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Yoël Perreau

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École doctorale CARNOT-PASTEUR

Sur quelques aspects non-linéaires et isométriques de la géometrie asymptotique des espaces de Banach

Thèse de doctorat de
l'Université Bourgogne Franche-Comté

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par

Yoël Perreau

Composition du jury :

Miguel Martín Suárez	University of Granada	Rapporteur
Beata Randrianantoanina	Miami University	Rapporteure
Frédéric Bayart	Université Clermont Auvergne	Examinateur
Anna Pelczar-Barwacz	Jagiellonian University	Examinatrice
Florent Baudier	Texas A&M University	Examinateur
Gilles Lancien	Université Bourgogne Franche-Comté	Directeur de thèse
Antonin Procházka	Université Bourgogne Franche-Comté	Directeur de thèse

Président du jury: Frédéric Bayart

Résumé

Cette thèse est consacrée à l'étude de la géométrie des espaces de Banach. Elle est composée de trois chapitres que nous allons brièvement décrire dans la suite du paragraphe, et est centrée autour deux grands thèmes de recherche, la géométrie non-linéaire et la géométrie isométrique.

Dans le premier chapitre, nous introduirons tout le bagage nécessaire en théorie asymptotique des espaces de Banach. On introduira les propriétés standards de lissité asymptotique uniforme et de convexité asymptotique uniforme, et on décrira un large éventail de leur utilisation. On présentera ensuite les résultats majeurs de théorie du renormage dans le contexte en relation avec l'indice de Szlenk, et on décrira quelques applications de cet invariant isomorphique en théorie linéaire des espaces de Banach. En particulier, on considérera la notion fondamentale d'arbres dans un espace de Banach et on fera un brève incursion en théorie des structures asymptotiques qui met en oeuvre dans le contexte de puissants outils combinatoires provenant de la théorie des jeux. On terminera le chapitre par une présentation de la convexité asymptotique moyennée sur laquelle on amènera quelques nouvelles observations.

Le deuxième chapitre est consacré à la géométrie non-linéaire. On introduira brièvement toutes les notions de base de la théorie, et en particulier les diverses notions de plongements métriques dont nous aurons l'usage. Le résultat principal de ce chapitre est l'extension d'un résultat dû à Baudier, Kalton et Lancien concernant le non-plongement de la famille $(T_N)_{N\geq 1}$ des arbres hyperboliques à branchements dénombrables dans le contexte des espaces réflexifs au contexte plus géneral des espaces quasi-réflexifs. On présentera ensuite une extension d'un résultat de plongement dû à Baudier et ses co-auteurs des graphes diamants à branchements dénombrables dans le contexte des espaces réflexifs qui possèdent une structure asymptotique spécifique au contexte des espaces duaux pour lesquels une condition similaire sur la structure asymptotique préfailbe est vérifiée. On terminera le chapitre par quelques commentaires sur les modèles étalés des espaces de Banach et leur application en théorie linéaire et non-linéaire, et on établira quelques liens avec l'indice de Szlenk et le plongement de certaines familles de graphes.

Le troisième chapitre est dédié à la géométrie isométrique, et plus particulièrement à l'étude des points de Daugavet et des points Δ d'un espace de Banach. On se focalisera plus spécifiquement sur l'influence de la géométrie asymptotique d'un espace de Banach sur l'existence de tels points. Les résultats principaux de cette étude sont qu'aucun espace asymptotiquement uniformément lisse et qu'aucun espace réflexif asymptotiquement uniformément convexe ne peut contenir de point Δ, et qu'aucun dual préfaiblement asymptotiquement uniformément convexe de peut contenir de point de Daugavet. On produira également des versions locales de ces résultats. On s'intéressera aussi à la question de l'existence de ces points dans les espaces super-réflexifs et on produira un critère qui garantit leur existence dans tout ultrapuissance d'un espace donné dans ce contexte. Finalement on montrera que l'espace de James JT construit sur un arbre dyadique n'admet pas de point Δ.

En appendice, on introduira brièvement l'indice ℓ_1^+ faible dû à Alspach Judd et Odell, et on utilisera cet indice pour prouver la détermination séparable de l'indice de Szlenk par quotients dans le cadre des espaces dits " weakly compactly generated ".

Abstract

This thesis is devoted to the study of the geometry of Banach spaces. It is composed of three main chapters that we will briefly describe in the present paragraph, and is directed towards two main fields of research, the non-linear geometry and the isometric geometry.

In Chapter 1 we introduce all the necessary background in the asymptotic theory of Banach spaces which, roughly speaking, considers geometric properties of Banach spaces up to finite dimensional subspaces. This chapter forms the foundations of the thesis as the material presented in it will be central for our studies both in the non-linear and in the isometric theory. We introduce the standard properties of uniform asymptotic smoothness and convexity and present a detailed investigation of their usage. We then present the major known results in the related renorming theory in relation to the Szlenk index and describe some of the applications of this isomorphic invariant to the linear theory of Banach spaces. In particular we consider the important notion of trees in a Banach space and make a brief incursion in the theory of asymptotic structures of Banach spaces which provide powerful combinatoric tools in the context coming from the theory of games. We end the chapter with a presentation of the so-called asymptotic midpoint uniform convexity and provide a few novel observations on the subject.

Chapter 2 is devoted to non linear-geometry. We briefly introduce all the necessary background in the theory, and in particular all the relevant notions of metric embeddings. The main result of this chapter is the extension of a result by Baudier, Kalton and Lancien concerning the non embeddability of the family $(T_N)_{N\geq 1}$ of hyperbolic countably branching trees in the context of reflexive Banach spaces to the context of quasi-reflexive Banach spaces. As a corollary we obtain the non embeddability of this family into James spaces J_p. We then present an extension of an embeddability result by Baudier and his co-authors of countably branching diamond graphs in the context of reflexive Banach spaces with specific asymptotic structural properties to the context of dual spaces in which the weak* asymptotic structure presents a similar behavior. We end the chapter with a few comments on the notion of spreading models in Banach spaces and their usage in the non-linear theory, and in particular study their relation with the Szlenk index and the embeddability of certain families of graphs.

Chapter 3 is devoted to isometric geometry, and more specifically to the study of the so-called Daugavet- and Δ-points of a Banach space. We focus primarily on the influence of the asymptotic geometry of the space on the existence of such points. The main results of this chapter are that asymptotically uniformly smooth Banach spaces as well as reflexive asymptotically uniformly convex Banach spaces do not admit Δ-points and that weak* asymptotically uniformly convex dual spaces do not admit Daugavet-points. We also provide local versions of those results. We then investigate the question of the existence of Daugavet- or Δ-points in super-reflexive Banach spaces and provide a criterion ensuring in this context the existence of such points in any utrapower of the space. Finally we show that the James tree space JT does not admit Δ-points.

In the short Appendix A we introduce the weakly null ℓ_1^+ index from Alspach Judd and Odell, and use this index to prove the determination of the Szlenk index by separable quotients for weakly compactly generated Banach spaces.

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Introduction

The systematic study of complete normed vector spaces goes back to Banach's thesis [Ban1] published in 1922. With the publication in 1932 of Banach's famous monograph [Ban2] the nowadays called *Banach spaces* became a natural and powerful framework for modern functional analysis. This thesis is devoted to the study of their geometry from a non-linear point of view and from an isometric point of view. Our notation is standard and follows those of classical textbooks such as [AK] or [FHH+]. In particular, if X is a Banach space, we will denote by B_X its closed unit ball, S_X its unit sphere and X^* its dual space. Unless otherwise stated, we will work with infinite dimensional spaces, we will implicitly assume that subspaces are closed subspaces, and we will consider real Banach spaces only. Also if M is a metric space, we will denote by $B(x, \varepsilon)$ the closed ball centered at a point $x \in M$ and of radius $\varepsilon > 0$.

Non-linear geometry and asymptotic theory

Every Banach space has a natural underlying complete metric structure, and the question of the classification of Banach spaces up to non-linear transformations (see Section 2.1 for precise definitions) turned out to be a very rich and still very active field of research. For an overview of some of the most standard results in the area we refer to the classical book [BL2]. Concerning the question of the preservation of (linear) geometric properties of Banach spaces under non-linear transformations, an important rigidity result was discovered by Ribe in [Rib1] for the so-called *local properties* of Banach spaces, which roughly speaking are properties which concerns finite dimensional subspaces of a given Banach space. Important examples of such are super-reflexivity, type or cotype.

Theorem (Ribe, 1976)**.** *Let X and Y be Banach spaces, and let us assume that X coarse-Lipschitz embeds into Y. Then X is crudely finitely representable in Y, that is there is a constant $C > 0$ such that for every finite dimensional subspace E of X there is a finite dimensional subspace F of Y and an isomorphism T from E onto F satisfying $\|T\| \, \|T^{-1}\| \leq C$.*

One of the major consequence of this result is that every local property is *stable* under coarse-Lipschitz embeddings, meaning that if a Banach space X coarse-Lipschitz embeds into a Banach space Y and if Y has a local property (P), then X also has property (P). For the convenience of the reader let us recall that a *coarse-Lipschitz* embedding is a metric embedding which behaves like a standard Lipschitz embedding for large distances. In particular any uniform

homeomorphism between Banach spaces X and Y, that is any uniformly continuous bijective map with a uniformly continuous inverse naturally gives rise to a coarse-Lipschitz embedding from X into Y and from Y into X. Let us also recall that Ribe also published in [Rib2] (1984) the first example of uniformly homeomorphic separable Banach spaces which are not linearly isomorphic, also proving that reflexivity is not preserved by uniform homeomorphisms.

After the discovery of this rigidity result it was speculated that local properties of Banach spaces could admit purely metric characterizations and that their definition could be extended to a general metric setting without any more mention of the linear structure. The first result in this direction is the famous following characterization of super-reflexivity obtained by Bourgain [Bou] which can be considered as the starting point of the nowadays called *Ribe program*. We refer to the surveys [Nao1] and [Nao2] for an overview of some the most important results in this area and for the surprisingly effective application of those results in computer science.

Theorem (Bourgain, 1986)**.** *A Banach space X is super-reflexive if and only if the family $(d_N)_{N\geq 1}$ of hyperbolic dyadic trees of finite height does not equi-Lipschitz embed into X.*

Similar characterizations of super-reflexivity where later obtained with other families of metric graphs. In particular Johnson and Schechtman provided in [JS] (2009) an analogue to Bourgain's result with the family of (2-branching) diamond graphs and implemented a short and elegant " self-improvement argument " for the non equi-Lipschitz embeddability of this family into uniformly convex spaces. A similar argument was also used by Kloeckner in [Klo] (2013) to provide a short proof of Bourgain's non equi-Lipschitz embeddability of the family $(d_N)_{N\geq 1}$ into uniformly convex spaces. It should be pointed out here that those characterizations are known to be completely independent since Ostrovskii proved in [Ost1] (2014) that the two considered families are in some sense incomparable (they do not uniformly bi-Lipschitz embed into each other). Also Swift provided in [Swi] (2018) a very large class of families of metric graphs for which an analogue of Bourgain's result can be stated, the families generated by a finitely branching (non trivial) bundle graph.

Another important strengthening of Bourgain's result was obtained by Baudier in [Bau1] (2007) where it is proved that a " barycentric gluing technique " inspired from Ribe's paper [Rib2] allows to embed the dyadic tree of infinite height d_∞ in any non super-reflexive space. Such technique was developed further in several papers and culminated in the following result from Ostrovskii proved in [Ost2] (with an unnecessary restriction to uniformly discrete metric spaces which was then removed from subsequent versions of the paper). For more details on the barycentric gluing technique and on its applications to metric geometry, we refer to the paper [Bau2].

Theorem (Ostrovskii, 2012)**.** *Let M be a locally finite metric space, that is a metric space in which all balls are finite. If the finite subsets of M uniformly bi-Lipschitz embed into a Banach space X, then M bi-Lipschitz embeds into X.*

In [BKL] an analogue of Bourgain's (and Baudier's) result was discovered in the parallel " asymptotic Ribe Program " in the context of the asymptotic geometry of Banach spaces by Baudier, Kalton and Lancien. Roughly speaking the *asymptotic theory* considers geometric

properties of Banach spaces up to finite dimensional spaces and the most important objects of the theory are (closed) subspaces of finite co-dimension or equivalently weakly open neighborhoods of 0 in a given Banach space. This theory will be thoroughly studied in Chapter 1 but it should be pointed out that there is no equivalent result to Ribe's rigidity theorem in the asymptotic setting, and although strong analogies with the local theory appeared over time there is no reason to believe that every asymptotic property should admit a purely metric characterization or even should be stable under non-linear transformations. Their result is the following.

Theorem (Baudier, Kalton, Lancien, 2010)**.** *Let X be a separable reflexive (infinite dimensional) Banach space. The following assertions are equivalent.*

1. *The Szlenk index of the space X and the Szlenk index of the space X^* are both equal to the first infinite ordinal ω.*
2. *The family $(T_N)_{N\geq 1}$ of countably branching hyperbolic trees of finite height does not equi-Lipschitz embed into X.*
3. *The countably branching tree T_∞ of infinite height does not bi-Lipschitz embed into X.*

Note that T_∞ is not locally finite so that the last statement of this theorem does not follow from Ostrovskii's result (although similar gluing arguments are involved). Up to our knowledge it is in fact still unknown whether the bi-Lipschitz embeddability of T_∞ is implied by the equi-Lipschitz embeddability of the familly $(T_N)_{N\geq 1}$. Also let us point out that one can easily get rid of the separability assumption in this result by using the separable determination of the Szlenk index. An alternative short proof of this result involving property (β) of Rolewicz (whose existence up to renorming is sometimes referred to as asymptotic super-reflexivity) and relying on a self improvement argument " à la Kloeckner " was also provided by Baudier and Zhang in [BZ] (2016).

We introduce all the necessary background on asymptotic geometry in Chapter 1. In particular we introduce the notions of asymptotic smoothness and asymptotic convexity, and present the well known duality between asymptotically smooth spaces and weak* asymptotically convex dual spaces. We then introduce the Szlenk index of a Banach space which plays a fundamental role in the linear and non-linear theory of Banach spaces and present the famous renorming result from Knaust, Odell and Schlumprecht [KOS] which can be formulated as follows.

Theorem (Knaust, Odell, Schlumprecht, 1999)**.** *Let X be a separable Banach space. The following assertions are equivalent.*

1. *The Szlenk index of X is equal to ω.*
2. *The space X admits an equivalent asymptotically uniformly smooth norm with power type.*

It is very natural to ask wether the reflexivity assumption can be dropped in the result from Baudier, Kalton and Lancien. Since T_∞ embeds into any Banach space with Szlenk index strictly greater than ω and into any Banach space whose dual has Szlenk index strictly greater than ω, the question is essentially brought back to the following.

Question. *Let X be a (separable) non-reflexive Banach space. If the Szlenk index of X and the Szlenk index of X^* are equal to ω, do we have that the family $(T_N)_{N\geq 1}$ fails to equi-Lipschitz embed into X?*

Let us point out that there is no hope of extending the proof from Baudier and Zhang in this context since property (β) implies reflexivity. We will provide an affirmative answer to this question in Section 2.2 in the context of quasi-reflexive Banach spaces. The following theorem will be the main result of the non-linear Chapter 2.

Theorem (P., 2019)**.** *Let X be a separable quasi-reflexive Banach space. If the Szlenk index of the space X and the Szlenk index of the space X^* are both equal to ω, then the family $(T_N)_{N\geq 1}$ does not equi-Lipschitz embed into X.*

As a consequence we extend the result from Baudier, Kalton and Lancien to the quasi-reflexive setting, and we obtain the non embeddability of T_∞ into James spaces J_p $(1 < p < \infty)$ which are known to be quasi-reflexive of order 1 and to satisfy $S_Z(J_p) = S_Z(J_p^*) = \omega$.

Let us briefly explain some of the key ideas behind this result. The main difficulty in the extension of the proof from [BKL] outside of the reflexive setting is the lack of weak sequential compactness. Indeed one of the main tools in this proof is the use of sharp martingale like estimates on certain weakly null trees which are provided by the asymptotic smoothness and the asymptotic convexity of the space. The implementation of such arguments requires a first crucial step of decomposition of metric trees into weakly null pieces by a process of successive extractions of weakly converging subsequences at each level of the trees. At this point, a very naive idea would be to embed our trees in the bidual of the target space instead of the space itself, and then to make use of the available weak* sequential compactness in order to do some extractions. Now since weak* asymptotic smoothness in a dual space is known to force reflexivity, one has to rely on the asymptotic smoothness of the original space in order to obtain estimates there. Yet by extracting in a bidual one could find himself far from the original space, and there is no reason why such estimates should be provided.

This is where the quasi-reflexive assumptions pops out. For the convenience of the reader let us recall that a Banach space X is said to be *quasi-reflexive* if it has finite co-dimension in its bidual. In other words the bidual of any quasi-reflexive space can be decomposed as a direct sum $X^{**} = X \oplus E$ where E is a finite-dimensional space. Since the unit ball of any finite dimensional space is compact, this assumption allows us to use a combinatorial result of concentration from Ramsey which guarantees that at each fixed level of the trees in X^{**} one can ask differences between every two points at this level of the tree to be as close as we wish to an element of the space X. Specific extractions then allow to obtain full subtrees for which every sequence obtained by taking the difference between two branches of the tree which " interlace " satisfy smoothness estimates. The origin of those ideas goes back to the paper [LR] from 2018 where Lancien and Raja used similar arguments in order to extend some previous results from Kalton and Randrianarivony ([KR], 2008) on concentration phenomenon for Hamming graphs to the quasi-reflexive setting. In particular their simple but key observation that smoothness estimates do still hold for weak* null sequences rooted at points of X plays an important role in the previously mentioned extractions.

Another important result in the " asymptotic Ribe program " was discovered by Baudier and his co-authors in [BCD+] (2017).

Theorem (Baudier, Causey, Dilworth, Kutzarova, Randrianarivony, Schlumprecht, Zhang)**.** *Let X be a separable reflexive (infinite dimensional) Banach space and let us assume that X has an unconditional asymptotic structure. Then the following assertions are equivalent.*

1. *The Szlenk index of the space X^* is equal to ω .*
2. *The family $(D_N)_{N\geq 1}$ of countably branching diamond graphs does not equi-Lipschitz embed into X.*

Part of the result is based on another self-improvement argument " à la Johnson and Schechtman " and states in full generality that the family $(D_N)_{N\geq 1}$ fails to equi-Lipschitz embed into any space which is asymptotic midpoint uniformly convex. This property was introduced in [DKR+] in 2016 by Kutzarova and her co-authors as an (isometrically) weaker version of asymptotic convexity, and is closely related to the α size (where α stands for the Kuratowski index of non-compactness) of so-called approximate midpoints in a given Banach space, a simple but very powerful tool introduced by Enflo in an unpublished paper in order to prove that the space L_1 does not coarse-Lipschitz embed into ℓ_1. We refer to [Ben, Section 5] for a proof of Enflo's result. Let us also mention Kalton's result [Kal2, Theorem 7.3] (2013) where similar ideas are used in order to provide lower estimates for the spreading models of any Banach space X which coarse-Lischpitz embeds into an asymptotic midpoint uniformly convex Banach space Y. In Chapter 1 Section 1.4 we introduce a natural weak* version of asymptotic midpoint convexity and make use of the renorming theorem from Knaust, Odell and Schlumprecht to prove that this property is equivalent, up to renorming and under specific asymptotic structural properties to the standard weak* asymptotic convexity. This is the following result.

Theorem (P., 2020)**.** *Let X be a separable Banach space and let us assume that X^* has a weak* unconditional asymptotic structure. Then the following assertions are equivalent.*

1. *The space X has Szlenk index ω.*
2. *The space X admits an equivalent norm whose dual norm is AUC^*.*
3. *The space X^* admits an equivalent $AMUC^*$ norm.*

Surprisingly we also stumbled on the fact that asymptotic midpoint convexity actually coincides with its weak* version on any dual space, thus obtaining a major difference with the standard asymptotic convexity. As a corollary we obtained the following result.

Theorem (P., 2020)**.** *There exists a Banach space X whose dual norm is $AMUC^*$ but which admits no equivalent AUC^* norm*

More specifically this space is the James tree space JT. Since the dual norm of this space is known to be asymptotically convex (this is a result by Girardi, see [Gir]), the question of the equivalence of asymptotic midpoint convexity and standard asymptotic convexity up to renorming remains open.

The asymptotic structural properties which appear in the above results will be presented in Chapter 1. Getting rid of those would certainly be an important breakthrough in the asymptotic linear and non-linear theory but seems to ask for completely new ideas. For our part, we build in Chapter 2 on the ideas from [BCD+] of using specific trees with a strong c_0 like structure on branches which arise from considerations on the asymptotic structure of the space implied by properties of the Szlenk index. We extend their result in the context of dual spaces under similar (weak*) asymptotic structural requierements.

Theorem (P., 2020)**.** *Let X be a separable Banach space with Szlenk index strictly greater than ω, and let us assume that X^* has a weak* unconditional asymptotic structure. Then the family $(D_N)_{N\geq 1}$ does equi-Lipschitz embed into X^*.*

Let us point out that the results from [BCD+] were also extended by Swift in [Swi] (2018) to families generated by non trivial countably branching bundle graphs, and our result can also be naturally extended to those families thanks to his investigations. Let us also mention that getting rid of separability assumptions in the characterization from [BCD+] cannot merely be done by passing to a separable subspace since one might loose the essential asymptotic structural properties of the dual space. It is thus required to pass to a separable quotient while preserving the Szlenk index of the original space. This question might be very difficult in general since it is already unknown whether separable quotients exists for arbitrary Banach spaces. Yet it is possible to do so in weakly compactly generated Banach spaces, and we provide a proof of this in the Appendix A by using the so-called weakly null ℓ_1^+ index introduced by Alspach, Judd and Odell in [AJO] (2005).

Theorem (P., 2020)**.** *Let X be a weakly compactly generated Banach spaces and let α be a countable ordinal. If the Szlenk index of X is strictly greater than α, then there exists a separable quotient of X with Szlenk index stricly greater than α.*

We end Chapter 2 and our non-linear studies with a few considerations on spreading models in relation with the Szlenk index and the embeddability of the above families. Most of the results of this section are certainly well known by experts in the field, but are often difficult to track down in the literature. Yet the following result is new.

Proposition (P., 2020)**.** *Let X be a Banach space. If X contains a c_0 spreading model, then the family $(D_N)_{N\geq 1}$ does equi-Lipschitz embed into X.*

The results announced in this section are based on the paper [Per1] and on the preprint [Per2].

Diameter two properties and isometric geometry

A powerful way of understanding some of the geometric characteristics of the unit ball B_X of a Banach space X from an isometric point of view is to look at the behavior of its *slices* which are obtained by intersecting it with half-spaces supported by continuous functionals. This apparently simple idea has lead to many deep and beautiful results in the field, at the interface between geometry, probability and analysis (see for example the monographs [DGZ],[DU],[Pis]). Historically two of the most studied properties in this field are the (isomorphic) Radon-Nikodým property (RNP in short) and the Daugavet property which, in terms of behavior of slices, lie at the opposite ends of the available spectrum. Indeed RNP spaces are known to have plenty of slices of small diameter (more precisely every bounded subset of an RNP space admits slices of arbitrarily small diameter, see [DU, Section VII. 6.] for a summary of the equivalent reformulations of the RNP) while it is known that all slices of the unit ball of a space with the Daugavet property have maximal diameter (that is diameter two).

For the convenience of the reader let us recall that a Banach space X is said to have the *Daugavet property* if every rank-1 operator T from X to X satisfies the *Daugavet identity* $\|I+T\| = 1 + \|T\|$. Note that the Daugavet identity is in fact satisfied in a space with the Daugavet property for a much wider class of operators, and just to name a few for compact operators, weakly compact operators, and operators not fixing a copy of ℓ_1. This property was named after Daugavet who proved in [Dau] (1963) that the space $C[0,1]$ has the Daugavet property. Other examples of spaces with the Daugavet property soon followed, for example $L_1[0,1]$ (Lozanovskii [Loz] 1966) or $L_\infty[0,1]$ (this is attributed to Pełczyński 1965 in [Wer]).

The Daugavet property is known to have a strong influence on the linear structure of a Banach space. For example, if X is a space with the Daugavet property, then X and its dual X^* fail to have the RNP, X contains an isomorphic copy of ℓ_1, X^{**} contains an isometric copy of $L_1[0,1]$ (this is a more recent result by Abrahamsen, Lima, Nygaard and Troyanski in [ALNT] (2016) concerning spaces with the strong diameter two property), and X does not linearly embed into a space with an unconditional basis. We refer to the survey [Wer] for more details and for references, as well as for further applications in operator theory. We focus on the following geometric characterization of the Daugavet property which first appeared in [KSSW].

Proposition (Kadets, Shvidkoy, Sirotkin, Werner, 1997)**.** *Let X be a Banach space. The following assertions are equivalent.*

1. *The space X has the Daugavet property.*
2. *For every $x \in S_X$ and every slice S of B_X we have $\sup_{y\in S} \|x-y\| = 2$.*

A direct consequence of this result is that, as mentioned, every slice of the unit ball of a space with the Daugavet property has diameter two and in fact every point contained in a slice of the unit ball of a space with the Daugavet property is diametral in this slice. With modern terminology, the Daugavet property implies the diametral local (or slice) diameter two property. It was also proved by Shvydkoy in [Shv] (2000) that the Daugavet property implies other stronger diameter two properties, where weakly open subsets of B_X or convex combinations

of slices of B_X are considered instead of slices. The study of diameter two properties and the comparison between them has known a rich development those last years, and we refer for example to [ABGLP], [ALN], [BGLPRZ] or [HLLN] for a few (far from exhaustive) results in this direction. Also let us refer to Pirk's thesis [Pir] for a nice presentation of the subject and in particular for a few concise diagrams presented in appendix to synthesize all the relations between those properties (and the known example to distinguish them) as well as stability results under absolute sums.

The above characterization naturally gave rise in the paper [AHLP] by Abrahamsen, Haller, Lima and Pirk in 2020 to the notion of Daugavet-points and to the associated more local weaker notion of Δ-points. To be precise, a *Daugavet-point* in a space X is a element of S_X satisfying $\sup_{y\in S}\|x-y\| = 2$ for every slice S of B_X and a Δ-*point* in X is an element of S_X satisfying $\sup_{y\in S}\|x-y\| = 2$ for every slice S of B_X containing the point x. Since their introduction those points have attracted a lot of attention and where systematically studied in classical Banach spaces. We will say more about this at the start of Chapter 3 but we refer for example to [ALM], [ALMT], [HPV], [JRZ] or [Vee] for results in this direction. In particular a few very surprising results were discovered there showing that even very strong linear or geometric properties do not provide an obstruction to the existence of Daugavet-points. The most striking examples are the ones from [ALMT] by Abrahamsen, Lima, Martiny and Troyanski (2021) where a space with an unconditional basis in which Daugavet-points are weakly dense is obtained, and from [Vee] by Veeorg (2021) where a Lipschitz free-space with the RNP and admitting a Daugavet-point is constructed.

In Chapter 3 we provide asymptotic geometric conditions which ensure that specific points in the unit sphere of a given Banach fail to be Δ-points. An important tool in this study is the *Kuratowski index of non compactness* which measures the smallest radius $\varepsilon > 0$ for which one can hope to cover a given set by finitely many balls of radius ε. Our main results are the following.

Theorem (Abrahamsen, Lima, Martiny, P., 2022)**.** *Let X be a Banach space and let $x \in S_X$. If x is an asymptotically smooth point, then x is not a Δ-point.*

Theorem (Abrahamsen, Lima, Martiny, P., 2022)**.** *Let X be a Banach space and let $x \in S_X$. If there is a norm one functional which is norming for x and which defines slices of B_X of arbitrarily small Kuratowski index, then x is not a Δ-point.*

As a consequence, we obtain that asymptotically smooth spaces as well as reflexive asymptotically convex spaces fail to contain Δ-points, providing several new examples of Banach spaces without Δ-points such as spaces with the property (M^*) from Kalton, or spaces with an FDD admitting block upper ℓ_p estimates such as the predual of the James tree space JT. We also obtain that weak* asymptotically convex dual spaces fail to admit Daugavet-points as a consequence of the following result.

Theorem (Abrahamsen, Aliaga, Lima, Martiny, P., Procházka,Veeorg, work in progress)**.** *Let X be a Banach space with the Krein Milman property (KMP in short). If every extreme point of B_X is a denting point of B_X (or equivalently a point of weak-to-norm continuity for the identity map), then X fails to contain Daugavet-points.*

As a consequence, the James tree space JT fails to contain Daugavet-points. In this specific case, we also improved by hand this result by showing that JT fails to contain Δ-points. The main idea in this proof is the observation that if one starts with a finitely supported element x, then there is only a finite number of combinations of segments in the tree for which x attains its norm. So defining linear functionals supported on those combinations and taking their average, one obtains a functional norming x which defines slices of the unit ball of JT in which every element y has a large enough part concentrated on the same segments as x, thus obtaining a bound on the norm of the difference $x - y$. Passing to elements with an infinite support then requires a few technical adjustments, but is essentially based on the fact that if one restricts arbitrary combinations of segments in the tree to a finite subtree, then one can only find finitely many combinations of " starting segments " in this subtree. Our result is the following.

Theorem (Abrahamsen, Lima, Martiny, P., 2022)**.** *The James tree space JT fails to contain Δ-points.*

We also provide a few results concerning weak* Daugavet- and Δ-points in dual spaces, a natural extension of the notion of Daugavet- and Δ-points in the context of dual spaces where only weak* closed slices are considered.

The results announced in this section are part of a joint work and appear in the preprint [ALMP]. I wish to express here all my gratitude to Trond Abrahamsen, Vegard Lima and Andre Martiny for their warm welcome in Kristiansand in fall 2021 where the majority of this work was done. The result concerning spaces with the KMP was obtained in a work in progress with the same collaborators and the additional participation of Antonin Procházka, Ramon Aliaga, and Triinu Veeorg.

Chapter 1

Asymptotic properties of Banach spaces

1.1 Asymptotic uniform properties of norms

Following the standard notation from [JLPS] we define the asymptotic moduli of smoothness and convexity of a Banach space originally introduced by Milman in [Mil].

Definition 1.1.1. Let X be a Banach space and let $\operatorname{cof}(X)$ be the set of all (closed) subspaces of finite co-dimension of X. For every $t \geq 0$ and for every $x \in S_X$, let

$$\overline{\rho}_X(t,x) = \inf_{Y \in \operatorname{cof}(X)} \sup_{y \in S_Y} \|x + ty\| - 1$$

and

$$\overline{\delta}_X(t,x) = \sup_{Y \in \operatorname{cof}(X)} \inf_{y \in S_Y} \|x + ty\| - 1.$$

The functions

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \overline{\rho}_X(t,x)$$

and

$$\overline{\delta}_X(t) = \inf_{x \in S_X} \overline{\delta}_X(t,x)$$

are respectively called *modulus of asymptotic smoothness* and *modulus of asymptotic convexity* of the space X.

Remark 1.1.2. By a simple convexity argument we also have

$$\overline{\rho}_X(t,x) = \inf_{Y \in \operatorname{cof}(X)} \sup_{y \in Y, \|y\| \leq 1} \|x + ty\| - 1$$

and

$$\overline{\delta}_X(t,x) = \sup_{Y \in \operatorname{cof}(X)} \inf_{y \in Y, \|y\| \geq 1} \|x + ty\| - 1.$$

It is then easy to check that the functions $\overline{\rho}_X$ and $\overline{\delta}_X$ are both non-decreasing and 1-Lipschitz, and that $\overline{\rho}_X$ is moreover convex. Also let us point out that if X_0 is a subspace of X, then one clearly has $\overline{\delta}_X \leq \overline{\delta}_{X_0}$ and $\overline{\rho}_X \geq \overline{\rho}_{X_0}$. Since ℓ_1 has the best asymptotic modulus of convexity and since every separable Banach space can be represented as a quotient of ℓ_1, see for example [FHH+, Theorem 5.1], the modulus of asymptotic convexity of the quotient of a given Banach space is not bounded below in general by the modulus of the space. However, the modulus of asymptotic smoothness of any quotient of X is always bounded above by the modulus of X, see [JLPS, Proposition 2.8].

One of the features of the moduli of asymptotic smoothness and convexity is that they provide respectively upper and lower control on weakly converging sequences in Banach spaces.

Lemma 1.1.3. *Let X be a Banach space. For every $t > 0$, for every $x \in S_X$, and for every weakly null sequence $(x_n)_{n\geq 1}$ in B_X (respectively with norm $\|x_n\| \geq 1$) we have*

$$\limsup \|x + tx_n\| \leq 1 + \overline{\rho}_X(t, x)$$

and

$$\liminf \|x + tx_n\| \geq 1 + \overline{\delta}_X(t, x).$$

Remark 1.1.4. In fact it is well known that the moduli of asymptotic smoothness and asymptotic convexity of a Banach space X can be computed directly from weakly converging sequences provided that the space X does not contain ℓ_1, and in particular when the space X is reflexive or the dual space X^* is separable. A proof of this fact can be found for the modulus of asymptotic smoothness in [Dut, Section 6.4, Lemma 37] in the context of spaces with separable duals. We will provide a proof below for spaces not containing ℓ_1.

On some occurrence it is useful to compute the two asymptotic moduli with weak neighborhoods of 0 instead of subspaces of finite co-dimension. We state the following lemma here for future reference, but let us first introduce some notation. For every $x \in X$ we denote by $\mathcal{V}_w(x)$ the set of all neigborhoods of x for the weak topology and we write

$$V_{x_1^*,\ldots,x_n^*;\varepsilon}(x) = \{y \in X : \ |\langle x_i^*, x - y\rangle| < \varepsilon \text{ for every } 1 \leq i \leq n\}$$

the fundamental weakly open neigborhoods of x. If $x = 0$ we write $V_{x_1^*,\ldots,x_n^*;\varepsilon}$ instead.

Lemma 1.1.5. *Let X be a Banach space. For every $t > 0$ and for every $x \in S_X$ we have*

$$\overline{\rho}_X(t, x) = \inf_{V \in \mathcal{V}_w(0)} \sup_{v \in V, \|v\|=1} \|x + tv\| - 1$$

and

$$\overline{\delta}_X(t, x) = \sup_{V \in \mathcal{V}_w(0)} \inf_{v \in V, \|v\|=1} \|x + tv\| - 1,$$

and once again we can equivalently test the preceding expressions on elements with norm $\|v\| \leq 1$ and respectively $\|v\| \geq 1$.

Proof. In both cases one of the inequality is obvious since every fundamental weak neighborhood $V = V_{x_1^*,\dots,x_n^*;\varepsilon}$ of 0 contains the subspace of finite co-dimension $Y = \cap_{i=1}^n \ker x_i^*$. For the other direction one simply needs to use the following well known observation: for every $\mu \in (0,t)$ and for every $Y \in \mathrm{cof}(X)$ we can find a $V \in \mathcal{V}_w(0)$ such that for every $v \in V$ with norm $\|v\| = t$ there is a $y \in Y$ with norm $\|y\| = t$ at distance less than μ form v.

To prove this observation fix some $\eta > 0$ to be chosen later and take an η-net $(y_i^*)_{i=1}^n$ in the unit sphere of $Y^\perp \equiv (X/Y)^*$. Then let $V = V_{y_1^*,\dots,y_n^*;\eta}$. For any $v \in V$ with norm $\|v\| = t$ pick $y^* \in S_{Y^\perp}$ such that $\|q(v)\| = |y^*(v)|$ where $q = X \to X/Y$ is the quotient map. Choosing a suitable $1 \le i_0 \le n$ we then have $\|q(v)\| \le \left|y_{i_0}^*(v)\right| + \left\|y^* - y_{i_0}^*\right\| \|v\| \le (1+t)\eta < \frac{\mu}{2}$ if η was chosen strictly smaller than $\frac{\mu}{2(1+t)}$. Since $\|q(v)\| = \mathrm{dist}(v,Y)$ we can find a $y \in Y$ such that $\|v - y\| \le \frac{\mu}{2}$. We then have $\|y\| \in \left[t - \frac{\mu}{2}, t + \frac{\mu}{2}\right]$ and as a consequence $\left\|v - \frac{ty}{\|y\|}\right\| \le \mu$. The conclusion follows.

□

With this reformulation, the following result is then immediate.

Corollary 1.1.6. *Let X be a Banach space. For every $t > 0$ and every $x \in S_X$ we have*

$$\overline{\rho}_X(t,x) = \sup\ (\limsup \|x + tx_a\| - 1)$$

and

$$\overline{\delta}_X(t,x) = \inf\ (\liminf \|x + tx_a\| - 1)$$

where the supremum and infimum are taken on all weakly null normalized nets (or equivalently on weakly null nets in B_X, and respectively with norm $\|x_a\| \ge 1$).

With this corollary in hands, the result mentioned in Remark 1.1.4 follows from profound results by Rosenthal concerning spaces which do not contain ℓ_1 and in particular from the following lemma.

Lemma 1.1.7. *Let X be a Banach space not containing ℓ_1 and let $(x_a)_{a\in\alpha}$ be a bounded net in X. If $(x_a)_{a\in\alpha}$ converges weakly to some element $x \in X$, then we can find a sequence $(x_n)_{n\ge1}$ in the set $\{x_a\}_{a\in\alpha}$ which converges weakly to x.*

Proof. Let X be a Banach space not containing ℓ_1. With the terminology from [GG], every bounded set in X is a Rosenthal set, that is a set in which every sequence admits a weakly Cauchy subsequence. Thus the non separable version of Rosenthal's result [Ros2, Theorem 3] obtained in [GG, Theorem 2.6] gives that every bounded subset of X is weakly sequentially dense in its weak closure.

□

So let us state the following result.

Lemma 1.1.8. *Let X be a Banach space not containing ℓ_1. Then*

$$\overline{\rho}_X(t,x) = \sup\ (\limsup \|x + tx_n\| - 1)$$

and

$$\overline{\delta}_X(t,x) = \inf\ (\liminf \|x + tx_n\| - 1)$$

where the supremum and infimum are taken on all weakly null normalized sequences (or equivalently on weakly null sequences in B_X, and respectively with norm $\|x_n\| \geq 1$).

For later reference let us also point out the following strengthening of Lemma 1.1.3 for the modulus of asymptotic smoothness which has been proved in [LR, Proposition 2.1] using Goldstine's theorem. It basically says that the modulus of asymptotic smoothness also provides upper control on weak* converging sequences in the bidual X^{**} of the space provided the limit point is an element of X.

Lemma 1.1.9. *Let X be a Banach space. For every $t \in (0,1)$, for every $x \in S_X$, and for every weak*-null sequence $(x_n^{**})_{n\geq 1}$ in $B_{X^{**}}$ we have*

$$\limsup \|x + tx_n^{**}\| \leq 1 + \overline{\rho}_X(t,x).$$

The following asymptotic properties naturally arise when one wants to obtain non-trivial estimates in all the above inequalities. Let us point out that we are doing a slight abuse of language and of notation here as well as in previous definitions since all this concerns specifically the underlying norm on the considered space and is not preserved by taking equivalent norms. We will follow the convention and keep it as it is whenever there is no risk of confusion.

Definition 1.1.10. We say that X is *uniformly asymptotically smooth* (AUS in short) if the modulus $\overline{\rho}_X$ satisfies $\lim_{t\to 0} \frac{\overline{\rho}_X(t)}{t} = 0$ and we say that X is *asymptotically uniformly convex* (AUC in short) if the modulus $\overline{\delta}_X$ satisfies $\overline{\delta}_X(t) > 0$ for every $t > 0$.

By Remark 1.1.2 asymptotic smoothness and convexity pass to subspaces and asymptotic smoothness passes to quotients. Moreover it is well known that property AUS and AUC are dual one to the other in the reflexive setting. In order to obtain a general duality result, we need to introduce the weak* version of the modulus of asymptotic convexity and the corresponding property AUC*.

Definition 1.1.11. Let X Banach space and let $|.|$ be an equivalent norm on X^*. Also let $\mathrm{cof}^*(X^*)$ be the set of all weak* closed subspaces of finite co-dimension of X^*. For every $t > 0$ and for every $x^* \in S_{|.|}$, let

$$\overline{\delta}^*_{|.|}(t,x^*) = \sup_{E\in \mathrm{cof}^*(X^*)} \inf_{e^*\in E, |e^*|=1} |x^* + te^*| - 1.$$

The function

$$\overline{\delta}^*_{X^*}(t) = \inf_{x^*\in S_{|.|}} \overline{\delta}^*_{|.|}(t,x^*)$$

is called modulus of weak* asymptotic convexity of $(X^*, |.|)$. We say that $|.|$ is weak* asymptotically uniformly convex (AUC* in short) if the modulus $\overline{\delta}^*_{|.|}$ satisfies $\overline{\delta}^*_{|.|}(t) > 0$ for every $t > 0$.

Since an equivalent norm on a dual space is not necessarily the dual norm of an equivalent norm on the space itself, it is formally required to define the modulus of weak* asymptotic convexity as above. If $|.| = \|.\|_{X^*}$ is the dual norm of the underlying norm on the space X, we will use the notations $\overline{\delta}^*_{X^*}(t, x^*)$ and $\overline{\delta}^*_{X^*}(t)$. In practice we can always get back to such a case for AUC* norms thanks to the following fact.

Lemma 1.1.12. *Let X be a Banach space and let $|.|$ be an equivalent AUC* norm on X^*. Then $|.|$ is the dual norm of an equivalent norm on X.*

Proof. It follows from the definition of the modulus $\overline{\delta}^*_{|.|}$ that any equivalent AUC* norm on X^* is weak* lower semicontinuous. It is well known that such norms are dual norms (see for example [DGZ, Chapter 1, Fact 5.4]).

