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THÈSE DE DOCTORAT DE

L'UNIVERSITÉ RENNES 2
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*Mathématiques, Télécommunications, Informatique, Signal, Systèmes,
Électronique*

Spécialité : *Mathématiques et leurs interactions*

Par

Émilien MANENT

**Propriétés de Feller, d'ergodicité et inférence non-paramétrique du
processus jump-move**

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Chapter 1

Introduction (en français)

1.1 Motivations et premiers exemples

1.1.1 Mise en contexte

Depuis à près d'un siècle les processus stochastiques de naissance et de mort ont été introduits pour modéliser l'évolution temporelle de la taille d'une population. La formalisation mathématique de ce modèle dans un cadre uniforme, lorsque les intensités de naissance et de mort sont les mêmes pour chaque individu, remonte au moins à Feller [Fel39]. Cette méthode issue de l'aléatoire permet de complexifier les méthodes déterministes parfois inadaptées notamment dans la description des phénomènes de démographie ou des modèles de prédation par exemple. Dans le même temps des scientifiques, qu'ils soient mathématiciens, physiciens ou biologistes, s'approprient ce cadre stochastique qu'ils ajustent selon la nature des phénomènes qu'ils considèrent : dynamique d'une population animale, évolution d'un ensemble de végétaux ou encore comportement de particules à l'échelle atomique. À titre d'exemple Kendall [Ken49] apporte une généralisation au modèle dans le cadre d'une population humaine pour que les intensités de naissance et de mort puissent prendre en compte le sexe et l'âge des individus, ce qui lui permet d'obtenir des résultats plus cohérents dans l'étude de l'évolution du nombre d'habitants.

Cependant dans de nombreuses applications, nous n'observons pas uniquement la taille d'une population dans le temps, mais nous avons accès à l'évolution spatio-temporelle des individus. Nous souhaitons tirer profit de ces observations complètes pour définir un modèle plus exhaustif et ainsi pouvoir décrire les différentes caractéristiques de la population considérée.

Pour décrire toutes ces dynamiques nous considérons un processus markovien plus général $(X_t)_{t \geq 0}$ construit comme une succession de déplacements continus et de sauts à valeurs dans un espace polonais localement compact E . Un tel processus sera appelé processus jump-move. L'espace d'état E n'est pas a priori un espace euclidien de dimension finie contrairement à ce qui est supposé dans les modèles standards de diffusions à sauts. Il existe de nombreux type de processus étudiés dans la littérature et qui peuvent être considérés comme des exemples du processus jump-move.

Nous citons en premier lieu les processus de Markov déterministes par morceaux, introduits par Davis [Dav84] comme un mélange de déplacements déterministes et de sauts aléatoires. L'intérêt de ce nouveau modèle était alors de pouvoir décrire une classe de phénomènes aléatoires non diffusifs. Une autre famille de processus utilisés pour modéliser des phénomènes de reproduction sont les processus de branchement. Dans ce contexte les individus disparaissent ou bien se divisent en plusieurs nouveaux éléments à chaque temps de saut aléatoire. Entre chaque saut les particules peuvent suivre une diffusion sur un espace euclidien. Les équations d'évolution des processus de branchements dans ce cadre sont établis dans [Sko64]. Les applications de cette classe de modèles sont importantes en biologie mais aussi en physique atomique pour décrire la propagation des neutrons dans le coeur d'une centrale nucléaire par exemple. Dans ce genre d'études il est intéressant d'établir le comportement en temps long de la population, notamment pour mettre en lumière les phénomènes d'extinction. À titre d'exemple, l'ouvrage [AN12] offre une présentation pertinente de ces problématiques. Notons que Löcherbach [Löc02] propose une généralisation de ces modèles considérant que les particules interagissent pendant la phase de déplacement.

Un cas particulier fondamental qui nous intéressera est lorsque l'espace d'état E s'écrit sous la forme $E = \cup_{n \geq 0} E_n$ où les E_n sont des espaces disjoints et pourront être interprétés comme l'ensemble des états d'une population à n individus. Dans cette optique l'ensemble E_0 sera constitué d'un unique élément noté \emptyset . De plus, dans ce cadre spécifique, les sauts du processus $(X_t)_{t \geq 0}$ auront lieu exclusivement de E_n vers E_{n+1} (représentant la naissance d'un individu) ou de E_n vers E_{n-1} (représentant la mort d'un individu). Une dynamique respectant ce cadre sera appelée processus de birth-death-move et a été introduite comme telle par Lavancier et Le Guével [LLG21]. Un cas particulier de ce modèle, plus précisément lorsque les individus restent immobiles entre les sauts, est présenté par Preston [Pre75] sous le nom de processus spatial de naissance et de mort.

Là encore les exemples de modèles pouvant être décrits comme des processus de birth-death-move sont riches. Dans [BM15] ces dynamiques sont traitées avec un point de vue centré sur chaque individu. Le processus considéré est alors à valeurs dans l'ensemble des mesures ponctuelles finies à support fermé dans \mathbb{R}^d . Cette formalisation permet d'obtenir une écriture de la dynamique sous forme d'une équation différentielle stochastique. Cette démarche aboutit en général à l'étude du comportement du processus lorsque le nombre d'individus tend vers l'infini, ce qui n'est pas l'objectif de notre travail. Cette formalisation sous forme de processus mesure est également présente dans [ÇK91] qui étudie les systèmes de particules qui évoluent selon un flot stochastique entre deux sauts. Une application pertinente de ce dernier modèle est l'analyse des flux des eaux usées.

Citons à présent deux applications qui pourraient être modélisées par un processus jump-move. Un premier exemple en microbiologie est présenté dans [LLG21]. L'observation des dynamiques de biomolécules au sein d'une cellule permet d'obtenir des séquences de données dont l'image de gauche de la figure 1 est issue. Après un traitement numérique on obtient l'image de droite de la figure 1 qui représente toutes les trajectoires détectées. Les trajectoires représentent les déplacements des

biomolécules qui se déplacent jusqu'à un certain instant où elles disparaissent, ce qui correspond à une mort. Lors de la séquence des points apparaissent : cela s'interprète comme la naissance d'une biomolécule. Toutes ces observations spatio-temporelles sont de nature aléatoire et liées aux interactions existantes entre chaque individu ce qui correspond bien à un modèle décrit par le processus birth-death-move.

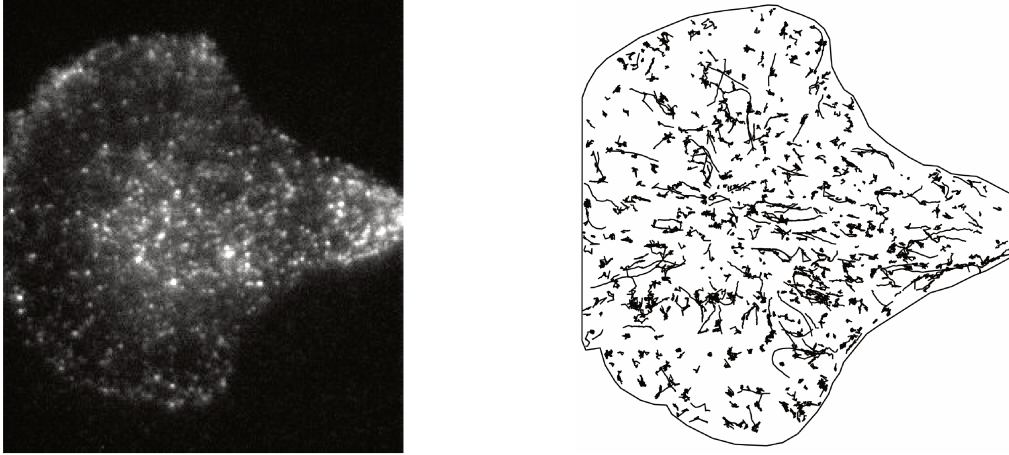


Figure 1: Dynamique des protéines de type Langerin près de la membrane d'une cellule lors du phénomène d'exocytose. La figure de gauche est issue d'une séquence d'images obtenues par microscopie par fluorescence. La figure de droite représente l'ensemble des trajectoires. Elle est obtenue par traitement informatique de la séquence d'images.

Citons une deuxième application liée à l'écologie et plus précisément à la gestion des forêts : les processus de croissance avec interaction présentés notamment dans [RS01] ou bien [HMP19]. Chaque individu consiste en un arbre localisé en un point de la forêt auquel est associée une marque positive qui représente la taille de la plante. Les arbres naissent et meurent à des instants aléatoires et les lieux de naissance sont en général attribués selon un processus ponctuel de Gibbs lié au potentiel d'interaction entre les arbres. Entre les sauts la position de chaque individu reste fixe tandis que la marque croît de manière continue et déterministe en interaction avec les marques des autres arbres. En général les marques sont données comme la solution d'une équation différentielle ordinaire liée au potentiel d'interaction. Cette exemple est donc bien décrit par un processus jump-move, le déplacement entre les sauts s'appliquant ici sur les marques. La figure 2 illustre une configuration obtenue avec ce modèle.

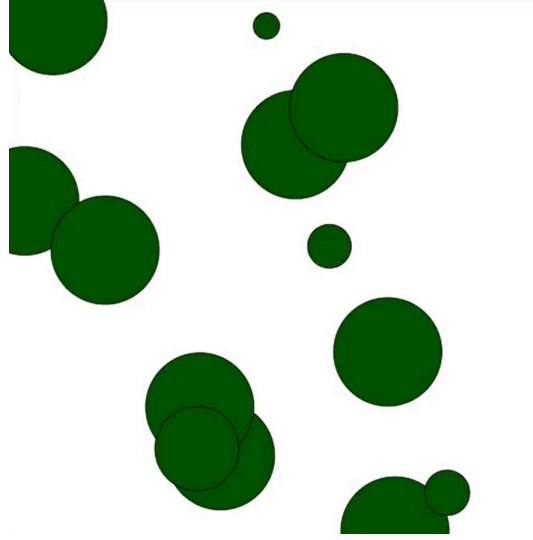


Figure 2: Extrait d'une simulation d'un processus de croissance avec interaction.

1.1.2 Objectifs mathématiques

Ce travail de thèse a deux principaux objectifs. Tout d'abord, il s'agit de définir et d'étudier des propriétés probabilistes du processus jump-move telles que son ergodicité. Ensuite, il consiste à estimer le noyau de probabilité de la loi des sauts du processus.

La première étape de ce travail, présentée au chapitre 2, est ainsi de définir le processus markovien jump-move et de développer des outils calculatoires : équation d'évolution de l'opérateur de transition, propriétés de Feller et expression du générateur infinitésimal. Dans le but d'avoir un cadre probabiliste complet pour mettre en place une démarche statistique nous obtiendrons lors du chapitre 3 des résultats d'ergodicité sur le processus birth-death-move. L'objectif est d'établir des résultats similaires à ceux prouvés pour le processus spatial de naissance et de mort (donc sans déplacement entre les sauts) par Preston [Pre75] et Møller [Møl89]. Les méthodes calculatoires développées dans ces articles ne s'appliquent pas dans le cadre où l'on ajoute un mouvement entre les sauts ce qui nécessitera d'introduire de nouveaux outils de calcul. Enfin le chapitre 4 de cette thèse porte sur l'estimation non-paramétrique du noyau de saut du processus jump-move, c'est-à-dire du noyau de probabilité caractérisant la loi des sauts. L'objectif dans ce travail est de retrouver des vitesses de convergence de l'estimateur proposé du même ordre que celles obtenues dans [LLG21] pour l'estimation de la fonction d'intensité du processus jump-move. Ici encore il nous faudra développer de nouveaux outils calculatoires car l'estimation du noyau de saut repose sur la loi conjointe de $(X_{t-}, X_t)_{t \geq 0}$ contrairement au cas de l'intensité qui est une fonction du processus $(X_t)_{t \geq 0}$.

À la lumière de ces objectifs mathématiques nous présenterons dans ce premier chapitre d'introduction les résultats existants pour le processus spatial de naissance et de mort (sans déplacement entre les sauts) en section 1.2 et pour l'estimation de l'intensité du processus jump-move $(X_t)_{t \geq 0}$ en section 1.3. La section 1.3 contient également la définition du processus jump-move $(X_t)_{t \geq 0}$. La section 1.4 quant à elle résume les principales contributions de la thèse.

1.2 Processus purement à sauts

Un exemple fondamental du processus général de jump-move est le cas où le déplacement entre les sauts est constant égal à la position initiale. Dans cette situation nous sommes ramenés à l'étude d'un processus de Markov purement à sauts. C'est cette théorie qui est utilisée par Preston [Pre75] pour prouver la bonne définition et le comportement asymptotique du processus spatial de naissance et de mort. Dans cette section nous présentons des résultats bien connus de ces processus que nous avons pour objectif de retrouver en ajoutant une étape de mouvement entre chaque saut.

1.2.1 Généralités sur les processus purement à sauts

Dans un premier temps rappelons comment sont définis les processus purement à sauts. Nous présentons également comment étudier leur dynamique en résolvant les équations d'évolution. Commençons par fixer un espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$ et un espace quelconque équipé d'une tribu (E, \mathcal{E}) .

Définition 1 (Feller [Fel71]). *Soit $\alpha : E \rightarrow \mathbb{R}_+$ et $K : E \times \mathcal{E} \rightarrow [0, 1]$ un noyau de probabilité. On dit que $(X_t)_{t \geq 0}$ est un processus de Markov à sauts si, étant donné $\{X_t = x\}$, pour $x \in E$ et $t \geq 0$, le temps d'attente du prochain saut suit une loi exponentielle de paramètre $\alpha(x)$, est indépendant du passé et la probabilité que ce saut soit vers un point de $A \in \mathcal{E}$ est $K(x, A)$. Dans ce cas α s'appelle l'intensité du processus et K le noyau de transition.*

Comme c'est l'usage pour les processus de Markov, on introduit l'opérateur de transition défini pour $t \geq 0$, $x \in E$ et $A \in \mathcal{E}$ par $Q_t(x, A) = \mathbb{P}(X_t \in A | X_0 = x)$.

Fixons pour la suite $x \in E$ et $A \in \mathcal{E}$. Par construction du processus $(X_t)_{t \geq 0}$ il est possible d'établir une équation d'évolution pour le processus appelée équation de Kolmogorov backward. On a pour tout $t \geq 0$,

$$Q_t(x, A) = e^{-\alpha(x)t} \mathbf{1}_A(x) + \alpha(x)e^{-\alpha(x)t} \int_0^t e^{\alpha(x)u} \int_E Q_u(y, A) K(x, dy) du.$$

Le processus purement à sauts sera considéré comme bien défini lorsque l'équation de Kolmogorov backward admet une unique solution à valeurs positives. Notons, pour $n \geq 0$, $Q_t^{(n)}(x, A)$ la probabilité de passer de l'état $X_0 = x$ à l'état $X_t \in A$ en au plus n sauts et définissons $Q_t^{(\infty)}(x, A) = \lim_{n \rightarrow \infty} Q_t^{(n)}(x, A)$. La résolution de l'équation de Kolmogorov backward est alors donnée par la proposition suivante.

Proposition 1 (Feller [Fel71]). *L'opérateur $Q_t^{(\infty)}$ est solution de l'équation de Kolmogorov backward. Si $Q_t^{(\infty)}$ est stochastique alors celle-ci en est l'unique solution à valeurs positives.*

Remarquons que la condition $Q_t^{(\infty)}(x, E) = 1$ est vérifiée en particulier lorsque la fonction d'intensité α est bornée. Dans le cas contraire la quantité $1 - Q_t^{(\infty)}(x, E)$ représente en quelque sorte la probabilité qu'une infinité de sauts ait lieu au temps $t \geq 0$ et donc le processus ne peut pas être défini saut après saut.

1.2.2 Exemple du processus spatial de naissance et de mort

Le processus simple de naissance et de mort

Le processus de naissance et de mort simple est un processus de Markov à sauts dont l'état représente le nombre d'individus vivants dans une population. C'est donc un processus à valeurs dans $E = \mathbb{N}$, dont les seules transitions possibles sont $n \rightarrow n + 1$ (naissance d'un individu) et $n \rightarrow n - 1$ (mort d'un individu) et les taux de naissance $(\beta_n)_{n \geq 0}$ et de mort $(\delta_n)_{n \geq 0}$ (avec $\delta_0 = 0$) ne dépendent que du nombre d'individus vivants.

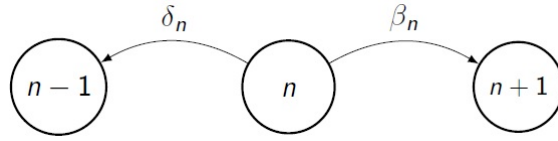


Figure 3: Processus simple de naissance et de mort.

En reprenant les notations de la définition 1 on établit que le processus simple de naissance et de mort est le processus purement à sauts sur \mathbb{N} associé à l'intensité $\alpha(n) = \beta_n + \delta_n$ et au noyau

$$K(n, A) = \frac{\beta_n}{\alpha(n)} \mathbf{1}_A(\{n + 1\}) + \frac{\delta_n}{\alpha(n)} \mathbf{1}_A(\{n - 1\}) \mathbf{1}_{n \geq 1},$$

pour $n \in \mathbb{N}$ et $A \in \mathcal{P}(\mathbb{N})$.

Dans le modèle du processus simple de naissance et de mort, les taux de transition ne dépendent que de la taille de la population. Bien qu'il possède de nombreuses applications, en théorie des files d'attente par exemple, ce cadre est trop grossier pour pouvoir décrire un certain nombre de phénomènes. Par exemple une population peut se développer selon un mécanisme de propagation ce qui nécessiterait un modèle dans lequel les nouveaux individus naissent proches de ceux déjà vivants. C'est avec cette ambition d'englober des modèles de population plus variées que Preston [Pre75] introduit le processus spatial de naissance et de mort dont la dynamique tient compte de la configuration spatiale de la population.

Notons tout de même pour finir que le processus simple de naissance et de mort à l'avantage d'être entièrement classifié. À titre d'exemple, des conditions explicites sur les taux de naissance et de mort sont obtenus dans [KM57] afin de décrire le comportement du processus (unicité de la solution de l'équation d'évolution de Kolmogorov backward, transience, récurrence, et temps de passage).

Le processus spatial de naissance et de mort [Pre75]

Le processus spatial de naissance et de mort est défini comme un processus purement à sauts. Ici l'espace d'états se décompose sous la forme : $E = \cup_{n=0}^{\infty} E_n$ où les $(E_n)_{n \geq 0}$ sont disjoints et où E_0 est constitué d'un unique élément noté \emptyset . Fixons \mathcal{E}_n une tribu

sur E_n et munissons E de la tribu engendrée par l'union des \mathcal{E}_n à savoir $\mathcal{E} = \bigvee_{n=0}^{\infty} \mathcal{E}_n$. Enfin pour $A \in \mathcal{E}$, on notera $A_n = A \cap E_n$.

Les taux de naissance et de mort sont deux fonctions mesurables β et δ définies de E vers \mathbb{R}_+ vérifiant $\delta(\emptyset) = 0$. L'intensité du processus est donc donnée par $\alpha = \beta + \delta$. Il reste à définir le noyau de transition K . Pour tout $n \geq 1$, soit $K_\beta^{(n)} : E_n \times \mathcal{E}_{n-1} \rightarrow [0, 1]$, et, pour $n \geq 0$, soit $K_\delta^{(n)} : E_n \times \mathcal{E}_{n+1} \rightarrow [0, 1]$ des noyaux de probabilité. On définit alors le noyau du processus $K : E \times \mathcal{E} \rightarrow [0, 1]$ par :

$$K(x, A) = \frac{\beta(x)}{\alpha(x)} K_\beta^{(n)}(x, A_{n+1}) + \frac{\delta(x)}{\alpha(x)} K_\delta^{(n)}(x, A_{n-1}) \mathbf{1}_{\mathbb{N}^*}(n),$$

pour tout $x \in E_n$. Remarquons que le processus simple de naissance et de mort est un processus spatial de naissance et de mort dans le cas où $E_n = \{n\}$ pour tout $n \geq 0$.

Pour cette dynamique l'équation d'évolution de Kolmogorov backward s'écrit donc pour tout $x \in E_n$ et $A \in \mathcal{E}$,

$$\begin{aligned} Q_t(x, A) &= e^{-(\delta(x)+\beta(x))t} \mathbf{1}_A(x) + \beta(x) \int_0^t e^{-(\beta(x)+\delta(x))(t-u)} \int_{E_{n+1}} Q_u(y, A) K_\beta^{(n)}(x, dy) du \\ &\quad + \delta(x) \int_0^t e^{-(\beta(x)+\delta(x))(t-u)} \int_{E_{n-1}} Q_t(y, A) K_\delta^{(n)}(x, dy) du \mathbf{1}_{\mathbb{N}^*}(n). \end{aligned}$$

Pour s'assurer que le processus n'explose pas en temps fini il est nécessaire de vérifier que cette équation admet une unique solution. C'est ce qu'explique le théorème suivant qui repose sur une technique de couplage.

Théorème 2 (Preston [Pre75]). *Pour $n \geq 0$ définissons les quantités $\beta_n = \sup_{x \in E_n} \beta(x)$ et $\delta_n = \inf_{x \in E_n} \delta(x)$. Supposons que $\beta_n < +\infty$ pour tout $n \geq 0$. Alors les conditions suivantes sont suffisantes pour que l'équation de Kolmogorov backward associée au processus spatial de naissance et mort admette une unique solution :*

1. *Il existe un rang $n_0 \geq 0$ tel que $n \geq n_0 \implies \beta_n = 0$.*
2. *Il existe un rang $n_0 \geq 0$ tel que $n \geq n_0 \implies \beta_n > 0$ et la série numérique $\sum_{n \geq n_0} w_n$ de terme général*

$$w_n = \frac{1}{\beta_n} + \frac{\delta_n}{\beta_n \beta_{n-1}} + \dots + \frac{\delta_n \dots \delta_{n_0+1}}{\beta_n \dots \beta_{n_0}} + \frac{\delta_n \dots \delta_{n_0}}{\beta_n \dots \beta_{n_0}}$$

diverge.

1.2.3 Comportement asymptotique du processus spatial de naissance et de mort

À des fins statistiques notamment, il est intéressant d'établir des conditions sous lesquelles le processus spatial de naissance et de mort admet une mesure invariante et à quelle vitesse la convergence vers celle-ci a lieu.

Le comportement asymptotique du processus spatial de naissance et de mort peut être décrit par la théorie du renouvellement. En effet son ergodicité sera caractérisée par des passages répétés en $E_0 = \{\emptyset\}$. En appliquant le théorème du renouvellement à un couplage entre un processus spatial de naissance et de mort et un processus simple de naissance et de mort bien choisi, Preston [Pre75] obtient des conditions sous lesquelles la mesure invariante est unique à partir des hypothèses, explicitées dans [KM57], sous lesquelles un processus simple de naissance et de mort est ergodique.

Soit $(X_t)_{t \geq 0}$ un processus spatial de naissance et de mort, tel qu'introduit dans la section 1.2.2, de taux respectifs β et δ . Une mesure de probabilité invariante pour le processus $(X_t)_{t \geq 0}$ est une mesure μ sur (E, \mathcal{E}) , de masse 1, et telle que pour tout $t \geq 0$ et $A \in \mathcal{E}$, $\mu(A) = \int_E Q_t(x, A) \mu(dx)$.

Théorème 3 (Preston [Pre75]). *Pour $n \geq 0$ définissons, comme dans le théorème 2, $\beta_n = \sup_{x \in E_n} \beta(x)$, $\delta_n = \inf_{x \in E_n} \delta(x)$ et supposons $\beta_n < +\infty$ pour tout $n \geq 0$. Si $\delta_n > 0$ pour tout $n \geq 1$ et que l'une des conditions suivantes est vérifiée :*

1. *il existe un rang $n_0 \geq 1$ tel que $\beta_n = 0$ pour tout $n \geq n_0$,*
2. *$\beta_n > 0$ pour tout $n \geq 1$, $\sum_{n=2}^{+\infty} \frac{\beta_1 \dots \beta_{n-1}}{\delta_1 \dots \delta_n} < +\infty$ et $\sum_{n=0}^{\infty} \frac{\delta_1 \dots \delta_n}{\beta_1 \dots \beta_n} = \infty$,*

alors $\lim_{t \rightarrow +\infty} Q_t(x, A)$ existe pour tout $x \in E$, $A \in \mathcal{E}$ et est indépendante de x .

De plus $\mu(A) := \lim_{t \rightarrow +\infty} Q_t(x, A)$ définit l'unique mesure de probabilité invariante pour le processus $(X_t)_{t \geq 0}$.

En s'inscrivant toujours dans une démarche d'inférence statistique il est souhaitable de connaître la vitesse à laquelle le processus converge vers la mesure invariante et sous quelles conditions. C'est un outil précieux pour étudier les propriétés asymptotiques d'estimateurs des données du modèle telles que l'intensité α ou le noyau de transition K . Des conditions sous lesquelles la convergence du processus spatial de naissance et de mort est géométrique sont données dans [Møl89]. Il était attendu d'obtenir une vitesse exponentielle car cela correspond au taux de convergence d'une chaîne de Markov comme l'explique l'ouvrage de référence de [MT09].

Théorème 4 (Møller [Møl89]). *Reprenons les notations du théorème 3. Supposons $\beta_n < +\infty$ pour tout $n \geq 0$ et $\delta_n > 0$ pour tout $n \geq 1$. Soient γ_1 et γ_2 deux mesures de probabilité sur (E, \mathcal{E}) satisfaisant l'une des conditions suivantes :*

- i) *la condition 1. du théorème 3 est vérifiée et $\gamma_k \left(\bigcup_{n=0}^{n_0} E_n \right) = 1$ pour $k = 1, 2$;*
- ii) *la condition 2. du théorème 3 est vérifiée et $\sum_{n=2}^{\infty} \gamma_k(E_n) \sqrt{\frac{\delta_1 \dots \delta_n}{\beta_1 \dots \beta_n}} < \infty$ pour $k = 1, 2$.*

Alors, il existe $c > 0$ et $0 < r < 1$ tels que :

$$\sup_{A \in \mathcal{E}} \left| \int Q_t(x, A) \gamma_1(dx) - \int Q_t(y, A) \gamma_2(dy) \right| \leq cr^t.$$

De plus, dans le cas où $i)$ est vérifiée, les constantes c et r peuvent être choisies indépendamment de γ_1 et γ_2 .

Un des objectifs des chapitres 2 et 3 est de retrouver des résultats similaires à ceux énoncés par les théorèmes 2, 3 et 4 dans le cas où l'on ajoute une étape de mouvement entre les sauts. Notons que ces problématiques sont abordées avec le déplacement entre les sauts dans [LLG21] sous l'hypothèse $i)$ du théorème 4. Nous y reviendrons plus en détail en section 1.3.2.

1.3 Processus jump-move

1.3.1 Processus jump-move : modèle et notations

À présent nous présentons et définissons le processus jump-move. Comme c'était le cas pour les processus purement à sauts présentés en section 1.2, le processus jump-move est construit de manière itérative comme une alternance de sauts et de déplacements continus. Cette construction, donnée dans [LLG21], est inspirée de la construction des processus de Markov déterministes par morceaux introduits par Davis [Dav84] et dont l'ouvrage [DSDZ15] donne une présentation détaillée ainsi qu'une méthode de simulation. Notons qu'une telle construction permet, comme pour les processus de Markov déterministes par morceaux, de déduire un algorithme de simulation du processus jump-move (voir également [LLG21]).

Soit E un espace polonais muni d'une tribu notée \mathcal{E} et d'une distance d . Soit (Ω, \mathcal{F}) un espace mesurable muni d'une famille de mesures de probabilité $(\mathbb{P}_x)_{x \in E}$.

Un processus jump-move $(X_t)_{t \geq 0}$ sur E est caractérisé par trois paramètres. Tout d'abord, nous nous donnons une fonction d'intensité $\alpha : E \rightarrow \mathbb{R}_+$ et un noyau de transition $K : E \times \mathcal{E} \rightarrow [0, 1]$ de la même manière que pour définir un processus de Markov purement à sauts vus en section 1.2. Le rôle de la fonction d'intensité sera donc de déterminer les temps de sauts tandis que le noyau K indiquera le lieu du saut. Enfin, l'étape de mouvement du processus entre chacun de ses sauts est donnée par l'intermédiaire d'un processus de Markov homogène à trajectoires continues $((Y_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ sur E , appelé move associé au processus $(X_t)_{t \geq 0}$ et dont la distribution associée correspond au noyau de transition $(Q_t^Y)_{t \geq 0}$ défini par $Q_t^Y(x, A) = \mathbb{P}_x(Y_t \in A)$ pour tout $x \in E$ et $A \in \mathcal{E}$.

Pour garantir la non-explosion en temps fini du processus $(X_t)_{t \geq 0}$ la fonction d'intensité α sera supposée bornée par une constante strictement positive $\alpha^* > 0$. De cette façon l'équation de Kolmogorov backward associée à $(X_t)_{t \geq 0}$ aura une unique solution et cela justifie donc une construction itérative, saut par saut. Cette présentation du processus a l'avantage de donner une intuition claire de la dynamique du processus.

Donnons nous une suite $\left((Y_t^{(j)})_{t \geq 0}\right)_{j \geq 0}$ de processus sur E indépendants et de même loi que $(Y_t)_{t \geq 0}$. En suivant la construction de [LLG21], le processus $(X_t)_{t \geq 0}$ peut être présenté de manière itérative en alternant les sauts et les mouvements continus donnés par la suite $\left((Y_t^{(j)})_{t \geq 0}\right)_{j \geq 0}$.

Étape 0 : L'état initial X_0 est distribué selon une loi de probabilité donnée μ_0 à valeurs dans E . Par commodité posons $T_0 = 0$.

Étape 1 :

- Sachant $X_0 = z_0$, $(Y_t^{(0)})_{t \geq 0}$ est généré conditionnellement à $Y_0^{(0)} = z_0$ selon le noyau de transition $(Q_t^Y(z_0, \cdot))_{t \geq 0}$.
- Sachant $X_0 = z_0$ et $(Y_t^{(0)})_{t \geq 0}$, le premier temps de saut τ_1 est généré selon la fonction de répartition F_1 définie sur \mathbb{R}_+ par :

$$F_1(t) = 1 - \exp\left(-\int_0^t \alpha(Y_u^{(0)}) \, du\right).$$

- Sachant $X_0 = z_0$, $(Y_t^{(0)})_{t \geq 0}$ et τ_1 , la position obtenue après le premier saut z_1 est générée selon le noyau de probabilité $K(Y_{\tau_1}^{(0)}, \cdot)$.
- Posons alors $T_1 = \tau_1$, $X_t = Y_t^{(0)}$ pour $t \in [0, T_1)$ et $X_{T_1} = z_1$.

De manière itérative pour $j \geq 1$,

Étape $j+1$:

- Sachant $X_{T_j} = z_j$, $(Y_t^{(j)})_{t \geq 0}$ est généré conditionnellement à $Y_0^{(j)} = z_j$ selon le noyau de transition $(Q_t^Y(z_j, \cdot))_{t \geq 0}$.
- Sachant $X_{T_j} = z_j$ et $(Y_t^{(j)})_{t \geq 0}$, l'inter-temps de saut τ_{j+1} est généré selon la fonction de répartition F_{j+1} définie sur \mathbb{R}_+ par

$$F_{j+1}(t) = 1 - \exp\left(-\int_0^t \alpha(Y_u^{(j)}) \, du\right).$$

- Sachant $X_{T_j} = z_j$, $(Y_t^{(j)})_{t \geq 0}$ et τ_{j+1} , la position obtenue après le $j+1$ -ème saut z_{j+1} est générée selon le noyau de probabilité $K(Y_{\tau_{j+1}}^{(j)}, \cdot)$.
- Enfin posons $T_{j+1} = T_j + \tau_{j+1}$, $X_t = Y_{t-T_j}^{(j)}$ pour tout $t \in [T_j, T_{j+1})$ et $X_{T_{j+1}} = z_{j+1}$.

Dans cette construction $(T_j)_{j \geq 1}$ constitue la suite des temps de sauts associée au processus $(X_t)_{t \geq 0}$. Le nombre de sauts du processus avant le temps $t \geq 0$ est donc donné par

$$N_t = \text{Card}\{j \geq 1, T_j \leq t\}.$$

Il est prouvé dans [LLG21] que le processus $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ ainsi construit est un processus de Markov pour la filtration naturelle. De plus, on peut quantifier la

non-explosion du processus en temps fini. En effet un calcul de [LLG21] montre que pour tout $n \geq 0$ et $t \geq 0$ on a :

$$\mathbb{P}(N_t > n) \leq \mathbb{P}(N_t^* > n)$$

où N_t^* est une variable aléatoire distribuée selon une loi de Poisson de paramètre α^*t . Cette comparaison du nombre de sauts avec une loi usuelle sera un atout calculatoire important lors de l'étude du processus jump-move.

1.3.2 Processus birth-death-move

Comme nous l'avons expliqué en section 1.1.1 le processus de birth-death-move est un cas particulier du processus jump-move. Dans ce cadre spécifique l'espace d'états E se décompose sous la forme $E = \cup_{n=0}^{+\infty} E_n$ où les espaces $(E_n)_{n \geq 0}$ sont des espaces polonais disjoints équipés d'une tribu notée \mathcal{E}_n . Ainsi E est muni de la tribu \mathcal{E} engendrée par les tribus \mathcal{E}_n à savoir :

$$\mathcal{E} = \sigma \left(\bigcup_{n=0}^{+\infty} \mathcal{E}_n \right).$$

Pour un $n \in \mathbb{N}$ fixé, l'espace E_n sera interprété comme l'ensemble des états possibles pris par la population lorsque celle-ci est de taille n . Cet exemple se veut tout de même général : E_n n'est pas forcément associé à la configuration spatiale des individus et il n'est pas de dimension finie a priori. La volonté de garder un cadre vaste est liée à la multitude des applications comme nous l'avons présenté en section 1.1.1. Avec ce raisonnement il est raisonnable de considérer que l'espace E_0 , qui correspond à l'état d'une population vide, est constitué d'un singleton que nous noterons \emptyset . Cette hypothèse naturelle est de plus importante d'un point de vue mathématique lorsque l'on souhaite étudier l'ergodicité du processus. Introduisons enfin la fonction étage $n(\cdot)$ sur E telle que pour tout $x \in E$ on ait $x \in E_{n(x)}$.

Chaque saut du processus de birth-death-move doit s'interpréter comme une naissance ou une mort c'est pourquoi la fonction d'intensité α se décompose dans ce cas particulier sous la forme $\alpha = \beta + \delta$ où $\beta : E \rightarrow \mathbb{R}_+$ est l'intensité de naissance et $\delta : E \rightarrow \mathbb{R}_+$ l'intensité de mort. Nous supposons ces deux fonctions continues et bornées sur E de manière à ce que α le soit également. Enfin l'état \emptyset représentant la population vide, il est supposé que $\delta(\emptyset) = 0$. De la même manière que dans la construction du processus spatial de naissance et mort vu en section 1.2.2 le noyau de saut K se décompose naturellement comme la combinaison convexe d'un noyau de naissance et d'un noyau de mort : pour tout $x \in E$ et $A \in \mathcal{E}$ on a

$$K(x, A) = \frac{\beta(x)}{\alpha(x)} K_\beta(x, A) + \frac{\delta(x)}{\alpha(x)} K_\delta(x, A)$$

où K_β et K_δ sont des noyaux de probabilité sur $E \times \mathcal{E}$ tels que

$$K_\beta(x, E_{n+1}) = \mathbf{1}_{E_n}(x) \quad \text{et} \quad K_\delta(x, E_n) = \mathbf{1}_{E_{n+1}}(x).$$

Enfin, le mouvement $(Y_t)_{t \geq 0}$ d'un processus de birth-death-move correspond au déplacement continu de l'état de la population entre chaque saut. Par conséquent $(Y_t)_{t \geq 0}$ est cantonné à l'étage donné par la condition initiale, c'est-à-dire $\mathbb{P}((Y_t)_{t \geq 0} \subset E_n | Y_0 = x) = \mathbf{1}_{E_n}(x)$ pour tout $x \in E$ et $n \geq 0$. Remarquons que lorsque le déplacement est constant, autrement dit si $\mathbb{P}(Y_t \in A | Y_0 = x) = \mathbf{1}_A(x)$ pour tout $t \geq 0$, $x \in E$ et $A \in \mathcal{E}$, alors le processus birth-death-move correspond au processus spatial de naissance et mort introduit par Preston [Pre75].

Le processus birth-death-move est étudié de manière probabiliste et inférentielle par Lavancier et Le Guével [LLG21] dans le cas où il existe un rang $n^* \geq 1$ tel que la fonction d'intensité de naissance β est identiquement nulle sur $\cup_{n=n^*}^{+\infty} E_n$. Cela correspond à étudier un processus birth-death-move à valeurs dans l'espace d'états $E^* = \cup_{n=0}^{n^*} E_n$ et donc à considérer que le nombre d'individus est borné. Sous cette hypothèse le processus birth-death-move admet une unique mesure invariante μ_∞ et il existe deux constantes $c > 0$ et $r > 0$ telles que pour tout temps $t \geq 0$ on ait

$$\sup_{x \in E} \sup_{A \in \mathcal{E}} |\mathbb{P}_x(X_t \in A) - \mu_\infty(A)| \leq c e^{-rt}. \quad (1.1)$$

L'uniformité des constantes c et r est alors un atout considérable pour quantifier les performances asymptotiques d'estimateurs de la fonction d'intensité α ou du noyau de transition K . L'article [LLG21] contient du reste une étude complète de l'estimation non paramétrique de α . Dans le cas général nous expliciterons des hypothèses sous lesquelles la convergence géométrique a lieu, comme cela est présenté pour le processus de naissance et de mort spatial dans le théorème 4. Cependant l'uniformité de la constante c en la condition initiale $x \in E$ est perdue. Supposer l'existence d'un tel étage palier n^* n'est a priori pas contraignant pour la plupart des applications statistiques. Cependant, dans un souci de généralité et afin de pouvoir décrire un plus grand nombre de modèles nous souhaitons lever cette hypothèse et obtenir des résultats d'ergodicité et de statistique dans ce cadre plus vaste. Nous conservons tout de même dans notre définition du processus l'hypothèse d'une intensité α bornée sans quoi cela nous semble difficile d'étudier le processus de jump-move dans son cadre général.

1.3.3 Utilisation pour l'étude de systèmes de particules dans \mathbb{R}^d

Le processus birth-death-move s'applique pour décrire l'évolution spatio-temporelle d'un système de particules lorsque l'étage E_n , pour $n \geq 0$, correspond à l'ensemble des configurations spatiales à n particules. Pour ce faire on considère un sous ensemble W fermé de \mathbb{R}^d , pour $d \geq 1$, muni d'une tribu \mathcal{B} . Le fermé W correspond alors au lieu de vie des particules. Soit $n \geq 1$. L'ensemble des configurations de W à n particules peut-être construit à partir de W^n en identifiant deux éléments (x_1, \dots, x_n) et (y_1, \dots, y_n) de $(W^n, \mathcal{B}^{\otimes n})$ s'il existe une permutation σ de $\llbracket 1, n \rrbracket$ telle que $x_i = y_{\sigma(i)}$ pour tout $i \in \llbracket 1, n \rrbracket$. En notant $\pi_n : (x_1, \dots, x_n) \rightarrow \{x_1, \dots, x_n\}$ la projection naturelle associée à cette identification, il vient $E_n = \pi_n(W^n)$. Dans ce cadre les noyaux de naissance et de mort ne sont pas quelconques afin que le saut affecte uniquement la particule qui apparait ou qui disparaît : la configuration des

autres particules doit être conservée. Ainsi le noyau de naissance K_β vérifie pour tout $n \geq 1$ et $x = \{x_1, \dots, x_n\} \in E_n$, $K_\beta(x, x \cup W) = 1$ où

$$x \cup W = \{x \cup \{\xi\}, \xi \in W\} = \{\{x_1, \dots, x_n, \xi\}, \xi \in W\},$$

et le noyau de mort K_δ est tel que $K_\delta(x, \{x \setminus x_i, i \in \llbracket 1, n \rrbracket\}) = 1$ où

$$x \setminus x_i = \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$$

pour tout $i \in \llbracket 1, n \rrbracket$.

Soit $n \geq 1$. En ce qui concerne le déplacement entre les sauts nous verrons au chapitre 2 qu'à tout processus de Markov sur W^n à trajectoires continues peut-être associé un processus de Markov sur E_n à trajectoires continues par l'intermédiaire de la projection π_n . Ainsi il est possible de considérer que le mouvement continu associé à l'ensemble des particules entre chaque saut est issu d'une équation différentielle stochastique telle qu'une équation de Langevin amortie sur \mathbb{R}^d par exemple qui modélise des interactions deux à deux entre les particules. Cela correspond au système d'équations différentielles stochastiques suivant :

$$dZ_{t,i} = - \sum_{j \neq i} g(Z_{t,i} - Z_{t,j}) dt + \sqrt{2\beta^{-1}} dB_{t,i}, \quad 1 \leq i \leq n,$$

où $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ est une fonction lipschitzienne, $\beta > 0$ et $\{B_{t,i}\}_{1 \leq i \leq n, n \geq 1}$, est une famille de n mouvements browniens indépendants sur \mathbb{R}^d . La figure 4 représente une configuration X_t du processus à un instant $t \geq 0$ donné qui est obtenue lorsque la fonction g est associée à un potentiel de Lennard-Jones répulsif.

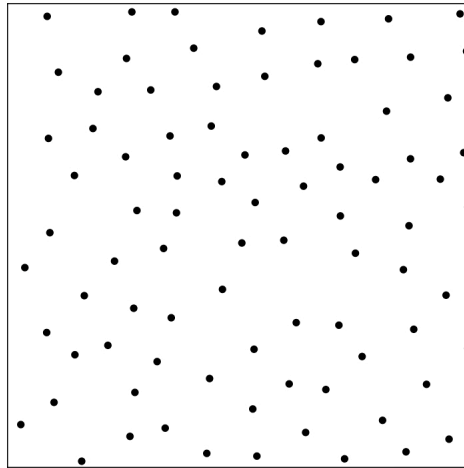


Figure 4: Exemple de configuration spatiale X_t à un instant $t \geq 0$ donné. Le déplacement entre les sauts est obtenu à partir d'une équation de Langevin amortie associée à un potentiel de Lennard-Jones répulsif.

Comme nous l'avons fait remarquer en section 1.1.1 il est possible de formaliser de tels systèmes de particules sur \mathbb{R}^d comme un processus à valeurs dans les mesures

à support fini. C'est le cas par exemple dans [BT11] où ce cadre est utilisé pour étudier des processus de branchements où les sujets suivent une diffusion de Feller entre chaque saut. Ce modèle s'applique alors à l'étude de la division de cellules infectées par un parasite. Cependant nous faisons le choix de conserver notre formalisme en partie car cette approche par processus à valeurs mesures semble dédiée principalement aux dynamiques individuelles sur un espace euclidien ce qui ne correspond pas au cadre de travail de cette thèse.

1.3.4 Inférence du processus jump-move

Un enjeu fondamental du travail de cette thèse, et de l'étude en général du processus de jump-move, est de proposer une estimation non-paramétrique de la fonction d'intensité α et du noyau de saut K du processus. En effet l'objectif premier de l'introduction du processus de jump-move est de pouvoir apporter des réponses mathématiques à des problématiques liées à des phénomènes biologiques ou physiques comme ceux présentés en section 1.1.1.

Tout d'abord lorsque nous considérons uniquement le processus simple de naissance et de mort, la procédure d'estimation porte sur les intensités de mort et de naissance $(\beta_n)_{n \geq 0}$ et $(\delta_n)_{n \geq 0}$. De nombreuses méthodes ont été développées pour répondre à ce problème dans la seconde moitié du vingtième siècle. À titre d'exemple [Dar56] propose une procédure paramétrique dans le cas simplifié où les intensités de naissance et de mort sont linéaires en le nombre d'individus à savoir qu'il existe des constantes strictement positives β_0 et δ_0 telles que pour tout $n \geq 1$, on ait $\beta_n = \beta_0 \times n$ et $\delta_n = \delta_0 \times n$. Le problème de l'estimation des fonctions d'intensité générales du processus simple de naissance et de mort a également été traité à de multiples reprises. Citons la méthode non-paramétrique mise en place dans [Wol65] et dans [Rey73] qui consiste simplement à estimer pour tout $n \geq 1$ l'intensité de naissance β_n (respectivement de mort δ_n) en comptant le nombre de naissances (respectivement de morts) observées lorsque la population est de taille n qui est ensuite normalisé par le temps total passé par la population à l'état $\{n\}$.

Notons également l'existence de nombreux travaux d'estimation sur des processus de naissance et de mort plus généraux. Par exemple [MS94] étudie les caractéristiques d'un processus spatial de naissance et de mort appliqué à l'étude du déplacement des dunes dans un désert. Également [Löc02] propose une méthode statistique paramétrique pour l'inférence d'un processus de branchement décrivant la dynamique spatio-temporelle d'une population. Des méthodes non paramétriques de choix de modèle sont parfois utilisées pour estimer l'intensité de processus spatio-temporels comme c'est le cas dans [RBS10] par exemple. Cependant cela nécessite en général de se restreindre au cadre d'un système de particules dans \mathbb{R}^d . En effet la méthode de projection avec pénalisation ne peut être mise en place que dans des cas simples ne serait-ce que pour pouvoir exhiber une famille totale explicite dans laquelle l'intensité peut se décomposer. Pour conserver un cadre le plus général possible nous utiliserons donc des méthodes non-paramétriques d'estimation à noyau. Ces méthodes sont classiques et adaptées à notre cas d'un espace d'état général qui n'est pas vectoriel a priori. Nous verrons tout de même que les résultats obtenus sont

satisfaisants. C'est d'ailleurs une méthode couramment utilisée, elle apparait par exemple pour l'estimation non-paramétrique de la densité conditionnelle des lieux de sauts ([Aza14]) ou de l'intensité des sauts ([ADGP14]) dans l'étude de processus de Markov déterministes par morceaux.

L'estimation des fonctions d'intensité α , β et δ du processus birth-death-move est traitée par Lavancier et Le Guével [LLG21] sous l'hypothèse de l'existence d'un nombre maximum n^* d'individus dont nous avons déjà parlé en section 1.3.2. La convergence vers la mesure invariante à vitesse exponentielle associée à des constantes uniformes donnée en (1.1) permet d'obtenir par la méthode des noyaux une vitesse de convergence standard sous des hypothèses assez larges. Dans cette thèse nous allons uniquement nous intéresser à l'estimation non paramétrique du noyau de transition K . En effet les méthodes utilisées dans [LLG21] pour estimer l'intensité se généralisent quitte à imposer des hypothèses plus restrictives. Cependant de nombreuses perspectives sont ouvertes pour l'estimation du noyau de transition K que ce soit dans un cadre très général d'un jump-move sur un espace quelconque E ou alors dans le cas particulier du processus de birth-death-move appliqué à l'étude d'un système de particules en interaction sur \mathbb{R}^d .

Pour terminer cette section énonçons un des résultats obtenu dans [LLG21]. Admettons que l'on observe le processus birth-death-move $(X_t)_{t \geq 0}$ sur un intervalle $[0, T]$ où $T > 0$. Faisons l'hypothèse que le nombre d'individus soit borné, c'est-à-dire qu'il existe un rang $n^* \geq 1$ tel que la fonction d'intensité de naissance β est identiquement nulle sur $\cup_{n=n^*}^{+\infty} E_n$ et on note donc μ_∞ l'unique mesure invariante du processus $(X_t)_{t \geq 0}$. Supposons que l'intensité α s'écrive sous la forme $\alpha(x) = \alpha_0(f(x))$ où $f : E \rightarrow \mathbb{R}^p$ (pour un certain $p \geq 1$) et α_0 est une fonction lipschitzienne sur \mathbb{R}^p . Supposons enfin que si Z est une variable aléatoire de loi μ_∞ alors $f(Z)$ est une variable aléatoire à densité continue et bornée sur \mathbb{R}^p . Étant donné k un noyau borné sur \mathbb{R} et tel que $\int_{\mathbb{R}} |t|^p k(|t|) dt < \infty$, on considère pour tout $x \in E$ l'estimateur $\hat{\alpha}(x)$ de $\alpha(x)$ défini par :

$$\hat{\alpha}(x) = \frac{1}{\int_0^T k\left(\frac{|f(X_s) - f(x)|}{h_T}\right) ds} \int_0^T k\left(\frac{|f(X_{s-}) - f(x)|}{h_T}\right) dN_s.$$

On a alors le résultat suivant :

Proposition 5 (Lavancier et Le Guével [LLG21]). *Sous les hypothèses énoncées ci-dessus, on a pour tout $\varepsilon > 0$ l'existence d'une constante $C > 0$ telle que :*

$$\mathbb{P}(|\hat{\alpha}(x) - \alpha(x)| > \varepsilon) \leq C \left(\frac{1}{Th_T^p} + h_T^2 \right).$$

Notre objectif dans le chapitre 4 est alors de retrouver une vitesse de convergence similaire dans l'estimation non-paramétrique du noyau de saut K .

1.4 Aperçu de la thèse

Nous terminons cette section avec une présentation des contributions mathématiques principales contenues dans cette thèse. La première section porte sur les aspects probabilistes des processus jump-move qui sont traités dans les chapitres 2 et 3. La deuxième section aborde l'inférence non-paramétrique du noyau de saut des processus jump-move dont les résultats sont énoncés au chapitre 4.

1.4.1 Chapitres 2 et 3 : propriétés probabilistes d'un processus jump-move

Article de référence : *Feller and ergodic properties of jump-move processes with applications to interacting particle systems.* [LLGM22]

Chapitre 2 : Feller properties of jump-move processes with applications to spatial particle systems.

Le chapitre 2 porte sur l'étude du processus jump-move en tant que processus Markovien : équation de Kolmogorov backward, propriétés de Feller et générateur infinitésimal. Nous commençons par formaliser la construction itérative du processus jump-move vue en section 1.3 et nous obtenons l'équation de Kolmogorov backward associée. Ayant supposé que l'intensité de saut α est bornée nous obtenons l'existence et l'unicité d'une solution à cette équation d'évolution. Dans la suite nous étudions les propriétés de Feller du processus jump-move $(X_t)_{t \geq 0}$. Pour ce faire introduisons l'opérateur de transition $(Q_t)_{t \geq 0}$ lié au processus markovien $(X_t)_{t \geq 0}$ et donné par

$$Q_t(x, A) = \mathbb{P}(X_t \in A | X_0 = x) = \mathbb{P}_x(X_t \in A)$$

pour tout $t \geq 0$, $x \in E$ et $A \in \mathcal{E}$. De même pour tout $f \in M_b(E)$, où $M_b(E)$ est l'ensemble des fonctions à valeurs réelles mesurables et bornées sur E , nous noterons

$$Q_t f(x) = \mathbb{E}_x[f(X_t)] = \int_E Q_t(x, dy) f(y).$$

De la même manière, posons $Q_t^Y f(x) = \mathbb{E}_x^Y[f(Y_t)]$. Pour rappel on dit que le processus $(X_t)_{t \geq 0}$ est dit Feller continu si pour tout $t \geq 0$ on a $Q_t C_b(E) \subset C_b(E)$, où $C_b(E)$ désigne l'ensemble des fonctions continues et bornées sur E . Également le processus $(X_t)_{t \geq 0}$ est dit Feller lorsque $\lim_{t \rightarrow 0} \|Q_t f - f\|_\infty = 0$ pour toute fonction $f \in C_0(E)$ et que pour tout $t \geq 0$ on a $Q_t C_0(E) \subset C_0(E)$, où $C_0(E)$ désigne l'ensemble des fonctions continues sur E et qui tendent vers 0 à l'infini. Nous avons obtenu une proposition à l'énoncé simple qui énonce que si le processus purement à sauts associé au noyau de transition K d'une part et le mouvement $(Y_t)_{t \geq 0}$ d'autre part sont tous les deux des processus Feller continus (respectivement Feller), alors $(X_t)_{t \geq 0}$ est Feller continu (respectivement Feller).

Théorème 6. *Soit $(X_t)_{t \geq 0}$ un processus jump-move sur un espace E polonais localement compact.*

1. *Si $Q_t^Y C_b(E) \subset C_b(E)$ et $K C_b(E) \subset C_b(E)$ alors $(X_t)_{t \geq 0}$ est un processus Feller continu.*

2. Si $Q_t^Y C_0(E) \subset C_0(E)$ et $K C_0(E) \subset C_0(E)$ alors $(X_t)_{t \geq 0}$ est un processus de Feller.

Il y a de multiples intérêts à étudier les propriétés de Feller continu et Feller. Tout d'abord elles impliquent chacune la propriété de Markov forte ce qui n'est pas acquis a priori pour le processus jump-move. D'autre part la propriété de Feller donne un cadre simple dans le calcul du générateur infinitésimal, que nous effectuerons tout de même en toute généralité. En effet si l'on suppose $Q_t^Y C_0(E) \subset C_0(E)$, pour tout $t \geq 0$, et $K C_0(E) \subset C_0(E)$ alors le domaine du générateur \mathcal{A} du processus $(X_t)_{t \geq 0}$ est égal au domaine du générateur \mathcal{A}^Y associé au mouvement $(Y_t)_{t \geq 0}$ et nous obtenons :

$$\mathcal{A} = \mathcal{A}^Y + \alpha(K - Id).$$

Autrement dit le générateur du jump-move process est la somme du générateur du mouvement $(Y_t)_{t \geq 0}$ et du générateur d'un processus de Markov purement à sauts d'intensité α et de noyau de transition K . Dans les preuves liées aux propriétés de Feller ainsi qu'au calcul du générateur nous nous sommes rendus compte de l'importance de supposer que l'intensité de saut α est bornée pour obtenir des résultats dans le contexte d'un processus jump-move le plus général possible. En effet les termes présents dans ces calculs révèlent une intrication entre les sauts et les déplacements continus. Ainsi utiliser la borne α^* permet de contrôler des quantités complexes. L'étude de ces différentes propriétés sans déplacement entre les sauts est beaucoup plus simple car l'équation de Kolmogorov backward devient alors dérivable et l'absence de borne supérieure pour α peut alors être plus facilement gérée : c'est le cas pour le processus spatial de naissance et de mort par exemple.

Nous appliquons ces résultats à des exemples dans le cas de systèmes de particules en interaction sur \mathbb{R}^d comme présenté en section 1.3.3. Étudier les propriétés de Feller nécessite alors de mettre en place une topologie sur cet espace des configurations spatiales car ces propriétés utilisent la notion de continuité et de compacité. On rappelle que dans ce cadre l'espace des configurations spatiales s'écrit $E = \cup_{n=0}^{\infty} E_n$ où $E_n = \pi_n(W^n)$ avec $\pi_n((x_1, \dots, x_n)) = \{x_1, \dots, x_n\}$ la projection naturelle. Nous avons choisi de munir l'espace E par la distance d_1 suivante : pour $x = \{x_1, \dots, x_{n(x)}\}$ et $y = \{y_1, \dots, y_{n(y)}\}$ tels que $n(x) \leq n(y)$ on pose

$$d_1(x, y) = \frac{1}{n(y)} \left(\min_{\sigma \in \mathcal{S}_{n(y)}} \sum_{i=1}^{n(x)} (\|x_i - y_{\sigma(i)}\| \wedge 1) + (n(y) - n(x)) \right)$$

où $\|\cdot\|$ est la norme euclidienne sur \mathbb{R}^d et où \mathcal{S}_n désigne l'ensemble des permutations sur $\{1, \dots, n\}$. Ainsi la distance d_1 se comporte par son premier terme comme la norme euclidienne sur chaque E_n et par son second terme elle pénalise l'écart relatif du nombre d'individus entre deux configurations spatiales. Une propriété fondamentale de d_1 est qu'elle rend continue la fonction $n(\cdot)$ qui à chaque configuration associe sa taille. Il aurait été possible de prendre une autre distance bornée vérifiant cette propriété pour obtenir des résultats topologiques similaires. Cependant nous montrerons que des distances qui rendent $n(\cdot)$ discontinue, comme la distance de Hausdorff, ne sont pas adaptées à notre modèle. Pour terminer ce résumé du chapitre 2, présentons un résultat fondamental pour l'étude de la propriété de Feller

dans ce contexte spatial : la description des compacts de E . On montrera le résultat suivant :

Proposition 7. *Supposons que W est un sous-ensemble fermé de \mathbb{R}^d .*

1. *Soit $n \geq 1$ et A un ensemble fermé de (E_n, d_1) . Alors A est un ensemble compact si et seulement si la propriété suivante est vérifiée :*

$$\forall w \in W, \exists R \geq 0, \text{ tel que } \forall x = \{x_1, \dots, x_n\} \in A, \max_{1 \leq k \leq n} \{\|x_k - w\|\} \leq R.$$

2. *Soit A un ensemble compact de E . Alors il existe un rang $n_0 \geq 0$ tel que*

$$A \subset \bigcup_{n=0}^{n_0} E_n.$$

La distance d_1 décrit donc bien le modèle que nous souhaitons à savoir qu'elle se comporte comme $\|\cdot\|$ sur chaque étage E_n et qu'elle pénalise la différence de taille entre les configurations.

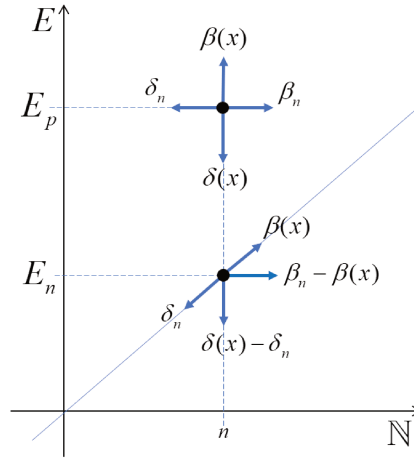
Chapitre 3 : Ergodic properties of birth-death-move processes with applications to pairwise interaction processes on \mathbb{R}^d .

Le chapitre 3 porte avant tout sur les propriétés ergodiques du processus birth-death-move. Comme nous avons pu l'évoquer précédemment un des objectifs est de retrouver des résultats similaires à ceux décrits pour le processus spatial de naissance et de mort en section 1.2.3. Cependant l'ajout d'une étape de mouvement continu entre les sauts pose de sérieux enjeux calculatoires. En effet la stratégie mise en place est de comparer le processus birth-death-move à un processus simple de naissance et de mort noté $(\eta_t)_{t \geq 0}$ et de taux définis pour $n \geq 0$ par $\beta_n = \sup_{x \in E_n} \beta(x)$ et $\delta_n = \inf_{x \in E_n} \delta(x)$. L'intérêt de ce procédé est que les résultats ergodiques liés au processus simple de naissance et de mort sont très bien connus. Pour pouvoir comparer le processus birth-death-move $(X_t)_{t \geq 0}$ et le processus simple de naissance et de mort $(\eta_t)_{t \geq 0}$ nous construisons un processus couplé $\check{X} = (X, \eta)$ de sorte que le processus $(\eta_t)_{t \geq 0}$ domine $(X_t)_{t \geq 0}$ au sens où

$$\forall x \in E, n(x) \leq n \implies \mathbb{P}_{(x,n)}(\check{X}_t \in \Gamma) = 0$$

avec $\Gamma = \{(y, m) \in E \times \mathbb{N}; n(y) > m\}$. Pour mieux visualiser le procédé de couplage la figure 5 représente les sauts possibles pour le processus $(\check{X}_t)_{t \geq 0}$, qui est alors un processus jump-move sur $E \times \mathbb{N}$.

Soulevons encore une fois la complexification calculatoire due à l'ajout du déplacement entre les sauts. En effet la non-dérivabilité de l'équation de Kolmogorov backward et l'intrication entre les sauts et les déplacements ont fortement compliqué la définition même du couplage. Nous avons donc de nouveau pu constater l'importance de considérer une intensité de saut bornée et nous avons dû utiliser la propriété de Feller pour arriver à nos fins ce qui impose des hypothèses sur le processus $(X_t)_{t \geq 0}$. Cependant une étude de plusieurs exemples présentés dans ce chapitre montrent que nous conservons une généralité certaine.

Figure 5: Transitions possibles pour le processus couplé $(\check{X}_t)_{t \geq 0}$.

Nous terminons le chapitre 3 en montrant qu'il est possible d'obtenir explicitement la mesure invariante du processus birth-death-move pour des systèmes de particules en interaction deux à deux sur \mathbb{R}^d . Pour ce faire nous considérons que le mouvement continu $(Y_t)_{t \geq 0}$ est issu d'une dynamique de Langevin comme présentée en section 1.3.3 basée sur un potentiel d'interaction répulsif noté V . De plus nous choisissons le noyau de saut de telle sorte que les naissances soient soumises au même potentiel V . Dans ce cas nous obtenons la convergence vers une mesure explicite qui correspond à la mesure de Gibbs associée à ce même potentiel. Les hypothèses sur V sont suffisamment larges pour englober les potentiels de Riesz, le potentiel de Strauss régularisé ou encore les potentiels de Lennard-Jones répulsifs. Ces résultats ouvrent potentiellement de nouvelles méthodes de simulation pour des mesures de Gibbs associées à de tels potentiels. Qualitativement nous pensons que le fait que le potentiel soit présent tant au niveau des sauts que du déplacement continu peut accélérer numériquement la convergence vers la mesure à simuler par rapport à des procédés plus classiques liés au processus purement à sauts.

1.4.2 Chapitre 4 : estimation du noyau de transition d'un processus jump-move

Article de référence : *Ce chapitre fait l'objet d'un article en cours de rédaction qui a vocation à être soumis pour publication.*

Le chapitre 4 porte sur l'estimation non-paramétrique du noyau de saut K d'un processus de jump-move par des méthodes à noyau. En effet nous avons déjà présenté en section 1.3.4 les résultats existants sur l'estimation non paramétrique de la fonction d'intensité α . La première étape de notre raisonnement est de procéder à l'estimation de $K(x, A)$ pour tout $x \in E$ et $A \in \mathcal{E}$ dans le cadre le plus général possible. Ne disposant pas de méthode calculatoire usuelle dans ce cas général, comme la méthode de projection par exemple, nous avons construit une martingale à partir du processus jump-move $(X_t)_{t \geq 0}$. Nous avons donc prouvé que dans le cas

où l'intensité α est bornée, le processus $(M_t(g))_{t \geq 0}$, où $g \in M_b(E \times E)$ défini par

$$M_t(g) = \int_0^t g(X_{s-}, X_s) dN_s - \int_0^t \alpha(X_s) \int_E g(X_s, z) K(X_s, dz) ds$$

est une martingale de carré intégrable pour la filtration continue à droite associée au processus. Cette propriété de martingale est intrinsèque à la construction du jump-move : elle n'est pas acquise de fait a priori car le processus $(g(X_{t-}, X_t))_{t \geq 0}$ n'est pas prévisible pour la filtration continue à droite. En particulier, remarquons que ce résultat implique que α correspond bien à l'intensité liée au processus à sauts $(N_t)_{t \geq 0}$.

