we can simplify this, on the one hand by considering that the generator f satisfies the condition of monotony in y following:*

$$\left(f(s, y^1, z) - f(s, y^2, z) \right) (y^1 - y^2) \leq K (y^1 - y^2)^2$$

and, the correspondence $y \mapsto f(s, y, z)$ is continuous, and, on the other hand the correspondence $z \mapsto f(s, y, z)$ is uniformly Lipschitz, then BSDE (1.4.3) admits a unique solution (cf Darling, RWR and Pardoux, E.: Backward SDE with random terminal time and applications to semilinear elliptic PDE. Ann. Proba. Vol. 25, No. 3, 1135-1159(1997) or see Pardoux, E.: BSDE's, weak convergence and homogenization of semilinear PDE's in Nonlinear analysis, Differential and Control, F. H. Clarke and R. J. Stern Eds, pp. 503-549, Kluwer Acad. Pub(1999)).

- (2) *We can also note that the ii) of the hypothesis **(H1)** can be further simplified by the quadratic growth of f i.e., $|f(t, y, z)| \leq C(1 + |y| + |z|^2)$ and the boundness of the terminal value ξ , under these conditions the BSDE admits a unique solution (c.f Kobylanski, M.: Backward Stochastic Differential Equations with Quadratic Growth, Annals of Probability 28, 558-602(2000)[50]).*

1.4.1.2 Picard iteration method

This method gives an approximation of the ODE defined as an integral by:

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t)) dt.$$

A primary approximation of the solution of $y(x)$ is given by the function $y_0(x) = y_0$. The sequence of successive Picard approximations is obtained from the formula:

$$y_n(x) = y_0 + \int_{x_0}^x f(t, y_{n-1}(t)) dt.$$

We obtain:

$$y_1(x) = y_0 + \int_{x_0}^x f(t, y_0(t)) dt$$

$$y_2(x) = y_0 + \int_{x_0}^x f(t, y_1(t)) dt$$

$$y_3(x) = y_0 + \int_{x_0}^x f(t, y_2(t)) dt$$

And so on.

Theorem 1.4.3. (Ghorai, [32]) *If the function f is Lipschitz, then the successive approximations $y_n(x)$ converge to a unique solution $y(x)$.*

Instead of using the fixed point theorem to show the existence and uniqueness of the solution of the BSDE (1.4.3), we can use a method of explicit approximation, better known under the name of Picard method of iterations [69].

Proposition 1.4.2. *Consider the Picard sequence (Y^k, Z^k) defined recursively by $(Y^0 = 0, Z^0 = 0)$, et*

$$-dY_t^{k+1} = f(t, Y^k, Z^k)dt - dZ_t^{*k+1}dB_t, \quad Y_T^{k+1} = \xi_T.$$

This sequence of (Y^k, Z^k) converges in $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$ and $dt \otimes d\mathbb{P}$ -a.s. to the solution (Y, Z) of the BSDE (1.4.3). Moreover, the sequence (Y^k) converges uniformly a.s. .

1.4.1.3 A priori estimate

We end this paragraph by giving a first estimate on the BSDE: it is in fact a question of studying the dependence of the solution of the BSDE on the data that are ξ and the process $\{f(t, 0, 0)\}_{0 \leq t \leq T}$.

Proposition 1.4.3. *Let (Y, Z) be a solution of the BSDE (1.4.3). Then there exists a constant C_u such that:*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] \leq C_u \mathbb{E} \left[|\xi_T|^2 + \int_0^T |f(t, 0, 0)|^2 dt \right].$$

1.4.2 Comparison theorem

1.4.2.1 Linear BSDE

In this paragraph we study the particular case of linear BSDE for which we are going to give a more or less explicit formula. We place ourselves in the case of $n = 1$; Y is therefore real and Z is a matrix of size $m \times 1$ i.e. a column vector of size m .

Proposition 1.4.4. *Let $\{(a_t, b_t)\}_{t \in [0, T]}$ a process with values in $\mathbb{R} \times \mathbb{R}^m$, progressively measurable and bounded. Let $\{c_t\}_{t \in [0, T]}$ an element of $\mathbb{H}^{2,1}$ and ξ a random variable, \mathcal{F}_T -measurable, square integral, with real values.*

Linear BSDE

$$Y_t = \xi + \int_t^T \{a_s Y_s + b_s Z_s + c_s\} ds - \int_t^T Z_s^* dB_s, \quad (1.4.5)$$

has a unique solution that verifies:

$$\forall t \in [0, T], Y_t = \Gamma_t^{-1} \mathbb{E} \left[\xi \Gamma_T + \int_t^T c_s \Gamma_s ds \mid \mathcal{F}_t \right], \quad (1.4.6)$$

with, $\forall t \in [0, T]$,

$$\Gamma_t = \exp \left\{ \int_0^t Z_s dB_s - \frac{1}{2} \int_0^t |b_s|^2 ds + \int_0^t a_s ds \right\}. \quad (1.4.7)$$

Remark 1.4.3. *Note that if $\xi \geq 0$ and $c \geq 0$ then the solution of the linear BSDE verifies $Y_t \geq 0$.*

This remark will allow us to obtain the comparison theorem.

The comparison theorem makes it possible to compare the solutions of two BSDE (in \mathbb{R}) as soon as we know how to compare the terminal conditions and the generators.

Theorem 1.4.4. *(Comparison theorem)*

*Suppose that $k = 1$ and that the hypothesis **(H1)** is verified. We denote by (Y, Z) and (Y', Z') the solutions of the corresponding BSDE. We also assume that \mathbb{P} -a.s. $\xi \leq \xi'$ and $f(t, Y_t, Z_t) \leq f'(t, Y_t, Z_t) dt \otimes \mathbb{P}$ -a.e. Then,*

$$\mathbb{P}\text{-p.s.}, \quad \forall t \in [0, T], \quad Y_t \leq Y'_t.$$

If more, $Y_0 = Y'_0$, then \mathbb{P} -a.s., $Y_t = Y'_t, \forall t \in [0, T]$ et $f(t, Y_t, Z_t) \leq f'(t, Y'_t, Z'_t) dt \otimes \mathbb{P}$ -a.e. In particular, as soon as $\mathbb{P}(\xi < \xi') > 0$ or $f(t, Y_t, Z_t) < f'(t, Y_t, Z_t)$ on a set of

$dt \otimes \mathbb{P}$ -measure strictly positive then $Y_0 < Y'_0$.

Remark 1.4.4. We can assume that $f(t, Y'_t, Z'_t) \leq f'(t, Y'_t, Z'_t)$ instead of $f(t, Y_t, Z_t) \leq f'(t, Y_t, Z_t)$

to get the previous result. It suffices to do a linearization starting from the writing

$$\begin{aligned} f'(s, Y'_s, Z'_s) - f(s, Y_s, Z_s) &= f'(s, Y'_s, Z'_s) - f(s, Y'_s, Z'_s) \quad (\text{assume positive here}) \\ &\quad + f(s, Y'_s, Z'_s) - f(s, Y_s, Z'_s) + f(s, Y_s, Z'_s) - f(s, Y_s, Z_s). \end{aligned}$$

1.4.3 BSDE with the non-Lipschitz generator in dimension 1

Let us now consider in dimension 1 an BSDE with a generator f which is just continuous; and under the hypothesis **(H2)**.

$$\text{(H2)} \begin{cases} i) \xi \in \mathbb{L}_T^{2,1} \text{ et } f(t, 0, 0)_{t \leq T} \in \mathbb{H}_T^{2,1} \\ ii) f \text{ is linear growth : there is a function } b_k(y, z) := k(|y| + |z|) \text{ a.s.} \\ \forall (y, z) |f(t, y, z)| \leq |f(t, 0, 0)| + k(|y| + |z|), \text{ } dt \otimes d\mathbb{P}\text{-a.s.} \\ iii) dt \otimes \mathbb{P}\text{-a.s., the function } (y, z) \mapsto f(t, y, z) \text{ is continuous.} \end{cases}$$

So, we have the following results.

Theorem 1.4.5. (Lepeltier and San Martin, [36])

Under the hypothesis **(H2)**, the BSDE associated with the parameters (f, ξ) admits a minimal solution (Y, Z) , i.e. If (Y', Z') is another solution of the BSDE (1.4.3), then,

$$Y \leq Y' \quad \mathbb{P}\text{-a.s.}$$

Proposition 1.4.5. (Hamadene, El Karoui, Matoussi, [36])

Let (Y, Z) the minimal solution of the BSDE (f, ξ) , where (f, ξ) satisfies the hypothesis **(H2)**. Consider (\bar{Y}, \bar{Z}) a solution of the BSDE $(\bar{f}, \bar{\xi})$.

If

$$\xi \leq \bar{\xi} \quad \mathbb{P}\text{-a.s.} \quad \text{and} \quad f(t, \bar{Y}, \bar{Z}) \leq \bar{f}(t, \bar{Y}, \bar{Z}) \quad dt \otimes \mathbb{P}\text{-a.e.}$$

then,

$$Y \leq \bar{Y} \quad \mathbb{P}\text{-a.s.}$$

Remark 1.4.5. As in the remark 1.4.4, if we use an increasing scheme, we build the maximal solution of the BSDE (1.4.3).

From these results on the BSDE, we will see in the next section the connection that exists between the BSDE and the PDE.

Indeed since their introduction by J. M. Bismut in 1973, BSDE have received a lot of attention in the probability literature.

The connection between BSDE and PDE will be made through the stochastic representation of the solutions of semi-linear and quasi-linear PDE and the generalization of

the Feynman-Kac formula.

1.5 Links between BSDEs and PDE: viscosity solution

Definition 1.5.1. Let u be a function defined on \mathbb{R}^d , with values in \mathbb{R} and sufficiently regular so that the following expressions make sense.

A **partial differential equation (PDE)** for the function u is a relation between u , the variables x_1, \dots, x_d and a finite number of partial differential of u ,

$$F(x_1, \dots, x_d, u, D_1 u, \dots, D_d u, D_1 D_1 u, D_1 D_2 u, \dots, D^\alpha u, \dots) = 0,$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$.

We say that u is the solution of the partial differential equation in $\Omega \subset \mathbb{R}^d$ if, after substitution of u and its partial differential, F vanishes for all $(x_1, \dots, x_d) \in \Omega$.

The order $m \in \mathbb{N}$ of a partial differential equation is that of the highest order partial differential.

A **system of partial differential equations** consists of several partial differential equations involving one or more unknown functions $(u_i)_{i=1, \dots, n}$.

A partial differential equation is **linear** if F is linear with respect to u and its partial differential. If m is the order of the partial differential equation, the equation is of the form

$$\sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha u(x) = B(x), \text{ with } |\alpha| = \alpha_1 + \dots + \alpha_d.$$

If $B = 0$ we have a **homogeneous** equation; otherwise it is **non-homogeneous** and $Lu = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha u$ is a linear differential operator.

A partial differential equation of order m is **quasi-linear** if F is linear in all partial differential of highest order, i.e of order m , the equation is then of the form,

$$\sum_{|\alpha|=m} A_\alpha(x, u, D^\beta u) D^\alpha u(x) = B(x, u, D^\beta u), \text{ with } \beta \in \mathbb{N}^d \text{ and } |\beta| \leq m.$$

A partial differential equation of order m is **semi-linear** if the equation is of the form

$$\sum_{|\alpha|=m} A_\alpha(x) D^\alpha u(x) = B(x, u, D^\beta u), \text{ with } \beta \in \mathbb{N}^d \text{ and } |\beta| \leq m.$$

The solution of a partial differential equation of order m generally depends on m arbitrary functions of $d - 1$ variables.

The general solution of an equation with partial differential is the one which allows to find all the solutions of the equation (except cases of singular solutions) by giving special

values to arbitrary functions.

To find particular solutions of a partial differential equation, from the general solution, we will impose restrictive conditions on the set of solutions.

The most frequent constraints are:

1. *initial conditions*: if u is a function of $(x, t) \in \mathbb{R}^d \times \mathbb{R}$,
We give $u(x, t_0) = \Phi_0(x)$ or $D_2^p u(x, t_0) = \Phi_p(x)$, we also speak of *Cauchy conditions*,
2. *conditions at the edge*,
3. *infinite conditions*,
4. *conditions on interfaces*.

Definition 1.5.2. After having taken our first steps in the theory, we will describe in this part how to classify all linear PDEs of order 2. We will have three types of ODE: **hyperbolic**, **parabolic** and **elliptical**.

We will restrict ourselves to the case where $m = 2$. Thus the PDEs that we will consider will be of the following form:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G, \quad (1.5.1)$$

and A, B, C, D, E, F and G are functions of x and y which do not cancel each other out simultaneously. We suppose also that u, A, B, C, D, E, F and G all have at least partial differential of order $m \leq 2$ continues on a domain \mathbb{D} of the plane x, y . The equation is called **hyperbolic** (respectively **parabolic**, **elliptic**) at the point $(x_0, y_0) \in \mathbb{D}$ iff $(B(x_0, y_0))^2 - 4A(x_0, y_0)C(x_0, y_0)$ is positive (respectively null, negative). If a PDE is hyperbolic (respectively parabolic, elliptic) for all the points (x_0, y_0) of the domain \mathbb{D} , then we say that it is hyperbolic (respectively parabolic, elliptic) on \mathbb{D} .

For equation

$$x^2 \frac{\partial^2 u}{\partial x^2} - xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = e^x,$$

then we have $A(x, y) = x^2$, $B(x, y) = -xy$, $C(x, y) = y^2$ and $B^2 - 4AC = (-xy)^2 - 4(x^2)(y^2) = -3x^2y^2$.

So if $x_0 = 0$ or $y_0 = 0$, this PDE is parabolic at the point (x_0, y_0) ; otherwise it is elliptical to the point (x_0, y_0) (ref., [9]).

Definition 1.5.3. (viscosity solutions)

The general framework for using viscosity solutions is a nonlinear partial differential equation defined in U open from \mathbb{R}^p of the form

$$F(x, u(x), Du(x), D^2u(x)) = 0, \quad x \in U \quad (1.5.2)$$

where Du is the differential of u , D^2u its second differential, $F : U \times \mathbb{R} \times \mathbb{R}^p \times S_p \rightarrow \mathbb{R}$ and S_p denote the positive symmetric $p \times p$ matrices. We could in fact extend the definition of viscosity solution to systems of the form

$$F_k(x, u_k(x), Du_k(x), D^2u_k(x)) = 0, \quad k = 1, \dots, n$$

when $u = (u_1, \dots, u_n)$. We will limit ourselves in here (reminders) to real-valued functions and refer to [64] for the general case.

We provide S_p with the usual partial order \leq on $S_p \times S_p$: $A \leq B$ iff $\forall x \in \mathbb{R}^p, x^*Ax \leq x^*Bx$.

Let us now give the usual hypotheses made on F . We will say that F is proper if

$$F(x, r, q, X) \leq F(x, s, q, X), \quad \forall r \leq s, \tag{1.5.3}$$

$$F(x, r, q, X) \leq F(x, s, q, Y), \quad \forall X \leq Y, \tag{1.5.4}$$

Let us now consider u of class C^2 on U (we will denote by $u \in C^2(U)$) classic solution of (1.5.2) and give we $\varphi \in C^2(U)$ and $x \in U$ local maximum of $u - \varphi$. By definition we have $x, Du(x) = D\varphi(x)$ and $D^2u(x) \leq D^2\varphi(x)$, then by the property (1.5.4), we get that

$$F(x, u(x), Du(x), D^2u(x)) \leq 0.$$

Similarly, if $\psi \in C^2(U)$ and $x \in U$ local minimum of $u - \psi$, then we have

$$F(x, u(x), D\psi(x), D^2\psi(x)) \geq 0.$$

This result therefore allows us to extend the notion of classical solution of the equation (1.5.2). For U open from \mathbb{R}^d , we denote by $C(U)$ the set of continuous real-valued functions on U . The previous reasoning then allows us to extend the notion of classical solution of the equation (1.5.2) to weak solutions which we will suppose to be continuous here (we can extend to semi-continuous functions): solutions of viscosity.

Definition 1.5.4. Soit $u \in C(U)$

- u is a viscosity sub-solution of (1.5.2) if for all $\varphi \in C^2(U)$, for all $x \in U$ local maximum from $u - \varphi$, we have $F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0$.
- u is a viscosity super-solution of (1.5.2) if for all $\varphi \in C^2(U)$, for all $x \in U$ local minimum from $u - \varphi$, we have $F(x, u(x), D\varphi(x), D^2\varphi(x)) \geq 0$.
- u is a solution of viscosity of (1.5.2) if it is both a sub-solution and a supersolution.

1.5.1 The Markovian framework

We consider here Markovian BSDEs: this is a very specific framework in which the dependence of the terminal condition and the generator of the SDE in the random ω is done through the solution of an SDE.

Assumptions

We consider two functions $b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ et $\sigma : [0, T] \times \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n$ continues. We assume that there exists a constant $K \geq 0$ such that, for $\forall t$, for $\forall x, x'$ of \mathbb{R}^n ,

1. $|b(t, x) - b(t, x')| + \|\sigma(t, x) - \sigma(t, x')\| \leq K|x - x'|$;
2. $|b(t, x)| + \|\sigma(t, x)\| \leq K(1 + |x|)$.

Under these assumptions, we can construct, given a real $t \in [0, T]$ and a random variable $\theta \in L^2(\mathcal{F}_t)$, $\{X_u^{t, \theta}\}_{t \leq u \leq T}$ the SDE solution

$$X_u^{t, \theta} = \theta + \int_t^u b(s, X_s^{t, \theta}) ds + \int_t^u \sigma(s, X_s^{t, \theta}) dB_s, \quad t \leq u \leq T; \quad (1.5.5)$$

we agree that, if $0 \leq u \leq t$, $X_u^{t, \theta} = \mathbb{E}(\theta \mid \mathcal{F}_u)$. Consider also two continuous functions $g : \mathbb{R}^k \rightarrow \mathbb{R}^k$ and

$$f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{k \times m} \rightarrow \mathbb{R}^k.$$

We assume that f and g verify the following hypotheses: there are two reals μ and $p \geq 1$ such that, for all (t, x, y, y', z, z') ,

1. $(y - y') \cdot (f(t, x, y, z) - f(t, x, y', z)) \leq \mu|y - y'|^2$;
2. $|f(t, x, y, z) - f(t, x, y, z')| \leq K\|z - z'\|$;
3. $|g(x)| + |f(t, x, y, z)| \leq K(1 + |x|^p + |y| + \|z\|)$.

Under these assumptions, if θ belongs to $L^{2p}(\mathcal{F}_t)$, we can solve the BSDE

$$Y_u^{t, \theta} = g(X_T^{t, \theta}) + \int_t^T f(s, X_s^{t, \theta}, Y_s^{t, \theta}, Z_s^{t, \theta}) ds - \int_t^T Z_s^{t, \theta} dB_s, \quad 0 \leq u \leq T. \quad (1.5.6)$$

In the following, we will assume that the previous hypotheses on the coefficients b, σ, f and g are satisfied. Sometimes we will make a stronger hypothesis on f and g assuming that g is K -Lipschitz and f is K -Lipschitz in x uniformly in (t, y, z) . In this case, $p = 1$. We will denote this hypothesis by (Lip).

Proposition 1.5.1. *If θ_n converges to θ in $L^{2p}(\mathcal{F}_t)$, then*

$$\mathbb{E} \left[\sup_{0 \leq u \leq T} |Y_u^{t, \theta} - Y_u^{t, \theta_n}|^2 + \int_0^T \|Z_s^{t, \theta} - Z_s^{t, \theta_n}\|^2 ds \right] \rightarrow 0, \quad \text{si } n \rightarrow +\infty.$$

Moreover, under the hypothesis (Lip), we have, if θ and θ' are \mathcal{F}_t -measurable and squared integrable,

$$\mathbb{E} \left[\sup_{0 \leq u \leq T} |Y_u^{t,\theta} - Y_u^{t,\theta'}|^2 + \int_0^T \|Z_s^{t,\theta} - Z_s^{t,\theta'}\|^2 ds \right] \leq C \mathbb{E} [|\theta - \theta'|].$$

Proposition 1.5.2. *Let $(t, x) \in [0, T] \times \mathbb{R}^n$. $\{(X_u^{t,x}, Y_u^{t,x})\}_{t \leq u \leq T}$ is adapted for filtration $\{\mathcal{F}_t^u\}_{t \leq u \leq T}$. In particular, $Y_t^{t,x}$ is deterministic. We can choose a version of $\{Z_u^{t,x}\}_{t \leq u \leq T}$ adapted for filtration $\{\mathcal{F}_t^u\}_{t \leq u \leq T}$.*

Since we know that $Y_t^{t,x}$ is a deterministic quantity, we can define a function by posing,

$$\forall (t, x) \in [0, T] \times \mathbb{R}^n, \quad u(t, x) := Y_t^{t,x}.$$

We have the increase and continuity of this function by the following proposition.

Proposition 1.5.3. *The function u is continuous and has polynomial growth. We have*

$$\forall (t, x) \in [0, T] \times \mathbb{R}^n, \quad |u(t, x)| \leq C(1 + |x|^p).$$

Under the (Lip) hypothesis, the function u furthermore verifies, for all $(t, x), (t', x')$,

$$|u(t, x) - u(t', x')| \leq C(|x - x'| + |t - t'|^{1/2}(1 + |x|)).$$

For the demonstration of the propositions 1.5.1, 1.5.2 and 1.5.3 the reader will be able to refer to (BRIAND, 2001, [16]).

Using the function $u(t, x) := Y_t^{t,x}$, we can establish the Markov property for the BSDE.

Theorem 1.5.1. *Soient $t \in [0, T]$ et $\theta \in L^{2p}(\mathcal{F}_t)$. We have*

$$\mathbb{P} - a.s. \quad Y_t^{t,\theta} = u(t, \theta) = Y_t^{t,\cdot} \circ \theta.$$

Corollary 1.5.1. *Let $t \in [0, T]$ and $\theta \in L^{2p}(\mathcal{F}_t)$. Then, we have, \mathbb{P} -a.s.,*

$$Y_s^{t,\theta} = u(s, X_s^{t,\theta}) \quad \forall s \in [t, T]$$

.

Remark 1.5.1. *We often use this result with θ constant equal to x . This Markov property, $Y_s^{t,x} = u(s, X_s^{t,x})$, plays an important role when we try to build the solution of an PDE using a BSDE: we will see it later.*

1.5.2 Feynman-Kac formula

In the previous paragraph, we saw that $Y_s^{t,x} = u(s, X_s^{t,x})$ where u is a deterministic function; we will see that u is a solution of a PDE. We denote by \mathcal{L} the following second order differential operator: for h regular.

$$\begin{aligned}\mathcal{L}(t, x) &= \frac{1}{2} \text{trace}(\sigma \sigma^*(t, x) D^2 h(t, x)) + b(t, x) \cdot \nabla h(t, x) \\ &= \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma_{i,j}^*(t, x) \partial x_i \partial x_j h(t, x)) + \sum_{i=1}^n b_i(t, x) \partial x_i h(t, x).\end{aligned}$$

The objective of this paragraph is to establish relations between $\{Y_s^{t,x}, Z_s^{t,x}\}_{0 \leq s \leq T}$ solution of the BSDE

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}) du - \int_s^T Z_u^{t,x} dB_u, \quad 0 \leq s \leq T \quad (1.5.7)$$

where $\{X_s^{t,x}\}_{0 \leq s \leq T}$ is the solution of the SDE, with the convention $X_s^{t,x} = x$ if $0 \leq s \leq t$

$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x}) du + \int_t^s \sigma(u, X_u^{t,x}) dB_u, \quad t \leq s \leq T \quad (1.5.8)$$

on the one hand and the solution v of the following parabolic PDE:

$$\partial_t v(t, x) + \mathcal{L}v(t, x) + f(t, x, v(t, x), Dv(t, x)\sigma(t, x)), \quad (t, x) \in]0, T[\times \mathbb{R}^n, \quad v(T, \cdot) = g. \quad (1.5.9)$$

Proposition 1.5.4. Suppose that the PDE (1.5.9) has a solution v , of class $C^{1,2}([0, T] \times \mathbb{R}^n)$, such that for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $|\nabla v(t, x)| \leq C(1 + |x|^q)$.

Then, for all $(t, x) \in [0, T] \times \mathbb{R}^n$, the BSDE solution (1.5.7), $\{Y_s^{t,x}, Z_s^{t,x}\}_{t \leq s \leq T}$, is given by the process couple $\{v(s, X_s^{t,x}), Dv\sigma(s, X_s^{t,x})\}_{t \leq s \leq T}$. In particular, we obtain the formula, $u(t, x) = Y_t^{t,x} = v(t, x)$.

The previous result gives a probabilistic representation formula for the solution of a Semi-linear parabolic PDE. This type of formula, known as the Feynman-Kac formula; following the work of Richard Feynman and Mark Kac; originally applied to linear problems. We get it as a corollary.

Corollary 1.5.2. Take $f(t, x, y, z) = c(t, x)y + h(t, x)$, where c and h are two continuous bounded functions. Under the previous assumptions, for all $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$v(t, x) = \mathbb{E} \left[g(X_T^{t,x}) \exp \left(\int_t^T c(s, X_s^{t,x}) ds \right) + \int_t^T h(s, X_s^{t,x}) \exp \left(\int_t^s c(u, X_u^{t,x}) du \right) ds \right].$$

Remark 1.5.2. In the last two statements, we have assumed the existence of a classical solution v and we have deduced from it the solution of the BSDE and the formula

$v(t, x) = Y_t^{t,x}$. If all the coefficients are regular, we can show that u which is defined by $u(t, x) = Y_t^{t,x}$ is a regular function which is PDE solution (1.5.9).

The approach that we had previously consists in studying the PDE, then to deduce the solutions of the BSDE. But we can also study the BSDE and deduce the construction of the solution of the PDE without assuming the regular coefficients. For this we will use the PDE viscosity solutions.

1.5.3 viscosity solution

Theorem 1.5.2. *The function $u(t, x) := Y_t^{t,x}$ is the viscosity solution of the EDP (1.5.9).*

The question that arises is can we always obtain a viscosity solution apart from the assumptions of Lipschitz and monotony on the generator.

The answer to this question will be provided in the next two chapters, where we deal with BSDE and RBSDE with jump coupled with parabolic PDEs.

In the next chapter we will deal with a larger class of BSDE, these are RBSDE jumping with a ℓ obstacle:

$$\left\{ \begin{array}{l} \text{(i)} \quad dY_s^{i;t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i;t,x})_{i=1,m}, Z_s^{i;t,x}, U_s^{i;t,x})ds \\ \quad \quad \quad -dK_s^{i;t,x} + Z_s^{i;t,x}dB_s + \int_{\mathbb{E}} U_s^{i;t,x}(e)\tilde{\mu}(ds, de), \quad s \leq T; \\ \text{(ii)} \quad Y_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i;t,x} - \ell(s, X_s^{t,x}))dK_s^{i;t,x} = 0; \\ \text{(iii)} \quad Y_T^{i;t,x} = g^i(X_T^{t,x}). \end{array} \right. \quad (1.5.10)$$

In [?], El Karoui et al. proved that if f is Lipschitz continues in (y, z) and both the terminal value g and the obstacle ℓ are integrable squares, then the solution of (0.0.137) exists and is unique in the absence of the jump part. They use two methods, penalization and the Snell envelope.

Later, jumping RBSDE were widely studied and some existence and uniqueness results have been established by many authors. We can cite ([67]), ([41]), ([27]). In ([67]), Tang and Li discussed one-dimensional RBSDE directed by a Brownian motion, Hamadene and Ouknine ([41]) and Essaky ([27]) studied the equation (0.0.137) in the one-dimensional case.

Our main goal is to use (0.0.137) for our system resolution equations with a bar-

rier(obstacle) ℓ , which is a function of $(t, x) \forall i \in \{1, \dots, m\}$,

$$\begin{cases} \min \left\{ u^i(t, x) - \ell(t, x); -\partial_t u^i(t, x) - b(t, x)^\top D_x u^i(t, x) \right. \\ \left. - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 u^i(t, x)) - K_i u^i(t, x) - h^{(i)}(t, x, u^i(t, x)), \right. \\ \left. (\sigma^\top D_x u^i(t, x), B_i u^i(t, x)) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ u^i(T, x) = g^i(x); \end{cases} \quad (1.5.11)$$

where B_i and K_i are defined as follows:

$$\begin{aligned} B_i u^i(t, x) &= \int_{\mathbb{E}} \gamma^i(t, x, e) (u^i(t, x + \beta(t, x, e)) - u^i(t, x)) \lambda(de); \\ K_i u^i(t, x) &= \int_{\mathbb{E}} (u^i(t, x + \beta(t, x, e)) - u^i(t, x) - \beta(t, x, e)^\top D_x u^i(t, x)) \lambda(de). \end{aligned} \quad (1.5.12)$$

First, we give the definitions of viscosity solution of IPDEs in sens of Barles and al., and in Hamadene and al. sens as given in [36, 37]. Our main result deal with this definition (Hamadene and al. sens).

Barles and al.'s definition for viscosity solution of IPDE (0.0.138)

Definition 1.5.1. We say that a family of deterministic functions $u = (u^i)_{i=1, m}$ which is continuous $\forall i \in \{1, \dots, m\}$, is a viscosity sub-solution (resp. super-solution) of the IPDE (0.0.138) if:

- (i) $\forall x \in \mathbb{R}^k$, $u^i(x, T) \leq g^i(x)$ (resp. $u^i(x, T) \geq g^i(x)$);
- (ii) For any $(t, x) \in [0, T] \times \mathbb{R}^k$ and any function ϕ of class $C^{1,2}([0, T] \times \mathbb{R}^k)$ such that (t, x) is a global maximum point of $u^i - \phi$ (resp. global minimum point of $u^i - \phi$) and $(u^i - \phi)(t, x) = 0$, one has

$$\min \begin{cases} u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) \\ -h^i(t, x, (u^j(t, x))_{j=1, m}, \sigma^\top(t, x)) D_x \phi(t, x), B_i \phi(t, x) \end{cases} \leq 0$$

(resp.

$$\min \begin{cases} u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) \\ -h^i(t, x, (u^j(t, x))_{j=1, m}, \sigma^\top(t, x)) D_x \phi(t, x), B_i \phi(t, x)(t, x) \end{cases} \geq 0.$$

The family $u = (u^i)_{i=1, m}$ is a viscosity solution of (0.0.138) if it is both a viscosity sub-solution and viscosity super-solution.

Note that $\mathcal{L}^X \phi(t, x) = b(t, x)^\top D_x \phi(t, x) + \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 \phi(t, x)) + K \phi(t, x)$;

where $K \phi(t, x) = \int_{\mathbb{E}} (\phi(t, x + \beta(t, x, e)) - \phi(t, x) - \beta(t, x, e)^\top D_x \phi(t, x)) \lambda(de)$.

Hamadene and al.'s definition for viscosity solution of IPDE (0.0.138)

Definition 1.5.2. We say that a family of deterministic functions $u = (u^i)_{i=1,m}$, which belongs to $\mathcal{U}[36]$, $\forall i \in \{1, \dots, m\}$, is a viscosity sub-solution (resp. super-solution) of the IPDE (0.0.138) if

- (i) For $\forall x \in \mathbb{R}^k$, $u^i(x, T) \leq g^i(x)$ (resp. $u^i(x, T) \geq g^i(x)$);
- (ii) For $\forall (t, x) \in [0, T] \times \mathbb{R}^k$ and any function ϕ of class $C^{1,2}([0, T] \times \mathbb{R}^k)$ such that (t, x) is a global maximum point of $u^i - \phi$ (resp. global minimum point of $u^i - \phi$) and $(u^i - \phi)(t, x) = 0$, one has

$$\min \left\{ u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) - h^i(t, x, (u^j(t, x))_{j=1,m}, \right. \quad (1.5.13)$$

$$\left. \sigma^\top(t, x) D_x \phi(t, x), B_i u^i(t, x) \right\} \leq 0, \quad (1.5.14)$$

$$\left(\text{resp. } \min \left\{ u^i(t, x) - \ell(t, x); -\partial_t \phi(t, x) - \mathcal{L}^X \phi(t, x) - h^i(t, x, (u^j(t, x))_{j=1,m}, \right. \quad (1.5.15)$$

$$\left. \sigma^\top(t, x) D_x \phi(t, x), B_i u^i(t, x) \right\} \geq 0 \right). \quad (1.5.16)$$

The family $u = (u^i)_{i=1,m}$ is a viscosity solution of (0.0.138) if it is both a viscosity sub-solution and viscosity super-solution.

Note that

$$\mathcal{L}^X \phi(t, x) = b(t, x)^\top D_x \phi(t, x) + \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 \phi(t, x)) + K \phi(t, x),$$

where

$$K \phi(t, x) = \int_{\mathbb{E}} (\phi(t, x + \beta(t, x, e)) - \phi(t, x) - \beta(t, x, e)^\top D_x \phi(t, x)) \lambda(de).$$

CHAPTER 2

REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL
EQUATION WITH JUMPS AND VISCOSITY SOLUTION OF
SECOND ORDER INTEGRO-PARTIAL DIFFERENTIAL
EQUATION WITHOUT MONOTONICITY CONDITION:CASE
WITH THE MEASURE OF LÉVY INFINITE

We consider the problem of viscosity solution of integro-partial differential equation (IPDE in short) with one obstacle via the solution of reflected backward stochastic differential equations (RBSDE in short) with jumps. We show the existence and uniqueness of a continuous viscosity solution of equation with non local terms, if the generator is not monotonous and Levy's measure is infinite.

2.1 Introduction

We consider the following system of integro-partial differential equation with one-obstacle ℓ , which is a function of (t, x) : $\forall i \in \{1, \dots, m\}$,

$$\left\{ \begin{array}{l} \min \left\{ u^i(t, x) - \ell(t, x); -\partial_t u^i(t, x) - b(t, x)^\top D_x u^i(t, x) \right. \\ \left. - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 u^i(t, x)) - K_i u^i(t, x) - h^{(i)}(t, x, u^i(t, x)), \right. \\ \left. (\sigma^\top D_x u^i)(t, x), B_i u^i(t, x) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ u^i(T, x) = g^i(x); \end{array} \right. \quad (2.1.1)$$

where the operators B_i and K_i are defined as follows:

$$\begin{aligned} B_i u^i(t, x) &= \int_{\mathbb{E}} \gamma^i(t, x, e) (u^i(t, x + \beta(t, x, e)) - u^i(t, x)) \lambda(\mathrm{d}e); \\ K_i u^i(t, x) &= \int_{\mathbb{E}} (u^i(t, x + \beta(t, x, e)) - u^i(t, x) - \beta(t, x, e)^\top D_x u^i(t, x)) \lambda(\mathrm{d}e) \end{aligned} \quad (2.1.2)$$

The resolution of (2.1.1) is in connection with the following system of backward stochastic differential equations with jumps and one-obstacle ℓ :

$$\left\{ \begin{array}{l} \text{(i)} \quad \mathrm{d}Y_s^{i;t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i;t,x})_{i=1,m}, Z_s^{i;t,x}, U_s^{i;t,x}) \mathrm{d}s \\ \quad \quad \quad - \mathrm{d}K_s^{i;t,x} + Z_s^{i;t,x} \mathrm{d}B_s + \int_{\mathbb{E}} U_s^{i;t,x}(e) \tilde{\mu}(\mathrm{d}s, \mathrm{d}e), \quad s \leq T; \\ \text{(ii)} \quad Y_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i;t,x} - \ell(s, X_s^{t,x})) \mathrm{d}K_s^{i;t,x} = 0; \\ \text{(iii)} \quad Y_T^{i;t,x} = g^i(X_T^{t,x}). \end{array} \right. \quad (2.1.3)$$

and the following standard stochastic differential equation of diffusion-jump type:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) \mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x}) \mathrm{d}B_r + \int_t^s \int_{\mathbb{E}} \beta(r, X_{r-}^{t,x}, e) \tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad (2.1.4)$$

for $s \in [t, T]$ and $X_s^{t,x} = x$ if $s \leq t$.

It is recalled that pioneering work was done for the resolution of (2.1.1), among these works we can mention those of Barles et al [8] if without obstacle, Harraj et al [42] in the case with two obstacles; with as common point the hypothesis of monotony on the generator and $\gamma \geq 0$. But recently Hamadène and Morlais relaxed these conditions with $\lambda(\cdot)$ finite [37].

In this work, we propose to solve (2.1.1) by relaxing the monotonicity of the generator and the positivity of γ and assuming that $\lambda = \infty$.

This article is organized as follows: in the next section, we give the notations and the assumptions of our objects; in Section 3, we recall a number of existing results; in Section 4, we build estimates and properties for a good resolution of our problem; Section 5 is reserved to give our main result and Section 6 is an extension of our result.

And in the end, classical definition of the concept of viscosity solution is put in Appendix.

2.2 Notations and Assumptions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ be a stochastic basis such that \mathcal{F}_0 contains all \mathbb{P} -null sets of \mathcal{F} , and $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$, $t \geq 0$, and we suppose that the filtration is generated by the

two mutually independent processes:

- (i) $B := (B_t)_{t \geq 0}$ a d -dimensional Brownian motion and
- (ii) a Poisson random measure μ on $\mathbb{R}^+ \times E$, where $E := \mathbb{R}^\ell - \{0\}$ is equipped with its Borel field \mathcal{E} ($\ell \geq 1$). The compensator $\nu(dt, de) = dt\lambda(de)$ is such that $\{\tilde{\mu}([0, t] \times A) = (\mu - \lambda)([0, t] \times A)\}_{t \geq 0}$ is a martingale for all $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. We also assume that λ is a σ -finite measure on (E, \mathcal{E}) , and integrates the function $(1 \wedge |e|^2)$ and $\lambda(E) = \infty$.

Let us now introduce the following spaces:

- (iii) \mathcal{P} (resp. \mathbf{P}) the field on $[0, T] \times \Omega$ of $\mathcal{F}_{t \leq T}$ -progressively measurable (resp. predictable) sets.
 - (iv) For $\kappa \geq 1$, $\mathbb{L}_\kappa^2(\lambda)$ the space of Borel measurable functions $\varphi := (\varphi(e))_{e \in E}$ from E into \mathbb{R}^κ such that $\|\varphi\|_{\mathbb{L}_\kappa^2(\lambda)}^2 = \int_E |\varphi(e)|_\kappa^2 \lambda(de) < \infty$; $\mathbb{L}_1^2(\lambda)$ will be simply denoted by $\mathbb{L}^2(\lambda)$;
 - (v) $\mathcal{S}^2(\mathbb{R}^\kappa)$ the space of rcll (for right continuous with left limits) \mathcal{P} -measurable and \mathbb{R}^κ -valued processes such that $\mathbb{E}[\sup_{s \leq T} |Y_s|^2] < \infty$; \mathcal{A}_c^2 is its subspace of continuous non-decreasing processes $(K_t)_{t \leq T}$ such that $K_0 = 0$ and $\mathbb{E}[(K_T)^2] < \infty$;
 - (vi) $\mathbb{H}^2(\mathbb{R}^{\kappa \times d})$ the space of processes $Z := (Z_s)_{s \leq T}$ which are \mathcal{P} -measurable, $\mathbb{R}^{\kappa \times d}$ -valued, and satisfying $\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty$;
 - (vii) $\mathbb{H}^2(\mathbb{L}_\kappa^2(\lambda))$ the space of processes $U := (U_s)_{s \leq T}$ which are \mathbf{P} -measurable, $\mathbb{L}_\kappa^2(\lambda)$ -valued and satisfying $\mathbb{E}\left[\int_0^T \|U_s(\omega)\|_{\mathbb{L}_\kappa^2(\lambda)}^2 ds\right] < \infty$;
 - (viii) Π_g the set of deterministic functions
- $\varpi : (t, x) \in [0, T] \times \mathbb{R}^\kappa \mapsto \varpi(t, x) \in \mathbb{R}$ of polynomial growth, that is, for which there exists two non-negative constants C and p such that for any $(t, x) \in [0, T] \times \mathbb{R}^\kappa$,

$$|\varpi(t, x)| \leq C(1 + |x|^p).$$

The subspace of Π_g of continuous functions will be denoted by Π_g^c ;

- (ix) \mathcal{U} the subclass of Π_g^c which consists of functions $\Phi : (t, x) \in [0, T] \times \mathbb{R}^\kappa \mapsto \mathbb{R}$ such that for some non-negative constants C and p , we have

$$|\Phi(t, x) - \Phi(t, x')| \leq C(1 + |x|^p + |x'|^p) |x - x'|, \text{ for any } t, x, x'.$$

- (x) For any process $\theta := (\theta_s)_{s \leq T}$ and $t \in (0, T]$, $\theta_{t-} = \lim_{s \nearrow t} \theta_s$ and

$$\Delta_t \theta = \theta_t - \theta_{t-}.$$

Now, let b and σ be the following functions:

$$b : (t, x) \in [0, T] \times \mathbb{R}^k \mapsto b(t, x) \in \mathbb{R}^k;$$

$$\sigma : (t, x) \in [0, T] \times \mathbb{R}^k \mapsto \sigma(t, x) \in \mathbb{R}^{k \times d}.$$

We assume that they are jointly continuous in (t, x) and Lipschitz continuous w.r.t. x uniformly in t , that is, there exists a constant C such that,

$$\forall (t, x, x') \in [0, T] \times \mathbb{R}^{k+k}, \quad |b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C |x - x'|. \quad (2.2.1)$$

Notice that by (2.2.1) and continuity, the functions b and σ are of linear growth, that is, there exists a constant C such that

$$\forall (t, x, x') \in [0, T] \times \mathbb{R}^{k+k}, \quad |b(t, x)| + |\sigma(t, x)| \leq C |1 + x|. \quad (2.2.2)$$

Let $\beta : (t, x, e) \in [0, T] \times \mathbb{R}^k \times E \mapsto \beta(t, x, e) \in \mathbb{R}^k$ be a measurable function, such that for some real constant C and for all $e \in E$, the followings hold:

- (i) $|\beta(t, x, e)| \leq C(1 \wedge |e|)$; (2.2.3)
- (ii) $|\beta(t, x, e) - \beta(t, x', e)| \leq C |x - x'| (1 \wedge |e|)$;
- (iii) the mapping $(t, x) \in [0, T] \times \mathbb{R}^k \mapsto \beta(t, x, e) \in \mathbb{R}^k$ is continuous for any $e \in E$.

