Well-posedness and long-time behaviour of the Stokes-transport equation

Antoine Leblond

Sorbonne Université

LJLL

Doctoral School **Sciences Mathématiques de Paris Centre**

University Department **Laboratoire Jacques-Louis Lions**

Thesis defended by **Antoine Leblond**

Defended on **November 10, 2023**

In order to become Doctor from Sorbonne Université

Academic Field **Applied Mathematics**

Well-posedness and long-time behaviour of the Stokes-transport equation

Thesis supervised by	Anne-Laure Dalibard	Supervisor
	Julien Guillod	Co-advisor

Committee members

Referee	Francisco Gancedo	Professor, Université de Séville (DMA)	
Examiners	Franck Sueur	Professor, Université de Bordeaux (IMB)	Committee President
	Vincent Duchêne	Junior Researcher, Université de Rennes 1 (IRMAR)	
	Amina Mecherbet	Associate Professor, Université Paris Cité (IMJ-PRG)	
	Ayman Moussa	Associate Professor, Sorbonne Université (LJLL)	
Guest	Peter Korn	Dr. Habil., Max-Planck Institute for Meteorology	
Supervisors	Anne-Laure Dalibard	Professor, Sorbonne Université (LJLL)	
	Julien Guillod	Associate Professor, Sorbonne Université (LJLL)	

Colophon

Doctoral dissertation entitled "Well-posedness and long-time behaviour of the Stokes-transport equation", written by Antoine Leblond, completed on November 12, 2023, typeset with the document preparation system LaTeX and the yathesis class dedicated to theses prepared in France.

Sorbonne Université

LJLL

École doctorale **Sciences Mathématiques de Paris Centre**

Unité de recherche **Laboratoire Jacques-Louis Lions**

Thèse présentée par **Antoine Leblond**

Soutenue le **10 novembre 2023**

En vue de l'obtention du grade de docteur de Sorbonne Université

Discipline **Mathématiques Appliquées**

Caractère bien posé et comportement en temps long de l'équation de Stokes-transport

Thèse dirigée par	Anne-Laure Dalibard	Directrice
	Julien Guillod	Co-encadrant

Composition du jury

Rapporteur	Francisco Gancedo	Professeur, Université de Séville (DMA)	
Examinateur·ice·s	Franck Sueur	Professeur, Université de Bordeaux (IMB)	Président du jury
	Vincent Duchêne	Chargé de recherche, Université de Rennes 1 (IRMAR)	
	Amina Mecherbet	MCF, Université Paris Cité (IMJ-PRG)	
	Ayman Moussa	MCF, Sorbonne Université (LJLL)	
Invité	Peter Korn	Dr. Habil., Max-Planck Institute for Meteorology	
Encadrement	Anne-Laure Dalibard	Professeure, Sorbonne Université (LJLL)	
	Julien Guillod	MCF, Sorbonne Université (LJLL)	

This thesis has been prepared at

Laboratoire Jacques-Louis Lions
Sorbonne Université
Campus Pierre et Marie Curie
4 place Jussieu
75005 Paris
France

☎ +33 1 44 27 42 98
Web Site `https://ljll.math.upmc.fr/`

À la mémoire de ma mère.

Remerciements

Merci. Merci mille fois, à toutes celles et tous ceux que je cite dans ces quelques lignes. Il m'est impossible de séparer clairement les remerciements formels et informels, tant, naturellement, mon parcours mathématique et mon cheminement personnel sont intriqués. Ces trois années parisiennes, intenses en tous points, ont succédé à des épisodes compliqués. Pendant ces trois années, j'ai profité de toutes les opportunités qu'ils m'étaient données pour me rétablir, me réparer et avancer. C'est dans un tel contexte que j'ai accompli cette thèse et me suis impliqué pour les autres comme je l'ai pu. Mon parcours, j'en suis fier, et je vous le dois à toutes et tous, professeur·e·s, collègues, famille, ami·e·s, et proches. Il m'est impossible de détailler pour chacun·e de façon exhaustives les raisons de ma reconnaissance ; il me faudrait pour cela un second manuscrit. Je me restreins donc à quelques énumérations, extractions et factorisations, dans un style plus sobre que j'aurais pu le souhaiter, encore que...

Je suis reconnaissant envers Anne-Laure Dalibard et Julien Guillod, sans qui cette thèse n'aurait tout simplement pas existé. En plus de me faire profiter de votre expertise scientifique, de votre patience face à mes questions naïves et mes tâtonnements, vous vous êtes montré·e·s très disponibles, à l'écoute dans mes nombreuses périodes de doutes, et vos conseils ont été plus qu'appropriés. Merci également pour les opportunités scientifiques dont vous m'avez permis de bénéficier, des présentations, séminaires et voyages dont j'ai pu profiter grâce à vous. J'ai énormément de chance d'avoir été votre élève.

I am grateful to all the members of this jury, for the role they played this day and during my thesis. Many thanks to Francisco Gancedo and Franck Sueur for taking on the meticulous task of reporting this manuscript. Thank you Vincent Duchêne, Amina Mecherbet and Ayman Moussa for agreeing to examine my thesis. I also thank Peter Korn for attending my defense and the trust he has placed in me for a post-doctorate.

Merci aux membres du laboratoire Jacques-Louis Lions d'en avoir fait un lieu de travail et de vie agréable, à son directeur Emmanuel Trélat, et à l'ensemble du personnel administratif. J'y salue notamment Barbara Gris, Fabien Vergnet, Kévin Le Balc'h, Jacques Sainte-Marie, Jean-Yves Chemin, Julien Salomon, Frédérique Charles, Laurent Boudin, Malika Larcher, Nina Aguillon et Valeria Banica. Enseigner à Sorbonne Université aura constitué une expérience très formatrice, grâce aux enseignant·e·s et aux étudiant·e·s avec et pour lesquel·le·s j'ai travaillé. Par ailleurs, l'équipe ANGE, l'ERC BLOC et l'ANR SingFlows m'ont permis de participer à divers événements scientifiques. Un merci un peu spécial, à Thomas Alazard, pour un déclic crucial bien qu'involontaire.

Ce doctorat conclut un cursus entamé il y a dix ans. Mes professeurs·e·s de classes préparatoires au Lycée Corneille m'ont laissé des souvenirs impérissables, qui m'influencent toujours lorsque j'enseigne à mon tour. Stéphane, merci pour cela, pour ton support d'alors, et le lien conservé ; merci aussi à Carole. Ont suivi quatre années à l'École Normale Supérieure de Rennes et à l'Université de Rennes 1, en somme d'excellentes conditions pour apprendre des mathématiques, et obtenir un financement de thèse. J'ai toujours plaisir à repasser à l'IRMAR, y recroiser les ancienn·e·s étudiant·e·s connus d'alors, et toquer aux portes de Benjamin Boutin, Nicolas Seguin, Miguel Rodrigues et Vincent Duchêne. François, merci infiniment pour nos longs

et précieux échanges. Pour boucler la boucle, je salue l'équipe de GR, Karima, Jonathan, Aghilas, Paco, Marius, l'équipe enseignante, et les élèves que j'y ai encadré, qui m'ont ell·eux-mêmes beaucoup appris.

Lors de mon bref investissement au sein du Collectif Doctoral, j'ai eu la chance de fréquenter des doctorant·e·s engagé·e·s et inspirant·e·s tel·le·s que Claire, Dorian, Eliott, Gwenaëlle, Louise, Mathilde, Romain, Sirine et Viviane.

Je dois bien des choses aux jeunes chercheurs et chercheuses que j'ai cotoyé·e·s au LJLL ou rencontré·e·s lors de conférences, pour beaucoup devenu·e·s des ami·e·s. Parmi celles et ceux m'ayant précédé·e·s au laboratoire, je salue Gabriela, Gontran, Idriss, Jean, Ludovic et Sylvain qui ont pris le temps de me faire part de leur expérience. Mentions particulières à : Adrien, dont j'admire l'intensité ; Agustín, gran artista, cocinero y amigo ; Alexiane, pour nos longues discussions ; Anatole, qui m'émerveille toujours avec ses cartes et sa finesse ; Chourouk, one of the bravest - and mischievious - person I ever met, choukran! ; David L., orateur génial ; Elena, à la générosité incommensurable ; Emily, for sharing so much ; Hezekiah, for kindly communicating a crucial reference at a critical time ; Lucas E., à qui je pourrais dédier tout un paragraphe - en bref, merci pour toutes nos discussions mathématiques comme non-mathématiques, ton support et ta prévenance ; Lucas P., toujours à la barre ; María, queen dont les sages conseils me suivent toujours ; Nicolaï, pour la confiance ; Pauline, joviale co-bureau intermittente ; Pierre, animateur universel, au GTT comme aux soirées ; Robin, fidèle ~~aux Sciences~~ au poste ; Sahar, in memory of our intense chess games ; Theresa, für die konvexe Integration and Deutsche Lektionen ; Thomas B., pour la complicité. Merci aussi à Emma et Juliette, avec qui j'ai eu l'honner de co-représenter les doctorant·e·s du laboratoire. Je salue l'énergie d'Antonio et d'Eva, avec qui j'ai co-représenté les doctorant·e·s de l'ED 386. Merci également à Alexandre, Annamaria, Antoine Lec., Arthur, Charles, Corentin, Geoffrey, Giorgia, Guillaume, Liangying, Lucas J., Matthieu, Roxane, Sonia et Yvonne.

Il y a aussi les ami·e·s grâce auxquell·e·s j'ai traversé des pays, des années, de mauvais mais aussi et surtout de bons moments. La bande née à Rennes et qui s'est agrandie depuis, avec Clarence, David X., Emeline, Louis, Maxence et Tom, pour tout ce que l'on partage ; Anaïs, maîtresse du jazz ; Camille, mon plus ancien ami ; Eva et Etienne, pour tous les moments que vous avez créés, les soirées à refaire le monde et votre culture inépuisable ; Kévin, dont les traits d'esprit uniques ne cesseront de m'émerveiller ; Julie, pour la constance ; Simon, pour la métaphysique ; Thomas C., que j'estime beaucoup ; Vickie, pour ta sagesse et ta clairvoyance.

For my new life, I thank Christina, who made my arrival in Germany more than smooth, and who introduced me to Edda und Nikos, die besten Gastgeber in Hamburg; Vielen Dank für eure herzlichen Empfang und unschätzbare Hilfe.

Lucie, je ne suis pas sûr qu'il existe de mots suffisants pour vous témoigner ma reconnaissance, et malgré cela vous me comprendrez.

Enfin, merci à ma famille. À mes parents, qui m'ont permis de suivre ma voie. À ma petite sœur, exceptionnelle, Louise, dont le soutien compte tellement ; merci Jean-Paul, de si bien l'accompagner, et bien sûr à votre petit Pharell, qui illumine maintenant toute notre famille. Si Maman était encore là, elle serait fière de tout cela, de nous, tu ne crois pas ?

Well-posedness and long-time behaviour of the Stokes-transport equation

Abstract

The Stokes-transport equation models an incompressible, viscous and inhomogeneous fluid, subject to gravity. It is a reduced model for oceanography and sedimentation. The density is transported by the velocity field, satisfying at any time the momentum balance between viscosity, pressure and gravity effects, namely the Stokes equation. In the first part, we establish the global well-posedness of this system in bounded domains and in the infinite channel, in the weak sense and for Lebesgue initial data. The unbounded channel case is solved in uniformly local Sobolev spaces, with solutions having infinite energy. These results are compared with previous works in the whole space and in the plane. In the second part, we focus on the long-time evolution of the solutions of the Stokes-transport equation in a periodic channel. We show that a class of monotonous stratified density profiles is stable for small and regular enough perturbations. We consider no-slip boundary conditions on the velocity field, which raises mathematical difficulties due to the presence of boundary effects. We obtain explicit algebraic convergence rates and show that the density rearranges vertically and monotonously, in line with the common intuition of sedimentation. We also give a refined description of the density profile, involving a boundary layer expansion in the vicinity of the boundaries. Besides, we extend a previous result obtained for a related problem, proving that any stationary profile is unstable in low regularity topologies. We also highlight properties compatible with the conjecture that the density always stratifies. In the last part, we undertake a numerical study of the evolution of graph density interfaces governed by the Stokes-transport equation. Several behaviours are observed, from the convergence toward the flat rest interface to the graph break. We compare our observations with existing theoretical results.

Keywords: incompressible viscous fluid, Stokes-transport, active scalar equation, global well-posedness, steady Stokes equation, transport equation, long-time behaviour, stability, rearrangement, boundary layers, numerical observation

---

Caractère bien posé et comportement en temps long de l'équation de Stokes-transport

Résumé

L'équation de Stokes-transport modélise un fluide incompressible, visqueux et inhomogène, soumis à la gravité. Il s'agit d'un modèle réduit d'océanographie et de sédimentation. La densité est transportée par le champ de vitesse du fluide, satisfaisant à tout instant l'équilibre entre les effets de viscosité, de pression et de gravité, d'après l'équation de Stokes. Dans la première partie, nous établissons le caractère bien posé de ce système dans les domaines bornés et dans un canal infini, au sens faible et pour des données intégrables. La cas du canal inclut des solutions d'énergie infinie, impliquant des espaces de fonctions uniformément localement Sobolev. Ces résultats sont comparés à des travaux antérieurs, dans l'espace et le plan. Dans la deuxième partie, nous nous concentrons sur le comportement en temps long des solutions de l'équation de Stokes-transport dans un canal périodique. Nous montrons qu'une classe de profils stratifiés est stable pour des perturbations assez petites et régulières. Nous supposons le fluide non-glissant aux bords, ce qui pose des problèmes particuliers dûs aux effets de bords induits. Nous obtenons des taux de convergence algébriques et montrons que la densité se réarrange verticalement et de façon monotone. Nous donnons également un développement de type couche limite du profil de densité à proximité des bords. En outre, nous prouvons, en adaptant un résultat antérieur, que tout profil stationnaire est instable pour des perturbations peu régulières. Nous mettons enfin en évidence des propriétés du système, compatibles avec la conjecture selon laquelle la densité tend toujours à se réordonner. Dans la dernière partie, nous menons une analyse numérique de l'évolution d'interfaces de densité de type graphe, gouvernée par l'équation de Stokes-transport. Plusieurs comportements sont observés, de la convergence vers l'équilibre plat à la rupture de graphe. Nous comparons nos observations à des résultats théoriques existants.

Mots clés : fluide visqueux incompressible, Stokes-transport, active scalar equation, caractère globalement bien-posé, équation de Stokes stationnaire, équation de transport, comportement en temps long, stabilité, réarrangement, couches limites, observation numérique

---

Laboratoire Jacques-Louis Lions
Sorbonne Université – Campus Pierre et Marie Curie – 4 place Jussieu – 75005 Paris – France

Foreword

Everyone possesses the intuition that heavy fluids sink. Nevertheless, the consequences of this rather simple principle lead to complex evolutions and behaviours. The aim of this thesis is to study various aspect of the Stokes-transport (ST) equation, which is a reduced system modeling this phenomenon and illustrating this complexity. The problem writes

$$\left\{\begin{array}{rcl} \partial_t \rho + \boldsymbol{u}\cdot\nabla\rho &=& 0 \\ -\Delta \boldsymbol{u} + \nabla p &=& -\rho \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} &=& 0 \\ \rho|_{t=0} &=& \rho_0, \end{array}\right. \tag{ST}$$

where ρ represents a fluid density, temperature or buoyancy, transported by a velocity field $\boldsymbol{u}$ satisfying at any time the balance between the viscosity effects, the pressure gradient ∇p, and the vertical gravity forcing $-\rho\boldsymbol{e}_z$. The fluid is also assumed incompressible through the divergence-free condition on $\boldsymbol{u}$. All the unknowns depend on time $t \geq 0$ and on a space variable $\boldsymbol{x} \in \Omega$ on some spatial domain Ω which will be specified later. The system must also be endowed with extra boundary conditions on $\boldsymbol{u}$, which are specified later on, depending on the type of spatial domain considered. This problem belongs to the class of active scalar equations, where the velocity field depends nonlocally on the quantity advected.

This manuscript is organised as follows. Chapter 1, the introduction, specifies the problem and its context in Section 1.1, and the three axes of our works, which are the well-posedness of (ST) in Chapter 2, the long-time behaviour of the solutions in Chapter 3 and a numerical study of the graph interface problem for (ST) in Chapter 3, respectively in Sections 1.2, 1.3 and 1.4. These latter sections contain summaries and states of the art of the aforementionned subjects, whereas the corresponding chapter are devoted to the author contributions. The two publications to which the author contributed are included, namely [Leb22] in Chapter 2 and [DGL23] in Chapter 3.

The Stokes-transport equation is a reduced model for oceanography, in particular for the Boussinesq approximation, adapted for fluid regims with small density variations. This system has also been derived in several ways from microscopic formulations of sedimenting particles in a viscous fluid. It also shares a similar active scalar equation structure with the Incompressible Porous Medium (IPM) equation and the Fractional Stokes-transport (FST) equation, the latter generalising the two others, see Section 1.1.

The Stokes-transport equation is globally well-posed in various frameworks, in a weak sense, for integrable data. A precise state of the art including synthetic overviews of the existing results is provided in Section 1.2. We establish in particular the well-posedness of this system in bounded domains of $\mathbf{R}^d, d \in \{1,2\}$ and in the infinite channel $\mathbf{R}\times(0,1)$ for data in L^∞, see Theorem 1.2.4, which is the main result of [Leb22], see Chapter 2. As a corollary, this results also holds in the periodic channel $\Omega = \mathbf{T}\times(0,1)$. The unbounded channel case is solved in uniformly local Sobolev spaces, with solutions having infinite energy. In bounded domains, refinements of this

result for L^q data with finite Lebesgue exponent q is obtained, global well-posedness holding for $q > d$ and existence for $q > 2d/(d+2)$. We also establish the global well-posedness of (ST) for regular data in H^m, see Theorem 1.2.5. These results and in particular the regularities at play are consistent with previous studies carried out in the whole space $\mathbf{R}^3$ and plane $\mathbf{R}^2$. These features are compared with those of (IPM) and (FST) in Subsection 1.2.3. We also investigate a hydrostatic approximation of this system, consisting in neglecting the vertical velocity dissipation and assuming that the pressure satisfies the hydrostatic balance, namely

$$\left\{ \begin{array}{rcl} -\Delta u_1 + \partial_x p & = & 0 \\ \partial_z p & = & -\rho. \end{array} \right.$$

This assumption is classical in geophysical fluids disciplines. The associated system is locally well-posed for regular data, see Theorem 1.2.7 and Subsection 1.2.4.

The long-time behaviour of the solutions in the periodic channel $\Omega = \mathbf{T} \times (0,1)$ is then investigated in Chapter 3, including the contribution [DGL23] written together with Dalibard and Guillod. A cornerstone of this analysis is the monotonicity of the potential energy of the system, defined as follows and which satisfies an energy balance with the viscosity dissipative effects,

$$E(t) = \int_\Omega z\rho(t,x,z)\,\mathrm{d}x\,\mathrm{d}z, \qquad \frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|\nabla \boldsymbol{u}\|_{L^2}^2.$$

The monotonicity of the potential energy suggests, according to the physical intuition of sedimentation, that the fluid always rearranges itself in a more stable way, with the heavy fluid under the lighter one. We investigate this conjecture and introduce the monotonous vertical rearrangement of functions on Ω. In particular we establish that the ω-set in H^{-1} of any initial data $\rho_0 \in L^\infty(\Omega)$ is never empty, and only contains stationary solutions of the problem which are the stratified profiles, see Subsection 1.3.1 and Section 3.6. The main part of the chapter is devoted to the demonstration of a stability result for the Stokes-transport equation for data $\rho_0 \in H^6(\Omega)$ close enough to linear profiles, see Theorem 1.3.5 and to the formation of density boundary layers, see Theorems 1.3.6 and 1.3.7, and Subsection 1.3.3. In this case, we prove that the density converges toward its monotonous vertical rearrangement. The first stability result extends previous works on asymptotic behaviour on (IPM), see Subsection 1.3.2. Some peculiar difficulties are raised in our case of interest, due to the no-slip condition imposed to the velocity of the fluid on the boundaries. Basically this assumption induces boundary effects, which raise mathematical obstacles to obtain decay estimates on the the solution. These difficulties are overcome by a thorough analysis for low order derivatives of the solutions. Moreover, properties and eigenfunction basis of the biharmonic operator with Dirichlet boundary conditions illustrate the reasons of these mathematical difficultes and boundary layers formation, see Section 3.8. We provide a refined description of the density profile, involving a boundary layer expansion in the viscinity of the boundaries. In particular, we obtain explicit decay rates on their typical size with respect to time. Besides, we also extend to (ST) a previous instability result of Kiselev & Yao on (IPM), and we obtain that any stationary profile for the system is unstable for some perturbation in H^{2^-}, see Theorem 1.3.9, Subsection 1.3.4 and Section 3.7. Note that there is no contradiction between these instability results and the previous stability statements, for which initial data have to be small in higher Sobolev spaces.

We study in Chapter 4 the graph interface problem governed by the Stokes-transport equation in the periodic channel $\Omega = \mathbf{T} \times (0,1)$. In all generalities, since (ST) is well-posed for initial data $\rho_0 \in L^\infty(\Omega)$, we can consider patch solutions, namely solutions associated to the indicator of a subdomain of Ω. These stay patches for all time, since the density is transported. For (ST), such patches having of Hölderian regularity $C^{1,\mu}, \mu \in (0,1)$ remains so, for all time, see

Subsection 1.4.2 for further references on this matter. We focus on patches on Ω associated to graphs $\eta : \mathbf{T} \to (0,1)$ modeling the interface between two immiscible fluids, in the following sense

$$\rho(x,z) = \mathbf{1}_{\{z<\eta(x)\}}.$$

This situation is illustrated in Figure 1.1. For an initial graph η_0, if the associated density solution to (ST) remains the indicator of a hypograph, the underlying graph $\eta(t)$ satisfies the following

$$\partial_t \eta(t,x) + u_1(t,x,\eta(t,x))\partial_x \eta(t,x) = u_2(t,x,\eta(t,x)), \quad \text{for } (t,x,z) \in [0,T) \times \Omega,$$

where $\boldsymbol{u} = (u_1, u_2)$ satisfies the Stokes equation with density source of the form above. An overview of this problem, tackled in $\mathbf{T} \times \mathbf{R}$, by Gancedo, Granero-Belinchón & Salguero is repported in Subsection 1.4.3. They proved that this problem is globally well-posed for data close enough in H^3 to a flat interface. We supplement this overview with a summary on the related Muskat problem. Our contribution lies in a few analytical remarks on the interface regularity and a discussion on Lyapunov functionals for this system, see Section 4.1, and in a numerical investigation on the evolution of the interface in $\mathbf{T} \times (0,1)$. We set an Arbitrary Lagrangian-Eulerian numerical scheme, consisting in solving the Stokes equation with a finite elements method and to move the mesh, including a discretisation of the interface, according to this velocity, see Subsection 4.2.2. Then we investigate some quantitative and qualitative properties of the interface evolution, including convergence rates compared to the theoretical results of Gancedo & Granero-Belinchón in Subsection 4.2.4, boundaries effects in Subsection 4.2.5, possible Lyapunov functionals in Subsection 4.2.7 and we observe a graph break in Subsection 4.2.8.

Avant-Propos

Tout un chacun a bien l'intuition que deux fluides de densités différentes, une fois mélangés, doivent se réordonner avec le plus lourd sous le plus léger. Néanmoins, ce principe relativement simple induit des évolutions et des comportements complexes. Cette thèse étudie différents aspects de l'équation de Stokes-transport (ST), qui est en particulier un modèle réduit pour la sédimentation, illustrant cette complexité. Le problème s'écrit

$$\left\{\begin{array}{rcl} \partial_t \rho + \boldsymbol{u}\cdot\nabla\rho &=& 0 \\ -\Delta \boldsymbol{u} + \nabla p &=& -\rho \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} &=& 0 \\ \rho|_{t=0} &=& \rho_0, \end{array}\right. \tag{ST}$$

où ρ représente la densité, température ou flottabilité d'un fluide, transporté par un champ de vitesse $\boldsymbol{u}$ satisfaisant à tout instant l'équilibre entre les effets de viscosité, le gradient de pression ∇p et les effets verticaux de la gravité $-\rho \boldsymbol{e}_z$. Le fluide est également supposé incompressible *via* la condition de divergence nulle sur $\boldsymbol{u}$. Toutes les inconnues dépendent du temps $t \geq 0$ et d'une variable d'espace $\boldsymbol{x}$ sur un domaine Ω précisé plus loin. Le système doit également être complété par des conditions de bords sur $\boldsymbol{u}$, selon le type de domaine considéré. Ce problème fait partie d'une classe d'équations de transport pour lesquelles le champ de vitesse dépend non-localement de la quantité advectée.

Ce manuscrit est organisé comme suit. Le Chapitre 1, introductif, précise le problème ainsi que les trois axes principaux de ce travail portant sur le caractère bien posé de (ST), associé à la Section 1.2 et au Chapitre 2, le comportement en temps long des solutions, traité à la Section 1.3 et au Chapitre 3, et une étude numérique de l'évolution d'un graphe d'interface pour (ST), voir la Section 1.4 et le Chapitre 4. Les deux publications auxquelles l'auteur a contribué sont incluses dans ce manuscrit, avec [Leb22] au Chapitre 2 et [DGL23] au Chapitre 3.

L'équation de Stokes-transport est un modèle océanographique réduit, à rapprocher du système de Boussinesq, ce dernier étant adapté à des régimes où la densité du fluide varie peu. Ce système a également été dérivé depuis plusieurs formulations microscopiques de particules sédimentant dans un fluide visqueux. Il partage une structure commune avec l'équation des milieux poreux (IPM) et l'équation de Stokes-transport Fractionnaire (FST), cette dernière généralisant les deux autres, voir Section 1.1.

L'équation de Stokes-transport est globalement bien posée dans plusieurs cas, au sens faible et pour des données intégrables. Un état de l'art est dressé et synthétise les résultats existants pour les divers domaines et régularités, voir Section 1.2. On établit en particulier le caractère bien posé de ce système dans les domaines bornés de $\mathbf{R}^d, d \in \{1, 2\}$ et dans le canal infini $\mathbf{R} \times (0, 1)$ pour des données L^∞. Ceci constitue le Théorème 1.2.4, le résultat principal de [Leb22], voir Chapitre 2. En corollaire, le même résultat se transpose au canal périodique $\Omega = \mathbf{T} \times (0, 1)$. Le cas du canal non-borné est traité grâce à des fonctions localement uniformément Sobolev afin d'inclure des solutions d'énergie infinie. Pour les domaines bornés nous précisons ce résultat

pour des données L^q, nous montrons que l'équation est toujours globalement bien posée lorsque $q > d$ et qu'il existe toujours des solutions lorsque $q > 2d/(d+2)$. Nous démontrons également le caractère bien posé de (ST) pour des données Sobolev, plus régulières, voir le Théorème 1.2.5. Ces résultats sont cohérents avec les travaux effectués pour les domaines $\mathbf{R}^3$ et $\mathbf{R}^2$, avec des régularités comparables. Ces propriétés sont également confrontées aux résultats disponibles pour (IPM) et (FST), voir la Sous-section 1.2.3. Nous considérons enfin une approximation hydrostatique de l'équation de Stokes-transport, qui consiste à négliger l'accélération verticale et à supposer que la pression satisfait l'équilibre hydrostatique à tout instant, c'est-à-dire

$$\left\{ \begin{array}{rcl} -\Delta u_1 + \partial_x p & = & 0 \\ \partial_z p & = & -\rho. \end{array} \right.$$

Cette hypothèse est classique lorsque l'on considère des fluides géophysiques, en particulier en météorologie et en climatologie. Nous montrons que le système associé est localement bien posé pour des données Sobolev, voir le Théorème 1.2.7 et la Sous-section 1.2.4.

Nous étudions le comportement en temps long des solutions dans le canal périodique $\Omega = \mathbf{T} \times (0,1)$, Chapitre 3, incluant la contribution [DGL23] co-écrite avec Anne-Laure Dalibard et Julien Guillod. Un point central de cette analyse est la monotonicité de l'énergie potentielle, définie comme suit et dont la décroissance correspond à la dissipation due aux effets de viscosité,

$$E(t) = \int_\Omega z\rho(t,x,z)\,\mathrm{d}x\,\mathrm{d}z, \qquad \frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|\nabla \boldsymbol{u}\|_{L^2}^2.$$

Cette monotonie suggère, conformément à l'intuition physique de la sédimentation, que le fluide se réordonne toujours verticalement en une configuration plus stable. Nous tentons d'apporter des réponses à cette conjecture, et pour ce faire introduisons le réarrangement vertical et monotone des fonctions sur Ω. En particulier, on établit que l'ensemble ω-limite dans la topologie de H^{-1} est, pour toute donnée initiale $\rho_0 \in L^\infty(\Omega)$, non-vide et ne contient que des profils de densités stationnaires, dont on vérifie qu'il s'agit des profils stratifiés, voir la Sous-section 1.3.1 et la Section 3.6. L'essentiel de ce chapitre est consacré à la démontration de la stabilité de l'équation de Stokes-transport pour des données $\rho_0 \in H^6(\Omega)$ suffisamment proches de profils linéaires, voir le Théorème 1.3.5, et de décrire la formation de couches limites, voir les Théorèmes 1.3.6 et 1.3.7, ainsi que la Sous-section 1.3.3. Dans ce cas, nous prouvons que la densité converge vers le réarrangement vertical monotone. Le résultat de stabilité se base sur des travaux antérieurs portant sur le comportement asymptotique des solutions de (IPM), voir la Sous-section 1.3.2. Des difficultés particulières apparaissent dans notre cas, à cause de la condition de non-glissement imposée pour le fluide sur les bords. Cette condition induit des effets de bords, qui posent des problèmes pour l'obtention d'estimations de décroissance des solutions. Ces difficultés sont surmontées par une analyse précise des dérivées d'ordre faible de ces solutions. De plus, des propriétés de l'opérateur biharmonique sur Ω pour les conditions de Dirichlet sont données, avec en particulier une description d'une base de fonctions propres adaptées, qui illustre les difficultés que posent ce problème, ainsi que l'apparition des couches limites, voir la Section 3.8. Nous donnons donc une description fine du profil de densité, impliquant un développement de type couches limites aux bords, avec une décroissance explicite de leur taille caractéristique. Par ailleurs, nous adaptons pour (ST) un résultat d'instabilité montré par Kiselev & Yao pour (IPM), et montrons que tout profil stationnaire est instable pour des perturbations peu régulières, en l'occurence dans $H^{2^-}(\Omega)$, voir le Théorème 1.3.9, la Sous-section 1.3.4 et la Section 3.7. Notons que ce résultat n'est pas contradictoire avec la stabilité évoquée ci-avant, pour laquelle les perturbations doivent être plus régulières et suffisamment petites.

Dans le Chapitre 4 nous rendons compte de notre étude numérique de l'évolution d'interface de densité de type graphe, gouvernée par l'équation de Stokes-transport, dans le canal périodique

$\Omega = \mathbf{T} \times (0,1)$. En toute généralité, puisque (ST) est bien posée pour des données initiales $\rho_0 \in L^\infty(\Omega)$, il est possible de considérer des patchs de densité, c'est à dire des profils correspondant à des indicatrices de sous-domaines de Ω. Les solutions associées restent des patchs puisque la densité est simplement transportée. Pour (ST), un patch dont l'interface est de régularité Höldérienne $C^{1,\mu}, \mu \in (0,1)$ préserve cette régularité pour tout temps, voir la Sous-section 1.4.2 à ce sujet. Nous nous concentrons sur les patchs associé à des graphes de fonctions $\eta : \mathbf{T} \to (0,1)$, modélisant l'interface entre deux fluides non-miscibles, de la forme

$$\rho(x,z) = \mathbf{1}_{\{z<\eta(x)\}}.$$

Cette situation est illustrée par la Figure 1.1. Pour une donnée initiale η_0, tant que la solution associée à (ST) est de la forme ci-dessus, le graphe η sous-jacent satisfait le système suivant

$$\partial_t \eta(t,x) + u_1(t,x,\eta(t,x))\partial_x \eta(t,x) = u_2(t,x,\eta(t,x)), \quad \text{pour } (t,x,z) \in [0,T) \times \Omega,$$

où $\boldsymbol{u} = (u_1, u_2)$ satisfait l'équation de Stokes avec un terme source de la forme ci-dessus. Un résumé de la contribution de Gancedo, Granero-Belinchón & Salguero, traitant ce problème sur $\mathbf{T} \times \mathbf{R}$, est donné à la Sous-section 1.4.3. Ils ont notamment prouvé que ce problème est globalement bien posé pour des données suffisamment proches du graphe plat, dans H^3. Nous présentons également une synthèse de résultats associé à des problèmes d'interface voisins, notamment le problème de Muskat. Notre contribution se porte sur quelques remarques théoriques sur la régularité de l'interface, sur les possibles fonctionnelles de Lyapunov de ce système, voir la Section 4.1, et sur une étude numérique de l'évolution des graphes d'interface. Nous avons mis en place un schéma impliquant une méthode Lagrangienne-Eulérienne, consistant à résoudre l'équation de Stokes grâce à des éléments finis adaptés et à transporter le maillage, incluant une discrétisation de l'interface, par le champ de vitesse obtenu, voir la Sous-section 4.2.2. De plus, nous observons quelques propriétés quantitatives et qualitives des interfaces simulées, en comparant notamment à la Sous-section 4.2.4 les taux de convergence observés à ceux, théoriques, obtenus par Gancedo & Granero-Belinchón, ainsi que des effets dûs à la présence de bords à la Sous-section 4.2.5, les possibles fonctionnelles de Lyapunov à la Sous-section 4.2.7, et enfin une rupture de graphe à la Sous-section 4.2.8.

Contents

Chapter 1

Introduction

Everyone possesses the intuition that heavy fluids sink. Nevertheless, the consequences of this rather simple principle lead to complex evolutions and behaviours. The Stokes-transport equation is a reduced system modeling this phenomenon, and illustrating this complexity. We present and motivate this equation in Section 1.1. In Section 1.2, we list the well-posedness studies around this equation, including our contributions on the subject, which are developed in Chapter 2. Section 1.3 is devoted to long-time behaviours for this system, including stability and instability results, developed in Chapter 3. Section 1.4, associated to Chapter 4, provides an overview of free interface problems related to the Stokes-transport equation, lists possible theoretical approaches to tackle this matter, and contains a numerical study of the graph interface evolution.