Nous commençons donc par estimer la quantité $K(x, A)$ par une méthode à noyau dans le cas où le processus $(X_t)_{t \geq 0}$ est observé continument sur un intervalle $[0, T]$ avec $T > 0$. Soit une fonction à noyau k_T définie sur $E \times E$, on définit l'estimateur $\hat{K}(x, A)$ de $K(x, A)$ par

$$\hat{K}(x, A) = \frac{\int_0^T k_T(x, X_{s-}) \mathbf{1}_A(X_s) dN_s}{\int_0^T k_T(x, X_{s-}) dN_s}.$$

Nous obtenons alors la convergence dans L^2 de cet estimateur, donc en particulier sa consistance, sous des conditions d'ergodicité du processus. La même étude est menée dans le cas où le noyau K admet une densité sur E , c'est-à-dire qu'il existe une mesure ν σ -finie sur E et une fonction f sur $E \times E$ tel que pour tout $x \in E$ et $A \in \mathcal{E}$ on ait $K(x, A) = \int_A f(x, y) d\nu(y)$. On obtiendra également sous certaines hypothèses la consistance faible d'un estimateur à noyau pour la quantité $f(x, y)$ avec $x, y \in E$.

Notons que lorsque l'estimation se fait sur un jeu de données réelles l'observation du processus ne peut pas être continue mais elle est discrète. Cela est traité dans le chapitre 4 dans le cas de l'estimation de $f(x, y)$. L'enjeu est alors de dominer l'écart entre l'estimateur $\hat{f}(x, y)$ lié aux observations continues et l'estimateur $\hat{f}_{(d)}(x, y)$ lié aux observations discrètes. Cet écart devient problématique lorsque la discrétisation ne nous permet pas d'observer plusieurs sauts entre chaque temps d'observation. Il est donc nécessaire que le processus n'explose pas pour contrôler le nombre de sauts entre deux temps d'observation lorsque le pas associé tend vers 0. Encore une fois l'hypothèse d'une intensité bornée est déterminante.

Ces procédures d'estimation liées au jump-move dans un cadre général vont pouvoir s'appliquer à l'estimation d'un noyau de transition lié à un processus de birth-death-move décrivant l'évolution d'un système de particules en interaction sur \mathbb{R}^d . Plaçons nous sous les hypothèses données en section 1.3.3. Le noyau de transition évalué en $(x, A) \in E \times \mathcal{E}$ se décompose sous la forme

$$K(x, A) = \frac{\beta(x)}{\alpha(x)} K_\beta(x, A) + \frac{\delta(x)}{\alpha(x)} K_\delta(x, A).$$

Le noyau K_δ correspond à la mort d'un individu de la population donc nous l'écrivons sous la forme

$$K_\delta(x, A) = \sum_{i=1}^{n(x)} \omega(x_i, x) \mathbf{1}_A(x \setminus x_i)$$

où ω est la fonction de poids associée aux morts. Supposons ensuite que les naissances ont lieu sur W selon une densité continue par rapport à la mesure de Lebesgue ce qui se traduit par

$$K_\beta(x, x \cup B) = \int_B f_\beta(x, \xi) \, d\xi$$

pour tout $B \subset W$ et avec f_β la densité associée. Nous avons alors pu constater que l'estimation de la fonction de poids w s'adapte immédiatement de l'estimation du noyau $K(x, A)$ du jump-move et que l'estimation de la densité de naissance f_β est une conséquence directe de la procédure d'estimation de la densité générale présentée dans le paragraphe précédent. Nous retrouvons alors des vitesses de convergence cohérentes avec celles obtenues pour l'intensité α dans [LLG21]. À titre d'exemple plaçons nous dans un cas simplifié où les naissances dépendent uniquement de l'espace sous-jacent W . Dans ce cas la densité f_β est uniquement définie sur W à savoir

$$K_\beta(x, x \cup B) = \int_B f_\beta(\xi) \, d\xi.$$

pour tout $x \in E$. Lorsque W est un fermé de \mathbb{R}^d avec $d \geq 1$ on estime la fonction de densité de naissance par

$$\hat{f}_\beta(\xi) = \frac{1}{c_T^d N_T^\beta} \int_0^T \kappa \left(\frac{|\xi - X_s \setminus X_{s-}|}{c_T} \right) \, dN_s^\beta,$$

où κ est un noyau borné sur \mathbb{R} , où $c_T \xrightarrow{T \rightarrow +\infty} 0$ et $(N_t^\beta)_{t \geq 0}$ est le processus à sauts qui compte le nombre de naissances du processus birth-death-move. Dans ce cas nous appliquons directement nos calculs sur les estimateurs dans le cas général pour obtenir la proposition suivante.

Proposition 8. *Supposons qu'il existe un rang $n^* \geq 1$ tel que la fonction d'intensité de naissance β est identiquement nulle sur $\cup_{n=n^*}^{+\infty} E_n$, que κ est à support compact et que f_β est γ -höldérienne pour un certain $\gamma > 0$. Soit $\varepsilon > 0$. Il existe une constante $C > 0$ et un temps $T_0 > 0$ tel que pour tout $T \geq T_0$,*

$$\mathbb{P} \left(|\hat{f}_\beta(\xi) - f_\beta(\xi)| > \varepsilon \right) \leq C \left(\frac{1}{c_T^d T} + c_T^{2\gamma} \right).$$

Cette proposition sera généralisée, comme tous les résultats de consistance des différents estimateurs que nous allons considérer, au cas où l'on ne suppose plus l'existence d'un nombre maximum d'individus n^* .

Chapter 2

Feller properties of jump-move processes with applications to spatial particle systems

This chapter constitutes the first part of the submitted article [LLGM22], so some considerations and definitions are redundant with Chapter 1. The main results are stated in Sections 2.2 and 2.3. Moreover Section 2.4 contains the proofs of these results and technical lemmas.

2.1 Introduction

In the spirit of jump-diffusion models, we consider Markov stochastic processes that alternate continuous motions and jumps in some locally compact Polish space E . We call these general processes jump-move processes. In this paper, the state-space E is typically not a finite-dimensional Euclidean space, contrary to standard jump-diffusion models. Many examples of such dynamics have been considered in the literature, including piecewise deterministic processes ([Dav84]), branching particle systems ([Sko64, AN12]), spatially structured population models ([BM15]) and some variations ([ÇK91, LÖc02]), to cite a few. A particular case that will be of special interest to us is when $E = \cup_{n \geq 0} E_n$ for some disjoint spaces E_n , E_0 consisting of a single element, and the jumps can only occur from E_n to E_{n+1} (like a birth) or from E_n to E_{n-1} (like a death). We call the latter specific dynamics a birth-death-move process ([Pre75, LLG21]). We will provide several illustrations in the particular case of interacting particles in \mathbb{R}^d , with births and deaths. These processes are observed in a various range of applications, including microbiology ([LLG21]), epidemiology ([MH17]) and ecology ([RS01, PG19]).

We start in Section 2.2 from a mechanistic general definition of jump-move processes, in the sense that we explicitly construct the process iteratively over time, which equivalently provides a simulation algorithm. This defines a Markov process $(X_t)_{t \geq 0}$, the jump intensity function of which reads $\alpha(X_t)$, for some continuous function α , and which follows between its jumps a continuous Markov motion on E . We then derive in Section 2.3 conditions ensuring that the transition kernel of $(X_t)_{t \geq 0}$ forms a Feller semigroup on $C_b(E)$ or on $C_0(E)$. We obtain the natural result that

if α is bounded, then a jump-move process is Feller whenever the transition kernel of the jumps and the transition kernel of the inter-jumps motion (i.e. the move part) are. Similarly, the infinitesimal generator is just the sum of the generator of the jumps and the generator of the move, the domain corresponding under mild conditions to the domain of the generator of the move.

We emphasise that the above results are very general in the sense that neither E is specified, nor the exact jump transition kernel, nor the form of the inter-jumps continuous Markov motion. Our only real working assumption is the boundedness of the intensity function α . We however provide many illustrations in the case where $(X_t)_{t \geq 0}$ represents the dynamics of a system of particles in \mathbb{R}^d , introduced in Section 2.2.4. In this situation, we consider continuous inter-jumps motions driven by deterministic growth-interacting dynamics, as already exploited in ecology ([RS01, HMP19]), or driven by interacting SDE systems, in particular overdamped Langevin dynamics, the Feller properties of which translate straightforwardly to the move part of $(X_t)_{t \geq 0}$. As to the jumps, they are continuous Feller in general, but not necessarily Feller on $C_0(E)$. The picture becomes however more intelligible when they only consist of births and deaths. We present in Section 2.3.3 general birth transition kernels that imply the Feller properties under mild assumptions. On the other hand, a simple uniform death kernel cannot be Feller on $C_0(E)$ in this setting, unless the particles are restricted to a compact subset of \mathbb{R}^d , a situation where any death kernel is Feller.

2.2 Jump-move processes

2.2.1 Iterative construction

Let E be a Polish space equipped with the Borel σ -algebra \mathcal{E} and a distance d . Let (Ω, \mathcal{F}) be a measurable space and $(\mathbb{P}_x)_{x \in E}$ a family of probability measures on (Ω, \mathcal{F}) . In order to define a jump-move process $(X_t)_{t \geq 0}$ on E , we need three ingredients:

1. An intensity function $\alpha : E \rightarrow \mathbb{R}_+$ that governs the inter-jumps waiting times.
2. A transition kernel K for the jumps, defined on $E \times \mathcal{E}$.
3. A continuous homogeneous Markov process $((Y_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ on E , the distribution of which will drive the inter-jumps motion of $(X_t)_{t \geq 0}$.

We will work throughout this paper under the assumption that $\alpha : E \rightarrow \mathbb{R}_+$ is continuous and bounded by $\alpha^* > 0$, i.e. for all $x \in E$

$$0 \leq \alpha(x) \leq \alpha^*. \quad (2.1)$$

We denote by $(Q_t^Y)_{t \geq 0}$ the transition kernel of $(Y_t)_{t \geq 0}$, given by

$$Q_t^Y(x, A) = \mathbb{P}_x(Y_t \in A), \quad x \in E, A \in \mathcal{E}.$$

The following iterative construction provides a clear intuition of the dynamics of the process $(X_t)_{t \geq 0}$. It follows closely the presentation in the supplementary

material of [LLG21], where an algorithm of simulation on a finite time interval is also derived.

Let $(Y_t^{(j)})_{t \geq 0}$, $j \geq 0$, be a sequence of processes on E identically distributed as $(Y_t)_{t \geq 0}$. Set $T_0 = 0$ and let $x_0 \in E$. Then $(X_t)_{t \geq 0}$ can be constructed as follows: for $j \geq 0$, iteratively do

- (i) Given $X_{T_j} = x_j$, generate $(Y_t^{(j)})_{t \geq 0}$ conditional on $Y_0^{(j)} = x_j$ according to the kernel $(Q_t^Y(x_j, \cdot))_{t \geq 0}$.
- (ii) Given $X_{T_j} = x_j$ and $(Y_t^{(j)})_{t \geq 0}$, generate τ_{j+1} according to the cumulative distribution function

$$F_{j+1}(t) = 1 - \exp\left(-\int_0^t \alpha(Y_u^{(j)}) \, du\right). \quad (2.2)$$

- (iii) Given $X_{T_j} = x_j$, $(Y_t^{(j)})_{t \geq 0}$ and τ_{j+1} , generate x_{j+1} according to the transition kernel $K(Y_{\tau_{j+1}}^{(j)}, \cdot)$.

- (iv) Set $T_{j+1} = T_j + \tau_{j+1}$, $X_t = (Y_{t-T_j}^{(j)})$ for $t \in [T_j, T_{j+1})$ and $X_{T_{j+1}} = x_{j+1}$.

We denote by $(\mathcal{F}_t^Y)_{t \geq 0}$ the natural filtration of $(Y_t)_{t \geq 0}$, i.e. $\mathcal{F}_t^Y = \sigma(Y_u, u \leq t)$, and by $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration of $(X_t)_{t \geq 0}$. We make these filtrations complete ([Bas11, Section 20.1]) and abusively use the same notation. The jump-move process $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ constructed above is a homogeneous Markov process with respect to $(\mathcal{F}_t)_{t \geq 0}$. The trajectories of $(X_t)_{t \geq 0}$ are continuous except at the jump times $(T_j)_{j \geq 1}$ where they are right-continuous. The specific form (2.2) implies that the law of the waiting time τ_j under \mathbb{P}_x is absolutely continuous with respect to the Lebesgue measure with density $\mathbb{E}_x \left[\alpha(Y_t^{(j-1)}) \exp\left(-\int_0^t \alpha(Y_u^{(j-1)}) \, du\right) \right] \mathbf{1}_{t > 0}$ as proved in Section 2.4.1. It also implies that the intensity of jumps is $\alpha(X_t)$. Denote by $N_t = \sum_{j \geq 0} \mathbf{1}_{T_j \leq t}$ the number of jumps before $t \geq 0$. Under the assumption (2.1), we have for any $n \geq 0$ and $t \geq 0$,

$$\mathbb{P}(N_t > n) \leq \mathbb{P}(N_t^* > n) \quad (2.3)$$

where N_t^* follows a Poisson distribution with rate $\alpha^* t$. This in particular implies that $(N_t)_{t \geq 0}$ is a non-explosive counting process. All the aforementioned properties of $(X_t)_{t \geq 0}$ are either immediate or verified in [LLG21].

Note that the above construction only implies the weak Markov property of $(X_t)_{t \geq 0}$ in general, at least because the process $(Y_t)_{t \geq 0}$ is only assumed to be a (weak) Markov process. A more abstract construction obtained by ‘‘piecing out’’ strong Markov processes is introduced in [INW68], leading to a strong Markov jump-move process. The strong Markov property can also be obtained in our case by strengthening the assumptions, see Section 2.3.1.

The transition kernel of $(X_t)_{t \geq 0}$ will be denoted, for any $t \geq 0$, $x \in E$ and $A \in \mathcal{E}$, by

$$Q_t(x, A) = \mathbb{P}(X_t \in A | X_0 = x) = \mathbb{P}_x(X_t \in A).$$

Also for $f \in M_b(E)$, where $M_b(E)$ is the set of real valued bounded and measurable functions on E , we will denote $Q_t f(x) = \mathbb{E}_x[f(X_t)] = \int_E Q_t(x, dy) f(y)$. Similarly we will write $Q_t^Y f(x) = \mathbb{E}_x^Y[f(Y_t)]$.

2.2.2 Special case of the birth-death-move process

A birth-death-move process is the particular case of a jump-move process where E takes the form $E = \bigcup_{n=0}^{\infty} E_n$, with $(E_n)_{n \geq 0}$ a sequence of disjoint Polish spaces, and where the jumps are only births and deaths. We assume that each E_n is equipped with the Borel σ -algebra \mathcal{E}_n , so that E is associated with the σ -field $\mathcal{E} = \sigma(\bigcup_{n=0}^{\infty} \mathcal{E}_n)$. We further assume that E_0 consists of a single element denoted by \emptyset . In this setting, the Markov process $(Y_t)_{t \geq 0}$ driving the motions of $(X_t)_{t \geq 0}$ is supposed to satisfy

$$\mathbb{P}_x((Y_t)_{t \geq 0} \subset E_n) = \mathbf{1}_{E_n}(x), \quad \forall x \in E, \forall n \geq 0.$$

We introduce a birth intensity function $\beta : E \rightarrow \mathbb{R}_+$ and a death intensity function $\delta : E \rightarrow \mathbb{R}_+$, both assumed to be continuous on E and satisfying $\alpha = \beta + \delta$. We prevent a death in E_0 by assuming that $\delta(\emptyset) = 0$. The probability transition kernel K for the jumps then reads, for any $x \in E$ and $A \in \mathcal{E}$,

$$K(x, A) = \frac{\beta(x)}{\alpha(x)} K_\beta(x, A) + \frac{\delta(x)}{\alpha(x)} K_\delta(x, A), \quad (2.4)$$

where $K_\beta : E \times \mathcal{E} \rightarrow [0, 1]$ is a probability transition kernel for a birth and $K_\delta : E \times \mathcal{E} \rightarrow [0, 1]$ is a probability transition kernel for a death. They satisfy, for $x \in E$ and $n \geq 0$,

$$K_\beta(x, E_{n+1}) = \mathbf{1}_{E_n}(x) \quad \text{and} \quad K_\delta(x, E_n) = \mathbf{1}_{E_{n+1}}(x).$$

Notice that a simple birth-death process is the particular case where $E = \mathbb{N}$, $E_n = \{n\}$ and the intensity functions β and δ are sequences.

For later purposes, when $E = \bigcup_{n=0}^{\infty} E_n$ as in the present section, we define the function $n(\cdot) : E \rightarrow \mathbb{N}$ by $n(x) = k$ when $x \in E_k$, so that $x \in E_{n(x)}$ is always satisfied.

2.2.3 Kolmogorov backward equation

The goal of this section is to present the Kolmogorov backward equation for the transition kernel of the general jump-move process $(X_t)_{t \geq 0}$ of Section 2.2.1, providing a more probabilistic viewpoint of its dynamics, and to show that the solution exists and is unique. To obtain these results we use similar methods as in [Fel71] for pure jump processes, see also [Pre75]. The key assumption is the boundedness (2.1) of the intensity α , which prevents the explosion of the process. The proofs are postponed to Section 2.4.2.

Theorem 9. *For all $x \in E$ and all $A \in \mathcal{E}$, the function $t \mapsto Q_t(x, A)$, for $t > 0$, satisfies the following Kolmogorov backward equation:*

$$\begin{aligned} Q_t(x, A) &= \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A} e^{-\int_0^t \alpha(Y_u) du} \right] \\ &+ \int_0^t \int_E Q_{t-s}(y, A) \mathbb{E}_x^Y \left[K(Y_s, dy) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds. \end{aligned} \quad (2.5)$$

With approximation arguments and using linearity, we can generalise (2.5) to the evolution equation of Q_t

$$Q_t f(x) = \mathbb{E}_x^Y \left[f(Y_t) e^{-\int_0^t \alpha(Y_u) du} \right] + \int_0^t \int_E Q_{t-s} f(y) \mathbb{E}_x^Y \left[K(Y_s, dy) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds \quad (2.6)$$

for any $f \in C_b(E)$.

In the case of the birth-death-move process of Section 2.2.2, the above equation reads, for $x \in E_n$,

$$Q_t(x, A) = \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A} e^{-\int_0^t \alpha(Y_u) du} \right] + \int_0^t \int_{E_{n+1}} Q_{t-s}(y, A) \mathbb{E}_x^Y \left[\beta(Y_s) K_\beta(Y_s, dy) e^{-\int_0^s \alpha(Y_u) du} \right] ds + \int_0^t \int_{E_{n-1}} Q_{t-s}(y, A) \mathbb{E}_x^Y \left[\delta(Y_s) K_\delta(Y_s, dy) e^{-\int_0^s \alpha(Y_u) du} \right] ds. \quad (2.7)$$

To show the existence of a unique solution to (2.5), let

$$Q_{t,p}(x, A) := \mathbb{P}_x(X_t \in A, T_p > t)$$

be the transition probability from state x to A in time t with at most p jumps. Notice that we can define $Q_{t,\infty} = \lim_{p \rightarrow \infty} Q_{t,p}$ because $Q_{t,p} \leq Q_{t,p+1} \leq 1$. We prove in the following proposition that $Q_{t,\infty}$ is the unique solution to (2.5) using a minimality argument as in [Fel71].

Proposition 10. *$Q_{t,\infty}$ is the unique sub-stochastic solution of (2.5), i.e. it is the unique solution satisfying $Q_t(x, E) \leq 1$ for all $x \in E$. Moreover $Q_{t,\infty}$ is stochastic, i.e. $Q_{t,\infty}(x, E) = 1$ for all $x \in E$.*

To conclude this section, we present an interpretation of $Q_{t,\infty}$ for the birth-death-move process of Section 2.2.2, which is much in the spirit of [Pre75]. We write $Q_{t,(p)}(x, A)$ for the transition probability from x to A in time t without having entered $\bigcup_{k=p+1}^\infty E_k$, that is

$$Q_{t,(p)}(x, A) = \mathbb{P}_x(X_t \in A, \forall s \in [0, t] \ n(X_s) \leq p).$$

We can also define $Q_{t,(\infty)}(x, A) = \lim_{p \rightarrow \infty} Q_{t,(p)}(x, A) \leq 1$ by monotonicity.

Proposition 11. *For all $x \in E$ and all $A \in \mathcal{E}$, $Q_{t,(\infty)}(x, A) = Q_{t,\infty}(x, A)$.*

2.2.4 Systems of interacting particles in \mathbb{R}^d

In this section, we focus on the dynamics of a system of interacting particles in \mathbb{R}^d and we provide general examples of birth kernels, death kernels and inter-jumps motions in this setting, that to our opinion constitute realistic models for applications and are actually already used in some domains. Some of them moreover

lead to an explicit Gibbs stationary measure of the dynamics, as we will show in Chapter 3. These running examples will serve in the rest of the paper to illustrate the theoretical results and make explicit our assumptions.

Let $W \subset \mathbb{R}^d$ be a closed set where the particles live, equipped with a σ -field \mathcal{B} . A collection of n particles in W is a point configuration for which the ordering does not matter. For this reason, for $n \geq 1$, we will identify two elements (x_1, \dots, x_n) and (y_1, \dots, y_n) of W^n if there exists a permutation σ of $\{1, \dots, n\}$ such that $x_i = y_{\sigma(i)}$ for any $1 \leq i \leq n$. Following [Pre75], [Löc02] and others, we thus define E_n as the space obtained by this identification. Specifically, denoting by $\pi_n : (x_1, \dots, x_n) \in W^n \mapsto \{x_1, \dots, x_n\}$ the associated projection, the space E_n corresponds for $n \geq 1$ to $E_n = \pi_n(W^n)$ equipped with the σ -field $\mathcal{E}_n = \pi_n(\mathcal{B}^{\otimes n})$, while $E_0 = \{\emptyset\}$ is just composed of the empty configuration. The general state space of a system of particles is then $E = \cup_{n \geq 0} E_n$ equipped with the σ -field $\mathcal{E} = \sigma(\cup_{n \geq 0} \mathcal{E}_n)$. This formalism allows us to go back and forth quite straightforwardly between the space E_n and the space W^n , the latter being in particular more usual to define the inter-jumps motion of n particles, as detailed below. Note that an alternative formalism consists in viewing a configuration of particles as a finite point measure in W , in which case E becomes the set of finite point measures in W , see for instance [Kal17]. We choose in this paper to adopt the former point of view. We denote by $\|\cdot\|$ the Euclidean norm on \mathbb{R}^d . If $x = \{x_1, \dots, x_n\} \in E_n$ and $\xi \in W$, $x \cup \xi$ stands for $\{x_1, \dots, x_n, \xi\} \in E_{n+1}$ and if $1 \leq i \leq n$, we write $x \setminus x_i$ for $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\} \in E_{n-1}$.

As long as we are concerned by continuous inter-jumps motions, we need to equip E with a distance. Following [SX08], we consider the distance d_1 defined for $x = \{x_1, \dots, x_{n(x)}\}$ and $y = \{y_1, \dots, y_{n(y)}\}$ in E such that $n(x) \leq n(y)$ by

$$d_1(x, y) = \frac{1}{n(y)} \left(\min_{\sigma \in \mathcal{S}_n(y)} \sum_{i=1}^{n(x)} (\|x_i - y_{\sigma(i)}\| \wedge 1) + (n(y) - n(x)) \right), \quad (2.8)$$

with $d_1(x, \emptyset) = 1$ and where \mathcal{S}_n denotes the set of permutations of $\{1, \dots, n\}$. Some topological properties of (E, d_1) are detailed in [SX08] and Section 2.4.5. For the purposes of this section, let us quote in particular that $n(\cdot) : (E, d_1) \rightarrow (\mathbb{N}, |\cdot|)$ is continuous and that π_n is continuous. Note that other distances than d_1 could have been chosen, provided these two last properties (at least) are preserved. Incidentally, the Hausdorff distance, which is a common choice of distance between random sets, does not satisfy these properties (see Section 2.4.5) and is not appropriate in our setting.

We now show how we can easily construct a continuous Markov process $(Y_t)_{t \geq 0}$ on E from continuous Markov processes on W^n for any $n \geq 1$. We focus on the case where for any $x \in E$ and $n \geq 0$, $\mathbb{P}_x((Y_t)_{t \geq 0} \subset E_n) = \mathbf{1}_{E_n}(x)$, as we required it for birth-death-move processes in Section 2.2.2. It is then enough to define a process $Y^{|n}$ on each E_n . To do so consider a continuous Markov process $(Z_t^{|n})_{t \geq 0}$ on W^n whose distribution is permutation equivariant with respect to its initial value $Z_0^{|n}$. This means that for any permutation $\sigma \in \mathcal{S}_n$, the law of $Z_t^{|n} = (Z_{t,1}^{|n}, \dots, Z_{t,n}^{|n})$ given $Z_0^{|n} = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$ is the same as the law of $(Z_{t,\sigma(1)}^{|n}, \dots, Z_{t,\sigma(n)}^{|n})$ given $Z_0^{|n} = (z_1, \dots, z_n)$. Let $x = \{x_1, \dots, x_n\} \in E_n$ and take the process $Z_t^{|n}$ with initial

state $Z_0^{|n} = (x_1, \dots, x_n)$. Note that from the previous permutation equivariance property the choice of ordering for the coordinates of this initial state does not matter, as it will become clear below. We finally define the process $Y_t^{|n}$ on E_n starting from x as

$$Y_t^{|n} = \pi_n(Z_t^{|n}) = \{Z_{t,1}^{|n}, \dots, Z_{t,n}^{|n}\}. \quad (2.9)$$

Note that the continuity of $t \rightarrow Y_t^{|n}$ (with respect to d_1) follows from the continuity of $t \rightarrow Z_t^{|n}$ and the continuity of π_n . The continuity of $t \rightarrow Y_t$ is then implied by the continuity of $n(\cdot)$.

With this construction, the transition kernel of Y reads, for any $f \in M_b(E)$,

$$\begin{aligned} Q_t^Y f(x) &= \sum_{n \geq 0} \mathbb{E} \left[f(Y_t^{|n}) \mid Y_0^{|n} = x \right] \mathbf{1}_{x \in E_n} \\ &= \sum_{n \geq 0} \mathbb{E} \left(f(\pi_n(Z_t^{|n})) \mid Z_0^{|n} = (x_1, \dots, x_n) \right) \mathbf{1}_{x \in E_n}, \end{aligned}$$

so that denoting by $Q_t^{Z^{|n}}$ the transition kernel of $Z^{|n}$ in W^n we have

$$Q_t^Y f(x) = \sum_{n \geq 0} Q_t^{Z^{|n}}(f \circ \pi_n)((x_1, \dots, x_n)) \mathbf{1}_{x \in E_n}.$$

Note that if we had chosen another ordering for the initial state, that is $Z_0^{|n} = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for some $\sigma \in \mathcal{S}_n$, then the transition kernel of Y would have remained the same, since by permutation equivariance

$$\begin{aligned} &\mathbb{E} \left(f(\pi_n(Z_t^{|n})) \mid Z_0^{|n} = (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \right) \\ &= \mathbb{E} \left(f(\pi_n(Z_{t,\sigma(1)}^{|n}, \dots, Z_{t,\sigma(n)}^{|n})) \mid Z_0^{|n} = (x_1, \dots, x_n) \right) \end{aligned}$$

which is $\mathbb{E} \left(f(\pi_n(Z_t^{|n})) \mid Z_0^{|n} = (x_1, \dots, x_n) \right)$.

We are now in position to present general examples of jump transition kernels and inter-jumps motions for a system of particles in W . The first example introduces a death transition kernel where an existing particle dies with a probability that may depend on the distance to the other particles. The next two examples focus on birth transition kernels, either driven by a mixture of densities around each particle, or by a Gibbs potential. The last two examples apply the above construction of $(Y_t)_{t \geq 0}$ on E to introduce inter-jumps Langevin diffusions and growth interaction processes.

Example 1 (death kernel): Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ be a continuous function and for $x = \{x_1, \dots, x_n\} \in E_n$, set $\omega(x_1, x) = 1$ if $n = 1$ and if $n \geq 2$, for any $i \in \{1, \dots, n\}$,

$$\omega(x_i, x) = \frac{1}{z(x)} \sum_{k \neq i} g(\|x_k - x_i\|),$$

with $z(x) = \sum_{i=1}^n \sum_{k \neq i} g(\|x_k - x_i\|)$. A general example of death transition kernel is

$$K_\delta(x, A) = \sum_{i=1}^{n(x)} \omega(x_i, x) \mathbf{1}_A(x \setminus x_i), \quad x \in E, \quad A \in \mathcal{E}.$$

The probability $\omega(x_i, x)$ that x_i disappears then depends on the distance between x_i and the other particles in x through g . Uniform deaths correspond to the particular case $\omega(x_i, x) = 1/n(x)$.

Example 2 (*birth kernel as a mixture*): Let φ be a density function on W , and $\phi_1 : W \rightarrow \mathbb{R}$ and $\phi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ be two continuous functions. We set for $x = \{x_1, \dots, x_{n(x)}\} \in E \setminus E_0$, $v(x_i, x) = \exp(\phi_1(x_i) + \sum_{k \neq i} \phi_2(\|x_k - x_i\|))$ and we consider the birth kernel defined for $\Lambda \subset W$ and $x \in E \setminus E_0$ by $K_\beta(\emptyset, \Lambda) = \int_\Lambda \varphi(\xi) d\xi$ and

$$K_\beta(x, \Lambda \cup x) = \frac{1}{n(x)} \sum_{i=1}^{n(x)} \frac{1}{z(x_i, x)} \int_\Lambda \varphi\left(\frac{\xi - x_i}{v(x_i, x)}\right) d\xi,$$

where $\Lambda \cup x = \{\{u\} \cup x, u \in \Lambda\}$ and $z(x_i, x) = \int_W \varphi((\xi - x_i)/v(x_i, x)) d\xi$. Note that $z(x_i, x) = v(x_i, x)^d$ if $W = \mathbb{R}^d$. It is easily checked that $K_\beta(x, E_{n+1}) = K_\beta(x, W \cup x) = 1$ for $x \in E_n$, and in particular this kernel is a genuine birth kernel in the sense that the transition from E_n to E_{n+1} is only due to the addition of a new particle, the existing ones remaining unchanged. Moreover, the new particle is distributed as a mixture of distributions driven by φ , each of them being centred at the existing particles. The term $v(x_i, x)$ quantifies the dispersion of births around the particle x_i and it depends on the distance between x_i and the other particles through ϕ_2 . A natural example is a mixture of isotropic Gaussian distributions on \mathbb{R}^d (restricted to W), respectively centred at x_i with standard deviation $v(x_i, x)$.

Example 3 (*birth kernel based on a Gibbs potential*): We introduce a measurable function $V : E \rightarrow \mathbb{R}$, so called a potential, satisfying

$$z(x) := \int_W \exp(-(V(x \cup \xi) - V(x))) d\xi < \infty$$

for all $x \in E$, and we consider the birth kernel defined for $\Lambda \subset W$ and $x \in E$ by

$$K_\beta(x, \Lambda \cup x) = \frac{1}{z(x)} \int_\Lambda e^{-(V(x \cup \xi) - V(x))} d\xi.$$

Note that $K_\beta(x, W \cup x) = 1$ for $x \in E$. With this kernel, given a configuration x , a new particle is more likely to appear in the vicinity of points $\xi \in W$ that make $V(x \cup \xi) - V(x)$ minimal. This kind of kernels K_β has been introduced in [Pre75] for spatial birth-death processes, the case of a birth-death-move process with no move. Their importance is due to the fact that the invariant measure of a spatial birth-death process associated to K_β , uniform deaths and specific birth and death intensities has been explicitly obtained in [Pre75] and corresponds to the Gibbs measure with potential V . This result is at the basis of perfect simulation of spatial Gibbs point process models, see [MW04]. We will similarly show in Chapter 3 that the same Gibbs measure is also invariant for a birth-death-move process associated to the same characteristics for the jumps and a well chosen inter-jumps move process $(Y_t)_{t \geq 0}$ constructed as in the next example.

Example 4 (*Langevin diffusions as inter-jumps motions*): Let $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a globally Lipschitz continuous function, $\beta > 0$ and $\{B_{t,i}\}_{1 \leq i \leq n}$, $n \geq 1$, a collection

of n independent Brownian motions on \mathbb{R}^d . We start from the following system of SDEs usually called overdamped Langevin equations

$$dZ_{t,i}^{|n} = - \sum_{j \neq i} g(Z_{t,i}^{|n} - Z_{t,j}^{|n}) dt + \sqrt{2\beta^{-1}} dB_{t,i}, \quad 1 \leq i \leq n.$$

For $z = (z_1, \dots, z_n) \in (\mathbb{R}^d)^n$, denoting by $\Phi_n : (\mathbb{R}^d)^n \rightarrow (\mathbb{R}^d)^n$ the function defined by $\Phi_n(z) = (\Phi_{n,1}(z), \dots, \Phi_{n,n}(z))$ with $\Phi_{n,i}(z) = \sum_{j \neq i} g(z_i - z_j)$, this system of SDEs reads

$$dZ_t^{|n} = -\Phi_n(Z_t^{|n}) dt + \sqrt{2\beta^{-1}} dB_t^{|n}, \quad (2.10)$$

where $B_t^{|n} = (B_{t,1}, \dots, B_{t,n})$. Since Φ_n is a permutation equivariant function, that is for any $\sigma \in \mathcal{S}_n$

$$\Phi_n(z_{\sigma(1)}, \dots, z_{\sigma(n)}) = (\Phi_{n,\sigma(1)}(z), \dots, \Phi_{n,\sigma(n)}(z)),$$

and since $B_t^{|n}$ is exchangeable, we can verify by writing (2.10) in integral form that the law of $Z_t^{|n}$ is permutation equivariant with respect to its initial state. So when $W = \mathbb{R}^d$, we can define each inter-jumps process $Y^{|n}$ in E_n from $Z^{|n}$ as in (2.9), yielding $(Y_t)_{t \geq 0}$ on E . The same construction can be generalised if $W \subsetneq \mathbb{R}^d$ by considering a Langevin equation with reflecting boundary conditions ([FG07]). This inter-jumps dynamics, associated with the birth kernel of Example 3 and a drift function g related to the potential V , converges to a Gibbs measure on W with potential V (see Chapter 3).

Example 5 (*Growth interaction processes*): This example is motivated by models used in ecology ([RS01, RCM09, Com09, HMP19]). Each particle consists of a plant located in $S \subset \mathbb{R}^d$ and associated with a positive mark, that typically represents the size of the plant, so that $W = S \times \mathbb{R}^+$ here. Births and deaths of plants occur according to a spatial birth and death process, while a deterministic growth applies to their mark. Specifically, when a plant appears, its mark is set to zero or generated according to a uniform distribution on $[0, \varepsilon]$ for some $\varepsilon > 0$ ([RS01]). Then the mark increases over time in interaction with the other marks. In order to formally define this inter-jumps dynamics, let us denote by $(U_i(t), m_i(t))_{t \geq 0}$, for $i = 1, \dots, n$, each component of the process $(Z_t^{|n})_{t \geq 0}$ where $U_i(t) \in S$ and $m_i(t) > 0$, so that $Z_t^{|n} \in W^n$. We introduce the system

$$\frac{dZ_t^{|n}}{dt} = \left((0, F_{1,n}(Z_t^{|n})), \dots, (0, F_{n,n}(Z_t^{|n})) \right), \quad (2.11)$$

where for all $1 \leq i \leq n$, $F_{i,n}$ is a function from W^n into \mathbb{R}_+ . We thus have $U_i(t) = U_i(0)$ for all i and the evolution of the marks $(m_1(t), \dots, m_n(t))$ is driven by a deterministic differential equation depending on $(U_1(0), \dots, U_n(0))$ as expected. To define $Y^{|n}$ by (2.9), we finally assume permutation equivariance, namely that $F_{\sigma(i),n}(z_1, \dots, z_n) = F_{i,n}(z_{\sigma(1)}, \dots, z_{\sigma(n)})$ for all i and all $\sigma \in \mathcal{S}_n$, which is satisfied for all examples in the aforementioned references.

2.3 Feller properties and infinitesimal generator

2.3.1 Feller properties

We assume henceforth that E is a locally compact Polish space. Let $C_b(E)$ be the set of continuous and bounded functions on E and $C_0(E)$ be the set of continuous functions that vanish at infinity in the sense that for all $\epsilon > 0$, there exists a compact set $B \in E$ such that $x \notin B \Rightarrow |f(x)| < \epsilon$.

Following [Dyn65] and [Øks13], we say that the jump-move process $(X_t)_{t \geq 0}$ on E with transition kernel Q_t is Feller continuous if $Q_t C_b(E) \subset C_b(E)$, and we say that it is Feller if both $\lim_{t \rightarrow 0} \|Q_t f - f\|_\infty = 0$ for any $f \in C_0(E)$ (strong continuity) and $Q_t C_0(E) \subset C_0(E)$.

The following proposition, proved in Section 2.4.3, provides information on the continuity property of Q_t when t goes to 0.

Proposition 12. *We have*

1. For any $f \in C_b(E)$ and any $x \in E$, $\lim_{t \rightarrow 0} Q_t f(x) = f(x)$.
2. Let $f \in M_b(E)$. Then $\lim_{t \rightarrow 0} \|Q_t f - f\|_\infty = 0$ iff $\lim_{t \rightarrow 0} \|Q_t^Y f - f\|_\infty = 0$.

By the second point above, the strong continuity of Q_t is implied by the strong continuity of Q_t^Y , which in turn holds automatically true if $Q_t^Y C_0(E) \subset C_0(E)$ by continuity of Y_t . We thus obtain the following natural conditions on the jump-move process on E to be Feller continuous or Feller. The proof is given in Section 2.4.3.

Theorem 13. *Let $(X_t)_{t \geq 0}$ be a general jump-move process on E .*

1. If $Q_t^Y C_b(E) \subset C_b(E)$ and $K C_b(E) \subset C_b(E)$ then $(X_t)_{t \geq 0}$ is a Feller continuous process.
2. If $Q_t^Y C_0(E) \subset C_0(E)$ and $K C_0(E) \subset C_0(E)$ then $(X_t)_{t \geq 0}$ is a Feller process.

We deduce in particular from this theorem that if $Q_t^Y C_b(E) \subset C_b(E)$ and $K C_b(E) \subset C_b(E)$ (or alternatively with $C_0(E)$ instead of $C_b(E)$), then $(X_t)_{t \geq 0}$ is a strong Markov process for the filtration $(\mathcal{F}_t)_{t \geq 0}$, a property implied by the Feller continuous and Feller properties ([Bas11]). The Feller property will also be useful to us in Chapter 3 to construct a coupling between a birth-death-move process and a simple birth-death process on \mathbb{N} , in a view to establish ergodic properties.

We investigate in Section 2.3.3 the conditions of Theorem 13 for the examples of dynamics of systems of interacting particles in \mathbb{R}^d introduced in Section 2.2.4. They turn out to be generally satisfied under mild conditions for these examples.

2.3.2 Infinitesimal generator

In this section we compute the infinitesimal generator associated to the jump-move process $(X_t)_{t \geq 0}$. We first introduce some notations and recall below the definition of the generator, see for instance [Dyn65]. In connection, remember that the family $(Q_t)_{t \geq 0}$ of transition operators is a semigroup on $(M_b(E), \|\cdot\|_\infty)$. If moreover the process $(X_t)_{t \geq 0}$ is Feller continuous (*resp.* Feller), then $(Q_t)_{t \geq 0}$ is a semigroup on $(C_b(E), \|\cdot\|_\infty)$ (*resp.* $(C_0(E), \|\cdot\|_\infty)$).

Definition 2. Let $L \subset M_b(E)$ and $(U_t)_{t \geq 0}$ be a semigroup on $(L, \|\cdot\|_\infty)$. We set

$$L_0 = \{f \in L : \lim_{t \rightarrow 0} \|U_t f - f\|_\infty = 0\}$$

and

$$\mathcal{D}_A = \{f \in L : \lim_{t \rightarrow 0} \frac{U_t f - f}{t} \text{ exists in } (L, \|\cdot\|_\infty)\}.$$

For $f \in \mathcal{D}_A$, define $\mathcal{A}f = \lim_{t \searrow 0} (U_t f - f)/t$. The operator $\mathcal{A} : \mathcal{D}_A \rightarrow L$ is called the infinitesimal generator associated to the semigroup $(U_t)_{t \geq 0}$ and \mathcal{D}_A is called the domain of the generator \mathcal{A} .

In the following we denote by L_0 (resp. L_0^Y) and \mathcal{A} (resp. \mathcal{A}^Y) the set as in Definition 2 and the infinitesimal operator associated to $(Q_t)_{t \geq 0}$ (resp. $(Q_t^Y)_{t \geq 0}$). Note that $L_0 = L_0^Y$ by Proposition 12.

Theorem 14. Let $(X_t)_{t \geq 0}$ be a general jump-move process on a Polish space E . Suppose that if $f \in L_0^Y$, then $\alpha \times f \in L_0^Y$ and $Kf \in L_0^Y$. Then $\mathcal{D}_A = \mathcal{D}_{A^Y}$ and for any $f \in \mathcal{D}_{A^Y}$,

$$\mathcal{A}f = \mathcal{A}^Y f + \alpha \times Kf - \alpha \times f.$$

This result, proved in Section 2.4.4, shows that the generator \mathcal{A} of the jump-move process $(X_t)_{t \geq 0}$ is just the sum of the generator of the move \mathcal{A}^Y and the generator of the jump, specifically of a pure jump Markov process with intensity α and transition kernel K , that is $\alpha \times (K - Id)$ (see [Fel71]).

Note that for a pure jump process, $Q_t^Y = Id$ for any $t \geq 0$, $L_0^Y = \mathcal{D}_{A^Y} = M_b(E)$ and $\mathcal{A}^Y \equiv 0$, so that all assumptions of Theorem 14 are trivially true in this setting. More generally, consider a jump-move process with a Feller inter-jumps process, i.e. $Q_t^Y C_0(E) \subset C_0(E)$, and a Feller jump transition, i.e. $K C_0(E) \subset C_0(E)$, so that $(X_t)_{t \geq 0}$ is Feller by Theorem 13, then we can take $L_0^Y = C_0(E)$ and again the assumptions of Theorem 14 are satisfied since α is bounded.

2.3.3 Application to systems of interacting particles in \mathbb{R}^d

We go back to the setting of Section 2.2.4 concerned with systems of interacting particles in $W \subset \mathbb{R}^d$, in order to inspect whether the examples of dynamics presented therein are (continuous) Feller or not. To do so and be able to check the conditions of Theorem 13, we need to first clarify what are the sets $C_b(E)$ and $C_0(E)$ in this framework. Remember that in this setting $E = \cup_{n \geq 0} E_n$ where $E_n = \pi_n(W^n)$ corresponds to the set of unordered n -tuples of W , and we have equipped E with the distance d_1 defined by (2.8). As a first result, it can be verified that (E, d_1) is a locally compact Polish space, see [SX08, Proposition 2.2] and Section 2.4.5. To characterise the elements of $C_b(E)$, we shall use the following proposition, proved in Section 2.4.5.

Proposition 15. Let $x \in E$ and $(x^{(p)})_{p \geq 1}$ a sequence such that $d_1(x^{(p)}, x) \rightarrow 0$ as $p \rightarrow \infty$. Then there exist $p_0 \geq 1$ such that for all $p \geq p_0$ $n(x^{(p)}) = n(x)$ and, when $n(x) \geq 1$, a sequence $(\sigma_p)_{p \geq p_0}$ of $\mathcal{S}_{n(x)}$ such that for any $i \in \{1, \dots, n(x)\}$,

$$\|x_{\sigma_p(i)}^{(p)} - x_i\| \xrightarrow[p \rightarrow \infty]{} 0. \quad (2.12)$$

On the other side, to deal with $C_0(E)$, we provide a characterization of the compact sets of each E_n , for $n \geq 1$, and an important property about the compact sets of E .

Proposition 16. *Suppose that W is a closed set of \mathbb{R}^d .*

1. *Let $n \geq 1$ and A be a closed subset of (E_n, d_1) . Then A is compact if and only if the following property holds:*

$$\forall w \in W, \exists R \geq 0, \text{ s.t. } \forall x = \{x_1, \dots, x_n\} \in A, \max_{1 \leq k \leq n} \{\|x_k - w\|\} \leq R.$$

2. *Let A be a compact set of E . Then there exists $n_0 \geq 0$ such that $A \subset \bigcup_{n=0}^{n_0} E_n$.*

The two previous propositions are the main tools to investigate the continuous Feller and Feller properties of the jump kernel K of a jump-move process. Concerning the inter-jumps move process $(Y_t)_{t \geq 0}$, remember that we can easily define it on each E_n from a continuous process $(Z_t^n)_{t \geq 0}$ on W^n by the projection (2.9). Similarly as for the continuity property discussed in Section 2.2.4, the Feller properties of $(Y_t)_{t \geq 0}$ on (E, d_1) inherit from that of $(Z_t^n)_{t \geq 0}$ on W^n .

Proposition 17. *Let $(Y_t)_{t \geq 0}$ be defined on E by (2.9), then if $(Z_t^n)_{t \geq 0}$ is a Feller continuous (resp. Feller) process on W^n for every $n \geq 1$ then $(Y_t)_{t \geq 0}$ is a Feller continuous (resp. Feller) process on E .*

By this result, standard inter-jumps motions are Feller continuous and Feller, as this is the case under mild assumptions for Examples 4 and 5 detailed below. Concerning the jump kernels, the global picture is as follows. They are generally Feller continuous, but not necessarily Feller even if the underlying space W is compact, as showed in the following example. However if we restrict ourselves to birth kernels, then they are generally Feller (see Examples 2 and 3 below). On the other hand, if we restrict ourselves to death kernels, then they are Feller if W is compact, but not otherwise, see Example 1 below. Remark that a birth-and-death jump kernel as in (2.4) is (continuous) Feller when the birth kernel K_β and the death kernel K_δ are. So it is generally continuous Feller, and if W is compact, it is generally Feller.

Let us make these informal claims more specific through some examples. The first one presents an example of jump kernel on a set W , possibly compact, that is continuous Feller but not Feller. The other ones correspond to the examples introduced in Section 2.2.4.

Example. Consider the jump kernel K defined for $f \in M_b(E)$ by

$$Kf(x) = \frac{1}{n(x)} \sum_{i=1}^{n(x)} f(\{x_i\})$$

for $x = \{x_1, \dots, x_{n(x)}\} \in E$, so that $K(x, E_1) = 1$ for any $x \in E$. Let $x^{(p)}$ be a sequence converging to x , from which we define p_0 and $(\sigma_p)_{p \geq p_0}$ as in Proposition 15. Let $f \in C_b(E)$. Then

$$Kf(x^{(p)}) = \frac{1}{n(x)} \sum_{i=1}^{n(x)} f(x_{\sigma_p(i)}^{(p)}) \xrightarrow{p \rightarrow \infty} \frac{1}{n(x)} \sum_{i=1}^{n(x)} f(\{x_i\}) = Kf(x),$$

which shows the continuous Feller property of K , i.e. $K C_b(E) \subset C_b(E)$. Let us now show that K is not Feller. Assume without loss of generality that $0 \in W$. Consider the function $f(x) = \max(1 - \|x\|, 0) \mathbf{1}_{n(x)=1}$, where we abusively write $\|x\| := \|x_1\|$ when $x = \{x_1\}$, $x_1 \in W$. Note that $f \in C_0(E)$. Let B be a compact subset of E . From Theorem 16 there exists $n_0 \geq 0$ such that $B \subset \bigcup_{n=0}^{n_0} E_n$. Choose $y = \{0, \dots, 0\} \in E_{n_0+1}$. Then $y \notin B$ but $Kf(y) = 1$ proving that $Kf \notin C_0(E)$.

Example 1 (continued) (*death kernel*): For the death kernel K_δ of this example, we have

$$(i) \quad K_\delta C_b(E) \subset C_b(E)$$

$$(ii) \quad K_\delta C_0(E) \subset C_0(E) \text{ if } W \text{ is compact, but not necessarily otherwise.}$$

To prove the first property, take $x \in E$, a sequence $(x^{(p)})_{p \geq 0}$ converging to x , and p_0 and $(\sigma_p)_{p \geq p_0}$ from Proposition 15. Then it is not difficult to verify that $\lim_{p \rightarrow \infty} w(x_{\sigma_p^{(i)}}^{(p)}, x^{(p)}) = w(x_i, x)$ by continuity of g . Moreover,

$$d_1 \left(x^{(p)} \setminus x_{\sigma_p^{(i)}}^{(p)}, x \setminus x_i \right) \leq \frac{1}{n-1} \sum_{j \neq i} \|x_{\sigma_p^{(j)}}^{(p)} - x_j\|$$

which shows that $x^{(p)} \setminus x_{\sigma_p^{(i)}}^{(p)} \xrightarrow{p \rightarrow \infty} x \setminus x_i$. Therefore, for any $f \in C_b(E)$,

$$\begin{aligned} \lim_{p \rightarrow \infty} K_\delta f(x^{(p)}) &= \lim_{p \rightarrow \infty} \sum_{i=1}^{n(x)} \omega(x_i^{(p)}, x^{(p)}) f(x^{(p)} \setminus x_i^{(p)}) \\ &= \lim_{p \rightarrow \infty} \sum_{i=1}^{n(x)} \omega(x_{\sigma_p^{(i)}}^{(p)}, x^{(p)}) f(x^{(p)} \setminus x_{\sigma_p^{(i)}}^{(p)}) \\ &= \sum_{i=1}^{n(x)} \omega(x_i, x) f(x \setminus x_i) \\ &= K_\delta f(x). \end{aligned}$$

Let us now consider the second claim (ii). Take $f \in C_0(E)$ and $\varepsilon > 0$. We fix A a compact set of (E, d_1) such that $|f(x)| < \varepsilon$ for $x \notin A$. By Proposition 16, $A \subset \bigcup_{n=0}^{n_0} E_n$ for some n_0 . As a straightforward consequence of Proposition 16 (see Corollary 32 in Section 2.4.5) the set $B := \bigcup_{n=0}^{n_0+1} E_n$ is a compact set when W is compact and it satisfies $K_\delta(x, A) = 0$ for $x \notin B$. This implies that for $x \notin B$,

$$\begin{aligned} |K_\delta f(x)| &\leq \left| \int_A f(y) K_\delta(x, dy) \right| + \left| \int_{A^c} f(y) K_\delta(x, dy) \right| \\ &\leq \|f\|_\infty K_\delta(x, A) + \varepsilon K_\delta(x, A^c) \leq \varepsilon, \end{aligned} \tag{2.13}$$

and so $K_\delta f \in C_0(E)$. Let us finally show that this result is not valid any more if W is not compact. Assume without loss of generality that $0 \in W$ and consider as in the previous example the function $f \in C_0(E)$ defined by $f(x) = \max(1 - \|x\|, 0) \mathbf{1}_{n(x)=1}$. Let B be any compact subset of E . Then $B_2 = B \cap E_2$ is compact because E_2 is closed and by Proposition 16, for any $x = \{x_1, x_2\} \in B_2$, there exists $R > 0$ such

that $\max\{\|x_1\|, \|x_2\|\} \leq R$. Take $y = \{0, y_2\}$ in E_2 such that $\|y_2\| > R + 1$, which is possible since W is not compact. Then $y \notin B$ but $K_\delta f(y) = w(y_2, y)$ proving that $Kf \notin C_0(E)$.

Example 2 (continued) (*birth kernel as a mixture*): For this example, we shall prove that if $\dot{W} \neq \emptyset$ and if the dispersion function v is continuous, then $K_\beta C_b(E) \subset C_b(E)$ and $K_\beta C_0(E) \subset C_0(E)$. Take $f \in C_b(E)$, $x \in E$ and a sequence $(x^{(p)})_{p \geq 0}$ converging to x , from which we define p_0 and $(\sigma_p)_{p \geq p_0}$ from Proposition 15. We have, for $p \geq p_0$,

$$\begin{aligned} K_\beta f(x^{(p)}) &= \frac{1}{n(x)} \sum_{i=1}^{n(x)} \frac{1}{z(x_{\sigma_p(i)}^{(p)}, x^{(p)})} \int_W f(x^{(p)} \cup \{\xi\}) \varphi \left(\frac{\xi - x_{\sigma_p(i)}^{(p)}}{v(x_{\sigma_p(i)}^{(p)}, x^{(p)})} \right) d\xi \\ &= \frac{1}{n(x)} \sum_{i=1}^{n(x)} \frac{\int_{\mathbb{R}^d} \mathbf{1}_W(x_{\sigma_p(i)}^{(p)} + v(x_{\sigma_p(i)}^{(p)}, x^{(p)})\xi) f(x^{(p)} \cup \{x_{\sigma_p(i)}^{(p)} + v(x_{\sigma_p(i)}^{(p)}, x^{(p)})\xi\}) \varphi(\xi) d\xi}{\int_{\mathbb{R}^d} \mathbf{1}_W(x_{\sigma_p(i)}^{(p)} + v(x_{\sigma_p(i)}^{(p)}, x^{(p)})\xi) \varphi(\xi) d\xi}. \end{aligned}$$

By continuity of v , the involved indicator functions tend to $\mathbf{1}_W(x_i + v(x_i, x)\xi)$ for any $x_i + v(x_i, x)\xi \in \dot{W}$. On the other hand, for any $i \in \{1, \dots, n(x)\}$ and any ξ ,

$$\begin{aligned} &d_1 \left(x^{(p)} \cup \{x_{\sigma_p(i)}^{(p)} + v(x_{\sigma_p(i)}^{(p)}, x^{(p)})\xi\}, x \cup \{x_i + v(x_i, x)\xi\} \right) \\ &\leq \frac{1}{n(x) + 1} \left(\sum_{j=1}^{n(x)} \|x_{\sigma_p(j)}^{(p)} - x_j\| + \|x_{\sigma_p(i)}^{(p)} + v(x_{\sigma_p(i)}^{(p)}, x^{(p)})\xi - x_i - v(x_i, x)\xi\| \right) \\ &\leq \frac{1}{n(x) + 1} \left(\sum_{j=1}^{n(x)} \|x_{\sigma_p(j)}^{(p)} - x_j\| + \|x_{\sigma_p(i)}^{(p)} - x_i\| + \|\xi\| |v(x_{\sigma_p(i)}^{(p)}, x^{(p)}) - v(x_i, x)| \right) \end{aligned}$$

which tends to 0 as $p \rightarrow \infty$. So by continuity of f , $f(x^{(p)} \cup \{x_{\sigma_p(i)}^{(p)} + v(x_{\sigma_p(i)}^{(p)}, x^{(p)})\xi\})$ tends to $f(x \cup \{x_i + v(x_i, x)\xi\})$. We conclude by the dominated convergence theorem, since f is bounded and φ is a density, that $K_\beta f(x^{(p)})$ converges to $K_\beta f(x)$ as $p \rightarrow \infty$, which proves that $K_\beta C_b(E) \subset C_b(E)$.

Let us now prove that $K_\beta C_0(E) \subset C_0(E)$. Let $f \in C_0(E)$ and $\varepsilon > 0$. We fix $A \subset E$ a compact set such that $x \notin A \Rightarrow |f(x)| < \varepsilon$. By Proposition 16, $A \subset \bigcup_{n=0}^{n_0} E_n$ for some n_0 . Letting $A_n = A \cap E_n$, for $n = 0, \dots, n_0$, we remark that A_n is a compact set because E_n is closed. By Proposition 16, there exists $R_n \geq 0$ such that for every $a = \{a_1, \dots, a_n\} \in A_n$, $\max_{1 \leq k \leq n} \|a_k\| \leq R_n$. Let now $B_n = \{x \in E_n, \sum_{k=1}^n \|x_k\|/n \leq R_n\}$ and $B = \bigcup_{n=0}^{n_0-1} B_n$. We can verify (see the proof of Proposition 16) that B_n is compact and so is B . We claim that if $x \notin B$, then $K_\beta(x, A) = 0$. Indeed, if $K_\beta(x, A) > 0$ then $K_\beta(x, A_n) > 0$ for some $n \in \{0, \dots, n_0\}$, but since $K_\beta(x, A_0) \leq K_\beta(x, \{\emptyset\}) = 0$ it cannot be $n = 0$. Now, for $n = 1, \dots, n_0$, $K_\beta(x, A_n) > 0$ implies that $x \in E_{n-1}$ and $A_n \subset \{z \cup x, z \in W\}$ since $K_\beta(x, W \cup x) = 1$. So $\max_{1 \leq k \leq n-1} \|x_k\| \leq R_n$, whereby $x \in B_{n-1}$. This shows that if $K_\beta(x, A) > 0$ then $x \in B$, as we claimed it. We deduce that for any $x \notin B$, $|K_\beta f(x)| \leq \varepsilon$ as in (2.13).

Example 3 (continued) (*birth kernel based on a Gibbs potential*): This birth kernel K_β is both Feller continuous and Feller, whenever the potential V is continuous and locally stable. By the latter, we mean that there exists $\psi \in L^1(W)$ such

that for any $x \in E$, $\exp(-(V(x \cup \xi) - V(x))) \leq \psi(\xi)$ ([MW04]). Under these conditions, we can prove similarly as in Example 2 that $K_\beta C_b(E) \subset C_b(E)$ by use of the dominated convergence theorem and that $K_\beta C_0(E) \subset C_0(E)$. Note that the examples of potentials considered in Chapter 3, leading to an invariant Gibbs measure, are continuous and locally stable.

Example 4 (continued) (*Langevin diffusions as inter-jumps motions*): The inter-jumps process $(Y_t)_{t \geq 0}$, defined through the stochastic differential equation (2.10), is a Feller continuous and a Feller process on E . This is due to the fact that g being globally Lipschitz, the function Φ_n in (2.10) is also globally Lipschitz for any $n \geq 1$, and so the solution $(Z_t^n)_{t \geq 0}$ of (2.10) is Feller continuous and Feller ([SP12]). The conclusion then follows from Proposition 17.

Example 5 (continued) (*Growth interaction processes*): In this example, the inter-jumps motion is driven by (2.11). If the functions $F_{1,n}, \dots, F_{n,n}$ are Lipschitz continuous, then $(Y_t)_{t \geq 0}$ is Feller continuous and Feller. Indeed, the solution of (2.11) is continuous in the initial condition Z_0^n under this assumption (see [Mar04]), implying the Feller continuity of $(Y_t)_{t \geq 0}$ by Proposition 17. Moreover, since the marks $m_i(t)$ in $(Z_t^n)_{t \geq 0}$ are all increasing functions, we have that $\|Z_t^n\| \geq \|Z_0^n\|$. Let $f \in C_0(W^n)$, $\epsilon > 0$ and $R > 0$ such that $\|x\| > R \Rightarrow |f(x)| < \epsilon$. Therefore if $\|Z_0^n\| > R$, we have $\|Z_t^n\| \geq R$ and so $f(Z_t^n) < \epsilon$, proving that Z_t^n is Feller and so is $(Y_t)_{t \geq 0}$ by Proposition 17.

Let again $(Y_t)_{t \geq 0}$ be the move process defined on each E_n from a continuous process $(Z_t^n)_{t \geq 0}$ on W^n by the projection (2.9). With the following Proposition we obtain informations on the domain of the generator associated to the move process $(Y_t)_{t \geq 0}$ from the domain of each $(Z_t^n)_{t \geq 0}$ for lack of knowing it perfectly. We introduce the notation $C_c(E)$ for the set of continuous functions with compact support.

Proposition 18. *Let $(Y_t)_{t \geq 0}$ the process defined on E by (2.9). If we suppose that for any $n \geq 1$ the process $(Z_t^n)_{t \geq 0}$ is a Feller process, then for any $f \in C_c(E)$ such that $f \circ \pi_n \in \mathcal{D}_{\mathcal{AZ}^n}$ for any $n \geq 1$, we get that $f \in \mathcal{D}_{\mathcal{AY}}$.*

2.4 Proofs and technical results

This section focuses on the proofs of the results previously stated in this chapter. It also contains properties of the sequence of the jumping times $(T_j)_{j \geq 0}$ associated to the jump-move process and describes some topological properties of the space E endowed with the distance d_1 in the case of interacting particles in \mathbb{R}^d , as introduced in Section 2.2.4.

2.4.1 Law of the jumping times

Here we check that the jumping times of the jump-move process are continuous random variables and we compute their densities.

Lemma 19. *Let $x \in E$.*

1. *For any $j \geq 1$, the variable τ_j has a density $\mu_{\tau_j}^x$ under \mathbb{P}_x given by:*

$$\mu_{\tau_j}^x(t) = \mathbb{E}_x \left[\alpha(Y_t^{(j-1)}) \exp \left(- \int_0^t \alpha(Y_u^{(j-1)}) du \right) \right] \mathbf{1}_{t>0}.$$

2. *Let $k \geq 1$. The vector (T_1, T_2, \dots, T_k) has a density under \mathbb{P}_x given by:*

$$\mu_{(T_1, \dots, T_k)}^x(t_1, \dots, t_k) = \mu_{\tau_1}^x(t_1) \prod_{j=2}^k \mu_{\tau_j}^x(t_j - t_{j-1})$$

In particular, T_j has a density under \mathbb{P}_x for any $j \geq 0$, which will be denoted $\mu_{T_j}^x$.

Proof of Lemma 19. Let h be a bounded continuous real function.

1. We use the fact that conditional on $(Y_u^{(j-1)})_{u \geq 0}$, τ_j has the density

$$t \geq 0 \mapsto \alpha(Y_t^{(j-1)}) e^{-\int_0^t \alpha(Y_u^{(j-1)}) du}$$

to obtain

$$\begin{aligned} \mathbb{E}_x[h(\tau_j)] &= \mathbb{E}_x \left[\mathbb{E}_x[h(\tau_j) | (Y_u^{(j-1)})_{u \geq 0}] \right] \\ &= \mathbb{E}_x \left[\int_0^{+\infty} h(t) \alpha(Y_t^{(j-1)}) e^{-\int_0^t \alpha(Y_u^{(j-1)}) du} dt \right] \\ &= \int_{\mathbb{R}} h(t) \mu_{\tau_j}^x(t) dt, \end{aligned}$$

by Fubini's theorem.