We are now going to introduce the objects which are specifically connected to the RBSDE with jumps we will deal with. Let ℓ the barrier of (2.1.3); $(g^i)_{i=1, m}$ and $(h^{(i)})_{i=1, m}$ be two functions defined as follows: for $i = 1, \dots, m$,

$$\begin{aligned} g^i : \mathbb{R}^k &\longrightarrow \mathbb{R}^m \\ x &\longmapsto g^i(x) \end{aligned}$$

and

$$\begin{aligned} h^{(i)} : [0, T] \times \mathbb{R}^{k+m+d+1} &\longrightarrow \mathbb{R} \\ (t, x, y, z, q) &\longmapsto h^{(i)}(t, x, y, z, q). \end{aligned}$$

Moreover, we assume that they satisfy

(H1) The reflecting barrier ℓ is real valued and \mathcal{P} -measurable process satisfying, $\ell \in \mathcal{U}$ that is, it is continuous and there exists constants C and p such that

$$|\ell(t, x) - \ell(t, x')| \leq C(1 + |x|^p + |x'|^p) |x - x'|, \quad \text{for any } t \geq 0, x, x'.$$

(H2) For any $i \in \{1, \dots, m\}$, the function g^i belongs to \mathcal{U} .

(H3) For any $i \in \{1, \dots, m\}$,

(i) the function $h^{(i)}$ is Lipschitz in (y, z, q) uniformly in (t, x) , that is, there exists a real constant C such that for any $(t, x) \in [0, T] \times \mathbb{R}^k$, (y, z, q) and (y', z', q') elements of \mathbb{R}^{m+d+1} ,

$$|h^{(i)}(t, x, y, z, q) - h^{(i)}(t, x, y', z', q')| \leq C(|y - y'| + |z - z'| + |q - q'|); \quad (2.2.4)$$

(ii) the $(t, x) \mapsto h^{(i)}(t, x, y, z, q)$, for fixed $(y, z, q) \in \mathbb{R}^{m+d+1}$, belongs uniformly to \mathcal{U} , that is, it is continuous and there exists constants C and p (which do not depend on (y, z, q)) such that

$$\left| h^{(i)}(t, x, y, z, q) - h^{(i)}(t, x', y, z, q) \right| \leq C(1 + |x|^p + |x'|^p) |x - x'|, \text{ for any } t \geq 0, x, x'. \quad (2.2.5)$$

Note that from [8] by Barles et al, the functions $h^i(t, x, (y^i)_{i=1,m}, z, q)$ are assumed to be increasing w.r.t y^k when the other components are fixed. But this assumption does not get involved in this article.

Next, let γ^i , $i = 1, \dots, m$ be Borel measurable functions defined from $[0, T] \times \mathbb{R}^k \times E$ into \mathbb{R} and satisfying the followings:

- (i) $|\gamma^i(t, x, e)| \leq C(1 \wedge |e|)$;
- (ii) $|\gamma^i(t, x, e) - \gamma^i(t, x', e)| \leq C(1 \wedge |e|) |x - x'| (1 + |x|^p + |x'|^p)$; (2.2.6)
- (iii) the mapping $t \in [0, T] \mapsto \gamma^i(t, x, e) \in \mathbb{R}$ is continuous for any $(x, e) \in \mathbb{R}^k \times E$.

Finally, we introduce the following functions $(f^{(i)})_{i=1,m}$ defined by

$$\forall (t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^{k+m+d} \times \mathbb{L}^2(\lambda),$$

$$f^{(i)}(t, x, y, z, \zeta) := h^{(i)} \left(t, x, y, z, \int_E \gamma^i(t, x, e) \zeta(e) \lambda(de) \right). \quad (2.2.7)$$

The functions $(f^{(i)})_{i=1,m}$, enjoy the two following properties:

(a) The function $f^{(i)}$ is Lipschitz in (y, z, ζ) uniformly in (t, x) , that is, there exists a real constant C such that

$$\left| f^{(i)}(t, x, y, z, \zeta) - f^{(i)}(t, x, y', z', \zeta') \right| \leq C(|y - y'| + |z - z'| + \|\zeta - \zeta'\|_{\mathbb{L}^2(\lambda)}), \quad (2.2.8)$$

because $h^{(i)}$ is uniformly Lipschitz in (y, z, q) and γ^i verifies (2.2.4)-(i);

(b) The function $(t, x) \in [0, T] \times \mathbb{R}^k \mapsto f^{(i)}(t, x, 0, 0, 0)$ belongs to Π_g^c , and then

$$\mathbb{E} \left[\int_0^T \left| f^{(i)}(r, X_r^{t,x}, 0, 0, 0) \right|^2 dr \right] < \infty.$$

Having defined our data and put our assumptions, we can look at the state of the art.

2.3 Preliminaires

2.3.1 A class of diffusion processes with jumps

Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and $(X_s^{t,x})_{s \leq T}$ be the stochastic process solution of (2.1.4). Under assumptions (2.2.1)–(2.2.3), the solution of Equation (2.1.4) exists and is unique (see [30] for more details). We state some properties of the process $\{(X_s^{t,x}), s \in [0, T]\}$, which can be found in [30].

Proposition 2.3.1. *For each $t \geq 0$, there exists a version of $\{(X_s^{t,x}), s \in [t, T]\}$ such that $s \rightarrow X_s^t$ is a $C^2(\mathbb{R}^d)$ -valued rcll process. Moreover, it satisfies the following estimates: $\forall p \geq 2$, $x, x' \in \mathbb{R}^d$ and $s \geq t$,*

$$\mathbb{E} \left[\sup_{t \leq r \leq s} |X_r^{t,x} - x|^p \right] \leq M_p(s-t)(1 + |x|^p), \quad (2.3.1)$$

$$\mathbb{E} \left[\sup_{t \leq r \leq s} |X_r^{t,x} - X_r^{t,x'} - (x - x')|^p \right] \leq M_p(s-t)(|x - x'|^p); \quad (2.3.2)$$

for some constant M_p .

2.3.2 Existence and uniqueness for a RBSDE with jumps

Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and we consider the following m -dimensional RBSDE with jumps:

$$\left\{ \begin{array}{l} \text{(i)} \quad \vec{Y}^{t,x} := (Y_s^{i,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad Z^{t,x} := (Z_s^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad K^{t,x} := (K_s^{i,t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad U^{t,x} := (U_s^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} \quad dY_s^{i,t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i,t,x})_{i=1,m}, Z_s^{i,t,x}, U_s^{i,t,x})ds - dK_s^{i,t,x} \\ \quad \quad \quad + Z_s^{i,t,x} dB_s + \int_{\mathbb{E}} U_s^{i,t,x}(e) \tilde{\mu}(ds, de), \quad s \leq T; \\ \text{(iii)} \quad Y_s^{i,t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i,t,x} - \ell(s, X_s^{t,x})) dK_s^{i,t,x} = 0, \end{array} \right. \quad (2.3.3)$$

where for any $i \in \{1, \dots, m\}$, $Z_s^{i,t,x}$ is the i th row of $Z_s^{t,x}$, $K_s^{i,t,x}$ is the i th component of $K_s^{t,x}$, and $U_s^{i,t,x}$ is the i th component of $U_s^{t,x}$.

The following result is related to the existence and uniqueness of a solution for the RBSDE with jumps (2.3.3).

Its proof is given in [38] using the penalization method (see p.5–12) and the Snell envelope method (see p.14–16).

Proposition 2.3.2. *Assume that assumptions (H1), (H2), and (H3) hold. Then, for any $(t, x) \in [0, T] \times \mathbb{R}^d$, the RBSDE (2.3.3) has an unique solution $(\vec{Y}^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x})$.*

Remark 2.3.1. *The solution of this RBSDE with jumps exist and is unique because of the followings:*

(i) $\mathbb{E}\left[\left|g(X_T^{t,x})\right|^2\right] < \infty$, be cause of polynomial growth of g and estimate (2.3.2) on $X^{t,x}$;

(ii) for any $i = 1, \dots, m$, $f^{(i)}$ verifies the properties (a)–(b) related to uniform Lipschitz w.r.t. (y, z, ζ) and $ds \otimes d\mathbb{P}$ -square integrability of the process $(f^{(i)}(s, X_s^{t,x}, 0, 0, 0))_{s \leq T}$.

2.3.3 Viscosity solutions of integro-differential partial equation with one obstacle

The following result on one obstacle is proved with two obstacles in Harraj et al (see [42], Theorem 4.6. p.47 by using (4.1) p.44), and it establishes the relationship between the solution of (2.3.3) and the one of system (2.1.1).

Proposition 2.3.3. *Assume that (H1), (H2), and (H3) are fulfilled. Then, there exists deterministic continuous functions $(u^i(t, x))_{i=1, m}$ which belong to Π_g such that for any $(t, x) \in [0, T] \times \mathbb{R}^k$, the solution of the RBSDE (2.3.3) verifies that*

$$\forall i \in \{1, \dots, m\}, \forall s \in [t, T], Y_s^{i;t,x} = u^i(s, X_s^{t,x}). \quad (2.3.4)$$

Moreover, if for any $i \in \{1, \dots, m\}$,

(i) $\gamma^i \geq 0$, and

(ii) for any fixed $(t, x, \vec{y}, z) \in [0, T] \times \mathbb{R}^{k+m+d}$, the mapping $(q \in \mathbb{R}) \mapsto h^{(i)}(t, x, \vec{y}, z, q) \in \mathbb{R}$ is non-decreasing, the function $(u^i)_{i=1, m}$ is a continuous viscosity solution of (2.1.1) (in Barles and al 's sense, see Definition 6.1 in Appendix). For the proof, see [42] for the same way.

Finally, the solution $(u^i)_{i=1, m}$ of (2.1.1) is unique in the class Π_g^c .

Remark 2.3.2 (see [42]). *Under the assumptions (H1), (H2), and (H3), there exists a unique viscosity solution of (2.1.1) in the class of functions satisfying*

$$\lim_{|x| \rightarrow +\infty} |u(t, x)| e^{-\tilde{A}[\log(|x|)]^2} = 0 \quad (2.3.5)$$

uniformly for $t \in [0, T]$, for some $\tilde{A} > 0$.

2.4 Estimates and Properties

In this section, we provide estimates for the functions $(u^i)_{i=1, m}$ defined in (2.3.4). Recall that $(\vec{Y}^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x}) := ((Y^{i;t,x})_{i=1, m}, (Z^{i;t,x})_{i=1, m}, (U^{i;t,x})_{i=1, m}, (K^{i;t,x})_{i=1, m})$ is the unique solution of the RBSDE with jumps (2.3.3).

Lemma 2.4.1. *Under assumptions (H1), (H2), and (H3), for any $p \geq 2$, there exists two non-negative constants C and ρ such that*

$$\mathbb{E} \left[\left\{ \int_0^T ds \left(\int_E |U_s^{t,x}(e)|^2 \lambda(de) \right) \right\}^{\frac{p}{2}} \right] = \mathbb{E} \left[\left\{ \int_0^T ds \|U_s^{t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}} \right] \leq C (1 + |x|^\rho). \quad (2.4.1)$$

Proof 2.4.1. *First, let us point out that as $X_s^{t,x} = x$ for $s \in [0, t]$, then the uniqueness of the solution of RBSDE of (2.3.3) implies that*

$$Z_s^{t,x} = U_s^{t,x} = K_s^{t,x} = 0, ds \otimes d\mathbf{P}\text{-a.e. on } [0, t] \times \Omega. \quad (2.4.2)$$

Next, let $p \geq 2$ be fixed. Using representation (2.3.4), for any $i \in \{1, \dots, m\}$ and $s \in [t, T]$, we have

$$\begin{aligned} Y_s^{i;t,x} &= g^i(X_T^{t,x}) + \int_s^T f^{(i)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i;t,x}, U_s^{i;t,x}) dr + K_T^{i;t,x} - K_s^{i;t,x} \\ &\quad - \int_s^T Z_r^{i;t,x} dB_r - \int_s^T \int_E U_r^{i;t,x}(e) \tilde{\mu}(dr, de). \end{aligned} \quad (2.4.3)$$

This implies that the system of RBSDEs with jumps (2.3.3) turns into a decoupled one because the equations in (2.4.3) are not related each other.

Next, for any $i = 1, \dots, m$, the functions u^i, g^i and $(t, x) \mapsto f^{(i)}(t, x, 0, 0, 0)$ are of polynomial growth and finally $y \mapsto f^{(i)}(t, x, y, 0, 0)$ is Lipschitz uniformly w.r.t. (t, x) . Then, for some C and $\rho \geq 0$, we have

$$\mathbb{E} \left[\left| g^i(X_T^{t,x}) \right|^p + \left(\int_s^T \left| f^{(i)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, 0, 0) \right|^2 dr \right)^{\frac{p}{2}} \right] \leq C (1 + |x|^\rho). \quad (2.4.4)$$

Let us now fix $i_0 \in \{1, \dots, m\}$, then for $\forall s \in [t, T]$,

$$\left\{ \begin{array}{l} \text{(i)} \quad Y_s^{i_0,t,x} \in \mathcal{S}^2(\mathbb{R}), \quad Z_s^{i_0,t,x} \in \mathbb{H}^2(\mathbb{R}^d), \quad K_s^{i_0,t,x} \in \mathcal{A}_c^2, \quad U_s^{i_0,t,x} \in \mathbb{H}^2(\mathbb{L}^2(\lambda)); \\ \text{(ii)} \quad Y_s^{i_0,t,x} = g^{i_0}(X_T^{t,x}) + \int_s^T f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) dr \\ \quad \quad \quad + K_T^{i_0,t,x} - K_s^{i_0,t,x} - \int_s^T Z_r^{i_0,t,x} dB_r - \int_s^T \int_E U_r^{i_0,t,x}(e) \tilde{\mu}(dr, de); \\ \text{(iii)} \quad Y_s^{i_0,t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i_0,t,x} - \ell(s, X_s^{t,x})) dK_s^{i_0,t,x} = 0. \end{array} \right.$$

Applying Itô formula to $|Y_s^{i_0,t,x}|^2$ between s and T , we have

$$\begin{aligned}
 & |Y_s^{i_0,t,x}|^2 + \int_s^T |Z_r^{i_0,t,x}|^2 dr + \sum_{s \leq r \leq T} (\Delta Y_r^{i_0,t,x})^2 \\
 = & |g^{i_0}(X_T^{t,x})|^2 + 2 \int_s^T Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) dr \\
 & + 2 \int_s^T Y_r^{i_0,t,x} dK_r^{i_0,t,x} - 2 \int_s^T \int_{\mathbb{E}} Y_r^{i_0,t,x} U_r^{i_0,t,x} \tilde{\mu}(dr, de) - 2 \int_s^T Y_r^{i_0,t,x} Z_r^{i_0,t,x} dB_r.
 \end{aligned} \tag{2.4.5}$$

Notice that $Y_r^{i_0,t,x} = u^{i_0}(r, X_r^{t,x})$ and we have $|u^{i_0}(r, X_r^{t,x})| \leq C(1 + |X_r^{t,x}|^q)$. Next, set $\Sigma = 1 + \sup_{s \leq T} |X_s^{t,x}|$. Therefore, $|Y_r^{i_0,t,x}| \leq C_q \Sigma^q$ and $|g^{i_0}(X_T^{t,x})| \leq C_q \Sigma^q$.

By raising to the power $\frac{p}{2}$ and then taking expectation, it follows from (2.4.5) and the fact that there exists $C > 0$ such that

$$\mathbb{E} \left[\left(\int_s^T \|U_r^{i_0,t,x}\|_{\mathbb{L}^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right] \leq C \mathbb{E} \left[\left(\sum_{s \leq r \leq T} (\Delta Y_r^{i_0,t,x})^2 \right)^{\frac{p}{2}} \right],$$

(see [51] p.28–45),

$$\begin{aligned}
 & \mathbb{E} [|Y_s^{i_0,t,x}|^p] + \mathbb{E} \left[\left(\int_s^T |Z_r^{i_0,t,x}|^2 dr \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[\left(\int_s^T \|U_r^{i_0,t,x}\|_{\mathbb{L}^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right] \\
 \leq & 5^{\frac{p}{2}-1} \mathbb{E} [|g^{i_0}(X_T^{t,x})|^p] \\
 & + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left(\int_s^T 2 |Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x})| dr \right)^{\frac{p}{2}} \right] \\
 & + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left| \int_s^T Y_r^{i_0,t,x} dK_r^{i_0,t,x} \right|^{\frac{p}{2}} \right] \\
 & + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left| \int_s^T \int_{\mathbb{E}} 2Y_r^{i_0,t,x} U_r^{i_0,t,x} \tilde{\mu}(dr, de) \right|^{\frac{p}{2}} \right] + 5^{\frac{p}{2}-1} \mathbb{E} \left[\left| \int_s^T 2Y_r^{i_0,t,x} Z_r^{i_0,t,x} dB_r \right|^{\frac{p}{2}} \right].
 \end{aligned} \tag{2.4.6}$$

For more comprehension, we adopt the following scripture for inequality (2.4.6);

$$\begin{aligned}
 & \mathbb{E} [|Y_s^{i_0,t,x}|^p] + \mathbb{E} \left[\left(\int_s^T |Z_r^{i_0,t,x}|^2 dr \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[\left(\int_s^T \|U_r^{i_0,t,x}\|_{\mathbb{L}^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right] \\
 \leq & 5^{\frac{p}{2}-1} \mathbb{E} [|g^{i_0}(X_T^{t,x})|^p] + 5^{\frac{p}{2}-1} T_1(s) + 5^{\frac{p}{2}-1} T_2(s) + 5^{\frac{p}{2}-1} T_3(s) + 5^{\frac{p}{2}-1} T_4(s)
 \end{aligned} \tag{2.4.7}$$

where

$$\begin{aligned}
 T_1(s) &= \mathbb{E} \left[\left(\int_s^T 2 \left| Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) \right| dr \right)^{\frac{p}{2}} \right]; \\
 T_2(s) &= \mathbb{E} \left[\left| \int_s^T Y_r^{i_0,t,x} dK_r^{i_0,t,x} \right|^{\frac{p}{2}} \right]; \\
 T_3(s) &= \mathbb{E} \left[\left| \int_s^T \int_{\mathbb{E}} 2Y_r^{i_0,t,x} U_r^{i_0,t,x} \tilde{\mu}(dr, de) \right|^{\frac{p}{2}} \right]; \\
 T_4(s) &= \mathbb{E} \left[\left| \int_s^T 2Y_r^{i_0,t,x} Z_r^{i_0,t,x} dB_r \right|^{\frac{p}{2}} \right].
 \end{aligned}$$

We will estimate $T_1(s)$, $T_2(s)$, $T_3(s)$, and $T_4(s)$, $\forall s \in [t, T]$.

(a) Before starting our estimations, let us linearize f with respect to $(u^j(X_r^{t,x}))_{j=1,m}$ and $Z_r^{i_0,t,x}$ that is

$$\begin{aligned}
 &f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) \\
 &= a_r^{t,x} Z_r^{i_0,t,x} + b_r^{t,x} (u^j(X_r^{t,x}))_{j=1,m} + f^{(i_0)}(r, X_r^{t,x}, 0, 0, U_r^{i_0,t,x}),
 \end{aligned}$$

where $a_r^{t,x}$ and $b_r^{t,x}$ are progressively measurable processes respectively bounded by the Lipschitz constants of f in $Z_r^{t,x}$ and $(u^j(X_r^{t,x}))_{j=1,m}$ that is, $|a_r^{t,x}| \leq C_Z$ and $|b_r^{t,x}| \leq \lambda_1$.

(b) We also take the fact that f is Lipschitz in $U_r^{i_0,t,x}$ that is, there exists a constant Lipschitz λ_2 such that $|f^{(i_0)}(r, X_r^{t,x}, 0, 0, U_r^{i_0,t,x})| \leq |f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0)| + \lambda_2 \|U_r^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}$.

By combining (a) and (b), we have

$$\begin{aligned}
 &|f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x})| \\
 &\leq |a_r^{t,x} Z_r^{i_0,t,x}| + |b_r^{t,x} (u^j(X_r^{t,x}))_{j=1,m}| + |f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0)| + \lambda_2 \|U_r^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}.
 \end{aligned} \tag{2.4.8}$$

Let us start our estimates.

For $T_1(s)$ By using (2.4.8), it follows that

$$\begin{aligned}
 &\int_s^T 2 \left| Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, (u^j(X_r^{t,x}))_{j=1,m}, Z_r^{i_0,t,x}, U_r^{i_0,t,x}) \right| dr \\
 &\leq \int_s^T 2 \left| Y_r^{i_0,t,x} (a_r^{t,x} Z_r^{i_0,t,x}) \right| dr + \int_s^T 2 \left| Y_r^{i_0,t,x} (b_r^{t,x} (u^j(X_r^{t,x}))_{j=1,m}) \right| dr \\
 &\quad + \lambda_2 \int_s^T 2 \left| Y_r^{i_0,t,x} (U_r^{i_0,t,x}) \right| dr + \int_s^T 2 \left| Y_r^{i_0,t,x} f^{(i_0)}(r, X_r^{t,x}, 0, 0, 0) \right| dr \\
 &\leq C_q^2 C_Z T \epsilon_1^{-1} \Sigma^{2q} + \epsilon_1 C_Z \int_s^T |Z_r^{i_0,t,x}|^2 dr + C_q^2 \lambda_1 T \epsilon_2^{-1} \Sigma^{2q} + \epsilon_2 \lambda_1 C_q^2 T \Sigma^{2q} \\
 &\quad + C_q^2 T \lambda_2 \epsilon_3^{-1} \Sigma^{2q} + \lambda_2 \epsilon_3 \int_s^T \|U_r^{i_0,t,x}\|_{\mathbb{L}_m^2(\lambda)}^2 dr + C_q^2 T \epsilon_4^{-1} \Sigma^{2q} + C_q^2 T \epsilon_4 \Sigma^{2q}.
 \end{aligned} \tag{2.4.9}$$

By raising to the power $\frac{p}{2}$ and then taking expectation, it follows that

$$\begin{aligned}
 T_1(s) &\leq 8^{\frac{p}{2}-1} C C_q^p \left\{ (C_Z T \epsilon_1^{-1})^{\frac{p}{2}} + (\lambda_1 T \epsilon_2^{-1})^{\frac{p}{2}} + C_q^p (\epsilon_2 \lambda_1 T)^{\frac{p}{2}} + (T \lambda_2 \epsilon_3^{-1})^{\frac{p}{2}} + (T \epsilon_4^{-1})^{\frac{p}{2}} \right. \\
 &\quad \left. + (T \epsilon_4)^{\frac{p}{2}} \right\} |1 + |x|^{pq}| + 8^{\frac{p}{2}-1} (\epsilon_1 C_Z)^{\frac{p}{2}} \mathbb{E} \left[\left(\int_s^T |Z_r^{i_0, t, x}|^2 dr \right)^{\frac{p}{2}} \right] \\
 &\quad + 8^{\frac{p}{2}-1} (\lambda_2 \epsilon_3)^{\frac{p}{2}} \mathbb{E} \left[\left(\int_s^T \|U_r^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 dr \right)^{\frac{p}{2}} \right]. \tag{2.4.10}
 \end{aligned}$$

Before estimating $T_2(s)$, let us first give an estimate of $\mathbb{E}[|K_T^{i_0, t, x} - K_s^{i_0, t, x}|^p]$, which will serve us in estimating of $T_2(s)$.

$$\begin{aligned}
 K_T^{i_0, t, x} - K_s^{i_0, t, x} &= Y_s^{i_0, t, x} - g^{i_0}(X_T^{t, x}) - \int_s^T f^{(i_0)}(r, X_r^{t, x}, (u^j(X_r^{t, x}))_{j=1, m}, Z_r^{i_0, t, x}, U_r^{i_0, t, x}) dr \\
 &\quad + \int_s^T Z_r^{i_0, t, x} dB_r + \int_s^T \int_{\mathbb{E}} U_r^{i_0, t, x}(e) \tilde{\mu}(dr, de).
 \end{aligned}$$

By (2.4.8) and Cauchy-Schwartz inequality, it follows that

$$\begin{aligned}
 |K_T^{i_0, t, x} - K_s^{i_0, t, x}| &\leq 2C_q \Sigma^q + C_q \lambda_1 T \Sigma^q + C_q T \Sigma^q + C_Z \left(\int_s^T |Z_r^{i_0, t, x}|^2 dr \right)^{\frac{1}{2}} \\
 &\quad + \lambda_2 \left(\int_s^T \|U_r^{i_0, t, x}\|^2 dr \right)^{\frac{1}{2}} + \int_s^T Z_r^{i_0, t, x} dB_r + \int_s^T \int_{\mathbb{E}} U_r^{i_0, t, x}(e) \tilde{\mu}(dr, de).
 \end{aligned}$$

By raising to the power p , and taking expectation and BDG inequality we have

$$\mathbb{E} \left[|K_T^{i_0, t, x} - K_s^{i_0, t, x}|^p \right] \leq 5^{p-1} C C_q^p \{2^p + (T \lambda_1)^p + T^p\} |1 + |x|^{pq}| \tag{2.4.11}$$

$$+ 5^{p-1} (C_Z^p + C_p) \mathbb{E} \left(\int_0^T |Z_s^{i_0, t, x}|^2 ds \right)^{\frac{p}{2}} \tag{2.4.12}$$

$$+ 5^{p-1} (\lambda_2^p + C_p) \mathbb{E} \left\{ \int_0^T ds \|U_s^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}}. \tag{2.4.13}$$

For $T_2(s)$

$$\begin{aligned}
 \int_s^T |Y_s^{i_0, t, x}| dK_s^{i_0, t, x} &\leq \int_s^T |(Y_s^{i_0, t, x} - \ell(s, X_s^{t, x}))| dK_s^{i_0, t, x} + \int_s^T |\ell(s, X_s^{t, x})| dK_s^{i_0, t, x} \\
 &\leq \sup_{s \leq T} |\ell(s, X_s^{t, x})| K_T^{i_0, t, x} \\
 &\leq \epsilon_5^{-1} \sup_{s \leq T} |\ell(s, X_s^{t, x})|^2 + \epsilon_5 (K_T^{i_0, t, x})^2 \\
 &\leq \epsilon_5^{-1} C_q^2 \Sigma^{2q} + \epsilon_5 (K_T^{i_0, t, x})^2. \tag{2.4.14}
 \end{aligned}$$

Using (2.4.11) and (2.4.14), it follows that

$$\begin{aligned}
 T_2(s) &\leq 2^{\frac{p}{2}-1} C C_q^p (\epsilon_5^{-1})^{\frac{p}{2}} |1 + |x|^{pq}| + 2^{\frac{p}{2}-1} (\epsilon_5)^{\frac{p}{2}} \mathbb{E} \left[(K_T^{i_0, t, x})^p \right] \\
 &\leq 2^{\frac{p}{2}-1} C C_q^p (\epsilon_5^{-1})^{\frac{p}{2}} |1 + |x|^{pq}| + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} C C_q^p \{2^p + (T\lambda_1)^p + T^p\} |1 + |x|^{pq}| \\
 &\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (C_Z^p + C_p) \mathbb{E} \left(\int_0^T |Z_s^{i_0, t, x}|^2 ds \right)^{\frac{p}{2}} \\
 &\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (\lambda_2^p + C_p) \mathbb{E} \left\{ \int_0^T ds \|U_s^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}} \\
 &\leq \left\{ 2^{\frac{p}{2}-1} C C_q^p (\epsilon_5^{-1})^{\frac{p}{2}} + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} C C_q^p (2^p + (T\lambda_1)^p + T^p) \right\} (1 + |x|^{pq}) \\
 &\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (C_Z^p + C_p) \mathbb{E} \left(\int_0^T |Z_s^{i_0, t, x}|^2 ds \right)^{\frac{p}{2}} \\
 &\quad + (\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (\lambda_2^p + C_p) \mathbb{E} \left\{ \int_0^T ds \|U_s^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{p}{2}}.
 \end{aligned}$$

For $T_3(s)$ By BDG inequality, we have

$$\begin{aligned}
 T_3(s) &\leq C_p \mathbb{E} \left(\int_0^T |Y_s^{i_0, t, x}|^2 |Z_s^{i_0, t, x}|^2 ds \right)^{\frac{p}{4}} \\
 &\leq C_p \mathbb{E} \left(\sup_{s \leq T} |Y_s^{i_0, t, x}|^2 \int_0^T |Z_s^{i_0, t, x}|^2 ds \right)^{\frac{p}{4}} \\
 &\leq C_p C_q^p \epsilon_6^{-1} (1 + |x|^{pq}) + C_p \epsilon_6 \mathbb{E} \left(\int_0^T |Z_s^{i_0, t, x}|^2 ds \right)^{\frac{p}{2}}. \quad (2.4.15)
 \end{aligned}$$

For $T_4(s)$ By BDG inequality, we obtain

$$\begin{aligned}
 T_4(s) &\leq C_p \mathbb{E} \left(\int_0^T |Y_s^{i_0, t, x}|^2 \|U_s^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 ds \right)^{\frac{p}{4}} \\
 &\leq C_p \mathbb{E} \left(\sup_{s \leq T} |Y_s^{i_0, t, x}|^2 \int_0^T \|U_s^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 ds \right)^{\frac{p}{4}} \\
 &\leq C_p C_q^p \epsilon_7^{-1} (1 + |x|^{pq}) + C_p \epsilon_7 \mathbb{E} \left(\int_0^T \|U_s^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 ds \right)^{\frac{p}{2}}. \quad (2.4.16)
 \end{aligned}$$

Finally, by taking estimation of $T_1(s)$, $T_2(s)$, $T_3(s)$, and $T_4(s)$, and choosing ϵ_1 , ϵ_2 , ϵ_3 , ϵ_4 , ϵ_5 , ϵ_6 , and ϵ_7 such that $\{(\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (C_Z^p + C_p) + C_p \epsilon_6\} < 1$, $\{(\epsilon_5)^{\frac{p}{2}} 2^{\frac{p}{2}-1} 7^{p-1} (\lambda_2^p + C_p) + C_p \epsilon_7\} < 1$, and the sum of all coefficients of $(1 + |x|^{pq})$ was small than 1.

It follows that

$$\mathbb{E} \left[\left\{ \int_0^T ds \|U_s^{i_0, t, x}\|_{\mathbb{L}_m^2(\lambda)}^2 \right\}^{\frac{\rho}{2}} \right] \leq C(1 + |x|^\rho), \quad (2.4.17)$$

where $\rho = pq$.

Finally, as $i_0 \in \{1, \dots, m\}$ is arbitrary, we then obtain estimate (2.4.1).

Proposition 2.4.2. For any $i = 1, \dots, m$, u^i belongs to \mathcal{U} .

Proof 2.4.2. Let x and x' be elements of \mathbb{R}^k . Let $(\vec{Y}^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x})$ (resp. $(\vec{Y}^{t,x'}, Z^{t,x'}, U^{t,x'}, K^{t,x'})$) be the solution of the RBSDE with jumps (2.3.3) associated with $f(s, X_s^{t,x}, y, \eta, \zeta, g(X_T^{t,x}))$ (resp. $f(s, X_s^{t,x'}, y, \eta, \zeta, g(X_T^{t,x'}))$). Applying Itô formula to $|\vec{Y}^{t,x} - \vec{Y}^{t,x'}|^2$ between s and T , we have

$$\begin{aligned} & |\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}|^2 + \int_s^T |\Delta Z_r|^2 dr + \sum_{s \leq r \leq T} (\Delta_r \vec{Y}_r^{t,x})^2 \\ &= |g(X_T^{t,x}) - g(X_T^{t,x'})|^2 + 2 \int_s^T \langle (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}), \Delta f(r) \rangle dr + 2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \\ & \quad - 2 \int_s^T \int_{\mathbb{E}} (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) (\Delta U_r(e)) \tilde{\mu}(dr, de) - 2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) (\Delta Z_r) dB_r, \end{aligned} \quad (2.4.18)$$

$$(2.4.19)$$

and taking expectation, we obtain the following result for $\forall s \in [t, T]$,

$$\begin{aligned} & \mathbb{E} \left[|\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}|^2 + \int_s^T |\Delta Z_r|^2 dr + \int_s^T \|\Delta U_r\|_{\mathbb{L}^2(\lambda)}^2 dr \right] \\ & \leq \mathbb{E} \left[|g(X_T^{t,x}) - g(X_T^{t,x'})|^2 + 2 \int_s^T \langle (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}), \Delta f(r) \rangle dr \right] \\ & \quad + \mathbb{E} \left[2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \right], \end{aligned} \quad (2.4.20)$$

$$(2.4.21)$$

where the processes $\Delta X_r, \Delta Y_r, \Delta f(r), \Delta K_r, \Delta Z_r, \Delta U_r$ and $\Delta \ell_r$ are defined as follows: $\forall r \in [t, T]$,

$$\begin{aligned} \Delta f(r) &:= ((\Delta f^{(i)}(r)))_{i=1, m} = (f^{(i)}(r, X_r^{i;t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, U_r^{i;t,x}) \\ & \quad - f^{(i)}(r, X_r^{i;t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, U_r^{i;t,x'}))_{i=1, m}, \end{aligned}$$

$$\Delta X_r = X_r^{t,x} - X_r^{t,x'}, \Delta Y(r) = \vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'} = (Y_r^{j;t,x} - Y_r^{j;t,x'})_{j=1, m},$$

$$\Delta K_r = K_r^{t,x} - K_r^{t,x'}, \Delta Z_r = Z_r^{t,x} - Z_r^{t,x'}, \Delta U_r = U_r^{t,x} - U_r^{t,x'}$$

and

$$\Delta \ell_r = (\ell(r, X_r^{t,x}) - \ell(r, X_r^{t,x'}))$$

$\langle \cdot, \cdot \rangle$ is the usual scalar product on \mathbb{R}^m .

Now, we will give an estimation of each three terms of the second member of inequality (2.4.20).

$$\begin{aligned}
 & \bullet \quad \mathbb{E} \left[\left| g(X_T^{t,x}) - g(X_T^{t,x'}) \right|^2 \right] \\
 & \leq C \mathbb{E} \left[\left| X_T^{t,x} - X_T^{t,x'} \right|^2 \left(1 + |X_T^{t,x}|^{2p} + |X_T^{t,x'}|^{2p} \right) \right] \quad (\text{for any } i \in \{1, \dots, m\}) \\
 & \qquad \qquad \qquad g^i \text{ belongs to } \mathcal{U} \\
 & \leq C \mathbb{E} \left[\left| (X_T^{t,x} - X_T^{t,x'} - (x - x')) + (x - x') \right|^2 (1 + |(X_T^{t,x} - x) + x|^{2p} \right. \right. \\
 & \quad \left. \left. + |(X_T^{t,x'} - x') + x'|^{2p} \right) \right]
 \end{aligned} \tag{2.4.22}$$

By triangle inequality and the fact of $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, Cauchy-Schwartz inequality, the relations (2.3.1) and (2.3.2) of Proposition 2.3.1

$$\mathbb{E} \left[\left| g(X_T^{t,x}) - g(X_T^{t,x'}) \right|^2 \right] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}), \tag{2.4.23}$$

$$\begin{aligned}
 & \bullet \text{ using (iii) of (2.3.3): } \mathbb{E} \left[2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \right] \text{ can be replaced by} \\
 & \quad \mathbb{E} \left[2 \int_s^T (\ell(r, X_r^{t,x}) - \ell(r, X_r^{t,x'})) d(\Delta K_r) \right].
 \end{aligned}$$

Indeed

$$\begin{aligned}
 & \mathbb{E} \left[2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \right] = \\
 & \mathbb{E} \left[2 \int_s^T \left((\vec{Y}_r^{t,x} - \ell(r, X_r^{t,x})) - (\vec{Y}_r^{t,x'} - \ell(r, X_r^{t,x'})) + (\ell(r, X_r^{t,x}) - \ell(r, X_r^{t,x'})) \right) d(\Delta K_r) \right].
 \end{aligned}$$

By linearity of integral and using (iii) of (2.3.3) we obtain our result.

Now, by (H1) and Cauchy-Schwartz inequality, we obtain

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} (\Delta \ell_t)^2 \right] \leq 2CC' |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}), \tag{2.4.24}$$

where $C' = \mathbb{E}[(\Delta K_T)^2]$.