1.1 The Stokes-transport equation

The Stokes-transport (ST) equation writes

$$\left\{\begin{array}{rcl} \partial_t \rho + \boldsymbol{u}\cdot\nabla\rho &=& 0 \\ -\Delta \boldsymbol{u} + \nabla p &=& -\rho \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} &=& 0, \end{array}\right. \tag{ST}$$

where ρ represents a fluid density or temperature, $(\boldsymbol{u}, p)$ its velocity and pressure fields, and $\boldsymbol{e}_z$ a unitary upward vertical vector. All the unknowns depend on time $t \geq 0$ and on a space variable $\boldsymbol{x} \in \Omega$ on some spatial domain Ω which will be specified later. The system must also be endowed with extra boundary conditions on $\boldsymbol{u}$, which are specified later on, depending on the type of spatial domain considered. This partial differential equation is a simplified model of ideal incompressible, viscous and inhomogeneous fluid. In Subsection 1.1.1, we explain to what extent (ST) is a reduced model for oceanography. Subsection 1.1.2 is dedicated to works about the derivation of (ST) from other models. In Subsection 1.1.3, we emphasise about the active scalar class of equation to which (ST) belongs. We also introduce there the Incompressible Porous Medium (IPM) equation and Fractional Stokes-transport (FST) equation, belonging to the same class of equation as (ST) and whose properties and techniques of study are comparable. Most of the previous works on (ST) were done on $\Omega = \mathbf{R}^3$. Our contributions focus on bounded domains of $\mathbf{R}^2$ and $\mathbf{R}^3$, on the infinite channel $\mathbf{R}\times(0,1)$ and the periodic channel $\mathbf{T}\times(0,1)$.

1.1.1 Reduced model for oceanography

The Stokes-transport equation is a reduced model for the Boussinesq equation. The Boussinesq approximation for the incompressible Navier-Stokes equation with inhomogeneous density writes

$$\left\{\begin{array}{rcl} \varrho_0(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \nabla p &=& -\rho g \boldsymbol{e}_z \\ \partial_t \rho + \boldsymbol{u}\cdot\nabla\rho &=& \kappa\Delta\rho \\ \operatorname{div}\boldsymbol{u} &=& 0, \end{array}\right.$$

where ρ denotes either the density, buoyancy or temperature variation around a constant ϱ_0, ν the viscosity, κ the density or thermal diffusivity, g the gravitational acceleration constant. It consists in neglecting the density variations in the inhomogeneous Navier-Stokes equation, while keeping the gravity effect $-\rho g\boldsymbol{e}_z$. Moreover, the density is still assumed to be advected by the fluid and possibly diffused when $\kappa > 0$. This system is known to be globally well-posed when at least one of the two coefficients ν and κ. The global regularity in the case $\kappa = \nu = 0$ remains unanswered. For an exhaustive review we refer to [Hu+16].

This equation is suitable for modeling the oceanic and atmospheric fluid evolutions. To give only one example, it is effectively used in global climate models such as the ICON-Earth System Model [Jun+22], implemented in the German operational forecast prediction. Of course more exhaustive systems of equations are considered, taking into account the evolution of the density, temperature, salinity, including thermodynamical state equations, ocean-atmosphere coupling, in different geometries, and other refinements. The interest of reduced models is to capture thoroughly enough the behaviour of the initial model, to ease the understanding of the problem and to reduce the simulation costs when time is an important factor, such as for weather forecast.

Let us return to the Boussinesq approximation. Depending on the parameters ratio, different regimes can be considered, and this system can be itself approximated by reduced models. In nondimensional form, the Boussinesq equation can be written as

$$\left\{\begin{array}{rcl} \frac{1}{\mathrm{Pr}}(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) - \Delta\boldsymbol{u} + \nabla p &=& -\mathrm{Ra}\rho \boldsymbol{e}_z \\ \partial_t \rho + \boldsymbol{u}\cdot\nabla\rho &=& \Delta\rho \\ \operatorname{div}\boldsymbol{u} &=& 0, \end{array}\right. \tag{1.1}$$

where $\mathrm{Pr} = \nu/\kappa$ and Ra are independent and dimensionless numbers, called after Prandtl and Rayleigh. The Rayleigh number is the ratio of the diffusion time scale over the advection time scale. The Prandtl number expresses the ratio between the viscosity and the diffusivity of the system. Assuming the diffusivity to be negligible and the viscosity effects importants, we formally consider the asymptotic $\mathrm{Pr} \to \infty$ and omit the self-advection term in (1.1). Neglecting accordingly the diffusion in the transport equation, we obtain (ST). Basically, the Stokes-transport equation models an inhomogeneous fluid influenced by gravity, with momentum balance between viscosity, pressure and gravity effects satisfied instantaneously and for all time. In other words it consist in neglecting the inertia of the system. This asymptotic regime is mentionned in [Doe+18], where the long-time behaviour of the Boussinesq equation is studied, as a reduced model of (1.1) for which the obtention of explicit rates for the long-time behaviour of the solutions might be eased.

The Stokes-transport equation has also been mathematically derived from a system of particle sedimenting in a Stokes fluid, and from a Vlasov-Stokes coupling, corresponding to the microscopic formulation of the aforementionned problem. These are reported in Subsection 1.1.2.

1.1.2 Derivation of the Stokes-transport equation

The works [Höf18a; Mec18] by Höfer and Mecherbet have been central in the development of this thesis. In these two articles, they mathematically derive the Stokes-transport equation from a microscopic formulation of sedimenting particles in a fluid. Since the techniques appearing in the proof have their counterpart in the analysis of the limit system (ST), we sketch the ideas of the demonstration below. In both works, the authors establish the well-posedness of (ST) in $\mathbf{R}^3$, which is detailed in Section 1.2. Note that Höfer also derived in [Höf18b] the Stokes-transport equation from a Vlasov–Stokes coupling, which corresponds to the kinetic formulation of the sedimentation.

Consider N spherical and solid particles of radii R sedimenting in a fluid filling the whole space $\mathbf{R}^3$. For i in $[\![1, N]\!]$ the particle i is centered at time t at position $\boldsymbol{X}_i(t)$ and occupies the domain $B_i = B_i(\boldsymbol{X}_i(t), R)$. The fluid is assumed to satify for all time the Stokes equation in the ambiant domain $\mathbf{R}^3 \setminus \cup_{i=1}^N \overline{B_i}$, subject to gravity with constant $g > 0$ and with constant fluid density ρ_{f},

$$\left\{ \begin{array}{rcll} -\Delta \boldsymbol{u} + \nabla p & = & -g\rho_{\mathrm{f}} \boldsymbol{e}_z & \\ \operatorname{div} \boldsymbol{u} & = & 0, & \end{array} \right. \qquad \text{on } \mathbf{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i}. \tag{1.2}$$

with $\boldsymbol{e}_z$ the unitary vertical upward vector. For clarity we drop the time dependency of the quantities. One has to specify boundary conditions on the spheres. The velocity of the fluid is asked to coincide with the velocity vector $\mathbf{V}_i$ of the particle i at its contact, and one ignores the possible rotation of the particle. In other words, the following is satisfied that for all time,

$$\boldsymbol{u}|_{\partial B_i}(t) = \mathbf{V}_i(t).$$

Conversely, the particles are transported by the fluid, and their respective positions $\boldsymbol{X}_i$ satisfy the ordinary differential equation

$$\frac{\mathrm{d}\boldsymbol{X}_i}{\mathrm{d}t} = \mathbf{V}_i.$$

The Stokes system also requires an assumption on the value of the velocity at infinity. A natural and convenient one is to assume that the velocity vanishes at infinity, meaning that the ambiant fluid is at rest far from the particles system,

$$\boldsymbol{u}(\boldsymbol{x}) \underset{|\boldsymbol{x}| \to \infty}{\longrightarrow} \mathbf{0}.$$

The inertialess assumption implies here that the gravity effect of a particle is balanced by the stress of the ambiant fluid on it, namely

$$-\frac{4\pi}{3} R^3 \rho_{\mathrm{p}} g \boldsymbol{e}_z + \int_{\partial B_i} (D(\boldsymbol{u}) - pI_3)\boldsymbol{n} \, \mathrm{d}\sigma_{\partial B_i} = \mathbf{0},$$

where ρ_{p} is the volumic density of a particle, $D(\boldsymbol{u}) - pI_3$ the Stokes stress tensor with $D(\boldsymbol{u}) = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T$ the symmetric gradient of $\boldsymbol{u}$, $\boldsymbol{n}$ the unitary outward normal vector and $\sigma_{\partial B_i}$ the surfacic measure on B_i.

Höfer shows this system is well-posed in the following sense. Consider a sequence of initial positions $(\boldsymbol{X}^0_{\varepsilon,i})_{i=1}^{N_\varepsilon}$ indexed by $\varepsilon \to 0$ with N_ε number of particles and their radii R_ε. This sequence has to satisfy three assumptions. The first one is related to the ratio between the number N_ε and the minimal distance between two particles at initial time. The second is about the order of volume fraction of particles $N_\varepsilon R_\varepsilon^3$. Both assumptions are necessary to ensure that the particles remain far away from each other to avoid possible wild solutions to the Stokes equation.

The third assumption concerns the lengthscale $(N_\varepsilon R_\varepsilon)^{-1/2}$ which determines the prevalence of the collective effect of the cloud of particles on one particle with respect to the self-interaction of a particle.

Under these assumptions, a proper rescaling of the mass density of the particles at step ε, namely

$$\rho_\varepsilon(t) := \sum_{i=1}^{N} \mathbf{1}_{B_i(t)},$$

satisfies, from the velocities compatibility assumption, the transport equation

$$\partial_t \rho_\varepsilon + \boldsymbol{u}_\varepsilon \cdot \nabla \rho_\varepsilon = 0 \tag{1.3}$$

where $\boldsymbol{u}_\varepsilon$ satisfies (1.2). Höfer proved that the coupling of (1.2) and (1.3) is well-posed and that ρ_ε converges strongly in some suitable weighted $L^\infty(\mathbf{R}^3)$ spaces toward some ρ, satisfying, with normalised constants here,

$$\begin{cases} \partial_t \rho + (c\boldsymbol{e}_z + \boldsymbol{u}) \cdot \nabla \rho &= 0 \\ -\Delta \boldsymbol{u} + \nabla p &= -\rho \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} &= 0 \\ \boldsymbol{u}(\boldsymbol{x}) &\underset{|\boldsymbol{x}|\to\infty}{\to} \mathbf{0}, \end{cases} \tag{1.4}$$

with c the celerity of a free falling particle, due to gravity. Notice that considering $\boldsymbol{u}' = \boldsymbol{u} + c\boldsymbol{e}_z$ would satisfy the same equation with $\boldsymbol{u}' \to c\boldsymbol{e}_z$ since $\Delta(c\boldsymbol{e}_z) = \mathbf{0}$, and consists in following the falling cloud of particle at its typical speed. In the following we will generically consider $c = 0$.

In [Mec18], Mecherbet investigates further this question, including in particular the rotation of particles, with the torque applied by the fluid to the i^{th} particle, which must cancel according to Newton's law, namely

$$\int_{\partial B_i} (\boldsymbol{x} - \boldsymbol{X}_i) \times ((D(\boldsymbol{u}) - pI_3)\boldsymbol{n}) \, \mathrm{d}\sigma_{\partial B_i}.$$

Together with the classical Newton dynamics for the particles, it determines the angular and linear velocities (V_i, Ω_i) of a particle, and is posed as the continuity condition of the velocity between the ambiant fluid and the surface of a particle

$$\boldsymbol{u}^N|_{\partial B_i} = \mathbf{V}_i + \boldsymbol{\Omega}_i \times (\boldsymbol{x} - \boldsymbol{X}_i) \quad \text{on } \partial B_i.$$

Mecherbet also consider smaller minimal distances between particles than in [Höf18a]. Similarily, she shows that, regarding its scaling, the solution to the system with particles converges toward the solution of (1.4). In this case ρ_ε converges toward ρ in Wasserstein distance.

1.1.3 Incompressible Porous Medium equation and Fractional Stokes-transport equation

The Stokes-transport equation belongs to the class of *active scalar equations*. An active scalar equation is a transport equation

$$\partial_t f + \boldsymbol{u} \cdot \nabla f = 0,$$

where the velocity field is induced by some possibly non-local operator applied to f. Among such equations, one finds the 2-dimensional Euler system in its vorticity formulation, namely

$$\begin{cases} \partial_t \omega + \boldsymbol{u} \cdot \nabla \omega &= 0 \\ \omega &= \nabla^\perp \boldsymbol{u}, \end{cases}$$

where the velocity field is recoverd from ω thanks to the Biot-Savart law $\boldsymbol{u} = \nabla^\perp (-\Delta)^{-1} \omega$ by solving a Poisson problem depending on the kind of spatial domain. This system generalises as

$$\boldsymbol{u} = \nabla^\perp (-\Delta)^s \omega,$$

for a range of exponents s. The specific case $s = 1/2$ corresponds to the Surface Quasi-Geostrophic equation, modelling large-scale fluid motions where the Coriolis force predominates over inertial forces. In the following, we also refer to the Fractional Stokes-transport (FST) equation.

In many cases, including the ones above, the operator relating the velocity field and the transported quantity is non-local. In the whole space, the solution writes as the convolution with a Green kernel,

$$\boldsymbol{u}(\boldsymbol{x}) = \int_{\mathbf{R}^3} G(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y},$$

and makes obvious the dependency of $\boldsymbol{u}$ regarding the source term ω on the whole space. This explicit expression for $\boldsymbol{u}$ in terms of the advected quantity f is convenient to analyse the system, as illustrated in the well-posedness proof for (ST) by Mecherbet, further detailed in Subsection 1.2.1.

The Stokes-transport system can be compared to the Incompressible Porous Medium equation (IPM) and to the Fractional Stokes-transport equation (FST), since they share a similar form. With normalised physical constants, the (IPM) system writes as follows,

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho &= 0 \\ \boldsymbol{u} + \nabla p &= -\rho \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} &= 0 \\ \rho|_{t=0} &= \rho_0. \end{cases} \tag{IPM}$$

This equation modelizes an incompressible fluid, either in porous media such as sandy ground, organic tissues or Hele-Shaw cells, a domain between two large plates separated by an infinitesimally small distance. In this situation the fluid satifies for all time Darcy's law instead of Stokes equation. We shall compare some of its features with (ST), since some approaches are similar, with different results. Indeed, the regularisation effect due to the Laplace operator in the Stokes system induces a smoother velocity field in the transport equation than in the Darcy case, but generates lower decay rates. This is detailed in Subsection 1.2.3.

In [Cob23], Cobb provides an exhaustive study of the fractional Stokes-transport system (FST), with a viscosity law given by the fractional Laplacian,

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho &= 0 \\ (-\Delta)^{\alpha/2} \boldsymbol{u} + \nabla p &= -\rho \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} &= 0. \end{cases} \tag{FST}$$

In the whole space $\mathbf{R}^d, d \geq 2$, the fractional Laplacian defines thanks to its Fourier symbol, for

instance on Sobolev functional spaces $H^s, s > \alpha$,

$$\forall f \in H^s(\mathbf{R}^d), \forall \boldsymbol{\xi} \in \mathbf{R}^d, \quad \mathcal{F}[(-\Delta)^{\alpha/2} f](\boldsymbol{\xi}) = |\boldsymbol{\xi}|^\alpha \mathcal{F}[f](\boldsymbol{\xi}).$$

In particular $\alpha = 0$ and $\alpha = 2$ correspond to (IPM) and (ST) respectivelly. Cobb addresses this problem for $\alpha \in [0, d]$. The study of this system is meant to reach a better understanding of the classical cases mentionned above. The existence and uniqueness of solutions is reported and compared to those of (ST) and (IPM) in Subsection 1.2.3, and summarised in Table 1.5.

1.2 Well-posedness of the Stokes-transport equation

In this Section we provide the background for the well-posedness results regarding (ST) obtained by the author, covered in Chapter 2. The state of the art at the time of writing this manuscript is summarised in Subsection 1.2.1, including the author's contributions. All these regularity results are synthesised in tables in Subsection 1.2.2, and also compared to the case of (IPM) and fractional Stokes-transport equation in Subsection 1.2.3. A different equation, named here *hydrostatic Stokes-transport equation* is also studied in the present manuscript, and its well-posedness is discussed in Subsection 1.2.4.

1.2.1 State of the art and contributions

Let us introduce the L^q spaces with finite moments.

Definition 1.2.1. *Let $\beta \geq 0$ and $1 \leq q \leq \infty$. Let us define $L^q_\beta(\Omega)$ the space of L^q functions with finite β moment in a given domain Ω as the set of measurable maps $f : \Omega \to \mathbf{R}$ satisfying either*

$$\|f\|^q_{L^q_\beta} := \int_\Omega (1 + |\boldsymbol{x}|^\beta)|f(\boldsymbol{x})|^q \, \mathrm{d}\boldsymbol{x} < \infty, \qquad \text{if } 1 \leq q < \infty,$$

or

$$\|f\|_{L^\infty_\beta} := \operatorname*{supess}_{\boldsymbol{x} \in \Omega} (1 + |\boldsymbol{x}|^\beta)|f(\boldsymbol{x}) < \infty, \qquad \text{if } q = \infty.$$

Let us denote $W^{k,q}_\beta(\Omega)$ the set of maps having derivatives of order less than or equal to k in $L^q_\beta(\Omega)$. Observe that $L^q_\alpha \subset L^q_\beta$ for any $0 \leq \alpha \leq \beta$.

The Stokes-transport system in $\mathbf{R}^3$ with vanishing velocity at infinity (1.4) (with $c = 0$) was shown to be well-posed for regular data by Höfer in [Höf18a, Theorem 9.2], in the sense reported here.

Theorem 1.2.2 (Höfer '18)**.** *Let $\rho_0 \in W^{1,\infty}_\beta(\mathbf{R}^3)$ for some $\beta > 2$. Then the Stokes-transport equation (1.4) admits a unique global solution*

$$\rho \in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}_+; L^\infty_\beta(\mathbf{R}^3)).$$

This problem is addressed by a fixed point argument, using the method of characteristics for the transport equation and the convolution representation of the Stokes equation solution. This scheme of proof is also adopted by Mecherbet in [Mec20, Theorem 1.1], with relaxed assumptions on the initial datum, requiring only integrable and bounded data with finite first moment and without assumption on the derivatives.

Theorem 1.2.3 (Mecherbet '20). *For any $\rho_0 \in L^1_1 \cap L^\infty(\mathbf{R}^3)$ the Stokes-transport equation (1.4) admits a unique and global weak solution in the following class of regularity*

$$(\rho, \boldsymbol{u}) \in L^\infty(\mathbf{R}_+; L^1 \cap L^\infty(\mathbf{R}^3)) \times L^\infty(\mathbf{R}_+; W^{1,1} \cap W^{1,\infty}(\mathbf{R}^3)).$$

In particular, Mecherbet takes advantage of an interplay between the stability estimates in the Wasserstein distance formalism for the transport equation, and the convolution representation of the solution to the Stokes equation. Indeed, on the one hand, two solutions $\boldsymbol{u}_i, i = 1, 2$ of the Stokes system with associated generic source terms $\boldsymbol{f}_i$ obey, for any compact $K \subset \mathbf{R}^3$ the following estimate

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{W^{1,1}(K)} \leq C_K W_1(\boldsymbol{f}_1, \boldsymbol{f}_2). \tag{1.5}$$

Let us report one computation justifying this relation. An optimal transport result such as [San14, Theorem 1.4] ensures in particular that for maps $\boldsymbol{f}_i \in L^1 \cap L^\infty(\mathbf{R}^3)$ with identical masses and seen as probability measures, there exists a transport map $\mathcal{T}$ such that one measure is the image measure of the other, meaning

$$\forall \varphi \text{ regular}, \quad \int_{\mathbf{R}^3} \varphi(\boldsymbol{y}) \boldsymbol{f}_1(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = \int_{\mathbf{R}^3} \varphi(\mathcal{T}(\boldsymbol{y})) \boldsymbol{f}_2(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}.$$

In this case, the Wasserstein distance between the measures computes as

$$W_1(\boldsymbol{f}_1, \boldsymbol{f}_2) = \int_{\mathbf{R}^3} |\mathcal{T}(\boldsymbol{y}) - \boldsymbol{y}| \boldsymbol{f}_1(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}. \tag{1.6}$$

Since the solution to Stokes equation on $\mathbf{R}^3$ writes as the convolution of the so-called Oseen tensor Φ, defined as

$$\Phi(\boldsymbol{x}) = \frac{1}{8\pi} \left(\frac{I_3}{|\boldsymbol{x}|} + \frac{\boldsymbol{x} \otimes \boldsymbol{x}}{|\boldsymbol{x}|^3} \right),$$

the difference of solutions to Stokes equation $\boldsymbol{u}_i$ with source terms $\boldsymbol{f}_i$ expresses as follows, using the aforementioned transport result

$$\begin{aligned} \forall \boldsymbol{x}, \quad \boldsymbol{u}_1(\boldsymbol{x}) - \boldsymbol{u}_2(\boldsymbol{x}) &= \int_{\mathbf{R}^3} \Phi(\boldsymbol{x} - \boldsymbol{y})(\boldsymbol{f}_1 - \boldsymbol{f}_2)(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \\ &= \int_{\mathbf{R}^3} (\Phi(\boldsymbol{x} - \boldsymbol{y}) - \Phi(\boldsymbol{x} - \mathcal{T}(\boldsymbol{y}))) \boldsymbol{f}_1(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}. \end{aligned}$$

Given the shape of the tensor Φ, it is possible to bound the right hand-side of the latter by (1.6) and to obtain estimate (1.5). To obtain a fixed point inequality, one finally requires a stability estimate on the solutions of the transport equation, measuring the distance between quantities transported by distinct velocity fields. Schematically, in this case, the distance between two density profiles ρ_i is bounded by a norm of the difference of the velocity fields, compatible with (1.5), with a factor $B(t)$ depending on time and such that $B(t) \to 0$ as $t \to 0$,

$$W_1(\rho_1, \rho_2) \leq B(t) \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|. \tag{1.7}$$

Besides, since the Stokes equation and the transport equation are well-posed for given data, Mecherbet constructed sequences of velocity fields $\boldsymbol{u}^N, N \in \mathbf{N}$ solving the Stokes-transport for all time with source terms $\rho^N \boldsymbol{e}_z$, where these ρ^N where obtained themselves as the solution to the transport equation for the field $\boldsymbol{u}^{N-1}$. This sequence is initialised thanks to the initial data

ρ_0. Combining both estimates (1.5) and (1.7) for these sequences provide

$$W_1(\rho^{N+1},\rho^N)(t) \leq B(t) W_1(\rho^N,\rho^{N-1})(t), \tag{1.8}$$

whence the contraction for t small enough such that $0 \leq B(t) < 1$.

Based on this scheme of proof, combining Stokes and transport estimates in a contraction, the author proved the well-posedness of (ST) on bounded domains of $\mathbf{R}^d$ for dimension $d = 2$ and 3, and in the infinite channel $\mathbf{R} \times (0,1)$. In this context, the velocity field requires boundary conditions. We impose the homogeneous Dirichlet boundary conditions $\boldsymbol{u} = \mathbf{0}$, also called no-slip boundary conditions. The result states as follows, and the dedicated article is [Leb22].

Theorem 1.2.4 (L. '20). *Let Ω be either a bounded domain of class C^2 of $\mathbf{R}^d$ for $d = 2, 3$, or $\mathbf{R} \times (0,1)$. For any $\rho_0 \in L^\infty(\Omega)$ there exists a unique solution $(\rho, \boldsymbol{u})$ to the Stokes-transport equation*

$$\left\{ \begin{array}{rcl} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho &=& 0 \\ -\Delta \boldsymbol{u} + \nabla p &=& -\rho \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} &=& 0 \\ \boldsymbol{u}|_{\partial\Omega} &=& \mathbf{0} \\ \rho|_{t=0} &=& \rho_0, \end{array} \right.$$

which belongs to

$$L^\infty(\mathbf{R}_+; L^\infty(\Omega)) \times L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega)).$$

A major difference between these cases and the whole space $\mathbf{R}^3$ is that the Green function associated to the Stokes equation is not explicit in general, and therefore less convenient to handle than the tensor Φ. Therefore, the interplay with the Wasserstein distance is not that clear. This result has been addressed by replacing estimate (1.6) by the estimate on the Stokes equation solution, and (1.7) by stability estimates in $H^{-1}(\Omega)$. Although the statement writes the same in both the bounded and unbounded cases, the latter requires to deal with functional spaces of infinite energy such as uniformly local Sobolev spaces. The article [Leb22] is included in Chapter 2.

Let us sketch its key steps here. As in [Mec20], we consider a Picard scheme and define sequences $(\rho^N)_N$ and $(\boldsymbol{u}^N)_N$ solving decoupled Stokes and transport equations. The goal is to obtain a contraction inequality satisfied by these sequences in the proper functional spaces, to obtain its convergence and provide a solution candidate. To do so, we combine two inequalities. Considering $\boldsymbol{u}_i, \rho_i, i = 1, 2$ satisfying the Stokes equation, the linearity of the problem ensures that

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{H^1} \lesssim \|\rho_1 - \rho_2\|_{H^{-1}}.$$

This inequality replaces (1.5) in our case. The second inequality shall replace the stability estimate (1.7). Since we do not have explicit Green function to represent the Stokes equation solution on bounded domains, the method performed by Mecherbet involving the Wasserstein distance expression (1.6) becomes more complex to implement. Actually, the stability estimate in the H^{-1} norm for the transport equation is a suitable substitute here, and we can obtain

$$\|\rho_1 - \rho_2\|_{L^\infty(0,T;H^{-1})} \lesssim T \|\rho_0\|_{L^\infty} e^{C\|\nabla \boldsymbol{u}_i\|_{L^\infty(0,T;L^\infty)}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,T;H^1)}.$$

if $\rho_i|_{t=0} = \rho_0$ is transported by $\boldsymbol{u}_i$. Applying and combining these two estimates to the sequence $(\rho^N)_N$ yields the following inequality, to compare with (1.8),

$$\|\rho^{N+1} - \rho^N\|_{L^\infty(0,T;H^{-1})} \lesssim B(T) \|\rho^N - \rho^{N-1}\|_{L^\infty(0,T;H^{-1})},$$

where $B(T) \to 0$ as $T \to 0$ is linked to the exponential factor above. Up to the choice of a small enough $T > 0$, we get the contraction. In the end, we get that $\boldsymbol{u}^N$ converges strongly in some Sobolev space, and that ρ^N converges weakly in L^∞. Note that, as in Mecherbet contribution, the Stokes-transport equation is understood in the weak sense. A couple $(\rho, \boldsymbol{u})$ in the aforementioned class is a *weak solution* on $[0,T), T > 0$ if $\operatorname{div} \boldsymbol{u} = 0$ and the following is satisfied for any test triplet $(\varphi, \boldsymbol{v}) \in C_c^\infty([0,T) \times \Omega)^{1+d}$ with $\operatorname{div} \boldsymbol{v} = 0$,

$$\begin{aligned} &\int_0^T \int_\Omega \rho(t,\boldsymbol{x})(\partial_t \varphi(t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}) \cdot \nabla \varphi(t,\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t = -\int_\Omega \rho(0,\boldsymbol{x})\varphi(0,\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, \\ &\int_0^T \int_\Omega \nabla \boldsymbol{u}(t,\boldsymbol{x}) : \nabla \boldsymbol{v}(t,\boldsymbol{x}) \cdot \boldsymbol{v}(t,\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t = 0. \end{aligned} \tag{1.9}$$

In particular, the weak versus strong convergence mentionned above is sufficient to pass to the limit in the decoupled problems, providing a solution to the Stokes-transport equation.

In the infinite channel $\mathbf{R} \times (0,1)$, one can expect solutions to be horizontally periodic, for instance. For such case, one cannot expect integrability of the data on the whole domain. Nevertheless, it is possible to consider data belonging to uniformly local spaces. For a classical Sobolev regularity $W^{m,q}$ with $m \in \mathbf{N}$ and $1 \leq q \leq \infty$, the associated uniformly local Sobolev space $W^{m,q}_{\mathrm{uloc}}(\Omega)$ is defined as the subspace of $W^{m,q}_{\mathrm{loc}}(\Omega)$ of maps f such that

$$\sup_{k \in \mathbf{Z}} \|f\|_{W^{m,q}((k,k+1)\times(0,1))} < \infty.$$

Basically, this norm measures the map variations in each segment $(k, k+1) \times (0,1)$, and requires this regularity to be uniformly bounded with respect to k. This constraint allows data with infinite energy, such as horizontally periodic (non-trivial) maps, and to transpose results valid in bounded domains to this unbounded case, such as functional spaces compact embeddings and Poincaré inequality. The method here consists in using the well-posedness of the Stokes problem on bounded domains of the form $(-n, n) \times (0,1)$ with $n \in \mathbf{N}$. Then, we obtain a solution on $\mathbf{R} \times (0,1)$ as a limit, by compactness and diagonal extraction arguments on (n,k). These considerations provide in the end $W^{2,q}_{\mathrm{uloc}}$ solutions to the Stokes problem, which is embbeded in $W^{1,\infty}$, and the characteristics method applies for the transport part. Once the well-posedness is obtained in this channel, the case of the periodic strip $\mathbf{T} \times (0,1)$ stems as a corollary. This later is extensively considered in our work, as it has a simple geometry, is compact and physically relevant as motivated in Chapter 3.

The Stokes-transport equation is also well-posed for Sobolev data.

Theorem 1.2.5 (L. '23)**.** *Let Ω be either $\mathbb{T} \times (0,1)$ or a simply connected compact subdomain of $\mathbf{R}^d, d = 2, 3$. Let $m \geq 3, \rho_0 \in H^m(\Omega)$ (and assume that Ω has regularity C^{m+2}). The Stokes-transport equation endowed with homogeneous Dirichlet boundary conditions $\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega$ has a unique global solution in the following class of regularity*

$$(\rho, \boldsymbol{u}) \in C(\mathbf{R}_+; H^m(\Omega)) \times C(\mathbf{R}_+; H^{m+2}(\Omega)).$$

Moreover, the solution obeys the following energy estimate

$$\|\rho(t)\|_{H^m} \leq \|\rho_0\|_{H^m} \exp\left(C \int_0^t (\|\nabla \boldsymbol{u}(s)\|_{L^\infty} + \|\nabla \rho(s)\|_{L^\infty})\, \mathrm{d}s \right). \tag{1.10}$$

The complete demonstration of this result can be found in Section 2.6. Formally, the energy estimate (1.10) shows that the Sobolev regularity is preserved as long as ρ and $\boldsymbol{u}$ remain

Lipschitz. Since H^m is embbeded in L^∞ for $m > 2$ in dimension 2, and from the uniform in time boundedness of the solution ensured by Theorem 1.2.4 for weak solutions,the Lipschitz regularity of $\boldsymbol{u}$ is ensured. This estimate is also instrumental to obtain existence of solutions by a compacness argument. Uniqueness comes from the ulterior result for L^∞ data in [Leb22].

In [HS21, Proposition 2.6] Höfer and Schubert relax the regularity assumption on the initial density and show that for any $\rho_0 \in \mathcal{P} \cap L^\infty(\mathbf{R}^3)$, where $\mathcal{P}(\mathbf{R}^3)$ denotes the space of probability densities, there exists a unique solution $(\rho, \boldsymbol{u}) \in L^\infty(\mathbf{R}_+; \mathcal{P} \cap L^\infty(\mathbf{R}^3)) \times L^\infty(\mathbf{R}_+; W^{2,q}(\mathbf{R}^3))$ for any $3 < q < \infty$.

This latter result has been refined by Mecherbet & Sueur in [MS22, Theorem 2.1], where the authors show in particular that for any $q \geq 3$, for any $\rho_0 \in \mathcal{P} \cap L^q(\mathbf{R}^3)$ the (ST) system is globally well-posed with solution satisfying (for $q > 3$) the following

$$(\rho, \boldsymbol{u}) \in L^\infty \cap C(\mathbf{R}_+; \mathcal{P} \cap L^q(\mathbf{R}^3)) \times L^\infty \cap C(\mathbf{R}_+; W^{2,q}(\mathbf{R}^3)).$$

The proof relies on the techniques introduced in [Mec20]. The critical case $q = 3$ is even more subtle and requires to deal with the log-Lipschitz character of the velocity field, see [MS22, Remark 2.1] for further details. This paper also addresses the analycity of the flow map and controllability aspects which are out of the scope of the present manuscript.

A natural question is whether or not the the $L^q, q > 1$ assumption can be lightened, and especially if measure initial data can be considered. Inversi showed in [Inv23] that solutions exist for $\rho_0 \in L^1(\mathbf{R}^3)$ only, in the Lagrangian sense. This notion of solution relies on the definition of a *Lagrangian flow* for a divergence-free velocity field $\boldsymbol{u} \in L^1_{\text{loc}}([0,T] \times \mathbf{R}^3)$. We call such a flow the Borel map $\boldsymbol{X}$, when it exists, satisfying in particular for all $s, t \in [0,T]$ that $\boldsymbol{x} \mapsto \boldsymbol{X}(t; s, \boldsymbol{x})$ is measure preserving and for almost every $\boldsymbol{x} \in \mathbf{R}^2$,

$$\boldsymbol{X}(t; s, \boldsymbol{x}) = \boldsymbol{x} + \int_s^t \boldsymbol{u}(\tau, \boldsymbol{X}(\tau; s, \boldsymbol{x}))\, \mathrm{d}\tau.$$

A map $\rho \in L^\infty([0,T]; L^1(\mathbf{R}^3))$ is called *Lagrangian solution* if it satisfies for all time $t \in [0,T]$,

$$\rho(t, \boldsymbol{X}(t; 0, \boldsymbol{x})) = \rho_0(\boldsymbol{x}) \quad \text{a.e. } \boldsymbol{x} \in \mathbf{R}^3.$$

In the end, Inversi states in [Inv23, Theorem 2.2] that given any $\rho_0 \in L^1(\mathbf{R}^3)$, there exists a Lagrangian solution $\rho \in L^\infty_{\text{loc}}(\mathbf{R}_+; L^1(\mathbf{R}^3))$ to (ST). Inversi also details the links between the Lagrangian and weak (also called distributional) type of solutions. In particular, if $\rho_0 \in L^1 \cap L^q(\mathbf{R}^3)$ for $q \geq 6/5$ and if an associated solution ρ satisfies an extra assumption we do not mention here, then it is a solution in the weak sense as in (1.9).

The author refined the proof of Theorem 1.2.5 and showed that the problem is actually well-posed for data in $L^q(\Omega)$ for q large enough, and that existence still holds for even more Lebesgue exponents.