2. Let $p \geq 1$ and $t_1, \dots, t_{p-1} \in \mathbb{R}_+$. Then

$$\begin{aligned} &\mathbb{E}_x[h(T_p) | T_{p-1} = t_{p-1}, \dots, T_1 = t_1] \\ &= \mathbb{E}_x \left[\mathbb{E}_x[h(T_p) | T_{p-1} = t_{p-1}, \dots, T_1 = t_1, (Y_u^{(p-1)})_{t \geq 0}] \right] \\ &= \mathbb{E}_x \left[\mathbb{E}_x[h(T_{p-1} + \tau_p) | T_{p-1} = t_{p-1}, \dots, T_1 = t_1, (Y_u^{(p-1)})_{t \geq 0}] \right] \\ &= \mathbb{E}_x \left[\int_0^{+\infty} h(t + t_{p-1}) \alpha(Y_t^{(p-1)}) e^{-\int_0^t \alpha(Y_u^{(p-1)}) du} dt \right] \\ &= \int_0^{+\infty} h(t) \mathbb{E}_x \left[\alpha(Y_{t-t_{p-1}}^{(p-1)}) e^{-\int_0^{t-t_{p-1}} \alpha(Y_u^{(p-1)}) du} \right] dt. \end{aligned}$$

Thus, given $T_{p-1} = t_{p-1}, \dots, T_1 = t_1$, the variable T_p has a density under \mathbb{P}_x given for $t > t_{p-1}$ by

$$\mu_{T_p | T_{p-1}=t_{p-1}, \dots, T_1=t_1}^x(t) = \mathbb{E}_x \left[\alpha(Y_{t-t_{p-1}}^{(p-1)}) e^{-\int_0^{t-t_{p-1}} \alpha(Y_u^{(p-1)}) du} \right]$$

i.e.

$$\mu_{T_p | T_{p-1}=t_{p-1}, \dots, T_1=t_1}^x(t) = \mu_{\tau_p}^x(t - t_{p-1}).$$

Now recall that T_1 has a density under \mathbb{P}_x given by

$$\mu_1^x(t) = \mathbb{E}_x \left[\alpha(Y_{t_1}^{(0)}) e^{-\int_0^{t_1} \alpha(Y_u^{(0)}) du} \right] \quad \text{for } t \geq 0.$$

Then (T_1, T_2, \dots, T_k) has a density under \mathbb{P}_x given by the formula

$$\begin{aligned} \mu_{(T_1, \dots, T_k)}^x(t_1, \dots, t_k) \\ = \mu_1^x(t_1) \mu_{T_2|T_1=t_1}^x(t_2) \mu_{T_3|T_2=t_2, T_1=t_1}^x(t_3) \cdots \mu_{T_k|T_{k-1}=t_{k-1}, \dots, T_1=t_1}^x(t_k). \end{aligned}$$

□

As already mentioned the process $(N_t)_{t \geq 0}$ is a non-explosive counting process. The intensity function α being bounded, (2.3) is a consequence of Lemma 19.

2.4.2 Proofs of Section 2.2.3 about the Kolmogorov backward equation

Proof of Theorem 9

On the one hand, for any $x \in E$, $t > 0$ and $A \in \mathcal{E}$,

$$\begin{aligned} \mathbb{P}_x(X_t \in A, \tau_1 > t) &= \mathbb{E}_x(\mathbf{1}_{X_t \in A} \mathbf{1}_{\tau_1 > t}) \\ &= \mathbb{E}_x \left[\mathbb{E}_x(\mathbf{1}_{X_t \in A} \mathbf{1}_{\tau_1 > t} | (Y_u^{(0)})_{u \geq 0}) \right] \\ &= \mathbb{E}_x \left[\mathbb{E}_x(\mathbf{1}_{Y_t^{(0)} \in A} \mathbf{1}_{\tau_1 > t} | Y^{(0)}) \right] \\ &= \mathbb{E}_x \left[\mathbf{1}_{Y_t^{(0)} \in A} \mathbb{E}_x(\mathbf{1}_{\tau_1 > t} | Y^{(0)}) \right] \\ &= \mathbb{E}_x \left[\mathbf{1}_{Y_t^{(0)} \in A} e^{-\int_0^t \alpha(Y_u^{(0)}) du} \right] \\ &= \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A} e^{-\int_0^t \alpha(Y_u) du} \right]. \end{aligned} \tag{2.14}$$

On the other hand, by construction of the process

$$\mathbb{E}_x[\mathbf{1}_{X_t \in A} | \mathcal{F}_{\tau_1}] \mathbf{1}_{\tau_1 \leq t} = Q_{t-\tau_1}(X_{\tau_1}, A) \mathbf{1}_{\tau_1 \leq t}$$

where $\mathcal{F}_{\tau_1} = \{F \in \mathcal{F} : F \cap \{\tau_1 \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$. Then

$$\begin{aligned} \mathbb{P}_x(X_t \in A, \tau_1 \leq t) &= \mathbb{E}_x[\mathbb{E}_x[\mathbf{1}_{X_t \in A} | \mathcal{F}_{\tau_1}] \mathbf{1}_{\tau_1 \leq t}] \\ &= \mathbb{E}_x[Q_{t-\tau_1}(X_{\tau_1}, A) \mathbf{1}_{\tau_1 \leq t}] \\ &= \mathbb{E}_x[\mathbb{E}_x[Q_{t-\tau_1}(X_{\tau_1}, A) \mathbf{1}_{\tau_1 \leq t} | \tau_1, Y^{(0)}]] \\ &= \mathbb{E}_x \left[\int_{y \in E} K(Y_{\tau_1}^{(0)}, dy) Q_{t-\tau_1}(y, A) \mathbf{1}_{\tau_1 \leq t} \right] \\ &= \mathbb{E}_x \left[\mathbb{E}_x \left[\int_{y \in E} K(Y_{\tau_1}^{(0)}, dy) Q_{t-\tau_1}(y, A) \mathbf{1}_{\tau_1 \leq t} | Y^{(0)} \right] \right] \\ &= \mathbb{E}_x \left[\int_0^t \int_{y \in E} K(Y_s^{(0)}, dy) Q_{t-s}(y, A) \alpha(Y_s^{(0)}) e^{-\int_0^s \alpha(Y_u^{(0)}) du} ds \right] \\ &= \int_0^t \int_E Q_{t-s}(y, A) \mathbb{E}_x \left[K(Y_s^{(0)}, dy) \alpha(Y_s^{(0)}) e^{-\int_0^s \alpha(Y_u^{(0)}) du} \right] ds \\ &= \int_0^t \int_E Q_{t-s}(y, A) \mathbb{E}_x^Y \left[K(Y_s, dy) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds. \end{aligned} \tag{2.15}$$

The result then follows gathering (2.14) and (2.15).

Proof of Proposition 10

The proof is made up from Lemmas 20, 21 and 22, the approach being similar to [Fel71]. In Lemma 20 we built a solution $Q_{t,\infty}(x, A)$ of (2.5) for any $x \in E$ and $A \in \mathcal{E}$, while Lemmas 21 and 22 will imply the unicity of the solution.

Lemma 20. *For all $x \in E$ and $A \in \mathcal{E}$, the function $t \in \mathbb{R}_+ \mapsto Q_{t,\infty}(x, A)$ is a solution of (2.5).*

Proof of Lemma 20. We will proceed as in the proof of Theorem 9. First

$$\mathbb{P}_x(X_t \in A, T_{p+1} > t, \tau_1 > t) = \mathbb{P}_x(X_t \in A, \tau_1 > t) = \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A} e^{-\int_0^t \alpha(Y_u) du} \right].$$

Secondly, if the process jumps once before t (at time τ_1) and is in A at time t with at most $p+1$ jumps, the process has at most p jumps after the time τ_1 . By construction of the process, the law of X_t given X_{τ_1} is the same as the one of $X_{t-\tau_1}$ given $X_0 = X_{\tau_1}$. We then obtain

$$\mathbb{E}_x[\mathbf{1}_{X_t \in A} \mathbf{1}_{\tau_1 \leq t < T_{p+1}} | \mathcal{F}_{\tau_1}] = Q_{t-\tau_1, p}(X_{\tau_1}, A) \mathbf{1}_{\tau_1 \leq t}.$$

This leads to

$$\begin{aligned} \mathbb{P}_x(X_t \in A, T_{p+1} > t, \tau_1 \leq t) &= \mathbb{E}_x[Q_{t-\tau_1, p}(X_{\tau_1}, A) \mathbf{1}_{\tau_1 \leq t}] \\ &= \mathbb{E}_x[\mathbb{E}_x[Q_{t-\tau_1, p}(X_{\tau_1}, A) \mathbf{1}_{\tau_1 \leq t} | Y^{(0)}, \tau_1]] \\ &= \mathbb{E}_x\left[\int_{y \in E} Q_{t-\tau_1, p}(y, A) \mathbf{1}_{\tau_1 \leq t} K(Y_{\tau_1}^{(0)}, dy)\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_x\left[\int_{y \in E} Q_{t-\tau_1, p}(y, A) \mathbf{1}_{\tau_1 \leq t} K(Y_{\tau_1}^{(0)}, dy) | Y^{(0)}\right]\right] \\ &= \mathbb{E}_x\left[\int_0^t \int_{y \in E} Q_{t-s, p}(y, A) \mathbf{1}_{\tau_1 \leq t} K(Y_s^{(0)}, dy) \alpha(Y_s^{(0)}) e^{-\int_0^s \alpha(Y_u^{(0)}) du}\right] \\ &= \int_0^t \int_E Q_{t-s, p}(y, A) \mathbb{E}_x^Y \left[K(Y_s, dy) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds. \end{aligned}$$

We then obtain the induction formula

$$\begin{aligned} Q_{t, p+1}(x, A) &= \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A} e^{-\int_0^t \alpha(Y_u) du} \right] \\ &\quad + \int_0^t \int_E Q_{t-s, p}(y, A) \mathbb{E}_x^Y \left[K(Y_s, dy) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds. \end{aligned}$$

This leads by monotone convergence to

$$\begin{aligned} Q_{t, \infty}(x, A) &= \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A} e^{-\int_0^t \alpha(Y_u) du} \right] \\ &\quad + \int_0^t \int_E Q_{t-s, \infty}(y, A) \mathbb{E}_x^Y \left[K(Y_s, dy) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds. \end{aligned}$$

□

Lemma 21. $Q_{t,\infty}$ is called the minimal solution of (2.5) in the sense that for any non-negative solution Q_t of (2.5), we have $Q_t \geq Q_{t,\infty}$.

Proof of Lemma 21. Let Q_t be a non-negative solution of (2.5). Then for any $x \in E$ and $A \in \mathcal{E}$

$$Q_t(x, A) \geq Q_{t,1}(x, A) = \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A} e^{-\int_0^t \alpha(Y_u) du} \right].$$

We then proceed by induction. If $Q_t \geq Q_{t,p}$ then

$$\begin{aligned} Q_t(x, A) &= \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A} e^{-\int_0^t \alpha(Y_u) du} \right] \\ &\quad + \int_0^t \int_E Q_{t-s}(y, A) \mathbb{E}_x^Y \left[K(Y_s, dy) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds \\ &\geq \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A} e^{-\int_0^t \alpha(Y_u) du} \right] \\ &\quad + \int_0^t \int_E Q_{t-s,p}(y, A) \mathbb{E}_x^Y \left[K(Y_s, dy) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds \\ &= Q_{t,p+1}(x, A). \end{aligned}$$

Finally $Q_t(x, A) \geq Q_{t,p}(x, A)$ for every $p \geq 1$ and the result follows by letting p go to infinity. \square

Lemma 22. The minimal solution $Q_{t,\infty}$ is stochastic, i.e. $Q_{t,\infty}(x, E) = 1$.

Proof of Lemma 22. Recall that α is bounded by $\alpha^* > 0$. It is then enough to show by induction that $Q_{t,p}(x, E) \geq 1 - (1 - e^{-\alpha^* t})^p$ for any $p \geq 1$. First

$$Q_{t,1}(x, E) = \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in E} e^{-\int_0^t \alpha(Y_u) du} \right] = \mathbb{E}_x^Y \left[e^{-\int_0^t \alpha(Y_u) du} \right] \geq \mathbb{E}_x \left(e^{-\alpha^* t} \right) = e^{-\alpha^* t}.$$

Then notice that

$$\mathbb{P}_x(\tau_1 \leq t) = \mathbb{E}_x^Y \left[1 - e^{-\int_0^t \alpha(Y_u) du} \right] \leq 1 - e^{-\alpha^* t}.$$

We then obtain by induction

$$\begin{aligned} Q_{t,p+1}(x, E) &= \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in E} e^{-\int_0^t \alpha(Y_u) du} \right] \\ &\quad + \int_0^t \int_E Q_{t-s,p}(y, A) \mathbb{E}_x^Y \left[K(Y_s, dy) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds \\ &\geq \mathbb{E}_x^Y \left[e^{-\int_0^t \alpha(Y_u) du} \right] \\ &\quad + \int_0^t \left(1 - (1 - e^{-\alpha^*(t-s)})^p \right) \mathbb{E}_x^Y \left[\int_E K(Y_s, dy) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds \\ &\geq \mathbb{E}_x^Y \left[e^{-\int_0^t \alpha(Y_u) du} \right] + \int_0^t \left(1 - (1 - e^{-\alpha^* t})^p \right) \mathbb{E}_x^Y \left[\alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du} \right] ds \\ &= \mathbb{P}_x(\tau_1 > t) + \left(1 - (1 - e^{-\alpha^* t})^p \right) \mathbb{P}_x(\tau_1 \leq t) \\ &= 1 - (1 - e^{-\alpha^* t})^p \mathbb{P}_x(\tau_1 \leq t) \geq 1 - (1 - e^{-\alpha^* t})^{p+1}. \end{aligned}$$

\square

By bringing together the last three lemmas it does not take long to prove Proposition 10. By Lemma 20, $Q_{t,\infty}$ is a solution of (2.5). We now prove the unicity. Let Q_t be a non-negative sub-stochastic solution of (2.5). Lemma 21 entails $Q_t(x, A) \geq Q_{t,\infty}(x, A)$ and $Q_t(x, E \setminus A) \geq Q_{t,\infty}(x, E \setminus A)$ for every $A \in \mathcal{E}$. We get then

$$1 \geq Q_t(x, E) = Q_t(x, A) + Q_t(x, E \setminus A) \geq Q_{t,\infty}(x, A) + Q_{t,\infty}(x, E \setminus A) = Q_{t,\infty}(x, E) = 1$$

by Lemma 22 so $Q_t(x, A) = Q_{t,\infty}(x, A)$ for every $A \in \mathcal{E}$.

Proof of Proposition 11

We first show that $Q_{t,(\infty)}$ is a solution of (2.5). Let $n \geq 0$, $x \in E_n$ and $p \geq n$. If there is no jump before t , then

$$\mathbb{P}_x(X_t \in A, \tau_1 > t, \forall s \in [0, t] n(X_s) \leq p) = \mathbb{P}_x(X_t \in A, \tau_1 > t).$$

By construction of the process, if the first jump before t is a death,

$$\mathbb{P}_x(X_t \in A, \forall s \in [0, t] n(X_s) \leq p \mid \mathcal{F}_{\tau_1}, \text{ a death occurs at } \tau_1) = Q_{t-\tau_1, (p)}(X_{\tau_1}, A),$$

and if the first jump before t is a birth,

$$\mathbb{P}_x(X_t \in A, \forall s \in [0, t] n(X_s) \leq p \mid \mathcal{F}_{\tau_1}, \text{ a birth occurs at } \tau_1) = Q_{t-\tau_1, (p)}(X_{\tau_1}, A) \mathbf{1}_{p > n}.$$

Following the same computations as in the proof of Theorem 9, we obtain

$$\begin{aligned} Q_{t, (p)}(x, A) &= \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A} e^{-\int_0^t \alpha(Y_u) du} \right] \\ &+ \int_0^t \int_{E_{n+1}} Q_{t-s, (p)}(y, A) \mathbb{E}_x^Y \left[\beta(Y_s) K_\beta(Y_s, dy) e^{-\int_0^s \alpha(Y_u) du} \right] ds \mathbf{1}_{p > n} \\ &+ \int_0^t \int_{E_{n-1}} Q_{t-s, (p)}(y, A) \mathbb{E}_x^Y \left[\delta(Y_s) K_\delta(Y_s, dy) e^{-\int_0^s \alpha(Y_u) du} \right] ds, \end{aligned}$$

and $Q_{t,(\infty)}(x, A)$ satisfies (2.5) by continuity of the probability. The proof is then complete thanks to the unicity of the solution to (2.5).

2.4.3 Proofs of Section 2.3.1 about Feller properties

Proof of Proposition 12

Both results of the proposition are based on the following calculation, for any $f \in M_b(E)$:

$$\begin{aligned} Q_t f(x) - f(x) &= \mathbb{E}_x^Y \left[f(Y_t) e^{-\int_0^t \alpha(Y_u) du} \right] - f(x) + \mathbb{E}_x [f(X_t) \mathbf{1}_{N_t \geq 1}] \\ &= Q_t^Y f(x) - f(x) + \mathbb{E}_x^Y \left[f(Y_t) \left(e^{-\int_0^t \alpha(Y_u) du} - 1 \right) \right] + \mathbb{E}_x [f(X_t) \mathbf{1}_{N_t \geq 1}]. \end{aligned}$$

The last two terms goes uniformly to 0 when $t \rightarrow 0$. Indeed,

$$\begin{aligned}
& \left| \mathbb{E}_x^Y \left[f(Y_t) \left(e^{-\int_0^t \alpha(Y_u) du} - 1 \right) \right] + \mathbb{E}_x [f(X_t) \mathbf{1}_{N_t \geq 1}] \right| \\
& \leq \|f\|_\infty \alpha^* t + \|f\|_\infty \mathbb{P}_x(N_t \geq 1) \\
& = \|f\|_\infty \alpha^* t + \|f\|_\infty \mathbb{E}_x^Y \left[\left(1 - e^{-\int_0^t \alpha(Y_u) du} \right) \right] \\
& \leq 2\alpha^* t \|f\|_\infty.
\end{aligned}$$

So we obtain directly the second point of the proposition. For the first point remark that when $f \in C_b(E)$, by continuity of $f \circ Y$ and the dominated convergence theorem, $\lim_{t \rightarrow 0} Q_t^Y f(x) = f(x)$.

Proof of Theorem 13 (part 1)

The proof of the Feller continuous property of $(X_t)_{t \geq 0}$ is based on the following Lemma 23 that exploits the Feller continuous property of Q_t^Y , and on Lemma 24 which in addition makes use of the Feller continuous property of the jump kernel K .

Lemma 23. *Assume that for any $t \geq 0$, $Q_t^Y C_b(E) \subset C_b(E)$. Then for any $p \geq 1$, $f_1, \dots, f_p \in C_b(E)$ and $0 \leq t_1 < \dots < t_p$ the function $x \mapsto \mathbb{E}_x^Y [f_1(Y_{t_1}) \dots f_p(Y_{t_p})]$ is continuous. Furthermore, for any $f \in C_b(E)$ the function $x \mapsto \mathbb{E}_x^Y [f(Y_t) e^{-\int_0^t \alpha(Y_u) du}]$ is continuous.*

Proof of Lemma 23. To prove the first statement, we proceed first by induction on $p \geq 1$. Since $x \mapsto \mathbb{E}_x^Y [f_1(Y_{t_1})] = Q_{t_1}^Y f_1(x)$, the property is satisfied for $p = 1$ because $Q_t^Y C_b(E) \subset C_b(E)$ for any $t \geq 0$ by assumption. Suppose now that the property is true for some $p \geq 1$. Let $f_1, \dots, f_{p+1} \in C_b(E)$ and $0 \leq t_1 < \dots < t_{p+1}$. Then

$$\begin{aligned}
\mathbb{E}_x^Y [f_1(Y_{t_1}) \dots f_{p+1}(Y_{t_{p+1}})] &= \mathbb{E}_x^Y \left[\mathbb{E}_x^Y (f_1(Y_{t_1}) \dots f_{p+1}(Y_{t_{p+1}}) \mid Y_{t_1}, \dots, Y_{t_p}) \right] \\
&= \mathbb{E}_x^Y [f_1(Y_{t_1}) \dots f_p(Y_{t_p}) \mathbb{E}_x^Y (f_{p+1}(Y_{t_{p+1}}) \mid Y_{t_p})] \\
&= \mathbb{E}_x^Y [f_1(Y_{t_1}) \dots f_p(Y_{t_p}) Q_{t_{p+1}-t_p}^Y f_{p+1}(Y_{t_p})].
\end{aligned}$$

The function $f_p \times Q_{t_{p+1}-t_p}^Y f_{p+1}$ is continuous by assumption so we can apply the induction hypothesis.

Regarding the second statement of the lemma, let us take $f \in C_b(E)$ and $t \geq 0$. We have

$$\begin{aligned}
\mathbb{E}_x^Y \left[f(Y_t) e^{-\int_0^t \alpha(Y_u) du} \right] &= \mathbb{E}_x^Y \left[f(Y_t) \sum_{k \geq 0} \frac{(-1)^k}{k!} \left(\int_0^t \alpha(Y_u) du \right)^k \right] \\
&= \sum_{k \geq 0} \frac{(-1)^k}{k!} \mathbb{E}_x^Y \left[f(Y_t) \left(\int_0^t \alpha(Y_u) du \right)^k \right] \\
&= \sum_{k \geq 0} \frac{(-1)^k}{k!} \mathbb{E}_x^Y \left[f(Y_t) \prod_{j=1}^k \int_0^t \alpha(Y_{u_j}) du_j \right],
\end{aligned}$$

that is:

$$\begin{aligned} \mathbb{E}_x^Y \left[f(Y_t) e^{-\int_0^t \alpha(Y_u) du} \right] \\ = \sum_{k \geq 0} \frac{(-1)^k}{k!} \int_{u_1=0}^t \cdots \int_{u_k=0}^t \mathbb{E}_x^Y [f(Y_t) \alpha(Y_{u_1}) \cdots \alpha(Y_{u_k})] du_1 \cdots du_k \end{aligned}$$

because $f \times \alpha^k$ is bounded. For any $u_1 \geq 0, \dots, u_k \geq 0$, the function

$$x \in E \mapsto \mathbb{E}_x^Y [f(Y_t) \alpha(Y_{u_1}) \cdots \alpha(Y_{u_k})]$$

is continuous by the first part of the proof and this expression is bounded uniformly in x by $\|f\|_\infty \times (\alpha^*)^k \in L^1([0, t]^k)$. Again, by normal convergence, we obtain the expected result. \square

Lemma 24. *Assume that $Q_t^Y C_b(E) \subset C_b(E)$ for any $t \geq 0$ and that $K C_b(E) \subset C_b(E)$. Let $t > 0$. Then for any $k \geq 1$, for any bounded measurable function φ on $E \times \mathbb{R}_+$ such that $\varphi(\cdot, u)$ is continuous for any $u \leq t$, the function $x \mapsto \mathbb{E}_x[\varphi(X_{T_k}, T_k) \mathbf{1}_{T_k \leq t}]$ is continuous.*

Proof of Lemma 24. We shall proceed by induction. For $k = 1$,

$$\begin{aligned} \mathbb{E}_x[\varphi(X_{T_1}, T_1) \mathbf{1}_{T_1 \leq t}] &= \mathbb{E}_x[\mathbf{1}_{T_1 \leq t} \mathbb{E}_x[\varphi(X_{T_1}, T_1) | Y^{(0)}, T_1]] \\ &= \mathbb{E}_x \left[\int_E K(Y_{T_1}^{(0)}, dz) \varphi(z, T_1) \mathbf{1}_{T_1 \leq t} \right] \\ &= \mathbb{E}_x \left[\int_E \mathbb{E}_x[K(Y_{T_1}^{(0)}, dz) \varphi(z, T_1) \mathbf{1}_{T_1 \leq t} | Y^{(0)}] \right] \\ &= \mathbb{E}_x \left[\int_E \int_0^t K(Y_{t_1}^{(0)}, dz) \varphi(z, t_1) \alpha(Y_{t_1}^{(0)}) e^{-\int_0^{t_1} \alpha(Y_u^{(0)}) du} dt_1 \right] \\ &= \int_0^t \mathbb{E}_x^Y [H(Y_{t_1}, t_1) \alpha(Y_{t_1}) e^{-\int_0^{t_1} \alpha(Y_u) du}] dt_1 \end{aligned}$$

where $H(x, u) = \int_E K(x, dz) \varphi(z, u)$. Since $z \mapsto \varphi(z, t_1)$ belongs to $C_b(E)$ for every $t_1 \leq t$, the Feller continuous property of K entails the continuity of $x \mapsto H(x, t_1)$ for every $t_1 \leq t$. Consequently the function

$$x \mapsto \mathbb{E}_x^Y [H(Y_{t_1}, t_1) \alpha(Y_{t_1}) e^{-\int_0^{t_1} \alpha(Y_u) du}]$$

is continuous for every t_1 by Lemma 23. The functions H and α being bounded, the dominated convergence theorem yields the continuity of $x \mapsto \mathbb{E}_x[\varphi(X_{T_1}, T_1) \mathbf{1}_{T_1 \leq t}]$, proving the statement for $k = 1$. Assume now that the property holds for $k \geq 1$. We compute similarly

$$\begin{aligned} \mathbb{E}_x[\varphi(X_{T_{k+1}}, T_{k+1}) \mathbf{1}_{T_{k+1} \leq t}] \\ = \mathbb{E}_x [\mathbb{E}_x \left[\int_E K(Y_{T_{k+1}-T_k}^{(k)}, dz) \varphi(z, T_{k+1}) \mathbf{1}_{T_{k+1} \leq t} | \mathcal{F}_{T_k}, Y^{(k)} \right]] \\ = \mathbb{E}_x [\mathbb{E}_{X_{T_k}}^Y \left[\int_0^{t-T_k} \int_E K(Y_\tau, dz) \varphi(z, \tau + T_k) \alpha(Y_\tau) e^{-\int_0^\tau \alpha(Y_u) du} \mathbf{1}_{T_k \leq t} \right]] \\ = \mathbb{E}_x [\tilde{\varphi}(X_{T_k}, T_k) \mathbf{1}_{T_k \leq t}], \end{aligned}$$

where

$$\begin{aligned}\tilde{\varphi}(x, u) &= \mathbb{E}_x^Y \left[\int_0^{t-u} \int_E K(Y_\tau, dz) \varphi(z, \tau + u) \alpha(Y_\tau) e^{-\int_0^\tau \alpha(Y_u) du} d\tau \right] \\ &= \int_0^{t-u} \mathbb{E}_x^Y [H(Y_\tau, \tau + u) \alpha(Y_\tau) e^{-\int_0^\tau \alpha(Y_u) du}] d\tau.\end{aligned}$$

By Lemma 23, $x \mapsto \mathbb{E}_x^Y [H(Y_\tau, \tau + u) \alpha(Y_\tau) e^{-\int_0^\tau \alpha(Y_u) du}]$ is continuous for each u, τ , so $\tilde{\varphi}(\cdot, u)$ is continuous for every $u \leq t$. We then obtain the result applying the induction hypothesis. \square

We are now in position to prove the first part of Theorem 13 about the Feller continuous property of $(X_t)_{t \geq 0}$. We compute for $t > 0$, $x \in E$ and $f \in C_b(E)$

$$\begin{aligned}Q_t f(x) &= \sum_{k=0}^{\infty} \mathbb{E}_x [f(X_t) \mathbf{1}_{N_t=k}] \\ &= \mathbb{E}_x^Y [f(Y_t) e^{-\int_0^t \alpha(Y_u) du}] + \sum_{k \geq 1} \mathbb{E}_x [f(X_t) \mathbf{1}_{T_k \leq t < T_{k+1}}] \\ &= \psi(x, t) + \sum_{k \geq 1} \mathbb{E}_x [f(X_t) \mathbf{1}_{T_{k+1} - T_k > t - T_k} \mathbf{1}_{T_k \leq t}]\end{aligned}$$

where $\psi(x, t) = \mathbb{E}_x^Y [f(Y_t) e^{-\int_0^t \alpha(Y_u) du}]$. We get from Lemma 23 that $\psi(\cdot, t)$ belongs to $C_b(E)$ for every $t > 0$. Then

$$\begin{aligned}\mathbb{E}_x [f(X_t) \mathbf{1}_{T_{k+1} - T_k > t - T_k} \mathbf{1}_{T_k \leq t}] &= \mathbb{E}_x [\mathbf{1}_{T_k \leq t} f(Y_{t-T_k}^{(k)}) \mathbb{E}_x [\mathbf{1}_{T_{k+1} - T_k > t - T_k} | \mathcal{F}_{T_k}, Y^{(k)}]] \\ &= \mathbb{E}_x [f(Y_{t-T_k}^{(k)}) e^{-\int_0^{t-T_k} \alpha(Y_u^{(k)}) du} \mathbf{1}_{T_k \leq t}] \\ &= \mathbb{E}_x [\mathbb{E}_{X_{T_k}}^Y [f(Y_{t-T_k}) e^{-\int_0^{t-T_k} \alpha(Y_u) du}] \mathbf{1}_{T_k \leq t}] \quad (2.16) \\ &= \mathbb{E}_x [\psi(X_{T_k}, t - T_k) \mathbf{1}_{T_k \leq t}],\end{aligned}$$

so Lemma 24 entails that $x \mapsto \mathbb{E}_x [\psi(X_{T_k}, t - T_k) \mathbf{1}_{T_k \leq t}]$ is continuous for every $k \geq 1$. The domination

$$\begin{aligned}|\mathbb{E}_x [\psi(X_{T_k}, t - T_k) \mathbf{1}_{T_k \leq t}]| &\leq \|f\|_\infty \mathbb{P}_x(T_k \leq t) \\ &\leq \|f\|_\infty \mathbb{P}(N_t^* \geq k)\end{aligned}$$

where $N^* t \sim \mathcal{P}(\alpha^* t)$ (by (2.3)) allows us to conclude that $x \mapsto Q_t f(x)$ is continuous.

Proof of Theorem 13 (part 2)

Our aim is to prove the Feller property of $(X_t)_{t \geq 0}$ assuming that for every $t > 0$, $Q_t^Y C_0(E) \subset C_0(E)$ and that $K C_0(E) \subset C_0(E)$. We follow the same steps as for the proof of Theorem 13 (part 1), by first inspecting the consequences of $Q_t^Y C_0(E) \subset C_0(E)$ in Lemma 25 and second the additional effect of $K C_0(E) \subset C_0(E)$ in Lemma 26.

Lemma 25. *Suppose that for every $t > 0$, $Q_t^Y C_0(E) \subset C_0(E)$. Then*

1. for any $f \in C_0(E)$, $\lim_{t \rightarrow 0} \|Q_t^Y f - f\|_\infty = 0$,

2. for any $t > 0$, $\sup_{s \in [0, t]} Q_s^Y C_0(E) \subset C_0(E)$,

3. for any $f \in C_0(E)$ the function $x \mapsto \mathbb{E}_x^Y [f(Y_t) e^{-\int_0^t \alpha(Y_u) du}]$ is continuous.

Proof of Lemma 25. By continuity of $(Y_t)_{t \geq 0}$, $\lim_{t \rightarrow 0} Q_t^Y f(x) = f(x)$ for every $f \in C_0(E)$ and every $x \in E$. As proved in [RY91], this is equivalent when $Q_t^Y C_0(E) \subset C_0(E)$ to $\lim_{t \rightarrow 0} \|Q_t^Y f - f\|_\infty = 0$, which proves the first statement of the lemma.

Concerning the second property, let $\varepsilon > 0$ and $f \in C_0(E)$. Fix $\eta(f) > 0$ such that for every $s < \eta(f)$, $\|Q_s^Y f - f\|_\infty \leq \varepsilon$ and $s(x) \in [0, t]$ satisfying

$$\sup_{s \in [0, t]} Q_s^Y f(x) = Q_{s(x)}^Y f(x).$$

Then we have

$$Q_{\frac{\lfloor 2^n s(x) \rfloor}{2^n} t}^Y f(x) \leq \max_{k=0, \dots, 2^n} Q_{\frac{kt}{2^n}}^Y f(x) \leq \sup_{s \in [0, t]} Q_s^Y f(x).$$

So

$$\begin{aligned} \left| \sup_{s \in [0, t]} Q_s^Y f(x) - \max_{k=0, \dots, 2^n} Q_{\frac{kt}{2^n}}^Y f(x) \right| &\leq \left| \sup_{s \in [0, t]} Q_s^Y f(x) - Q_{\frac{\lfloor 2^n s(x) \rfloor}{2^n} t}^Y f(x) \right| \\ &= \left| Q_{s(x)}^Y f(x) - Q_{\frac{\lfloor 2^n s(x) \rfloor}{2^n} t}^Y f(x) \right| \\ &= \left| Q_{\frac{\lfloor 2^n s(x) \rfloor}{2^n} t}^Y (Q_{s(x) - \frac{\lfloor 2^n s(x) \rfloor}{2^n} t}^Y f(x) - f(x)) \right| \\ &\leq \|Q_{s(x) - \frac{\lfloor 2^n s(x) \rfloor}{2^n} t}^Y f - f\|_\infty \\ &\leq \varepsilon \end{aligned}$$

whenever $t2^{-n} \leq \eta(f)$. This leads to

$$\lim_{n \rightarrow \infty} \left\| \sup_{s \in [0, t]} Q_s^Y f - \max_{k=0, \dots, 2^n} Q_{kt2^{-n}}^Y f \right\|_\infty = 0.$$

Since $\max_{k=0, \dots, 2^n} Q_{kt2^{-n}}^Y f \in C_0(E)$ for $f \in C_0(E)$ by assumption and $C_0(E)$ is a closed subset of $\mathcal{M}_b(E)$ for $\|\cdot\|_\infty$, we deduce that $\sup_{s \in [0, t]} Q_s^Y f \in C_0(E)$.

We finally prove the third point of the lemma in a similar way as in the proof of Lemma 23. First we show by induction on $p \geq 1$ that for any $t \geq 0$ and $0 \leq u_1 \leq \dots \leq u_p \leq t$ and $f \in C_0(E)$ the function $x \mapsto \mathbb{E}_x^Y [f(Y_t) \alpha(Y_{u_1}) \dots \alpha(Y_{u_p})]$ is in $C_0(E)$. Indeed for $p = 1$

$$\begin{aligned} \mathbb{E}_x [f(Y_t) \alpha(Y_{u_1})] &= \mathbb{E}_x [\alpha(Y_{u_1}) \mathbb{E}_x [f(Y_t) | \mathcal{F}_{u_1}]] \\ &= \mathbb{E}_x [\alpha(Y_{u_1}) Q_{t-u_1}^Y f(Y_{u_1})] \\ &= Q_{u_1}^Y (\alpha \times Q_{t-u_1}^Y f)(x) \end{aligned}$$

and $Q_{u_1}^Y (\alpha \times Q_{t-u_1}^Y f) \in C_0(E)$ by assumption. For the induction step we just write

$$\mathbb{E}_x^Y [f(Y_t) \alpha(Y_{u_1}) \dots \alpha(Y_{u_p}) \alpha(Y_{u_{p+1}})] = \mathbb{E}_x^Y [\alpha(Y_{u_1}) \dots \alpha(Y_{u_p}) (Q_{t-u_{p+1}}^Y f \times \alpha)(Y_{u_{p+1}})]$$

that is in $C_0(E)$ by assumption and the induction hypothesis. We then obtain the continuity of the function

$$x \mapsto \mathbb{E}_x^Y [f(Y_t) e^{-\int_0^t \alpha(Y_u) du}]$$

similarly as in the proof of Lemma 23. \square

Lemma 26. *Assume that for every $t > 0$, $Q_t^Y C_0(E) \subset C_0(E)$ and that $K C_0(E) \subset C_0(E)$. Let $t > 0$. Then for every $k \geq 1$ and all $g \in C_0(E)$, $x \mapsto \mathbb{E}_x[g(X_{T_k}) \mathbf{1}_{T_k \leq t}]$ vanishes at infinity.*

Proof of Lemma 26. Let us prove the result by induction. For $k = 1$,

$$\begin{aligned} |\mathbb{E}_x[g(X_{T_1}) \mathbf{1}_{T_1 \leq t}]| &= \left| \int_0^t \mathbb{E}_x[Kg(Y_s) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du}] ds \right| \\ &\leq \alpha^* \int_0^t \mathbb{E}_x[K|g|(Y_s)] ds \\ &\leq \alpha^* t \sup_{s \in [0, t]} Q_s^Y K|g|(x). \end{aligned}$$

Since $K C_0(E) \subset C_0(E)$, the function $K|g|$ belongs to $C_0(E)$, so we get by Lemma 25 $\sup_{s \in [0, t]} Q_s^Y K|g| \in C_0(E)$. This entails in particular that $x \mapsto \mathbb{E}_x[g(X_{T_1}) \mathbf{1}_{T_1 \leq t}]$ vanishes at infinity. Let now $k \geq 1$ and assume that $x \mapsto \mathbb{E}_x[g(X_{T_k}) \mathbf{1}_{T_k \leq t}]$ vanishes at infinity. We compute similarly

$$\mathbb{E}_x[g(X_{T_{k+1}}) \mathbf{1}_{T_{k+1} \leq t}] = \mathbb{E}_x[\mathbb{E}_{X_{T_k}}^Y [\int_0^{t-T_k} \int_E K(Y_s, dz) g(z) \alpha(Y_s) e^{-\int_0^s \alpha(Y_u) du}] \mathbf{1}_{T_k \leq t}]$$

and

$$\begin{aligned} |\mathbb{E}_x[g(X_{T_{k+1}}) \mathbf{1}_{T_{k+1} \leq t}]| &\leq \alpha^* \mathbb{E}_x[\mathbb{E}_{X_{T_k}} [\int_0^t K|g|(Y_s) ds] \mathbf{1}_{T_k \leq t}] \\ &= \alpha^* \mathbb{E}_x[\int_0^t Q_s^Y K|g|(X_{T_k}) ds \mathbf{1}_{T_k \leq t}] \\ &\leq \alpha^* t \mathbb{E}_x[\sup_{s \in [0, t]} Q_s^Y K|g|(X_{T_k}) \mathbf{1}_{T_k \leq t}]. \end{aligned}$$

Since $\sup_{s \in [0, t]} Q_s^Y K|g| \in C_0(E)$, the result follows from the induction hypothesis. \square

In order to prove Theorem 13 (part 2), first remark that $x \mapsto Q_t f(x)$ is continuous for $f \in C_0(E)$ and any $t \geq 0$. This follows by the same arguments as in the proof of Theorem 13 (part 1) taking $f \in C_0(E)$. Indeed, using the same notation as in the proof of Lemma 24, we obtain that the function $H(\cdot, u)$ belongs to $C_0(E)$ for any $u \leq t$ by the assumption $K C_0(E) \subset C_0(E)$ and Lemma 25 (item 3.). The conclusion of Lemma 24 then follows by the same proof, using Lemma 25 instead of Lemma 23. Similarly, the proof of Theorem 13 (part 1) with the same substitution entails that $Q_t f \in C_b(E)$.

The strong continuity of Q_t follows by Proposition 12 and the first statement of Lemma 25.

It remains to prove that $x \mapsto Q_t f(x)$ vanishes at infinity. By the same decomposition of $Q_t f$ as in the proof of Theorem 13 (part 1), we obtain using in particular (2.16) that for any $j \geq 1$

$$|Q_t f(x)| \leq Q_t^Y |f|(x) + \sum_{k=1}^j \mathbb{E}_x \left[\sup_{s \in [0, t]} Q_s^Y |f|(X_{T_k}) \mathbf{1}_{T_k \leq t} \right] + \|f\|_\infty \mathbb{P}(N_t^* \geq j) \quad (2.17)$$

where $N_t^* \sim \mathcal{P}(\alpha^* t)$. Let $\varepsilon > 0$. First, $Q_t^Y |f| \in C_0(E)$ by assumption, so that $Q_t^Y |f|(x) \leq \varepsilon/3$ for x outside a compact set. Second, since $\lim_{j \rightarrow \infty} \mathbb{P}(N_t^* \geq j) = 0$, there exists $j_0 \geq 1$ such that $\|f\|_\infty \mathbb{P}(N_t^* \geq j) \leq \varepsilon/3$. Third, Lemma 26 entails that for every $k \leq j_0$ the function

$$x \mapsto \mathbb{E}_x \left[\sup_{s \in [0, t]} Q_s^Y |f|(X_{T_k}) \mathbf{1}_{T_k \leq t} \right]$$

vanishes at infinity because $\sup_{s \in [0, t]} Q_s^Y |f| \in C_0(E)$ by Lemma 25. It is therefore bounded by ε/j_0 for x outside a compact set. Combining these three results in (2.17) concludes the proof.

2.4.4 Proof of Theorem 14 about the infinitesimal generator

Let $f \in L_0^Y$, $x \in E$ and $h > 0$. We decompose $\frac{1}{h}(Q_h f(x) - f(x))$ as

$$\begin{aligned} & \frac{1}{h}(Q_h f(x) - f(x)) \\ &= \frac{1}{h} \left(\mathbb{E}_x^Y \left[f(Y_h) e^{-\int_0^h \alpha(Y_u) du} \right] - f(x) + \mathbb{E}_x [f(X_h) \mathbf{1}_{N_h=1}] + \mathbb{E}_x [f(X_h) \mathbf{1}_{N_h \geq 2}] \right) \\ &= \mathbb{E}_x^Y \left[\frac{f(Y_h) - f(x)}{h} \right] + T(x), \end{aligned}$$

where

$$\begin{aligned} T(x) = & -\frac{1}{h} \mathbb{E}_x^Y \left[f(Y_h) \int_0^h \alpha(Y_u) du \right] + \frac{1}{h} \mathbb{E}_x^Y \left[f(Y_h) \left(e^{-\int_0^h \alpha(Y_u) du} - 1 + \int_0^h \alpha(Y_u) du \right) \right] \\ & + \frac{1}{h} \mathbb{E}_x [f(X_h) \mathbf{1}_{N_h=1}] + \frac{1}{h} \mathbb{E}_x [f(X_h) \mathbf{1}_{N_h \geq 2}]. \quad (2.18) \end{aligned}$$

To prove the theorem, we thus need to show that for any $f \in L_0^Y$

$$\sup_{x \in E} |T(x) + \alpha(x)f(x) - \alpha(x)Kf(x)| \xrightarrow{h \searrow 0} 0.$$

Following (2.18), we denote $T(x) = T_1(x) + T_2(x) + T_3(x) + T_4(x)$ and we shall prove that

$$\sup_{x \in E} |T_1(x) + \alpha(x)f(x)| \xrightarrow{h \searrow 0} 0, \quad (2.19)$$

$$\sup_{x \in E} |T_2(x)| \xrightarrow{h \searrow 0} 0, \quad (2.20)$$

$$\sup_{x \in E} |T_3(x) - \alpha(x)Kf(x)| \xrightarrow{h \searrow 0} 0, \quad (2.21)$$

$$\sup_{x \in E} |T_4(x)| \xrightarrow{h \searrow 0} 0. \quad (2.22)$$

For (2.19), we compute for $h > 0$ and $x \in E$,

$$\begin{aligned} T_1(x) + \alpha(x)f(x) &= \alpha(x)f(x) - \frac{1}{h} \mathbb{E}_x^Y \left[f(Y_h) \int_0^h \alpha(Y_u) du \right] \\ &= \frac{1}{h} \int_0^h \mathbb{E}_x^Y [\alpha(x)f(x) - f(Y_h)\alpha(Y_u)] du \\ &= \frac{1}{h} \int_0^h \mathbb{E}_x^Y [\mathbb{E}_x^Y [\alpha(x)f(x) - f(Y_h)\alpha(Y_u) | Y_u]] du \\ &= \frac{1}{h} \int_0^h \mathbb{E}_x^Y [\alpha(x)f(x) - Q_{h-u}^Y f(Y_u)\alpha(Y_u)] du \\ &= \frac{1}{h} \int_0^h \mathbb{E}_x^Y [\alpha(x)f(x) - f(Y_u)\alpha(Y_u)] du + \frac{1}{h} \int_0^h \mathbb{E}_x^Y [f(Y_u)\alpha(Y_u) - Q_{h-u}^Y f(Y_u)\alpha(Y_u)] du \\ &= \frac{1}{h} \int_0^h (f \times \alpha - Q_u^Y(f \times \alpha))(x) du + \frac{1}{h} \int_0^h \mathbb{E}_x^Y [\alpha(Y_u) (f - Q_{h-u}^Y f)(Y_u)] du \\ &= \int_0^1 (f \times \alpha - Q_{hv}^Y(f \times \alpha))(x) dv + \int_0^1 \mathbb{E}_x^Y [\alpha(Y_{hv}) (f - Q_{h(1-v)}^Y f)(Y_{hv})] dv. \end{aligned}$$

So,

$$|T_1(x) + \alpha(x)f(x)| \leq \int_0^1 \|Q_{hv}^Y(f \times \alpha) - f \times \alpha\|_\infty dv + \alpha^* \int_0^1 \|Q_{h(1-v)}^Y f - f\|_\infty dv,$$

that does not depend on $x \in E$ and converges to zero when $h \searrow 0$ by the dominated convergence theorem, the fact that $f \in L_0^Y$ and the assumption $\alpha \times f \in L_0^Y$. This proves (2.19).

Now for $f \in L_0^Y$ and $x \in E$

$$|T_2(x)| \leq \frac{\|f\|_\infty \mathbb{E}_x^Y \left[\left(\int_0^h \alpha(Y_u) du \right)^2 \right]}{2h} \leq \frac{\|f\|_\infty (\alpha^* h)^2}{2h} = \frac{\|f\|_\infty (\alpha^*)^2}{2} h,$$

that does not depend on $x \in E$ and converges to zero when $h \searrow 0$, proving (2.20).

For (2.21), we have for any $f \in L_0^Y$,

$$\begin{aligned} T_3(x) &= \frac{1}{h} \mathbb{E}_x [f(X_h) \mathbf{1}_{\tau_1 \leq h} \mathbf{1}_{\tau_2 > h - \tau_1}] \\ &= \frac{1}{h} \mathbb{E}_x \left[\mathbb{E}_x [f(X_h) \mathbf{1}_{\tau_1 \leq h} \mathbf{1}_{\tau_2 > h - \tau_1} | \mathcal{F}_{\tau_1}, Y^{(1)}] \right] \\ &= \frac{1}{h} \mathbb{E}_x \left[f(Y_{h-\tau_1}^{(1)}) \mathbf{1}_{\tau_1 \leq h} \mathbb{P}_x (\tau_2 > h - \tau_1 | \mathcal{F}_{\tau_1}, Y^{(1)}) \right] \\ &= \frac{1}{h} \mathbb{E}_x \left[f(Y_{h-\tau_1}^{(1)}) \mathbf{1}_{\tau_1 \leq h} e^{-\int_0^{h-\tau_1} \alpha(Y_u^{(1)}) du} \right] \\ &= \frac{1}{h} \mathbb{E}_x \left[\mathbf{1}_{\tau_1 \leq h} \mathbb{E}_x \left[f(Y_{h-\tau_1}^{(1)}) e^{-\int_0^{h-\tau_1} \alpha(Y_u^{(1)}) du} | \mathcal{F}_{\tau_1} \right] \right] \\ &= \frac{1}{h} \mathbb{E}_x \left[\mathbf{1}_{\tau_1 \leq h} \mathbb{E}_{X_{\tau_1}}^Y \left[f(Y_{h-\tau_1}) e^{-\int_0^{h-\tau_1} \alpha(Y_u) du} \right] \right] \\ &= \frac{1}{h} \mathbb{E}_x \left[\mathbf{1}_{\tau_1 \leq h} \mathbb{E}_{X_{\tau_1}}^Y [f(Y_{h-\tau_1})] \right] + \frac{1}{h} \mathbb{E}_x \left[\mathbf{1}_{\tau_1 \leq h} \mathbb{E}_{X_{\tau_1}}^Y \left[f(Y_{h-\tau_1}) \left(e^{-\int_0^{h-\tau_1} \alpha(Y_u) du} - 1 \right) \right] \right]. \end{aligned} \quad (2.23)$$

The second term above converges uniformly to 0 when $h \searrow 0$ because

$$\begin{aligned}
& \left| \frac{1}{h} \mathbb{E}_x \left[\mathbf{1}_{\tau_1 \leq h} \mathbb{E}_{X_{\tau_1}}^Y \left[f(Y_{h-\tau_1}) \left(e^{-\int_0^{h-\tau_1} \alpha(Y_u) du} - 1 \right) \right] \right] \right| \\
& \leq \mathbb{E}_x \left[\frac{\mathbf{1}_{\tau_1 \leq h} \|f\|_\infty}{h} \mathbb{E}_{X_{\tau_1}}^Y \left| \int_0^{h-\tau_1} \alpha(Y_u) du \right| \right] \\
& \leq \frac{h\alpha^* \|f\|_\infty}{h} \mathbb{P}_x(\tau_1 \leq h) \\
& = \frac{h\alpha^* \|f\|_\infty}{h} \mathbb{E}_x^Y \left(1 - e^{-\int_0^h \alpha(Y_u) du} \right) \\
& \leq \alpha^* \|f\|_\infty \mathbb{E}_x^Y \left(\int_0^h \alpha(Y_u) du \right) \\
& \leq (\alpha^*)^2 \|f\|_\infty h.
\end{aligned}$$

Let us now consider the first term in (2.23) and prove that it converges uniformly to $\alpha(x)Kf(x)$.

$$\begin{aligned}
& \frac{1}{h} \mathbb{E}_x \left[\mathbf{1}_{\tau_1 \leq h} \mathbb{E}_{X_{\tau_1}}^Y [f(Y_{h-\tau_1})] \right] \\
& = \frac{1}{h} \mathbb{E}_x \left[\mathbf{1}_{\tau_1 \leq h} \mathbb{E}_x \left[\mathbb{E}_{X_{\tau_1}}^Y [f(Y_{h-\tau_1})] \mid Y^{(0)}, \tau_1 \right] \right] \\
& = \frac{1}{h} \mathbb{E}_x \left[\mathbf{1}_{\tau_1 \leq h} \int_E \mathbb{E}_z^Y [f(Y_{h-\tau_1})] K(Y_{\tau_1}^{(0)}, dz) \right] \\
& = \frac{1}{h} \mathbb{E}_x \left[\mathbf{1}_{\tau_1 \leq h} \int_E Q_{h-\tau_1}^Y f(z) K(Y_{\tau_1}^{(0)}, dz) \right] \\
& = \frac{1}{h} \mathbb{E}_x \left[\mathbb{E}_x \left[\mathbf{1}_{\tau_1 \leq h} \int_E Q_{h-\tau_1}^Y f(z) K(Y_{\tau_1}^{(0)}, dz) \mid Y^{(0)} \right] \right] \\
& = \frac{1}{h} \mathbb{E}_x \left[\int_0^h \int_E Q_{h-s}^Y f(z) K(Y_s^{(0)}, dz) \alpha(Y_s^{(0)}) e^{-\int_0^s \alpha(Y_u^{(0)}) du} ds \right] \\
& = \mathbb{E}_x^Y \left[\int_0^1 \int_E Q_{h(1-v)}^Y f(z) K(Y_{hv}, dz) \alpha(Y_{hv}) e^{-\int_0^{hv} \alpha(Y_u) du} dv \right] \\
& = \mathbb{E}_x^Y \left[\int_0^1 \int_E Q_{h(1-v)}^Y f(z) K(Y_{hv}, dz) \alpha(Y_{hv}) dv \right] \\
& \quad + \mathbb{E}_x^Y \left[\int_0^1 \int_E Q_{h(1-v)}^Y f(z) K(Y_{hv}, dz) \alpha(Y_{hv}) \left(e^{-\int_0^{hv} \alpha(Y_u) du} - 1 \right) dv \right].
\end{aligned}$$

On one hand,

$$\begin{aligned}
& \left| \mathbb{E}_x^Y \left[\int_0^1 \int_E Q_{h(1-v)}^Y f(z) K(Y_{hv}, dz) \alpha(Y_{hv}) \left(e^{-\int_0^{hv} \alpha(Y_u) du} - 1 \right) ds \right] \right| \\
& \leq \alpha^* \|f\|_\infty \mathbb{E}_x^Y \left[\int_0^1 \int_0^{hv} \alpha(Y_u) du dv \right] \\
& \leq (\alpha^*)^2 \|f\|_\infty h,
\end{aligned}$$

which tends uniformly to 0 when $h \searrow 0$. And on the other hand,

$$\begin{aligned}
& \left| \mathbb{E}_x^Y \left[\int_0^1 \int_E Q_{h(1-v)}^Y f(z) K(Y_{hv}, dz) \alpha(Y_{hv}) dv \right] - \alpha(x) Kf(x) \right| \\
& \leq \int_0^1 \left| \mathbb{E}_x^Y \left[\alpha(Y_{hv}) K Q_{h(1-v)}^Y f(Y_{hv}) - \alpha(Y_{hv}) Kf(Y_{hv}) \right] \right| dv \\
& \quad + \int_0^1 \left| \mathbb{E}_x^Y \left[\alpha(Y_{hv}) Kf(Y_{hv}) - \alpha(x) Kf(x) \right] \right| dv \\
& \leq \alpha^* \int_0^1 \|K Q_{h(1-v)}^Y f - Kf\|_\infty dv + \int_0^1 |Q_{hv}^Y(\alpha \times Kf)(x) - (\alpha \times Kf)(x)| dv \\
& \leq \alpha^* \int_0^1 \|Q_{h(1-v)}^Y f - f\|_\infty dv + \int_0^1 \|Q_{hv}^Y(\alpha \times Kf) - (\alpha \times Kf)\|_\infty dv,
\end{aligned}$$

converges to 0 when $h \searrow 0$ by the dominated convergence theorem and the fact that $f \in L_0^Y$ and $\alpha \times Kf \in L_0^Y$. The latter is implied by the fact that by assumption $g := Kf \in L_0^Y$, implying $\alpha \times g \in L_0^Y$. This proves (2.21).

To complete the proof, it remains to remark that (2.22) follows from the following, using (2.3),

$$\begin{aligned}
|T_4(x)| & \leq \frac{\|f\|_\infty}{h} \mathbb{P}_x(N_h \geq 2) \\
& \leq \frac{\|f\|_\infty}{h} (1 - e^{-\alpha^* h} - \alpha^* h e^{-\alpha^* h}) \\
& = \frac{\|f\|_\infty}{h} \left(\frac{(\alpha^* h)^2}{2} + o_{h \searrow 0}(h^2) \right) \\
& = \frac{\|f\|_\infty (\alpha^*)^2}{2} h + o_{h \searrow 0}(h).
\end{aligned}$$

2.4.5 Topological results for systems of interacting particles in \mathbb{R}^d

We detail the topological properties of the state space E for systems of interacting particles in $W \subset \mathbb{R}^d$, introduced in Section 2.2.4. Remember that in this setting $E = \cup_{n=0}^\infty E_n$ where $E_n = \pi_n(W^n)$ with $\pi_n((x_1, \dots, x_n)) = \{x_1, \dots, x_n\}$, and we have equipped the space E with the distance d_1 defined for $x = \{x_1, \dots, x_{n(x)}\}$ and $y = \{y_1, \dots, y_{n(y)}\}$ in E such that $n(x) \leq n(y)$ by

$$d_1(x, y) = \frac{1}{n(y)} \left(\min_{\sigma \in \mathcal{S}_n(y)} \sum_{i=1}^{n(x)} (\|x_i - y_{\sigma(i)}\| \wedge 1) + (n(y) - n(x)) \right),$$

with $d_1(x, \emptyset) = 1$ and where \mathcal{S}_n denotes the set of permutations of $\{1, \dots, n\}$.

We verify in this section that if W is a closed subset of \mathbb{R}^d (possibly $W = \mathbb{R}^d$), then (E, d_1) is a locally compact and complete set, strengthening results already obtained in [SX08]. We also show that $n(\cdot)$ and $\pi_n(\cdot)$ are continuous under this topology, as claimed in Section 2.2.4. We continue with the proof of Proposition 15, which clarifies the meaning of converging sequences in (E, d_1) , and of Proposition 16 that describes the compact sets of E_n and E , along with some useful corollaries. We

finally show that the Hausdorff distance is not appropriate in our setting, not the least because it does not make $n(\cdot)$ continuous.

In the following, we will often use in an equal way the spaces $(\mathbb{R}^{nd}, \|\cdot\|)$ and $(\mathbb{R}^d)^n, \|\cdot\|_n$ where

$$\|x\|_n = \frac{1}{n} \sum_{i=1}^n \|x_i\|.$$

Indeed, introducing the natural bijection $\psi_n : z \in \mathbb{R}^{nd} \mapsto (z_1, \dots, z_n) \in (\mathbb{R}^d)^n$ we observe that for any $z \in \mathbb{R}^{nd}$, $\|z\|/n \leq \|\psi_n(z)\|_n \leq \|z\|/\sqrt{n}$ by the Cauchy-Schwarz inequality. The norms being equivalent, we henceforth abusively confuse z and $\psi_n(z)$. Similarly, any function from \mathbb{R}^{nd} to \mathbb{R}^d can be seen as a function from $(\mathbb{R}^d)^n$ to \mathbb{R}^d and we will confuse the two points of view.

We start in the following lemmas with the continuity of $n(\cdot)$ and $\pi_n(\cdot)$. We will use the following straightforward property, for all $x, y \in E$,

$$d_1(x, y) \geq \frac{|n(y) - n(x)|}{n(x) \vee n(y)}. \quad (2.24)$$

Lemma 27. *The function $n(\cdot) : (E, d_1) \rightarrow (\mathbb{N}, |\cdot|)$ is continuous.*

Proof of Lemma 27. Take $x \in E$ and a sequence $(x^{(p)})_{p \geq 0}$ such that $d_1(x^{(p)}, x) \rightarrow 0$ as $p \rightarrow \infty$. Assume that the sequence $(n(x^{(p)}))_{p \geq 0}$ is not bounded. We then may define a subsequence $(n(x^{(p')}))_{p' \geq 0}$ such that $n(x^{(p')}) \rightarrow \infty$, and by (2.24) we obtain

$$d_1(x, x^{(p')}) \geq \frac{|n(x) - n(x^{(p')})|}{n(x) \vee n(x^{(p')})} \xrightarrow{p' \rightarrow \infty} 1,$$

which is a contradiction. The sequence $(n(x^{(p)}))_{p \geq 0}$ is therefore bounded by some $M > 0$, which gives again by (2.24)

$$|n(x^{(p)}) - n(x)| \leq (M \vee n(x)) d_1(x^{(p)}, x) \xrightarrow{p \rightarrow \infty} 0,$$

that is

$$n(x^{(p)}) \xrightarrow{p \rightarrow \infty} n(x).$$

□

Lemma 28. *The projection $\pi_n : (W^n, \|\cdot\|_n) \rightarrow (E_n, d_1)$ is continuous.*

Proof of Lemma 28. Let $x, y \in W^n$. Then

$$\begin{aligned} d_1(\pi_n(x), \pi_n(y)) &= \frac{1}{n} \left(\min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n (\|x_i - y_{\sigma(i)}\| \wedge 1) \right) \\ &\leq \frac{1}{n} \sum_{i=1}^n (\|x_i - y_i\| \wedge 1) \\ &\leq \|x - y\|_n. \end{aligned}$$

□

From Lemma 28 we deduce that (E, d_1) is a locally compact space.

Corollary 29. *Let W a closed subset of \mathbb{R}^d . Then (E, d_1) is a locally compact space.*

Proof of Corollary 29. First recall that $d_1(x, \emptyset) = 1$ so $\{\emptyset\}$ is a compact neighborhood of \emptyset . Now take $x = \{x_1, \dots, x_n\} \in E_n$ with $n \geq 1$. The space W^n is locally compact so there exists $K \subset W^n$ a compact neighborhood of (x_1, \dots, x_n) . Now set $\tilde{K} = \pi_n(K)$. Then, $x \in \tilde{K}$ and \tilde{K} is a compact set by Lemma 28. We show that there is an open set containing x which is included in \tilde{K} . By definition there exists $\varepsilon \in (0, \frac{1}{2})$ such that $B_{\|\cdot\|_n}((x_1, \dots, x_n), \varepsilon) \cap W^n \subset K$, where $B_{\|\cdot\|_n}((x_1, \dots, x_n), \varepsilon)$ is the open ball centred at (x_1, \dots, x_n) with radius ε for the norm $\|\cdot\|_n$. If $z \in B_{d_1}(x, \varepsilon) \cap E_n$ there exists $\sigma \in \mathcal{S}_n$ such that

$$\frac{1}{n} \sum_{i=1}^n \|x_i - z_{\sigma(i)}\| < \varepsilon,$$

so $z = \pi_n((z_{\sigma(1)}, \dots, z_{\sigma(n)}))$ and $(z_{\sigma(1)}, \dots, z_{\sigma(n)}) \in B_{\|\cdot\|_n}((x_1, \dots, x_n), \varepsilon) \cap W^n$. To sum up,

$$B_{d_1}(x, \varepsilon) \cap E_n \subset \pi_n \left(B_{\|\cdot\|_n}((x_1, \dots, x_n), \varepsilon) \cap W^n \right) \subset \tilde{K},$$

so \tilde{K} is a compact neighborhood of x in E_n and so in E . \square

A further consequence of Lemma 28 is the following result, that will turn to be useful when considering the Feller continuous property of a process on E .

Corollary 30. *If $f \in C_b(E)$ then for any $n \geq 1$, $f \circ \pi_n \in C_b(W^n)$.*

Proof of Corollary 30. For any $n \geq 1$ and $f \in C_b(E)$, the function $f \circ \pi_n$ is well-defined on W^n , continuous as the composition of two continuous functions and bounded by $\|f\|_\infty$. \square

Let us now prove that (E, d_1) is a complete space.

Proposition 31. *Suppose that W is closed. Then (E, d_1) is a complete space and for any $n \geq 1$, (E_n, d_1) is also complete.*

Proof of Proposition 31. Let $(x^{(p)})_{p \geq 0}$ be a Cauchy sequence in (E, d_1) . First, we show that the sequence $(n(x^{(p)}))_{p \geq 0}$ is constant for p large enough. Fix $\varepsilon \in (0, 1)$. There exists $q \geq 0$ such that for any $p \geq q$, $d_1(x^{(p)}, x^{(q)}) < \varepsilon$, so by (2.24)

$$\left| n(x^{(p)}) - n(x^{(q)}) \right| \leq (n(x^{(p)}) \vee n(x^{(q)})) \varepsilon \leq (n(x^{(p)}) + n(x^{(q)})) \varepsilon,$$

implying that $(1 - \varepsilon) n(x^{(p)}) \leq (1 + \varepsilon) n(x^{(q)})$ and $n(x^{(p)}) \leq n(x^{(q)})(1 + \varepsilon)/(1 - \varepsilon)$. This entails that the sequence $(n(x^{(p)}))_{p \geq 0}$ is bounded by some $N_0 > 0$. Now take $\varepsilon \in (0, 1)$ and $p_1 \geq 0$ such that for any $p \geq p_1$, $d_1(x^{(p)}, x^{(p_1)}) < \varepsilon/N_0$. Write $n = n(x^{(p_1)})$ for short. Then by (2.24) one has for any $p \geq p_1$

$$\left| n(x^{(p)}) - n \right| \leq (n(x^{(p)}) \vee n) d_1(x^{(p)}, x^{(p_1)}) \leq N_0 d_1(x^{(p)}, x^{(p_1)}) \leq \varepsilon < 1,$$

which implies that $n(x^{(p)}) = n$ for all $p \geq p_1$.