□

From the definition of the moduli, it is clear that $\overline{\delta}^*_{|.|} \leq \overline{\delta}_{|.|}$ for every equivalent norm $|.|$ on X^* and thus any AUC* norm is AUC. The converse does not hold true in general, as witnesses the dual of the James Tree space JT. Indeed, JT^* is AUC by a result from [Gir] but is known to admit no equivalent AUC* norm by an argument that we will encounter later (JT is not Asplund and thus does not admit an equivalent AUS norm, see Remark 1.1.15).

The modulus of weak* asymptotic convexity shares similar properties with its weak version. In particular it is 1-Lipschitz, non-decreasing, and can be expressed as in Lemma 1.1.5 using weak* neighborhoods of 0 instead of weak* closed subspaces of finite co-dimension. As a consequence it can be computed using weak* convergent nets as in Corollary 1.1.6 and it gives lower control on weak* null sequences in X^*. We gather those results in the following lemma. In this context we will denote by $\mathcal{V}_{w^*}(x^*)$ the set of all neighborhoods of an element x^* of X^* for the weak* topology and we will write $V_{x_1,\dots,x_n;\varepsilon}(x^*)$ and $V_{x_1,\dots,x_n;\varepsilon}$ the fundamental weak* open neighborhoods of x^* and of 0.

Lemma 1.1.13. *Let X be a Banach space and let $|.|$ be an equivalent norm on X^*. For every $t > 0$ and for every $x^* \in S_{|.|}$ we have:*

$$\overline{\delta}^*_{|.|}(t, x^*) = \sup_{V \in \mathcal{V}_{w^*}(0)} \inf_{v^* \in V, |v|=1} |x^* + tv^*| - 1,$$

and we can equivalently consider elements with norm $|v^*| \geq 1$ in the previous expression. For every weak null sequence $(x^*_n)_{n\geq 1}$ with norm $|x^*_n| \geq 1$, we have*

$$\liminf |x^* + tx^*_n| \geq 1 + \overline{\delta}^*_{|.|}(t, x^*).$$

Furthermore, if X is separable, then

$$\overline{\delta}^*_{|.|}(t, x^*) = \inf\left(\liminf |x^* + tx^*_n| - 1\right)$$

where the infimum is taken on all weak null normalized sequences (or equivalently on weak* null sequences with norm $|x^*_n| \geq 1$).*

The key point here is the well known fact that $(X^*/E)^* \equiv E^\top$ where $E^\top$ is the pre-orthogonal of E in X whenever E is a weak* closed subspace of X^*.

The following duality result was proved in [GKL2, Proposition 2.6] for separable spaces and in [DKLR, Proposition 2.1] in the general setting.

Proposition 1.1.14. *Let X be a Banach space and let $\sigma, \tau \in (0,1)$.*

1. *If $\overline{\rho}_X(\sigma) < \sigma\tau$, then $\overline{\delta}^*_{X^*}(6\tau) \geq \sigma\tau$.*
2. *If $\overline{\delta}^*_{X^*}(\tau) > \sigma\tau$, then $\overline{\rho}_X(\sigma) \leq \sigma\tau$.*

An immediate consequence of this result is that X is AUS if and only if X^* is AUC*. In particular a reflexive Banach space X is AUC if and only if its dual space is AUS.

Remark 1.1.15. Let us point out that this duality result does not hold true in general if one consider the sequential versions of the asymptotic properties of smoothness and (weak*) convexity which are defined by asking the exact same above conditions on the sequential moduli instead of the generalized ones. Indeed the space ℓ_1 has the Schur property (every weakly convergent sequence in ℓ_1 is norm convergent) and thus is sequential AUS, but its dual space ℓ_∞ does not admit an equivalent sequential AUC* norm. Indeed, such norm would be AUC* since ℓ_1 is separable, and thus would be by Lemma 1.1.12 and by duality the dual norm of an equivalent AUS norm on ℓ_1. Now ℓ_1 is not Asplund since it is separable with a non-separable dual and [JLPS, Proposition 2.4] tells us that $\overline{\rho}_{|.|}(t) = t$ for every $t > 0$ and for every equivalent norm $|.|$ on ℓ_1. Yet this sequential duality does hold true for reflexive spaces or for separable spaces not containing ℓ_1 since the sequential and generalized moduli coincide in this context.

At this point the reader might wonder why we did not bother introducing a weak* version of the modulus of asymptotic smoothness. This is because the existence of an equivalent AUS norm on a dual space X^* which is a dual norm forces X to be reflexive. This follows from the duality AUS/AUC* and the following result, which is proved for example in [CL1, Proposition 2.6].

Lemma 1.1.16. *Let us assume that the bidual norm $\|.\|_{X^{**}}$ is AUC*. Then the space X is reflexive.*

Let us now give the following fundamental examples.

Example 1.1.17. Let $p \in (1,\infty)$ and take $t \in (0,1)$. Using a gliding hump argument, one can show that $\overline{\rho}_{c_0}(t) = 0$, $\overline{\rho}_{\ell_p}(t) = \overline{\delta}_{\ell_p}(t) = (1+t^p)^{\frac{1}{p}} - 1$ and $\overline{\delta}^*_{\ell_1}(t) = t$ for the weak* topology $\sigma(l_1, c_0)$. Thus c_0 has the best modulus of asymptotic smoothness while ℓ_1 has the best modulus of (weak*) asymptotic convexity. Since asymptotic moduli do not see spaces of finite dimension, the same actually goes for $\left(\sum_{n\geq 1} F_n\right)_X$ where the F_n are finite dimensional spaces and X is any of the above space.

This motivates the following definitions.

Definition 1.1.18. Let X be a Banach space. We say that X is *p-AUS* or AUS with *power type* p for some $p \in (1,\infty)$ if there is a constant $C > 0$ such that $\overline{\rho}_X(t) \leq Ct^p$ for every $t \in (0,1)$. Also we say that X is *asymptotically uniformly flat* (AUF in short) if there is a $t_0 > 0$ such that $\overline{\rho}_X(t_0) = 0$. We say that X (respectively X^*) is *p-AUC* (respectively *p-AUC**) for some $p \in [1,\infty)$ if there is a constant $c > 0$ such that $\overline{\delta}_X(t) \geq ct^p$ (respectively $\overline{\delta}^*_{X^*}(t) \geq ct^p$) for every $t \in (0,1)$.

With the preceding computations c_0 is AUF, ℓ_p is p-AUS and p-AUC for every $p \in (1,\infty)$, and ℓ_1 is 1-AUC*.

An immediate consequence of Proposition 1.1.14 is that a Banach space X is p-AUS for some $p \in (1,\infty)$ if and only if its dual space is p^*-AUC* where p^* is the conjugate exponent of p (that is $p^* \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{p^*} = 1$). Also X is AUF if and only if X^* is 1-AUC*. Those definitions capture the asymptotic behavior of c_0, ℓ_1 and ℓ_p spaces.

Lemma 1.1.19. *Let X be a Banach space. If X is p-AUS for some $p \in (1,\infty)$ then there is a constant $C > 0$ such that for every $x \in X$ and for every weakly null sequence $(x_n)_{n\geq 1}$ in X we have*

$$\limsup \|x + x_n\|^p \leq \|x\|^p + C \limsup \|x_n\|^p .$$

By Lemma 1.1.9 this also applies to weak null sequences in X^{**}. Next if X is q-AUC for some $q \in [1,\infty)$, then there is a constant $c > 0$ such that for every $x \in X$ and for every weak-null sequence $(x_n)_{n\geq 1}$ in X we have*

$$\liminf \|x + x_n\|^q \geq \|x\|^q + c \liminf \|x_n\|^q .$$

A similar statement holds for q-AUC duals.*

Let us point out that this is known to imply the following proposition. The proof can be done by a direct induction but then involves a few technical tricks. A more elegant proof can be done by using the so called Orlicz iterated norms associated with the asymptotic moduli. We will introduce those later in the text (Chapter 2, Section 2.1) but we refer the interested reader to [Kal2, Lemma 4.3 and related] or [LPP, Lemma 3.4 and related] for definitions and a few basic results. Note that there is no way here to obtain a similar result in general in the bidual of the space for the AUS part since each step would require limit points to be in the space itself. Skirting around this obstruction will be a key point in the forthcoming non-linear section when we will be working in the bidual of non-reflexive AUS spaces. Let us also point out that this proposition is connected to the so called p-Banach-Saks and p-co-Banach-Saks property, see again [Kal2, Proposition 4.6 and related].

Proposition 1.1.20. *Let X be a Banach space. If X is p-AUS for some $p \in (1,\infty)$ or if X is AUF, then there is a constant $C > 0$ such that for every weakly null sequence $(x_n)_{n\geq 1}$ in X we can extract a subsequence $(x_{n_i})_{i\geq 1}$ for which we have the following property: for every $k \geq 1$ and for every $n_1 < \cdots < n_k$ we have*

$$\left\| \sum_{i=1}^{k} x_{n_i} \right\|^p \leq C \sum_{i=1}^{k} \|x_{n_i}\|^p ,$$

and respectively

$$\left\| \sum_{i=1}^{k} x_{n_i} \right\| \leq C \max_{1 \leq i \leq k} \|x_{n_i}\| .$$

Next if X is q-AUC for some $q \in [1, \infty)$, then there is a constant $c > 0$ such that for every weak-null sequence $(x_n)_{n \geq 1}$ in X we can extract a subsequence $(x_{n_i})_{i \geq 1}$ for which we have the following property: for every $k \geq 1$ and for every $n_1 < \cdots < n_k$ we have

$$\left\| \sum_{i=1}^{k} x_{n_i} \right\|^q \geq c \sum_{i=1}^{k} \|x_{n_i}\|^q .$$

A similar statement holds for q-AUC* duals.

Remark 1.1.21. It is well known that every separable AUF space is isomorphic to a subspace of c_0 (see [GKL1, Section 2] in which the result is stated with a different terminology and [GKL1, Section 4] for some non-separable versions, and see [JLPS, Theorem 2.9] for an easier proof and a few historical remarks). Subspaces of c_0 are not necessarily isomorphic to quotients of c_0, but if one fixes a sequence $(F_n)_{n \geq 1}$ of finite dimensional spaces which is dense in Banach-Mazur distance in the space of all finite dimensional spaces, then it is known since the results from [JZ] that subspaces of $\mathcal{U}_\infty = \left(\sum_{n \geq 1} F_n\right)_{c_0}$ (and in particular subspaces of c_0) are isomorphic to quotients of $\mathcal{U}_\infty$. By duality, we then immediately obtain that if the dual of a separable space is 1-AUC*, then it has to be isomorphic to a weak* closed subspace of $\left(\sum_{n \geq 1} F_n^*\right)_{\ell_1}$. A remarkable isometric version of those results can be obtained by applying the results from [KW] which are based on property (m_∞) from Kalton (we provide a proof below for the sake of completeness). Also we refer to the paper [CL2] for very recent analogue results obtained for finite exponents p. The optimality of the standard asymptotic convexity modulus can also be formulated in two different ways: isometrically with the condition $\overline{\delta}_X(t) = t$ for every $t > 0$, and isomorphically with 1-AUC. In [PRZ, Proposition 4.8] the isometric optimality is shown to have implications on some metric properties of infinite subsets of the space and thus on the octahedrally of the associated Lipschitz-free space. It is asked (Remark 4.10) whether every such space has to embed almost isometrically into an ℓ_1-sum of finite dimensional spaces. Up to our knowledge the question is still open in general, but we have the following weak* result.

Theorem 1.1.22. *Let X be a separable Banach space and let us assume that $\overline{\delta}^*_{X^*}(t) = t$ for every $t \in (0, 1)$. Then X^* embeds almost isometrically into an ℓ_1-sum of finite dimensional spaces.*

Proof. Let us assume that $\overline{\delta}^*_{X^*}(t) = t$ for every $t \in (0, 1)$. We will prove that the space X has property (m_∞) as introduced in [Kal1], that is that for every $x \in X$ and every weakly null sequence $(x_n)_{n \geq 1}$ in X we have $\limsup \|x + x_n\| = \max\{\|x\|, \limsup \|x_n\|\}$.

First let us use duality results to prove that $\overline{\rho}_X(t) = 0$ for every $t \in (0, 1)$. For any fixed $t \in (0, 1)$ and $\varepsilon \in (0, t)$ we have $\overline{\delta}^*_{X^*}\left(\frac{\varepsilon}{t}\right) = \frac{\varepsilon}{t} > \varepsilon = t \times \frac{\varepsilon}{t}$ so that $\overline{\rho}_X(t) \leq t \times \frac{\varepsilon}{t} = \varepsilon$ by Proposition 1.1.14. Letting ε go to 0 we obtain $\overline{\rho}_X(t) = 0$ as desired. Also observe that this

implies that $\overline{\rho}_X(1) = 0$ since $\overline{\rho}_X$ is continuous, and thus that $\overline{\rho}_X(t) = \overline{\rho}_X(t) - \overline{\rho}_X(1) \leq t - 1$ for every $t \geq 1$ since $\overline{\rho}_X$ is 1-Lipschitz as mentioned in Remark 1.1.2.

Now let us fix a non-zero point $x \in X$ and let us fix a weakly null sequence $(x_n)_{n\geq 1}$ in X. A direct reformulation of Lemma 1.1.3 then yields

$$\limsup \|x + x_n\| \leq \|x\| \left(1 + \overline{\rho}_X\left(\frac{\limsup \|x_n\|}{\|x\|}\right)\right),$$

and by the above properties of $\overline{\rho}_X$ we easily obtain $\limsup \|x + x_n\| \leq \max\{\|x\|, \limsup \|x_n\|\}$.

By weak lower continuity of the norm, we have $\|x\| \leq \liminf \|x + x_n\|$ and we may thus assume that $\alpha = \limsup \|x_n\| > 0$. By using standard Mazur techniques presented for example in [AK, Chapter 1, Section 1.5] we can then extract for every $\varepsilon > 0$ a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ such the sequence $(y_k)_{k\geq 1}$ defined by $y_1 = \frac{x}{\|x\|}$ and $y_k = \frac{x_{n_{k-1}}}{\alpha}$ for every $k \geq 2$ is $(1+\varepsilon)$ basic. Then

$$\frac{1}{1+\varepsilon}\max\{\|x\|, \alpha\} \leq \|\|x\| y_1 + \alpha y_k\| = \left\|x + x_{n_{k-1}}\right\|$$

for every $k \geq 2$ so that

$$\frac{1}{1+\varepsilon}\max\{\|x\|, \alpha\} \leq \limsup \left\|x + x_{n_{k-1}}\right\| \leq \limsup \|x + x_n\|.$$

Letting ε go to 0 we obtain $\limsup \|x + x_n\| \geq \max\{\|x\|, \limsup \|x_n\|\}$ and X has property (m_∞) as announced.

To conclude we appeal to [KW, Theorem 3.2] which states that for any separable Banach space with property (m_∞) and for every $\varepsilon > 0$ there exists a quotient of the space $\left(\sum_{n\geq 1} F_n\right)_{c_0}$ which is $(1+\varepsilon)$ isomorphic to X, where $(F_n)_{n\geq 1}$ is any fixed sequence of finite dimensional spaces which is dense in Banach-Mazur distance in the space of all finite dimensional spaces. □

Let us conclude the section by giving a few more examples.

Example 1.1.23. It is shown in [JLPS, Proposition 2.3] that the moduli of asymptotic smoothness and convexity of a Banach space X are respectively controlled by the standard modulus of smoothness ρ_X of X and by the modulus of convexity δ_X of X. More precisely, $\rho_X \geq \frac{1}{2}\overline{\rho}_X$ and $\delta_X \leq \overline{\delta}_X$ on $(0,1)$. As a consequence any uniformly smooth space is AUS and any uniformly convex space is AUC. In particular the classical $L_p[0,1]$ spaces for $p \in (1,\infty)$ are both AUS and AUC with power types determined in [Han]. Also see [BCL] for discussions about the optimality of those results.

Observe that Example 1.1.17 already provides easy examples of reflexive Banach spaces which are p-AUS and p-AUC but not super-reflexive, such as $\left(\sum_{n\geq 1} \ell_\infty^n\right)_{\ell_p}$. In the following we denote by c_{00} the space of all eventually null sequences of real numbers and we denote by $(e_i)_{i\geq 1}$ the canonical unit vector basis of c_{00}.

Example 1.1.24. Let $p \in (1, \infty)$. For every $x \in c_{00}$, we introduce the quantity

$$\|x\|_{B_p}^p = \sup \left\{ \sum_{i=1}^n \|E_i x\|_{\ell_1}^p : \ E_1 < \cdots < E_n \subset \mathbb{N} \text{ and } |E_i| \leq E_i \text{ for every } 1 \leq i \leq n \right\}$$

where Ex stands for the projection of the vector x on the space $[e_n]_{n \in E}$ for every non empty subset E of $\mathbb{N}$ and $E_1 < \cdots < E_n$ means that $\max E_i < \min E_{i+1}$ for every $1 \leq i \leq n-1$. The Baernstein space B_p is the completion of the space $(c_{00}, \|.\|_{B_p})$. It is known to be reflexive, p-AUC (the unit vector basis $(e_i)_{i \geq 1}$ of c_{00} forms a monotone unconditional basis of B_p and satisfies lower ℓ_p estimates, see Example 1.1.26), but not super reflexive since failing the Banach-Saks property. Basic properties of the Baernstein space $B = B_2$ are presented in the introductory Chapter 0 of [CS, Construction 0.9].

Example 1.1.25. Let $p \in (1, \infty)$. For every $x \in c_{00}$ we introduce the p-variation of x

$$\|x\|_{J_p}^p = \sup_{k \geq 1} \sup_{1 \leq n_1 < \cdots < n_k} \sum_{i=1}^{k-1} \left| x_{n_{i+1}} - x_{n_i} \right|^p .$$

The James space J_p is the completion of the space $(c_{00}, \|.\|_{J_p})$. It is known to be quasi-reflexive of order 1 (i.e. to satisfy $\dim J_p^{**}/J_p = 1$) and to admit an equivalent p-AUS norm. It is also known that its dual J_p^* admits an equivalent p^*-AUS norm where p^* is the conjugate exponent of p, and that its bidual J_p^{**} also admits an equivalent p-AUS norm. We refer to [LPP, Section 5] for more information and for references, and to [Net, Section 2.3] for the construction of the equivalent norms.

Example 1.1.26. Let X be a Banach space with an FDD E. We say that E admits block upper ℓ_p estimates for some $p \in (1, \infty)$ if there is a constant $C > 0$ such that for every finite blocking F of E and for every $x_1, \ldots x_N$ in the blocking we have $\left\| \sum_{n=1}^N x_n \right\|^p \leq C \sum_{n=1}^N \|x_n\|^p$. We say that E admits block lower ℓ_q estimates for some $q \in (1, \infty)$ if there is a constant $c > 0$ such that for every finite blocking F of E and for every $x_1, \ldots x_N$ in the blocking we have $\left\| \sum_{n=1}^N x_n \right\|^q \geq c \sum_{n=1}^N \|x_n\|^q$. It is well known that an FDD admitting upper ℓ_p estimates is shrinking while an FDD admitting lower ℓ_q estimates is boundedly complete (we refer to [LT, Section 1.g] for definitions in this context). The latter can be proved by using the following criterion (whose extension to the FDD context is straightforward), which is left as an exercise in [AK, exercice 3.8] and whose proof is can be found in [Cau1, proposition 3.1]: an FDD E is boundedly complete if and only if $\sup_N \left\| \sum_{n=1}^N x_n \right\| = \infty$ for any sequence $(x_n)_{n \geq 1}$ bounded away from 0 in a blocking F of E. The former is then obtained by duality. Applying [GLR, corollary 2.4] we then have that any space with a monotone FDD admitting block upper ℓ_p estimates is p-AUS while any space admitting a monotone FDD with block lower ℓ_q estimates is q-AUC* as the dual of the space of $Y = \overline{\text{span}}\{\text{rng } (P_i^E)^*, \ i \geq 1\}$ where the P_i^E are the projections associated with the FDD E. In particular we will see later in Chapter 2 Section 3.4 that the James tree space JT is 2-AUC*.

Remark 1.1.27. As already mentioned, spaces which are not Asplund do not admit equivalent AUS norms, and in particular both ℓ_1 and JT do not admit equivalent AUS norms. It is also known that any AUC space (respectively any AUC* dual) satisfies the point of continuity property (PCP) [JLPS, Proposition 2.6] (respectively the weak* PCP) and furthermore that any equivalent norm $|.|$ on a space without the PCP satisfy $\delta_{|.|}(\frac{1}{2}) = 0$ [Gir, Proposition 6]. In particular the spaces c_0, ℓ_∞, $L_1[0,1]$ and $C[0,1]$ do not admit equivalent AUC norms since they all fail the PCP in a strong way: their norms satisfy the diameter two property (i.e. every weakly open subset in the unit ball of those spaces has diameter two).

1.2 Szlenk index and trees in Banach spaces

The Szlenk index is a fundamental object in the linear theory of Banach spaces which was introduced in [Szl] (in a slightly different form which coincides with the nowadays standard following definition provided that the considered space does not contain ℓ_1) in order to prove that there is no separable reflexive universal Banach space for the class of separable reflexive Banach spaces. It is an ordinal which is obtained by measuring the speed of a certain " peeling procedure ".

Generally speaking, implementing a peeling procedure on a topological space X consists in constructing for every $\varepsilon > 0$ and for every ordinal $\alpha \geq 1$ a function f_ε^α which associates to any given closed subset K of X a closed subset $f_\varepsilon^\alpha(K)$ of K, such that the family $(f_\varepsilon^\alpha)_{\varepsilon,\alpha}$ is decreasing with respect to the parameter ε and with respect to the parameter α. In other words, if one fixes a closed subset K of X, then $f_\varepsilon^\alpha(K)$ gets bigger as ε goes to 0 for any fixed α and $f_\varepsilon^\alpha(K)$ gets smaller as α increases for any fixed ε. The standard way of constructing such a family is to start by fixing a derivation on X, that is to provide for every $\varepsilon > 0$ a function f_ε which associate to any given closed subset K of X a closed subset $f_\varepsilon(K)$ of K, such that the family $(f_\varepsilon)_{\varepsilon>0}$ is decreasing with respect to the parameter ε. Then one obtains a peeling procedure inductively by defining, for a fixed $\varepsilon > 0$, $f_\varepsilon^1 = f_\varepsilon$, $f_\varepsilon^{\alpha+1} = f_\varepsilon \circ f_\varepsilon^\alpha$ for any ordinal $\alpha \geq 1$, and $f_\varepsilon^\beta = \bigcap_{\alpha<\beta} f_\varepsilon^\alpha$ for any limit ordinal β. For a fixed closed subset K of X, the index for the peeling procedure $(f_\varepsilon^\alpha)_{\varepsilon,\alpha}$ or for the derivation $(f_\varepsilon)_{\varepsilon>0}$ for the set K is then defined as $I(K) = \sup_{\varepsilon>0} I(K,\varepsilon)$ where $I(K,\varepsilon)$ is equal to the minimum of the set $\{\alpha \geq 1 : \ f_\varepsilon^\alpha(K) = \emptyset\}$ if such an ordinal exists and otherwise is given an abstract ∞ value.

Let X be a Banach space, let K be a weak*-compact subset of X^* and fix $\varepsilon > 0$. Denote by $\mathcal{V}$ the set of all weak*-open subsets V of X^* satisfying diam $(V \cap K) \leq \varepsilon$ and define $s_\varepsilon(K) = K \setminus \left(\bigcup_{V\in\mathcal{V}} V\right)$. Observe that

$$s_\varepsilon(K) = \{x^* \in K : \ \forall V \in \mathcal{V}_{w^*}(x^*), \ \operatorname{diam} V \cap K > \varepsilon\}.$$

The *Szlenk index* of the set K is the index associated with the derivation $(s_\varepsilon)_{\varepsilon>0}$ and it is denoted by $S_Z(K) = \sup_{\varepsilon>0} S_Z(K,\varepsilon)$.

Definition 1.2.1. The *Szlenk index* of a Banach space X is defined as $S_Z(X) = S_Z(B_{X^*})$. We also write $S_Z(X,\varepsilon) = S_Z(B_{X^*},\varepsilon)$.

For convenience we gather a few basic properties of the Szlenk index. An extensive study of its properties and of its usage both in the linear and non-linear theory can be found in the survey [Lan2].

Proposition 1.2.2. *Let X be a Banach space.*

1. *The space X is finite dimensional if and only if $S_Z(X) = 1$. Moreover, if X is infinite dimensional, then $S_Z(X) \geq \omega$ where ω is the first infinite ordinal.*

2. *The Szlenk index is invariant under linear isomorphisms. Moreover, if Y is a subspace of X, then $S_Z(Y) \leq S_Z(X)$ and $S_Z(X/Y) \leq S_Z(X)$.*

3. *The ordinal $S_Z(X, \varepsilon)$ is never a limit ordinal. In particular $S_Z(X) \leq \omega$ if and only if $S_Z(X, \varepsilon) < \omega$ for every $\varepsilon > 0$.*

4. *The function $S_Z(X, .)$ is submultiplicative: for every $\varepsilon, \varepsilon' > 0$, we have $S_Z(X, \varepsilon\varepsilon') \leq S_Z(X, \varepsilon)S_Z(X, \varepsilon')$. In particular if $S_Z(X) = \omega$ then there is a constant $C > 0$ and a $p \in [1, \infty)$ such that $S_Z(X, \varepsilon) \leq C\varepsilon^p$ for every $\varepsilon > 0$.*

It is also mentionned in [Lan2, Proposition 3.3] that the Szlenk index is equal to ω^α for some ordinal α whenever it has an ordinal value (i.e. is not equal to ∞). The set of all possible values for the Szlenk index of a Banach space is completely described in [Cau2, Theorem 1.5], and this result is refined in [CL1] by using a construction from Lindenstrauss, which for any given separable space X provides a Banach space Z satisfying $Z^{**}/Z \simeq X$ and $S_Z(Z) = S_Z(Z^*) = \omega$, the latter being proved in [CL1, Theorem 1.2]. In this dissertation we will mainly focus on the ω value, and we will refer the reader to appropriate references for higher ordinals when generalizations exist.

Note that the Szlenk index coincides with the *Oscillation index* of the identity map $id : (B_{X^*}, w^*) \to (B_{X^*}, \|.\|)$. Thus the standard characterization of Banach spaces admitting separable duals can be written as follows (see [Lan2, Theorem 3.8] for more details).

Theorem 1.2.3. *Let X be a separable Banach space. The following assertions are equivalent.*

1. *The space X^* is separable.*

2. *The Szlenk index $S_Z(X)$ is countable, that is $S_Z(X) < \omega_1$ where ω_1 is the first uncountable ordinal.*

3. *The identity map from (B_{X^*}, w^*) to $(B_{X^*}, \|.\|)$ is Baire-1.*

In fact the Szlenk index plays a fundamental role in the Asplund theory, see for example the monograph [DGZ].

Theorem 1.2.4. *Let X be a Banach space. The following assertions are equivalent.*

1. *The space X is Asplund.*

2. *Every bounded non-empty subset of X^* admits non-empty weak* slices of arbitrarily small diameter.*

3. *The space X^* has the Radon-Nikodým property (RNP).*

4. *Every separable subspace of X has a separable dual.*

5. *Every bounded non-empty subset of X^* admits non-empty relatively weak* open subsets of arbitrarily small diameter (X^* satisfy the weak* PCP).*

6. *The Szlenk index has an ordinal value, that is $S_Z(X) < \infty$.*

Remark 1.2.5. Let us point out that the above assertions are not equivalent to the space X^* having the PCP property. Indeed the James tree space JT is not Asplund since it is separable and has a non-separable dual, but its dual space JT^* is AUC [Gir] and thus has the PCP, see Remark 1.1.27.

The following theorem is one of the fundamental results in the asymptotic theory of Banach spaces. The original proof from [KOS] is based on deep studies of the asymptotic structure of Banach spaces with an FDD, a concept originally introduced in [MMTJ]. We will present an elementary introduction to this theory in the next section.

Theorem 1.2.6 (UKK* renorming, [KOS])**.** *Let X be an infinite dimensional separable Banach space. The following assertions are equivalent.*

1. *The space X admits an equivalent norm whose dual norm is UKK^*.*

2. *The space X satisfy $S_Z(X) = \omega$.*

3. *The space X admits an equivalent norm whose dual norm is UKK^* with power type modulus.*

See bellow for the definition of the UKK* property. In fact a stronger structural result was proved in [KOS] and can be reformulated as follows.

Theorem 1.2.7. *Let X be a separable infinite dimensional Banach space with $S_Z(X) = \omega$. Then there exists a Banach space Y with a shrinking FDD satisfying block upper ℓ_p estimates for some $p \in (1, \infty)$ such that X is isomorphic to a quotient of Y.*

An alternative proof of the renorming Theorem 1.2.6 was given in [GKL2] based on tight estimates on the so called *convex Szlenk index* $C_Z(X) = C_Z(B_{X^*})$ associated with the derivation given by $c_\varepsilon = \overline{\operatorname{conv}}\, s_\varepsilon$, and on considerations about the maximal height of certain weakly null trees in the unit ball of the space. In particular the following result concerning the best possible power type for the UKK* renorming was obtained.

Theorem 1.2.8 ([GKL2], Theorem 4.8)**.** *Let X be an infinite dimensional separable Banach space with $S_Z(X) = \omega$ and let $p_X = \lim_{\varepsilon\to 0} \frac{\log S_Z(X,\varepsilon)}{|\log \varepsilon|}$ be the Szlenk power type of X. Then for every $p > p_X$ there is an equivalent norm on X whose dual norm is UKK* with power type p.*

Remark 1.2.9. The *Szlenk power type* can be also expressed as

$$p_X = \inf\left\{p \geq 1 : \sup_{\varepsilon>0} \varepsilon^p S_Z(X,\varepsilon) \text{ is finite}\right\}.$$

In a way Theorem 1.2.8 is optimal since the small loss on the exponent cannot be relaxed in general, even in the reflexive setting. Indeed was pointed out in [KOS, Remark 7.2] that the Tsirelson's space is known to admits equivalent p-UKK norms for every $p > 1$ but to admit no equivalent 1-UKK norm. We provide a short proof of this fact here even though the necessary tools will be introduced a bit later.

Proof. Let T be the space constructed by Figiel and Johnson (see [CS, Chapter 1]) and which corresponds to the dual of the original Tsirelson's space. We claim that for any weakly null bounded tree $(x_s)_{s\in T_n}$ in T rooted at 0 we can find a branch $(x_t)_{t\leq s}$ where $s \in T_n$ is of maximal length such that $\left\|\sum_{t\leq s} x_t\right\| \geq \frac{1}{4}\sum_{t\leq s}\|x_t\|$. Roughly the idea is to use a gliding hump argument to construct a sequence $(y_i)_{i=1}^n$ in T such that $n \leq \operatorname{supp} y_1 < \cdots < \operatorname{supp} y_n$ and $\|y_i\| \simeq \left\|x_{s_{|i}}\right\|$ for every $1 \leq i \leq n$, and then to use the fundamental property of T, [CS, Proposition I.2], which then implies $\|\sum_{i=1}^n y_i\| \geq \frac{1}{2}\sum_{i=1}^n \|y_i\|$. From the claim and from Lemma 1.2.17 it follows that $S_Z(T^*,\varepsilon) \lesssim \varepsilon^{-1}$ so that T^* has Szlenk index ω and Szlenk power type $p_{T^*} = 1$. Now T^* is reflexive so it cannot be isomorphic to a subspace of c_0 (see for example [AK, Proposition 2.2.1]) and thus it cannot have any equivalent AUF norm. By duality, T has no 1-AUC (or equivalently since T is reflexive no 1-UKK) norm.

□

Those results where extended to the non-separable setting in [Raj] where a different geometric approach of renorming based on "slow slicing methods" involving the Kuratowski index on non-compactness is also proposed. In [LPR] general measures of non-compactness are considered and applications to higher ordinals are given (in the separable case). In particular the following [LPR, Corollary 5.1] was obtained, filling the gaps in previous results from [HS].

Theorem 1.2.10. *Let X be a separable Banach space. Then $C_Z(X) = S_Z(X)$.*

Those results where extended to the non-separable setting and for operators in [CD1] and further generalizations concerning higher ordinal Szlenk power type where then obtained in [Cau5].

The *uniform Kadec-Klee property* (UKK in short) was first introduced in [Huf]. Let us recall that X is UKK if for every $\varepsilon > 0$ we can find a $\delta > 0$ such that for every element $x \in X$, if there exists an ε-separated sequence $(x_n)_{n\geq 1}$ in B_X which converges weakly to x , then x has to be in $(1-\delta)B_X$. Let us also recall that a sequence $(x_n)_{n\geq 1}$ is said to be *ε-separated* if

$\operatorname{sep}\{x_n\}_{n\geq 1} = \inf_{n\neq m} \|x_n - x_m\|$ is bigger than or equal to ε. Looking at the best possible $\delta(\varepsilon)$ in the above definition one can define a UKK modulus as in [KOS] and consider the notion of power type UKK. The following lemma is well known but the proof is scattered in the literature. We provide a sketch of it bellow for completeness.

Lemma 1.2.11. *Let X be a Banach space. The following assertions are equivalent.*

1. *The space X is UKK.*
2. *For every $\varepsilon > 0$ we can find a $\delta > 0$ such that for every element $x \in B_X$, if there exists $(x_n)_{n\geq 1}$ in B_X which converges weakly to x and satisfies $\|x - x_n\| \geq \varepsilon$ for every $n \geq 1$ then x has to be in $(1-\delta)B_X$.*
3. *For every $t > 0$ we can find a $\delta > 0$ such that for every $x \in S_X$ and for every weak-null sequence $(x_n)_{n\geq 1}$ in X with norm $\|x_n\| \geq 1$, we have $\liminf \|x + tx_n\| \geq 1 + \delta$.*

Proof. (1) $\implies$ (2) Fix $\varepsilon > 0$ and let $(x_n)_{n\geq 1}$ be a sequence in B_X converging weakly to some $x \in B_X$ and satisfying $\|x - x_n\| \geq \varepsilon$ for every $n \geq 1$. For any fixed $n_0 \geq 1$ we have, by lower semicontinuity of the norm, $\liminf \|x_n - x_{n_0}\| \geq \|x - x_{n_0}\| \geq \varepsilon$ so we can find some $N \geq 1$ such that $\|x_n - x_{n_0}\| \geq \frac{\varepsilon}{2}$ for every $n \geq N$. Building on this observation we construct inductively a subsequence $(x_{n_k})_{k\geq 1}$ which is $\frac{\varepsilon}{2}$-separated.

(2) $\implies$ (3) Goes by contraposition. Fix $t > 0$, let $x \in S_X$, and let $(x_n)_{n\geq 1}$ be a weak-null sequence with norm $\|x_n\| \geq 1$ such that $\liminf \|x + tx_n\| < 1 + \delta$ for some $\delta \in (0, 1)$. Then let us assume as we may that $\|x + tx_n\| \leq 1 + \delta$ for every $n \geq 1$. We define $z_n = \frac{x+tx_n}{1+\delta}$ for every $n \geq 1$ and $z = \frac{x}{1+\delta}$. Then the sequence $(z_n)_{n\geq 1} \subset B_X$ converges weakly to z, and we have $\|z - z_n\| \geq \frac{\|tx_n\|}{1+\delta} \geq \frac{t}{2}$ for every $n \geq 1$ and $\|z\| = \frac{1}{1+\delta} = 1 - \frac{\delta}{1+\delta} \geq 1 - \frac{\delta}{2}$.

(3) $\implies$ (1) Let $x \in B_X$ and let $(x_n)_{n\geq 1}$ be an ε-separated sequence in B_X converging weakly to x. We define $z = \frac{x}{\|x\|}$ and $z_n = \frac{x-x_n}{\|x\|}$ for every $n \geq 1$. The sequence $(z_n)_{n\geq 1}$ converges weakly to 0 and we have $\|z_n - z_m\| = \frac{\|x_n - x_m\|}{\|x\|} \geq \varepsilon$ for every distinct $n, m \geq 1$ so that $\|z_n\| \geq \frac{\varepsilon}{2}$ or $\|z_m\| \geq \frac{\varepsilon}{2}$. Passing to a subsequence we may assume that $\|z_n\| \geq \frac{\varepsilon}{2}$ for every $n \geq 1$. Then observe that $\|z + z_n\| = \frac{\|x_n\|}{\|x\|} \leq \frac{1}{\|x\|}$ for every $n \geq 1$ so that $\liminf \|z + z_n\| \leq \frac{1}{\|x\|}$. □

A direct consequence of this lemma is that a norm is UKK if and only if it is sequentially AUC, and it is moreover contained in the above proof that the corresponding moduli are equivalent. By the results mentioned in Section 1.1 (Lemma 1.1.8) UKK is equivalent to AUC for spaces not containing ℓ_1. In full generality, AUC is equivalent to the following generalized version of UKK property introduced in [Lan1].

Lemma 1.2.12. *Let X be a Banach space. Then X is AUC if and only if it satisfies the following property: for all $\varepsilon > 0$ there is a $\Delta > 0$ such that for all $x \in B_X$, if $\|x\| > 1 - \Delta$ then there is a weak-neighborhood V of x such that* $\operatorname{diam} V \cap B_X \leq \varepsilon$.

Proof. Using Lemma 1.1.5 the property " X is AUC " can be reformulated in the following way: for every $t > 0$ there is a $\delta > 0$ such that for every $x \in S_X$ there exists a weak neighborhood $V \in \mathcal{V}_w(0)$ of 0 such that $\|x + tv\| \geq 1 + \delta$ for every $v \in V$ with norm $\|v\| \geq 1$. Also the generalized UKK property is clearly equivalent to the following: for all $\varepsilon > 0$ there is a $\Delta > 0$ such that for all $x \in B_X$, if $\|x\| > 1 - \Delta$ then there is a weak-neighborhood V of 0 such that $x + v \notin B_X$ for every $v \in V$ with norm $\|v\| > \varepsilon$. Thus the proof (by contraposition in both directions) is brought back to straightforward rescaling arguments as in the proof of Lemma 1.2.11.

□

Remark 1.2.13. Using the derivation given by

$$w - s_\varepsilon(F) = \{x \in F : \ \forall V \in \mathcal{V}_w(x), \ \operatorname{diam} V \cap B_X > \varepsilon\}$$

for any weakly closed bounded subset F of X we define another peeling index which is sometimes referred to as the *weak Szlenk index* of X and that we denote $w - S_Z(X)$. Although this index shares most of the basic properties of the Szlenk index, lack of weak-compactness in the non-reflexive setting provides a strong obstruction to the extension for this index of most of the more involved applications of the Szlenk index. For example the condition $w - S_Z(X) < \infty$ is known to be equivalent to the PCP but as already mentioned does not imply the RNP (Remark 1.2.5). The above Lemma 1.2.12 can be reformulated as follows: AUC is equivalent to the following: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $w - s_\varepsilon(B_X) \subset (1-\delta)B_X$. Iterating, this implies that $w - S_Z(X, \varepsilon)$ is finite for every $\varepsilon > 0$ in any AUC space and thus we have that AUC implies $w - S_Z(X) \leq \omega$ (and in particular AUC implies PCP). However, the renorming question " Does a Banach space X with weak Szlenk index ω admit an equivalent AUC norm? " is still unanswered. Let us also mention here that, as pointed out in [DGK, Proposition 2 and following comment], (sequential) UKK does not imply the PCP in general: the Bourgain-Rosenthal space constructed in [BR] has the Schur property and thus is (sequential) UKK but fails the PCP. In particular this space is an example of a space with a sequential AUC norm but with no equivalent AUC norm.

As for the AUC* property we define a weak* version of the UKK property in dual spaces (the UKK* property) by replacing the weak topology in the UKK definition by the weak* topology. Proving an analogue of Lemma 1.2.11 in this context and appealing to Lemma 1.1.13 we then obtain that UKK* is equivalent to AUC* in duals of separable spaces. We also have an analogue of Lemma 1.2.12 in this context, and this result can even be further improved by using the weak* compactness of the unit ball of dual spaces.

Lemma 1.2.14. *Let X be a Banach space. The following properties are equivalent.*

1. *The space X^* is AUC*.*
2. *For all $\varepsilon > 0$ there is a $\Delta > 0$ such that for all $x^* \in B_{X^*}$, if $\|x^*\| > 1 - \Delta$ then there is a weak*-neighborhood V of x^* such that $\operatorname{diam} V \cap B_{X^*} \leq \varepsilon$. In other words $s_\varepsilon(B_{X^*}) \subset (1 - \Delta)B_{X^*}$.*

3. *For all $\varepsilon > 0$ there is a $\Delta > 0$ such that for all $x^* \in B_{X^*}$, if $\|x^*\| > 1 - \Delta$ then there is a weak* slice $S(x, \delta)$ of B_{X^*} which contains x^* and whose Kuratowski index of non-compactness $\alpha(S(x, \delta))$ is smaller than ε.*

Before proving this lemma, let us recall that a weak* (open) slice of a bounded convex set C in X^* is a subset of the form $S(x, \delta; C) = \{x^* \in C : \ x^*(x) > \sup_{y^* \in C} y^*(x) - \delta\}$ where $x \in S_X$ and $\delta > 0$. If $C = B_{X^*}$ we simply write $S(x, \delta)$. Also let us recall that the *Kuratowski index* of a subset A of a Banach space X is defined as $\alpha(A) = \inf\{\varepsilon > 0 : A$ is covered by a finite number of balls of radius $\varepsilon\}$. We will give more details about slices and about the Kuratowski index in Chapter 3. Let us sketch the proof of Lemma 1.2.14.