Theorem 1.2.6 (L. '23)**.** *Let Ω be either $\mathbb{T} \times (0,1)$ or simply connected compact subdomain of $\mathbf{R}^d, d = 2, 3$ regular enough. Let $q \in (d, \infty)$. For any $\rho_0 \in L^q(\Omega)$ the (ST) equation has a unique weak solution $(\rho, \boldsymbol{u})$ in the class*

$$L^\infty(\mathbf{R}_+; L^q(\Omega)) \times L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega)).$$

Existence still holds for $q > 2d/(d+2)$.

Note that in dimension $d = 2$, the existence holds for any $q > 1$. In [MS22], Mecherbet & Sueur wonder if, in dimension 3, their $L^q, q > 3$ regularity result adapts in bounded domains,

which is the case. The criterion $q > 2d/(d+2)$, equal to $6/5$ for $d = 3$, is also consistent with the results of Inversi regarding the distributional solutions for $q > 6/5$. This is the minimum regularity in order to ensure the good definition of the integrals in (1.9).

In [Gra22], Grayer II shows that (ST) is well-posed in $\mathbf{R}^2$ for initial data in $L^1 \cap L^\infty(\mathbf{R}^2)$ that are compactly supported. This latter assumption is required to come around the Green function G associated to the Stokes equation in dimension 2, since it increases logarithmically as

$$|G(\boldsymbol{x} - \boldsymbol{y})| \lesssim 1 + |\ln |\boldsymbol{x} - \boldsymbol{y}||.$$

Hence since $\boldsymbol{u}$ given as the convolution $G * (\rho \boldsymbol{e}_z)$ is not well-defined a priori for general maps $\rho \in L^q(\mathbf{R}^2)$. To circumvent this obstacle, Grayer II considers initial data that are not only $L^1 \cap L^\infty(\mathbf{R}^2)$ but also compactly supported. Its analysis relies on a Yudovich-like approach, with Calderón–Zygmund type results on Green kernels.

During the very last days of the writing of this thesis, Grayer II kindly brought the paper [AYM00] to the author's attention. This paper seems to have remained unknown to all the aforementioned references, until recently. This might be explained by the different terminologies used to designate (ST). In this paper published in 2000, Antontsev et al. studied (ST) on C^2 bounded domains in $\mathbf{R}^d$ for $d \geq 2$. They focused on piecewise constant density profiles, with two values, modeling two immiscible fluids. In particular, they proved that if the interface Σ_0 between these two fluids is initially C^2, then the interface remains $C^{1,\mu}$ for any $\mu \in (0,1)$ and for all time, see Subsection 1.4.2 on this topic. In the mean time, they established the global well-posedness of (ST) for piecewise constant data, relying on a fixed point argument, proving the existence of solution for smooth data and then using compactness to show the result for low regular data. Some of the ingredients, in particular regarding the existence of solutions, are the same than in the proof of Theorem 1.2.4. They moreover detailed the regularity of the solution, with, for arbitrary $q > d$ and $\mu = 1 - d/q$,

$$\boldsymbol{u} \in L^\infty(\mathbf{R}_+; W^{2,q}(\Omega)) \cap L^\infty(\mathbf{R}_+; C^{1,\mu}(\Omega)) \cap C^{0,\mu}(\mathbf{R}_+; C^{1,\mu}(\Omega))$$

and for any $T > 0$,

$$\rho \in L^\infty(0,T; BV(\Omega)) \cap BV((0,T) \times \Omega).$$

The Höder regularity of $\boldsymbol{u}$ is actually a direct consequence of $\boldsymbol{u} \in W^{2,q}(\Omega)$ and Morrey's inequality, see Subsection 1.4.2. The bounded variations regularity of ρ means that $\nabla \rho$ and $\partial_t \rho$ remain measures for all time. The main difference with [Leb22] are the followings. We prove the global well-posedness of (ST) for any L^∞ data, we use a thorougher stability estimate in H^{-1} and we extend the result to the unbounded channel $\mathbf{R} \times (0,1)$.

1.2.2 Overview

The aforementionned well-posedness results on the Stokes-transport equation are synthesised in the following tables. Table 1.1 is dedicated to the results in the whole space $\mathbf{R}^3$, since most of the other domain cases present coherent with it. Table 1.2 gathers the results for other domains, especially bounded and 2-dimensional cases, including all the author contributions. The characteristics (char.) of the solutions are specified, in particular existence ($\exists$), uniqueness (!) and global existence (gl.). We note $\Omega \Subset \mathbf{R}^d$ when Ω is a compact subdomain of $\mathbf{R}^d$. Functional spaces are essentially defined throughout Subsection 1.2.1.

Reference	ρ_0 datum	char.	$\begin{matrix}\rho\\ \boldsymbol{u}\end{matrix}$ regularity
[Höf18a]	$W^{1,\infty}_\beta, \beta > 2$	$\exists!$ gl.	$L^\infty(\mathbf{R}_+; L^\infty_\beta)$ $L^\infty(\mathbf{R}_+; W^{1,\infty})$
[Mec18]	$L^1_1 \cap L^\infty$	$\exists!$ gl.	$L^\infty(\mathbf{R}_+; L^1 \cap L^\infty)$ $L^\infty(\mathbf{R}_+; W^{1,1} \cap W^{1,\infty})$
[HS21]	$\mathcal{P} \cap L^\infty$	$\exists!$ gl.	$L^\infty(\mathbf{R}_+; L^\infty)$ $L^\infty(\mathbf{R}_+; W^{2,q})$, $\forall q \in (3, \infty)$
[MS22]	$\mathcal{P} \cap L^q, q \geq 3$	$\exists!$ gl.	$L^\infty(\mathbf{R}_+; \mathcal{P} \cap L^q)$ $L^\infty(\mathbf{R}_+; W^{2,q}))$, $(q > 3)$
[Inv23]	L^1	$\exists$ gl.	$L^\infty(\mathbf{R}_+; L^1)$, (Lagrangian) $L^\infty_{\mathrm{loc}}(\mathbf{R}_+; W^{1,q} + W^{1,\infty}), \forall q \in [1, 3/2)$

Table 1.1: Synthesis of well-posedness results for (ST) in $\mathbf{R}^3$.

Reference	Domain	ρ_0 datum	char.	$\begin{matrix}\rho\\ \boldsymbol{u}\end{matrix}$ regularity
[Leb22]	$\Omega \Subset \mathbf{R}^{d=2,3}$	L^∞	$\exists!$ gl.	$L^\infty(\mathbf{R}_+; L^\infty)$ $L^\infty(\mathbf{R}_+; W^{1,\infty})$
L. Chap 2	$\Omega \Subset \mathbf{R}^{d=2,3}$	$L^q, q > d$	$\exists!$ gl.	$L^\infty(\mathbf{R}_+; L^q)$ $L^\infty(\mathbf{R}_+; W^{1,\infty})$
L. Chap 2	$\Omega \Subset \mathbf{R}^{d=2,3}$	$L^q, d \geq q > 2d/(d+2)$	$\exists$ gl.	$L^\infty(\mathbf{R}_+; L^q)$ $L^\infty(\mathbf{R}_+; W^{1,q})$
[Leb22]	$\mathbf{R} \times (0,1)$	L^∞	$\exists!$ gl.	$L^\infty(\mathbf{R}_+; L^\infty)$ $L^\infty(\mathbf{R}_+; W^{1,\infty})$
L. Chap 2	$\Omega, \mathbf{T} \times (0,1)$	$H^k, k \geq 3$	$\exists!$ gl.	$C(\mathbf{R}_+; H^k)$ $C(\mathbf{R}_+; H^{k+2})$
[Gra22]	$\mathbf{R}^2$	$L^1 \cap L^\infty_{\mathrm{c}}$	$\exists!$ gl.	$L^\infty(\mathbf{R}_+; L^1 \cap L^\infty)$ $L^\infty(\mathbf{R}_+; \dot{C}^{1,\mu}), \forall \mu \in (0,1)$

Table 1.2: Synthesis of well-posedness results for (ST) in bounded and 2d domains.

1.2.3 Comparison between (ST), (IPM) and (FST)

The Incompressible Porous Medium (IPM) equation is introduced in Subsection 1.1.3. We provide here a concise summary of its known well- and ill-posedness results, compiled in Table 1.3. We compare these features to those of (ST) in the synthetic Table 1.4. In addition, we complete this study by citing the work of [Cob23] about the fractional Stokes-transport equation, generalising both (IPM) and (ST), also introduced in Subsection 1.1.3.

The (IPM) equation is locally well-posed in $\mathbf{R}^2$ for Sobolev data, see [CGO07]. Their results rely on a reformulation of the problem, in particular by using Green kernel representations of the stream function and the velocity field. Indeed, considering the rotational of the Darcy equation, and since $\boldsymbol{u} = \nabla^\perp \psi$ for some stream function ψ by divergence-free condtion, one gets

$$\Delta \psi = -\partial_x \rho.$$

In $\mathbf{R}^2$, the convolution representation of the solution writes

$$\psi(t,\boldsymbol{x}) = -\frac{1}{2\pi}\int_{\mathbf{R}^2} \ln|\boldsymbol{x}-\boldsymbol{y}|\partial_x\rho(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

Considering the perpendicular gradient of the latter expression, a kernel representation stems for the velocity field, and provides a convenient mapping to apply a Picard theorem argument, leading to the local well-posedness of the problem.

Lack of uniqueness has been proved for this equation. In [CFG11, Theorem 5.2], Córdoba, Faraco and Gancedo showed that (IPM) with initial datum $\rho_0 = 0$ admits, for every $T > 0$ infinitely many non-trivial weak solutions in $L^\infty(0,T;L^\infty(\mathbf{T}^2))$. Their demonstrations involves De Lellis–Székelyhidi convex integration approach to construct multiple solutions. In their case, the constructed solutions satisfy moreover the following growth, comparable to some other long-time behaviour results discussed in Section 1.3.

$$\forall s > 0, \qquad \limsup_{t\to\infty} \|\rho(t)\|_{H^s(t)} = \infty.$$

In [Szé12, Theorem 1.1], Székelyhidi showed a similar result with initial data of the form

$$(x,z) \in (-1,1)^2, \qquad \rho_0(x,z) = \begin{cases} +1 & z > 0 \\ -1 & z < 0. \end{cases}$$

However global well-posedness for general regular data remains to this day an open question, some results have been obtained for specific initial profiles. For stationary profiles proportional to the depth of the domain, typically as the following

$$\Theta(x,z) = -z,$$

it has been shown that the (IPM) equation admits global solutions for initial small Sobolev perturbations. This result has been first obtained in [Elg17] and refined in [CCL19a]. The methods and mechanisms at play here are introduced in Section 1.3 and detailed in Chapter 3, since it concerns the asymptotic behaviour of the system.

The aforementionned results are compiled in the Table 1.3. In addition to the characteristics introduced for Figure 1.1 and 1.2, we emphasise on lack of uniqueness ($\not\exists$) and local existence (loc.). The space $X^m(\Omega) \subset H^m(\Omega)$, motivated by Theorem 1.3.4, defines as

$$X^m(\Omega) = \{\theta \in H^m(\Omega) : \partial_{\boldsymbol{n}}^{2n}\theta|_{\partial\Omega} = 0, 0 \le n \le (m-1)/2\}.$$

Reference	Domain	ρ_0 datum	char.	ρ regularity
[CGO07]	$\mathbf{R}^2$	Sobolev	$\exists!$ loc.	Sobolev
[CFG11]	$\mathbf{T}^2$	0	$\exists \not!$ loc.	$L^\infty_{\mathrm{loc}}(\mathbf{R}_+;L^\infty)$
[Szé12]	$(-1,1)^2$	$\pm\mathbf{1}_{\{\pm z>0\}}$	$\exists \not!$ loc.	$L^\infty_{\mathrm{loc}}(\mathbf{R}_+;L^\infty)$
[Elg17]	$\mathbf{R}^2, \mathbf{T}^2$	$\|\rho_0-\Theta\|_{H^m} < \varepsilon, m \ge 20$	$\exists!$ gl.	$L^\infty(\mathbf{R}_+;H^3)$
[CCL19a]	$\mathbf{T}\times(-1,1)$	$\|\rho_0-\Theta\|_{X^m} < \varepsilon, m \ge 10$	$\exists!$ gl.	$L^\infty(\mathbf{R}_+;H^3)$

Table 1.3: Synthesis of well-posedness for (IPM) equation.

The a priori estimate for the (IPM) system writes just as (1.10). One key difference with the Stokes-transport system lies in the regularisation of the velocity with respect to the density.

For Darcy equation, $\boldsymbol{u}$ has basically the same regularity as ρ, whereas Stokes equation provides elliptic regularisation implying in particular $\|\nabla \boldsymbol{u}\|_{L^\infty} \lesssim \|\rho\|_{L^\infty}$. In this latter case, a bounded density ensures a Lipschitz velocity field favorable to the transport, which is not the case in (IPM).

	L^∞ (IPM)	datum (ST)	H^m (IPM)	datum (ST)	H^m, Θ pert. (IPM)
Existence	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
Uniqueness	$\times$	$\surd$	$\surd$	$\surd$	$\surd$
Global	$(\surd)$	$\surd$	**?**	$\surd$	$\surd$

Table 1.4: Compared known well-posedness results of (IPM) and (ST) equation, overview

Regarding the fractional Stokes-transport equation, Cobb provided in [Cob23] an exhaustive study of the its well-posedness, providing sufficient conditions for existence, uniqueness and globality of solutions. Its study occur in the whole space $\mathbf{R}^d$ for $d \geq 2$. Its results are consistents with what we know about (IPM) and (ST), and the date regularity are optimized. We synthesise its main results in Table 1.5.

[Cob23, Theorem]	α	$\rho_0 \in L^q$	$\cap X$	char.	ρ regularity
1.1	$(0,d)$	$q \in \left(\frac{2}{1+\frac{\alpha}{d}}, \frac{d}{\alpha}\right)$	$\cdot$	$\exists$ gl.	$L^\infty(\mathbf{R}_+; L^q)$
1.3	$[0,d)$	$q \in [1, d/\alpha)$	$B^s_{\infty,1}, s \geq 1-\alpha$	$\exists!$ loc.	$L^\infty(0,T; L^q \cap B^s_{\infty,1})$
1.5	$(1,d)$	$q = d/(\alpha-1)$	$L^r, r \in (1, d/\alpha)$	$\exists!$ gl.	$L^\infty(\mathbf{R}_+; L^q \cap L^r)$
1.6	$[1,d]$	$q = d/\alpha$	$\dot{B}^0_{q,1} \cap B^0_{r,1}, r = \frac{d}{\alpha-1}$	$\exists!$ gl.	$L^\infty(\mathbf{R}_+; L^q \cap X)$

Table 1.5: Synthesis of Cobb well-posedness results of F(ST) in $\mathbf{R}^d, d \geq 2$ from [Cob23]

The regularity of ρ is dictated by the fractional Stokes equation associated Green kernel, which has a certain decay at infinity and a singularity at the origin. These two features require ρ to belong to several L^q spaces or subspaces noted $L^q \cap X$ in Table 1.5. It was the motive for the assumption $\rho \in L^1 \cap L^\infty$ in [Höf18a; Mec18]. Moreover, uniqueness of the solution stems for Lipschitz velocity fields, which property is eventually ensured up to more regularity constraint on ρ. Nethertheless, log-Lipschitz velocity can be dealt with, as in [MS22]. The local well-posedness [Cob23, Theorem 1.3] is consistent with the local well-posedness of (IPM) when $\alpha = 0$, with no Sobolev assumption required on the initial data, here. The global well-posedness result from [MS22] for data $\rho_0 \in L^1 \cap L^q(\mathbf{R}^3), q \geq 3$, addressed with different techniques, coresponds to a critical case that [Cob23, Theorem 1.5] does not cover, with same Lebesgue exponents. [Cob23, Theorem 1.6] does circumvent it, including the Stokes-transport case $\alpha = d$. These results confirm the intuition that the larger α is, the better the problem is posed. Cobb also captures some optimal assumptions on the functional data spaces, with in particular some Besov regularity. Note that another constraint comes from the non-linearity $\boldsymbol{u} \cdot \nabla \rho$, which requires $\rho \boldsymbol{u}$ to be at least integrable in space to satisfy the weak formulation (1.9) of the equation. Regarding these distinct requirements, assuming data in L^1 only is never enough to obtain the weak well-posedness of the problem, and requires another notion of solution, which is dealt in [Inv23].

1.2.4 Hydrostatic Stokes-transport equation

When modeling an ocean motion, it is common to proceed to the *hydrostatic approximation*, consisting in neglection the vertical acceleration of the fluid, inducing the hydrostatic balance

between the pressure and the density

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho.$$

More precise formal derivations are detailed in [Ped13] for instance. This approximation is called hydrostatic in reference to the case of a fluid at rest, with velocity $\boldsymbol{u} = \mathbf{0}$, where only the pressure gradient and the external forcing remain. In the present case where the external force correspond to the gravity effect, it provides indeed

$$\nabla p = -\rho \boldsymbol{e}_z.$$

This equation also allows us to identify the stationary states of the system, see Subsection 3.6.2.

This approximation is made for a wide class of problems generically called *Primitive equations*, modelling the ocean in various regimes, with different effects and forcings. These are paramount for weather and climate prediction. Some seminal works on these equations are [LTW92a; LTW92b], where the authors initiated a program in view of studying different approximations and formulations including the temperature and salinity of the ocean, in addition to the velocity, pressure and density. In works such as [Zia95; HTZ02] obtain the well-posedness of Stokes type problems related to these primitive equations. Cao and Titi proved the global well-posedness of various viscous primitive equations such as in [CT07] where the system includes Coriolis forcing, anisotropic viscosity and temperature diffusivity. We also mention recent works on this topic such as [Kor21; KT23], where Korn (resp. Korn & Titi) prove the global well-posedness of some ocean primitive equations. The models involved are actually used to make weather forecasts or simulate ocean circulation on the globe, are based on hydrostatic Boussinesq equations, supplemented with more physical variable, such as salinity and temperature evolution, and closed with some specific equation of state.

For simplicity we consider this equation in the periodic strip $\Omega = \mathbf{T} \times (0,1)$. One major advantage of the hydrostatic approximation is to easily compute the pressure, since it simply requires to integrate the density vertically and to use a pressure of reference $p_0 = p_0(x)$ on one of the boundaries, generically

$$p(x,z) = p_0(x) - \int_0^z \rho(x,\zeta)\,\mathrm{d}\zeta.$$

This computation is less expensive than solving the nonlocal equation satisfied by the pressure in general, such as

$$\Delta p = -\partial_z \rho,$$

in our case, obtained as the divergence of the Stokes equation.

The counterpart of this assumption lies in a loss of regularity regarding the velocity. Indeed, since the approximation consists in omitting some vertical velocity term in the considered system, one computes it by integrating the incompressible condition, up to some boundary assumption

$$u_2(x,z) = -\int_0^z \partial_x u_1(x,\zeta)\,\mathrm{d}\zeta.$$

Whence a priori the vertical velocity u_2 possesses one order less of derivation than the horizontal one u_1. In the case of the Stokes equation, we consider the system

$$\left\{\begin{array}{rcl} -\Delta u_1 + \partial_x p &=& 0 \\ \partial_z p &=& -\rho \\ \operatorname{div} \boldsymbol{u} &=& 0, \end{array}\right.$$

with no-slip boundary condition and a zero average on Ω assumption on u_1, detailed in Subsection 2.7.1,

$$u_2|_{\partial\Omega} = 0, \quad \partial_z u_1|_{\partial\Omega} = 0, \quad \langle u_1 \rangle_\Omega = 0. \tag{1.11}$$

In this particular case, we obtain that the velocity field has essentially the same regularity as the density field

$$\|\boldsymbol{u}\|_{H^m} \lesssim \|\rho\|_{H^m},$$

which contrasts with the regularisation effect of the original Stokes equation where in general

$$\|\boldsymbol{u}\|_{H^{m+2}} \lesssim \|\rho\|_{H^m}.$$

This is an obstacle to establish the global well-posedness of the associated *hydrostatic Stokes-transport* equation that we define as

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho &= 0 \\ -\Delta u_1 + \partial_x p &= 0 \\ \partial_z p &= -\rho \\ \operatorname{div} \boldsymbol{u} &= 0. \end{cases} \tag{1.12}$$

completed with conditions (1.11). Nevertheless, we obtained the local well-posedness of this system for regular enough data, stated here on the periodic strip $\Omega = \mathbf{T} \times (0,1)$.

Theorem 1.2.7. *Let $m \geq 3$. For any $\rho_0 \in H^m(\Omega)$ there exists a time $T(\|\rho_0\|_{H^3}) > 0$ such that the hydrostatic Stokes-transport system (1.12) is locally well-posed, with solution*

$$(\rho, \boldsymbol{u}) \in C(0,T; H^m(\Omega)) \times C(0,T; H^m(\Omega)),$$

obeying moreover

$$\forall t \in [0,T), \quad \|\rho(t)\|_{H^m} \leq \frac{\|\rho_0\|_{H^m}}{1 - C\|\rho_0\|_{H^3} t}.$$

1.3 Long-time behaviour for the Stokes-transport equation

Since the Stokes-transport is globally well-posed in the frameworks presented in Section 1.2, we can wonder about the long-time behaviour of its solutions. As this equation involves density, it is relevant to consider its *potential energy*,

$$E(t) := \int_\Omega z\rho(t).$$

The natural energy estimates for the Stokes equation and the transport equation once combined for the Stokes-transport system provide the following energy balance, for any $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -\|\nabla \boldsymbol{u}(t)\|_{L^2}^2. \tag{1.13}$$

This observation already tells us that the potential energy is decreasing, that the evolution of the system is monotonous and that the fluid rearranges itself. Natural questions may arise. Does the solution converges for large time? If so, can we identify the limit state? What are the stationary profiles of the system? If the convergence occurs, what is the decay rate? We partially answer these questions in Chapter 3. In Subsection 1.3.1 we establish long-time properties of the Stokes-transport equation solution for generic data in L^∞. In particular, we discuss the conjecture that

the solution always reorders itself vertically, and we use some rearrangement theory. Subsection 1.3.2 gathers the previous long-time behaviour studies for the Incompressible Porous Medium equation. These works have been fundamental for our contribution on the asymptotic behaviour of the Stokes-transport equation in a channel [DGL23]. In this latter article, we consider a class of stable initial data in Sobolev spaces for which we obtained explicit convergence rates and a description of the asymptotic state, as the vertical rearrangement of the initial data. We also provide a description of boundary layers formation occurring due to the boundary conditions imposed on the fluid. These works are introduced in Subsection 1.3.3. Besides, we extend a method developed by Kiselev & Yao in [KY23] about (IPM), showing that all stationary density profiles are unstable for low regularity perturbations, for the Stokes-transport equation. The main difference with this previous work is that we know that (ST) is globally, and the solution grows with rates distinct from the ones obtained for (IPM). Finally, Subsection 1.3.5 contains a selection of long-time behaviour results for the Boussinesq equation, for initial data near stratified density and zero velocity field stationary solutions. We compare these results with the ones obtained for the Stokes-transport system.

1.3.1 General considerations and rearrangement

The stationary profiles for the Stokes-transport equation are exactly the stratified functions, *i.e.* the functions depending only on the vertical coordinate. This is shown thanks to the monotonicity of the potential energy given by (1.13). The physical intuition suggests that the fluid should sedimentate and reorder in a monotonous way, where density should be increasing with the depth. We recall and adapt the notion of function rearrangement, and define the *vertical rearrangement* in the periodic channel $\Omega = \mathbf{T} \times (0,1)$. For general data $\rho_0 \in L^\infty(\Omega)$, we established that its ω-set in the H^{-1} topology, *i.e.* the set of accumulation points of the density trajectory, is never empty, and can only contain stationary solutions of the problem. We also obtained that the density indeed converges in H^{-1} toward the vertical rearrangement ρ_0^* of its initial datum ρ_0 if and only if the potential energy converges toward the potential energy of ρ_0^*. Besides, the energy balance (1.13) implies that $\boldsymbol{u} \in L^2(\mathbf{R}_+; H^1(\Omega))$. We compare this regularity to some natural sufficient condition ensuring the strong convergence of the transport equation solution in L^2. We will see that this latter condition is satisfied for data close enough to the hydrostatic profile $\Theta(z) = 1 - z$, in Subsection 1.3.3, and that we can prove that the solution indeed converges toward the rearrangement in this case.

1.3.2 Previous works about the stability for the (IPM) equation

An overview of the Incompressible Porous Medium equation well-posedness and some comparisons with the Stokes-transport and Fractional Stokes-transport are given in Subsection 1.2.3. In particular, it is not known to this date whether the (IPM) equation is globally well-posed in time, even for regular initial data. For Sobolev data, for instance, only local well-posedness results are available, see [CGO07].

In [Elg17; CCL19a], Elgindi and Castro, Córdoba & Lear prove existence of global solutions for the Incompressible Porous Medium equation for specific initial data. They consider perturbations of the following affine stationary solution of the form

$$\Theta(z) = 1 - z.$$

This profile is *stratified*, in the sense that it depends only on the vertical coordinate. It is also monotonous, increasing with the depth. Intuitively, we expect such a density distribution to be stable in a system modelling sedimentation. This is proven in the aforementioned works. Let

us define θ as the difference between the density ρ and the background profile Θ, such that it splits as $\rho = \Theta + \theta$. We will call θ the "perturbation" of the stationary profile Θ. Plugging this decomposition in the transport equation and given an initial perturbation θ_0, we obtain that θ satisfies the following

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = u_2, \quad \theta|_{t=0} = \theta_0, \tag{1.14}$$

where $\boldsymbol{u} = (u_1, u_2)$. We report here the stability results obtained by Elgindi, referenced as [Elg17, Theorems 1.3 and 1.4], and a refinement obtain by Ma in [Ma23, Theorem 1.1]. All the notations are unified in order to ease the results comparison.

Theorem 1.3.1 (Elgindi '18)**.** *Let us consider the domain $\mathbf{R}^2$. There exists $\varepsilon_0 > 0$ such that for any initial data $\rho_0 = \Theta + \theta_0$ satisfying*

$$\|\theta_0\|_{W^{4,1}} + \|\theta_0\|_{H^s} \leq \varepsilon,$$

for $0 < \varepsilon \leq \varepsilon_0$ and $s \geq 20$, the solution to (IPM) exists globally and obeys for all $t \geq 0$ the following estimates

$$\|\theta(t)\|_{H^3} \lesssim \frac{\varepsilon}{t^{1/4}}, \qquad \|u_1(t)\|_{H^3} \lesssim \frac{\varepsilon}{t^{3/4}}, \quad \|u_2(t)\|_{H^3} \lesssim \frac{\varepsilon}{t^{5/4}}.$$

Theorem 1.3.2 (Ma '23)**.** *Let $s > 0$ large enough and $\theta_0 \in H^s \cap W^{7,1}(\mathbf{R}^2)$. There exists $\varepsilon_0 > 0$ such that if $\|\theta_0\|_{H^s} + \|\theta_0\|_{W^{7,1}} \leq \varepsilon \leq \varepsilon_0$ then the solution to (IPM) exists globally and obeys the following estimates*

$$\|\theta(t)\|_{W^{1,\infty}} \lesssim \frac{\varepsilon}{t^{1/2}}, \quad \|u_1(t)\|_{W^{1,\infty}} \lesssim \frac{\varepsilon}{t}, \quad \|u_2(t)\|_{W^{1,\infty}} \lesssim \frac{\varepsilon}{t^{3/2}}.$$

Theorem 1.3.3 (Elgindi '18)**.** *Let us consider the domain $\mathbb{T}^2$. There exists $\varepsilon_0 > 0$ such that for any initial data $\rho_0 = \Theta + \theta_0$ satisfying*

$$\|\theta_0\|_{H^s} \leq \varepsilon,$$

for $0 < \varepsilon \leq \varepsilon_0$ and $s \geq 20$, the solution to (IPM) exists globally and obeys for all $t \geq 0$ the following estimates

$$\|\theta(t)\|_{H^{20}} \leq 2\varepsilon, \quad \|\boldsymbol{u}(t)\|_{H^3} \lesssim \frac{\varepsilon}{t^{5/2}}.$$

Note that these results prove that the solutions exist globally in time for such initial data. In $\mathbf{R}^2$, Theorem 1.3.1 means that the background profile Θ is asymptotically stable, in the sense that $\rho(t)$ converges toward Θ. This latter convergence occur in $H^3_{\text{loc}}(\mathbf{R}^2)$, since Θ is not integrable on the whole space. Nevertheless, the initial perturbation θ_0 belongs to $H^s(\mathbf{R}^2)$ for some $s \geq 20$, this regularity is propagated in time, and θ decays in $H^3(\mathbf{R}^2)$ with explicit rate. In $\mathbf{T}^2$, the conclusion of the Theorem 1.3.3 is slightly different. The whole perturbation, initially small, remains small for all time, so Θ is stable in the sense that the solution remains in its vicinity forever. Moreover, the velocity field belongs to $L^1(\mathbf{R}_+; H^3(\mathbf{T}^2))$ which ensures that the perturbation θ converges, see Proposition 3.6.7. Since the pure transport preserves the level sets measures of ρ when $\boldsymbol{u}$ is divergence-free, the distribution function of ρ is constant with respect to time. Hence if ρ_0 has a different distribution function than Θ, it is not possible for ρ to converge toward Θ, even in Lebesgue spaces.

Another similar stability result has been obtained in [CCL19a, Theorem 5.1], in a periodic

channel (renormalised here) $\Omega = \mathbf{T} \times (0,1)$. Let us introduce the decomposition $\theta = \bar{\theta} + \theta'$ where

$$\bar{\theta}(z) := \frac{1}{2\pi} \int_{\mathbf{T}} \theta(x', z) \, \mathrm{d}x', \qquad \theta' := \theta(x, z) - \bar{\theta}(z).$$

The term $\bar{\theta}$ is the horizontal average of θ. Note that θ' has zero horizontal average for any $z \in (0,1)$. Since we expect the density to rearrange in a stratified state, the limit should depend only on z, and therefore θ', capturing all the horizontal dependency of θ, should decay. Projecting equation (1.14) provide an evolution equation for θ' and $\bar{\theta}$. These are formalised in the following result, with explicit decay rates. In particular the following holds for $s \geq 10$ and a convenient choice of γ. This makes a difference with the initial data smallness assumption in H^{20} required by Elgindi.

Theorem 1.3.4 (Castro, Córdoba & Lear '19)**.** *Let us consider the domain $\Omega = \mathbf{T} \times (0,1)$. There exists $\varepsilon_0 > 0$ and $\gamma > 4$ such that for any initial data $\rho_0 = \Theta + \theta_0$ satisfying*

$$\|\theta_0\|_{H^s} \leq \varepsilon, \qquad \partial_{\boldsymbol{n}}^{2n} \theta_0|_{\partial\Omega} = 0, \quad 0 \leq n \leq (s-1)/2, \tag{1.15}$$

where $0 < \varepsilon \leq \varepsilon_0$ and $s > 5 + \gamma$, the solution to (IPM) exists globally and obeys the following estimates

$$\|\theta'(t)\|_{H^3} \lesssim \frac{\varepsilon}{(1+t)^{\gamma/4}}, \quad \|\bar{\theta}(t)\|_{H^s} \leq 2\varepsilon.$$

The convergence of the density $\rho(t) = \Theta + \bar{\theta}(t) + \theta'(t)$ is a consequence related to the integrability in time of $\boldsymbol{u}$, as above. Therefore, it is enough to control θ' to bound $\boldsymbol{u}$, since stratified source terms do not contribute to the Darcy equation solutions,

$$\|\boldsymbol{u}\|_{H^3} \lesssim \|\theta'\|_{H^3}.$$

Since the parameter γ and the Sobolev exponent s can be arbitrary large here, the decay can be arbitrary quick, provided the initial datum has a small H^s norm. The same conclusion holds in for the cases handled by Elgindi, althought it does not explicitly appear in these statement.

The Theorem 1.3.4 has the same structure as Theorems 1.3.1 and 1.3.3. Assume the initial perturbation θ_0 small in some high order Sobolev spaces, then the perturbation remains small and the density ρ converges toward some asymptotic state in lower order Sobolev spaces. In the periodic strip, Castro, Córdoba & Lear also assumed their data to have vanishing normal derivatives on the boundaries, which somehow imitates the periodic case $\mathbf{T}^2$. In the following Subsection 1.3.3, Dalibard, Guillod & the author obtained a similar stability result as Theorem 1.3.4 for the Stokes-transport equation, in [DGL23]. When replacing Darcy equation by Stokes equation, one extra boundary condition is required on the velocity, and we explain further why this condition choice impacts greatly the analysis. In particular, it it not possible anymore to obtain arbitrary large decays, due to some obstacle to the establishment of energy estimates at arbitrary order. This is a counterpart to getting rid of the vanishing derivatives conditions (1.15).

We sketch the main ideas of the above theorems proofs here, as they share the same core. This main scheme is updated to establish a similar result for the Stokes-transport equation in [DGL23].