• To complete our estimation of (2.4.20), we need to deal with $\mathbb{E} \left[2 \int_s^T \langle \vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}, \Delta f(r) \rangle dr \right]$. Taking into account the expression of $f^{(i)}$ given by (2.2.7), we then split

$\Delta f(r)$ in the follows way: for $r \leq T$,

$$\begin{aligned}\Delta f(r) &= (\Delta f(r))_{i=1,m} = \Delta_1(r) + \Delta_2(r) + \Delta_3(r) + \Delta_4(r) \\ &= (\Delta_1^i(r) + \Delta_2^i(r) + \Delta_3^i(r) + \Delta_4^i(r))_{i=1,m},\end{aligned}$$

where for any $i = 1, \dots, m$,

$$\begin{aligned}\Delta_1^i(r) &= h^{(i)}\left(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e)\right) \\ &\quad - h^{(i)}\left(r, X_r^{t,x'}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e)\right); \\ \Delta_2^i(r) &= h^{(i)}\left(r, X_r^{t,x'}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e)\right) \\ &\quad - h^{(i)}\left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e)\right); \\ \Delta_3^i(r) &= h^{(i)}\left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e)\right) \\ &\quad - h^{(i)}\left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e)\right); \\ \Delta_4^i(r) &= h^{(i)}\left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e)\right) \\ &\quad - h^{(i)}\left(r, X_r^{t,x'}, \vec{Y}_r^{t,x'}, Z_r^{i;t,x'}, \int_E \gamma^i(r, X_r^{t,x'}, e) U_r^{i;t,x'}(e) \lambda(\mathrm{d}e)\right).\end{aligned}$$

By Cauchy-Schwartz inequality, the inequality $2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$, relation (2.11), and estimate (2.3.2), we have

$$\begin{aligned}& \mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_1(r) \rangle \mathrm{d}r \right] \tag{2.4.25} \\ & \leq \mathbb{E} \left[\frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 \mathrm{d}r + C^2 \epsilon \int_s^T |X_r^{t,x} - X_r^{t,x'}|^2 (1 + |X_r^{t,x}|^p + |X_r^{t,x'}|^p)^2 \mathrm{d}r \right] \\ & \leq \mathbb{E} \left[\frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 \mathrm{d}r \right] + C^2 \epsilon |x - x'|^2 (1 + |x|^p + |x'|^p)^2. \tag{2.4.26}\end{aligned}$$

Besides, as $h^{(i)}$ is Lipschitz w.r.t. (y, z, q) , then

$$\mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_2(r) \rangle \mathrm{d}r \right] \leq 2C \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 \mathrm{d}r \right] \tag{2.4.27}$$

and

$$\mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_3(r) \rangle \mathrm{d}r \right] \leq \mathbb{E} \left[\frac{1}{\epsilon} \int_s^T |\Delta Y(r)|^2 \mathrm{d}r + C^2 \epsilon \int_s^T |\Delta Z(r)|^2 \mathrm{d}r \right]. \tag{2.4.28}$$

It remains to obtain a control of the last term. But for any $s \in [t, T]$, we have

$$\begin{aligned} & \mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_4(r) \rangle dr \right] \\ & \leq 2C \mathbb{E} \left[\int_s^T |\Delta Y(r)| dr \times \left| \int_E \left(\gamma(r, X_r^{t,x}, e) U_r^{t,x}(e) - \gamma(r, X_r^{t,x'}, e) U_r^{t,x'}(e) \right) \lambda(de) \right| \right]. \end{aligned} \quad (2.4.29)$$

Next, by splitting the crossing terms as follows:

$$\begin{aligned} & \gamma(r, X_r^{t,x}, e) U_r^{t,x}(e) - \gamma(r, X_r^{t,x'}, e) U_r^{t,x'}(e) \\ & = \Delta U_s(e) \gamma(s, X_s^{t,x}, e) + U_s^{t,x'} \left(\gamma(s, X_s^{t,x}, e) - \gamma(s, X_s^{t,x'}, e) \right) \end{aligned}$$

and setting $\Delta \gamma_s(e) := \left(\gamma(s, X_s^{t,x}, e) - \gamma(s, X_s^{t,x'}, e) \right)$, we obtain

$$\begin{aligned} & \mathbb{E} \left[2 \int_s^T \langle \Delta Y(r), \Delta_4(r) \rangle dr \right] \\ & \leq 2C \mathbb{E} \left[\int_s^T |\Delta Y(r)| \times \left(\int_E (|U_r^{t,x'}(e) \Delta \gamma_r(e)| + |\Delta U_r(e) \gamma(r, X_r^{t,x}, e)|) \lambda(de) \right) dr \right] \\ & \leq \frac{2}{\epsilon} \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right] + C^2 \epsilon \mathbb{E} \left[\int_s^T \left(\int_E (|U_r^{t,x'}(e) \Delta \gamma_r(e)| \lambda(de))^2 dr \right) \right] \\ & \quad + C^2 \epsilon \mathbb{E} \left[\int_s^T \left(\int_E (|\Delta U_r(e) \gamma(r, X_r^{t,x}, e)| \lambda(de))^2 dr \right) \right]. \end{aligned} \quad (2.4.31)$$

By Cauchy-Schwartz inequality, (2.2.6) and (2.3.2), and the result of Lemma 2.4.1, it holds that

$$\begin{aligned} & \mathbb{E} \left[\int_s^T \left(\int_E (|U_r^{t,x'}(e) \Delta \gamma_r(e)| \lambda(de))^2 dr \right) \right] \\ & \leq \mathbb{E} \left[\int_s^T dr \left(\int_E |U_r^{t,x'}(e)|^2 \lambda(de) \right) \left(\int_E |\Delta \gamma_r(e)|^2 \lambda(de) \right) \right] \\ & \leq C \mathbb{E} \left[\left\{ \sup_{r \in [t, T]} |X_r^{t,x} - X_r^{t,x'}|^2 (1 + \sup_{r \in [t, T]} |X_r^{t,x}|^p + |X_r^{t,x'}|^p)^2 dr \right\} \right] \\ & \quad \times \mathbb{E} \left[\int_s^T dr \left(\int_E |U_r^{t,x'}(e)|^2 \lambda(de) \right) \right] \\ & \leq C \sqrt{\mathbb{E} \left[\left\{ \sup_{r \in [t, T]} |X_r^{t,x} - X_r^{t,x'}|^4 (1 + \sup_{r \in [t, T]} |X_r^{t,x}|^p + |X_r^{t,x'}|^p)^4 \right\} \right]} \\ & \quad \times \sqrt{\mathbb{E} \left[\left\{ \int_s^T dr \left(\int_E |U_r^{t,x'}(e)|^2 \lambda(de) \right) \right\}^2 \right]} \\ & \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}). \end{aligned} \quad (2.4.33)$$

For some exponent p , on the other hand using once more Cauchy-Schwartz inequality and (2.2.6)-(i), we get

$$\mathbb{E} \left[\int_s^T \left(\int_E (|\Delta U_r(e) \gamma(r, X_r^{t,x}, e)| \lambda(de)) \right)^2 dr \right] \quad (2.4.34)$$

$$\begin{aligned} &\leq \mathbb{E} \left[\int_s^T dr \left(\int_E (|\Delta U_r(e)|^2 \lambda(de)) \right) \left(\int_E |\gamma(r, X_r^{t,x}, e)|^2 \lambda(de) \right) \right] \\ &\leq C \mathbb{E} \left[\int_s^T dr \left(\int_E (|\Delta U_r(e)|^2 \lambda(de)) \right) \right]. \end{aligned} \quad (2.4.35)$$

Taking now into account inequalities (2.4.25)–(2.4.34), we obtain

$$\begin{aligned} &\mathbb{E} \left[|\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}|^2 + \int_s^T |\Delta Z_r|^2 dr + \int_s^T \|\Delta U_r\|_{\mathbb{L}^2(\lambda)}^2 dr \right] \\ &\leq \mathbb{E} \left[|g(X_T^{t,x}) - g(X_T^{t,x'})|^2 + 2 \int_s^T \langle (\vec{Y}_s^{t,x} - \vec{Y}_s^{t,x'}), \Delta f(r) \rangle dr \right] \\ &\quad + \mathbb{E} \left[2 \int_s^T (\vec{Y}_r^{t,x} - \vec{Y}_r^{t,x'}) d(\Delta K_r) \right] \\ &\leq |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}) (C + 2CC' + C^2\epsilon + C^3\epsilon) + \left(\frac{3}{\epsilon} + 2C \right) \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right] \\ &\quad + C^2\epsilon \mathbb{E} \left[\int_s^T |\Delta Z(r)|^2 dr \right] + C^3\epsilon \mathbb{E} \left[\int_s^T dr \left(\int_E (|\Delta U_r(e)|^2 \lambda(de)) \right) \right]. \end{aligned}$$

Choosing now ϵ small enough, we deduce the existence of a constant $C \geq 0$ such that for any $s \in [t, T]$,

$$\mathbb{E} [|\Delta Y(s)|^2] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}) + \mathbb{E} \left[\int_s^T |\Delta Y(r)|^2 dr \right]$$

and by Gronwall lemma, this implies that for any $s \in [t, T]$,

$$\mathbb{E} [|\Delta Y(s)|^2] \leq C |x - x'|^2 (1 + |x|^{2p} + |x'|^{2p}).$$

Finally, taking $s = t$ and considering (2.3.4), we obtain the desired result.

Remark 2.4.1. This result give also estimate of U , where we use the function $(h^{(i)})_{1 \leq i \leq m}$, contrary in estimate (2.4.1).

Corollary 2.4.3. For $u^i \in \mathcal{U}$ ($i = 1, \dots, m$), $B_i u^i$ defined in (2.1.2) is well posed because the functions β and $(\gamma_i)_{i=1,m}$ verify (2.2.3) and (2.2.6), respectively.

Proof 2.4.3. The main point to notice is that λ integrates $(1 \wedge |e|^p)$, $\forall p \geq 2$.

We have

$$\begin{aligned}
 |B_i u^i(t, x)| &\leq \int_E |\gamma^i(t, x, e)| \times |(u^i(t, x + \beta(t, x, e)) - u^i(t, x))| \lambda(\mathrm{d}e) \\
 &\leq \int_E C(1 \wedge |e|) |\beta(t, x, e)| (1 + |x + \beta(t, x, e)|^p + |x|^p) \lambda(\mathrm{d}e) \\
 &\leq C^2(1 + |x|^p(1 + 2^{p-1})) \int_E C(1 \wedge |e|^2) \lambda(\mathrm{d}e) \\
 &\quad + (2^{\frac{p-1}{p}} C^{\frac{p+2}{p}})^p \int_E C(1 \wedge |e|^p) \lambda(\mathrm{d}e).
 \end{aligned}$$

Which finish the proof.

Now, by remark 2.4.1, the last estimate of U confirm the following result.

Proposition 2.4.4. *For any $i = 1, \dots, m$, $(t, x) \in [0, T] \times \mathbb{R}^k$, we have*

$$U_s^{i;t,x}(e) = u^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u^i(s, X_{s-}^{t,x}), \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}s \otimes \mathrm{d}\lambda\text{-a.e. on } \Omega \times [t, T] \times E. \quad (2.4.36)$$

Proof 2.4.4. *First, note that because the measure λ is not finite, then we cannot use the same technique as in [37], where the authors use the jumps of processes and (2.3.4).*

In our case, $U^{i;t,x}$ is only square integrable and not necessarily integrable w.r.t. $\mathrm{d}\mathbb{P} \otimes \mathrm{d}s \otimes \mathrm{d}\lambda$. Therefore, we first begin by truncating the Lévy measure as the same way in [36].

Step 1 Truncation of the Lévy measure

For any $k^ \geq 1$, let us first introduce a new Poisson random measure μ_{k^*} (obtained from the truncation of μ) and its associated compensator ν_{k^*} as follows:*

$$\mu_{k^*}(\mathrm{d}s, \mathrm{d}e) = 1_{\{|e| \geq \frac{1}{k^*}\}} \mu(\mathrm{d}s, \mathrm{d}e) \quad \text{and} \quad \nu_{k^*}(\mathrm{d}s, \mathrm{d}e) = 1_{\{|e| \geq \frac{1}{k^*}\}} \nu(\mathrm{d}s, \mathrm{d}e).$$

Which means that, as usual, $\tilde{\mu}_{k^}(\mathrm{d}s, \mathrm{d}e) := (\mu_{k^*} - \nu_{k^*})(\mathrm{d}s, \mathrm{d}e)$ is the associated random martingale measure.*

The main point to notice is that

$$\begin{aligned}
 \lambda_{k^*}(E) &= \int_E \lambda_{k^*}(\mathrm{d}e) = \int_E 1_{\{|e| \geq \frac{1}{k^*}\}} \lambda(\mathrm{d}e) \\
 &= \int_{\{|e| \geq \frac{1}{k^*}\}} \lambda(\mathrm{d}e) = \lambda(\{|e| \geq \frac{1}{k^*}\}) < \infty.
 \end{aligned} \quad (2.4.37)$$

As in [36], let us introduce the process ${}^{k^}X^{t,x}$ solving the following standard SDE of*

jump-diffusion type:

$$\begin{aligned}
 {}^{k^*}X_s^{t,x} &= x + \int_t^s b(r, {}^{k^*}X_r^{t,x})dr + \int_t^s \sigma(r, {}^{k^*}X_r^{t,x})dB_r \\
 &\quad + \int_t^s \int_E \beta(r, {}^{k^*}X_{r-}^{t,x}, e)\tilde{\mu}_{k^*}(dr, de), \\
 t \leq s \leq T; \quad &{}^{k^*}X_r^{t,x} = x \text{ if } s \leq t.
 \end{aligned} \tag{2.4.38}$$

Note that Because of the assumptions on b , σ , β , the process ${}^{k^*}X^{t,x}$ exists and is unique. Moreover, it satisfies the same estimates as in (2.3.2) because λ_{k^*} is just a truncation at the origin of λ , which integrates $(1 \wedge |e|^2)_{e \in E}$.

On the other hand, let us consider the following Markovian RBSDE with jumps

$$\left\{ \begin{array}{l}
 \text{(i)} \quad \mathbb{E} \left[\sup_{s \leq T} |{}^{k^*}Y_s^{t,x}|^2 + \int_s^T |{}^{k^*}Z_r^{t,x}|^2 dr + \int_s^T \|{}^{k^*}U_r^{t,x}\|_{\mathbb{L}^2(\lambda_{k^*})}^2 dr \right] < \infty \\
 \text{(ii)} \quad {}^{k^*}Y^{t,x} := ({}^{k^*}Y^{i,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad {}^{k^*}Z^{t,x} := ({}^{k^*}Z^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\
 \quad {}^{k^*}K^{t,x} := ({}^{k^*}K^{i,t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad {}^{k^*}U^{t,x} := ({}^{k^*}U^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda_{k^*})); \\
 \text{(iii)} \quad {}^{k^*}Y_s^{t,x} = g({}^{k^*}X_T^{t,x}) + \int_s^T f_{\mu_{k^*}}(r, {}^{k^*}X_r^{t,x}, {}^{k^*}Y_r^{t,x}, {}^{k^*}Z_r^{t,x}, {}^{k^*}U_r^{t,x})dr + {}^{k^*}K_T^{t,x} - {}^{k^*}K_s^{t,x} \\
 \quad \quad - \int_s^T \left\{ {}^{k^*}Z_r^{t,x} dB_r + \int_E U_r^{t,x}(e)\tilde{\mu}_{k^*}(dr, de) \right\}, \quad s \leq T; \\
 \text{(iv)} \quad {}^{k^*}Y_s^{i,t,x} \geq \ell(s, {}^{k^*}X_s^{t,x}) \text{ and } \int_0^T ({}^{k^*}Y_s^{i,t,x} - \ell(s, {}^{k^*}X_s^{t,x})) d({}^{k^*}K_s^{i,t,x}) = 0.
 \end{array} \right. \tag{2.4.39}$$

Finally, let us introduce the following functions $(f^{(i)})_{i=1,m}$ defined as follows: for $\forall (t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{L}_m^2(\lambda_{k^*})$,

$$\begin{aligned}
 f_{\mu_{k^*}}(t, x, y, z, \zeta) &= (f_{\mu_{k^*}}^{(i)}(t, x, y, z_i, \zeta_i))_{i=1,m} \\
 &:= \left(h^{(i)} \left(t, x, y, z, \int_E \gamma^i(t, x, e)\zeta_i(e)\lambda_{k^*}(de) \right) \right)_{i=1,m}.
 \end{aligned}$$

First, let us emphasize that this latter RBSDE is related to the filtration $(\mathcal{F}_s^{k^*})_{s \leq T}$, generated by the Brownian motion and the independent random measure μ_{k^*} . However, this point does not raise major issues because for any $s \leq T$, $\mathcal{F}_s^{k^*} \subset \mathcal{F}_s$ and by the relationship between μ and μ_{k^*} .

Next, by the properties of the functions b , σ , β and by the same opinions of Propositions 2.3.2 and 2.3.3, there exists an unique quadruple $({}^{k^*}Y^{t,x}, {}^{k^*}K^{t,x}, {}^{k^*}Z^{t,x}, {}^{k^*}U^{t,x})$ solving (2.4.39) and there also exists a function u^{k^*} from $[0, T] \times \mathbb{R}^k$ into \mathbb{R}^m of Π_g^c such that

$$\forall s \in [t, T], \quad {}^{k^*}Y^{t,x} := u^{k^*}(s, {}^{k^*}X_s^{t,x}), \quad \mathbb{P} - a.s. \tag{2.4.40}$$

Moreover, as in Proposition 2.4.2, there exists positive constants C and p which do not

depend on k^* such that for $\forall t, x, x'$,

$$|u^{k^*}(t, x) - u^{k^*}(t, x')| \leq C|x - x'| (1 + |x|^p + |x'|^p). \quad (2.4.41)$$

Finally, as λ_{k^*} is finite, then we have the following relationship between the process ${}^{k^*}U^{t,x} := ({}^{k^*}U^{i;t,x})_{i=1,m}$ and the deterministic functions $u^{k^*} := (u_i^{k^*})_{i=1,m}$ (see [37]): $\forall i = 1, \dots, m$, we have

$$\begin{aligned} {}^{k^*}U_s^{i;t,x}(e) &= u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x} + \beta(s, {}^{k^*}X_{s-}^{t,x}, e)) - u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x}), \\ &\quad d\mathbb{P} \otimes ds \otimes d\lambda_{k^*}\text{-a.e. on } \Omega \times [t, T] \times E. \end{aligned}$$

This is mainly due to the fact that ${}^{k^*}U^{t,x}$ belongs to $\mathbb{L}^1 \cap \mathbb{L}^2(d\lambda_s \otimes d\mathbb{P} \otimes d\lambda_{k^*})$ because $\lambda_{k^*}(E) < \infty$ and then we can split the stochastic integral w.r.t. $\tilde{\mu}_{k^*}$ in (2.4.39). Therefore, for all $i = 1, \dots, m$, we have

$$\begin{aligned} {}^{k^*}U_s^{i;t,x}(e)1_{\{|e| \geq \frac{1}{k^*}\}} &= (u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x} + \beta(s, {}^{k^*}X_{s-}^{t,x}, e)) - u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x}))1_{\{|e| \geq \frac{1}{k^*}\}}, \\ &\quad d\mathbb{P} \otimes ds \otimes d\lambda_{k^*}\text{-a.e. on } \Omega \times [t, T] \times E. \end{aligned} \quad (2.4.42)$$

Step 2 Convergence of the auxiliary processes

Let us now prove the following convergence result:

$$\begin{aligned} &\mathbb{E} \left[\sup_{s \leq T} |Y_s^{t,x} - {}^{k^*}Y_s^{t,x}|^2 + (K_T^{t,x} - {}^{k^*}K_T^{t,x})^2 + \int_0^T |Z_s^{t,x} - {}^{k^*}Z_s^{t,x}|^2 ds \right. \\ &\quad \left. + \int_0^T ds \int_E \lambda(de) |U_s^{t,x}(e) - {}^{k^*}U_s^{t,x}(e)1_{\{|e| \geq \frac{1}{k^*}\}}|^2 \right] \xrightarrow{k^* \rightarrow +\infty} 0, \end{aligned} \quad (2.4.43)$$

where $(Y^{t,x}, K^{t,x}, Z^{t,x}, U^{t,x})$ is solution of the RBSDE with jumps (2.3.3).

First, note that the following convergence result was established in [36]

$$\mathbb{E} \left[\sup_{s \leq T} |X_s^{t,x} - {}^{k^*}X_s^{t,x}|^2 \right] \xrightarrow{k^* \rightarrow +\infty} 0. \quad (2.4.44)$$

We now focus on (2.4.43). Note that we can apply Ito's formula, even if the RBSDEs are related to filtrations and Poisson random measures which are not the same, because of the followings:

- (i) $\mathcal{F}_s^{k^*} \subset \mathcal{F}_s$, $\forall s \leq T$;
- (ii) for any $s \leq T$, $\int_0^s \int_E U^{i;t,x}(e) \tilde{\mu}_{k^*}(dr, de) = \int_0^s \int_E U^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k^*}\}} \tilde{\mu}(dr, de)$ and then the first $(\mathcal{F}_s^{k^*})_{s \leq T}$ -martingale is also an $(\mathcal{F}_s)_{s \leq T}$ -martingale.

For $\forall s \in [0, T]$, we have

$$\begin{aligned}
 & \left| \vec{Y}_s^{t,x} - {}^{k^*} Y_s^{t,x} \right|^2 + \int_0^T \left| Z_s^{t,x} - {}^{k^*} Z_s^{t,x} \right|^2 ds + \sum_{s \leq r \leq T} ({}^{k^*} \Delta_r \vec{Y}_r^{t,x})^2 \\
 = & \left| g(X_T^{t,x}) - g({}^{k^*} X_T^{t,x}) \right|^2 + 2 \int_s^T \left(\vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} \right) \times {}^{k^*} \Delta f(r) dr \\
 & + 2 \int_s^T \left(\vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} \right) d({}^{k^*} \Delta K_r) - 2 \int_s^T \int_{\mathbb{E}} \left(\vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} \right) ({}^{k^*} \Delta U_r(e)) \tilde{\mu}(dr, de) \\
 & - 2 \int_s^T \left(\vec{Y}_r^{t,x} - {}^{k^*} \vec{Y}_r^{t,x} \right) ({}^{k^*} \Delta Z_r) dB_r;
 \end{aligned}$$

and taking expectation, we obtain, for $\forall s \in [t, T]$,

$$\mathbb{E} \left[\left| \vec{Y}_s^{t,x} - {}^{k^*} Y_s^{t,x} \right|^2 + \left| {}^{k^*} \Delta K_T \right|^2 \right] \quad (2.4.45)$$

$$\begin{aligned}
 & + \int_0^T \left\{ \left| Z_s^{t,x} - {}^{k^*} Z_s^{t,x} \right|^2 + \int_{\mathbb{E}} \left| U_s^{t,x} - {}^{k^*} U_s^{t,x} 1_{\{|e| \geq \frac{1}{k^*}\}} \right|^2 \lambda(de) \right\} ds \\
 \leq & \mathbb{E} \left[\left| g(X_T^{t,x}) - g({}^{k^*} X_T^{t,x}) \right|^2 + 2 \int_s^T \left(\vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} \right) \times {}^{k^*} \Delta f(r) dr \right] \quad (2.4.46)
 \end{aligned}$$

$$+ \mathbb{E} \left[\sup_{s \leq T} \left| {}^{k^*} \Delta \ell_s \right|^2 \right], \quad (2.4.47)$$

where the processes ${}^{k^*} \Delta X_r$, ${}^{k^*} \Delta Y_r$, ${}^{k^*} \Delta f(r)$, ${}^{k^*} \Delta K_r$, ${}^{k^*} \Delta Z_r$, ${}^{k^*} \Delta U_r$, and ${}^{k^*} \Delta \ell_r$ are defined as follows, for $\forall r \in [0, T]$,

$$\begin{aligned}
 {}^{k^*} \Delta f(r) := & \left(({}^{k^*} \Delta f^{(i)}(r))_{i=1,m} = (f^{(i)}(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, U_r^{i;t,x}) \right. \\
 & \left. - f_{k^*}^{(i)}(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{t,x}, {}^{k^*} U_r^{t,x}) \right)_{i=1,m},
 \end{aligned}$$

$${}^{k^*} \Delta X_r = X_r^{t,x} - {}^{k^*} X_r^{t,x}, \quad {}^{k^*} \Delta Y(r) = \vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} = (Y_r^{j;t,x} - {}^{k^*} Y_r^{j;t,x})_{j=1,m},$$

$${}^{k^*} \Delta K_r = K_r^{t,x} - {}^{k^*} K_r^{t,x}, \quad {}^{k^*} \Delta Z_r = Z_r^{t,x} - {}^{k^*} Z_r^{t,x}, \quad {}^{k^*} \Delta U_r = U_r^{t,x} - {}^{k^*} U_r^{t,x} 1_{\{|e| \geq \frac{1}{k^*}\}}$$

and

$${}^{k^*} \Delta \ell_r = \left(\ell(r, X_r^{t,x}) - \ell(r, {}^{k^*} X_r^{t,x}) \right).$$

Next, let us set, for $r \leq T$,

$$\begin{aligned}
 {}^{k^*} \Delta f(r) & = (f(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{t,x}, U_r^{t,x}) - f_{k^*}(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{t,x}, {}^{k^*} U_r^{t,x})) \\
 & = A(r) + B(r) + C(r) + D(r),
 \end{aligned}$$

where for any $i = 1, \dots, m$,

$$\begin{aligned}
 A(r) &= \left(h^{(i)} \left(r, X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right. \\
 &\quad \left. - h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right)_{i=1,m}, \\
 B(r) &= \left(h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, \vec{Y}_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right. \\
 &\quad \left. - h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right)_{i=1,m}, \\
 C(r) &= \left(h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right. \\
 &\quad \left. - h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right)_{i=1,m}, \\
 D(r) &= \left(h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \right. \\
 &\quad \left. - h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{i;t,x}, \int_E \gamma^i(r, {}^{k^*} X_r^{t,x}, e) {}^{k^*} U_r^{i;t,x}(e) \lambda_{k^*}(\mathrm{d}e) \right) \right)_{i=1,m}.
 \end{aligned}$$

By (2.4.44) and the fact of $g \in \mathcal{U}$ and $\ell \in \mathcal{U}$, we have

$$\mathbb{E} \left[\left| g(X_T^{t,x}) - g({}^{k^*} X_T^{t,x}) \right|^2 \right]_{k^* \rightarrow +\infty} \rightarrow 0 \quad (2.4.48)$$

and

$$\mathbb{E} \left[\sup_{s \leq T} \left| \ell(X_s^{t,x}) - \ell({}^{k^*} X_s^{t,x}) \right|^2 \right]_{k^* \rightarrow +\infty} \rightarrow 0. \quad (2.4.49)$$

Now, we will interest to $\mathbb{E} \left[\int_s^T \left(\vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} \right) \times {}^{k^*} \Delta f(r) \mathrm{d}r \right]$ for get (2.4.43).

By (2.2) and (2.2.4), for $\forall r \in [0, T]$, we have

$$\begin{aligned}
 |A(r)| &\leq C \left| X_r^{t,x} - {}^{k^*} X_r^{t,x} \right| \left(1 + \left| X_r^{t,x} \right|^p + \left| {}^{k^*} X_r^{t,x} \right|^p \right), \\
 |B(r)| &\leq C \left| \vec{Y}_r^{t,x} - {}^{k^*} Y_r^{t,x} \right| \text{ and } |C(r)| \leq \left| Z_r^{t,x} - {}^{k^*} Z_r^{t,x} \right|;
 \end{aligned} \quad (2.4.50)$$

where C is a constant. Finally let us deal with $D(r)$, which is more involved. First, note that $D(r) = (D_i(r))_{i=1,m}$, where

$$\begin{aligned}
 D_i(r) &= h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(\mathrm{d}e) \right) \\
 &\quad - h^{(i)} \left(r, {}^{k^*} X_r^{t,x}, {}^{k^*} Y_r^{t,x}, {}^{k^*} Z_r^{i;t,x}, \int_E \gamma^i(r, {}^{k^*} X_r^{t,x}, e) {}^{k^*} U_r^{i;t,x}(e) \lambda_{k^*}(\mathrm{d}e) \right).
 \end{aligned}$$

But as $h^{(i)}$ is Lipschitz w.r.t to the last component q , then,

$$\begin{aligned}
 |D(r)|^2 &\leq C \left\{ \int_E \left| \gamma^i(r, X_r^{t,x}, e) U_r^{i;t,x}(e) - \gamma^i(r, {}^{k^*} X_r^{t,x}, e) {}^{k^*} U_r^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k^*}\}} \right|^2 \lambda(de) \right\} \\
 &\leq C \left\{ \left\{ \int_E \left| \gamma^i(r, X_r^{t,x}, e) - \gamma^i(r, {}^{k^*} X_r^{t,x}, e) \right| \left| U_r^{i;t,x}(e) \right| \lambda(de) \right\}^2 \right. \\
 &\quad \left. + \left\{ \int_E \left| \gamma^i(r, X_r^{t,x}, e) \right| \left| U_r^{i;t,x}(e) - {}^{k^*} U_r^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k^*}\}} \right| \lambda(de) \right\}^2 \right\} \\
 &\leq C \left\{ |X_r^{t,x} - {}^{k^*} X_r^{t,x}| (1 + |X_r^{t,x}|^p + |{}^{k^*} X_r^{t,x}|^p) \left| \int_E U_r^{i;t,x}(e) \lambda(de) \right|^2 \right. \\
 &\quad \left. + C \int_E (1 \wedge |e|) \left| U_r^{i;t,x}(e) - {}^{k^*} U_r^{i;t,x}(e) 1_{\{|e| \geq \frac{1}{k^*}\}} \right|^2 \lambda(de), \right. \quad (2.4.51)
 \end{aligned}$$

and (2.4.45) become, by using the majorations obtained in (2.4.50) and (2.4.51),

$$\begin{aligned}
 &\mathbb{E} \left[\left| \vec{Y}_s^{t,x} - {}^{k^*} Y_s^{t,x} \right|^2 + \left| {}^{k^*} \Delta K_T \right|^2 \right] \quad (2.4.52) \\
 &+ \int_0^T \left\{ \left| Z_s^{t,x} - {}^{k^*} Z_s^{t,x} \right|^2 + \int_E \left| U_s^{t,x} - {}^{k^*} U_s^{t,x} 1_{\{|e| \geq \frac{1}{k^*}\}} \right|^2 \lambda(de) \right\} ds \\
 &\leq \mathbb{E} \left[\left| g(X_T^{t,x}) - g({}^{k^*} X_T^{t,x}) \right|^2 \right] + \mathbb{E} \left[\sup_{s \leq T} \left| \ell(X_s^{t,x}) - \ell({}^{k^*} X_s^{t,x}) \right|^2 \right] \\
 &+ C \mathbb{E} \left[\int_s^T \left| \vec{Y}_s^{t,x} - {}^{k^*} Y_s^{t,x} \right|^2 \right] + C \mathbb{E} \left[\int_0^T \left| X_r^{t,x} - {}^{k^*} X_r^{t,x} \right|^2 (1 + |X_r^{t,x}|^p + |{}^{k^*} X_r^{t,x}|^p)^2 dr \right] \\
 &+ C \mathbb{E} \left[\int_0^T dr \left\{ \left| X_r^{t,x} - {}^{k^*} X_r^{t,x} \right| (1 + |X_r^{t,x}|^p + |{}^{k^*} X_r^{t,x}|^p) \int_E U_r^{i;t,x}(e) \lambda(de) \right\}^2 \right]. \quad (2.4.53)
 \end{aligned}$$

The two first terms converge to 0 by (2.4.48) and (2.4.49).

For the fourth term, we have

$$\begin{aligned}
 &\mathbb{E} \left[\int_0^T \left| X_r^{t,x} - {}^{k^*} X_r^{t,x} \right|^2 (1 + |X_r^{t,x}|^p + |{}^{k^*} X_r^{t,x}|^p)^2 dr \right] \\
 &\leq \mathbb{E} \left[\sup_{r \leq T} \left| X_r^{t,x} - {}^{k^*} X_r^{t,x} \right|^2 \int_0^T (1 + |X_r^{t,x}|^p + |{}^{k^*} X_r^{t,x}|^p)^2 dr \right] \\
 &\leq \left\{ \mathbb{E} \left[\sup_{r \leq T} \left| X_r^{t,x} - {}^{k^*} X_r^{t,x} \right|^2 \right] \right\}^{\frac{1}{2}} \\
 &\quad \times \left\{ \mathbb{E} \left[\left(\int_0^T (1 + |X_r^{t,x}|^p + |{}^{k^*} X_r^{t,x}|^p)^2 \left| X_r^{t,x} - {}^{k^*} X_r^{t,x} \right|^2 dr \right)^2 \right] \right\}^{\frac{1}{2}}.
 \end{aligned}$$

The first factor in the right-hand side of this inequality goes to 0 when $k^* \rightarrow \infty$ because of (2.4.44) and the second factor is uniformly bounded by the uniform estimates (2.3.2) of $X^{t,x}$ and ${}^{k^*} X^{t,x}$.

Note also that the last term converge to 0 when $k^* \rightarrow \infty$, it is a consequence of (2.4.44), the fact that ${}^{k^*}X^{t,x}$ verifies estimates (2.3.2) uniformly, the Cauchy-Schwartz inequality (used twice) and finally (2.4.1) of Lemma 4.1. Then, by Gronwall's lemma, we deduce first that for any $s \leq T$,

$$\mathbb{E} \left[\left| \bar{Y}_s^{t,x} - {}^{k^*}Y_s^{t,x} \right|^2 \right]_{k^* \rightarrow +\infty} \rightarrow 0, \quad (2.4.54)$$

and taking $s = t$, we obtain $u^{k^*}(t, x) \xrightarrow{k^* \rightarrow +\infty} u(t, x)$. As $(t, x) \in [0, T] \times \mathbb{R}^k$ is arbitrary, then $u^{k^*} \xrightarrow{k^* \rightarrow +\infty} u$ pointwisely.

Next, going back to (2.4.52), taking the limit w.r.t k^* and using the uniform polynomial growth of u^{k^*} and the Lebesgue dominated convergence theorem as well, we obtain

$$\mathbb{E} \left[\int_t^T \int_E \left| U_s^{t,x} - {}^{k^*}U_s^{t,x} 1_{\{|e| \geq \frac{1}{k^*}\}} \right|^2 \lambda(de) ds \right]_{k^* \rightarrow +\infty} \rightarrow 0. \quad (2.4.55)$$

Step 3 Conclusion

First, note that by (2.4.41) and the pointwise convergence of $(u^{k^*})_{k^*}$ to u , if $(x_{k^*})_{k^*}$ is a sequence of \mathbb{R}^k , which converge to x , then $((u^{k^*}(t, x_{k^*}))_{k^*})$ converge to $u(t, x)$.

Now, let us consider a subsequence which we still denote by $\{k^*\}$, such that

$$\sup_{s \leq T} \left| X_s^{t,x} - {}^{k^*}X_s^{t,x} \right|^2_{k^* \rightarrow +\infty} \rightarrow 0,$$

\mathbb{P} -a.s. (and then $\left| X_{s-}^{t,x} - {}^{k^*}X_{s-}^{t,x} \right|_{k^* \rightarrow +\infty} \rightarrow 0$ since $\left| X_{s-}^{t,x} - {}^{k^*}X_{s-}^{t,x} \right| \leq \sup_{s \leq T} \left| X_s^{t,x} - {}^{k^*}X_s^{t,x} \right|^2$). By (2.4.44), this subsequence exists. As the mapping $x \mapsto \beta(t, x, e)$ is Lipschitz, then the sequence

$$\begin{aligned} \left({}^{k^*}U_s^{t,x}(e) 1_{\{|e| \geq \frac{1}{k^*}\}} \right)_{k^*} = & \left(\{ u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x} + \beta(s, {}^{k^*}X_{s-}^{t,x}, e)) \right. \\ & \left. - u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x}) \} 1_{\{|e| \geq \frac{1}{k^*}\}} \right)_{k^* \geq 1} \end{aligned} \quad (2.4.56)$$

$$\begin{aligned} \left(\{ u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x} + \beta(s, {}^{k^*}X_{s-}^{t,x}, e)) - u_i^{k^*}(s, {}^{k^*}X_{s-}^{t,x}) \} 1_{\{|e| \geq \frac{1}{k^*}\}} \right)_{k^* \geq 1} & \xrightarrow{k^* \rightarrow +\infty} \\ (u_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u_i(s, X_{s-}^{t,x})) & \quad (2.4.57) \\ d\mathbb{P} \otimes ds \otimes d\lambda - a.e. \quad \text{on } \Omega \times [t, T] \times E, & \end{aligned}$$

for any $i = 1, \dots, m$. Finally, from (2.4.55), we deduce that

$$U_s^{t,x}(e) = (u_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u_i(s, X_{s-}^{t,x})), \quad \text{on } \Omega \times [t, T] \times E, \quad (2.4.58)$$

which is the desired result.

2.5 Main Result

Theorem 2.5.1. *Under assumptions (H1), (H2), and (H3), IPDE (2.1.1) has unique solution, which is the m -tuple of functions $(u^i)_{i=1,m}$ defined in Proposition 2.3.3 by (2.3.4).*

Proof 2.5.1. Step 1 *Existence*

Assume that assumptions (H1), (H2) and (H3) are fulfilled, then the following multi-dimensional RBSDEs with jumps

$$\left\{ \begin{array}{l} \text{(i)} \quad \underline{Y}^{\vec{t},x} := (\underline{Y}^{i;t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \underline{Z}^{t,x} := (\underline{Z}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad \underline{K}^{t,x} := (\underline{K}^{i;t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad \underline{U}^{t,x} := (\underline{U}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} \quad \underline{Y}_s^{i;t,x} = g^i(X_T^{t,x}) + \underline{K}_T^{i;t,x} - \underline{K}_s^{i;t,x} - \int_s^T \underline{Z}^{i;t,x} dB_r - \int_s^T \int_{\mathbb{E}} \underline{U}_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\ \quad + \int_s^T h^{(i)} \left(r, X_r^{t,x}, \underline{Y}_r^{i;t,x}, \underline{Z}_r^{i;t,x}, \right. \\ \quad \left. \int_{\mathbb{E}} \gamma^i(t, X_r^{t,x}, e) \{ (u^i(t, X_{r-}^{t,x} + \beta(t, X_{r-}^{t,x}, e)) - u^i(t, X_{r-}^{t,x})) \} \lambda(de) \right) dr; \\ \text{(iii)} \quad \underline{Y}_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (\underline{Y}_s^{i;t,x} - \ell(s, X_s^{t,x})) d\underline{K}_s^{i;t,x} = 0 \end{array} \right. \quad (2.5.1)$$

has unique solution $(\underline{Y}, \underline{Z}, \underline{K}, \underline{U})$. Next, as for any $i = 1, \dots, m$, u^i belongs to \mathcal{U} , then by (2.3.4) Proposition 2.3.3, there exists a family of deterministic continuous functions of polynomial growth $(\underline{u}^i)_{i=1,m}$ that fact for any $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\forall s \in [t, T], \underline{Y}_s^{i;t,x} = \underline{u}^i(s, X_s^{t,x}).$$

Such that by the same proposition, the family $(\underline{u}^i)_{i=1,m}$ is a viscosity solution of the following system:

$$\left\{ \begin{array}{l} \min \left\{ \underline{u}^i(t, x) - \ell(t, x); -\partial_t \underline{u}^i(t, x) - b(t, x)^\top D_x \underline{u}^i(t, x) \right. \\ \quad \left. - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 \underline{u}^i(t, x)) - K_i \underline{u}^i(t, x) - h^{(i)}(t, x, (\underline{u}^j(t, x))_{j=1,m}), \right. \\ \quad \left. (\sigma^\top D_x \underline{u}^i)(t, x), B_i \underline{u}^i(t, x) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ \underline{u}^i(T, x) = g^i(x). \end{array} \right. \quad (2.5.2)$$

Now, know that the family $(\underline{u}^i)_{i=1,m}$ is a viscosity solution; our main objective is to found relation between $(\underline{u}^i)_{i=1,m}$ and $(u^i)_{i=1,m}$ which is defined in (2.3.4).

For this, let us consider the system of RBSDE with jumps

$$\left\{ \begin{array}{l} \text{(i)} \quad \vec{Y}^{t,x} := (Y^{i;t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad Z^{t,x} := (Z^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad \quad K^{t,x} := (K^{i;t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad U^{t,x} := (U^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} \quad Y_s^{i;t,x} = g^i(X_T^{t,x}) + K_T^{i;t,x} - K_s^{i;t,x} - \int_s^T Z^{i;t,x} dB_r - \int_s^T \int_E U_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\ \quad \quad \quad + \int_s^T h^{(i)}(r, X_r^{t,x}, Y_r^{i;t,x}, Z_r^{i;t,x}, \int_E \gamma^i(t, X_r^{t,x}, e) U_r^{i;t,x}(e) \lambda(de)) dr; \\ \text{(iii)} \quad Y_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i;t,x} - \ell(s, X_s^{t,x})) dK_s^{i;t,x} = 0. \end{array} \right. \quad (2.5.3)$$

By the uniqueness of the solution of the RBSDEs with jumps (1.5.15), then for $\forall s \in [t, T]$ and $\forall i \in \{1, \dots, m\}$, $\underline{Y}_s^{i;t,x} = Y_s^{i;t,x}$.

Therefore, $\underline{u}^i = u^i$, such that by (2.4.55), we obtain

$$U_s^{t,x}(e) = (u_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u_i(s, X_{s-}^{t,x})), \quad \text{on } \Omega \times [t, T] \times E,$$

which give the viscosity solution in the sense of Definition 1.5.2 (see [36]) by plugging (2.4.56) in $h^{(i)}$ of (2.5.2).

Step 2 Uniqueness

For the uniqueness, let $(\bar{u}^i)_{i=1,m}$ be another family of \mathcal{U} which is the solution viscosity of system (2.1.1) in the sense of Definition 1.5.2 and we consider RBSDE with jumps defined with \bar{u}^i .

$$\left\{ \begin{array}{l} \text{(i)} \quad \vec{\bar{Y}}^{t,x} := (\bar{Y}^{i;t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \vec{\bar{Z}}^{t,x} := (\bar{Z}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad \quad \bar{K}^{t,x} := (\bar{K}^{i;t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad \bar{U}^{t,x} := (\bar{U}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} \quad \bar{Y}_s^{i;t,x} = g^i(X_T^{t,x}) + \bar{K}_T^{i;t,x} - \bar{K}_s^{i;t,x} - \int_s^T \bar{Z}^{i;t,x} dB_r - \int_s^T \int_E \bar{U}_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\ \quad \quad \quad + \int_s^T h^{(i)} \left(r, X_r^{t,x}, \bar{Y}_r^{i;t,x}, \bar{Z}_r^{i;t,x}, \right. \\ \quad \quad \quad \left. \int_E \gamma^i(t, X_r^{t,x}, e) (\bar{u}_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - \bar{u}_i(s, X_{s-}^{t,x})) \lambda(de) \right) dr; \\ \text{(iii)} \quad \bar{Y}_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (\bar{Y}_s^{i;t,x} - \ell(s, X_s^{t,x})) d\bar{K}_s^{i;t,x} = 0. \end{array} \right. \quad (2.5.4)$$

By Feynman Kac formula, $\bar{u}^i(s, X_s^{t,x}) = Y_s^{i;t,x}$, where $Y_s^{i;t,x}$ satisfies the RBSDE with jumps (2.1.3) associated to IPDE (2.1.1).