Proof of Lemma 1.2.14. The equivalence between (1) and (2) goes as above by appealing to Lemma 1.1.13. A detailed proof can also be found in [Net, Section 1.3.5, Proposition 1.3.33].

(2) $\implies$ (3) Take any x^* in B_{X^*} with norm $\|x^*\| > 1 - \Delta$. Using the weak* version of Hahn-Banach separation theorem (see for example [FHH+, Corollary 3.34]) we can find a weak* slice $S(x, \delta)$ of B_{X^*} containing x and whose closure $\overline{S}(x, \delta)$ does not intersect $(1-\Delta)B_{X^*}$. Now if any point in $\overline{S}(x, \delta)$ is contained in a weak* open set of diameter less than ε, so one can extract by weak* compactness a finite subcover of $\overline{S}(x, \delta)$ and thus get $\alpha(S(x, \delta)) \leq \varepsilon$.

(3) $\implies$ (2) Let us assume that $x^* \in B_{X^*}$ belongs to a weak* slice $S(x, \delta)$ of Kuratowski index smaller than ε. Then take $x_1^*, \dots x_n^* \in X^*$ such that $S(x, \delta) \subset \bigcup_{i=1}^n B(x_i^*, \varepsilon)$ and let $I = \{1 \leq i \leq n : \ x^* \notin B(x_i^*, \varepsilon)\}$. The set $U = S(x, \delta) \setminus \bigcup_{i \in I} B(x_i^*, \varepsilon)$ is a relatively open subset of B_{X^*} for the weak* topology and we have $\|u^* - v^*\| \leq \|u^* - x^*\| + \|x^* - v^*\| \leq 4\varepsilon$ for every $u^*, v^* \in U$ since $U \subset \bigcup_{i \notin I} B(x_i^*, \varepsilon)$ and since each of the remaining balls contains x^*.

□

Remark 1.2.15. If we consider the derivation given by

$$k_\varepsilon(K) = \{x^* \in K : \ \forall S(x, \delta; K) \text{ weak* slice of } K \text{ containing } x^*, \ \alpha(S(x, \delta; K)) > \varepsilon\}$$

we obtain an index called the *weak* Kuratowski index* of X denoted by $K(X)$, which coincides with the convex Szlenk index $C_Z(X)$ (see [HL, Proposition 4.8]). We do not know whether there is an example of a Banach space for which the weak version of this result fails. We also don't know of any example of a space being AUC but not KUC (that is such that for every $\varepsilon > 0$ there is a $\Delta > 0$ such that $w - k_\varepsilon(X) \subset (1 - \Delta)B_X$). Note that any point in the unit sphere of a KUC space has to be " quasi-denting " that is to be contained in slices of arbitrarily small Kuratowski index. Finding a point in the unit sphere of JT^* which is not quasi-denting would thus provide a counter-example.

As hinted above, the notion of tree plays an important role in general index theory as well as in the asymptotic theory. For simplicity we will only work with countably branching trees and most of the time we will only consider trees of finite height. For non-separable spaces or for higher ordinals more general definitions are needed, and we refer the interested reader to [Cau4] for such considerations.

For every $n \in [1, \omega]$, let $\mathbb{N}^n$ be the set of all sequences of elements of $\mathbb{N}$ of length n. For all $N \geq 1$, let $T_N = \{\emptyset\} \cup \bigcup_{n=1}^N \mathbb{N}^n$ be the *countably branching tree* of height N. Also let $T_\infty = \bigcup_{N \geq 1} T_N$. We will use the following notation.

1. For every non empty $s \in T_\infty$, $|s|$ is the *length* of the sequence s. Also $|\emptyset| := 0$.

2. For every non empty $s = (s_1, \dots, s_n)$ and $t = (t_1, \dots, t_m)$ in T_∞, $s \frown t$ is the *concatenation* of the sequences s and t that is $s \frown t := (s_1, \dots, s_n, t_1, \dots, t_m)$. Also $s \frown \emptyset := s$ and $\emptyset \frown t := t$.

3. For every non empty $s = (s_1, \dots, s_n) \in T_\infty$ (respectively $s = (s_i)_{i \geq 1} \in \mathbb{N}^\omega$), $s_{|k}$ is the *troncation* of the sequence s at level k, that is $s_{|k} := (s_1, \dots, s_k)$ for every $1 \leq k \leq n$ (respectively for every $k \geq 1$) and $s_{|0} := \emptyset$. Also, $s^- := s_{||s|-1}$ is the predecessor of s.

4. For all $s, t \in T_\infty \cup \mathbb{N}^\omega$, we write $s \leq t$ whenever the sequence t is an *extension* of the sequence s and we say that s is an ancestor of t. This defines an ordering of $T_\infty \cup \mathbb{N}^\omega$.

For a given $N \in [1, \infty]$, a *subtree* T of T_N is a non empty subset of T_N containing all the predecessors of its elements, and a *full subtree* of T_N is a subtree T of T_N such that all the sets $T \cap \{s \frown n,\ n \in \mathbb{N}\}$ with $s \in T \cap T_{N-1}$ ($s \in T$ if $N = \infty$) are infinite. Those sets will be called the forks of the tree T.

For any finite $N \geq 1$ and any subtree T of T_N, a *leaf* of T is a maximal element of T (in particular an element of height N if T is a full subtree of T_N). If T is a subtree of T_∞ then a leaf of T is either a maximal element of T or an element $s \in \mathbb{N}^\omega$ such that $s_{|k} \in T$ for every $k \geq 1$.

Let X be a Banach space. A *tree* in X is a subset of X indexed by a full subtree T of T_N or of T_∞. In the first case we say that the tree is of finite height N, and in the second case we say that it is of infinite height. Let $(x_s)_{s \in T}$ be a tree in X. We will use the following denomination.

1. The *root* of the tree is the element $x_\emptyset$.

2. A *fork* is a set of the form $\{x_{s \frown n} :\ n \in \mathbb{N} \text{ and } s \frown n \in T\}$ where s is an element of T which is not a leaf.

3. A *branch* is a set of the form $\{x_s,\ s \leq t\}$ where $t \in T$ is a leaf. If we are working with an unrooted tree, we omit $x_\emptyset$ in this definition.

We say that the tree is *bounded* (respectively *normalized*) if the corresponding subset is bounded (respectively is contained in the unit sphere of X with the eventual exception of its root 0). We say that it is *weakly null* if every fork forms a weakly null sequence in X, that is if $w - \lim_n x_{s \frown n} = 0$ for all $s \in T$. For any $\varepsilon > 0$ we will call *weakly convergent ε-climbing tree* in X any tree $(z_t)_{t \in T}$ obtained by summing the elements of a weakly null tree $(x_s)_{s \in T}$ in X rooted at 0 and satisfying $\|x_s\| \geq \varepsilon$ for every non empty $s \in T$ along a branch of T, that is given by $z_t = \sum_{s \leq t} x_s$ for every $t \in T$. We define in the same way the notion of *weak* null trees* and *weak* convergent ε-climbing trees* in a dual space.

Remark 1.2.16. For some statements (for example for tree reformulations of asymptotic structural properties in Section 1.3), it will be required not to include roots of trees. In theory an *unrooted tree* is simply a tree deprived of its root. Yet for convenience we will implicitly put

value 0 to $x_\emptyset$ whenever we speak about unrooted trees so as to extend all the above definitions in a natural way and to avoid heavy $T\backslash\{\emptyset\}$ kind of notations. The only notable exceptions are that normalized unrooted trees will be trees $(x_s)_{s\in T}$ in X with $x_\emptyset = 0$ and $\|x_s\| = 1$ for every non empty sequence $s \in T$, and that branches of unrooted trees will not involve the root anymore.

If X is a separable Banach space, then the Szlenk derivation $s_\varepsilon(K)$ is equivalent to the derivation

$$l_\varepsilon(K) = \{x^* \in K : \exists (z_n^*)_{n\geq 1} \subset K : \forall n \geq 1, \|z_n^* - x^*\| \geq \varepsilon, \text{ and } z_n^* \xrightarrow{w^*} x^*\}.$$

In other words, an element x^* of K is in $s_\varepsilon(K)$ if and only if we can find a weak* null sequence $(x_n^*)_{n\geq 1}$ in X^* such that $\|x_n^*\| \geq \varepsilon$ and $x^* + x_n^* \in K$ for every $n \geq 1$. Iterating, an element x^* of K belongs to $s_\varepsilon^N(K)$ (respectively to $s_\varepsilon^\omega(K)$) if and only if we can find a weak* convergent ε-climbing tree of height N (respectively of infinite height) rooted at the point x^* (i.e. translated at x^*) which is contained in K. Building on this observation, we obtain the following.

Lemma 1.2.17. *Let X be a separable Banach space. Then $S_Z(X) > \omega$ if and only if there exists $\varepsilon > 0$ such that for every $N \geq 1$ we can find a weak* convergent ε-climbing tree of height N in B_{X^*}. In particular, if X has Szlenk index ω, then $S_Z(X, \varepsilon)$ is equivalent to the integer $N(\varepsilon)$ corresponding to the maximal height of weak* convergent ε-climbing trees in B_{X^*}.*

Proof. In the proof of [Lan2, Proposition 3.3], the following identity is mentioned: $\frac{1}{2}B_{X^*} + \frac{1}{2}s_\varepsilon^\alpha(B_{X^*}) \subset s_{\frac{\varepsilon}{2}}^\alpha(B_{X^*})$ for any ordinal α and for every $\varepsilon > 0$. In particular $0 \in s_{\frac{\varepsilon}{2}}^\alpha(B_{X^*})$ whenever $s_\varepsilon^\alpha(B_{X^*})$ is non-empty.

If $S_Z(X) > \omega$, then there is an $\varepsilon > 0$ such that $S_Z(X, \varepsilon) > \omega$, and thus $s_\varepsilon^N(B_{X^*}) \neq \emptyset$ for every $N \geq 1$. By the above observations we can find weak* convergent $\frac{\varepsilon}{2}$-climbing trees of height N in B_{X^*} for every $N \geq 1$.

The converse follows from the fact that $S_Z(X, \varepsilon)$ is never a limit ordinal: if $s_\varepsilon^N(B_{X^*}) \neq \emptyset$ for every $N \geq 1$, then $S_Z(X, \varepsilon) > \omega$. □

Remark 1.2.18. Since reflexive spaces do not contain ℓ_1 and since the weak and weak* topology coincide for duals of reflexive spaces, the derivations s_ε and l_ε are also equivalent in this context by Lemma 1.1.7 and thus the above observations as well as Lemma 1.2.17 extend to non-separable reflexive spaces.

Let us now introduce the notion of tree estimates in Banach spaces.

Definition 1.2.19. Let X be a Banach space, and let $p \in [1, \infty]$.

1. We say that X satisfies the *upper ℓ_p finite tree property* if there is a constant $C > 0$ such that for every finite $n \geq 1$ and for every weakly null normalized tree $(x_s)_{s\in T_n}$ of height n in X we can find a leaf s of T_n such that $\left\|\sum_{i=0}^n a_i x_{s_{|i}}\right\| \leq C \|a\|_{\ell_p}$ for every $a = (a_i)_{i\geq 0} \in \mathbb{R}^n$.

2. We say that X satisfies the *upper ℓ_p infinite tree property* if there is a constant $C > 0$ such that for every weakly null normalized tree $(x_s)_{s\in T_\infty}$ of infinite height in X we can find a leaf $s \in \mathbb{N}^\omega$ of T_∞ such that $\left\|\sum_{i\geq 0} a_i x_{s_{|i}}\right\| \leq C \|a\|_{\ell_p}$ for every $a = (a_i)_{i\geq 0} \in c_{00}$.

Remark 1.2.20. Let us recall one of the corollaries of the famous combinatorial theorem from Ramsey. For every $n \geq 1$ and for every non empty subset A of X, if for every tree $(x_s)_{s\in T_n}$ in X we can find a leaf s of T_n such that $x_s \in A$, then we can find for every tree $(x_s)_{s\in T_n}$ in X a full subtree T of T_n such that for every leaf s of T, $x_s \in A$.

In particular, if a space satisfies the upper ℓ_p finite tree property, then one can extract for every weakly null normalized tree in X of finite height a full subtree for which all the branches admit the above upper ℓ_p estimates. For trees of infinite height this requires more delicate combinatorial considerations, and we refer to [CD2].

For spaces not containing ℓ_1, Banach spaces with the upper ℓ_p finite (respectively infinite) tree property belong to the class A_p (respectively T_p) from [CFL]. Using the tools mentioned before Proposition 1.1.20 it is not difficult to prove that any p-AUS space (respectively any AUF space) satisfies the upper ℓ_p finite tree property for $1 < p < \infty$ (respectively for $p = \infty$). Passing to infinite tree requires more involved combinatorial tools, but let us point out that [Cau3] contains the following result.

Theorem 1.2.21. *Let X be a Banach space. Then the following assertions are equivalent.*

1. *The space X is in the class T_p.*
2. *The space X admits an equivalent p-AUS norm.*

Note that it is known that the upper ℓ_p finite tree property does not force a space to admit an equivalent p-AUS norm (as witnesses the dual of the q-convexification $(T_q)^*$ of Figiel and Johnson's construction of the Tsirelson's space, see [CS, Paragraph X.E]), and in fact it is known that $T_p \underset{\neq}{\subset} A_p \underset{\neq}{\subset} \bigcap_{q>p} T_p = \bigcap_{q>p} A_p$. However, the following renorming theorem was obtained in [CFL].

Theorem 1.2.22. *Let X be a Banach space. Then the following assertions are equivalent.*

1. *The space X is in the class A_p.*
2. *For every $t_0 > 0$, the space X admits an equivalent norm $|.|$ for which there is a constant $C > 0$ such that for every $t \geq t_0$, $\overline{\rho}_{|.|}(t) \leq Ct^p$.*

Similarly we can define lower tree properties in Banach spaces.

Definition 1.2.23. Let X be a Banach space and let $q \in [1, \infty]$.

1. We say that X satisfies the *lower ℓ_q finite tree property* if there is a constant $c > 0$ such that for every finite $n \geq 1$ and for every weakly null normalized tree $(x_s)_{s\in T_n}$ of height n in X we can find a leaf s of T_n such that $\left\|\sum_{i=0}^{n} a_i x_{s_{|i}}\right\| \geq c \|a\|_{\ell_q}$ for every $a = (a_i)_{i\geq 0} \in \mathbb{R}^n$.

2. We say that X satisfies the *lower ℓ_q infinite tree property* if there is a constant $C > 0$ such that for every weakly null normalized tree $(x_s)_{s\in T_\infty}$ of infinite height in X we can find a leaf $s \in \mathbb{N}^\omega$ of T_∞ such that $\left\|\sum_{i\geq 0} a_i x_{s_{|i}}\right\| \geq c \|a\|_{\ell_q}$ for every $a = (a_i)_{i\geq 0} \in c_{00}$.

In this context it also makes sense to replace the weak topology by the weak* topology in order to define the *weak* lower ℓ_q finite and infinite tree properties* in X^*.

Remark 1.2.24. Clearly upper ℓ_1 tree estimates are trivial. The fact that lower ℓ_∞ tree estimates are also trivial is less obvious at first sight, but follows easily from the fact that one can always extract from a weak* null normalized tree a full subtree which forms, for a suitable ordering, a basic sequence (with basic constant as close to 1 as one wishes). We state a precise statement in Chapter 2, Lemma 2.3.12.

It is again not difficult to show that any q-AUC space satisfies the lower ℓ_q finite tree property for $1 \leq q \leq \infty$ (and we have a similar version for the weak* topology), and it also seems to be known to imply the lower ℓ_q infinite tree property. We couldn't find a suitable reference for the latter, and up to our knowledge no analogue of the above theory has been fully developed for (weak*) lower tree estimates yet.

Remark 1.2.25. In [OS, Theorem 4.1] it is proved that any separable reflexive space satisfying both the upper and lower ℓ_p infinite tree property for some $p \in (1, \infty)$ has to be isomorphic to a subspace of an ℓ_p-sum of finite dimensional spaces. As a corollary it is then proved that the upper ℓ_p infinite tree property and the lower ℓ_{p^*} infinite tree property are dual to one another in the separable reflexive setting [OS, Corollary 4.6]. We do not know if the latter extends to a more general setting (with the weak* topology for lower estimates in X^*) and we also do not know if there is a counter example for the first statements for finite tree properties. A consequence of those results is that any separable reflexive space admitting an equivalent p-AUS norm and an equivalent p-AUC norm for some $p \in (1, \infty)$ has to be isomorphic to a subspace of an ℓ_p-sum of finite dimensional spaces (also see [JLPS, Proposition 2.11] for an alternative proof). However, let us observe that the James space J_p is a non-reflexive separable space with an equivalent p-AUS norm and such that J_p^{**} admits an equivalent p-AUC* norm, see Example 1.1.25 and references therein, so that those kind of conditions do not imply reflexivity.

We end the section with a few remarks on separable determination. The proof of the following result is sketched in [Lan2, Theorem 3.11] for the dentability index and can easily be adapted to the Szlenk index.

Proposition 1.2.26. *Let X be a Banach space and let $\alpha < \omega_1$ be a countable ordinal. If $S_Z(X) > \alpha$, then there is a separable subspace Y of X with Szlenk index $S_Z(Y) > \alpha$.*

The following result is proved in [DKLR, Proposition 3.1].

Proposition 1.2.27. *Let X be a reflexive Banach space and let $\alpha < \omega_1$ be a countable ordinal. If $S_Z(X) > \alpha$, then there is a subspace Y of X such that X/Y is separable and has Szlenk index $S_Z(X/Y) > \alpha$.*

We will extend this result to weakly compactly generated (WCG in short) spaces in Appendix A by using the so called ℓ_1^+ weakly null index from [AJO].

Proposition 1.2.28. *Let X be a WCG Banach space and let $\alpha < \omega_1$ be a countable ordinal. If $S_Z(X) > \alpha$, then there is a subspace Y of X such that X/Y is separable and has Szlenk index $S_Z(X/Y) > \alpha$.*

1.3 Asymptotic structure

We give a brief overview of the so called *asymptotic structure theory* of Banach spaces introduced in [MMTJ]. Although we will not dwelve into it, this theory is closely related to the theory of games in Banach spaces. We refer to [AH, Chapter 3] for basics in the theory of games and to [Pro, Chapter 2] and references therein for a few applications in Banach space geometry. A brief description of the translation of the properties of this section in the language of games can be found for example in [BLMS, Remark 3.2] or in [BCD+, Paragraph 3.2].

Let X be a Banach space. For every $n \geq 1$, let $\mathcal{E}_n$ be the set of all triplets $(E, \|.\|_E, (e_i)_{i=1}^n)$ where $(E, \|.\|_E)$ is an n-dimensional normed space and $(e_i)_{i=1}^n$ is a normalized basis of E. When there is no risk of confusion, we will write $E = (E, \|.\|_E, (e_i)_{i=1}^n) \in \mathcal{E}_n$ but one should keep in mind that there is an underlying norm and a fixed underlying normalized basis in the space E.

We say that a space $E \in \mathcal{E}_n$ *is in the nth asymptotic structure of X up to a constant $C \geq 1$* and we write $E \in \{X\}_{n,C}$ if we have the following property:

$$\forall \varepsilon > 0,\ \forall X_1 \in \mathrm{cof}(X),\ \exists x_1 \in S_{X_1} :\ \ldots,\ \forall X_n \in \mathrm{cof}(X),\ \exists x_n \in S_{X_n} :$$
$$(x_i)_{i=1}^n \text{ is } C(1+\varepsilon)\text{-equivalent to } (e_i)_{i=1}^n.$$

If $C = 1$, we say that E *is in the nth asymptotic structure of X* and we write $E \in \{X\}_n$.

We say that X is *asymptotically ℓ_p* (respectively *asymptotically c_0*) or that X *has an ℓ_p (respectively c_0) asymptotic structure* if there is a constant $C > 0$ such that for every $n \geq 1$ and for every $E \in \{X\}_n$, the underlying basis $(e_i)_{i=1}^n$ of E is C-equivalent to the unit vector basis of the space ℓ_p^n (respectively ℓ_∞^n). Also we say that X is *asymptotically unconditional* or that X *has an unconditional asymptotic structure* if there is a constant $C > 0$ such that for every $n \geq 1$ and for every $E \in \{X\}_n$, the underlying basis $(e_i)_{i=1}^n$ of E is C-unconditional.

It is well known that a space X is asymptotically ℓ_p if and only if there is a constant $C > 0$ such that

$$\exists X_1 \in \mathrm{cof}(X) :\ \forall x_1 \in S_{X_1},\ \ldots,\ \exists X_n \in \mathrm{cof}(X) :\ \forall x_n \in S_{X_n},$$
$$(x_i)_{i=1}^n \text{ is } C\text{-equivalent to the unit vector basis of } \ell_p^n. \qquad (\Psi)$$

The same goes for asymptotically c_0 spaces and asymptotically unconditional spaces. This becomes very clear once those kind of properties are reformulated in terms of generalized trees.

Let $\mathcal{Y} := \mathrm{cof}(X)$ and for every $n \geq 1$ let $\mathcal{Y}^{\leq n}$ be the set of finite non-empty sequences of elements of $\mathcal{Y}$ of length at most n. We will say that a generalized tree $(x_y)_{y \in \mathcal{Y}^{\leq n}}$ in X is

weakly null if $x_y \in X_k$ for every $y = (X_1, \dots, X_k) \in \mathcal{Y}^{\leq n}$. This terminology comes from the fact that any node $(x_{y^\frown Z})_{Z \in \mathcal{Y}}$ in such a tree forms a weakly null net. The following lemma is easily proved by induction on n.

Lemma 1.3.1. *Let X be a Banach space.*

1. *For every $E \in \mathcal{E}_n$, $E \in \{X\}_{n,C}$ if and only if we can find for every $\varepsilon > 0$ a weakly null generalized unrooted tree $(x_y)_{y \in \mathcal{Y}^{\leq n}}$ in S_X of height n whose branches are all $C(1+\varepsilon)$ equivalent to the underlying basis $(e_i)_{i=1}^n$ of E.*

2. *A Banach space X satisfies the above property (Ψ) if and only if there is a constant $C \geq 1$ such that for every $n \geq 1$ and for every weakly null generalized unrooted tree $(x_y)_{y \in \mathcal{Y}^{\leq n}}$ in S_X of height n, we can find a branch which is C-equivalent to the unit vector basis of ℓ_p.*

The next lemma then follows from Lemma 1.1.7 and the fact that the distance $d(x_n, Y)$ from a weakly null normalized sequence $(x_n)_{n \geq 1}$ in X to any fixed $Y \in \mathrm{cof}(X)$ goes to 0, see for example [BLMS, Lemma 3.4 and 3.5].

Lemma 1.3.2. *Let X be a Banach space which does not contain ℓ_1.*

1. *For every $E \in \mathcal{E}_n$, $E \in \{X\}_{n,C}$ if and only if we can find for every $\varepsilon > 0$ a weakly null normalized unrooted tree $(x_s)_{s \in T_n}$ in X of height n whose branches are all $C(1+\varepsilon)$ equivalent to $(e_i)_{i=1}^n$.*

2. *A Banach space X is asymptotically ℓ_p if and only if there is a constant $C \geq 1$ such that for every $n \geq 1$ and for every weakly null normalized unrooted tree $(x_s)_{s \in T_n}$ in X of height n, we can find a branch which is C-equivalent to the unit vector basis of ℓ_p.*

We have a similar characterization for asymptotically c_0 and asymptotically unconditional spaces.

Remark 1.3.3. A space satisfying the second part of point (2) is sometimes referred to as having the *finite ℓ_p (respectively c_0 or unconditional) tree property.*

Example 1.3.4. Let $p \in (1, \infty)$. Using a gliding hump argument, one can show that any space X which is p-AUS renormable and p-AUC renormable is asymptotically ℓ_p. Also any space which is 1-AUC renormable is asymptotically ℓ_1. In particular any ℓ_p-sum of finite dimensional spaces is asymptotically ℓ_p for $1 \leq p < \infty$ and the p-James space J_p is asymptotically ℓ_p for every $1 < p < \infty$. We refer to [OSZ] for results concerning the structure of asymptotically ℓ_p reflexive spaces.

Example 1.3.5. Using again a gliding hump argument, we can prove that any space X with an unconditional basis is asymptotically unconditional. As mentioned in [BCD+, Remark 3.4] the latter is a strictly weaker property: the Argyros-Delyianni space constructed in [AD, Section 3] is asymptotically ℓ_1 but does not contain any unconditional basic sequence.

Remark 1.3.6. A consequence of results from [Cau6] is that a Banach space is asymptotically c_0 if and only if it has summable Szlenk index as introduced in [KOS, Definition 6.6], also see [GKL2, Theorem 4.10]. Looking back at the proof following Remark 1.2.9, Lemma 1.3.2 combined with a small refinement of the argument there shows that the space T constructed by Figiel and Johnson is asymptotically ℓ_1 and that the original Tsirelson's space T^* has summable Szlenk index and thus is asymptotically c_0. Also see [BLMS, Inequality (6) and related] for a direct argument.

Let us recall two famous results in Banach space theory. For those we refer respectively to the papers [Jam] and [Ros1].

Theorem 1.3.7 (James, non-distordability of c_0)**.** *For all $m \geq 1$, $C \geq 1$ and $\varepsilon > 0$, there is an $n \geq 1$ such that: any basic sequence $(e_i)_{i=1}^n$ of length n which is C-equivalent to the unit vector basis of ℓ_∞^n admits a block basis of length m which is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_∞^m.*

Theorem 1.3.8 (Krivine/Rosenthal)**.** *Let $1 \leq p < \infty$. For all $m \geq 1$, $C \geq 1$ and $\varepsilon > 0$, there is an $n \geq 1$ such that: any basic sequence $(e_i)_{i=1}^n$ of length n which is C-equivalent to the unit vector basis of ℓ_p^n admits a block basis of length m which is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^m.*

We will use these results and the preceding tree characterizations to optimize constants in the asymptotic structure for spaces not containing ℓ_1. This is stated in full generality in [BCD+, Lemma 3.1] but a few details are missing in the proof, and in particular we don't know if the key stabilization argument below (Claim 1) can be obtained without passing to countably branching trees and using Ramsey's theorem.

Lemma 1.3.9. *Let X be a Banach space which does not contain ℓ_1 and let $1 \leq p \leq \infty$. If there is a constant $C \geq 1$ such that $\ell_p^n \in \{X\}_{n,C}$ for all $n \geq 1$, then $\ell_p^n \in \{X\}_n$ for all $n \geq 1$.*

Proof. For the purpose of this proof we will say that two elements $E, F \in \mathcal{E}_n$ are C-equivalent for some $C \geq 1$ if the underlying basis $(e_i)_{i=1}^n$ of E is C-equivalent to the underlying basis $(f_i)_{i=1}^n$ of F. As for the standard Banach-Mazur distance between n-dimensional normed spaces, this allows to define a distance on $\mathcal{E}_n$ which turns $\mathcal{E}_n$ into a compact metric space.

Let us assume that X does not contain ℓ_1 and that there is a constant $C > 0$ such that for every $n \geq 1$, $\ell_p^n \in \{X\}_{n,C}$. Then we have the following.

Claim 1. For every $n \geq 1$ we can find a space $E_n \in \mathcal{E}_n$ which is $2C$ equivalent to ℓ_p^n and belongs to $\{X\}_n$.

Proof of Claim 1. This is a standard stabilization argument. Let us fix $n \geq 1$ and $\varepsilon \in (0,1)$. By compactness of $\mathcal{E}_n$ we can find a finite collection $\mathcal{F}_\varepsilon$ of elements of $\mathcal{E}_n$ such that every space $E \in \mathcal{E}_n$ which is $2C$-equivalent to ℓ_p^n has to be $(1+\varepsilon)$ equivalent to some space in $\mathcal{F}_\varepsilon$. Since X does not contain ℓ_1 we can find by Lemma 1.3.2 a weakly null normalized unrooted tree

$(x_s)_{s\in T_n}$ in X whose branches are all $2C$-equivalent to the canonical basis of ℓ_p^n. So for every branch β of the tree there is a space $E_\beta \in \mathcal{E}_\varepsilon$ which is $(1+\varepsilon)$-equivalent to the linear span of β. By Ramsey's theorem we may assume up to the extraction of a full subtree that there is a space $E_\varepsilon \in \mathcal{E}_\varepsilon$ such that the linear span of every branch β of the tree is $(1+\varepsilon)$-equivalent to E_ε.

Now let us fix any sequence $(\varepsilon_i)_{i\geq 1}$ in $(0,1)$ converging to 0. Using sequential compactness we can assume up to the extraction of a subsequence that there is a space $E \in \mathcal{E}_n$ such that the sequence $(E_{\varepsilon_i})_{i\geq 1}$ converges to E. Then E is $2C$-equivalent to ℓ_p^n and we can find for every $\varepsilon > 0$ a weakly-null normalized unrooted tree in X whose branches are all $(1+\varepsilon)$-equivalent to the underlying basis $(e_i)_{i\geq 1}$ of E. By Lemma 1.3.2 $E \in \{X\}_n$, and we are done.

□

Now let us fix $m \geq 1$ and $\varepsilon > 0$. By Lemma 1.3.1 we need to provide a generalized weakly null unrooted tree $(x_y)_{y\in\mathcal{Y}^{\leq m}}$ in S_X whose branches are all $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^m. So let us fix any $\eta > 0$ to be chosen later. By applying either Theorem 1.3.7 or Theorem 1.3.8, we can find an $n \geq 1$ such that every normalized basic sequence of length n which is $2C$ equivalent to the canonical basis of ℓ_p^n admits a normalized block basis of length m which is $(1+\eta)$-equivalent to the canonical basis of ℓ_p^m. Now use Claim 1 to find a space $E_n \in \mathcal{E}_n$ which is $2C$ equivalent to ℓ_p^n and belongs to $\{X\}_n$ and use the preceding statement to pick a normalized block basis $(f_j)_{j=1}^m$ of the underlying basis $(e_i)_{i=1}^n$ of E_n which is $(1+\eta)$-equivalent to the canonical basis of ℓ_p^m. We need the following.

Claim 2. The space $F = ([f_j]_{j=1}^m, \|.\|_E, (f_j)_{j=1}^m)$ is in $\{X\}_m$.

Proof. Proof of Claim 2.

Fix some $\delta > 0$. Since $E_n \in \{X\}_n$ we can find a weakly null generalized unrooted tree $(x_y)_{y\in\mathcal{Y}^{\leq n}}$ in S_X whose branches are all $(1+\delta)$-equivalent to $(e_i)_{i=1}^n$. For every $1 \leq j \leq m$ let us introduce $n_j = \min(\operatorname{supp} f_j)$ and $l_j = |\operatorname{supp} f_j|$. For every $y = (X_1, \ldots, X_j) \in \mathcal{Y}^{\leq m}$ we define inductively $y_0 = y$ and $y_i = y_{i-1} \frown X_j$ for every $1 \leq i \leq l_j - 1$ and we consider $z_y = \sum_{i=0}^{l_j-1} e^*_{n_j+i}(f_j) x_{y_i}$. Then $z_y \in X_j$ so that $(z_y)_{y\in\mathcal{Y}^{\leq m}}$ is a weakly null generalized tree, and by construction every branch of the tree is $(1+\delta)$-equivalent to $(f_j)_{j=1}^m$. Using a standard perturbation argument we then get a normalized weakly null generalized unrooted tree whose branches are all $(1+2\delta)$-equivalent to $(f_j)_{j=1}^m$, and the conclusion follows by letting δ go to 0.

□

Now by Claim 2 we can find a weakly null generalized tree $(x_y)_{y\in\mathcal{Y}^{\leq m}}$ in S_X whose branches are all $(1+\eta)$-equivalent to $(f_j)_{j=1}^m$ and thus $(1+\eta)^2$-equivalent to the unit vector basis of ℓ_p^m. The conclusion follows if η was initially chosen so that $(1+\eta)^2 \leq (1+\varepsilon)$.

□

Replacing finite co-dimenisonal subspaces by weak* closed finite co-dimensional subspaces we define in a similar way the nth *weak* asymptotic structure* of a dual space $\{X^*\}_{n,C}^{w^*}$ and $\{X^*\}_n^{w^*}$ and the notion of *weak* asymptotically ℓ_p, c_0 or unconditional spaces*. Then we have the following results.

Lemma 1.3.10. *Let X be a separable Banach space. For all $n \geq 1$ and for all n-dimensional space E with a normalized basis $(e_i)_{i=1}^n$, E is in the nth weak* asymptotic structure of X^* up to a constant $C \geq 1$ if and only if for all $\varepsilon > 0$, there is a weak*-null normalized unrooted tree of height n whose branches are all $C(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^n$.*

Lemma 1.3.11. *Let X be a separable Banach space and let $1 \leq p \leq \infty$. If there is a constant $C \geq 1$ such that the space ℓ_p^n is in the nth weak* asymptotic structure of X^* up to C for all $n \geq 1$, then $\ell_p^n \in \{X^*\}_n^{w^*}$ for all $n \geq 1$.*

Lemma 1.3.12. *Let X be a separable Banach space. Then X^* is weak* asymptotically ℓ_p (respectively c_0 or unconditional) if and only if it has the weak* ℓ_p (respectively c_0 or unconditional) finite tree property.*

For the sake of completeness, let us provide a proof for Lemma 1.3.10 . This will be the occasion to introduce the following concept which allows to implement inductive proofs for trees. We say that an enumeration $(\sigma_i)_{i\geq 1}$ of a tree T is a compatible linear ordering of T if it satisfies the following condition: for every $i \geq 1$ and for every ancestor s of σ_i, there is a $j < i$ such that $s = \sigma_j$.

Proof of Lemma 1.3.10. Let X be a separable Banach space, let us fix some $n \geq 1$ and some space $E \in \mathcal{E}_n$, and let us first assume that $E \in \{X^*\}_{n,C}^{w^*}$. Since X is separable, we can choose a dense sequence $(z_k)_{k\geq 1}$ in S_X. For all $k \geq 1$, let us define the weak* closed subspace of finite codimension $Z_k = \bigcap_{i=1}^k \ker z_i$ of X^* where z_i is identified with its image in X^{**}. Then we have the property that any normalized sequence $(z_k^*)_{k\geq 1}$ in S_{X^*} such that $z_k^* \in Z_k$ for all $k \geq 1$ is weak*-null. Let us fix $\varepsilon > 0$. Using the assumption on E, we will build a normalized unrooted tree $(z_s^*)_{s\in T_n}$ in S_{X^*} such that:

1. For all $s \in T_{n-1}$, for all $k \in \mathbb{N}$, $z_{s\frown k}^* \in Z_k$.
2. All the branches of $(z_s^*)_{s\in T_n}$ are $C(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^n$.

This gives the desired tree by the choice of the subspaces Z_k. In order to do this construction let us introduce a compatible linear ordering $(\sigma_i)_{i\geq 1}$ of $T_n \backslash \{\emptyset\}$. By an easy induction we provide a sequence $(z_{\sigma_i}^*)_{i\geq 1}$ in S_{X^*} such that:

1. If $\sigma_i = \sigma_i^- \frown k$ for some $k \in \mathbb{N}$ then $z_{\sigma_i} \in Z_k$.
2. If $l = |\sigma_i| = n$, then $(z_\sigma^*,\ \sigma \leq \sigma_i)$ is $C(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^n$.
3. If $l = |\sigma_i| < n$ it satisfies:

$$\forall X_{l+1} \in \mathrm{cof}^*(X^*),\ \exists x_{l+1}^* \in S_{X_{l+1}},\ \ldots,\ \forall X_n \in \mathrm{cof}^*(X^*),\ \exists x_n^* \in S_{X_n}:$$
$$(z_\sigma^*,\ \sigma \leq \sigma_i) \frown (x_i^*)_{i=l+1}^n \text{ is } C(1+\varepsilon)\text{-equivalent to } (e_i)_{i=1}^n.$$

Second, let us fix $\varepsilon > 0$ and let us take $\delta > 0$ such that: for every normalized sequences $(v_i)_{i=1}^n$ and $(w_i)_{i=1}^n$ in X^*, if $(v_i)_{i=1}^n$ is $C(1+\delta)$-equivalent to $(e_i)_{i=1}^n$ and if $\|v_i - w_i\| \leq \delta$ for all $i \leq n$, then $(w_i)_{i=1}^n$ is $C(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^n$. Then let us assume that there is a weak*-null normalized unrooted tree $(z_s^*)_{s\in T_n}$ in S_{X^*} whose branches are all $C(1+\delta)$-equivalent to $(e_i)_{i=1}^n$. Let us recall the following well known property: if Z is a weak* closed subspace of finite codimension of X^* and if $(z_k^*)_{k\geq 1}$ is normalized and weak*-null, then the distance $\operatorname{dist}(z_k^*, S_Z)$ goes to 0. Using this, a straightforward induction shows that the following property holds for all $1 \leq j \leq n$:

$$\forall X_1 \in \operatorname{cof}(X^*),\ \exists x_1^* \in S_{X_1},\ \ldots,\ \forall X_j \in \operatorname{cof}(X^*),\ \exists x_j^* \in S_{X_j},\ \exists s \in T_n,\ |s| = j :$$
$$\forall 1 \leq i \leq j,\ \left\| x_i^* - z_{s_{|i}}^* \right\| \leq \delta$$

where the X_i are weak* closed subspaces of finite co-dimension of Y. By our choice of δ and by the properties of the branches of our tree, we get the desired result.

□

We end the section with a few results concerning the asymptotic structure of a space X satisfying $S_Z(X) > \omega$ and of its dual space under some separability assumptions. We start by providing a small improvement of [BKL, Proposition 2.2] which rests on the following classical result from Mazur.

Lemma 1.3.13 (Mazur)**.** *Let $(x_k^*)_{k\geq 1}$ be a weak*-null sequence in X^* such that $\|x_k^*\| \geq 1$ for all $k \geq 1$, and let F be a finite subset of X^*. Then there is a sequence $(x_k)_{k\geq 1}$ in S_X such that $x_k \in \bigcap_{y^*\in F} \ker y^*$ for all $k \geq 1$ and $\liminf x_k^*(x_k) \geq \frac{1}{2}$.*

Proposition 1.3.14. *Let X be a separable Banach space with $S_Z(X) > \omega$. For all $n \geq 1$ and for all δ, there exist a weak*-null bounded tree $(x_s^*)_{s\in T_n}$ in X^* and a normalized tree $(x_s)_{s\in T_n}$ in X such that:*

1. *For all $s \in T_n \backslash \{\emptyset\}$, $\|x_s^*\| \geq 1$ and $\left\|\sum_{t\leq s} x_t^*\right\| \leq 3$.*
2. *For all $s \in T_n$, $x_s^*(x_s) \geq \frac{1}{3}\|x_s^*\|$.*
3. *For all $s \neq t$ in T_n, $|x_s^*(x_t)| \leq \delta$.*

Moreover, if X has a separable dual, then one can ask the tree $(x_s)_{s\in T_n}$ to be weakly null.

Proof. For separable spaces, this is [BKL, Proposition 3.2]. So let us assume that X has a separable dual and satisfies $S_Z(X) > \omega$, and let us pick a dense sequence $(y_k^*)_{n\geq 1}$ in S_{X^*}. We define $Y_k = \bigcap_{i=1}^k \ker y_i^*$. Then any normalized sequence $(x_k)_{k\geq 1}$ in X such that $x_k \in Y_k$ for all $k \geq 1$ is weakly-null. Now let us fix $n \geq 1$ and $\delta > 0$.

It is explained in [BKL] that the submultiplicativity of the Szlenk index forces $S_Z(X, \frac{1}{3}) > \omega$, and since X is separable we can find as in Lemma 1.2.17 a weak*-null bounded tree $(x_s^*)_{s\in T_n}$ in X^* rooted at 0 such that for all $s \in T_n \backslash \{\emptyset\}$, $\|x_s^*\| \geq 1$ and $\left\|\sum_{t\leq s} x_t^*\right\| \leq 3$. Let us pick a compatible linear ordering $(\sigma_i)_{i\geq 1}$ of T_N and let us assume as we may that this ordering also satisfies the following property: if $\sigma_i = \sigma_{i_0} \frown m$ and $\sigma_j = \sigma_{i_0} \frown m'$ with $m < m'$ then $i < j$. By induction, we can build a sequence $(\theta_i)_{i\geq 1}$ in T_n and a sequence $(x_{\theta_i})_{i\geq 1}$ in S_X such that:

1. $\theta_1 = \emptyset$

2. If $\sigma_i^- = \sigma_{i_0}$ then $\theta_i = \theta_{i_0} \frown m'$ for some $m' \in \mathbb{N}$ bigger than $\max\{l : \exists j < i : \theta_j = \theta_{i_0} \frown l\}$ and $x_{\theta_i} \in Y_m$.