Essence of the stability proofs for (IPM) The background state Θ induces a stabilising mechanism. Indeed, the energy balance in L^2 on θ writes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^2}^2 = \int_\Omega u_2\theta = -\|\nabla \boldsymbol{u}\|_{L^2}^2,$$

where the nonlinear advection term does not contribute since $\boldsymbol{u}$ is divergence-free, and the latter equality arises by combining transport and Darcy equation estimates, just as in the potential energy balance (1.13). The calculations for (ST) are detailed in Subsection 3.6.2. Actually, the potential energy appears explicitly here, since

$$\|\theta\|_{L^2}^2 = \|\rho - \Theta\|_{L^2}^2 = \|\rho\|_{L^2}^2 - 2\int_\Omega (1-z)\rho + \|\Theta\|_{L^2}^2.$$

Because $\|\rho(t)\|_{L^2}, \|\Theta\|_{L^2}$ and $\int_\Omega \rho(t)$ are constants, which is a major difference with $\|\theta\|_{L^2}$, we have exactly

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t}E(t).$$

Computing thoroughly the energy estimates for higher order Sobolev norms (see [CCL19a, Theorem 5.2]) leads to inequalities of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^m}^2 \lesssim \|\nabla u_2\|_{L^\infty}\|\theta\|_{H^m}^2 - (1 - C\|\theta\|_{H^m})\|\boldsymbol{u}\|_{H^m}^2.$$

Note that this estimate is valid for arbitrary large m thanks to the data vanishing normal derivatives, cancelling all boundary integrals when performing integrations by parts. We can read in this estimate that if $\|\nabla u_2\|_{L^\infty}$ is integrable in time, and if $\|\theta\|_{H^m}$ is small enough, then $\|\theta\|_{H^m}$ remain small, by a Grönwall argument. To obtain decay rates and the convergence of the solution toward a stratified state, one splits the modified transport equation (1.14) into two equations on θ' and $\bar{\theta}$ respectivelly. The estimates on θ' stem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2} \lesssim -(1 - C\|\partial_z\bar{\theta}\|_{H^2})\|\boldsymbol{u}\|_{L^2}^2.$$

Once again, if θ is small enough in some appropriate Sobolev spaces, then so is $\bar{\theta}$, and $\|\theta'\|_{L^2}$ is decreasing. To obtain algebraic decay rates, we need to relate the norms of $\boldsymbol{u}$ and θ'. A priori,$\boldsymbol{u}$ is controlled by θ', either for the Darcy equation or the Stokes equation, so without specifying the functional spaces, we have in general

$$\|\boldsymbol{u}\| \lesssim \|\theta'\|. \tag{1.16}$$

Here we need the inequality in the other way. To do so, Castro, Córdoba & Lear establish some interpolation inequality, using the Fourier decomposition of the functions and relying once again on the vanishing normal derivatives as in (1.15). In the course of their demonstration, this interpolation estimate takes the form

$$\frac{\|\theta'(t)\|_{L^2}^2}{(1+t)^{1/2}} \lesssim \|\boldsymbol{u}(t)\|_{L^2}^2 + \frac{\|\theta'(t)\|_{H^\alpha}^2}{(1+t)^{(1+\alpha)/2}}. \tag{1.17}$$

Here $\boldsymbol{u}$ partially controls θ' in L^2, as a reciprocal of (1.16). The weight involves in this inequality depends here on time, and we see that higher the Sobolev regularity, higher the decay in the right hand-side. Combining the latter estimates, and applying a Grönwall lemma, we eventually

get

$$\|\theta'(t)\|_{L^2} \lesssim \frac{\|\theta'\|_{L^\infty(0,t;H^\alpha)}}{(1+t)^{\alpha/4}}. \tag{1.18}$$

Such estimates also hold for the H^3 norm of θ' instead of L^2. Up to prove the smallness of θ' in $H^\alpha(\Omega)$ for all time, we get a decay estimate on θ'. Since α is arbitrary, one choses it so that the quantities of interest are integrable in time, ensuring the convergence of the solution. A bootstrap argument is performed to prove that all the aforementioned assumptions are satisfied at the same time.

1.3.3 Contribution: Stability for the Stokes-transport system in a channel

In this subsection we consider the spatial domain $\Omega = \mathbf{T} \times (0,1)$ and positive densities, without loss of generality. We obtained the stability of the stationary profile $\Theta(z) = 1 - z$ for small perturbations in Sobolev spaces.

Theorem 1.3.5 (Dalibard, Guillod & L. '23)**.** *Let $\theta_0 \in H^6(\Omega)$ such that $\theta_0|_{\partial\Omega} = \partial_{\boldsymbol{n}}\theta_0|_{\partial\Omega} = 0$, $\partial_z^2\bar{\theta}_0|_{\partial\Omega} = 0$, and set $\rho_0 = \Theta + \theta_0$. There exists $\varepsilon_0 > 0$ such that if $\|\theta_0\|_{H^6} \leq \varepsilon \leq \varepsilon_0$ the solution of the Stokes-transport equation with no-slip boundary condition and initial datum ρ_0 satifies*

$$\|\partial_x^3\theta'(t)\|_{L^2} \lesssim \frac{\varepsilon}{1+t}, \quad \|\partial_x\theta'(t)\|_{H^4} \lesssim \varepsilon, \quad \|\bar{\theta}(t)\|_{H^5} \lesssim \varepsilon, \quad \forall t > 0. \tag{1.19}$$

Moreover, the density converges in H^m for $m < 4$ toward the vertical rearrangement of ρ_0:

$$\rho_\infty(z) := \int_0^\infty \mathbf{1}_{0\leq z\leq |\{\rho_0 > \lambda\}|}\, \mathrm{d}\lambda.$$

We only required the initial perturbation θ_0 to have its trace and its first normal derivative trace vanishing, so that Θ and ρ_0 share the same boundary conditions. For a linearised version of the system, that this condition can be lifted, and give rise to boundary layers that are described in Theorem 1.3.6. In the nonlinear setting, we provide a more complete description of the solution involving these boundary conditions, see Theorem 1.3.7.

We provide the outlines of the proof of Theorem 1.3.5, based on the sketch for (IPM) reported hereabove, and we emphasise on the differences induced by the boundary condition choice on $\boldsymbol{u}$ for the Stokes-transport equation. Even though the outlines of the proof are the same for [Elg17; CCL19a; DGL23], the framework and tools are different. In [Elg17], Elgindi relies on the Fourier analysis on the full space $\mathbf{R}^2$ and Fourier series on the torus $\mathbf{T}^2$, in particular to obtain interpolation estimates. In [CCL19a], Castro, Córdoba & Lear adapt the Fourier basis to the periodic channel. In both cases, the boundaries are either non-existent or do not raise any obstacle. In [DGL23], due to the second order operator in the Stokes equation and the Dirichlet boundary conditions, the boundaries cannot be neglected anymore, and the Fourier approach does not fit. Note that we have identified a basis adapted to our problem in Subsection 3.8, whose features illustrate the difficulties raised by the boundaries. In particular, energy estimates for arbitrary orders such as (1.18) inducing arbitrary algebraic decay rates for (IPM) solutions, as pointed out rightafter Theorem 1.3.4, does not hold anymore because of the boundaries and the Dirichlet boundary condition.

Specificities induced by the Dirichlet condition for (ST) We focus here on the main differences and limitations induced by the homogeneous Dirichlet boundary condition $\boldsymbol{u}|_{\partial\Omega} = 0$.

Let us assume the slip boundary condition on $\boldsymbol{u}$, namely

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\quad \partial_{\boldsymbol{n}}\boldsymbol{u}=0,\quad \text{on } \partial\Omega.$$

If one assumed the initial datum θ_0 to satisfy a null normal derivatives trace condition as (1.15), this property would be transfered to θ for all time. Therefore, all the above computations can be performed in the same way, up to some powers adjustments, the integrations by parts do not induce any boundary integral, and the interpolation estimate (1.17) does adapt. In the end the same proof as [CCL19a] would provide a similar result for (ST).

With the Dirichlet boundary condition, no such null trace can be propagated, except the first two ones

$$\theta=0,\quad \partial_{\boldsymbol{n}}\theta=0,\quad \text{on } \partial\Omega.$$

Therefore it is only possible to rely on the energy estimates such as (1.18) for low derivative orders, and to hope having a sufficient decay on $\|\theta'\|_{L^2}$ to ensure convergence of the whole solution. In this case, the above approach provides

$$\|\theta'\|_{L^2}\lesssim\frac{\|\partial_x\theta_0'\|_{H^4}}{1+t}. \tag{1.20}$$

Note that an interplay between horizontal derivatives and the isotropic regularity occurs due to the underlying operator, appearing in the linearised system,

$$\partial_t\theta'=u_2,\qquad -\Delta\boldsymbol{u}+\nabla p=-\theta'\boldsymbol{e}_z. \tag{1.21}$$

At this point, it is convenient to express the velocity $\boldsymbol{u}$ in term of its stream function ψ, satisfying $\boldsymbol{u}=\nabla^\perp\psi$, and the Stokes equation with Dirichlet boundary conditions rewrites

$$\Delta^2\psi=\partial_x\theta',\quad \psi=\partial_n\psi=0 \text{ on } \partial\Omega. \tag{1.22}$$

Let us denote by $\Delta^{-2}\partial_x\theta'$ the solution to (1.22). Remark that Δ^{-2} is distinct from the iteration of the inverse Laplace operator, namely $(\Delta^{-1})^2$, due to the distinct boundary conditions. Then the linearised system also writes

$$\partial_t\theta'=\partial_x\psi=\Delta^{-2}\partial_x^2\theta'.$$

More generally, we observe in the analysis that each elliptic regularity gain $(\Delta^{-2})^{1/2}$ is paired with one horizontal derivative.

Back to (1.20), we need to control the H^4 norm of $\partial_x\theta'$ to ensure that $\|\theta'\|_{L^2}$ decays indeed as $(1+t)^{-1}$, which is possible to show without integrations by parts, and by controlling that the whole perturbation θ does not grow too fast in H^6. The last step of the proof consists in bringing the previous linear analysis into the full nonlinear system. Intuitively, the strategy is the following: denote by $(0,T^*)$ the maximal time interval over which $\|\theta'\|_{L^2}\leq B(1+t)^{-1}$ and $\|\theta'\|_{H^4}\leq B$ are valid with a constant B. In fact more estimates need to be included in the boostrap argument for technical reasons, see (3.18). On this time interval, the quadratic terms can be treated perturbatively, provided $\|\theta_0\|_{H^4}$ is sufficiently small. Hence the bootstrap estimates hold with a constant which is better than B, and thus $T^*=\infty$. It follows that θ' converges towards zero in L^2, and that the time derivative of $\bar\theta$ is integrable. Hence $\bar\theta$ has a limit in L^2 as $t\to\infty$. This is the main part of the proof which is detailed in Subsection 3.2.3.

Asymptotic profile identification. Since θ' converges to zero in any H^m for $m < 4$ as $t \to \infty$ and $\bar{\theta}$ has a limit in L^2 as $t \to \infty$, the whole density $\rho = \Theta + \theta = \Theta + \bar{\theta} + \theta'$ converges to some limit ρ_∞ in L^2 and ρ_∞ depends only on z. Since θ is small compared to Θ, $\partial_z \theta$ is small, hence ρ_∞ is strictly decreasing with respect to z as is Θ. The transport of the density by the divergence-free field $\boldsymbol{u}$ ensures that the level sets of ρ are preserved by the time evolution, and by strong convergence this is also the case for the limit ρ_∞. According to rearrangement theory, ρ_∞ is therefore a rearrangement of ρ_0. One can show that there exists a unique decreasing vertical rearrangement of ρ_0, hence ρ_∞ is uniquely determined. This part of the proof is detailed in Subsection 3.2.4.

Biharmonic operator and adapted basis During the analysis of the stability problem, the Stokes equation reduces to a bilaplacian equation. Depending on the boundary condition for the velocity required in the Stokes system, the bilaplacian equation is also endowed with different boundary conditions. It appears that considering slip boundary conditions, also called Neumann boundary conditions, leads to similar analysis and stability result as in the work [CCL19a], with the same proof. On the contrary, the no-slip assumption, also called Dirichlet condition, breaks the aforementioned proof at different steps, which we detailed all along our work. We provide a selfcontained proof of the classical estimates on the solutions of the bilaplacian equation. Moreover, it seemed convenient to study closer this bilaplacian operator endowed with Dirichlet boundary conditions, and to obtain an explicit eigenfunction basis to adapt the Fourier analysis of Castro, Córdoba & Lear. This basis is described in Section 3.8. The structure of the eigenfunctions, based on the Fourier basis with addititional terms rectifying the trace, is relatable to the boundary layer formation stated in Theorems 3.1.2 and 3.1.3.

Linear asymptotic expansion for non-vanishing perturbation on $\partial\Omega$ The previous result is only valid under the assumption that the perturbation and its normal derivative are vanishing on $\partial\Omega$ *i.e.* when $\rho_0 - \Theta \in H^2_0(\Omega)$. If the perturbation does not vanish on the boundary, this question is non trivial even for the linearized equations around $\Theta = 1 - z$:

$$\begin{cases} \partial_t \theta - \boldsymbol{u} \cdot \boldsymbol{e}_z = 0 \\ -\Delta \boldsymbol{u} + \nabla p = -\theta \boldsymbol{e}_z \\ \qquad \operatorname{div} \boldsymbol{u} = 0 \\ \qquad \theta|_{t=0} = \theta_0. \end{cases} \tag{1.23}$$

It turns out that θ vanishes as $t \to \infty$ but with a much slower rate than in the case when $\theta_0 = \partial_{\boldsymbol{n}} \theta_0 = 0$ on $\partial\Omega$. This is due to the formation of boundary layers of typical size $t^{-1/4}$ as $t \to \infty$, in the vicinity of $z = 0$ and $z = 1$. More precisely, we will prove the following result in Section 3.3:

Theorem 1.3.6 (DGL. '23). *Let $\theta_0 \in H^s(\Omega)$ for some s sufficiently large. Then the solution of (1.23) satisfies:*

$$\theta = \bar{\theta}_0 + \theta^{\mathrm{BL}} + O(t^{-1}) \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad t \to \infty,$$

where $\bar{\theta}_0(z) = \frac{1}{2\pi} \int_0^{2\pi} \theta_0(x, z) \, \mathrm{d}x$ is the horizontal average of the initial data and θ^{BL} is the boundary layer part whose leading terms are:

$$\theta^{\mathrm{BL}} = \Theta^0_{\mathrm{top}}(x, t^{1/4}(1 - z)) + \Theta^0_{\mathrm{bot}}(x, t^{1/4} z) + l.o.t$$

with Θ^0_{top} *and* Θ^1_{bot} *decaying exponentially as* $Z = t^{1/4}z \to \infty$. *Note that* $\|\theta^{\text{BL}}\|_{L^2(\Omega)} \lesssim (1+t)^{-1/8}$.

Non-linear asymptotic expansion For the nonlinear problem, more complex boundary layers seems to develop when the perturbation is non-zero on $\partial\Omega$. To avoid such complications, we go back to the case where $\rho_0 - \Theta \in H^2_0(\Omega)$. The previous results will then allow us to derive uniform bounds in $H^8(\Omega)$, modulo some boundary layer terms:

Theorem 1.3.7 (DGL. '23)**.** *There exists* $\varepsilon_0 > 0$ *small such that for any* $\rho_0 \in H^{14}(\Omega)$ *satisfying* $\|\rho_0 - \Theta\|_{H^{14}} \leq \varepsilon_0$ *and* $\rho_0 - \Theta \in H^2_0(\Omega)$, *then the solution* ρ *of (ST) satisfies:*

$$\rho = \rho_\infty + \rho_{\text{BL}} + O(t^{-2}) \quad \textit{in} \quad L^2(\Omega) \quad \textit{as} \quad t \to \infty,$$

where ρ_{BL} *is the boundary layer part:*

$$\rho_{\text{BL}} = \frac{1}{t}\Theta_{\text{top}}(x, t^{1/4}(1-z)) + \frac{1}{t}\Theta_{\text{bot}}(x, t^{1/4}z) + l.o.t.$$

with Θ_{top} *and* Θ_{bot} *decaying exponentially as* $Z = t^{1/4}z \to \infty$.

We note that $\|\rho_{\text{BL}}\|_{L^2(\Omega)} \lesssim (1+t)^{-9/8}$ so this result strongly suggests that the optimal decay of $\rho - \rho_\infty$ is like $t^{-9/8}$ in $L^2(\Omega)$, which is close to the rate t^{-1} obtained in Theorem 1.3.5. Indeed, it would be very surprising but not excluded that the non-linear dynamics drive the system to the case where these boundary layers terms always vanish.

It seems there are no major obstacles in obtaining a similar non-linear result when the perturbation do not vanish on the boundary, except technicalities due to the fact that boundary layer equations will be non-linear. The present result proved in Section 3.4 requires only linear boundary layer and is already quite technical.

1.3.4 Low regularity instability for (IPM) and (ST)

We saw that for regular enough small initial data θ_0, assumed small, (IPM) and (ST) admit global solutions near $z \mapsto 1 - z$. In [KY23], Kiselev & Yao show that there exist initial data arbitrary close to any stationary profile Θ in $H^{2-\gamma}$ for any fixed $2 > \gamma > 0$, such that the perturbation has algebraic growths. In the periodic channel, their statement writes as follows, with adapted notations from [KY23, Theoram 1.5, Corollary 4.2].

Theorem 1.3.8 (Kiselev & Yao '21)**.** *Let* $\Theta \in C^\infty(\Omega)$ *be a stationary solution of (IPM). For any* $\varepsilon > 0$ *and* $2 > \gamma > 0$, *there exists a perturbation* $\theta_0 \in C^\infty(\Omega)$ *satisfying*

$$\|\theta_0\|_{H^{2-\gamma}} \leq \varepsilon,$$

such that the associated solution $\rho = \Theta + \theta$, *provided it exists globally in time, obeys*

$$\limsup_{t\to\infty} t^{-s/2}\|\rho(t) - \Theta\|_{\dot{H}^s} = \infty, \qquad \textit{for all } s > 0.$$

The statements in $\mathbf{R}^2$ and $\mathbf{T}^2$ are essentially the same, with different growth rates and a vertical symmetry assumption specific to the torus case. Note that this theorem relies on the global existence of smooth solutions, which remains an open question. The growth formulated here means that, there exists time extractions such that the norm of the perturbation grows at least polynomially with respect to time. The proof of this result relies in particular on the

monotonicity of the potential energy and on a geometric argument. Let us assume the density ρ_0 to be constant on a closed curve. Since the transport is pure, this density is transported together with the curve. Under one extra assumption, it happens that the curve shall squash, which implies the growth of the derivatives.

The result of Kiselev & Yao adapts to the Stokes-transport equation.

Theorem 1.3.9 (L.). *Let $\Theta \in C^\infty(\Omega)$ be a stationary solution for (ST). For any $\varepsilon > 0$ and $2 > \gamma > 0$, there exists a perturbation $\theta_0 \in C^\infty(\Omega)$ satisfying*

$$\|\theta_0\|_{H^{2-\gamma}} \leq \varepsilon,$$

such that the solution associated to $\rho_0 = \Theta + \theta_0$ obeys

$$\limsup_{t\to\infty} t^{-s/4}\|\rho(t) - \Theta\|_{\dot{H}^s} = \infty, \qquad \text{for all } s > 0.$$

In this case, the solution exists globally since the problem is well-posed. The growth rates are different, due to the laplacian regularization provided by the Stokes equation, which modifies the velocity regularity. This is the main difference with the (IPM) equation, since the proof comes to study closed curves geometry evolution under the flow, with the same transport equation in both cases. The proof of Theorem 1.3.9 is given in Section 3.7. Note that there is no contradiction between these instability results and the previous stability statements, for which initial data have to be small in higher Sobolev spaces. This observations raises the question of the existence of a Sobolev regularity threshold such that the profiles are unstable (resp. stable) when less (resp. more) regular than this threshold.

1.3.5 Long-time behaviour for the Boussinesq equation

The goal of this section is to compare the asymptotic behaviours observed for (IPM) and (ST), described in the present section, with the ones identified in the literature for the Boussinesq equation. We consider the Boussinesq equation with no diffusion,

$$\left\{\begin{array}{rcl} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p &=& -\rho \boldsymbol{e}_z \\ \partial_t \rho + \boldsymbol{u}\cdot\nabla\rho &=& 0 \\ \operatorname{div}\boldsymbol{u} &=& \mathbf{0}. \end{array}\right. \tag{1.24}$$

This equation is often endowed with the slip or no-slip boundary conditions, respectively

$$\left\{\begin{array}{l} \boldsymbol{u}\cdot\boldsymbol{n} = 0 \\ \partial_{\boldsymbol{n}}\boldsymbol{u} = 0, \end{array}\right. \quad \text{or} \quad \boldsymbol{u} = \mathbf{0} \quad \text{on } \partial\Omega.$$

We refer to [Hu+16] for a comprehensive review of recent results on the global existence and persistence of regularity for the 2-dimensional Boussinesq equation, in particular with viscosity and zero diffusivity as in (1.24). This equation is globally well-posed in Sobolev spaces, see for instance the following result from [HKZ13, Theorem 2.1].

Theorem 1.3.10 (Hu, Kukavica & Ziane '13). *Let $\Omega \subset \mathbf{R}^2$ be a smooth, bounded and connected open set. Let $\boldsymbol{u}_0 \in H^2(\Omega)$ such that $\operatorname{div}\boldsymbol{u}_0 = 0$, and $\rho_0 \in H^1(\Omega)$. Then there exists a unique global solution $(\rho, \boldsymbol{u})$ for (1.24) endowed with the no-slip boundary condition, such that*

$$(\rho, \boldsymbol{u}) \in L^\infty(\mathbf{R}_+; H^1(\Omega)) \times \left(L^\infty(\mathbf{R}_+; H^2(\Omega)) \cap L^2_{\mathrm{loc}}(\mathbf{R}_+; H^3(\Omega))\right).$$

Note that the energy balance of the system for the Boussinesq equation states

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}\|_{L^2}^2 + E(t)\right) = -\|\nabla \boldsymbol{u}\|_{L^2}^2.$$

In particular, it is not ensured that the potential energy decays monotonously, contrary to the Stokes-transport solutions. One can consider an initial datum with a localised density and upward velocity, which should increase at least locally in time the potential energy, surely compensated by a kinetic energy decay. Nevertheless, [LPZ11, Theorem 1.1] ensures that the kinectic energy $\|\boldsymbol{u}\|_{L^2}^2$ of the system is uniformly bounded, for initial data $(\rho_0, \boldsymbol{u}_0) \in H^3 \times H^3(\Omega)$ on bounded 2-dimensional domains.

In [Doe+18], Doering, Wu, Zhao & Zheng show that global well-posedness holds for (1.24) endowed with slip boundary conditions, on bounded domains $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$ in the class $C^{1+\mu}, \mu \in (0,1)$. The authors also study the evolution of density ρ near stationary profiles Θ, with perturbation $\theta = \rho - \Theta$ according to our notations. In [Doe+18, Theorem 1.2], the authors consider perturbations $(\rho_0, \boldsymbol{u}_0)$ of the steady state $(\Theta, \mathbf{0})$ with Θ proportional to $z \mapsto -z$ up to an additive constant. In this framework, they prove that for $(\theta_0, \boldsymbol{u}_0) \in H^1 \times H^2(\Omega)$, divergence-free and with boundary compatibility conditions, the global solution $(\theta, \boldsymbol{u})$ satisfies

$$\|\boldsymbol{u}\|_{H^1} \to 0, \quad \|\partial_t \boldsymbol{u}\|_{L^2} \to 0, \quad \|\theta\|_{L^2} \to c_\infty, \quad \|\nabla p + \theta \boldsymbol{e}_z\|_{H^{-1}} \to 0, \tag{1.25}$$

where c_∞ is a constant in $\left(0, \sqrt{\|\boldsymbol{u}_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2}\right)$. The same result has been obtained by [Tao+20, Theorem 1.4] in $\mathbf{T}^2$ for data only in $\theta_0 \in L^\infty(\mathbf{T}^2)$ and $\boldsymbol{u}_0 \in H^2(\mathbf{T}^2)$. These properties are obtained for the Stokes-transport equation, and the latter convergence in H^{-1} has implications on the ω-set for the density, see Subsection 3.6.3. Doering et al. point out several missing informations regarding their result. No decay rates are available. The convergence of θ and its limit are unclear, although the authors opinion is that the vertical rearrangement of θ_0 is a privileged candidate. Also, they mention the Boussinesq equation in the infinite Prandtl number asymptotic, corresponding to the Stokes-transport equation, as a reduced model facilitating the obtainment of decay rates and identification of the asymptotic profile. Chapter 3 answers both these questions. Indeed, we obtained explicit convergence rates for small enough perturbations of the affine profile $\Theta(z) = 1 - z$, and were able to identify the asymptotic profile as the vertical rearrangement of the initial datum.

For generic initial data, the convergence toward an asymptotic state is unclear. In particular, [AKZ23, Theorem 2.1] states that for any $(\rho_0, \boldsymbol{u}_0) \in H^1(\Omega) \times H^2(\Omega)$ with $\operatorname{div} \boldsymbol{u}_0 = 0$, on bounded and smooth $\Omega \subset \mathbf{R}^2$, the solution satisfies in addition to (1.25) the following growth control,

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \quad \|\rho(t)\|_{H^1} \leq C_\varepsilon e^{\varepsilon t}.$$

This result can be compared with the instability Theorem 1.3.8 from [KY23] for (IPM) and the stability Proposition 1.3.9 for (ST), where existence of solutions admiting some algebraic growth of the density in Sobolev spaces, in particular in H^1. Note that a growth in H^1 does not exclude a convergence in L^2.

The Boussinesq equation solutions can also present algebraic growths, in Sobolev norms, similarily to Theorem [KY23] for (IPM), for data satisfying certain geometric assumptions, see [KPY22, Theorem 1.1] by Kiselev, Park & Yao. The growths have distinct exponent rates, and the initial data for the Boussinesq equation are assumed smooth and compactly supported, whereas the data were in $H^{2-\gamma}, 0 < \gamma < 2$. The proof relies on similar ideas as for (IPM).

In [BFL17], Biswas, Foias & Larios describe properties of the attractor of the system (1.24), and identify some of its steady states in $\mathbf{T}^2$. Let us recall that the steady states for the Stokes-

transport equation are of the form

$$(\rho, \boldsymbol{u}) = (\Theta(z), \mathbf{0}), \quad \Theta \in L^\infty(0,1).$$

These are also steady states for the Boussinesq equation. But there exist more steady states with nonzero velocity fields, for instance. Biswas et al. also identified steady solutions of the form

$$(\rho, \boldsymbol{u}) = (\rho(z), (u_1(z), 0)), \qquad (\rho, \boldsymbol{u}) = (\rho(x), (0, u_2(x))), \tag{1.26}$$

which are respectively density and horizontal velocity depending only on the vertical coordinate (shear flow), and density and vertical velocity depending only on the horizontal coordinate. These latter solutions are admissible since the spatial domain is periodic. Also, in the Stokes-transport case, any nonstratified state induces a nonzero horizontal velocity, and a stratified density does not contribute to the velocity. This explains why (1.26) are not steady solutions for (ST).

More stability results around specific velocity flows are studied, but we restrain our overview to the case $\boldsymbol{u} = 0$, since it is the only stable one for (ST) and (IPM).

In [CCL19b], Castro, Córdoba & Lear proved the analogous stability result as Theorem 1.3.4, for a Boussinesq equation with velocity damping, where the momentum equation is

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u} + \nabla p = -\rho \boldsymbol{e}_z.$$

This system corresponds to the (IPM) equation supplemented with the fluid self-advection. It differs from the classical Boussinesq equation(1.24) by a damping term $\boldsymbol{u}$ replacing the viscosity effect $-\Delta \boldsymbol{u}$. This can be seen a limiting case of fractional dissipation on the velocity.

Theorem 1.3.11 (Castro, Córdoba & Lear '19)**.** *Let us consider the domain $\Omega = \mathbf{T} \times (0,1)$. There exists $\varepsilon_0 > 0$ and $\gamma > 4$ such that for any initial data $\rho_0 = \Theta + \theta_0$ and $\boldsymbol{u}_0$ having some energy, involving the $H^{s+1}, s \geq 6 + 2\gamma$, norm of the data, smaller than $0 < \varepsilon \leq \varepsilon_0$ and satisfying*

$$\partial_{\boldsymbol{n}}^{2n} \theta_0|_{\partial\Omega} = 0, \quad 0 \leq n \leq (s-1)/2,$$

with a similar condition on $\boldsymbol{u}_0$, the solution to (IPM) exists globally and obeys

$$\|\theta'(t)\|_{H^4} + \|\boldsymbol{u}(t)\|_{H^4} \lesssim \frac{\varepsilon}{(1+t)^{\gamma/4}}, \quad \|\bar{\theta}(t)\|_{H^{s+1}} \leq 6\varepsilon^2.$$

The proof of this results is based on the same scheme as for the (IPM) case, made more technical due to the self-advection term.

It is natural to wonder if the Stokes-transport stability in a channel with Dirichlet boundary conditions as stated in Theorem 1.3.5 holds for the Boussinesq equation. Let us recall some decay rates for the Stokes-transport stability described Subsection 1.3.3,

$$\|\theta'\|_{L^2} \lesssim \frac{1}{1+t}, \quad \|\nabla\theta'\|_{L^2} \lesssim \frac{1}{(1+t)^{3/4}}, \quad \|\partial_t\theta'\|_{L^2} \lesssim \frac{1}{(1+t)^2}.$$

The algebraic decay reduces by $1/4$ for each spatial derivative order, and increases by 1 for time derivatives. Regarding the velocity field solving the Stokes equation, the same principes hold with a power shift due to the elliptic regularisation, as follows,

$$\|\boldsymbol{u}\|_{L^2} \lesssim \frac{1}{(1+t)^{7/4}}, \quad \|\nabla\boldsymbol{u}\|_{L^2} \lesssim \frac{1}{(1+t)^{3/2}}, \quad \|\partial_t\boldsymbol{u}\|_{L^2} \lesssim \frac{1}{(1+t)^{11/4}},$$

$$\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{L^2} \lesssim \|\boldsymbol{u}\|_{L^\infty}\|\nabla\boldsymbol{u}\|_{L^2} \lesssim \underbrace{\|\boldsymbol{u}\|_{L^2}^{1/2}\|\boldsymbol{u}\|_{H^2}^{1/2}\|\nabla\boldsymbol{u}\|_{L^2}}_{\frac{1}{2}\cdot\frac{7}{4}+\frac{1}{2}\cdot\frac{5}{4}+\frac{3}{2}} \lesssim \frac{1}{(1+t)^3}.$$

Schematically, we can expect to absorb the self-advection term effects since it should decay faster than the density perturbation θ', and consider it perturbatively,

$$-\Delta\boldsymbol{u}+\nabla p = -\theta'\boldsymbol{e}_z - \partial_t\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}.$$

Hence, up to the technicalities inherent to the self-advection term, we expect the hydrostatic rest state $(\rho,\boldsymbol{u}) = (\Theta(z)=1-z,\boldsymbol{0})$ to be stable for the Boussinesq equation, in the sense of Theorem 1.3.5.

1.4 Numerical study of the graph interface evolution

The aim of Chapter 4 is to provide a study of some graph interface evolution governed by the Stokes-transport equation. We sketch the outlines of this problem in Subsection 1.4.1. In Subsection 1.4.2 we briefly discuss the evolution of density patches and summarise some works of Mecherbet and Grayer II on this matter. In Subsection 1.4.3 we summarise works on free interfaces governed by Stokes flows, and we give some context about related free interfaces problems. In particular we mention the Muskat problem, since it corresponds to the free interface evolution driven by (IPM). Several usual mathematical approaches for such problems are also described. We establish some properties for our problem of interest, summarised in Subsection , including discussions about monotonous functionals for the interface problem. Finally, our main contribution consists in numerical simulations and observations for the evolution of graph interfaces in the horizontally periodic channel, also summarised in Subsection 1.4.4.

1.4.1 Outlines of the problem

We consider the evolution of a graph interface separating two subdomains on which the fluid density is constant. Consider a generic 2-dimensional domain $I\times J$, where I,J are some intervals, on which we consider the Stokes equation with a source term of the form $-\rho_0\boldsymbol{e}_z$. The typical case we are looking at is the graph interface case, where the initial density is given as a patch associated to an hypograph. In other words, consider a map $\eta_0 : x\in I\to\mathbf{R}$ where I is typically $\mathbf{R}$ or $\mathbf{T}$, and set the initial density profile as the indicator of the hypograph of η_0, namely

$$\rho_0 : (x,z)\in I\times J \longmapsto \mathbf{1}_{\{z<\eta_0(x)\}}.$$

Some typical domains $I\times J$ to consider are the whole space $\mathbf{R}^2$, the half-space $\mathbf{R}^2_+$, the infinite channel $\mathbf{R}\times(0,1)$ and in the periodic channel $\mathbf{T}\times(0,1)$ and the periodic plane $\mathbf{T}\times\mathbf{R}$. In the present work, we will focus on the horizontaly periodic channel $\Omega=\mathbf{T}\times(0,1)$. A lot of the general considerations discussed in the following remain true in other cases. See Figure 1.1 for an illustration of ρ_0 in this case.

Let us recall that (ST) is well-posed for L^∞ initial data in bounded domains, in the infinite channel and periodic strip, and in the whole space $\mathbf{R}^3$ for datum $\rho_0\in L^1\cap L^\infty(\mathbf{R}^3)$ and in $\mathbf{R}^2$ for $\rho_0\in L^1\cap L^\infty(\mathbf{R}^2)$ with compact support. In any of these frameworks, a *density patch* associated to a connected subdomain P_0 remains a patch for all time. Indeed, since the transport is pure and the velocity field is divergence-free, we know that the solution of (ST) is the pushforward of the initial datum by the flow map $\boldsymbol{X}$ as defined in Subsection 2.2.2, so $\rho(t)=\mathbf{1}_{P_0}\circ\boldsymbol{X}(t)^{-1}=\mathbf{1}_{\boldsymbol{X}(t;P_0)}$.

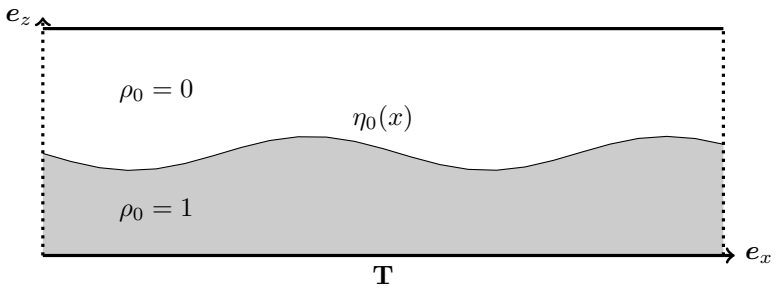

Figure 1.1: Example of a patch associated to an hypograph on $\mathbf{T} \times (0,1)$.

Moreover, the flow preserves the topology of the boundary ∂P_0 and the area of the patch. Some general patch problems are discussed in Subsection 1.4.2.

In the case of a graph interface, *i.e.* a patch corresponding to some hypograph, it is natural to wonder if the interface remains a graph, locally or globally in time, and for which class of regularity. Let us assume there exists a function $\eta : (t,x) \in [0,T) \times I \mapsto \eta(t,x) \in J$ such that

$$\rho(t,x,z) = \mathbf{1}_{\{z<\eta(t,x)\}}.$$

Then, ρ satisfies (ST) if and only if η satisfy the following system

$$\left\{ \begin{array}{rcll} \partial_t \eta + u_1(t,x,\eta(t,x))\partial_x \eta(t,x) & = & u_2(t,x,\eta(t,x)), & \text{on } [0,T) \times I, \\ -\Delta \boldsymbol{u} + \nabla p & = & -\mathbf{1}_{\{z<\eta(t,x)\}} \boldsymbol{e}_z, & \text{on } [0,T) \times \Omega, \\ \operatorname{div} \boldsymbol{u} & = & 0, & \text{on } [0,T) \times \Omega. \end{array} \right. \tag{1.27}$$

This system is obtained in Subsection 4.1.2. The first equation is very classical in the free surface literature, and simply means that the graph is transported by the velocity field on the interface $\Sigma(t) := \{(x, \eta(t,x)) : x \in I\}$. Observe that the velocity field $\boldsymbol{u}$ is given as the solution to the Stokes equation with a L^∞ datum, whose regularity is not correlated to the graph of η, which restricts the maximal regularity we can expect on $\boldsymbol{u}$. This fact is precised in Subsection 1.4.2 and in Section 4.1.