Second, we may fix $p_2 \geq 0$ such that $d_1(x^{(p)}, x^{(q)}) \leq \varepsilon$ for any $p, q \geq p_2$. Finally let $p_0 = \max(p_1, p_2)$, so that for all $p, q \geq p_0$,

$$d_1(x^{(p)}, x^{(q)}) = \frac{1}{n} \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n \|x_i^{(p)} - x_{\sigma(i)}^{(q)}\| \leq \varepsilon.$$

In particular for $q = p_0$, this leads to $\min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n \|x_i^{(p_0)} - x_{\sigma(i)}^{(p)}\|/n \leq \varepsilon$ for any $p \geq p_0$. The minimum over σ is reached for some $\sigma_{p_0, p} \in \mathcal{S}_n$, so that we may define the sequence $(\hat{x}^{(p)})_{p \geq p_0}$ in W^n by $\hat{x}^{(p)} = (x_{\sigma_{p_0, p}(1)}^{(p)}, \dots, x_{\sigma_{p_0, p}(n)}^{(p)})$ satisfying $\|\hat{x}^{(p)} - x^{(p_0)}\|_n \leq \varepsilon$ for all $p \geq p_0$. Then for $p, q \geq p_0$, $\|\hat{x}^{(p)} - \hat{x}^{(q)}\|_n \leq 2\varepsilon$. This proves that the sequence $(\hat{x}^{(p)})_{p \geq p_0}$ is a Cauchy sequence in the finite dimensional vector space $((\mathbb{R}^d)^n, \|\cdot\|_n)$, implying its convergence to some $\hat{x} \in W^n$ because W is a closed set. Finally for $p \geq p_0$

$$d_1(x^{(p)}, \pi_n(\hat{x})) = d_1(\pi_n(\hat{x}^{(p)}), \pi_n(\hat{x})) \leq \|\hat{x}^{(p)} - \hat{x}\|_n \leq 2\varepsilon,$$

which proves that $(x^{(p)})_{p \geq 0}$ converges to $\pi_n(\hat{x})$ in E , and so (E, d_1) is complete.

Finally for any $n \geq 1$, (E_n, d_1) is also complete as a closed subset of (E, d_1) by continuity of $n(\cdot)$. \square

Proof of Proposition 15

Let $x \in E$ and set $n = n(x)$. By Lemma 27, if $x^{(p)}$ converges to x as $p \rightarrow \infty$, i.e. $d_1(x^{(p)}, x) \rightarrow 0$, then $n(x^{(p)})$ tends to n , which means that there exists $p_0 \geq 1$ such that $n(x^{(p)}) = n$ for all $p \geq p_0$. From the definition of d_1 , for any $p \geq p_0$ there exists a permutation $\sigma_p \in \mathcal{S}_n$ satisfying

$$d_1(x^{(p)}, x) = \frac{1}{n} \sum_{i=1}^n (\|x_i - x_{\sigma_p(i)}^{(p)}\| \wedge 1).$$

Assume that there exists $i \in \{1, \dots, n\}$ such that $\limsup_{p \rightarrow \infty} \|x_i - x_{\sigma_p(i)}^{(p)}\| > 0$. We then may fix $\eta > 0$ and a subsequence $(\varphi(p))_{p \geq p_0}$, both depending on i , such that for every $p \geq p_0$, $\|x_i - x_{\sigma_{\varphi(p)}(i)}^{(\varphi(p))}\| \geq \eta$. This implies $d_1(x^{(\varphi(p))}, x) \geq (\eta \wedge 1)/n$ and $\limsup_{p \rightarrow \infty} d_1(x^{(p)}, x) > 0$ which is a contradiction. Finally, for every $i = 1, \dots, n$, $\limsup_{p \rightarrow \infty} \|x_i - x_{\sigma_p(i)}^{(p)}\| = 0$, proving the result.

Proof of Proposition 16 and corollaries

In order to prove this proposition, we first recall the following definitions and results (see e.g. [Bou66]):

- A finite subset L of a metric space (X, d) is called an ε -net, for $\varepsilon > 0$, if the following property is satisfied:

$$\forall x \in X, \exists l \in L, \text{ s.t. } d(x, l) \leq \varepsilon.$$

- A metric space (X, d) is said to be totally bounded if it contains an ε -net for any $\varepsilon > 0$.

- Let (X, d) a metric space. Then (X, d) is compact if and only if (X, d) is totally bounded and complete.

To prove the first statement of the proposition, let A be a closed subset of (E_n, d_1) . We start by assuming that we may fix $\varepsilon \in (0, 1/n)$ and $w \in W$ such that

$$\forall R > 0, \exists x = \{x_1, \dots, x_n\} \in A, \max_{1 \leq k \leq n} \{\|x_k - w\|\} > R + n\varepsilon, \quad (2.25)$$

and we show that A is not a compact set because it does not contain any ε -net. Take $L = \{l^{(1)}, \dots, l^{(N)}\}$ a finite subset of A and let us define

$$R_0 = \max_{1 \leq i \leq N} \max_{1 \leq k \leq n} \{\|l_k^{(i)} - w\|\}.$$

By (2.25) we may define $x \in A$ and $1 \leq j \leq n$ such that

$$\|x_j - w\| = \max_{1 \leq k \leq n} \{\|x_k - w\|\} > R_0 + n\varepsilon.$$

This leads for all $\sigma \in \mathcal{S}_n$ and $1 \leq i \leq N$ to

$$\|x_j - l_{\sigma(j)}^{(i)}\| \geq \left| \|x_j - w\| - \|l_{\sigma(j)}^{(i)} - w\| \right| = \|x_j - w\| - \|l_{\sigma(j)}^{(i)} - w\| > n\varepsilon$$

and for any $1 \leq i \leq N$

$$\begin{aligned} d_1(x, l^{(i)}) &= \frac{1}{n} \min_{\sigma \in \mathcal{S}_n} \sum_{k=1}^n \left(\|x_k - l_{\sigma(k)}^{(i)}\| \wedge 1 \right) \\ &\geq \frac{1}{n} \min_{\sigma \in \mathcal{S}_n} \left(\|x_j - l_{\sigma(j)}^{(i)}\| \wedge 1 \right) \\ &> \frac{n\varepsilon}{n} = \varepsilon. \end{aligned}$$

Therefore L cannot be an ε -net and A cannot be a compact set.

Let us now prove the converse. Fix $w \in W$ and assume that there exists a positive R such that for all $x \in A$,

$$\max_{1 \leq k \leq n} \{\|x_k - w\|\} \leq R.$$

Under this assumption, A is a subset of

$$C := \left\{ x \in E_n, \frac{1}{n} \sum_{k=1}^n \|x_k - w\| \leq R \right\}.$$

Let us show that C is a compact set. To this end we define $\mathbf{w} = (w, \dots, w) \in W^n$ and write $\bar{B}_{\|\cdot\|_n}(\mathbf{w}, R)$ for the closed ball of radius R and center \mathbf{w} for the norm $\|\cdot\|_n$ on the finite dimensional vector space $(\mathbb{R}^d)^n$. The closed set $\bar{B}_{\|\cdot\|_n}(\mathbf{w}, R) \cap W^n$ is then a compact set of W^n and by continuity of the projection π_n , we get that $\pi_n(\bar{B}_{\|\cdot\|_n}(\mathbf{w}, R) \cap W^n)$ is a compact set of E_n . In order to conclude the proof let us show that $\pi_n(\bar{B}_{\|\cdot\|_n}(\mathbf{w}, R) \cap W^n) = C$. First, if $x = \{x_1, \dots, x_n\} \in C$ then

$\hat{x} = (x_1, \dots, x_n) \in \bar{B}_{\|\cdot\|_n}(\mathbf{w}, R) \cap W^n$ and $\pi_n(\hat{x}) = x$. Second, if $x = \pi_n(\hat{x})$ with $\hat{x} = (x_1, \dots, x_n) \in \bar{B}_{\|\cdot\|_n}(\mathbf{w}, R) \cap W^n$, then

$$\frac{1}{n} \sum_{k=1}^n \|x_k - w\| = \|\hat{x} - \mathbf{w}\|_n \leq R$$

which proves the claim. The set C is then compact and so is A because it is a closed set.

Let us finally prove the second statement of Proposition 16 by contradiction. Let A be a compact subset of E and suppose that $\mathcal{P} = \{p \geq 0, A \cap E_p \neq \emptyset\}$ is infinite. Then we can construct a sequence $(y_p)_{p \in \mathcal{P}}$ with $y_p \in A \cap E_p$. But A is a compact set so there exists a subsequence $(y_{p'})_{p' \in \mathcal{P}}$ which converges to some $y \in E$ when $p' \rightarrow \infty$. But by Lemma 27, $n(y_{p'}) = p' \rightarrow n(y)$ as $p' \rightarrow \infty$ which is absurd, concluding the proof.

We end this section with two corollaries of Proposition 16.

Corollary 32. *If W is a compact set, then (E_n, d_1) is a compact set for any $n \geq 1$.*

Proof of Corollary 32. W is a compact set of \mathbb{R}^d so it is bounded, i.e. we may fix a non-negative R such that $\|w\| \leq R$ for any $w \in W$. Let $w \in W$ and $x \in E_n$. Then

$$\max_{1 \leq k \leq n} \{\|x_k - w\|\} \leq \max_{1 \leq k \leq n} \|x_k\| + \|w\| \leq 2R.$$

E_n is therefore a compact set by the first statement of Proposition 16. \square

Corollary 33. *If $f \in C_0(E)$ then for any $n \geq 1$, $f \circ \pi_n \in C_0(W^n)$.*

Proof of Corollary 33. Take $f \in C_0(E)$ and $\varepsilon > 0$. There exists a compact set $B \subset E$ such that if $x \notin B$ then $|f(x)| < \varepsilon$. In this case $B_n := B \cap E_n$ is a compact set because E_n is closed so by Proposition 16 there exists $w \in W$ and $R \geq 0$ such that for any $x = \{x_1, \dots, x_n\} \in B_n$, $\max_{1 \leq k \leq n} \|x_k - w\| \leq R$. Then for any $z \notin \bar{B}_{\|\cdot\|_n}(w, R/n)$ we get $|f \circ \pi_n(z)| < \varepsilon$. \square

The Hausdorff distance is not appropriate

For systems of particles in \mathbb{R}^d , we have equipped E with the distance d_1 defined in (2.8). A common alternative distance between random sets is the Hausdorff distance defined for $x = \{x_1, \dots, x_{n(x)}\}$ and $y = \{y_1, \dots, y_{n(y)}\}$ in E by

$$d_H(x, y) = \max \left\{ \max_{1 \leq i \leq n(x)} \min_{1 \leq j \leq n(y)} \|x_i - y_j\|, \max_{1 \leq j \leq n(y)} \min_{1 \leq i \leq n(x)} \|x_i - y_j\| \right\}.$$

Yet we show in this section that this distance does not make the function $n(\cdot)$ continuous, which has serious consequences on the structure of $C_b(E)$ with this topology. In particular, we show that a simple uniform death kernel is not even Feller continuous in this setting.

As a preliminary, for the Hausdorff distance to be a proper distance, we must focus on simple point configurations only. We therefore consider for any $n \geq 1$

$$\tilde{W}_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n, \quad i \neq j \implies x_i \neq x_j\},$$

and the state space is

$$\tilde{E} = \bigcup_{n \geq 0} \tilde{E}_n,$$

where $\tilde{E}_n = \tilde{\pi}_n(\tilde{W}_n)$ and $\tilde{\pi}_n$ is the same projection function as in Section 2.2.4 but defined on \tilde{W}_n .

Lemma 34. *The Hausdorff distance d_H is a proper distance function on \tilde{E} .*

Proof of Lemma 34. Symmetry is obvious and triangle inequality is well known for d_H . We only prove the identity of indiscernibles. Let $x = \{x_1, \dots, x_{n(x)}\}$ and $y = \{y_1, \dots, y_{n(y)}\}$ in \tilde{E} satisfying $d_H(x, y) = 0$. This implies

$$\min_{1 \leq j \leq n(y)} \|x_i - y_j\| = 0$$

for any $i \in \{1, \dots, n(x)\}$, leading for any $i \in \{1, \dots, n(x)\}$ to the existence of an index $j \in \{1, \dots, n(y)\}$ such that $x_i = y_j$. Since x and y are simple, we deduce that $n(y) \geq n(x)$. We obtain similarly $n(x) \geq n(y)$ and then $n(x) = n(y)$. We then may define a permutation $\sigma \in \mathcal{S}_n$ such that for all $i \in \{1, \dots, n(x)\}$, $x_i = y_{\sigma(i)}$ which means that $x = y$ in \tilde{E} . \square

We now verify that $n(\cdot)$ is not continuous for this topology.

Lemma 35. *Assume that $\mathring{W} \neq \emptyset$. Then the function $n(\cdot)$ is not continuous on (\tilde{E}, d_H) .*

Proof of Lemma 35. Assume without loss of generality that $0 \in \mathring{W}$. Let $k \geq 1$ and $y \in \mathbb{R}^d$ such that $\|y\| = 1/k$. Take k large enough so that $y \in W$. Then $|n(\{0, y\}) - n(\{0\})| = 1$ and $d_H(\{0, y\}, \{0\}) = 1/k \rightarrow 0$ as $k \rightarrow \infty$, proving the result. \square

This result reveals a singularity caused by the distance d_H . As a consequence, a simple uniform death kernel is not even Feller continuous, as proved in the following lemma.

Lemma 36. *Assume that $\mathring{W} \neq \emptyset$ and consider for $f \in M_b(\tilde{E})$ the kernel*

$$Kf(x) = \frac{1}{n(x)} \sum_{i=1}^{n(x)} f(x \setminus x_i).$$

Then $KC_b(\tilde{E})$ is not included in $C_b(\tilde{E})$, i.e. K is not Feller continuous.

Proof. Consider the function $f(x) = \max_{1 \leq i \leq n(x)} x_{i,1} \wedge 1$ where $x_{i,1}$ is the first coordinate of $x_i \in W$. This function is bounded and satisfies for any $x, y \in \tilde{E}$,

$$|f(x) - f(y)| \leq \left| \max_{1 \leq i \leq n(x)} x_{i,1} - \max_{1 \leq j \leq n(y)} y_{j,1} \right|, \quad (2.26)$$

for any $x, y \in \tilde{E}$. Let us show that the latter bound is lower than $d_H(x, y)$. Let $\mathcal{I}_0 = \operatorname{argmax}_{1 \leq i \leq n(x)} x_{i,1}$ and $\mathcal{J}_0 = \operatorname{argmax}_{1 \leq j \leq n(y)} y_{j,1}$. This follows from the fact that for any $i_0 \in \mathcal{I}_0$ and $j_0 \in \mathcal{J}_0$,

$$d_H(x, y) \geq \max_{1 \leq i \leq n(x)} \min_{1 \leq j \leq n(y)} \|x_i - y_j\| \geq \min_{1 \leq j \leq n(y)} \|x_{i_0} - y_j\| \geq \min_{1 \leq j \leq n(y)} |x_{i_0,1} - y_{j,1}|,$$

that is $d_H(x, y) \geq |x_{i_0} - y_{j_0}|$. So by (2.26) $|f(x) - f(y)| \leq d_H(x, y)$, proving that $f \in C_b(\tilde{E})$.

Assume without loss of generality that $0 \in \overset{\circ}{W}$. Let $a \in W$, $a \neq 0$, and $a_k = (1/k, 0, \dots, 0) \in \mathbb{R}^d$ with k large enough to ensure $a_k \in W$. Consider the sequence $x^{(k)} = \{0, a, a_k\}$ and let $x = \{0, a\}$ so that $d_H(x^{(k)}, x) = 1/k$ tends to 0 as $k \rightarrow \infty$. On the one hand,

$$Kf(x^{(k)}) = \frac{1}{3} [f(\{0, a_k\}) + f(\{a, a_k\}) + f(\{0, a\})] = \frac{(1/k) + (1/k) \vee a_1 + a_1}{3} \xrightarrow[k \rightarrow \infty]{} \frac{2a_1}{3},$$

and on the other hand,

$$Kf(x) = \frac{1}{2} (f(\{0\}) + f(\{a\})) = \frac{a_1}{2}$$

whereby $Kf \notin C_b(\tilde{E})$. □

2.4.6 Proof of Propositions 17 and 18

Proof of Proposition 17

First we show that if $(Z_t^{|n})_{t \geq 0}$ is a Feller continuous process on W^n for every $n \geq 1$ then $(Y_t)_{t \geq 0}$ is a Feller continuous process on E . Indeed, let $x \in E$ and a sequence $(x^{(p)})_{p \geq 0}$ converging to x . By Proposition 15 we may fix $p_0 \geq 1$ such that $n(x^{(p)}) = n(x) := n$ for any $p \geq p_0$ and a sequence of permutations σ_p of $\{1, \dots, n\}$ such that for any $1 \leq i \leq n$, $x_{\sigma_p(i)}^{(p)} \rightarrow x_i$ as $p \rightarrow \infty$. We then obtain for any $f \in C_b(E)$ and $p \geq p_0$, using the permutation equivariance property of $(Z_t^{|n})_{t \geq 0}$ (that allows us to arbitrarily choose the ordering of its initial value), the continuity of its transition kernel, and Corollary 30:

$$\begin{aligned} \mathbb{E}(f(Y_t) | Y_0 = x^{(p)}) &= \mathbb{E}(f(Y_t^{|n}) | Y_0 = x^{(p)}) \\ &= \mathbb{E}(f \circ \pi_n(Z_t^{|n}) | Z_0^{|n} = (x_{\sigma_p(1)}^{(p)}, \dots, x_{\sigma_p(n)}^{(p)})) \\ &\xrightarrow[p \rightarrow \infty]{} \mathbb{E}(f \circ \pi_n(Z_t^{|n}) | Z_0^{|n} = (x_1, \dots, x_n)) \\ &= \mathbb{E}(f(Y_t) | Y_0 = x). \end{aligned}$$

Second, let us prove that if $(Z_t^{|n})_{t \geq 0}$ is a Feller process on W^n for every $n \geq 1$ then $(Y_t)_{t \geq 0}$ is a Feller process on E . Let $f \in C_0(E)$. We start by the strong continuity. Take $\varepsilon > 0$. By the second statement of Proposition 16 there exists $n_0 \geq 0$ such that $n(x) > n_0 \Rightarrow |f(x)| < \frac{\varepsilon}{4}$. So for any $x \in E$,

$$\begin{aligned} &|Q_t^Y f(x) - f(x)| \\ &\leq |Q_t^Y f(x) - f(x)| \mathbf{1}_{n(x) \leq n_0} + \mathbb{E}_x[|f(Y_t)|] \mathbf{1}_{n(x) > n_0} + f(x) \mathbf{1}_{n(x) > n_0} \\ &\leq \sum_{n=0}^{n_0} |Q_t^{Y^{|n}} f(x) - f(x)| \mathbf{1}_{x \in E_n} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &\leq \sum_{n=1}^{n_0} |\mathbb{E}(f(\pi_n(Z_t^{|n})) | Z_0^{|n} = (x_1, \dots, x_n)) - f(\pi_n((x_1, \dots, x_n)))| \mathbf{1}_{x \in E_n} + \frac{\varepsilon}{2} \\ &\leq \sum_{n=1}^{n_0} \|Q_t^{Z^{|n}}(f \circ \pi_n) - f \circ \pi_n\|_\infty + \frac{\varepsilon}{2}. \end{aligned}$$

By Corollary 33, for any $n = 1, \dots, n_0$, there exists $t_n > 0$ such that

$$t \in (0, t_n) \implies \|Q_t^{Z|n}(f \circ \pi_n) - f \circ \pi_n\|_\infty < \frac{\varepsilon}{2n_0}.$$

So for any $t \in (0, t(\varepsilon))$ where $t(\varepsilon) = \min_{1 \leq n \leq n_0} t_n$, we get $\|Q_t^Y f - f\|_\infty < \varepsilon$, which proves the strong continuity of Q_t^Y at 0.

It remains to show that $Q_t^Y C_0(E) \subset C_0(E)$. Continuity follows from above. Take now $f \in C_0(E)$ and fix $\varepsilon > 0$ and $B \subset E$ a compact set such that $x \notin B \implies |f(x)| < \frac{\varepsilon}{2}$. By Proposition 16 there exists $n_0 \geq 0$ such that $x \in B \implies n(x) \leq n_0$. Also by Corollary 33 we can fix for any $n = 1, \dots, n_0$ a compact set A_n of W^n such that $z \notin A_n \implies |Q_t^{Z|n}(f \circ \pi_n)(z)| < \varepsilon/(2n_0)$. Then, $A = \bigcup_{n=1}^{n_0} \pi_n(A_n)$ is a compact set of E and for any $x \notin \{\emptyset\} \cup A \cup B$

$$\|Q_t^Y f(x)\| \leq \sum_{n=1}^{n_0} \|Q_t^{Z|n}(f \circ \pi_n)((x_1, \dots, x_n))\| + \frac{\varepsilon}{2} \leq \varepsilon.$$

Proof of Proposition 18

For $f \in C_c(E)$ there exists $n_0 \geq 0$ such that $n(x) > n_0 \implies |f(x)| = 0$ by Proposition 16. So,

$$\begin{aligned} & \left| \frac{Q_t^Y f(x) - f(x)}{t} - \sum_{n=1}^{n_0} \mathcal{A}^{Z|n}(f \circ \pi_n)(x) \mathbf{1}_{E_n}(x) \right| \\ & \leq \sum_{n=1}^{n_0} \left| \frac{Q_t^Y f(x) - f(x)}{t} - \mathcal{A}^{Z|n}(f \circ \pi_n)(x) \right| \mathbf{1}_{E_n}(x) \\ & = \sum_{n=1}^{n_0} \left| \frac{Q_t^{Y,n} f(x) - f(x)}{t} - \mathcal{A}^{Z|n}(f \circ \pi_n)(x) \right| \mathbf{1}_{E_n}(x) \\ & = \sum_{n=1}^{n_0} \left| \frac{\mathbb{E} \left(f(\pi_n(Z_t^{|n})) \mid Z_0^{|n} = (x_1, \dots, x_n) \right) - f(\pi_n((x_1, \dots, x_n)))}{t} - \mathcal{A}^{Z|n}(f \circ \pi_n)(x) \right| \mathbf{1}_{E_n}(x) \\ & \leq \sum_{n=1}^{n_0} \left\| \frac{Q_t^{Z,n}(f \circ \pi_n) - f \circ \pi_n}{t} - \mathcal{A}^{Z|n}(f \circ \pi_n) \right\|_\infty \xrightarrow[t \searrow 0]{} 0, \end{aligned}$$

that proves the result.

Chapter 3

Ergodic properties of birth-death-move processes with applications to pairwise interaction processes on \mathbb{R}^d

This chapter constitutes the second part of the submitted article [LLGM22] augmented by some detailed proofs. The notations used in this chapter are the same as in Chapter 2. The main results are stated in Sections 3.1 and 3.2. Moreover Section 3.3 contains the proofs of these results and technical lemmas.

3.1 Ergodic properties of birth-death-move processes

In this chapter we focus on birth-death-move processes as described in Section 2.2.2. Accordingly, the state space is $E = \bigcup_{n=0}^{\infty} E_n$ where $(E_n)_{n \geq 1}$ is a sequence of disjoint locally compact Polish spaces with $E_0 = \{\emptyset\}$ in such a way that E is a locally Polish space, and the jump-kernel K reads as in (2.4). Remember that in this setting the jump intensity function is $\alpha = \beta + \delta$ where β and δ are the birth and death intensity functions. We introduce the following notation

$$\beta_n = \sup_{x \in E_n} \beta(x), \quad \delta_n = \inf_{x \in E_n} \delta(x) \quad \text{and} \quad \alpha_n = \beta_n + \delta_n. \quad (3.1)$$

We present in Section 3.1.1 conditions on the sequences (β_n) and (δ_n) ensuring the convergence of the birth-death-move process towards a unique invariant probability measure. Section 3.1.2 provides a geometric rate of convergence and we characterize some invariant measures in Section 3.1.3.

3.1.1 Convergence to an invariant measure

Let $(X_t)_{t \geq 0}$ be a birth-death-move process as in Section 2.2.2. Inspired by [Pre75], the first step to establish the ergodic properties of $(X_t)_{t \geq 0}$ is to construct a coupling

between $(X_t)_{t \geq 0}$ and a simple birth-death process $(\eta_t)_{t \geq 0}$ on \mathbb{N} with birth rates β_n and death rates δ_n . This coupling is detailed in Section 3.3.1. In a nutshell, we built a jump-move process $\check{C}_t = (X'_t, \eta'_t)$ so that the properties stated in the following theorem are satisfied, in particular we have the equality in distribution $(X'_t)_{t \geq 0} = (X_t)_{t \geq 0}$ and $(\eta'_t)_{t \geq 0} = (\eta_t)_{t \geq 0}$. We denote by \check{Q}_t the transition kernel of \check{C}_t and by $\mathcal{P}(\mathbb{N})$ the power set of \mathbb{N} .

Theorem 37. *Suppose that $(Y_t)_{t \geq 0}$ is a Feller process and that $KC_0(E) \subset C_0(E)$. Then for any $t \geq 0$, $(x, n) \in E \times \mathbb{N}$, $A \in \mathcal{E}$ and $S \in \mathcal{P}(\mathbb{N})$ we have:*

1. $\check{Q}_t((x, n); E \times S) = q_t(n, S)$,
2. $\check{Q}_t((x, n); A \times \mathbb{N}) = Q_t(x, A)$,
3. if $x \in E$ with $n(x) \leq n$, then $\check{Q}_t((x, n); \Gamma) = 0$ where

$$\Gamma = \{(y, m) \in E \times \mathbb{N}; n(y) > m\}.$$

This theorem is proved in Section 3.3.1. When the move $(Y_t)_{t \geq 0}$ is constant, which is the setting in [Pre75], then the proof is easy under (2.1) by use of the derivative form of the Kolmogorov backward equation. In the general case of a birth-death-move process, this strategy does not work anymore and the proof becomes more challenging. We managed to do it by exploiting the generator of $(X_t)_{t \geq 0}$, see Theorem 14, which explains the Feller conditions in Theorem 37.

We deduce from the third point of Theorem 37 that for any $x \in E_m$ with $m \leq n$, then

$$\mathbb{P}_{(x,n)} \left((\check{C}_s)_{s \geq 0} \subset \Gamma^c \right) = 1.$$

This means that the simple process $(\eta_t)_{t \geq 0}$ converges more slowly to the state 0 than $(X_t)_{t \geq 0}$ converges to the state \emptyset . We can thus build upon the renewal theory ([Fel71]) to prove that \emptyset is an ergodic state for $(X_t)_{t \geq 0}$ whenever 0 is an ergodic state for $(\eta_t)_{t \geq 0}$. Conditions ensuring the latter are either (3.2) or (3.3) below ([KM57]), so that we obtain the following, proved in Section 3.3.3.

Theorem 38. *Suppose that $(Y_t)_{t \geq 0}$ is a Feller process and that $KC_0(E) \subset C_0(E)$. Suppose that $\delta_n > 0$ for all $n \geq 1$ and one of the following conditions holds:*

$$(i) \text{ there exists } n_0 \geq 1 \text{ such that } \beta_n = 0 \text{ for any } n \geq n_0, \quad (3.2)$$

$$(ii) \beta_n > 0 \text{ for all } n \geq 1, \quad \sum_{n=2}^{\infty} \frac{\beta_1 \dots \beta_{n-1}}{\delta_1 \dots \delta_n} < \infty \text{ and } \sum_{n=1}^{\infty} \frac{\delta_1 \dots \delta_n}{\beta_1 \dots \beta_n} = \infty. \quad (3.3)$$

Then, $\mu(A) := \lim_{t \rightarrow \infty} Q_t(x, A)$ exists for all $x \in E$ and $A \in \mathcal{E}$, and is independent of x . Moreover μ is a probability measure on (E, \mathcal{E}) and it is the unique invariant probability measure for the process, i.e. such that $\mu(A) = \int_E Q_t(x, A) \mu(dx)$ for any $A \in \mathcal{E}$ and $t \geq 0$.

3.1.2 Rate of convergence

Based on the coupling given by Theorem 37, and under the assumptions of Theorem 38, the rate of convergence of Q_t towards the invariant measure μ follows from the rate of convergence of the simple birth-death process η towards its invariant distribution. This is proven and exploited in [Møl89] in the case of spatial birth-death processes (without move), based upon the coupling constructed in [Pre75]. Since Theorem 37 extends this coupling, we deduce in the following theorem the same rates of convergence as in [Møl89]. The proof follows the same lines as the one in [Møl89], but we provide some details and complements in Section 3.3.4.

Theorem 39. *Suppose that $(Y_t)_{t \geq 0}$ is a Feller process and that $KC_0(E) \subset C_0(E)$. Let γ_1 and γ_2 be two probability measures on (E, \mathcal{E}) , such that one of the two following conditions holds:*

$$(i) \text{ (3.2) holds true and for } k = 1, 2, \gamma_k \left(\bigcup_{n=0}^{n_0} E_n \right) = 1, \quad (3.4)$$

$$(ii) \text{ (3.3) holds true and for } k = 1, 2, \sum_{n=2}^{\infty} \gamma_k(E_n) \sqrt{\frac{\delta_1 \dots \delta_n}{\beta_1 \dots \beta_{n-1}}} < \infty. \quad (3.5)$$

Then there exist real constants $c > 0$ and $0 < r < 1$ such that for any $t \geq 0$

$$\sup_{A \in \mathcal{E}} \left| \int_E Q_t(x, A) \gamma_1(dx) - \int_E Q_t(y, A) \gamma_2(dy) \right| \leq cr^t.$$

Moreover when the condition (3.4) holds, the constants c and r can be chosen independently of γ_1 and γ_2 .

This result is declined into several particular cases in [Møl89, Corollary 3.1], that are also valid in our setting. In particular, when γ_1 corresponds to the invariant measure μ obtained in Theorem 38, and γ_2 is a point measure, the assumptions (3.4) and (3.5) simplify and we get the following corollary.

Corollary 40. *Suppose that $(Y_t)_{t \geq 0}$ is a Feller process and that $KC_0(E) \subset C_0(E)$. Assume either (3.2) or (3.3) along with*

$$\sum_{n=2}^{\infty} \sqrt{\frac{\beta_1 \dots \beta_{n-1}}{\delta_1 \dots \delta_n}} < \infty \quad \text{and} \quad \exists N \geq 0, \text{ s.t. } \forall n \geq N, \beta_n \leq \delta_{n+1}. \quad (3.6)$$

Denote by μ the invariant measure given by Theorem 38. Then for any $y \in E$, there exist $c(y) > 0$ and $0 < r < 1$ (independent of y) such that

$$\sup_{A \in \mathcal{E}} |\mu(A) - Q_t(y, A)| \leq c(y) r^t. \quad (3.7)$$

Moreover the function $c(\cdot)$ satisfies

$$\int_E c(y) d\mu(y) < +\infty. \quad (3.8)$$

The inequality (3.8) will be useful in Chapter 4 when estimating the transition kernel of a birth-death-move process.

3.1.3 Characterisation of some invariant measures

In general the invariant measure μ of a birth-death-move process $(X_t)_{t \geq 0}$, provided it exists, can be a very complicated distribution that mixes the repartition in E due to births and deaths of points, including the probability to be in E_n for each n , with the average distribution on each E_n due to the move process Y . In particular, note that according to Theorem 38, Y does not need to be a stationary process for $(X_t)_{t \geq 0}$ to converge to an invariant measure. Heuristically, this is because the move process is always eventually “killed” by a return to \emptyset of $(X_t)_{t \geq 0}$ under the hypotheses of Theorem 38.

The situation becomes more intelligible when Y admits an invariant measure that is compatible with the jumps of $(X_t)_{t \geq 0}$, as formalised in the next proposition.

Proposition 41. *Suppose that $(Y_t)_{t \geq 0}$ is a Feller process and that $KC_0(E) \subset C_0(E)$. Assume moreover that there exists a finite measure μ on E such that for any $f \in \mathcal{D}_{\mathcal{A}^Y}$,*

$$\int_{E_n} \mathcal{A}^Y f(x) \, d\mu|_{E_n}(x) = 0, \quad \forall n \geq 0, \quad (3.9)$$

$$\text{and} \quad \int_E (\alpha(x)Kf(x) - \alpha(x)f(x)) \, d\mu(x) = 0. \quad (3.10)$$

Then for any $f \in \mathcal{D}_{\mathcal{A}^Y}$, $\int_E \mathcal{A}f(x) \, d\mu(x) = 0$.

Proof of Proposition 41. By Theorem 14, for any $f \in \mathcal{D}_{\mathcal{A}^Y}$,

$$\begin{aligned} \int_E \mathcal{A}f(x) \, d\mu(x) &= \int_E (\alpha(x)Kf(x) - \alpha(x)f(x)) \, d\mu(x) + \int_E \mathcal{A}^Y f(x) \, d\mu(x) \\ &= \sum_{n \geq 0} \int_{E_n} \mathcal{A}^Y f(x) \, d\mu|_{E_n}(x) = 0. \end{aligned}$$

□

This proposition will be useful to characterize the invariant measure of the birth-death-move processes considered in Section 3.2. Indeed, suppose that the hypotheses of Theorem 38 are satisfied. Then $(X_t)_{t \geq 0}$ converges to a unique invariant measure ν . Suppose moreover that the pure jump Markov process with intensity α and transition kernel K admits some invariant measure μ , and that for any $n \geq 0$, $\mu|_{E_n}$ is also invariant for the move process $Y|_n$ on E_n . Then by Proposition 41 and the unicity of ν , we have that $\nu = \mu$.

3.2 Application to pairwise interaction processes on \mathbb{R}^d

We present in this section examples of birth-death-move processes, defined through a pairwise potential function V on a compact set $W \subset \mathbb{R}^d$, that converge to the Gibbs probability measure associated to V . The specificity is that we make compatible the jumps dynamics with the inter-jumps diffusion, so that Proposition 41 applies and allows us to characterize this Gibbs measure as the invariant measure.

When there is no inter-jumps motion, this type of convergence is proved in [Pre75] and is a prerequisite for perfect simulation of spatial Gibbs point process models (see [MW04]). However the weakness of this approach is that for rigid interactions (as for instance induced by a Lennard-Jones or a Riesz potential, see the examples below), the dynamics based on spatial births and deaths may mix poorly, so that the convergence to the associated Gibbs measure may be very slow. Adding inter-jumps motions that do not affect the stationary measure, as carried out in this section, may alleviate this issue.

Let $W := I_1 \times \cdots \times I_d$ where, for $i \in \{1, \dots, d\}$, I_i is a compact interval of \mathbb{R} . Define $\tilde{W}_n = \{(x_1, \dots, x_n) \in (\mathring{W})^n, i \neq j \Rightarrow x_i \neq x_j\}$. As in Section 2.2.4, we let $E_0 = \{\emptyset\}$, $E_n = \pi_n(\tilde{W}_n)$ for $n \geq 1$, and $E = \bigcup_{n=0}^{\infty} E_n$.

We consider a pairwise potential function $V : E \rightarrow \mathbb{R} \cup \{\infty\}$, in the sense that there exist $a > 0$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ satisfying $\phi(\xi) = \phi(-\xi)$ for all $\xi \in \mathbb{R}^d$ such that for any $x = \{x_1, \dots, x_n\} \in E_n$,

$$V(x) = a n(x) + \sum_{1 \leq i \neq j \leq n} \phi(x_i - x_j),$$

when $n \geq 2$, while $V(\{\emptyset\}) = 0$ and $V(\{\xi\}) = a$ for $\xi \in W$. Let $\phi_0 : (0, \infty) \rightarrow \mathbb{R}_+$ be a decreasing function with $\phi_0(r) \rightarrow \infty$ as $r \rightarrow 0$. We will assume the following on ϕ .

(A) The potential is locally stable, i.e there exists $\psi : W \rightarrow \mathbb{R}_+$ integrable such that:

$$\forall n \geq 1, \forall x \in E_n, \forall \xi \in W, \exp\left(-\sum_{i=1}^n \phi(x_i - \xi)\right) \leq \psi(\xi).$$

(B) ϕ is bounded, or otherwise there exists $r_1 > 0$ such that $\phi(\xi) \geq \phi_0(\|\xi\|)$ for all $\|\xi\| < r_1$.

(C) ϕ is weakly differentiable on $\mathbb{R}^d \setminus \{0\}$, $\exp(-\phi)$ is weakly differentiable on \mathbb{R}^d and for any $p > d$ we have $e^{-\phi} \nabla \phi \in L_{loc}^p$.

Let us present some examples of pairwise potentials ϕ that fulfil these assumptions. These are standard instances used in spatial statistics and statistical mechanics.

Example. (*repulsive Lennard-Jones potential*): For $\xi \in \mathbb{R}^d$, $\phi(\xi) = c\|\xi\|^{-12}$ with $c > 0$. This potential verifies condition (A) with $\psi \equiv 1$ and condition (B). It is moreover differentiable on $\mathbb{R}^d \setminus \{0\}$ and for any $\xi \in \mathbb{R}^d \setminus \{0\}$, $\nabla \phi(\xi) = -12c\xi/\|\xi\|^{14}$. We deduce that the function $e^{-\phi} \nabla \phi$ can be extended to a continuous function on \mathbb{R}^d by setting $(e^{-\phi} \nabla \phi)(0) = 0$. As a consequence the condition (C) is satisfied.

Example. (*Riesz potential*): It is defined on $\mathbb{R}^d \setminus \{0\}$ by $\phi(\xi) = c\|\xi\|^{\alpha-d}$ for $c > 0$ and $0 < \alpha < d$. As in the previous example, we obtain that ϕ satisfies the conditions (A), (B) and (C).

Example. (*soft-core potential*): $\phi(\xi) = -\ln(1 - \exp(-c\|\xi\|^2))$ for $c > 0$. Again this potential verifies condition (A) with $\psi \equiv 1$ and condition (B). Moreover, we compute for $\xi \in \mathbb{R}^d \setminus \{0\}$, $\nabla \phi(\xi) = -(2ce^{-c\|\xi\|^2})(1 - e^{-c\|\xi\|^2})^{-1} \xi$. As $\|\nabla \phi(\xi)\| \sim 1/(c\|\xi\|)$ as

$\|\xi\| \rightarrow 0$, we also obtain that the function $e^{-\phi} \nabla \phi$ can be extended to a continuous function on \mathbb{R}^d and condition (C) follows.

Example. (*regularised Strauss potential*): For $R > 0$ and $\gamma \geq 0$, the standard Strauss potential corresponds to $\phi(\xi) = \gamma \mathbf{1}_{\|\xi\| < R}$. We consider a regularised version by introducing a parameter $0 < \epsilon < R$, so that $\phi(\xi) = \gamma$ if $\|\xi\| \leq R - \epsilon$, $\phi(\xi) = 0$ if $\|\xi\| \geq R + \epsilon$, and ϕ is interpolated between $R - \epsilon$ and $R + \epsilon$ in such a way that it is differentiable. With this regularised version, ϕ satisfies the conditions (A) with $\psi \equiv 1$, (B) and (C).

Based on a potential V as above, we construct a birth-death-move process $(X_t)_{t \geq 0}$ with the following characteristics. The birth transition kernel is given as in Example 3 by

$$K_\beta(x, \Lambda \cup x) = \frac{1}{z(x)} \int_\Lambda e^{-(V(x \cup \xi) - V(x))} d\xi,$$

for any $x \in E$ and $\Lambda \subset W$, where $z(x) = \int_W \exp(-(V(x \cup \xi) - V(x))) d\xi$. Note that by the local stability assumption (A), $z(x) < \infty$ for any $x \in E$. The death transition kernel is just the uniform kernel, a particular case of Example 1, i.e.

$$K_\delta f(x) = \frac{1}{n(x)} \sum_{i=1}^{n(x)} f(x \setminus x_i)$$

for any $f \in M_b(E)$ and $x = \{x_1, \dots, x_{n(x)}\} \in E$. For the birth and death intensity functions, we take

$$\beta(x) = \frac{z(x)}{n(x) \vee 1} \quad \text{and} \quad \delta(x) = \mathbf{1}_{n(x) \geq 1},$$

for any $x \in E$. Finally, for the move process, we start with the following Langevin diffusion on \tilde{W}_n :

$$dZ_{t,i}^n = - \sum_{j \neq i} \nabla \phi(Z_{t,i}^n - Z_{t,j}^n) dt + \sqrt{2} dB_{t,i}, \quad 1 \leq i \leq n,$$

with reflecting boundary conditions ([FG07]), and we deduce the move process Y on E as in Example 4.

Proposition 42. *The birth-death-move process $(X_t)_{t \geq 0}$ defined above is a Feller process and converges towards the invariant Gibbs probability measure on W with potential V , i.e. the measure having a density proportional to $\exp(-V(x))$ with respect to the unit rate Poisson point process on W .*

Proof of Proposition 42. First note that by the local stability assumption (A),

$$\beta(x) \leq \frac{e^{-c} \|\psi\|_1}{n(x) \vee 1}$$

where $\|\psi\|_1 = \int_W \psi(\xi) d\xi$, so that $\alpha(x) = \beta(x) + \delta(x)$ is uniformly bounded as required by (2.1).

Under the assumptions (B) and (C), it is proved in [FG07] that the process $(Z_t^n)_{t \geq 0}$ is a well defined Markov process on \tilde{W}_n and it is a Feller process. By

Proposition 17, Y is then a Feller process on E . On the other hand the jump transition kernel K given by (2.4) satisfies $KC_0(E) \subset C_0(E)$, as verified in Examples 1 and 3 in Section 2.3.3 since W is compact. We thus obtain by Theorem 13 that $(X_t)_{t \geq 0}$ is a Feller process. Moreover, by (A) we have that for all $n \geq 1$, $\beta_n \leq e^{-c} \|\psi\|_1/n$, so that (3.3) is verified. All assumptions of Theorem 38 are satisfied, which implies that $(X_t)_{t \geq 0}$ converges to a unique invariant probability measure as $t \rightarrow \infty$.

It remains to characterize this invariant measure. The choice of β , δ , K_β and K_δ satisfy the conditions of [Pre75, Theorem 8.1], see also [MW04], which implies that the invariant measure μ for the birth-death process (without move) having the previous characteristics is the one claimed in the proposition. We deduce that (3.10) holds true. On the other hand, [FG07] proved under (B) and (C) that $(Z_t^n)_{t \geq 0}$ converges towards the invariant measure on \tilde{W}_n with a density (with respect to the Lebesgue measure) proportional to $\exp(-\sum_{1 \leq i \neq j \leq n} \phi(x_i - x_j))$. After projection on E_n , this means that (3.9) follows, with the same measure μ as before. Proposition 41 then applies and μ is the invariant measure of $(X_t)_{t \geq 0}$. \square

3.3 Proofs and technical results

This section focuses on the proofs of the results previously stated in this chapter. It also contains all the construction of the coupling of birth-death-move processes with the associated outcomes and proofs.

3.3.1 Coupling of birth-death-move processes

Construction of the coupling

We start from a birth-death-move process $(X_t)_{t \geq 0}$ as defined in Section 2.2.2. We consider a simple birth-death process $(\eta_t)_{t \geq 0}$ on \mathbb{N} with birth rate β_n and death rate δ_n given by (3.1). Note that $(\eta_t)_{t \geq 0}$ can be viewed as a birth-death-move process on \mathbb{N} having a constant move process $y_t = y_0$, for all $t \geq 0$. We denote by $(t_j)_{j \geq 1}$ the jump times of $(\eta_t)_{t \geq 0}$ and by $n_t := \sum_{j \geq 1} \mathbf{1}_{t_j \leq t}$ the number of jumps before $t \geq 0$. We also denote by q_t the transition kernel of $(\eta_t)_{t \geq 0}$, i.e. $q_t(n, S) = \mathbb{P}(\eta_t \in S | \eta_0 = n)$ for any $n \in \mathbb{N}$ and $S \in \mathcal{P}(\mathbb{N})$.

We define the coupled process $\check{C} = (X', \eta')$ as a jump-move process on the state space $\check{E} = E \times \mathbb{N}$ equipped with the σ -algebra $\check{\mathcal{E}} = \mathcal{E} \otimes \mathcal{P}(\mathbb{N})$. Denoting by d the distance on E , we also equip \check{E} with the distance

$$\check{d}((x, k); (y, n)) := d(x, y) + \frac{|n - k|}{n \wedge k} \mathbf{1}_{nk \neq 0}.$$

To fully characterize \check{C} , we now specify its jump intensity function $\check{\alpha}$, its jump kernel \check{K} and its inter-jump move process \check{Y} .

The intensity function $\check{\alpha} : E \times \mathbb{N} \rightarrow \mathbb{R}_+$ is given by

$$\check{\alpha}(x, n) = \begin{cases} \beta(x) + \delta(x) + \beta_n + \delta_n & \text{if } x \in E_m, m \neq n, \\ \beta_n + \delta(x) & \text{if } x \in E_n. \end{cases}$$

One can easily check that $\check{\alpha}$ is a continuous function on \check{E} , bounded by $2\alpha^*$.

The transition kernel $\check{K} : \check{E} \times \check{\mathcal{E}} \rightarrow [0, 1]$ takes the same specific form as in [Pre75]:

1. If $x \in E_m$, $m \neq n$:

$$\check{K}((x, n); A \times \{n\}) = \frac{\alpha(x)}{\check{\alpha}(x, n)} K(x, A);$$

$$\check{K}((x, n); \{x\} \times \{n+1\}) = \frac{\beta_n}{\check{\alpha}(x, n)};$$

$$\check{K}((x, n); \{x\} \times \{n-1\}) = \frac{\delta_n}{\check{\alpha}(x, n)}.$$

2. If $x \in E_n$:

$$\check{K}((x, n); A_{n+1} \times \{n+1\}) = \frac{\beta(x)}{\check{\alpha}(x, n)} K_\beta(x, A_{n+1});$$

$$\check{K}((x, n); \{x\} \times \{n+1\}) = \frac{\beta_n - \beta(x)}{\check{\alpha}(x, n)};$$

$$\check{K}((x, n); A_{n-1} \times \{n-1\}) = \frac{\delta_n}{\check{\alpha}(x, n)} K_\delta(x, A_{n-1});$$

$$\check{K}((x, n); A_{n-1} \times \{n\}) = \frac{\delta(x) - \delta_n}{\check{\alpha}(x, n)} K_\delta(x, A_{n-1}).$$

The inter-jump move process \check{Y} is finally obtained by an independent coupling of $(Y_t)_{t \geq 0}$ and $(y_t)_{t \geq 0}$, specifically its transition kernel $Q_t^{\check{Y}}$ is given for any $(x, p) \in \check{E}$ and $A \times S \in \check{\mathcal{E}}$ by

$$\begin{aligned} Q_t^{\check{Y}}((x, p); A \times S) &= \mathbb{P}(\check{Y}_t \in A \times S | \check{Y}_0 = (x, p)) \\ &= \mathbb{P}(Y_t \in A | Y_0 = x) \mathbf{1}_S(p) \\ &= Q_t^Y(x, A) \mathbf{1}_S(p). \end{aligned} \tag{3.11}$$

This means that $\check{Y}_t = (Y'_t, y'_t) = (Y'_t, y'_0)$ for any $t \geq 0$, where $(Y'_t)_{t \geq 0}$ and $(y'_t)_{t \geq 0}$ are independent and follow the same distribution as $(Y_t)_{t \geq 0}$ and $(y_t)_{t \geq 0}$, respectively. Since Y is a continuous Markov process for the distance d , we can choose a version of Y' such that \check{Y} is also continuous for \check{d} . Remark moreover that $(\check{Y}_t)_{t \geq 0}$ satisfies

$$\mathbb{P}((\check{Y}_t)_{t \geq 0} \subset E_n \times \{k\} | \check{Y}_0 = (x, p)) = \mathbf{1}_{E_n}(x) \mathbf{1}_{k=p}, \quad \forall x \in E, \forall n \geq 0.$$

Given $\check{\alpha}$, \check{K} and \check{Y} as above, the jump-move process \check{C} is well defined and can be constructed as in Section 2.2.1. We denote by \check{Q}_t its transition kernel, by $(\check{T}_j)_{j \geq 1}$ its jump times and by $\check{N}_t := \sum_{j \geq 1} \mathbf{1}_{\check{T}_j \leq t}$ the number of jumps before $t \geq 0$. We also set $\check{\tau}_j = \check{T}_j - \check{T}_{j-1}$. The fact that \check{C} defines a relevant coupling of X with η is the object of Theorem 37.

Proof of Theorem 37

To prove the first point of the theorem, we use the following results, proved in Section 3.3.2. Fix $(x, n) \in E \times \mathbb{N}$ and $q \geq 0$, and let

$$\psi_q : t \in \mathbb{R}_+ \mapsto \check{Q}_t((x, n), E \times \{q\}).$$

Lemma 43. *For any $(x, n) \in E \times \mathbb{N}$ and $q \geq 0$, ψ_q is a continuous function.*

Lemma 44. *For any $(x, n) \in E \times \mathbb{N}$ and $q \geq 0$, ψ_q is right differentiable and satisfies :*

$$\frac{\partial_+}{\partial t} \psi_q(t) = -\alpha_q \psi_q(t) + \beta_{q-1} \psi_{q-1}(t) + \delta_{q+1} \psi_{q+1}(t).$$

Corollary 45. *For any $(x, n) \in E \times \mathbb{N}$ and $q \geq 0$,*

$$\psi_q(t) = \mathbf{1}_q(n) + \int_0^t (-\alpha_q \psi_q(s) + \beta_{q-1} \psi_{q-1}(s) + \delta_{q+1} \psi_{q+1}(s)) \, ds$$

and in particular ψ_q is differentiable.

Let now $w_s(x, n) = \check{Q}_{t-s}(\mathbf{1}_E \times q_s \mathbf{1}_{\{q\}})(x, n)$ for $s \in [0, t]$. Then using Corollary 45, we have:

Lemma 46. *For any $(x, n) \in E \times \mathbb{N}$ and $q \geq 0$, $s \mapsto w_s$ is differentiable on $[0, t]$ and $\partial w_s / \partial s \equiv 0$.*

Since $w_0(x, n) = \check{Q}_t((x, n); E \times \{q\})$ and $w_t(x, n) = q_t(n, \{q\})$, Lemma 46 entails that these two quantities are equal. The first part of Theorem 37 then follows from the decomposition:

$$q_t(n, S) = \sum_{q \in S} q_t(n, \{q\}) = \sum_{q \in S} \check{Q}_t((x, n); E \times \{q\}) = \check{Q}_t((x, n); E \times S).$$

We turn to the proof of the second point of Theorem 37. Similarly as for the first part, it is based on the following results proved in Section 3.3.2. For $(x, n) \in E \times \mathbb{N}$ and $f \in C_0(E)$, we set

$$\psi_f : t \in \mathbb{R}_+ \mapsto \check{Q}_t(f \times \mathbf{1}_{\mathbb{N}})(x, n).$$

Lemma 47. *Suppose that $(Y_t)_{t \geq 0}$ is a Feller process. Then for any $(x, n) \in E \times \mathbb{N}$ and any $f \in C_0(E)$, ψ_f is a continuous function.*

Lemma 48. *Suppose that $(Y_t)_{t \geq 0}$ is a Feller process and that $KC_0(E) \subset C_0(E)$. Then for any $(x, n) \in E \times \mathbb{N}$ and any $f \in \mathcal{D}_{\mathcal{A}}$ the function ψ_f is right-differentiable and satisfies:*

$$\frac{\partial_+}{\partial t} \psi_f(t) = \psi_{\mathcal{A}f}(t), \quad (3.12)$$

where \mathcal{A} is the infinitesimal generator of X given by Theorem 14.

Corollary 49. *Suppose that $(Y_t)_{t \geq 0}$ is a Feller process and that $KC_0(E) \subset C_0(E)$. Then for any $(x, n) \in E \times \mathbb{N}$ and any $f \in \mathcal{D}_{\mathcal{A}^Y}$,*

$$\psi_f(t) = f(x) + \int_0^t \psi_{\mathcal{A}f}(s) \, ds, \quad (3.13)$$

and in particular the function ψ_f is differentiable with derivative corresponding to (3.12).

By the Dynkin theorem, the second point of Theorem 37 is implied by the equality $\check{Q}_t((x, n); U \times \mathbb{N}) = Q_t(x, U)$ for any open set $U \subset E$, or equivalently

$$\check{Q}_t(g \times \mathbf{1}_{\mathbb{N}})(x, n) = Q_t(g)(x) \quad (3.14)$$

for $g = \mathbf{1}_U$. We first prove (3.14) for $g \in \mathcal{D}_{\mathcal{A}^Y}$, second for $g \in C_0(E)$, before getting the result for $g = \mathbf{1}_U$.

Let $g \in \mathcal{D}_{\mathcal{A}^Y}$ and for $s \in [0, t]$, define $v_s(x, n) = \psi_{Q_s g}(t-s) = \check{Q}_{t-s}(Q_s g \times \mathbf{1}_{\mathbb{N}})(x, n)$. We shall prove that $s \mapsto v_s$ is differentiable with $v'_s = 0$. For any $h \in \mathbb{R}$, write $(v_{s+h}(x, n) - v_s(x, n))/h = A_1 + A_2 + A_3$ with

$$\begin{aligned} A_1 &= \frac{1}{h} \left(\psi_{Q_{s+h}g}(t-s-h) - \psi_{Q_s g}(t-s-h) \right) - \psi_{\mathcal{A}Q_s g}(t-s-h), \\ A_2 &= \psi_{\mathcal{A}Q_s g}(t-s-h) - \psi_{\mathcal{A}Q_s g}(t-s), \\ A_3 &= \frac{1}{h} \left(\psi_{Q_s g}(t-s-h) - \psi_{Q_s g}(t-s) \right) + \psi_{\mathcal{A}Q_s g}(t-s). \end{aligned}$$

We know by Theorem 14, with $L_0^Y = C_0(E)$, that $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}^Y}$ and since $Q_s \mathcal{D}_{\mathcal{A}} \subset \mathcal{D}_{\mathcal{A}}$ ([Dyn65, Chapter 1, §2]), we deduce from Corollary 49 that A_3 tends to $-\partial \psi_{Q_s g}(t-s)/\partial t + \psi_{\mathcal{A}Q_s g}(t-s) = 0$ as $h \rightarrow 0$. Regarding A_2 , note that $Q_s g \in \mathcal{D}_{\mathcal{A}}$ implies that $\mathcal{A}Q_s g \in C_0(E)$ ([Dyn65, Chapter 1, §2]), so that Lemma 47 applies and $A_2 \rightarrow 0$ as $h \rightarrow 0$. Concerning A_1 , using the linearity of $\psi_f(t)$ in f , we can write

$$|A_1| = |\psi_{(Q_{s+h}-Q_s)g/h - \mathcal{A}Q_s g}(t-s-h)| \leq \|(Q_{s+h} - Q_s)g/h - \mathcal{A}Q_s g\|_{\infty}$$

which also tends to 0 as $h \rightarrow 0$ ([Dyn65, Chapter 1, §2]). We therefore obtain that $v'_s = 0$ and so $v_t(x, n) = (Q_t g \times \mathbf{1}_{\mathbb{N}})(x, n) = \check{Q}_t(g \times \mathbf{1}_{\mathbb{N}})(x, n) = v_0(x, n)$, proving (3.14) when $g \in \mathcal{D}_{\mathcal{A}^Y}$.

Now let $g \in C_0(E)$. By our assumptions and Theorem 13, $(X_t)_{t \geq 0}$ is Feller, which implies that $C_0(E) = \overline{\mathcal{D}_{\mathcal{A}}}$ ([Dyn65]). So there exists a sequence of functions $(g_p)_{p \geq 0}$ in $\mathcal{D}_{\mathcal{A}^Y}$ so that $\|g_p - g\|_{\infty} \rightarrow 0$ as $p \rightarrow \infty$. The two linear operators $f \in M_b(E) \mapsto \check{Q}_t(f \times \mathbf{1}_{\mathbb{N}})$ and $f \in M_b(E) \mapsto Q_t(f)$ being bounded, we can take the limit in (3.14) when applied to g_p to get the same relation for $g \in C_0(E)$.

Finally take $U \subset E$ an open subset. Then define for any $p \geq 0$ the function

$$\phi_p : x \in E \mapsto \frac{d(x, E \setminus U)}{d(x, E \setminus U) + d(x, U_p)},$$

where $U_p = \{y \in E, d(y, E \setminus U) \geq 1/p\}$. Then $\phi_p \in C_0(E)$ for any $p \geq 0$, and for any $x \in E$ we have $\phi_p(x) \rightarrow \mathbf{1}_U(x)$ as $p \rightarrow \infty$. Taking the limit we obtain by the

dominated convergence theorem the relation (3.14) for $g = \mathbf{1}_U$, concluding the proof of the second statement of Theorem 37.

We finish by the proof of the third point of Theorem 37. Let $x \in E$ and $n \in \mathbb{N}$ such that $n(x) \leq n$. We show by induction on $k \geq 0$ that $\mathbb{P}_{(x,n)}(\check{C}_{\check{T}_k} \in \Gamma) = 0$. If $k = 0$, then $\mathbb{P}_{(x,n)}(\check{C}_{\check{T}_0} \in \Gamma) = \mathbf{1}_{(x,n) \in \Gamma} = 0$. Suppose next that there exists $k \geq 0$ such that $\mathbb{P}_{(x,n)}(\check{C}_{\check{T}_k} \in \Gamma) = 0$, then

$$\begin{aligned}
\mathbb{P}_{(x,n)}(\check{C}_{\check{T}_{k+1}} \in \Gamma) &= \mathbb{E}_{(x,n)} \left[\mathbb{E}_{(x,n)} \left(\mathbf{1}_{\check{C}_{\check{T}_{k+1}} \in \Gamma} \mid \check{C}_{\check{T}_k}, (\check{Y}_t^{(k)})_{t \geq 0}, \check{\tau}_{k+1} \right) \right] \\
&= \mathbb{E}_{(x,n)} \left[\check{K}(\check{Y}_{\check{\tau}_{k+1}}^{(k)}, \Gamma) \right] \\
&= \mathbb{E}_{(x,n)} \left[\check{K}(\check{Y}_{\check{\tau}_{k+1}}^{(k)}, \Gamma) \mathbf{1}_{\check{Y}_{\check{\tau}_{k+1}}^{(k)} \in \Gamma} \right] + \mathbb{E}_{(x,n)} \left[\check{K}(\check{Y}_{\check{\tau}_{k+1}}^{(k)}, \Gamma) \mathbf{1}_{\check{Y}_{\check{\tau}_{k+1}}^{(k)} \in \Gamma^c} \right] \\
&= \mathbb{E}_{(x,n)} \left[\check{K}(\check{Y}_{\check{\tau}_{k+1}}^{(k)}, \Gamma) \mathbf{1}_{\check{Y}_{\check{\tau}_{k+1}}^{(k)} \in \Gamma} \right] \quad (\text{by definition of } \check{K}) \\
&= \mathbb{E}_{(x,n)} \left[\check{K}(\check{Y}_{\check{\tau}_{k+1}}^{(k)}, \Gamma) \mathbf{1}_{\check{C}_{\check{T}_k} \in \Gamma} \right] \quad (\text{for any } t \geq 0, \check{C}_{\check{T}_k} \in \Gamma \Leftrightarrow \check{Y}_t^{(k)} \in \Gamma) \\
&\leq \mathbb{E}_{(x,n)} \left[\mathbf{1}_{\check{C}_{\check{T}_k} \in \Gamma} \right] = \mathbb{P}_{(x,n)} \left[\check{C}_{\check{T}_k} \in \Gamma \right] = 0,
\end{aligned}$$

which proves the induction step. To conclude, recall that $\mathbb{P}_{(x,n)}(\check{N}_t < \infty) = 1$ and notice that due to the form of Γ one has $\{\check{C}_t \in \Gamma\} = \{\check{C}_{\check{T}_{\check{N}_t}} \in \Gamma\}$ for any $t \geq 0$. Then

$$\check{Q}_t((x, n); \Gamma) = \mathbb{P}_{(x,n)}(\check{C}_t \in \Gamma) = \mathbb{P}_{(x,n)}(\check{C}_{\check{T}_{\check{N}_t}} \in \Gamma) = \sum_{k=0}^{\infty} \mathbb{P}_{(x,n)}(\check{C}_{\check{T}_k} \in \Gamma, \check{N}_t = k) = 0.$$

3.3.2 Proofs of lemmas relating to the coupling

Proof of Lemma 43

First note that for any $x \in E$, $n \geq 0$ and $h > 0$ one has

$$\mathbb{P}_{(x,n)}(\check{T}_1 \leq h) = \mathbb{E}_{(x,n)} \left(1 - e^{-\int_0^h \check{\alpha}(\check{Y}_u) du} \right) \leq \mathbb{E}_{(x,n)} \left(\int_0^h \check{\alpha}(\check{Y}_u) du \right) \leq 2\alpha^* h.$$

Next take $h > 0$. Then

$$\begin{aligned}
\psi_q(t+h) - \psi_q(t) &= \mathbb{E}_{(x,n)} \left[\mathbf{1}_E(X'_{t+h}) \mathbf{1}_{\{q\}}(\eta'_{t+h}) - \mathbf{1}_E(X'_t) \mathbf{1}_{\{q\}}(\eta'_t) \right] \\
&= \mathbb{E}_{(x,n)} \left[\mathbb{E}_{(x,n)} \left(\mathbf{1}_E(X'_{t+h}) \mathbf{1}_{\{q\}}(\eta'_{t+h}) \mid \check{\mathcal{F}}_t \right) - \mathbf{1}_E(X'_t) \mathbf{1}_{\{q\}}(\eta'_t) \right] \\
&= \mathbb{E}_{(x,n)} \left[\check{Q}_h((X'_t, \eta'_t); E \times \{q\}) - \mathbf{1}_E(X'_t) \mathbf{1}_{\{q\}}(\eta'_t) \right] \\
&= \sum_{k \geq 0} \mathbb{E}_{(x,n)} \left[(\check{Q}_h((X'_t, \eta'_t); E \times \{q\}) - \mathbf{1}_E(X'_t) \mathbf{1}_{\{q\}}(\eta'_t)) \mathbf{1}_{\eta'_t = k} \right]. \\
&= \sum_{k \geq 0} \mathbb{E}_{(x,n)} \left[\check{Q}_h((X'_t, k); E \times \{q\}) - \mathbf{1}_{\{q\}}(k) \mid \eta'_t = k \right] \mathbb{P}_{(x,n)}(\eta'_t = k).
\end{aligned} \tag{3.15}$$

For any $k \geq 0$ and $y \in E$

$$\begin{aligned}
& \left| \check{Q}_h((y, k), E \times \{q\}) - \mathbf{1}_{\{q\}}(k) \right| \\
&= \left| \mathbb{E}_{(y,k)} \left(\mathbf{1}_E(X'_h) \mathbf{1}_{\{q\}}(\eta'_h) \right) - \mathbf{1}_{\{q\}}(k) \right| \\
&= \left| \mathbb{E}_{(y,k)} \left(\mathbf{1}_{\{q\}}(\eta'_h) \mathbf{1}_{\check{T}_1 > h} \right) + \mathbb{E}_{(y,k)} \left(\mathbf{1}_{\{q\}}(\eta'_h) \mathbf{1}_{\check{T}_1 \leq h} \right) - \mathbf{1}_{\{q\}}(k) \right| \\
&\leq \mathbb{E}_{(y,k)} \left| \left(\mathbf{1}_{\{q\}}(\eta'_h) - \mathbf{1}_{\{q\}}(k) \right) \mathbf{1}_{\check{T}_1 \leq h} \right| + \mathbb{E}_{(y,k)} \left| \left(\mathbf{1}_{\{q\}}(\eta'_h) - \mathbf{1}_{\{q\}}(k) \right) \mathbf{1}_{\check{T}_1 > h} \right| \\
&\leq 2\mathbb{P}_{(x,n)}(\check{T}_1 \leq h) + \mathbb{E}_{(y,k)} \left| \left(\mathbf{1}_{\{q\}}(k) - \mathbf{1}_{\{q\}}(k) \right) \mathbf{1}_{\check{T}_1 > h} \right| \\
&\leq 4\alpha^* h,
\end{aligned}$$

whereby

$$|\psi_q(t+h) - \psi_q(t)| \leq \sum_{k \geq 0} 4\alpha^* h \mathbb{P}_{(x,n)}(\eta'_t = k) = 4\alpha^* h \xrightarrow{h \searrow 0} 0.$$

On the other hand, we obtain with the same calculations for $h \in [0, t]$

$$\begin{aligned}
& \psi_q(t) - \psi_q(t-h) \\
&= \mathbb{E}_{(x,n)} \left[\mathbb{E}_{(x,n)} \left(\mathbf{1}_E(X'_t) \mathbf{1}_{\{q\}}(\eta'_t) \middle| \check{\mathcal{F}}_{t-h} \right) - \mathbf{1}_E(X'_{t-h}) \mathbf{1}_{\{q\}}(\eta'_{t-h}) \right] \\
&= \mathbb{E}_{(x,n)} \left[\check{Q}_h((X'_{t-h}, \eta'_{t-h}); E \times \{q\}) - \mathbf{1}_E(X'_{t-h}) \mathbf{1}_{\{q\}}(\eta'_{t-h}) \right] \\
&= \sum_{k \geq 0} \mathbb{E}_{(x,n)} \left[\check{Q}_h((X'_{t-h}, k); E \times \{q\}) - \mathbf{1}_{\{q\}}(k) \middle| \eta'_{t-h} = k \right] \check{Q}_{t-h}((x, n); E \times \{k\}) \\
&\leq 4\alpha^* h \xrightarrow{h \searrow 0} 0.
\end{aligned}$$

Therefore the function $t \in \mathbb{R}_+ \mapsto \psi_q(t)$ is continuous.