3. For all $i < j$, $\left|x^*_{\theta_j}(x_{\theta_i})\right| \leq \delta$ and $x^*_{\theta_i}(x_{\theta_j}) = 0$.

4. For all $i \geq 1$, $x^*_{\theta_i}(x_{\theta_i}) \geq \frac{1}{3}\left\|x^*_{\theta_i}\right\|$.

For $i = 1$, just let $\theta_1 = \emptyset$ and pick any $x_\emptyset$ in S_X such that $x^*_\emptyset(x_\emptyset) \geq \frac{1}{3}\left\|x^*_\emptyset\right\|$. Now let us assume that $\theta_1, \dots \theta_i$ and $x_{\theta_1}, \dots, x_{\theta_i}$ have been chosen with the required properties. Since we are working with a compatible linear ordering, there is some $i_0 \leq i$ and such that $\sigma^-_{i+1} = \sigma_{i_0}$. Since $(x^*_{\theta_{i_0} \frown k})_{k \geq 1}$ is weak*-null there is a $K \in \mathbb{N}$ such that $\left|x^*_{\theta_{i_0} \frown k}(x_{\theta_j})\right| \leq \delta$ for all $j \leq i$ and for all $k \geq K$. Moreover, we can apply Mazur's lemma to this sequence with $F = \{y^*_1, \dots, y^*_m, x^*_{\theta_1}, \dots, x^*_{\theta_i}\}$ and take any m' big enough in order to get a element $x_{\theta_{i_0} \frown m'} \in S_{Y_m}$ such that $x^*_{\theta_{i_0} \frown m'}(x_{\theta_{i_0} \frown m'}) \geq \frac{1}{3}\left\|x^*_{\theta_{i_0} \frown m'}\right\|$ and $x^*_j(x_{\theta_{i_0} \frown m'}) = 0$ for all $j \leq i$ and also m' bigger than K and bigger than $\max\{l : \exists j < i : \theta_j = \theta_{i_0} \frown l\}$. We conclude by putting $\theta_{i+1} = \theta_{i_0} \frown m'$. Then $T = \{\theta_i,\ i \geq 1\}$ defines a full subtree of T_n and for every node $(x_{\theta_{i_0} \frown m'_k})_{k \geq 1}$ of T there is a strictly increasing sequence $(m_k)_{k \geq 1}$ in $\mathbb{N}$ such that $x_{\theta_{i_0} \frown m'_k} \in S_{Y_{m_k}}$. Consequently, $(x_s)_{s \in T}$ is a weakly-null tree in X and satisfies all the required properties together with $(x^*_s)_{s \in T}$.

□

Using those biorthogonal trees, we can prove the following. This is an extension of a reflexive result contained in the proof of [BCD+, Lemma 3.5].

Theorem 1.3.15. *Let X^* be the dual of a separable Banach space X with $S_Z(X) > \omega$, and assume that X^* is weak* asymptotically unconditional. Then $\ell^n_\infty \in \{X^*\}^{w^*}_n$ for every $n \geq 1$.*

Proof. Let us assume that X is separable and that X^* is asymptotically unconditional. By Lemma 1.3.12 and by Ramsey's combinatorial theorem there is a constant $C \geq 1$ such that any weak* null normalized unrooted tree of finite height in X^* has a full subtree whose branches are all C unconditional. Moreover it is sufficient by Lemma 1.3.10 and Lemma 1.3.11 to show that there is a constant $D \geq 1$ such that for all $n \geq 1$, there is a weak* null normalized tree of height n in X^* whose branches are all D-equivalent to the unit vector basis of ℓ^n_∞. Let us fix some $n \geq 1$ and let us fix $\delta > 0$ to be chosen later. By proposition 1.3.14, there exist a weak*-null bounded tree $(x^*_s)_{s \in T_n}$ in X^* and a normalized tree $(x_s)_{s \in T_n}$ in X such that:

1. For all $s \in T_n \backslash \{\emptyset\}$, $\|x^*_s\| \geq 1$ and $\left\|\sum_{t \leq s} x^*_t\right\| \leq 3$.

2. For all $s \in T_n$, $x^*_s(x_s) \geq \frac{1}{3}\|x^*_s\|$.

3. For all $s \neq t$ in T_n, $|x^*_s(x_t)| \leq \delta$.

As mentioned one can assume, up to passing to some full subtree, that all banches of $(x_s^*)_{s\in T_n}$ are C-unconditional. For all $s \in T_n$, let $y_s = \frac{x_s^*}{\|x_s^*\|}$. Since the function $(a_0, \dots, a_{|s|}) \in [-1,\ 1]^{|s|} \mapsto \left\|\sum_{t\leq s} a_t x_t^*\right\|$ is continuous and convex we have:

$$\begin{aligned}
\max\left\{\left\|\sum_{t\leq s} a_t y_t\right\|,\ a_t \in [-1,\ 1]\right\} &\leq \max\left\{\left\|\sum_{t\leq s} a_t x_t^*\right\|,\ a_t \in [-1,\ 1]\right\} \\
&= \max\left\{\left\|\sum_{t\leq s} a_t x_t^*\right\|,\ a_t \in \{-1,\ 1\}\right\} \\
&\leq C\left\|\sum_{t\leq s} x_t^*\right\| \\
&\leq 3C.
\end{aligned}$$

Thus, for all $s \in T_n$ with $|s| = n$ and for all $a_0, \dots, a_n \in \mathbb{R}$, we have:

$$\left\|\sum_{t\leq s} a_{|t|} y_t\right\| \leq 3C \max_{0\leq i\leq n} |a_i|.$$

As mentioned in Remark 1.2.24 lower ℓ_∞ tree estimates are trivial so up to the extraction of a full subtree we have the desired result (with D as close as $3C$ as we wish). □

Theorem 1.3.16. *Let X be an asymptotically unconditional Banach space with a separable dual, and with $S_Z(X) > \omega$. Then $\ell_1^n \in \{X\}_n$ for every $n \geq 1$.*

Proof. Let us assume that X is an asymptotically unconditional Banach space with a separable dual. Then X does not contain ℓ_1 and by Lemma 1.3.2 there is a constant $C \geq 1$ such that any weak-null normalized tree of finite height in X has a full subtree whose branches are all C unconditional. Moreover it is sufficient by Lemma 1.3.2 and by Lemma 1.3.9 to show that there is a constant $D \geq 1$ such that for all $n \geq 1$, there is a weak-null normalized tree of height n in X whose branches are all D-equivalent to the unit vector basis of ℓ_1^n. Let us fix some $n \geq 1$ and let us fix $\delta > 0$ to be chosen later. By proposition 1.3.14, there is a weak*-null bounded tree $(x_s^*)_{s\in T_n}$ in X^* and a normalized weak-null tree $(x_s)_{s\in T_n}$ in X such that:

1. For all $s \in T_n\backslash\{\emptyset\}$, $\|x_s^*\| \geq 1$ and $\left\|\sum_{t\leq s} x_t^*\right\| \leq 3$.
2. For all $s \in T_n$, $x_s^*(x_s) \geq \frac{1}{3}\|x_s^*\|$.
3. For all $s \neq t$ in T_n, $|x_s^*(x_t)| \leq \delta$.

As mentioned one can assume, up to passing to some full subtree, that all branches of $(x_s)_{s\in T_n}$ are C-unconditional. Since $(x_s)_{s\in T_N}$ is normalized, a simple triangular inequality yields:

$$\left\|\sum_{t\leq s} a_{|t|} x_t\right\| \leq \sum_{i=0}^{n} |a_i|$$

for all $a_0, \dots, a_n \in \mathbb{R}$ and for all $s \in T_n$ of length n.

Moreover,

$$\begin{aligned}\left\|\sum_{t\leq s} a_{|t|}x_t\right\| &\geq \frac{1}{C}\left\|\sum_{t\leq s} \operatorname{sign} a_{|t|}a_{|t|}x_t\right\| \\ &\geq \frac{1}{3C}\left\langle \sum_{t\leq s} \left|a_{|t|}\right| x_t \right\rangle \sum_{t\leq s} x_t^* \\ &\geq \left(\frac{1}{9C} - n\delta\right)\sum_{i=0}^{n} |a_i| \\ &\geq \frac{1}{D}\sum_{i=0}^{n} |a_i|\end{aligned}$$

for any chosen constant $D > 9C$ if δ was initially chosen small enough.

□

Remark 1.3.17. Looking back at the characterization from Lemma 1.2.17 of the value ω of the Szlenk index of separable Banach spaces in terms of weak* convergent ε-climbing trees, we would like to point out that the condition $S_Z(X) = \omega$ can be characterized in general by the following asymptotic games: for every $\varepsilon > 0$ there is an $N \geq 1$ such that for every $n \geq N$,

$$\exists X_1 \in \operatorname{cof}^*(X^*) : \forall x_1^* \in X_1^\varepsilon, \ \dots, \ \exists X_n \in \operatorname{cof}^*(X^*) : \ \forall x_n^* \in X_n^\varepsilon,$$
$$\sum_{i=1}^{n} x_i^* \text{ does not belong to } B_{X^*}$$

where $E^\varepsilon = \{x \in E : \ \|x\| \geq \varepsilon\}$ for any $E \in \operatorname{cof}^*(X^*)$.

1.4 Asymptotic midpoint convexity

Following [DKR+] we introduce a modulus of asymptotic midpoint convexity.

Definition 1.4.1. Let X be a Banach space. For every $t > 0$ and for every $x \in S_X$, let

$$\widehat{\delta}_X(t, x) = \sup_{Y \in \operatorname{cof}(X)} \inf_{y \in S_Y} \max\{\|x + ty\|, \|x - ty\|\} - 1.$$

The function

$$\widehat{\delta}_X(t) = \inf_{x \in S_X} \widehat{\delta}_X(t, x)$$

is called *modulus of asymptotic midpoint convexity* of X.

Note that this modulus is equivalent to the one introduced in [Kal2] where an actual average is taken. This modulus shares a lot of the properties of the asymptotic modulus of convexity. We list a few of them below, their proof can be directly imported from Section 1.1.

Lemma 1.4.2. *Let X be a Banach space and take $t > 0$ and $x \in S_X$.*

1. *We have*
$$\widehat{\delta}_X(t,x) = \sup_{Y \in \mathrm{cof}(X)} \inf_{y \in Y, \|y\| \geq 1} \max\{\|x+ty\|, \|x-ty\|\} - 1.$$
In particular the function $\widehat{\delta}_X$ is non-decreasing and 1-Lipschitz.

2. *We have*
$$\widehat{\delta}_X(t,x) = \sup_{V \in \mathcal{V}_w(0)} \inf_{v \in V, \|v\|=1} \max\{\|x+tv\|, \|x-tv\|\} - 1,$$
and we can equivalently take $\|v\| \geq 1$ in the above expression.

3. *For every weakly null sequence $(x_n)_{n\geq 1}$ in X with norm $\|x_n\| \geq 1$ we have*
$$\liminf \max\{\|x+tx_n\|, \|x-tx_n\|\} \geq 1 + \widehat{\delta}_X(t,x).$$

4. *If the space X does not contain ℓ_1, then*
$$\widehat{\delta}_X(t,x) = \inf\left(\liminf \max\{\|x+tx_n\|, \|x-tx_n\|\} - 1\right)$$
where the infimum is taken on all weakly null normalized sequences (or equivalently on weakly null sequences with norm $\|x_n\| \geq 1$).

We now introduce the asymptotic property associated with the modulus $\widehat{\delta}_X$.

Definition 1.4.3. Let X be a Banach space. We say that X is *asymptotically midpoint uniformly convex* (AMUC in short) if $\widehat{\delta}_X(t) > 0$ for every $t > 0$.

It is clear from the definition that $\overline{\delta}_X \leq \widehat{\delta}_X$ so that any AUC space is AMUC. Let us point out that [DKR+, Theorem 2.4] provided a renorming of ℓ_2 which is AMUC but not AUC. However, the question of the equivalence between AUC and AMUC up to renorming is still open, see below for a few partial results. Also note that (4) shows the equivalence between AMUC and sequential AMUC for spaces not containing ℓ_1. A different proof of this fact was provided in [DKR+, Corollary 2.3].

Remark 1.4.4. A modulus of sign-asymptotic smoothness is also considered in [DKR+, Section 4] and a corresponding SAUS property is introduced. Yet unlike the above mentioned example, a general reverse inequality between the two moduli is obtained (with however a loss on power types) and the equivalence in every Banach space between AUS and SAUS is proved.

The main reason for the introduction of the modulus of asymptotic midpoint convexity is that is provides control on the Kuratowski index of sets of *approximate midpoints* in a Banach space. Let us recall the following definition. Let M be a metric space and let $x, y \in M$. For every $\delta > 0$ we denote by
$$\mathrm{Mid}(x,y;\delta) = \left\{ z \in M : \max\{d(x,z), d(y,z)\} \leq \frac{1+\delta}{2} d(x,y) \right\}$$

the set of δ-approximate midpoints between x and y. It appears that the comparison of the size of those sets in different Banach spaces yields simple powerful arguments for preventing the existence of certain type of non-linear embeddings between those spaces. We will say more on this in Chapter 2 Section 2.3 but we refer to [KR, Section 3], [BCD+, Section 4], [Kal2, Section 7] or [Net, Chapters 2 and 3] for examples of usage of midpoint techniques in non-linear geometry. The following simple result was first proved in [Laa, Section 5].

Lemma 1.4.5. *A Banach space X is uniformly convex if and only if* $\operatorname{diam}(Mid\,(x,-x;\delta))$ *tends uniformly to* 0 *on* S_X.

The asymptotic analogue was proved in [DKR+, Theorem 2.1].

Lemma 1.4.6. *A Banach space X is AMUC if and only if $\alpha\,(Mid\,(x,-x;\delta))$ tends uniformly to 0 on S_X.*

We recall that the *Kuratowski measure of non-compactness* α of a set A is the infimum of all $\varepsilon > 0$ such that A can be covered by a finite number of balls of diameter less than ε. Let us also point out that the modulus of uniform convexity (respectively of asymptotic midpoint convexity) is equivalent to the function $f(t) = \sup\{\delta > 0 :\ \forall x \in S_X,\ \operatorname{diam}(\operatorname{Mid}\,(x,-x;\delta)) < t\}$ (respectively $\widehat{f}(t) = \sup\{\delta > 0 :\ \forall x \in S_X,\ \alpha(\operatorname{Mid}\,(x,-x;\delta)) < t\}$).

Now let us introduce the weak* version of property AMUC in dual spaces.

Definition 1.4.7. Let X be a Banach space and let $|.|$ be an equivalent norm on X^*. For every $t > 0$ and for every $x^* \in S_{|.|}$, let

$$\widehat{\delta}^*_{|.|}\,(t,x^*) = \sup_{E\in \operatorname{cof}^*(X^*)}\ \inf_{e^*\in E, |e^*|=1} \max\{\|x^*+te^*\|\,,\|x^*-te^*\|\} - 1.$$

The function

$$\widehat{\delta}^*_{|.|}\,(t) = \inf_{x^*\in S_{|.|}} \widehat{\delta}^*_X\,(t,x^*)$$

is called *modulus of weak* asymptotic midpoint convexity* of $(X^*,|.|)$. We say that $|.|$ is *weak* asymptotically midpoint uniformly convex* (AMUC* in short) if $\widehat{\delta}^*_{|.|}\,(t) > 0$ for all $t > 0$.

Note that we clearly have $\overline{\delta}^*_X \leq \widehat{\delta}^*_X$ so that any AUC* norm is AMUC*. Also $\widehat{\delta}^*_X \leq \widehat{\delta}_X$ and any AMUC* norm is AMUC. An analogue of Lemma 1.4.2 is available in this context, replacing the condition X does not contain ℓ_1 by X is separable in the last statement.

Remark 1.4.8. Unlike Lemma 1.1.12 it is not clear that any equivalent AMUC* norm on a dual space has to be weak* lower continuous. Thus we don't know if every equivalent AMUC* norm on a dual space has to be a dual norm. Also there is no clear duality result in this context. Indeed the example from [DKR+] of a norm which is AMUC but not AUC combined with the equivalence of properties SAUS and AUS mentioned in Remark 1.4.4 show that AMUC* cannot be the dual property of SAUS even in the reflexive setting.

We will now show that the existence of an equivalent AMUC* on a dual space X^* implies some tree property in X^*, and take advantage of tree reformulations of the condition $S_Z(X) = \omega$ from Section 1.1 and of the renorming theorem Theorem 1.2.6 to obtain the equivalence between AMUC* and AUC* up to renorming for duals of separable spaces under some specific condition on the weak* asymptotic structure. Let $(z_t^*)_{t\in T_n}$ be a weak* convergent ε-climbing tree in X^*, that is $z_t^* = \sum_{s\leq t} x_s^*$ for every $t \in T_n$ where $(x_s^*)_{s\in T_n}$ is a weak* null tree in X^* satisfying $x_\emptyset^* = 0$ and $\|x_s^*\| \geq \varepsilon$ for every non empty sequence $s \in T_n$. We say that $(z_t^*)_{t\in T_n}$ is *absolutely contained* in B_{X^*} is for every leaf t of T_n and for every $(\varepsilon_i)_{i=1}^n$ in $\{-1,1\}^n$ we have $\sum_{i=1}^n \varepsilon_i x_{t|i}^* \in B_{X^*}$. Then we introduce the following property.

Definition 1.4.9. Let X be a Banach space. We say that X^* has the *infinite weak* absolute climbing tree property* (property $(*)$ in short) if there is an $\varepsilon > 0$ such that for every $n \geq 1$, we can find a weak* convergent ε-climbing tree of height n in X^* which is absolutely contained in B_{X^*}.

We have the following result.

Proposition 1.4.10. *Let X be a Banach space and let $|.|$ be an equivalent AMUC* norm on X^*. Then X^* fails property $(*)$. In other words: for every $\varepsilon > 0$ there is an integer $N \geq 1$ such that for every $n \geq N$ and for every weak* null tree $(x_s^*)_{s\in T_n}$ in X^* with $x_\emptyset^* = 0$ and $\|x_s^*\| \geq \varepsilon$ for every $s \in T_n$ non empty, there exists a leaf $t \in T_n$ and a sequence of signs $(\varepsilon_i)_{i=1}^n$ in $\{-1,1\}$ such that $\left\|\sum_{i=1}^n \varepsilon_i x_{t_{|i}}^*\right\| > 1$.*

Proof. Let us assume that X^* has property $(*)$. Then there exists $\varepsilon > 0$ such that for every $n \geq 1$ one can find a weak* convergent ε-climbing tree of height n which is absolutely contained in B_{X^*}. Fix n big enough for our later purpose and fix a weak* null tree $(x_s^*)_{s\in T_n}$ in X^* with $x_\emptyset^* = 0$ and $\|x_s^*\| \geq \varepsilon$ for every $s \in T_n$ non empty such that for every leaf $t \in T_n$ and every sequence of signs $(\varepsilon_i)_{i=1}^n$ in $\{-1,1\}$ we have $\left\|\sum_{i=1}^n \varepsilon_i x_{t_{|i}}^*\right\| \leq 1$. Note that by weak* lower continuity of the norm, one has in fact $\left\|\sum_{i=1}^k \varepsilon_i x_{t_{|i}}^*\right\| \leq 1$ for every $1 \leq k \leq n$ and for every leaf t of T_k.

Now let us assume that X^* is AMUC* and let $\delta = \widehat{\delta}_{X^*}^*(\varepsilon) > 0$. By the weak* analogue of Lemma 1.4.2 we have that for every $x^* \in B_{X^*}$ and for every weak* null sequence $(x_k^*)_{k\geq 1}$ in X^* with $\|x_k^*\| \geq \varepsilon$ for every $k \geq 1$ one has

$$\liminf \max\{\|x^* + x_k^*\|, \|x^* - x_k^*\|\} \geq (1+\delta)\|x^*\|.$$

Thus working by induction in the weak* null tree $(x_s^*)_{s\in T_n}$ we obtain for any fixed x^* in the first level of the tree a leaf $s \in T_n$ and a sequence of signs $(\varepsilon_i)_{i=2}^n$ in $\{-1,1\}^{n-1}$ such that $\left\|x^* + \sum_{i=2}^n \varepsilon_i x_{s_{|i}}^*\right\| \geq \|x^*\|(1+\delta)^n \geq \varepsilon(1+\delta)^n$. Now this is strictly bigger than 1 if n was initially chosen larger than some constant depending on ε, and we obtain a contradiction.

□

Using this we can prove the following.

Theorem 1.4.11. *Let X be a separable space and let us assume that X^* is weak* asymptotically unconditional. Then the following properties are equivalent.*

1. *The space X has Szlenk index $S_Z(X) = \omega$.*
2. *The space X admits an equivalent norm whose dual norm is AUC*.*
3. *The space X^* admits an equivalent AMUC* norm.*

Proof. (1) $\implies$ (2) comes from Theorem 1.2.6 and from a weak* version of Lemma 1.2.11 and (2) $\implies$ (3) is clear.

For (3) $\implies$ (1) recall that the dual X^* of a separable space with a weak* unconditional asymptotic structure has the weak* unconditional finite tree property (Lemma 1.3.12): there is a constant $C \geq 1$ such that one can extract a full subtree whose branches are all C-unconditional of any weak* null normalized tree in X^*. So if X^* is AMUC*, Proposition 1.4.10 provides for every $\varepsilon > 0$ an integer $N = N(\varepsilon) \geq 1$ such that: for every $n \geq N$ and for every weak* null tree $(x_s^*)_{s \in T_n}$ in X^* such that $x_\emptyset^* = 0$ and $\|x_s^*\| \geq \frac{\varepsilon}{C}$ for all $s \in T_n$ non-empty, there is a leaf $s \in T_n$ and a sequence of signs $(\varepsilon_i)_{i=1}^n$ in $\{-1, 1\}^n$ such that $\left\|\sum_{i=1}^n \varepsilon_i x_{s_{|i}}^*\right\| > 1$. Now

$$C \left\|\sum_{i=1}^n x_{s_{|i}}^*\right\| \geq \left\|\sum_{i=1}^n \varepsilon_i x_{s_{|i}}^*\right\| > 1,$$

so that $0 \notin s_\varepsilon^N(B_{X^*})$ and $s_{2\varepsilon}^N(B_{X^*}) = \emptyset$ by Lemma 1.2.17. In other words, $S_Z(X, 2\varepsilon) \leq N(\varepsilon)$ for every $\varepsilon > 0$ and it follows that $S_Z(X) \leq \omega$. □

Remark 1.4.12. Working with the weak topology we can prove in a similar way that any asymptotically unconditional Banach space X which is AMUC has weak Szlenk index $w - S_Z(X) = \omega$. In particular such a space has the PCP. We do not know if any AMUC space has the PCP. A counterexample to that would in particular provide an AMUC space with no equivalent AUC norm. Let us point out that for several reasons, the predual of the James tree space JT_∞ introduced in [GM1] and constructed on a countably branching tree would be a natural candidate to investigate for this problem. Indeed it is known that $(JT_\infty)_*$ fails the PCP (see for example [GM1, Proposition IV.2]) but satisfies a weaker convex PCP property [GMS], and from the work of [Gir] we know that the original binary version JT of the James tree space has an AUC (pre)dual.

Using the midpoint characterization of AMUC we show that the property coincides with its weak* version on dual spaces.

Proposition 1.4.13. *Let X be a Banach space. Then X^* is AMUC if and only if it is AMUC*.*

Proof. From the definition of the moduli any AMUC* norm is AMUC on a dual space.

Now let us assume that X^* is not AMUC*. Following [DKR+, Theorem 2.1] we will prove that $\alpha\left(\mathrm{Mid}\left(x^*, -x^*; \delta\right)\right)$ does not tend uniformly to 0 on S_{X^*}. Since X^* is not AMUC* we can find some $t \in (0,1)$ such that for every $\delta > 0$, we can find $x^*_\delta \in S_{X^*}$ such that:

$$\forall Y \in \mathrm{cof}^*(X^*),\ \exists y^* \in S_Y :\ \max\{\|x^*_\delta + ty^*\|, \|x^*_\delta - ty^*\|\} \leq 1 + \delta,$$

that is to say $ty^* \in \mathrm{Mid}\left(x^*_\delta, -x^*_\delta; \delta\right)$.

Fix $\delta > 0$ and pick $x^* = x^*_\delta$ satisfying the above property. Then one can choose inductively a sequence $(y^*_k)_{k\geq 1}$ in S_{X^*}, and a sequence $(y_k)_{k\geq 1}$ in S_X such that

1. $y^*_k(y_k) \geq \frac{1}{2}$ and $y^*_k \in \bigcap_{i=1}^{k-1} \ker y_i$
2. $ty^*_k \in \mathrm{Mid}\left(x^*, -x^*; \delta\right)$.

Then we have $\|ty^*_k - ty^*_l\| \geq (ty^*_k - ty^*_l)(y_k) \geq \frac{t}{2}$ for every $k, l \geq 1$ with $k > l$ and thus $\alpha\left(\mathrm{Mid}\left(x, -x; \delta\right)\right) \geq \alpha\left(\{ty^*_k\}_{k\geq 1}\right) \geq \frac{t}{2}$. By Lemma 1.4.6, X^* is not AMUC.

□

Remark 1.4.14. It is not clear that the same can be proved for equivalent AMUC* norms on X^* which are not dual norms (if such norms exists).

As a consequence we obtain the following example.

Corollary 1.4.15. *There exists a separable Banach space whose dual norm is AMUC* but which admits no equivalent norm whose dual norm is AUC*. In particular AMUC* does not imply the weak* PCP.*

Proof. Let JT be the James tree-space. It was proved in [Gir] that the dual JT^* of JT is AUC. In particular JT* is AMUC and by the preceding result it is also AMUC*. But since JT is not Asplund it admits no equivalent AUS norm (see Remark 1.1.15) and thus no equivalent norm whose dual norm is AUC* by duality (in fact there is no equivalent AUC* norm on JT* by Lemma 1.1.12).

□

Remark 1.4.16. In terms of trees, this means that B_{JT^*} contains weak* convergent ε-climbing trees of arbitrarily high height for every $\varepsilon > 0$ (and in fact of arbitrarily high ordinal height with trees from Appendix A) while there is an $\varepsilon > 0$ and an $N \geq 1$ such that any ε-climbing tree of height $n \geq N$ leaves B_{JT^*} at least in one direction (that is for at least one choice of signs as in property $(*)$).

To conclude this section, let us make a few last remarks concerning property $(*)$. Looking back at Proposition 1.3.14 and Theorem 1.3.15 let us observe that property $(*)$ contains all the necessary conditions to prove the following result. In fact the assumption " X^* asymptotically unconditional " seems to be only an artificial condition which allows to recover property $(*)$ from the assuption $S_Z(X) > \omega$. We do not know if a Banach space with property $(*)$ has to be asymptotically unconditional.

Theorem 1.4.17. *Let X be a separable Banach space and let us assume that X^* has property $(*)$. Then $\ell_\infty^n \in \{X^*\}_n^{w^*}$ for every $n \geq 1$.*

Also observe as in Remark 1.3.17 that the following asymptotic games would correspond to a generalized version of the negation of property $(*)$: for every $\varepsilon > 0$ there is an $N \geq 1$ such that for every $n \geq N$,

$$\begin{gathered}\exists X_1 \in \mathrm{cof}^*(X^*) : \ \forall x_1^* \in X_1^\varepsilon, \ \exists X_2 \in \mathrm{cof}^*(X^*), \ \exists \varepsilon_1 \in \{-1,1\} : \ \forall x_2 \in X_2^\varepsilon, \ \cdots : \\ \exists X_n \in \mathrm{cof}^*(X^*), \ \exists \varepsilon_{n-1} \in \{-1,1\} : \ \forall x_n^* \in X_n^\varepsilon, \ \exists \varepsilon_n \in \{-1,1\} : \\ x_1^* + \sum_{i=2}^n \varepsilon_{i-1} x_i^* \text{ does not belong to } B_{X^*}\end{gathered}$$

where $E^\varepsilon = \{x \in E : \ \|x\| \geq \varepsilon\}$ for any $E \in \mathrm{cof}^*(X^*)$.

It is not difficult using the weak* analogue of Lemma 1.4.2 and a rescaling argument as in Lemma 1.2.14 the prove the following result.

Lemma 1.4.18. *A dual space X^* is AMUC* if and only if it satisfies the following property: for all $\varepsilon > 0$ there is a $\Delta > 0$ such that for all $x^* \in B_{X^*}$, if $\|x^*\| > 1 - \Delta$ then there is a weak*-neighborhood V of x^* such that for every $y^* \in X^*$ with norm $\|y^*\| \geq \varepsilon$ we have $x^* + y^* \notin V \cap B_{X^*}$ or $x^* - y^* \notin V \cap B_{X^*}$.*

However it is not clear how one could define a suitable derivation in order to obtain a peeling index which would be related to property $(*)$ for duals of separable spaces.

Chapter 2

Non-linear geometry of Banach spaces

2.1 Hyperbolic countably branching trees

Let us start by recalling a few definitions about metric embeddings. For a more complete presentation of the theory we refer to the monograph [Ost3, Chapter 1].

Definition 2.1.1. Let M and N be two metric spaces, and let f be a map from M to N. For every $t \geq 0$, let

$$\rho_f(t) := \inf\{d_N(f(x), f(y)) : \ x, y \in M \text{ and } d_M(x, y) \geq t\}$$

with the convention $\inf \emptyset := \infty$, and let

$$\omega_f(t) := \sup\{d_N(f(x), f(y)) : \ x, y \in M \text{ and } d_M(x, y) \leq t\}.$$

The function ρ_f is called *modulus of compression* of f, and the function ω_f is called *modulus of expension* of f.

Remark 2.1.2. Note that for every x and y in M we have

$$\rho_f(d_M(x, y)) \leq d_N(f(x), f(y)) \leq \omega_f(d_M(x, y)).$$

In particular, the map f is uniformly continuous if and only if $\lim_{t\to 0} \omega_f(t) = 0$ and f is Lipschitz (respectively coarse-Lipschitz) if and only if ω_f is bounded above by a linear function (respectively an affine function).

We will consider the following metric embeddings.

Definition 2.1.3. Let M and N be two metric spaces, and let f be a map from M to N.

1. We say that f is a *Lipschitz or bi-Lipschitz embedding* if we can find constants $A, B > 0$ such that for every $t \geq 0$ we have $At \leq \rho_f(t)$ and $\omega_f(t) \leq Bt$.

2. We say that f is a *coarse-Lipschitz embedding* if we can find constants $A, B, C > 0$ such that for every $t \geq 0$ we have $At - C \leq \rho_f(t)$ and $\omega_f(t) \leq Bt + C$.

3. We say that f is a *coarse embedding* if $\lim_{t\to\infty} \rho_f(t) = \infty$ and if $\omega_f(t) < \infty$ for every $t \geq 0$.

Remark 2.1.4. Clearly a map f is a Lipschitz embedding if and only if it is Lipschitz, injective, and has a Lipschitz inverse, and in practice we can write this in the following way: there exists constants $a, b > 0$ such that for every x and y in M we have

$$ad_M(x, y) \leq d_N(f(x), f(y)) \leq bd_M(x, y).$$

The *distortion* of a Lipschitz embedding f is defined as the product of the Lipschitz constant of f by the Lipschitz constant of its inverse.

Also if a map f is a coarse-Lipschitz embedding then there exists constants $a, b, \theta > 0$ such that for every x and y in M satisfying $d_M(x, y) \geq \theta$, we have

$$ad_M(x, y) \leq d_N(f(x), f(y)) \leq bd_M(x, y)$$

so a coarse-Lipschitz embedding behaves like a bi-Lipschitz embedding for " large distances ".

We will say that M *Lipschitz embeds* (or *bi-Lipschitz embeds*) into N if there is a Lipschitz embedding $f : M \to N$. We will use the same terminology for other kind of embeddings. Also we say that M Lipschitz embeds into N *with distortion* D for a constant $D \geq 1$ if there is a Lipschitz embedding $f : M \to N$ with distortion smaller than or equal to D.

For families of metric spaces, we will need the following definition.

Definition 2.1.5. Let $(M_i)_{i\in I}$ be a family of metric spaces and let N be a metric space. We say that $(M_i)_{i\in I}$ *equi-Lipschitz embeds* into N if there is a constant $D \geq 1$ such that each M_i embeds into N with distortion D. Also we say that $(M_i)_{i\in I}$ *equi-coarsely embeds* into N if there exists functions $\rho, \omega : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ and maps $f_i : M_i \to N$ such that $\lim_{t\to\infty} \rho(t) = \infty$ and $\omega(t) < \infty$ for every $t \geq 0$, and $\rho(t) \leq \rho_{f_i}(t)$ and $\omega_{f_i}(t) \leq \omega(t)$ for every $i \in I$ and for every $t \geq 0$.

Now let us recall that for every $N \geq 1$, $T_N := \{\emptyset\} \cup \bigcup_{n=1}^{N} \mathbb{N}^n$ is the countably branching tree of height N, and $T_\infty := \bigcup_{N\geq 1} T_N$ if the countably branching tree of infinite height. We will use notation from Chapter 1 Section 1.2 and take advantage of the ordering of T_∞ introduced there to turn T_∞ into a metric space. For every sequences s, t in T_∞ let us denote by $a_{s,t}$ the greatest common ancestor of s and t. We define

$$d(s, t) := d(a_{s,t}, s) + d(a_{s,t}, t) = |s| + |t| - 2\,|a_{s,t}|\,.$$

It is easy to check that d is a distance on T_∞, and it induces a distance on every T_N that we will still denote by d. This distance is sometimes referred to as the *hyperbolic distance* on T_∞ because it is equal to the graph distance associated with the following natural graph structure on T_∞: $G = (T_\infty, E)$ where $E = \{\{s, s^-\} : \ s \in T_\infty \text{ non empty}\}$. We refer again to [Ost3] for a detailed presentation of the theory of metric graphs.

Let us now give two simple but fundamental examples of bi-Lipschitz embeddings of T_∞ into Banach spaces.

Example 2.1.6. (1) Let $X =: \ell_1(T_\infty)$, that is

$$X := \left\{ x = (x_s)_{s \in T_\infty} \subset \mathbb{R} : \ \|x\|_1 = \sum_{s \in T_\infty} |x_s| < \infty \right\}$$

and let $(e_s)_{s \in T_\infty}$ be the unit vector basis of X. We introduce the map $f : T_\infty \to X$ given by

$$f(s) := \sum_{t \leq s} e_t$$

for every $s \in T_\infty$. For every distinct $s, t \in T_\infty$ we have

$$f(s) - f(t) = \sum_{a_{s,t} < u \leq s} e_u - \sum_{a_{s,t} < v \leq t} e_v,$$

so that

$$\|f(s) - f(t)\|_1 = d(a_{s,t}, s) + d(a_{s,t}, t) = d(s, t),$$

and the map f is an isometric embedding.

(2) Let $Y := c_0(T_\infty)$ be the completion of the space $(c_{00}(T_\infty), \|.\|_\infty)$ and let $(\sigma_s)_{s \in T_\infty}$ be the sequence of elements of $c_0(T_\infty)$ defined by

$$\sigma_s = \sum_{t \leq s} e_t$$

for every $s \in T_\infty$ where $(e_s)_{s \in T_\infty}$ is the unit vector basis of Y. We introduce the map $g : T_\infty \to Y$ given by

$$g(s) := \sum_{t \leq s} \sigma_t = \sum_{t \leq s} \sum_{u \leq t} e_u$$

for every $s \in T_\infty$. For every distinct $s, t \in T_\infty$ and for every $u \in [a_{s,t}, s] \cup [a_{s,t}, t]$, let us write $d_u = d(a_{s,t}, u)$. We have

$$g(s) - g(t) = \Sigma_s - \Sigma_t$$

where $\Sigma_s = \sum_{a_{s,t} < u \leq s} \sigma_u$ has coordinate 0 everywhere, except on the segment $[\emptyset, s]$ where its coordinates are given by $\Sigma_{s,u} = d_s$ if $u \leq a_{s,t}$ and $\Sigma_{s,u} = d_s - d_u + 1$ if $a_{s,t} < u \leq s$. Similarly $\Sigma_t = \sum_{a_{s,t} < v \leq t} \sigma_v$ has coordinate 0 everywhere, except on the segment $[\emptyset, t]$ where its coordinates are given by $\Sigma_{t,v} = d_t$ if $v \leq a_{s,t}$ and $\Sigma_{t,v} = d_t - d_v + 1$ if $a_{s,t} < v \leq t$. It then follows that

$$\|g(s) - g(t)\|_\infty \leq \|\Sigma_s\|_\infty + \|\Sigma_t\|_\infty = d_s + d_t = d(s, t)$$

and

$$\|g(s) - g(t)\|_\infty \geq \max\{d_s, d_t\} \geq \frac{1}{2} d(s, t)$$

looking at the successor of $a_{s,t}$ on the segments $[a_{s,t}, s]$ and on the segment $[a_{s,t}, t]$, so that T_∞ bi-Lipschitz embeds in Y with distortion 2.

Building on those examples and using the bi-orthogonal trees that we mentioned in Chapter 1 Section 1.2 (Proposition 1.3.14), Baudier Kalton and Lancien proved in [BKL] the following result.

Theorem 2.1.7. *Let X be a separable Banach space. If $S_Z(X) > \omega$ or if $S_Z(X^*) > \omega$, then the family $(T_N)_{N\geq 1}$ equi-Lipschitz embeds into X and into X^*.*

Remark 2.1.8. More precisely, it is proved in [BKL, Proposition 2.3] that there is a universal constant $C \geq 1$ such that T_N bi-Lipschitz embeds into X and into X^* with distortion C whenever X is a separable Banach space satisfying $S_Z(X) > \omega$. Then it is proved in [BKL, Proposition 2.5] that a " lift up argument " can be implemented using Goldstine's theorem to show that T_N bi-Lipschitz embeds into X with distortion C whenever X is a separable Banach for which $S_Z(X^*) > \omega$.

Those results are respectively improved in [BKL, Theorem 2.4] using a Gluing technique similar to the one from [Bau1], and in [BKL, Theorem 2.6] using a much more technical argument, and we in fact have the following.

Theorem 2.1.9. *Let X be a separable Banach space. If $S_Z(X) > \omega$ or if $S_Z(X^*) > \omega$, then T_∞ bi-Lipschitz embeds into X and into X^*.*

Note that the separable determination of the Szlenk index allows to forget about the separability assumption in the above results. In [BKL, Theorem 3.1] the converse of Theorem 2.1.7 was obtained for reflexive Banach spaces. The proof there is done by contradiction and is based on one side on a certain *tree decomposition* which is done by a process of successive extractions of weakly convergent subtrees (for which weak sequential compactness is key), and on the other side on sharp martingale like computations in those trees based on upper and lower tree estimates coming from the asymptotic renorming theory.

Theorem 2.1.10. *Let X be a reflexive Banach space. If $S_Z(X) \leq \omega$ and $S_Z(X^*) \leq \omega$, then the family $(T_N)_{N\geq 1}$ does not equi-Lipschitz embed into X.*

Combining the above results thus yield the following.

Theorem 2.1.11. *Let X be a reflexive Banach space. The following assertions are equivalent.*

1. *The space X satisfies $S_Z(X) \leq \omega$ and $S_Z(X^*) \leq \omega$.*
2. *The family $(T_N)_{N\geq 1}$ does not equi-Lipschitz embed into X.*
3. *The metric space T_∞ does not bi-Lipschitz embed into X.*

In particular the space T_∞ does not bi-Lipschitz embed into any ℓ_p-sum of finite dimensional spaces nor into $L_p[0,1]$ for $1 < p < \infty$. Using a self-improvement argument " à la Kloeckner " based the property (β) of Rolewicz, Baudier and Zhang gave in [BZ] a short elegant proof of Theorem 2.1.10, which is very similar to the one of Kloeckner for Bourgain's characterization

of super-reflexivity from [Klo], and which moreover has the advantage of providing a lower bound on the distortion of the T_N's into any Banach space X with a power type (β) modulus. However, note that this argument cannot be extended to a more general non-reflexive setting since property (β) implies reflexivity. In fact the extension of Theorem 2.1.10 that we will provide below shows that the non bi-Lipschitz embeddability of T_∞ (or the non equi-Lipschitz embeddability of the family $(T_N)_{N\geq 1}$) cannot provide a purely metric characterization of property (β) since it does not force reflexivity. We refer to [Zha] and to [BG] for recent progress on the question of providing purely metric characterizations of property (β).

Remark 2.1.12. Concerning weaker types of embeddings, let us point out that it was proved in [BLS, Theorem 3.6] that a Banach space X has finite dimension if and only if the family $(T_N)_{N\geq 1}$ does not equi-coarsely embed into X if and only if T_∞ does not coarsely embeds into X, thus providing a purely metric characterization of finite dimensionality.

Let us recall that a Banach space X is quasi-reflexive if the quotient X^{**}/X is of finite dimension or equivalently if there is a finite dimensional space E such that $X^{**} = X \oplus E$. We will provide the following improvement of Theorem 2.1.10.

Theorem 2.1.13. *Let X be a quasi-reflexive Banach space. If $S_Z(X) \leq \omega$ and $S_Z(X^*) \leq \omega$, then the family $(T_N)_{N\geq 1}$ does not equi-Lipschitz embed into X.*

As a direct consequence of this result, we obtain the following (see Example 1.1.25 for definitions).

Theorem 2.1.14. *Let $p \in (1,\infty)$. The family $(T_N)_{N\geq 1}$ does not equi-Lipschitz embed in the James space $\mathcal{J}_p$ and it does not equi-Lipschitz embed in its dual $\mathcal{J}_p^*$.*

So as mentioned above, the non equi-Lipschitz embeddability of the family $(T_N)_{N\geq 1}$ into a space X does thus not force the reflexivity of X.