1.4.2 About density patches

We[1] call *density patch* any function ρ of the form $\mathbf{1}_P$ where P is a connected and non-negligible spatial subdomain. Considering such an initial datum $\rho_0 = \mathbf{1}_{P_0}$ for the Stokes-transport equation, we know that the associated solution is the pushforward of ρ_0 by the flow $\boldsymbol{X}$ associated to the velocity field. In particular, ρ remains a patch for all time, since $\rho(t) = \mathbf{1}_{\boldsymbol{X}(t;P_0)}$, and we define its support as $P(t) = \boldsymbol{X}(t;P_0)$. Since $\boldsymbol{u}$ is divergence-free, the flow is measure preserving, and we have $|P(t)| = |P_0|$ for all time t. The regularity of the interfaces of such patches is often described thanks to Hölder spaces $C^{k,\mu}$, associated to the following semi-noms and norms, for $k \in \mathbf{N}, \mu \in (0,1)$,

$$\|\gamma\|_{\dot{C}^{k,\mu}} = \sup_{\boldsymbol{x} \neq \boldsymbol{y}} \frac{|\nabla^k \gamma(\boldsymbol{x}) - \nabla^k \gamma(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^\mu}, \quad \|\gamma\|_{C^{k,\mu}} = \max_{\ell=0}^{k} \|\nabla^\ell \gamma\|_{L^\infty} + \|\gamma\|_{\dot{C}^{k,\mu}}.$$

[1] Other definitions might require the subdomain P to be simply connected, which is not the case for hypograph patches in the periodic channel $\Omega = \mathbf{T} \times (0,1)$, for instance.

The situation of density pockets for (ST) has been considered by Mecherbet in [Mec20] and by Grayer II in [Gra22].

In [Mec20], Mecherbet investigates the sedimentation of a droplet in a Stokes flow in $\mathbf{R}^3$, governed by (ST). She derives an interface evolution equation for patches homeomorphic to the sphere, described thanks to a spherical parametrisation. Denoting by $(\theta, \phi) \mapsto r(\theta, \phi)$ this parametrisation, this evolution equation shares a similar form as (1.27), with two operators A_i,

$$\partial_t r + \partial_\theta A_1[r] = A_2[r], \quad r|_{t=0} = r_0.$$

Mecherbet establishes the local well-posedness of this system, with solutions in $C^{0,1}$. In particular, she proves that the shape of a spherical interface is preserved. Her proof relies on the property that the normal component of the fluid velocity is constant on the sphere, and corresponds to the velocity fall. This is relatable to the modeling of the fluid-particle system from which (ST) has been derived by Höfer and Mecherbet, where such an assumption is made at the scale of a particle, see Subsection 1.1.2. Finally, Mecherbet performed a numerical study of this interface evolution, and observed stationary sphere shape.

In [Gra22], Grayer II investigates the regularity of density patches in $\mathbf{R}^2$. He starts by proving the well-posedness of (ST) in $\mathbf{R}^2$ for compactly supported initial data $\rho_0 \in L^1 \cap L^\infty(\mathbf{R}^2)$, see [Gra22, Theorem 3]. From here, it is possible to consider initial density patches associated to simply connected and compact subdomains $P_0 \subset \mathbf{R}^2$. In [Gra22, Theorem 5], Grayer II states that for any $k \in \{0, 1, 2\}$ and $\mu \in (0, 1)$ the Hölder regularity $C^{k,\mu}$ of $\partial P(t)$ is preserved for all time.

In [AYM00], Antontsev et al. prove that the interface of a multiple patch initial data, such that the initial interface Σ_0 is C^2, remain $C^{1,\mu}, \mu \in (0, 1)$ for all time. See the end of Subsection 1.2.1 on the belated rediscovery of this reference.

The preservation of Hölder regularity $C^{k,\mu}$ comes straighforwardly from the description of the patch at time t in terms of the flow $\boldsymbol{X}$ and P_0, with in particular

$$\partial P(t) = \boldsymbol{X}(t; \partial P_0).$$

The regularity of $\boldsymbol{X}(t) \in C^{1,\lambda}$ can be deduced from $\boldsymbol{u} \in C^{1,\lambda}$, for any $\lambda \in (0, 1)$. The regularity $\partial P_0 \in C^{1,\mu}$ is equivalent to have existence of a parametrisation $\gamma_0 : [0, 1] \mapsto \partial P_0$ for regularity $C^{1,\mu}$ of the curve ∂P_0. With this description, $\partial P(t)$ parametrises by $\gamma(t) = X(t; \gamma_0)$. Hence, $\gamma(t)$ belongs to $C^{1,\min(\lambda,\mu)}$, see Lemma 4.1.3, hence to $C^{1,\mu}$ since λ is arbitrary. See Subsection 4.1.4 for further details on this matter. Here in $\mathbf{R}^2$, since the density belongs to $L^1 \cap L^2(\mathbf{R}^2)$ the velocity is $\dot{C}^{1,\mu}$, and the above argument allows to prove that $C^{1,\mu}$ is preserved. These are also the main arguments used in [AYM00] to prove that patch solutions for (ST) in bounded domains remains $C^{1,\mu}, \mu \in (0, 1)$ for all time, if the initial Σ_0 is C^2. Since the regularity of the velocity is a priori limited, we cannot expect to propagate higher regularity order with this approach. To prove that $\partial P(t)$ remains $C^{2,\mu}$, Grayer II deals with more subtle geometric properties of the explicit formula for the velocity field $\boldsymbol{u}$ in terms of the kernel associated to the Stokes equation, including symmetry and cancellation properties. These techniques come from the seminal work of Yudovich [Yud63] on vorticity patches solution for the 2-dimensional Euler equation, see also [Che93] on this topic. Besides, Grayer II runs a numerical simulation to observe the evolution of a circle interface. He observes the formation of corner-like singularities arising from this initial disk patch. Besides, he shows that the integral expression of the initial vorticity associated with a disk patch initial datum is not symmetric, suggesting that the interface does indeed not remain a circle. This result is to compare with the stationary sphere investigated by Mecherbet.

1.4.3 Related graph interfaces problems and techniques

In the present subsection we provide a summary of some works about graph interface problems related to the Stokes-transport equation. To our knowledge, only one contribution [GGS22] exists about the graph interface problem (1.27), governed by (ST). They analyse the case of the periodic domain $\mathbf{T}\times\mathbf{R}$. Other problems concerning an interface evolution in a Stokes fluid governed by capillary forces are tackled in the literature. Besides, the graph interface problem has been extensively studied for the Incompressible Porous Medium system, better known as the Muskat problem. These three types of problems are respectively covered in the following paragraphs. We insist on two different techniques involved to address such matters, which are the contour dynamic approach and the Arbitrary Lagrangian-Eulerian formulations. These could in particular be carried out to address the evolution of interfaces for (ST) in other domains than $\mathbf{T}\times\mathbf{R}$.

Stokes-transport graph interface In [GGS22], Gancedo, Granero-Belinchón and Salguero address the free interface problem for gravity Stokes flow, corresponding to the system (1.27) on the 2-dimensional horizontally periodic domain $\mathbf{T}\times\mathbf{R}$. In all generalities, they express the interface as a curve, and consider the particular case of a graph. We will stick to the graph point of view in the following. With our notations, once the constants are renormalised, the equation writes as follows.

$$\partial_t\eta = \boldsymbol{u}|_\eta\cdot\boldsymbol{n}_\Sigma\langle\partial_x\eta\rangle, \qquad \text{on } \mathbf{T}, \tag{1.28a}$$
$$-\Delta\boldsymbol{u}^\pm+\nabla p^\pm = -\rho^\pm\boldsymbol{e}_z, \qquad \text{in } \Omega^\pm(t), \tag{1.28b}$$
$$\operatorname{div}\boldsymbol{u}^\pm = 0, \qquad \text{in } \Omega^\pm(t), \tag{1.28c}$$
$$[\![\nabla\boldsymbol{u}^\pm+(\nabla\boldsymbol{u}^\pm)^\perp-2p^\pm I]\!]\boldsymbol{n}_\Sigma = \mathbf{0}, \qquad \text{on } \Sigma(t), \tag{1.28d}$$
$$[\![\boldsymbol{u}]\!] = 0, \qquad \text{on } \Sigma(t), \tag{1.28e}$$
$$\eta|_{t=0} = \eta_0, \tag{1.28f}$$

where $\rho^+=0,\rho^-=1$, with $\Sigma(t)=\{(x,\eta(t,x)) : x\in\mathbf{T}\}$ is the interface, $\langle\partial_x\eta\rangle=\sqrt{1+|\partial_x\eta|^2}$ and $[\![f]\!]:=f^+_{|\Sigma}-f^-_{|\Sigma}$ denotes the jump of the quantity f at the interface. Here equation (1.28a) corresponds to the formulation (4.9) of the graph interface evolution, presented at the end of Subsection 4.1.2. The graph is assumed to have zero average, so η corresponds to ζ in this case. Equations (1.28b) and (1.28c) are restrictions of the Stokes equations on the subdomains $\Omega^\pm$, completed with the continuity of the stress tensor (1.28d) and velocity field (1.28e) at the interface. These conditions are equivalent to the Stokes equation considered on the whole domain. We know in particular that the velocity field and the Stokes stress tensor are continous for bounded source terms. The Stokes tensor is

$$\frac{\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\perp}{2}-pI,$$

such that the Stokes equation rewrites

$$-\operatorname{div}\left(\frac{\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\perp}{2}-pI\right)=-\rho\boldsymbol{e}_z.$$

We report here [GGS22, Theorem 1], which contains both qualitative and quantitative results, which we compare our numerical observations to.

Theorem 1.4.1 (Gancedo, Granero-Belinchón & Salguero, '22). *Let $\rho^- > \rho^+$ and $3/2 < m < 2$. There exists $\varepsilon_0 > 0$ such that for any $\zeta_0 \in H^3(\mathbf{T})$ satisfying*

$$\|\zeta_0\|_{H^3} < \varepsilon_0,$$

there exists a unique global solution $\zeta \in L^\infty(\mathbf{R}_+; H^3(\mathbf{T}))$ to (1.28), which moreover obeys

$$(1+t)^m \|\zeta\|_{L^2} + \|\zeta\|_{H^3} \lesssim \|\zeta_0\|_{H^3}.$$

In the first place, Gancedo, Granero-Belinchón & Salguero prove that any generic curve in $C^{1,\mu}$ for $\mu \in (0,1)$ remains a curve at least for short time, whether $\rho^- > \rho^+$ or $\rho^- < \rho^+$. The case investigated here has the heavy fluid below the light one, which should be stable according to the physical intuition. Then, Theorem 4.2.1 means that the flat profile is asymptotically stable for small enough perturbations in H^3. The proof of this theorem relies on a contour dynamic formulation of this problem. We summarise the principle of this method in the following. They also provided a second global existence theorem for solutions of (1.28) in another class of regularity, in which the solution decays exponentially, see [GGS22, Theorems 2 and 3]. In the unstable case $\rho^- < \rho^+$ where the heavy fluid is above, they prove that there exist graph interfaces that grow exponentially in some analytic norm.

The contour dynamic approach The principle of this method is to express the evolution of the interface thanks to an integral equation of the form

$$\partial_t \eta = \mathcal{I}[\eta],$$

where $\mathcal{I}[\eta]$ in an integral with respect to the horizontal variable, here on $\mathbf{T}$, whose integrand involves η and possibly its derivatives. Such a formula can be obtained by tracing back to some Green kernel for the elliptic problem satisfied by the velocity. In the present case, recall that $\boldsymbol{u} = \nabla^\perp \psi$ with ψ satisfying

$$\Delta^2 \psi = \nabla^\perp \cdot (-\rho \boldsymbol{e}_z).$$

Considering the Green kernel associated to the bilaplacian, denoted here K, such that $\Delta^2 K = \delta_{\mathbf{0}}$ on $\mathbf{T} \times \mathbf{R}$, the stream function expresses as

$$\psi(\boldsymbol{x}) = \int_{\mathbf{T}\times\mathbf{R}} K(\boldsymbol{x} - \boldsymbol{y}) \nabla^\perp \cdot (-\rho(\boldsymbol{y}) \boldsymbol{e}_z) \, \mathrm{d}\boldsymbol{y}.$$

From here and in the following, we adopt the notation $\boldsymbol{x} = (x, z)$ and $\boldsymbol{y} = (y_1, y_2)$. The expression of K is obtained by a decomposition in the Fourier basis, horizontally, as follows $K(x,z) = \sum_{n\in\mathbf{Z}} \beta_n(z) e^{inx}$, and to obtain $(\beta_n)_{n\in\mathbf{N}}$ as explicit solutions of some ordinary differential equations of order 4. See the parallel with the eigenfunctions analysis for the bilaplacian equation, in Subsection 3.8.3. An integration by parts provides

$$\psi(\boldsymbol{x}) = \int_{\mathbf{T}\times\mathbf{R}} \Psi(\boldsymbol{x} - \boldsymbol{y}) \rho \boldsymbol{e}_z \, \mathrm{d}\boldsymbol{y}, \qquad \Psi = \nabla^\perp K. \tag{1.29}$$

Using here that ρ is piecewise constant, we can write that $\rho(y_1, y_2)\boldsymbol{e}_z = \nabla(\rho^\pm z)$ for $(y_1, y_2) \in \Omega^\pm$. Substituting this expression in (1.29) and performing an integration by parts, we obtain that ψ expresses as a boundary integral, since the main integrals vanish because $\nabla\Psi = \nabla \cdot \nabla^\perp K = 0$. In the end, expliciting the boundary integral, recalling that $\boldsymbol{u} = \nabla^\perp \psi$ and considering its trace

on $\{\boldsymbol{x} = (x, \eta(x))\}$, we get

$$\boldsymbol{u}(x,\eta(x)) = (\rho^- - \rho^+) \int_{\mathbf{T}} S(x-y, \eta(x)-\eta(y)) \cdot (-\partial_x \eta(y), 1)\eta(y)\,\mathrm{d}y,$$

with an explicit matrix S of size 2×2,

$$S(\boldsymbol{y}) = \frac{1}{8\pi}\left(\log(2(\cosh(y_2)-\cos(y_1)))I_2 - \frac{y_2}{\cosh(y_2)-\cos(y_1)}\begin{pmatrix} -\sinh(y_2) & \sin(y_1) \\ \sin(y_1) & \sinh(y_2)\end{pmatrix}\right).$$

Using that the normal velocity of the interface corresponds to the velocity $\boldsymbol{u}$, see (4.9), we can relate $\partial_t \eta$ and the above, and obtain

$$\partial_t \eta(t,x) = (\rho^- - \rho^+) \int_{\mathbf{T}} S(x-y, \eta(t,x)-\eta(t,y)) \cdot (-\partial_x \eta(t,y), 1)\eta(t,y)\,\mathrm{d}y \cdot (-\partial_x \eta(t,x), 1). \quad (1.30)$$

Of course all the above computations have to be performed carefully, and we must check that all the integrals are properly defined. This latter integral, corresponding to the aforementioned $\mathcal{I}[\eta]$, depends in a highly nonlinear way on η. Existence and uniqueness of a solution to (1.30) in short time is obtained thanks to a Picard fixed-point theorem applyied to I, on a suitable Banach space. The associated linear operator is dissipative, which is linked to the stability provided by having $\rho^- > \rho^+$. The existence of global solutions is obtained by a bootstrap argument. The dynamic contour method applies to a broad familly of interface problems, such as the Muskat problem mentionned thereafter. The integral formulation varies from one problem to another due to the different elliptic equations and spatial domains, and on the associated Green kernels.

Stokes with capillarity governed graph interface We discuss now the evolution of a graph interface in a Stokes fluid governed by capillarity effects, without gravity. We keep the same notations as above. In this paragraph, the spatial domain is $\mathbf{R}^2$. The problem formulates similarly to (1.28), without density and gravity terms and with a different interface condition for the tensor.

$$\partial_t \eta = \boldsymbol{u}|_\eta \cdot \boldsymbol{n}_\Sigma \langle \partial_x \eta \rangle, \qquad \text{on } \mathbf{R}, \quad (1.31a)$$

$$-\nu^\pm \Delta \boldsymbol{u}^\pm + \nabla p^\pm = \mathbf{0}, \qquad \text{in } \Omega^\pm(t), \quad (1.31b)$$

$$\operatorname{div} \boldsymbol{u}^\pm = 0, \qquad \text{in } \Omega^\pm(t), \quad (1.31c)$$

$$[\![\nabla \boldsymbol{u}^\pm + (\nabla \boldsymbol{u}^\pm)^\perp - 2p^\pm I]\!]\boldsymbol{n}_\Sigma = -\frac{\partial_x^2 \eta}{\langle \partial_x \eta \rangle^3}\boldsymbol{n}_\Sigma, \qquad \text{on } \Sigma(t), \quad (1.31d)$$

$$[\![\boldsymbol{u}]\!] = 0, \qquad \text{on } \Sigma(t), \quad (1.31e)$$

$$\eta|_{t=0} = \eta_0, \quad (1.31f)$$

where $-\frac{\partial_x^2 \eta}{\langle \partial_x \eta \rangle^3}$ is the curvature of η, which models surface tension effects on the interface between the two fluids of respective viscosities $\nu^\pm$ in $\Omega^\pm$.

The first work on this topic is [BD98], in which Badea & Duchon prove the global existence of solutions for small data in a space of Fourier transform with bounded measures. Recently, Matioc & Prokert proved in [MP21; MP22] that given any $\eta_0 \in H^m(\mathbf{R}), m \in \left(\frac{3}{2}, 2\right)$, problem (1.31) admits a unique maximal solution $\eta \in C(0, T^*; H^m(\mathbf{R})) \cap C^1(0, T^*; H^{m-1}(\mathbf{R}))$. Their two papers address respectively the cases $\nu^- = \nu^+$ and $\nu^- \neq \nu^+$.

The approach of Matioc & Prokert meets the contour dynamic method applied above by

Gancedo et al. It consist in expressing the data in an integral form, as in (1.30). This system admits a regularising effect due to the surface tension, modelled thanks to the curvature, which is not present in the gravity driven case.

Muskat problem The Muskat problem models the evolution of the interface between two fluids satisfying Darcy's law. It derives from the Incompressible Porous Medium equation, just as the gravity driven Stokes interface problem derives from the Stokes-transport equation. This problem has been extensively studied for years, long before the interface evolution for (ST), which only begins to be tackled. We refer to [Gan17; GL20] for exhaustive reviews on this topic. The system writes, on $\mathbf{R}^2$, as follows,

$$\partial_t \eta = \boldsymbol{u}|_\eta \cdot \boldsymbol{n}_\Sigma \langle \partial_x \eta \rangle, \qquad \text{on } \mathbf{R}, \tag{1.32a}$$

$$-\nu^\pm \boldsymbol{u}^\pm + \nabla p^\pm = -\rho^\pm \boldsymbol{e}_z \qquad \text{in } \Omega^\pm(t), \tag{1.32b}$$

$$\operatorname{div} \boldsymbol{u}^\pm = 0, \qquad \text{in } \Omega^\pm(t), \tag{1.32c}$$

$$[\![\boldsymbol{u}]\!] = 0, \qquad \text{on } \Sigma(t), \tag{1.32d}$$

$$\eta|_{t=0} = \eta_0. \tag{1.32e}$$

The difference with (1.28) is that the Stokes equation (1.28b) is replaced by Darcy's law (1.32b), and the Stokes tensor continuity condition (1.28d) is no longer relevant. By the same contour dynamic approach described above, the evolution of η is given by

$$\partial_t \eta(t,x) = \frac{\rho}{2\pi} \int_{\mathbf{R}} \frac{(\partial_x \eta(t,x) - \partial_x \eta(t,x-y))y}{y^2 + (\eta(t,x) - \eta(t,x-y))^2} \, \mathrm{d}y, \tag{1.33}$$

where the integral is understood in the principal value sense.

In Darcy's law, the velocity field is basically as regular as its density data, which is L^∞ for patches. From here, it is challenging to analyse the evolution of the interface under such a flow. Regarding the Stokes driven interfaces, the velocity is more regular, basically $W^{2,q}$, which eases the analysis. In return, the contour dynamic formulation for the Stokes case is more tedious than for Muskat, due to the respective kernels at play.

For instance, Córdoba & Gancedo prove in [CG07] that this equation is locally well-posed for initial data $\eta_0 \in L^2 \cap \dot{H}^3(\mathbf{R})$. Other proofs of this result have been given, see references therein [GL20]. Besides, the system satisfies a maximum principle, in the sense that for $\eta_0 \in H^3$, either on $\mathbf{T}$ or $\mathbf{R}$, the unique solution to (1.32) satisfies as long as it exists,

$$\|\eta(t)\|_{L^\infty} \leq \|\eta_0\|_{L^\infty}.$$

This has been proven by Córdoba & Gancedo in [CG09], using crucially the contour integral formulation. Global existence of weak solutions for Lipschitz initial data satisfying $\|\eta_0\|_{L^\infty}$ and $\|\partial_x \eta_0\|_{L^\infty} < 1$ has been established in [Con+12]. Alazard & Nguyen established in [AN21] the first global well-posedness result for non-Lipschitz data in $H^{3/2}$ with fractional logarithmic correction.

In [GGI12], Córdoba, Granero-Belinchón & Orive-Illera compare the properties of the Muskat problem in $\mathbf{R}^2$, with infinite depth, and in the confined channel $\mathbf{R} \times (-1,1)$. In particular, they establish a maximum principle for η, in the sense that $\|\partial_x \eta\|_{L^\infty} \leq \|\partial_x \eta_0\|_{L^\infty}$ for all time, under a slope condition on the intial data. In particular, the proof of this principle relies strongly on the explicit contour integral formulation, which is more complicated for the Stokes case, compare (1.30) and (1.33). This principle is investigated for (ST) in Subsection 4.2.6. Moreover, they obtained numerical evidences that the confined problem is more singular than the infinite depth

one, with initial data developing singularities in the first case, whereas it becomes a graph in the second. We can wonder to what extent such differences are transposable to the Stokes case. See our numerical observations on the boundary influence for (ST) in Subsections 1.4.4 and 4.2.5 on this matter.

The Arbitrary Lagrangian-Eulerian method Another possible approach for tackling such interface problems is the so called Arbitrary Lagrangian-Eulerian method. It consists in mapping the subdomains $\Omega^\pm$ to $\mathbf{R}^2_\pm := \mathbf{R}\times\mathbf{R}_\pm$ respectively, thanks to time-dependent pull-back map designed after η, therefore depending on time, to obtain an equivalent problem with a flat interface. The interface dependency is then moved in additional coefficients appearing in the system due to the change of variable. This method is for instance perfomed by Cheng, Granero-Belinchón & Shkoller in [CGS16] to prove the well-posedness of the Muskat problem with H^2 initial data.

Let us sketch the main ideas here. We call harmonic extension of η on $\mathbf{R}^2_+$ the solution φ^+ to the problem

$$\left\{\begin{array}{rcll}\Delta\varphi^+ &=& 0, & \text{in } \mathbf{R}^2_+,\\ \varphi^+ &=& \eta, & \text{on } \{z=0\}.\end{array}\right.$$

Define the following function,

$$\begin{array}{rccc}\Phi^+ : & \mathbf{R}^2_+ & \longrightarrow & \Omega^+\\ & (x,z) & \longmapsto & (x, z+\varphi(x)),\end{array}$$

which maps $\mathbf{R}^2_+$ to Ω^+ with in particular $\Phi^+(x,0)=(x,\eta(x))$. This extension is also quite regular, as it is harmonic. Let us admit that Φ^+ is a diffeomorphism under some smallness assumption on η in suitable Sobolev spaces. Let us define $(\boldsymbol{v}^+, q^+)$ as the pull back of $(\boldsymbol{u}^+, p^+)$, in the sense

$$\boldsymbol{v}^+ = \boldsymbol{u}^+\circ\Phi^+, \quad q^+ = p^+\circ\Phi^+.$$

From the fact that $(\boldsymbol{u}^+, p^+)$ satisfies (1.32), we can get

$$\partial_t\eta = \nabla\Phi^+((\nabla\Phi^+)^{-1})^\perp\boldsymbol{v}\cdot\boldsymbol{e}_z, \qquad \text{on } \{z=0\}, \tag{1.34a}$$

$$\nu^+\boldsymbol{v}^+ + ((\nabla\Phi^+)^{-1})^\perp\nabla q^+ = -\rho^+\boldsymbol{e}_z, \qquad \text{in } \mathbf{R}^2_+, \tag{1.34b}$$

$$(\nabla\Phi^+)^{-1} : \nabla\boldsymbol{v}^+ = 0, \qquad \text{in } \mathbf{R}^2_+, \tag{1.34c}$$

equivalent to (1.32a), (1.32b) and (1.32c) respectively. The same operation can be performed on $\mathbf{R}^2_-$. This equation has a similar structure to Darcy's law. The idea here is to obtain a problem equivalent to (1.32), but on the fixed domains $\mathbf{R}^2_\pm$, on which systems such as Darcy or Stokes equations solutions can be estimated more easily and explicited thanks to kernels. The price to pay is the introduction of variable coefficients from $\nabla\Phi$. Actually, other transformations of this kind can be considered, see [CGS16] to recover a divergence-free vector field instead of the twisted divergence condition (1.34c), for instance, to ease computations of certain estimates. Also, Φ is a priori only ensured to be a diffeomorphism for small times, according to the aforementioned smallness assumption on η. Global existence is recovered thanks to a bootstrap argument ensuring this condition to remain satisfied for all time.

For the Stokes equation, this method seems more intricated since the change of variable $\boldsymbol{u}^\pm\circ\Phi^\pm = \boldsymbol{v}^\pm$ induces more complex coefficients, depending on derivatives of $\Phi^{\pm 1}$, due to the Laplace operator.

1.4.4 Contribution: Analytical remarks and numerical study

In Chapter 4, we discuss a few analytical aspects of the graph interface evolution equation (1.27) in the horizontal periodic channel $\mathbf{T} \times (0,1)$, and we perform a numerical study of this problem.

In Section 4.2, we derive formally (1.27) from (ST), assuming the density is of the form $\rho(t) = \mathbf{1}_{\{z<\eta(t,x)\}}$ at least for short time. We also consider different formulations of this problem. By an elementary geometric argument, we obtain that an initial datum $\rho_0 = \mathbf{1}_{\{z<\eta_0\}}$ with η_0 Lipschitz remains a patch of the form $\rho(t) = \mathbf{1}_{\{z<\eta(t,x)\}}$, at least locally in time. We also discuss the lack of interplay between the regularity of η and $\boldsymbol{u}$. Indeed, $\boldsymbol{u}$ is given as the solution of the Stokes equation with the bounded datum $\mathbf{1}_{\{z<\eta\}}$, therefore the regularity of $\boldsymbol{u}$ is only a priori $W^{2,q}$ for $q \in (2,\infty)$, without any link with the regularity of η as a graph. We also emphasise that, even though the interface itself remains $C^{1,\mu}, \mu \in (0,1)$, according to Subsection 1.4.2, it does not prevent a regularity break from the graph point of view. We also discuss about the possible Lyapunov functionals for this equation, *i.e.* functionals I such that any solution η of the problem satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} I(\eta(t)) \leq 0.$$

Such functionals are usually useful to address the long-time behaviour of partial differential equations. Formally, only $\eta \mapsto \|\eta\|_{L^2}^2$ is proven to be a Lyapunov functional for the system, and is related to the potential energy monotonicity, see (1.13). Other functionals are considered, and numerical counter-example are found, see below.

In Section 4.2, we first describe the numerical scheme we set to simulate the evolution of (1.27). Since we are interested in patch data, we chose to implement the whole density transport, thanks to an Arbitrary Lagrangian-Eulerian method. The principle is the following: for a given density, we solve the steady Stokes-equation thanks to a suitable finite elements method, and we transport the density in the direction of the velocity field, for a short time step, before repeating the process. The interface of the density is discretised in segments and completed in a mesh for the whole domain, and the density represented as a valuation function on the cells. This way, the patch character of the density is respected and preserved, and we can expect the scheme to be stable for regular enough interfaces, for instance Hölderian. Also, this method does not require the patch to be associated to an hypograph, which allows graph breaks if there were to occur.

After justifying our parameters settings for this scheme, we can plot the evolution of some graph interfaces, such as in Figure 1.2.

We observe here that the interface converges toward the flat rest state, according to the intuition. This convergence is also numerically verified. In particular, we compare the evolution of several quantities such as the $\dot{H}^m$ norms of η for $m \in \{0,1,2,3\}$ to the rates ensured by Theorem 1.4.1 of Gancedo, Granero-Belinchón & Salguero [GGS22] for small enough and regular enough initial data. Some of our tests clearly present faster decays, and some others are slower or even nonmonotonous, providing counter-examples ruling out some Lyapunov functionals candidates. The length of the curve, which is a Lyapunov functional for some interface equations, is in particular disqualified for this problem, as we exhibit a numerical example for which this quantity is not monotonous. For each example we considerd, we only observed decreasing entropy, defined as $\eta \mapsto \int_{\mathbf{T}} \eta \log(\eta)$, which we therefore cannot exclude as a Lyapunov functional. The maximum principle is also investigated in Subsection 4.2.6. We also observe that graph breaks can occur, even for smooth initial data, see Figure 1.3.

We also highlight some boundary effects, inducing corner-like formation for graphs initially smooth, by comparing the evolution of several data being vertical translations of each other. Such behaviours can not occur in the unbounded domain $\mathbf{T} \times \mathbf{R}$, vertically invariant. Nethertheless, some local singularity formation as such do not stop the interface to converge toward the flat

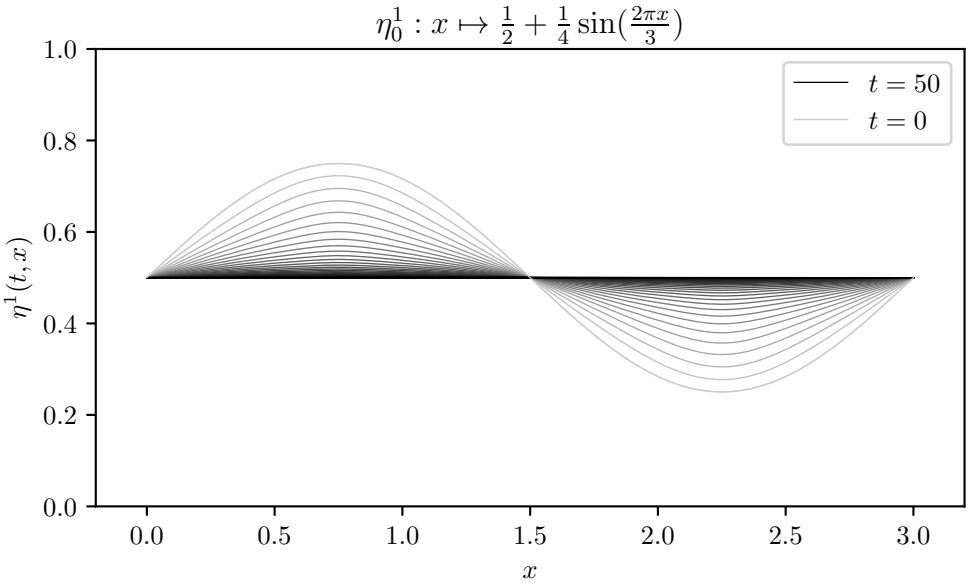

Figure 1.2: Graph evolution for an initial sine interface.

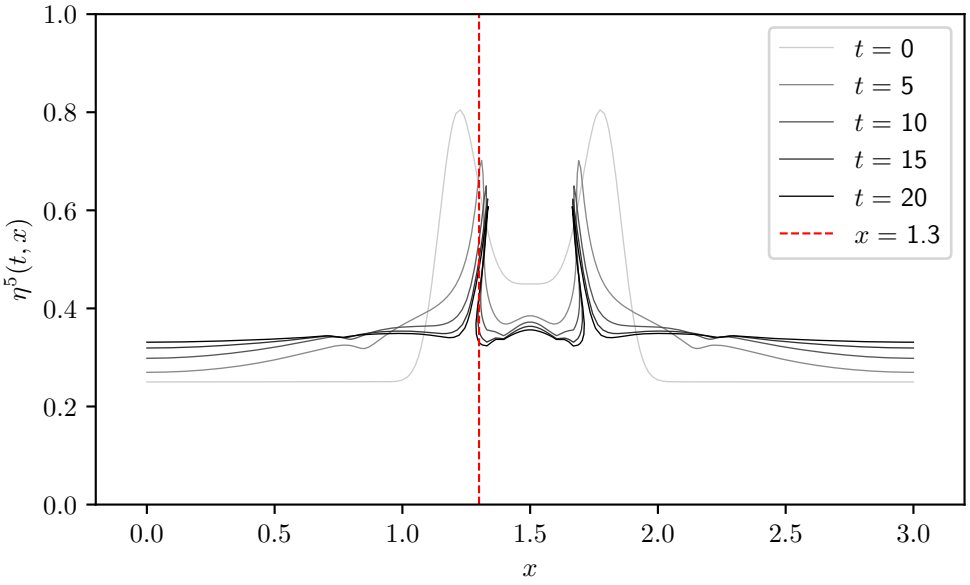

Figure 1.3: Example of a graph break.

state in L^2.

Chapter 2

Well-posedness of the Stokes-transport equation

This chapter presents the contributions of the author regarding the well-posedness of the Stokes-transport equation. The article [Leb22] is reproduced from Section 2.1 to 2.4, establishing the weak and global well-posedness of the Stokes-transport equation in bounded domains in dimension 2 and 3 spaces, as in the infinite strip $\mathbf{R} \times (0,1)$. The Section 2.5 supplements this article by improving these results, demonstrating the well-posedness of the problem for L^q data for q larger than the space dimension, and showing that solutions exist for a wider range of Lebesgue exponents. The strong well-posedness of (ST), for Sobolev data, fits in Section 2.6. A hydrostatic version of the Stokes-transport equation is also considered in Section 2.7, shown to be locally well-posed.