Proof of Lemma 44

Take $h > 0$. Recall from (3.15) that

$$\frac{1}{h} (\psi_q(t+h) - \psi_q(t)) \tag{3.16}$$

$$= \frac{1}{h} \sum_{k \geq 0} \mathbb{E}_{(x,n)} \left[\check{Q}_h((X'_t, k); E \times \{q\}) - \mathbf{1}_{\{q\}}(k) \middle| \eta'_t = k \right] \check{Q}_t((x, n); E \times \{k\}). \tag{3.17}$$

For any $y \in E$ and $k \geq 0$

$$\check{Q}_h((y, k), E \times \{q\}) - \mathbf{1}_{\{q\}}(k) = \mathbb{E}_{(y,k)} \left(\mathbf{1}_{\{q\}}(\eta'_h) - \mathbf{1}_{\{q\}}(k) \right) = A_1(h) + A_2(h) + A_3(h), \tag{3.18}$$

where

$$\begin{aligned}
A_1(h) &= \mathbb{E}_{(y,k)} \left(\left(\mathbf{1}_{\{q\}}(\eta'_h) - \mathbf{1}_{\{q\}}(k) \right) \mathbf{1}_{\check{T}_1 > h} \right), \\
A_2(h) &= \mathbb{E}_{(y,k)} \left(\left(\mathbf{1}_{\{q\}}(\eta'_h) - \mathbf{1}_{\{q\}}(k) \right) \mathbf{1}_{\check{N}_h = 1} \right), \\
A_3(h) &= \mathbb{E}_{(y,k)} \left(\left(\mathbf{1}_{\{q\}}(\eta'_h) - \mathbf{1}_{\{q\}}(k) \right) \mathbf{1}_{\check{T}_2 < h} \right).
\end{aligned}$$

Let us treat each term separately.

First,

$$A_1(h) = \mathbb{E}_{(y,k)} \left((\mathbf{1}_{\{q\}}(\eta'_h) - \mathbf{1}_{\{q\}}(k)) \mathbf{1}_{\check{\tau}_1 > h} \right) = 0.$$

Second, $A_2(h)$ reads

$$\begin{aligned} & \mathbb{E}_{(y,k)} \left((\mathbf{1}_{\{q\}}(\eta'_h) - \mathbf{1}_{\{q\}}(k)) \mathbf{1}_{\check{\tau}_1 \leq h} \mathbf{1}_{\check{\tau}_2 > h - \check{\tau}_1} \right) \\ &= \mathbb{E}_{(y,k)} \left[\mathbb{E}_{(y,k)} \left[(\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \mathbf{1}_{\check{\tau}_1 \leq h} \mathbf{1}_{\check{\tau}_2 > h - \check{\tau}_1} \mid \check{\mathcal{F}}_{\check{\tau}_1}, \check{Y}^{(1)} \right] \right] \\ &= \mathbb{E}_{(y,k)} \left[(\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{P}_{(y,k)} \left(\check{\tau}_2 > h - \check{\tau}_1 \mid \check{\mathcal{F}}_{\check{\tau}_1}, \check{Y}^{(1)} \right) \right] \\ &= \mathbb{E}_{(y,k)} \left[(\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \mathbf{1}_{\check{\tau}_1 \leq h} e^{-\int_0^{h-\check{\tau}_1} \check{\alpha}(\check{Y}_u^{(1)}) du} \right] \\ &= \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} (\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \mathbb{E}_{(y,k)} \left[e^{-\int_0^{h-\check{\tau}_1} \check{\alpha}(\check{Y}_u^{(1)}) du} \mid \check{\mathcal{F}}_{\check{\tau}_1} \right] \right] \\ &= \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} (\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} \left[e^{-\int_0^{h-\check{\tau}_1} \check{\alpha}(\check{Y}_u) du} \right] \right] \\ &= \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} (\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} \left[e^{-\int_0^{h-\check{\tau}_1} \check{\alpha}(\check{Y}_u) du} - 1 \right] \right] \\ &\quad + \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} (\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \right]. \end{aligned}$$

For the first term above,

$$\begin{aligned} & \frac{1}{h} \left| \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} (\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} \left[e^{-\int_0^{h-\check{\tau}_1} \check{\alpha}(\check{Y}_u) du} - 1 \right] \right] \right| \\ &\leq 4\alpha^* \mathbb{E}_{(y,k)}(\mathbf{1}_{\check{\tau}_1 \leq h}) \\ &\leq 8(\alpha^*)^2 h. \end{aligned}$$

For the second term, we have

$$\mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} (\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \right] \tag{3.19}$$

$$\begin{aligned} &= \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k) \mid \check{Y}^{(0)}, \check{\tau}_1 \right] \right] \\ &= \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \left(\int_{z_1 \in E} \check{K}((Y'_{\check{\tau}_1})^{(0)}, k); dz_1 \times \{q\} - \mathbf{1}_{\{q\}}(k) \right) \right] \\ &= \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \left(\check{K}((Y'_{\check{\tau}_1})^{(0)}, k); E \times \{q\} - \mathbf{1}_{\{q\}}(k) \right) \right]. \end{aligned} \tag{3.20}$$

Following the definition of \check{K} in Section 3.3.1, this formula takes two forms depending on whether $y \notin E_k$ or $y \in E_k$. If $y \notin E_k$, then

$$\begin{aligned} & \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} (\mathbf{1}_{\{q\}}(\eta'_{\check{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \right] = \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \left(\frac{\alpha(Y'_{\check{\tau}_1})^{(0)}}{\check{\alpha}(Y'_{\check{\tau}_1})^{(0)}, k} - 1 \right) \mathbf{1}_{\{q\}}(k) \right] \\ &+ \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \frac{\beta_k}{\check{\alpha}(Y'_{\check{\tau}_1})^{(0)}, k} \right] \mathbf{1}_{\{q-1\}}(k) + \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \frac{\delta_k}{\check{\alpha}(Y'_{\check{\tau}_1})^{(0)}, k} \right] \mathbf{1}_{\{q+1\}}(k), \end{aligned}$$

that is

$$\begin{aligned} & \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} (\mathbf{1}_{\{q\}}(\eta'_{\tilde{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \right] \\ &= \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \frac{1}{\check{\alpha}(Y'_{\tilde{\tau}_1}(0), k)} \right] \left(-\alpha_k \mathbf{1}_{\{q\}}(k) + \beta_k \mathbf{1}_{\{q-1\}}(k) + \delta_k \mathbf{1}_{\{q+1\}}(k) \right). \end{aligned} \quad (3.21)$$

Since

$$\left| \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \frac{1}{\check{\alpha}(Y'_{\tilde{\tau}_1}(0), k)} \right] - 1 \right| = \left| \frac{1}{h} \mathbb{E}_{(y,k)}^{\check{Y}} \left[\int_0^h \left(e^{-\int_0^s \check{\alpha}(Y'_u, k) du} - 1 \right) ds \right] \right| \leq 2\alpha^* h,$$

we then conclude that

$$\sup_{k \geq 0, y \notin E_k} \left| \frac{A_2(h)}{h} + \alpha_k \mathbf{1}_{\{q\}}(k) - \beta_k \mathbf{1}_{\{q-1\}}(k) - \delta_k \mathbf{1}_{\{q+1\}}(k) \right| \xrightarrow{h \searrow 0} 0. \quad (3.22)$$

If now $y \in E_k$, we obtain from (3.19)

$$\begin{aligned} & \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} (\mathbf{1}_{\{q\}}(\eta'_{\tilde{\tau}_1}) - \mathbf{1}_{\{q\}}(k)) \right] \\ &= \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \frac{\beta(Y'_{\tilde{\tau}_1}(0))}{\check{\alpha}(Y'_{\tilde{\tau}_1}(0), k)} \right] \mathbf{1}_{\{q-1\}}(k) + \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \frac{\beta_k - \beta(Y'_{\tilde{\tau}_1}(0))}{\check{\alpha}(Y'_{\tilde{\tau}_1}(0), k)} \right] \mathbf{1}_{\{q-1\}}(k) \\ &+ \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \frac{\delta_k}{\check{\alpha}(Y'_{\tilde{\tau}_1}(0), k)} \right] \mathbf{1}_{\{q+1\}}(k) + \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \left(\frac{\delta(Y'_{\tilde{\tau}_1}(0)) - \delta_k}{\check{\alpha}(Y'_{\tilde{\tau}_1}(0), k)} - 1 \right) \right] \mathbf{1}_{\{q\}}(k), \end{aligned}$$

that is the same expression as (3.21). The convergence (3.22) then remains true when the supremum is taken over $y \in E_k$, and so over $y \in E$, i.e.

$$\sup_{k \geq 0, y \in E} \left| \frac{A_2(h)}{h} + \alpha_k \mathbf{1}_{\{q\}}(k) - \beta_k \mathbf{1}_{\{q-1\}}(k) - \delta_k \mathbf{1}_{\{q+1\}}(k) \right| \xrightarrow{h \searrow 0} 0.$$

Third, for $A_3(h)$ in (3.18), we have, using (2.3) and denoting $\check{N}_h^* \sim \mathcal{P}(2\alpha^* h)$,

$$\begin{aligned} \frac{1}{h} |A_3(h)| &\leq \frac{2}{h} \mathbb{P}_{(y,k)} (\check{N}_h \geq 2) \\ &\leq \frac{2}{h} \mathbb{P} (\check{N}_h^* \geq 2) \\ &\leq \frac{2}{h} (1 - e^{-2\alpha^* h} - 2\alpha^* h e^{-2\alpha^* h}) \\ &= 4(\alpha^*)^2 h + \underset{h \searrow 0}{o}(h). \end{aligned}$$

Combining the results for $A_1(h)$, $A_2(h)$ and $A_3(h)$ in (3.18), we get

$$\sup_{(y,k) \in \check{E}} \left| \frac{1}{h} \left(\check{Q}_h((y, k); E \times \{q\}) - \mathbf{1}_{\{q\}}(k) \right) + \mathbf{1}_{\{q\}}(k) \alpha_k - \beta_k \mathbf{1}_{\{q-1\}}(k) - \delta_k \mathbf{1}_{\{q+1\}}(k) \right| \xrightarrow{h \searrow 0} 0.$$

Finally, coming back to (3.16), we obtain by uniform convergence, for any $x \in E$,

$$\frac{1}{h} (\psi_q(t+h) - \psi_q(t)) \xrightarrow{h \searrow 0} \sum_{k \geq 0} \left\{ -\alpha_k \mathbf{1}_q(k) + \beta_k \mathbf{1}_{\{q-1\}}(k) + \delta_k \mathbf{1}_{\{q+1\}}(k) \right\} \check{Q}_t((x, n); E \times \{k\}),$$

where the limit reads $-\alpha_q \psi_q(t) + \beta_{q-1} \psi_{q-1}(t) + \delta_{q+1} \psi_{q+1}(t)$ using the convention $\beta_{-1} = 0$.

Proof of Corollary 45

For $t \geq 0$, define

$$G(t) = \psi_q(t) - \mathbf{1}_q(n) - \int_0^t (-\alpha_q \psi_q(s) + \beta_{q-1} \psi_{q-1}(s) + \delta_{q+1} \psi_{q+1}(s)) ds.$$

Then G is continuous, right-differentiable on \mathbb{R}_+ by Lemma 44, and $\partial_+ G(t)/\partial t = 0$. So G is constant. But $G(0) = 0$ because $s \mapsto \psi_q(s)$ is bounded on \mathbb{R}_+ for any $q \geq 0$. As a consequence we obtain

$$\psi_q(t) = \mathbf{1}_q(n) + \int_0^t (-\alpha_q \psi_q(s) + \beta_{q-1} \psi_{q-1}(s) + \delta_{q+1} \psi_{q+1}(s)) ds.$$

In particular the integrand is continuous by Lemma 43 so ψ_q is differentiable.

Proof of Lemma 46

Let us develop $q_s \mathbf{1}_{\{q\}}$ as

$$q_s \mathbf{1}_{\{q\}} = \sum_{k \geq 0} q_s(k, \{q\}) \mathbf{1}_{\{k\}} = \sum_{k \geq 0} \mathbb{P}_k(\eta_s = q) \mathbf{1}_{\{k\}}. \quad (3.23)$$

Take $p > q$. Then using (2.3) by denoting $n_t^* \sim \mathcal{P}(\alpha^* t)$

$$\begin{aligned} \sum_{k=p}^{\infty} \mathbb{P}_k(\eta_s = q) &\leq \sum_{k=p}^{\infty} \mathbb{P}_k(n_s > k - q) \\ &\leq \sum_{k=p}^{\infty} \mathbb{P}_k(n_t > k - q) \\ &\leq \sum_{k=p}^{\infty} \mathbb{P}(n_t^* > k - q) \\ &= \sum_{j=p-q}^{\infty} \mathbb{P}(n_t^* > j) \xrightarrow[p \rightarrow \infty]{} 0, \end{aligned}$$

because $\mathbb{E}(n_t^*) < \infty$. Coming back to (3.23), we thus have that for any $\varepsilon > 0$, there exists $p \geq 0$ such that any $d \geq p$ satisfies

$$\sup_{s \in [0, t]} \left\| q_s \mathbf{1}_{\{q\}} - \sum_{k=0}^d q_s(k, \{q\}) \mathbf{1}_{\{k\}} \right\|_{\infty} < \varepsilon. \quad (3.24)$$

Since \check{Q}_t is a continuous linear operator on $M_b(E \times \mathbb{N})$

$$\begin{aligned} w_s &= \check{Q}_{t-s}(\mathbf{1}_E \times q_s \mathbf{1}_{\{q\}}) \\ &= \check{Q}_{t-s} \left(\mathbf{1}_E \times \lim_{p \rightarrow \infty} \sum_{k=0}^p q_s(k, \{q\}) \mathbf{1}_{\{k\}} \right) \\ &= \lim_{p \rightarrow \infty} \check{Q}_{t-s} \left(\mathbf{1}_E \times \sum_{k=0}^p q_s(k, \{q\}) \mathbf{1}_{\{k\}} \right) \\ &= \lim_{p \rightarrow \infty} \sum_{k=0}^p q_s(k, \{q\}) \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k\}}) \\ &= \sum_{k=0}^{\infty} q_s(k, \{q\}) \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k\}}). \end{aligned} \quad (3.25)$$

Let $\phi_k(s) = q_s(k, \{q\})\check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k\}})$. From the Kolmogorov backward equation (2.7), we obtain that

$$\partial q_s(k, \{q\})/\partial s = -\alpha_k q_s(k, \{q\}) + \beta_k q_s(k+1, \{q\}) + \delta_k q_s(k-1, \{q\}).$$

Using in addition Corollary 45, we deduce that ϕ_k is differentiable and

$$\begin{aligned} \phi'_k(s) &= [-\alpha_k q_s(k, \{q\}) + \beta_k q_s(k+1, \{q\}) + \delta_k q_s(k-1, \{q\})]\check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k\}}) \\ &+ q_s(k, \{q\}) \left[\alpha_k \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k\}}) - \beta_{k-1} \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k-1\}}) - \delta_{k+1} \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k+1\}}) \right]. \end{aligned}$$

Since

$$\sup_{s \in [0, t]} \left\| \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k\}}) \right\|_{\infty} \leq 1$$

and

$$\sup_{s \in [0, t]} \left\| \alpha_k \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k\}}) - \beta_{k-1} \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k-1\}}) - \delta_{k+1} \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k+1\}}) \right\|_{\infty} \leq 3\alpha^*,$$

we can show similarly as for (3.24) that

$$\sup_{s \in [0, t]} \left\| \sum_{k \geq p} \phi'_k(s) \right\|_{\infty} \xrightarrow{p \rightarrow \infty} 0.$$

Since by (3.25) $w_s = \sum_{k \geq 0} \phi_k(s)$, we deduce that w_s is differentiable on $[0, t]$ and

$$\begin{aligned} \frac{\partial}{\partial s} w_s &= \sum_{k=0}^{\infty} \phi'_k(s) \\ &= \sum_{k=0}^{\infty} \left[\beta_k q_s(k+1, \{q\}) \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k\}}) - \beta_{k-1} q_s(k, \{q\}) \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k-1\}}) \right] \\ &\quad + \sum_{k=0}^{\infty} \left[\delta_k q_s(k-1, \{q\}) \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k\}}) - \delta_{k+1} q_s(k, \{q\}) \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k+1\}}) \right], \end{aligned}$$

where $\beta_{-1} = \delta_0 = 0$. The first of these two telescoping series vanishes because $\beta_{-1} = 0$ and

$$\left\| \beta_k q_s(k+1, \{q\}) \check{Q}_{t-s}(\mathbf{1}_E \times \mathbf{1}_{\{k\}}) \right\|_{\infty} \leq \alpha^* q_s(k+1, \{q\}) \leq \alpha^* \mathbb{P}(n_t^* > k+1-q) \rightarrow 0.$$

The second series vanishes by similar arguments and we have $\partial w_s / \partial s \equiv 0$.

Proof of Lemma 47

Let $h > 0$, then

$$\begin{aligned} \psi_f(t+h) - \psi_f(t) &= \mathbb{E}_{(x,n)} \left(f(X'_{t+h}) \mathbf{1}_{\mathbb{N}}(\eta'_{t+h}) \right) - \mathbb{E}_{(x,n)} \left(f(X'_t) \mathbf{1}_{\mathbb{N}}(\eta'_t) \right) \\ &= \mathbb{E}_{(x,n)} \left[\mathbb{E}_{(x,n)} \left(f(X'_{t+h}) \mathbf{1}_{\mathbb{N}}(\eta'_{t+h}) \mid \check{\mathcal{F}}_t \right) \right] - \mathbb{E}_{(x,n)} \left(f(X'_t) \mathbf{1}_{\mathbb{N}}(\eta'_t) \right) \\ &= \mathbb{E}_{(x,n)} \left[\check{Q}_h(f \times \mathbf{1}_{\mathbb{N}})(X'_t, \eta'_t) - f(X'_t) \mathbf{1}_{\mathbb{N}}(\eta'_t) \right]. \end{aligned} \quad (3.26)$$

For any $y \in E$ and $k \in \mathbb{N}$ one has

$$\begin{aligned}
& \left| \check{Q}_h(f \times \mathbf{1}_{\mathbb{N}})(y, k) - f(y)\mathbf{1}_{\mathbb{N}}(k) \right| \\
&= \left| \mathbb{E}_{(y,k)}(f(X'_h)) - f(y) \right| \\
&= \left| \mathbb{E}_{(y,k)} \left(f(X'_h)(\mathbf{1}_{\check{T}_1 \leq h} + \mathbf{1}_{\check{T}_1 > h}) \right) - f(y)\mathbb{E}_{(y,k)} \left(\mathbf{1}_{\check{T}_1 \leq h} + \mathbf{1}_{\check{T}_1 > h} \right) \right| \\
&\leq 2\|f\|_{\infty} \mathbb{P}_{(y,k)}(\check{T}_1 \leq h) + \left| \mathbb{E}_{(y,k)} \left((f(Y'_h)^{(0)} - f(y))(\mathbf{1} - \mathbf{1}_{\check{T}_1 \leq h}) \right) \right| \\
&\leq 8\alpha^* \|f\|_{\infty} h + \left| \mathbb{E}_y^Y [f(Y_h) - f(y)] \right| \\
&\leq 8\alpha^* \|f\|_{\infty} h + \left\| Q_h^Y f - f \right\|_{\infty},
\end{aligned}$$

where we have used (3.11) in the second last step. Coming back to (3.26), we deduce that

$$|\psi_f(t+h) - \psi_f(t)| \leq 8\alpha^* \|f\|_{\infty} h + \left\| Q_h^Y f - f \right\|_{\infty}.$$

As a Feller process, $(Y_t)_{t \geq 0}$ is strongly continuous at 0, whereby $\psi_f(t+h) \rightarrow \psi_f(t)$ as $h \searrow 0$.

On the other hand, for $h \in [0, t]$,

$$\begin{aligned}
|\psi_f(t) - \psi_f(t-h)| &= \left| \mathbb{E}_{(x,n)}(f(X'_t)\mathbf{1}_{\mathbb{N}}(\eta'_t)) - \mathbb{E}_{(x,n)}(f(X'_{t-h})\mathbf{1}_{\mathbb{N}}(\eta'_{t-h})) \right| \\
&= \left| \mathbb{E}_{(x,n)} \left[\mathbb{E}_{(x,n)} \left(f(X'_t)\mathbf{1}_{\mathbb{N}}(\eta'_t) \mid \check{\mathcal{F}}_{t-h} \right) \right] - \mathbb{E}_{(x,n)}(f(X'_{t-h})\mathbf{1}_{\mathbb{N}}(\eta'_{t-h})) \right| \\
&= \left| \mathbb{E}_{(x,n)} \left[\check{Q}_h(f \times \mathbf{1}_{\mathbb{N}})(X'_{t-h}, \eta'_{t-h}) - f(X'_{t-h})\mathbf{1}_{\mathbb{N}}(\eta'_{t-h}) \right] \right| \\
&\leq 8\alpha^* \|f\|_{\infty} h + \left\| Q_h^Y f - f \right\|_{\infty},
\end{aligned}$$

and we conclude similarly that $\psi_f(t-h) \rightarrow \psi_f(t)$ as $h \searrow 0$.

Proof of Lemma 48

Let $h > 0$. For any $t \geq 0$

$$\begin{aligned}
& \left| \frac{\psi_f(t+h) - \psi_f(t)}{h} - \psi_{\mathcal{A}f}(t) \right| \\
&= \left| \check{Q}_t \left(\frac{\check{Q}_h(f \times \mathbf{1}_{\mathbb{N}})(x, n) - f(x)\mathbf{1}_{\mathbb{N}}(n)}{h} - \mathcal{A}f(x) \times \mathbf{1}_{\mathbb{N}}(n) \right) \right| \\
&\leq \sup_{(y,k) \in \check{E}} \left| \frac{\check{Q}_h(f \times \mathbf{1}_{\mathbb{N}})(y, k) - f(y)\mathbf{1}_{\mathbb{N}}(k)}{h} - \mathcal{A}f(y) \times \mathbf{1}_{\mathbb{N}}(k) \right|.
\end{aligned}$$

The proof thus consists in showing that

$$\sup_{(y,k) \in \check{E}} \left| \frac{\check{Q}_h(f \times \mathbf{1}_{\mathbb{N}})(y, k) - f(y)\mathbf{1}_{\mathbb{N}}(k)}{h} - \mathcal{A}f(y) \times \mathbf{1}_{\mathbb{N}}(k) \right| \xrightarrow{h \searrow 0} 0.$$

For any $h > 0$, $y \in E$ and $k \geq 0$

$$\begin{aligned} & \frac{1}{h} \left(\check{Q}_h(f \times \mathbf{1}_{\mathbb{N}})(y, k) - f(y) \mathbf{1}_{\mathbb{N}}(k) \right) \\ &= \mathbb{E}_{(y,k)} \left[\frac{f(X'_h) - f(y)}{h} \right] \\ &= \frac{1}{h} \left(\mathbb{E}_{(y,k)} \left[f(X'_h) \mathbf{1}_{\check{T}_1 > h} \right] + \mathbb{E}_{(y,k)} \left[f(X'_h) \mathbf{1}_{\check{N}_h = 1} \right] + \mathbb{E}_{(y,k)} \left[f(X'_h) \mathbf{1}_{\check{T}_2 < h} \right] - f(y) \right). \end{aligned}$$

But

$$\begin{aligned} & \frac{1}{h} \left(\mathbb{E}_{(y,k)} \left[f(X'_h) \mathbf{1}_{\check{T}_1 > h} \right] - f(y) \right) \\ &= \frac{1}{h} \left(\mathbb{E}_{(y,k)} \left[f(Y'_h)^{(0)} \mathbf{1}_{\check{T}_1 > h} \right] - f(y) \right) \\ &= \frac{1}{h} \left(\mathbb{E}_{(y,k)} \left[f(Y'_h)^{(0)} \mathbb{P}_{(y,k)} \left(\check{T}_1 > h \mid \check{Y}^{(0)} \right) \right] - f(y) \right) \\ &= \frac{1}{h} \left(\mathbb{E}_{(y,k)} \left[f(Y'_h)^{(0)} e^{-\int_0^h \check{\alpha}(Y'_u)^{(0), k} du} \right] - f(y) \right) \\ &= \mathbb{E}_y^Y \left[\frac{f(Y_h) - f(y)}{h} \right] + \frac{1}{h} \mathbb{E}_{(y,k)}^{\check{Y}} \left[f(Y'_h) \left(e^{-\int_0^h \check{\alpha}(Y'_u, k) du} - 1 + \int_0^h \check{\alpha}(Y'_u, k) du \right) \right] \\ &\quad - \frac{1}{h} \mathbb{E}_{(y,k)}^{\check{Y}} \left[\int_0^h f(Y'_h) \check{\alpha}(Y'_u, k) du \right]. \end{aligned}$$

Using this result and the expression of $\mathcal{A}f(y)$ from Theorem 14, we can write

$$\begin{aligned} & \frac{1}{h} \left(\check{Q}_h(f \times \mathbf{1}_{\mathbb{N}})(y, k) - f(y) \mathbf{1}_{\mathbb{N}}(k) \right) - \mathcal{A}f(y) \\ &= \frac{1}{h} \left(\check{Q}_h(f \times \mathbf{1}_{\mathbb{N}})(y, k) - f(y) \mathbf{1}_{\mathbb{N}}(k) \right) - \mathcal{A}^Y f(y) + \alpha(y) f(y) - \alpha(y) K f(y) \\ &= A_1(h) + A_2(h) + A_3(h) + A_4(h) + A_5(h), \end{aligned}$$

where

$$\begin{aligned} A_1(h) &= \mathbb{E}_y^Y \left[\frac{f(Y_h) - f(y)}{h} \right] - \mathcal{A}^Y f(y), \\ A_2(h) &= \check{\alpha}(y, k) f(y) - \frac{1}{h} \mathbb{E}_{(y,k)}^{\check{Y}} \left[\int_0^h f(Y'_h) \check{\alpha}(Y'_u, k) du \right], \\ A_3(h) &= \frac{1}{h} \mathbb{E}_{(y,k)}^{\check{Y}} \left[f(Y'_h) \left(e^{-\int_0^h \check{\alpha}(Y'_u, k) du} - 1 + \int_0^h \check{\alpha}(Y'_u, k) du \right) \right], \\ A_4(h) &= \frac{1}{h} \mathbb{E}_{(y,k)} \left[f(X'_h) \mathbf{1}_{\check{N}_h = 1} \right] - \alpha(y) K f(y) + (\alpha(y) - \check{\alpha}(y, k)) f(y), \\ A_5(h) &= \frac{1}{h} \mathbb{E}_{(y,k)} \left[f(X'_h) \mathbf{1}_{\check{T}_2 < h} \right]. \end{aligned}$$

The end of the proof consists in proving that each of these five terms tends uniformly to 0 as $h \searrow 0$.

For the first one, note that

$$\mathbb{E}_y^Y \left[\frac{f(Y_h) - f(y)}{h} \right] = \frac{Q_h^Y f(y) - f(y)}{h},$$

and since $f \in \mathcal{D}_A^Y$, by the definition of \mathcal{A}^Y , $\sup_{(y,k) \in \check{E}} |A_1(h)|$ tends to 0 as $h \rightarrow 0$.

To show that $\sup_{(y,k) \in \check{E}} |A_2(h)| \xrightarrow{h \searrow 0} 0$, we consider two cases whether $y \notin E_k$ or $y \in E_k$. First suppose that $y \notin E_k$. Then $\check{\alpha}(y, k) = \alpha(y) + \alpha_k$ and

$$\begin{aligned} A_2(h) &= \frac{1}{h} \int_0^h \mathbb{E}_{(y,k)}^{\check{Y}} \left[\check{\alpha}(y, k) f(y) - f(Y'_h) \check{\alpha}(Y'_u, k) \right] du \\ &= \alpha_k \mathbb{E}_y^Y [f(y) - f(Y_h)] + \frac{1}{h} \int_0^h \mathbb{E}_y^Y [\alpha(y) f(y) - f(Y_h) \alpha(Y_u)] du \end{aligned}$$

where the switch from $\mathbb{E}_{(y,k)}^{\check{Y}}$ to \mathbb{E}_y^Y is a consequence of (3.11), specifically the bivariate generalization of it. Therefore,

$$\begin{aligned} A_2(h) &= \alpha_k (f(y) - Q_h^Y f(y)) + \frac{1}{h} \int_0^h \mathbb{E}_y^Y \left[\mathbb{E}_y^Y [\alpha(y) f(y) - f(Y_h) \alpha(Y_u) | Y_u] \right] du \\ &= \alpha_k (f(y) - Q_h^Y f(y)) + \frac{1}{h} \int_0^h \mathbb{E}_y^Y [\alpha(y) f(y) - Q_{h-u}^Y f(Y_u) \alpha(Y_u)] du \\ &= \alpha_k (f(y) - Q_h^Y f(y)) + \frac{1}{h} \int_0^h \mathbb{E}_y^Y [\alpha(y) f(y) - f(Y_u) \alpha(Y_u)] du \\ &\quad + \frac{1}{h} \int_0^h \mathbb{E}_y^Y [f(Y_u) \alpha(Y_u) - Q_{h-u}^Y f(Y_u) \alpha(Y_u)] du \\ &= \alpha_k (f(y) - Q_h^Y f(y)) + \frac{1}{h} \int_0^h (f \times \alpha - Q_u^Y (f \times \alpha)) (y) du \\ &\quad + \frac{1}{h} \int_0^h \mathbb{E}_y^Y [\alpha(Y_u) (f - Q_{h-u}^Y f) (Y_u)] du \\ &= \alpha_k (f(y) - Q_h^Y f(y)) + \int_0^1 (f \times \alpha - Q_{hv}^Y (f \times \alpha)) (y) dv \\ &\quad + \int_0^1 \mathbb{E}_y^Y [\alpha(Y_{hv}) (f - Q_{h(1-v)}^Y f) (Y_{hv})] dv. \end{aligned}$$

So when $y \notin E_k$,

$$|A_2(h)| \leq \alpha^* \|Q_h^Y f - f\|_\infty + \int_0^1 \|Q_{hv}^Y (f \times \alpha) - f \times \alpha\|_\infty dv + \alpha^* \int_0^1 \|Q_{h(1-v)}^Y f - f\|_\infty dv, \quad (3.27)$$

that does not depend on $(y, k) \in \check{E}$ and converges to zero when h goes to 0 by the dominated convergence theorem because $f \in \mathcal{D}_A^Y \subset C_0(E)$. When $y \in E_k$, $\check{\alpha}(y, k) = \beta_k + \delta(y)$ and we obtain with the same computations the same inequality (3.27), leading to the same convergence. So $\sup_{(y,k) \in \check{E}} |A_2(h)| \xrightarrow{h \searrow 0} 0$.

Regarding $A_3(h)$, its uniform convergence towards 0 is easily obtained from

$$|A_3(h)| \leq \frac{\|f\|_\infty}{2h} \mathbb{E}_{(y,k)}^{\check{Y}} \left[\left(\int_0^h \check{\alpha}(Y'_u, k) du \right)^2 \right] \leq \frac{\|f\|_\infty (2\alpha^* h)^2}{2h} = 2h \|f\|_\infty (\alpha^*)^2.$$

Let us now prove that $\sup_{(y,k) \in \check{E}} |A_4(h)| \xrightarrow{h \searrow 0} 0$. We compute

$$\begin{aligned}
& \frac{1}{h} \mathbb{E}_{(y,k)} \left[f(X'_h) \mathbf{1}_{\check{N}_t=1} \right] = \frac{1}{h} \mathbb{E}_{(y,k)} \left[f(X'_h) \mathbf{1}_{\check{\tau}_1 \leq h} \mathbf{1}_{\check{\tau}_2 > h - \check{\tau}_1} \right] \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbb{E}_{(y,k)} \left[f(X'_h) \mathbf{1}_{\check{\tau}_1 \leq h} \mathbf{1}_{\check{\tau}_2 > h - \check{\tau}_1} \mid \check{\mathcal{F}}_{\check{\tau}_1}, \check{Y}^{(1)} \right] \right] \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[f(Y'_{h-\check{\tau}_1}) \mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{P}_{(y,k)} \left(\check{\tau}_2 > h - \check{\tau}_1 \mid \check{\mathcal{F}}_{\check{\tau}_1}, \check{Y}^{(1)} \right) \right] \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[f(Y'_{h-\check{\tau}_1}) \mathbf{1}_{\check{\tau}_1 \leq h} e^{-\int_0^{h-\check{\tau}_1} \check{\alpha}(\check{Y}_u^{(1)}) du} \right] \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{E}_{(y,k)} \left[f(Y'_{h-\check{\tau}_1}) e^{-\int_0^{h-\check{\tau}_1} \check{\alpha}(\check{Y}_u^{(1)}) du} \mid \check{\mathcal{F}}_{\check{\tau}_1} \right] \right] \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} \left[f(Y'_{h-\check{\tau}_1}) e^{-\int_0^{h-\check{\tau}_1} \check{\alpha}(\check{Y}_u) du} \right] \right] \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} \left[f(Y'_{h-\check{\tau}_1}) \right] \right] \\
&\quad + \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} \left[f(Y'_{h-\check{\tau}_1}) \left(e^{-\int_0^{h-\check{\tau}_1} \check{\alpha}(\check{Y}_u) du} - 1 \right) \right] \right].
\end{aligned}$$

The second term converges uniformly to 0 because its norm is bounded by

$$\mathbb{E}_{(y,k)} \left[\frac{\mathbf{1}_{\check{\tau}_1 \leq h} \|f\|_{\infty}}{h} \mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} \left| \int_0^{h-\check{\tau}_1} \check{\alpha}(\check{Y}_u) du \right| \right] \leq \frac{2h\alpha^* \|f\|_{\infty}}{h} \mathbb{P}_{(y,k)}(\check{\tau}_1 \leq h) \leq 4h(\alpha^*)^2 \|f\|_{\infty}.$$

Let us prove that the first term converges uniformly to $\alpha(y)Kf(y) - (\alpha(y) - \check{\alpha}(y,k))f(y)$, proving that $\sup_{(y,k) \in \check{E}} |A_4(h)| \xrightarrow{h \searrow 0} 0$. We have

$$\begin{aligned}
& \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} \left[f(Y'_{h-\check{\tau}_1}) \right] \right] \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{E}_{(y,k)} \left[\mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} \left[f(Y'_{h-\check{\tau}_1}) \right] \mid \check{Y}^{(0)}, \check{\tau}_1 \right] \right] \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \int_{z_1 \in E} \sum_{q \geq 0} \mathbb{E}_{(z_1, q)}^{\check{Y}} \left[f(Y'_{h-\check{\tau}_1}) \right] \check{K} \left((Y'_{\check{\tau}_1}^{(0)}, k); dz_1 \times \{q\} \right) \right].
\end{aligned}$$

We separate as before the cases $y \notin E_k$ and $y \in E_k$. If $y \notin E_k$, we obtain

$$\begin{aligned}
& \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} \left[f(Y'_{h-\check{\tau}_1}) \right] \right] \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \frac{\alpha(Y'_{\check{\tau}_1}^{(0)})}{\check{\alpha}(Y'_{\check{\tau}_1}^{(0)}, k)} \int_E \mathbb{E}_{(z_1, k)}^{\check{Y}} \left[f(Y'_{h-\check{\tau}_1}) \right] K(Y'_{\check{\tau}_1}^{(0)}, dz_1) \right] \\
&\quad + \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \frac{\beta_k}{\check{\alpha}(Y'_{\check{\tau}_1}^{(0)}, k)} \mathbb{E}_{(Y'_{\check{\tau}_1}^{(0)}, k+1)}^{\check{Y}} \left[f(Y'_{h-\check{\tau}_1}) \right] \right] \\
&\quad + \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \frac{\delta_k}{\check{\alpha}(Y'_{\check{\tau}_1}^{(0)}, k)} \mathbb{E}_{(Y'_{\check{\tau}_1}^{(0)}, k-1)}^{\check{Y}} \left[f(Y'_{h-\check{\tau}_1}) \right] \right]. \tag{3.28}
\end{aligned}$$

Let us show that the first term in (3.28) converges to $\alpha(y)Kf(y)$:

$$\begin{aligned}
& \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \frac{\alpha(Y'_{\tilde{\tau}_1})}{\check{\alpha}(Y'_{\tilde{\tau}_1}, k)} \int_E \mathbb{E}_{(z_1, k)}^{\check{Y}} [f(Y'_{h-\tilde{\tau}_1})] K(Y'_{\tilde{\tau}_1}, dz_1) \right] - \alpha(y)Kf(y) \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \frac{\alpha(Y'_{\tilde{\tau}_1})}{\check{\alpha}(Y'_{\tilde{\tau}_1}, k)} \int_E Q_{h-\tilde{\tau}_1}^{\check{Y}}(f \times \mathbf{1}_{\mathbb{N}})(z_1, k) K(Y'_{\tilde{\tau}_1}, dz_1) \right] - \alpha(y)Kf(y) \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \frac{\alpha(Y'_{\tilde{\tau}_1})}{\check{\alpha}(Y'_{\tilde{\tau}_1}, k)} \int_E Q_{h-\tilde{\tau}_1}^{\check{Y}}(f \times \mathbf{1}_{\mathbb{N}})(z_1, k) K(Y'_{\tilde{\tau}_1}, dz_1) \middle| \check{Y}^{(0)} \right] \right] \\
&\quad - \alpha(y)Kf(y) \\
&= \frac{1}{h} \mathbb{E}_{(y,k)}^{\check{Y}} \left[\int_0^h \alpha(Y'_s) \int_E Q_{h-s}^Y f(z_1) K(Y'_s, dz_1) e^{-\int_0^s \check{\alpha}(Y'_u, k) du} ds \right] - \alpha(y)Kf(y) \\
&= \mathbb{E}_{(y,k)}^{\check{Y}} \left[\int_0^1 \alpha(Y'_{hv}) \int_E Q_{h(1-v)}^Y f(z_1) K(Y'_{hv}, dz_1) e^{-\int_0^{hv} \check{\alpha}(Y'_u, k) du} dv \right] - \alpha(y)Kf(y) \\
&= \mathbb{E}_{(y,k)}^{\check{Y}} \left[\int_0^1 \alpha(Y'_{hv}) \int_E Q_{h(1-v)}^Y f(z_1) K(Y'_{hv}, dz_1) dv \right] - \alpha(y)Kf(y) \\
&\quad + \mathbb{E}_{(y,k)}^{\check{Y}} \left[\int_0^1 \alpha(Y'_{hv}) \int_E Q_{h(1-v)}^Y f(z_1) K(Y'_{hv}, dz_1) \left(e^{-\int_0^{hv} \check{\alpha}(Y'_u, k) du} - 1 \right) dv \right]
\end{aligned}$$

But for the last term,

$$\begin{aligned}
& \left| \mathbb{E}_{(y,k)}^{\check{Y}} \left[\int_0^1 \alpha(Y'_{hv}) \int_E Q_{h(1-v)}^Y f(z_1) K(Y'_{hv}, dz_1) \left(e^{-\int_0^{hv} \check{\alpha}(Y'_u, k) du} - 1 \right) dv \right] \right| \\
&\leq \alpha^* \|f\|_{\infty} \mathbb{E}_{(y,k)}^{\check{Y}} \left[\int_0^1 \int_0^{hv} \check{\alpha}(Y'_u, k) du dv \right] \leq 2h(\alpha^*)^2 \|f\|_{\infty}
\end{aligned}$$

and from (the bivariate version of) (3.11), we can replace $\mathbb{E}_{(y,k)}^{\check{Y}}$ by \mathbb{E}_y^Y in the other term to get

$$\begin{aligned}
& \left| \mathbb{E}_y^Y \left[\int_0^1 \alpha(Y_{hv}) \int_E Q_{h(1-v)}^Y f(z_1) K(Y_{hv}, dz_1) dv \right] - \alpha(y)Kf(y) \right| \\
&\leq \int_0^1 \left| \mathbb{E}_y^Y \left[\alpha(Y_{hv}) K Q_{h(1-v)}^Y f(Y_{hv}) - \alpha(Y_{hv}) Kf(Y_{hv}) \right] \right| dv \\
&\quad + \int_0^1 \left| \mathbb{E}_y^Y \left[\alpha(Y_{hv}) Kf(Y_{hv}) - \alpha(y)Kf(y) \right] \right| dv \\
&\leq \alpha^* \int_0^1 \|K Q_{h(1-v)}^Y f - Kf\|_{\infty} dv + \int_0^1 \left| Q_{hv}^Y(\alpha \times Kf)(y) - (\alpha \times Kf)(y) \right| dv \\
&\leq \alpha^* \int_0^1 \|Q_{h(1-v)}^Y f - f\|_{\infty} dv + \int_0^1 \|Q_{hv}^Y(\alpha \times Kf) - (\alpha \times Kf)\|_{\infty} dv,
\end{aligned}$$

converges to 0 when h goes to 0 by the dominated convergence theorem, using the fact that $f \in \mathcal{D}_A^Y \subset C_0(E)$ and $KC_0(E) \subset C_0(E)$. This proves the convergence to $\alpha(y)Kf(y)$ of the first term in (3.28). Concerning the second and third terms in (3.28), their sum converges to $(\beta_k + \delta_k)f(y) = (\check{\alpha}(y, k) - \alpha(y))f(y)$. Indeed for any

q ,

$$\begin{aligned}
& \frac{1}{h} \left| \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \frac{1}{\check{\alpha}(Y'_{\check{\tau}_1(0)}, k)} \mathbb{E}_{(Y'_{\check{\tau}_1(0)}, q)}^{\check{Y}} [f(Y'_{h-\check{\tau}_1})] \right] - f(y) \right| \\
&= \frac{1}{h} \left| \int_0^h \mathbb{E}_{(y,k)}^{\check{Y}} \left[\mathbb{E}_{Y'_s}^Y [f(Y_{h-s})] e^{-\int_0^s \check{\alpha}(Y'_u, k) du} \right] ds - f(y) \right| \\
&= \left| \int_0^1 \mathbb{E}_{(y,k)}^{\check{Y}} \left[\mathbb{E}_{Y'_{hv}}^Y [f(Y_{h(1-v)})] e^{-\int_0^{hv} \check{\alpha}(Y'_u, k) du} \right] dv - f(y) \right| \\
&\leq \left| \int_0^1 \mathbb{E}_{(y,k)}^{\check{Y}} \left[\mathbb{E}_{Y'_{hv}}^Y [f(Y_{h(1-v)})] \left(e^{-\int_0^{hv} \check{\alpha}(Y'_u, k) du} - 1 \right) \right] dv \right| \\
&\quad + \left| \int_0^1 \mathbb{E}_y^Y \left[\mathbb{E}_{Y'_{hv}}^Y [f(Y_{h(1-v)})] \right] dv - f(y) \right| \\
&\leq \int_0^1 \|f\|_\infty \mathbb{E}_{(y,k)}^{\check{Y}} \left[\left| \int_0^{hv} \check{\alpha}(Y'_u, k) du \right| \right] dv + \int_0^1 \left| \mathbb{E}_y^Y [Q_{h(1-v)}^Y f(Y_{hv})] - f(y) \right| dv \\
&\leq \alpha^* \|f\|_\infty h + \int_0^1 |Q_h^Y f(y) - f(y)| dv \\
&\leq \alpha^* \|f\|_\infty h + \|Q_h^Y f - f\|_\infty,
\end{aligned}$$

that converges to 0 when h goes to 0 by the dominated convergence theorem because $f \in \mathcal{D}_A^Y \subset C_0(E)$. This completes the proof of the claimed convergence of $A_4(h)$ when $y \notin E_k$.

Suppose now that $y \in E_k$. The development made in (3.28) becomes in this case

$$\begin{aligned}
& \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \mathbb{E}_{\check{C}_{\check{\tau}_1}}^{\check{Y}} [f(Y_{h-\check{\tau}_1})] \right] \\
&= \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \frac{\beta(Y'_{\check{\tau}_1(0)})}{\check{\alpha}(Y'_{\check{\tau}_1(0)}, k)} \int_E \mathbb{E}_{(z_1, k+1)}^{\check{Y}} [f(Y'_{h-\check{\tau}_1})] K_\beta(Y'_{\check{\tau}_1(0)}, dz_1) \right] \\
&\quad + \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \frac{\beta_k - \beta(Y'_{\check{\tau}_1(0)})}{\check{\alpha}(Y'_{\check{\tau}_1(0)}, k)} \mathbb{E}_{(Y'_{\check{\tau}_1(0)}, k+1)}^{\check{Y}} [f(Y'_{h-\check{\tau}_1})] \right] \\
&\quad + \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \frac{\delta_k}{\check{\alpha}(Y'_{\check{\tau}_1(0)}, k)} \int_E \mathbb{E}_{(Y'_{\check{\tau}_1(0)}, k-1)}^{\check{Y}} [f(Y'_{h-\check{\tau}_1})] K_\delta(Y'_{\check{\tau}_1(0)}, dz_1) \right] \\
&\quad + \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\check{\tau}_1 \leq h} \frac{\delta(Y'_{\check{\tau}_1(0)}) - \delta_k}{\check{\alpha}(Y'_{\check{\tau}_1(0)}, k)} \int_E \mathbb{E}_{(Y'_{\check{\tau}_1(0)}, k)}^{\check{Y}} [f(Y'_{h-\check{\tau}_1})] K_\delta(Y'_{\check{\tau}_1(0)}, dz_1) \right].
\end{aligned} \tag{3.29}$$

The first, third and fourth terms above can be treated exactly as the first term in (3.28) to prove that they converge uniformly towards $\beta(y)K_\beta f(y)$, $\delta_k K_\delta f(y)$ and $(\delta(y) - \delta_k)K_\delta f(y)$, respectively, the sum of which is $\alpha(y)Kf(y)$. For the second

term, we compute

$$\begin{aligned}
& \left| \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \frac{\beta(Y'_{\tilde{\tau}_1})}{\check{\alpha}(Y'_{\tilde{\tau}_1}, k)} \mathbb{E}_{(Y'_{\tilde{\tau}_1}, k+1)}^{\check{Y}} \left[f(Y'_{h-\tilde{\tau}_1}) \right] \right] - \beta(y)f(y) \right| \\
&= \left| \frac{1}{h} \mathbb{E}_{(y,k)} \left[\mathbb{E}_{(y,k)} \left[\mathbf{1}_{\tilde{\tau}_1 \leq h} \frac{\beta(Y'_{\tilde{\tau}_1})}{\check{\alpha}(Y'_{\tilde{\tau}_1}, k)} \mathbb{E}_{(Y'_{\tilde{\tau}_1}, k+1)}^{\check{Y}} \left[f(Y'_{h-\tilde{\tau}_1}) \right] \mid \check{Y}^{(0)} \right] \right] - \beta(y)f(y) \right| \\
&= \left| \frac{1}{h} \int_0^h \mathbb{E}_{(y,k)}^{\check{Y}} \left[\beta(Y'_s) \mathbb{E}_{Y'_s}^Y \left[f(Y_{h-s}) \right] e^{-\int_0^s \check{\alpha}(Y'_u, k+1) du} \right] ds - \beta(y)f(y) \right| \\
&= \left| \int_0^1 \mathbb{E}_{(y,k)}^{\check{Y}} \left[\beta(Y'_{hv}) \mathbb{E}_{Y'_{hv}}^Y \left[f(Y_{h(1-v)}) \right] e^{-\int_0^{hv} \check{\alpha}(Y'_u, k+1) du} \right] dv - \beta(y)f(y) \right| \\
&\leq \left| \int_0^1 \mathbb{E}_{(y,k)}^{\check{Y}} \left[\beta(Y'_{hv}) \mathbb{E}_{Y'_{hv}}^Y \left[f(Y_{h(1-v)}) \right] \left(e^{-\int_0^{hv} \check{\alpha}(Y'_u, k+1) du} - 1 \right) \right] dv \right| \\
&\quad + \left| \int_0^1 \mathbb{E}_y^Y \left[\beta(Y_{hv}) \mathbb{E}_{Y_{hv}}^Y \left[f(Y_{h(1-v)}) \right] \right] dv - \beta(y)f(y) \right| \\
&\leq \alpha^* \|f\|_\infty \int_0^1 \int_0^{hv} \mathbb{E}_y^Y \left[\check{\alpha}(Y_u, k+1) \right] du dv \\
&\quad + \left| \int_0^1 \mathbb{E}_y^Y \left[\beta(Y_{hv}) Q_{h(1-v)}^Y f(Y_{hv}) \right] dv - \beta(y)f(y) \right| \\
&\leq (\alpha^*)^2 \|f\|_\infty h + \left| \int_0^1 \mathbb{E}_y^Y \left[\beta(Y_{hv}) Q_{h(1-v)}^Y f(Y_{hv}) - \beta(Y_{hv})f(Y_{hv}) \right] dv \right| \\
&\quad + \left| \int_0^1 \mathbb{E}_y^Y \left[\beta(Y_{hv}) f(Y_{hv}) \right] dv - \beta(y)f(y) \right| \\
&\leq (\alpha^*)^2 \|f\|_\infty h + \alpha^* \int_0^1 \|Q_{h(1-v)}^Y f - f\|_\infty dv + \int_0^1 \|Q_{hv}^Y(\beta \times f) - (\beta \times f)\|_\infty dv,
\end{aligned}$$

that converges to 0 when h goes to 0 by the dominated convergence theorem because $f \in \mathcal{D}_A^Y \subset C_0(E)$. Given this result and the convergence already proven for the second term in (3.28), we deduce that the second term in (3.29) converges uniformly in (y, k) to $(\beta_k - \beta(y))f(y) = (\check{\alpha}(y, k) - \alpha(y))f(y)$.

The study for $y \notin E_k$ and $y \in E_k$ yields to the same convergence results, so in conclusion

$$\sup_{(y,k) \in E \times \mathbb{N}} \left| \mathbb{E}_{(y,k)} \left(f(X'_h) \mathbf{1}_{\check{N}_h=1} \right) + (\alpha(y) - \check{\alpha}(y, k))f(y) - \alpha(y)Kf(y) \right| \xrightarrow{h \searrow 0} 0,$$

that is $\sup_{(y,k) \in \check{E}} |A_4(h)| \xrightarrow{h \searrow 0} 0$.

To finish the proof, it remains to handle $A_5(h)$ using (2.3) where $\check{N}_h^* \sim \mathcal{P}(2\alpha^*h)$

$$\begin{aligned}
|A_5(h)| &\leq \frac{\|f\|_\infty}{h} \mathbb{P}_{(y,k)} \left(\check{N}_h \geq 2 \right) \\
&\leq \frac{\|f\|_\infty}{h} \mathbb{P} \left(\check{N}_h^* \geq 2 \right) \\
&\leq \frac{\|f\|_\infty}{h} \left(1 - e^{-2\alpha^*h} - 2\alpha^*h e^{-2\alpha^*h} \right) \\
&= \frac{\|f\|_\infty}{h} \left(\frac{(2\alpha^*h)^2}{2} + o_{h \searrow 0}(h^2) \right) \\
&= 2\|f\|_\infty (\alpha^*)^2 h + o_{h \searrow 0}(h),
\end{aligned}$$

that converges uniformly to 0 when h goes to 0.

Proof of Corollary 49

Let

$$G(t) = \psi_f(t) - f(x) - \int_0^t \psi_{\mathcal{A}f}(s) ds.$$

This function is continuous, right-differentiable on \mathbb{R}_+ from Lemmas 47 and 48, and $\partial_+ G(t)/\partial t = 0$. So G is constant. But $G(0) = 0$ because $s \geq 0 \mapsto \psi_{\mathcal{A}f}(s)$ is bounded. As a consequence, we obtain (3.12). Moreover $\mathcal{A}f \in C_0(E)$ so by Lemma 47 the function $s \geq 0 \mapsto \psi_{\mathcal{A}f}(s)$ is continuous, and by (3.12) we deduce that ψ_f is differentiable.

3.3.3 Proof of Theorem 38

We recall and complete notations introduced in Section 3.3.1 regarding the coupling between X and η . The coupled process is $\check{C} = (X', \eta')$, where from Theorem 37, X' and η' have the same distributions as X and η . We denote by T_j and t_j the jump times of X and η . Similarly we denote by T'_j and t'_j the jump times of X' and η' . To prove Theorem 38, we start with the following lemma where $s_0 := \inf\{t \geq t_1, \eta_t = 0\}$ is the time of the first return of η in the state 0 and $S_\emptyset := \inf\{t \geq T_1, X_t = \emptyset\}$ is the time of the first return of $(X_t)_{t \geq 0}$ in the state \emptyset .

Lemma 50. *Suppose that 0 is an ergodic state for the simple process η , that is $\mathbb{E}_0(s_0) < \infty$. Then $\lim_{t \rightarrow \infty} Q_t(\emptyset, A)$ exists for all $A \in \mathcal{E}$. Suppose moreover that for all $n \geq 0$, $\mathbb{E}_n(s_0) < \infty$. Then, $\lim_{t \rightarrow \infty} Q_t(x, A)$ exists for all $x \in E$, $A \in \mathcal{E}$, and is independent of x .*

Proof. Let $\check{s}_0 := \inf\{t \geq t'_1, \check{C}_t \in E \times \{0\}\}$. Using the first statement of Theorem 37, we can prove that $\mathbb{P}_{(\emptyset, 0)}(\check{s}_0 > t) = \mathbb{P}_0(s_0 > t)$. Similarly, by the second statement of this theorem, $\mathbb{P}_{(\emptyset, 0)}(\check{S}_\emptyset > t) = \mathbb{P}_\emptyset(S_\emptyset > t)$ where $\check{S}_\emptyset := \inf\{t \geq T'_1, \check{C}_t \in \{\emptyset\} \times \mathbb{N}\}$. We thus have

$$\mathbb{P}_\emptyset(S_\emptyset > t) = \mathbb{P}_{(\emptyset, 0)}(\check{S}_\emptyset > t) \leq \mathbb{P}_{(\emptyset, 0)}(\check{s}_0 > t) = \mathbb{P}_0(s_0 > t),$$

where the inequality comes from the third statement of Theorem 37.

By the assumptions of Lemma 50, this implies that $S_\emptyset < \infty$ $\mathbb{P}_\emptyset - a.s.$ and that

$$\mathbb{E}_\emptyset(S_\emptyset) = \int_0^\infty \mathbb{P}_\emptyset(S_\emptyset > t) dt \leq \int_0^\infty \mathbb{P}_0(s_0 > t) dt < \infty,$$

proving that \emptyset is an ergodic state for the process $(X_t)_{t \geq 0}$. Note moreover that S_\emptyset has a density with respect to the Lebesgue measure, that we denote by μ_\emptyset . This comes from the fact that τ_j has a density for any j , so does T_j , whereby given a Lebesgue null set $I \in \mathcal{B}(\mathbb{R})$, $\mathbb{P}_\emptyset(S_\emptyset \in I) \leq \sum_{j=1}^\infty \mathbb{P}_\emptyset(T_j \in I) = 0$.

We have the following equation

$$\begin{aligned} Q_t(\emptyset, A) &= \mathbb{P}_\emptyset(X_t \in A, S_\emptyset > t) + \int_0^t \mathbb{P}_\emptyset(X_t \in A, S_\emptyset \in ds) \\ &= \mathbb{P}_\emptyset(X_t \in A, S_\emptyset > t) + \int_0^t \mathbb{P}_\emptyset(X_t \in A | S_\emptyset = s) \mu_\emptyset(s) ds \\ &= \mathbb{P}_\emptyset(X_t \in A, S_\emptyset > t) + \int_0^t Q_{t-s}(\emptyset, A) \mu_\emptyset(s) ds. \end{aligned}$$

This is a renewal equation and we may apply the renewal theorem given in [Fel71, Chapter XI]. To this end, denote by $\mathcal{Z}(t) = Q_t(\emptyset, A)$, $\xi(t) = \mathbb{P}_\emptyset(X_t \in A, S_\emptyset > t)$ and $F\{I\} = \mathbb{P}_\emptyset(S_\emptyset \in I)$. Remark that \mathcal{Z} is bounded, ξ is non-negative, bounded by 1 and directly Riemann integrable on \mathbb{R}_+ because it is dominated by the monotone integrable function $t \mapsto \mathbb{P}_\emptyset(S_\emptyset > t)$. Moreover, $0 < \mathbb{E}_\emptyset(S_\emptyset) < \infty$ and since S_\emptyset has a density, F is not arithmetic. Then, by the renewal theorem, we obtain:

$$Q_t(\emptyset, A) = \mathcal{Z}(t) \xrightarrow{t \rightarrow \infty} \frac{1}{\mathbb{E}_\emptyset(S_\emptyset)} \int_0^\infty \xi(u) du = \frac{1}{\mathbb{E}_\emptyset(S_\emptyset)} \int_0^\infty \mathbb{P}_\emptyset(X_u \in A, S_\emptyset > u) du \quad (3.30)$$

which proves the first statement of Lemma 50.

Let now turn to the second part of Lemma 50. Let $x \in E_n$. By the arguments as in the beginning of the proof, we get that $S_\emptyset < \infty$, $\mathbb{P}_x - a.s.$ and that $\mathbb{E}_x(S_\emptyset) \leq \mathbb{E}_n(s_0) < \infty$. We have

$$\begin{aligned} Q_t(x, A) &= \mathbb{P}_x(X_t \in A) \\ &= \mathbb{P}_x(X_t \in A, S_\emptyset > t) + \int_0^t \mathbb{P}_x(X_t \in A | S_\emptyset = s) \mu_\emptyset(s) ds \\ &= \mathbb{P}_x(X_t \in A, S_\emptyset > t) + \int_0^t \mathbb{P}_\emptyset(X_{t-s} \in A) \mu_\emptyset(s) ds \\ &= \mathbb{P}_x(X_t \in A, S_\emptyset > t) + \int_0^t Q_{t-s}(\emptyset, A) \mu_\emptyset(s) ds. \end{aligned}$$

The first term tends to 0 as $t \rightarrow \infty$ because it is dominated by $\mathbb{P}_x(S_\emptyset > t)$ and we know that $\mathbb{P}_x(S_\emptyset < \infty) = 1$. For the second term, for all $s \geq 0$, we have by (3.30)

$$Q_{t-s}(\emptyset, A) \mathbf{1}_{[0,t]}(s) \xrightarrow{t \rightarrow \infty} \frac{1}{\mathbb{E}_\emptyset(S_\emptyset)} \int_0^\infty \mathbb{P}_\emptyset(X_u \in A, S_\emptyset > u) du.$$

Moreover $|Q_{t-s}(\emptyset, A) \mathbf{1}_{[0,t]}(s) \mu_\emptyset(s)| \leq \mu_\emptyset(s)$ which is integrable. So by the dominated convergence theorem,

$$Q_t(x, A) \xrightarrow{t \rightarrow \infty} \frac{1}{\mathbb{E}_\emptyset(S_\emptyset)} \int_0^\infty \mathbb{P}_\emptyset(X_u \in A, S_\emptyset > u) du$$

which is independent of x . □

We are now in position to prove Theorem 38. The conditions (3.2) or (3.3) of [KM57] imply the assumptions made in Lemma 50. We then deduce that $\mu(A) := \lim_{t \rightarrow \infty} Q_t(x, A)$ exists for all $x \in E$ and $A \in \mathcal{E}$, and is independent of x . It is a probability measure because for any $t \geq 0$ and $x \in E$, $Q_t(x, \cdot)$ is a probability measure.

Let us prove that μ is an invariant measure. The previous convergence reads

$$\int_E f(y)Q_s(x, dy) \xrightarrow{s \rightarrow \infty} \int_E f(y)\mu(dy). \quad (3.31)$$

where $f = \mathbf{1}_A$ with $A \in \mathcal{E}$. It is not difficult to extend it to any step function and by limiting arguments to any $f \in \mathcal{M}_b^+(E)$. By the Markov property, for all $t, s \geq$, $x \in E$ and $A \in \mathcal{E}$,

$$Q_{t+s}(x, A) = \int_E Q_t(y, A) Q_s(x, dy).$$

Letting s tend to ∞ , we obtain that the left hand side converges to $\mu(A)$, while for the right hand side, we may apply (3.31) to $f = Q_t(\cdot, A) \in \mathcal{M}_b^+(E)$ to finally obtain

$$\mu(A) = \int_E Q_t(y, A)\mu(dy).$$

Finally, if ν is a probability measure on E , such that for any $A \in \mathcal{E}$

$$\nu(A) = \int_E Q_t(y, A)\nu(dy),$$

then as $Q_t(x, A) \leq 1$, taking $t \rightarrow \infty$, we get by the dominated convergence theorem

$$\nu(A) = \int_E \mu(A)\nu(dy) = \mu(A).$$

Hence μ is the unique invariant probability measure.