Remark 2.1.15. In view of previous results, an immediate consequence of Theorem 2.1.13 is that the characterization from Theorem 2.1.11 extends to the quasi-reflexive setting. In particular this imply that the condition $S_Z(X) \leq \omega$ and $S_Z(X^*) \leq \omega$ is stable in the class of quasi-reflexive Banach spaces under coarse-Lipschitz embeddings, that is if X is a quasi-reflexive Banach space and if X coarse-Lipschitz embeds into a quasi-reflexive Banach space Y satisfying $S_Z(Y) \leq \omega$ and $S_Z(Y^*) \leq \omega$, then X also has to satisfy $S_Z(X) \leq \omega$ and $S_Z(X^*) \leq \omega$. In [BKL, Theorem 4.3], it is proved that the condition X reflexive, $S_Z(X) \leq \omega$ and $S_Z(X^*) \leq \omega$ is stable under coarse-Lipschitz embeddings without any further restriction to a subclass of Banach spaces. To shed some light on the peculiarity of such result, let us recall that reflexivity (and even the condition X reflexive and $S_Z(X^*) \leq \omega$) is not stable under coarse-Lipschitz embeddings. Indeed Ribe showed in [Rib2] that the space $E = \left(\sum_{n\geq 1} \ell_{p_n}\right)_{\ell_2}$ where $(p_n)_{n\geq 1}$ is a sequence of real numbers strictly decreasing to 1 is uniformly homeomorphic to $E \oplus \ell_1$ so that $E \oplus \ell_1$ coarse-Lipchitz embeds into E. However, E is reflexive, 2-AUC and satisfies $S_Z(E) = \omega^2$ (this can be proved by direct computations, or can be seen as a direct consequence of [Bro, Theorem 2.11]).

We do not know if [BKL, Theorem 4.3] extends to the quasi-reflexive setting. Yet let us point out that this result follows from Theorem 2.1.11 and from [BKL, Theorem 4.1] which states that if a separable Banach space X coarse-Lipschitz embeds into a separable reflexive Banach space Y with $S_Z(Y) = \omega$, then X has to be reflexive. It is far from clear that the latter extends to the quasi-reflexive setting and in fact a few results from the litterature tends to point towards the negative. Indeed [BKL, Theorem 4.1] essentially relies on James characterization of reflexivity and on a concentration phenomenon for Hamming graphs from [KR, Theorem 4.2], which can be reformulated using the terminology from [Fov] as follows: any p-AUS reflexive Banach space has HFC_p. In [LR] it is proved that this concentration result somehow extends to the quasi-reflexive setting: any p-AUS quasi-reflexive Banach space has HIC_p. Now it is known that HFC_p implies reflexivity (this is contained in the proof of [BKL, Theorem 4.1]). However, the stability result from [Fov] of concentration properties for Hamming graphs under ℓ_p-sums of Banach spaces showed that the space $\ell_2(J)$ satisfies HIC_2 while being non quasi-reflexive.

The proof of Theorem 2.1.13 will be postponed to Section 2.2. It will closely follow the proof from [BKL] but will need a few non trivial adjustments because of the lack of compactness in this setting. In particular the development of specific upper tree estimates will be required. We end the present section by presenting all the necessary tools for this purpose.

First we recall a few basic notions coming from the theory of Orlicz sequence spaces. Following [LT, Chapter 4] we say that a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an *Orlicz function* if it is convex, continuous, non-decreasing, and satisfies $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. For an Orlicz function Φ we introduce the space

$$\ell_\Phi := \left\{ (x_n)_{n\geq 1} \subset \mathbb{R} : \ \exists t > 0 : \ \sum_{n\geq 1} \Phi\left(\frac{|x_n|}{t}\right) < \infty \right\}$$

and the space

$$h_\Phi := \left\{ (x_n)_{n\geq 1} \subset \mathbb{R} : \ \forall t > 0 : \ \sum_{n\geq 1} \Phi\left(\frac{|x_n|}{t}\right) < \infty \right\}.$$

For every $x := (x_n)_{n\geq 1}$ in ℓ_Φ we introduce the quantity

$$\|x\|_{\ell_\Phi} := \inf \left\{ t > 0 : \ \sum_{n\geq 1} \Phi\left(\frac{|x_n|}{t}\right) \leq 1 \right\}.$$

This defines a norm on ℓ_Φ which turns ℓ_Φ into a Banach space called *Orlicz sequence space* associated with Φ. The space h_Φ is easily seen to be a closed subspace of ℓ_Φ and the unit vector basis of c_{00} is known to be a symmetric basis of h_Φ.

Example 2.1.16. (1) Let Φ be an Orlicz function. If there exists a $t > 0$ such that $\Phi(t) = 0$, then Φ is called *degenerate*, and in this case the space ℓ_Φ is isomorphic to ℓ_∞ and the space h_Φ is isomorphic to c_0.

(2) For every $1 \leq p < \infty$, the function $t \to t^p$ is an Orlicz function and the associated Orlicz sequence space is known to be isometric to the space ℓ_p.

Note that in general the space h_Φ is a proper subspace of ℓ_Φ. In fact the two spaces coincide if and only if the function Φ satisfies the so called Δ_2 condition at 0, see [LT, Proposition 4.a.4]. Also note that if two Orlicz functions coincide on an interval $[0, t_0]$ for some $t_0 > 0$, then the associated Orlicz sequence spaces are isomorphic. In particular we have the following easy lemma.

Lemma 2.1.17. *Let Φ and Ψ be two Orlicz functions. If there is a constant $C > 0$ such that $\Phi(t) \leq C\Psi(t)$ for every $t \in [0, 1]$, then there is a constant $A > 0$ such that $\|x\|_{\ell_\Phi} \leq A \|x\|_{\ell_\Psi}$ for every $x \in c_{00}$.*

Now let Φ be a non degenerate Lipschitz Orlicz function. Then the limit $\alpha := \lim_{t\to\infty} \frac{\Phi(t)}{t}$ exists and we define a function $N^\Phi : \mathbb{R}^2 \to \mathbb{R}_+$ by letting

$$N^\Phi(x, y) := |x| \left(1 + \Phi\left(\frac{|y|}{|x|}\right)\right)$$

for every $x \neq 0$ and $y \in \mathbb{R}$ and

$$N^\Phi(0, y) := \alpha |y| .$$

We then define inductively functions $N_k^\Phi : \mathbb{R}^k \to \mathbb{R}_+$ by letting first $N_1^\Phi(.) := |.|$, and then

$$N_k^\Phi(x_1, \dots x_k) := N^\Phi(N_{k-1}^\Phi(x_1, \dots, x_{k-1}), x_k)$$

for every $k \geq 2$ and every $x_1, \dots, x_k \in \mathbb{R}$. For every $k \geq 1$, the function N_k^Φ is a norm on $\mathbb{R}^k$, and for every $x := (x_n)_{n\geq 1}$ in c_{00} we then define the quantity

$$\|x\|_{\Lambda_\Phi} := \sup_{k\geq 1} N_k^\Phi(x_1, \dots, x_k).$$

This defines a norm on c_{00}, and the *iterated Orlicz space associated to* Φ is the completion Λ_Φ of the space $(c_{00}, \|.\|_{\Lambda_\Phi})$. The space Λ_Φ is known to be isomorphic to h_Φ and we have the following result, see [Kal2, Lemma 4.3].

Lemma 2.1.18. *Let Φ be a non degenerate Lipschitz Orlicz function. For every $x \in c_{00}$ we have*

$$\frac{1}{2} \|x\|_{\ell_\Phi} \leq \|x\|_{\Lambda_\Phi} \leq e \|x\|_{\ell_\Phi} .$$

Let X be a Banach space and let us assume that X is not AUF. Then we have $\overline{\rho}_X(t) > 0$ for all $t > 0$ by the results mentioned in Remark 1.1.21. So combined with the basic properties of $\overline{\rho}_X$ mentioned in Remark 1.1.2, we have that $\overline{\rho}_X$ is a non degenerate 1-Lipschitz Orlicz function.

Now let us assume that the space X is *AUC*. Then the function $\overline{\delta}_X$ satisfies the condition $\overline{\delta}_X(t) > 0$ for all $t > 0$. Since $\overline{\delta}_X$ need not be convex but is such that $t \to \frac{\overline{\delta}_X(t)}{t}$ is non decreasing, we introduce, following [Kal2, Section 3], an auxiliary function $\delta(t) = \int_0^t \frac{\overline{\delta}_X(s)}{s} ds$ which turns out to be a 1-Lipschitz non degenerate Orlicz function, and to satisfy $\frac{1}{2}\overline{\delta}_X \leq \delta \leq \overline{\delta}_X$. Then combining the above Lemma 2.1.17 and Lemma 2.1.18 we have the following.

Lemma 2.1.19. *Let X be an infinite dimensional Banach space.*

1. *If X is $p-AUS$ for some $p \in (1,\infty)$ and if it is not AUF, there is a constant $A > 0$ such that:*
$$\forall v \in c_{00},\ \|v\|_{\Lambda_{\overline{\rho}_X}} \leq A \|v\|_{l_p}.$$

2. *If X is $q-AUC$ for some $q \in (1,\infty)$, there is a constant $a > 0$ such that:*
$$\forall v \in c_{00},\ \|v\|_{\Lambda_\delta} \geq a \|v\|_{l_p}.$$

We have a similar result for $q-AUC^*$ duals with the Orlicz function $\delta^*(t) = \int_0^t \frac{\overline{\delta}^*_{X^*}(s)}{s} ds$.

As mentioned in the previous Chapter 1 the theory of Orlicz spaces allows to implement short elegant proofs of the fact that any p-AUS space satisfies the upper ℓ_p finite tree property, as well as other related results. Indeed this is a consequence of the following result (that we only state with countably branching trees for convenience, but which extends easily to the generalized trees from [CFL]) combined with the estimates from Lemma 2.1.19.

Proposition 2.1.20. *Let X be a Banach space. For every weakly null normalized unrooted tree $(x_s)_{s\in T_N}$ in X of finite height $N \geq 1$, we can find a full subtree T of T_N such that for every leaf s of T we have*
$$\frac{1}{2}\|a\|_{\Lambda_\delta} \leq \left\|\sum_{i=1}^N a_i x_{s|i}\right\| \leq 2\|a\|_{\Lambda_{\overline{\rho}_X}}$$
for every $a = (a_i)_{i=11}^N$ in $\mathbb{R}^N$. In a dual space we have similar lower estimates for weak* null normalized trees with $\|.\|_{\Lambda_{\delta^*}}$.

Remark 2.1.21. Passing to trees of infinite height requires more sophisticated combinatorial Ramsey like theorems. For such considerations we refer to [Cau3].

For convenience we also state here a simplified version of the key Proposition 2.2.4 from the next Section 2.2 which gives a weaker version of upper tree estimates in the bidual of any quasi-reflexive space holding for differences of interlaced branches in weak* null normalized trees only but still relying on the modulus of asymptotic smoothness of the space itself. By *interlaced leaves* of T_N we mean any two leaves s, t of T_N which satisfy either $s_1 < t_1 < \cdots < s_N < t_N$ or $t_1 < s_1 < \cdots < t_N < s_N$.

Proposition 2.1.22. *Let X be a quasi-reflexive Banach space. For every weak* null normalized unrooted tree $(x_s^{**})_{s\in T_N}$ of finite height $N \geq 1$ in X^{**} we can find a full subtree T of T_N such that for any two interlaced leaves s and t of T we have*
$$\left\|\sum_{i=1}^N a_i(x_{s|i}^{**} - x_{t|i}^{**})\right\| \leq 4\left\|\widetilde{a}\right\|_{\Lambda_{\overline{\rho}_X}}$$
for every $a = (a_i)_{i=1}^N$ in $\mathbb{R}^N$ with $\widetilde{a} = \left(a_i \left\|x_{s|i}^{**} - x_{t|i}^{**}\right\|\right)_{i=1}^N$.

The proof of those is essentially contained for a fixed $a \in \mathbb{R}^N$ in the proof of Proposition 2.2.3 and respectively of Proposition 2.2.4 from the next section. Further Ramsey arguments combined with standard compactness tricks allow to recover a subtree which does the job for every $a \in \mathbb{R}^N$. We leave the details to the reader. Yet let us already mention that the key ingredient in the proof of Proposition 2.1.22 is the following concentration theorem from Ramsey.

Theorem 2.1.23. *Let K be a compact metric space and let $(x_s)_{s\in\mathbb{N}^n}$ be a family of elements of K indexed by $\mathbb{N}^n$. For every $\varepsilon > 0$ there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that for every elements s and t of $\mathbb{M}^n$ we have $d(s,t) \leq \varepsilon$.*

The main idea is then to use the standard norm compactness for finite dimensional balls in order to show that the differences between any two elements of a weak* null tree in the bidual of a quasi-reflexive space have to be close, once the tree is property concentrated, to points of the space X, thus allowing to apply Lemma 1.1.9 and to obtain estimates from the modulus of asymptotic smoothness of X.

Although the above result could be proved directly by using sequential moduli, the proof of Proposition 2.2.3 and of Proposition 2.2.4 will require a few more technical tricks. For this purpose we introduce some convenient inequalities using the asymptotic uniform moduli from Section 1.1. Let X be a Banach space. The following result follows immediately from Lemma 1.1.5.

Proposition 2.1.24. *Fix $x^* \in S_{X^*}$, $\sigma \geq 0$ and $\varepsilon > 0$. There is a weak* neighborhood V of 0 such that*

$$\forall y^* \in V,\ \|y^*\| \geq \sigma \implies \|x^* + y^*\| \geq 1 + \overline{\delta}^*_{X^*}(\sigma) - \varepsilon.$$

Using this, we obtain the following.

Lemma 2.1.25. *Fix $x^* \in S_{X^*}$, $R > 0$ and $\varepsilon > 0$. There is a weak* neighborhood V of 0 such that*

$$\forall y^* \in V \cap RB_{X^*},\ \|x^* + y^*\| \geq 1 + \overline{\delta}^*_X(\|y^*\|) - \varepsilon.$$

Proof. Fix $\eta \in (0,1)$ and take a finite η-net $(\sigma_i)_{1\leq i\leq n}$ in $[0,R]$ containing 0. Applying the preceding proposition we get a weak* neighborhood V of 0 such that

$$\forall i \in \{1,\dots,n\},\ \forall y^* \in V,\ \|y^*\| \geq \sigma_i \implies \|x^* + y^*\| \geq 1 + \overline{\delta}^*_X(\sigma_i) - \eta.$$

Now take $y^* \in V$ with $\|y^*\| \leq R$. We can find some $1 \leq i_0 \leq n$ such that $\sigma_{i_0} \leq \|y^*\| \leq \sigma_{i_0} + \eta$. Applying the preceding inequality we get

$$\begin{aligned}\|x^* + y^*\| &\geq 1 + \overline{\delta}^*_X(\|y^*\|) + \overline{\delta}^*_X(\sigma_{i_0}) - \overline{\delta}^*_X(\|y^*\|) - \eta \\ &\geq 1 + \overline{\delta}^*_X(\|y^*\|) - \omega_{\overline{\delta}^*_{X^*}}(\eta) - \eta\end{aligned}$$

where $\omega_{\overline{\delta}^*_{X^*}}$ is the modulus of continuity of the function $\overline{\delta}^*_X$. The result follows because $\overline{\delta}^*_X$ is uniformly continuous.

□

After a straightforward non-sequential generalization of the result from [LR, Proposition 2.1] (Lemma 1.1.9) we have the following.

Proposition 2.1.26. *Fix $x \in S_X$, $\sigma \geq 0$ and $\varepsilon > 0$. There is a weak*-neighborhood V of 0 in X^{**} such that*

$$\forall y^{**} \in V,\ \|y^{**}\| \leq \sigma;\ \|x + y^{**}\| \leq 1 + \overline{\rho}_X(\sigma) + \varepsilon.$$

Then as above we can prove the following result.

Lemma 2.1.27. *Fix $x \in S_X$, $R \geq 0$ and $\varepsilon > 0$. There is a weak*-neighborhood V of 0 in X^{**} such that*

$$\forall y^{**} \in V \cap RB_{X^{**}},\ \|x + y^{**}\| \leq 1 + \overline{\rho}_X(\|y^{**}\|) + \varepsilon.$$

2.2 Non embeddability of T_∞ into certain quasi-reflexive Banach spaces.

This section is devoted to the proof of the main result of this chapter.

Theorem 2.2.1. *Let X be a quasi-reflexive Banach space satisfying $S_Z(X) \leq \omega$ and $S_Z(X^*) \leq \omega$. Then the family $(T_N)_{N\geq 1}$ does not equi-Lipschitz embed into X.*

In order to prove our result, let us consider a quasi-reflexive infinite dimensional Banach space X and let us assume that $(T_N)_{N\geq 1}$ equi-Lipschitz embeds into X. Then we may assume that there exist a constant $c > 0$ and functions $f_N : T_N \to X$ with $f_N(\emptyset) = 0$ such that

$$\forall N \geq 1,\ \forall s, t \in T_N,\ d(s,t) \leq \|f_N(s) - f_N(t)\| \leq cd(s,t).$$

Considering the closed linear span of $\bigcup_{N\geq 1} f_N(T_N)$ in X, we may further assume that X and therefore all its iterated duals are separable.

Now suppose that X satisfies $S_Z(X) \leq \omega$ and $S_Z(X^*) \leq \omega$. By Theorem 1.2.6 and by duality, we may assume that $\|.\|_X$ is $p-AUS$ and that the dual space X^* admits an equivalent q^*-AUS norm $|.|$ for some $p, q^* \in (1,\infty)$. As mentioned we get that the dual norm $|.|$ on X^{**} is $q-AUC^*$ where q is the conjugate exponent of q^*. We may also assume that

$$|.| \leq \|.\| \leq e\,|.|$$

for some constant $e \geq 1$ on X^{**}.

Let us fix some $N \geq 1$ which is to be determined later and let us write $f = f_N$. We will be considering the function f as a function with values in X^{**}. For all $t \in T_N,\ t \neq \emptyset$, we put

$$z(t) = f(t) - f(t^-).$$

Note that $\|z(t)\| \leq c$ for every $t \in T_N$. Hence, using weak* sequential compactness and passing to a full subtree, we may assume that for all $1 \leq j \leq N$ and for all $\forall t \in T_{N-j}$, the iterated weak* limit

$$\partial_j z(t) = w^* - \lim_{n_1} \ \dots\ w^* - \lim_{n_j} z(t \frown (n_1, \dots, n_j))$$

is well defined. We also denote $\partial_0 z(t) = z(t)$. Note that $\|\partial_j z(t)\| \leq c$ by lower semi-continuity of the norm. For all $1 \leq j \leq k \leq N$ and for all $t \in T_N$ of length $|t| \geq j$, we introduce

$$z_{k,j}(t) = \partial_{k-j} z\left(t_{|j}\right) - \partial_{k-j+1} z\left(t_{|j-1}\right).$$

Also, let $z_{k,0}(t) = \partial_k z(\emptyset)$. Note that $z_{k,j}(t)$ only depends on the j first coordinates of the sequence t and that $\|z_{k,j}(t)\| \leq 2c$. Moreover, we have the following properties. The proof of these results is straightforward but we will apply them often in the sequel.

Proposition 2.2.2. *For all $t \in T_N,\ t \neq \emptyset$, we have:*

1. $f(t) = \sum_{k=1}^{|t|} z(t_{|k}) = \sum_{k=1}^{|t|} \sum_{j=0}^{k} z_{k,j}(t)$
2. $\forall 1 \leq l \leq k \leq |t|\,,\ \left\|\sum_{j=0}^{l} z_{k,j}(t)\right\| \leq c$
3. $\forall 1 \leq j \leq k \leq N,\ j \leq |t|,\ w^* - \lim_n z_{k,j}\left(t_{|j-1} \frown n\right) = 0.$

Now let us assume that $N = 3Q^M$ for some $Q > 3$ and $M \geq 1$. Then for all $1 \leq k \leq N$, there is a unique $1 \leq M_k \leq M+1$ such that $Q^{M_k - 1} \leq k < Q^{M_k}$. Thus we can define exponentially decreasing indices $\alpha_{k,0} = k$, $\alpha_{k,r} = k - Q^r$ for $1 \leq r < M_k$ and $\alpha_{k,M_k} = -1$. We consider block functions $w_{k,r}$ defined on the roof of the tree T_N by

$$w_{k,r}(t) = \sum_{j=\alpha_{k,r}+1}^{\alpha_{k,r-1}} z_{k,j}(t).$$

Our goal in the sequel will be to give upper and lower estimates of the quantity

$$\sum_{k=1}^{N} \sum_{r=1}^{M_k} \|w_{k,r}(t)\|$$

in a certain full subtree in order to get a contradiction when Q and M are sufficiently big.

In the reflexive case, it is possible to get such estimates using the result from [KOS] (Theorem 1.2.7) where the space is embedded into a Banach space admitting a finite dimensional decomposition in which nice upper and lower ℓ_p and ℓ_q estimates hold. We will replace this result in our setting by the two following propositions.

Proposition 2.2.3. *There is a constant $a > 0$ such that for all $\eta > 0$, there is a full subtree T of T_N such that for all $1 \leq L \leq N$, for all $0 \leq i_1 \leq j_1 < \cdots < i_L \leq j_L \leq N$ and for all $N \geq k_i \geq j_i$, we have*

$$\left|\sum_{l=1}^{L} B_l(t)\right| \geq a \left(\sum_{l=1}^{L} |B_l(t)|^q\right)^{\frac{1}{q}} - \eta$$

whenever t is an element of T of length $|t| \geq j_L$, where B_l is the block function defined by

$$B_l(t) = \sum_{j=i_l}^{j_l} z_{k_l,j}(t).$$

Proposition 2.2.4. *There is a constant $A > 0$ such that for all $\eta > 0$, there is a full subtree T of T_{2N} such that for all $1 \leq L \leq N$, for all $0 \leq i_1 \leq j_1 < \cdots < i_L \leq j_L \leq N$ and for all $N \geq k_i, \geq j_i$, we have*

$$\left\| \sum_{l=1}^{L} B_l(s) - B_l(t) \right\| \leq A \left(\sum_{l=1}^{L} \left\| \sum_{j=i_l}^{j_l} B_l(s) - B_l(t) \right\|^p \right)^{\frac{1}{p}} + \eta$$

whenever s and t are two elements of T_N of length $|t| = |s| \geq j_L$ such that the interlaced sequence $s \propto t = (s_1, t_1, s_2, \dots)$ belongs to T, where B_l is the block function defined by

$$B_l(t) = \sum_{j=i_l}^{j_l} z_{k_l,j}(t).$$

The proof of these two propositions will be postponed to the end of the section to make the reading lighter. We turn to the proof of the main result.

Proof of 2.2.1. Fix $\eta > 0$ and assume that the two propositions are satisfied respectively on the whole T_N and T_{2N} for this constant.

First, we apply Proposition 2.2.3 for every $1 \leq k \leq N$ to the block functions $w_{k,r}$ with r running from 1 to M_k. We get

$$\left(\sum_{r=1}^{M_k} |w_{k,r}(t)|^q \right)^{\frac{1}{q}} \leq \frac{1}{a} \left(\left| \sum_{r=1}^{M_k} w_{k,r}(t) \right| + \eta \right) \leq \frac{c+\eta}{a}$$

for every $t \in T_N$ of length N because

$$\sum_{r=1}^{M_k} w_{k,r}(t) = \sum_{j=0}^{k} z_{k,j}(t)$$

is of norm at most c. Then assuming that $\eta \leq c$ and using Hölder's inequality, we get

$$\sum_{k=1}^{N} \sum_{r=1}^{M_k} |w_{k,r}(t)| \leq \frac{2c}{a} (M+1)^{\frac{1}{q^*}} N.$$

So

$$\sum_{k=1}^{N} \sum_{r=1}^{M_k} \|w_{k,r}(t)\| \leq (M+1)^{\frac{1}{q^*}} Q^{M+1}$$

if Q is was chosen bigger than $\frac{6ce}{a}$.

Second, we want to get an estimate from below. To do that, we will use some computation tricks. We start with an easy lemma.

Lemma 2.2.5. *Let $1 \leq m \leq M$. For all $Q^m \leq l \leq N - Q^m$ and for all $s, t \in T_N$ such that $|s| = |t| \geq l + Q^m$ and $|a_{s,t}| = l$, we have*

$$\left\| \sum_{k=l+1}^{l+Q^m} \sum_{r=1}^{m} w_{k,r}(s) - w_{k,r}(t) \right\| \geq 2Q^m.$$

Note that the condition $l \geq Q^m$ is crucial in order to ensure that the $w_{k,r}$ appearing in the sums are well defined.

Proof. Let us recall that $z_{k,j}$ only depends on the j first coordinates of the sequence. So if we take $s, t \in T_N$ satisfying the properties of the lemma and if we take $1 \leq j \leq l$, then $z_{k,j}(s) = z_{k,j}(t)$ whenever $j \leq k \leq N$. Thus for every $1 \leq l \leq L \leq |s|$ we have

$$f\left(s_{|L}\right) - f\left(t_{|L}\right) = \sum_{k=l+1}^{L} \sum_{j=0}^{k} z_{k,j}(s) - z_{k,j}(t),$$

and thus, we get

$$\left\| \sum_{k=l+1}^{L} \sum_{j=0}^{k} z_{k,j}(s) - z_{k,j}(t) \right\| \geq 2(L - l).$$

Moreover, we have

$$\begin{aligned} \sum_{k=l+1}^{l+Q^m} \sum_{r=1}^{m} w_{k,r}(s) - w_{k,r}(t) &= \sum_{k=l+1}^{l+Q^m} \sum_{j=\alpha_{k,m}+1}^{\alpha_{k,0}} z_{k,j}(s) - z_{k,j}(t) \\ &= \sum_{k=l+1}^{l+Q^m} \sum_{j=k-Q^m+1}^{k} z_{k,j}(s) - z_{k,j}(t) \\ &= \sum_{k=l+1}^{l+Q^m} \sum_{j=0}^{k} z_{k,j}(s) - z_{k,j}(t) \end{aligned}$$

because $k - Q^m + 1 \leq l + 1$ whenever $l + 1 \leq k \leq l + Q^m$. Combining the two facts, we get the desired result.

□

Next fix $1 \leq m \leq M$, $Q^m \leq l \leq N - Q^m$ and $l + 1 \leq k \leq N - (Q - 1)Q^{m-1}$. For every $0 \leq n \leq Q - 1$ and for every $t \in T_N$ of length N, we have

$$\sum_{r=1}^{m-1} w_{k+nQ^{m-1},r}(t) = \sum_{j=k+(n-1)Q^{m-1}+1}^{k+nQ^{m-1}} z_{k+nQ^{m-1},j}(t).$$

In particular,

$$\left\| \sum_{r=1}^{m-1} w_{k+nQ^{m-1},r}(t) \right\| \leq \left\| \sum_{j=0}^{k+(n-1)Q^{m-1}} z_{k+nQ^{m-1},j}(t) \right\| + \left\| \sum_{j=0}^{k+nQ^{m-1}} z_{k+nQ^{m-1},j}(t) \right\| \leq 2c.$$

Now, we apply Proposition 2.2.4 to the block functions $\sum_{r=1}^{m-1} w_{k+nQ^{m-1},r}$ with n running from 0 to $Q-1$. We get

$$\begin{aligned}\left\|\sum_{n=0}^{Q-1}\sum_{r=1}^{m-1} w_{k+nQ^{m-1},r}(t) - w_{k+nQ^{m-1},r}(s)\right\| &\leq A\left(\sum_{n=0}^{Q-1}\left\|\sum_{r=1}^{m-1} w_{k+nQ^{m-1},r}(t) - w_{k+nQ^{m-1},r}(s)\right\|^p\right)^{\frac{1}{p}} + \eta \\ &\leq 4cAQ^{\frac{1}{p}} + \eta\end{aligned}$$

for all interlacing sequences $s, t \in T_N$ of length N.

Thus, assuming that $\eta \leq 4cA$ and summing over k, we get

$$\begin{aligned}\left\|\sum_{k=l+1}^{l+Q^m}\sum_{r=1}^{m-1} w_{k,r}(s) - w_{k,r}(t)\right\| &= \left\|\sum_{k=l+1}^{l+Q^{m-1}}\sum_{n=0}^{Q-1}\sum_{r=1}^{m-1} w_{k+nQ^{m-1},r}(s) - w_{k+nQ^{m-1},r}(t)\right\| \\ &\leq 8cAQ^{\frac{1}{p}}Q^{m-1} \\ &\leq Q^m\end{aligned}$$

if Q was chosen bigger than $(8cA)^{p^*}$, where p^* is the conjugate exponent of p.

Combining this and the lemma, we get that whenever we take interlacing $s, t \in T_N$ of length N satisfying $|a_{s,t}| = l$, we have

$$\left\|\sum_{k=l+1}^{l+Q^m} w_{k,m}(s) - w_{k,m}(t)\right\| \geq Q^m,$$

and thus at least one of the quantities $\left\|\sum_{k=l+1}^{l+Q^m} w_{k,m}(s)\right\|$ or $\left\|\sum_{k=l+1}^{l+Q^m} w_{k,m}(t)\right\|$ is bigger than $\frac{1}{2}Q^m$. Then, using Ramsey's theorem, it is easy to get a full subtree of T_N where this inequality holds for every sequence of length N.

Consequently, we can assume up to the successive extraction of finitely many full subtrees that for all $t \in T_N$ of length N, for all $1 \leq m \leq M$ and for all $Q^m \leq l \leq N - Q^m$, we have

$$\left\|\sum_{k=l+1}^{l+Q^m} w_{k,m}(t)\right\| \geq \frac{1}{2}Q^m.$$

Now take $1 \leq \gamma \leq Q^{M-m}$ and let $l = \gamma Q^m$. Then

$$\sum_{k=l+1}^{l+Q^m} w_{k,m}(t) = \sum_{k=\gamma Q^m+1}^{(\gamma+1)Q^m} w_{k,m}(t).$$

Using the preceding inequality and summing over γ we get

$$\sum_{\gamma=1}^{Q^{M-m}}\left\|\sum_{k=\gamma Q^m+1}^{(\gamma+1)Q^m} w_{k,m}(t)\right\| \geq \frac{Q^M}{2}.$$

Thus, by the triangle inequality,

$$\sum_{k=Q^m+1}^{N} \|w_{k,m}(t)\| \geq \frac{Q^M}{2}.$$

Finally, let us recall that $k \geq Q^m$ implies $M_k > m$. Thus, after reordering, we obtain

$$\sum_{k=1}^{N}\sum_{m=1}^{M_k} \|w_{k,m}(t)\| \geq \sum_{m=1}^{M}\sum_{k=Q^m+1}^{N} \|w_{k,m}(t)\| \geq M\frac{Q^M}{2}.$$

Gathering the two estimates, we get that if Q is bigger than some constant depending only on a, A, e, c and p^*, we have

$$M\frac{Q^M}{2} \leq (M+1)^{\frac{1}{q^*}} Q^{M+1}.$$

This gives a contradiction for M large enough.

□

We end this section with the proof of Proposition 2.2.3 and Proposition 2.2.4.

Proof of Proposition 2.2.3. Let δ^* be the Orlicz function introduced in the preceding section and which is equivalent to the weak* modulus of asymptotic convexity of $(X^{**}, |.|)$. We will show by induction on L that for all $\xi > 0$, there is a full subtree T of T_N such that for all $1 \leq L \leq N$ and for all choices of block functions $B_1, \ldots, B_L$ as in the statement of the proposition, we have

$$\left|\sum_{l=1}^{L} B_l(t)\right| \geq N_L^{\delta^*}\Big(|B_1(t)|, \ldots, |B_L(t)|\Big) - \xi$$

for all $t \in T$ of length $|t| \geq j_L$, where j_L corresponds to the maximal "height" of the block function B_L. The conclusion will then follow from Lemma 2.1.19 applied to the space $(X^{**}, |.|)$. Note that all our blocks $B_l(t)$ are of norm $|.|$ at most $R = 2Nc$.

For $L = 1$, the property is satisfied on the whole T_N for all choices of $\xi > 0$ because $N_1^{\delta^*} = |.|$ by convention.

Now, suppose that our property it is satisfied for all choices of $\xi > 0$ for a given $1 \leq L \leq N-1$. Fix $\eta > 0$. By the uniform continuity of $N_2^{\delta^*}$, we can find a $\nu > 0$ such that $\left|N_2^{\delta^*}(u) - N_2^{\delta^*}(v)\right| \leq \frac{\eta}{2}$ whenever $\|u - v\|_1 \leq \nu$ in $\mathbb{R}^2$. For our later use, we assume that $\nu \leq \frac{\eta}{2}$. We may assume that the inequalities for L block functions are satisfied on the whole T_N for the constant $\xi = \nu$.

First fix L block functions $B_1, \ldots, B_L$ with $j_L \leq N-1$ and fix $t \in T_N$ with $|t| = j_L$. Assuming that $\sum_{l=1}^{L} B_l(t) \neq 0$, we apply Lemma 2.1.25. There is a weak* neighborhood $V = V_{x_1^*,\ldots,x_m^*;\varepsilon}$ of

0 such that for all $x^{**} \in V \cap RB_{(X^{**},|\cdot|)}$ we have

$$\left|\sum_{l=1}^{L} B_l(t) + x^{**}\right| \geq \left|\sum_{l=1}^{L} B_l(t)\right| \left(1 + \overline{\delta}^*_{X^*}\left(\frac{|x^{**}|}{\left|\sum_{l=1}^{L} B_l(t)\right|}\right)\right) - \frac{\eta}{2}$$
$$= N_2^{\delta^*}\left(\left|\sum_{l=1}^{L} B_l(t)\right|, x^{**}\right) - \frac{\eta}{2}$$

From this, we easily deduce, using the inequality for L block functions, the definition of $N_{L+1}^{\delta^*}$ and our choice of ν that

$$\left|\sum_{l=1}^{L} B_l(t) + x^{**}\right| \geq N_{L+1}^{\delta^*}\Big(|B_1(t)|, \ldots, |B_L(t)|, |x^{**}|\Big) - \eta$$

whenever $x^{**} \in V$ and $|x^{**}| \leq R$.

Our goal now is to extract a full subtree over the sequence t which is fully contained in the weak*-neighborhood V. We know that $w^* - \lim z_{k,j_L+1}(t \frown n) = 0$ for every $N \geq k \geq j_L + 1$. Thus we can find some $N_1 \geq 1$ such that

$$\forall n_1 \geq N_1,\ \forall N \geq k \geq j_L + 1,\ z_{k,j_L+1}(t \frown n_1) \in V_{x_1^*,\ldots,x_m^*;\frac{\varepsilon}{2}}.$$

Then fix some $n_1 \geq N_1$. Again, we know that $w^* - \lim z_{k,j_L+2}(t \frown (n_1, n)) = 0$ for every $N \geq k \geq j_L + 2$. Thus we can find some $N_2(n_1) \geq 1$ such that

$$\forall n_2 \geq N_2(n_1),\ \forall N \geq k \geq j_L + 2,\ z_{k,j_L+2}(t \frown (n_1, n_2)) \in V_{x_1^*,\ldots,x_m^*;\frac{\varepsilon}{4}}.$$

Iterating this procedure, we obtain a full subtree $T^{(t)}$ of T_{N-j_L} such that

$$\forall (n_1, \ldots, n_j) \in T^{(t)},\ \forall N \geq k \geq j_L + j,\ z_{k,j_L+j}(t \frown (n_1, \ldots, n_j)) \in V_{x_1^*,\ldots,x_m^*;\frac{\varepsilon}{2^j}}.$$

Consequently, this subtree satisfies: for all choices of block function B_{L+1}, for all $s \in T^{(t)}$ such that $|t \frown s| \geq j_{L+1}$, $B_{L+1}(t \frown s) \in V$ and thus

$$\left|\sum_{l=1}^{L} B_l(t) + B_{L+1}(t \frown s)\right| \geq N_{L+1}\Big(|B_1(t)|, \ldots, |B_L(t)|, |B_{L+1}(t \frown s)|\Big) - \eta.$$

Note that $B_l(t) = B_l(t \frown s)$ for all $1 \leq l \leq L$ because the block function B_l only depends on the $j_l \leq j_L$ first coordinates of a sequence. Thus, by "gluing" every $T^{(t)}$ over the corresponding point t, we get a full subtree T of T_N satisfying the required property for our initial choice of block functions $B_1, \ldots, B_L$.

Since choosing L block functions is equivalent to choosing integers $0 \leq i_1 \leq j_1 < \cdots < i_L \leq j_L \leq N - 1$ and $N \geq k_i \geq j_i$ a finite number of successive extractions will give us the desired inequality on a full subtree of T_N for every choice of $L + 1$ block functions $B_1, \ldots, B_{L+1}$.

□

Proof of Proposition 2.2.4. Again, we will show by induction on L that for all $\xi > 0$, there is a full subtree T of T_{2N} such that for all $1 \leq L \leq N$ and for all choices of block functions $B_1, \dots, B_L$, we have

$$\left\| \sum_{l=1}^{L} D_l(s,t) \right\| \leq N_L^{\overline{\rho}_X}\Big(\|D_1(s,t)\|, \dots, \|D_L(s,t)\| \Big) + \xi$$

whenever $s, t \in T_N$ are of length $|t| = |s| \geq j_L$ and satisfies $s \propto t = (s_1, t_1, s_2, \dots) \in T$, where $D_l(s,t)$ is defined as the difference $B_l(s) - B_l(t)$. Note that these objects are all of norm at most $R = 4Nc$.

Again, this is clear for $L = 1$. Suppose that the property is satisfied for all choices of ξ for some $1 \leq L \leq N-1$. Fix $\eta > 0$. By the uniform continuity of $N_2^{\overline{\rho}_X}$, we can find a $\nu > 0$ such that $\left| N_2^{\overline{\rho}_X}(u) - N_2^{\overline{\rho}_X}(v) \right| \leq \frac{\eta}{4}$ whenever $\|u - v\|_1 \leq \nu$ in $\mathbb{R}^2$. For our later use, we assume that $\nu \leq \frac{\eta}{4}$. Again, we may assume that the inequalities for L differences of block functions are satisfied for every $s, t \in T_N$ of same length for the constant ν.

At this point we appeal to the quasi-reflexive assumption and to Ramsey's concentration theorem. Since X is quasi-reflexive, there is a space E of finite dimension such that: $X^{**} = X \oplus E$. For all $u \in T_N$, let $z_{k,j}(u) = x_{k,j}(u) + e_{k,j}(u)$ be the associated decomposition in this sum. Also denote by $D_l^{(X)}$ and $D_l^{(E)}$ the projections of the functions D_l respectively on X and on E. By Ramsey's concentration theorem (Theorem 2.1.23), we may assume after passing to a full subtree that for all $1 \leq j \leq k \leq N$ and for all $u, v \in T_N$ of length N we have

$$\|e_{k,j}(u) - e_{k,j}(v)\| \leq \frac{\nu}{N}.$$

Note that this inequality holds in fact whenever $e_{k,j}(u)$ and $e_{k,j}(v)$ are defined since they only depends on the j first coordinates of u and v.

Now fix L block functions $B_1, \dots, B_L$ and fix $w \in T_{2N}$, $|w| = 2j_L$. Also take $s, t \in T_N$ with $|s| = |t| = j_L$ such that $s \propto t = w$.

Again, assuming that $\sum_{l=1}^{L} D_l^{(X)}(s,t) \neq 0$ and applying Lemma 2.1.27 there is a weak* neighborhood $V = V_{x_1^*, \dots, x_m^*; \varepsilon}$ of 0 such that for all $x^{**} \in V \cap RB_{X^{**}}$, we have

$$\begin{aligned}
\left\| \sum_{l=1}^{L} D_l^{(X)}(s,t) + x^{**} \right\| &\leq \left\| \sum_{l=1}^{L} D_l^{(X)}(s,t) \right\| \left(1 + \overline{\rho}_X \left(\frac{\|x^{**}\|}{\left\| \sum_{l=1}^{L} D_l^{(X)}(s,t) \right\|} \right) \right) + \frac{\eta}{4} \\
&= N_2^{\overline{\rho}_X} \left(\left\| \sum_{l=1}^{L} D_l^{(X)}(s,t) \right\|, \|x^{**}\| \right) + \frac{\eta}{4}
\end{aligned}$$

Now, we have $\left\| \sum_{l=1}^{L} D_l^{(E)}(s,t) \right\| \leq \nu$ thanks to the concentration in E obtained before.

Thus

$$\begin{aligned}\left\|\sum_{l=1}^{L} D_l(s,t)+x^{**}\right\| &\leq N_2^{\bar{\rho}_X}\left(\left\|\sum_{l=1}^{L} D_l(s,t)\right\|+\nu, \|x^{**}\|\right)+\nu+\frac{\eta}{4}\\ &\leq N_2^{\bar{\rho}_X}\left(\left\|\sum_{l=1}^{L} D_l(s,t)\right\|, \|x^{**}\|\right)+\frac{3\eta}{4}\\ &\leq N_{L+1}^{\bar{\rho}_X}\Big(\|D_1(s,t)\|,\dots,\|D_L(s,t)\|,\|x^{**}\|\Big)+\eta\end{aligned}$$

whenever $x^{**}\in V$ and $|x^{**}|\leq R$ using the inequality for L difference functions, the definition of $N_{L+1}^{\bar{\rho}_X}$ and the choice of ν.

Using the same arguments as in the proof of the first proposition, we can find a full subtree $T^{(w)}\subset T_{N-j_L}$ such that for all $u=(u_1,\dots,u_j)\in T^{(w)}$ and for all $k\geq j_L+j$, $z_{k,j_L+j}(s\frown u)\in V_{x_1^*,\dots,x_m^*;\frac{\varepsilon}{4J}}$ and $z_{k,j_L+j}(t\frown u)\in V_{x_1^*,\dots,x_m^*;\frac{\varepsilon}{4J}}$.

Thus, for all choices of block function B_{L+1}, for all $u,v\in T^{(w)}$ such that $|s\frown u|=|t\frown v|\geq j_{L+1}$, $D_{L+1}(s\frown u,t\frown v)\in V$ and so

$$\left\|\sum_{l=1}^{L} D_l(s,t)+D_{L+1}(s\frown u,t\frown v)\right\|\leq N_{L+1}^{\bar{\rho}_X}\Big(\|D_1(s,t)\|,\dots,\|D_L(s,t)\|,\|D_{L+1}(s\frown u,t\frown v)\|\Big)+\eta.$$

Noting that the function D_l only depends on the $j_l\leq j_L$ first coordinates of both sequences, and considering the full subtree $T_2^{(w)}$ of $T_{2(N-j_L)}$ for which each sequence of even length is obtained by interlacing two sequences of $T^{(w)}$, we can conclude in the same way as in the preceding proof.