2.1 Introduction

This contribution is dedicated to the study of the following coupling between the transport equation and the Stokes equation,

$$\left\{\begin{array}{rcl} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho &=& 0 \\ -\Delta \boldsymbol{u} + \nabla p &=& -\rho \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} &=& 0 \\ \rho|_{t=0} &=& \rho_0, \end{array}\right. \tag{1.0}$$

where ρ, $\boldsymbol{u}$ and p respectively stand for the density, velocity and pressure of a fluid, ρ_0 is the initial density profile and $\boldsymbol{e}_z$ is the vertical upward unitary vector. This system is a model of evolution of an incompressible and viscous fluid having an inhomogeneous density or buoyancy subject to the gravity directed by $-\boldsymbol{e}_z$. It differs from the classical Boussinesq equation by neglecting the velocity self-advection term and the diffusion of the density. It is especially derived as the mesoscopic model of a cloud of inertialess particles sedimenting in a Stokes fluid, see for instance [Höf18a; Mec18]. This system belongs to a broad family of transport equations with non-local velocity field, the active scalar equations, meaning that $\boldsymbol{u}$ depends on ρ in a non-local way. The vorticity equation, the surface quasi-geostrophic equation and the incompressible porous medium equation (IPM) are examples of extensively studied systems having this structure, see [BG15] for an overview. In particular, (IPM) is a well-known model for the evolution of an incompressible inhomogeneous fluid inside a porous material, which writes as the system (1.0) where one replaces

the Stokes equation by Darcy's law, namely

$$\boldsymbol{u} + \nabla p = -\rho \boldsymbol{e}_z.$$

To our knowledge, (1.0) has been shown to be well-posed in the whole space $\mathbf{R}^3$ by Höfer [Höf18a] and Mecherbet [Mec20]. In particular, Mecherbet proved that for any $\rho_0 \in L^1 \cap L^\infty$ having a finite first moment, the system (1.0) admits a unique global solution,

$$(\rho, \boldsymbol{u}) \in L^\infty(\mathbf{R}_+; L^1(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)) \times L^\infty(\mathbf{R}_+; W^{1,\infty}(\mathbf{R}^3)).$$

Later, Höfer and Schubert [HS21, Theorem 2.7] proved a similar well-posedness result on $\mathbf{R}^3$ for any probability measure ρ_0 being in L^∞, without the first moment hypothesis. We thereafter state the well-posedness of (1.0) for initial data ρ_0 in $L^\infty(\Omega)$, for any regular enough bounded subdomain Ω of $\mathbf{R}^2$ and $\mathbf{R}^3$, as well as in the infinite strip $\mathbf{R} \times (0,1)$. These results stand out with the ill-posedness of IPM, for which uniqueness fails on various domains for weak solutions associated to L^∞ initial data, see [CFG11; Szé12]. Even for Sobolev data, the question of global in time well-posedness of IPM is still open, see [KY23], although the particular case of a small and smooth perturbation of a stationary density profile does lead to a global solution, as proved in [CCL19a].

In both Höfer and Mecherbet works, the proof of existence of a solution relies on a fixed point argument, respectively a contracting map and a Picard iteration. The latter consists in solving successively the Stokes and the transport equations, providing function sequences that appear to be convergent. This convergence is proven thanks to a stability estimate on the solutions of the transport equation, combined with an energy estimate for the Stokes equation. In the full space $\mathbf{R}^3$, one can express the solution of the Stokes equation as the convolution with the adequate Green kernel. Thanks to this explicit formula, Mecherbet established the stability estimate on solutions of the Stokes equation, controlled by the Wasserstein distance between their source terms. Combined with a stability estimate for the transport equation, also expressed with the Wasserstein distance, this provides the contraction inequality allowing to apply the Picard argument. In particular, the Wasserstein distance allows to state the stability estimate without any derivability assumption on the density. In the following, we will emphasize the fact that the shape of the spatial domain on which one solves (1.0) strongly conditions the way one have to address the Stokes part. Apart from that, the classical transport theory is not as much sensitive to the geometry of the domain, as long as the velocity field is bounded and Lipschitz. Moreover it appears that ρ is the push-forward of ρ_0 by the flow of $\boldsymbol{u}$. In particular, if ρ_0 is a patch, the density ρ will remain a patch at all times.

In this Chapter, we follow a strategy similar to the one adopted by Mecherbet [Mec20] in order to prove the well-posedness of (1.0) on bounded domains of $\mathbf{R}^2$ and $\mathbf{R}^3$. Since the Green kernel associated to the Stokes equation is no longer explicit for such domains in general, the remaining exploitable tools are the energy estimate and the elliptic gain of regularity due to Stokes equation. We also replaced the Wasserstein distance, which was convinient in $\mathbf{R}^3$ in which case there exists an explicit Green kernel, by some negative Sobolev norms. Notice that both metrics are relatable, see for instance [San15, Subsection 5.5.2]. We also impose the Dirichlet boundary condition on the velocity, considering more precisely the system

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho &= 0 & \text{in } \mathbf{R}_+ \times \Omega, \\ -\Delta \boldsymbol{u} + \nabla p &= -\rho \boldsymbol{e}_z & \text{in } \mathbf{R}_+ \times \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 & \text{in } \mathbf{R}_+ \times \Omega, \\ \boldsymbol{u} &= \mathbf{0} & \text{in } \mathbf{R}_+ \times \partial\Omega, \\ \rho|_{t=0} &= \rho_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

Our first result is the following well-posedness theorem for this system, which comes together with a stability estimate left in Proposition 2.2.7.

Theorem 2.1.1. *Let Ω be a bounded domain of class C^2 of $\mathbf{R}^d$ for $d = 2$ or 3. For any $\rho_0 \in L^\infty(\Omega)$ there exists a unique solution $(\rho, \boldsymbol{u})$ to (2.1) in*

$$L^\infty(\mathbf{R}_+; L^\infty(\Omega)) \times L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega)).$$

We further deal with the infinite strip domain $\mathbf{R} \times (0,1)$. Such an unbounded domain in the horizontal direction is mathematically convenient for modelling an oceanic profile, for which the typical height is very small compared to its typical length. In this framework the system is a toy model for the evolution of the density, salinity or temperature of the sea water in an ocean. The main difference with the bounded domain case lies in the analysis of the Stokes equation. Working with L^∞ densities requires to deal with velocity fields that are not square-integrable. To overcome this difficulty we work with Kato spaces, also called uniformly local Sobolev spaces. Using some classical tools we provide a proof of the well-posedness of the Stokes equation in this framework, since we could not find this result in the literature. In particular we prove that a L^∞ datum induces a $W^{1,\infty}$ solution, as in the bounded domain case, which will provide handy Lipschitz velocity fields to deal further with the transport part. The method consists in considering an increasing covering of bounded open subsets of the strip, and to solve the Stokes problem on each of these subdomains. This provides a sequence of functions on $\mathbf{R} \times (0,1)$. By managing carefully the interior estimates of the elements of this sequence, we prove its boundedness in the uniformly locally H^1 function space, denoted H^1_{uloc} and defined in paragraph 2.3.1. Therefore we obtain a H^1_{uloc} solution by a compact argument. Uniqueness comes from the well-posedness in the classical H^1 framework. Using the elliptic regularity of the Stokes equation in bounded domains, we prove the H^2_{uloc} regularity of the solution on the strip. Hence we obtain a leverage to establish existence of a unique solution in $W^{2,q}_{\mathrm{uloc}}$ for L^q_{uloc} data for exponents $2 < q < \infty$, and to conclude to the well-posedness of the problem in $W^{1,\infty}$ for L^∞ data by Sobolev embeddings. The proof of the well-posedness of the coupling then lies in the extension of results used in the bounded domain case to the infinite strip through the uniformly local topology. Precisely, we consider in $\Omega = \mathbf{R} \times (0,1)$ the system

$$\left\{\begin{array}{rcll} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho &=& 0 & \text{in } \mathbf{R}_+ \times \Omega, \\ -\Delta \boldsymbol{u} + \nabla p &=& -\rho \boldsymbol{e}_z & \text{in } \mathbf{R}_+ \times \Omega, \\ \operatorname{div} \boldsymbol{u} &=& 0 & \text{in } \mathbf{R}_+ \times \Omega, \\ \boldsymbol{u} &=& \mathbf{0} & \text{in } \mathbf{R}_+ \times \partial\Omega, \\ \int u_1 \, \mathrm{d}z &=& 0 & \text{in } \mathbf{R}_+ \times \mathbf{R}, \\ \rho|_{t=0} &=& \rho_0 & \text{in } \Omega, \end{array}\right. \tag{2.2}$$

where the extra condition on the flux $\int u_1 \, \mathrm{d}z$ is introduced and discussed in paragraph 2.3.1. In the end, the well-posedness of (2.2) almost writes as in Theorem 2.1.1, with a similar stability result stated in Proposition 2.3.10.

Theorem 2.1.2. *Let $\Omega = \mathbf{R} \times (0,1)$. For any $\rho_0 \in L^\infty(\Omega)$ there exists a unique solution $(\rho, \boldsymbol{u})$ to (2.2) in*

$$L^\infty(\mathbf{R}_+; L^\infty(\Omega)) \times L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega)).$$

Regarding the oceanographic considerations mentioned for the infinite strip, it is rather natural to wonder if this result extends to the case of the layer domain $\mathbf{R}^2 \times (0,1)$. It appears that the Stokes problem in this unbounded domain raises some additional difficulties, among those finding a functional space in which the problem admits a unique solution, in accordance with

the handling of the interior estimates mentioned in the previous paragraph, see Remark 2.3.6 for further details. We abstain from considering this case in the present work.

The paper splits in Section 2.2, dedicated to the bounded domain case, and Section 2.3, dedicated to the infinite strip case. Both sections are ordered in the same way. Subsections 2.2.1 and 2.3.1 recall and prove the necessary prerequisites about the Stokes equation, including the well-posedness complete proof in the infinite strip in Section 2.3. Subsections 2.2.2 and 2.3.2 contain preliminary results concerning the transport theory, and in particular the stability estimates proofs. Subsections 2.2.3 and 2.3.3 are dedicated to the proofs of Theorem 2.1.1 and Theorem 2.1.2, respectively. Ultimately, we state and prove the stability estimates of the coupling in Subsections 2.2.4 and 2.3.4.

Definitions and notations

The dimension d is always 2 or 3. A *domain* Ω is a non-empty open and simply connected subset of $\mathbf{R}^d$. The Bochner spaces are denoted by $L^q([0,T);B(\Omega))$ with $1 \leq q \leq \infty$, $T \in [0,\infty]$ and $B(\Omega)$ a Banach or a Fréchet space of functions defined in Ω. It is endowed with its classical norm denoted here $\|\cdot\|_{L^q(0,T;B)}$, the space domain being specified only when differing from the whole domain Ω. A vector valued map is denoted by a bold symbol, implicitly of size d. We note $\boldsymbol{u}|_{\partial\Omega}$ the trace of $\boldsymbol{u}$ on the boundary of a domain Ω, when the boundary is regular enough to define it. We sometimes write $f \equiv c$ to signify that a function f is equal to some constant with respect to time. We denote by C any non-negative constant that is adjusted from one line to another, independent of the data and we specify its eventual space or exponent dependencies when necessary. We sometimes write $f \lesssim g$, meaning there exists such a constant C for which $f \leq Cg$, as well as $f \simeq g$, meaning $f \lesssim g$ and $g \lesssim f$.

2.2 Well-posedness of the coupling in a bounded domain

In this section, and unless stated otherwise, Ω denotes a bounded domain of $\mathbf{R}^d$ with Lipschitz boundary.

2.2.1 Preliminaries on the Stokes problem in a bounded domain

Let us recall the Stokes problem on Ω,

$$\left\{\begin{array}{rcll} -\Delta \boldsymbol{u} + \nabla p & = & \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} & = & g & \text{in } \Omega, \\ \boldsymbol{u} & = & \boldsymbol{0} & \text{on } \partial\Omega, \end{array}\right. \tag{2.3}$$

where $\boldsymbol{f}$ and g belong to functional spaces specified further. Notice that it is necessary for g to satisfy the following compatibility condition due to the homogeneous boundary condition assumption,

$$\int_\Omega g = \int_\Omega \operatorname{div} \boldsymbol{u} = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} = 0. \tag{2.4}$$

The well-posedness of this problem is well known for Sobolev data, see in particular [Gal11, Theorem IV.6.1 and Exercise I.6.3], reported below. We especially use that L^∞ data induce $W^{1,\infty}$ solutions, see the following corollary, which will provide velocity fields easy to deal with in the transport part.

Theorem 2.2.1 (Galdi). *Let Ω be a bounded domain of $\mathbf{R}^d$ of class C^2 and let $1 < q < \infty$.*

1. For any $\boldsymbol{f} \in L^q(\Omega)$ and $g \in W^{1,q}(\Omega)$ satisfying (2.4), there exists a unique[1] *pair $(\boldsymbol{u}, p)$ in $W^{2,q}(\Omega) \times \left(W^{1,q}(\Omega)/\mathbf{R}\right)$ satisfying (2.3), which moreover obeys the inequality*

$$\|\boldsymbol{u}\|_{W^{2,q}} + \|p\|_{W^{1,q}} \leq C\left(\|\boldsymbol{f}\|_{L^q} + \|g\|_{W^{1,q}}\right); \tag{2.5}$$

2. For any $\boldsymbol{f} \in W^{-1,q}(\Omega)$ there exists a unique pair $(\boldsymbol{u}, p)$ in $W^{1,q}(\Omega) \times (L^q(\Omega)/\mathbf{R})$ satisfying (2.3) with $g = 0$, which moreover obeys the inequality

$$\|\boldsymbol{u}\|_{W^{1,q}} + \|p\|_{L^q} \leq C\|\boldsymbol{f}\|_{W^{-1,q}}. \tag{2.6}$$

The L^∞ to $W^{1,\infty}$ regularity of the problem is deduced from the Sobolev embeddings of $W^{2,4}(\Omega)$ in $W^{1,\infty}(\Omega)$ and $W^{1,4}(\Omega)$ in $L^\infty(\Omega)$ for $d = 2$ and 3.

Corollary 2.2.2. *Let Ω be a bounded domain of $\mathbf{R}^d$ of class C^2. For any $\boldsymbol{f} \in L^\infty(\Omega)$ and $g \in W^{1,\infty}(\Omega)$ there exists a unique pair $(\boldsymbol{u}, p)$ in $W^{1,\infty}(\Omega) \times (L^\infty(\Omega)/\mathbf{R})$ satisfying (2.3), which moreover obeys the inequality*

$$\|\boldsymbol{u}\|_{W^{1,\infty}} + \|p\|_{L^\infty} \leq C\left(\|\boldsymbol{f}\|_{L^\infty} + \|g\|_{W^{1,\infty}}\right). \tag{2.7}$$

2.2.2 Preliminaries on the transport equation and stability estimates

Let us consider the transport equation, in the weak sense, for a given vector field $\boldsymbol{u} \in L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega))$ satisfying the homogeneous Dirichlet condition $\boldsymbol{u}|_{\partial\Omega} \equiv 0$,

$$\left\{ \begin{array}{rcll} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho & = & 0 & \text{in } \mathbf{R}_+ \times \Omega, \\ \rho(0, \cdot) & = & \rho_0 & \text{in } \Omega. \end{array} \right. \tag{2.8}$$

We recall the definition of the *characteristic (map)* or *flow* $\boldsymbol{X}$ associated to the vector field $\boldsymbol{u}$, as the solution of

$$\forall s, t \in \mathbf{R}_+, \forall \boldsymbol{x} \in \Omega, \quad \left\{ \begin{array}{rcl} \partial_t \boldsymbol{X}(t; s, \boldsymbol{x}) & = & \boldsymbol{u}(t, \boldsymbol{X}(t; s, \boldsymbol{x})) \\ \boldsymbol{X}(s; s, \boldsymbol{x}) & = & \boldsymbol{x}. \end{array} \right.$$

The Cauchy-Lipschitz theory ensures that $\boldsymbol{X}$ is well defined, and that for any $s, t \in \mathbf{R}^+$, $\boldsymbol{X}(t; s, \cdot)$ is a homeomorphism from Ω onto itself, satisfying the composition principle

$$\forall r, s, t \in \mathbf{R}_+, \quad \boldsymbol{X}(t; s, \cdot) \circ \boldsymbol{X}(s; r, \cdot) = \boldsymbol{X}(t; r, \cdot).$$

In particular we have the relation $\boldsymbol{X}(t; 0, \cdot)^{-1} = \boldsymbol{X}(0; t, \cdot)$. From now on, we use indifferently the following notations

$$\forall t \in R_+, \quad \boldsymbol{X}(t) = \boldsymbol{X}(t; \cdot) = \boldsymbol{X}(t; 0, \cdot), \quad \boldsymbol{X}(-t) := \boldsymbol{X}(0; t, \cdot).$$

Let us enumerate a few classical properties of the flow. These are elementary consequences of Duhamel formula and Grönwall inequality.

Lemma 2.2.3. *Let $\boldsymbol{u} \in L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega))$ with $\boldsymbol{u}|_{\partial\Omega} \equiv \mathbf{0}$. The associated characteristic map $\boldsymbol{X}$ satisfies,*

$$\forall t \in \mathbf{R}, \forall \boldsymbol{x}, \boldsymbol{y} \in \Omega, \quad |\boldsymbol{X}(t; \boldsymbol{x}) - \boldsymbol{X}(t; \boldsymbol{y})| \leq e^{C|t|\|\nabla \boldsymbol{u}\|_{L^\infty(0,t;L^\infty)}} |\boldsymbol{x} - \boldsymbol{y}|. \tag{2.9}$$

[1] Unless explicitly stated otherwise, the canonical representative of p considered is the one having zero average over Ω.

In particular $\boldsymbol{X}(t)$ is bi-Lipschitz for any t. Recall that Liouville theorem ensures that if $\operatorname{div} \boldsymbol{u} \equiv 0$ the jacobian determinant of $\boldsymbol{X}(t)$ is identically equal to 1 with respect to t. Besides, let us introduce a classical stability estimate on the characteristics.

Lemma 2.2.4. *Let $\boldsymbol{u}_i \in L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega))$ with $\boldsymbol{u}_i|_{\partial\Omega} \equiv \mathbf{0}$ and $\operatorname{div} \boldsymbol{u}_i \equiv 0$ for $i = 1, 2$. If Ω is bounded, the associated characteristic maps $\boldsymbol{X}_i$ satisfy, for any $1 \leq q \leq \infty$,*

$$\forall t \in \mathbf{R}_+, \quad \|\boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)\|_{L^q} \leq t e^{Ct\|\nabla \boldsymbol{u}_1\|_{L^\infty(0,t;L^\infty)}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,t;L^q)}. \tag{2.10}$$

If $q = \infty$, the inequality holds true for unbounded Ω.

Proof. Let us consider Ω bounded. From Duhamel formula we write for any $t \in \mathbf{R}_+$ and $\boldsymbol{x} \in \Omega$,

$$\begin{aligned}
\boldsymbol{X}_1(t,\boldsymbol{x}) - \boldsymbol{X}_2(t,\boldsymbol{x}) &= \int_0^t \boldsymbol{u}_1(\tau, \boldsymbol{X}_1(\tau;\boldsymbol{x})) - \boldsymbol{u}_2(\tau, \boldsymbol{X}_2(\tau;\boldsymbol{x}))\,\mathrm{d}\tau \\
&= \int_0^t \boldsymbol{u}_1(\tau, \boldsymbol{X}_1(\tau;\boldsymbol{x})) - \boldsymbol{u}_1(\tau, \boldsymbol{X}_2(\tau;\boldsymbol{x}))\,\mathrm{d}\tau \\
&\quad + \int_0^t \boldsymbol{u}_1(\tau, \boldsymbol{X}_2(\tau;\boldsymbol{x})) - \boldsymbol{u}_2(\tau, \boldsymbol{X}_2(\tau;\boldsymbol{x}))\,\mathrm{d}\tau.
\end{aligned}$$

The Lipschitz regularity of $\boldsymbol{u}_i$ and the Minkowski inequality provide, for any $1 \leq q \leq \infty$,

$$\begin{aligned}
\|\boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)\|_{L^q} \leq C\|\nabla \boldsymbol{u}\|_{L^\infty(0,t;L^\infty)} &\int_0^t \|\boldsymbol{X}_1(\tau) - \boldsymbol{X}_2(\tau)\|_{L^q}\,\mathrm{d}\tau \\
&+ \int_0^t \|\boldsymbol{u}_1(\tau, \boldsymbol{X}_2(\tau)) - \boldsymbol{u}_2(\tau, \boldsymbol{X}_2(\tau))\|_{L^q}\,\mathrm{d}\tau.
\end{aligned}$$

As a consequence of Liouville theorem, one has for $1 \leq q < \infty$ and $\tau \in [0,t]$,

$$\|\boldsymbol{u}_1(\tau, \boldsymbol{X}_2(\tau)) - \boldsymbol{u}_2(\tau, \boldsymbol{X}_2(\tau))\|_{L^q} = \|\boldsymbol{u}_1(\tau,\cdot) - \boldsymbol{u}_2(\tau,\cdot))\|_{L^q}.$$

The case $q = \infty$ holds naturally true. Then we have

$$\|\boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)\|_{L^q} \leq C\|\nabla \boldsymbol{u}_1\|_{L^\infty(0,t;L^\infty)} \int_0^t \|\boldsymbol{X}_1(\tau) - \boldsymbol{X}_2(\tau)\|_{L^q}\,\mathrm{d}\tau + t\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,t;L^q)},$$

which yields (2.10) by Grönwall inequality. □

The classical characteristics method provides the well-posedness of the transport equation.

Proposition 2.2.5. *Let Ω be a Lipschitz domain of $\mathbf{R}^d$, not necessarily bounded. Let $\boldsymbol{u} \in L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega))$ with $\boldsymbol{u}|_{\partial\Omega} \equiv \mathbf{0}$, $\operatorname{div} \boldsymbol{u} = 0$ and let $\rho_0 \in L^\infty(\Omega)$. There exists a unique ρ in $L^\infty(\mathbf{R}_+; L^\infty(\Omega))$ satisfying (2.8), which is moreover the push-forward of ρ_0 by the characteristic $\boldsymbol{X}$ of $\boldsymbol{u}$, namely*

$$\forall t \in \mathbf{R}_+, \quad \rho(t) = \rho_0 \circ \boldsymbol{X}(-t).$$

In particular the L^q norm of the solution ρ is constant in time, for any $1 \leq q \leq \infty$. Besides, we state the following estimate for the evolution of the difference of two solutions of (2.8) associated to distinct velocity fields.

Proposition 2.2.6. *Let $\boldsymbol{u}_i \in L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega))$, with $\boldsymbol{u}_i|_{\partial\Omega} \equiv \mathbf{0}$ and $\operatorname{div} \boldsymbol{u}_i \equiv 0$, for $i = 1, 2$. Let $\rho_0 \in L^\infty(\Omega)$ and ρ_i be the solutions of (2.8) associated to $\boldsymbol{u}_i$ with initial datum ρ_0. For any $1 < q < \infty$ there exists $\bar{T}(\|\nabla \boldsymbol{u}_i\|_{L^\infty}) > 0$ and $C(\Omega, q) > 0$ such that for any $T \in [0, \bar{T}]$,*

$$\|\rho_1 - \rho_2\|_{L^\infty(0,T;W^{-1,q})} \leq C\|\rho_0\|_{L^\infty} T e^{CT\|\nabla \boldsymbol{u}_1\|_{L^\infty(0,T;L^\infty)}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,T;L^q)}. \tag{2.11}$$

Proof. Let $t \in \mathbf{R}_+$ and $\varphi \in C_c^\infty(\Omega)$. Since the vector fields $\boldsymbol{u}_i$ are divergence-free, the Liouville theorem ensures the following change of variable,

$$\begin{aligned} I_\varphi &:= \int_\Omega \big(\rho_1(t,\boldsymbol{x}) - \rho_2(t,\boldsymbol{x})\big)\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega \big(\rho_0(\boldsymbol{X}_1(-t;\boldsymbol{x})) - \rho_0(\boldsymbol{X}_2(-t;\boldsymbol{x}))\big)\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega \rho_0(\boldsymbol{x})\big(\varphi(\boldsymbol{X}_1(t;\boldsymbol{x})) - \varphi(\boldsymbol{X}_2(t;\boldsymbol{x}))\big)\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Since φ is smooth, we can write

$$I_\varphi = \int_\Omega \rho_0(\boldsymbol{x})\big(\boldsymbol{X}_1(t;\boldsymbol{x}) - \boldsymbol{X}_2(t;\boldsymbol{x})\big) \cdot \int_0^1 \nabla\varphi(\boldsymbol{X}^\lambda(t;\boldsymbol{x}))\,\mathrm{d}\lambda\,\mathrm{d}\boldsymbol{x},$$

where we set $\boldsymbol{X}^\lambda(t;\boldsymbol{x}) := \lambda\boldsymbol{X}_1(t;\boldsymbol{x}) + (1-\lambda)\boldsymbol{X}_2(t;\boldsymbol{x})$. Then, Hölder's inequality provides

$$|I_\varphi| \leq \|\rho_0\|_{L^\infty}\|\boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)\|_{L^q} \int_0^1 \|\nabla\varphi(\boldsymbol{X}^\lambda(t))\|_{L^{q'}}\,\mathrm{d}\lambda. \tag{2.12}$$

Let us show that $\boldsymbol{X}^\lambda(t)$ is bi-Lipschitz from Ω onto its range. Consider the derivative of the Duhamel formula satisfied by $\boldsymbol{X}_i$,

$$\nabla\boldsymbol{X}_i(t,\cdot) - I_d = \int_0^t \nabla\boldsymbol{u}_i(\tau, \boldsymbol{X}_i(\tau,\cdot)) \cdot \nabla\boldsymbol{X}_i(\tau,\cdot)\,\mathrm{d}\tau$$

and deduce thanks to inequality (2.9) the uniform estimate

$$\|\nabla\boldsymbol{X}_i(t) - I_d\|_{L^\infty} \leq C\left(e^{Ct\|\nabla\boldsymbol{u}_i\|_{L^\infty(0,t;L^\infty)}} - 1\right). \tag{2.13}$$

Therefore for some arbitrary constant $C > 1$ there exists $\bar{T}(\|\nabla\boldsymbol{u}_i\|_{L^\infty}) > 0$ such that

$$\forall t \in [0, \bar{T}], \quad C^{-1} \leq \det \nabla\boldsymbol{X}_i(t) \leq C,$$

and such that the Lipschitz constants of $\boldsymbol{X}_i(t)$ are uniformly bounded with respect to t, by a constant smaller than 1, as follows,

$$L_{\bar{T}} := \max_i \sup_{t\in[0,\bar{T}]} \operatorname{Lip}(\boldsymbol{X}_i(t) - \mathrm{Id}) < 1.$$

The latter inequality ensures the injectivity of $\boldsymbol{X}^\lambda(t)$, since for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, the equality $\boldsymbol{X}^\lambda(t;\boldsymbol{x}) = \boldsymbol{X}^\lambda(t;\boldsymbol{y})$ is equivalent to

$$\boldsymbol{x} - \boldsymbol{y} = \lambda\Big(\boldsymbol{X}_1(t;\boldsymbol{y}) - \boldsymbol{y} - \big(\boldsymbol{X}_1(t;\boldsymbol{x}) - \boldsymbol{x}\big)\Big) + (1-\lambda)\Big(\boldsymbol{X}_2(t;\boldsymbol{y}) - \boldsymbol{y} - \big(\boldsymbol{X}_2(t;x) - \boldsymbol{x}\big)\Big),$$

and implies

$$|\boldsymbol{x}-\boldsymbol{y}| \leq L_{\bar{T}}|\boldsymbol{x}-\boldsymbol{y}|,$$

so that $\boldsymbol{x}=\boldsymbol{y}$ since $L_{\bar{T}}<1$. In the end we have proved that $\boldsymbol{X}^{\lambda}(t)$ is bi-Lipschitz for any $\lambda \in [0,1]$ and $t \in [0,\bar{T}]$, with a uniform bound on its jacobian determinant, independent of λ and t. Therefore we have for any $1<q<\infty$,

$$\|\nabla\varphi(\boldsymbol{X}^{\lambda}(t))\|_{L^{q'}} \leq C_q\|\nabla\varphi\|_{L^{q'}}. \tag{2.14}$$

Combining (2.12) and (2.14) leads for any $t \in [0,\bar{T}]$ to

$$\int_{\Omega}(\rho_1(t,\boldsymbol{x})-\rho_2(t,\boldsymbol{x}))\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \leq C\|\rho_0\|_{L^\infty}\|\boldsymbol{X}_1(t)-\boldsymbol{X}_2(t)\|_{L^q}\|\varphi\|_{W^{1,q'}}.$$

Plugging the stability estimate (2.10) and taking the supremum over the test functions provides the following bound on the negative Sobolev norm, for any $t \in [0,\bar{T}]$,

$$\|\rho_1(t)-\rho_2(t)\|_{W^{-1,q}} \leq C\|\rho_0\|_{L^\infty}te^{Ct\|\nabla\boldsymbol{u}_1\|_{L^\infty(0,t;L^\infty)}}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^\infty(0,t;L^q)}.$$

It remains to consider the supremum over $t \in [0,T]$ for any $T \leq \bar{T}$ to get the result. □

2.2.3 Proof of Theorem 2.1.1

The strategy of the proof is inspired from the one adopted by Mecherbet in [**mecherbet**] for the space domain $\mathbf{R}^3$. In this work Mecherbet solves successively Stokes and transport problems, providing a contracting sequence of velocity fields and density profiles. Here the contracting property is obtained by combination of the Stokes estimate from Theorem 2.2.1 and the stability estimate for the transport from Proposition 2.2.6. Both interplay in the case of the whole space thanks to the Green kernel, also called *Oseen tensor*, of the Stokes equation and the stability estimates formulated with the Wasserstein distance. Since we work in general open bounded domains, we do not use Green kernels but only rely on the variational estimates. We also replace the Wasserstein distance, handled by Mecherbet, by negative Sobolev norms. Both are known to be related, see [San15, Subsection 5.5.2], and in our case present the same asset to allow the statement of stability estimates without any derivability assumption concerning the density.

Local existence: Set $\rho^0 \equiv \rho_0$ in $L^\infty(\mathbf{R}_+;L^\infty(\Omega))$. Theorem 2.2.1 and Proposition 2.2.5 allow to define the following sequences by induction on $N \in \mathbf{N}$,

$$\rho^N \in L^\infty(\mathbf{R}_+;L^\infty(\Omega)), \quad \boldsymbol{u}^N \in L^\infty(\mathbf{R}_+;W^{1,\infty}(\Omega)),$$

satisfying for any $N \in \mathbf{N}$ the Stokes problem

$$\left\{\begin{array}{rcll} -\Delta\boldsymbol{u}^N+\nabla p^N &=& -\rho^N\boldsymbol{e}_z & \text{in } \mathbf{R}_+\times\Omega,\\ \operatorname{div}\boldsymbol{u}^N &=& 0 & \text{in } \mathbf{R}_+\times\Omega,\\ \boldsymbol{u}^N &=& \boldsymbol{0} & \text{in } \mathbf{R}_+\times\partial\Omega, \end{array}\right. \tag{2.15}$$

and the transport equation

$$\left\{\begin{array}{rcll} \partial_t\rho^{N+1}+\boldsymbol{u}^N\cdot\nabla\rho^{N+1} &=& 0 & \text{in } \mathbf{R}_+\times\Omega,\\ \rho^{N+1}|_{t=0} &=& \rho_0 & \text{in } \Omega. \end{array}\right. \tag{2.16}$$

Let us denote $B := C\|\rho_0\|_{L^\infty}$ with adjustable constant. Since ρ^N is the push-forward of ρ_0 by the flow of $\boldsymbol{u}^{N-1}$, we have the uniform bound

$$\forall N, \quad \|\rho^N\|_{L^\infty(\mathbf{R}_+;L^\infty)} = \|\rho_0\|_{L^\infty} \leq B.$$

Using allthemore Stokes estimate (2.7), we obtain

$$\forall N, \quad \|\boldsymbol{u}^N\|_{L^\infty(\mathbf{R}_+;W^{1,\infty})} \leq C\|\rho^N\|_{L^\infty(\mathbf{R}_+;L^\infty)} \leq B.$$

Hence $\rho^N, \boldsymbol{u}^N$ and $\nabla \boldsymbol{u}^N$ converge in $\mathrm{w}^* - L^\infty(\mathbf{R}_+ \times \Omega)$ up to the extraction of subsequences. Besides, estimates from Proposition 2.2.6 and Theorem 2.2.1 ensure that there exists $\bar{T}(\|\rho_0\|_{L^\infty}) > 0$ such that

$$\begin{aligned} \forall T \in [0, \bar{T}], \quad \|\rho^{N+1} - \rho^N\|_{L^\infty(0,T;H^{-1})} &\leq BTe^{BT}\|\boldsymbol{u}^N - \boldsymbol{u}^{N-1}\|_{L^\infty(0,T;H^1)} \\ &\leq BTe^{BT}\|\rho^N - \rho^{N-1}\|_{L^\infty(0,T;H^{-1})}. \end{aligned} \tag{2.17}$$

We see that for $T > 0$ small enough we have $BTe^{BT} < 1$ so that $(\rho^N)_N$ is a Cauchy sequence in $L^\infty(0,T;H^{-1}(\Omega))$. As a consequence of the Stokes estimate (2.5), we have that $(\boldsymbol{u}^N)_N$ is also a Cauchy sequence, in $L^\infty(0,T;H^1(\Omega))$. Its limit belongs to $L^\infty(0,T;W^{1,\infty}(\Omega))$ since it converges for the weak* topology. In particular, $\boldsymbol{u}^N$ converges in $L^1(0,T;W^{1,1}(\Omega))$, which together with the weak* convergence of $(\rho^N)_N$ allows to pass to the limit in the weak formulations of both (2.15) and (2.16). Therefore the limit $(\rho, \boldsymbol{u})$ satisfies (2.1) on a short time, with regularity

$$L^\infty(0,T;L^\infty(\Omega)) \times L^\infty(0,T;W^{1,\infty}(\Omega)).$$

Local uniqueness : Let $(\rho_i, \boldsymbol{u}_i)$ be two such solutions of (2.1). The estimate (2.17) adapts in

$$\|\rho_1 - \rho_2\|_{L^\infty(0,T;H^{-1})} \leq BTe^{BT}\|\rho_1 - \rho_2\|_{L^\infty(0,T;H^{-1})},$$

up to the choice of a smaller $T > 0$. We deduce that $\rho_1 = \rho_2$ on $[0,T]$, and that $\boldsymbol{u}_1 = \boldsymbol{u}_2$ thanks to the Stokes estimate.