3.3.4 Proofs of Section 3.1.2

Elements of the proof of Theorem 39

We show in the following lemma that the proof of Theorem 39 boils down to investigating the waiting time until two independent simple processes are both at the state 0. Moreover [Møl89] shows that under the conditions of Theorem 39 the tail of this waiting time is subgeometric. Proving the next lemma then proves Theorem 39. As in the proof of Theorem 38 we use the coupling of birth-death-move processes introduced in Section 3.3.1.

Lemma 51. *Let $(X_t^{(1)})_{t \geq 0}$ and $(X_t^{(2)})_{t \geq 0}$ be two identical birth-death-move processes on E such that the jump transition kernel satisfies $KC_0(E) \subset C_0(E)$ and that the associated move process is a Feller process. The initial distribution of $(X_t^{(1)})_{t \geq 0}$ is denoted by μ_1 and the initial distribution of $(X_t^{(2)})_{t \geq 0}$ is denoted by μ_2 . We also denote $(\eta_t^{(1)})_{t \geq 0}$ and $(\eta_t^{(2)})_{t \geq 0}$ two simple birth-death processes associated with birth and death rates defined by (3.1) and with initial distributions ν_1 and ν_2 given by $\nu_i(n) = \mu_i(E_n)$ for any $n \geq 0$ and $i = 1, 2$. Let us define the stopping times*

$$T = \inf\{t \geq 0, X_t^{(1)} = X_t^{(2)} = \emptyset\},$$

and

$$\tau = \inf\{t \geq 0, \eta_t^{(1)} = \eta_t^{(2)} = 0\}.$$

Finally we denote by $\mathbb{P}_{\mu_1 \times \mu_2}$ the probability measure under which the processes $(X_t^{(1)})_{t \geq 0}$ and $(X_t^{(2)})_{t \geq 0}$ are independent and $\mathbb{P}_{\nu_1 \times \nu_2}$ the probability measure under which the processes $(\eta_t^{(1)})_{t \geq 0}$ and $(\eta_t^{(2)})_{t \geq 0}$ are independent. Then we get for all $A \in \mathcal{E}$:

$$\left| \int_E Q_t(x_1, A) d\mu_1(x_1) - \int_E Q_t(x_2, A) d\mu_2(x_2) \right| \leq 2\mathbb{P}_{\mu_1 \times \mu_2}(T > t) \leq 2\mathbb{P}_{\nu_1 \times \nu_2}(\tau > t).$$

Proof of Lemma 51. The process defined for any $t \geq 0$ by $Z_t := (X_t^{(1)}, X_t^{(2)})$ is a Feller process by [Kal97], Chapter 19, Exercise 10. Let $t \geq 0$. The stopping time $T \wedge t$ is bounded and the processes $(X_t^{(1)})_{t \geq 0}$ and $(X_t^{(2)})_{t \geq 0}$ are non-explosive Feller processes. This entails by [RY91], Chapter 3, for any $A \in \mathcal{E}$:

$$\begin{aligned} \mathbb{P}_{\mu_1 \times \mu_2}(X_t^{(1)} \in A, T \leq t) &= \mathbb{E}_{\mu_1 \times \mu_2}[\mathbb{E}_{\mu_1 \times \mu_2}[\mathbf{1}_A(X_t^{(1)}) | \mathcal{F}_T] \mathbf{1}_{T \leq t}] \\ &= \mathbb{E}_{\mu_1 \times \mu_2}[\mathbb{E}_{\emptyset \times \emptyset}[\mathbf{1}_A(X_{t-T}^{(1)})] \mathbf{1}_{T \leq t}] \\ &= \mathbb{E}_{\mu_1 \times \mu_2}[\mathbb{E}_{\emptyset}[\mathbf{1}_A(X_{t-T}^{(1)})] \mathbf{1}_{T \leq t}] \\ &= \mathbb{E}_{\mu_1 \times \mu_2}[\mathbb{E}_{\emptyset}[\mathbf{1}_A(X_{t-T}^{(2)})] \mathbf{1}_{T \leq t}] \\ &= \mathbb{P}_{\mu_1 \times \mu_2}(X_t^{(2)} \in A, T \leq t). \end{aligned}$$

Then,

$$\begin{aligned} &\left| \int_E Q_t(x_1, A) d\mu_1(x_1) - \int_E Q_t(x_2, A) d\mu_2(x_2) \right| \\ &= \left| \mathbb{P}_{\mu_1 \times \mu_2}(X_t^{(1)} \in A) - \mathbb{P}_{\mu_1 \times \mu_2}(X_t^{(2)} \in A) \right| \\ &= \left| \mathbb{P}_{\mu_1 \times \mu_2}(X_t^{(1)} \in A, T \leq t) + \mathbb{P}_{\mu_1 \times \mu_2}(X_t^{(1)} \in A, T > t) - \mathbb{P}_{\mu_1 \times \mu_2}(X_t^{(2)} \in A, T \leq t) \right. \\ &\quad \left. - \mathbb{P}_{\mu_1 \times \mu_2}(X_t^{(2)} \in A, T > t) \right| \\ &= \left| \mathbb{P}_{\mu_1 \times \mu_2}(X_t^{(1)} \in A, T > t) - \mathbb{P}_{\mu_1 \times \mu_2}(X_t^{(2)} \in A, T > t) \right| \\ &\leq 2\mathbb{P}_{\mu_1 \times \mu_2}(T > t). \end{aligned}$$

For $i = 1, 2$ we denote by $\check{C}_t^{(i)} = (X_t^{(i)}, \eta_t^{(i)})$ the coupling defined in Section 3.3.1. Let us introduce for $i = 1, 2$:

$$\check{T} = \inf\{t \geq 0, \forall i = 1, 2 \check{C}_t^{(i)} \in \{\emptyset\} \times \mathbb{N}\}$$

and

$$\check{\tau} = \inf\{t \geq 0, \forall i = 1, 2 \check{C}_t^{(i)} \in E \times \{0\}\}.$$

We derive from Theorem 37 the following computations:

$$\begin{aligned} \mathbb{P}_{\mu_1 \times \mu_2}(T > t) &= \int_E \int_E \mathbb{P}_{x_1 \times x_2}(T > t) d\mu_1(x_1) d\mu_2(x_2) \\ &\leq \int_E \int_E \mathbb{P}_{(x_1, n(x_1)) \times (x_2, n(x_2))}(\check{T} > t) d\mu_1(x_1) d\mu_2(x_2) \\ &\leq \int_E \int_E \mathbb{P}_{(x_1, n(x_1)) \times (x_2, n(x_2))}(\check{\tau} > t) d\mu_1(x_1) d\mu_2(x_2) \\ &= \int_E \int_E \mathbb{P}_{n(x_1) \times n(x_2)}(\tau > t) d\mu_1(x_1) d\mu_2(x_2) \\ &= \mathbb{P}_{\nu_1 \times \nu_2}(\tau > t). \end{aligned}$$

□

Proof of Corollary 40

Let us suppose that the hypotheses of Theorem 38 hold and we denote by μ the unique invariant probability measure for the birth-death-move process. In this case the simple birth-death process of birth rates $(\beta_n)_{n \geq 0}$ and death rates $(\delta_n)_{n \geq 0}$ defined by (3.1) admits a unique invariant distribution $(\rho\pi_n)_{n \geq 0}$ given by [KM57]:

$$\pi_0 = 1, \quad \pi_n = \frac{\beta_0 \cdots \beta_{n-1}}{\delta_1 \cdots \delta_n} \quad \text{for } n \geq 1,$$

and

$$\rho = \frac{1}{\sum_{n=0}^{\infty} \pi_n}.$$

Moreover the invariant distribution for the simple process dominates the invariant distribution for the birth-death-move process, that is the point of the following proposition.

Lemma 52. *Suppose that the move process $(Y_t)_{t \geq 0}$ is a Feller process and that $KC_0(E) \subset C_0(E)$. Suppose that $\delta_n > 0$ for all $n \geq 1$ and that (3.2) or (3.3) holds. Then for all $m \geq 0$,*

$$\sum_{n=m}^{\infty} \mu(E_n) \leq \rho \sum_{n=m}^{\infty} \pi_n.$$

Proof of Lemma 52. Let us consider the coupled process $\check{X}_t = (X_t, \eta_t)$ defined in Section 3.3.1. Recall that we denote

$$\Gamma = \{(y, m) \in E \times \mathbb{N}; y \in E_k \text{ et } k > m\}.$$

Applying Theorem 37 we get

$$\begin{aligned} \sum_{n=0}^m q_t(j, n) &= \sum_{n=0}^m \mathbb{P}_j(\eta_t = n) \\ &= \mathbb{P}_j(\eta_t \in \llbracket 0, m \rrbracket) \\ &= \mathbb{P}_{(j,x)}(\eta_t \in \llbracket 0, m \rrbracket, X_t \in E) \\ &= \mathbb{P}_{(j,x)}(\eta_t \in \llbracket 0, m \rrbracket, X_t \in E, \check{X}_t \in \Gamma^c) \\ &= \mathbb{P}_{(j,x)}(\eta_t \in \llbracket 0, m \rrbracket, X_t \in \bigcup_{n=0}^m E_n, \check{X}_t \in \Gamma^c) \\ &= \mathbb{P}_{(j,x)}(\eta_t \in \llbracket 0, m \rrbracket, X_t \in \bigcup_{n=0}^m E_n) \\ &\leq \mathbb{P}_{(j,x)}(\eta_t \in \mathbb{N}, X_t \in \bigcup_{n=0}^m E_n) \\ &= \mathbb{P}_x(X_t \in \bigcup_{n=0}^m E_n) \quad (\text{by Theorem 37}) \\ &= \sum_{n=0}^m \mathbb{P}_x(X_t \in E_n) \\ &= \sum_{n=0}^m Q_t(x, E_n). \end{aligned}$$

Taking $t \rightarrow +\infty$ we get for any $m \geq 0$:

$$\sum_{n=0}^m \mu(E_n) \geq \rho \sum_{n=0}^m \pi_n,$$

that is

$$1 - \sum_{n=m}^{\infty} \mu(E_n) \geq 1 - \rho \sum_{n=m}^{\infty} \pi_n,$$

and then

$$\sum_{n=m}^{\infty} \mu(E_n) \leq \rho \sum_{n=m}^{\infty} \pi_n.$$

□

In order to prove Corollary 40 we have to show that under (3.6) the invariant probability measure μ satisfies the condition (3.5).

Set $U_n = \sum_{k=n}^{+\infty} \mu(E_k)$ and $P_n = \sum_{k=n}^{+\infty} \pi_k$. Let us fix $n_0 \in \mathbb{N}$ such that $\left(\frac{1}{\sqrt{\pi_n}}\right)_{n \geq n_0}$ is an increasing sequence. Moreover, as $\pi_n \xrightarrow{n \rightarrow +\infty} 0$ there exists $n_1 \geq 0$ such that for any $n \geq n_1$ one has $\pi_n \leq \sqrt{\pi_n}$. Take $N > \max(n_0, n_1)$. Then:

$$\begin{aligned} \sum_{k=n_0}^N \frac{\mu(E_k)}{\sqrt{\pi_k}} &= \sum_{k=n_0}^N \frac{U_{k-1} - U_k}{\sqrt{\pi_k}} \\ &= \sum_{k=n_0}^N \frac{U_{k-1}}{\sqrt{\pi_k}} - \sum_{k=n_0}^N \frac{U_k}{\sqrt{\pi_k}} \\ &= \sum_{k=n_0-1}^{N-1} \frac{U_k}{\sqrt{\pi_{k+1}}} - \sum_{k=n_0}^N \frac{U_k}{\sqrt{\pi_k}} \\ &= \sum_{k=n_0}^{N-1} U_k \left(\frac{1}{\sqrt{\pi_{k+1}}} - \frac{1}{\sqrt{\pi_k}} \right) - \frac{U_N}{\sqrt{\pi_N}} + \frac{U_{n_0-1}}{\sqrt{\pi_{n_0}}} \end{aligned}$$

Applying Lemma 52 we have that

$$\frac{U_N}{\sqrt{\pi_N}} \leq \rho \frac{P_N}{\sqrt{\pi_N}} \leq \rho \sum_{k=N}^{+\infty} \sqrt{\pi_k} \xrightarrow{N \rightarrow +\infty} 0.$$

Again by Lemma 52,

$$\sum_{k=n_0}^{N-1} U_k \left(\frac{1}{\sqrt{\pi_{k+1}}} - \frac{1}{\sqrt{\pi_k}} \right) \leq \rho \sum_{k=n_0}^{N-1} P_k \left(\frac{1}{\sqrt{\pi_{k+1}}} - \frac{1}{\sqrt{\pi_k}} \right)$$

We compute this term:

$$\begin{aligned} \sum_{k=n_0}^{N-1} P_k \left(\frac{1}{\sqrt{\pi_{k+1}}} - \frac{1}{\sqrt{\pi_k}} \right) &= \sum_{k=n_0}^{N-1} \frac{P_k}{\sqrt{\pi_{k+1}}} - \sum_{k=n_0}^{N-1} \frac{P_k}{\sqrt{\pi_k}} \\ &= \sum_{k=n_0+1}^N \frac{P_{k-1}}{\sqrt{\pi_k}} - \sum_{k=n_0}^{N-1} \frac{P_k}{\sqrt{\pi_k}} \\ &= \sum_{k=n_0+1}^{N-1} \frac{P_{k-1} - P_k}{\sqrt{\pi_k}} + \frac{P_{N-1}}{\sqrt{\pi_N}} - \frac{P_{n_0}}{\sqrt{\pi_{n_0}}}. \end{aligned}$$

Yet $\sum_{k=n_0+1}^{N-1} \frac{P_{k-1} - P_k}{\sqrt{\pi_k}} = \sum_{k=n_0+1}^{N-1} \sqrt{\pi_k}$ is a convergent series. Finally:

$$\frac{P_{N-1}}{\sqrt{\pi_N}} = \frac{\pi_{N-1}}{\sqrt{\pi_N}} + \frac{P_N}{\sqrt{\pi_N}}$$

where $\frac{\pi_{N-1}}{\sqrt{\pi_N}} \leq 1$ and $\frac{P_N}{\sqrt{\pi_N}} \xrightarrow{N \rightarrow +\infty} 0$.

By comparison test for positive series we conclude that $\sum_{k \geq n_0} \frac{\mu(E_k)}{\sqrt{\pi_k}}$ is a convergent

series and so is $\sum_{k \geq 2} \frac{\mu(E_k)}{\sqrt{\pi_k}}$.

To end the proof, we observe in the computations in the proof of Theorem 3.1 of [Møl89] that the exponential rate r does not depend on the initial state $y \in E$. Moreover the constant $c(y)$ is under the form

$$c_1 + \frac{c_2}{\sqrt{\pi_{n(y)}}}.$$

In particular it satisfies

$$\int_E c(y) d\mu(y) = c_1 + c_2 \sum_{k \geq 0} \frac{\mu(E_k)}{\sqrt{\pi_k}} < +\infty. \quad (3.32)$$

Chapter 4

Non-parametric estimation of the transition kernel of jump-move processes

In this chapter we proceed to the non-parametric estimation of the transition kernel K of a jump-move process. For this purpose we use kernel methods. In Section 4.1 we present theoretical results of the jump-move process useful for the study of the estimators considered in the sequel. Then, in Sections 4.2, 4.3 and 4.4 we investigate the estimation of K when the jump-move process $(X_t)_{t \geq 0}$ is continuously observed on $[0, T]$. We handle this problem in a general setting, when K admits a density and also in the special case of the birth-death-move process. Finally Section 4.5 deals with the estimation of the density of K from discrete observations of the process on $[0, T]$. The notations introduced in the previous chapters will be used without reminder.

4.1 Preliminary results

This section contains technical results that will be useful for the theoretical study of the forthcoming non-parametric estimators of the transition kernel of a jump-move process. In particular we exhibit an important martingale to this end.

In this section we do not suppose that the intensity function α is bounded: we only assume that the counting process $(N_t)_{t \geq 0}$ is non-explosive, that is $N_t < \infty$ for any $t \geq 0$, \mathbb{P} -almost surely.

4.1.1 Filtrations associated to the jump-move process

For any $t > 0$, let us denote $\mathcal{F}_t^0 = \sigma(X_u, u \leq t)$ the natural filtration associated to the jump-move process $(X_t)_{t \geq 0}$. Following [Bas11], Chapter 20, we make this filtration complete setting $\mathcal{F}_t = \sigma(F_t^0 \cup \mathcal{N})$ where \mathcal{N} is the collection of sets that are \mathbb{P}_x -null for every $x \in E$. It was proven in [LLG21] that $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ is a Markov process with respect to $(\mathcal{F}_t)_{t > 0}$.

To consider martingale properties of the process we need to introduce the right-

continuous filtration

$$\mathcal{F}_{t^+} = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}.$$

Finally we denote for an initial probability measure μ_0 on E , $t \geq 0$ and $f \in M_b(E)$,

$$\mathbb{E}_{\mu_0}[f(X_t)] = \int_E \mathbb{E}_x[f(X_t)] d\mu_0(x).$$

In the following we omit the dependency on the initial distribution μ_0 when it is not relevant and simply write \mathbb{E} instead of \mathbb{E}_{μ_0} .

4.1.2 A martingale associated to the jump-move process

Let $g \in M_b(E \times E)$. We consider in this section the process $(M_t^*)_{t \geq 0}$ defined by

$$M_t^* = \int_0^t g(X_{s-}, X_s) dN_s - \int_0^t \alpha(X_s) \int_E g(X_s, z) K(X_s, dz) ds \quad (4.1)$$

for $t \geq 0$.

Lemma 53. *The process $(M_t^*)_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$ -local-martingale.*

Proof of Lemma 53. Let $t \geq 0$ and $k \geq 1$. We show that for any $x \in E$, $s \in [0, t]$ and $B \in \mathcal{F}_s$,

$$\mathbb{E}_x \left[(M_{t \wedge T_k}^* - M_{t \wedge T_{k-1}}^*) \mathbf{1}_B \right] = \mathbb{E}_x \left[(M_{s \wedge T_k}^* - M_{s \wedge T_{k-1}}^*) \mathbf{1}_B \right]. \quad (4.2)$$

By definition

$$\mathbb{E}_x \left[(M_{t \wedge T_k}^* - M_{t \wedge T_{k-1}}^* - M_{s \wedge T_k}^* + M_{s \wedge T_{k-1}}^*) \mathbf{1}_B \right] = \mathbb{E}_x \left[\mathbf{1}_B \left(\int_{s \wedge T_k}^{t \wedge T_k} dM_u^* - \int_{s \wedge T_{k-1}}^{t \wedge T_{k-1}} dM_u^* \right) \right].$$

We decompose the right-hand term into

$$\mathbb{E}_x \left[\mathbf{1}_B \left(\int_{s \wedge T_k}^{t \wedge T_k} dM_u^* - \int_{s \wedge T_{k-1}}^{t \wedge T_{k-1}} dM_u^* \right) \right] = D_1 + D_2 + D_3 + D_4 \quad (4.3)$$

where

$$\begin{aligned} D_1 &= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s > T_{k-1}} \mathbf{1}_{s < T_k < t} \int_s^{T_k} dM_u^* \right], \\ D_2 &= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s > T_{k-1}} \mathbf{1}_{T_k > t} \int_s^t dM_u^* \right], \\ D_3 &= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k < t} \int_{T_{k-1}}^{T_k} dM_u^* \right], \\ D_4 &= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k > t} \int_{T_{k-1}}^t dM_u^* \right]. \end{aligned}$$

We start by computing D_1 :

$$\begin{aligned}
D_1 &= \mathbb{E}_x [\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbf{1}_{s<T_k} \times \\
&\quad \left(\int_s^{T_k} g(X_{u^-}, X_u) dN_u - \int_s^{T_k} \alpha(X_u) \int_E g(X_u, z) K(X_u, dz) du \right)] \\
&= \mathbb{E}_x [\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbf{1}_{s<T_k} g(Y_{\tau_k}^{(k-1)}, X_{T_k})] \\
&\quad - \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbf{1}_{s<T_k} \int_s^{T_k} \alpha(X_u) \int_E g(X_u, z) K(X_u, dz) du \right]. \quad (4.4)
\end{aligned}$$

First,

$$\begin{aligned}
&\mathbb{E}_x [\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbf{1}_{s<T_k} g(Y_{\tau_k}^{(k-1)}, X_{T_k})] \\
&= \mathbb{E}_x [\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbf{1}_{s<T_k} \mathbb{E}_x [g(Y_{\tau_k}^{(k-1)}, X_{T_k}) | \mathcal{F}_{T_{k-1}}, Y^{(k-1)}, T_k]] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbf{1}_{s<T_k} \int_E g(Y_{\tau_k}^{(k-1)}, z) K(Y_{\tau_k}^{(k-1)}, dz) \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbb{E}_x \left[\int_E g(Y_{\tau_k}^{(k-1)}, z) K(Y_{\tau_k}^{(k-1)}, dz) \mathbf{1}_{s<T_k} | \mathcal{F}_{T_{k-1}}, Y^{(k-1)} \right] \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \int_{s-T_{k-1}}^{t-T_{k-1}} \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) \alpha(Y_u^{(k-1)}) e^{-\int_0^u \alpha(Y_v^{(k-1)}) dv} du \right] \\
&= \mathbb{E}_x [\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \times \\
&\quad \int_{s-T_{k-1}}^{t-T_{k-1}} \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) \alpha(Y_u^{(k-1)}) \mathbb{E}_x [\mathbf{1}_{\tau_k > u} | \mathcal{F}_{T_{k-1}}, Y^{(k-1)}] du].
\end{aligned}$$

Now we compute the second term in the decomposition (4.4).

$$\begin{aligned}
&\mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbf{1}_{s<T_k} \int_s^{T_k} \alpha(X_u) \int_E g(X_u, z) K(X_u, dz) du \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbf{1}_{s<T_k} \int_{s-T_{k-1}}^{T_k-T_{k-1}} \alpha(Y_u^{(k-1)}) \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbf{1}_{s<T_k} \int_{s-T_{k-1}}^{t-T_{k-1}} \mathbf{1}_{u \leq \tau_k} \alpha(Y_u^{(k-1)}) \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du \right] \\
&= \mathbb{E}_x [\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \int_{s-T_{k-1}}^{t-T_{k-1}} \mathbb{E}_x [\mathbf{1}_{u < \tau_k} | \mathcal{F}_{T_{k-1}}, Y^{(k-1)}] \alpha(Y_u^{(k-1)}) \times \\
&\quad \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du].
\end{aligned}$$

We then obtain

$$\begin{aligned}
D_1 &= \mathbb{E}_x [\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \int_{s-T_{k-1}}^{t-T_{k-1}} \mathbb{E}_x [(\mathbf{1}_{u < \tau_k} - \mathbf{1}_{u < \tau_k < t-T_{k-1}}) | \mathcal{F}_{T_{k-1}}, Y^{(k-1)}] \alpha(Y_u^{(k-1)}) \times \\
&\quad \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du] \\
&= \mathbb{E}_x [\mathbf{1}_B \mathbf{1}_{s>T_{k-1}} \mathbb{E}_x [\mathbf{1}_{t-T_{k-1} < \tau_k} | \mathcal{F}_{T_{k-1}}, Y^{(k-1)}] \times \\
&\quad \int_{s-T_{k-1}}^{t-T_{k-1}} \alpha(Y_u^{(k-1)}) \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du]. \quad (4.5)
\end{aligned}$$

Let us compute D_2 .

$$\begin{aligned}
D_2 &= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s > T_{k-1}} \mathbf{1}_{T_k > t} \int_s^t dM_u^* \right] \\
&= -\mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s > T_{k-1}} \mathbf{1}_{T_k > t} \int_s^t \alpha(X_u) \int_E g(X_u, z) K(X_u, dz) du \right] \\
&= -\mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s > T_{k-1}} \mathbf{1}_{T_k > t} \int_{s-T_{k-1}}^{t-T_{k-1}} \alpha(Y_u^{(k-1)}) \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du \right] \\
&= -\mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s > T_{k-1}} \mathbb{E}_x \left[\mathbf{1}_{\tau_k > t - T_{k-1}} \middle| \mathcal{F}_{T_{k-1}}, Y^{(k-1)} \right] \times \right. \\
&\quad \left. \int_{s-T_{k-1}}^{t-T_{k-1}} \alpha(Y_u^{(k-1)}) \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du \right] \\
&= -D_1
\end{aligned}$$

by (4.5).

To compute D_3 we shall write:

$$\begin{aligned}
D_3 &= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k < t} \times \right. \\
&\quad \left. \left(\int_{T_{k-1}}^{T_k} g(X_{u-}, X_u) dN_u - \int_{T_{k-1}}^{T_k} \alpha(X_u) \int_E g(X_u, z) K(X_u, dz) du \right) \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k < t} g(Y_{\tau_k}^{(k)}, X_{T_k}) \right] \\
&\quad - \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s > T_{k-1}} \mathbf{1}_{s < T_k < t} \int_{T_{k-1}}^{T_k} \alpha(X_u) \int_E g(X_u, z) K(X_u, dz) du \right]. \quad (4.6)
\end{aligned}$$

Firstly,

$$\begin{aligned}
&\mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k < t} g(Y_{\tau_k}^{(k-1)}, X_{T_k}) \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k < t} \mathbb{E}_x \left[g(Y_{\tau_k}^{(k-1)}, X_{T_k}) \middle| \mathcal{F}_{T_{k-1}}, Y^{(k-1)}, T_k \right] \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k < t} \int_E g(Y_{\tau_k}^{(k-1)}, z) K(Y_{\tau_k}^{(k-1)}, dz) \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbb{E}_x \left[\int_E g(Y_{\tau_k}^{(k-1)}, z) K(Y_{\tau_k}^{(k-1)}, dz) \mathbf{1}_{T_k < t} \middle| \mathcal{F}_{T_{k-1}}, Y^{(k-1)} \right] \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \times \right. \\
&\quad \left. \int_0^{t-T_{k-1}} \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) \alpha(Y_u^{(k-1)}) e^{-\int_0^u \alpha(Y_v^{(k-1)}) dv} du \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \times \right. \\
&\quad \left. \int_0^{t-T_{k-1}} \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) \alpha(Y_u^{(k-1)}) \mathbb{E}_x \left[\mathbf{1}_{\tau_k > u} \middle| \mathcal{F}_{T_{k-1}}, Y^{(k-1)} \right] du \right].
\end{aligned}$$

The second term in the decomposition (4.6) is

$$\begin{aligned}
& \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s > T_{k-1}} \mathbf{1}_{s < T_k < t} \int_{T_{k-1}}^{T_k} \alpha(X_u) \int_E g(X_u, z) K(X_u, dz) du \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k < t} \int_0^{T_k - T_{k-1}} \alpha(Y_u^{(k-1)}) \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k < t} \int_0^{t - T_{k-1}} \mathbf{1}_{u < \tau_k} \alpha(Y_u^{(k-1)}) \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \int_0^{t - T_{k-1}} \alpha(Y_u^{(k-1)}) \times \right. \\
&\quad \left. \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) \mathbb{E}_x \left[\mathbf{1}_{u < \tau_k < t - T_{k-1}} \middle| \mathcal{F}_{T_{k-1}}, Y^{(k-1)} \right] du \right].
\end{aligned}$$

We then obtain the value of D_3 that is

$$\begin{aligned}
D_3 &= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \int_0^{t - T_{k-1}} \alpha(Y_u^{(k-1)}) \times \right. \\
&\quad \left. \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) \mathbb{E}_x \left[(\mathbf{1}_{u < \tau_k} - \mathbf{1}_{u < \tau_k < t - T_{k-1}}) \middle| \mathcal{F}_{T_{k-1}}, Y^{(k-1)} \right] du \right] \\
&= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbb{E}_x \left[\mathbf{1}_{\tau_k > t - T_{k-1}} \middle| \mathcal{F}_{T_{k-1}}, Y^{(k-1)} \right] \times \right. \\
&\quad \left. \int_0^{t - T_{k-1}} \alpha(Y_u^{(k-1)}) \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du \right]. \tag{4.7}
\end{aligned}$$

It remains to compute D_4 . We get

$$\begin{aligned}
D_4 &= \mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k > t} \int_{T_{k-1}}^t dM_u^* \right] \\
&= -\mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k > t} \int_{T_{k-1}}^t \alpha(X_u) \int_E g(X_u, z) K(X_u, dz) du \right] \\
&= -\mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbf{1}_{T_k > t} \int_0^{t - T_{k-1}} \alpha(Y_u^{(k-1)}) \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du \right] \\
&= -\mathbb{E}_x \left[\mathbf{1}_B \mathbf{1}_{s < T_{k-1} < t} \mathbb{E}_x \left[\mathbf{1}_{\tau_k > t - T_{k-1}} \middle| \mathcal{F}_{T_{k-1}}, Y^{(k-1)} \right] \times \right. \\
&\quad \left. \int_0^{t - T_{k-1}} \alpha(Y_u^{(k-1)}) \int_E g(Y_u^{(k-1)}, z) K(Y_u^{(k-1)}, dz) du \right] \\
&= -D_3,
\end{aligned}$$

by (4.7) so (4.2) is proved.

From that result we get for any $n \geq 1$ and $B \in \mathcal{F}_s$

$$\begin{aligned}
\mathbb{E}_x(\mathbf{1}_B M_{t \wedge T_n}^*) &= \sum_{k=1}^n \mathbb{E}_x \left[\mathbf{1}_B (M_{t \wedge T_k}^* - M_{t \wedge T_{k-1}}^*) \right] \\
&= \sum_{k=1}^n \mathbb{E}_x \left[\mathbf{1}_B (M_{s \wedge T_k}^* - M_{s \wedge T_{k-1}}^*) \right] \\
&= \mathbb{E}_x[\mathbf{1}_B M_{s \wedge T_n}^*].
\end{aligned}$$

Integrating over the law of X_0 we get for any $n \geq 1$ and $B \in \mathcal{F}_s$

$$\mathbb{E}[\mathbf{1}_B M_{t \wedge T_n}^*] = \mathbb{E}[\mathbf{1}_B M_{s \wedge T_n}^*].$$

The counting process $(N_t)_{t \geq 0}$ is non-explosive so the sequence of jumping times $(T_n)_{n \geq 1}$ diverges to $+\infty$, \mathbb{P} -a.s. Therefore the lemma is proved. \square

Lemma 54. *Suppose that $\mathbb{E} \left[\int_0^t \alpha(X_s) ds \right] < \infty$ for any $t \geq 0$. Then $(M_t^*)_{t \geq 0}$ is a $(\mathcal{F}_{t+})_{t \geq 0}$ -martingale.*

Proof of Lemma 54. First we show that $(M_t^*)_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$ -martingale. Taking $g \equiv 1$ in Lemma 53 we get by [Bré81], Chapter 2, that $\alpha(X_t)$ is the \mathcal{F}_t -intensity of N_t . If we suppose moreover that $\mathbb{E} \left[\int_0^t \alpha(X_s) ds \right] < +\infty$ for any $t \geq 0$ we deduce again from [Bré81], Chapter 2, that $N_t - \int_0^t \alpha(X_s) ds$ is a \mathcal{F}_t -martingale. In particular we obtain $\mathbb{E}[N_t] = \mathbb{E} \left[\int_0^t \alpha(X_s) ds \right]$. Almost surely

$$\begin{aligned} \left| \mathbf{1}_B M_{t \wedge T_n}^* \right| &\leq \left| \int_0^t g(X_{s-}, X_s) \mathbf{1}_{s \leq T_n} dN_s \right| + \left| \int_0^t \alpha(X_s) \int_E g(X_s, z) K(X_s, dz) \mathbf{1}_{s \leq T_n} ds \right| \\ &\leq \|g\|_\infty N_t + \|g\|_\infty \int_0^t \alpha(X_s) \int_E K(X_s, dz) ds \\ &= \|g\|_\infty N_t + \|g\|_\infty \int_0^t \alpha(X_s) ds \end{aligned}$$

which is integrable with respect to \mathbb{P} . Using the dominated convergence theorem we then obtain

$$\begin{aligned} \mathbb{E}[\mathbf{1}_B M_t^*] &= \lim_{n \rightarrow +\infty} \mathbb{E}[\mathbf{1}_B M_{t \wedge T_n}^*] \\ &= \lim_{n \rightarrow +\infty} \mathbb{E}[\mathbf{1}_B M_{s \wedge T_n}^*] \\ &= \mathbb{E}[\mathbf{1}_B M_s^*]. \end{aligned}$$

Now we can prove the lemma. Let $t > s \geq 0$ and B in \mathcal{F}_{s+} . By definition of this filtration we get for any $\varepsilon > 0$ that $B \in \mathcal{F}_{s+\varepsilon}$. So

$$\forall \varepsilon \in (0, t - s), \quad \mathbb{E}[M_t^* \mathbf{1}_B] = \mathbb{E}[M_{s+\varepsilon}^* \mathbf{1}_B].$$

Note that for any $\varepsilon \in (0, t - s)$, $|M_{s+\varepsilon}^*| \leq \|g\|_\infty \left(N_t + \int_0^t \alpha(X_s) ds \right)$ and

$$\mathbb{E} \left[N_t + \int_0^t \alpha(X_v) dv \right] = 2 \mathbb{E} \left[\int_0^t \alpha(X_v) dv \right] < +\infty.$$

The process $(M_u^*)_{u \geq 0}$ is right-continuous so applying the dominated convergence theorem we obtain

$$\mathbb{E}[M_s^* \mathbf{1}_B] = \mathbb{E} \left[\lim_{\varepsilon \searrow 0} M_{s+\varepsilon}^* \mathbf{1}_B \right] = \lim_{\varepsilon \searrow 0} \mathbb{E}[M_{s+\varepsilon}^* \mathbf{1}_B] = \mathbb{E}[M_t^* \mathbf{1}_B].$$

\square

Corollary 55. *Suppose that there exists $\alpha^* > 0$ such that for any $x \in E$, $\alpha(x) \leq \alpha^*$. Then for any $t \geq 0$,*

$$\mathbb{E}[M_t^{*2}] = \mathbb{E} \left[\int_0^t \alpha(X_s) \int_E g^2(X_s, z) K(X_s, dz) ds \right].$$

Proof of Corollary 55. First note that for any $t \geq 0$,

$$\mathbb{E} \left[\int_0^t \alpha(X_s) ds \right] \leq \alpha^* t < \infty$$

so $(M_t^*)_{t \geq 0}$ is a $(\mathcal{F}_{t+})_{t \geq 0}$ -martingale.

We fix $T > 0$ and we show that $(M_t^*)_{t \in [0, T]}$ is a square integrable martingale. Recall from [Bas11], Chapter 9, that it suffices to prove that

$$\sup_{t \in [0, T]} \mathbb{E}[M_t^{*2}] < \infty.$$

We compute for $t \in [0, T]$

$$\begin{aligned} \mathbb{E}[M_t^{*2}] &\leq 2 \mathbb{E} \left[\left(\int_0^t g(X_{s-}, X_s) dN_s \right)^2 \right] + 2 \mathbb{E} \left[\left(\int_0^t \alpha(X_s) \int_E g(X_s, z) K(X_s, dz) ds \right)^2 \right] \\ &\leq 2 \|g\|_\infty^2 \mathbb{E}[N_t^2] + 2t^2 \alpha^{*2} \|g\|_\infty^2 \\ &\leq 2 \|g\|_\infty^2 \left(\mathbb{E}[N_T^2] + T^2 \alpha^{*2} \right). \end{aligned}$$

As α is bounded by α^* we obtain by (2.3) that

$$\mathbb{E}[N_T^2] = \sum_{n \geq 0} \mathbb{P}(N_T^2 > n) \leq \sum_{n \geq 0} \mathbb{P}(N_T^{*2} > n) \leq \mathbb{E}[N_T^{*2}]$$

where $N_T^* \sim \mathcal{P}(\alpha^* T)$. Then $(M_t^*)_{t \in [0, T]}$ is a square integrable martingale and its continuous part is null so applying [Bas11], Chapter 17, we obtain for $t \in [0, T]$

$$\begin{aligned} \mathbb{E}[M_t^{*2}] &= \mathbb{E} \left[\sum_{s \leq t} |\Delta M_s^*|^2 \right] \\ &= \mathbb{E} \left[\int_0^t g^2(X_{s-}, X_s) dN_s \right] \\ &= \mathbb{E} \left[\int_0^t \alpha(X_s) \int_E g^2(X_s, z) K(X_s, dz) ds \right]. \end{aligned}$$

using Lemma 54 with the function g^2 . □

4.1.3 Ergodicity

We show in this section that if the jump-move process converges with an exponential rate to the invariant measure then we control for large $T \geq 0$ quantities of the form $\mathbb{E}_x(\int_0^T g(X_s) ds)$ and $\mathbb{V}_x(\int_0^T g(X_s) ds)$ for bounded functions $g : E \rightarrow \mathbb{R}$. This will be useful to bound the mean squared error of the considered estimators in the next sections.

Let us introduce the following hypothesis:

(H1) $(X_t)_{t \geq 0}$ admits a unique invariant measure μ_∞ and there exist $r > 0$ and $c : E \rightarrow \mathbb{R}_+^*$ such that for any $x \in E$, $g \in M_b(E)$ and $t > 0$,

$$\left| \mathbb{E}_x[g(X_t)] - \int_E g(z) \mu_\infty(dz) \right| \leq \|g\|_\infty c(x) e^{-rt}.$$

One can integrate in time the condition (H1) to obtain the following proposition.

Proposition 56. *Suppose (H1). Let $g \in \mathcal{M}_b(E)$. For any $x \in E$ and $t > 0$ we get*

$$\left| \mathbb{E}_x \left[\frac{1}{t} \int_0^t g(X_s) ds \right] - \int_E g(z) \mu_\infty(dz) \right| \leq \frac{c(x) \|g\|_\infty}{rt}. \quad (4.8)$$

Proof of Proposition 56. Take $g \in M_b(E)$ then

$$\begin{aligned} \left| \mathbb{E}_x \left[\int_0^t g(X_s) ds \right] - t \int_E g(z) \mu_\infty(dz) \right| &\leq \int_0^t \left| \int_E g(z) Q_s(x, dz) - \int_E g(z) d\mu_\infty(z) \right| ds \\ &\leq c(x) \|g\|_\infty \int_0^t e^{-rs} ds \\ &= \frac{c(x) \|g\|_\infty}{r} (1 - e^{-rt}) \\ &\leq \frac{c(x) \|g\|_\infty}{r}. \end{aligned}$$

□

Note that when $c(\cdot)$ is bounded by $c^* > 0$, as it is the case for a birth-death-move process when condition (3.4) of Theorem 39 is satisfied, we get for any $g \in \mathcal{M}_b(E)$

$$\left| \mathbb{E} \left[\frac{1}{t} \int_0^t g(X_s) ds \right] - \int_E g(z) \mu_\infty(dz) \right| \leq \frac{c^* \|g\|_\infty}{rt}. \quad (4.9)$$

From Proposition 56 we can compare a quantity of the form $\mathbb{E}_x \left[\int_0^t k(X_s, t) ds \right]$ to the integral of the functional $k(\cdot, t)$ with respect to the invariant measure μ_∞ . This result will be useful to study the asymptotic behaviour of the estimators introduced in the following sections.

Corollary 57. *Suppose (H1). Let $x \in E$ and $k \in M_b(E \times \mathbb{R}_+, \mathbb{R}_+)$.*

i) There exists $c_0(x)$ such that for any $t \geq 0$

$$\mathbb{E}_x \left[\int_0^t k(X_s, t) ds \right] \leq c_0(x) \left(1 + t \int_E k(z, t) \mu_\infty(dz) \right).$$

ii) If

$$t \int_E k(z, t) d\mu_\infty(z) \xrightarrow{t \rightarrow +\infty} +\infty,$$

then there exist $t_0(x) \geq 0$, $c_m(x) > 0$ and $c_M(x) > 0$ such that for any $t > t_0(x)$

$$c_m(x) t \int_E k(z, t) \mu_\infty(dz) \leq \mathbb{E}_x \left[\int_0^t k(X_s, t) ds \right] \leq c_M(x) t \int_E k(z, t) \mu_\infty(dz).$$

Moreover, when the function $c(\cdot)$ given by (H1) is bounded then the constants t_0 , c_0 , c_m and c_M can be chosen independently of the initial state x .

Proof of Corollary 57. Suppose that (H1) is satisfied. Let $t \geq 0$. Applying (4.8) to $z \in E \mapsto k(z, t)$ we get:

$$\left| \mathbb{E}_x \left[\int_0^t k(X_s, t) ds \right] - t \int_E k(z, t) d\mu_\infty(z) \right| \leq \frac{c(x) \|k\|_\infty}{r},$$

that proves *i)* with $c_0(x) = \max \left(1, \frac{c(x) \|k\|_\infty}{r} \right)$. We also deduce that when

$$t \int_E k(z, t) d\mu_\infty(z) \xrightarrow{t \rightarrow +\infty} +\infty$$

we get

$$\mathbb{E}_x \left[\int_0^t k(X_s, t) ds \right] = t \int_E k(z, t) d\mu_\infty(z) + \underset{t \rightarrow +\infty}{O} (1)$$

that implies *ii)*. The proof when the function $c(\cdot)$ is bounded is the same using (4.9). \square

Using Proposition 56 one can find a bound on the variance of the process $\int_0^t g(X_s) ds$. In the special case when the function $c(\cdot)$ given in (H1) is bounded, the bound obtained is thinner. This result will be relevant to control the mean squared error of the estimators introduced in the following.

Proposition 58. *Suppose (H1). For any $x \in E$ and $g \in M_b(E)$ we get for any $t \geq 0$,*

$$\mathbb{V}_x \left(\int_0^t g(X_s) ds \right) \leq \frac{2 \|g\|_\infty}{r} \left(3 c(x) \|g\|_\infty + \int_E g(z) c(z) d\mu_\infty(z) \right) t. \quad (4.10)$$

Moreover if $c(\cdot)$ is bounded by a positive constant c^* then for any $g \in M_b^+(E)$ and $t \geq 0$

$$\mathbb{V} \left(\int_0^t g(X_s) ds \right) \leq \frac{8 c^* \|g\|_\infty}{r} \mathbb{E} \left[\int_0^t g(X_s) ds \right]. \quad (4.11)$$

Proof of Proposition 58. The special case when $c(\cdot)$ is bounded is proved in [LLG21].

For the general case, let us write

$$\begin{aligned} & \mathbb{V}_x \left(\int_0^t g(X_s) ds \right) \\ &= 2 \int_0^t \int_s^t \mathbb{E}_x [g(X_s) g(X_u)] du ds - 2 \int_0^t \int_s^t \mathbb{E}_x [g(X_s)] \mathbb{E}_x [g(X_u)] du ds \\ &= 2 \int_0^t \int_s^t \mathbb{E}_x [g(X_s) \mathbb{E}_{X_s} [g(X_{u-s})]] du ds - 2 \int_0^t \int_s^t \mathbb{E}_x [g(X_s)] \mathbb{E}_x [g(X_u)] du ds \end{aligned}$$

by Markov property. Therefore

$$\begin{aligned} & \mathbb{V}_x \left(\int_0^t g(X_s) ds \right) \\ &= 2 \int_0^t \int_0^{t-s} \mathbb{E}_x [g(X_s) \mathbb{E}_{X_s} [g(X_u)]] du ds - 2 \int_0^t \int_0^{t-s} \mathbb{E}_x [g(X_s)] \mathbb{E}_x [g(X_{u+s})] du ds \\ &= 2 \int_0^t \int_0^{t-s} \mathbb{E}_x [g(X_s) (\mathbb{E}_{X_s} [g(X_u)] - \mathbb{E}_x [g(X_{u+s})])] du ds. \end{aligned}$$

We set for $s \in [0, t]$ and $u \in [0, t - s]$,

$$h_{x,s,u} : y \in E \mapsto \mathbb{E}_y[g(X_u)] - \mathbb{E}_x[g(X_{u+s})].$$

Remark that for any $y \in E$, $|h_{x,s,u}(y)| \leq 2\|g\|_\infty$ so that $h_{x,s,u} \in M_b(E)$. We decompose

$$\begin{aligned} \mathbb{V}_x \left(\int_0^t g(X_s) ds \right) &= 2 \int_0^t \int_0^{t-s} \left(\mathbb{E}_x[g(X_s) h_{x,s,u}(X_s)] - \int_E g(z) h_{x,s,u}(z) d\mu_\infty(z) \right) du ds \\ &\quad + 2 \int_0^t \int_0^{t-s} \int_E g(z) h_{x,s,u}(z) d\mu_\infty(z) du ds. \end{aligned}$$

First

$$\begin{aligned} &\left| 2 \int_0^t \int_0^{t-s} \left(\mathbb{E}_x[g(X_s) h_{x,s,u}(X_s)] - \int_E g(z) h_{x,s,u}(z) d\mu_\infty(z) \right) du ds \right| \\ &\leq 2 \int_0^t \int_0^{t-s} \left| \mathbb{E}_x[g(X_s) h_{x,s,u}(X_s)] - \int_E g(z) h_{x,s,u}(z) d\mu_\infty(z) \right| du ds \\ &\leq 2 \int_0^t \int_0^{t-s} c(x) \|g \times h_{x,s,u}\|_\infty e^{-rs} du ds, \end{aligned}$$

where $c(x) > 0$ and $r > 0$ are given by (H1). Hence

$$\begin{aligned} &\left| 2 \int_0^t \int_0^{t-s} \left(\mathbb{E}_x[g(X_s) h_{x,s,u}(X_s)] - \int_E g(z) h_{x,s,u}(z) d\mu_\infty(z) \right) du ds \right| \\ &\leq 4\|g\|_\infty^2 c(x) \int_0^t (t-s) e^{-rs} ds \\ &\leq 4\|g\|_\infty^2 c(x) t \int_0^t e^{-rs} ds \\ &\leq \frac{4\|g\|_\infty^2 c(x)}{r} t. \end{aligned}$$

Secondly, applying Proposition 56 we get

$$\begin{aligned} &\left| 2 \int_0^t \int_0^{t-s} \int_E g(z) h_{x,s,u}(z) d\mu_\infty(z) du ds \right| \\ &\leq 2 \int_0^t \int_E g(z) \left| \int_0^{t-s} \mathbb{E}_z[g(X_u)] du - (t-s) \int_E g(y) d\mu_\infty(y) \right| d\mu_\infty(z) ds \\ &\quad + 2 \int_0^t \int_E g(z) \int_0^{t-s} \left| \mathbb{E}_x[g(X_{u+s})] - \int_E g(y) d\mu_\infty(y) \right| du d\mu_\infty(z) ds \\ &\leq 2 \int_0^t \int_E g(z) \frac{c(z) \|g\|_\infty}{r} d\mu_\infty(z) ds + 2\|g\|_\infty \int_0^t \int_0^{t-s} c(x) \|g\|_\infty e^{-rs} e^{-ru} du ds \\ &\leq 2t \frac{\|g\|_\infty}{r} \int_E g(z) c(z) d\mu_\infty(z) + 2c(x) \|g\|_\infty^2 t \int_0^t e^{-rs} ds \\ &\leq 2t \frac{\|g\|_\infty}{r} \int_E g(z) c(z) d\mu_\infty(z) + 2c(x) \frac{\|g\|_\infty^2}{r} t. \end{aligned}$$

□

4.2 Estimation of the transition kernel in continuous time

In this section we proceed to the non-parametric estimation of the transition kernel K of a jump-move process in a general setting. The goal is to obtain the rate of convergence of the mean squared error of the estimator.

Henceforth the intensity function α is supposed to be bounded by $\alpha^* > 0$.

4.2.1 Kernel estimation

Let $(k_t)_{t \geq 0}$ be a family of non-negative functions on $E \times E$. Intuitively, for large $t > 0$, $k_t(x, y)$ will quantify the proximity between the two states $x \in E$ and $y \in E$. Let us suppose that there exists k^* such that

$$\forall (x, y) \in E \times E, \forall t \geq 0, k_t(x, y) \leq k^*.$$

We observe the jump-move process $(X_t)_{t \geq 0}$ in continuous time on $[0, T]$ for $T > 0$. For $x \in E$ and $A \in \mathcal{E}$, we define an estimator of $K(x, A)$ by

$$\hat{K}(x, A) = \frac{\int_0^T k_T(x, X_{s-}) \mathbf{1}_A(X_s) dN_s}{\int_0^T k_T(x, X_{s-}) dN_s}, \quad (4.12)$$

with the convention $\frac{0}{0} = 0$.

Let us introduce the quantities

$$R_T = \int_0^T k_T(x, X_s) \alpha(X_s) (K(X_s, A) - K(x, A)) ds$$

and

$$A_T = \int_0^T k_T(x, X_s) \alpha(X_s) ds.$$

In the following proposition we give an upper bound for the mean squared error of $\hat{K}(x, A)$ with a term depending on the moments of A_T and R_T .

Proposition 59. *For any $T > 0$, $x \in E$ and $A \in \mathcal{E}$,*

$$\mathbb{E} \left[|\hat{K}(x, A) - K(x, A)|^2 \right] \leq \frac{32k^*}{\mathbb{E}[A_T]} + \frac{8 \mathbb{V}(A_T)}{\mathbb{E}[A_T]^2} + \frac{12 \mathbb{E}[R_T^2]}{\mathbb{E}[A_T]^2}.$$

Proof of Proposition 59. One has the following decomposition

$$\begin{aligned} & \hat{K}(x, A) - K(x, A) \\ &= \frac{\int_0^T k_T(x, X_{s-}) \mathbf{1}_A(X_s) dN_s - \int_0^T k_T(x, X_s) K(X_s, A) \alpha(X_s) ds}{Z_T} \\ & \quad + \frac{\int_0^T k_T(x, X_s) [K(X_s, A) - K(x, A)] \alpha(X_s) ds}{Z_T} - K(x, A) \frac{Z_T - A_T}{Z_T} \\ &= \frac{M_T + R_T - K(x, A)(Z_T - A_T)}{Z_T} \end{aligned}$$

where

$$M_T = \int_0^T k_T(x, X_{s-}) \mathbf{1}_A(X_s) dN_s - \int_0^T k_T(x, X_s) \alpha(X_s) K(X_s, A) ds$$

and $Z_T = \int_0^T k_T(x, X_{s-}) dN_s$. Applying Lemma 54 with $g(y, z) = k_T(x, y)$ we obtain that $\mathbb{E}[Z_T] = \mathbb{E}[A_T]$ so we can write

$$\hat{K}(x, A) - K(x, A) = \frac{\mathbb{E}[Z_T]}{Z_T} \left(\frac{M_T}{\mathbb{E}[A_T]} + \frac{R_T}{\mathbb{E}[A_T]} - K(x, A) \frac{Z_T - A_T}{\mathbb{E}[A_T]} \right). \quad (4.13)$$

We compute

$$\begin{aligned} & \mathbb{E}[|\hat{K}(x, A) - K(x, A)|^2] \\ &= \mathbb{E} \left[|\hat{K}(x, A) - K(x, A)|^2 \mathbf{1}_{Z_T < \frac{\mathbb{E}[Z_T]}{2}} \right] + \mathbb{E} \left[|\hat{K}(x, A) - K(x, A)|^2 \mathbf{1}_{Z_T \geq \frac{\mathbb{E}[Z_T]}{2}} \right] \\ &\leq \mathbb{P} \left(Z_T < \frac{\mathbb{E}[Z_T]}{2} \right) + \mathbb{E} \left[4 \left(\frac{M_T}{\mathbb{E}[A_T]} + \frac{R_T}{\mathbb{E}[A_T]} - K(x, A) \frac{Z_T - A_T}{\mathbb{E}[A_T]} \right)^2 \right] \\ &\leq \mathbb{P} \left(|Z_T - \mathbb{E}[Z_T]| > \frac{\mathbb{E}[Z_T]}{2} \right) + \frac{12 \mathbb{E}[M_T^2]}{\mathbb{E}[A_T]^2} + \frac{12 \mathbb{E}[R_T^2]}{\mathbb{E}[A_T]^2} + \frac{12}{\mathbb{E}[A_T]^2} \mathbb{E} \left[(Z_T - A_T)^2 \right]. \end{aligned} \quad (4.14)$$

By Markov's inequality

$$\mathbb{P} \left(|Z_T - \mathbb{E}[Z_T]| > \frac{\mathbb{E}[Z_T]}{2} \right) \leq \frac{4 \mathbb{V}(Z_T)}{\mathbb{E}[Z_T]^2} = \frac{4 \mathbb{V}(Z_T)}{\mathbb{E}[A_T]^2}.$$

Let us now compute the variance term

$$\begin{aligned} \mathbb{V}(Z_T) &\leq 2 (\mathbb{V}(Z_T - A_T) + \mathbb{V}(A_T)) \\ &= 2 (\mathbb{E} [(Z_T - A_T)^2] + \mathbb{V}(A_T)) \\ &= 2 \left(\mathbb{E} \left[\int_0^T k_T^2(x, X_s) \alpha(X_s) ds \right] + \mathbb{V}(A_T) \right) \\ &\leq 2 (k^* \mathbb{E}[A_T] + \mathbb{V}(A_T)), \end{aligned}$$

by Corollary 55 applied with

$$g(y, z) = k_T(x, y),$$

which leads to

$$\mathbb{P} \left(|Z_T - \mathbb{E}[Z_T]| > \frac{\mathbb{E}[Z_T]}{2} \right) \leq 8 \left(\frac{k^*}{\mathbb{E}[A_T]} + \frac{\mathbb{V}(A_T)}{\mathbb{E}[A_T]^2} \right).$$

As viewed in the previous computations

$$\frac{12}{\mathbb{E}[A_T]^2} \mathbb{E} [(Z_T - A_T)^2] = \frac{12}{\mathbb{E}[A_T]^2} \mathbb{E} \left[\int_0^T k_T(x, X_s)^2 \alpha(X_s) ds \right] \leq \frac{12k^*}{\mathbb{E}[A_T]}.$$

Applying Corollary 55 with $g(y, z) = k_T(x, y) \mathbf{1}_A(z)$ we get

$$\mathbb{E}[M_T^2] = \mathbb{E} \left[\int_0^T k_T(x, X_s)^2 \alpha(X_s) K(X_s, A) ds \right] \leq k^* \mathbb{E}[A_T],$$

hence

$$\frac{12 \mathbb{E}[M_T^2]}{\mathbb{E}[A_T]^2} \leq \frac{12 k^*}{\mathbb{E}[A_T]}.$$

□

4.2.2 Asymptotic behaviour

Now we control the mean squared error of the estimator $\hat{K}(x, A)$ defined by (4.12) when T is large using ergodicity results on the process under (H1). In the special case when the function $c(\cdot)$ given in (H1) is bounded we obtain a better rate of convergence and we need weaker hypotheses.

Let us define for $x \in E$,

$$v_T(x) = \int_E k_T(x, z) \alpha(z) d\mu_\infty(z)$$

and

$$w_T(x, A) = \frac{1}{v_T(x)} \int_E k_T(x, z) \alpha(z) |K(z, A) - K(x, A)| d\mu_\infty(z).$$

Theorem 60. *Let $x \in E$ and $A \in \mathcal{E}$. Suppose (H1) and that $Tv_T(x) \xrightarrow{T \rightarrow +\infty} +\infty$.*

We have the two following results on the estimator $\hat{K}(x, A)$ of $K(x, A)$ defined by (4.12).

i) Suppose that there exists a time $T_0 \geq 0$ such that

$$\sup_{T \geq T_0} \int_E k_T(x, z) c(z) \alpha(z) d\mu_\infty(z) < +\infty. \quad (4.15)$$

Then, for any $x_0 \in E$, there exist a constant $c_0(x_0) > 0$ and a time $t_0(x_0) \geq 0$ such that for any $T \geq t_0(x_0)$,

$$\mathbb{E}_{x_0} (|\hat{K}(x, A) - K(x, A)|^2) \leq c_0(x_0) \left(\frac{1}{Tv_T(x)^2} + w_T(x, A)^2 \right). \quad (4.16)$$

ii) Suppose that the function $c(\cdot)$ given in (H1) is bounded. Then there exist a constant $c_0 > 0$ and a time $t_0 \geq 0$ such that for any $T \geq t_0$,

$$\mathbb{E} (|\hat{K}(x, A) - K(x, A)|^2) \leq c_0 \left(\frac{1}{Tv_T(x)} + w_T(x, A)^2 \right). \quad (4.17)$$

Note that the hypothesis of *ii*) is stronger than that of *i*). Indeed when $c(\cdot)$ is bounded by some positive constant c^* then the condition (4.15) is satisfied because

$$\int_E k_T(x, z)c(z)\alpha(z) d\mu_\infty(z) \leq \alpha^* c^* k^*.$$

Actually since $|k_T(z, z')| \leq k^*$ for any $(z, z') \in E^2$, (4.15) is fulfilled for a birth-death-move process satisfying (H1). This will be detailed in Section 4.4.1.

Moreover, $T \mapsto v_T(x)$ being bounded, the rate of convergence given by (4.17) is better than in (4.16). Indeed, one has

$$\frac{1}{Tv_T(x)} = \frac{v_T(x)}{Tv_T(x)^2} \leq \frac{k^* \alpha^*}{Tv_T(x)^2}.$$

Proof of Theorem 60. In this proof $c_0(x_0), t_0(x_0), c_0$ and t_0 are generic positive constants that may change from line to line. Since $Tv_T(x) \xrightarrow{T \rightarrow +\infty} +\infty$, we get for any $T \geq t_0, v_T(x) > 0$.

Let us prove part *i*) of the theorem. Recall that

$$A_T = \int_0^T k_T(x, X_s)\alpha(X_s) ds.$$

Applying Corollary 57 *ii*) with $k : (z, t) \in E \times \mathbb{R}_+ \mapsto k_t(x, z)\alpha(z)$ we obtain for any $T \geq t_0(x_0)$,

$$\mathbb{E}_{x_0}[A_T] \geq c_0(x_0)Tv_T(x).$$

By Proposition 58 and (4.15) one can fix $\kappa > 0$ such that for $T \geq T_0$,

$$\mathbb{V}_{x_0}(A_T) \leq \frac{2k^* \alpha^*}{r} (3c(x_0)k^* \alpha^* + \kappa) T.$$

Then, for $T \geq t_0(x_0)$,

$$\begin{aligned} \frac{1}{\mathbb{E}_{x_0}[A_T]} + \frac{\mathbb{V}_{x_0}(A_T)}{\mathbb{E}_{x_0}[A_T]^2} &\leq \frac{1}{\mathbb{E}_{x_0}[A_T]} + \frac{2k^* \alpha^*}{r} (3c(x_0)k^* \alpha^* + \kappa) \frac{T}{\mathbb{E}_{x_0}[A_T]^2} \\ &\leq c_0(x_0) \left(\frac{1}{Tv_T(x)} + \frac{1}{Tv_T(x)^2} \right) \\ &\leq \frac{c_0(x_0)}{Tv_T(x)^2}. \end{aligned}$$

To finish let us compute the rate of convergence of the term $\frac{\mathbb{E}_{x_0}[R_T^2]}{\mathbb{E}_{x_0}[A_T]^2}$. We set

$$I_T = \int_0^T k_T(x, X_s)\alpha(X_s)K(X_s, A) ds.$$

One has

$$\begin{aligned} \mathbb{E}_{x_0}[R_T^2] &= \mathbb{E}_{x_0} \left[\left(\int_0^T k_T(x, X_s)\alpha(X_s)(K(X_s, A) - K(x, A)) ds \right)^2 \right] \\ &= \mathbb{E}_{x_0} \left[(I_T - K(x, A)A_T)^2 \right] \\ &= \mathbb{E}_{x_0} \left[(I_T - \mathbb{E}_{x_0}[I_T] + K(x, A)\mathbb{E}_{x_0}[A_T] - K(x, A)A_T + \mathbb{E}_{x_0}[I_T - K(x, A)A_T])^2 \right] \\ &\leq 3\mathbb{E}_{x_0} \left[(I_T - \mathbb{E}[I_T])^2 \right] + 3\mathbb{E}_{x_0} \left[(A_T - \mathbb{E}[A_T])^2 \right] + 3\mathbb{E}_{x_0} \left[(I_T - K(x, A)A_T)^2 \right] \\ &= 3(\mathbb{V}_{x_0}(I_T) + \mathbb{V}_{x_0}(A_T)) + 3\mathbb{E}_{x_0} [I_T - K(x, A)A_T]^2. \end{aligned}$$

For any $z \in E$ note that $k_T(x, z)\alpha(z)K(z, A) \leq k_T(x, z)\alpha(z)$ so Proposition 58 and (4.15) lead to

$$\mathbb{V}_{x_0}(I_T) + \mathbb{V}_{x_0}(A_T) \leq \frac{2k^*\alpha^*}{r} (3c(x_0)k^*\alpha^* + \kappa) T.$$

Applying Corollary 57 *i*) with

$$k : (z, t) \in E \times \mathbb{R}_+ \mapsto k_t(x, z)\alpha(z)|K(z, A) - K(x, A)|$$

we get for $T \geq t_0(x_0)$

$$\begin{aligned} \mathbb{E}_{x_0} [I_T - K(x, A)A_T]^2 &= \mathbb{E}_{x_0} \left[\int_0^T k_T(x, X_s)\alpha(X_s)(K(X_s, A) - K(x, A)) ds \right]^2 \\ &\leq \mathbb{E}_{x_0} \left[\int_0^T k_T(x, X_s)\alpha(X_s)|K(X_s, A) - K(x, A)| ds \right]^2 \\ &\leq c_0(x_0) \left(T \int_E k_T(x, z)\alpha(z)|K(z, A) - K(x, A)| d\mu_\infty(z) + 1 \right)^2 \\ &= c_0(x_0) \left(1 + (Tv_T(x))^2 w_T(x, A)^2 \right). \end{aligned}$$

We then obtain for $T \geq t_0(x_0)$

$$\begin{aligned} \frac{\mathbb{E}_{x_0}[R_T^2]}{\mathbb{E}_{x_0}[A_T]^2} &\leq c_0(x_0) \frac{1 + T + T^2 v_T(x)^2 w_T(x, A)^2}{\mathbb{E}_{x_0}[A_T]^2} \\ &\leq c_0(x_0) \left(\frac{1}{(Tv_T(x))^2} + \frac{1}{Tv_T(x)^2} + w_T^2(x, A) \right) \\ &\leq c_0(x_0) \left(\frac{1}{Tv_T(x)^2} + w_T^2(x, A) \right). \end{aligned}$$

The proof of *ii*) is very similar to the one of *i*). We obtain for any $T \geq t_0$,

$$\begin{aligned} \frac{1}{\mathbb{E}[A_T]} + \frac{\mathbb{V}(A_T)}{\mathbb{E}[A_T]^2} &\leq \frac{1}{\mathbb{E}[A_T]} + \frac{c_0 \mathbb{E}[A_T]}{\mathbb{E}[A_T]^2} \\ &\leq \frac{c_0}{Tv_T(x)} \end{aligned}$$

by applying successively Proposition 58 and Corollary 57. As in the proof of *i*) we get for $T \geq t_0$,

$$\begin{aligned} \frac{\mathbb{E}[R_T^2]}{\mathbb{E}[A_T]^2} &\leq \frac{3\mathbb{V}(I_T) + 3\mathbb{V}(A_T) + 3\mathbb{E}[I_T - K(x, A)A_T]^2}{\mathbb{E}[A_T]^2} \\ &\leq c_0 \frac{1 + \mathbb{E}[A_T] + T^2 v_T(x)^2 w_T(x, A)^2}{\mathbb{E}[A_T]^2} \\ &\leq c_0 \left(\frac{1}{(Tv_T(x))^2} + \frac{1}{Tv_T(x)} + w_T(x, A)^2 \right) \\ &\leq c_0 \left(\frac{1}{Tv_T(x)} + w_T(x, A)^2 \right). \end{aligned}$$

We then conclude by Proposition 59. □

4.3 Estimation on continuous time of the density of the transition kernel

In this section we assume that the transition kernel can be written as

$$K(x, A) = \int_A f(x, y) d\nu(y) \quad (4.18)$$

for $A \subset E$ and $x \in E$ where ν is a σ -finite measure on E and $f : E^2 \rightarrow \mathbb{R}_+$ satisfies for any $x \in E$, $\int_E f(x, y) d\nu(y) = 1$. The purpose of this section is to estimate the density function f in each point $(x, y) \in E \times E$.