□

2.3 Diamond graphs

We define and study the metric geometry of another important family of metric graphs, the *countably branching diamond graphs*. Informally their construction is done by induction by the following procedure. The first diamond D_1 is the graph consisting in two distinguished vertices, a top and a bottom, which are connected by edges to countably many vertices in-between. Then the diamond D_{N+1} of depth $N+1$ is constructed for any $N\geq 1$ by replacing all the edges of the diamond of depth N by a copy of D_1. The metric on D_N is then define as the usual hyperbolic or graph distance. Note that in order to compute the distance between two vertices in D_N there are always two possible shortest path to consider: one top-bottom and the other bottom-top. We refer to [BCD+, Section 2] for a formal construction, as well as a non-recursive construction giving a precise expression for the distance on D_N.

In [BCD+] it is proved that those graphs equi-Lipschitz embed into any Banach space which contains a family of trees with a strong c_0 behavior, the so called good ℓ_∞-trees. So let us recall the following definition.

Definition 2.3.1. Let X be a Banach space, let $(\sigma_i)_{i\geq 1}$ be a linear compatible ordering of T_n in the sense of Section 1.3, let $p \in [1,\ \infty]$, and let $C, D \geq 1$. A normalized unrooted tree $(x_s)_{s\in T_n}$ in X is called a (C, D) *good ℓ_p-tree of height n* if if satisfies the two following properties:

1. All the branches of the tree are C^2-equivalent to the unit vector basis of ℓ_p^n.
2. The sequence $(x_{\sigma_i})_{i\geq 1}$ is D-basic.

We say that X *contains good ℓ_p-trees of arbitrary height almost isometrically* if for every $n \geq 1$ and for every $\varepsilon > 0$, X contains a $(1+\varepsilon, 1+\varepsilon)$ good ℓ_p-tree of height n.

The following result is [BCD+, Theorem 3.1].

Theorem 2.3.2. *Let X be a Banach space. If X contains good ℓ_∞-trees of arbitrary height almost isometrically, then for every $\varepsilon > 0$ and for every $N \geq 1$ the diamond D_N bi-Lipschitz embeds into X with distortion $6+\varepsilon$. If moreover the good trees are bi-monotone for a well chosen linear compatible ordering, then the distortion can be improved up to $3+\varepsilon$.*

Example 2.3.3. The space $c_0 \equiv c_0(T_\infty)$ contains bi-monotone good $(1,1)$ ℓ_∞-trees of arbitrary height, and as a consequence the diamond D_N bi-Lipschitz embeds into c_0 (and in fact into c_0^+ following the proof of [BCD+]) with distortion 3 for every $N \geq 1$. Note that by doing so one recovers the optimal constant obtained in [Pel] concerning the best possible distortion of arbitrary separable metric spaces into c_0+ and providing a sharp version of the famous result from the paper [Aha] of Aharoni. However, [KL] states that distortion 2 can be attained for embeddings into c_0, and we refer to this paper for more details on the subject.

In [BCD+, Section 3.2] the authors then looked for sufficient conditions ensuring that a Banach space would contain good ℓ_∞-trees of arbitrary height almost isometrically. Based on results coming from the theory of asymptotic structure exposed in Section 1.3 they obtained the following result.

Theorem 2.3.4. *Let X be a separable reflexive Banach space with an unconditional asymptotic structure. If $S_Z(X^*) > \omega$, then X contains good ℓ_∞-trees of arbitrary height almost isometrically.*

Our goal in this section is to extend this result to a a larger setting, namely to duals of - separable Banach spaces with a Szlenk index strictly greater than ω - which admit a weak* unconditional asymptotic structure. Before explaining the ins and outs of those results, let us present the non-embeddability results obtained in [BCD+, Section 4].

Theorem 2.3.5. *Let X be a Banach space with property AMUC. Then the family $(D_N)_{N\geq 1}$ does not equi-Lipschitz embed into X.*

This result heavily relies on considerations about the size of approximate midpoints given by property AMUC, and is again based on a self improvement argument " à la Johnson and Schechtman ". A precise lower estimates on the distortion of diamonds is moreover obtained when the space X has a modulus of asymptotic midpoint convexity with power type. Combined with Theorem 2.3.2, Theorem 2.3.4, and with renorming theorems from Section 1.2 they obtained the following characterization.

Theorem 2.3.6. *Let X be a separable reflexive (infinite dimensional) Banach with an unconditional asymptotic structure. Then the following assertions are equivalent.*

1. *The Szlenk index of X^* is equal to ω.*
2. *The space X admits an equivalent AUC norm.*
3. *The space X admits an equivalent AMUC norm.*
4. *The family $(D_N)_{N\geq 1}$ does not equi-Lipschitz embed into X.*

Note that the separability assumption can be removed in this result by passing to a separable quotient using Proposition 1.2.27.

Remark 2.3.7. It is also known that D_N bi-Lipschitz embeds into the space L_1 with distortion 2 for every $N \geq 1$ and that this distortion is optimal. We refer to [BCD+, Section 3.3] for comments on this point and for a constructive approach involving Bernouilli random variables. Let us also point out that a transfer result from dyadic diamond in a space X to countably branching diamonds in $L_p(X)$ is obtained in [BCD+, Section 3.4] using similar tools, and is then applied to obtain a new renoming result for $L_p(X)$ in [BCD+, Corollary 5.5].

Actually a more general result in obtained in [BCD+, Section 4] since it is proved that no family generated by a (non trivial) countably branching bundle graph can equi-Lipschitz embed into a Banach space with property AMUC. Let us recall the following definition.

Definition 2.3.8. A *top-bottom graph* is a graph with two distinguished vertices, one designated as the top and the other as the bottom. A *countably branching bundle graph* is a top-bottom graph which can be formed, starting by a path of length 1, by a finite sequence of the following operations.

1. Parallel composition: given two countably branching bundle graphs, identify the top of one with the bottom of the other and let the bottom of the first (respectively top of the second) be the bottom (respectively the top) of the new graph.
2. Series composition: take countably many copies of a countably branching bundle graph and identify all the bottoms (respectively all the tops) with each other.

A *non-trivial* countably branching bundle graph is a bundle graph obtained by such a sequence with at least one series composition. Every bundle graph is endowed with its graph distance.

We refer to [Swi, Section 2] for a non recursive definition of bundle graphs and for explicit formula for distances in them. We also refer to [Swi, Section 6] for a formal definition of the operation $\oslash$ which consists in replacing every edge of some countably branching bundle graph by another countably branching bundle graph. As for the diamond graphs we define inductively for every (non trivial) countably branching bundle graph G the family $(G^{\oslash N})_{N\geq 1}$ of *countably branching bundle graphs generated by* G by $G^{\oslash 1} = G$ and $G^{\oslash(N+1)} = G \oslash G^{\oslash N}$ for every $N \geq 1$.

The result from [BCD+, Theorem 4.1] is the following.

Theorem 2.3.9. *Let G be a non-trivial countably branching bundle graph and let X be a Banach space with property AMUC. Then the family $(G^{\oslash N})_{N\geq 1}$ does not equi-Lipschitz embed into X.*

In [Swi], Swift generalized all the above mentioned results from [BCD+] to countably branching bundle graphs. We refer to his paper for precise statements.

We end the section by proving the following extension of Theorem 2.3.4

Proposition 2.3.10. *Let X be a separable Banach space with $S_Z(X) > \omega$ and let us assume that X^* has a weak* unconditional asymptotic structure. Then X^* contains good ℓ_∞-trees of arbitrary height almost isometrically.*

Combined with the results from [Swi] and the renorming results from Section 1.2 this yields the following.

Theorem 2.3.11. *Let X be a separable Banach space and let us assume that X^* has a weak* unconditional asymptotic structure. Also let G be any non-trivial countably branching bundle graph. Then the following assertions are equivalent.*

1. *The space X satisfies $S_Z(X) \leq \omega$.*
2. *The space X admits an equivalent norm whose dual norm is AUC*.*
3. *The space X^* admits an equivalent AMUC norm.*
4. *The family $(G^{\oslash N})_{N\geq 1}$ does not equi-Lipschitz embed into X^*.*

Since Mazur's work it is a well known fact that it is possible to extract a basic subsequence from every weakly null normalized sequence. A simple extension of Mazur's proof using the concept of linear compatible ordering allows to prove the following tree version of this result.

Lemma 2.3.12. *Let X be a Banach space and let $(x_s)_{s\in T_n}$ be a weakly null normalized tree in X. For all $\delta \in (0,1)$ there is a full subtree T of T_n and a compatible linear ordering $(\sigma_i)_{i\geq 1}$ of T such that the sequence $(x_{\sigma_i})_{i\geq 1}$ is $(1+\delta)$-basic.*

The same goes for weak* null trees in a dual space and in this case it is even possible to do a bit better by extracting a weak* basic full subtree in the sense of [JR] . The following is thus a direct consequence of Lemma 1.3.10.

Lemma 2.3.13. *Let X be a separable Banach space, let $p \in [1,\infty]$, and let us assume that $\ell_p^n \in \{X^*\}_n^{w^*}$ for every $n \geq 1$. Then X^* contains good ℓ_p-trees of arbitrary height almost isometrically.*

Thus as a consequence of Theorem 1.3.15 we obtain.

Proposition 2.3.14. *Let X be a separable Banach space satisfying $S_Z(X) > \omega$, and let us assume that X^* has a weak* unconditional asymptotic structure. Then X^* contains good ℓ_∞-trees of arbitrary height almost isometrically.*

Also note that by Theorem 1.3.16 we also have.

Proposition 2.3.15. *Let X be a Banach space with separable dual, with an unconditional asymptotic structure and with $S_Z(X) > \omega$. Then X contains good ℓ_1-trees of arbitrary height almost isometrically.*

Finally note that property $(*)$ defines in Definition 1.4.9 is actually sufficient thanks to Theorem 1.4.17 to obtain the conclusions from Proposition 2.3.14, and by the results from Swift we obtain the following result.

Proposition 2.3.16. *Let X be a separable Banach space and let us assume that X^* has property $(*)$. Then every non-trivial countably branching bundle graph embeds bi-Lipschitz into X^* with distortion less than $6 + \varepsilon$.*

2.4 Spreading models

In this section we briefly introduce the important notion of spreading models of a Banach space and gather a few results concerning the embeddability of the family $(T_N)_{N\geq 1}$ into a Banach space in the presence of an ℓ_1 or a c_0 spreading model.

Definition 2.4.1. Let X be a Banach space. By using Ramsey's theorem, one can show that for every bounded sequence $(x_n)_{n\geq 1} \subset X$ there a subsequence $(y_n)_{n\geq 1}$ such that for all $k \geq 1$ and for all $a_1, \dots, a_k \in \mathbb{R}$ the limit $\lim_{n_1<\cdots<n_k} \left\|\sum_{i=1}^k a_i y_{n_i}\right\|$ exists. Let $(e_i)_{i\geq 1}$ be the unit vector basis of c_{00}. If the sequence $(y_n)_{n\geq 1}$ is not convergent the quantity

$$\left\|\sum_{i=1}^k a_i e_i\right\| = \lim_{n_1<\cdots<n_k} \left\|\sum_{i=1}^k a_i y_{n_i}\right\|$$

defines a norm on c_{00}. The completion of the space $(c_{00}, \|.\|)$ is called *spreading model* associated with the fundamental sequence $(e_i)_{i\geq 1}$ and *generated* by the sequence $(y_n)_{n\geq 1}$. Note that the fundamental sequence is *spreading* in the sense that for all $k \geq 1$, for all $a_1, \dots, a_k \in \mathbb{R}$ and for all $1 \leq n_1 < \cdots < n_k$ we have the norm equality $\left\|\sum_{i=1}^k a_i e_i\right\| = \left\|\sum_{i=1}^k a_i e_{n_i}\right\|$. For a Banach space E we shall say that X has an E *spreading model* if X has a spreading model isomorphic to E.

We refer to [BL1] for a detailed presentation of the theory of spreading models. In particular [BL1, Chapter 4] presents plenty of examples of classifications up to isomorphisms of the spreading models of classical Banach spaces. In view of the construction of spreading models, the following result directly follows from consideration on Orlicz spaces as in the end of Section 2.1 and is a sharp version of Proposition 1.1.20.

Theorem 2.4.2. *Let X be a Banach space. There is a constant $C > 0$ such that for every spreading model E of X generated by a weakly null sequence and for every $n \geq 1$ we have*

$$\frac{1}{C}\left\|\sum_{i=1}^{n} e_i\right\|_{\ell_\delta} \leq \left\|\sum_{i=1}^{n} e_i\right\|_{E} \leq C\left\|\sum_{i=1}^{n} e_i\right\|_{\ell_{\overline{\rho}_X}}$$

where δ is the Orlicz function considered at the end of Section 2.1 and which is equivalent to the modulus of asymptotic convexity $\overline{\delta}_X$.

Remark 2.4.3. Let us point out that striking analogues of this result where obtained in the non-linear theory in [KR] and [Kal2] for coarse-Lipschitz embeddings, in particular see [Kal2, Theorem 7.4 and following remark]. Those are particularly interesting in the theory since they provide easy obstructions to the coarse-Lipschitz embeddability of some Banach spaces into others depending on the behavior of their respective spreading models.

We will now focus on ℓ_1 and c_0 spreading models. A direct consequence of Theorem 2.4.2 (eventually combined with some considerations on the classification of Orlicz sequence spaces) is that any AUS Banach space (respectively any AUC Banach space) fails to admit an ℓ_1 (respectively a c_0) spreading model generated by a weakly null sequence. We will explain shortly after how to get rid of this assumption. For now let us state the following two results.

Proposition 2.4.4. *If X has an ℓ_1 spreading model then for every $\varepsilon > 0$ there is a normalized sequence $(x_k)_{k\geq 1}$ in X such that for every $k \geq 1$, for every sequence $a = (a_i)_{i=1}^k$ in $\mathbb{R}^k$ and for every integers $n_1, \ldots, n_k$ satisfying $k \leq n_1 < \cdots < n_k$ we have*

$$\frac{1}{1+\varepsilon}\|a\|_1 \leq \left\|\sum_{i=1}^{n} a_i x_{n_i}\right\| \leq (1+\varepsilon)\|a\|_1 .$$

Proposition 2.4.5. *If X has a c_0 spreading model then for every $\varepsilon > 0$ there is a normalized sequence $(x_k)_{k\geq 1}$ in X such that for every $k \geq 1$, for every sequence $a = (a_i)_{i=1}^k$ in $\mathbb{R}^k$ and for every integers $n_1, \ldots, n_k$ satisfying $k \leq n_1 < \cdots < n_k$ we have*

$$\frac{1}{1+\varepsilon}\|a\|_\infty \leq \left\|\sum_{i=1}^{n} a_i x_{n_i}\right\| \leq (1+\varepsilon)\|a\|_\infty .$$

Remark 2.4.6. Observe that the sequences $(x_k)_{k\geq 1}$ obtained in Proposition 2.4.5 have to be weakly null. Indeed if we assume that for a fixed $\varepsilon > 0$ the corresponding sequence $(x_k)_{k\geq 1}$ fails to be weakly null, then up to the extraction of a subsequence we can assume that there is a norm one functional $f \in X^*$ and a constant $\delta > 0$ such that for every $k \geq 1$ we have $f(x_k) \geq \delta$. But then $n\delta \leq f\left(\sum_{i=1}^n x_{k+i-1}\right) \leq \|\sum_{i=1}^n x_{k+i-1}\| \leq (1+\varepsilon)$ for every $n \geq 1$. A contradiction. In particular any AUC Banach space fails to admit a c_0 spreading model. Also note that by [Kal2, Proposition 4.5] we actually have that any AMUC Banach space fails to admit a c_0 spreading model.

Using Proposition 2.4.5 it is proved in [BLS, Proposition 3.4] by a direct construction that the family $(T_N)_{N\geq 1}$ equi-Lipschitz embeds into any Banach space X which admits a c_0 spreading model. A similar construction can easily be done for ℓ_1 spreading models using Proposition 2.4.4 but we can actually prove stronger statements.

Theorem 2.4.7. *Let X be a Banach space. If X has an ℓ_1 spreading model, then $S_Z(X) > \omega$.*

Proof. Let us assume that $S_Z(X) \leq \omega$ and that X admits an ℓ_1 spreading model E generated by some sequence $(x_k)_{k\geq 1}$. By Proposition 2.4.4 we may assume that the fundamental sequence $(e_i)_{i\geq 1}$ of E is actually equivalent to the unit vector basis of ℓ_1. Also X fails to contain ℓ_1 since $S_Z(X) \leq \omega$ by assumption, and we may assume by Rosenthal's ℓ_1 theorem that the sequence $(x_k)_{k\geq 1}$ is weakly Cauchy. Thus the sequence $(x_{2k} - x_{2k-1})_{k\geq 1}$ is weakly null, and it generates a spreading model F which is also isomorphic to ℓ_1. Indeed if we denote by $(f_j)_{j\geq 1}$ the fundamental sequence of F we have

$$\left\| \sum_{j=1}^{m} a_j f_j \right\|_F = \left\| \sum_{i=1}^{2m} (-1)^{i+1} a_{2i-1} e_{2i-1} \right\|_E$$

for every sequence $(a_j)_{j=1}^m$ in $\mathbb{R}^m$ so that $(f_j)_{j\geq 1}$ is also equivalent to the unit vector basis of ℓ_1. Now by the renorming theorems from Section 1.2 we may also assume that X is AUS, and this contradicts Theorem 2.4.2.

□

In [BL1, Chapter 3 Section 4] a duality result between ℓ_1 and c_0 spreading models is presented. In particular it is shown that if a quotient of a space X has a c_0 spreading model, then the dual space X^* has an ℓ_1 spreading model. The following result is thus a direct consequence of Theorem 2.4.7

Corollary 2.4.8. *Let X be a Banach space. If X has a c_0 spreading model, then $S_Z(X^*) > \omega$.*

In particular the following is then a direct consequence of Theorem 2.1.9.

Theorem 2.4.9. *Let X be a Banach space. If X has an ℓ_1 spreading model or if X has a c_0 spreading model, then T_∞ bi-Lipschitz embeds into X and into X^*.*

Remark 2.4.10. The space $E = \left(\sum_{n\geq 1} \ell_{p_n}\right)_{\ell_2}$ where $(p_n)_{n\geq 1}$ is a sequence of real numbers strictly decreasing towards 1 already encountered in Remark 2.1.15 and its dual provide easy examples showing that the converse of any of the above results fails even in the reflexive setting. Indeed, using a gliding hump argument, it is not difficult to prove that the property of admitting no ℓ_1 spreading model is stable by taking ℓ_2 sums. Thus the space E does not admit any ℓ_1 spreading model. Since it satisfies $S_Z(E^*) = \omega$ it also fails to admit c_0 spreading models. Yet $S_Z(E) = \omega^2$ so that T_∞ does bi-Lipschitz embed into E.

Actually we can slightly improve Corollary 2.4.8.

Proposition 2.4.11. *Let X be a Banach space. If X has a c_0 spreading model, then $\ell_\infty^n \in \{X\}_n$ for all $n \geq 1$.*

Proof. Fix $n \geq 1$ and $\varepsilon > 0$, and take a normalized (weakly null) sequence $(x_k)_{k\geq 1}$ as in Proposition 2.4.5. Then for every $s = (s_1, \ldots, s_k) \in T_n$ non empty, let $y_s = x_{\phi(s)}$ where $\phi(s) = n + \sum_{i=1}^k s_i$. Then $(x_s)_{s\in T_n}$ is a weakly null normalized unrooted tree and since $n \leq \phi(s_{|1}) < \cdots < \phi(s)$ for every leaf s of T_n, we have that every branch of this unrooted tree is $(1+\varepsilon)^2$ equivalent to the unit vector basis of ℓ_∞^n. The conclusion follows from Lemma 1.3.1. □

Since we are working directly with trees in the preceding Proposition, we actually have that any Banach space admitting a c_0 spreading model contains good ℓ_∞-trees of arbitrary height almost isometrically, and thus by the results from Section 2.3 we obtain.

Proposition 2.4.12. *Let X be a Banach space. If X has a c_0 spreading model, then for every $\varepsilon > 0$ every non-trivial countably branching bundle graph embeds into X with distortion $6 + \varepsilon$.*

Chapter 3

Daugavet and Δ-points in Banach spaces

3.1 Daugavet- and Delta-points

Daugavet- and Δ-points first appeared in [AHLP] as natural pointwise versions of geometric characterizations of the Daugavet property [Shv, Lemma 2] and of the so called spaces with bad projections [IK, Theorem 1.4] (also known as spaces with the diametral local diameter two property (DLD2P) [BGLPRZ]).

Definition 3.1.1. Let X be a Banach space, let $x \in S_X$, and let $\varepsilon > 0$. We introduce

$$\Delta_\varepsilon^X(x) := \{y \in B_X : \ \|x - y\| \geq 2 - \varepsilon\},$$

and we will simply write $\Delta_\varepsilon(x) := \Delta_\varepsilon^X(x)$ when there is no risk of confusion.

1. We say that x is a $(2-\varepsilon)$ *Daugavet-point* if $B_X = \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$, and we say that x is a *Daugavet-point* if it is a $(2-\varepsilon)$ Daugavet-point for every $\varepsilon > 0$.

2. We say that x is a $(2-\varepsilon)$ Δ-*point* if $x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$, and we say that x is a Δ-*point* if it is a $(2-\varepsilon)$ Δ-point for every $\varepsilon > 0$.

We recall that for any $x^* \in S_{X^*}$ and $\delta > 0$ we define a slice of B_X by

$$S(x^*, \delta) := \{y \in B_X : x^*(y) > 1 - \delta\}.$$

The corresponding closed slice is denoted

$$\overline{S}(x^*, \delta) := \{y \in B_X : x^*(y) \geq 1 - \delta\}.$$

Remark 3.1.2. Slices and closed slices of B_X can also be defined using non-zero functionals in X^* by replacing the 1 above by the norm of the corresponding functional. Now if x^* is a non-zero functional we have $S(x^*, \delta) = S\left(\frac{x^*}{\|x^*\|}, \frac{\delta}{\|x^*\|}\right)$ and we would like to point out that working

with non-normalized functionals can cause delicate computational problems (see e.g. the proof of Theorem 3.2.6). We will avoid them as much as possible. We will also informally say that a Banach space has *thin slices of arbitrary large diameter* when the diameter of $S(x^*, \delta)$ is as close as we want to 2 for small δ.

For every $x \in X$, let us write $D(x) := \{x^* \in S_{X^*} : \ x^*(x) = \|x\|\}$. We state the following lemma here for future reference.

Lemma 3.1.3. *Let $x \in S_X$. For every $n \geq 1$, $\delta > 0$, and $x_1^*, \ldots, x_n^* \in D(x)$, we have:*

$$S\left(\frac{1}{n}\sum_{i=1}^{n} x_i^*, \frac{\delta}{n}\right) \subset \bigcap_{i=1}^{n} S(x_i^*, \delta).$$

Proof. Let $x^* = \frac{1}{n}\sum_{i=1}^{n} x_i^*$. Since every x_i^* is in $D(x)$, we immediately obtain that x^* is in $D(x)$ and in particular that $\|x^*\| = 1$. Now if $y \in S\left(x^*, \frac{\delta}{n}\right)$, we have, for every $1 \leq i_0 \leq n$,

$$x_{i_0}^*(y) = nx^*(y) - \sum_{\substack{i=1 \\ i \neq i_0}}^{n} x_i^*(y) > n - \frac{n\delta}{n} - (n-1) = 1 - \delta,$$

and the conclusion follows. □

Note that the fact that every functional x_i^* attains its norm at x is crucial here to guarantee that the average of the x_i^* remains a norm one functional.

Using Hahn–Banach separation we get the following well known lemma that we will use without reference.

Lemma 3.1.4. *Let X be a Banach space, let $x \in S_X$, and let $\varepsilon > 0$.*

1. *The point x is a $(2-\varepsilon)$ Daugavet-point if and only if for all $\delta > 0$ and all $x^* \in S_{X^*}$, there exists $y \in S(x^*, \delta)$ such that $\|x - y\| \geq 2 - \varepsilon$.*

2. *The point x is a $(2-\varepsilon)$ Δ-point if and only if for all $\delta > 0$ and all $x^* \in S_{X^*}$ such that $x \in S(x^*, \delta)$, there exists $y \in S(x^*, \delta)$ such that $\|x - y\| \geq 2 - \varepsilon$.*

It is also well known that this result can be strenghened in the following way, see for example [JRZ, Lemma 2.2].

Lemma 3.1.5. *Let X be a Banach space and let $x \in S_X$.*

1. *If x is a Daugavet-point, then for all $\varepsilon > 0$, and for all $\delta > 0$ and all $x^* \in S_{X^*}$, there exists $\eta > 0$ and $y^* \in S_{X^*}$ such that $S(y^*, \eta) \subset S(x^*, \delta)$ and $\|x - y\| \geq 2 - \varepsilon$ for every $y \in S(y^*, \eta)$.*

2. *If x is a Δ-point, then for all $\varepsilon > 0$, and for all $\delta > 0$ and all $x^* \in S_{X^*}$ such that $x \in S(x^*, \delta)$, there exists $\eta > 0$ and $y^* \in S_{X^*}$ such that $S(y^*, \eta) \subset S(x^*, \delta)$ and $\|x - y\| \geq 2-\varepsilon$ for every $y \in S(y^*, \eta)$.*

As observed in [JRZ, Remark 2.4] this implies the following.

Corollary 3.1.6. *Let X be a Banach space, let $x \in S_X$, and let A be a subset of B_X such that $\overline{\operatorname{conv}} A = B_X$.*

1. *The point x is a Daugavet-point if and only if for all $\varepsilon > 0$, and for all $\delta > 0$ and all $x^* \in S_{X^*}$, there exists $y \in S(x^*, \delta) \cap A$ such that $\|x - y\| \geq 2 - \varepsilon$.*

2. *The point x is a Δ-point if and only if for all $\varepsilon > 0$, and for all $\delta > 0$ and all $x^* \in S_{X^*}$ such that $x \in S(x^*, \delta)$, there exists $y \in S(x^*, \delta) \cap A$ such that $\|x - y\| \geq 2 - \varepsilon$.*

In [AHLP] it is proved that Daugavet- and Δ-points coincide for $L_1(\mu)$ spaces and $C(K)$ spaces (as well as in some Müntz spaces) and precise characterizations are given for those points in relation to the support of the the considered function. However, it is proved there that the two notions behave quite differently with respect to sums of Banach spaces. In particular, ℓ_p sums of Banach spaces never contain Daugavet-points for $1 < p < \infty$ while $C[0,1] \oplus_{\ell_2} C[0,1]$ has the DLD2P and every point of its unit sphere is a Δ-point. This work was then pursued in [HPV], and much more subtle behavior where obtained for Daugavet-points in comparison with the Daugavet property which is known to be persevered only by ℓ_∞ and ℓ_1 sums among all absolute sums. We also refer to [Pir, Section 3.3] for an overview of the question. In [ALMT, ALM] other examples of Δ-points which are not Daugavet-points where provided and [JRZ] showed that examples of this sort are very natural and easy to construct in the context of Lipchitz free spaces.

We also introduce a weak* version of Daugavet- and Δ-points in dual spaces.

Definition 3.1.7. Let X be a Banach space, let $x^* \in S_{X^*}$, and let $\varepsilon > 0$.

1. We say that x^* is a $(2-\varepsilon)$ *weak* Daugavet-point* if $B_X = \overline{\operatorname{conv}}^{w^*} \Delta_\varepsilon(x^*)$. We say that x^* is a *weak* Daugavet-point* if it is a $(2-\varepsilon)$ weak* Daugavet-point for every $\varepsilon > 0$.

2. We say that x^* is a $(2-\varepsilon)$ Δ*-point* if $x^* \in \overline{\operatorname{conv}}^{w^*} \Delta_\varepsilon(x^*)$. We say that x is a *weak** Δ*-point* if it is a $(2-\varepsilon)$ weak* Δ-point for every $\varepsilon > 0$.

For any set $A \in X^*$, the norm closure $\overline{A}^{\|\cdot\|}$ of A is contained in the weak* closure $\overline{A}^{w^*}$ of A (in short $\overline{A}^{\|\cdot\|} \subset \overline{A}^{w^*}$), so that any $(2-\varepsilon)$ Daugavet-point (respectively $(2-\varepsilon)$ Δ-point) is a $(2-\varepsilon)$ weak* Daugavet-point (respectively a $(2-\varepsilon)$ weak* Δ-point). The converse is known to be false. Indeed it is pointed out in [Wer] that the Daugavet property does not pass from X to X^* (although it is equivalent to every point in S_{X^*} being a weak* Daugavet-point) since $C[0,1]^*$ fails the Daugavet property. As a consequence this space must contain points which are weak* Daugavet-points but not Daugavet-points.

We recall that for any $x \in S_X$ and $\delta > 0$ we define a weak* slice of B_{X^*} by

$$S(x, \delta) := \{y^* \in B_{X^*} : y^*(x) > 1 - \delta\}.$$

The corresponding closed slice is denoted

$$\overline{S}(x, \delta) := \{y^* \in B_{X^*} : y^*(x) \geq 1 - \delta\}.$$

Using the Hahn-Banach separation theorem for the weak* topology (see for example [FHH+, Corollary 3.34]), we obtain a weak* analogue of Lemma 3.1.4, Lemma 3.1.5 and Corollary 3.1.6 with weak* slices instead of slices and a subset A of X^* satisfying $\overline{\text{conv}}^{w^*} A = B_{X^*}$.

Note that if Y is a subspace of X, then for every $x \in S_Y$ we have $\Delta_\varepsilon^Y(x) \subset \Delta_\varepsilon^X(x)$ so that if x is a $(2-\varepsilon)$ Δ-point in Y, then x is also a $(2-\varepsilon)$ Δ-point in X. The same goes for $(2-\varepsilon)$ weak* Δ-points if $x^* \in S_E$ where E is a weak* closed subset of X^*.

Remark 3.1.8. Note that this fails for Daugavet-points since the ℓ_2 sum of any two Banach spaces does not contain such point.

In particular, if $x \in S_X$ is a $(2-\varepsilon)$ Δ-point, then x is a $(2-\varepsilon)$ Δ-point in X^{**}. By Goldstine's theorem we have the following.

Lemma 3.1.9. *Let X be a Banach space, let $x \in S_X$, and let $\varepsilon > 0$. If x is a $(2-\varepsilon)$ Daugavet-point, then x is a $(2-\varepsilon)$ weak* Daugavet-point in X^{**}.*

Proof. Let us assume that $x \in S_X$ is a Daugavet-point. Then

$$B_X = \overline{\text{conv}}\, \Delta_\varepsilon^X(x) \subset \overline{\text{conv}}^{\|\cdot\|}\, \Delta_\varepsilon^{X^{**}}(x) \subset \overline{\text{conv}}^{w^*}\, \Delta_\varepsilon^{X^{**}}(x).$$

As a consequence $B_{X^{**}} = \overline{B_X}^{w^*} \subset \overline{\text{conv}}^{w^*}\, \Delta_\varepsilon^{X^{**}}(x)$ and x is a weak* $(2-\varepsilon)$ Daugavet-point in X^{**}.

□

The following result can be extracted from the proof of [AHLP, Lemma 3.5]. It relies on the principle of local reflexivity, and was used there to show that Daugavet- and Δ-points also coincide in $L_1(\mu)$ preduals.

Lemma 3.1.10. *Let X be a Banach space, let $x, y \in S_X$, and let $\varepsilon > 0$. If $y \in \overline{\text{conv}}\, \Delta_\varepsilon^{X^{**}}(x)$, then $y \in \overline{\text{conv}}\, \Delta_{\varepsilon'}^X(x)$ for every $\varepsilon' > \varepsilon$.*

As a consequence an element $x \in S_X$ is a Δ point in X if and only if it is a Δ-point in X^{**}, and it is a Daugavet point in X if and only if $B_X \subset \overline{\text{conv}}\, \Delta_\varepsilon^{X^{**}}$ for every $\varepsilon > 0$.

We recall that a point $x \in B_X$ is *denting* if it is contained in slices of B_X of arbitrarily small diameter. A point $x^* \in B_{X^*}$ is called *weak* denting* if it is contained in weak* slices of B_{X^*} of arbitrarily small diameter. It was observed in [JRZ, Proposition 3.1] any Daugavet-point in a space X has to be at distance 2 from every denting-point of B_X. In the same paper it is proved that the converse does hold true for Lipschitz free spaces over compact metric spaces, and this was extended in [Vee] to general metric spaces. Let us point out that this does also hold true in RNP spaces.

Lemma 3.1.11. *Let X be a Banach space with the Radon-Nikodým property. Then a point $x \in B_X$ is a Daugavet-point if and only if it is at distance 2 from every denting-point in B_X.*

Proof. It is a well known fact that every closed bounded non empty subset in an RNP space is equal to the closure of the convex hull of its denting points. We refer to the monograph [DU] for considerations about the Radon-Nikodým property, and in particular to [DU, Section VII.6.] for a summary of the equivalent reformulations of this property.

So if X has the RNP, then any slice of B_X contains a denting-point of B_X, and it follows immediately that any point which is at distance two from every denting-point in B_X has to be a Daugavet-point.

□

In particular observe that if $x \in S_X$ is a Daugavet-point and if X has the RNP, then for every $\delta > 0$ and every $x^* \in S_{X^*}$, there is a $y \in S(x^*, \delta)$ such that $\|x - y\| = 2$. The weak* analogue of this result follows from Theorem 1.2.4.

In view of the previous observations, the example from [Vee] of a Lipchitz free space satisfying the RNP and admitting a Daugavet-point is particularly surprising. In the same lines let us point out that there exists a Banach space with a 1-unconditional basis such that the set of Daugavet-points are weakly dense in the unit ball [ALMT, Theorem 4.7].

A Banach space X is said to be *uniformly non-square* if there exists $\varepsilon > 0$ such that for all $x, y \in B_X$ we have either $\|\frac{1}{2}(x + y)\| \leq 1 - \varepsilon$ or $\|\frac{1}{2}(x - y)\| \leq 1 - \varepsilon$. Uniformly non-square spaces were introduced by James in [Jam]. It is well known that uniformly non-square spaces are super-reflexive and that uniformly convex and uniformly smooth spaces are uniformly non-square (see e.g. [KMT, Proposition 1]). The following simple observations characterizes uniformly non-square spaces in terms of slices.

Proposition 3.1.12. *Let X be a Banach space. The following assertions are equivalent.*

1. *The space X is uniformly non-square.*
2. *There exists $\varepsilon > 0$ such that for all $x^* \in S_{X^*}$ the diameter of $S(x^*, \varepsilon)$ is smaller than $2 - 2\varepsilon$.*

Proof. (1) $\implies$ (2) Let $\varepsilon > 0$ be such that for all $x, y \in B_X$ we have either $\|\frac{1}{2}(x + y)\| \leq 1 - \varepsilon$ or $\|\frac{1}{2}(x - y)\| \leq 1 - \varepsilon$. If $x^* \in S_{X^*}$ and $x, y \in S(x^*, \varepsilon)$, then $\|x + y\| \geq x^*(x + y) \geq 2 - 2\varepsilon$, so that $\|x - y\| \leq 2 - 2\varepsilon$.

(2) $\implies$ (1) If X is not uniformly non-square, then for every $\varepsilon > 0$ there exists $x, y \in B_X$ such that $\|x+y\| > 2-\varepsilon$ and $\|x-y\| > 2-\varepsilon$. Take $x^* \in S_{X^*}$ such that $x^*(x+y) > 2-\varepsilon$. Then $x, y \in S(x^*, \varepsilon)$ and $\|x-y\| > 2-\varepsilon$. $\square$

If x is a Δ-point then for all $\delta > 0$ and $x^* \in S_{X^*}$ with $x^*(x) > 1-\delta$ we have diam $S(x^*, \delta) = 2$ so we immediately get the following.

Corollary 3.1.13. *Let X be a uniformly non-square Banach space. Then X does not admit Δ-points.*

If a Banach space X is *not* uniformly non-square then for every $\varepsilon > 0$ there is a slice $S(x^*, \varepsilon)$, $x^* \in S_{X^*}$, with diameter strictly greater than $2-2\varepsilon$, so X admits thin slices of arbitrarily large diameter. Observe that if X is a dual space, then we may assume that the functional x^* in the proof of (2) $\implies$ (1) in Proposition 3.1.12 is weak* continuous. Hence a dual space which is not uniformly non-square has a unit ball with small weak* slices of arbitrarily large diameter. These simple observations show that for spaces that are not uniformly non-square there is no hope of ruling out Δ-points by some upper bound on the diameter of slices of the unit ball. Note that any Banach space X of dimension greater or equal to 2 has an equivalent norm $|\cdot|$ such that $(X, |\cdot|)$ is *not* uniformly non-square [KMT, Corollary 1]. In particular, Corollary 3.1.13 does not rule out Δ-points in super-reflexive or finite dimensional spaces. In particular it is natural to ask the following.

Question 3.1.14. Do all finite dimensional spaces fail to contain Δ-points?

Question 3.1.15. Do all super-reflexive Banach spaces fail to contain Δ-points?

We will provide a positive answer to Question 3.1.14 in Section 3.3, but we will first end the present section by an attempt to simplify Question 3.1.15. As we have seen, even super-reflexive spaces whose norm is not uniformly non-square have thin slices of the unit ball with diameter arbitrary close to 2. However, for a super-reflexive space X we know that the dual of an ultrapower $X^{\mathcal{U}}$ is isometric to $(X^*)^{\mathcal{U}}$, and this opens the door to using ultrafilter limits. Since $X^{\mathcal{U}}$ is also super-reflexive we do not leave the context of super-reflexive spaces when passing to ultrapowers. Motivated by this we introduce the following definitions which are a weakening of the notions of Daugavet- and Δ-points.

Definition 3.1.16. We say that X *admits almost Daugavet-points* if it admits a $(2-\varepsilon)$ Daugavet-point for every $\varepsilon > 0$, and we say that X *admits almost Δ-points* if it admits a $(2-\varepsilon)$ Δ-point for every $\varepsilon > 0$.

For our later purpose let us say that a Banach space X *contains ℓ_p^n's uniformly* for some $1 \le p \le \infty$ if for every $\varepsilon > 0$ and every $n \ge 1$ there exist $x_1, \dots, x_n \in B_X$ such that for every sequence $a = (a_i)_{i=1}^n$ in $\mathbb{R}^n$ we have

$$(1+\varepsilon)^{-1}\|a\|_p \le \left\| \sum_{i=1}^n a_i x_i \right\| \le \|a\|_p.$$

We will start by studying the existence of almost Daugavet- and Δ-points in some classical spaces. We highlight the following observation, which although obvious from the definition will provide an easy way of constructing almost Δ-points.

Observation 3.1.17. Let $\varepsilon > 0$. If $x \in \operatorname{conv} \Delta_\varepsilon(x)$, then x is a $(2-\varepsilon)$ Δ-point.

Building on this observation, we can prove the two following lemmas.

Lemma 3.1.18. *If a Banach space X contains ℓ_1^n's uniformly, then X admits almost Δ-points.*

Proof. Take $\varepsilon > 0$ and $n \geq 1$, and find $x_1, \dots, x_n \in B_X$ from the definition of containing ℓ_1^n's uniformly. Let $x = \frac{1}{n}\sum_{i=1}^n x_i$. Then $\|x\| \leq 1$ and for each $1 \leq j \leq n$ we have

$$\|x - x_j\| = \left\| \sum_{i \neq j} \frac{1}{n} x_i + \left(\frac{1}{n} - 1\right) x_j \right\| \geq (1+\varepsilon)^{-1} \left(2 - \frac{2}{n}\right)$$

which can be made as close to 2 as we like. The conclusion follows from Observation 3.1.17. □

Lemma 3.1.19. *If a Banach space X contains ℓ_∞^n's uniformly, then X admits almost Δ-points.*

Proof. Since $\ell_1^n \subset \ell_\infty^{2n}$ for every $n \geq 1$, the conclusion immediately follows from Lemma 3.1.18. Still let us propose a constructive argument for that.

Take $\varepsilon > 0$ and $n \geq 1$, and find $x_1, \dots, x_n \in B_X$ from the definition of containing ℓ_∞^n's uniformly. For every $1 \leq i \leq n$ we define

$$y_i = -2x_i + \sum_{k=1}^n x_k.$$

Then we let $x = \frac{1}{n}\sum_{i=1}^n y_i$, that is

$$x = \left(1 - \frac{2}{n}\right) \sum_{k=1}^n x_k.$$

Then $\|x\| \leq 1$ and for every $1 \leq j \leq n$ we have

$$\|y_j - x\| = \left\| \sum_{k \neq j} \frac{2}{n} x_k + \left(\frac{2}{n} - 2\right) x_j \right\| \geq (1+\varepsilon)^{-1} \left(2 - \frac{2}{n}\right)$$

which can be made as close to 2 as we like. Again, the conclusion follows from Observation 3.1.17. □

The two lemmas above can be formulated as follows: if X does not have finite co-type or does not have non-trivial type, then X admits almost Δ-points. In particular the spaces c_0 and ℓ_1 both admits almost Δ-points. Let us recall that they do not admit Δ-points (by e.g. [ALMT, Theorem 2.17]). For those spaces we can say more.