Globality: By existence and uniqueness of a solution to (2.1) locally in time, we know that there exists a unique maximal solution $(\rho, \boldsymbol{u})$ on some interval $[0, T^*)$ with $T^* \in [0, \infty]$. Remark that T^* depends only on $\|\rho_0\|_{L^\infty}$. Since $\|\rho\|_{L^\infty} \equiv \|\rho_0\|_{L^\infty}$, a classical continuation argument implies that $T^* = \infty$, which proves that the solution is global.

2.2.4 Stability estimate for the system in a bounded domain

We prove a stability estimate for the Stokes-transport system, inherited from Proposition 2.2.6.

Proposition 2.2.7. *Let $\rho_{0,i} \in L^\infty(\Omega)$ and set ρ_i the solution of (2.1) with initial datum $\rho_{0,i}$ for $i = 1, 2$. For any $1 < q < \infty$ there exists $C(\Omega, q, \|\rho_{0,i}\|_{L^\infty}) > 0$ such that*

$$\forall T \in \mathbf{R}_+, \quad \|\rho_1 - \rho_2\|_{L^\infty(0,T;W^{-1,q})} \leq Ce^{CT}\|\rho_{0,1} - \rho_{0,2}\|_{W^{-1,q}}. \tag{2.18}$$

Proof. Let us set $\boldsymbol{u}_i$ the velocity fields associated to ρ_i for $i = 1, 2$. Set $\rho_{1,2}$ the solution of (2.8) with initial datum $\rho_{0,1}$ and vector field $\boldsymbol{u}_2$. Hence, consider the triangular inequality

$$\forall T \in \mathbf{R}_+, \quad \|\rho_1 - \rho_2\|_{L^\infty(0,T;W^{-1,q})} \leq \underbrace{\|\rho_1 - \rho_{1,2}\|_{L^\infty(0,T;W^{-1,q})}}_{I_1} + \underbrace{\|\rho_{1,2} - \rho_2\|_{L^\infty(0,T;W^{-1,q})}}_{I_2}$$

From Proposition 2.2.6 and estimate (2.7) we know that there exists some $\bar{T}(\|\rho_{0,1}\|_{L^\infty}) > 0$ and $C(\Omega, q) > 0$ such that

$$\forall T \in [0, \bar{T}], \quad I_1 \leq CT\|\rho_{0,1}\|_{L^\infty} e^{CT\|\nabla \boldsymbol{u}_1\|_{L^\infty(0,T;L^\infty)}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,T;L^q)}.$$

Let us denote $B := C \max_i \|\rho_{0,i}\|_{L^\infty}$. Stokes estimates (2.6) and (2.7) respectively provide here

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,T;L^q)} \leq C\|\rho_1 - \rho_2\|_{L^\infty(0,T;W^{-1,q})}, \qquad \|\nabla \boldsymbol{u}_1\|_{L^\infty(0,T;L^\infty)} \leq B,$$

which yields

$$\forall T \in [0, \bar{T}], \quad I_1 \leq BTe^{BT}\|\rho_1 - \rho_2\|_{L^\infty(0,T;W^{-1,q})}.$$

To bound I_2 let us apply again Liouville theorem for any $t \in \mathbf{R}_+$ and $\varphi \in C_c^\infty(\Omega)$, to get

$$\int_\Omega \big(\rho_{1,2}(t,\boldsymbol{x}) - \rho_2(t,\boldsymbol{x})\big)\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega (\rho_{0,1}(\boldsymbol{x}) - \rho_{0,2}(\boldsymbol{x}))\varphi(\boldsymbol{X}_2(t;\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}.$$

Now we have, by definition of the Sobolev norm,

$$\int_\Omega \big(\rho_{1,2}(t,\boldsymbol{x}) - \rho_2(t,\boldsymbol{x})\big)\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \leq \|\rho_{0,1} - \rho_{0,2}\|_{W^{-1,q}} \|\varphi(\boldsymbol{X}_2(t))\|_{W^{1,q'}}.$$

From estimates (2.9) and (2.7) together with the bound $\|\nabla \boldsymbol{X}_i\|_{L^\infty(0,t;L^\infty)} \leq Ce^{Bt}$ provided by Lemma 2.2.3, it follows

$$\|\varphi(\boldsymbol{X}_2(t))\|_{W^{1,q'}} \leq Ce^{Bt}\|\varphi\|_{W^{1,q'}}.$$

Passing to the supremum over the test functions gives

$$I_2 \leq Ce^{BT}\|\rho_{0,1} - \rho_{0,2}\|_{W^{-1,q}}.$$

In the end we have shown that for any $T \in [0, \bar{T}]$,

$$\|\rho_1 - \rho_2\|_{L^\infty(0,T;W^{-1,q})} \leq BTe^{BT}\|\rho_1 - \rho_2\|_{L^\infty(0,T;W^{-1,q})} + Ce^{BT}\|\rho_{0,1} - \rho_{0,2}\|_{L^\infty(0,T;W^{-1,q})}.$$

Up to choice of a smaller $\bar{T}(\|\rho_{0,i}\|_{L^\infty}) > 0$, now depending on both $\|\rho_{0,i}\|_{L^\infty}$, we have

$$\forall T \in [0, \bar{T}], \quad \|\rho_1 - \rho_2\|_{L^\infty(0,T;W^{-1,q})} \leq \underbrace{\tfrac{Ce^{B\bar{T}}}{1 - B\bar{T}e^{B\bar{T}}}}_{\bar{C}} \|\rho_{0,1} - \rho_{0,2}\|_{W^{-1,q}},$$

Notice that the choice of $\bar{T}$ depends only on B, and recall that $\|\rho_i(t)\|_{L^\infty} = \|\rho_{0,i}\|_{L^\infty}$ for any $t \in \mathbf{R}_+$. Therefore, one obtains by induction

$$\forall T \in \mathbf{R}_+, \quad \|\rho_1 - \rho_2\|_{L^\infty(0,T;W^{-1,q})} \leq \bar{C}^{\lceil T/\bar{T} \rceil}\|\rho_{0,1} - \rho_{0,2}\|_{W^{-1,q}},$$

which also writes (2.18). □

2.3 Well-posedness of the system in the infinite strip

In this section, Ω stands for the *infinite strip* $\mathbf{R} \times (0,1)$. We denote by $(\boldsymbol{e}_x, \boldsymbol{e}_z)$ the canonical base of $\mathbf{R}^2$ in which $\boldsymbol{u}$ has coordinates (u_1, u_2).

Regarding our problem, the transport theory does not depend on the nature of the domain, and the related results presented in the previous section are still valid in the strip. The main difference lies in the tools and methods required to solve the Stokes equation in Ω. In particular, we state that this equation is still well-posed for L^∞ data, with $W^{1,\infty}$ solution. To do so, we consider *Kato spaces*, also known as *uniformly local Sobolev spaces*. This framework allows to consider uniformly bounded densities ρ and non globally integrable velocity fields $\boldsymbol{u}$, having infinite energy $\|\boldsymbol{u}\|_{H^1}$, but admitting locally a finite energy, uniformly bounded with respect to any compact stallion subdomain of the strip. We first solve Stokes equation in a L^2 framework, then recover elliptic regularity and use Sobolev injection to prove its well-posedness for bounded data. Even if it requires known methods, we did not find the precise proof of this latter result in the uniformly local framework.

The Subsection 2.3.1 is dedicated to the statements and proofs related to the Stokes equation. In particular we introduce Kato spaces in paragraph 2.3.1, then discuss the flux condition in paragraph 2.3.1 and finally prove the related well-posedness theorems in paragraph 2.3.1. The Subsection 2.3.2 concerns the stability estimate for the transport in the strip. Subsection 2.3.3 contains the proof of the well-posedness of the Stokes-transport coupling, and we state its stability estimate in Subsection 2.3.4.

2.3.1 The Stokes problem in the strip

The functional spaces

Let us set the following subdomains of Ω, for any $k \in \mathbf{Z}$,

$$U_k = \{(x,z) \in \Omega : k < x < k+1\}, \qquad U_k^* = \{(x,z) \in \Omega : k-1 < x < k+2\}.$$

Define a smooth map $\chi : \Omega \to [0,1]$, depending on x only, equal to 1 in U_0 and to 0 outside U_0^*. Set its translations $\chi_k := \chi(\cdot - k\boldsymbol{e}_x)$ so that χ_k is equal to 1 in U_k and supported in U_k^*. For convenience we choose χ so that $\sum_{k=-\infty}^{+\infty} \chi_k = 2$ in Ω. Let us set for $m \in \mathbf{Z}$ and $1 \leq q \leq \infty$ the uniformly local norm

$$\forall u \in W^{m,q}_{\mathrm{loc}}(\Omega), \qquad \|u\|_{W^{m,q}_{\mathrm{uloc}}} := \sup_{k \in \mathbf{Z}} \|\chi_k u\|_{W^{m,q}},$$

and define the *Kato space* as the set of locally Sobolev maps having a finite uniformly local norm,

$$W^{m,q}_{\mathrm{uloc}}(\Omega) := \{u \in W^{m,q}_{\mathrm{loc}}(\Omega) : \|u\|_{W^{m,q}_{\mathrm{uloc}}} < \infty\}.$$

This is a Banach space, that does not depend on the choice of χ, see for instance [ABZ16, §2.2]. The following result provides in particular some handy norms equivalences.

Lemma 2.3.1. *For any $m \in \mathbf{N}$ and $1 < q \leq \infty$, the following quantities are equivalent,*

$$u \in W^{m,q}_{\mathrm{loc}}(\Omega), \quad \sup_{k \in \mathbf{Z}} \|u\|_{W^{m,q}(U_k)} \simeq \sup_{k \in \mathbf{Z}} \|u\|_{W^{m,q}(U_k^*)} \simeq \|u\|_{W^{m,q}_{\mathrm{uloc}}}.$$

For any $m \in \mathbf{N}^*$ and $1 < q < \infty$, the following quantities are comparable,

$$u \in W^{-m,q}_{\mathrm{loc}}(\Omega), \quad \sup_{k \in \mathbf{Z}} \|u\|_{W^{-m,q}(U_k)} \lesssim \sup_{k \in \mathbf{Z}} \|u\|_{W^{-m,q}(U_k^*)} \simeq \|u\|_{W^{-m,q}_{\mathrm{uloc}}}.$$

Although this result presents no difficulty, we provide a short proof and a comment, about the missing inequality for the last comparison, in appendix 2.4.1, for sake of completeness.

Flux condition

In general, the homogeneous Stokes system as formulated in (2.3) admits non-trivial solutions in domains with unbounded boundaries, called *Poiseuille solutions*, see for instance [Gal11, Section IV]. In our case, these are described as follows

$$\boldsymbol{u}_\phi(x,z) = \begin{pmatrix} 6\phi z(1-z) \\ 0 \end{pmatrix}, \quad p_\phi(x,z) = 12\phi x, \qquad \phi \in \mathbf{R}.$$

Let us introduce the definition of the *flux* of $\boldsymbol{u}$ throw the section of abscissa $x \in \mathbf{R}$ of Ω,

$$\int u_1 \,\mathrm{d}z := \int_0^1 u_1(x,z)\,\mathrm{d}z = \int_0^1 \boldsymbol{u}(x,z)\cdot \boldsymbol{e}_x \,\mathrm{d}z.$$

Notice that the divergence-free and the homogeneous Dirichlet condition ensure that any solution of (2.3) on Ω has flux independent of x, which we will denote further by $\int u_1 \,\mathrm{d}z$. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^1 u_1(x,z)\,\mathrm{d}z\right] = \int_0^1 \partial_x u_1(x,z)\,\mathrm{d}z = -\int_0^1 \partial_z u_2(x,z)\,\mathrm{d}z = u_2(x,0) - u_2(x,1) = 0. \quad (2.19)$$

For instance, the flux of the Poiseuille solution $\boldsymbol{u}_\phi$ is ϕ. We will see that a choice of flux value prescribes a unique Poiseuille solution, and provides uniqueness of a solution in Kato spaces. Since the Stokes equation is linear, we can choose one value without loss of generality. From now on we consider the Stokes problem with the zero flux condition,

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 & \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} & \text{on } \partial\Omega, \\ \int u_1 \,\mathrm{d}z &= 0 & \text{in } \mathbf{R}. \end{cases} \quad (2.20)$$

Well-posedness of the Stokes problem in the strip

We show that the system (2.20) is well-posed for $H^{-1}_{\mathrm{uloc}}(\Omega)$ and $L^2_{\mathrm{uloc}}(\Omega)$ data $\boldsymbol{f}$, with some elliptic regularity gain. We then deduce that this system is also well-posed for $L^\infty(\Omega)$ data with $W^{1,\infty}(\Omega)$ solutions, by adapting the steps of the proof of the bounded domain case. The general technique presented here is originally due to Ladyzhenskaya and Solonnikov [LS83]. A proof in a framework closer to ours can be found in [GM10, Theorem 3]. Although these are classical tools, we have not found the proof of this result in the literature.

Theorem 2.3.2. *Let $\boldsymbol{f} \in H^{-1}_{\mathrm{uloc}}(\Omega)$. There exists a unique $\boldsymbol{u}$ in $H^1_{\mathrm{uloc}}(\Omega)$ satisfying (2.20), which moreover obeys the inequality*

$$\|\boldsymbol{u}\|_{H^1_{\mathrm{uloc}}} \le C\|\boldsymbol{f}\|_{H^{-1}_{\mathrm{uloc}}}. \quad (2.21)$$

Let us introduce a few more notations. Set for any $k \in \mathbf{Z}$,

$$\Omega_k := \{(x,z) \in \Omega : -k < x < k\}.$$

Let us define for any $k \in \mathbf{N}^*$ a smooth map $\eta_k : \Omega \to [0,1]$, depending on x only, equal to 1 in Ω_k and supported in Ω_{k+1}. Remark that its derivatives are supported in $\Omega_{k+1}\backslash\Omega_k$. We can choose η_k such that there exists a constant $C > 0$ independent of k and satisfying

$$\|\eta_k'\|_{L^\infty} + \|\eta_k''\|_{L^\infty} \le C.$$

Finally, let us observe the following estimate linking uniformly local and classical Sobolev norms over Ω_n. We report its proof in appendix 2.4.2.

Lemma 2.3.3. *Let $n \in \mathbf{N}^*$. There exists a constant $C > 0$ such that for any f in $L^2_{\mathrm{uloc}}(\Omega)$, resp. in $H^{-1}_{\mathrm{uloc}}(\Omega)$, one has*

$$\|f\|_{L^2(\Omega_n)} \leq Cn^{1/2}\|f\|_{L^2_{\mathrm{uloc}}}, \quad \textit{resp. } \|f\|_{H^{-1}(\Omega_n)} \leq Cn^{1/2}\|f\|_{H^{-1}_{\mathrm{uloc}}}.$$

Proof of Theorem 2.3.2. Let us set for any $n \in \mathbf{N}^*$ the unique couple $(\boldsymbol{u}_n, p_n)$ in $H^1(\Omega_n) \times \left(L^2(\Omega_n)/\mathbf{R}\right)$ satisfying the system

$$\left\{ \begin{array}{rcll} -\Delta \boldsymbol{u}_n + \nabla p_n & = & \boldsymbol{f} & \text{in } \Omega_n, \\ \operatorname{div} \boldsymbol{u}_n & = & 0 & \text{in } \Omega_n, \\ \boldsymbol{u}_n & = & \mathbf{0} & \text{on } \partial\Omega_n, \end{array} \right. \tag{2.22}$$

existence and uniqueness of which is ensured by [BF12, Theorem IV.5.1]. Then define for any integers $1 \leq k \leq n$ the energy of $\boldsymbol{u}_n$ on the subdomain Ω_k,

$$E_{n,k} := \|\boldsymbol{u}_n\|^2_{H^1(\Omega_k)} = \int_{-k}^{k}\int_0^1 |\nabla \boldsymbol{u}_n|^2 + |\boldsymbol{u}_n|^2.$$

By evaluating the variational formulation of (2.22) in the test function $\boldsymbol{u}_n$ we find

$$\int_{-n}^{n}\int_0^1 |\nabla \boldsymbol{u}_n|^2 = \langle \boldsymbol{f}, \boldsymbol{u}_n\rangle_{\Omega_n} \leq \|\boldsymbol{f}\|_{H^{-1}(\Omega_n)}\|\boldsymbol{u}_n\|_{H^1(\Omega_n)}.$$

Using Lemma 2.3.3 and Poincaré's inequality, for which the constant involved can be chosen independent of n, one finds

$$E_{n,n} \leq Cn\|\boldsymbol{f}\|^2_{H^{-1}_{\mathrm{uloc}}}.$$

Our goal is to show that there exists $C > 0$ independent of n and $\boldsymbol{f}$, such that

$$E_{n,1} = \|\boldsymbol{u}_n\|^2_{H^1(\Omega_1)} \leq C\|\boldsymbol{f}\|^2_{H^{-1}_{\mathrm{uloc}}}. \tag{2.23}$$

This allows to conclude to the existence of a solution $\boldsymbol{u} \in H^1_{\mathrm{uloc}}(\Omega)$ obeying the estimate (2.21), by translation invariance of the domain and compactness considerations. To prove (2.23) we will fix $n \in \mathbf{N}^*$ and show by descending induction over k that there exists $C > 0$ and $k_0 \in \mathbf{N}^*$ independent of $\boldsymbol{u}$, f and n such that

$$\forall k, k_0 \leq k \leq n, \quad E_{n,k} \leq Ck\|\boldsymbol{f}\|^2_{H^{-1}_{\mathrm{uloc}}}.$$

Let us evaluate the variational formulation of (2.22) in the test function $\eta_k \boldsymbol{u}_n$, which yields

$$\int_{\Omega_n} \eta_k |\nabla \boldsymbol{u}_n|^2 = \langle \boldsymbol{f}, \eta_k \boldsymbol{u}_n\rangle_{\Omega_{k+1}} - \int_{\Omega_n} \eta_k' \boldsymbol{u}_n \cdot \partial_x \boldsymbol{u}_n + \int_{\Omega_n} \eta_k' p_n u_{n,1}. \tag{2.24}$$

By Poincaré's inequality we bound from below the left hand side by $E_{n,k}$, up to a multiplicative constant. Let us bound from above all the right hand side terms. Lemma 2.3.3 provides

$$\langle \boldsymbol{f}, \eta_k \boldsymbol{u}_n\rangle_{\Omega_{k+1}} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega_{k+1})}\|\boldsymbol{u}_n\|_{H^1(\Omega_{k+1})} \leq C(k+1)^{1/2}\|\boldsymbol{f}\|_{H^{-1}_{\mathrm{uloc}}} E^{1/2}_{n,k+1}. \tag{2.25}$$

Since η_k' is supported in $\Omega_{k+1}\backslash\Omega_k$ and uniformly bounded independently of k, we have

$$\left|\int_{\Omega_n} \eta_k' \boldsymbol{u}_n \cdot \partial_x \boldsymbol{u}_n\right| \leq C \int_{\Omega_{k+1}\backslash\Omega_k} |\nabla \boldsymbol{u}_n|^2 + |\boldsymbol{u}_n|^2 = C(E_{n,k+1} - E_{n,k}). \tag{2.26}$$

Let us split the remaining integral of (2.24) as follows,

$$\int_{\Omega_n} \eta_k' p_n u_{n,1} = \int_{U_k} \eta_k' p_n u_{n,1} + \int_{-U_k} \eta_k' p_n u_{n,1}.$$

Remark that

$$\int_{U_k} \eta_k' u_{n,1} = \int_k^{k+1} \eta_k' \int u_{n,1}\,\mathrm{d}z\,\mathrm{d}x = 0,$$

since the flux $\int \boldsymbol{u}_{n,1}\,\mathrm{d}z$ is independent of x for the same reason as in (2.19), and equal to 0 because of the homogenous Dirichlet condition in $\{x = \pm n\}$. Hence, let us denote by $\langle p_n\rangle_{U_k}$ the average of p_n over U_k, and find

$$\int_{U_k} \eta_k' p_n u_{n,1} = \int_{U_k} \eta_k' (p_n - \langle p_n\rangle_{U_k}) u_{n,1} \leq C\|p_n - \langle p_n\rangle_{U_k}\|_{L^2(U_k)} \|\boldsymbol{u}_n\|_{L^2(U_k)}.$$

Let us apply Nečas inequality, see [BF12, Lemma IV.1.9], and get

$$\begin{aligned}\|p_n - \langle p_n\rangle_{U_k}\|_{L^2(U_k)} &\leq C\|\nabla p_n\|_{H^{-1}(U_k)} \\ &\leq C\|\Delta \boldsymbol{u}_n + \boldsymbol{f}\|_{H^{-1}(U_k)} \\ &\leq C\left(\|\nabla \boldsymbol{u}_n\|_{L^2(U_k)} + \|\boldsymbol{f}\|_{H^{-1}(U_k)}\right).\end{aligned}$$

Hence we deduce, using Lemma 2.3.1, that

$$\left|\int_{U_k} \eta_k' p_n u_{n,1}\right| \leq C\big((E_{n,k+1} - E_{n,k})^{1/2} + \|\boldsymbol{f}\|_{H^{-1}_{\mathrm{uloc}}}\big) E_{n,k}^{1/2}. \tag{2.27}$$

The very same considerations hold true for the integral over $-U_k$. Bounding (2.24) thanks to (2.25), (2.26) and (2.27), plus applying Young's inequality, we obtain that for any integers k, n such that $1 \leq k \leq n$,

$$E_{n,k} \leq C\left(E_{n,k+1} - E_{n,k} + (k+1)\|\boldsymbol{f}\|^2_{H^{-1}_{\mathrm{uloc}}}\right). \tag{2.28}$$

This relation implies (2.23), as stated in the following lemma, proven apart in appendix 2.4.3.

Lemma 2.3.4. *Let $(E_{n,k})_{k,n}$ be a non-negative family indexed by all couples $(k,n) \in \mathbf{N}^2$ satisfying $1 \leq k \leq n$, non-decreasing with respect to k, obeying (2.28) and such that*

$$\forall n \in \mathbf{N}^*, \quad E_{n,n} \leq Cn\|\boldsymbol{f}\|^2_{H^{-1}_{\mathrm{uloc}}}.$$

There exists $C_0 > 0$ and $k_0 \in \mathbf{N}^*$ independent of $\boldsymbol{f}$ such that for any $k, n \in \mathbf{N}$ satisfying $k_0 \leq k \leq n$, we have

$$E_{n,k} \leq C_0 k\|\boldsymbol{f}\|^2_{H^{-1}_{\mathrm{uloc}}}.$$

This result implies the expected inequality,

$$\forall n \geq k_0, \quad E_{n,1} \leq E_{n,k_0} \leq C_0 k_0 \|\boldsymbol{f}\|^2_{H^{-1}_{\mathrm{uloc}}}.$$

By extending $\boldsymbol{f}|_{\Omega_n}$ and $\boldsymbol{u}_n$ to Ω by $\mathbf{0}$ outside Ω_n, we can perform a similar analysis and find the same energy estimates over each subdomain U_ℓ, namely,

$$\forall n \geq k_0, \forall \ell \in \mathbf{Z}, \quad \|\boldsymbol{u}_n\|^2_{H^1(U_\ell)} \leq C\|\boldsymbol{f}\|^2_{H^{-1}_{\mathrm{uloc}}},$$

where $C = C_0 k_0$ with C_0 and k_0 independent of n and ℓ. Therefore, for any $n \geq k_0$, $\boldsymbol{u}_n$ belongs to $H^1_{\mathrm{uloc}}(\Omega)$ and satisfies

$$\|\boldsymbol{u}_n\|_{H^1_{\mathrm{uloc}}} \leq C\|\boldsymbol{f}\|_{H^{-1}_{\mathrm{uloc}}}.$$

Since bounded subsets of $H^1(U_\ell)$ are weakly relatively compact, there exists a subsequence of $(\boldsymbol{u}_n)_n$ converging weakly in $H^1_{\mathrm{loc}}(\Omega)$ toward some $\boldsymbol{u} \in H^1_{\mathrm{uloc}}(\Omega)$, with $\boldsymbol{u}$ satisfying estimate (2.21). The limit also verifies $\int u_1 \,\mathrm{d}z = 0$ since every $\boldsymbol{u}_n$ has zero flux. Hence, it is a solution of (2.20).

To prove uniqueness of such an element, let us consider some $\boldsymbol{u} \in H^1_{\mathrm{uloc}}(\Omega)$ satisfying (2.20) with $\boldsymbol{f} = \mathbf{0}$. Define the energy $E_k := \|\boldsymbol{u}\|^2_{H^1(\Omega_k)}$ and proceed to the same computations as previously to find

$$E_k \leq C(E_{k+1} - E_k + 1).$$

Notice that the zero flux condition is used here to bound the pressure term and obtain this estimate. Since $E_{k+1} - E_k$ is bounded by $\|\boldsymbol{u}\|^2_{H^1_{\mathrm{uloc}}}$, we have

$$\forall k \in \mathbf{Z}, \quad E_k \leq C(\|\boldsymbol{u}\|^2_{H^1_{\mathrm{uloc}}} + 1) < \infty,$$

which means that $\boldsymbol{u}$ belongs to $H^1(\Omega)$. Since the Stokes system with Dirichlet boundary condition is well-posed in $H^1(\Omega)$ in any domain Ω of $\mathbf{R}^2$ bounded in one direction, see for instance [Tem79, Theorem 2.1], we necessarily have $\boldsymbol{u} = \mathbf{0}$, which allows to conclude to the uniqueness and the proof. □

Remark 2.3.5. *The pressure does not belong to $L^2_{\mathrm{uloc}}(\Omega)$ in general; observe for instance the following triplet, satisfying (2.20),*

$$\boldsymbol{f} = \boldsymbol{e}_x, \quad \boldsymbol{u} = \mathbf{0}, \quad p(x,z) = x.$$

Nevertheless, we have thanks to Nečas inequality some similar estimate as (2.21) on the pressure,

$$\sup_{k \in \mathbf{Z}} \|p - \langle p \rangle_{U_k}\|_{L^2(U_k)} \leq C\|\boldsymbol{f}\|_{L^2_{\mathrm{uloc}}}.$$

Remark 2.3.6. *This proof does not adapt straightforwardly to the case of the layer domain $\mathbf{R}^2 \times (0,1)$. The first issue one needs to deal with is to determine conditions on $\boldsymbol{u}$ ensuring uniqueness of a solution. Also, the descending induction on the energy estimates no longer holds in this form. Indeed, one needs to replace the slices $[k, k+1] \times (0,1)$ in $\mathbf{R} \times (0,1)$ by chunks $[k,k+1] \times [\ell, \ell+1] \times (0,1)$ in $\mathbf{R}^2 \times (0,1)$. To bound the energy on $[-k,k]^2 \times (0,1)$ by the energy on $[-(k+1), k+1]^2 \times (0,1)$ makes appear some quadratic terms in k in (2.28) instead of the linear ones present for the strip case, making the induction fail. Under different boundary assumptions, it is however possible to adapt it in a non-trivial way and to conclude; see for instance [DP14, Section 3].*

Since $L^\infty(\Omega) \subset L^2_{\mathrm{uloc}}(\Omega) \subset H^{-1}_{\mathrm{uloc}}(\Omega)$, we already have existence of a solution to (2.20) for L^∞

data. Recall that we need to establish the $W^{1,\infty}$ regularity of this solution. We first show that the system satisfies some elliptic regularity property in the hilbertian framework.

Theorem 2.3.7. *Let $\boldsymbol{f} \in L^2_{\mathrm{uloc}}(\Omega)$. The associated solution $\boldsymbol{u} \in H^1_{\mathrm{uloc}}(\Omega)$ of (2.20) belongs to $H^2_{\mathrm{uloc}}(\Omega)$ and obeys the inequality*

$$\|\boldsymbol{u}\|_{H^2_{\mathrm{uloc}}} \leq C\|\boldsymbol{f}\|_{L^2_{\mathrm{uloc}}}. \tag{2.29}$$

Proof. The demonstration consists in truncating the global solution $\boldsymbol{u}$ within some bounded subdomains, and to use the elliptic regularity in these bounded domains provided by Theorem 2.2.1. Let $\boldsymbol{f} \in L^2_{\mathrm{uloc}}(\Omega)$ and set $(\boldsymbol{u}, p) \in H^1_{\mathrm{uloc}}(\Omega) \times \left(L^2_{\mathrm{loc}}(\Omega)/\mathbf{R}\right)$, the associated solution to (2.20). For any $k \in \mathbf{Z}$, set $\boldsymbol{u}_k := \chi_k \boldsymbol{u}$ and $q_k := \chi_k(p - \langle p \rangle_{U_k^*})$, which satisfy the system

$$\left\{ \begin{array}{rcll} -\Delta \boldsymbol{u}_k + \nabla q_k & = & \boldsymbol{F}_k & \text{in } \tilde{U}_k, \\ \operatorname{div} \boldsymbol{u}_k & = & \chi_k' u_1 & \text{in } \tilde{U}_k, \\ \boldsymbol{u}_k & = & \boldsymbol{0} & \text{on } \partial \tilde{U}_k, \end{array} \right. \tag{2.30}$$

where we set

$$\boldsymbol{F}_k := \chi_k \boldsymbol{f} - 2\chi_k' \partial_x \boldsymbol{u} - \chi_k'' \boldsymbol{u} + \chi_k' \left(p - \langle p \rangle_{U_k^*}\right) \boldsymbol{e}_x$$

for any smooth bounded subdomain $\tilde{U}_k$ of Ω containing U_k^*. Let us set $(\tilde{U}_k)_k$ a family of such domains given by the choice of a smooth $\tilde{U}_0$ containing U_0^* and its translations $\tilde{U}_k = \tilde{U}_0 + k\boldsymbol{e}_x$. The regularity of $\boldsymbol{u}$ and p implies that $\boldsymbol{F}_k$ belongs to $L^2(\tilde{U}_k)$ and that $\chi_k' u_1$ satisfies the compatibility condition (2.4). Therefore, Theorem 2.2.1 ensures that $\boldsymbol{u}_k$ is the only solution of (2.30) on $\tilde{U}_k$, with estimate

$$\|\boldsymbol{u}_k\|_{H^2(\tilde{U}_k)} \leq C\|\boldsymbol{F}_k\|_{L^2(\tilde{U}_k)}, \tag{2.31}$$

where the constant $C > 0$ can be chosen independent of k since the subdomains $\tilde{U}_k$ are translations of each other. A few computations lead to

$$\|\boldsymbol{F}_k\|_{L^2(\tilde{U}_k)} \leq C(\|\boldsymbol{u}\|_{H^1_{\mathrm{uloc}}} + \|\boldsymbol{f}\|_{L^2_{\mathrm{uloc}}} + \|p - \langle p \rangle_{U_k^*}\|_{L^2(U_k^*)}). \tag{2.32}$$

Nečas inequality and Lemma 2.3.1 provide

$$\begin{aligned} \|p - \langle p \rangle_{U_k^*}\|_{L^2(U_k^*)} &\leq C\|\nabla p\|_{H^{-1}(U_k^*)} \\ &\leq C\|\Delta \boldsymbol{u} + \boldsymbol{f}\|_{H^{-1}(U_k^*)} \\ &\leq C(\|\boldsymbol{u}\|_{H^1_{\mathrm{uloc}}} + \|\boldsymbol{f}\|_{L^2_{\mathrm{uloc}}}). \end{aligned}$$

The latter estimate combined with (2.21) and (2.32) in (2.31) leads to

$$\|\boldsymbol{u}\|_{H^2(U_k)} \leq \|\boldsymbol{u}_k\|_{H^2(\tilde{U}_k)} \leq C\|\boldsymbol{f}\|_{L^2_{\mathrm{uloc}}},$$

which proves that $\boldsymbol{u}$ belongs to $H^2_{\mathrm{uloc}}(\Omega)$ and satisfies inequality (2.29). □

From the latter result and Sobolev embeddings we obtain existence of $W^{1,q}_{\mathrm{uloc}}$ solutions for L^q data. Then elliptic regularity is once again recovered and we show that these solutions are in $W^{2,q}_{\mathrm{uloc}}$. Finally Sobolev embeddings once again yield a unique solution in $W^{1,\infty}$ for L^∞ data.

Theorem 2.3.8. *Let $\boldsymbol{f} \in L^\infty(\Omega)$. There exists a unique $\boldsymbol{u} \in W^{1,\infty}(\Omega)$ satisfying (2.20), which obeys*

$$\|\boldsymbol{u}\|_{W^{1,\infty}} \leq C\|\boldsymbol{f}\|_{L^\infty}. \tag{2.33}$$

Proof. Let $\boldsymbol{f} \in L^\infty(\Omega)$. We always have

$$\|\boldsymbol{f}\|_{L^2_{\mathrm{uloc}}} \leq C\|\boldsymbol{f}\|_{L^\infty}.$$

The Sobolev embeddings in bounded domains adapts into the continuous inclusion

$$H^2_{\mathrm{uloc}}(\Omega) \hookrightarrow W^{1,q}_{\mathrm{uloc}}(\Omega), \quad 2 \leq q < \infty.$$

Since Theorem 2.3.7 ensures the existence of a solution to (2.20) in $H^2_{\mathrm{uloc}}(\Omega)$, we also have existence of a solution in $W^{1,q}_{\mathrm{uloc}}(\Omega)$. Besides, the inclusions of Lebesgue spaces imply

$$W^{1,q}_{\mathrm{uloc}}(\Omega) \hookrightarrow H^1_{\mathrm{uloc}}(\Omega), \quad 2 \leq q < \infty.$$

Hence, the uniqueness of a solution in H^1_{uloc}, ensured by Theorem 2.3.2, implies that there exists at most one solution of (2.20) in $W^{1,q}_{\mathrm{uloc}}(\Omega)$. In the end, (2.20) admits a unique solution $\boldsymbol{u} \in W^{1,q}_{\mathrm{uloc}}(\Omega)$ for $2 \leq q < \infty$, with estimate

$$\|\boldsymbol{u}\|_{W^{1,q}_{\mathrm{uloc}}} \leq C_q\|\boldsymbol{f}\|_{L^\infty}.$$

Now the method is exactly the same as in Theorem 2.3.7 to prove that $\boldsymbol{u}$ belongs to $W^{2,q}_{\mathrm{uloc}}(\Omega)$. To do so, the only extra result we require is Nečas inequality in the general L^q framework, see [Gal11, Ex. III.3.4, p. 175], which provides the very same pressure estimates as for $q = 2$. In the end we obtain the well-posedness of the problem in $W^{2,q}_{\mathrm{uloc}}(\Omega)$, with estimate

$$\|\boldsymbol{u}\|_{W^{2,q}_{\mathrm{uloc}}} \leq C\|\boldsymbol{f}\|_{L^\infty}, \quad 2 \leq q < \infty.$$

Now use that

$$W^{2,4}_{\mathrm{uloc}}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \hookrightarrow W^{1,4}_{\mathrm{uloc}}(\Omega).$$

As previously, the first embedding provides existence of a solution $\boldsymbol{u} \in W^{1,\infty}(\Omega)$, together with estimate (2.33), and the second one ensures uniqueness. □

2.3.2 Stability estimate for the transport in the strip

The transport equation (2.8) is still well-posed on $\Omega = \mathbf{R} \times (0,1)$ and Proposition 2.2.5 still applies. The lemmas related to the properties of the characteristics are also valid still, up to minor adaptations mentioned when required in the following. The only adaptation demanding particular attention is the stability estimate from Proposition 2.2.6, stated as follows.