4.3.1 Density estimator

We observe the jump-move process $(X_t)_{t \geq 0}$ in continuous time on $[0, T]$ for $T > 0$.

Let us consider for $i = 1, 2$, $\kappa_i \in M_b(\mathbb{R}_+, \mathbb{R}_+)$. We set $(h_t)_{t \geq 0}$ and $(c_t)_{t \geq 0}$ such that for any $t \geq 0$, $h_t > 0$ and $c_t > 0$. Finally we introduce a pseudo-distance d on E .

We suppose that for any $z \in E$ and $t \geq 0$ we can define

$$\xi_t(z) := \int_E \kappa_2 \left(\frac{d(z, z')}{c_t} \right) d\nu(z') < +\infty. \quad (4.19)$$

We fix $(x, y) \in E^2$ and we estimate $f(x, y)$ by

$$\hat{f}(x, y) = \frac{1}{Z_T^*} \times \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \quad (4.20)$$

where

$$Z_T^* = \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \xi_T(X_s) dN_s.$$

Let us introduce the following quantities:

$$\begin{aligned} M_T &= \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \\ &\quad - \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) f(X_s, z) d\nu(z) \alpha(X_s) ds, \end{aligned}$$

$$R_T = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \left(\kappa_2 \left(\frac{d(y, z)}{c_T} \right) - f(x, y) \xi_T(z) \right) f(X_s, z) d\nu(z) \alpha(X_s) ds,$$

$$\text{and } A_T^* = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \alpha(X_s) \int_E \xi_T(z) f(X_s, z) d\nu(z) ds.$$

To control $|\hat{f}(x, y) - f(x, y)|$ we require the function $\xi_T : z \in E \mapsto \xi_T(z)$ to be bounded.

(H2) For any $T > 0$ there exists $\xi_T^* > 0$ such that for any $z \in E$ the quantity $\xi_T(z)$ defined by (4.19) satisfies $\xi_T(z) \leq \xi_T^*$.

Proposition 61. *Let $(x, y) \in E^2$. Suppose (H2). For any $\varepsilon > 0$ and $T \geq 0$ the estimator $\hat{f}(x, y)$ given by (4.20) satisfies*

$$\mathbb{P}\left(\left|\hat{f}(x, y) - f(x, y)\right| > \varepsilon\right) \leq 4\|\kappa_1\|_\infty \left(2 + \frac{9f(x, y)^2}{\varepsilon^2}\right) \frac{\xi_T^*}{\mathbb{E}[A_T^*]} + \frac{8\mathbb{V}(A_T^*)}{\mathbb{E}[A_T^*]^2} + \frac{36(\mathbb{E}[R_T^2] + \mathbb{E}[M_T^2])}{\mathbb{E}[A_T^*]^2}. \quad (4.21)$$

Proof of Proposition 61. By (4.18) one can rewrite the $(\mathcal{F}_{t+})_{t \geq 0}$ -martingale defined by (4.1) associated to some $g \in M_b(E)$ by making visible the density f , that is the matter of the following lemma.

Lemma 62. *Let $g \in \mathcal{M}_b(E \times E)$. Suppose that the transition kernel K can be written under the form given by (4.18). Let us consider the process $(M_t^*)_{t \geq 0}$ defined by*

$$M_t^* = \int_0^t g(X_{s-}, X_s) dN_s - \int_0^t \alpha(X_s) \int_E g(X_s, z) f(X_s, z) d\nu(z) ds.$$

Then $(M_t^)_{t \geq 0}$ is a $(\mathcal{F}_{t+})_{t \geq 0}$ -martingale satisfying*

$$\mathbb{E}[M_t^{*2}] = \mathbb{E}\left[\int_0^t \alpha(X_s) \int_E g^2(X_s, z) f(X_s, z) d\nu(z) ds\right]. \quad (4.22)$$

Proof. Since the intensity function α is bounded, the $(\mathcal{F}_{t+})_{t \geq 0}$ -martingale of Lemma 54 is given by

$$\begin{aligned} & \int_0^t g(X_{s-}, X_s) dN_s - \int_0^t \alpha(X_s) \int_E g(X_s, z) K(X_s, dz) ds \\ &= \int_0^t g(X_{s-}, X_s) dN_s - \int_0^t \alpha(X_s) \int_E g(X_s, z) f(X_s, z) d\nu(z) ds \\ &= M_t^*. \end{aligned}$$

Then (4.22) is obtained by applying Corollary 55 with M_t^* . \square

We use the same decomposition as in (4.13) to obtain

$$\hat{f}(x, z) - f(x, z) = \frac{M_T + R_T - f(x, y)(Z_T^* - A_T^*)}{Z_T^*}.$$

Let $\varepsilon > 0$. We get

$$\begin{aligned} & \mathbb{P}\left(\left|\hat{f}(x, y) - f(x, y)\right| > \varepsilon\right) \\ & \leq \mathbb{P}\left(Z_T^* < \frac{\mathbb{E}[Z_T^*]}{2}\right) + \mathbb{P}\left(\left|M_T + R_T - f(x, y)(Z_T^* - A_T^*)\right| > \frac{\varepsilon}{2} \mathbb{E}[Z_T^*]\right) \\ & \leq \mathbb{P}\left(\left|Z_T^* - \mathbb{E}[Z_T^*]\right| > \frac{\mathbb{E}[Z_T^*]}{2}\right) + \mathbb{P}\left(\left|M_T\right| > \frac{\varepsilon}{6} \mathbb{E}[Z_T^*]\right) + \mathbb{P}\left(\left|R_T\right| > \frac{\varepsilon}{6} \mathbb{E}[Z_T^*]\right) \\ & \quad + \mathbb{P}\left(f(x, y)\left|Z_T^* - A_T^*\right| > \frac{\varepsilon}{6} \mathbb{E}[Z_T^*]\right) \\ & \leq \frac{4\mathbb{V}(Z_T^*)}{\mathbb{E}[Z_T^*]^2} + \frac{36(\mathbb{E}[M_T^2] + \mathbb{E}[R_T^2] + f(x, y)^2 \mathbb{E}[(Z_T^* - A_T^*)^2])}{\varepsilon^2 \mathbb{E}[Z_T^*]^2}, \end{aligned}$$

by Markov's inequality. First

$$\frac{4}{\mathbb{E}[Z_T^*]^2} \mathbb{V}(Z_T^*) \leq \frac{8}{\mathbb{E}[A_T^*]^2} \mathbb{E}[(Z_T^* - A_T^*)^2] + \frac{8}{\mathbb{E}[A_T^*]^2} \mathbb{V}(A_T^*).$$

because we get $\mathbb{E}[Z_T] = \mathbb{E}[A_T^*]$ applying Lemma 62 with

$$g : (z, z') \in E \mapsto \kappa_1 \left(\frac{d(x, z)}{h_T} \right) \xi_T(z').$$

With (4.22) we can compute

$$\begin{aligned} \mathbb{E}[(Z_T^* - A_T^*)^2] &= \mathbb{E} \left[\int_0^T \kappa_1^2 \left(\frac{d(x, X_s)}{h_T} \right) \alpha(X_s) \int_E \xi_T(z)^2 f(X_s, z) d\nu(z) ds \right] \\ &\leq \|\kappa_1\|_\infty \xi_T^* \mathbb{E}[A_T^*], \end{aligned}$$

so

$$\frac{4\mathbb{V}(Z_T^*)}{\mathbb{E}[Z_T^*]^2} \leq \frac{8\|\kappa_1\|_\infty \xi_T^*}{\mathbb{E}[A_T^*]} + \frac{8\mathbb{V}(A_T^*)}{\mathbb{E}[A_T^*]^2}.$$

We also just proved:

$$36 f(x, y)^2 \frac{\mathbb{E}[(Z_T^* - A_T^*)^2]}{\varepsilon^2 \mathbb{E}[Z_T^*]^2} \leq 36 f(x, y)^2 \|\kappa_1\|_\infty \frac{\xi_T^*}{\varepsilon^2 \mathbb{E}[A_T^*]}.$$

□

4.3.2 Asymptotic results

Now we control the rate of convergence to zero of the upper-bound given by (4.21) when T is large using ergodicity results on the process.

For simplicity in the calculations we suppose the kernel functions to be supported on a bounded set for the pseudo-distance d .

(H3) For $i = 1, 2$, κ_i is compactly supported.

We also require the density function f to be Hölder continuous on E with respect to the pseudo-distance d .

(H4) There exist $\gamma > 0$ and $a \geq 0$ such that

$$\forall (z_1, z_2, z'_1, z'_2) \in E^4, \quad |f(z_1, z'_1) - f(z_2, z'_2)| \leq a (d(z_1, z_2) + d(z'_1, z'_2))^\gamma.$$

Let us define

$$v_T^*(x) = \int_E \kappa_1 \left(\frac{d(x, z)}{h_T} \right) \alpha(z) \int_E \xi_T(z') f(z, z') d\nu(z') d\mu_\infty(z),$$

$$v_T(x) = \int_E \kappa_1 \left(\frac{d(x, z)}{h_T} \right) \alpha(z) d\mu_\infty(z),$$

$$\text{and } w_T(x, y) = \frac{1}{v_T^*(x)} \int_E \kappa_1 \left(\frac{d(x, z)}{h_T} \right) \alpha(z) \int_E |\xi_T(z') - \xi_T(y)| f(z, z') d\nu(z') d\mu_\infty(z).$$

Theorem 63. Let $(x, y) \in E^2$. Suppose (H1), (H2), (H3) and (H4). Let us suppose that $h_T \xrightarrow{T \rightarrow +\infty} 0$, $c_T \xrightarrow{T \rightarrow +\infty} 0$ and $\xi_T^* \xrightarrow{T \rightarrow +\infty} 0$. We have the two following results on the estimator $\hat{f}(x, y)$ of $f(x, y)$ defined by (4.20).

i) Suppose that $\int_E c(z) d\mu_\infty(z) < \infty$ and $Tv_T^*(x)^2 \xrightarrow{T \rightarrow +\infty} +\infty$. Let $x_0 \in E$ and $\varepsilon > 0$. There exist $t_0(x_0) > 0$ and $c_0(x_0) > 0$ such that for any $T \geq t_0(x_0)$

$$\begin{aligned} \mathbb{P}_{x_0} \left(\left| \hat{f}(x, y) - f(x, y) \right| > \varepsilon \right) \\ \leq c_0(x_0) \left(\frac{\xi_T^{*2}}{Tv_T^*(x)^2} + \frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2} + w_T(x, y)^2 + (h_T + c_T)^{2\gamma} \frac{\xi_T^{*2} v_T(x)^2}{v_T^*(x)^2} \right). \end{aligned} \quad (4.23)$$

ii) Suppose that $c(\cdot)$ is bounded and $Tv_T^*(x) \xrightarrow{T \rightarrow +\infty} +\infty$. Let $\varepsilon > 0$. There exist $t_0 > 0$ and a constant $c_0 > 0$ such that for any $T \geq t_0$

$$\mathbb{P} \left(\left| \hat{f}(x, y) - f(x, y) \right| > \varepsilon \right) \leq c_0 \left(\frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2} + w_T(x, y)^2 + (h_T + c_T)^{2\gamma} \frac{\xi_T^{*2} v_T(x)^2}{v_T^*(x)^2} \right). \quad (4.24)$$

The hypothesis of ii) is stronger than that of i) because when $c(\cdot)$ is bounded by $c^* > 0$ then $\int_E c(z) d\mu_\infty(z) \leq c^*$. Moreover if $Tv_T^*(x)^2 \xrightarrow{T \rightarrow +\infty} +\infty$ we show that $Tv_T^*(x) \xrightarrow{T \rightarrow +\infty} +\infty$. Indeed applying (H2) we obtain $v_T^*(x) \leq \xi_T^* \|\kappa_1\|_\infty \alpha^*$. Fixing $t_0 > 0$ such that $T \geq t_0 \implies v_T^*(x) > 0$, we get for $T \geq t_0$:

$$Tv_T^*(x) = \frac{Tv_T^*(x)^2}{v_T^*(x)} \geq \frac{Tv_T^*(x)^2}{\xi_T^* \|\kappa_1\|_\infty \alpha^*} \xrightarrow{T \rightarrow +\infty} +\infty.$$

However the rate of convergence obtained in ii) is the same or greater than in i) depending on ξ_T^* . In the spatial case studied in Section 4.4, $\xi_T(\cdot)$ will be a constant function for any $T > 0$ that is $\xi_T(\cdot) \equiv \xi_T^*$. This will simplify the rate of convergence because in this case $v_T^*(x) = \xi_T^* v_T(x)$ and $w_T(x) = 0$.

Proof of Theorem 63. Let us introduce the quantity

$$A_T = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \alpha(X_s) ds.$$

We begin by finding a bound for the variances of A_T and A_T^* using Proposition 58, which is the aim of the following lemma.

Lemma 64. Suppose (H1) with $\int_E c(z) d\mu_\infty(z) < \infty$ and (H2). Then for any $x_0 \in E$ there exists $c_0(x_0) > 0$ such that for all $T > 0$,

$$\mathbb{V}_{x_0}(A_T^*) \leq c_0(x_0) \xi_T^{*2} T \quad \text{and} \quad \mathbb{V}_{x_0}(A_T) \leq c_0(x_0) T. \quad (4.25)$$

Moreover when $c(\cdot)$ is bounded on E then there exists $c_0 > 0$ such that for any $T > 0$,

$$\mathbb{V}(A_T^*) \leq c_0 \xi_T^* \mathbb{E}[A_T^*] \quad \text{and} \quad \mathbb{V}(A_T) \leq c_0 \mathbb{E}[A_T]. \quad (4.26)$$

Proof of Lemma 64. Let us fix $T > 0$. Recall that

$$A_T^* = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \alpha(X_s) \int_E \xi_T(z) f(X_s, z) d\nu(z) ds$$

and

$$A_T = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \alpha(X_s) ds.$$

We first prove (4.25). We introduce the functions g and g^* defined on E by

$$g^*(\chi) = \kappa_1 \left(\frac{d(x, \chi)}{h_T} \right) \alpha(\chi) \int_E \xi_T(z) f(\chi, z) d\nu(z)$$

and

$$g(\chi) = \kappa_1 \left(\frac{d(x, \chi)}{h_T} \right) \alpha(\chi)$$

for any $\chi \in E$. Note that g and g^* are measurable and bounded functions on E with

$$\|g^*\|_\infty \leq \|\kappa_1\|_\infty \alpha^* \xi_T^* \quad \text{and} \quad \|g\|_\infty \leq \|\kappa_1\|_\infty \alpha^*,$$

by (H2). Then by Proposition 58 we obtain for any $x_0 \in E$,

$$\begin{aligned} \mathbb{V}_{x_0}(A_T^*) &= \mathbb{V}_{x_0} \left(\int_0^T g^*(X_s) ds \right) \\ &\leq \frac{2 \|\kappa_1\|_\infty \alpha^* \xi_T^*}{r} \left(3c(x_0) \|\kappa_1\|_\infty \alpha^* \xi_T^* + \|\kappa_1\|_\infty \alpha^* \xi_T^* \int_E c(z) d\mu_\infty(z) \right) T \\ &\leq c_0(x_0) \xi_T^{*2} T \end{aligned}$$

for some $c_0(x_0) > 0$ independent of $T > 0$. Using again Proposition 58 we obtain for any $x_0 \in E$,

$$\begin{aligned} \mathbb{V}_{x_0}(A_T) &= \mathbb{V}_{x_0} \left(\int_0^T g(X_s) ds \right) \\ &\leq \frac{2 \|\kappa_1\|_\infty \alpha^*}{r} \left(3c(x_0) \|\kappa_1\|_\infty \alpha^* + \|\kappa_1\|_\infty \alpha^* \int_E c(z) d\mu_\infty(z) \right) T \\ &\leq c_0(x_0) T \end{aligned}$$

for some $c_0(x_0) > 0$ independent of $T > 0$.

Now we consider the special case when $c(\cdot)$ is bounded. By Proposition 58 we introduce $c_0 > 0$ such that

$$\mathbb{V}(A_T^*) \leq c_0 \|g^*\|_\infty \mathbb{E}[A_T^*] \leq c_0 \|\kappa_1\|_\infty \alpha^* \xi_T^* \mathbb{E}[A_T^*],$$

and

$$\mathbb{V}(A_T) \leq c_0 \|g\|_\infty \mathbb{E}[A_T] \leq c_0 \|\kappa_1\|_\infty \alpha^* \mathbb{E}[A_T]$$

which proves (4.26). □

Let us continue the proof of Theorem 63. Here $c_0(x_0)$, $t_0(x_0)$, c_0 and t_0 are generic positive constants that may change from line to line. In both cases $Tv_T^*(x) \xrightarrow{T \rightarrow +\infty} +\infty$ so we get $v_T^*(x) > 0$ for $t \geq t_0$.

First let us prove *i*). We compute the rate of convergence of the terms that appear in (4.21). Recall that

$$A_T^* = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \alpha(X_s) \int_E \xi_T(z) f(X_s, z) d\nu(z) ds.$$

By Corollary 57 *ii*) we get for $T \geq t_0(x_0)$,

$$\frac{\xi_T^*}{\mathbb{E}_{x_0}[A_T^*]} \leq \frac{c_0(x_0) \xi_T^*}{Tv_T^*(x)} \leq \frac{c_0(x_0) \xi_T^{*2}}{Tv_T^*(x)^2}$$

because $v_T^*(x) \leq \alpha^* \|\kappa_1\|_\infty \xi_T^*$. Applying Lemma 64 and Corollary 57 *ii*) we obtain for $T \geq t_0(x_0)$,

$$\frac{\mathbb{V}_{x_0}(A_T^*)}{\mathbb{E}_{x_0}[A_T^*]^2} \leq c_0(x_0) \frac{\xi_T^{*2} T}{\mathbb{E}_{x_0}[A_T^*]^2} \leq c_0(x_0) \frac{\xi_T^{*2}}{Tv_T^*(x)^2}.$$

We now dominate the term associated to the martingale M_T . We apply Lemma 62 with the function $g(z, z') = \kappa_1 \left(\frac{d(x, z)}{h_T} \right) \kappa_2 \left(\frac{d(y, z')}{c_T} \right)$ to obtain

$$\begin{aligned} & \mathbb{E}[M_T^2] \\ &= \mathbb{E} \left[\int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right)^2 \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right)^2 f(X_s, z) d\nu(z) \alpha(X_s) ds \right] \\ &\leq \|\kappa_1\|_\infty \|\kappa_2\|_\infty \left(f(x, y) \mathbb{E} \left[\int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) d\nu(z) \alpha(X_s) ds \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) |f(X_s, z) - f(x, y)| d\nu(z) \alpha(X_s) ds \right] \right) \\ &\leq c_0 f(x, y) \mathbb{E} \left[\xi_T(y) \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \alpha(X_s) ds \right] \\ &\quad + a c_0 \mathbb{E} \left[\int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) (d(X_s, x) + d(z, y))^\gamma d\nu(z) \alpha(X_s) ds \right] \end{aligned}$$

by (H4). Applying now (H3) we introduce $r_0 > 0$ such that $\text{supp}(\kappa_i) \subset [0, r_0]$ for $i = 1, 2$. Then

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) (d(X_s, x) + d(z, y))^\gamma d\nu(z) \alpha(X_s) ds \right] \\ &\leq c_0 \mathbb{E} \left[\int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) (d(X_s, x) + r_0 c_T)^\gamma d\nu(z) \alpha(X_s) ds \right] \\ &\leq c_0 \mathbb{E} \left[\int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) (r_0 h_T + r_0 c_T)^\gamma \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) d\nu(z) \alpha(X_s) ds \right] \\ &\leq c_0 (r_0 h_T + r_0 c_T)^\gamma \mathbb{E} \left[\xi_T(y) \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \alpha(X_s) ds \right]. \end{aligned}$$

In particular for $T \geq t_0$

$$\mathbb{E}[M_T^2] \leq c_0 \mathbb{E} \left[\xi_T(y) \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \alpha(X_s) ds \right] \leq c_0 \xi_T^* \mathbb{E}[A_T]. \quad (4.27)$$

Applying (4.27) and Corollary 57 *ii*) we obtain for $T \geq t_0(x_0)$,

$$\frac{\mathbb{E}_{x_0}[M_T^2]}{\mathbb{E}_{x_0}[A_T^*]^2} \leq \frac{\xi_T^* \mathbb{E}_{x_0}[A_T]}{\mathbb{E}_{x_0}[A_T^*]^2} \leq c_0(x_0) \frac{\xi_T^* T v_T(x)}{T^2 v_T^*(x)^2} = c_0(x_0) \frac{\xi_T^* v_T(x)}{T v_T^*(x)^2}.$$

It remains to control $\mathbb{E}_{x_0}[R_T^2]$. We write

$$\mathbb{E}_{x_0}[R_T^2] \leq 2 (\mathbb{E}_{x_0}[C_T^2] + \mathbb{E}_{x_0}[D_T^2])$$

where

$$C_T = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \left(\kappa_2 \left(\frac{d(y, z)}{c_T} \right) - \xi_T(y) f(x, y) \right) f(X_s, z) d\nu(z) \alpha(X_s) ds,$$

and

$$D_T = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E (\xi_T(y) f(x, y) - \xi_T(z) f(x, y)) f(X_s, z) d\nu(z) \alpha(X_s) ds.$$

First let us bound $\mathbb{E}_{x_0}[C_T^2]$. We get

$$\begin{aligned} |C_T| &\leq \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \left| \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) f(X_s, z) d\nu(z) - \xi_T(y) f(x, y) \right| \alpha(X_s) ds \\ &= \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \times \\ &\quad \left| \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) f(X_s, z) d\nu(z) - f(x, y) \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) d\nu(z) \right| \alpha(X_s) ds \\ &\leq \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) |f(X_s, z) - f(x, y)| d\nu(z) \alpha(X_s) ds \\ &\leq a \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) (d(x, X_s) + d(y, z))^\gamma d\nu(z) \alpha(X_s) ds \end{aligned}$$

by (H4). Applying again (H3) we get

$$\begin{aligned} |C_T| &\leq a \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E \kappa_2 \left(\frac{d(y, z)}{c_T} \right) (r_0(c_T + h_T))^\gamma d\nu(z) \alpha(X_s) ds \\ &\leq a r_0^\gamma (c_T + h_T)^\gamma \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \xi_T(y) \alpha(X_s) ds \\ &\leq a r_0^\gamma (c_T + h_T)^\gamma \xi_T^* A_T. \end{aligned} \quad (4.28)$$

To conclude we notice by Corollary 57 *ii*) and Lemma 64 that for $T \geq t_0(x_0)$:

$$\begin{aligned} \frac{\mathbb{E}_{x_0}[A_T^2]}{\mathbb{E}_{x_0}[A_T^*]^2} &= \frac{\mathbb{E}_{x_0}[A_T]^2}{\mathbb{E}_{x_0}[A_T^*]^2} + \frac{\mathbb{V}_{x_0}(A_T)}{\mathbb{E}_{x_0}[A_T^*]^2} \\ &\leq c_0(x_0) \left(\frac{v_T(x)^2}{v_T^*(x)^2} + \frac{1}{Tv_T^*(x)^2} \right) \\ &\leq c_0(x_0) \left(\frac{v_T(x)^2}{v_T^*(x)^2} + \frac{1}{Tv_T(x)^2} \frac{v_T(x)^2}{v_T^*(x)^2} \right) \\ &\leq c_0(x_0) \frac{v_T(x)^2}{v_T^*(x)^2}, \end{aligned}$$

the last inequality coming from the fact that

$$Tv_T(x)^2 \geq \frac{1}{\xi_T^{*2}} Tv_T^*(x)^2 \xrightarrow{T \rightarrow +\infty} +\infty.$$

Then for $T \geq t_0(x_0)$,

$$\frac{\mathbb{E}_{x_0}[C_T^2]}{\mathbb{E}_{x_0}[A_T^*]^2} \leq c_0(x_0) (c_T + h_T)^{2\gamma} \frac{\xi_T^{*2} v_T(x)^2}{v_T^*(x)^2}.$$

We end the proof of (4.23) by finding a bound to $\mathbb{E}_{x_0}[D_T^2]$. Remark that

$$\begin{aligned} D_T &= \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_E (\xi_T(y)f(x, y) - \xi_T(z)f(x, y)) f(X_s, z) d\nu(z) \alpha(X_s) ds \\ &= f(x, y) (\xi_T(y)A_T - A_T^*). \end{aligned}$$

We compute

$$\begin{aligned} &\mathbb{E}_{x_0}[D_T^2] \\ &= f(x, y)^2 \mathbb{E}_{x_0} \left[(\xi_T(y)A_T - \xi_T(y) \mathbb{E}_{x_0}[A_T] + \mathbb{E}_{x_0}[A_T^*] - A_T^* + \xi_T(y)A_T - A_T^*)^2 \right] \\ &\leq 3f(x, y)^2 \left(\xi_T^{*2} \mathbb{V}_{x_0}(A_T) + \mathbb{V}_{x_0}(A_T^*) + \mathbb{E}_{x_0} [\xi_T(y)A_T - A_T^*]^2 \right) \\ &\leq c_0(x_0) \xi_T^{*2} T + 3f(x, y)^2 \mathbb{E}_{x_0} [\xi_T(y)A_T - A_T^*]^2 \end{aligned} \tag{4.29}$$

by Lemma 64. Applying Corollary 57 *i*) to

$$k : (z, t) \mapsto \kappa_1 \left(\frac{d(x, z)}{h_T} \right) \alpha(z) \int_E |\xi_T(y) - \xi_T(z')| f(z, z') d\nu(z')$$

we get for $T \geq t_0(x_0)$:

$$\begin{aligned} &\mathbb{E}_{x_0} [\xi_T(y)A_T - A_T^*]^2 \\ &\leq \mathbb{E}_{x_0} \left[\int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \alpha(X_s) \int_E |\xi_T(y) - \xi_T(z)| f(X_s, z) d\nu(z) \right]^2 \\ &\leq c_0(x_0) (1 + Tv_T^*(x)w_T(x, y))^2. \end{aligned}$$

So we obtain for $T \geq t_0(x_0)$:

$$\frac{\mathbb{E}_{x_0}[D_T^2]}{\mathbb{E}_{x_0}[A_T^*]^2} \leq c_0(x_0) \left(\frac{\xi_T^{*2}}{Tv_T^*(x)^2} + w_T(x, y)^2 + \frac{1}{T^2v_T^*(x)^2} \right).$$

Finally we observe that for $T \geq t_0(x_0)$

$$\frac{1}{T^2v_T^*(x)^2} \leq c_0(x_0) \frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2}$$

since

$$\xi_T^* v_T(x) \geq v_T^*(x) \geq \frac{Tv_T^*(x)}{T} \geq \frac{c_0(x)}{T}$$

for large T .

Now we prove *ii*). We follow the same way as in the one of *i*). We obtain by Corollary 57 *ii*) and by Lemma 64 for $T \geq t_0$:

$$\begin{aligned} \frac{\xi_T^*}{\mathbb{E}[A_T^*]} &\leq c_0 \frac{\xi_T^*}{Tv_T^*(x)} \\ &\leq c_0 \frac{v_T^*(x)}{v_T(x)} \times \frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2} \\ &\leq c_0 \xi_T^* \frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2} \\ &\leq c_0 \frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2} \end{aligned}$$

and

$$\begin{aligned} \frac{\mathbb{V}(A_T^*)}{\mathbb{E}[A_T^*]^2} &\leq c_0 \frac{\xi_T^*}{\mathbb{E}[A_T^*]} \\ &\leq c_0 \frac{\xi_T^*}{Tv_T^*(x)} \\ &\leq c_0 \frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2}. \end{aligned}$$

Applying (4.27) we also obtain for $T \geq t_0$:

$$\frac{\mathbb{E}[M_T^2]}{\mathbb{E}[A_T^*]^2} \leq \frac{\xi_T^* \mathbb{E}[A_T]}{\mathbb{E}[A_T^*]^2} \leq c_0 \frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2}.$$

As in (4.28) we obtain

$$\frac{\mathbb{E}[C_T^2]}{\mathbb{E}[A_T^*]^2} \leq c_0 (h_T + c_T)^{2\gamma} \frac{\xi_T^{*2} \mathbb{E}[A_T^2]}{\mathbb{E}[A_T^*]^2},$$

where

$$\begin{aligned} \frac{\mathbb{E}[A_T^2]}{\mathbb{E}[A_T^*]^2} &\leq c_0 \left(\frac{v_T(x)^2}{v_T^*(x)^2} + \frac{v_T(x)}{Tv_T^*(x)^2} \right) \\ &\leq c_0 \frac{v_T(x)^2}{v_T^*(x)^2} \end{aligned}$$

for $T \geq t_0$. Indeed

$$\begin{aligned} \frac{v_T(x)}{Tv_T^*(x)^2} &= \frac{1}{Tv_T(x)} \times \frac{v_T(x)^2}{v_T^*(x)^2} \\ &\leq \frac{\xi_T^*}{Tv_T^*(x)} \times \frac{v_T(x)^2}{v_T^*(x)^2} \\ &= \underset{T \rightarrow +\infty}{o} \left(\frac{v_T(x)^2}{v_T^*(x)^2} \right) \end{aligned}$$

as $Tv_T^*(x) \xrightarrow{T \rightarrow +\infty} +\infty$. Finally, we use Lemma 64 as in (4.29) and Corollary 57 *i*) to obtain

$$\begin{aligned} \frac{\mathbb{E}[D_T^2]}{\mathbb{E}[A_T]^2} &\leq \frac{3c_0}{T^2v_T^*(x)^2} \left(\xi_T^{*2} \mathbb{V}(A_T) + \mathbb{V}(A_T^*) + \mathbb{E}[\xi_T(y)A_T - A_T^*]^2 \right) \\ &\leq c_0 \left(\frac{\xi_T^{*2} v_T(x) + \xi_T^* v_T^*(x)}{Tv_T^*(x)^2} + w_T(x, y)^2 + \frac{1}{T^2v_T^*(x)^2} \right) \\ &\leq c_0 \left(\xi_T^* \frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2} + w_T(x, y)^2 \right) \\ &\leq c_0 \left(\frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2} + w_T(x, y)^2 \right) \end{aligned}$$

for $T \geq t_0$. □

4.4 Special case of the spatial birth-death-move process

In this section we use the methods introduced in Sections 4.2 and 4.3 to construct consistent estimators of the birth kernel and the death kernel of the birth-death-move process on \mathbb{R}^d .

Let $W \subset \mathbb{R}^d$ a closed set. Here we consider a spatial birth-death-move process defined on $E = \sqcup_{n \geq 0} E_n$ where $E_0 = \{\emptyset\}$ and $E_n = \pi_n(W^n)$ where π_n is the natural projection that identifies two elements $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ of W^n if there exists a permutation σ of $\{1, \dots, n\}$ such that $x_i = y_{\sigma(i)}$ for any $i \in \llbracket 1, n \rrbracket$.

Let $x \in E$. The transition kernel $K(x, \cdot)$ can be decomposed in this case as

$$K(x, A) = \frac{\beta(x)}{\alpha(x)} K_\beta(x, A) + \frac{\delta(x)}{\alpha(x)} K_\delta(x, A)$$

for all $A \subset E$. Here $K_\beta(x, \cdot)$ and $K_\delta(x, \cdot)$ are two probability transition kernels on E satisfying $K_\beta(x, x \cup W) = 1$ and $\sum_{i=1}^{n(x)} K_\delta(x, x \setminus x_i) = 1$. For $t \geq 0$ we denote by N_t^β the number of births that occurs in $[0, t]$ and N_t^δ the number of deaths that occurs in $[0, t]$.

Let us write the death kernel under the form

$$K_\delta(x, A) = \sum_{i=1}^{n(x)} \omega(x_i, x) \mathbf{1}_A(x \setminus x_i),$$

where $\omega : W \times E \rightarrow [0, 1]$.

We also suppose that K_β is written

$$K_\beta(x, x \cup B) = \int_B f_\beta(x, \xi) d\xi \quad (4.30)$$

for $B \subset W$ and $x \in E$ where $f_\beta : E \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ satisfies for any $x \in E$, $\int_W f_\beta(x, \xi) d\xi = 1$ and $f_\beta(x, \xi) > 0 \implies \xi \in W$.

In the following we estimate the functions f_β and ω .

4.4.1 Ergodicity of birth-death-move processes

First of all let us recall some ergodicity results for the birth-death-move process obtained in Chapter 3. In particular we check that (H1) is satisfied and that $c(\cdot)$ is integrable over μ_∞ . One can then apply the results of Sections 4.2.2 and 4.3.2 to estimate the functions w and f_β .

Lemma 65. *Hypothesis (H1) is equivalent to equation (3.7).*

Proof of Lemma 65. Suppose (H1) and let $A \in \mathcal{E}$. Then taking $g = \mathbf{1}_A$ in (H1) we get

$$|Q_t(x, A) - \mu_\infty(A)| \leq c(x)e^{-rt}$$

that is (3.7).

Conversely, suppose that (3.7) is satisfied. If $g = \mathbf{1}_A$ with $A \in \mathcal{E}$ then there exists $c_0(x) > 0$ and $r_0 > 0$ such that

$$\left| \int_E g(y)Q_t(x, dy) - \int_E g(y)\mu_\infty(dy) \right| = |Q_t(x, A) - \mu_\infty(A)| \leq c_0(x)e^{-r_0 t}.$$

If g is a measurable function taking values in $[0, 1]$ then g is approximated for $n \geq 1$ by the step function $g_n(z) = 2^{-n} \lfloor 2^n g(z) \rfloor$ and $\|g - g_n\|_\infty \leq 2^{-n}$. Let us introduce for any $j \in \{0, 2^n\}$

$$A_{j,n} = \{z \in E, j2^{-n} \leq g(z) < (j+1)2^{-n}\}$$

so that $g_n(z) = 2^{-n}j$ for any $z \in A_{j,n}$. Remark that $E = \bigsqcup_{j=0}^{2^n} A_{j,n}$ so

$$\begin{aligned} \left| \int_E g_n(y)Q_t(x, dy) - \int_E g_n(y)\mu_\infty(dy) \right| &\leq \sum_{j=0}^{2^n} 2^{-n}j |Q_t(x, A_{j,n}) - \mu_\infty(A_{j,n})| \\ &\leq 2^{-n}c_0(x)e^{-r_0 t} \sum_{j=0}^{2^n} j \leq \frac{2^n + 1}{2} c_0(x)e^{-r_0 t}. \end{aligned}$$

Then for any $n \geq 1$,

$$\begin{aligned} \left| \int_E g(y)Q_t(x, dy) - \int_E g(y)\mu_\infty(dy) \right| &\leq 2\|g_n - g\|_\infty + \frac{2^n + 1}{2} c_0(x)e^{-r_0 t} \\ &\leq 2^{-n+1} + \frac{2^n + 1}{2} c_0(x)e^{-r_0 t}. \end{aligned}$$

Let us introduce $\theta = \frac{\log(2e^{\frac{r_0 t}{2}} - 1)}{\log(2)}$ and $n_0 = \lfloor \theta \rfloor$. Then

$$\frac{2^{n_0} + 1}{2} c_0(x) e^{-r_0 t} \leq \frac{2^\theta + 1}{2} c_0(x) e^{-r_0 t} = c_0(x) e^{-\frac{r_0 t}{2}},$$

and

$$2^{-n_0+1} \leq 2^{-(\theta-1)-1} = 4 \times 2^{-\theta} = 4 \frac{e^{-\frac{r_0 t}{2}}}{2 - e^{-\frac{r_0 t}{2}}} \leq 4 e^{-\frac{r_0 t}{2}}.$$

So we obtain (H1) with $c(x) = 4 + c_0(x)$ and $r = \frac{r_0}{2}$ for any g taking values in $[0, 1]$. We get that (H1) is true for any positive bounded function g applying the last proof to $\frac{1}{\|g\|_\infty} g$. Finally the result to $g \in M_b(E)$ is obtained by considering the decomposition $g = g_+ - g_-$ where g_+ and g_- respectively denote the positive and negative part of g . \square

Proposition 66. *Let $(X_t)_{t \geq 0}$ a birth-death-move process such that the move $(Y_t)_{t \geq 0}$ is a Feller process and that $KC_0(E) \subset C_0(E)$. Assume either (3.2) or (3.3) along with*

$$\sum_{n=2}^{\infty} \sqrt{\frac{\beta_1 \dots \beta_{n-1}}{\delta_1 \dots \delta_n}} < \infty \quad \text{and} \quad \exists N \geq 0, \text{ s.t. } \forall n \geq N, \beta_n \leq \delta_{n+1}.$$

Then (H1) is satisfied and the function $c(\cdot)$ of (H1) satisfies

$$\int_E c(y) d\mu_\infty(y) < \infty \tag{4.31}$$

where μ_∞ denote the invariant probability measure of $(X_t)_{t \geq 0}$.

Proof of Proposition 66. Hypothesis (H1) is satisfied applying consecutively Corollary 40 and Lemma 65. We observe in the proof of Lemma 65 that the function $c(\cdot)$ of (H1) and the one of Corollary 40 have a linear relation. Since the function $c(\cdot)$ of Corollary 40 satisfies (4.31), then it is also true for the one of (H1). \square

4.4.2 Estimation of the birth density function

In this section we do the estimation of the birth density function f_β of a birth-death-move process defined by (4.30). In this context we observe the process on $[0, T]$ so we can apply the methods introduced in Section 4.3.

Asymptotic results

We take back the pseudo-distance d on E , the functions $\kappa_i \in M_b(\mathbb{R}_+, \mathbb{R}_+)$ introduced in Section 4.3.1 and the associated positive numbers $(c_i)_{i \geq 0}$ and $(h_t)_{t \geq 0}$. For $x \in E$ and $\xi \in W$ we estimate $f_\beta(x, \xi)$ by the estimator $\hat{f}_\beta(x, \xi)$:

$$\hat{f}_\beta(x, \xi) = \frac{\int_0^T \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \kappa_2 \left(\frac{|\xi - X_s \setminus X_{s^-}|}{c_T} \right) dN_s^\beta}{c_T^d \int_0^T \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) dN_s^\beta}. \tag{4.32}$$

To ensure that the estimator is well normalized we suppose that κ_2 satisfies the following assumption.

(H5) The function $\kappa_2 \in M_b(\mathbb{R}_+, \mathbb{R}_+)$ satisfies $\int_{\mathbb{R}^d} \kappa_2(|u|) du = 1$.

When (H5) is satisfied we have indeed

$$\begin{aligned} \int_{\mathbb{R}^d} \hat{f}_\beta(x, \xi) d\xi &= \frac{\int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \int_{\mathbb{R}^d} \kappa_2 \left(\frac{|\xi - X_s \setminus X_{s-}|}{c_T} \right) d\xi dN_s^\beta}{c_T^d \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) dN_s^\beta} \\ &= \frac{\int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) c_T^d \int_{\mathbb{R}^d} \kappa_2(|u|) d\xi dN_s^\beta}{c_T^d \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) dN_s^\beta} \\ &= 1. \end{aligned}$$

We consider a function ξ_T on \mathbb{R}^d similar to the one defined in Section 4.3 by (4.19):

$$\xi_T(z) = \int_{\mathbb{R}^d} \kappa_2 \left(\frac{|z - z'|}{c_T} \right) dz' = c_T^d \int_{\mathbb{R}^d} \kappa_2(|u|) du = c_T^d$$

for any $z \in \mathbb{R}^d$. Finally we add an hypothesis of regularity on the density function f_β to compute the rate of convergence of the estimator $\hat{f}_\beta(x, \xi)$. This is very similar to (H4).

(H4') There exist $\gamma > 0$ and $a \geq 0$ such that for any $(x_1, x_2) \in E \times E$ and $(\xi_1, \xi_2) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$|f_\beta(x_1, \xi_1) - f_\beta(x_2, \xi_2)| \leq a(d(x_1, x_2) + |\xi_1 - \xi_2|)^\gamma.$$

We will obtain in the following theorem simpler rates of convergence than in Theorem 63 since the function ξ_T is constant on \mathbb{R}^d . In this model we are also able to compute the mean squared error because $\hat{f}(x, \xi)$ is almost surely bounded but the obtained rate for

$$\mathbb{E} \left[\left| \hat{f}_\beta(x, \xi) - f_\beta(x, \xi) \right|^2 \right]$$

is rougher than for

$$\mathbb{P} \left(\left| \hat{f}_\beta(x, \xi) - f_\beta(x, \xi) \right| > \varepsilon \right).$$

Finally we define $v_T(x) = \int_E \kappa_1 \left(\frac{d(x, z)}{h_T} \right) \beta(z) d\mu_\infty(z)$ which corresponds to the quantity introduced before Theorem 63.

Theorem 67. *Let $(x, \xi) \in E \times W$. Suppose (H1), (H3), (H4') and (H5). Suppose that $c_T \xrightarrow{T \rightarrow +\infty} 0$ and $h_T \xrightarrow{T \rightarrow +\infty} 0$.*

i) Let $x_0 \in E$ and $\varepsilon > 0$. If $Tv_T(x)^2 \xrightarrow{T \rightarrow +\infty} +\infty$, then there exist $t_0(x_0) > 0$ and a constant $c_0(x_0) > 0$ such that for any $T \geq t_0(x_0)$

$$\mathbb{P}_{x_0} \left(\left| \hat{f}_\beta(x, \xi) - f_\beta(x, \xi) \right| > \varepsilon \right) \leq c_0(x_0) \left(\frac{1}{c_T^d Tv_T(x)} + \frac{1}{Tv_T(x)^2} + (c_T + h_T)^{2\gamma} \right) \quad (4.33)$$

and

$$\mathbb{E}_{x_0} \left[\left| \hat{f}_\beta(x, \xi) - f_\beta(x, \xi) \right|^2 \right] \leq c_0(x_0) \left(\frac{1}{c_T^{2d} Tv_T(x)^2} + (c_T + h_T)^{2\gamma} \right). \quad (4.34)$$

ii) Suppose that $c(\cdot)$ is bounded and $Tv_T(x) \xrightarrow{T \rightarrow +\infty} +\infty$. Let $\varepsilon > 0$. Then there exist $t_0 > 0$ and a positive constant $c_0 > 0$ such that for any $T \geq t_0$

$$\mathbb{P} \left(\left| \hat{f}_\beta(x, \xi) - f_\beta(x, \xi) \right| > \varepsilon \right) \leq c_0 \left(\frac{1}{c_T^d Tv_T(x)} + (c_T + h_T)^{2\gamma} \right) \quad (4.35)$$

and

$$\mathbb{E} \left[\left| \hat{f}_\beta(x, \xi) - f_\beta(x, \xi) \right|^2 \right] \leq c_0 \left(\frac{1}{c_T^{2d} Tv_T(x)} + (c_T + h_T)^{2\gamma} \right). \quad (4.36)$$

Proof of Theorem 67. Let us write

$$\hat{f}_\beta(x, \xi) - f_\beta(x, \xi) = \frac{M_T + R_T - c_T^d f_\beta(x, \xi)(Z_T - A_T)}{c_T^d Z_T}$$

where

$$Z_T = \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) dN_s^\beta,$$

$$A_T = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \beta(X_s) ds,$$

$$\begin{aligned} M_T = & \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \kappa_2 \left(\frac{|\xi - X_s \setminus X_{s-}|}{c_T} \right) dN_s^\beta \\ & - \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \int_W \kappa_2 \left(\frac{|\xi - z|}{c_T} \right) f_\beta(X_s, z) dz \beta(X_s) ds, \end{aligned}$$

and

$$R_T = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{h_T} \right) \left[\int_W \kappa_2 \left(\frac{|\xi - z|}{c_T} \right) f_\beta(X_s, z) dz - c_T^d f_\beta(x, \xi) \right] \beta(X_s) ds.$$

Firstly we obtain a non asymptotic bound of

$$\mathbb{P}\left(\left|\hat{f}_\beta(x, \xi) - f_\beta(x, x_i)\right| > \varepsilon\right)$$

and

$$\mathbb{E}\left[\left|\hat{f}_\beta(x, \xi) - f_\beta(x, \xi)\right|^2\right]$$

applying the proof of Proposition 61. This is the aim of the following lemma.

Lemma 68. *Let $(x, \xi) \in E \times W$. Suppose (H5). Then for any $\varepsilon > 0$ the estimator $\hat{f}(x, \xi)$ defined by (4.32) satisfies*

$$\mathbb{P}\left(\left|\hat{f}_\beta(x, \xi) - f_\beta(x, x_i)\right| > \varepsilon\right) \leq c_0 \left(\frac{1}{\mathbb{E}[A_T]} + \frac{1}{c_T^d \mathbb{E}[A_T]} + \frac{\mathbb{V}(A_T)}{\mathbb{E}[A_T]^2} + \frac{\mathbb{E}[R_T^2]}{c_T^{2d} \mathbb{E}[A_T]^2} \right), \quad (4.37)$$

and

$$\begin{aligned} & \mathbb{E}\left[\left|\hat{f}_\beta(x, \xi) - f_\beta(x, \xi)\right|^2\right] \\ & \leq c_0 \left(\frac{1}{\mathbb{E}[A_T]} + \frac{1}{c_T^d \mathbb{E}[A_T]} + \frac{1}{c_T^{2d} \mathbb{E}[A_T]} + \frac{\mathbb{E}[R_T^2]}{c_T^{2d} \mathbb{E}[A_T]^2} + \frac{\mathbb{V}(A_T)}{\mathbb{E}[A_T]^2} + \frac{\mathbb{V}(A_T)}{c_T^{2d} \mathbb{E}[A_T]^2} \right). \end{aligned} \quad (4.38)$$

for a given positive constants c_0 .

Proof. Since $\xi_T \equiv c_T^d$ we can apply the proof of Proposition 61. The calculations are readily the same so we obtain:

$$\mathbb{P}\left(\left|\hat{f}_\beta(x, \xi) - f_\beta(x, x_i)\right| > \varepsilon\right) \leq c_0 \left(\frac{c_T^d}{c_T^d \mathbb{E}[A_T]} + \frac{\mathbb{E}[M_T^2]}{c_T^{2d} \mathbb{E}[A_T]^2} + \frac{\mathbb{V}(A_T)}{\mathbb{E}[A_T]^2} + \frac{\mathbb{E}[R_T^2]}{c_T^{2d} \mathbb{E}[A_T]^2} \right).$$

Thanks to the convolution product in \mathbb{R}^d we can compute $\mathbb{E}[M_T^2]$. We use the proof of Lemma 62 to have:

$$\begin{aligned} \mathbb{E}[M_T^2] &= \mathbb{E}\left[\int_0^T \kappa_1\left(\frac{d(x, X_s)}{h_T}\right)^2 \int_W \kappa_2\left(\frac{|\xi - z|}{c_T}\right)^2 f_\beta(X_s, z) dz \beta(X_s) ds\right] \\ &= c_T^d \mathbb{E}\left[\int_0^T \kappa_1\left(\frac{d(x, X_s)}{h_T}\right)^2 \int_W \kappa_2(|u|)^2 f_\beta(X_s, \xi + c_T u) du \beta(X_s) ds\right] \\ &\leq c_T^d \|\kappa_1\|_\infty \|\kappa_2\|_\infty^2 \mathbb{E}\left[\int_0^T \kappa_1\left(\frac{d(x, X_s)}{h_T}\right) \int_W f_\beta(X_s, \xi + c_T u) du \beta(X_s) ds\right] \\ &\leq c_T^d \|\kappa_1\|_\infty \|\kappa_2\|_\infty^2 \mathbb{E}\left[\int_0^T \kappa_1\left(\frac{d(x, X_s)}{h_T}\right) \beta(X_s) ds\right] = c_T^d \mathbb{E}[A_T]. \end{aligned}$$

Therefore

$$\frac{\mathbb{E}[M_T^2]}{c_T^{2d} \mathbb{E}[A_T]^2} \leq \|\kappa_1\|_\infty \|\kappa_2\|_\infty^2 \frac{1}{c_T^d \mathbb{E}[A_T]}$$

which proves (4.37).

Remark that almost surely, $|\hat{f}_\beta(x, \xi)| \leq \frac{\|\kappa_2\|_\infty}{c_T^d}$. By taking back the calculations of the proof of Proposition 61 we get

$$\begin{aligned} & \mathbb{E} \left[\left| \hat{f}_\beta(x, \xi) - f_\beta(x, \xi) \right|^2 \right] \\ & \leq \mathbb{E} \left[\left| \hat{f}_\beta(x, \xi) - f_\beta(x, \xi) \right|^2 \mathbf{1}_{Z_T > \frac{\mathbb{E}[Z_T]}{2}} \right] + \left(f_\beta(x, \xi) + \frac{\|\kappa_2\|_\infty}{c_T^d} \right)^2 \mathbb{P} \left(Z_T < \frac{\mathbb{E}[Z_T]}{2} \right) \\ & \leq \frac{4}{c_T^{2d} \mathbb{E}[Z_T]^2} \mathbb{E} \left[\left| M_T + R_T - c_T^d f_\beta(x, \xi)(Z_T - A_T) \right|^2 \right] \\ & \quad + 2 \left(f_\beta(x, \xi)^2 + \frac{\|\kappa_2\|_\infty^2}{c_T^{2d}} \right) \left(\frac{\|\kappa_1\|_\infty}{\mathbb{E}[A_T]} + \frac{8 \mathbb{V}(A_T)}{\mathbb{E}[A_T]^2} \right). \end{aligned}$$

Finally,

$$\begin{aligned} & \frac{4}{c_T^{2d} \mathbb{E}[Z_T]^2} \mathbb{E} \left[\left| M_T + R_T - c_T^d f_\beta(x, \xi)(Z_T - A_T) \right|^2 \right] \\ & \leq \frac{12}{c_T^{2d} \mathbb{E}[A_T]^2} \left(\mathbb{E}[M_T^2] + \mathbb{E}[R_T^2] + c_T^{2d} f_\beta(x, \xi)^2 \mathbb{E}[(Z_T - A_T)^2] \right) \\ & \leq \frac{12 \|\kappa_1\|_\infty \|\kappa_2\|_\infty^2}{c_T^d \mathbb{E}[A_T]} + \frac{12 \mathbb{E}[R_T^2]}{c_T^{2d} \mathbb{E}[A_T]^2} + \frac{12 f_\beta(x, \xi)^2 \|\kappa_1\|_\infty}{\mathbb{E}[A_T]}. \end{aligned}$$

□

Let us continue the proof of Theorem 67. Here $c_0(x_0), t_0(x_0), c_0$ and t_0 are generic positive constants that may change from line to line. In both cases $Tv_T(x) \xrightarrow{T \rightarrow +\infty} +\infty$ so we get $v_T(x) > 0$ for $t \geq t_0$.

We start proving *i*). Let $x_0 \in E$. The function ξ_T is in particular bounded by c_T^d with $c_T^d \xrightarrow{T \rightarrow +\infty} 0$ and $\int_E c(z) d\mu_\infty(z) < \infty$ by Proposition 66. Therefore we can use the calculations of the proof of Theorem 63 *i*). As $\xi_T(z) \equiv c_T^d$ here the computation of $\mathbb{E}_{x_0}[R_T^2]$ is easier. Indeed taking back the notations introduced in the proof of Theorem 63, $R_T = C_T$. So

$$\frac{\mathbb{E}_{x_0}[R_T^2]}{c_T^{2d} \mathbb{E}_{x_0}[A_T]^2} \leq c_0(x_0) (h_T + c_T)^{2\gamma} \frac{c_T^{2d} v_T(x)^2}{c_T^{2d} v_T(x)^2}.$$

The other calculations are same as in Theorem 63 so (4.33) is obtained.

To prove (4.34) we apply the proof of Theorem 63 *i*) to the two added terms

$$\frac{1}{c_T^{2d} \mathbb{E}_{x_0}[A_T]} \text{ and } \frac{\mathbb{V}_{x_0}(A_T)}{c_T^{2d} \mathbb{E}_{x_0}[A_T]^2}$$

which are respectively bounded by

$$\frac{c_0(x_0)}{c_T^{2d} T v_T(x)}$$

and

$$\frac{c_0(x_0)}{c_T^{2d} T v_T(x)^2}.$$

To conclude, since $c_T \xrightarrow{T \rightarrow +\infty} 0$ and $|v_T(x)| \leq \beta^* \|\kappa_1\|_\infty$, we get for $t \geq t_0(x_0)$

$$\frac{1}{c_T^d T v_T(x)} + \frac{1}{c_T^{2d} T v_T(x)} + \frac{1}{T v_T^2(x)} \leq \frac{c_0(x_0)}{c_T^{2d} T v_T(x)^2}.$$

Now we establish the asymptotic behavior of $\hat{f}(x, y)$ when $c(\cdot)$ is bounded. The domination (4.35) is a direct consequence of the proof of Theorem 63 *ii*).

To prove (4.36) we also apply the proof of Theorem 63 to the two added terms

$$\frac{1}{c_T^{2d} \mathbb{E}_{x_0}[A_T]} \text{ and } \frac{\mathbb{V}_{x_0}(A_T)}{c_T^{2d} \mathbb{E}_{x_0}[A_T]^2}$$

which are respectively bounded by

$$\frac{c_0(x_0)}{c_T^{2d} T v_T(x)}$$

and

$$\frac{c_0(x_0)}{c_T^{2d} T v_T(x)}.$$

We can conclude since $c_T \xrightarrow{T \rightarrow +\infty} 0$. □

A particular form of the birth transition density

An interesting special case is when the births do not depend on the existing particles but only on the spatial location so that $f_\beta(x, \xi) = f_\beta(\xi)$. In this setting, we estimate $f_\beta(\xi)$ for any $\xi \in W$ by

$$\hat{f}_\beta(\xi) = \frac{1}{c_T^d N_T^\beta} \int_0^T \kappa_2 \left(\frac{|\xi - X_s \setminus X_{s-}|}{c_T} \right) dN_s^\beta.$$

We suppose through the following assumption that \emptyset is not an absorbing state for the birth-death-move process $(X_t)_{t \geq 0}$. Indeed, if the trajectories of $(X_t)_{t \geq 0}$ get stuck in \emptyset , then it will no longer be possible to estimate the distribution of the births.

(H6) The quantity $\beta_0 = \beta(\{\emptyset\})$ is positive.

We obtain from Proposition 68 the rate of convergence of $\hat{f}_\beta(\xi)$. In this particular case the obtained rate is simpler: it only depends on T and c_T .

Corollary 69. *Suppose (H1), (H3), (H4'), (H5) and (H6). Let $x_0 \in E$ and $\varepsilon > 0$. Suppose $c_T \xrightarrow{T \rightarrow +\infty} 0$. There exist a constant $c_0(x_0) > 0$ and a time $t_0(x_0) > 0$ such that for any $T \geq t_0(x_0)$*

$$\mathbb{P}_{x_0} \left(\left| \hat{f}_\beta(\xi) - f_\beta(\xi) \right| > \varepsilon \right) \leq c_0(x_0) \left(\frac{1}{c_T^d T} + c_T^{2\gamma} \right), \quad (4.39)$$

and

$$\mathbb{E}_{x_0} \left[\left| \hat{f}_\beta(\xi) - f_\beta(\xi) \right|^2 \right] \leq c_0(x_0) \left(\frac{1}{c_T^{2d} T} + c_T^{2\gamma} \right).$$

If we suppose moreover that the function $c(\cdot)$ is bounded then the constant $c_0(x_0)$ is uniform in $x_0 \in E$.

In (4.39), the optimal rate $T^{-2\gamma/(2\gamma+d)}$ is obtained for the choice $c_T = T^{-1/(2\gamma+d)}$ and corresponds to the standard minimax rate of convergence for the non-parametric estimation of a γ -Hölder continuous density function on \mathbb{R}^d .

Proof of Corollary 69. Here $c_0(x_0), t_0(x_0), c_0$ and t_0 are generic positive constants that may change from line to line. Follow the same proof as Theorem 67 writting

$$\hat{f}_\beta(\xi) - f_\beta(\xi) = \frac{M_T + R_T - f_\beta(\xi)(N_T^\beta - A_T)}{N_T^\beta}$$

where

$$A_T = \int_0^T \beta(X_s) ds,$$

$$M_T = \frac{1}{c_T^d} \int_0^T \kappa_2 \left(\frac{|\xi - X_s \setminus X_{s-}|}{c_T} \right) dN_s^\beta - \frac{1}{c_T^d} \int_0^T \int_W \kappa_2 \left(\frac{|\xi - z|}{c_T} \right) f_\beta(X_s, z) dz \beta(X_s) ds,$$

and

$$R_T = \int_0^T \left[\frac{1}{c_T^d} \int_W \kappa_2 \left(\frac{|\xi - z|}{c_T} \right) f_\beta(z) dz - f_\beta(\xi) \right] \beta(X_s) ds.$$

Here the quantity $v_T(x)$ introduced in Theorem 67 doesn't depend on x and T because

$$\mathbb{E} \left[\int_0^T \beta(X_s) ds \right]_{T \rightarrow +\infty} \sim T \int_E \beta(z) d\mu_\infty(z).$$

Let us check that $\int_E \beta(z) d\mu_\infty(z) > 0$. We get

$$\int_E \beta(z) d\mu_\infty(z) \geq \beta(\emptyset) \mu(\emptyset) = \beta_0 \mu(\emptyset) \geq \beta_0 \rho > 0$$

applying Lemma 52 and where ρ is defined in Section 3.3.4. As a consequence we get for $T \geq t_0(x_0)$

$$\frac{\mathbb{V}_{x_0}(A_T)}{\mathbb{E}_{x_0}[A_T]^2} \leq \frac{c_0(x_0)}{T}$$

whether $c(\cdot)$ is bounded or not. Finally it is easier to control $\mathbb{E}_{x_0}[R_T^2]$ because

$$\begin{aligned} |R_T| &\leq \int_0^T \int_W \kappa_2(u) |f_\beta(\xi + uc_T) - f_\beta(\xi)| du \beta(X_s) ds \\ &\leq c_0 c_T^\gamma \int_0^T \beta(X_s) ds. \end{aligned}$$

□

4.4.3 Estimation of the death kernel

Recall that we suppose that the death kernel can be written as

$$K_\delta(x, A) = \sum_{i=1}^{n(x)} \omega(x_i, x) \mathbf{1}_A(x \setminus x_i), \quad (4.40)$$

where $\omega : W \times E \rightarrow [0, 1]$ satisfies $\sum_{\xi \in x} \omega(\xi, x) = 1$. We take back the pseudo-distance d on E , the function $\kappa_1 \in M_b(\mathbb{R}_+, \mathbb{R}_+)$ introduced in Section 4.3.1 and the associated positive numbers $(c_t)_{t \geq 0}$. For $x = \{x_1, \dots, x_{n(x)}\}$ and $i \in \llbracket 1, n(x) \rrbracket$ we estimate $\omega(x_i, x)$ by

$$\hat{\omega}(x_i, x) = \frac{\int_0^T \kappa_1 \left(\frac{d(x, X_{s^-})}{c_T} \right) \mathbf{1}_{\{|x_i - X_{s^-} \setminus X_s| = \min_{1 \leq j \leq n(x)} |x_j - X_{s^-} \setminus X_s|\}} dN_s^\delta}{\int_0^T \kappa_1 \left(\frac{d(x, X_{s^-})}{c_T} \right) dN_s^\delta} \quad (4.41)$$

where $|\cdot|$ is the euclidian norm on \mathbb{R}^d . In the numerator of $\hat{\omega}(x_i, x)$ we are looking to the configurations X_{s^-} before a death which are close to x . For these configurations, we count 1 if the killed particle $X_{s^-} \setminus X_s$ is the one closest to x_i and 0 otherwise. The denominator, in turn, simply counts the number of deaths that occurs for a configuration X_{s^-} close to x . Thus $\hat{\omega}(x_i, x)$ properly estimates the probability that x_i is killed when the process is in the configuration x . In the sequel we introduce for any $t \geq 0$ the notation

$$X_t = \{X_{t,1}, X_{t,2}, \dots, X_{t,n(X_t)}\}.$$

We aim to control the asymptotic behavior of the mean squared error of $\hat{\omega}(x_i, x)$ when T goes to $+\infty$. In this setting we are able to apply the results of Section 4.2. For $x \in E$ and $i \in \llbracket 1, n(x) \rrbracket$ we define the quantities

$$v_T(x) = \int_E \kappa_1 \left(\frac{d(x, z)}{c_T} \right) \delta(z) d\mu_\infty(z)$$

and

$$w_T(x_i, x) = \frac{1}{v_T(x)} \int_E \kappa_1 \left(\frac{d(x, z)}{c_T} \right) \delta(z) \times \left(\sum_{k=1}^{n(z)} \mathbf{1}_{\{|x_i - z_k| = \min_{1 \leq j \leq n(x)} |x_j - z_k|\}} \omega(z, z_k) - \omega(x_i, x) \right) d\mu_\infty(z).$$

It is not easy to interpret the obtained rate $w_T(x_i, x)$ in a general case. In particular the nature of this term is related to the properties of the pseudo-distance d . It will be shown in Proposition 73 that with $d = d_1$ and $\omega(\cdot, \cdot)$ smooth we obtain $w_T(x_i, x) = \underset{T \rightarrow +\infty}{O} (c_T^\gamma)$ for some $\gamma > 0$.