Since that the RBSDE with jumps (2.5.2) has solution and it is unique by assuming that (H1), (H2), and (H3) are verified. By (2.3.4) Proposition 2.3.3, there exists a family of deterministic continuous functions of polynomial growth $(v^i)_{i=1,m}$ that fact for

any $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\forall s \in [t, T], \quad \bar{Y}_s^{i;t,x} = v^i(s, X_s^{t,x}).$$

Such that by the same proposition, the family $(v^i)_{i=1,m}$ is a viscosity solution of the following system:

$$\left\{ \begin{array}{l} \min \left\{ v^i(t, x) - \ell(t, x); -\partial_t v^i(t, x) - b(t, x)^\top D_x v^i(t, x) \right. \\ \left. - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 v^i(t, x)) - K_i v^i(t, x) - h^{(i)}(t, x, (v^j(t, x))_{j=1,m}), \right. \\ \left. (\sigma^\top D_x v^i)(t, x), B_i \bar{u}^i(t, x) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ u^i(T, x) = g^i(x). \end{array} \right. \quad (2.5.5)$$

By the uniqueness of solution of (2.5.3), \bar{u}^i is viscosity solution of (2.5.4), and by Proposition 2.3.3 $v^i = \bar{u}^i$ for $\forall i \in \{1, \dots, m\}$.

Now, for completing our proof, we show that on $\Omega \times [t, T] \times E$, $ds \otimes d\mathbb{P} \otimes d\lambda$ -a.e., for $\forall i \in \{1, \dots, m\}$, we have

$$\begin{aligned} \bar{U}_s^{i;t,x}(e) &= (v^i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - v^i(s, X_{s-}^{t,x})) \\ &= (\bar{u}_i(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - \bar{u}_i(s, X_{s-}^{t,x})). \end{aligned} \quad (2.5.6)$$

By Remark 3.4 in [36], let us consider $(x_{k^*})_{k^* \geq 1}$, a sequence of \mathbb{R}^k , which converges to $x \in \mathbb{R}^k$ and the two following RBSDE with jumps (adaptation is w.r.t. \mathcal{F}^{k^*}):

$$\left\{ \begin{array}{l} \text{(i)} \quad \bar{Y}^{k^*,t,x} := (\bar{Y}^{i;k^*,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \bar{Z}^{k^*,t,x} := (\bar{Z}^{i;k^*,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad \bar{K}^{k^*,t,x} := (\bar{K}^{i;k^*,t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad \bar{U}^{k^*,t,x} := (\bar{U}^{i;k^*,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} \quad \bar{Y}_s^{i;k^*,t,x} = g^i(X_T^{k^*,t,x}) + \bar{K}_T^{i;k^*,t,x} - \bar{K}_s^{i;k^*,t,x} - \int_s^T \bar{Z}_r^{i;k^*,t,x} dB_r \\ \quad - \int_s^T \int_E \bar{U}_r^{i;k^*,t,x}(e) \tilde{\mu}(dr, de) + \int_s^T h^{(i)}\left(r, X_r^{k^*,t,x}, \bar{Y}_r^{i;k^*,t,x}, \bar{Z}_r^{i;k^*,t,x}, \right. \\ \quad \left. \int_E \gamma^i(t, X_r^{k^*,t,x_{k^*}}, e)(\bar{u}_i(s, X_{s-}^{k^*,t,x_{k^*}} + \beta(s, X_{s-}^{k^*,t,x_{k^*}}, e)) \right. \\ \quad \left. - \bar{u}_i(s, X_{s-}^{k^*,t,x_{k^*}})) \lambda(de) \right) dr; \\ \text{(iii)} \quad \bar{Y}_s^{i;k^*,t,x_{k^*}} \geq \ell(s, X_s^{k^*,t,x_{k^*}}) \text{ and } \int_0^T (\bar{Y}_s^{i;k^*,t,x_{k^*}} - \ell(s, X_s^{k^*,t,x_{k^*}})) d\bar{K}_s^{i;k^*,t,x_{k^*}} = 0; \end{array} \right. \quad (2.5.7)$$

and

$$\left\{ \begin{array}{l}
 \text{(i)} \quad \vec{Y}^{\bar{k}^*, t, x_{k^*}} := (\bar{Y}^{i; k^*, t, x_{k^*}})_{i=1, m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \vec{Z}^{\bar{k}^*, t, x_{k^*}} := (\bar{Z}^{i; k^*, t, x_{k^*}})_{i=1, m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\
 \vec{K}^{\bar{k}^*, t, x_{k^*}} := (\bar{K}^{i; k^*, t, x_{k^*}})_{i=1, m} \in \mathcal{A}_c^2, \quad \vec{U}^{\bar{k}^*, t, x_{k^*}} := (\bar{U}^{i; k^*, t, x_{k^*}})_{i=1, m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\
 \text{(ii)} \quad \bar{Y}_s^{i; k^*, t, x_{k^*}} = g^i(X_T^{k^*, t, x_{k^*}}) + \bar{K}_T^{i; k^*, t, x_{k^*}} - \bar{K}_s^{i; k^*, t, x_{k^*}} - \int_s^T \bar{Z}^{i; k^*, t, x_{k^*}} dB_r \\
 - \int_s^T \int_E \bar{U}_r^{i; k^*, t, x_{k^*}}(e) \tilde{\mu}(dr, de) + \int_s^T h^{(i)}\left(r, X_r^{k^*, t, x_{k^*}}, \bar{Y}_r^{i; k^*, t, x_{k^*}}, \right. \\
 \left. \bar{Z}_r^{i; k^*, t, x_{k^*}}, \int_E \gamma^i(t, X_r^{k^*, t, x_{k^*}}, e)(\bar{u}_i(s, X_{s-}^{k^*, t, x_{k^*}}) \right. \\
 \left. + \beta(s, X_{s-}^{k^*, t, x_{k^*}}, e)) - \bar{u}_i(s, X_{s-}^{k^*, t, x_{k^*}}) \right) \lambda(de) dr; \\
 \text{(iii)} \quad \bar{Y}_s^{i; k^*, t, x_{k^*}} \geq \ell(s, X_s^{k^*, t, x_{k^*}}) \text{ and } \int_0^T (\bar{Y}_s^{i; k^*, t, x_{k^*}} - \ell(s, X_s^{k^*, t, x_{k^*}})) d\bar{K}_s^{i; k^*, t, x_{k^*}} = 0.
 \end{array} \right. \quad (2.5.8)$$

By Proof of Step 2 of Proposition 2.4.4, $(\bar{Y}^{i; k^*, t, x}, \bar{K}^{i; k^*, t, x}, \bar{Z}^{i; k^*, t, x}, \bar{U}^{i; k^*, t, x} 1_{\{|e| \geq \frac{1}{k^*}\}})_{k^*}$ converge to $(\bar{Y}^{i; t, x}, \bar{K}^{i; t, x}, \bar{Z}^{i; t, x}, \bar{U}^{i; t, x})$ in $\mathcal{S}^2(\mathbb{R}) \times \mathcal{A}_c^2 \times \mathbb{H}^2(\mathbb{R}^{\kappa \times d}) \times \mathbb{H}^2(\mathbb{L}^2(\lambda))$.

Let $((v_{i=1, m}^{k^*}))_{k^* \geq 1}$ be the sequence of continuous deterministic functions, such that for any $t \leq T$ and $s \in [t, T]$,

$$\bar{Y}_s^{i; k^*, t, x} = \bar{v}_i^{k^*}(s, {}^{k^*}X_s^{t, x}) \text{ and } \bar{Y}_s^{i; k^*, t, x_{k^*}} = \bar{v}_i^{k^*}(s, {}^{k^*}X_s^{t, x_{k^*}}) \quad \forall i = 1, \dots, m.$$

Such that by Proof of Proposition 2.4.4 in Step 1 and step 2, we have respectively:

(i) $\bar{U}_s^{i; k^*, t, x}(e) = (v^i(s, {}^{k^*}X_{s-}^{t, x} + \beta(s, {}^{k^*}X_{s-}^{t, x}, e)) - v^i(s, {}^{k^*}X_{s-}^{t, x})), ds \otimes d\mathbb{P} \otimes d\lambda_{k^*}$ -a.e on $[t, T] \times \Omega \times E$;

(ii) the sequence $((v_{i=1, m}^{k^*}))_{k^* \geq 1}$ converge to $v^i(t, x)$ by using (2.4.54).

So that $x_{k^*} \rightarrow_{k^*} x$, where we take the following estimation which is obtained by Ito's formula and the properties of $h^{(i)}$:

$$\mathbb{E} \left[\left| \vec{Y}_s^{\bar{k}^*, t, x_{k^*}} - Y_s^{k^*, t, x} \right|^2 + \left| \vec{K}_T^{\bar{k}^*, t, x_{k^*}} - K_T^{k^*, t, x} \right|^2 \right] \quad (2.5.9)$$

$$+ \int_0^T \left\{ \left| Z_s^{k^*, t, x_{k^*}} - Z_s^{k^*, t, x} \right|^2 + \int_E \left| U_s^{k^*, t, x_{k^*}} - U_s^{k^*, t, x} \right|^2 \lambda_{k^*}(de) \right\} ds \Big] \\
 \leq \mathbb{E} \left[\left| g({}^{k^*}X_T^{t, x_{k^*}}) - g({}^{k^*}X_T^{t, x}) \right|^2 \right] + \mathbb{E} \left[\sup_{s \leq T} \left| \ell({}^{k^*}X_s^{t, x_{k^*}}) - \ell({}^{k^*}X_s^{t, x}) \right|^2 \right] \quad (2.5.10)$$

$$+ C \mathbb{E} \left[\int_s^T \left| \vec{Y}_r^{\bar{k}^*, t, x_{k^*}} - \vec{Y}_r^{k^*, t, x} \right|^2 dr \right] \\
 + C \mathbb{E} \left[\int_0^T \left| {}^{k^*}X_r^{t, x_{k^*}} - {}^{k^*}X_r^{t, x} \right|^2 (1 + \left| {}^{k^*}X_r^{t, x_{k^*}} \right|^p + \left| {}^{k^*}X_r^{t, x} \right|^p) dr \right] \\
 + C \sum_{i=1, m} \mathbb{E} \left[\int_s^T \left| B_i \bar{u}^i(r, {}^{k^*}X_r^{t, x_{k^*}}) - B_i \bar{u}^i(r, {}^{k^*}X_r^{t, x}) \right|^2 dr \right]. \quad (2.5.11)$$

Next, using (2.4.48) and (2.4.49), the continuity of the function $(t, x) \mapsto B_i \bar{u}^i(t, x)$, and the fact of it belonging to Π_g , and in the other hand, the majoration of the fourth term of (2.4.52); we can use Gronwall's lemma, for $s = t, \forall i = 1, \dots, m$,

$$v_i^{k^*}(t, x_{k^*}) \longrightarrow_{k^*} v_i^{k^*}(t, x).$$

Therefore, by (i)–(ii), we have, for any $i = 1, \dots, m$,

$$\bar{U}_s^{i;t,x}(e) = (v^i(s, X_{s^-}^{t,x} + \beta(s, X_{s^-}^{t,x}, e)) - v^i(s, X_{s^-}^{t,x})) \quad (2.5.12)$$

$$ds \otimes d\mathbb{P} \otimes d\lambda\text{-a.e. in } [t, T] \times \Omega \times E, \quad \forall i \in \{1, \dots, m\}. \quad (2.5.13)$$

By this result, we can replace $(\bar{u}_i(s, X_{s^-}^{t,x} + \beta(s, X_{s^-}^{t,x}, e)) - \bar{u}_i(s, X_{s^-}^{t,x}))$ by $\bar{U}_s^{i;t,x}(e)$ in (2.5.6), we deduce that the quadruple $(\bar{Y}^{t,x}, \bar{K}^{t,x}, \bar{Z}^{t,x}, \bar{U}^{t,x})$ verifies that for $\forall i \in \{1, \dots, m\}$,

$$\left\{ \begin{array}{l} \text{(i)} \quad \bar{Y}^{\rightarrow t,x} := (\bar{Y}^{i;t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad \bar{Z}^{t,x} := (\bar{Z}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad \bar{K}^{t,x} := (\bar{K}^{i;t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad \bar{U}^{t,x} := (\bar{U}^{i;t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \text{(ii)} \quad \bar{Y}_s^{i;t,x} = g^i(X_T^{t,x}) + \bar{K}_T^{i;t,x} - \bar{K}_s^{i;t,x} - \int_s^T \bar{Z}_r^{i;t,x} dB_r - \int_s^T \int_E \bar{U}_r^{i;t,x}(e) \tilde{\mu}(dr, de) \\ \quad + \int_s^T h^{(i)}(r, X_r^{t,x}, \bar{Y}_r^{i;t,x}, \bar{Z}_r^{i;t,x}, \int_E \gamma^i(r, X_r^{t,x}, e) \bar{U}_r^{i;t,x} \lambda(de)) dr; \\ \text{(iii)} \quad \bar{Y}_s^{i;t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (\bar{Y}_s^{i;t,x} - \ell(s, X_s^{t,x})) d\bar{K}_s^{i;t,x} = 0. \end{array} \right. \quad (2.5.14)$$

It follows that

$$\bar{Y}^{i;t,x} = Y^{i;t,x}, \text{ for } \forall i \in \{1, \dots, m\}.$$

With the uniqueness of solution (2.5.4), we have $u^i = \bar{u}^i = v^i$, which means that the solution of (2.1.1) in the sense of Definition 1.5.2 is unique inside the class \mathcal{U} .

2.6 Extension

In this section, we will redefine the function $h^{(i)}$ as a function of $\|U^{i,t,x}\|_{\mathbb{L}^2(\lambda)}$ ($\forall i \in \{1, \dots, m\}$). And to show that the results of the previous section remain valid, let us consider the functions $f^{(i)}$, for any $i \in \{1, \dots, m\}$, defined by

$$\forall (t, x, y, z, \zeta) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{m+d} \times \mathbb{L}^2(\lambda)$$

and

$$f^{(i)}(t, x, y, z, \zeta) = h^{(i)}(t, x, y, z, \|\zeta\|_{\mathbb{L}^2(\lambda)}),$$

where the functions $(h^{(i)})_{i=1,m}$ are the same as that defined in Section 2.

We recall that the result of Theorem 2.5.1 is obtained by having mainly $U_s^{t,x}(e) = (u^i(s, X_{s^-}^{t,x} + \beta(s, X_{s^-}^{t,x}, e)) - u^i(s, X_{s^-}^{t,x}))$; this makes it possible to have Definition 1.5.2 by passing through a modification of the expression of $B_i u^i$ ($\forall i \in \{1, \dots, m\}$).

We show that $\|U_s^{i,t,x}(e)\|_{\mathbb{L}^2(\lambda)}^2 = \|(u^i(s, X_{s^-}^{t,x} + \beta(s, X_{s^-}^{t,x}, e)) - u^i(s, X_{s^-}^{t,x}))\|_{\mathbb{L}^2(\lambda)}^2$ and that in this case, $B_i u^i$ ($\forall i \in \{1, \dots, m\}$) is well.

Now, let $(t, x) \in [0, T] \times \mathbb{R}^d$ and let us consider the following m -dimensional RBSDE with jumps:

$$\left\{ \begin{array}{l} \text{(i)} \quad \vec{Y}^{t,x} := (Y^{i,t,x})_{i=1,m} \in \mathcal{S}^2(\mathbb{R}^m), \quad Z^{t,x} := (Z^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{R}^{m \times d}), \\ \quad K^{t,x} := (K^{i,t,x})_{i=1,m} \in \mathcal{A}_c^2, \quad U^{t,x} := (U^{i,t,x})_{i=1,m} \in \mathbb{H}^2(\mathbb{L}_m^2(\lambda)); \\ \quad \forall i \in \{1, \dots, m\} \quad Y_T^{i,t,x} = g^i(X_T^{t,x}) \text{ and;} \\ \text{(ii)} \quad dY_s^{i,t,x} = -f^{(i)}(s, X_s^{t,x}, (Y_s^{i,t,x})_{i=1,m}, Z_s^{i,t,x}, \|U_s^{t,x}(e)\|_{\mathbb{L}^2(\lambda)}) ds \\ \quad - dK_s^{i,t,x} + Z_s^{i,t,x} dB_s + \int_{\mathbb{E}} U_s^{i,t,x}(e) \tilde{\mu}(ds, de), \quad s \leq T; \\ \text{(iii)} \quad Y_s^{i,t,x} \geq \ell(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{i,t,x} - \ell(s, X_s^{t,x})) dK_s^{i,t,x} = 0. \end{array} \right. \quad (2.6.1)$$

Assuming that (H1), (H2), and (H3) are verified and by (2.3.4) in Proposition 2.3.3, there exists a family of deterministic continuous functions of polynomial growth $(w^i)_{i=1,m}$ that fact for any $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\forall s \in [t, T], \quad Y_s^{i,t,x} = w^i(s, X_s^{t,x}).$$

Such that by the same proposition, the family $(w^i)_{i=1,m}$ is a viscosity solution of the following system:

$$\left\{ \begin{array}{l} \min \left\{ w^i(t, x) - \ell(t, x); -\partial_t w^i(t, x) - b(t, x)^\top D_x w^i(t, x) \right. \\ \quad \left. - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) D_{xx}^2 w^i(t, x)) - K_i w^i(t, x) - h^{(i)}(t, x, (w^j(t, x))_{j=1,m}), \right. \\ \quad \left. (\sigma^\top D_x w^i)(t, x), B_i w^i(t, x) \right\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^k; \\ w^i(T, x) = g^i(x). \end{array} \right. \quad (2.6.2)$$

Indeed, using Lemma 2.4.1 and the fact that U^{i,t,x_k} converges to $U^{i,t,x}$ ($\forall i \in \{1, \dots, m\}$) when $x_k \rightarrow_k x$, we deduce that $\|U_s^{i,t,x_k}(e)\|_{\mathbb{L}^2(\lambda)} \rightarrow_k \|U_s^{i,t,x}(e)\|_{\mathbb{L}^2(\lambda)}$.

Moreover, from property of $h^{(i)}$ and the proof of Step 2 in Theorem 2.5.1 (viscosity solution uniqueness),

$$\|U_s^{i,t,x}(e)\|_{\mathbb{L}^2(\lambda)}^2 = \|(w^i(s, X_{s^-}^{t,x} + \beta(s, X_{s^-}^{t,x}, e)) - w^i(s, X_{s^-}^{t,x}))\|_{\mathbb{L}^2(\lambda)}^2,$$

from where

$$B_i w^i = \left\{ \int_E |(w^i(s, X_{s^-}^{t,x} + \beta(s, X_{s^-}^{t,x}, e)) - w^i(s, X_{s^-}^{t,x}))|^2 \lambda(de) \right\}^{\frac{1}{2}}.$$

Because of Corollary 2.4.3, we deduce that $B_i w^i$ ($\forall i \in \{1, \dots, m\}$) is well defined.

CHAPTER 3

BSDES DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH TIME DELAYED GENERATORS

This chapter deals with a class of backward stochastic differential equations driven by fractional Brownian motion (with Hurst parameter H greater than $1/2$) with time delayed generators. In this type of equation, a generator at time t can depend on the values of a solution in the past, weighted with a time delay function, for instance, of the moving average type. We establish an existence and uniqueness result of solutions for a sufficiently small time horizon or for a sufficiently small Lipschitz constant of a generator. The stochastic integral used throughout the chapter is the divergence operator type integral.

3.1 Introduction

Nonlinear Backward stochastic differential equations (BSDEs in short) have been first introduced by Pardoux and Peng [57, 58] in order to give a probabilistic interpretation for the solutions of semi-linear Partial differential equations (PDEs). Since this first result, it has been widely recognized that BSDEs provide a useful framework for formulating a lot of mathematical problems such as used in financial mathematics, optimal control, stochastic games and partial differential equations. We also mention that, following Pardoux and Peng [57], many papers were devoted to improving the results of Pardoux and Peng [57] by weakening the Lipschitz conditions on coefficients (for example, see Aidara [2], Wang and Huang [75], Aidara and Sow [6], Mao [53], Aidara and Sagna [4]). Based on the above important applications, specially in the field of Finance,

and optimal control, recently in [21], Delong and Imkeller introduced BSDEs with time delayed generators defined by

$$Y(t) = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z(s) dW_s \quad 0 \leq t \leq T, \quad (3.1.1)$$

where for all $t \in [0, T]$,

$$(Y_t, Z_t) = (Y(t+u), Z(t+u))_{-T \leq u \leq 0} \quad (3.1.2)$$

represents all the past of the process solution until t . An existence and uniqueness result under a small global Lipschitz K or a small horizon time T has been derived. This restrictive is inevitable, since authors with two examples, proved that this result can not be extended in the general case. However, for some special class of generators, existence and uniqueness result may still hold for arbitrary both time horizon and Lipschitz constant.

On the other hand, several authors investigate BSDEs with respect to fractional Brownian motion (fractional BSDEs in short). In [11], Bender gave one of the earliest result on fractional BSDEs. The author established an explicit solution of a class of linear fractional BSDEs with arbitrary Hurst parameter H . This is done essentially by means of solution of a specific linear parabolic PDE. There are two major obstacles depending on the properties of fractional Brownian motion: Firstly, the fractional Brownian motion is not a semimartingale except for the case of Brownian motion ($H = 1/2$), hence the classical Itô calculus based on semimartingales cannot be transposed directly to the fractional case. Secondly, there is no martingale representation theorem with respect to the fractional Brownian motion. Studing nonlinear fractional BSDEs, Hu and Peng [45] overcame successfully the second obstacle in the case $H > 1/2$ by means of the quasi-conditional expectation. The authors prove existence and uniqueness of the solution but with some restrictive assumptions on the generator. In this same spirit, Maticiuc and Nie [54] interesting in backward stochastic variational inequalities, improved this first result by weakening the required condition on the drift of the stochastic equation.

In this chapter, our aim is to generalize the result established in [22] to the following fractional BSDE:

$$Y(t) = \xi + \int_t^T f(s, \eta_s, Y_s, Z_s) ds - \int_t^T Z(s) dB_s^H \quad 0 \leq t \leq T, \quad (3.1.3)$$

where B^H is a fractional Brownian motion with Hurst parameter greater than $1/2$, η is a stochastic process given by $\eta_t = \eta_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dB_s^H$, and $(Y_t, Z_t) = (Y(t+u), Z(t+u))_{-T \leq u \leq 0}$. The stochastic integral used in above equation is the divergence-type integral. By the help of the fixed point principle, we prove we

establish our main result.

The chapter is organized as follows. In section 2, we recall some important results on fractional stochastic calculus. In section 3, we establish an existence and uniqueness theorem for a class of fractional BSDEs with time delayed generators.

3.2 Fractional stochastic calculus

Let Ω be a non-empty set, \mathcal{F} a σ -algebra of sets of Ω and \mathbb{P} a probability measure defined on \mathcal{F} . The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ defines a probability space and \mathbb{E} the mathematical expectation with respect to the probability measure \mathbb{P} .

Let us recall that, for $H \in (0, 1)$, a fractional Brownian motion $(B^H(t))_{t \geq 0}$ with Hurst parameter H is a continuous and centered Gaussian process with covariance

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0.$$

Denote $\phi(x) = H(2H - 1)|x|^{2H-2}$, $x \in \mathbb{R}$. Let ξ and η be measurable functions on $[0, T]$. Define $\langle \eta, \xi \rangle_t = \int_0^t \int_0^t \phi(u - v)\xi(u)\eta(v)dudv$ and $\|\xi\|_t^2 = \langle \xi, \xi \rangle_t$. Note that, for any $t \in [0, T]$, $\langle \eta, \xi \rangle_t$ is a Hilbert scalar product. Let \mathcal{H} be the completion of the set of measurable functions such that $\|\xi\|_t^2 < +\infty$.

For a polynomial function of k variables f , let \mathcal{P}_T be the set of all polynomials of fractional Brownian motion of the form

$$F(\omega) = f \left(\int_0^T \xi_1(t)dB_t^H(\omega), \int_0^T \xi_2(t)dB_t^H(\omega), \dots, \int_0^T \xi_k(t)dB_t^H(\omega) \right)$$

where $(\xi_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{H} such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$. The Malliavin derivative of $F \in \mathcal{P}_T$ is given by

$$D_s^H F = \sum_{i=1}^k \frac{\partial f}{\partial x_i} \left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \dots, \int_0^T \xi_k(t)dB_t^H \right) \xi_i(s), \quad 0 \leq s \leq T.$$

It is well known that the divergence operator $D^H : L^2(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\Omega, \mathcal{F}, \mathcal{H})$ is closable. Hence we can consider the space $\mathbb{D}_{1,2}$ is the completion of \mathcal{P}_T with the norm

$$\|F\|_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}\|D_s^H F\|_T^2.$$

We define $\mathbb{D}_s^H F = \int_0^T \phi(s - r)D_r^H F dr$ and denote by $\mathbb{L}_H^{1,2}$ the space of all stochastic

processes $F : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathcal{H}$ such that

$$\mathbb{E} \left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 ds dt \right) < +\infty.$$

We have the following (see [43, Proposition 6.25])

Theorem 3.2.1. *If $F \in \mathbb{L}_H^{1,2}$, then the Itô-Skorohod type stochastic integral $\int_0^T F_s dB_s^H$ exists in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and satisfies*

$$\mathbb{E} \left(\int_0^T F_s dB_s^H \right) = 0 \quad \text{and} \quad \mathbb{E} \left(\int_0^T F_s dB_s^H \right)^2 = \mathbb{E} \left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_t^H F_s \mathbb{D}_s^H F_t ds dt \right).$$

The following integration by parts given in [43, Theorem 11.1] will be useful in the sequel

Theorem 3.2.2. *Let $f_i(s), g_i(s)$, $i = 1, 2$, are in $\mathbb{D}_{1,2}$ and $\mathbb{E} \int_0^T (|f_i(s)| + |g_i(s)|) ds < +\infty$. Assume that $\mathbb{D}_t^H f_1(s)$ and $\mathbb{D}_t^H f_2(s)$ are continuously differentiable w.r.t $(s, t) \in [0, T] \times [0, T]$ for almost all $\omega \in \Omega$. Suppose that $\mathbb{E} \int_0^T \int_0^T |\mathbb{D}_t^H f_i(s)|^2 ds dt < +\infty$ and denote*

$$X_i(t) = \int_0^t g_i(s) ds + \int_0^t f_i(s) dB_s^H, \quad i = 1, 2.$$

Then we have for $0 \leq t \leq T$,

$$\begin{aligned} X_1(t)X_2(t) &= \int_0^t X_1(s)g_2(s)ds + \int_0^t X_1(s)f_2(s)dB_s^H + \int_0^t X_2(s)g_1(s)ds \\ &\quad + \int_0^t X_2(s)f_1(s)dB_s^H + \int_0^t \mathbb{D}_s^H X_1(s)f_2(s)ds + \int_0^t \mathbb{D}_s^H X_2(s)f_1(s)ds. \end{aligned}$$

Let us finish this section by giving an Itô formula for the divergence type integral (see [43, Theorem 10.3]).

Theorem 3.2.3. (see [26]) *Let μ and σ be stochastics differentials functions. If*

$$X(t) = X(0) + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dB_s^H, \quad 0 \leq t \leq T,$$

we have

$$\begin{aligned} \varphi(t, X(t)) &= \varphi(0, X(0)) + \int_0^t \frac{\partial \varphi}{\partial s}(s, X(s))ds + \int_0^t \frac{\partial \varphi}{\partial x}(s, X(s))\mu(X(s)) ds \quad (3.2.1) \\ &\quad + \int_0^t \frac{\partial \varphi}{\partial x}(s, X(s))\sigma(X(s))dB_s^H + \int_0^t \frac{\partial^2 \varphi}{\partial x^2}(s, X(s))\sigma(X(s))\mathbb{D}_s^H X_s ds. \end{aligned}$$

where

$$\begin{aligned} \mathbb{D}_s^H X_t &= \int_0^t \mu'(X(r)) \mathbb{D}_s^H X_r + \int_0^t \sigma'(X(r)) \mathbb{D}_s^H X_r \\ &\quad + H(2H-1) \int_0^t \sigma(X(s)) |s-r|^{2H-2} dr. \end{aligned}$$

Thanks to these preliminary results, we are now in position to investigate our main subject.

3.3 FBSDE with time delayed generators

3.3.1 Definitions and notations

Let us consider the following process

$$\eta_t = \eta_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dB_s^H \quad 0 \leq t \leq T,$$

where η_0 , b and σ satisfy:

- $\eta_0 \in \mathbb{R}$ is a given constant;
- $b, \sigma : [0, T] \rightarrow \mathbf{R}$ are continuous deterministic, σ is differentiable and $\sigma(t) \neq 0, t \in [0, T]$.

Note that, since

$$\|\sigma\|_t^2 = H(2H-1) \int_0^t \int_0^t |u-v|^{2H-2} \sigma(u)\sigma(v) dudv,$$

we have

$$\frac{d}{dt} (\|\sigma\|_t^2) = \sigma(t)\hat{\sigma}(t) > 0 \quad \text{where } \hat{\sigma}(t) = \int_0^t \phi(t-v)\sigma(v)dv, \quad 0 \leq t \leq T.$$

We introduce the following sets (where \mathbb{E} denotes the mathematical expectation with respect to the probability measure \mathbb{P}):

- $\mathcal{C}_{pol}^{1,2}([0, T] \times \mathbb{R})$ is the space of all $\mathcal{C}^{1,2}$ -functions over $[0, T] \times \mathbb{R}$, which together with their derivatives are of polynomial growth,
- $\mathcal{V}_{[0, T]} = \left\{ Y(\cdot) = \psi(\cdot, \eta); \quad \psi \in \mathcal{C}_{pol}^{1,2}([0, T] \times \mathbb{R}), \quad \frac{\partial \psi}{\partial t} \text{ is bounded, } t \in [0, T] \right\}$,
- $\tilde{\mathcal{V}}_T^{H, \beta}$ the completion of $\mathcal{V}_{[0, T]}$ under the following norm (where $\beta > 0$)

$$\left\| Y \right\|_{\tilde{\mathcal{V}}_T^{H, \beta}} = \left(\int_0^T t^{2H-1} \mathbb{E}[e^{\beta t} |Y(t)|^2] dt \right)^{1/2} = \left(\int_0^T t^{2H-1} \mathbb{E}[e^{\beta t} |\psi(t, \eta_t)|^2] dt \right)^{1/2}.$$

- $\mathcal{B}^2([0, T], \mathbb{R}) = \tilde{\mathcal{V}}_{[0, T]}^{1/2, \beta} \times \tilde{\mathcal{V}}_{[0, T]}^{H, \beta}$ is a Banach space with the norm

$$\left\| (Y, Z) \right\|_{\mathcal{B}^2}^2 = \left\| Y \right\|_{\tilde{\mathcal{V}}_T^{1/2, \beta}}^2 + \left\| Z \right\|_{\tilde{\mathcal{V}}_T^{H, \beta}}^2$$

- Finally let $\mathbf{L}_{-T}^2(\mathbb{R})$ denote the space of measurable functions $\varphi : [-T, 0] \rightarrow \mathbb{R}$ satisfying

$$\int_{-T}^0 |\varphi(t)|^2 dt < +\infty.$$

This space is to be understood as the space \mathcal{B}_v in [35] (see p.4) and the space \mathcal{B} in [77] (see p.7) Moreover in the hypothesis **(H2)** it appears clearly the difference between $y(t+u)$ and $y(t)$ or between $z(t+u)$ and $z(t)$ this is further reinforced in the proof of our main result

The goal of this paper is to study backward stochastic differential equations driven by fractional Brownian motion with time delayed generators which dynamics is described by

$$Y(t) = \xi + \int_t^T f(r, \eta_s, Y_s, Z_s) ds - \int_t^T Z(s) dB_s^H, \quad 0 \leq t \leq T, \quad (3.3.1)$$

where the generator f at time $s \in [0, T]$ depends on the past values of the solution denoted by $Y_s := Y(s+u)_{-T \leq u \leq 0}$ and $Z_s := Z(s+u)_{-T \leq u \leq 0}$. We always set $Z(t) = 0$ and $Y(t) = Y(0)$ for $t < 0$.

More precisely we establish an existence and uniqueness result for (3.3.1) under the following assumptions:

- **(H1)**: $\xi = h(\eta_T)$ for some function h with bounded derivative, $\mathbb{E} [e^{\beta T} |\xi|^2] < +\infty$;
- **(H2)**: the generator $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbf{L}_{-T}^2(\mathbb{R}) \times \mathbf{L}_{-T}^2(\mathbb{R}) \rightarrow \mathbb{R}$ is product measurable, \mathbb{F} -adapted and Lipschitz continuous in the sense that for some probability measure α on $([-T, 0] \times \mathcal{B}([-T, 0]))$

$$|f(t, x, y_t, z_t) - f(t, x, y'_t, z'_t)|^2 \leq K \int_{-T}^0 (|y(t+u) - y'(t+u)|^2 + |z(t+u) - z'(t+u)|^2) \alpha(du),$$

for $\mathbb{P} \otimes \lambda$ -a.e. $(\omega, t) \in \Omega \times [0, T]$ and for any $x \in \mathbb{R}$; $y, z, y', z' \in \mathbf{L}_{-T}^2(\mathbb{R})$.

- **(H3)**: $f(s, \cdot, \cdot, \cdot) = 0$ for $t < 0$.

Definition 3.3.1. A couple of processes (Y, Z) is called a solution to the fractional BSDE (3.3.1) with parameters (ξ, f) , if it satisfies eq.(3.3.1) with $(Y, Z) \in \mathcal{B}^2([0, T], \mathbb{R})$.

3.3.2 Existence and uniqueness of solution

Thanks to this result, we have the representation given in [5, Proposition 3.6].

Proposition 3.3.2. *Let $(Y_t, Z_t)_{0 \leq t \leq T}$ be a solution of the fractional BSDE (3.3.1).*

Then :

(i) *We have the stochastic representation*

$$\mathbb{D}_t^H Y_t = \frac{\hat{\sigma}(t)}{\sigma(t)} Z_t, \quad 0 \leq t \leq T,$$

(ii) *There exists a constant $M > 0$ such that*

$$\frac{t^{2H-1}}{M} \leq \frac{\hat{\sigma}(t)}{\sigma(t)} \leq M t^{2H-1}, \quad 0 \leq t \leq T.$$

Comment on the proposition 3.3.2:

- *The (i): $\mathbb{D}_t^H Y_t = \frac{\hat{\sigma}(t)}{\sigma(t)} Z_t$, $0 \leq t \leq T$, is proved for $t \in [0, T]$. But for the case of "time delayed", we have $t+u < 0 \Rightarrow Z(t+u) = 0$; so for $t+u \in [0, T]$, $\mathbb{D}_t^H Y_t = \frac{\hat{\sigma}(t)}{\sigma(t)} Z_t$,*
- *The (ii): $\frac{t^{2H-1}}{M} \leq \frac{\hat{\sigma}(t)}{\sigma(t)} \leq M t^{2H-1}$, $0 \leq t \leq T$. was obtained for $t \in [0, T]$ and its use is combined with $\mathbb{D}_t^H Y_t = \frac{\hat{\sigma}(t)}{\sigma(t)} Z_t$, (see [54]) which is valid for $t+u \in [0, T]$ so the (ii) remains valid for $t+u \in [0, T]$*

We are in position to prove our main result.

Theorem 3.3.3. *Assume that (H1), (H2) and (H3) hold. For a sufficiently small time horizon T or for a sufficiently small Lipschitz constant K , the backward stochastic differential equation (3.3.1) has a unique solution $(Y, Z) \in \mathcal{B}^2([0, T], \mathbb{R})$.*

Proof. To prove the existence and uniqueness of a solution, we follow the classical idea by constructing a Picard scheme and show its convergence. To this end, we consider now the sequence $(Y^n, Z^n)_{n \geq 0}$ given by

$$\begin{cases} Y^0(t) = Z^0(t) = 0; \\ Y^{n+1}(t) = \xi + \int_t^T f(s, \eta_s, Y_s^n, Z_s^n) ds - \int_t^T Z^{n+1}(t) dB_s^H, \quad 0 \leq t \leq T. \end{cases} \quad (3.3.2)$$

Given $(Y^n, Z^n)_{n \geq 0} \in \mathcal{B}^2([0, T], \mathbb{R})$, equation (3.3.2) has an unique solution $(Y^{n+1}, Z^{n+1})_{n \geq 0} \in \mathcal{B}^2([0, T], \mathbb{R})$ (see, [22] p.1517 and [46] p.57-58).

Let us define for a process $\delta \in \{Y, Z\}$, $n, m \geq 1$, $\bar{\delta}^{n+1} = \delta^{n+1} - \delta^n$.

It is easily seen that the pair of processes $(\bar{Y}^{n+1}(t), \bar{Z}^{n+1}(t))_{0 \leq t \leq T}$ solves the FBSDE

$$\bar{Y}^{n+1}(t) = \int_t^T \left(f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right) ds - \int_t^T \bar{Z}^{n+1}(s) dB_s^H, \quad 0 \leq t \leq T.$$

By the Itô formula, we have

$$\begin{aligned} d|\bar{Y}_t^{n+1}|^2 &= -2\bar{Y}_t^{n+1} \left(f(t, \eta_t, Y_t^n, Z_t^n) - f(t, \eta_t, Y_t^{n-1}, Z_t^{n-1}) \right) dt \\ &\quad + 2\bar{Y}_t^{n+1} Z_t^{n+1} dB_t^H + 2\mathbb{D}_t^H \bar{Y}_t^{n+1} \bar{Z}_t^{n+1} dt. \end{aligned}$$

Applying again Itô's formula to $e^{\beta t} |\bar{Y}^{n+1}(t)|^2$, (where β will be chosen later), we obtain for $n \geq 1$ and $0 \leq t \leq T$,

$$\begin{aligned} e^{\beta t} |\bar{Y}^{n+1}(t)|^2 &= 2 \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \left(f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right) ds \\ &\quad - 2 \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s) dB_s^H - 2 \int_t^T e^{\beta s} \mathbb{D}_s^H \bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s) ds \\ &\quad - \beta \int_t^T e^{\beta s} |\bar{Y}^{n+1}(s)|^2 ds. \end{aligned}$$

Using the fact that $(\bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s))_{t \leq s \leq T} \in \tilde{\mathcal{V}}_{[0,T]}^{1/2, \beta} \times \tilde{\mathcal{V}}_{[0,T]}^{H, \beta}$ and $\mathcal{V}_{[0,T]} \subset \mathbb{L}_H^{1,2}$ (see lemma 8 in [54]) which implies in particular $F_s = e^{\beta s} \bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s) \in \mathbb{L}_H^{1,2}$. Then by theorem 3.2.1, we have $\mathbb{E} \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s) dB_s^H = 0$.

Take mathematical expectation on both sides, then we have

$$\begin{aligned} \mathbb{E} \left(e^{\beta t} |\bar{Y}^{n+1}(t)|^2 \right) &= 2\mathbb{E} \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \left(f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right) ds \\ &\quad - 2\mathbb{E} \int_t^T e^{\beta s} \mathbb{D}_s^H \bar{Y}^{n+1}(s) \bar{Z}^{n+1}(s) ds - \beta \mathbb{E} \int_t^T e^{\beta s} |\bar{Y}^{n+1}(s)|^2 ds. \end{aligned} \quad (3.3.3)$$

It is known that $\mathbb{D}_s^H \bar{Y}^{n+1}(s) = \frac{\hat{\sigma}(s)}{\sigma(s)} \bar{Z}^{n+1}(s)$ and there exists $M > 0$ such that for all $t \in [0, T]$, $\frac{t^{2H-1}}{M} \leq \frac{\hat{\sigma}(t)}{\sigma(t)} \leq Mt^{2H-1}$.

Hence, we deduce from (3.3.3)

$$\begin{aligned} \mathbb{E} \left(e^{\beta t} |\bar{Y}^{n+1}(t)|^2 + \beta \int_t^T e^{\beta s} |\bar{Y}^{n+1}(s)|^2 ds + \frac{2}{M} \int_t^T e^{\beta s} s^{2H-1} |\bar{Z}^{n+1}(s)|^2 ds \right) \\ \leq 2\mathbb{E} \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \left(f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right) ds. \end{aligned} \quad (3.3.4)$$

Using the standard estimates $2ab \leq \lambda a^2 + b^2/\lambda$ for $\lambda > 0$, we deduce that

$$\begin{aligned} 2\mathbb{E} \int_t^T e^{\beta s} \bar{Y}^{n+1}(s) \left(f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right) ds &\leq \lambda \mathbb{E} \int_t^T e^{\beta s} |\bar{Y}^{n+1}(s)|^2 ds \\ &\quad + \frac{1}{\lambda} \mathbb{E} \int_t^T e^{\beta s} \left| f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right|^2 ds, \end{aligned}$$

which implies

$$\begin{aligned}
 & (\beta - \lambda) \mathbb{E} \int_0^T e^{\beta s} |\overline{Y}^{n+1}(s)|^2 ds + \frac{2}{M} \mathbb{E} \int_0^T e^{\beta s} s^{2H-1} |\overline{Z}^{n+1}(s)|^2 ds \\
 & \leq \frac{1}{\lambda} \mathbb{E} \int_0^T e^{\beta s} \left| f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right|^2 ds.
 \end{aligned} \tag{3.3.5}$$

Using the assumption **(H2)**, we have

$$\begin{aligned}
 & \frac{1}{\lambda} \mathbb{E} \int_0^T e^{\beta s} \left| f(s, \eta_s, Y_s^n, Z_s^n) - f(s, \eta_s, Y_s^{n-1}, Z_s^{n-1}) \right|^2 ds \\
 & \leq \frac{K}{\lambda} \mathbb{E} \int_0^T e^{\beta s} \left[\int_{-T}^0 \left(|\overline{Y}^n(s+u)|^2 + |\overline{Z}^n(s+u)|^2 \right) \alpha(du) \right] ds \\
 & \leq \frac{K}{\lambda} (I_1 + I_2),
 \end{aligned} \tag{3.3.6}$$

where $I_1 := \mathbb{E} \int_0^T e^{\beta s} \int_{-T}^0 |\overline{Y}^n(s+u)|^2 \alpha(du) ds$ and $I_2 := \mathbb{E} \int_0^T e^{\beta s} \int_{-T}^0 |\overline{Z}^n(s+u)|^2 \alpha(du) ds$.