Lemma 3.1.20. *The space c_0 does not admit almost Daugavet-points.*

Proof. Let $x = (x_i)_{i\geq 1} \in S_{c_0}$. Given $\delta \in (0,1)$ there exists a finite non-empty set $J \subset \mathbb{N}$ of cardinality $n \geq 1$ such that $|x_j| \geq 1-\delta$ for every $j \in J$ and $|x_i| < 1-\delta$ for every $i \in \mathbb{N}\setminus J$. Now let $x^* = \frac{1}{n}\sum_{j\in J} \operatorname{sign}(x_j)e_j^* \in S_{\ell_1}$. By Lemma 3.1.3 we have $S(x^*, \frac{\delta}{n}) \subset \bigcap_{j\in J} S(\operatorname{sign}(x_j)e_j^*, \delta)$ so if $y = (y_i)_{i\geq 1}$ is in $S(x^*, \frac{\delta}{n})$, then it satisfies $|x_j - y_j| \leq \delta$ for every $j \in J$, and thus $\|x-y\| \leq \max\{\delta,\ 2-\delta\}$. The conclusion follows. □

Lemma 3.1.21. *The space ℓ_1 admits almost Daugavet-points.*

Proof. Fix $n \geq 1$ and let $x = \frac{1}{n}\sum_{i=1}^n e_i \in S_{\ell_1}$ where (e_i) is the unit vector basis of ℓ_1. Then x is a $(2-\frac{2}{n})$ Δ-point by Observation 3.1.17 and we will show that it is in fact a $(2-\frac{2}{n})$ Daugavet-point. Indeed fix $x^* \in S_{\ell_\infty}$ and $\delta > 0$. Since $\sup_i |x^*(e_i)| = 1$ we can find some i_0 such that $|x^*(e_{i_0})| > 1-\delta$ that is $e_{i_0} \in S(x^*, \delta)$ or $-e_{i_0} \in S(x^*, \delta)$. Now it is easy to check that $\|x \pm e_{i_0}\| \geq 2 - \frac{2}{n}$ so we are done. □

Remark 3.1.22. We will see in Theorem 3.3.3 that if a Banach space X has Rolewicz' property (α), then X has no Δ-points. But X can contain almost Δ-points. Indeed, let T be the Tsirelson space. Even though T fails Rolewicz' property (α) there exists an equivalent norm $|\cdot|$, so that $(T, |\cdot|)$ has Rolewicz' property (α) by [Mon, Theorems 3 and 4]. Since T contains ℓ_1^n's uniformly the same holds for $(T, |\cdot|)$ by James' ℓ_1-distortion theorem. Thus $(T, |\cdot|)$ admits almost Δ-points by Lemma 3.1.18.

The following result, whose proof is clear from Proposition 3.1.12, covers a lot of classical norms, and in particular uniformly smooth and uniformly convex ones.

Proposition 3.1.23. *If X is uniformly non-square, then X does not admit almost Δ-points.*

By Proposition 3.1.12 again, the unit ball of every non uniformly non-square norm admits thin slices of diameter arbitrarily close to 2, so ruling out almost Daugavet- and Δ-points is also non-trivial even in super-reflexive spaces. The following result is the main reason for the introduction of the notions almost Daugavet- and almost Δ-points.

Proposition 3.1.24. *Let X be a super-reflexive Banach space and let $\mathcal{U}$ be some free ultrafilter on $\mathbb{N}$. If X admits almost Δ- (resp. Daugavet-) points, then there exists $x \in X^{\mathcal{U}}$ with $\|x\| = 1$ such that for any slice S of $B_{X^{\mathcal{U}}}$ containing x (resp. any slice S of $B_{X^{\mathcal{U}}}$) there exists $y \in S$ with $\|y\| = 1$ and $\|x-y\| = 2$.*

In particular, if there exists a super-reflexive Banach space which admits almost Δ- (resp. Daugavet-) point, then there exists a super-reflexive Banach space with a Δ- (resp. Daugavet-) point.

Proof. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Since X is super-reflexive we have $(X^{\mathcal{U}})^* = (X^*)^{\mathcal{U}}$ (see e.g. [Hei, Proposition 7.1]). For each $n \in \mathbb{N}$ choose a $(2-\frac{1}{n})$ Δ-point $x_n \in B_X$. Let $x = (x_n)_{\mathcal{U}}$. We have $\|x_n\| \geq 1 - \frac{1}{n}$ hence $\|x\| = \lim_{\mathcal{U}} \|x_n\| = 1$.

Let $x^* = (x_n^*)_{\mathcal{U}}$ with $\|x^*\| = 1$ and $\delta > 0$. Assume that $x \in S(x^*, \delta)$. This means that there is an $\eta > 0$ such that $x^*(x) = \lim_{\mathcal{U}} x_n^*(x_n) > 1 - \delta + \eta$ and thus there is a set $A \in \mathcal{U}$ such that

$$x_n^*(x_n) > 1 - \delta + \eta = \|x_n^*\| - (\|x_n^*\| - (1 - \delta + \eta)).$$

for every $n \in A$. In particular we have $\|x_n^*\| > 1 - \delta + \eta$ for these n and

$$x_n \in S_n := \{y \in B_X : x_n^*(y) > \|x_n^*\| - (\|x_n^*\| - (1 - \delta + \eta))\}.$$

We can then use the definition to find $y_n \in S_n$ with $\|x_n - y_n\| \geq 2 - \frac{1}{n}$. For $n \notin A$ we just let $y_n = 0$.

Now $y = (y_n) \in X^{\mathcal{U}}$ and

$$\|y\| = \lim_{\mathcal{U}} \|y_n\| \geq \lim_{\mathcal{U}} (\|x_n - y_n\| - \|x_n\|) \geq \lim_{\mathcal{U}} (2 - \frac{1}{n} - 1) = 1.$$

Finally we check that y is in the slice $S(x^*, \delta)$

$$x^*(y) = \lim_{\mathcal{U}} x_n^*(y_n) = \lim_{\mathcal{U}} \|x_n^*\| \lim_{\mathcal{U}} \frac{x_n^*}{\|x_n^*\|}(y_n) \geq 1 - \delta + \eta > 1 - \delta.$$

For the in particular part we just note that ultrapowers of super-reflexive spaces are super-reflexive by finite representability of $X^{\mathcal{U}}$ in X. □

By Proposition 3.1.24 we can thus restate Question 3.1.15 in the following way.

Question 3.1.25. Is it possible to find a super-reflexive space admitting almost Δ-points or almost Daugavet-points?

In particular it would be interesting to investigate the following question.

Question 3.1.26. Is it possible to find a renorming of ℓ_2 with almost Δ-points or almost Daugavet points?

Also note that proving that super-reflexive spaces do not admit Δ- (or Daugavet-) points in general would immediately imply by Proposition 3.1.24 that super-reflexive spaces do not admit almost Δ- (or Daugavet-) points.

3.2 Asymptotic smoothness and Δ-points

The goal of this section is to show that Banach spaces with an asymptotic uniformly smooth norm do not admit Δ-points. Let us recall that for every $x \in X$, we write $D(x) := \{x^* \in S_{X^*} : x^*(x) = \|x\|\}$. Also let us recall that if X is a Banach space and A is a bounded subset of X, we denote by $\alpha(A)$ the *Kuratowski measure of non-compactness* of A which is defined as the

infimum of $\varepsilon > 0$ such that A can be covered by a finite number of sets with diameters less than ε, that is,

$$\alpha(A) := \inf\left\{\varepsilon > 0 : A \subset \bigcup_{i=1}^{n} A_i, A_i \subset X, \operatorname{diam}(A_i) < \varepsilon, i = 1, 2, \ldots, n\right\}.$$

Our first result connects the pointwise modulus of asymptotic smoothness $\overline{\rho}_X(t, x)$ at $x \in S_X$ with the measure of non-compactness of the slices defined by x. In his thesis Dutrieux gave a proof that a separable Banach space is AUS if and only if its dual is weak* uniformly Kadec–Klee [Dut, Proposition 36]. Our proof of the following proposition follows closely part of the proof given by Dutrieux, but we do not assume separability.

Proposition 3.2.1. *Let X be a Banach space, and fix $x \in S_X$ and $\varepsilon > 0$. If there exists $t > 0$ such that $\bar{\rho}_X(t, x)/t < \varepsilon$, then we can find $\delta > 0$ such that $\alpha(S(x, \delta)) < 4\varepsilon$.*

To prove this proposition we will need two well known lemmas. The first lemma is from [JLPS, Lemma 2.13]. A nice different proof can be found in the thesis of Dutrieux [Dut, Lemma 38].

Lemma 3.2.2. *Let X be a Banach space. For all $Y \in \operatorname{cof}(X)$ and all $\varepsilon > 0$, there exists a compact set K_ε such that $B_X \subset K_\varepsilon + (2 + \varepsilon)B_Y$.*

The proof of the next lemma is simple using contradiction, so we skip it.

Lemma 3.2.3. *Let X be a Banach space, let $x^* \in B_{X^*}$, and let $\varepsilon > 0$. If $\limsup_\alpha \|x^* - x^*_\alpha\| < \varepsilon$ whenever $(x^*_\alpha) \subseteq B_{X^*}$ and $x^*_\alpha \overset{w^*}{\to} x^*$, then there exists a weak*-neighborhood V of x^* such that $V \cap B_{X^*} \subseteq B(x^*, \varepsilon)$.*

Proof of Proposition 3.2.1. Let us assume that $\bar{\rho}_X(t, x)/t < \varepsilon$ and let us write $\delta_x = \varepsilon t - \bar{\rho}_X(t, x)$. We then have the following.

Claim. Take any $\delta < \delta_x$. For every $x^* \in S(x, \delta)$ and every net $(x^*_\alpha) \subset B_{X^*}$ such that $x^*_\alpha \to x^*$ weak* we have $\limsup_\alpha \|x^* - x^*_\alpha\| < 2\varepsilon$.

Before proving the claim let us see how to finish the proof. Pick any $\delta < \delta_x$ and take δ' such that $\delta < \delta' < \delta_x$. By the claim and by Lemma 3.2.3 we can find for every $x^* \in S(x, \delta')$ a weak*-neighborhood V_{x^*} of x^* with $V_{x^*} \cap B_{X^*} \subseteq B(x^*, 2\varepsilon)$. Then $(V_{x^*})_{x^* \in S(x,\delta')}$ is an open cover of $S(x, \delta')$ and therefore an open cover of the weak* compact set $\bar{S}(x, \delta)$. By compactness there is a finite subcover.

Proof of the claim. Fix some $\delta < \delta_x$ and let $x^* \in S(x, \delta)$ and $(x^*_\alpha) \subset B_{X^*}$ such that $x^*_\alpha \to x^*$ weak*. We will show that $L = \limsup_\alpha \|x^* - x^*_\alpha\| < 2\varepsilon$. Again pick any δ' such that $\delta < \delta' < \delta_x$. By definition of δ_x we then have $\bar{\rho}_X(t, x) < \varepsilon t - \delta'$ so there exists $Z \in \operatorname{cof}(X)$ such that

$$\sup_{z \in B_Z} \|x + tz\| \leq 1 + \varepsilon t - \delta'.$$

Now let $Y = Z \cap \ker(x^*)$ and let $\eta > 0$. By Lemma 3.2.2 there exists a compact set K in X such that $B_X \subseteq K + (2+\eta)B_Y$.

By compactness of K and boundedness of (x^*_α) we have that $x^*_\alpha \to x^*$ uniformly on K. We may therefore choose β such that

- $|\langle x^*_\beta - x^*, k\rangle| < \eta$ for all $k \in K$;
- $x^*_\beta(x) > 1 - \delta$;
- $|\|x^*_\beta - x^*\| - L| < \eta$.

Choose $x_\beta \in S_X$ such that $\langle x^*_\beta - x^*, x_\beta\rangle > L - \eta$. Now write $x_\beta = k_\beta + (2+\eta)y_\beta$ with $k_\beta \in K$ and $y_\beta \in B_Y$. We get

$$x^*_\beta(y_\beta) = \langle x^*_\beta - x^*, y_\beta\rangle = \frac{\langle x^*_\beta - x^*, x_\beta\rangle - \langle x^*_\beta - x^*, k_\beta\rangle}{2+\eta} > \frac{L - 2\eta}{2+\eta}$$

since $y_\beta \in \ker(x^*)$. Therefore

$$1 - \delta + \frac{t(L-2\eta)}{2+\eta} < \langle x^*_\beta, x\rangle + \frac{t(L-2\eta)}{2+\eta} < \langle x^*_\beta, x + ty_\beta\rangle \leq \|x + ty_\beta\| \leq 1 + \varepsilon t - \delta'.$$

Finally

$$L - 2\eta < (2+\eta)(\varepsilon - \theta)$$

with $\theta = \frac{\delta' - \delta}{t} > 0$. Since $\eta > 0$ was arbitrary we get $L \leq 2(\varepsilon - \theta) < 2\varepsilon$ as desired. □

As an immediate corollary we get.

Corollary 3.2.4. *Let X be a Banach space and let $x \in S_X$ be an asymptotically smooth point, that is a point for which $\lim_{t\to 0} \frac{\overline{\rho}_X(t,x)}{t} = 0$. Then $\lim_{\delta\to 0} \alpha(S(x,\delta)) = 0$.*

We will now show that this condition on the Kuratowski index of the slices $S(x,\delta)$ prevents the point x to be a Δ-point. In fact it appears that this is the case as soon as one of the slices $S(x,\delta)$ admits a non-trivial covering by finitely many balls. In order to prove this result, we will need the following refinement of Lemma 3.1.3.

Lemma 3.2.5. *Let $n \geq 1$, $x^*_1, \ldots, x^*_n \in S_{X^*}$, and $\delta > 0$. Define $\delta' := \frac{\delta}{2n}$. If $\left\|\frac{1}{n}\sum_{i=1}^n x^*_i\right\| > 1 - \delta'$, then*

$$S\left(\frac{1}{n}\sum_{i=1}^n x^*_i, \delta'\right) \subset \bigcap_{i=1}^n S(x^*_i, \delta).$$

The proof is similar to the one of Lemma 3.1.3 and we leave the details to the reader. We now state and proof the announced result.

Theorem 3.2.6. *Let X be a Banach space and let $x \in S_X$. If there exists a $\delta > 0$ such that $\alpha(S(x,\delta)) < 2$, then x is not a Δ-point.*

Proof. Take $\delta_0 > 0$ and $\varepsilon \in (0,2)$ such that $\alpha(S(x,\delta_0)) < \varepsilon$, pick any $\delta \leq \delta_0$ such that $\varepsilon + \delta < 2$, and let $\eta = 2 - (\varepsilon + \delta)$. We have $\alpha(S(x,\delta)) \leq \alpha(S(x,\delta_0)) < \varepsilon$, so we can find $n \geq 1$ and non-empty subsets $A_1, \ldots, A_n$ of X^* with diameter smaller than ε such that $S(x,\delta) \subset \bigcup_{i=1}^n A_i$. By an easy refinement we obtain the following.

Claim. For every $\delta' \leq \delta$, we can find $1 \leq m \leq n$ and $y_1^*, \ldots, y_m^* \in X^*$ such that

1. $y_j^* \in S_{X^*} \cap S(x, \delta')$ for every $1 \leq j \leq m$;
2. $S_{X^*} \cap S(x, \delta') \subset \bigcup_{j=1}^m B(y_j^*, \varepsilon)$.

Proof of the Claim. For any $\delta' \leq \delta$, we have $S_{X^*} \cap S(x, \delta') \subset \bigcup_{i=1}^n A_i$. Now this set is not empty since it contains $D(x)$, and the set $J = \{1 \leq i \leq n : \ (S_{X^*} \cap S(x, \delta')) \cap A_i \neq \emptyset\}$ thus has cardinality $|J| = m$ for some $1 \leq m \leq n$.

Picking for every $j \in J$ an element $y_j^* \in (S_{X^*} \cap S(x, \delta')) \cap A_j$, we obtain $S_{X^*} \cap S(x, \delta') \subset \bigcup_{j \in J} A_j \subset \bigcup_{j \in J} B(y_j^*, \varepsilon)$ since A_j has diameter smaller than ε, and the conclusion follows (relabelling the y_j^*'s if necessary). □

Now let us find $1 \leq m \leq n$ and $y_1^*, \ldots y_m^* \in S_{X^*}$ as in the Claim for $\delta' = \frac{\delta}{2n}$, and let us define $y^* = \frac{1}{m} \sum_{j=1}^m y_j^*$. Since $y_j^*(x) > 1 - \delta'$ for every $1 \leq j \leq m$, we have $y^*(x) > 1 - \delta' \geq \|y^*\| - \delta'$ so that $x \in S(y^*, \delta')$ and $\|y^*\| > 1 - \delta' \geq 1 - \frac{\delta}{2m}$. It thus follows from Lemma 3.2.5 that

$$S\left(y^*, \frac{\delta}{2m}\right) \subset \bigcap_{j=1}^m S(y_j^*, \delta)$$

and as a consequence we have

$$x \in S(y^*, \delta') \subset \bigcap_{j=1}^m S(y_j^*, \delta).$$

To conclude, let us take $y \in S(y^*, \delta')$, and let us take $z^* \in S_{X^*}$ such that $\|x - y\| = z^*(x - y)$. We distinguish two cases.

Case 1: If $z^*(x) > 1 - \delta'$, then

$$z^* \in S_{X^*} \cap S(x, \delta') \subset \bigcup_{j=1}^m B(y_j^*, \varepsilon)$$

so there exists $j_0 \in \{1, \ldots, m\}$ such that $\|z^* - y_{j_0}^*\| \leq \varepsilon$. Now we have $y \in S(y_{j_0}^*, \delta)$ by the above inclusion so that

$$\|x - y\| = z^*(x) + (y_{j_0}^* - z^*)(y) - y_{j_0}^*(y) \leq 1 + \varepsilon - (1 - \delta) = \varepsilon + \delta = 2 - \eta.$$

Case 2: If $z^*(x) \leq 1 - \delta'$, we have

$$\|x - y\| = z^*(x) - z^*(y) \leq 1 - \delta' + 1 = 2 - \delta'.$$

Combining the two cases we obtain $\|x - y\| \leq \max\{2 - \delta', 2 - \eta\}$ and x cannot be a Δ-point. □

Combining Theorem 3.2.6 and Proposition 3.2.1 we then get

Proposition 3.2.7. *Let X be a Banach space and let $x \in S_X$ satisfy $\overline{\rho}_X(t,x) < \frac{t}{2}$ for some $t > 0$. Then x is not a Δ-point.*

In particular no asymptotically smooth point can be a Δ-point and we obtain the following result.

Theorem 3.2.8. *Let X be an AUS Banach space. Then X does not admit a Δ-point.*

Since property AUS passes to subspaces and quotients (see Remark 1.1.2) we do have that any subspace or quotient of c_0 fails to contain Δ-points. More generally any subspace of quotient of a Banach space with property (M^*) fails to contain Δ-points.

Recall that a Banach space has Kalton's property (M) if whenever $x, y \in X$ with $\|x\| = \|y\|$ and (x_α) is a bounded weakly null net in X, then

$$\limsup_\alpha \|x + x_\alpha\| = \limsup_\alpha \|y + x_\alpha\|.$$

Similarly X has property (M^*) if whenever $x^*, y^* \in X^*$ with $\|x^*\| = \|y^*\|$ and (x^*_α) is a bounded weak* null net in X^*, then

$$\limsup_\alpha \|x^* + x^*_\alpha\| = \limsup_\alpha \|y^* + x^*_\alpha\|.$$

If X has property (M^*), then X has property (M) and X is an M-ideal in X^{**} (see e.g. [HWW, Proposition VI.4.15]) that is X is M-embedded. In particular, X is an Asplund space (see e.g. [HWW, Theorem III.3.1]). Also it is well known that property (M^*) is inherited by both subspaces and quotients (see e.g. [Oja2]) and by chasing references we find that the following proposition holds.

Proposition 3.2.9. *Assume a Banach space X has property (M). The following are equivalent:*

1. *X is AUS;*
2. *X contains no copy of ℓ_1;*
3. *X has property (M^*).*

Proof. 1 $\Rightarrow$ 2. If X is AUS, then X is Asplund (see e.g. [JLPS, Proposition 2.4]). Hence X contains no copy of ℓ_1.

2 $\Rightarrow$ 3. If X contains no copy of ℓ_1, then no separable subspace of X can contain ℓ_1. Clearly every separable closed subspace of X has property (M) (both net and sequential version, see [Oja1, Proposition 1]) and then they all have property (M^*) (both net and sequential version) by Theorem 2.6 in [KW]. Finally X has property (M^*) if every separable closed subspace does [Oja2, Proposition 3.1].

3 $\Rightarrow$ 1. Dutta and Godard [DG] proved that if X is a separable Banach space with property (M^*), then X is AUS. However, using property (M^*) and Proposition 2.2 in [GLR] one finds $\bar{\rho}_X(t,x) = \bar{\rho}_X(t)$ for all $x \in S_X$ and their proof also works in the non-separable case. $\square$

Note that there are M-embedded spaces which are not AUS. For example the Schreier space $\mathcal{S}$ is not AUS since it does not have property (M^*). Indeed, if a Banach space X has property (M^*), then the relative norm and weak* topologies on S_{X^*} coincide (see e.g. [HWW, Proposition VI.4.15]). But if (e_i) is the unit vector basis in $\mathcal{S}$ and (e_i^*) the the biorthogonal functionals in the dual, then $e_2^* + e_i^* \in S_{\mathcal{S}^*}$ and converges weak* to e_2^*, but not in norm. Note however that $\mathcal{S}$ does admit Δ-points by Proposition 2.15 in [ALMT].

Looking back at Example 1.1.26 we also have that any subspace or quotient of a Banach space with an FDD admitting block upper ℓ_p or of the predual of a Banach space with an FDD admitting lower ℓ_q estimates fails to contain Δ-points. This applies in particular to subspaces and quotients of the predual JT_* of the James tree space (see Section 3.4) or of duals B_p^* of Baernstein spaces (see Example 1.1.24).

3.3 Asymptotic convexity

The main result of this section is that reflexive AUC spaces do not admit Δ-points. The proof uses the characterization of reflexive AUC spaces in terms of the measure of non-compactness of slices and relies on the following result of Kuratowski (see e.g. [BM, p. 151])

Lemma 3.3.1. *Let (M, d) be complete metric space. If (F_n) is a decreasing sequence of non-empty, closed, and bounded subsets of M such that $\lim_n \alpha(F_n) = 0$, then the intersection $F_\infty = \bigcap_{n=1}^\infty F_n$ is a non-empty compact subset of M.*

A Banach space X has *Rolewicz' property* (α) if for every $x^* \in S_{X^*}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(S(x^*, \delta)) \leq \varepsilon$. We say that X has *uniform property* (α) if the same δ works for all $x^* \in S_{X^*}$. These properties were introduced by Rolewicz in [Rol]. If X has Rolewicz' property (α), then it is reflexive (see e.g. [Mon]).

Implicit in Rolewicz [Rol, Theorem 3] is the result that X is AUC and reflexive if and only if X has uniform property (α) (Rolewicz uses the term "X is Δ-uniformly convex" instead of X is AUC and reflexive). Note that this is contained in Lemma 1.2.14. It is also known that X is AUC and reflexive if and only if X is nearly uniformly convex (NUC) [Huf]. The difference between these two types of uniform convexity is that they use different (but equivalent) measures of non-compactness.

We also note that for dual spaces we have that if X^* is AUC* then X^* is has *weak** uniform property (α), that is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(S(x, \delta)) \leq \varepsilon$ for all $x \in S_X$ (this is Lemma 1.2.14). Our Corollary 3.2.4 is a pointwise version of this result. Lennard [Len, Proposition 1.3] states that X^* has the weak* version of the uniform Kadec–Klee property if and only if X^* has weak* uniform property (α). Note that Lennard credits this result to Sims and he uses a different measure of non-compactness.

Let us first prove a pointwise version of the main result.

Theorem 3.3.2. *Let X be a Banach space and $x \in S_X$. If there exists $x_0^* \in D(x)$ such that $\lim_{\delta \to 0} \alpha(S(x_0^*, \delta)) = 0$, then x is not a Δ-point.*

Proof. Let $x \in S_X$ be such that there exists $x_0^* \in D(x)$ with $\lim_{\delta\to 0} \alpha(S(x_0^*, \delta)) = 0$ and let us assume for contradiction that x is a Δ-point.

Define a set of functionals norming x by

$$D_0(x) := \left\{ f_{x^*} := \frac{x^* + x_0^*}{2} : x^* \in D(x) \right\}.$$

If $\delta > 0$ and $f = f_{x^*} \in D_0(x)$, then by Lemma 3.1.3

$$S\left(f, \frac{\delta}{2}\right) = S\left(\frac{x^* + x_0^*}{2}, \frac{\delta}{2}\right) \subset S(x^*, \delta) \cap S(x_0^*, \delta)$$

and therefore $\lim_{\delta\to 0} \alpha(\overline{S}(f, \delta)) = \lim_{\delta\to 0} \alpha(S(f, \delta)) = 0$ for all $f \in D_0(x)$.

Given $f \in D_0(x)$ and a sequence $(\delta_n) \subseteq (0, 1)$ decreasing to 0 we define for each n

$$F_n^f := \overline{S}(f, \delta_n) \cap \Delta_{\delta_n}(x).$$

Each F_n^f is a closed non-empty subset of B_X with $\alpha(F_n^f) \to 0$ hence $F^f = \cap_n F_n^f$ is non-empty and compact for every $f \in D_0(x)$ by Lemma 3.3.1. Using a compactness argument we can then prove the following.

Claim.

$$F := \bigcap_{f \in D_0(x)} F^f \neq \emptyset.$$

Before proving this claim let us see that it will give us the desired conclusion. Indeed, if $z \in F$, then $\|x - z\| = 2$ and for all $f \in D_0(x)$ we must have $f(z) = 1$, in particular $x_0^*(z) = 1$. But this is nonsense since for any $x^* \in S_{X^*}$ with $x^*(x - z) = 2$ we must have $x^*(x) = 1$ and $x^*(z) = -1$, so

$$1 = f_{x^*}(x) = \frac{x^* + x_0^*}{2}(z) = \frac{-1 + 1}{2} = 0.$$

This contradiction shows that x is not a Δ-point. To finish the proof we only need to prove the claim that $F \neq \emptyset$.

Proof of the Claim. It is enough to show that $(F^f)_{f \in D_0(x)}$ has the finite intersection property since all the sets F^f are compact and non-empty. So let $f_1, \ldots, f_k \in D_0(x)$, which means $f_j = \frac{x_j^* + x_0^*}{2}$ for $x_j^* \in D(x)$, and define $f = \frac{1}{k}\sum_{j=1}^k f_j = \frac{1}{2}\left(x_0^* + \frac{1}{k}\sum_{j=1}^k x_j^*\right)$.

Clearly $f \in D_0(x)$. By Lemma 3.1.3 we have for all $\delta > 0$

$$S\left(f, \frac{\delta}{k}\right) \subset \bigcap_{j=1}^k S(f_j, \delta)$$

and hence

$$S\left(f, \frac{\delta}{k}\right) \cap \Delta_{\frac{\delta}{k}}(x) \subset \bigcap_{j=1}^k S(f_j, \delta) \cap \Delta_\delta(x).$$

Since (δ_n) is decreasing to 0 there must for any n exist m with $\delta_m < \delta_n/k$. By the above inclusion we thus get

$$F_m^f \subset \bigcap_{j=1}^{k} F_n^{f_j}$$

and hence by commutativity of intersections

$$\emptyset \neq F^f = \bigcap_n F_n^f \subseteq \bigcap_{j=1}^{k} \bigcap_n F_n^{f_j} = \bigcap_{j=1}^{k} F^{f_j}$$

and the claim is proved.

□

From Theorem 3.3.2 we immediately get.

Theorem 3.3.3. *If X has Rolewicz' property (α), then X does not have Δ-points.*

As we noted above a Banach space X is reflexive and AUC if and only if it has uniform property (α). Also finite-dimensional spaces are trivially AUC since for example $\alpha(S(x^*, \delta)) = 0$ for slices of B_X in finite-dimensional spaces.

Theorem 3.3.4. *If X is reflexive and AUC, then X does not have Δ-points.*
In particular, if X is finite-dimensional then X does not have Δ-points.

Remark 3.3.5. Let X be a Banach space such that for every $x \in S_X$ there exists $x^* \in D(x)$ with $\lim_{\delta \to 0} \alpha(S(x^*, \delta)) = 0$. Then by Theorem 3.3.2 X does not admit a Δ-point. Note that unlike Rolewicz' property (α) (see [Rol]) this property does not imply reflexivity.

Indeed, every separable Banach space has an equivalent locally uniformly rotund renorming and if (the norm of) X is locally uniformly rotund, then every $x \in S_X$ is strongly exposed by $x^* \in D(x)$ so that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $S(x^*, \delta)$ has diameter less than ε. In particular, $\alpha(S(x^*, \delta)) < \varepsilon$.

Together with the results from Section 3.1 we also obtain.

Corollary 3.3.6. *If X is finite dimensional, then it does not admit almost Δ-points.*

Proof. This is an immediate consequence of Proposition 3.1.24 since finite dimensional spaces do not admit Δ-points and since any ultrapower of a finite dimensional space is trivially identified with the space itself (under the diagonal map). □

Using the duality AUS/AUC in reflexive spaces we can in fact combine Theorem 3.2.8 and 3.3.4.

Corollary 3.3.7. *If X is reflexive and AUC, then neither X nor X^* admit Δ-points.*

In particular we can apply this result to Baernstein's spaces B_p $(1 < p < \infty)$ from Example 1.1.24. Let us recall that the space $B = B_2$ was originally introduced in [Bae] as an example of a reflexive space failing the Banach–Saks property. It is known to have a normalized (unconditional) basis with block lower ℓ_2 estimates and thus to be 2-AUC (see [Par, Theorem 3] with the NUC terminology). Also the optimal modulus of near convexity of B has been estimated in [BOS]. From our preceding results the space B_p and its dual space B_p^* both fail to have Δ-points.

Here is a pointwise application of Theorem 3.3.2.

Corollary 3.3.8. *Let X be a Banach space and let $x^* \in S_{X^*}$ be a norm one functional which attains its norm at some $x \in S_X$. If $\lim_{t\to 0} \frac{\overline{\rho}(t,x)}{t} = 0$, then x^* is not a Δ-point.*

Proof. By Corollary 3.2.4 we have $\lim_{\delta\to 0} \alpha(S(x,\delta)) = 0$ and the conclusion follows directly from Theorem 3.3.2 since $x \in D(x^*)$. □

Let X be a Banach space. An element $x \in B_X$ is said to be a *quasi denting-point* if given $\varepsilon > 0$, there exists $x^* \in S_{X^*}$ and $\delta > 0$ with $x \in S(x^*, \delta)$ such that $\alpha(S(x^*, \delta)) < \varepsilon$. Recall that x is a *denting-point* if the above can be strengthened to $S(x^*, \delta) \subset B(x, \varepsilon)$. In dual spaces one can similarly define weak* (quasi) denting-points by requiring x^* to be weak* continuous. Giles and Moors [GM2] introduced quasi denting-points (under the name α denting-points) see e.g. [MMT] or [Tro].

Let X be a Banach space such that the dual is AUC*, then every $x^* \in S_{X^*}$ is a quasi denting-point. This is essentially contained in e.g. [HL, Proposition 4.8], but we include the straightforward argument. As mentioned in Section 1.2, AUC* is equivalent to the following generalized version of UKK*: for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x^* \in B_{X^*}$ and all weak* neighborhoods V of x^* satisfy $\operatorname{diam}(V \cap B_{X^*}) > \varepsilon$, then $\|x^*\| \leq 1 - \delta$. Now fix $\varepsilon > 0$ and choose $\delta > 0$ as above. If $x^* \in S_{X^*}$, then we can find $x \in S_X$ with $x^*(x) > 1 - \delta$. Let $0 < \delta' < \delta$. If $y^* \in S(x, \delta)$, then $\|y^*\| > 1 - \delta$ and there exists a weak* open neighborhood V_{y^*} of y^* with $\operatorname{diam}(V_{y^*} \cap B_{X^*}) \leq \varepsilon$. We therefore have an open cover of the weak* compact set $\bar{S}(x, \delta')$ and by compactness we have a finite cover and hence x^* is a weak* quasi denting-point.

If the dual X^* is AUC*, then X is AUS and by Corollary 3.3.8 no norm-attaining $x^* \in S_{X^*}$ can be a Δ-point, but we do not know if a weak* quasi-denting point, or more generally a quasi denting-point, can be a Daugavet-point or a Δ-point. For non-reflexive AUC spaces, we do not even know if every element of the unit sphere is quasi denting.

We end the section with a few more partial results concerning AUC* duals. We recall that X has the *Krein-Milman property* (KMP in short) if every non empty bounded closed convex set in X has an extreme point.

Proposition 3.3.9. *Let X be a Banach space with the KMP and assume that every extreme point of B_X is denting. Then X does not admit Daugavet-points.*

Proof. For a point $x \in B_X$ we denote

$$F_x := \bigcap_{f \in D(x)} \{y \in B_X : f(y) = f(x)\}.$$

This is a non-empty closed convex bounded set, so it admits an extreme point $y \in F_x$. Moreover it is an extremal set of B_X so it follows that $y \in \text{ext}(B_X)$. Now our hypothesis yields that $y \in dent(B_X)$ so if x is a Daugavet point, we get $\|x - y\| = 2$. Let $f \in S_{X^*}$ be norming for $x - y$. Then we must have $f(x) = 1$ and $f(y) = -1$. But since $y \in F_x$ we also have $f(y) = 1$, which is a contradiction. □

Remark 3.3.10. Let us recall that a point $x \in B_X$ is a *point of weak-to-norm continuity* (in short point of continuity) if the identity map $id_X : (B_X, \sigma(X, X^*)) \to (B_X, \|.\|)$ is continuous at the point x. Lin, Lin, and Troyanski proved in [LLT] that an extreme point of continuity in a bounded closed convex subset of a Banach space is a denting point. Thus one can replace denting points by points of continuity on the unit sphere in the assumptions in Proposition 3.3.9.

We have a weak* analogue of Proposition 3.3.9 for weak* Daugavet-points when every extreme point of B_{X^*} is weak* denting, or equivalently is a point of weak*-to-norm continuity. Note that Huff and Morris showed in [HM] that KMP and RNP are equivalent for dual spaces. In particular we obtain the following result.

Corollary 3.3.11. *Let X be a Banach space and let us assume that X^* is AUC*. Then X^* does not admit weak* Daugavet-points.*

Proof. By Theorem 1.2.4 a dual AUC* space has the RNP and thus the KMP. Moreover any point $x^* \in S_{X^*}$ in such space is a point of weak*-to-norm continuity so that every extreme point in B_{X^*} is weak* denting (this is an easy consequence of Choquet's lemma). The conclusion follows from the weak* version of the above proposition.

□

3.4 Delta-points in the James tree space

Let $T = \{\emptyset\} \cup \bigcup_{n\geq 1}\{0,1\}^n$ be the infinite binary tree and let us denote by $\leq$ the natural ordering on T. A totally ordered subset S of T is called a segment if it satisfies:

$$\forall s, t \in S,\ [s,t] = \{u \in T : s \leq u \leq t\} \subset S.$$

An infinite segment of T is also called a branch of T. Let us denote by $\mathcal{F}$ the set of all finite families of disjoint segments in T. The James tree space JT is define as follows:

$$JT = \left\{ x = (x_s)_{s\in T} \subset \mathbb{R} :\ \|x\|_{JT}^2 = \sup_{F\in\mathcal{F}} \sum_{S\in F} x_S^2 < \infty \right\}$$

where $x_S = \sum_{s\in S} x_s$ for every non-empty segment S and $x_\emptyset = 0$.

It is well known that JT is a Banach space and that the set of canonical unit vectors $\{e_t\}_{t\in T}$ of $c_{00}(T)$ forms, for the lexicographic order, a normalized monotone boundedly complete basis of JT. Moreover, it is also known that the closed linear span $JT_* = [e_t^*]_{t\in T}$ of the set of biorthogonal functionals in JT^* is a unique isometric predual of JT. If S is a segment of T

or if β is a branch of T we will sometimes refer to the set $\{e_s\}_{s\in S}$ as a segment of JT and the set $\{e_t\}_{t\in\beta}$ as a branch of JT. From the definition of the norm, it is easy to show that $\|x+y\|_{JT}^2 \geq \|x\|_{JT}^2 + \|y\|_{JT}^2$ whenever $x, y \in JT$ have totally disconnected supports (that is if $\operatorname{conv}(\operatorname{supp}_T x) \cap \operatorname{conv}(\operatorname{supp}_T y) = \emptyset$). Since we are working with the lexicographic order on T, this applies whenever $\operatorname{supp} x < \operatorname{supp} y$ with respect to the ordering of the basis, and $\{e_t\}_{t\in T}$ thus satisfies block lower ℓ_2 estimates as stated in Example 1.1.26. It follows that JT is 2-AUC* and Corollary 3.3.11 combined with Theorem 3.2.8 applied to the 2-AUS JT_* yields the following result.

Theorem 3.4.1. *The space JT does not admit weak* Daugavet-points. The predual JT_* of the James tree space does not admit Δ-points.*

Let us also emphasize that $\|x\|_{JT} \geq \|x\|_{\ell_2}$ for every $x \in JT$. It is also worth mentioning that if one consider the equivalent norm $\|x\|^2 = \sup_{S_1<\cdots<S_n} \sum_{i=1}^n x_{S_i}^2$ on the James space J, where the S_i are segments of $\mathbb{N}$, then one obtains a Banach space isometric to the closed linear span $[e_t]_{t\in\beta}$ of any branch of JT. In particular all the results we will obtain for the space JT will also apply to $(J, \|.\|)$.

As we see from Proposition 3.1.12 the unit ball of spaces that are not uniformly non-square contain thin slices of diameter arbitrary close to 2. Let us start by illustrating this by exhibiting slices of diameter 2 in JT.

Proposition 3.4.2. *For every $\delta > 0$ we can find some $x^* \in S_{JT^*}$ such that* $\operatorname{diam} S(x^*, \delta) = 2$ *and the diameter is attained.*

Proof. For convenience let us work in the space $(J, \|.\|)$ introduced above. Doing the same construction on any branch $(x_t)_{t\in\beta}$ of JT would do the work for JT.

Fix $\delta > 0$, fix $n \geq 1$, and let

$$x = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{i+1} e_{2i-1} = \frac{1}{\sqrt{n}}(1, 0, -1, 0, \ldots, -1, 0, 0, 0, \ldots)$$

and

$$y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{i} e_{2i} = \frac{1}{\sqrt{n}}(0, -1, 0, 1, \ldots, 0, 1, 0, 0, \ldots)$$

so that

$$x - y = \frac{1}{\sqrt{n}}(1, 1, -1, -1, \ldots, -1, -1, 0, 0, \ldots)$$

and

$$x + y = \frac{1}{\sqrt{n}}(1, -1, -1, 1, 1, \ldots, -1, -1, 1, 0, 0, \ldots).$$

It is easy to check that $\|x\| = \|y\| = 1$, $\|x - y\| = 2$, and $\|x + y\| = \sqrt{\frac{1+(n-1)\times 2^2+1}{n}} = \sqrt{4 - \frac{2}{n}}$.

By assuming that n was chosen large enough so that $\sqrt{4-\frac{2}{n}} > 2-\delta$ and by choosing a norming functional $x^* \in S_{J^*}$ for $x+y$ we then have $x^*(x+y) = \|x+y\| > 2-\delta$ and this implies that $x, y \in S(x^*, \delta)$. Since $\|x-y\| = 2$ the conclusion follows. □

We start by showing that the elements of the basis of JT are weak* denting (and even strongly exposed by an element of the predual) and then use this to give a simple proof of the fact that JT does not admit weak* Daugavet-points.

Lemma 3.4.3. *For each $t \in T$, $\lim_{\delta\to 0} \operatorname{diam} S(e_t^*, \delta) = 0$.*

Proof. Let us fix $t \in T$ and $\delta > 0$, and let us take $y \in S(e_t^*, \delta)$. By the triangle inequality, we have

$$\|y - e_t\|_{JT} \leq \|y - y_t e_t\|_{JT} + (1 - y_t) \leq \|y - y_t e_t\|_{JT} + \delta$$

so we only need to estimate $\|y - y_t e_t\|_{JT}$.

If t is not in the support of a family $F \in \mathcal{F}$, that is if $t \notin \bigcup_{S\in F} S$, then the segment $\{t\}$ is disjoint from all the segments in F and we have

$$y_t^2 + \sum_{S\in F} (y - y_t e_t)_S^2 = y_t^2 + \sum_{S\in F} y_S^2 \leq \|y\|_{JT}^2 \leq 1$$

so that

$$\sum_{S\in F} (y - y_t e_t)_S^2 \leq 1 - (1-\delta)^2 \leq 2\delta.$$

Now if we take any segment S of T containing t, we can write $S = S^- \cup \{t\} \cup S^+$ with $S^- < \{t\} < S^+$ segments of T, and by the preceding computations we have

$$(y - y_t e_t)_S^2 \leq y_{S^-}^2 + y_{S^+}^2 + 2\sqrt{y_{S^-}^2 y_{S^+}^2} \leq 6\delta.$$

By combining the two observations we get $\sum_{S\in F} (y - y_t e_t)_S^2 \leq 8\delta$ for every $F \in \mathcal{F}$, that is $\|y - y_t e_t\|_{JT}^2 \leq 8\delta$. The conclusion follows. □

Corollary 3.4.4. *The space JT does not admit weak* Daugavet-points.*

Proof. Let us fix $x \in S_{JT}$. If $x = e_t$ for some $t \in T$ then x is a weak* denting point by the preceding lemma and x cannot be a weak* Daugavet-point. So let us assume that x is not of that form and let us take $t \in T$ such that $x_t \neq 0$ and $x_s = 0$ for every $s < t$.