Theorem 70. *Suppose (H1) and (H3). Suppose moreover that $c_T \xrightarrow{T \rightarrow +\infty} 0$. Let $x \in E$ and $i \in \llbracket 1, n(x) \rrbracket$. The estimator $\hat{\omega}(x_i, x)$ defined by (4.41) satisfies the following properties.*

i) Let $x_0 \in E$. If $Tv_T(x)^2 \xrightarrow{T \rightarrow +\infty} +\infty$ then there exist a constant $c_0(x_0) > 0$ and a time $t_0(x_0)$ such that for any $T \geq t_0(x_0)$

$$\mathbb{E}_{x_0}[|\hat{\omega}(x_i, x) - \omega(x_i, x)|^2] \leq c_0(x_0) \left(\frac{1}{Tv_T(x)^2} + w_T(x_i, x)^2 \right).$$

ii) Suppose that $c(\cdot)$ is bounded and $Tv_T(x) \xrightarrow{T \rightarrow +\infty} +\infty$. Then there exist a constant $c_0 > 0$ and a time t_0 such that for any $T \geq t_0$

$$\mathbb{E}[|\hat{\omega}(x_i, x) - \omega(x_i, x)|^2] \leq c_0 \left(\frac{1}{Tv_T(x)} + w_T(x_i, x)^2 \right).$$

Proof of Theorem 70. Our aim is to apply Proposition 59 and Theorem 60. The first step is to rewrite the martingale defined by (4.1) associated to some $g \in M_b(E)$ by making visible the function ω .

Lemma 71. Let $g \in \mathcal{M}_b(E^2)$. Suppose that the death kernel K_δ can be written under the form given by (4.40). Let us consider the process $(M_t^*)_{t \geq 0}$ defined by

$$M_t^* = \int_0^t g(X_{s-}, X_s) dN_s^\delta - \int_0^t \delta(X_s) \sum_{k=1}^{n(X_s)} g(X_s, X_s \setminus X_{s,k}) \omega(X_{s,k}, X_s) ds.$$

Then $(M_t^*)_{t \geq 0}$ is a $(\mathcal{F}_{t+})_{t \geq 0}$ -martingale satisfying

$$\mathbb{E}[M_t^{*2}] = \mathbb{E} \left[\int_0^t \delta(X_s) \sum_{k=1}^{n(X_s)} g(X_s, X_s \setminus X_{s,k})^2 \omega(X_{s,k}, X_s) ds \right].$$

Proof. Since δ is bounded we obtain by Lemma 54 that

$$\begin{aligned} M_t^* &= \int_0^t g(X_{s-}, X_s) dN_s^\delta - \int_0^t \delta(X_s) \int_E g(X_s, z) K(X_s, dz) ds \\ &= \int_0^t g(X_{s-}, X_s) dN_s^\delta - \int_0^t \delta(X_s) \sum_{k=1}^{n(X_s)} g(X_s, X_s \setminus X_{s,k}) \omega(X_{s,k}, X_s) ds \end{aligned}$$

is a $(\mathcal{F}_{t+})_{t \geq 0}$ -martingale. In this case Corollary 55 is rewritten as

$$\mathbb{E}[M_t^{*2}] = \mathbb{E} \left[\int_0^t \delta(X_s) \sum_{k=1}^{n(X_s)} g(X_s, X_s \setminus X_{s,k})^2 \omega(X_{s,k}, X_s) ds \right].$$

□

First we obtain a non asymptotic bound for the mean squared error

$$\mathbb{E}[|\hat{\omega}(x_i, x) - \omega(x_i, x)|^2].$$

This is the purpose of the following lemma which is based on Proposition 68. Let us introduce the quantities

$$A_T = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{c_T} \right) \delta(X_s) ds$$

and

$$R_T = \int_0^T \kappa_1 \left(\frac{d(x, X_s)}{c_T} \right) \delta(X_s) \times \left(\sum_{k=1}^{n(X_s)} \mathbf{1}_{\{|x_i - X_{s,k}| = \min_{1 \leq j \leq n(x)} |x_j - X_{s,k}|\}} \omega(X_{s,k}, X_s) - \omega(x_i, x) \right) ds.$$

Lemma 72. *Let $T > 0$, $x = \{x_1, x_2, \dots, x_{n(x)}\}$ and $i \in \llbracket 1, n(x) \rrbracket$. Then*

$$\mathbb{E}[|\hat{\omega}(x_i, x) - \omega(x_i, x)|^2] \leq \frac{32}{\mathbb{E}[A_T]} + 8 \frac{\mathbb{V}(A_T)}{\mathbb{E}[A_T]^2} + 12 \frac{\mathbb{E}[R_T^2]}{\mathbb{E}[A_T]^2}.$$

Proof. We decompose $\hat{\omega}(x_i, x)$ in the same spirit as before:

$$\hat{\omega}(x_i, x) - \omega(x_i, x) = \frac{M_T + R_T - \omega(x_i, x)(Z_T - A_T)}{Z_T}$$

with

$$Z_T = \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{c_T} \right) dN_s^\delta$$

and

$$M_T = \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{c_T} \right) \mathbf{1}_{\{|x_i - X_{s-} \setminus X_s| = \min_{1 \leq j \leq n(x)} |x_j - X_{s-} \setminus X_s|\}} dN_s^\delta - \int_0^T \delta(X_s) \kappa_1 \left(\frac{d(x, X_s)}{c_T} \right) \sum_{k=1}^{n(X_s)} \mathbf{1}_{\{|x_i - X_{s,k}| = \min_{1 \leq j \leq n(x)} |x_j - X_{s,k}|\}} \omega(X_{s,k}, X_s) ds.$$

Only the domination of $\mathbb{E}[M_T^2]$ differs from the proof of Proposition 59. Therefore we directly obtain

$$\mathbb{E}[|\hat{\omega}(x_i, x) - \omega(x_i, x)|^2] \leq \frac{8\|\kappa_3\|_\infty}{\mathbb{E}[A_T]} + \frac{8\mathbb{V}(A_T)}{\mathbb{E}[A_T]^2} + \frac{12\mathbb{E}[M_T^2]}{\mathbb{E}[A_T]^2} + \frac{12\mathbb{E}[R_T^2]}{\mathbb{E}[A_T]^2} + \frac{12\|\kappa_3\|_\infty}{\mathbb{E}[A_T]}.$$

To compute the mean square of the martingale we apply Lemma 71 to

$$g(y, z) = \kappa_1 \left(\frac{d(x, y)}{c_T} \right) \mathbf{1}_{\{|x_i - y \setminus z| = \min_{1 \leq j \leq n(x)} |x_j - y \setminus z|\}}.$$

Thus we get

$$\begin{aligned} \mathbb{E}[M_T^2] &= \mathbb{E} \left[\int_0^T \kappa_3 \left(\frac{d(x, X_s)}{k_T} \right)^2 \delta(X_s) \sum_{k=1}^{n(X_s)} \mathbf{1}_{\{|x_i - X_{s,k}| = \min_{1 \leq j \leq n(x)} |x_j - X_{s,k}|\}} \omega(X_{s,k}, X_s) ds \right] \\ &\leq \mathbb{E} \left[\int_0^T \kappa_3 \left(\frac{d(x, X_s)}{k_T} \right)^2 \delta(X_s) \sum_{k=1}^{n(X_s)} \omega(X_{s,k}, X_s) ds \right] \\ &= \mathbb{E} \left[\int_0^T \kappa_3 \left(\frac{d(x, X_s)}{k_T} \right)^2 \delta(X_s) ds \right] \\ &\leq \|\kappa_3\|_\infty \mathbb{E}[A_T]. \end{aligned}$$

□

Let us pursue the proof of Theorem 70. Here $c_0(x_0), t_0(x_0), c_0$ and t_0 are generic positive constants that may change from line to line. In both cases $Tv_T(x) \xrightarrow{T \rightarrow +\infty} +\infty$ so we get $v_T(x) > 0$ for $t \geq t_0$.

We start proving *i*). Let $x_0 \in E$. We apply Theorem 60 *i*). The condition (4.15) is satisfied by Proposition 66. Taking back the proof of Theorem 60 *i*) we get immediatly for $t \geq t_0(x_0)$:

$$\frac{1}{\mathbb{E}_{x_0}[A_T]} + \frac{\mathbb{V}_{x_0}(A_T)}{\mathbb{E}_{x_0}[A_T^2]} \leq \frac{c_0(x_0)}{Tv_T(x)^2}.$$

In the same way as in the proof of Theorem 60 and applying Corollary 57 we obtain for $t \geq t_0(x_0)$

$$\frac{\mathbb{E}_{x_0}[R_T^2]}{\mathbb{E}_{x_0}[A_T]^2} \leq c_0(x_0) \frac{1 + T^2v_T(x)^2w_T(x)^2}{T^2v_T(x)^2}.$$

Since

$$\frac{1}{T^2v_T(x)^2} \leq \frac{c_0}{Tv_T^2(x)},$$

i) is proven.

To prove *ii*) we apply the proof of Theorem 60 *ii*) in the same way. □

In the following proposition we expose the behaviour of $w_T(x_i, x)$ when d is the distance d_1 introduced by (2.8). We also make the following regularity assumption on $\omega(., .)$.

(H7) There exist $a > 0$ and $\gamma > 0$ such that for any $y = \{y_1, y_2, \dots, y_{n(y)}\}$ and $z = \{z_1, z_2, \dots, z_{n(z)}\} \in E$,

$$|\omega(y_k, y) - \omega(z_j, z)| \leq a (d_1(y, z) + |y_k - z_j|)^\gamma, \quad \forall k \in \llbracket 1, n(y) \rrbracket, \quad \forall j \in \llbracket 1, n(z) \rrbracket.$$

Proposition 73. *Let $x = \{x_1, x_2, \dots, x_{n(x)}\} \in E$ and $i \in \llbracket 1, n(x) \rrbracket$. Suppose (H3), (H7) and that the pseudo-distance d used in the estimator $\hat{\omega}(x_i, x)$ is the distance d_1 defined by (2.8). Suppose moreover that $c_T \xrightarrow{T \rightarrow +\infty} 0$. Then there exist a positive constant c_0 and a time t_0 such that*

$$|w_T(x_i, x)| \leq c_0 c_T^\gamma$$

for $T \geq t_0$.

Proof of Proposition 73. In this proof $t_0 > 0$ is a generic positive constant that may change from line to line.

In a first step we show that for $T \geq t_0$ the integral term appearing in $w_T(x_i, x)$ is reduced to an integral on $E_{n(x)}$. This is a similar result than Proposition 16 because κ_1 is compactly supported. Therefore we shall prove that for any $z \in E$,

$$n(z) \neq n(x) \implies \forall T \geq t_0, \kappa_1 \left(\frac{d_1(x, z)}{c_T} \right) = 0. \tag{4.42}$$

Let $z \in E$ such that $n(z) < n(x)$. By (2.8) we get

$$d_1(x, z) \geq \frac{n(x) - n(z)}{n(x)} \geq \frac{1}{n(x)}.$$

Also, for $z \in E$ with $n(z) > n(x)$ we have

$$d_1(x, z) \geq \frac{n(x) - n(z)}{n(z)} \geq \frac{1}{n(x) + 1}.$$

We introduce $r_0 > 0$ such that $r > r_0 \implies \kappa_1(r) = 0$. Since $c_T \xrightarrow{T \rightarrow +\infty} 0$, we get $c_T \leq \frac{1}{2r_0(n(x) + 1)}$ for $T \geq t_0$. Therefore for any $z \notin E_{n(x)}$, $\frac{d_1(x, z)}{c_T} \geq 2r_0$ and then $\kappa_1\left(\frac{d_1(x, z)}{c_T}\right) = 0$. To conclude we obtain for $T \geq t_0$,

$$\begin{aligned} w_T(x_i, x) &= \frac{1}{v_T(x)} \int_{E_{n(x)}} \kappa_1\left(\frac{d(x, z)}{c_T}\right) \delta(z) \\ &\quad \times \left(\sum_{k=1}^{n(x)} \mathbf{1}_{\{|x_i - z_k| = \min_{1 \leq j \leq n(x)} |x_j - z_k|\}} \omega(z_k, z) - \omega(x_i, x) \right) d\mu_\infty(z). \end{aligned}$$

The second step consists in proving for $z \in E_{n(x)}$ and $T \geq t_0$

$$d_1(x, z) \leq r_0 c_T \implies \sum_{k=1}^{n(x)} \mathbf{1}_{\{|x_i - z_k| = \min_{1 \leq j \leq n(x)} |x_j - z_k|\}} = 1. \quad (4.43)$$

Since $c_T \xrightarrow{T \rightarrow +\infty} 0$, we get for $T \geq t_0$:

$$c_T \leq \frac{1}{3n(x)r_0} \min_{1 \leq j \neq k \leq n(x)} |x_k - x_j|.$$

Let $z \in E_{n(x)}$ such that $d_1(x, z) \leq r_0 c_T$ (if not then $\kappa_1(d_1(x, y)/c_T) = 0$). By (2.8) we fix a permutation σ of $\{1, 2, \dots, n(x)\}$ such that

$$\frac{1}{n(x)} \sum_{i=1}^{n(x)} |x_i - z_{\sigma(i)}| \leq \frac{1}{3n(x)} \min_{1 \leq j \neq k \leq n(x)} |x_k - x_j|$$

and thus

$$|x_i - z_{\sigma(i)}| \leq \frac{1}{3} \min_{1 \leq j \neq k \leq n(x)} |x_k - x_j|.$$

Applying the reverse triangle inequality we directly get for any $k \in \llbracket 1, n(x) \rrbracket$,

$$k = \sigma(i) \Leftrightarrow |x_i - z_k| = \min_{1 \leq j \leq n(x)} |x_j - z_k| \quad (4.44)$$

that proves (4.43).

The quantity $w_T(x_i, x)$ then satisfies for $T \geq t_0$:

$$\begin{aligned}
|w_T(x_i, x)| &= \left| \frac{1}{v_T(x)} \int_{E_n(x)} \kappa_1 \left(\frac{d(x, z)}{c_T} \right) \delta(z) \right. \\
&\quad \left. \times \sum_{k=1}^{n(x)} \mathbf{1}_{\{|x_i - z_k| = \min_{1 \leq j \leq n(x)} |x_j - z_k|\}} (\omega(z_k, z) - \omega(x_i, x)) \, d\mu_\infty(z) \right| \\
&\leq \frac{1}{v_T(x)} \int_{E_n(x)} \kappa_1 \left(\frac{d(x, z)}{c_T} \right) \delta(z) \\
&\quad \times \sum_{k=1}^{n(x)} \mathbf{1}_{\{|x_i - z_k| = \min_{1 \leq j \leq n(x)} |x_j - z_k|\}} |\omega(z_k, z) - \omega(x_i, x)| \, d\mu_\infty(z) \\
&\leq \frac{1}{v_T(x)} \int_{E_n(x)} \kappa_1 \left(\frac{d(x, z)}{c_T} \right) \delta(z) \\
&\quad \times \sum_{k=1}^{n(x)} \mathbf{1}_{\{|x_i - z_k| = \min_{1 \leq j \leq n(x)} |x_j - z_k|\}} |\omega(z_{\sigma(i)}, z) - \omega(x_i, x)| \, d\mu_\infty(z)
\end{aligned}$$

where $\sigma(\cdot)$ satisfies (4.44). Using (H7) we get for $T \geq t_0$:

$$\begin{aligned}
|w_T(x_i, x)| &\leq \frac{a}{v_T(x)} \int_{E_n(x)} \kappa_1 \left(\frac{d(x, z)}{c_T} \right) \delta(z) \\
&\quad \times \sum_{k=1}^{n(x)} \mathbf{1}_{\{|x_i - z_k| = \min_{1 \leq j \leq n(x)} |x_j - z_k|\}} (r_0 c_T + r_0 c_T n(x))^\gamma \, d\mu_\infty(z) \\
&\leq \frac{a}{v_T(x)} (r_0 c_T (n(x) + 1))^\gamma \int_{E_n(x)} \kappa_1 \left(\frac{d(x, z)}{c_T} \right) \delta(z) \, d\mu_\infty(z).
\end{aligned}$$

Finally we get for $T \geq t_0$,

$$\int_{E_n(x)} \kappa_1 \left(\frac{d(x, z)}{c_T} \right) \delta(z) \, d\mu_\infty(z) = \int_E \kappa_1 \left(\frac{d(x, z)}{c_T} \right) \delta(z) \, d\mu_\infty(z) = v_T(x)$$

by (4.42). □

4.5 A discrete time estimator for the density

In this section we place ourselves in the setting of Section 4.3: we suppose that the transition kernel K is written under the form

$$K(x, A) = \int_A f(x, y) \, d\nu(y)$$

for $A \subset E$, $x \in E$, ν a σ -finite measure on E and $f : E^2 \rightarrow \mathbb{R}_+$ satisfies $\int_E f(x, y) \, d\nu(y) = 1$ for any $x \in E$. We take back the notations used in Section 4.3. However, in this section, we estimate the transition density f from discrete time observations. We then get $m \geq 1$ such that the process is observed at times $t_1 = 0, t_2, \dots, t_m = T$. Let us denote $\Delta t_j = t_j - t_{j-1}$, $\Delta_m = \max_{1 \leq j \leq m} \Delta t_j$ and

$\Delta N_{t_j} = N_{t_j} - N_{t_{j-1}}$. We suppose that the quantity $\frac{m\Delta_m}{T}$ is uniformly bounded. In this setting we thus have

$$T \leq m\Delta_m \leq c_0 T$$

for some $c_0 > 0$. For $j \in \llbracket 1, m \rrbracket$, let D_j an observable approximation of ΔN_{t_j} satisfying the following assumption.

(H8) For $j \in \llbracket 1, m \rrbracket$, $D_j = \Delta N_{t_j}$ when $\Delta N_{t_j} \leq 1$ and $D_j \leq \Delta N_{t_j}$ otherwise.

Let $(x, y) \in E^2$. We estimate the density function $f(x, y)$ by

$$\hat{f}_d(x, y) = \frac{\sum_{j=0}^{m-1} \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) D_{j+1}}{\sum_{j=0}^{m-1} \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) D_{j+1}}. \quad (4.45)$$

Let us finally introduce the following hypothesis on the move process $(Y_t)_{t \geq 0}$.

(H9) For $T \geq 0$, let g_T be either $\kappa_1(d(x, \cdot)/h_T)$, $\kappa_2(d(y, \cdot)/c_T)$ or $\xi_T(\cdot)$. We suppose that there exist $l_T(x, y) \geq 0$ and $a > 0$ such that for any $u, v \geq 0$ and $z \in E$,

$$|u - v| \leq \Delta_m \implies \mathbb{E}_z^Y [|g_T(Y_u) - g_T(Y_v)|] \leq l_T(x, y) |u - v|^a$$

and

$$l_T(x, y) \frac{\Delta_m^a}{v_T^*(x)^2} \xrightarrow{T \rightarrow +\infty} 0.$$

Theorem 74. *Let $(x, y) \in E^2$. Suppose (H1), (H2), (H3), (H4), (H8) and (H9). Let us suppose that $h_T \xrightarrow{T \rightarrow +\infty} 0$, $c_T \xrightarrow{T \rightarrow +\infty} 0$, $\xi_T^* \xrightarrow{T \rightarrow +\infty} 0$ and $\Delta_m \xrightarrow{m \rightarrow +\infty} 0$. We have the two following results on the discrete estimator $\hat{f}_d(x, y)$ of $f(x, y)$ defined by (4.45).*

i) *Suppose that $\int_E c(z) d\mu_\infty(z) < \infty$ and $Tv_T^*(x)^2 \xrightarrow{T \rightarrow +\infty} +\infty$. Let $x_0 \in E$ and $\varepsilon > 0$. There exist $t_0(x_0) > 0$, $m_0 \in \mathbb{N}^*$ and $c_0(x_0) > 0$ such that for any $T \geq t_0(x_0)$ and $m \geq m_0$,*

$$\mathbb{P}_{x_0} \left(\left| \hat{f}_d(x, y) - f(x, y) \right| > \varepsilon \right) \leq c_0(x_0) \left(\frac{\xi_T^{*2}}{Tv_T^*(x)^2} + \frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2} + w_T(x, y)^2 + (h_T + c_T)^{2\gamma} \frac{\xi_T^{*2} v_T(x)^2}{v_T^*(x)^2} + \frac{l_T(x, y) \Delta_m^a}{v_T^*(x)^2} + \frac{\xi_T^* \Delta_m}{v_T^*(x)^2} \right).$$

ii) *Suppose that $c(\cdot)$ is bounded and $Tv_T^*(x) \xrightarrow{T \rightarrow +\infty} +\infty$. Let $\varepsilon > 0$. There exist $t_0 > 0$, $m_0 \geq 1$ and $c_0 > 0$ such that for any $T \geq t_0$ and $m \geq m_0$,*

$$\mathbb{P} \left(\left| \hat{f}_d(x, y) - f(x, y) \right| > \varepsilon \right) \leq c_0 \left(\frac{\xi_T^* v_T(x)}{Tv_T^*(x)^2} + w_T(x, y)^2 + (h_T + c_T)^{2\gamma} \frac{\xi_T^{*2} v_T(x)^2}{v_T^*(x)^2} + \frac{l_T(x, y) \Delta_m^a}{v_T^*(x)^2} + \frac{\xi_T^* \Delta_m}{v_T^*(x)^2} \right).$$

Proof of Theorem 74. In the following lemma we present a control of the number of jumps that could occur in the intervals $(t_{j-1}, t_j]$, $j \in \llbracket 1, m \rrbracket$.

Lemma 75. *For any $j \in \llbracket 1, m \rrbracket$ and $i \geq 0$ we get*

$$\mathbb{P}(\Delta N_{t_j} = i) \leq \frac{(\alpha^* \Delta t_j)^i}{i!}. \quad (4.46)$$

If moreover $\Delta_m \xrightarrow{m \rightarrow +\infty} 0$, then there exist $c_0 > 0$ and $m_0 \in \mathbb{N}^*$ such that

$$\mathbb{E} \left[\sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} \right] \leq c_0 m \Delta_m^2 \quad (4.47)$$

for any $m \geq m_0$.

Proof. We first prove (4.46). Let $j \in \llbracket 1, m \rrbracket$. Using Markov property of the counting process $(N_t)_{t \geq 0}$ we write

$$\begin{aligned} \mathbb{P}(\Delta N_{t_j} = i) &= \mathbb{E} \left[\mathbb{E} \left[\mathbf{1}_{N_{t_j} = N_{t_{j-1}} + i} \mid \mathcal{F}_{t_{j-1}} \right] \right] \\ &= \mathbb{E} \left[\mathbb{E}_{X_{t_{j-1}}} \left[\mathbf{1}_{N_{\Delta t_j} = i} \right] \right] \\ &\leq \sup_{x \in E} \mathbb{P}_x(N_{\Delta t_j} = i). \end{aligned}$$

It suffices to show for any $t \geq 0$ and $i \in \mathbb{N}$,

$$\sup_{x \in E} \mathbb{P}_x(N_t = i) \leq \frac{(\alpha^* t)^i}{i!}.$$

For $i = 0$ it is immediate because for any $x \in E$, $\mathbb{P}_x(N_t = 0) \leq 1$. Let $i \geq 1$. Then

$$\mathbb{P}_x(N_t = i) = \mathbb{E}_x \left[\mathbf{1}_{N_t = i} \right] = \mathbb{E}_x \left[\mathbf{1}_{T_1 < \dots < T_i \leq t < T_{i+1}} \right].$$

Recall from Lemma 19 that the vector (T_1, \dots, T_{i+1}) admits a density under \mathbb{P}_x and then:

$$\begin{aligned} &\mathbb{E}_x \left[\mathbf{1}_{T_1 < \dots < T_i \leq t < T_{i+1}} \right] \\ &= \int_{[0, t]^i} \int_t^{+\infty} \mathbb{E}_x \left[\alpha(Y_{t_1}^{(0)}) e^{-\int_0^{t_1} \alpha(Y_u^{(0)}) du} \right] \prod_{j=2}^{i+1} \mathbb{E}_x \left[\alpha(Y_{t_j - t_{j-1}}^{(j-1)}) e^{-\int_0^{t_j - t_{j-1}} \alpha(Y_u^{(j-1)}) du} \right] \\ &\quad \times \mathbf{1}_{t_1 < \dots < t_{i+1}} dt_{i+1} \dots dt_1 \\ &= \int_{[0, t]^i} \mathbb{E}_x \left[\alpha(Y_{t_1}^{(0)}) e^{-\int_0^{t_1} \alpha(Y_u^{(0)}) du} \right] \prod_{j=2}^i \mathbb{E}_x \left[\alpha(Y_{t_j - t_{j-1}}^{(j-1)}) e^{-\int_0^{t_j - t_{j-1}} \alpha(Y_u^{(j-1)}) du} \right] \\ &\quad \times \int_{t-t_i}^{+\infty} \mathbb{E}_x \left[\alpha(Y_s^i) e^{-\int_0^s \alpha(Y_u^i) du} \right] ds \mathbf{1}_{t_1 < \dots < t_i} dt_i \dots dt_1 \\ &\leq (\alpha^*)^i \int_{[0, t]^i} \mathbb{E}_x \left[e^{-\int_0^{t-t_i} \alpha(Y_u^{(i)}) du} \right] \mathbf{1}_{t_1 < \dots < t_i} dt_i \dots dt_1 \\ &\leq (\alpha^*)^i \int_{[0, t]^i} \mathbf{1}_{t_1 < \dots < t_i} dt_1 \dots dt_i \\ &= (\alpha^*)^i \frac{t^i}{i!}, \end{aligned}$$

the last equality coming from the fact that the density of the joint probability distribution of the order statistics associated to a statistical sample of i uniform variables on $[0, t]$ is given by

$$(t_1, \dots, t_i) \in [0, t]^i \mapsto \frac{i!}{i^i} \mathbf{1}_{t_1 < \dots < t_i}.$$

We now prove (4.47) applying (4.46):

$$\begin{aligned} \mathbb{E} \left[\sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} \right] &= \sum_{j=0}^{m-1} \sum_{k \geq 2} k \mathbb{P}(\Delta N_{t_{j+1}} = k) \\ &= \sum_{j=1}^m \sum_{k \geq 2} k \mathbb{P}(\Delta N_{t_j} = k) \\ &\leq \sum_{j=1}^m \sum_{k \geq 2} \frac{(\alpha^* \Delta_{t_j})^k}{(k-1)!} \\ &\leq \alpha^* \Delta_m \sum_{j=1}^m \sum_{k \geq 1} \frac{(\alpha^* \Delta_m)^k}{k!} \\ &= \alpha^* \Delta_m m (e^{\alpha^* \Delta_m} - 1) \\ &\leq c_0 m \Delta_m^2 \end{aligned}$$

for sufficiently large $m \geq 1$ since $\Delta_m \xrightarrow{m \rightarrow +\infty} 0$. □

Let $\varepsilon > 0$. Note that

$$\begin{aligned} \mathbb{P} \left(|\hat{f}_d(x, y) - f(x, y)| > \varepsilon \right) &\leq \mathbb{P} \left(|\hat{f}(x, y) - f(x, y)| > \varepsilon/2 \right) \\ &\quad + \mathbb{P} \left(|\hat{f}_d(x, y) - \hat{f}(x, y)| > \varepsilon/2 \right). \end{aligned}$$

The asymptotic behaviour of

$$\mathbb{P} \left(|\hat{f}(x, y) - f(x, y)| > \varepsilon/2 \right)$$

is given by Theorem 63 so we only have to work on

$$\mathbb{P} \left(|\hat{f}_d(x, y) - \hat{f}(x, y)| > \varepsilon/2 \right).$$

We introduce the following quantities

$$Z_T^* = \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \xi_T(X_s) dN_s,$$

and

$$Z_T^d = \sum_{j=0}^{m-1} \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) D_{j+1}.$$

We decompose

$$\begin{aligned}
& |\hat{f}_d(x, y) - \hat{f}(x, y)| \\
&= \left| \frac{\sum_{j=0}^{m-1} \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) D_{j+1}}{Z_T^d} \right. \\
&\quad \left. - \frac{\int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s}{Z_T^*} \right| \\
&\leq \frac{1}{Z_T^d} \left| \sum_{j=0}^{m-1} \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) (D_{j+1} - \Delta N_{t_{j+1}}) \right| \\
&\quad + \frac{1}{Z_T^d} \left| \sum_{j=0}^{m-1} \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) \Delta N_{t_{j+1}} \right. \\
&\quad \quad \left. - \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \right| \\
&\quad + \frac{|Z_T^* - Z_T^d|}{Z_T^* Z_T^d} \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s. \tag{4.48}
\end{aligned}$$

We find a bound to these three terms. First

$$\begin{aligned}
& \frac{1}{Z_T^d} \left| \sum_{j=0}^{m-1} \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) (D_{j+1} - \Delta N_{t_{j+1}}) \right| \\
&\leq \frac{\|\kappa_1\|_\infty \|\kappa_2\|_\infty}{Z_T^d} \sum_{j=0}^{m-1} |D_{j+1} - \Delta N_{t_{j+1}}| \\
&\leq \frac{2\|\kappa_1\|_\infty \|\kappa_2\|_\infty}{Z_T^d} \sum_{j=0}^{m-1} \Delta N_{t_{j+1}} \mathbf{1}_{\Delta N_{t_{j+1}} \geq 2}. \tag{4.49}
\end{aligned}$$

The second term is

$$\begin{aligned}
& \frac{1}{Z_T^d} \left| \sum_{j=0}^{m-1} \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) \Delta N_{t_{j+1}} \right. \\
&\quad \left. - \int_0^T \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \right| \\
&\leq \frac{1}{Z_T^d} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) \Delta N_{t_{j+1}} \right. \\
&\quad \left. - \int_{t_j}^{t_{j+1}} \kappa_1 \left(\frac{d(x, X_{s-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \right|.
\end{aligned}$$

For $j \in \llbracket 0, m-1 \rrbracket$ we compute the term within the sum according to the value of

$\Delta N_{t_{j+1}}$. When there is no jump in $(t_j, t_{j+1}]$ we get

$$\left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) \Delta N_{t_{j+1}} - \int_{t_j}^{t_{j+1}} \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \right| \mathbf{1}_{\Delta N_{t_{j+1}}=0} = 0.$$

If it occurs exactly one jump in $(t_j, t_{j+1}]$ we write

$$\begin{aligned} & \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) \Delta N_{t_{j+1}} - \int_{t_j}^{t_{j+1}} \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} \\ &= \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \kappa_2 \left(\frac{1}{c_T} d(y, X_{T_{N_{t_{j+1}}}}) \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1}. \end{aligned}$$

In the case where there are at least two jumps in $(t_j, t_{j+1}]$ we obtain

$$\begin{aligned} & \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) \Delta N_{t_{j+1}} - \int_{t_j}^{t_{j+1}} \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \right| \mathbf{1}_{\Delta N_{t_{j+1}} \geq 2} \\ & \leq 2 \|\kappa_1\|_\infty \|\kappa_2\|_\infty \Delta N_{t_{j+1}} \mathbf{1}_{\Delta N_{t_{j+1}} \geq 2}. \end{aligned}$$

To sum up we just proved:

$$\begin{aligned} & \frac{1}{Z_T^d} \left| \sum_{j=0}^{m-1} \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) \Delta N_{t_{j+1}} - \int_0^T \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \right| \\ & \leq \frac{1}{Z_T^d} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \kappa_2 \left(\frac{1}{c_T} d(y, X_{T_{N_{t_{j+1}}}}) \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} \\ & \quad + \frac{2 \|\kappa_1\|_\infty \|\kappa_2\|_\infty}{Z_T^d} \sum_{j=0}^{m-1} \Delta N_{t_{j+1}} \mathbf{1}_{\Delta N_{t_{j+1}} \geq 2}. \end{aligned} \tag{4.50}$$

To control the last term of (4.48) we first write:

$$\frac{|Z_T^* - Z_T^d|}{Z_T^* Z_T^d} \int_0^T \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \leq \|\kappa_1\|_\infty \|\kappa_2\|_\infty \frac{|Z_T^* - Z_T^d|}{Z_T^* Z_T^d} N_T.$$

It remains to bound $|Z_T^* - Z_T^d|$. We write:

$$|Z_T^* - Z_T^d| \leq \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) D_{j+1} - \int_{t_j}^{t_{j+1}} \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \xi_T(X_s) dN_s \right|.$$

Proceeding in the same way as for the second term of (4.48) we get for any $j \in \llbracket 0, n(x) \rrbracket$:

$$\left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) D_{j+1} - \int_{t_j}^{t_{j+1}} \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \xi_T(X_s) dN_s \right| \mathbf{1}_{\Delta N_{t_{j+1}}=0} = 0,$$

$$\begin{aligned} & \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) D_{j+1} - \int_{t_j}^{t_{j+1}} \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \xi_T(X_s) dN_s \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} \\ &= \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \xi_T \left(X_{T_{N_{t_{j+1}}}} \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1}, \end{aligned}$$

and

$$\begin{aligned} & \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) D_{j+1} - \int_{t_j}^{t_{j+1}} \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \xi_T(X_s) dN_s \right| \mathbf{1}_{\Delta N_{t_{j+1}} \geq 2} \\ & \leq 2 \|\kappa_1\|_\infty \xi_T^* \Delta N_{t_{j+1}} \mathbf{1}_{\Delta N_{t_{j+1}} \geq 2}. \end{aligned}$$

We finally have

$$\begin{aligned} & \frac{|Z_T^* - Z_T^d|}{Z_T^* Z_T^d} \int_0^T \kappa_1 \left(\frac{d(x, X_{s^-})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_s)}{c_T} \right) dN_s \\ & \leq \frac{\|\kappa_1\|_\infty \|\kappa_2\|_\infty N_T}{Z_T^* Z_T^d} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) \right. \\ & \quad \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \xi_T \left(X_{T_{N_{t_{j+1}}}} \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} \\ & + \frac{2 \xi_T^* \|\kappa_1\|_\infty^2 \|\kappa_2\|_\infty N_T}{Z_T^* Z_T^d} \sum_{j=0}^{m-1} \Delta N_{t_{j+1}} \mathbf{1}_{\Delta N_{t_{j+1}} \geq 2}. \end{aligned} \quad (4.51)$$

By bringing together (4.49), (4.50) and (4.51), then (4.48) becomes

$$\begin{aligned} & |\hat{f}_d(x, y) - \hat{f}(x, y)| \\ & \leq \left(\frac{4 \|\kappa_1\|_\infty \|\kappa_2\|_\infty}{Z_T^d} + \frac{2 \xi_T^* \|\kappa_1\|_\infty^2 \|\kappa_2\|_\infty N_T}{Z_T^* Z_T^d} \right) \sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} \\ & + \frac{1}{Z_T^d} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) \right. \\ & \quad \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \kappa_2 \left(\frac{1}{c_T} d(y, X_{T_{N_{t_{j+1}}}}) \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} \\ & + \frac{\|\kappa_1\|_\infty \|\kappa_2\|_\infty N_T}{Z_T^* Z_T^d} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) \right. \\ & \quad \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \xi_T \left(X_{T_{N_{t_{j+1}}}} \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1}. \end{aligned} \quad (4.52)$$

Therefore

$$\mathbb{P} \left(|\hat{f}_d(x, y) - \hat{f}(x, y)| > \varepsilon/2 \right) \leq P_1 + P_2 + P_3 + P_4 + P_5$$

where

$$P_1 = \mathbb{P} \left(Z_T^d < \frac{Tv_T^*(x)}{4} \right),$$

$$P_2 = \mathbb{P} \left(\frac{16 \|\kappa_1\|_\infty \|\kappa_2\|_\infty}{Tv_T^*(x)} \sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} > \varepsilon/8 \right),$$

$$P_3 = \mathbb{P} \left(\frac{8 \xi_T^* \|\kappa_1\|_\infty^2 \|\kappa_2\|_\infty N_T}{Z_T Tv_T^*(x)} \sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} > \varepsilon/8 \right),$$

$$P_4 = \mathbb{P} \left(\frac{4}{Tv_T^*(x)} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-} } \right) \kappa_2 \left(\frac{1}{c_T} d(y, X_{T_{N_{t_{j+1}}} } \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} > \varepsilon/8 \right),$$

and

$$P_5 = \mathbb{P} \left(\frac{4 \|\kappa_1\|_\infty \|\kappa_2\|_\infty N_T}{Z_T^* Tv_T^*(x)} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-} } \right) \xi_T(X_{T_{N_{t_{j+1}}} } \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} > \varepsilon/8 \right).$$

We now present through the following lemma two consequences of (H9) that will be essential to study the previous quantities P_1, P_4 and P_5 .

Lemma 76. *Suppose (H9). Let $j \in \llbracket 1, m-1 \rrbracket$. Then*

$$\mathbb{E} \left[\left| g_T(X_{t_j}) - g_T(X_{T_{N_{t_{j+1}}}^-} } \right) \mathbf{1}_{\Delta N_{t_{j+1}}=1} \right] \leq l_T(x, y) \alpha^* \Delta_m^{a+1}, \quad (4.53)$$

and

$$\mathbb{E} \left[\left| g_T(X_{t_{j+1}}) - g_T(X_{T_{N_{t_{j+1}}} } \right) \mathbf{1}_{\Delta N_{t_{j+1}}=1} \right] \leq l_T(x, y) \alpha^* \Delta_m^{a+1}. \quad (4.54)$$

Proof. We start proving (4.53). Using Markov property we write:

$$\begin{aligned} & \mathbb{E} \left[\left| g_T(X_{t_j}) - g_T(X_{T_{N_{t_{j+1}}}^-} } \right) \mathbf{1}_{\Delta N_{t_{j+1}}=1} \right] \\ &= \mathbb{E} \left[\mathbb{E}_{X_{t_j}} \left[\left| g_T(X_0) - g_T(X_{T_1^-} } \right) \mathbf{1}_{T_1 < \Delta t_{j+1}} \mathbf{1}_{T_2 > \Delta t_{j+1}} \right] \right] \\ &\leq \mathbb{E} \left[\mathbb{E}_{X_{t_j}} \left[\left| g_T(Y_0^{(0)}) - g_T(Y_{T_1}^{(0)}) \right| \mathbf{1}_{T_1 < \Delta t_{j+1}} \right] \right] \\ &= \mathbb{E} \left[\mathbb{E}_{X_{t_j}} \left[\int_0^{\Delta t_{j+1}} \alpha(Y_u^{(0)}) e^{-\int_0^u \alpha(Y_v^{(0)}) dv} \left| g_T(Y_0^{(0)}) - g_T(Y_u^{(0)}) \right| du \right] \right] \\ &\leq \mathbb{E} \left[\alpha^* \int_0^{\Delta t_{j+1}} \mathbb{E}_{X_{t_j}}^Y \left[\left| g_T(Y_0) - g_T(Y_u) \right| \right] du \right] \\ &\leq l_T(x, y) \alpha^* \Delta_m^{a+1}. \end{aligned}$$

We use the same strategy to prove (4.54):

$$\begin{aligned}
& \mathbb{E} \left[\left| g_T(X_{t_{j+1}}) - g_T(X_{T_{N_{t_{j+1}}}}) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} \right] \\
&= \mathbb{E} \left[\mathbb{E}_{X_{t_j}} \left[\left| g_T(X_{\Delta t_{j+1}}) - g_T(X_{T_1}) \right| \mathbf{1}_{T_1 < \Delta t_{j+1}} \mathbf{1}_{T_2 > \Delta t_{j+1}} \right] \right] \\
&= \mathbb{E} \left[\mathbb{E}_{X_{t_j}} \left[\left| g_T(Y_{\Delta t_{j+1}-T_1}^{(1)}) - g_T(Y_0^{(1)}) \right| \mathbf{1}_{T_1 < \Delta t_{j+1}} \mathbf{1}_{T_2 > \Delta t_{j+1}} \right] \right] \\
&\leq \mathbb{E} \left[\mathbb{E}_{X_{t_j}} \left[\left| g_T(Y_{\Delta t_{j+1}-T_1}^{(1)}) - g_T(Y_0^{(1)}) \right| \mathbf{1}_{T_1 < \Delta t_{j+1}} \right] \right] \\
&= \mathbb{E} \left[\mathbb{E}_{X_{t_j}} \left[\mathbb{E}_{X_{T_1}}^Y \left[\left| g_T(Y_{\Delta t_{j+1}-T_1}) - g_T(Y_0) \right| \mathbf{1}_{T_1 < \Delta t_{j+1}} \right] \right] \right] \\
&\leq l_T(x, y) \Delta_m^a \mathbb{E} \left[\mathbb{E}_{X_{t_j}} \left[\mathbf{1}_{T_1 < \Delta t_{j+1}} \right] \right] \\
&\leq l_T(x, y) \Delta_m^{a+1} \alpha^*.
\end{aligned}$$

□

From there $c_0(x_0)$, $t_0(x_0)$, c_0 , t_0 and m_0 are generic positive constants that may change from line to line. We begin by computing P_1 . We write

$$\begin{aligned}
& \mathbb{P} \left(Z_T^d < \frac{Tv_T^*(x)}{4} \right) \\
&\leq \mathbb{P} \left(Z_T^d < \frac{Tv_T^*(x)}{4}, Z_T^* \geq \frac{Tv_T^*(x)}{2} \right) + \mathbb{P} \left(Z_T^d < \frac{Tv_T^*(x)}{4}, Z_T^* < \frac{Tv_T^*(x)}{2} \right) \\
&\leq \mathbb{P} \left(|Z_T^d - Z_T^*| > \frac{Tv_T^*(x)}{4} \right) + \mathbb{P} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right) \\
&\leq \frac{4 \mathbb{E} \left[|Z_T^d - Z_T^*| \right]}{Tv_T^*(x)} + \mathbb{P} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right).
\end{aligned} \tag{4.55}$$

In the setting of *i*), using Propositions 56 and 58, we get

$$\begin{aligned}
\mathbb{P}_{x_0} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right) &\leq \mathbb{P}_{x_0} \left(|Z_T^* - Tv_T^*(x)| > \frac{Tv_T^*(x)}{2} \right) \\
&\leq \frac{4 \mathbb{E} \left[(Z_T^* - Tv_T^*(x))^2 \right]}{T^2 v_T^*(x)^2} \\
&= \frac{4 \mathbb{V}_{x_0}(Z_T^*)}{T^2 v_T^*(x)^2} + \frac{4 (\mathbb{E}_{x_0}[Z_T^*] - Tv_T^*(x))^2}{T^2 v_T^*(x)^2} \\
&\leq c_0(x_0) \frac{\xi_T^{*2}}{Tv_T^*(x)^2} + \frac{c_0(x_0) \xi_T^{*2}}{T^2 v_T^*(x)^2} \\
&\leq c_0(x_0) \frac{\xi_T^{*2}}{Tv_T^*(x)^2}
\end{aligned} \tag{4.56}$$

for $T \geq t_0(x_0)$. On the other hand, when $c(\cdot)$ is bounded, we get

$$\begin{aligned} \mathbb{P}\left(Z_T < \frac{Tv_T^*(x)}{2}\right) &\leq c_0 \left(\frac{\xi_T^* \mathbb{E}[Z_T^*]}{T^2 v_T^*(x)^2} + \frac{\xi_T^{*2}}{T^2 v_T^*(x)^2} \right) \\ &\leq c_0 \left(\frac{\xi_T^*}{Tv_T^*(x)} + \frac{\xi_T^{*2}}{T^2 v_T^*(x)^2} \right) \\ &\leq c_0 \frac{\xi_T^*}{Tv_T^*(x)} \end{aligned} \quad (4.57)$$

applying Corollary 57 *ii*). We find an asymptotic bound for the first term of the decomposition (4.55). As in (4.51) we obtain

$$\begin{aligned} &\mathbb{E}\left[|Z_T^d - Z_T^*|\right] \\ &\leq \sum_{j=0}^{m-1} \mathbb{E}\left[\left|\kappa_1\left(\frac{d(x, X_{t_j})}{h_T}\right)\xi_T(X_{t_{j+1}}) - \kappa_1\left(\frac{1}{h_T}d(x, X_{T_{N_{t_{j+1}}}^-})}\right)\xi_T(X_{T_{N_{t_{j+1}}}})\right|\mathbf{1}_{\Delta N_{t_{j+1}}=1}\right] \\ &\quad + 2\|\kappa_1\|_\infty \xi_T^* \sum_{j=0}^{m-1} \mathbb{E}\left[\Delta N_{t_{j+1}} \mathbf{1}_{\Delta N_{t_{j+1}} \geq 2}\right] \end{aligned}$$

Let $j \in \llbracket 0, m-1 \rrbracket$. Then applying Lemma 76 we get

$$\begin{aligned} &\mathbb{E}\left[\left|\kappa_1\left(\frac{d(x, X_{t_j})}{h_T}\right)\xi_T(X_{t_{j+1}}) - \kappa_1\left(\frac{1}{h_T}d(x, X_{T_{N_{t_{j+1}}}^-})}\right)\xi_T(X_{T_{N_{t_{j+1}}}})\right|\mathbf{1}_{\Delta N_{t_{j+1}}=1}\right] \\ &\leq \xi_T^* \mathbb{E}\left[\left|\kappa_1\left(\frac{d(x, X_{t_j})}{h_T}\right) - \kappa_1\left(\frac{1}{h_T}d(x, X_{T_{N_{t_{j+1}}}^-})}\right)\right|\mathbf{1}_{\Delta N_{t_{j+1}}=1}\right] \\ &\quad + \|\kappa_1\|_\infty \mathbb{E}\left[\left|\xi_T(X_{t_{j+1}}) - \xi_T(X_{T_{N_{t_{j+1}}}})\right|\mathbf{1}_{\Delta N_{t_{j+1}}=1}\right] \\ &\leq (\xi_T^* + \|\kappa_1\|_\infty) l_T(x, y) \alpha^* \Delta_m^{a+1} \\ &\leq c_0 l_T(x, y) \Delta_m^{a+1} \end{aligned}$$

for $T \geq t_0$. By Lemma 75 we then have for $T \geq t_0$ and $m \geq m_0$

$$\mathbb{E}\left[|Z_T^d - Z_T^*|\right] \leq c_0 l_T(x, y) m \Delta_m^{a+1} + c_0 \xi_T^* m \Delta_m^2.$$

We conclude in the setting of *i*) with (4.55) and (4.56):

$$\begin{aligned} \mathbb{P}_{x_0}\left(Z_T^d < \frac{Tv_T^*(x)}{4}\right) &\leq \frac{c_0 m}{Tv_T^*(x)} (\xi_T^* \Delta_m^2 + l_T(x, y) \Delta_m^{a+1}) + c_0(x_0) \frac{\xi_T^{*2}}{Tv_T^*(x)^2} \\ &\leq \frac{c_0}{v_T^*(x)} (\xi_T^* \Delta_m + l_T(x, y) \Delta_m^a) + c_0(x_0) \frac{\xi_T^{*2}}{Tv_T^*(x)^2} \end{aligned}$$

for $m \geq m_0$ and $T \geq t_0(x_0)$. When $c(\cdot)$ is bounded this becomes by (4.57)

$$\mathbb{P}\left(Z_T^d < \frac{Tv_T^*(x)}{4}\right) \leq \frac{c_0}{v_T^*(x)} (\xi_T^* \Delta_m + l_T(x, y) \Delta_m^a) + c_0 \frac{\xi_T^*}{Tv_T^*(x)}$$

for $m \geq m_0$ and $T \geq t_0$.

Let us compute P_2 . We obtain by Lemma 75:

$$\begin{aligned}
P_2 &= \mathbb{P} \left(\frac{16 \|\kappa_1\|_\infty \|\kappa_2\|_\infty}{Tv_T^*(x)} \sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} > \varepsilon/8 \right) \\
&\leq \frac{128 \|\kappa_1\|_\infty \|\kappa_2\|_\infty}{\varepsilon Tv_T^*(x)} \mathbb{E} \left[\sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} \right] \\
&\leq \frac{c_0}{Tv_T^*(x)} m \Delta_m^2 \\
&\leq \frac{c_0}{v_T^*(x)} \Delta_m
\end{aligned}$$

for $m \geq m_0$.

Let us now consider P_3 . First note that applying Corollary 55 with $g \equiv 1$ we get

$$\mathbb{V}(N_T) = \mathbb{E} \left[\int_0^T \alpha(X_s) ds \right] \leq \alpha^* T.$$

Therefore, since $\mathbb{E}[N_T] \leq \alpha^* T$,

$$\mathbb{P}(N_T > 2\alpha^* T) \leq \mathbb{P}(|N_T - \mathbb{E}[N_T]| > \alpha^* T) \leq \frac{\mathbb{V}(N_T)}{\alpha^{*2} T^2} \leq \frac{1}{\alpha^* T}.$$

We compute P_3 :

$$\begin{aligned}
&\mathbb{P} \left(\frac{8 \xi_T^* \|\kappa_1\|_\infty^2 \|\kappa_2\|_\infty N_T}{Z_T^* Tv_T^*(x)} \sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} > \varepsilon/8 \right) \\
&\leq \mathbb{P} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right) + \mathbb{P}(N_T > 2\alpha^* T) \\
&\quad + \mathbb{P} \left(\frac{32 \alpha^* \xi_T^* \|\kappa_1\|_\infty^2 \|\kappa_2\|_\infty}{Tv_T^*(x)^2} \sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} > \varepsilon/8 \right) \\
&\leq \mathbb{P} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right) + \frac{1}{\alpha^* T} + \frac{256 \alpha^* \xi_T^* \|\kappa_1\|_\infty^2 \|\kappa_2\|_\infty}{\varepsilon Tv_T^*(x)^2} \mathbb{E} \left[\sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} \right] \\
&\leq \mathbb{P} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right) + \frac{c_0}{T} + c_0 \frac{\xi_T^*}{Tv_T^*(x)^2} m \Delta_m^2 \\
&\leq \mathbb{P} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right) + \frac{c_0}{T} + c_0 \frac{\xi_T^*}{v_T^*(x)^2} \Delta_m
\end{aligned}$$

for $m \geq m_0$. By (4.56), we then have in the setting of i):

$$\begin{aligned}
&\mathbb{P}_{x_0} \left(\frac{8 \xi_T^* \|\kappa_1\|_\infty^2 \|\kappa_2\|_\infty N_T}{Z_T^* Tv_T^*(x)} \sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} > \varepsilon/8 \right) \\
&\leq c_0 \left(\frac{1}{T} + \frac{\xi_T^*}{v_T^*(x)^2} \Delta_m \right) + c_0(x_0) \frac{\xi_T^{*2}}{Tv_T^*(x)^2}
\end{aligned}$$

for $m \geq m_0$ and $T \geq t_0(x_0)$. In the special case when $c(\cdot)$ is bounded this becomes for $m \geq m_0$ and $T \geq t_0$,

$$\begin{aligned} \mathbb{P} \left(\frac{8 \xi_T^* \|\kappa_1\|_\infty^2 \|\kappa_2\|_\infty N_T}{Z_T^* T v_T^*(x)} \sum_{j=0}^{m-1} \Delta N_{T_{j+1}} \mathbf{1}_{\Delta N_{T_{j+1}} \geq 2} > \varepsilon/8 \right) \\ \leq c_0 \left(\frac{1}{T} + \frac{\xi_T^*}{v_T^*(x)^2} \Delta_m + \frac{\xi_T^*}{T v_T^*(x)} \right) \end{aligned}$$

by (4.57).

We compute P_4 :

$$\begin{aligned} P_4 &= \mathbb{P} \left(\frac{4}{T v_T^*(x)} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) \right. \right. \\ &\quad \left. \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \kappa_2 \left(\frac{1}{c_T} d(y, X_{T_{N_{t_{j+1}}}}) \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} > \varepsilon/8 \right) \\ &\leq \frac{32}{\varepsilon T v_T^*(x)} \sum_{j=0}^{m-1} \mathbb{E} \left[\left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) \right. \right. \\ &\quad \left. \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \kappa_2 \left(\frac{1}{c_T} d(y, X_{T_{N_{t_{j+1}}}}) \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} \right] \\ &\leq \frac{32 \|\kappa_2\|_\infty}{\varepsilon T v_T^*(x)} \sum_{j=0}^{m-1} \mathbb{E} \left[\left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} \right] \\ &\quad + \frac{32 \|\kappa_1\|_\infty}{\varepsilon T v_T^*(x)} \sum_{j=0}^{m-1} \mathbb{E} \left[\left| \kappa_2 \left(\frac{d(y, X_{t_{j+1}})}{c_T} \right) - \kappa_2 \left(\frac{1}{c_T} d(y, X_{T_{N_{t_{j+1}}}}) \right) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} \right] \\ &\leq c_0 \frac{l_T(x, y) m \Delta_m^{a+1}}{T v_T^*(x)} \end{aligned}$$

by Lemma 76. To conclude

$$P_4 \leq c_0 \frac{l_T(x, y) \Delta_m^a}{v_T^*(x)}.$$

In the calculus of P_5 we use the same technique as is P_3 for the control of Z_T^* and N_T so we write:

$$\begin{aligned} P_5 &= \mathbb{P} \left(\frac{4 \|\kappa_1\|_\infty \|\kappa_2\|_\infty N_T}{Z_T^* T v_T^*(x)} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) \right. \right. \\ &\quad \left. \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \xi_T(X_{T_{N_{t_{j+1}}}}) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} > \varepsilon/8 \right) \\ &\leq \mathbb{P} \left(Z_T^* < \frac{T v_T^*(x)}{2} \right) + \mathbb{P}(N_T > 2\alpha^* T) \\ &\quad + \mathbb{P} \left(\frac{16 \|\kappa_1\|_\infty \|\kappa_2\|_\infty \alpha^*}{T v_T^*(x)^2} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) \right. \right. \\ &\quad \left. \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \xi_T(X_{T_{N_{t_{j+1}}}}) \right| \mathbf{1}_{\Delta N_{t_{j+1}}=1} > \varepsilon/8 \right). \end{aligned}$$

Applying Lemma 76 we obtain:

$$\begin{aligned}
P_5 &\leq \mathbb{P} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right) + \frac{c_0}{T} \\
&\quad + \frac{128 \|\kappa_1\|_\infty \|\kappa_2\|_\infty \alpha^*}{\varepsilon T v_T^*(x)^2} \sum_{j=0}^{m-1} \mathbb{E} \left[\left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) \right. \right. \\
&\quad \quad \quad \left. \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \xi_T \left(X_{T_{N_{t_{j+1}}}^-} \right) \right| \mathbf{1}_{\Delta_{N_{t_{j+1}}}=1} \right] \\
&\leq \mathbb{P} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right) + \frac{c_0}{T} \\
&\quad + \xi_T^* \frac{128 \|\kappa_1\|_\infty \|\kappa_2\|_\infty \alpha^*}{\varepsilon T v_T^*(x)^2} \sum_{j=0}^{m-1} \mathbb{E} \left[\left| \kappa_1 \left(\frac{d(x, X_{t_j})}{c_T} \right) \right. \right. \\
&\quad \quad \quad \left. \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \right| \mathbf{1}_{\Delta_{N_{t_{j+1}}}=1} \right] \\
&\quad + \frac{128 \|\kappa_1\|_\infty^2 \|\kappa_2\|_\infty \alpha^*}{\varepsilon T v_T^*(x)^2} \sum_{j=0}^{m-1} \mathbb{E} \left[\left| \xi_T(X_{t_{j+1}}) - \xi_T \left(X_{T_{N_{t_{j+1}}}^-} \right) \right| \mathbf{1}_{\Delta_{N_{t_{j+1}}}=1} \right] \\
&\leq \mathbb{P} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right) + \frac{c_0}{T} + \frac{c_0}{T v_T^*(x)^2} l_T(x, y) m \Delta_m^{a+1} \\
&\leq \mathbb{P} \left(Z_T^* < \frac{Tv_T^*(x)}{2} \right) + \frac{c_0}{T} + \frac{c_0}{v_T^*(x)^2} l_T(x, y) \Delta_m^a.
\end{aligned}$$

In the setting of i) we then get by (4.56):

$$\begin{aligned}
\mathbb{P}_{x_0} &\left(\frac{4 \|\kappa_1\|_\infty \|\kappa_2\|_\infty N_T}{Z_T^* T v_T^*(x)} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) \right. \right. \\
&\quad \quad \quad \left. \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \xi_T \left(X_{T_{N_{t_{j+1}}}^-} \right) \right| \mathbf{1}_{\Delta_{N_{t_{j+1}}}=1} > \varepsilon/8 \right) \\
&\leq c_0(x_0) \frac{\xi_T^{*2}}{T v_T^*(x)^2} + c_0 \left(\frac{1}{T} + \frac{l_T(x, y) \Delta_m^a}{v_T^*(x)^2} \right)
\end{aligned}$$

for $T \geq t_0(x_0)$. In the special case when $c(\cdot)$ is bounded this becomes for $T \geq t_0$,

$$\begin{aligned}
\mathbb{P}_{x_0} &\left(\frac{4 \|\kappa_1\|_\infty \|\kappa_2\|_\infty N_T}{Z_T^* T v_T^*(x)} \sum_{j=0}^{m-1} \left| \kappa_1 \left(\frac{d(x, X_{t_j})}{h_T} \right) \xi_T(X_{t_{j+1}}) \right. \right. \\
&\quad \quad \quad \left. \left. - \kappa_1 \left(\frac{1}{h_T} d(x, X_{T_{N_{t_{j+1}}}^-}) \right) \xi_T \left(X_{T_{N_{t_{j+1}}}^-} \right) \right| \mathbf{1}_{\Delta_{N_{t_{j+1}}}=1} > \varepsilon/8 \right) \\
&\leq c_0 \left(\frac{\xi_T^*}{T v_T^*(x)} + \frac{1}{T} + \frac{l_T(x, y) \Delta_m^a}{v_T^*(x)^2} \right),
\end{aligned}$$

by (4.57).

In conclusion, applying Theorem 63 *i*), we obtain under the hypotheses of *i*):

$$\begin{aligned}
& \mathbb{P}_{x_0} \left(|\hat{f}_a(x, y) - f(x, y)| > \varepsilon \right) \\
& \leq c_0(x_0) \left(\frac{\xi_T^{*2}}{T v_T^*(x)^2} + \frac{\xi_T^* v_T(x)}{T v_T^*(x)^2} + w_T(x, y)^2 + (h_T + c_T)^{2\gamma} \frac{\xi_T^{*2} v_T(x)^2}{v_T^*(x)^2} \right. \\
& \quad \left. + \frac{1}{T} + \frac{l_T(x, y) \Delta_m^a}{v_T^*(x)^2} + \frac{\xi_T^* \Delta_m}{v_T^*(x)^2} + \frac{\Delta_m}{v_T^*(x)} \right) \\
& \leq c_0(x_0) \left(\frac{\xi_T^{*2}}{T v_T^*(x)^2} + \frac{\xi_T^* v_T(x)}{T v_T^*(x)^2} + w_T(x, y)^2 + (h_T + c_T)^{2\gamma} \frac{\xi_T^{*2} v_T(x)^2}{v_T^*(x)^2} \right. \\
& \quad \left. + \frac{l_T(x, y) \Delta_m^a}{v_T^*(x)^2} + \frac{\xi_T^* \Delta_m}{v_T^*(x)^2} \right)
\end{aligned}$$

for $m \geq m_0$ and $T \geq t_0(x_0)$. Also, $c(\cdot)$ is bounded, we get by Theorem 63 *ii*):

$$\begin{aligned}
& \mathbb{P} \left(|\hat{f}_a(x, y) - f(x, y)| > \varepsilon \right) \\
& \leq c_0 \left(\frac{\xi_T^* v_T(x)}{T v_T^*(x)^2} + w_T(x, y)^2 + (h_T + c_T)^{2\gamma} \frac{\xi_T^{*2} v_T(x)^2}{v_T^*(x)^2} \right. \\
& \quad \left. + \frac{1}{T} + \frac{l_T(x, y) \Delta_m^a}{v_T^*(x)^2} + \frac{\xi_T^* \Delta_m}{v_T^*(x)^2} + \frac{\Delta_m}{v_T^*(x)} + \frac{\xi_T^*}{T v_T^*(x)} \right) \\
& \leq c_0 \left(\frac{\xi_T^* v_T(x)}{T v_T^*(x)^2} + w_T(x, y)^2 + (h_T + c_T)^{2\gamma} \frac{\xi_T^{*2} v_T(x)^2}{v_T^*(x)^2} \right. \\
& \quad \left. + \frac{l_T(x, y) \Delta_m^a}{v_T^*(x)^2} + \frac{\xi_T^* \Delta_m}{v_T^*(x)^2} \right)
\end{aligned}$$

for $m \geq m_0$ and $T \geq t_0(x_0)$. □

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Titre : Propriétés de Feller, d'ergodicité et inférence non-paramétrique du processus jump-move.

Mot clés : processus de Feller, processus birth-death-move, couplage, mesure de Gibbs, estimation par noyau

Résumé : Cette thèse consiste à étudier les processus jump-move qui ont, entre autres, pour vocation de modéliser des dynamiques de populations spatiales. Leur dynamique est constituée de mouvements aléatoires markoviens continus et de sauts en alternance, généralisant des modèles tels que les processus de Markov déterministes par morceaux. Obtenir des résultats statistiques ne se faisant pas sans une étude préalable des propriétés probabilistes du processus observé, nous commençons par étudier en détail le caractère Feller des processus jump-move, avec comme horizon l'application à des systèmes de particules dans \mathbb{R}^d . Ensuite, nous nous intéressons au comportement en temps long d'un certain type de processus jump-move, ap-

pelés processus birth-death-move spatiaux. Nous obtenons des critères sur les taux de naissances et morts pour avoir une vitesse de convergence géométrique vers une mesure stationnaire. Nous terminons par l'étude de méthodes d'estimation non-paramétriques du noyau de sauts d'un processus jump-move. La convergence en norme L^2 de l'estimateur considéré est alors obtenue dans le cadre de l'observation d'une trajectoire du processus en temps continu. Des résultats dérivés sont alors établis dans les cas spécifiques d'un noyau général à densité, dans le cas particulier du processus birth-death-move spatial et enfin lorsque l'observation du processus se fait de façon discrète.

Title: Feller property, ergodicity and non-parametric inference of the jump-move process.

Keywords: Feller processes, birth-death-move processes, coupling, Gibbs measures, kernel estimator

Abstract: This thesis consists in studying jump-move processes with a view to modelling the spatio-temporal dynamics of a system of particles. A jump-move process alternates between continuous Markovian random motions and random jumps, generalizing models such as piecewise deterministic Markov processes. Since statistical results cannot be obtained without a prior study of the probabilistic properties, we start by studying in detail the Feller properties of jump-move processes with a particular focus on particle systems in \mathbb{R}^d . Then, we are interested in the long time behavior of a type of jump-move processes, called spatial

birth-death-move processes. We obtain criteria on the birth and death rates to have a geometric convergence rate to a stationary measure. We conclude by studying non-parametric estimation methods of the jump transition kernel of a jump-move process. The convergence in L^2 norm of the considered estimators is then obtained when the trajectory of the process is observed in continuous time. Derived results are then established first in the specific case of a general density kernel, second in the particular case of the spatial birth-death-move process and finally when the observation of the process is in discrete time.