Using Fubini's Theorem leads to

$$\begin{aligned}
 I_1 & := \mathbb{E} \int_0^T e^{\beta s} \int_{-T}^0 |\overline{Y}^n(s+u)|^2 \alpha(du) ds \leq \mathbb{E} \int_{-T}^0 e^{-\beta u} \left(\int_0^T e^{\beta(s+u)} |\overline{Y}^n(s+u)|^2 ds \right) \alpha(du) \\
 & \leq \int_{-T}^0 e^{-\beta u} \mathbb{E} \left(\int_u^{T+u} e^{\beta(v)} |\overline{Y}^n(v)|^2 dv \right) \alpha(du) \\
 & \leq \frac{e^{\beta T}}{\beta} \left\| \overline{Y}^n \right\|_{\widetilde{\mathcal{V}}_T^{\frac{1}{2}, \beta}}^2;
 \end{aligned} \tag{3.3.7}$$

and

$$\begin{aligned}
 I_2 & := \mathbb{E} \int_0^T e^{\beta s} \int_{-T}^0 |\overline{Z}^n(s+u)|^2 \alpha(du) ds \leq \mathbb{E} \int_{-T}^0 e^{-\beta u} \left(\int_0^T e^{\beta(s+u)} |\overline{Z}^n(s+u)|^2 ds \right) \alpha(du) \\
 & \leq \int_{-T}^0 e^{-\beta u} \mathbb{E} \left(\int_u^{T+u} \frac{1}{v^{2H-1}} e^{\beta(v)} v^{2H-1} |\overline{Z}^n(v)|^2 dv \right) \alpha(du) \\
 & \leq \int_{-T}^0 e^{-\beta u} \sup_{u \leq v \leq T+u} \left(\frac{1}{v^{2H-1}} \right) \alpha(du) \mathbb{E} \left(\int_u^{T+u} e^{\beta(v)} v^{2H-1} |\overline{Z}^n(v)|^2 dv \right) \\
 & \leq \int_{-T}^0 \frac{e^{-\beta u}}{u^{2H-1}} \alpha(du) \left\| \overline{Z}^n \right\|_{\widetilde{\mathcal{V}}_T^{H, \beta}}^2;
 \end{aligned} \tag{3.3.8}$$

where $\sup_{u \leq v \leq T+u} \left(\frac{1}{v^{2H-1}} \right) \leq \frac{1}{u^{2H-1}}$ (recall that $1/2 \leq H \leq 1$).

By using Hölder's inequality, we deduce that

$$\begin{aligned}
 I_2 &\leq \left(\int_{-T}^0 e^{-2\beta u} \alpha(du) \right)^{\frac{1}{2}} \left(\int_{-T}^0 u^{2(1-2H)} \alpha(du) \right)^{\frac{1}{2}} \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 \\
 &\leq \left(\frac{e^{2\beta T}}{2\beta} \right)^{\frac{1}{2}} \sqrt{T^{2(1-2H)}} \times \sqrt{T} \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 \\
 &\leq \frac{e^{\beta T} T^{1-2H} \times \sqrt{T}}{\sqrt{2\beta}} \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2; \text{ where } \sup_{-T \leq u \leq 0} (u^{2(1-2H)}) \leq T^{2(1-2H)}
 \end{aligned}$$

Putting pieces together, we deduce from (3.3.5) that

$$\begin{aligned}
 (\beta - \lambda) \left\| \bar{Y}^{n+1} \right\|_{\tilde{\mathcal{V}}_T^{\frac{1}{2},\beta}}^2 + \frac{2}{M} \left\| \bar{Z}^{n+1} \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 &\leq \frac{K e^{\beta T}}{\lambda \beta} \left\| \bar{Y}^n \right\|_{\tilde{\mathcal{V}}_T^{\frac{1}{2},\beta}}^2 + \frac{K e^{\beta T} T^{1-2H} \times \sqrt{T}}{\lambda \sqrt{2\beta}} \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 \\
 &\leq \frac{K C e^2}{\beta \lambda} \left(\left\| \bar{Y}^n \right\|_{\tilde{\mathcal{V}}_T^{\frac{1}{2},\beta}}^2 + \left\| \bar{Z}^n \right\|_{\tilde{\mathcal{V}}_T^{H,\beta}}^2 \right), \quad (3.3.9)
 \end{aligned}$$

where putting $T = \frac{2}{\beta}$ and $C = \max \{1, T^{1-2H}\}$.

Hence, if we choose $\lambda = \lambda_0 = \frac{1}{T}$ satisfying $\beta - \lambda_0 = \frac{2}{M}$, then we have

$$\left\| (\bar{Y}^{n+1}, \bar{Z}^{n+1}) \right\|_{\mathcal{B}^2}^2 \leq \frac{T^3 K C e^2}{2} \left\| (\bar{Y}^n, \bar{Z}^n) \right\|_{\mathcal{B}^2}^2. \quad (3.3.10)$$

For a sufficiently small T or for a sufficiently small K and assume that $c = \frac{T^3 K C e^2}{2} < 1$. Thus, we have, for $n \geq 2$,

$$\left\| (\bar{Y}^n, \bar{Z}^n) \right\|_{\mathcal{B}^2}^2 \leq c^{n-1} \left\| (\bar{Y}^1, \bar{Z}^1) \right\|_{\mathcal{B}^2}^2. \quad (3.3.11)$$

Since $\left\| (\bar{Y}^1, \bar{Z}^1) \right\|_{\mathcal{B}^2}^2$ is finite, it follows immediately that $(Y^n - Y^1, Z^n - Z^1)$ converge in the space $\mathcal{B}^2([0, T], \mathbb{R})$ to some (U, V) . We deduce that Y^n converge to Y (see [17], proof of theorem 6.3)

$$Y(t) = \xi + \int_t^T f(r, \eta_s, Y_s, Z_s) ds - \int_t^T Z(s) dB_s^H, \quad 0 \leq t \leq T.$$

By taking (Y_1, Z_1) and (Y_2, Z_2) two solutions of equation 3.1.3, with the same arguments for existence proof, repeating the above technique we obtain a uniqueness of the solution of FBSDE with time delayed generators on $[0, T]$. \square

3.3.3 Example

Consider the equation

$$Y(t) = \xi + \int_t^T K(Y(s-T) + Z(s-T)) ds - \int_t^T Z(s) dB_s^H, \quad 0 \leq t \leq T, \quad (3.3.12)$$

According to [44] p.12 the process $B_t^H = \int_t^T Z_H(t, s) dW_s$, $\forall H \in (0, 1)$ and $dB_t^H = Z_H(t, s) dW_t$. The equation (3.3.12) is equivalent to

$$Y(t) = \xi + \int_t^T K(Y(s-T) + Z(s-T)) ds - \int_t^T Z_H(t, s) dW_s, \quad 0 \leq t \leq T, \quad (3.3.13)$$

Using the notions of the previous section, (3.3.13) can be rewritten as

$$Y(t) = \xi + \int_t^T \int_{-T}^0 K(Y(s-T) + Z(s-T)) \mathbf{1}_{[0, \infty[} \alpha(du) ds - \int_t^T Z_H(t, s) dW_s, \quad 0 \leq t \leq T, \quad (3.3.14)$$

with Dirac measure α concentrated at the point T . Equation (3.3.14) is clearly equivalent to

$$Y(t) = \xi + K(T-t)Y(0) - \int_t^T Z_H(t, s) dW_s, \quad 0 \leq t \leq T, \quad (3.3.15)$$

For $t = 0$ we arrive at

$$(1 - TK)Y(0) = \xi - \int_0^T Z_H(0, s) dW_s, \quad 0 \leq t \leq T,$$

and Applying the mathematical expectation on both sides produces the condition

$$\mathbb{E}[\xi] = (1 - TK)Y(0)$$

We consider $TK < 1$. Define Z as the unique square integrable process from the martingale representation of $\xi \in \mathbb{L}^2(\mathbf{R})$ given by

$$\xi = \mathbb{E}[\xi] + \int_0^T Z_H(0, s) dW_s$$

and the process Y , according to (3.3.15), by

$$\begin{aligned} Y(t) &= \mathbb{E}[\xi] + (T-t)Y(0) + \int_0^t Z_H(t,s)dW_s \\ &= (1-tK)Y(0) + \int_0^t Z_H(t,s)dW_s \\ &= \frac{1-tK}{1-TK}\mathbb{E}[\xi] + \int_0^t Z_H(t,s)dW_s \end{aligned} \tag{3.3.16}$$

The pair $(Y, Z) \in \mathcal{B}^2([0, T], \mathbb{R})$ is the unique solution of (3.3.12) on $[0, T]$. In fact, suppose there were another solution (\tilde{Y}, \tilde{Z}) of (3.3.12) on $[0, T]$.

Writing the difference of (3.3.15) for the two solutions, and using $(1 - TK)Y(0) = \mathbb{E}[\xi] = (1 - TK)\tilde{Y}(0)$ for getting $Y(0) = \tilde{Y}(0)$, we obtain

$$\int_0^t (Z_H(t, s) - \tilde{Z}_H(t, s)) dW_s = 0 \quad \mathbb{P}.a.s$$

hence $\tilde{Z} = Z$, whence finally $\tilde{Y} = Y$.

CHAPTER 4

GENERALIZED ANTICIPATED BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS: WITH NON-LIPSCHITZ CONDITION

In this work, we study the anticipated generalized backward doubly stochastic differential equations (AGBDSDEs) whose coefficients satisfy a non-Lipschitz condition. An existence and uniqueness theorem is formulated and proved for this equation. A comparison theorem for this type of AGBDSDEs is also reported.

4.1 Introduction

Generalized Backward Stochastic differential equations (GBSDEs in short) has been considered in [59]. These stochastic equations can be seen as an extension of non linear BSDE which involves an integral with respect to an increasing process. They provide probabilistic representation of viscosity solution of both parabolic and elliptic PDE with Neumann boundary condition.

On the other hand, anticipated backward stochastic differential equations (ABSDEs in short) has been introduced by Peng and Yang [60]. The authors obtained the existence and uniqueness result for the solution of ABSDEs under Lipschitz conditions on the coefficients. After this result, many works on the theory of ABSDEs appeared. Among them, Zhu and Han [82] studied an anticipated backward doubly stochastic differential equations (ABDSDEs in short). In this work, the authors proved the existence and uniqueness of the solution for ABDSDEs under Lipschitz conditions. Recently, Wang

and Jiaxin [73] extended Peng and Yang's results to generalized anticipated backward doubly stochastic differential equations (GABDSDEs in short):

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) ds + \int_t^T g(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) dB_s \\ \quad + \int_t^T h(s, Y_s, Y_{s+\delta(s)}) d\Lambda_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \\ Y_t = \xi_t, \quad Z_t = \eta_t, \quad t \in [T, T + K]. \end{cases} \quad (4.1.1)$$

where $\delta, \zeta : [0, T] \rightarrow \mathbb{R}_+$ are continuous functions and $\{\eta_t\}_{0 \leq t \leq T}$ is a solution of a stochastic differential equation with reflection and $\{\Lambda_t\}_{0 \leq t \leq T}$ is an increasing process. This equation was first studied in [76] where under Lipschitz conditions, the author proved existence and uniqueness of solution. Inspired by the works of Borkowska [13] and Sow [5], we are interesting in this paper in extending this result with weak assumption on the drift. The key point is a iterated scheme on a suitable sequence.

The paper is organized as follows. In Section 2, The existence and uniqueness of solutions for GBDSDEs as well as a comparison theorem are obtained. Finally, in Section 3, we prove the existence and uniqueness of solution for GABDSDEs with coefficients satisfying rather weaker conditions. The comparison theorem for GABDSDEs are also studied.

4.2 Preliminaries

Let Ω be a non-empty set, \mathcal{F} a σ -algebra of sets of Ω and \mathbb{P} a probability measure defined on \mathcal{F} . The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ defines a probability space, which is assumed to be complete. For a fix real $0 < T \leq \infty$, we assume given two mutually independent processes:

- a ℓ -dimensional Brownian motion $(B_t)_{0 \leq t \leq T}$,
- a d -dimensional Brownian motion $(W_t)_{0 \leq t \leq T}$.

For any $t \in [0, T + K]$, we define:

$$\mathcal{F}_t = \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,T+K}^B, \quad 0 \leq t \leq T + K,$$

where for any process $\{\phi_t\}_{t \geq 0}$, $\mathcal{F}_{s,t}^\phi = \sigma\{\phi_r - \phi_s, s \leq r \leq t\} \vee \mathcal{N}$, $\mathcal{F}_t^\phi = \mathcal{F}_{0,t}^\phi$. \mathcal{N} denotes the class of \mathbb{P} -null sets of \mathcal{F} . Note that $(\mathcal{F}_t)_{0 \leq t \leq T}$ does not constitute a classical filtration.

Let Λ_t be a continuous, increasing and \mathcal{F}_t -adapted process on $[0, T + K]$ with $\Lambda_0 = 0$. For $k \in \mathbb{N}^*$ $\mu > 0$ and $\beta > 0$, we consider the following sets (where \mathbb{E} denotes the

mathematical expectation with respect to the probability measure \mathbb{P}):

- $L^2(\mathcal{F}_T, \mathbb{R}^k)$ the space of \mathcal{F}_T -measurable random variable such that

$$\mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 \right] < +\infty;$$

- $\mathcal{S}_{[0,T]}^2(\mathcal{F}, \mathbb{R}^k)$ the space of \mathcal{F}_t -adapted càdlàg processes

$$\Psi : [0, T] \times \Omega \longrightarrow \mathbb{R}^k, \quad \|\Psi\|_{\mathcal{S}^2}^2 = \mathbb{E} \left(\sup_{0 \leq t \leq T} e^{\mu t + \beta \Lambda_t} |\Psi_t|^2 \right) < +\infty;$$

- $\mathcal{M}_{[0,T]}^2(\mathcal{F}, \mathbb{R}^{k \times d})$ the space of \mathcal{F}_t -progressively measurable processes

$$\Psi : [0, T] \times \Omega \longrightarrow \mathbb{R}^k, \quad \|\Psi\|_{\mathcal{M}^2(\mathbb{R}^{k \times d})}^2 = \mathbb{E} \int_0^T e^{\mu t + \beta \Lambda_t} |\Psi_t|^2 dt < +\infty;$$

- $\mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k) = \mathcal{S}_{[0,T]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[0,T]}^2(\mathcal{F}, \mathbb{R}^{k \times d})$ endowed with the norm

$$\|(Y, Z)\|_{\mathcal{B}_{\mathcal{F}}^2([0,T], \mathbb{R}^k)}^2 = \|Y\|_{\mathcal{S}^2(\beta, \mathbb{R}^k)}^2 + \|Z\|_{\mathcal{M}^2(\beta, \mathbb{R}^{k \times d})}^2;$$

- \mathbf{S} be the set of all nondecreasing, continuous and concave function $\rho(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\rho(0) = 0$; one can find a pair of positive constants a and b such that

$$\rho(v) \leq a + bv \text{ and } \int_{0^+} \frac{du}{\rho(u)} = +\infty.$$

For notational simplicity we assume for all $x, y \in \mathbb{R}^k$ we denote by $|x|$ the Euclidean norm of x and denote by $\langle x, y \rangle$ the Euclidean inner product.

Given ξ a measurable real valued random variable and the functions

$$F, G : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow L^2(\mathcal{F}_t, \mathbb{R}^k) \quad \text{and} \quad H : \Omega \times [0, T] \times \mathbb{R}^k \rightarrow L^2(\mathcal{F}_t, \mathbb{R}^k).$$

Then, according to Theorem 2.1 in [14], the following GBDSDE with parameters (ξ, F, G, H, Λ) ,

$$Y_t = \xi_T + \int_t^T F(s, Y_s, Z_s) ds + \int_t^T G(s, Y_s, Z_s) dB_s + \int_t^T H(s, Y_s) d\Lambda_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (4.2.1)$$

admits a unique solution $(Y, Z) \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$ under the Lipschitz conditions.

For each $t \in [0, T]$, we assume given three functions:

$$\begin{aligned} f &: \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^{k \times d}) \rightarrow L^2(\mathcal{F}_t, \mathbb{R}^k), \\ g &: \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^{k \times d}) \rightarrow L^2(\mathcal{F}_t, \mathbb{R}^k), \\ h &: \Omega \times [0, T] \times \mathbb{R}^k \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \rightarrow L^2(\mathcal{F}_t, \mathbb{R}^k). \end{aligned}$$

We are interested in the AGBDSDEs

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) ds + \int_t^T g(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) dB_s \\ \quad + \int_t^T h(s, Y_s, Y_{s+\delta(s)}) d\Lambda_s - \int_t^T Z_s dW_s, \quad t \in [0, T] \\ Y_t = \xi_t, \quad Z_t = \eta_t, \quad t \in [T, T+K], \end{cases} \quad (4.2.2)$$

where K is a positive constant, $\delta, \zeta : [0, T] \rightarrow \mathbb{R}_+$ are continuous functions and for $\sigma(\cdot) = \delta(\cdot), \zeta(\cdot)$: such that:

(A1): there exists a constant $K \geq 0$ such that for all $t \in [0, T]$, $t + \sigma(t) \leq T + K$;

(A2): There exists $M \geq 0$ such that for $0 \leq t \leq T$ and non negative integrable function Φ ,

$$\int_t^T \Phi(s + \sigma(s)) ds \leq M \int_t^{T+K} \Phi(s) ds \quad \text{and} \quad \int_t^T \Phi(s + \sigma(s)) d\Lambda_s \leq M \int_t^{T+K} \Phi(s) d\Lambda_s.$$

Remark 4.2.1. If the assumption **(A2)** holds, then for all $t \in [0, T]$, $\beta > 0$, $\mu > 0$ and nonnegative integrable function $\Phi(\cdot)$, we have:

$$\int_t^T e^{\mu s + \beta \Lambda_s} \Phi(s + \sigma(s)) ds \leq M \int_t^{T+K} e^{\mu s + \beta \Lambda_s} \Phi(s) ds, \quad \sigma \in \{\delta, \zeta\}.$$

In fact,

$$\begin{aligned} \int_t^T e^{\mu s + \beta \Lambda_s} \Phi(s + \sigma(s)) ds &= \int_t^T e^{-\mu \sigma(s)} e^{\mu(s+\sigma(s)) + \beta \Lambda_s} \Phi(s + \sigma(s)) ds \\ &\leq \int_t^T e^{-\mu \sigma(s)} e^{\mu(s+\sigma(s)) + \beta \Lambda_{(s+\sigma(s))}} \Phi(s + \sigma(s)) ds \\ &\leq M \int_t^{T+K} e^{\mu s + \beta \Lambda_s} \Phi(s) ds. \end{aligned}$$

Similarly, we have

$$\int_t^T e^{\mu s + \beta \Lambda_s} \Phi(s + \sigma(s)) d\Lambda_s \leq M \int_t^{T+K} e^{\mu s + \beta \Lambda_s} \Phi(s) d\Lambda_s.$$

Let us recall the following result given in [\[\[73\], Theorem 1\]](#):

Proposition 4.2.1. *Assume that there exists $K > 0$ such that f and g are K -Lipschitz functions and h satisfies the monotonicity condition. Then Eq.(4.1.1) has a unique solution $(Y_t, Z_t) \in \mathcal{B}_{\mathcal{F}}^2([0, T + K], \mathbb{R}^k)$.*

Before investigating the solution for the AGBDSDE (4.1.1) and establishing its comparison theorem, we give some results of GBDSDE (4.2.1), which will play a key role in what follows.

4.3 Generalized BDSDEs

In this section, we are interested in the following generalized backward doubly stochastic differential equations

$$Y_t = \xi_T + \int_t^T F(s, Y_s, Z_s) ds + \int_t^T G(s, Y_s, Z_s) dB_s + \int_t^T H(s, Y_s) d\Lambda_s - \int_t^T Z_t dW_s, \quad 0 \leq t \leq T. \quad (4.3.1)$$

We assume that there exists $\rho \in \mathbf{S}$ such that ξ, F, G and H satisfy assumptions (H1):

(H1.1) *There exists $C > 0$, for all $t \in [0, T]$, $(t, y, z), (t, y', z') \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$, we have*

$$\begin{aligned} \left| F(t, y, z) - F(t, y', z') \right|^2 \vee \left| G(t, y, z) - G(t, y', z') \right|^2 &\leq C \left(\rho (|y - y'|^2) + |z - z'|^2 \right), \\ \left\langle y - y', H(t, y) - H(t, y') \right\rangle &\leq C |y - y'|^2. \end{aligned}$$

(H1.2) *The integrability condition holds*

$$\mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + \int_0^T e^{\mu s + \beta \Lambda_s} |F(s, 0, 0)|^2 ds + \int_0^T e^{\mu s + \beta \Lambda_s} |G(s, 0, 0)|^2 ds \right] < +\infty.$$

Definition 4.3.1. *A pair of processes (Y, Z) is called a solution to GBDSDEs (4.3.1) if $(Y_t, Z_t) \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$ and satisfies eq.(4.3.1).*

4.3.1 Existence and uniqueness of solution

To prove the existence of a solution, let us consider the sequence of processes $(Y_t^n, Z_t^n)_{n \geq 0}$ given by

$$\begin{cases} Y_t^0 = 0; \forall 0 \leq t \leq T, \\ Y_t^n = \xi_T + \int_t^T F(s, Y_s^{n-1}, Z_s^n) ds + \int_t^T G(s, Y_s^{n-1}, Z_s^n) dB_s + \int_t^T H(s, Y_s^n) d\Lambda_s \\ \quad - \int_t^T Z_s^n dW_s. \end{cases} \quad (4.3.2)$$

In order to obtain the existence and uniqueness result, we need the following result

Lemma 4.3.1. *Assume that the assumptions **(H1.1)** and **(H1.2)** are true and let $\xi_T \in L^2(\mathcal{F}_T, \mathbb{R}^k)$. There exists a positive constant $C_{4.3.1}$ such that*

$$\sup_{n \geq 0} \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 \right] \leq C_{4.3.1} (1 + \mathbb{E}[X_T]), \quad t \in [0, T], \quad (4.3.3)$$

where

$$X_T = e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + \int_0^T e^{\mu s + \beta \Lambda_s} (|F(s, 0, 0)|^2 + |G(s, 0, 0)|^2) ds. \quad (4.3.4)$$

Proof. Itô's formula applied to $e^{\mu t + \beta \Lambda_t} |Y_t^n|^2$, we obtain

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 (\mu ds + \beta d\Lambda_s) \right] \\ &= \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, F(s, Y_s^{n-1}, Z_s^n) \rangle ds \right] \\ &+ 2\mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, H(s, Y_s^n) \rangle d\Lambda_s \right. \\ & \quad \left. + \int_t^T e^{\mu s + \beta \Lambda_s} \|G(s, Y_s^{n-1}, Z_s^n)\|^2 ds \right]. \end{aligned} \quad (4.3.5)$$

Using the inequality $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ (where $\varepsilon > 0$ will be chosen later), we deduce from assumptions **(H1.1)** and **(H1.2)**:

- $2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, F(s, Y_s^{n-1}, Z_s^n) \rangle ds \leq \lambda \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 ds$
 $+ \frac{1}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |F(s, 0, 0)|^2 ds$
 $+ \frac{C}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\rho(|Y_s^{n-1}|^2) + |Z_s^n|^2) ds;$
- $\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \|G(s, Y_s^{n-1}, Z_s^n)\|^2 ds \leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |G(s, 0, 0)|^2 ds$
 $+ 2C \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\rho(|Y_s^{n-1}|^2) + |Z_s^n|^2) ds;$
- $2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, H(s, Y_s^n) \rangle d\Lambda_s \leq 2C \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 d\Lambda_s.$

Putting pieces together, we deduce from (4.3.5) that

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 \left((\mu - \lambda) ds + (\beta - 2C) d\Lambda_s \right) \right] \\ & \leq C \left(\frac{2\lambda + 1}{\lambda} \right) \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} \left(\rho(|Y_s^{n-1}|^2) + |Z_s^n|^2 \right) ds \right] + C_0 \mathbb{E}[X_T], \quad t \in [0, T]; \end{aligned}$$

where X_T is given by eq.(4.3.4) and C_0 is a positive constant.

Choose $\lambda = \lambda_0$ such that $\lambda_0 = \frac{2c}{1-4C} \leq \frac{2\mu-1}{2}$ and β such that $\beta \geq 2C + \frac{1}{2}$. Hence there exists a positive constant $C' > 0$ (whose value may change from line to line) such that

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \frac{1}{2} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 (ds + d\Lambda_s) + \frac{1}{2} \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds \right] \\ & \leq C' \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \rho(|Y_s^{n-1}|^2) ds + C' \mathbb{E}[X_T], \quad t \in [0, T]; \end{aligned}$$

In particular, putting $q_n(t) = \sup_{n \in \mathbb{N}} \mathbb{E}[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2]$, we have

$$q_n(t) \leq C'(1 + \mathbb{E}[X_T]) + C' \int_t^T q_n(s) ds, \quad t \in [0, T].$$

Gronwall's inequality yields

$$\sup_{n \in \mathbb{N}} \mathbb{E}[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2] \leq C_{4.3.1} (1 + \mathbb{E}[X_T]), \quad t \in [0, T].$$

This immediately gives 4.3.3. □

The following theorem is the main result of this section. In this theorem, with the help of Lemma 4.3.1, we can now prove our existence and uniqueness theorem.

Theorem 4.3.1. *Assume that the assumptions (H1.1) and (H1.2) are true and let $\xi_T \in L^2(\mathcal{F}_T, \mathbb{R}^k)$. Then the GBDSDE (4.3.1) has a unique solution $(Y_t, Z_t)_{0 \leq t \leq T} \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$.*

Proof. (i) Existence. We consider the sequence defined in (4.3.2). For a process $\varrho \in \{Y, Z\}$ and $n \in \mathbb{N}, m \in \mathbb{N}$, we define $\bar{\varrho}_t^{n,m} = \varrho_t^n - \varrho_t^m$, and for $\Psi \in \{F, G\}$

$$\begin{aligned}\Delta\Psi^{(n,m)}(s) &= \Psi(s, Y_s^{n-1}, Z_s^n) - \Psi(s, Y_s^{m-1}, Z_s^m) \\ \Delta H^{(n,m)}(s) &= H(s, Y_s^n) - H(s, Y_s^m).\end{aligned}$$

Then, it is obvious that $(\bar{Y}_t^{n,m}, \bar{Z}_t^{n,m})_{0 \leq t \leq T}$ solves the following Generalized BDSDE

$$\bar{Y}_t^{n,m} = \int_t^T \Delta f^{(n,m)}(s) ds + \int_t^T \Delta g^{(n,m)}(s) dB_s + \int_t^T \Delta h^{(n,m)}(s) d\Lambda_s - \int_t^T \bar{Z}_s^{n,m} dW_s, \quad t \in [0, T]. \quad (4.3.6)$$

By the same computations as in Lemma 4.3.1, we have

$$\begin{aligned}\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 \right] + \frac{1}{2} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s^{n,m}|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s^{n,m}|^2 ds \right] \\ \leq C' \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \rho(|\bar{Y}_s^{n-1, m-1}|^2) ds, \quad t \in [0, T]\end{aligned} \quad (4.3.7)$$

Set

$$q(t) = \lim_{n, m \rightarrow \infty} \sup \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 \right], \quad t \in [0, T].$$

Then by (4.3.7), Fatou's lemma and the fact that ρ is an increasing continuous concave function, we get

$$q(t) \leq C' \int_t^T \rho(q(s)) ds, \quad t \in [0, T].$$

Therefore, we can use Bihari's inequality to get $q(t) = 0$, i.e.

$$\lim_{n, m \rightarrow \infty} \sup \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 \right] = 0, \quad t \in [0, T]$$

So, from inequality (4.3.7), we obtain for $t \in [0, T]$,

$$\lim_{n, m \rightarrow \infty} \mathbb{E} \left(e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s^{n,m}|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s^{n,m}|^2 ds \right) = 0,$$

Then there exists $(Y_t, Z_t)_{0 \leq t \leq T} \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(e^{\mu t + \beta \Lambda_t} |Y_t^n - Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n - Y_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n - Z_s|^2 ds \right) = 0.$$

Letting $n \rightarrow +\infty$, we obtain

$$Y_t = \xi_T + \int_t^T F(s, Y_s, Z_s) ds + \int_t^T G(s, Y_s, Z_s) dB_s + \int_t^T H(s, Y_s) d\Lambda_s - \int_t^T Z_s dW_s, \\ \forall t \in [0, T].$$

This shows that $(Y_t, Z_t)_{0 \leq t \leq T} \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$ solves Generalized BDSDE (4.3.1). The proof of existence is complete.

(ii) Uniqueness. Let $(Y^i, Z^i) \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R}^k)$, $i = 1, 2$ be any two solutions. Define $\bar{Y}_t = Y_t^1 - Y_t^2$, $\bar{Z}_t = Z_t^1 - Z_t^2$ and for a function $\Psi \in \{F, G\}$,

$$\Delta\Psi(s) = \Psi(s, Y_s^1, Z_s^1) - \Psi(s, Y_s^2, Z_s^2) \quad \text{and} \quad \Delta H(s) = H(s, Y_s^1) - H(s, Y_s^2).$$

We consider the following equation

$$\bar{Y}_t = \int_t^T \Delta F(s) ds + \int_t^T \Delta G(s) dB_s + \int_t^T \Delta H(s) d\Lambda_s - \int_t^T \bar{Z}_s dW_s, \quad t \in [0, T]. \quad (4.3.8)$$

By the same computations as in Lemma 4.3.1, we obtain

$$\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 \right] + \frac{1}{2} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s|^2 ds \right] \leq \\ C' \mathbb{E} \int_t^T \rho(e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2) ds. \quad (4.3.9)$$

For $t \in [0, T]$, this leads to

$$\mathbb{E} [e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2] \leq C' \mathbb{E} \int_t^T \rho(e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2) ds,$$

Using Fubini's Theorem and Jensen's inequality leads to

$$\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 \right] \leq C' \int_t^T \rho \left(\mathbb{E} \left[e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 \right] \right) ds, \quad t \in [0, T].$$

Then we can use Bihari's inequality to obtain $\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 \right] = 0$, for all $t \in [0, T]$. This implies $\bar{Z}_t = 0$, for all $t \in [0, T]$. \square

4.3.2 Comparison Theorem for GBDSDEs

In this subsection, our objective is to establish the comparison theorem for the following 1-dimensional Generalized BDSDEs under non-Lipschitz coefficients, for $t \in [0, T]$

$$Y_t^{(i)} = \xi_T^i + \int_t^T F^i(s, Y_s^{(i)}, Z_s^{(i)}) ds + \int_t^T G^i(s, Y_s^{(i)}, Z_s^{(i)}) dB_s + \int_t^T H^i(s, Y_s^{(i)}) d\Lambda_s - \int_t^T Z_s^{(i)} dW_s. \quad (4.3.10)$$

Then by Theorem 4.3.1, Eq.(4.3.10) has a unique solution $(Y_t^{(i)}, Z_t^{(i)})_{0 \leq t \leq T} \in \mathcal{B}_{\mathcal{F}}^2([0, T], \mathbb{R})$.

Let us give a comparison theorem of GBDSDEs under non-Lipschitz assumptions, which will play a key role in what follows.

Theorem 4.3.2. Let $(Y^{(1)}, Z^{(1)})$ and $(Y^{(2)}, Z^{(2)})$ be solutions of 1-dimensional GBDSDEs (4.3.10) respectively. Assume that F^1, F^2, G^1, G^2, H^1 and H^2 satisfy **(H1)** and $\xi_T^{(1)}, \xi_T^{(2)} \in L^2(\mathcal{F}_T, \mathbb{R})$. We suppose that:

- (i) $\xi_T^{(1)} \geq \xi_T^{(2)}$, a.s.;
- (ii) For all $t \in [0, T]$, we have
 - $F^1(t, Y^{(1)}, Z^{(1)}) \geq F^2(t, Y^{(1)}, Z^{(1)})$ or $F^1(t, Y^{(2)}, Z^{(2)}) \geq F^2(t, Y^{(2)}, Z^{(2)})$, a.s.;
 - $G^1(t, Y^{(1)}, Z^{(1)}) \geq G^2(t, Y^{(1)}, Z^{(1)})$ or $G^1(t, Y^{(2)}, Z^{(2)}) \geq G^2(t, Y^{(2)}, Z^{(2)})$, a.s.;
 - $H^1(t, Y^{(1)}) \geq H^2(t, Y^{(1)})$ or $H^1(t, Y^{(2)}) \geq H^2(t, Y^{(2)})$, a.s.;

Then

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad a.s., \quad t \in [0, T].$$

Proof. For all $t \in [0, T]$, we denote

$$\begin{aligned} \widehat{Y}_t &= Y_t^{(2)} - Y_t^{(1)}, \quad \widehat{Z}_t = Z_t^{(2)} - Z_t^{(1)}, \quad \widehat{\xi}_t = \xi_t^{(2)} - \xi_t^{(1)}; \\ \widehat{F}(t) &= F^2(t, Y_t^{(2)}, Z_t^{(2)}) - F^1(t, Y_t^{(1)}, Z_t^{(1)}), \quad \widehat{G}(t) = G^2(t, Y_t^{(2)}, Z_t^{(2)}) - G^1(t, Y_t^{(1)}, Z_t^{(1)}) \\ \text{and } \widehat{H}(t) &= H^2(t, Y_t^{(2)}) - H^1(t, Y_t^{(1)}). \end{aligned}$$

Then $(\widehat{Y}, \widehat{Z})$ satisfies

$$\widehat{Y}_t = \widehat{\xi}_T + \int_t^T \widehat{F}(s) ds + \int_t^T \widehat{G}(s) dB_s + \int_t^T \widehat{H}(s) d\Lambda_s - \int_t^T \widehat{Z}_s dW_s, \quad t \in [0, T]. \quad (4.3.11)$$

Apply the extension of the Itô formula to $e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2$, we have

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 (\mu ds + \beta d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} |\widehat{Z}_s|^2 ds \right] \\ &= 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \widehat{F}(s) ds + 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \widehat{H}(s) d\Lambda_s \\ & \quad + \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} |\widehat{G}(s)|^2 ds. \end{aligned} \quad (4.3.12)$$

In view of **(H1)**, linearization, Young's inequality and Jensen's inequality, for any $\lambda > 0$, we have:

- $$\begin{aligned} 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+| \widehat{F}(s) ds &\leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+| \left(F^2(t, Y_t^{(2)}, Z_t^{(2)}) \right. \\ & \quad \left. - F^1(t, Y_t^{(1)}, Z_t^{(1)}) \right) ds \\ &\leq \frac{C}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 ds + \lambda \int_t^T \rho \left(\mathbb{E} \left[e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \right] \right) ds \\ & \quad + \lambda \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} |\widehat{Z}_s|^2 ds; \end{aligned}$$
- $$\begin{aligned} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} |\widehat{G}(s)|^2 ds &\leq C \int_t^T \rho \left(\mathbb{E} \left[e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \right] \right) ds \\ & \quad + C \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} |\widehat{Z}_s|^2 ds; \end{aligned}$$
- $$\begin{aligned} 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+| \widehat{H}(s) d\Lambda_s &\leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \left(H^1(t, Y_t^{(2)}) - H^1(t, Y_t^{(1)}) \right) d\Lambda_s \\ &\leq C \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 d\Lambda_s. \end{aligned}$$

Then, thanks to the above inequalities, we obtain

$$\begin{aligned} & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \left(\left(\mu - \frac{C}{\lambda} \right) ds + (\beta - C) d\Lambda_s \right) \right] \\ &+ (1 - c - \lambda) \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} |\widehat{Z}_s|^2 ds \leq (C + \lambda) \int_t^T \rho \left(\mathbb{E} \left[e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \right] \right) ds \end{aligned} \quad (4.3.13)$$

Choosing $\beta \geq C$ and $\lambda = \lambda_0$ such that $\lambda_0 + C \leq 1$ and $\frac{C}{\lambda_0} \leq \mu$, we have

$$\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 \right] \leq (C + \lambda) \int_t^T \rho \left(\mathbb{E} \left[e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 \right] \right) ds, \quad t \in [0, T]. \quad (4.3.14)$$

By the same argument in the proof of uniqueness of Theorem 4.3.1, we have

$$\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 \right] = 0, \text{ for all } t \in [0, T]. \text{ This implies}$$

$$Y_t^{(1)} \geq Y_t^{(2)}, \text{ a.s., } t \in [0, T].$$

□

Thanks to these preliminary results, we are now in position to investigate our main subject.

4.4 Anticipated GBDSDEs

In this section, we will mainly study the following equation

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) ds + \int_t^T g(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) dB_s \\ \quad + \int_t^T h(s, Y_s, Y_{s+\delta(s)}) d\Lambda_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \\ Y_t = \xi_t, \quad Z_t = \eta_t, \quad t \in [T, T + K]. \end{cases} \quad (4.4.1)$$

Definition 4.4.1. A pair of processes (Y, Z) is called a solution to AGBDSDEs (4.4.1) if $(Y_t, Z_t) \in \mathcal{B}_{\mathcal{F}}^2([0, T + K], \mathbb{R}^k)$ and satisfies eq.(4.4.1).

We assume that there exists $\rho \in \mathbf{S}$ such that ξ, f, g and h satisfy assumptions **(H2)**:

(H2.1) for all $t \in [0, T]$, $r, \bar{r} \in [t, T + K]$, $(t, y, z, \theta(r), \varphi(\bar{r}))$, $(t, y', z', \theta'(r), \varphi'(\bar{r})) \in$

$$[0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[t, T+K]}^2(\mathcal{F}, \mathbb{R}^{k \times d}),$$

$$\begin{aligned} \left| f(t, y, z, \theta(r), \varphi(\bar{r})) - f(t, y', z', \theta'(r), \varphi'(\bar{r})) \right|^2 &\leq C\rho(|y - y'|^2) + \mathbb{E}^{\mathcal{F}_t}[\rho(|\theta(r) - \theta'(r)|^2)] \\ &\quad + C|z - z'|^2 + \mathbb{E}^{\mathcal{F}_t}[|\varphi(\bar{r}) - \varphi'(\bar{r})|^2], \end{aligned}$$

$$\begin{aligned} \left| g(t, y, z, \theta(r), \varphi(\bar{r})) - g(t, y', z', \theta'(r), \varphi'(\bar{r})) \right|^2 &\leq C\rho(|y - y'|^2) + \mathbb{E}^{\mathcal{F}_t}[\rho(|\theta(r) - \theta'(r)|^2)] \\ &\quad + \alpha_1|z - z'|^2 + \alpha_2\mathbb{E}^{\mathcal{F}_t}[|\varphi(\bar{r}) - \varphi'(\bar{r})|^2];, \end{aligned}$$

$$\left\langle y - y', h(t, y, \theta(r)) - h(t, y', \theta'(r)) \right\rangle \leq C|y - y'|^2 + \mathbb{E}^{\mathcal{F}_t}[|\theta(r) - \theta'(r)|^2],$$

where $C > 0$, $\alpha_1 \in (0, \frac{1}{2})$ and $\alpha_2 \in (0, \frac{1}{M})$ satisfying $\alpha_1 + \alpha_2 M \in (0, \frac{1}{2})$.

(H2.2) For all $s \in [0, T]$, $f(s, 0) = f(s, 0, 0, 0, 0)$ and the integrability condition holds

$$\begin{aligned} \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + \int_0^T e^{\mu s + \beta \Lambda_s} |f(s, 0)|^2 ds \right. \\ \left. + \int_T^{T+K} e^{\mu s + \beta \Lambda_s} \left(\rho(|\xi_s|^2)(ds + d\Lambda_s) + |\eta_s|^2 \right) ds \right] < +\infty. \end{aligned}$$

4.4.1 Existence and uniqueness of solution

To prove the existence of a solution, let us consider the sequence of processes $(Y_t^n, Z_t^n)_{n \geq 0}$ given by

$$\begin{cases} Y_t^0 = 0; \quad \forall 0 \leq t \leq T, \\ Y_t^n = \xi_T + \int_t^T f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) ds + \int_t^T g(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) dB_s \\ \quad + \int_t^T h(s, Y_s^n, Y_{s+\delta(s)}^n) d\Lambda_s - \int_t^T Z_s^n dW_s \quad t \in [0, T], \\ Y_t^n = \xi_t, \quad Z_t^n = \eta_t, \quad \forall t \in [T, T+K]. \end{cases} \quad (4.4.2)$$

Thanks to [[73], Theorem 1], this sequence is well defined. In what follows, we establish two results which will be useful in the sequel.