Now let us write $x_t = \theta\alpha$ with $\alpha \in (0,1)$ and $\theta \in \{-1, 1\}$. Because of the choice of t we clearly have

$$\|x - x_t e_t\|_{JT}^2 = \sup_{F\in\mathcal{F}_t} \sum_{S\in F}^{n} x_S^2$$

where $\mathcal{F}_t$ is the set of finite families of disjoint segments of T not intersecting $[\emptyset, t]$. Since

$$x_t^2 + \sum_{S\in F} x_S^2 \leq \|x\|_{JT}^2$$

for every such family we obtain

$$\|x - x_t e_t\|_{JT}^2 \leq 1 - \alpha^2 < 1.$$

Thus

$$\|x - \theta e_t\|_{JT} \leq \|x - x_t e_t\|_{JT} + (1 - \alpha) \leq 2 - \alpha$$

so that x is at distance strictly less than 2 from a (weak*) denting point of JT. The conclusion follows from a weak* version of [JRZ, Proposition 3.1]. □

Now we will show that JT does not admit Δ-points. For this purpose let us introduce some more notations. Let us write $\mathcal{F}_\infty$ for the set of (finite or infinite) families of disjoint segments in T. Note that every infinite $F \in \mathcal{F}_\infty$ has to be countable. For any segment S of T, let us write $\mathbb{1}_S = \sum_{s \in S} e_s^*$. A molecule in JT^* is an element of the form $\sum_{i \geq 1} \lambda_i \mathbb{1}_{S_i}$ where $\{S_i\} \in \mathcal{F}_\infty$ and $\lambda = (\lambda_i) \in B_{\ell_2}$. The molecule is finite if the family $\{S_i\}$ contains only finitely many non-empty segments, and we will write $\mathcal{M}$ (resp. $\mathcal{M}_\infty$) for the set of finite (resp. finite or infinite) molecules of JT^*.

Molecules play an important role in the study of the space JT because they turn computations in JT into computations in ℓ_2 and because of the following result due to Schachermayer [Sch, Proposition 2.2].

Theorem 3.4.5. *The unit ball B_{JT^*} of the dual of JT is the norm closed convex hull of $\mathcal{M}$.*

Remark 3.4.6. It is also known that $\mathcal{M}_\infty$ is the weak* closure of $\mathcal{M}$ in B_{JT^*}.

We will use some specific molecules to provide norming functionals for elements of S_{JT}. For $x \in S_{JT}$ and $F \in \mathcal{F}_\infty$, let us write $m_{x,F} = \sum_{S \in F} x_S \mathbb{1}_S$. Since $\sum_{S \in F} x_S^2 \leq \|x\|_{JT} = 1$, those elements are molecules in JT^*. Moreover, we have the following result.

Lemma 3.4.7. *Let us assume that $\sum_{S \in F} x_S^2 = \|x\|_{JT} = 1$. Then the molecule $m_{x,F}$ belongs to $D(x)$ (that is, it is a norm one functional and it norms x). Moreover, if $y \in S(m_{x,F}, \delta)$ for some $\delta > 0$, then we have*

$$\sum_{S \in F} (x_S - y_S)^2 \leq 2\delta.$$

Proof. The key observation here is that for every $y \in JT$, we have

$$m_{x,F}(y) = \sum_{S \in F} x_S y_S = \langle (x_S)_{S \in F}, (y_S)_{S \in F} \rangle_{\ell_2}.$$

In particular we have, by the Cauchy–Schwartz inequality,

$$|m_{x,F}(y)| \leq \|(x_S)_{S \in F}\|_{\ell_2} \|(y_S)_{S \in F}\|_{\ell_2} \leq \|x\|_{JT} \|y\|_{JT},$$

so $\|m_{x,F}\|_{JT^*} \leq 1$. By assumption $m_{x,F}(x) = \|(x_S)_{S \in F}\|_{\ell_2}^2 = 1$ and the first part of the lemma follows.

Now $y \in S(m_{x,F}, \delta)$ if and only if $\langle (x_S)_{S\in F}, (y_S)_{S\in F}\rangle_{\ell_2} > 1-\delta$ and thus every y in $S(m_{x,F}, \delta)$ satisfy

$$\begin{aligned}\|(x_S - y_S)_{S\in F}\|_{\ell_2}^2 &= \|(x_S)_{S\in F}\|_{\ell_2}^2 + \|(x_S)_{S\in F}\|_{\ell_2}^2 - 2\langle (x_S)_{S\in F}, (y_S)_{S\in F}\rangle_{\ell_2} \\ &\leq 2 - 2(1-\delta) = 2\delta.\end{aligned}$$

□

Note that it is not obvious at first that such a norm attaining family exists. We will first do a warm up with finitely supported elements in JT for which this is obvious, and then prove it in a lemma.

Proposition 3.4.8. *Let $x \in S_{JT}$ be an element of finite support. Then x is not a weak* Δ-point.*

Proof. Let $x \in S_{JT}$ be an element of finite support Σ and let $\mathcal{F}_\Sigma$ be the set of families of disjoint segments of the convex hull of Σ in T (that is the smaller subset of T containing $[s,t]$ for every $s \leq t$ in Σ). Then $\mathcal{F}_\Sigma$ is finite and we have

$$\|x\|_{JT}^2 = \max_{F\in\mathcal{F}_\Sigma} \sum_{S\in F} x_S^2.$$

Now let $\mathcal{D} = \{F \in \mathcal{F}_\Sigma : \sum_{S\in F} x_S^2 = 1\}$. The set $\mathcal{D}$ is a non-empty subset of $\mathcal{F}_\Sigma$ and since this set is finite we can find a constant $\eta_x > 0$ such that for every $F \in \mathcal{F}_\Sigma \backslash \mathcal{D}$ we have

$$\sum_{S\in F} x_S^2 < (1-\eta_x)^2.$$

Let us introduce $x^* = \frac{1}{|\mathcal{D}|}\sum_{F\in\mathcal{D}} m_{x,F}$ where $m_{x,F}$ is the molecule associated to x and F. For every $F \in \mathcal{D}$, the molecule $m_{x,F}$ is in $D(x)$ by the preceding lemma, and by Lemma 3.1.3 we know that x^* is in $D(x)$ and that for every choice of a $\delta > 0$ we have

$$S\left(x^*, \frac{\delta}{n}\right) \subset \bigcap_{F\in\mathcal{D}} S(m_{x,F}, \delta).$$

To conclude choose any $\delta \in (0,1)$ and let us take $y \in S\left(x^*, \frac{\delta}{n}\right)$. If $F \in \mathcal{F}$ does not belong to $\mathcal{D}$, then by Minkowski's inequality we have

$$\sum_{S\in F}(x_S - y_S)^2 \leq \left(\sqrt{\sum_{S\in F} x_S^2} + \sqrt{\sum_{S\in F} y_S^2}\right)^2 \leq (2-\eta_x)^2.$$

Now if $F \in \mathcal{D}$ we have $y \in S(m_{x,F}, \delta)$ and by the preceding lemma we get

$$\sum_{S\in F}(x_S - y_S)^2 \leq 2\delta.$$

Finally $\|x-y\|_{JT} \leq \max\{2-\eta_x, \sqrt{2\delta}\}$ and this quantity is strictly less than 2 since $\delta < 1$. To conclude, note that $x^* \in JT_*$ since all the segments involved in the construction are finite. □

To tackle the other elements of JT we first need to ensure the existence of norm attaining (possibly infinite) families.

Lemma 3.4.9. *Let $x \in S_{JT}$. Then there is an $F \in \mathcal{F}_\infty$ such that $\sum_{S\in F} x_S^2 = 1$.*

Proof. Let $x \in S_{JT}$. By Krein–Milman x attains its norm on an extreme point $x^* \in B_{JT^*}$. By Milman's converse, Lemma 3.4.5 and Remark 3.4.6, $x^* \in M_\infty$, so we can write $x^* = \sum_{i\geq 1} \lambda_i \mathbb{1}_{S_i}$ for some $\lambda = (\lambda_i)$ in B_{ℓ_2} and $\{S_i\}$ in $\mathcal{F}_\infty$. To conclude, consider $\mu = (x_{S_i})$ in B_{ℓ_2} and observe that

$$\langle \lambda, \mu \rangle_{\ell_2} = x^*(x) = 1,$$

so λ strongly exposes μ in B_{ℓ_2}. Thus $\lambda = \mu$ and $\|\mu\|_{\ell_2} = 1$ which is precisely the desired result. □

For every $x \in S_{JT}$ let us introduce $\mathcal{D}(x) = \{F \in \mathcal{F}_\infty : \sum_{S\in F} x_S^2 = 1\}$. By the preceding lemma $\mathcal{D}(x)$ is non-empty, but it does not need to be finite. To get around this problem we will consider the restriction of families in $\mathcal{D}(x)$ to a subtree of finite level. So for every $N \geq 1$ let us write $T_N = \text{level}[0, N] = \{\emptyset\} \cup \bigcup_{n=1}^N \{0,1\}^n$ the binary tree of height N and let us write $\mathcal{D}_N(x) = \{F \cap T_N : F \in \mathcal{D}(x)\}$ where $F \cap T_N = \{S \cap T_N : S \in F\}$. Then $\mathcal{D}_N(x)$ is a finite non-empty set and we have, similarly to the finite support case, the following lemma.

Lemma 3.4.10. *For every $x \in S_{JT}$ and for every $N \geq 1$, there is a constant $\eta_{x,N} > 0$ such that $\sum_{S\in F} x_S^2 < (1 - \eta_{x,N})^2$ for every F in $\mathcal{F}$ for which $F \cap T_N$ does not belong to $\mathcal{D}_N(x)$.*

Proof. If this was not true, then since $\mathcal{F} \cap T_N$ is also finite we could find a family $F \in \mathcal{F}$ of segments of T_N not belonging to $\mathcal{D}_N(x)$ and a sequence $(F_i)_{i\geq 1} \subset \mathcal{F}$ such that $F_i \cap T_N = F$ and $\sum_{S\in F_i} x_S^2 > \left(1 - \frac{1}{i}\right)^2$ for every $i \geq 1$. Using a compactness argument (weak* extractions for the sequence of corresponding molecules in B_{JT^*} and metrizability of the weak* topology on B_{JT^*}) we would then obtain a family $G \in \mathcal{F}_\infty$ for which $\sum_{S\in G} x_S^2 = 1$ and such that $G \cap T_N = F$. But then this would mean that $F \in \mathcal{D}_N(x)$ and we would get a contradiction. □

We will need a last easy fact which states that any $x \in S_{JT}$ has its norm almost concentrated on a subtree of finite level. The proof is elementary and comes from the definition of the norm of JT.

Lemma 3.4.11. *Let $x \in S_{JT}$ and let $\varepsilon > 0$. There is an $N \geq 1$ such that for every family $F \in \mathcal{F}_\infty$ of segments of T which do not intersect T_N one has $\sum_{S\in F} x_S^2 \leq \varepsilon^2$.*

With those tools in hand we can now prove the main result of this section.

Theorem 3.4.12. *Let $x \in S_{JT}$. Then x is not a Δ-point.*

Proof. Let us fix some $x \in S_{JT}$ and let us assume that x has infinite support. Let us also fix some $\varepsilon > 0$ and let us take some $N \geq 1$ for which x is almost concentrated on T_N in the sense of the previous lemma.

For every F in $\mathcal{D}_N(x)$ we pick a representative family $F_\mathcal{R}$ in $\mathcal{D}(x)$ which satisfy $F_\mathcal{R} \cap T_N = F$. This means that for every $S \in F$ there is (a unique) $S_\mathcal{R} \in F_\mathcal{R}$ such that $S_\mathcal{R} \cap T_N = S$. For

every such F we define $x_F^* = m_{x,F_{\mathcal{R}}}$ to be the molecule associated to x and $F_{\mathcal{R}}$ and we let x^* be the average of the x_F^*s with $F \in \mathcal{D}_N(x)$. Since each x_F^* is in $D(x)$, Lemma 3.1.3 tells us that $x^* \in D(x)$ and that

$$S\left(x^*, \frac{\varepsilon^2}{2n}\right) \subset \bigcap_{F \in \mathcal{D}_N(x)} S\left(x_F^*, \frac{\varepsilon^2}{2}\right)$$

for $n = |\mathcal{D}_N(x)|$. We now want to get a uniform bound for $\|x - y\|_{JT}$ on $S\left(x^*, \frac{\varepsilon^2}{2n}\right)$.

So let us take $y \in S\left(x^*, \frac{\varepsilon^2}{2n}\right)$ and let us fix some $G \in \mathcal{F}$. First observe that Lemma 3.4.10 allows us to get rid of the case $G \cap T_N \notin \mathcal{D}_N(x)$ exactly as in the finite support proof because it yields

$$\sum_{S \in G} (x_S - y_S)^2 \leq (2 - \eta_{x,N})^2.$$

So let us assume that $F = G \cap T_N$ belongs to $\mathcal{D}_N(x)$. We will split the family G into 4 disjoint subfamilies G_i, $1 \leq i \leq 4$, and we will estimate separately the sums $\sum_{S \in G_i} (x_S - y_S)^2$. For this we will use repeatedly the two following inequalities.

Claim A**.** Let H be a subfamily of $F_{\mathcal{R}}$. Then

$$\sum_{S \in H} (x_S - y_S)^2 \leq \varepsilon^2.$$

Proof of claim A. Since segments in H are in $F_{\mathcal{R}}$ and since $y \in S\left(x_F^*, \frac{\varepsilon^2}{2}\right)$ with $x_F^* = m_{x,F_{\mathcal{R}}}$ Lemma 3.4.7 yields

$$\sum_{S \in H} (x_S - y_S)^2 \leq \sum_{S \in F_{\mathcal{R}}} (x_S - y_S)^2 \leq \frac{2\varepsilon^2}{2} = \varepsilon^2.$$

□

Claim B**.** Let H be a family of disjoint segments of T which do not intersect T_N. Then

$$\sum_{S \in H} (x_S - y_S)^2 \leq \sum_{S \in H} y_S^2 + \varepsilon^2 + 2\varepsilon.$$

Proof of claim B. Since segments in H do not intersect T_N we have, by our initial choice of N, $\sum_{S \in H} x_S^2 \leq \varepsilon^2$ and so using Minkowski's inequality

$$\begin{aligned}\sum_{S \in H} (x_S - y_S)^2 &\leq \sum_{S \in H} x_S^2 + \sum_{S \in H} y_S^2 + 2\sqrt{\sum_{S \in H} x_S^2}\sqrt{\sum_{S \in H} y_S^2} \\ &\leq \sum_{S \in H} y_S^2 + \varepsilon^2 + 2\varepsilon.\end{aligned}$$

□

Now let us split the family G and let us start the computations of the corresponding sums.

Claim 1. Let $G_1 = \{S \in G : \ S \subset T_{N-1}\}$. There is a $\gamma_1 = \gamma_1(\varepsilon)$ such that

$$\sum_{S \in G_1} (x_S - y_S)^2 \leq \gamma_1.$$

Proof of Claim 1. Since segments in G_1 are contained in T_{N-1}, every segment S in G_1 has to be equal to its representative $S_{\mathcal{R}}$ because $S_{\mathcal{R}} \cap T_N = S$. This means that G_1 is a subfamily of $F_{\mathcal{R}}$ and the result follows directly from Claim A. □

Claim 2. Let $G_2 = \{S \in G : \ S \subset T \backslash T_N\}$. There is a $\gamma_2 = \gamma_2(\varepsilon)$ such that

$$\sum_{S \in G_2} (x_S - y_S)^2 \leq \sum_{S \in G_2} y_S^2 + \gamma_2.$$

Proof of Claim 2. Since segments in G_2 do not intersect T_N the result follows directly from Claim B. □

The remaining segments of G are those that intersect the Nth level of T. For those, we will distinguish between segments S such that S and its representative $S_{\mathcal{R}}$ are on the same branch of T and those for which S and $S_{\mathcal{R}}$ split at some higher level. So let G_3 be the subset of the remaining segments of G satisfying the first condition and let G_4 be the subset of those satisfying the second condition.

Note that if S is a segment in G_3, then since S and $S_{\mathcal{R}}$ are on the same branch and are equal in T_N we have either $S \subset S_{\mathcal{R}}$ or $S_{\mathcal{R}} \subset S$ and the complements are in either case segments which do not intersect T_N. Thus.

Claim 3. Let $G_{3,1} = \{S \in G_3 : \ S = S_{\mathcal{R}}\}$, let $G_{3,2} = \{S \in G_3 : \ S \subsetneqq S_{\mathcal{R}}\}$ and let $G_{3,3} = \{S \in G_3 : \ S_{\mathcal{R}} \subsetneqq S\}$. There is a $\gamma_3 = \gamma_3(\varepsilon)$ such that

$$\sum_{S \in G_3} (x_S - y_S)^2 \leq \sum_{S \in G_{3,2}} y_{S_{\mathcal{R}} \backslash S}^2 + \sum_{S \in G_{3,3}} y_{S \backslash S_{\mathcal{R}}}^2 + \gamma_3.$$

Proof of Claim 3. By definition $G_{3,1}$ is a subfamily of $F_{\mathcal{R}}$ so Claim A yields

$$\sum_{S \in G_{3,1}} (x_S - y_S)^2 \leq \varepsilon^2.$$

Now let us deal with $G_{3,2}$. By Claim A we know that

$$\sum_{S \in G_{3,2}} (x_{S_{\mathcal{R}}} - y_{S_{\mathcal{R}}})^2 \leq \varepsilon^2.$$

Moreover, for every $S \in G_{3,2}$, we know that the segment $S_{\mathcal{R}} \backslash S$ does not intersect T_N and thus Claim B yields

$$\sum_{S \in G_{3,2}} (x_{S_{\mathcal{R}} \backslash S} - y_{S_{\mathcal{R}} \backslash S})^2 \leq \sum_{S \in G_{3,2}} y_{S_{\mathcal{R}} \backslash S}^2 + \varepsilon^2 + 2\varepsilon.$$

So finally we get

$$\begin{aligned}\sum_{S\in G_{3,2}}(x_S-y_S)^2 &= \sum_{S\in G_{3,2}}[(x_{S_\mathcal{R}}-y_{S_\mathcal{R}})-(x_{S_\mathcal{R}\setminus S}-y_{S_\mathcal{R}\setminus S})]^2\\ &\le \sum_{S\in G_{3,2}}(x_{S_\mathcal{R}}-y_{S_\mathcal{R}})^2+\sum_{S\in G_{3,2}}(x_{S_\mathcal{R}\setminus S}-y_{S_\mathcal{R}\setminus S})^2\\ &\qquad +2\sqrt{\sum_{S\in G_{3,2}}(x_{S_\mathcal{R}}-y_{S_\mathcal{R}})^2}\sqrt{\sum_{S\in G_{3,2}}(x_{S_\mathcal{R}\setminus S}-y_{S_\mathcal{R}\setminus S})^2}\\ &\le \sum_{S\in G_{3,2}}y^2_{S_\mathcal{R}\setminus S}+\varepsilon^2+(\varepsilon^2+2\varepsilon)+4\varepsilon.\end{aligned}$$

Similar computations allow us to deal with $G_{3,3}$ and the conclusion follows by combining the 3 inequalities. □

Finally let us take $S\in G_4$. Since S and its representative $S_\mathcal{R}$ are equal on T_N the segments $S^-=S\setminus(S\cap S_\mathcal{R})$ and $S^-_\mathcal{R}=S_\mathcal{R}\setminus(S\cap S_\mathcal{R})$ do not intersect T_N (and are non-empty). Thus.

Claim 4. There is a $\gamma_4=\gamma_4(\varepsilon)$ such that

$$\sum_{S\in G_4}(x_S-y_S)^2\le 1+\sum_{S\in G_4}y^2_{S^-}+\sum_{S\in G_4}y^2_{S^-_\mathcal{R}}+\gamma_4.$$

Proof of Claim 4. Using again Claim A and Claim B, we have

$$a=\sum_{S\in G_4}(x_{S_\mathcal{R}}-y_{S_\mathcal{R}})^2\le\varepsilon^2,$$

$$b=\sum_{S\in G_4}(x_{S^-}-y_{S^-})^2\le\sum_{S\in G_4}y^2_{S^-}+\varepsilon^2+2\varepsilon,$$

and

$$c=\sum_{S\in G_4}(x_{S^-_\mathcal{R}}-y_{S^-_\mathcal{R}})^2\le\sum_{S\in G_4}y^2_{S^-_\mathcal{R}}+\varepsilon^2+2\varepsilon.$$

Thus we get, using the $\sqrt{u+v}\le\sqrt{u}+\sqrt{v}$ identity for the last term:

$$\begin{aligned}\sum_{S\in G_4}(x_S-y_S)^2 &= \sum_{S\in G_4}[(x_{S_\mathcal{R}}-y_{S_\mathcal{R}})-(x_{S^-_\mathcal{R}}-y_{S^-_\mathcal{R}})+(x_{S^-}-y_{S^-})]^2\\ &\le a+b+c+2\sqrt{ab}+2\sqrt{ac}+2\sqrt{bc}\\ &\le \varepsilon^2+\sum_{S\in G_4}y^2_{S^-}+\sum_{S\in G_4}y^2_{S^-_\mathcal{R}}+2(\varepsilon^2+2\varepsilon)+4\sqrt{\varepsilon^2(1+\varepsilon^2+2\varepsilon)}\\ &\qquad +2\sqrt{\sum_{S\in G_4}y^2_{S^-}}\sqrt{\sum_{S\in G_4}y^2_{S^-_\mathcal{R}}}+2\sqrt{2(\varepsilon^2+2\varepsilon)+(\varepsilon^2+2\varepsilon)^2}\end{aligned}$$

so it only remains to show that

$$2\sqrt{\sum_{S\in G_4} y_{S^-}^2}\sqrt{\sum_{S\in G_4} y_{S_{\mathcal{R}}^-}^2} \leq 1.$$

To show this, observe that if you take two different segments S and T in G_4, then you obviously have $S\cap T=\emptyset$ and $S_{\mathcal{R}}\cap T_{\mathcal{R}}=\emptyset$ since we work with families of disjoint segments, but you also have $S\cap T_{\mathcal{R}}=\emptyset$ and $T\cap S_{\mathcal{R}}=\emptyset$ since S and T have disjoint starting parts in T_N. Consequently:

$$\sum_{S\in G_4} y_{S^-}^2 + \sum_{S\in G_4} y_{S_{\mathcal{R}}^-}^2 \leq \|y\|_{JT}^2 = 1.$$

The conclusion follows since the function $f(x)=x\sqrt{1-x^2}$ has maximum $\frac{1}{2}$ on $[0,1]$ (attained at $\frac{1}{\sqrt{2}}$). □

Now letting $\gamma=\sum_{i=1}^4 \gamma_i$ and combining the results from the 4 claims, we obtain:

$$\begin{aligned}\sum_{S\in G}(x_S-y_S)^2 \leq 1 &+ \sum_{S\in G_2} y_S^2 + \sum_{S\in G_{3,3}} y_{S\setminus S_{\mathcal{R}}}^2 + \sum_{S\in G_4} y_{S^-}^2 \\ &+ \sum_{S\in G_{3,2}} y_{S_{\mathcal{R}}\setminus S}^2 + \sum_{S\in G_4} y_{S_{\mathcal{R}}^-}^2 + \gamma,\end{aligned}$$

and since we work with families of disjoint segments we get

$$\sum_{S\in G}(x_S-y_S)^2 \leq 3+\gamma.$$

Finally, $\|x-y\|_{JT} \leq \max\{2-\eta_{x,N}\sqrt{3+\gamma}\}$ and this quantity is strictly less than 2 if we take a small enough ε since γ goes to 0 as ε goes to 0. □

Remark 3.4.13. Here the segments involved can be infinite so the functional x^* we slice with does not have to be in JT_*. This raises the following question.

Question 3.4.14. Does JT admit weak* Δ-points?

From the work of [Gir] we know that the dual of the James tree space JT^* is AUC. Now JT does not admit an equivalent norm whose dual norm is AUC*. Indeed such norm would be AUS by the asymptotic duality. This is impossible because JT is not Asplund (it is separable while JT^* is not), see [JLPS, Proposition 2.4]. In view of those observations JT^* would be a natural candidate for our study and the question of the existence of Δ- or Daugavet-points there, as well as the question of the existence of non quasi-denting points would be particularly relevant. Unfortunately it seems that computations are still out of reach in this context, although the use of molecules facilitates them, and we where only able to obtain a few partial results even while trying to restrict ourselves to J^*.

Lemma 3.4.15. *Let $s, t \in T$ be two distinct points. For any $\varepsilon > 0$ and $z^* \in JT^*$ with $\{s,t\} \cap \operatorname{supp}(z^*) = \emptyset$ we have*

$$\|e_t^* - \varepsilon e_s^* + z^*\| > 1.$$

Proof. Let $x^* = e_t^* - \varepsilon e_s^* + z^*$ for some $\varepsilon > 0$ and z^* supported in $T \backslash \{s,t\}$. Assume first that $\varepsilon \geq 1$, and let $\alpha = 1/\sqrt{2}$ and $x = \alpha e_t - \alpha e_s$. We have $\|x\| = 1$ and

$$x^*(x) = \alpha + \varepsilon\alpha \geq 2/\sqrt{2} > 1.$$

Next if $\varepsilon < 1$ we let $\alpha = \sqrt{1-\varepsilon^2}$ and $x = \alpha e_t - \varepsilon e_s$. We have $\|x\| = 1$ since $\alpha^2 + \varepsilon^2 = 1$ and $(\alpha - \varepsilon)^2 \leq 1 - 2\alpha\varepsilon < 1$. Now

$$x^*(x) = \alpha + \varepsilon^2 > \alpha^2 + \varepsilon^2 = 1.$$

In both cases, $\|x^*\| > 1$. □

Lemma 3.4.16. *In JT^* every basis vector e_t^* is an extreme point of B_{JT^*}, and therefore a weak* denting point.*

Proof. Assume that x^* and y^* are norm one elements such that $e_t^* = \frac{x^*+y^*}{2}$. Then we have $x^*(e_t) = y^*(e_t) = 1$ and $x^*(e_s) = -y^*(e_s)$ for all $s \neq t$. If we have $x^*(e_s) < 0$ for some $s \neq t$, then with $z^* = x^* - (e_t^* + x^*(e_s)e_s^*)$ we get

$$\|x^*\| = \|e_t^* + x^*(e_s)e_s^* + z^*\| > 1$$

by Lemma 3.4.15. Hence $x^* = y^* = e_t^*$.

By Proposition 3.d.19 in [FGdB] we know that e_t^* is a point of weak* to norm continuity on the unit ball of JT^* so the conclusion follows by applying Choquet's lemma (see for example [FHH+, Lemma 3.69]) which tells that the weak* slices form a neighborhood basis of e_t^*. Indeed the continuity of the identity map at e_t^* then ensures that any ball around e_t^* contains a weak* slice containing e_t^*. □

Corollary 3.4.17. *No molecule in JT^* is a weak* Daugavet point.*

Proof. Let $x^* = \sum_{i\geq 1} \lambda_i \mathbb{1}_{S_i}$ where S_i are disjoint segments of T and $\sum_{i\geq 1} \lambda_i^2 \leq 1$. If there is an $i_0 \geq 1$ such that $\lambda_{i_0} = 1$, then $x^* = \mathbb{1}_{S_{i_0}}$. Since the biorthogonal functionals are (weak*) denting points we may assume that S_{i_0} contains at least two points of T. Now if we let s_{i_0} be the starting point of this segment, and if we let $T_{i_0} = S_{i_0} \backslash \{s_{i_0}\}$, we have $\left\|x^* - e_{s_{i_0}}^*\right\| = \left\|\mathbb{1}_{T_{i_0}}\right\| = 1$ so x^* is at distance strictly less than 2 from a (weak*) denting point in JT^* and thus cannot be a weak* Daugavet point.

Next let us assume that $\lambda_i \in (0,1)$ for every $i \geq 1$ and let us fix any $i_0 \geq 1$. We define as above s_{i_0} and $T_{s_{i_0}}$ (which might eventually be empty) and we let $T_i = S_i$ for any $i \neq i_0$. Since the T_i are disjoint segments of T we have $\left\|x^* - e_{s_{i_0}}^*\right\| \leq 1 - |\lambda_{i_0}| + \left\|x^* - \lambda_{s_{i_0}} e_{s_{i_0}}^*\right\| = 1 - |\lambda_{i_0}| + \left\|\sum_{i\geq 1} \lambda_i \mathbb{1}_{T_i}\right\| \leq 2 - \lambda_{s_{i_0}} < 2$ and the conclusion follows as above. □

In light of the above we ask:

Question 3.4.18. Can molecules in JT^* be Δ-points?

Note that the observation from [JRZ, Remark 2.4] applied to the set of molecules in JT^* which satisfies as mentioned $\overline{\operatorname{conv}}\,\mathcal{M} = B_{X^*}$ gives the following simplification.

Lemma 3.4.19. *Let $x^* \in S_{JT^*}$. Then x^* is a Δ-point if and only if we can find, for every $\varepsilon > 0$ and for every slice S of B_{JT^*}, some $m \in \mathcal{M} \cap S$ such that $\|x - m\| \geq 2 - \varepsilon$.*

Although it is possible to do some computations in very specific cases (for example when x^* is a molecule supported on two segments) it seems to be difficult to estimate the distance between two molecules in general even when they are supported on the same branch of T. Moreover, it is not completely trivial to distinguish norm one molecules in $\mathcal{M}$ and to find suitable norming elements in JT.

Finally let us mention that with techniques similar to those for JT, it is possible to prove that also J with the equivalent norms $\|\cdot\|_{\mathcal{J}}$ and $\|\cdot\|_0$ as given on p. 62 in [AK], fail to contain Δ-points. However, if we replace the binary tree T by the countably branching tree $T_\infty = \{\emptyset\} \cup \bigcup_{n\geq 1} \mathbb{N}^n$ in the definition of the James tree space we obtain the Banach space JT_∞ originally introduced in [GM1] which shares some of the basic properties of JT, but presents a few striking dissimilarities. Indeed it is proved in [GM1] that the predual of JT_∞ fails the PCP and as a consequence admits no equivalent AUC norm (in fact $\overline{\delta}_{|.|}(\frac{1}{2}) = 0$ for any equivalent norm $|.|$ on $(JT_\infty)_*$, see [Gir]) while it satisfy the so called convex PCP, see [GMS]. For our study we have as before that JT_∞ is 2-AUC* and admits no weak* Daugavet-points, but our proof of non-existence of Δ-points fails for infinitely supported elements since restricting the support of such element to a finite level of the tree does not necessarily provide finitely many nodes of T_∞ anymore. It is thus natural to ask the following.

Question 3.4.20. Does JT_∞ admit Δ-points?

Appendix A

The weakly null ℓ_1^+ index from Alspach Judd and Odell

As mentioned at the end of Section 1.2, the Szlenk index of any Banach space is separably determined (Proposition 1.2.26). This result proved to be useful in Chapter 2 since it allowed to extend for free a couple of linear and non-linear results from the separable setting, where some practical tools such as sequences or trees are readily available, to the non-separable setting.

However, as pointed out in Section 2.3, the separable determination of the Szlenk index by subspaces does not allow to get rid of separability assumptions in Theorem 2.3.11 since we might loose all the information on the asymptotic structure of the dual space X^* by passing to a subspace of X. For the original result from [BCD+] the determination of the Szlenk index by separable quotients in reflexive Banach spaces (Proposition 1.2.27) was used in order to overcome this hindrance. In the present appendix, we extend Proposition 1.2.27 to a larger setting, namely the class of weakly compactly generated Banach spaces, and as an application get rid of separability assumptions in Theorem 2.3.11 in this context. The main tool in our proof is the so called weakly null ℓ_1^+ index introduced in [AJO] and that we will define in the following section.

To conclude this short introductory paragraph, let us point out that the classical problem of the existence of separable quotients in Banach spaces is still open, and we refer to the surveys [Muj] and [FKLPS] for more details on the subject. In particular the extension of Proposition 1.2.27 to a fully general setting might be very difficult.

A.1 The weakly null ℓ_1^+ index

In order to introduce this index, we need to consider a more general definition for trees. We essentially follow [AJO, Section 2].

Definition A.1.1. We will call *tree* any non empty partially ordered set $(T, \leq)$ for which all the subsets of the form $\{y \in T : \ y \leq x\}$ with $x \in T$ are linearly ordered and finite. For any tree T we will use the following denomination.

1. A *root* of T is a minimal element of T and a *leaf* of T is a maximal element of T.

2. If an element $x \in T$ is not a root of T, then the *predecessor* of x is the maximal element of the set $\{y \in T : \ y < x\}$.

3. If an element $x \in T$ is not a leaf of T, then any minimal element of the set $\{y \in T : y > x\}$ is an *immediate successor* of x. The set of all immediate successors of x is called a *fork* of T, and we say that T is *countably branching* if every fork of T is countable.

4. A *branch* of T is a maximal linearly ordered subset of T. We say that T is *well founded* if it does not admit any infinite branch.

For any well founded tree T, we define the *derived tree* $D(T) := T \backslash L(T)$ where $L(T)$ is the set of all leaves of T. We also let $D(\emptyset) := \emptyset$. Then we define inductively trees T^α for any ordinal α by letting $T^0 := T$, $T^{\alpha+1} := D(T^\alpha)$ for every ordinal $\alpha \geq 0$, and $T^\beta := \bigcap_{\alpha<\beta} T^\alpha$ if β is a limit ordinal. The *order* of a tree T is defined as the minimum of the set $\{\alpha \geq 0 : \ T^\alpha = \emptyset\}$ if such an ordinal exists, and otherwise is given an abstract ∞ value.

Example A.1.2. The countably branching trees T_N from Section 1.2 are well founded trees, and the order of T_N is $N+1$ for every $N \geq 1$. However T_∞ is not well founded. In order to obtain well founded countably branching tree of higher order, we construct inductively subtrees T_α of T_∞ for every $\alpha < \omega_1$ by letting $T_0 := \{\emptyset\}$, $T_{\alpha+1} := \{\emptyset\} \cup \bigcup_{N\geq 1} N \frown T_\alpha$ for every ordinal $\alpha \geq 0$ where $N \frown T_\alpha := \{N \frown t, \ t \in T_\alpha\}$, and if β is a limit ordinal, we pick any ordering $(\alpha_N)_{N\geq 1}$ of $\{\alpha : \ 1 \leq \alpha < \beta\}$ and let $T_\beta := \{\emptyset\} \cup \bigcup_{N\geq 1} N \frown T_{\alpha_N}$. Then every T_α is well founded, has a single root $\emptyset$, and has order $\alpha+1$. Also note that T_α is a minimal countably branching tree of order $\alpha+1$ with the terminology from [AJO, Definition 2.3].

As is Section 1.2 a *tree* in a Banach space X will simply be a family $(x_s)_{s\in T}$ of elements of X indexed by a tree T, and the terminology used there extends naturally in the present context.

Following [AJO, Definition 3.6] we say that a finite sequence $(x_i)_{i=1}^n$ in a Banach space X is a K ℓ_1^+*-sequence* if it is normalized, K basic, and satisfies

$$\frac{1}{K}\sum_{i=1}^n a_i \leq \left\| \sum_{i=1}^n a_i x_i \right\|$$

for every sequence $(a_i)_{i=1}^n \in (\mathbb{R}_+)^n$ of positive real numbers. Also we will say that a tree in X is a K ℓ_1^+*-tree* if every branch of the tree forms a K ℓ_1^+-sequence. We define the weakly null ℓ_1^+ index of X by

$$I_w^+(X) := \sup_{K\geq 1} I_w^+(X,K)$$

where $I_w^+(X,K)$ is defined as the supremum of the set of all ordinals α for which there exists a countably branching weakly null K ℓ_1^+-tree of order α in X.

Remark A.1.3. The ordinal $I_w^+(X,K)$ is never a limit ordinal, and by minimality of the trees T_α it is also equal to the supremum of the set of all ordinals α for which there exists a weakly null K ℓ_1^+-tree indexed by T_α in X.

One of the main results from [AJO] is that the weakly null ℓ_1^+ index coincide with the Szlenk index for Banach spaces not containing ℓ_1. We sketch some of the main ideas of the proof below.

Theorem A.1.4. *Let X be a Banach space not containing ℓ_1. Then $S_Z(X) = I_w^+(X)$.*

Sketch of proof. As mentioned above, the Szlenk index is separably determined and it is clear that this does also hold true for the weakly null ℓ_1^+ index. Thus we may assume that the space X is separable. Now if X is separable and does not contain ℓ_1, then by [AJO, Theorem 3.14] we have that $I_w^+(X)$ is countable if and only if X^* is separable, and by Theorem 1.2.3 we know that this is equivalent to having $S_Z(X)$ countable. So we may also assume that X^* is separable.

Let us explain how the inequality $S_Z(X) \leq I_w^+(X)$ can be proved. First note that since X^* is assumed to be separable, the weakly null ℓ_1^+ index of X has to be of the form ω^α for some countable ordinal $\alpha < \omega_1$ by [AJO, Theorem 3.22]. As mentioned in Section 1.2 the same goes for the Szlenk index of X. Thus it is sufficient for our purpose to show that for any fixed countable ordinal $\alpha < \omega_1$, if $S_Z(X) > \omega^\alpha$ then $I_w^+(X) > \omega^\alpha$.

If such an α is fixed, and if we assume that $S_Z(X) > \omega^\alpha$, then following the proof of Lemma 1.2.17 we can use the separability assumption on X to show that X^* contains for some $\varepsilon > 0$ weak* convergent ε-climbing trees of order β for every $\beta \leq \omega^\alpha$. Next using Mazur's Lemma 1.3.13 we can construct as in Proposition 1.3.14 weakly null bi-orthogonal trees of order β in X and X^*, and using those trees we follow the proof of Theorem 1.3.16 to construct weakly null trees of order β whose branches all form K ℓ_1^+ sequences for some $K \geq 1$ independent of β. This yields $I_w^+(X, K) > \omega^\alpha$ since $I_w^+(X, K)$ is never a limit ordinal, and the conclusion follows.

The converse inequality follows from [AJO, Proposition 4.10]: if X contains a weakly null K ℓ_1^+-tree of order α, then $l^\alpha_{\frac{1}{K}} \neq \emptyset$ where l^α_ε is the derivation introduced just before Lemma 1.2.17.

□

A.2 Determination of the Szlenk index by separable quotients for weakly compactly generated Banach spaces

Let us recall that a Banach space X is *weakly compactly generated* (WCG in short) if there is a weakly compact subset K of X such that $\overline{\text{span}}(K) = X$. As we shall see the main reason for working with WCG Banach spaces is that this property passes to quotients and that the density of any WCG Banach space is equal to the weak* density of its dual. This can be found for example in [FHH$^+$, Section 13.1]. As mentioned, we will prove the following result.

Theorem A.2.1. *Let X be a WCG Banach space and let $\alpha < \omega_1$ be a countable ordinal. If $S_Z(X) > \alpha$ then there is a subspace Y of X such that the quotient $Z = X/Y$ is separable and satisfies $S_Z(Z) > \alpha$.*

Proof. Let us start by assuming that X does not contain ℓ_1. Then $I_w^+(X) = S_Z(X) > \alpha$ and we can find a weakly null K ℓ_1^+-tree $(x_s)_{s\in T_\alpha}$ of order α in X which witnesses this inequality. For every element s in T_α pick a norming functional f_s of x_s and for every branch β of T_α and

every sequence $a = (a_s)_{s\in\beta}$ in $\mathbb{Q}_+$ pick a norming functional $g_{a,\beta}$ of $\sum_{s\in\beta} a_s x_s$. Note that the set of all branches of T_α is countable because T_α is well founded. Then let F be the set of all functionals f_s and $g_{a,\beta}$ and let $Y = F^\top$ be the pre-orthogonal of F. It is a well known fact that the dual of the quotient $Z = X/Y$ is isometric to the weak* closure $\overline{F}^{w^*}$ of F in X^* which is weak* separable by construction. Now Z is WCG as quotient of a WCG space and thus it is separable since its density is the same as the weak* density of its dual. Moreover the tree $(Q(x_s))_{s\in T_\alpha}$ where Q is the quotient map is normalized in Z thanks to the functionals f_s and has branches which form $K - \ell_1^+$ sequences over $\mathbb{Q}$ thus also over $\mathbb{R}$ thanks to the functionals $g_{a,\beta}$. It follows that the weakly null ℓ_1^+ index of Z will also be strictly greater than α. Now Z cannot contain ℓ_1 as a quotient of a space not containing it (it is well known that ℓ_1 can be lifted from quotients) so $S_Z(Z) = I^+(Z) > \alpha$ and we are done.

Now if X contains ℓ_1 the same idea allows to build a separable quotient of X which also contains ℓ_1 and this is enough to conclude.

□

This allows to give a non-separable version of Theorem 2.3.11 for WCG Banach spaces.

Theorem A.2.2. *Let X be a WCG Banach space and let us assume that X^* has a weak* asymptotic unconditional structure. Also let G be any non-trivial countably branching bundle graph. Then the following assertions are equivalent.*

1. *The space X satisfies $S_Z(X) \leq \omega$.*
2. *The space X admits an equivalent norm whose dual norm is AUC*.*
3. *The space X^* admits an equivalent AMUC norm.*
4. *The family $(G^{\oslash N})_{N\geq 1}$ does not equi-Lipschitz embed into X^*.*

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Université Bourgogne Franche-Comté
32, avenue de l'Observatoire
25000 Besançon

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