Lemma 4.4.1. Assume that the assumptions **(A1)**, **(A2)** and **(H2.1)** and **(H2.2)** are true and let $\xi_T \in L^2(\mathcal{F}_T, \mathbb{R}^k)$. Then for any $(\xi, \eta) \in \mathcal{S}_{[T, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[T, T+K]}^2(\mathcal{F}, \mathbb{R}^{k \times d})$. There exists a positive constant $C_{4.4.1}$ such that

$$\sup_{n \geq 0} \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 \right] \leq C_{4.4.1} (1 + \mathbb{E}[\Gamma_T]), \quad t \in [0, T+K], \quad (4.4.3)$$

where

$$\begin{aligned} \Gamma_T &= e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + \int_T^{T+K} e^{\mu s + \beta \Lambda_s} (\rho(|\xi_s|^2) + |\eta_s|^2) ds \\ &\quad + \int_T^{T+K} e^{\mu s + \beta \Lambda_s} |\xi_s|^2 d\Lambda_s + \int_0^T e^{\mu s + \beta \Lambda_s} |f(s, 0)|^2 ds. \end{aligned} \quad (4.4.4)$$

Proof. Itô's formula applied to $e^{\mu t + \beta \Lambda_t} |Y_t^n|^2$, we obtain

$$\begin{aligned} &\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 (\mu ds + \beta \Lambda_s) \right] \\ &= \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) \rangle ds \right] \\ &\quad + 2 \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, h(s, Y_s^n, Y_{s+\delta(s)}^n) \rangle d\Lambda_s \right. \\ &\quad \left. + \int_t^T e^{\mu s + \beta \Lambda_s} |g(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n)|^2 ds \right]. \end{aligned} \quad (4.4.5)$$

Using the inequality $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ (where $\varepsilon > 0$ will be chosen later), we deduce from assumptions **(H2.1)**, **(H2.2)**, **(A1)** and **(A2)**:

- $$\begin{aligned} 2 \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, f(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n) \rangle ds &\leq 2\lambda \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 ds \\ &\quad + \frac{1}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |f(s, 0)|^2 ds + \frac{C}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\rho(|Y_s^{n-1}|^2) + |Z_s^n|^2) ds \\ &\quad + \frac{C}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\mathbb{E}^{\mathcal{F}_s} [\rho(|Y_{s+\delta(s)}^{n-1}|^2)] + \mathbb{E}^{\mathcal{F}_s} [|Z_{s+\zeta(s)}^n|^2]) ds \\ &\leq 2\lambda \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 ds + \frac{C+M}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\rho(|Y_s^{n-1}|^2) + |Z_s^n|^2) ds \\ &\quad + \mathbb{E} \int_T^{T+K} e^{\mu s + \beta \Lambda_s} (\rho(|\xi_s|^2) + |\eta_s|^2) ds; \end{aligned}$$
- $$\begin{aligned} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |g(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n)|^2 ds &\leq \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (C\rho(|Y_s^{n-1}|^2) + \alpha_1 |Z_s^n|^2) ds \\ &\quad + \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (\mathbb{E}^{\mathcal{F}_s} [\rho(|Y_{s+\delta(s)}^{n-1}|^2)] + \alpha_2 \mathbb{E}^{\mathcal{F}_s} [|Z_{s+\zeta(s)}^n|^2]) ds \\ &\leq \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} ((C+M)\rho(|Y_s^{n-1}|^2) + (\alpha_1 + \alpha_2 M)|Z_s^n|^2) ds \\ &\quad + \mathbb{E} \int_T^{T+K} e^{\mu s + \beta \Lambda_s} (\rho(|\xi_s|^2) + \alpha_2 |\eta_s|^2) ds; \end{aligned}$$

$$\begin{aligned}
 \bullet \quad & 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \langle Y_s^n, h(s, Y_s^n, Y_{s+\delta(s)}^n) \rangle d\Lambda_s \\
 & \leq 2C\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 d\Lambda_s + 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbb{E}^{\mathcal{F}_s} \left[|Y_{s+\delta(s)}^n|^2 \right] d\Lambda_s \\
 & \leq 2(C+M)\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 d\Lambda_s + \mathbb{E} \int_T^{T+K} e^{\mu s + \beta \Lambda_s} |\xi_s|^2 d\Lambda_s
 \end{aligned}$$

Putting pieces together, we deduce from (4.4.5) that

$$\begin{aligned}
 & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds \right. \\
 & \quad \left. + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 \left((\mu - 2\lambda) ds + [\beta - 2(C+M)] d\Lambda_s \right) \right] \\
 & \leq \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} \left(\frac{(C+M)(1+\lambda)}{\lambda} \rho(|Y_s^{n-1}|^2) + \frac{C+M+\lambda(\alpha_1+\alpha_2 M)}{\lambda} |Z_s^n|^2 \right) ds \right] \\
 & \quad + C_0 \mathbb{E} [\Gamma_T];
 \end{aligned}$$

where Γ_T is given by (4.4.4) and C_0 is a positive constant.

Choose $\lambda = \lambda_0$ such that $\lambda_0 = \frac{2(1+c)}{1-2(\alpha_1+\alpha_2 M)} \leq \frac{2\mu-1}{4}$ and β such that $\beta - 2(M+c) \geq \frac{1}{2}$. Hence there exists a positive constant $C' > 0$ (whose value may change from line to line) such that

$$\begin{aligned}
 & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t^n|^2 + \frac{1}{2} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s^n|^2 (ds + d\Lambda_s) + \frac{1}{2} \int_t^T e^{\mu s + \beta \Lambda_s} |Z_s^n|^2 ds \right] \\
 & \leq C' \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \rho(|Y_s^{n-1}|^2) ds + C' \mathbb{E} [\Gamma_T], \quad t \in [0, T+K].
 \end{aligned}$$

Similarly to the proof of Lemma 4.3.1, we can prove that

$$\sup_{n \in \mathbb{N}} \mathbb{E} [e^{\mu t + \beta \Lambda_t} |Y_t^n|^2] \leq C_{4.4.1} (1 + \mathbb{E}[\Gamma_T]), \quad t \in [0, T+K].$$

□

With the help of Lemma 4.4.1, we can establish the following existence and uniqueness theorem in this part.

Theorem 4.4.1. *Assume that the assumptions (A1), (A2), (H2.1) and (H2.2) are true and let $\xi_T \in L^2(\mathcal{F}_T, \mathbb{R}^k)$. Then for any $(\xi, \eta) \in \mathcal{S}_{[T, T+K]}^2(\mathcal{F}, \mathbb{R}^k) \times \mathcal{M}_{[T, T+K]}^2(\mathcal{F}, \mathbb{R}^{k \times d})$ the AGBDSDE (4.4.1) has a unique solution $(Y_t, Z_t)_{0 \leq t \leq T+K} \in \mathcal{B}_{\mathcal{F}}^2([0, T+K], \mathbb{R}^k)$.*

Proof. (i) Existence. We consider the sequence defined in (4.4.2). For a process

$\varrho \in \{Y, Z\}$ and $n \in \mathbb{N}, m \in \mathbb{N}$, we define $\bar{\varrho}_t^{n,m} = \varrho_t^n - \varrho_t^m$, and for $\Psi \in \{f, g\}$

$$\begin{aligned}\Delta\Psi^{(n,m)}(s) &= \Psi\left(s, Y_s^{n-1}, Z_s^n, Y_{s+\delta(s)}^{n-1}, Z_{s+\zeta(s)}^n\right) - \Psi\left(s, Y_s^{m-1}, Z_s^m, Y_{s+\delta(s)}^{m-1}, Z_{s+\zeta(s)}^m\right) \\ \Delta h^{(n,m)}(s) &= h\left(s, Y_s^n, Y_{s+\delta(s)}^n\right) - h\left(s, Y_s^m, Y_{s+\delta(s)}^m\right).\end{aligned}$$

Then, it is obvious that $(\bar{Y}_t^{n,m}, \bar{Z}_t^{n,m})_{0 \leq t \leq T+K}$ solves the following AGBDSDE

$$\begin{cases} \forall t \in [0, T] \\ \bar{Y}_t^{n,m} = \int_t^T \Delta f^{(n,m)}(s) ds + \int_t^T \Delta g^{(n,m)}(s) dB_s + \int_t^T \Delta h^{(n,m)}(s) d\Lambda_s - \int_t^T \bar{Z}_s^{n,m} dW_s, \\ \bar{Y}_t^{n,m} = 0, \quad \bar{Z}_t^{n,m} = 0, \quad t \in [T, T+K]. \end{cases},$$

By the same computations as in Lemma 4.4.1, we have

$$\begin{aligned}\mathbb{E}\left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2\right] + \frac{1}{2} \mathbb{E}\left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s^{n,m}|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s^{n,m}|^2 ds\right] \\ \leq C' \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \rho(|\bar{Y}_r^{n-1, m-1}|^2) dr, \quad t \in [0, T+K].\end{aligned}\quad (4.4.6)$$

Set

$$q(t) = \limsup_{n,m \rightarrow \infty} \mathbb{E}\left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2\right], \quad t \in [0, T+K].$$

Then by (4.4.6), Fatou's lemma and the fact that ρ is an increasing continuous concave function, we get

$$q(t) \leq C' \int_t^{T+K} \rho(q(s)) ds, \quad t \in [0, T+K].$$

Therefore, we can use Bihari's inequality to get $q(t) = 0$, i.e.

$$\limsup_{n,m \rightarrow \infty} \mathbb{E}\left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2\right] = 0, \quad t \in [0, T+K].$$

So, from inequality (4.4.6), we obtain, $\forall t \in [0, T+K]$

$$\lim_{n,m \rightarrow \infty} \mathbb{E}\left(e^{\mu t + \beta \Lambda_t} |\bar{Y}_t^{n,m}|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s^{n,m}|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s^{n,m}|^2 ds\right) = 0.$$

Then there exists $(Y_t, Z_t)_{0 \leq t \leq T+K} \in \mathcal{B}_{\mathcal{F}}^2([0, T+K], \mathbb{R}^k)$ such that

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathbb{E}\left(e^{\mu t + \beta \Lambda_t} |Y_t^n - Y_t|^2 + \int_t^{T+K} e^{\mu s + \beta \Lambda_s} |Y_s^n - Y_s|^2 (ds + d\Lambda_s) \right. \\ \left. + \int_t^{T+K} e^{\mu s + \beta \Lambda_s} |Z_s^n - Z_s|^2 ds\right) = 0.\end{aligned}$$

Letting $n \rightarrow +\infty$ in (4.4.2), we obtain

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) ds + \int_t^T g(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)}) dB_s \\ \quad + \int_t^T h(s, Y_s, Y_{s+\delta(s)}) d\Lambda_s - \int_t^T Z_s dW_s \quad \forall t \in [0, T], \\ Y_t = \xi_t, \quad Z_t = \eta_t, \quad \forall t \in [T, T+K]. \end{cases}$$

This shows that $(Y_t, Z_t)_{0 \leq t \leq T+K} \in \mathcal{B}_{\mathcal{F}}^2([0, T+K], \mathbb{R}^k)$ solves AGBDSDE (4.4.1). The proof of existence is complete.

(ii) Uniqueness. Let $(Y^i, Z^i) \in \mathcal{B}_{\mathcal{F}}^2([0, T+K], \mathbb{R}^k)$, $i = 1, 2$ be any two solutions. Define $\bar{Y}_t = Y_t^1 - Y_t^2$, $\bar{Z}_t = Z_t^1 - Z_t^2$ and for a function $\Psi \in \{f, g\}$,

$$\begin{aligned} \Delta\Psi(s) &= \Psi(s, Y_s^1, Z_s^1, Y_{s+\delta(s)}^1, Z_{s+\zeta(s)}^1) - \Psi(s, Y_s^2, Z_s^2, Y_{s+\delta(s)}^2, Z_{s+\zeta(s)}^2), \quad \text{and} \\ \Delta h(s) &= h(s, Y_s^1, Y_{s+\delta(s)}^1) - h(s, Y_s^2, Y_{s+\delta(s)}^2) \end{aligned}$$

We consider the following equation

$$\begin{cases} \bar{Y}_t = \int_t^T \Delta f(s) ds + \int_t^T \Delta g(s) dB_s + \int_t^T \Delta h(s) d\Lambda_s - \int_t^T \bar{Z}_s dW_s, \quad t \in [0, T], \\ \bar{Y}_t = 0, \quad \bar{Z}_t = 0, \quad t \in [T, T+K]. \end{cases} \quad (4.4.7)$$

By the same computations as in Lemma 4.4.1, we obtain

$$\begin{aligned} \mathbb{E} [e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2] + \frac{1}{2} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_s|^2 ds \right] \\ \leq C' \mathbb{E} \int_t^T \rho(e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2) ds. \end{aligned} \quad (4.4.8)$$

This leads to

$$\mathbb{E} [e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2] \leq C' \mathbb{E} \int_t^{T+K} \rho(e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2) ds.$$

Using Fubini's Theorem and Jensen's inequality leads to

$$\mathbb{E} [e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2] \leq C' \int_t^{T+K} \rho \left(\mathbb{E} [e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2] \right) ds, \quad t \in [0, T+K].$$

Then we can use Bihari's inequality to obtain $\mathbb{E} [e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2] = 0$, for all $t \in [0, T+K]$. This implies $\bar{Z}_t = 0$. \square

4.4.2 Comparison Theorem for Anticipated GBDSDEs

In this subsection, We will give the comparison theorem of the following 1-dimensional anticipated GBDSDEs under non-Lipshitz coefficients:

$$\left\{ \begin{array}{l} 0 \leq t \leq T, \\ Y_t^{(i)} = \xi_T^i + \int_t^T f^i(s, Y_s^{(i)}, Z_s^{(i)}, Y_{s+\delta(s)}^{(i)}, Z_{s+\zeta^{(i)}(s)}^{(i)}) ds + \int_t^T g^i(s, Y_s^{(i)}, Z_s^{(i)}, Y_{s+\delta(s)}^{(i)}, Z_{s+\zeta^{(i)}(s)}^{(i)}) dB_s \\ \quad + \int_t^T h^i(s, Y_s^{(i)}, Y_{s+\delta(s)}^{(i)}) d\Lambda_s - \int_t^T Z_s^{(i)} dW_s, \\ Y_t^{(i)} = \xi_t^{(i)}, \quad Z_t^{(i)} = \eta_t^{(i)}, \quad T \leq t \leq T + K \end{array} \right. \quad (4.4.9)$$

where $i = 1, 2$, and **(A1)**, **(A2)** and **(H)** hold. Then by Theorem 4.4.1, (4.4.9) has a unique solution.

Theorem 4.4.2. Let $(Y^{(1)}, Z^{(1)})$ and $(Y^{(2)}, Z^{(2)})$ be solutions of AGBDSDEs respectively. Under the assumption **(H1)**, assume that:

(i) $\xi_t^{(1)} \geq \xi_t^{(2)}$, a.s., $t \in [T, T + K]$;

(ii) For all $t \in [0, T]$, we have

- $f^1(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(1)}(t)}^{(1)}) \geq f^2(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(2)}(t)}^{(2)})$, a.s.,
- $G^1(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_{t+\zeta^{(1)}(t)}^{(1)}) = G^2(t, Y_t^{(1)}, Z_t^{(1)}, Y_{t+\delta(t)}^{(2)}, Z_{t+\zeta^{(2)}(t)}^{(2)})$, a.s.,
- $h^1(t, Y_t^{(1)}, Y_{t+\delta(t)}^{(1)}) \geq h^2(t, Y_t^{(1)}, Y_{t+\delta(t)}^{(2)})$, a.s.

(iii) for all $t \in [0, T]$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and $\varphi(\bar{r}) \in \mathcal{M}_{[0, T]}^2(\mathcal{F}, \mathbb{R}^d)$, $\bar{r} \in [t, T + K]$, $f^2(t, y, z, \cdot, \varphi(\bar{r}))$ is nondecreasing.

Then

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad a.s., \quad t \in [T, T + K].$$

Proof. For $i = 1, 2$ and for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, we denote:

- $F^i(t, y, z) = f^i(t, y, z, Y_{t+\delta(t)}^{(i)}, Z_{t+\zeta^{(i)}(t)}^{(i)})$,
- $G^i(t, y, z) = g^i(t, y, z, Y_{t+\delta(t)}^{(i)}, Z_{r+\zeta^{(i)}(r)}^{(i)})$,
- $H^i(t, y) = h^i(t, y, z, Y_{t+\delta(t)}^{(i)})$.

Then $(Y^{(i)}, Z^{(i)})$ is also the unique solution of the following equation

$$\begin{aligned}
 Y_t^{(i)} = \xi_T^{(i)} + \int_t^T F^i(s, Y_s^{(i)}, Z_s^{(i)})ds + \int_t^T G^i(s, Y_s^{(i)}, Z_s^{(i)})dB_s + \int_t^T H^i(s, Y_s^{(i)})d\Lambda_s \\
 - \int_t^T Z_s^{(i)}dW_s, \quad t \in [0, T].
 \end{aligned} \tag{4.4.10}$$

Let us introduce the following equation

$$\begin{cases}
 Y_t^{(3)} = \xi_T^{(2)} + \int_t^T f^2(s, Y_s^{(3)}, Z_s^{(3)}, Y_{s+\delta(s)}, Z_{s+\zeta^{(2)}(s)})ds + \int_t^T G^2(s, Y_s^{(3)}, Z_s^{(3)})dB_s \\
 \quad + \int_t^T H^2(s, Y_s^{(3)})d\Lambda_s - \int_t^T Z_s^{(3)}dW_s, \quad t \in [0, T], \\
 Y_t^{(3)} = \xi_t^{(2)}, \quad Z_t^{(3)} = \eta_t^{(2)}, \quad t \in [T, T+K].
 \end{cases}$$

If we set $\tilde{F}^2(s, y, z) = f^2(s, y, z, Y_{s+\delta(s)}^{(1)}, Z_{s+\zeta^{(2)}(s)}^{(2)})$, then this equation is equivalent to

$$\begin{cases}
 Y_t^{(3)} = \xi_T^{(2)} + \int_t^T \tilde{F}^2(s, Y_s^{(3)}, Z_s^{(3)})ds + \int_t^T G^2(s, Y_s^{(3)}, Z_s^{(3)})dB_s \\
 \quad + \int_t^T H^2(s, Y_s^{(3)})d\Lambda_s - \int_t^T Z_s^{(3)}dW_s, \quad t \in [0, T], \\
 Y_t^{(3)} = \xi_t^{(2)}, \quad Z_t^{(3)} = \eta_t^{(2)}, \quad t \in [T, T+K],
 \end{cases}$$

which admits a unique solution $(Y^{(3)}, Z^{(3)}) \in \mathcal{S}_{[0, T+K]}^2(\mathcal{F}, \mathbb{R}) \times \mathcal{M}_{[0, T+K]}^2(\mathcal{F}, \mathbb{R}^d)$, by Theorem 4.4.1. Moreover $(Y^{(3)}, Z^{(3)})$ also uniquely solves

$$\begin{aligned}
 Y_t^{(3)} = \xi_T^{(2)} + \int_t^T \tilde{F}^2(s, Y_s^{(3)}, Z_s^{(3)})ds + \int_t^T G^2(s, Y_s^{(3)}, Z_s^{(3)})dB_s \\
 + \int_t^T H^2(s, Y_s^{(3)})d\Lambda_s - \int_t^T Z_s^{(3)}dW_s, \quad t \in [0, T].
 \end{aligned}$$

From the assumptions **(i)** and **(ii)**, for $t \in [0, T]$, it follows that $\xi_t^{(1)} \geq \xi_t^{(2)}$, a.s., $F^1(t, Y_t^{(1)}, Z_t^{(1)}) \geq \tilde{F}^2(t, Y_t^{(1)}, Z_t^{(1)})$, $G^1(t, Y_t^{(1)}, Z_t^{(1)}) \geq G^2(t, Y_t^{(1)}, Z_t^{(1)})$ and $H^1(t, Y_t^{(1)}) \geq H^2(t, Y_t^{(1)})$. So Theorem 4.3.2 can lead to $Y_t^{(1)} \geq Y_t^{(3)}$, a.s., for all $t \in [0, T]$, which implies

$$Y_t^{(1)} \geq Y_t^{(3)}, \text{ a.s.}, \quad t \in [0, T+K].$$

Similarly, we obtain the following equation

$$\begin{cases} Y_t^{(4)} = \xi_T^{(2)} + \int_t^T f^2(s, Y_s^{(4)}, Z_s^{(4)}, Y_{s+\delta(s)}^{(3)}, Z_{s+\zeta^{(2)}(s)}^{(2)})ds + \int_t^T G^2(s, Y_s^{(4)}, Z_s^{(4)})dB_s \\ \quad + \int_t^T H^2(s, Y_s^{(4)})d\Lambda_s - \int_t^T Z_s^{(4)}dW_s, \quad t \in [0, T], \\ Y_t^{(4)} = \xi_t^{(2)}, \quad Z_t^{(4)} = \eta_t^{(2)}, \quad t \in [T, T + K]. \end{cases}$$

With a similar argument and the assumption **iii)** on the function f^2 , we have

$$Y_t^{(3)} \geq Y_t^{(4)}, \quad a.s., \quad t \in [0, T + K].$$

Now, for $m = 5, 6, \dots$, we introduce the following equation

$$\begin{cases} Y_t^{(m)} = \xi_T^{(2)} + \int_t^T f^2(s, Y_s^{(m)}, Z_s^{(m)}, Y_{s+\delta(s)}^{(m-1)}, Z_{s+\zeta^{(2)}(s)}^{(2)})ds + \int_t^T G^2(s, Y_s^{(m)}, Z_s^{(m)})dB_s \\ \quad + \int_t^T H^2(s, Y_s^{(m)})d\Lambda_s - \int_t^T Z_s^{(m)}dW_s, \quad t \in [0, T], \\ Y_t^{(m)} = \xi_t^{(2)}, \quad Z_t^{(m)} = \eta_t^{(2)}, \quad t \in [T, T + K]. \end{cases}$$

Thanks to Theorem 4.3.2, by induction we have

$$Y_t^{(4)} \geq Y_t^{(5)} \geq Y_t^{(6)} \geq \dots \geq Y_t^{(m-1)} \geq Y_t^{(m)}, \quad a.s., \quad t \in [0, T + K].$$

Similarly to the proof of Theorem 4.4.1, we can prove that $(Y_t^{(m)}, Z_t^{(m)})$ converges in $\mathcal{B}_{\mathcal{F}}^2([0, T + K], \mathbb{R})$ to $(Y_t^{(2)}, Z_t^{(2)})$. Since for all $t \in [0, T + k]$,

$$Y_t^{(4)} \geq Y_t^{(5)} \geq Y_t^{(6)} \geq \dots \geq Y_t^{(2)}, \quad a.s.,$$

it holds immediately that

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad a.s., \quad t \in [0, T + K].$$

□

CHAPTER 5

GENERALIZED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY TWO MUTUALLY INDEPENDENT FRACTIONAL BROWNIAN MOTIONS

This chapter deals with a class of generalized backward stochastic differential equation driven by two mutually independent fractional Brownian motions (FGBSDEs in short). The existence and uniqueness of solutions for FGBSDE as well as a comparison theorem are obtained. Moreover, we show the connection between this solution and the solution of partial differential equation.

This chapter is devoted to the following equation called fractional generalized BSDE (FGBSDE in short):

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds + \int_t^T g(s, \eta_s, Y_s) d\Lambda_s \quad (5.0.1)$$
$$- \int_t^T Z_{1,s} dB_{1,s}^{H_1} - \int_t^T Z_{2,s} dB_{2,s}^{H_2}, \quad t \in [0, T], \quad (5.0.2)$$

where $\{B_{1,t}^{H_1}, t \in [0, T]\}$ and $\{B_{2,t}^{H_2}, t \in [0, T]\}$ are two mutually independent fractional Brownian motions and $\{\Lambda_t, t \in [0, T]\}$ is a continuous real valued increasing process. The novelty in these type of stochastic equations lies in the fact of the presence of a new term in the BSDE, namely an integral with respect to a continuous increasing process. Inspired by the works of Aidara and Sagna [3] and Borkowska [13], we prove that kind

of equation has a unique solution and establish a comparison theorem. Moreover, we show a connection with appropriate PDEs.

This chapter is organized as follows: In Section 2, we introduce some preliminaries, before studying the solvability of our equation under Lipschitz conditions on the generator in Section 3. Using this result, we prove establish a comparison theorem in Section 4. Finally, Section 5 is devoted to the connection between FGBSDE and PDE.

5.1 Preliminaries

5.1.1 Fractional Stochastic calculus

We assume that there are two mutually independent fractional Brownian motions $B^H \in \{B_1^{H_1}, B_2^{H_2}\}$ with Hurst parameter $H \geq \frac{1}{2}$.

Let Ω be a non-empty set, \mathcal{F} a σ -algebra of sets Ω , \mathbb{P} a probability measure defined on \mathcal{F} and $\{\mathcal{F}_t, t \in [0, T]\}$ a σ -algebra generated by both fractional Brownian motions. The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ defines a probability space and \mathbb{E} the mathematical expectation with respect to the probability measure \mathbb{P} .

The fractional Brownian motion B^H is a zero mean Gaussian process with the covariance function

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0.$$

Denote

$$\phi(t, s) = H(2H - 1)|t - s|^{2H-2}, \quad (t, s) \in \mathbb{R}^2.$$

Let ξ and η be measurable functions on $[0, T]$. It has been defined

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \phi(u, v) \xi(u) \eta(v) du dv \quad \text{and} \quad \|\xi\|_t^2 = \langle \xi, \xi \rangle_t.$$

Note that, for any $t \in [0, T]$, $\langle \xi, \eta \rangle_t$ is a Hilbert scalar product. Let \mathcal{H} be the completion of the set of continuous functions under this Hilbert norm $\|\cdot\|_t$ and $(\xi_n)_n$ be a sequence in \mathcal{H} such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$.

Let \mathcal{P}_T^H be the set of all polynomials of fractional Brownian motion $(B_t^H)_{t \geq 0}$. Namely, \mathcal{P}_T^H contains all elements of the form

$$F(\omega) = f \left(\int_0^T \xi_1(t) dB_t^H, \int_0^T \xi_2(t) dB_t^H, \dots, \int_0^T \xi_n(t) dB_t^H \right)$$

where f is a polynomial function of n variables.

The Malliavin derivative D_t^H of F is given by

$$D_s^H F = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \left(\int_0^T \xi_1(t) dB_t^H, \int_0^T \xi_2(t) dB_t^H, \dots, \int_0^T \xi_n(t) dB_t^H \right) \xi_j(s), \quad 0 \leq s \leq T.$$

Now, we introduce the Malliavin ϕ -derivative \mathbb{D}_t^H of F by

$$\mathbb{D}_t^H F = \int_0^T \phi(t, s) D_s^H F ds.$$

We have the following (see[[43], Proposition 6.25]):

Theorem 5.1.1. *Let $F : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathcal{H}$ be a stochastic processes such that*

$$\mathbb{E} \left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 ds dt \right) < +\infty.$$

Then, the Itô-Skorohod type stochastic integral denoted by $\int_0^T F_s dB_s^H$ exists in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and satisfies

$$\mathbb{E} \left(\int_0^T F_s dB_s^H \right) = 0 \quad \text{and} \quad \mathbb{E} \left(\int_0^T F_s dB_s^H \right)^2 = \mathbb{E} \left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s ds dt \right).$$

Let us recall the fractional Itô formula (see[[29], Theorem 3.1]).

Theorem 5.1.2. *Let $\sigma_1, \sigma_2 \in \mathcal{H}$ be deterministic continuous functions. Denote*

$$X_t = X_0 + \int_0^t \alpha(s) ds + \int_0^t \sigma_1(s) dB_{1,s}^{H_1} + \int_0^t \sigma_2(s) dB_{2,s}^{H_2},$$

where X_0 is a constant, $\alpha(t)$ is a deterministic function with $\int_0^t |\alpha(s)| ds < +\infty$.

Let $F(t, x)$ be continuously differentiable with respect to t and twice continuously differentiable with respect to x . Then

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) dX_s \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s) \left[\frac{d}{ds} \|\sigma_1\|_s^2 + \frac{d}{ds} \|\sigma_2\|_s^2 \right] ds, \quad 0 \leq t \leq T. \end{aligned}$$

Let us finish this section by giving a fractional Itô chain rule (see[[29], Theorem 3.2]).

Theorem 5.1.3. *Assume that for $j = 1, 2$, the processes μ_j, α_j and ϑ_j , satisfy*

$$\mathbb{E} \left[\int_0^T \mu_j^2(s) ds + \int_0^T \alpha_j^2(s) ds + \int_0^T \vartheta_j^2(s) ds \right] < +\infty.$$

Suppose that $\mathbb{D}_t^{H_1}\alpha_j(s)$ and $\mathbb{D}_t^{H_2}\vartheta_j(s)$ are continuously differentiable with respect to $(s, t) \in [0, T]^2$ for almost all $\omega \in \Omega$. Let X_t and Y_t be two processes satisfying

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu_1(s)ds + \int_0^t \alpha_1(s)dB_{1,s}^{H_1} + \int_0^t \vartheta_1(s)dB_{2,s}^{H_2}, & 0 \leq t \leq T, \\ Y_t &= Y_0 + \int_0^t \mu_2(s)ds + \int_0^t \alpha_2(s)dB_{1,s}^{H_1} + \int_0^t \vartheta_2(s)dB_{2,s}^{H_2}, & 0 \leq t \leq T. \end{aligned}$$

If the following conditions hold:

$$\mathbb{E} \left[\int_0^T |\mathbb{D}_t^{H_1}\alpha_i(s)|^2 ds dt \right] < +\infty \quad \text{and} \quad \mathbb{E} \left[\int_0^T |\mathbb{D}_t^{H_2}\vartheta_i(s)|^2 ds dt \right] < +\infty$$

then

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s \\ &\quad + \int_0^t \left[\alpha_1(s) \mathbb{D}_s^{H_1} Y_s + \alpha_2(s) \mathbb{D}_s^{H_1} X_s + \vartheta_1(s) \mathbb{D}_s^{H_2} Y_s + \vartheta_2(s) \mathbb{D}_s^{H_2} X_s \right] ds, \end{aligned}$$

which may be written formally as

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \left[\alpha_1(t) \mathbb{D}_t^{H_1} Y_t + \alpha_2(t) \mathbb{D}_t^{H_1} X_t + \vartheta_1(t) \mathbb{D}_t^{H_2} Y_t + \vartheta_2(t) \mathbb{D}_t^{H_2} X_t \right] dt.$$

5.1.2 Definitions and notations

Let G be an open connected subset of \mathbb{R}^d such that for some $\ell \in \mathcal{C}^2(\mathbb{R}^d)$, $G = \{x; \ell(x) > 0\}$ and $\partial G = \{x; \ell(x) = 0\}$ and $|\nabla \ell(x)| = 1$ for $x \in \partial G$.

Let $\eta_0 \in G$ and (η_t, Λ_t) be a solution of the following reflected SDE with respect to fractional Brownian motions

$$\eta_t = \eta_0 + \int_0^t b(s)ds + \int_0^t \nabla \ell(s) d\Lambda_s + \int_0^t \sigma_1(s) dB_{1,s}^{H_1} + \int_0^t \sigma_2(s) dB_{2,s}^{H_2}, \quad t \in [0, T], \quad (5.1.1)$$

where the coefficients η_0 , b , σ_1 and σ_2 satisfy:

- η_0 is a given constant and $b : [0, T] \rightarrow \mathbb{R}$ is deterministic continuous function;
- $\sigma_1, \sigma_2 : [0, T] \rightarrow \mathbb{R}$ are deterministic differentiable continuous functions and $\sigma_1(t) \neq 0$, $\sigma_2(t) \neq 0$ such that

$$|\sigma|_t^2 = \|\sigma_1\|_t^2 + \|\sigma_2\|_t^2, \quad t \in [0, T], \quad (5.1.2)$$

where $\|\sigma_i\|_t^2 = H_i(2H_i - 1) \int_0^t \int_0^t |u - v|^{2H_i - 2} \sigma_i(u) \sigma_i(v) dudv$, $i = 1, 2$.

- Λ is a nondecreasing process, $\Lambda_0 = 0$, and $\int_0^T (\eta_t - a) d\Lambda_t \leq 0$ for any $a \in G$.

The next Remark will be useful in the sequel.

Remark 5.1.1. For $i = 1, 2$, the function $|\sigma|_t^2$ defined by eq.(5.1.2) is continuously differentiable with respect to t on $[0, T]$, and for a suitable constant $C_0 > 0$,

$$\inf_{t \in [0, T]} \frac{\hat{\sigma}_i(t)}{\sigma_i(t)} \geq C_0 \quad \text{where} \quad \hat{\sigma}_i(t) = \int_0^t \phi(t, v) \sigma_i(v) dv.$$

Given ξ a measurable real valued random variable and the functions

$$f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R},$$

we consider the following GFBSDE with parameters (ξ, f, g, Λ)

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds + \int_t^T g(s, \eta_s, Y_s) d\Lambda_s - \int_t^T Z_{1,s} dB_{1,s}^{H_1} - \int_t^T Z_{2,s} dB_{2,s}^{H_2}, \quad t \in [0, T]. \quad (5.1.3)$$

Before giving the definition of the solution for the above equation, we introduce the following sets (where $\mu, \beta > 0$):

- $\mathcal{C}_{pol}^{1,2}([0, T] \times \mathbb{R})$ is the space of all $\mathcal{C}^{1,2}$ -functions over $[0, T] \times \mathbb{R}$, which together with their derivatives are of polynomial growth,
- $\mathcal{V}_{[0, T]} = \left\{ Y = \psi(\cdot, \eta); \quad \psi \in \mathcal{C}_{pol}^{1,2}([0, T] \times \mathbb{R}), \quad \frac{\partial \psi}{\partial t} \text{ is bounded, } t \in [0, T] \right\}$,
- $\tilde{\mathcal{V}}_{[0, T]}^2$ and $\tilde{\mathcal{V}}_{([0, T], \Lambda)}^2$ the completion of $\mathcal{V}_{[0, T]}$ under the following norm respectively

$$\|Y\|_{\tilde{\mathcal{V}}_{[0, T]}^2}^2 = \mathbb{E} \int_0^T e^{\mu t + \beta \Lambda_t} |Y_t|^2 dt \quad \text{and} \quad \|Y\|_{\tilde{\mathcal{V}}_{([0, T], \Lambda)}^2}^2 = \mathbb{E} \int_0^T e^{\mu t + \beta \Lambda_t} |Y_t|^2 d\Lambda_t.$$

- $\mathcal{B}^2([0, T], \Lambda) = \left(\tilde{\mathcal{V}}_{[0, T]}^2 \cap \tilde{\mathcal{V}}_{([0, T], \Lambda)}^2 \right) \times \tilde{\mathcal{V}}_{[0, T]}^2 \times \tilde{\mathcal{V}}_{[0, T]}^2$ endowed with the norm

$$\left\| (Y, Z_1, Z_2) \right\|_{\mathcal{B}^2([0, T], \Lambda)}^2 = \mathbb{E} \left[\int_0^T e^{\mu t + \beta \Lambda_t} |Y_t|^2 (dt + d\Lambda_t) + \int_0^T e^{\mu t + \beta \Lambda_t} (|Z_{1,t}|^2 + |Z_{2,t}|^2) dt \right]$$

Definition 5.1.1. A triplet of processes $(Y_t, Z_{1,t}, Z_{2,t})_{0 \leq t \leq T}$ is called a solution to fractional GBSDE (5.1.3), if $(Y_t, Z_{1,t}, Z_{2,t})_{0 \leq t \leq T} \in \mathcal{B}^2([0, T], \Lambda)$ and satisfies eq.(5.1.3).

The next proposition will be useful in the sequel.

Proposition 5.1.1. Let $(Y_t, Z_{1,t}, Z_{2,t})_{0 \leq t \leq T} \in \mathcal{B}^2([0, T], \Lambda)$ be a solution of the fractional GBSDE (5.1.3). Then for almost $t \in [0, T]$,

$$\mathbb{D}_t^{H_1} Y_t = \frac{\hat{\sigma}_1(t)}{\sigma_1(t)} Z_{1,t} \quad \text{and} \quad \mathbb{D}_t^{H_2} Y_t = \frac{\hat{\sigma}_2(t)}{\sigma_2(t)} Z_{2,t}.$$

5.2 Fractional GBSDE

5.2.1 Existence and uniqueness of solution

We assume that the coefficients f and g of the GFBSDE and ξ satisfy the following assumptions:

(H1) $\xi_T = h(\eta_T)$, where $h : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function and $\mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi_T|^2 \right] < +\infty$.

(H2) There exists a constant $K > 0$ such that for all $t \in [0, T]$, $x \in \mathbb{R}$, $(y, y') \in \mathbb{R}^2$, $(z_1, z'_1) \in \mathbb{R}^2$, $(z_2, z'_2) \in \mathbb{R}^2$, we have

- $|f(t, x, y, z_1, z_2) - f(t, x, y', z'_1, z'_2)|^2 \leq K(|y - y'|^2 + |z_1 - z'_1|^2 + |z_2 - z'_2|^2)$;
- $|g(t, x, y) - g(t, x, y')|^2 \leq K|y - y'|^2$

We first establish an a priori estimate on the solution.

Proposition 5.2.1. Under the conditions **(H1)** and **(H2)**, if $(Y, Z_1, Z_2) \in \mathcal{B}^2([0, T], \Lambda)$ is a solution of Eq. (5.1.3), then there exists a constant C such that, for all $t \in [0, T]$

$$\begin{aligned} & \mathbb{E} \left(e^{\mu t + \beta \Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_{1,s}|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |Z_{2,s}|^2 ds \right) \\ & \leq C \mathbb{E} \left(e^{\mu T + \beta \Lambda_T} |\xi_T|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |f(s, \eta, 0, 0, 0)|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |g(s, \eta, 0)|^2 d\Lambda_s \right). \end{aligned}$$

Proof. By C we will denote a constant which may vary from line to line. From the Itô formula,

$$\begin{aligned} e^{\mu t + \beta \Lambda_t} |Y_t|^2 &= e^{\mu T + \beta \Lambda_T} |\xi|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds \\ & \quad + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s - 2 \int_t^T e^{\mu s + \beta \Lambda_s} Z_{1,s} \mathbb{D}_s^{H_1} Y_s ds \\ & \quad - 2 \int_t^T e^{\mu s + \beta \Lambda_s} Z_{2,s} \mathbb{D}_s^{H_2} Y_s ds - 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s Z_{1,s} dB_{1,s}^{H_1} \\ & \quad - 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s Z_{2,s} dB_{2,s}^{H_2} - \mu \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 ds - \beta \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 d\Lambda_s \end{aligned}$$

Take mathematical expectation on both sides, then we have

$$\begin{aligned}
 & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 (\mu ds + \beta d\Lambda_s) \right. \\
 & \quad \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} \left(Z_{1,s} \mathbb{D}_s^{H_1} Y_s + Z_{2,s} \mathbb{D}_s^{H_2} Y_s \right) ds \right] \\
 &= \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds \right. \\
 & \quad \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s \right] \\
 & \quad - 2 \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} Y_s Z_{1,s} dB_{1,s}^{H_1} + \int_t^T e^{\mu s + \beta \Lambda_s} Y_s Z_{2,s} dB_{2,s}^{H_2} \right]
 \end{aligned}$$

Using the fact that $\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} Y_s Z_{i,s} dB_{i,s}^{H_i} = 0$. Hence we deduce

$$\begin{aligned}
 & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 (\mu ds + \beta d\Lambda_s) \right. \\
 & \quad \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} \left(Z_{1,s} \mathbb{D}_s^{H_1} Y_s + Z_{2,s} \mathbb{D}_s^{H_2} Y_s \right) ds \right] \\
 &= \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds \right. \\
 & \quad \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s \right].
 \end{aligned}$$

It is known that, by Proposition 5.1.1, $\mathbb{D}_s^{H_1} Y_s = \frac{\widehat{\sigma}_1(s)}{\sigma_1(s)} Z_{1,s}$ and $\mathbb{D}_s^{H_2} Y_s = \frac{\widehat{\sigma}_2(s)}{\sigma_2(s)} Z_{2,s}$. By Remark 5.1.1, we obtain

$$\begin{aligned}
 & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 (\mu ds + \beta d\Lambda_s) + 2C_0 \int_t^T e^{\mu s + \beta \Lambda_s} \left(|Z_{1,s}|^2 + |Z_{2,s}|^2 \right) ds \right] \\
 & \leq \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi|^2 + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds \right. \\
 & \quad \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s \right]. \tag{5.2.1}
 \end{aligned}$$

Using standard estimates $2ab \leq \lambda a^2 + \frac{1}{\lambda} b^2$ (where $\lambda > 0$) and assumption **(H2)**, we obtain

- $$\begin{aligned}
 & \bullet \quad 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} Y_s f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds \\
 & \leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s| \left(|f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) \right. \\
 & \quad \left. - f(s, \eta_s, 0, 0, 0) + |f(s, \eta_s, 0, 0, 0)| \right) ds \\
 & \leq \frac{K + 2\lambda^2}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 ds + \frac{K}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} (|Z_{1,s}|^2 + |Z_{2,s}|^2) ds \\
 & \quad + \frac{1}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |f(s, \eta_s, 0, 0, 0)|^2 ds
 \end{aligned}$$
- $$\begin{aligned}
 & \bullet \quad 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} Y_s g(s, \eta_s, Y_s) d\Lambda_s \leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s| \left(|g(s, \eta_s, Y_s) - g(s, \eta_s, 0)| \right. \\
 & \quad \left. + |g(s, \eta_s, 0)| \right) d\Lambda_s \\
 & \leq \frac{K + 2\lambda^2}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 d\Lambda_s \\
 & \quad + \frac{1}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |g(s, 0)|^2 d\Lambda_s.
 \end{aligned}$$

Choosing $\lambda = \lambda_0$ such that $\mu, \beta \geq \frac{K + 2\lambda_0^2 + \lambda_0}{\lambda_0}$ and $C_0 \geq \frac{K + \lambda_0}{2\lambda_0}$, we deduce from (5.2.1)

$$\begin{aligned}
 & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |Y_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |Y_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} (|Z_{1,s}|^2 + |Z_{2,s}|^2) ds \right] \\
 & \leq C \mathbb{E} \left[e^{\mu T + \beta \Lambda_T} |\xi|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |f(s, \eta_s, 0, 0, 0)|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |g(s, 0)|^2 d\Lambda_s \right].
 \end{aligned}$$

□

The main result of this subsection is the following theorem:

Theorem 5.2.1. *Assume that the assumptions **(H1)** and **(H2)** are true. Then the fractional GBSDE (5.1.3) has a unique solution $(Y_t, Z_{1,t}, Z_{2,t})_{0 \leq t \leq T} \in \mathcal{B}^2([0, T], \Lambda)$.*

Proof. Let us consider the mapping $\Gamma : \mathcal{B}^2([0, T], \Lambda) \rightarrow \mathcal{B}^2([0, T], \Lambda)$ driven by $(U, V_1, V_2) \mapsto \Gamma(U, V_1, V_2) = (Y, Z_1, Z_2)$. We will show that the mapping Γ is a contraction, where (Y, Z_1, Z_2) is a solution of the following fractional GBSDE:

$$\begin{aligned}
 Y_t = \xi + \int_t^T f(s, \eta_s, U_s, V_{1,s}, V_{2,s}) ds + \int_t^T g(s, \eta_s, U_s) d\Lambda_s - \int_t^T Z_{1,s} dB_{1,s}^{H_1} \\
 - \int_t^T Z_{2,s} dB_{2,s}^{H_2}, \quad t \in [0, T].
 \end{aligned} \tag{5.2.2}$$

Let us define for a process $\delta \in \{U, V_1, V_2, Y, Z_1, Z_2\}$, $\bar{\delta} = \delta - \delta'$ and the functions

$$\Delta f(s) = f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) - f(s, \eta_s, Y'_s, Z'_{1,s}, Z'_{2,s}), \quad \Delta g(s) = g(s, \eta_s, Y_s) - g(s, \eta_s, Y'_s).$$

Then, the triplet $(\bar{Y}, \bar{Z}_1, \bar{Z}_2)$ solves the fractional GBSDE

$$\bar{Y}_t = \int_t^T \Delta f(s) ds + \int_t^T \Delta g(s) d\Lambda_s - \int_t^T \bar{Z}_{1,s} dB_{1,s}^{H_1} - \int_t^T \bar{Z}_{2,s} dB_{2,s}^{H_2}, \quad t \in [0, T].$$

Applying Itô formula to $e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2$, we obtain that

$$\begin{aligned} e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 &= 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \Delta f(s) ds + 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \Delta g(s) d\Lambda_s \\ &\quad - 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Z}_{1,s} \mathbb{D}_s^{H_1} \bar{Y}_s ds - 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Z}_{2,s} \mathbb{D}_s^{H_2} \bar{Y}_s ds \\ &\quad - 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \bar{Z}_{1,s} dB_{1,s}^{H_1} - 2 \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \bar{Z}_{2,s} dB_{2,s}^{H_2} \\ &\quad - \mu \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 ds - \beta \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 d\Lambda_s \end{aligned}$$

Take mathematical expectation on both sides, then we have

$$\begin{aligned} \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 (\mu ds + \beta d\Lambda_s) \right. \\ \left. + 2 \int_t^T e^{\mu s + \beta \Lambda_s} (\bar{Z}_{1,s} \mathbb{D}_s^{H_1} \bar{Y}_s + \bar{Z}_{2,s} \mathbb{D}_s^{H_2} \bar{Y}_s) ds \right] \\ = 2\mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \Delta f(s) ds + \int_t^T e^{\mu s + \beta \Lambda_s} \bar{Y}_s \Delta g(s) d\Lambda_s \right]. \end{aligned}$$

By the same computations as in Proposition 5.2.1, we deduce that

$$\begin{aligned} \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\bar{Y}_t|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 ((\mu - \lambda) ds + (\beta - \lambda) d\Lambda_s) \right. \\ \left. + 2C_0 \int_t^T e^{\mu s + \beta \Lambda_s} (|\bar{Z}_{1,s}|^2 + |\bar{Z}_{2,s}|^2) ds \right] \\ \leq \frac{K}{\lambda} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{U}_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} (|\bar{V}_{1,s}|^2 + |\bar{V}_{2,s}|^2) ds \right], \quad t \in [0, T]. \end{aligned} \tag{5.2.3}$$

Taking $\lambda = \lambda_0$ such that $\frac{1}{2} \min \{\mu - \lambda_0, \beta - \lambda_0\} \geq 2C_0 = \frac{4K}{3\lambda_0}$, we get

$$\begin{aligned} & \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Y}_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_{1,s}|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{Z}_{2,s}|^2 ds \right] \\ & \leq \frac{3}{4} \mathbb{E} \left[\int_t^T e^{\mu s + \beta \Lambda_s} |\bar{U}_s|^2 (ds + d\Lambda_s) + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{V}_{1,s}|^2 ds + \int_t^T e^{\mu s + \beta \Lambda_s} |\bar{V}_{2,s}|^2 ds \right], \\ & \quad t \in [0, T]. \end{aligned}$$

Thus, the mapping $(U, V_1, V_2) \mapsto \Gamma(U, V_1, V_2) = (Y, Z_1, Z_2)$ determined by the fractional GBSDE (5.1.3) is a strict contraction on $\mathcal{B}^2([0, T], \Lambda)$. Using the fixed point principle, we deduce that the solution to the fractional GBSDE (5.1.3) exists and is unique. This completes the proof. \square

5.2.2 Comparison Theorem

In this section, our objective is to establish the comparison theorem for the following fractional GBSDEs, for all $t \in [0, T]$,

$$Y_t^{(i)} = h^i(\eta_T) + \int_t^T f^i(s, \eta_s, Y_s^{(i)}, Z_{1,s}^{(i)}, Z_{2,s}^{(i)}) ds + \int_t^T g^i(s, \eta_s, Y_s^{(i)}) d\Lambda_s - \int_t^T Z_{1,s}^{(i)} dB_s^{H_1} - \int_t^T Z_{2,s}^{(i)} dB_s^{H_2}, \quad (5.2.4)$$

where $i = 1, 2$. Then by Theorem 5.2.1, Eq.(5.2.4) has a unique solution $(Y^{(i)}, Z_1^{(i)}, Z_2^{(i)}) \in \mathcal{B}^2([0, T], \Lambda)$.

Let us give a comparison theorem of GBSDEs under Lipschitz assumptions.

Theorem 5.2.2. For $i = 1, 2$, assume that f^i and g^i satisfy **(H1)**. Let h^i be continuously differentiable. Let $(Y^{(i)}, Z_1^{(i)}, Z_2^{(i)})$ be solutions of fractional GBSDEs (5.2.4) respectively. We suppose that:

(i) $h^1(\eta_T) \geq h^2(\eta_T)$, a.s.;

(ii) For all $t \in [0, T]$, $f^1(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}) \geq f^2(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)})$ or

$$f^1(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) \geq f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) \quad \text{a.s.}$$

,

(iii) For all $t \in [0, T]$, $g^1(t, \eta_t, Y_t^{(1)}) \geq g^2(t, \eta_t, Y_t^{(1)})$ or

$$g^1(t, \eta_t, Y_t^{(2)}) \geq g^2(t, \eta_t, Y_t^{(2)}) \quad \text{a.s.}$$

,

Then

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad \text{a.s.}, \quad t \in [0, T].$$

Proof. For all $t \in [0, T]$, we denote

$$\begin{aligned}\widehat{Y}_t &= Y_t^{(2)} - Y_t^{(1)}, \quad \widehat{Z}_{1,t} = Z_{1,t}^{(2)} - Z_{1,t}^{(1)}, \quad \widehat{Z}_{2,t} = Z_{2,t}^{(2)} - Z_{2,t}^{(1)}, \quad \widehat{h}(\eta_T) = h^2(\eta_T) - h^1(\eta_T); \\ \Delta f(t) &= f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) - f^1(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}), \\ \Delta g(t) &= g^2(t, \eta_t, Y_t^{(2)}) - g^1(t, \eta_t, Y_t^{(1)})\end{aligned}$$

Then $(\widehat{Y}, \widehat{Z}_1, \widehat{Z}_2)$ satisfies

$$\widehat{Y}_t = \widehat{h}(\eta_T) + \int_t^T \Delta f(s) ds + \int_t^T \Delta g(s) d\Lambda_s - \int_t^T \widehat{Z}_{1,s} dB_s^{H_1} - \int_t^T \widehat{Z}_{2,s} dB_s^{H_2}, \quad t \in [0, T]. \quad (5.2.5)$$

Apply the extension of the Itô formula to $e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2$, we have

$$\begin{aligned}\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 (\mu ds + \beta d\Lambda_s) \right. \\ \left. + 2C_0 \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} \left(|\widehat{Z}_{1,s}|^2 + |\widehat{Z}_{2,s}|^2 \right) ds \right] \\ \leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \Delta f(s) ds + 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \Delta g(s) d\Lambda_s\end{aligned} \quad (5.2.6)$$

In view of **(H1)** and **ii)**, Young's inequality and Jensen's inequality, for any $\lambda > 0$, we have:

- $$\begin{aligned}2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \Delta f(s) ds &= 2\mathbb{E} \left(\int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \left(f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) \right. \right. \\ &\quad \left. \left. - f^1(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}) \right) ds \right) \\ &\leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \left(f^2(t, \eta_t, Y_t^{(2)}, Z_{1,t}^{(2)}, Z_{2,t}^{(2)}) - f^2(t, \eta_t, Y_t^{(1)}, Z_{1,t}^{(1)}, Z_{2,t}^{(1)}) \right) ds \\ &\leq \frac{\lambda^2 + K}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 ds + \frac{K}{\lambda} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} \left(|\widehat{Z}_{1,s}|^2 + |\widehat{Z}_{2,s}|^2 \right) ds\end{aligned}$$
- $$\begin{aligned}2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \Delta g(s) d\Lambda_s &= 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \left(g^2(t, \eta_t, Y_t^{(2)}) - g^1(t, \eta_t, Y_t^{(1)}) \right) d\Lambda_s \\ &\leq 2\mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} \widehat{Y}_s^+ \left(g^2(t, \eta_t, Y_t^{(2)}) - g^2(t, \eta_t, Y_t^{(1)}) \right) d\Lambda_s \\ &\leq \frac{\lambda^2 + K}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 d\Lambda_s\end{aligned}$$

Then, thanks to the above inequalities, we obtain

$$\begin{aligned}
 & \mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 + \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 (\mu ds + \beta d\Lambda_s) \right. \\
 & \quad \left. + 2C_0 \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} (|\widehat{Z}_{1,s}|^2 + |\widehat{Z}_{2,s}|^2) ds \right] \\
 & \leq \frac{\lambda^2 + K}{\lambda} \mathbb{E} \int_t^T e^{\mu s + \beta \Lambda_s} |\widehat{Y}_s^+|^2 (ds + d\Lambda_s) + \frac{K}{\lambda} \int_t^T e^{\mu s + \beta \Lambda_s} \mathbf{1}_{\{\widehat{Y}_s > 0\}} (|\widehat{Z}_{1,s}|^2 + |\widehat{Z}_{2,s}|^2) ds
 \end{aligned} \tag{5.2.7}$$

Choosing $\lambda = \lambda_0$ such that $\min\{\mu, \beta\} \geq \frac{\lambda_0^2 + K}{\lambda_0}$ and $\lambda_0 \geq \frac{K}{2C_0}$, we have

$$\mathbb{E} \left[e^{\mu t + \beta \Lambda_t} |\widehat{Y}_t^+|^2 \right] \leq 0, \quad t \in [0, T].$$

This implies

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad a.s., \quad t \in [0, T].$$

This completes the proof of the theorem. \square

5.2.3 Connection with partial differential equations

We consider following fractional GBSDEs

$$\begin{cases} dY_t = -f(t, \eta_t, Y_t, Z_{1,t}, Z_{2,t}) dt - g(t, \eta_t, Y_t) d\Lambda_t + Z_{1,t} dB_t^{H_1} + Z_{2,t} dB_t^{H_2} \\ Y_T = \xi_T. \end{cases} \tag{5.2.8}$$

We denote

$$\tilde{\sigma}(t) = \left[\frac{d}{ds} \|\sigma_1\|_s^2 + \frac{d}{ds} \|\sigma_2\|_s^2 \right]$$

The function $u(t, x)$ satisfies the following partial differential equation (PDE in short)

$$\begin{cases} u'_t(t, x) + \frac{1}{2} \tilde{\sigma}(t) u''_{xx}(t, x) + b(t) u'_x(t, x) + f(t, x, u(t, x), \sigma_1(t) u'_x(t, x), \sigma_2(t) u'_x(t, x)) = 0; \\ \nabla \ell(t) u'_x(t, x) + g(t, x, u(t, x)) = 0; \\ u(T, x) = h(x) \end{cases} \tag{5.2.9}$$

Now we give the following theorem

Theorem 5.2.3. *If the PDE (5.2.9) has a solution $u(t, x)$ which is continuously differentiable in t and twice is continuously differentiable in x , then*

$$(Y_t, Z_{1,t}, Z_{2,t}) = (u(t, \eta_t), \sigma_1(t) u'_x(t, \eta_t), \sigma_2(t) u'_x(t, \eta_t))$$

satisfies the fractional GBSDE (5.2.8).

Proof. From the fractional Itô formula (Theorem 5.1.2), we get

$$\begin{aligned} du(t, \eta_t) &= u'_t(t, \eta_t)dt + u'_x(t, \eta_t) \left[b(t)dt + \nabla \ell(t)d\Lambda_t + \sigma_1(t)dB_{1,t}^{H_1} + \sigma_2(t)dB_{2,t}^{H_2} \right] \\ &\quad + \frac{1}{2} \tilde{\sigma}(t)u''_{xx}(t, \eta_t)dt \\ &= \left[u'_t(t, \eta_t) + b(t)u'_x(t, \eta_t) + \frac{1}{2} \tilde{\sigma}(t)u''_{xx}(t, \eta_t) \right] dt + u'_x(t, \eta_t) \nabla \ell(t)d\Lambda_t \\ &\quad + \sigma_1(t)u'_x(t, \eta_t)dB_{1,t}^{H_1} + \sigma_2(t)u'_x(t, \eta_t)dB_{2,t}^{H_2}. \end{aligned}$$

Since $u(t, x)$ satisfies the PDE (5.2.9), we have

$$\begin{aligned} du(t, \eta_t) &= -f(t, \eta_t, u(t, \eta_t), \sigma_1(t)u'_x(t, \eta_t), \sigma_2(t)u'_x(t, \eta_t))dt - g(t, \eta_t, u(t, \eta_t))d\Lambda_t \\ &\quad + \sigma_1(t)u'_x(t, \eta_t)dB_{1,t}^{H_1} + \sigma_2(t)u'_x(t, \eta_t)dB_{2,t}^{H_2}. \end{aligned}$$

Thus, the proof is complete. □

Remark 5.2.1. *From the above proof, we also see that if the nonlinear partial differential equation (5.2.9) has a unique solution, then the fractional generalized backward stochastic differential equation (5.2.8) also has a unique solution.*

CONCLUSION

For this work four main results were established; one around RBSDE with jumps and the other around BSDE driven by the fractional Brownian motion.

And for each results the main conditions were not the same (use of the class \mathcal{U} for the first result), the Mao's condition (non-Lipschitz): conditions that apply to the generator.

Temporal conditions also marked the realization of this research work with in particular the cases of Anticipated Generalized Backward Doubly Stochastic Differential Equations and the case of time delayed generators. Therefore, it appears that for the resolution of nonlinear BSDE we can do without the usual conditions (Lipschitz, monotony).

It should be noted that all the four classes of differential equations which have been studied during this thesis; existence and uniqueness of solution have been established. It is in this same spirit that it was possible to have a theorem of comparison in the AGBSDE.

The first result was the opportunity to prove the existence and uniqueness of viscosity solution for a class of non-lipschitz function and not satisfying the monotonicity condition. It was necessary to use a truncation of Levy's infinite measure to find the case treated in [37]. It should also be noted that an extension of the main result was established including the L^2_λ norm, where the latter main result stays true.

Starting from this first result we investigated the BSDE directed by a fractional brownian motion with a delay time. This result allowed us to generalize the result of [22] where we had to show the existence and uniqueness of solution by using Picard iteration techniques. For this result an illustrative example was given in the linear case.

The third result was an opportunity to investigate the generalized BSDE coupled with the comparison theorem. For this result one of the main hypotheses was the substitution of the Lipschitz condition by the Mao condition with in particular the use of the Bihari lemma to be able to establish our estimates in order to show the existence and unique-

ness of the solution. This result had two major moments: that of the establishment of the existence and uniqueness of Generalized BDSDEs as well as the obtaining of the comparison theorem; and that of the existence and uniqueness of solution for anticipated GBDSDE where we have anticipation times unlike the second result where we had delay times. The comparison theorem was also established for this class of BSDE, this result was shown by doing a concatenation, to then proceed by convergence.

The fourth and final result revolves around a class of BSDEs driven by two mutually independent fractional Brownians motion. It was shown the existence and uniqueness of solution with also the establishment of the comparison theorem.

After this result on the existence and uniqueness of solution, we established a Connection with partial differential equations.

Ultimately, this work was an opportunity to explore a wide range of concepts and results on BSDE. This pushed us to want to invest in further research in the field.

It seems important to us to see what would become of our results in the case of two barriers as defined in [19], [7], [39] and [40]. Moreover, this has been invested for some time with a main idea of obtaining the a priori estimate like that of 2.4.1 and we hope to achieve the expected results.

A study is also being carried out within the team to see a possible application of our results to the result "backward stochastic differential equations with time delayed generators " by Lukasz Delong and Peter Imkeller [22] and to become Monia Karouf's paper [48] constitutes a privileged point of support to reach the term.

For other research perspectives we intend to explore the domain of the statistic of Poisson process and we have project " On the nonparametric hypothesis testing for the intensity function of the Poisson process" and we want explore the stochastic optimal control.

Moreover, the works of Jingtao Shi [63] and those of Xunjing and Yong [79], [78]) are really worth exploring.

This will be a perfect transition to see through the stochastic zero-sum game, the connection between BSDE and game theory.

Moreover, we have exposed in the annex these two subjects of research perspectives to clarify our first steps.

Also, in perspective, given that the natural filtration of the Fractionnal Brownian Motion (FBM) is the same as that of the Brownian motion which defines the FBM, it would be interesting to see if there is a possibility of reducing the study of BSDE directed by a FBM to the case classical (BSDE driven by a Volterra-type Brownian motion).

ANNEX A

BSDE SIMULATION

Implementations used with R Software

Simulation of the process trajectory of the Black-Scholes equation

```
#####
```

```
T=1
n=1000
dt=T/n
dW=sqrt(dt)*rnorm(n)
parametre=c(0.75,0.1)
X=numeric(n)
X[1]=1
for(i in 1:n){
  X[i+1]=X[i]+parametre[1]*X[i]*dt+parametre[2]*X[i]*dW[i]
}
pdf(file="black.pdf")
```

```
plot(X,xlab="",ylab="",type="l",sub="",main="Simulation de la trajectoire du processus
de l'équation de Black-Schole",col=2)
schole=dev.off()
```

trajectory of the process representing the course of action in the Black-Scholes model

```
#####
```

```

T=2
n=5000
dt=T/n
dW=sqrt(dt)*rnorm(n)
parametre=c(0.5,1)
para=c(0.5,-1)
X=numeric(n)
Y=numeric(n)
X[1]=1
Y[1]=1
for(i in 1:n){
  X[i+1]=X[i]+parametre[2]*X[i]*dt+parametre[1]*X[i]*dW[i]
  Y[i+1]=Y[i]+para[2]*Y[i]*dt+para[1]*Y[i]*dW[i]
}
pdf(file="action.pdf")

plot(X,xlab="",ylab="",type="l",sub="",main="trajectoire du processus représentant le
cours de l'action dans le modèle de Black-Scholes",col="green")
lines(Y,col="brown")
legend(x=1000,y=6,legend=c("dérive positif","dérive négatif"),col=c("green","brown"),lty=1)
acte=dev.off()

```

Euler scheme for SDE

Let b and σ be coefficients which satisfy the same Lipschitz conditions, $n \in \mathbb{N}^*$ and for all integer $k \in \{0, \dots, n\}$, note $t_k = \frac{kT}{n}$. Let us define by induction the process X^n over $[0, T]$ (Euler scheme of X of step $h = T/n$) by setting :

$$X_0^n = x, \tag{A.3.1}$$

$$X_{t_{k+1}}^n = X_{t_k}^n + \sigma(t_k, X_{t_k}^n)(B_{t_{k+1}} - B_{t_k}) + b(t_k, X_{t_k}^n)h, \quad \forall 0 \leq k < n. \tag{A.3.2}$$

BSDE temporal discretization

$$Y_T^n = h(X_T^n),$$

$$Z_{t_k}^n = h^{-1} \mathbb{E} \left[Y_{t_{k+1}}^n (B_{t_{k+1}} - B_{t_k}) \mid \mathcal{F}_{t_k} \right], \quad (\text{A.3.3})$$

$$Y_{t_k}^n = \mathbb{E} \left[Y_{t_{k+1}}^n + hf(t_k, X_{t_k}^n, Y_{t_k}^n, Z_{t_k}^n) \mid \mathcal{F}_{t_k} \right]. \quad (\text{A.3.4})$$

ANNEX B

RESEARCH PERSPECTIVES

B.1 Links between BSDE and Optimal Stochastic Control

To see how the BSDE appears naturally in the stochastic version of the Pontryagin Maximum Principle, let us briefly recall the deterministic version: let the dynamics of x governed by an ordinary controlled differential equation

$$x(t) = f(x(t), u(t)), \quad x(0) = x_0$$

and consider a profit function

$$J(u) = \int_0^T r(t, x(t), u(t)) dt + g(x(T))$$

that we want to maximize over the set of admissible controls. Let $u_0(t)$, $t \in [0, T]$ be an optimal control, i.e. $J(u_0) \geq J(u')$ for any u' admissible¹. Let us introduce a disturbed version of u , $u_\epsilon = u_0 + \epsilon u$ for some u (it is not sure that there is an admissible u_ϵ defined like this, but we argue it formally). Denote by x_0 the solution u corresponding to 0, and x_ϵ that corresponding to u_ϵ . In writing

$$Y(t) = \left. \frac{d}{d\epsilon} x_\epsilon(t) \right|_{\epsilon=0} .$$

¹admissible has the following meaning in this document: applicable to the considered function.

Then we get

$$\begin{aligned}
 Y(t) &= \lim_{\epsilon \rightarrow 0} \frac{x_\epsilon(t) - x_0(t)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \frac{d}{ds} (x_\epsilon(s) - x_0(s)) ds \\
 &= \int_0^t \lim_{\epsilon \rightarrow 0} \frac{f(x_\epsilon(s), u_\epsilon(s)) - f(x_0(s), u_0(s))}{\epsilon} ds \\
 &= \int_0^t (D_x f(x_0(s), u_0(s))y(s) + D_u f(x_0(s), u_0(s))u(s)) ds .
 \end{aligned}$$

So that

$$y(t) = D_x f(x_0(t), u_0(t))y(t) + D_u f(x_0(t), u_0(t))u(t), \quad Y(0) = 0.$$

Let's vary the control and see how the gain changes:

$$\left. \frac{d}{d\epsilon} J(u_\epsilon) \right|_{\epsilon=0} = \int_0^T (D_x r(t, x_0(t), u_0(t))y(t) + D_u r(t, x_0(t), u_0(t))u(t)) dt + \underbrace{D_x g(x_0(T))y(T)}_{=0}. \quad (\text{B.1.1})$$

Because $\epsilon \rightarrow J(u_\epsilon)$ has a local maximum in 0. We would like to have terms of the form $\int (\dots)u(t) dt$, because if this is zero for all eligible commands u , and there is "enough" admissible checks, (\dots) must already be zero. Let's introduce an assistant variable p_0 which will take care of the badterms:

$$\begin{aligned}
 p_0(t) &= -D_x f(t, x_0(t), u_0(t))^* p_0(t) - D_x r(t, x_0(t), u_0(t))^* \\
 p_0(T) &= D_x g(x_0(T))^*.
 \end{aligned}$$

We get with this choice of p_0 and that $y(0) = 0$:

$$D_x g(x_0(T))^* y(T) = p_0(t)^* y(t) - p_0(0)^* y(0) \quad (\text{B.1.2})$$

$$\begin{aligned}
 &= \int_0^T (P_0(t)^* y(t) + p_0(t)^* y(t)) dt \\
 &= \int_0^T ((-p_0^* D_x f - D_x r)y + p_0^* (D_x f y + D_u f u)) dt \\
 &= \int_0^T (p_0^* D_u f u - D_x r y) dt \quad (\text{B.1.3})
 \end{aligned}$$

the combination of (B.1.1) and (B.1.2) gives

$$\left. \frac{d}{d\epsilon} J(u_\epsilon) \right|_{\epsilon=0} = \int_0^T (p_0^* D_u f - D_x r)u dt$$

such as heuristically ² $p_0^* D_u f - D_u r = 0$. But if we define the **Hamiltonian**

$$H(t, u, x, p) = p^* f(t, x, u) + r(t, x, u).$$

This is just $D_u H$. This heuristically implies that $\forall t, u \rightarrow H(t, u, x_0(t), p_0(t))$ has a local extremum at $u_0(t)$. To see that this should in fact be a maximum, note that for $\epsilon \geq 0$:

$$0 \geq J(u_\epsilon) - J(u_0) \simeq \epsilon \int_0^T (p_0 D_u f - D_u r) dt = \int_0^T D_u H(u_\epsilon - u_0) dt .$$

We obtain the Principle of the maximum of Pontryagin.

Theorem B.1.1. *Let u_0 be an optimal control, and x_0 the associated trajectory.*

Then under suitable assumptions there exists p_0 such that

$$\begin{aligned} \frac{d}{dt} x_0(t) &= D_p H(t, u_0(t), x_0(t), p_0(t)), & x_0(0) &= x_0 \\ \frac{d}{dt} p_0(t) &= -D_x H(t, u_0(t), x_0(t), p_0(t)), & p_0(T) &= D_x g(x_0(T))^* . \end{aligned}$$

And for all $t \in [0, T]$,

$$H(t, u_0(t), x_0(t), p_0(t)) = \max_{u \in U} H(t, u, x_0(t), p_0(t)).$$

The correspondence $t \rightarrow H(t, u_0(t), x_0(t), p_0(t))$ is constant. How can we find a candidate for optimal control? Well, to solve for all t, x, p

$$u(t, x, p) = \max_{u \in U} H(t, u, x, p).$$

Then solve

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} D_p H(t, u(t, x(t), p(t)), x(t), p(t)) \\ -D_x H(t, u(t, x(t), p(t)), x(t), p(t)) \end{pmatrix}, \quad x_0(0) = x_0 \quad p(T) = D_x g(x(T)).$$

Finally $u_0(t) = u(t, x(t), p(t))$.

In the stochastic case we consider the following command problem:

$$dX_t = b(X_t, u_t)dt + \sigma(X_t, u_t)dB_t, \quad X_0 = x_0.$$

Where $X_t \in \mathbb{R}^m$, B is a standard Brownian motion of dimension n and u is a predictable

²Which proceeds by successive approaches gradually eliminating the alternatives and keeping only a restricted range of solutions tending towards the optimal one.

control process which takes its values in a certain set U . We want to maximize

$$J(u) = \mathbb{E} \left(\int_0^T r(s, X_s, u_s) ds + g(X_T) \right).$$

For some measurable $r: \mathbb{R}_+ \times \mathbb{R}^m \times U \rightarrow \mathbb{R}$ and g concave: $\mathbb{R}^m \rightarrow \mathbb{R}$. It is possible to derive the principle of the maximum in the same sense as in the case of deterministic. This is done in (Yong and Zhou (1999), [79]), but it is very technical, its basic argument being a stochastic Taylor expansion. Here, we start with a well-chosen Hamiltonian, and show the principle of the maximum under a certain concavity condition, using only part integration.

This proof is less intuitive, which is why it was necessary to include the deterministic case.

Suppose there is $C > 0$ a.s.

- $|b(x, u) - b(y, u)| + |\sigma(x, u) - \sigma(y, u)| \leq L|x - y|$,
- $r(\cdot, \cdot, u)$ is a continuous function of (t, x) for all u ,
- g is concave and class C^1
- $|r(t, x, u)| + |g(x)| \leq C(1 + |x|^2)$ for all t, x, u ,
- b, σ and r are all differentiable in x .

Define the generalized Hamiltonian

$$H : \mathbb{R}_+ \times \mathbb{R}^m \times U \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$$

$$H(t, x, u, y, z) = b(x, u)^*y + \text{tr}(\sigma(x, u)^*z) + r(t, x, u).$$

And let the assistant (Y, Z) be the solution of

$$-dY_t = D_x H(t, X_t, u_t, Y_t, Z_t)^* dt - Z_t dB_t, \quad Y_T = D_x g(X_T)^*. \quad (\text{B.1.4})$$

Theorem B.1.2. Let u be a control process. Suppose that $(\widehat{Y}, \widehat{Z})$ solution of the BSDE (B.1.4).

Suppose that for all $t \in [0, T]$

$$H(t, \widehat{X}_t, \widehat{u}_t, \widehat{Y}_t, \widehat{Z}_t) = \max_{u \in U} H(t, \widehat{X}_t, u_t, \widehat{Y}_t, \widehat{Z}_t).$$

And that

$$(x, u) \rightarrow H(t, x, u, \widehat{Y}_t, \widehat{Z}_t) \quad (\text{B.1.5})$$

is a concave function for all $t \in [0, T]$. Then \widehat{u} is optimal, i.e.

$$J(\widehat{u}) = \max_{u \in U} J(u).$$

Proof B.1.1. (*Sketch of proof*)

Let u be another command process, and denote by X , Y , Z the associated processes. Just show $J(\hat{u}) \geq J(u)$.

$$J(\hat{u}) - J(u) = \mathbb{E} \left(\int_0^T r(t, \hat{u}_t, \hat{x}_t) - r(t, u_t, x_t) + g(\widehat{Y}_T) - g(Y_T) dt \right). \quad (\text{B.1.6})$$

Let's write this expression again

$$\begin{aligned} r(t, \hat{u}_t, \hat{x}_t) - r(t, u_t, x_t) &= H(t, \widehat{X}_t, \widehat{u}_t, \widehat{Y}_t, \widehat{Z}_t) - H(t, X_t, u_t, \widehat{Y}_t, \widehat{Z}_t) \\ &\quad - [b(\widehat{X}_t, \widehat{u}_t) - b(X_t, u_t)] \widehat{Y}_t \\ &\quad - \text{tr} \left[(\sigma(\widehat{X}_t, \widehat{u}_t) - \sigma(X_t, u_t)) \widehat{Z}_t \right]. \end{aligned} \quad (\text{B.1.7})$$

On the other hand, because g is concave and differentiable:

$$\begin{aligned} \mathbb{E} \left(g(\widehat{X}_T) - g(X_T) \right) &\geq \mathbb{E} \left(D_x g(\widehat{X}_T) (\widehat{X}_T - X_T) \right) = \mathbb{E} \left(\widehat{Y}_T^* (\widehat{X}_T - X_T) \right) = \\ &\mathbb{E} \left(\int_0^T -D_x H(t, \widehat{X}_t, \widehat{u}_t, \widehat{Y}_t, \widehat{Z}_t) (\widehat{X}_t - X_t) dt + \int_0^T \widehat{Y}_t [b(\widehat{X}_t, \widehat{u}_t) - b(X_t, u_t)] dt \right. \\ &\quad \left. + \int_0^T \text{tr} \left[\widehat{Z}_t (\sigma(\widehat{X}_t, \widehat{u}_t) - \sigma(X_t, u_t)) \right] dt \right). \end{aligned} \quad (\text{B.1.8})$$

When we use part integration, the fact that $X_0 = \widehat{X}_0$, and the expectation of stochastic integrals should be nothing. From the combination of (B.1.6), (B.1.7) and (B.1.8) we get

$$\begin{aligned} J(\hat{u}) - J(u) &\geq \mathbb{E} \left(\int_0^T \left[H(t, \widehat{X}_t, \widehat{u}_t, \widehat{Y}_t, \widehat{Z}_t) - H(t, X_t, u_t, \widehat{Y}_t, \widehat{Z}_t) \right. \right. \\ &\quad \left. \left. - D_x H(t, \widehat{X}_t, \widehat{u}_t, \widehat{Y}_t, \widehat{Z}_t) (\widehat{X}_t - X_t) \right] dt \right). \end{aligned}$$

Due to the concavity (B.1.5),

$$\begin{aligned} H(t, \widehat{X}_t, \widehat{u}_t, \widehat{Y}_t, \widehat{Z}_t) - H(t, X_t, u_t, \widehat{Y}_t, \widehat{Z}_t) &\geq D_x H(t, \widehat{X}_t, \widehat{u}_t, \widehat{Y}_t, \widehat{Z}_t) (\widehat{X}_t - X_t) \\ &\quad + D_u H(t, \widehat{X}_t, \widehat{u}_t, \widehat{Y}_t, \widehat{Z}_t) (\widehat{u}_t - u_t). \end{aligned}$$

Which implies,

$$J(\widehat{u}) - J(u) \geq \mathbb{E} \left(\int_0^T -D_u H(t, \widehat{X}_t, \widehat{u}_t, \widehat{Y}_t, \widehat{Z}_t) (\widehat{u}_t - u_t) dt \right).$$

Because $u \rightarrow H(t, \widehat{X}_t, \widehat{u}_t, \widehat{Y}_t, \widehat{Z}_t)$ takes its maximum value in \widehat{u}_t , which is always positive. Hence the result.

Remark B.1.1. Note however that to obtain a candidate for optimal control, in general, we need to solve an SDE coupled with an BSDE, where the upstream components (Y, Z) influence the dynamics of the X component of the SDE,

1. solve $u(t, x, y, z) = \max_{u \in U} H(t, x, u, y, z)$ pour tout t, x, y, z ,
2. solve an SDE coupled with an BSDE

$$\begin{aligned} dX_t &= b(X_t, u(t, X_t, Y_t, Z_t))dt + \sigma(X_t, u(t, X_t, Y_t, Z_t))dB_t, & X_0 &= x_0 \\ -dY_t &= H(t, X_t, u(t, X_t, Y_t, Z_t), Y_t, Z_t)dt - Z_t dB_t, & Y_T &= D_x g(X_T)^*, \end{aligned}$$

3. put $u_t = u(t, X_t, Y_t, Z_t)$.

Let us end this section with the following theorem from Hamadene and Lepeltier in Zero-sum stochastic differential games, *Systems and Control Letter*, but also Peng and Quenez [24].

Theorem B.1.3. Under the previous assumptions, for any admissible control $u \in U$, the Hamiltonian process $H(t, x, z, u)$ and the maximal Hamiltonian process $H^*(t, x, z)$:

$$H^*(t, x, z) = \sup_{u \in U} H(t, x, z, u)$$

define a family of BSDE with the terminal condition $g(X_T)$, and convex or linear coefficients satisfying **(H1)**. The associated solutions are denoted by (Y_t^u, Z_t^u) and (Y_t^*, Z_t^*) . In addition, there is a measurable control process $u^*(t, x, z)$ such that for all $t \in [0, T]$ and for all $z \in \mathbb{R}^n$, $H^*(t, x, z) = H(t, x, z, u^*(t, x, z))$. The process u_t^* is an optimal control, and that for $t \leq T$,

$$Y_t^* = Y_t^{u^*} = \sup_{u \in U} Y_t^{u^*}, \text{ in particular } Y_0^* = \sup_{u \in U} J(u).$$

We recall that the stochastic optimal control theory has been well developed in [79], [78]

B.2 Links between BSDE and Game Theory

Stochastic games model the interaction between decision makers who can influence their environment. These games were first introduced and studied by Lloyd Shapley (1953) in [62]. Since then, the literature has not stops growing.

In a stochastic game, players face potentially different goals. They must ensure a good payment today while maintaining a high payment expectation for tomorrow.

Stochastic games use a wide variety of mathematical tools. We will present here some classic results, mainly for zero-sum games.

Zero-sum stochastic differential games

A zero-sum stochastic differential game is generally a game, where two opposing players control a diffusive quantity, while they observe each other .

We consider a situation where two agents, denoted by $c1$ and $c2$, operate on the \mathbf{S} system, but present advantages antagonists. To control the system the agent $c1$ (resp. $c2$) acts with a stochastic process P -measurable $u := (u_t)_{t \leq T}$ (resp. $v := (v_t)_{t \leq T}$) taking values in a compact metric space U (resp. V). The set of admissible controls for $c1$ (resp. $c2$) is denoted by \mathcal{U} (resp. \mathcal{V}). When the agents act on \mathbf{S} with a pair of strategies $(u, v) \in \mathcal{U} \times \mathcal{V}$, the dynamics of the system set to same law as that of $(x_t)_{t \leq T}$ under the probability $\mathbb{P}^{u,v}$ defined by its density with respect to \mathbb{P} as follows:

$$\frac{d\mathbb{P}^{u,v}}{d\mathbb{P}} = \varepsilon \left(\int_0^T \sigma^{-1}(s, x) f(s, x, u_s, v_s) dB_s \right).$$

The function f is measurable, continuous with respect to x , and sub-linearly with respect to (u, v) . As in the adjustment of the command, the interventions of the controllers generate a drift in the dynamics of the evolution of the system. More precisely,

$$dx_t = f(t, x, u_t, v_t)dt + \sigma(t, x)dB_t^{u,v}, \quad t \leq T; \quad x_0 = x,$$

where $B^{u,v}$ is a Brownian motion under $d\mathbb{P}^{u,v}$.

Between the agents $c1$ and $c2$, depending on the pair of strategies used (u, v) , there is a gain $J(u, v)$, which is a r The compensation for $c1$ and the cost for $c2$. Its expression is given by

$$J(u, v) := \mathbb{E}^{u,v} \left[\int_0^T h(s, x, u_s, v_s) ds + \Psi(x_T) \right].$$

Where h is a bounded measurable function. Moreover, it is continuous with respect to (u, v) .

The problem is now to find an equitable strategy (u^*, v^*) for the two agents, that is to say a strategy which satisfied for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$:

$$J(u, v^*) \leq J(u^*, v^*) \leq J(u^*, v).$$

This is a zero-sum game problem and (u^*, v^*) is called a saddle-point for the game. For a suitable pair of strategies (u, v) , we define the Hamiltonian of the game to be zero sum $H(t, x, z, u_t, v_t) := z\sigma^{-1}(t, x)f(t, x, u_t, v_t) + h(t, x, u_t, v_t)$. Note that H is an affine function, uniformly Lipschitz with respect to z .

Now suppose:

(J1): the function H satisfies the following Isaac condition:

$$\inf_{v \in V} \sup_{u \in U} H(t, x, z, u, v) = \sup_{u \in U} \inf_{v \in V} H(t, x, z, u, v).$$

Under **(J1)**, there are two measurable functions $u^*(t, x, z)$ and $v^*(t, x, z)$ taking their values respectively in U and V such that:

- (i) $\sup_{u \in U} \inf_{v \in V} H(t, x, z, u, v) = H(t, x, z, (u^*, v^*)(t, x, z))$,
- (ii) for all $(u, v) \in U \times V$, for all t, z :

$$H(t, x, z, u, v^*(t, x, z)) \leq H(t, x, z, (u^*, v^*)(t, x, z)) \leq H(t, x, z, u^*(t, x, z), v).$$

We have the following result related to the existence of a saddle point for the zero-sum stochastic differential game.

Theorem B.2.1. (Hamadene and Lepeltier)

Suppose that **(J1)** is satisfied. Then there exists a pair (Y_t^*, Z_t^*) the solution of the BSDE associated with $(H(t, x, z, u^*(t, x, z), v^*(t, x, z)), \Psi(x_T))$, i.e.

$$-dY_t^* = H(t, x, Z_t^*, (u^*, v^*)(t, x, Z_t^*))dt - Z_t^*dB_t, \quad Y_T^* = \Psi(x_T).$$

In addition, the pair of strategies $(u^*, v^*) := (u^*(t, x, Z_t^*), v^*(t, x, Z_t^*))$ is an equilibrium point for the zero-sum stochastic differential game and

$$J(u^*, v^*) = Y_0^* = \inf_{v \in \mathcal{V}} \sup_{u \in \mathcal{U}} J(u, v) = \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{V}} J(u, v),$$

Y_0^* is the value of the game.

Proof B.2.1. (Sketch of proof, [36])

For the sake of simplicity, we do not consider here the case of bounded g . The existence of (Y^*, Z^*) is a direct consequence of theorem 2.1.2. The function which associates

$z \rightarrow H(t, x, z, u^*(t, x, z), v^*(t, x, z))$, is uniformly Lipschitz continues using the point i).
Also, we have $Y_0 = J^*(u^*, v^*)$.

Now, if u is another command for c_1 , then there exists (Y^u, Z^u) solution of the BSDE associated with $(H(t, x, z, u_t, v^*(t, x, Z_t^*)), \Psi(x_T))$ and $Y_0^u = J(u, v^*)$. On the other hand, also the comparison theorem (2.1.4) and point ii) implies that $Y_0^u \leq Y_0^*$. Let,

$J(u, v^*) \leq J(u^*, v^*)$. Symmetrically, we can prove that,

$J(u^*, v^*) \leq J(u^*, v)$ for any strategy v admissible for c_2 . Now (u^*, v^*) is a saddle point for the game.

The fact that Y_0^* is the value of the bound inf as well as of the sup bound of the two inequalities $Y_0^* \leq \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{V}} J(u, v)$ and $Y_0^* \geq \inf_{v \in \mathcal{V}} \sup_{u \in \mathcal{U}} J(u, v)$. Therefore, we have the equality in the previous inequalities, since (u^*, v^*) is point -saddle for the zero-sum game.

APPENDIX A

PROOFS FOR SOME THEOREMS AND PROPOSITIONS IN THE FIRST CHAPTER

A.1 The proofs

Proof A.1.1. [lemma 1.1.1]

By the Itô formula used over the time interval $[t_0, t]$, we have

$$\begin{aligned} u(t, X_t^{t_0, x_0}) - u(t_0, x_0) &= \int_0^t \frac{\partial u(s, X_s^{t_0, x_0})}{\partial s} ds + \int_0^t \frac{\partial u(s, X_s^{t_0, x_0})}{\partial x} dX_s^{t_0, x_0} \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 u(s, X_s^{t_0, x_0})}{\partial x^2} d \langle X_s^{t_0, x_0} \rangle \end{aligned}$$

Using the fact that X is a solution of (1.1.1), we therefore find that

$$\begin{aligned} u(t, X_t^{t_0, x_0}) - u(t_0, x_0) &= \int_0^t \left(\frac{\partial u(s, X_s^{t_0, x_0})}{\partial s} + f(s, X_s^{t_0, x_0}) \frac{\partial u(s, X_s^{t_0, x_0})}{\partial x} \right. \\ &\quad \left. + \frac{1}{2} g^2(s, X_s^{t_0, x_0}) \frac{\partial^2 u(s, X_s^{t_0, x_0})}{\partial x^2} \right) ds \\ &\quad + \int_0^t g(s, X_s^{t_0, x_0}) \frac{\partial u(s, X_s^{t_0, x_0})}{\partial x} dB_s \\ &= \int_0^t g(s, X_s^{t_0, x_0}) \frac{\partial u(s, X_s^{t_0, x_0})}{\partial x} dB_s. \end{aligned}$$

Hence by using the fact that u is solution of the equation (1.1.7) and the fourth

point of the property 1.1.1 on the stochastic integral then $(u(t, X_t^{t_0, x_0}), t \in [t_0, T])$ is a martingale.

Proof A.1.2. [proposition 1.4.1]

Note first that Y_0 is deterministic according to the equality $Y_t = \xi$ established above; taking into account the same thing done for Y_t adapted.

Let us now start from the following relation:

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s) ds - \int_0^t Z_s^* dB_s, \quad 0 \leq t \leq T, \quad (R)$$

the triangular inequality applied to (R) implies

$$|Y_t| \leq |Y_0| + \int_0^t |f(s, Y_s, Z_s)| ds + \sup_{0 \leq t \leq T} \left| \int_0^t Z_s^* dB_s \right|,$$

the assumption (1.4.4) implies

$$|Y_t| \leq |Y_0| + \int_0^t (f_s + \lambda \|Z_s\|) ds + \sup_{0 \leq t \leq T} \left| \int_0^t Z_s^* dB_s \right| + \lambda \int_0^t |Y_s| ds.$$

Let's pose

$$\zeta = |Y_0| + \int_0^t (f_s + \lambda \|Z_s\|) ds + \sup_{0 \leq t \leq T} \left| \int_0^t Z_s^* dB_s \right|.$$

By assumption, $Z \in \mathbb{H}_T^{2,1}(\mathbb{R}^{m \times n})$ and therefore, via Doob's inequality, the third term is square integrable; it is the same for $\{f_t\}$ and Y_0 is deterministic therefore of internal square integrable; it follows that ζ is an integrable square random variable.

As $Y \in \mathbb{H}_T^{2,1}$ therefore by the Gronwall lemma $\sup_{0 \leq t \leq T} |Y_t| \leq \zeta e^{\lambda t}$ which shows that Y belongs to \mathcal{S}_n^2 .

Lemma A.1.1. Let $Y \in \mathcal{S}_n^2$ and $Z \in \mathbb{H}_T^{2,1}(\mathbb{R}^{m \times n})$. Then $\left\{ \int_0^t Y_s \cdot Z_s^* dB_s, \quad t \in [0, T] \right\}$ is a uniformly integrated martingale.

Proof A.1.3. We have $Y_t \cdot Z_t^*$ is progressively measurable so

$$\left\{ \int_0^t Y_s \cdot Z_s^* dB_s, \quad t \in [0, T] \right\}$$

is a martingale.

Now let's prove the integrability uniform.

BDG inequalities give

$$\begin{aligned}
 \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t Y_s \cdot Z_s^* dB_s \right| \right] &\leq C \mathbb{E} \left[\left(\int_0^t |Y_s|^2 \|Z_s^*\|^2 ds \right)^{\frac{1}{2}} \right] \\
 &\leq C \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t| \left(\int_0^t \|Z_s^*\|^2 ds \right)^{\frac{1}{2}} \right] \\
 &\leq \frac{C}{2} \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] + \mathbb{E} \left[\int_0^t \|Z_s^*\|^2 ds \right] \right) \quad \text{since } ab \leq \frac{a^2}{2} + \frac{b^2}{2} \\
 &\leq C' \left(\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] + \mathbb{E} \left[\int_0^t \|Z_s^*\|^2 ds \right] \right) \quad \text{with } C' = \frac{C}{2}.
 \end{aligned}$$

However, this last quantity is finite by hypothesis; hence the result.

Proof A.1.4. [theorem 1.4.2]

(a) Let's start first with the case where f does not depend on (y, z) i.e.

$$f(t, y, z) := f(t) = f(t, 0, 0) \in \mathbb{H}_T^{2,n}.$$

Suppose that (Y, Z) is a solution of :

$$Y_t = \xi + \int_t^T f(s) ds - \int_t^T Z_s^* dB_s, \quad \text{with } Z \in \mathbb{H}_T^{2,1}(\mathbb{R}^{m \times n}).$$

If we take the conditional expectation knowing \mathcal{F}_t , we have

$$Y_t = \mathbb{E} \left(\xi + \int_t^T f(s) ds \mid \mathcal{F}_t \right),$$

because $\int_0^t Z_s^* dB_s$, $t \in [0, T]$, is martingale of zero expectation.

We therefore define Y using the previous formula and we still have to find Z . Note that, according to Fubini's theorem¹ [31], as f is progressively measurable, $\int_0^t f(s) ds$ is a process adapted for filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ in fact in \mathcal{S}_n^2 since $f(t, 0, 0)$ is squared integrable. We have then, for $\forall t \in [0, T]$,

$$Y_t = \mathbb{E} \left(\xi + \int_0^T f(s) ds \mid \mathcal{F}_t \right) - \int_0^t f(s) ds := M_t - \int_0^t f(s) ds.$$

¹Let (E_1, T_1, m_1) and (E_2, T_2, m_2) des espaces mesurés σ -finis. We note (E, T, m) product space. Let f be a function T -measurable from E in \mathbb{R} (i.e. $f \in \mathcal{M}(E, T)$) and integrable for the measure m , i.e. $f \in \mathcal{L}_{\mathbb{R}}^1(E, T, m)$. Then :

$$\int \left(\int f(x_1, x_2) dm_2(x_2) \right) dm_1(x_1) = \int \left(\int f(x_1, x_2) dm_1(x_1) \right) dm_2(x_2)$$

M is a Brownian martingale; via the theorem of representation of Brownian martingales we construct a process Z belonging to $\mathbb{H}_T^{2,1}(\mathbb{R}^{m \times n})$ such that

$$Y_t = M_t - \int_0^t f(s) ds = M_0 + \int_0^t Z_s^* dB_s - \int_0^t f(s) ds.$$

(Y, Z) thus constructed is a solution of the BSDE studied since $Y_T = \xi$,

$$\begin{aligned} Y_t - \xi &= M_0 + \int_0^t Z_s^* dB_s - \int_0^t f(s) ds - \left(M_0 + \int_0^T Z_s^* dB_s - \int_0^T f(s) ds \right) \\ &= \int_t^T f(s) ds - \int_t^T Z_s^* dB_s. \end{aligned}$$

The uniqueness follows from the construction made via the representation theory of Brownian martingales for solutions verifying $Z \in \mathbb{H}_T^{2,m \times n}$.

- (b) We now consider the case where the generator is Lipschitzian at (x, y) . As in the case of deterministic, the solution is obtained by applying the fixed point theorem in Banach space $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$ by building an application Ψ of $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$ in itself so that $(Y, Z) \in \mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$ is solution of the BSDE (1.4.3) iff it is a fixed point of Ψ . For (U, V) element of $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$, we define $(Y, Z) = \Psi(U, V)$ as being the solution of the BSDE

$$Y_t = \xi + \int_t^T f(s, U_s, V_s) ds - \int_t^T Z_s^* dB_s, \quad 0 \leq t \leq T.$$

Notice that this last BSDE has a unique solution which is in $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$. Indeed, let $F_s = f(s, U_s, V_s)$. This process belongs to $\mathbb{H}_T^{2,n}$ since, f being Lipschitz,

$$|F_s| \leq |f(s, 0, 0)| + \lambda |U_s| + \lambda \|V_s\|$$

and these last three processes are square integrable. Therefore, we can apply the result of (a) to obtain a unique solution (Y, Z) such that $Z \in \mathbb{H}_T^{2,m \times n}$. (Y, Z) belongs to $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$: The integrity of Z is obtained by construction and, according to Proposition 1.4.1, Y belongs to S_n^2 .

The application Ψ of $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$ in itself is therefore well-defined.

Let (U, V) and (U', V') two elements of $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$ such as :

$(Y, Z) = \Psi(U, \mathit{mathrm}V)$, $(Y', Z') = \Psi(U', V')$. Note $y = Y - Y'$ and $z = \mathit{mathrm}Z - Z'$. We have, $y_T = 0$ and

$$dy_t = -\{f(t, U_t, V_t) - f(t, U'_t, V'_t)\}dt + z_t dB_t.$$

We apply Itô's formula $e^{\alpha t} |y_t|^2$ to get:

$$d(e^{\alpha t} |y_t|^2) = \alpha e^{\alpha t} |y_t|^2 - 2e^{\alpha t} y_t \cdot \{f(t, U_t, V_t) - f(t, U'_t, V'_t)\} dt + 2e^{\alpha t} y_t \cdot z_t dB_t + e^{\alpha t} \|z_t\|^2 dt.$$

Consequently, by integrating between t and T , we obtain

$$\begin{aligned} e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds &= \int_t^T e^{\alpha s} \left(-\alpha |y_s|^2 + 2y_s \cdot \{f(s, U_s, V_s) - f(s, U'_s, V'_s)\} \right) ds \\ &\quad - \int_t^T 2e^{\alpha s} y_s \cdot z_s dB_s. \end{aligned}$$

And, since f is Lipschitz, it comes, noting u and v for $U - U'$ and $V - V'$ respectively,

$$\begin{aligned} e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds &\leq \int_t^T e^{\alpha s} \left(-\alpha |y_s|^2 + 2\lambda |y_s| |u_s| + 2\lambda |y_s| \|v_s\| \right) ds \\ &\quad - \int_t^T 2e^{\alpha s} y_s \cdot z_s dB_s. \end{aligned}$$

For all $\epsilon > 0$, we have $2ab \leq a^2/\epsilon + \epsilon b^2$ and therefore, the previous inequality gives

$$\begin{aligned} e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds &\leq \int_t^T e^{\alpha s} \left(-\alpha + 2\lambda^2/\epsilon \right) |y_s|^2 ds - \int_t^T 2e^{\alpha s} y_s \cdot z_s dB_s \\ &\quad \epsilon \int_t^T e^{\alpha s} \left(|u_s|^2 + \|v_s\|^2 \right) ds, \end{aligned}$$

and taking $\alpha = 2\lambda^2/\epsilon$, we have, noting $R_\epsilon = \epsilon \int_t^T e^{\alpha s} \left(|u_s|^2 + \|v_s\|^2 \right) ds$.

$$\forall t \in [0, T], \quad e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds \leq R_\epsilon - 2 \int_t^T e^{\alpha s} y_s \cdot z_s dB_s. \quad (\text{A.1.1})$$

By the Lemma [A.1.1](#), the local martingale $\left\{ \int_0^T e^{\alpha s} y_s \cdot z_s dB_s \right\}_{0 \leq t \leq T}$ is a zero martingale in 0 since Y, Y' belong to S_n^2 and Z, Z' belong to $\mathbb{H}_T^{2, m \times n}$. In particular, taking the expectation, which starts the stochastic integral via the previous remark, we obtain, for $t = 0$,

$$\mathbb{E} \left[\int_0^T e^{\alpha s} \|z_s\|^2 ds \right] \leq \mathbb{E} [R_\epsilon]. \quad (\text{A.1.2})$$

Returning to the inequality [\(A.1.1\)](#), the BDG inequalities provide, with C universal,

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} e^{\alpha t} |y_t|^2 \right] &\leq \mathbb{E} [R_\epsilon] + \text{CE} \left[\left(\int_0^t e^{\alpha s} |Y_s|^2 \|Z_s^*\|^2 ds \right)^{1/2} \right] \\ &\leq \mathbb{E} [R_\epsilon] + \text{CE} \left[\sup_{0 \leq t \leq T} e^{\alpha s/2} |Y_t| \left(\int_0^t e^{\alpha s} \|Z_s^*\|^2 ds \right)^{1/2} \right]. \end{aligned}$$

Since $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} e^{\alpha t} |y_t|^2 \right] \leq \mathbb{E} [R_\epsilon] + \frac{1}{2} \mathbb{E} \left[\sup_{0 \leq t \leq T} e^{\alpha s} |Y_t|^2 \right] + \frac{C^2}{2} \mathbb{E} \left[\int_0^t \|Z_s^*\|^2 ds \right].$$

Considering the inequality (A.1.2), we finally get

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} e^{\alpha t} |y_t|^2 + \int_0^T e^{\alpha s} \|z_s\|^2 ds \right] \leq (3 + C^2) \mathbb{E} [R_\epsilon]. \quad (\text{A.1.3})$$

And therefore, returning to the definition of R_ϵ ,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} e^{\alpha t} |y_t|^2 + \int_0^T e^{\alpha s} \|z_s\|^2 ds \right] \leq \epsilon(3 + C^2)(1 \vee T) \mathbb{E} \left[\sup_{0 \leq t \leq T} e^{\alpha t} |u_t|^2 + \int_0^T e^{\alpha s} \|v_s\|^2 ds \right]. \quad (\text{A.1.4})$$

Take ϵ such that $\epsilon(3 + C^2)(1 \vee T) = 1/2$, so that the application Ψ is then a strict contraction of $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$ in itself if we equip it with the standard

$$\|(U, V)\|_\alpha = \mathbb{E} \left[\sup_{0 \leq t \leq T} e^{\alpha t} |U_t|^2 + \int_0^T e^{\alpha s} \|V_s\|^2 ds \right]^{1/2},$$

which makes it a Banach space, this last norm being equivalent to the usual norm corresponding to the case $\alpha = 0$.

Ψ therefore has a unique fixed point, which ensures the existence and uniqueness of a solution of the BSDE (1.4.3) in $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$.

Proof A.1.5. [proposition 1.4.2]

Let the Picard sequence (Y^k, Z^k) be defined recursively by $(Y^0 = 0, Z^0 = 0)$, and

$$-dY_t^{k+1} = f(t, Y^k, Z^k)dt - dZ_t^{*k+1}dB_t, \quad Y_T^{k+1} = \xi_T.$$

Let $\delta^k(Y) = Y^k - Y^{k-1}$, $\delta^k(Z) = Z^k - Z^{k-1}$ two processes resulting from the variation between two iterations. Then, by applying the previous estimate (A.1.4) to $(u_t^k, v_t^k) = (Y^{k-1}, Z^{k-1})$ and $(Y^{k-1}, Z^{k-1}) = (Y^{(u^k, v^k)}, Z^{(u^k, v^k)})$, we obtain

$$\|(\delta^{k+1}(Y), \delta^{k+1}(Z))\|_\alpha^2 \leq \epsilon(3 + C^2)(1 \vee T) \|(\delta^k(Y), \delta^k(Z))\|_\alpha^2 \leq \eta^k \|((Y)^1, (Z)^1)\|_\alpha^2 \quad (\text{A.1.5})$$

where $\eta = \epsilon(3 + C^2)(1 \vee T) < 1$. The norm of the variations between two iterations is a convergent geometric series. So the two series $(Y^{k+1} - Y^k)$ and $(Z^{k+1} - Z^k)$ converge in Banach space $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$. In particular, the sequence of partial sums of the series converges in $\mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$ and $dt \otimes d\mathbb{P}$ -a.s. .

Moreover, it is possible to obtain the same type of estimate for the norm associated with the semimartingale $\delta^k(Y)$ defined by $\|\delta^k(Y)\|_\infty^2 := \mathbb{E} \left[\sup_{0 \leq t \leq T} |\delta_t^k(Y)|^2 \right]$.

First, let us observe that for any process of Itô $\Phi_t = \Phi_0 + \int_0^t \phi_s ds + \int_0^t \sigma_s^* dB_s$ with $(\phi, \sigma) \in \mathbb{H}_T^{2,n} \times \mathbb{H}_T^{2,m \times n}$, the maximum inequality of the martingales ensures that

$$\left\| \int_0^t \sigma_s^* dB_s \right\|_\infty^2 := \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \sigma_s^* dB_s \right|^2 \right] \leq 4\mathbb{E} \left[\int_0^t |\sigma_s|^2 ds \right] := 4\|\sigma\|_{\mathbb{H}_T^{2,m \times n}}^2.$$

Since, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \leq 2(a^2 + b^2 + c^2)$.

The same estimate for the process of Itô Φ gives:

$$\begin{aligned} \|\Phi\|_\infty^2 &\leq 2 \left(\mathbb{E} [|\Phi_0|^2] + \mathbb{E} \left[\left(\int_0^T |\phi_s| ds \right)^2 \right] + \left\| \int_0^T \sigma_s^* dB_s \right\|_\infty^2 \right) \\ &\leq 2\mathbb{E} [|\Phi_0|^2] + 2T\|\phi\|_{\mathbb{H}_T^{2,n}}^2 + 8\|\sigma\|_{\mathbb{H}_T^{2,m \times n}}^2. \end{aligned}$$

Using the representation of semi-martingale $\Phi_t = \Phi_T - \int_t^T \phi_s ds - \int_t^T \sigma_s dB_s$, the same estimate can be used, given by the inequalities:

$$\left| \int_t^T \sigma_s dB_s \right|^2 \leq 2 \left(\left| \int_0^T \sigma_s dB_s \right|^2 + \left| \int_0^t \sigma_s dB_s \right|^2 \right)$$

and

$$\|\Phi\|_\infty^2 \leq 2 \left(\mathbb{E}[|\Phi_T|^2] + 2T\|\phi\|_{\mathbb{H}_T^{2,n}}^2 + 10\|\sigma\|_{\mathbb{H}_T^{2,m \times n}}^2 \right).$$

So, let's apply this last inequality to the semimartingale $\delta^{k+1}(Y)$ with the decomposition $\phi_t^k = f(t, Y^k, Z^k) - f(t, Y^{k-1}, Z^{k-1})$ and $\sigma_t^k = Z^k - Z^{k-1} = \delta_t^k(Z)$. In addition f is Lipschitz with a Lipschitz constant C , $\|\phi\|_2 \leq C\|(\delta^k(Y), \delta^k(Z))\|_2$. So we get,

$$\|\delta^{k+1}(Y)\|_\infty^2 \leq 2TC\|(\delta^k(Y), \delta^k(Z))\|_2^2 + 20\|\delta^k(Z)\|_2^2 \leq K\|(\delta^k(Y), \delta^k(Z))\|_2^2 \leq K\eta^{k-1}$$

where K is a universal constant. This last inequality is given by the equation (A.1.5) in terms of α -norm which is equivalent to the norm $\|\cdot\|_{\mathbb{H}_T^{2,n}}$. Therefore, the series $\delta^{k+1}(Y)$ is uniformly convergent.

Proof A.1.6. [proposition 1.4.4]

Let's start by noting that the process Γ verifies:

$$d\Gamma_t = \Gamma_t(a_t dt + b_t \cdot dB_t), \quad \Gamma_0 = 1.$$

On the other hand, since b is bounded, Doob's inequality shows that Γ belongs to S^2 . Moreover, the hypotheses of this proposition ensure the existence of a unique solution (Y, Z) to the linear BSDEs; it suffices to set $f(t, y, z) = a_t y + b_t z + c_t$ and to verify that **(H1)** is satisfied. Y belongs to S^2 by Proposition 1.4.1.

The formula of integration by parts gives

$$d\Gamma_t Y_t = \Gamma_t dY_t + Y_t d\Gamma_t + d\langle \Gamma, Y \rangle_t = -\Gamma_t c_t dt + \Gamma_t Z_t dB_t + \Gamma_t Y_t b_t \cdot dB_t,$$

which shows that the process $\Gamma_t Y_t + \int_0^t \Gamma_s c_s ds$ is a local martingale which is in fact a martingale because $c \in \mathbb{H}^{2,1}$ and Γ, Y are in S^2 .

As a result,

$$\Gamma_t Y_t + \int_0^t \Gamma_s c_s ds = \mathbb{E} \left[\Gamma_T Y_T + \int_0^T \Gamma_s c_s ds \mid \mathcal{F}_t \right],$$

which gives the formula announced.

Proof A.1.7. [theorem 1.4.4]

The proof is carried out by linearization which makes it possible to reduce to linear EDSRs. We are looking for an equation satisfied by $U = Y - Y'$; we have noting $V = Z - Z'$ and $\zeta = \xi - \xi'$,

$$U_t = \zeta + \int_t^T (f'(s, Y'_s, Z'_s) - f(s, Y_s, Z_s)) ds - \int_t^T V_s dB_s.$$

We split the increase of f into three pieces by writing

$$\begin{aligned} f'(s, Y'_s, Z'_s) - f(s, Y_s, Z_s) &= f'(s, Y'_s, Z'_s) - f'(s, Y_s, Z'_s) + f'(s, Y_s, Z'_s) - f'(s, Y_s, Z_s) + \\ &\quad f'(s, Y_s, Z_s) - f(s, Y_s, Z_s) \quad (\text{which is positive here}). \end{aligned}$$

We introduce two processes a and b : a is real-valued and b is a vector (column) of dimension m . We pose

$$a_s = \frac{f'(s, Y'_s, Z'_s) - f'(s, Y_s, Z'_s)}{U_s}, \quad \text{if } U_s \neq 0, \text{ and } a_s = 0 \text{ otherwise.}$$

To define b , we must introduce another notation: for $0 \leq i \leq m$, Z_s is the row whose last m_i components are those of Z'_s and the i first those of Z_s . For $1 \leq i \leq m$, we pose

$$b_s^i = \frac{f'(s, Y_s, Z_s^{i-1}) - f'(s, Y_s, Z_s^i)}{V_s^i}, \quad \text{if } V_s^i \neq 0, \text{ and } b_s^i = 0 \text{ otherwise.}$$

Note that, since f is Lipschitz, these two processes are progressively measurable and bounded. With these notations, we have,

$$U_t = \xi + \int_t^T \{a_s U_s + b_s V_s + c_s\} ds - \int_t^T V_s dB_s,$$

where $c_s = f'(s, Y_s, Z_s) - f(s, Y_s, Z_s)$. By hypothesis, we have $\xi \geq 0$ and $c_s \geq 0$. Using the "explicit" formula for linear BSDEs-Proposition 1.4.4-, we have, for $t \in [0, T]$,

$$U_t = \Gamma_t^{-1} \mathbb{E} \left[\xi \Gamma_T + \int_t^T c_s \Gamma_s ds \mid \mathcal{F}_t \right],$$

with, for $0 \leq s \leq T$,

$$\Gamma_s = \exp \left\{ \int_0^s Z_u dB_u - \frac{1}{2} \int_0^s |b_u|^2 du + \int_0^s a_u du \right\}.$$

As already mentioned in the remark following Proposition 1.4.4, this formula shows that $U_t \geq 0$, as soon as $\xi \geq 0$ and $c_s \geq 0$. For the second part of the result, if in addition $U_0 = 0$ we have

$$0 = \mathbb{E} \left[\xi \Gamma_T + \int_0^T c_s \Gamma_s ds \right],$$

and the integrated random variable is positive. Therefore, it is zero \mathbb{P} -a.s. which ends the proof of this theorem by noting that in this case $\xi = 0$ and $c_s = 0$.

Proof A.1.8. [theorem 1.5.1]

First, suppose that θ is a tiered random variable: $\theta = \sum_{i=1}^l x_i 1_{A_i}$ where $(A_i)_{1 \leq i \leq l}$ is a \mathcal{F}_t -measurable partition of Ω and $x_i \in \mathbb{R}^n$ for $\forall i = 1 \dots l$. Note $(X_s^i, Y_s^i, Z_s^i)_{0 \leq s \leq T}$ in the place of $(X_s^{t,x_i}, Y_s^{t,x_i}, Z_s^{t,x_i})_{0 \leq s \leq T}$. We have then, for $t \leq s \leq T$,

$$X_s^{t\theta} = \sum_{i=1}^l 1_{A_i} X_s^i, \quad Y_s^{t,\theta} = \sum_{i=1}^l 1_{A_i} Y_s^i, \quad Z_s^{t,\theta} = \sum_{i=1}^l 1_{A_i} Z_s^i$$

Indeed, by definition, for $\forall i$, if $s \geq t$,

$$X_s^i = x_i + \int_t^s b(u, X_u^i) du + \int_t^s \sigma(u, X_u^i) dB_u,$$

and, by multiplying by 1_{A_i} and taking the sum over i , we get, the indicatrices fit inside the stochastic integral since $A_i \in \mathcal{F}_t$.

$$\sum_{i=1}^l 1_{A_i} X_s^i = \theta + \int_t^s \sum_{i=1}^l 1_{A_i} b(u, X_u^i) du + \int_t^s \sum_{i=1}^l 1_{A_i} \sigma(u, X_u^i) dB_u;$$

or we have $\sum_{i=1}^l 1_{A_i} H(T_i) = H(\sum_{i=1}^l 1_{A_i} T_i)$. It follows that

$$\sum_{i=1}^l 1_{A_i} X_s^i = \theta + \int_t^s \sum_{i=1}^l 1_{A_i} b(u, \sum_{i=1}^l 1_{A_i} X_u^i) du + \int_t^s \sum_{i=1}^l 1_{A_i} \sigma(u, \sum_{i=1}^l 1_{A_i} X_u^i) dB_u.$$

By uniqueness of the SDE solutions with Lipschitz coefficients, we have

$$\mathbb{P} - a.s., \quad \forall t \leq s \leq T, \quad X_s^{t,\theta} = \sum_{i=1}^l 1_{A_i} X_s^i.$$

In the same way, we have, for $\forall i$, if $t \leq s \leq T$,

$$Y_s^i = g(X_T^i) + \int_s^T f(u, X_u^i, Y_u^i, Z_u^i) du - \int_s^T Z_u^i dB_u,$$

and multiplying by 1_{A_i} and summing up i , we show, using the relation already established for the diffusion that $(\sum_{i=1}^l 1_{A_i}, Y_u^i, \sum_{i=1}^l 1_{A_i} Z_u^i)$ is solution on $[t, T]$ of the BSDE

$$Y_s' = g(X_T^{t,\theta}) + \int_s^T f(u, X_u^{t,\theta}, Y_u', Z_u') du - \int_s^T Z_u' dB_u,$$

whose $(Y_s^{t,\theta}, Z_s^{t,\theta})$ is also solution by definition. The uniqueness of the BSDE solutions give

$$\mathbb{P} - a.s., \quad \forall t \leq s \leq T, \quad Y_s^{t,\theta} = \sum_{i=1}^l 1_{A_i} Y_s^i, \quad Z_s^{t,\theta} = \sum_{i=1}^l 1_{A_i} Z_s^i, \quad \mathbb{P} \otimes m - a.e. \text{ on } \omega \times [t, T].$$

In particular, for $s = t$, we have, returning to the initial notations, $\mathbb{P} - a.s.$, $\forall t \leq s \leq T$

$$, \quad Y_t^{t,\theta} = \sum_{i=1}^l 1_{A_i} Y_t^i = \sum_{i=1}^l 1_{A_i} Y_t^{t,x_i} = \sum_{i=1}^l 1_{A_i} u(t, x_i) = u(t, \sum_{i=1}^l 1_{A_i} x_i) = u(t, \theta).$$

Before dealing with the general case, let's take a look at what happens under the (Lip) hypothesis. Consider a sequence of θ_n , \mathcal{F}_t -tiered, which converges to θ in $L^2(\mathcal{F}_t)$. Proposition 1.5.1 provides the estimate

$$\mathbb{E} [|Y_t^{t,\theta_n} - Y_t^{t,\theta}|^2] \leq C \mathbb{E} [|\theta_n - \theta|^2].$$

And, according to Proposition 1.5.3, $u(t, \cdot)$ is Lipschitzian which implies that

$$\mathbb{E} [|u(t, \theta_n) - u(t, \theta)|^2] \leq C \mathbb{E} [|\theta_n - \theta|^2].$$

However, according to the previous step, for any integer n , $u(t, \theta_n) = Y_t^{t,\theta_n}$; we can directly deduce that $u(t, \theta) = Y_t^{t,\theta}$, $\mathbb{P} - a.s.$.

In the general case, consider a sequence of va θ_n , \mathcal{F}_t -tiered, which converges to θ in $L^{2p}(\mathcal{F}_t)$. Proposition 1.5.1 always implies that

$$Y_t^{t,\theta_n} \longrightarrow Y_t^{t,\theta}, \quad \text{in } L^2, \quad \text{if } n \longrightarrow +\infty.$$

And, moreover, the continuity of the map $u(t, \cdot)$ implies, in particular, the convergence of $u(t, \theta_n)$ to $u(t, \theta)$ in probability. In addition $|u(t, x)|$ is bounded by $C(1 + |x|^p)$. As a result,

$$|u(t, \theta_n)|^2 \leq C'(1 + |\theta_n|^{2p}),$$

which shows that $(|u(t, \theta_n)|^2)_n$ is an equi-integrated sequence; so $u(t, \theta_n)$ converges to $u(t, \theta)$ in probability and in L^2 . Since $(|u(t, \theta_n)|^2)_n = Y_t^{t,\theta_n}$, \mathbb{P} -a.s., we obtain $u(t, \theta) = Y_t^{t,\theta}$, \mathbb{P} -a.s.

Hence the result.

Proof A.1.9. [proposition 1.5.4]

The growth condition on the gradient of v ensures that $Dv\sigma(r, X_r^{t,x})$ is an integral square process. Moreover, since v is regular, we obtain by applying Itô's formula,

$$dv(r, X_r^{t,x}) = \partial_t v(r, X_r^{t,x})dr + \nabla v(r, X_r^{t,x}) \cdot dX_r^{t,x} + \frac{1}{2} \text{trace}(\sigma\sigma^*(r, X_r^{t,x})D^2v(r, X_r^{t,x}))dr.$$

Or again, using $dX_r^{t,x}$ for $r \geq t$, and the definition of \mathcal{L} ,

$$dv(r, X_r^{t,x}) = \{\partial_t v(r, X_r^{t,x}) + \mathcal{L}v(r, X_r^{t,x})\}dr + Dv\sigma(r, X_r^{t,x})dB_r.$$

Or v is PDE solution (1.5.9), therefore $\partial_t v + \mathcal{L}v = -f$; it comes then

$$dv(r, X_r^{t,x}) = -f(r, X_r^{t,x}, v(r, X_r^{t,x}), Dv\sigma(r, X_r^{t,x})) + Dv\sigma(r, X_r^{t,x})dB_r.$$

We conclude by integrating s to T , noting that $v(T, X_T^{t,x}) = g(X_T^{t,x})$.

Proof A.1.10. [corollary 1.5.2]

From the proposition 1.5.4, we know that,

$v(t, x) = Y_t^{t,x}$. But when the function f takes the form $f(t, x, y, z) = c(t, x)y + h(t, x)$, the BSDE we need to solve is a Linear BSDE. We then use the formula seen on the linear BSDEs to conclude.

Proof A.1.11. [theorem 1.5.2]

Notice that the function u is continuous and furthermore verifies $u(T, \cdot) = g$. We let us only show that u is a sub-solution (the proof of u super-solution is the same).

So let φ be a function of class $C^{1,2}$ such that $u - \varphi$ has in (t_0, x_0) a local maximum ($0 < t_0 < T$). Even if it means replacing φ by $\varphi - \varphi(t_0, x_0) + u(t_0, x_0)$, operation which does not affect not the derivatives of φ , we can assume that $\varphi(t_0, x_0) = u(t_0, x_0)$. We

have to show that

$$\varphi(t_0, x_0)' + \mathcal{L}\varphi(t_0, x_0) + f(t_0, x_0, u(t_0, x_0), D\varphi\sigma(t_0, x_0)) \geq 0.$$

Suppose the opposite i.e. there exists $\delta > 0$ such that

$$\varphi(t_0, x_0)' + \mathcal{L}\varphi(t_0, x_0) + f(t_0, x_0, u(t_0, x_0), D\varphi\sigma(t_0, x_0)) = -\delta < 0.$$

$u - \varphi$ has a local maximum in (t_0, x_0) which is zero so by continuity, there exists $0 < \alpha \leq T - t_0$ such that, if $t_0 \leq t \leq t_0 + \alpha$ and $|x - x_0| \leq \alpha$,

$$u(t, x) \leq \varphi(t, x), \quad \text{and,} \quad \varphi(t, x)' + \mathcal{L}\varphi(t, x) + f(t, x, u(t, x), D\varphi\sigma(t, x)) \leq -\delta/2.$$

Consider the stopping time and $\tau = \inf\{u \geq t_0; |X_u^{t_0, x_0} - x_0| > \alpha\} \wedge t_0 + \alpha$. As X^{t_0, x_0} is a continuous process we have $|X_\tau^{t_0, x_0} - x_0| \leq \alpha$.

Itô's formula applied to $\varphi(s, X_s^{t_0, x_0})$ give

$$d\varphi(s, X_s^{t_0, x_0}) = \{\varphi(s, X_s^{t_0, x_0})' + \mathcal{L}\varphi(s, X_s^{t_0, x_0})\}ds + D\varphi\sigma(s, X_s^{t_0, x_0})dB_s.$$

And, by integrating $u \wedge \tau$ to $(t_0 + \alpha) \wedge \tau = \tau$ for $t_0 \leq u \leq t_0 + \alpha$, we obtain

$$\varphi(u \wedge \tau, X_{u \wedge \tau}^{t_0, x_0}) = \varphi(\tau, X_\tau^{t_0, x_0}) - \int_{u \wedge \tau}^{\tau} \{\varphi' + \mathcal{L}\varphi\}(s, X_s^{t_0, x_0}) ds - \int_{u \wedge \tau}^{\tau} D\varphi\sigma(s, X_s^{t_0, x_0})dB_s.$$

Note, for $t_0 \leq u \leq t_0 + \alpha$, $Y'_u = \varphi(u \wedge \tau, X_{u \wedge \tau}^{t_0, x_0})$ and $Z'_u = 1_{u \leq \tau} D\varphi\sigma(s, X_s^{t_0, x_0})$; the previous equality takes the form

$$Y'_u = \varphi(\tau, X_\tau^{t_0, x_0}) + \int_u^{t_0 + \alpha} -1_{u \leq \tau} \{\varphi' + \mathcal{L}\varphi\}(s, X_s^{t_0, x_0}) ds - \int_u^{t_0 + \alpha} Z'_s dB_s, \quad t_0 \leq u \leq t_0 + \alpha.$$

Similarly, if we pose, for $t_0 \leq u \leq t_0 + \alpha$, $Y_u = Y_{u \wedge \tau}^{t_0, x_0}$ and $Z_u = 1_{u \leq \tau} Z_u^{t_0, x_0}$, we have

$$Y_u = Y_{t_0 + \alpha} + \int_u^{t_0 + \alpha} 1_{u \leq \tau} f(s, X_s^{t_0, x_0}, Y_s, Z_s) ds - \int_u^{t_0 + \alpha} Z_s dB_s, \quad t_0 \leq u \leq t_0 + \alpha.$$

Now the Markov property implies that, \mathbb{P} -a.s. for $\forall t_0 \leq u \leq t_0 + \alpha$,

$$Y_s^{t_0, x_0} = u(s, X_s^{t_0, x_0}) \text{ from which we deduce that } Y_{t_0 + \alpha} = Y_\tau^{t_0, x_0} = u(\tau, X_\tau^{t_0, x_0}).$$

The previous equality then becomes

$$Y_u = u(\tau, X_\tau^{t_0, x_0}) + \int_u^{t_0 + \alpha} 1_{u \leq \tau} f(s, X_s^{t_0, x_0}, u(s, X_s^{t_0, x_0}), Z_s) ds - \int_u^{t_0 + \alpha} Z_s dB_s, \quad t_0 \leq u \leq t_0 + \alpha.$$

We will apply the comparison theorem to $(Y'_u, Z'_u)_u$ and $(Y_u, Z_u)_u$ which are BSDE so-

lutions. By definition of τ , we have $u(\tau, X_\tau^{t_0, x_0}) \leq \varphi(\tau, X_\tau^{t_0, x_0})$ and

$$\begin{aligned} 1_{u \leq \tau} f(s, X_s^{t_0, x_0}, u(s, X_s^{t_0, x_0}), Z'_s) &= 1_{u \leq \tau} f(s, X_s^{t_0, x_0}, u(s, X_s^{t_0, x_0}), D\varphi\sigma(s, X_s^{t_0, x_0})) \\ &\leq -1_{u \leq \tau} \{\varphi' + \mathcal{L}\varphi\}(s, X_s^{t_0, x_0}). \end{aligned}$$

In addition, we have, still by definition of τ ,

$$\mathbb{E} \left[\int_u^{t_0 + \alpha} -1_{u \leq \tau} (\varphi' + \mathcal{L}\varphi + f)(s, X_s^{t_0, x_0}, u(s, X_s^{t_0, x_0}), D\varphi\sigma(s, X_s^{t_0, x_0})) ds \right] \geq \mathbb{E} [\tau - t_0] \delta / 2.$$

This quantity is strictly positive. Indeed, $\delta > 0$ and $\tau > t_0$ because $|X_t^{t_0, x_0} - x_0| = 0$. We can therefore apply the strict version of the comparison theorem, to obtain $u(t_0, x_0) = Y_{t_0} < Y'_{t_0} = \varphi(t_0, x_0)$. This is impossible since $u(t_0, x_0) = \varphi(t_0, x_0)$. u is therefore indeed a sub-solution.

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