Proposition 2.3.9. *Let $\boldsymbol{u}_i \in L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega))$ with $\boldsymbol{u}_i|_{\partial\Omega} \equiv \mathbf{0}$ and $\operatorname{div} \boldsymbol{u}_i \equiv 0$, for $i = 1, 2$. Let $\rho_0 \in L^\infty(\Omega)$ and set ρ_i the solution of (2.8) associated to $\boldsymbol{u}_i$ with initial datum ρ_0. There exists $\bar{T}(\|\nabla \boldsymbol{u}_i\|_{L^\infty}) > 0$ such that for any $T \in [0, \bar{T}]$,*

$$\|\rho_1 - \rho_2\|_{L^\infty(0,T;H^{-1}_{\mathrm{uloc}})} \leq BT(1+MT)^{1/2} e^{CT\|\nabla \boldsymbol{u}_1\|_{L^\infty(0,T;L^\infty)}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,T;L^2_{\mathrm{uloc}})},$$

where $B := C\|\rho_0\|_{L^\infty}$ and $M := \max_i \|\boldsymbol{u}_i\|_{L^\infty}$.

Proof. The goal is to bound the following quantity for any test function $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$ uniformly in $k \in \mathbf{Z}$ and with respect to $t \in [0, \bar{T}]$ where $\bar{T}$ is determined further. To apply Liouville theorem

gives

$$\begin{aligned} I_{\varphi,k} &:= \int_\Omega \big(\rho_1(t,\boldsymbol{x}) - \rho_2(t,\boldsymbol{x})\big)(\chi_k\varphi)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega \rho_0(\boldsymbol{x})\big((\chi_k\varphi)(\boldsymbol{X}_1(t;\boldsymbol{x})) - (\chi_k\varphi)(\boldsymbol{X}_2(t;\boldsymbol{x}))\big)\,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega \rho_0(\boldsymbol{x})(\boldsymbol{X}_1(t;\boldsymbol{x}) - \boldsymbol{X}_2(t;\boldsymbol{x}))\cdot\int_0^1 \nabla(\chi_k\varphi)(\boldsymbol{X}^\lambda(t;\boldsymbol{x}))\,\mathrm{d}\lambda\,\mathrm{d}\boldsymbol{x}, \end{aligned}$$

where $\boldsymbol{X}^\lambda(t;\boldsymbol{x}) := \lambda\boldsymbol{X}_1(t;\boldsymbol{x}) + (1-\lambda)\boldsymbol{X}_2(t;\boldsymbol{x})$. Since ρ_i are the push-forwards of ρ_0 by $\boldsymbol{X}_i$, the respective transports occur at finite speed, bounded by M. Since χ_k is supported in U_k^*, the support of $(\chi_k\varphi)\circ\boldsymbol{X}_i(t)$ is included in

$$U_k^{M,t} := \{k-1-Mt < x < k+2+Mt\}.$$

Hence Hölder's inequality applies as follows

$$|I_{\varphi,k}| \le \|\rho_0\|_{L^\infty}\|\boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)\|_{L^2(U_k^{M,t})}\int_0^1 \|\nabla(\chi_k\varphi)(\boldsymbol{X}^\lambda(t))\|_{L^2}\,\mathrm{d}\lambda.$$

We saw in Proposition 2.2.6 that there exists $\bar{T}(\|\nabla\boldsymbol{u}_i\|_{L^\infty}) > 0$ such that $\boldsymbol{X}^\lambda(t)$ performs a change of variable with jacobian determinant uniformly bounded with respect to $t\in[0,\bar{T}]$ and $\lambda\in[0,1]$, meaning there exists a constant $C>0$ such that

$$\forall t\in[0,\bar{T}], \lambda\in[0,1], \qquad \|\nabla(\chi_k\varphi)(\boldsymbol{X}^\lambda(t))\|_{L^2} \le C\|\nabla(\chi_k\varphi)\|_{L^2} \le C_\chi\|\varphi\|_{H^1}.$$

Besides, Lemma 2.2.4 applies on the bounded domain $U_k^{M,t}$, providing

$$\|\boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)\|_{L^2(U_k^{M,t})} \le te^{Ct\|\nabla\boldsymbol{u}_1\|_{L^\infty(0,t;L^\infty)}}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,t;L^2(U_k^{M,t}))}.$$

From considerations similar to those of Lemma 2.3.3 we obtain

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,t;L^2(U_k^{M,t}))} \le C(1+Mt)^{1/2}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,t;L^2_{\mathrm{uloc}})},$$

with finite right hand side, since $\boldsymbol{u}_i \in L^\infty(\mathbf{R}_+;W^{1,\infty}(\Omega))$. Combining these last equalities lead to the following estimate, for any $t\in[0,\bar{T}]$,

$$|I_{\varphi,k}| \le C\|\rho_0\|_{L^\infty}t(1+Mt)^{1/2}e^{Ct\|\nabla\boldsymbol{u}_1\|_{L^\infty(\mathbf{R}_+;L^\infty)}}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,t;L^2_{\mathrm{uloc}})}\|\varphi\|_{H^1}.$$

Taking the supremum over the test functions φ, $k\in\mathbf{Z}$ and $t\in[0,T]$, we see that for any $T\in[0,\bar{T}]$ we have

$$\|\rho_1 - \rho_2\|_{L^\infty(0,T;H^{-1}_{\mathrm{uloc}})} \le BT(1+MT)^{1/2}e^{CT\|\nabla\boldsymbol{u}_1\|_{L^\infty(0,T;L^\infty)}}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,T;L^2_{\mathrm{uloc}})}.$$

□

2.3.3 Proof of Theorem 2.1.2

The proof essentially follows the same path as in Theorem 2.1.1. For this reason we recall briefly the similar steps and focus on the parts that differ from this former case.

Local existence : Set $\rho^0 \equiv \rho_0$ in $L^\infty(\mathbf{R}_+; L^\infty(\Omega))$. Define, thanks to Proposition 2.2.5 and Theorem 2.3.8, the following sequences

$$\forall N \in \mathbf{N}, \quad \rho^N \in L^\infty(\mathbf{R}_+; L^\infty(\Omega)), \quad \boldsymbol{u}^N \in L^\infty(\mathbf{R}_+, W^{1,\infty}(\Omega)),$$

satisfying the partial problems

$$\left\{ \begin{array}{rcll} \partial_t \rho^{N+1} + \boldsymbol{u}^N \cdot \nabla \rho^{N+1} & = & 0 & \text{in } \mathbf{R}_+ \times \Omega, \\ \rho^{N+1}(0,\cdot) & = & \rho_0 & \text{in } \Omega, \end{array} \right. \tag{2.34}$$

and

$$\left\{ \begin{array}{rcll} -\Delta \boldsymbol{u}^N + \nabla p^N & = & -\rho^N \boldsymbol{e}_z & \text{in } \mathbf{R}_+ \times \Omega, \\ \operatorname{div} \boldsymbol{u}^N & = & 0 & \text{in } \mathbf{R}_+ \times \Omega, \\ \boldsymbol{u}^N & = & \mathbf{0} & \text{in } \mathbf{R}_+ \times \partial\Omega, \\ \int u_1^N & = & 0 & \text{in } \mathbf{R}_+. \end{array} \right. \tag{2.35}$$

The uniform bounds, with $B := C\|\rho_0\|_{L^\infty}$, remain true,

$$\|\rho^N\|_{L^\infty(\mathbf{R}_+, L^\infty)} \leq B, \quad \|\boldsymbol{u}^N\|_{L^\infty(\mathbf{R}_+; W^{1,\infty})} \leq B. \tag{2.36}$$

Therefore we still have weak* convergence of $\rho^N, \boldsymbol{u}^N$ and $\nabla \boldsymbol{u}^N$, up to the extraction of subsequences. Beside, Proposition 2.3.9 ensures the existence of a $\bar{T}(\|\rho_0\|_{L^\infty}) > 0$ such that for any $T \in [0, \bar{T}]$,

$$\|\rho^{N+1} - \rho^N\|_{L^\infty(0,T;H^{-1}_{\text{uloc}})} \leq BT(1+BT)^{1/2} e^{BT} \|\rho^N - \rho^{N-1}\|_{L^\infty(0,T;H^{-1}_{\text{uloc}})},$$

where we have plugged (2.36). Therefore, up to the choice of a small enough $T(\|\rho_0\|_{L^\infty}) > 0$, $(\rho^N)_N$ is a Cauchy sequence in $L^\infty(0,T;H^{-1}_{\text{uloc}}(\Omega))$, which implies that $(\boldsymbol{u}^N)_N$ is also a Cauchy sequence in $L^\infty(0,T;H^1_{\text{uloc}}(\Omega))$, with limit denoted $\boldsymbol{u}$. The weak* convergence of $(\boldsymbol{u}^N)_N$ and $(\nabla \boldsymbol{u}^N)_N$ ensures that $\boldsymbol{u}$ also belongs to $L^\infty(0,T;W^{1,\infty}(\Omega))$. In particular, $\boldsymbol{u}^N$ and its derivatives converge in $L^1_{\text{loc}}([0,T] \times \Omega)$, which, together with the weak* convergence of $(\rho^N)_N$, is enough to pass to the limit in the weak formulation of partial problems (2.34) and (2.35). We obtain a local in time solution $(\rho, \boldsymbol{u})$ of (2.20), with regularity

$$L^\infty(0,T;L^\infty(\Omega)) \times L^\infty(0,T;W^{1,\infty}(\Omega)).$$

Local uniqueness : Let $(\rho_i, \boldsymbol{u}_i)$ be two such solutions of (2.20). The contraction adapts thanks to Proposition 2.3.9 in

$$\|\rho_1 - \rho_2\|_{L^\infty(0,T;H^{-1}_{\text{uloc}})} \leq BT(1+BT)^{1/2} e^{BT} \|\rho_1 - \rho_2\|_{L^\infty(0,T;H^{-1}_{\text{uloc}})},$$

which implies uniqueness for $T(\|\rho_0\|_{L^\infty}) > 0$ small enough.

Globality : The extension proves just as in the bounded case, see the proof of Theorem 2.1.1.

2.3.4 Stability estimate for the system in the strip

The result and the proof are identical to the ones of Proposition 2.2.7, replacing the stability estimate (2.18) of the bounded case by the one in the strip from Proposition 2.3.9.

Proposition 2.3.10. *Let $\rho_{0,i} \in L^\infty(\Omega)$ and ρ_i be the solution of (2.1) with initial datum $\rho_{0,i}$, for $i = 1, 2$. There exists $C = C(\Omega, \|\rho_{0,i}\|_{L^\infty}) > 0$ such that*

$$\forall T \in \mathbf{R}_+, \quad \|\rho_1 - \rho_2\|_{L^\infty(0,T;H^{-1}_{\mathrm{uloc}})} \leq Ce^{CT}\|\rho_{0,1} - \rho_{0,2}\|_{H^{-1}_{\mathrm{uloc}}}. \tag{2.37}$$

2.4 Appendix

2.4.1 Proof of Lemma 2.3.1

Let $m \in \mathbf{N}, k \in \mathbf{Z}, 1 < q < \infty$ and $u \in W^{m,q}_{\mathrm{uloc}}(\Omega)$. Since $\chi_k = 1$ on U_k,

$$\|u\|_{W^{m,q}(U_k)} \leq \|\chi_k u\|_{W^{m,q}}.$$

The support of χ_k being U_k^*, one has

$$\|\chi_k u\|_{W^{m,q}} \leq C_{\chi,m}\|u\|_{W^{m,q}(U_k^*)}.$$

One can split this last norm as follows

$$\|u\|_{W^{m,q}(U_k^*)} \leq \sum_{\ell=k-1}^{k+1} \|u\|_{W^{m,q}(U_\ell)}.$$

These three inequalities prove the first assertion,

$$\sup_{k\in\mathbf{Z}} \|u\|_{W^{m,q}(U_k)} \simeq \sup_{k\in\mathbf{Z}} \|u\|_{W^{m,q}(U_k^*)} \simeq \|u\|_{W^{m,q}_{\mathrm{uloc}}}.$$

Let $m \in \mathbf{N}^*$ and $1 < q < \infty$. For readability we adopt the following notations in the rest of this proof. For any $u \in W^{-m,q}(U)$ and $\varphi \in W_0^{m,q'}(U)$ where U is a subdomain of Ω, denote

$$\|u\|_U := \|u\|_{W^{-m,q}(U)}, \quad \|u\|_{\mathrm{uloc}} := \|u\|_{W^{-m,q}_{\mathrm{uloc}}}, \qquad \|\varphi\|_{0,U} = \|\varphi\|_{W_0^{m,q'}(U)}$$

and the duality brackets

$$\langle u, \varphi\rangle_U = \langle u, \varphi\rangle_{W^{-m,q}(U), W_0^{m,q'}(U)}.$$

The inclusion $\{\varphi \in W_0^{m,q}(U_k) : \|\varphi\|_{0,U_k} = 1\} \subset \{\varphi \in W_0^{m,q}(U_k^*) : \|\varphi\|_{0,U_k^*} = 1\}$ provides the first inequality

$$\|u\|_{W^{-m,q}(U_k)} \leq \|u\|_{W^{-m,q}(U_k^*)}.$$

Let us show the remaining direct inequality. We use the definition of the dual norm, and recall that $(\chi_k)_k$ is a partition of the unity, up to a factor 2. Also notice that a product $\chi_\ell\varphi$ with $\varphi \in W_0^{m,q'}(U_k^*)$ has possibly non-empty support only if $|\ell - k| \leq 3$. These remarks justify each

step of the following computations, for any $k \in \mathbf{Z}$,

$$\begin{aligned}
\|u\|_{U_k^*} &= \sup_{\|\varphi\|_{0,U_k^*}=1} \langle u, \varphi\rangle_{U_k^*} \\
&\simeq \sup_{\|\varphi\|_{0,U_k^*}=1} \sum |_{\ell-k|\leq 3} \langle u, \chi_\ell \varphi\rangle_{U_k^*} \\
&\leq \sup_{\|\varphi\|_{0,U_k^*}=1} \sum |_{\ell-k|\leq 3} \langle \chi_\ell u, \varphi\rangle_{U_k^*} \\
&\leq \sup_{\|\varphi\|_{0,\Omega}=1} \sum |_{\ell-k|\leq 3} \langle \chi_\ell u, \varphi\rangle_{\Omega}; \\
&\lesssim \|u\|_{\mathrm{uloc}}.
\end{aligned}$$

Finally, the reciprocal inequality is proved by noticing that $\chi_k\varphi$ belongs to $W_0^{m,q'}(U_k^*)$ for any $\varphi \in W_0^{m,q'}(\Omega)$;

$$\begin{aligned}
\|\chi_k u\|_\Omega &= \sup_{\|\varphi\|_{0,\Omega}=1} \langle u, \chi_k\varphi\rangle_\Omega \\
&\leq \sup_{\|\varphi\|_{0,\Omega}=1} \|u\|_{U_k^*} \|\chi_k\varphi\|_{0,U_k^*} \\
&\leq C(\chi, m)\|u\|_{U_k^*} \sup_{\|\varphi\|_{0,\Omega}=1} \|\varphi\|_{U_k^*} \\
&\leq C\|u\|_{U_k^*}.
\end{aligned}$$

□

Remark 2.4.1. *For $m \in \mathbf{N}^*$, we do not have in general*

$$\sup_{k\in\mathbf{Z}} \|u\|_{W^{-m,q}(U_k)} \gtrsim \sup_{k\in\mathbf{Z}} \|u\|_{W^{-m,q}(U_k^*)}.$$

Indeed, consider the Dirac mass $\delta_{(0,1/2)}$ belonging to $H^{-2}(\Omega)$ and therefore to $H^{-2}_{\mathrm{uloc}}(\Omega)$, with $\|\delta\|_{H^{-2}_{\mathrm{uloc}}} > 0$. Nevertheless, for any $k \in \mathbf{Z}$ we have

$$\forall \varphi \in H_0^2(U_k), \quad \langle \delta, \varphi\rangle_{U_k} = 0.$$

The reason is that the support of an element of the negative Sobolev spaces can be included in the complementary of $\cup_k U_k$. This does not happen when the considered subdomains family covers the whole domain, as does $(U_k^*)_k$.

2.4.2 Proof of Lemma 2.3.3

The case $f \in L^2_{\mathrm{uloc}}(\Omega_n)$ is straightforward,

$$\|f\|^2_{L^2(\Omega_n)} = \sum_{k=-n}^{n-1} \|f\|^2_{L^2(U_k)} \leq 2n\|f\|^2_{L^2_{\mathrm{uloc}}}.$$

The case $f \in H^{-1}_{\mathrm{uloc}}(\Omega_n)$ requires a little more care. We use notations from the proof of Lemma 2.3.1. Notice that $\sum_{\ell=-n-1}^{n} \chi_\ell = 2$ on Ω_n, so for any $\varphi \in H^1_0(\Omega_n)$ we have

$$\begin{aligned}
\langle f, \varphi \rangle_{\Omega_n} &\simeq \sum_{\ell=-n-1}^{n} \langle f, \chi_\ell \varphi \rangle_{U_\ell^*} \\
&\lesssim \sum_{\ell=-n-1}^{n} \|f\|_{H^{-1}(U_\ell^*)} \|\chi_\ell \varphi\|_{H^1} \\
&\lesssim \|f\|_{H^{-1}_{\mathrm{uloc}}} \sum_{\ell=-n-1}^{n} \|\chi_\ell \varphi\|_{H^1} \\
&\lesssim \|f\|_{H^{-1}_{\mathrm{uloc}}} \sum_{\ell=-n-1}^{n} \|\varphi\|_{H^1(U_\ell^*)} \\
&\lesssim \|f\|_{H^{-1}_{\mathrm{uloc}}} (2n+2)^{1/2} \left(\sum_{\ell=-n-1}^{n} \|\varphi\|^2_{H^1(U_\ell^*)} \right)^{1/2},
\end{aligned}$$

where we used Lemma 2.3.1. Now, bound $2n+2$ by $4n$ and notice that the last sum is equivalent to $\|\varphi\|_{H^1}$ to complete the proof. □

2.4.3 Proof of Lemma 2.3.4

Set $(E_{n,k})_{n,k}$ a family of positive real numbers, indexed by the couples $(n,k) \in \mathbf{N}^2$ such that $1 \leq k \leq n$, non-decreasing according to k for a fixed n, obeying

$$\forall 1 \leq k \leq n, \quad E_{n,k} \leq C\left(E_{n,k+1} - E_{n,k} + F(k+1)\right) \tag{2.38}$$

and satisfying

$$\forall n, \quad E_{n,n} \leq CFn, \tag{2.39}$$

where F is a constant playing the role of $\|\boldsymbol{f}\|^2_{H^{-1}_{\mathrm{uloc}}}$. Let us show that there exists $\alpha > 0$ and $k_0 \in \mathbf{N}^*$ such that

$$\forall k, n, \quad k_0 \leq k \leq n \quad \Longrightarrow \quad E_k \leq \alpha F k. \tag{2.40}$$

By (2.39) we already know that (2.40) is satisfied for any $n \in \mathbf{N}^*$ and $k = n$, with $\alpha = C$. For a fixed n, let k_0 be the greatest index such that (2.40) is not satisfied, meaning

$$E_{n,k_0} > \alpha F k_0. \tag{2.41}$$

Therefore, plugging (2.41) in (2.38) and using the definition of k_0 provides

$$\alpha(1+C)Fk_0 \leq CF(\alpha+1)(k_0+1),$$

which is equivalent to

$$\frac{k_0}{k_0+1} \leq \frac{C}{C+1}\left(1 + \frac{1}{\alpha}\right) =: C_\alpha.$$

Up to the choice of a greater α, we can assume that $C_\alpha < 1$. This implies that $k_0 \leq \frac{C_\alpha}{1-C_\alpha}$, independently of n and F. Therefore, we conclude that for any k, n such that $k_0 \leq k \leq n$ we

have

$$E_k \leq \alpha F k.$$

□

2.5 Well-posedness of the Stokes-transport equation for Lebesgue data

2.5.1 Well-posedness in L^q for $q > d$

The Stokes-transport equation well-posedness result on bounded domains for $\rho_0 \in L^\infty$ can be adapted for Lebesgue data $\rho_0 \in L^q(\Omega)$ with finite exponents $q \in (d, \infty)$, as follows.

Theorem 2.5.1. *Let Ω be either $\mathbb{T}\times(0,1)$ or simply connected compact subdomain of $\mathbf{R}^d, d = 2, 3$ of class C^2. Let $q \in (d, \infty)$. For any $\rho_0 \in L^q(\Omega)$ the ST equation has a unique weak solution $(\rho, \boldsymbol{u})$ in the class*

$$L^\infty(\mathbf{R}_+; L^q(\Omega)) \times L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega)). \tag{2.42}$$

In [MS22] Mecherbet & Sueur show that Stokes-transport is well-posed in the whole space $\mathbf{R}^3$ for data $\rho_0 \in L^q(\mathbf{R}^3)$ with $q \geq 3$, with similar regularity than (2.42) when $q > 3$ (the critical exponent $q = 3$ requires more attention). They raise whether this result holds in bounded domain, which is therefore the case. Note that the critical exponent $q > d$ is consistent whether the domain is bounded or not.

We adjust the proof of Theorem 2.1.1. We choose $q > d$ so that $W^{2,q}(\Omega)$ embeds continuously in $W^{1,\infty}(\Omega)$, hence for any solution $\boldsymbol{u}$ to the Stokes equation with source term $\rho \in L^q(\Omega)$ we have

$$\|\boldsymbol{u}\|_{W^{1,\infty}} \lesssim \|\boldsymbol{u}\|_{W^{2,q}} \lesssim \|\rho\|_{L^q}.$$

Since solutions to the transport equation have constant L^q norm, this inequality ensures that the velocity fields considered are still Lipschitz and that the characteristic method applies just as in the previous cases.

We can define a sequence $(\rho^N)_N$ and $(\boldsymbol{u}^N)_N$ as follows. Let us define $\rho^0 : t \in \mathbf{R}_+ \mapsto \rho_0 \in L^q(\Omega)$. Now iterate as follows. If $\rho^N \in L^\infty(\mathbf{R}_+; L^q(\Omega))$ there exists a unique $\boldsymbol{u}^N \in L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega))$ satisfying for all time the Stokes system (2.1). If $\boldsymbol{u}^N \in L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega))$, the characteristics method provides a unique solution $\rho^{N+1} \in L^\infty(\mathbf{R}_+; L^q(\Omega))$.

Now we need to adapt the stability estimate (2.11) in order to show that the sequences are of Cauchy type in convenient spaces. The difference lies in the treatment of a generalised Hölder estimate considering within the proof, requiring to bound ρ_0 in L^q for q finite instead of L^∞, which modify the space in which the other quantities are estimated. Then, it remains to check thanks to the convenient embeding estimates that the stability estimate can be closed in view of the contraction argument performed in the well-posedness proof.

Lemma 2.5.2. *Let $\rho_0 \in L^q(\Omega), q > d$, $\boldsymbol{u}_i \in L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega))$ given and $\rho_i \in L^\infty(\mathbf{R}_+; L^q(\Omega))$ the associated solutions to the transport equation with initial datum ρ_0. Then for any $1 < r, s < \infty$ satisfying $\frac{1}{q} + \frac{1}{r} = \frac{1}{s}$ we have*

$$\|\rho_1 - \rho_2\|_{L^\infty_T W^{-1,s}} \lesssim \|\rho_0\|_{L^q} e^{CT\|\nabla \boldsymbol{u}_1\|_{L^\infty_T W^{1,\infty}}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty_T L^r}. \tag{2.43}$$

Proof. As in Proposition 2.2.6, we get for any $\varphi \in C^\infty_c(\Omega)$ the following equality in view of an

estimate in a dual Sobolev space, for any $t \geq 0$,

$$\int_\Omega (\rho_1(t) - \rho_2(t))\varphi = \int_\Omega \rho_0(\boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)) \cdot \int_0^1 \nabla\varphi(\boldsymbol{X}^\lambda(t))\,\mathrm{d}\lambda.$$

Now we estimate this latter expression by the generalised Hölder inequality,

$$\int_\Omega (\rho_1(t) - \rho_2(t))\varphi \lesssim \|\rho_0\|_{L^q} \|\boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)\|_{L^r} \|\nabla\varphi\|_{L^{s'}}, \tag{2.44}$$

where the exponents satisfy

$$1 < q, r, s' < \infty, \qquad \frac{1}{q} + \frac{1}{r} + \frac{1}{s'} = 1 \quad \Leftrightarrow \quad \frac{1}{q} + \frac{1}{r} = \frac{1}{s}.$$

For any such set of exponent, we consider the supremum on the test function $\varphi \in W_0^{1,s'}(\Omega)$ and get

$$\forall T > 0, \quad \|\rho_1 - \rho_2\|_{L^\infty(0,T;W^{-1,s})} \lesssim \|\rho_0\|_{L^q} \|\boldsymbol{X}_1 - \boldsymbol{X}_2\|_{L^\infty(0,T;L^r)}. \tag{2.45}$$

Besides, let us recall that for any $r \in (1,\infty)$ we showed in Lemma 2.2.4 that

$$\forall T \geq 0, \quad \|\boldsymbol{X}_1 - \boldsymbol{X}_2\|_{L^\infty(0,T;L^r)} \lesssim T e^{CT\|\nabla \boldsymbol{u}_1\|_{L^\infty(0,T;W^{1,\infty})}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^\infty(0,T;L^r)},$$

which concludes the proof. □

In order to close estimate (2.45) we need the following inequality for the Stokes system,

$$\|\boldsymbol{u}\|_{L^r} \lesssim \|\rho\|_{W^{-1,s}}.$$

This is achieved by considering the embedding $W^{1,s} \hookrightarrow L^r$ occurring if

$$1 - \frac{d}{s} > -\frac{d}{r} \quad \Leftrightarrow \quad \frac{1}{d} + \frac{1}{r} > \frac{1}{s}.$$

Whence, first by embedding, then by Stokes solution estimate, we get

$$\|\boldsymbol{u}\|_{L^r} \lesssim \|\boldsymbol{u}\|_{W^{1,s}} \lesssim \|\rho\|_{W^{-1,s}}. \tag{2.46}$$

Remark 2.5.3. *If we consider $s = r$ then $q = \infty$ remains the only option in (2.44), and we would fall back in the previous proof.*

It remains to check that all the exponents constraints are compatible in view of obtaining (2.43). We need

$$q > d, \quad \frac{1}{d} + \frac{1}{r} > \frac{1}{s}, \quad \frac{1}{q} + \frac{1}{r} = \frac{1}{s},$$

which is actually satisfied for any $q > 2$ with $r \in (1,\infty)$ since

$$\frac{1}{d} > \frac{1}{q} \quad \Rightarrow \quad \frac{1}{d} + \frac{1}{r} > \frac{1}{q} + \frac{1}{r} = \frac{1}{s},$$

where s deduces from q, r.

Now from (2.43) and (2.46) we can deduce that for all $N \in \mathbf{N}$ density difference satisfies

$$\begin{aligned}\|\rho^{N+1} - \rho^N\|_{L^\infty(0,T;W^{-1,s})} &\lesssim T\|\rho_0\|_{L^q} e^{CT\|\nabla \boldsymbol{u}^N\|_{L^\infty(0,T;L^\infty)}} \|\boldsymbol{u}^N - \boldsymbol{u}^{N-1}\|_{L^\infty(0,T;L^r)} \\ &\lesssim T\|\rho_0\|_{L^q} e^{CT\|\rho_0\|_{L^q}} \|\rho^N - \rho^{N-1}\|_{L^\infty(0,T;W^{-1,s})}.\end{aligned}$$

Hence for $T(\|\rho_0\|_{L^q}) > 0$, $(\rho^N)_N$ is a Cauchy sequence in $L^\infty(0,T;W^{-1,s}(\Omega))$, therefore $(\boldsymbol{u}^N)_N$ is a Cauchy sequence in $L^\infty(0,T;W^{1,s})$, and in particular $(\boldsymbol{u}^N)_N, (\nabla \boldsymbol{u}^N)_N$ are of Cauchy type in $L^s((0,T)\times\Omega)$. We infer there exists some $\boldsymbol{u}$ such that $\boldsymbol{u} \in L^s((0,T)\times\Omega)$ with $\boldsymbol{u} \in L^s((0,T)\times\Omega))$ is the limit of $(\boldsymbol{u}^N)_N$ in $W^{1,s}(\Omega)$. Besides, since $\|\rho^N(t)\|_{L^q} = \|\rho_0\|_{L^q}$ for all N and t, we have

$$\|\rho^N\|_{L^q((0,T)\times\Omega)} \leq T^{1/q}\|\rho_0\|_{L^q}.$$

In particular $(\rho^N)_N$ converges weakly in L^q, up to extractation, toward some $\rho \in L^q((0,T)\times\Omega)$. Hence by strong and weak convergence of $(\boldsymbol{u}^N)_N$ and $(\rho^N)_N$ respectively, we can pass to the limit in the intermediate problem weak formulation of the transport equation, *i.e.* for any $\varphi \in C_c^\infty([0,T)\times\Omega)$,

$$\int_0^T \int_\Omega \rho^{N+1}(t,\boldsymbol{x})(\partial_t\varphi(t,\boldsymbol{x}) + \boldsymbol{u}^N(t,\boldsymbol{x})\cdot\nabla\varphi(t,\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = -\int_\Omega \rho_0(\boldsymbol{x})\varphi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

In the same fashion the limit satisfies the Stokes weak formulation. Hence $(\rho,\boldsymbol{u})$ is a local weak solution. Uniqueness deduces from the stability estimate (2.43). Moreover, since the existence time T depends only on the L^q norm of the initial datum, and since this L^q norm is preserved along time for the density, then we can extend the solution for all time. By (2.5) the velocity field satisfies

$$\|\boldsymbol{u}\|_{L^\infty(\mathbf{R}_+;W^{1,\infty})} \lesssim \|\rho_0\|_{L^q},$$

which concludes the proof of Theorem 2.5.1.

2.5.2 Existence for lower Lebesgue exponents

We obtain existence of solutions for initial data in Lebesgue spaces for a range of exponents lower than the dimension.

Proposition 2.5.4. *Let $\Omega \subset \mathbf{R}^d, d = 2,3$ of class C^2 and let $q > 2d/(d+2)$. Then for any $\rho_0 \in L^q(\Omega)$ and $T > 0$ there exists at least one weak solution to the Stokes-transport system on $[0,T)$. Since $T > 0$ is arbitrary, there exists global solutions. These solutions belong to the following functional space*

$$(\rho,\boldsymbol{u}) \in L^\infty(\mathbf{R}_+;L^q(\Omega)) \times L^\infty(\mathbf{R}_+;W^{2,q}(\Omega)).$$

In the former case, $\rho \in L^\infty$ ensured $\boldsymbol{u}$ to be Lipschitz, ensuring the proper definition of the flow map $\boldsymbol{X}$ and in particular the uniqueness of the solution. When $d \geq q > 2d/(d+2)$, the velocity field belongs a priori to $W^{2,q}(\Omega)$ which does not embed in $W^{1,\infty}$ anymore. Note that for $d = 2$ the result holds for any $q > 1$. In dimension $d = 3$, we have a threshold $q > 6/5$. Note that the present result is also consistent with [Inv23, Corollary 2.3] in which Inversi details the nature of the renormalised solution to the Stokes-transport system in $\mathbf{R}^3$ for data in $L^q(\mathbf{R}^3)$, which are weak (or distributional) solutions if $q > 6/5$. Addressing the case $q < 2d/(d+2)$ might require an other approach to overcome this difficulty.

Proof. Let us consider a sequence of initial data $\rho_{0,n} \in L^\infty(\Omega)$ converging in L^q toward ρ_0. For

instance, the sequence defined by $\rho_{0,n} = \rho_0 \mathbf{1}_{\{|\rho_0|<n\}}$ suits this property. By Theorem 2.1.1 we know that for all $n \in \mathbf{N}$ there exists a global solution to the ST problem with initial datum $\rho_{0,n}$, such that

$$(\rho_n, \boldsymbol{u}_n) \in L^\infty(\mathbf{R}_+; L^\infty(\Omega)) \times L^\infty(\mathbf{R}_+; W^{1,\infty}(\Omega)).$$

In particular, the sequence $(\rho_n)_n$ is uniformly bounded as follows

$$\|\rho_n\|_{L^q([0,T]\times\Omega)} \leq T^{1/q}\|\rho_{0,n}\|_{L^q} \leq T^{1/q}\|\rho_0\|_{L^q}.$$

Hence the exists an extraction such that ρ_n converges weakly in L^q toward some $\rho \in L^q([0,T]\times\Omega)$. Now a strong convergence of $(\boldsymbol{u}_n)_n$ would be enough to pass to the limit in the weak formulation of the problem, in particular, for any $\varphi \in C_c^\infty([0,T)\times\Omega)$, the following

$$\int\int_{[0,T)\times\Omega} \rho_n(t,\boldsymbol{x})(\partial_t\varphi(t,\boldsymbol{x}) + \boldsymbol{u}_n(t,\boldsymbol{x})\cdot\nabla\varphi(t,\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = -\int_\Omega \rho_0(\boldsymbol{x})\varphi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

We know that

$$\|\boldsymbol{u}_n\|_{W^{2,q}} \lesssim \|\rho_{0,n}\|_{L^q} \leq \|\rho_0\|_{L^q}.$$

An extraction provides a weak convergence of $(\boldsymbol{u}_n)_n$ in $L^q(0,T;W^{2,q}(\Omega))$. In particular the Kondrachov embedding theorem ensures that if

$$2 - \frac{d}{q} > -\frac{d}{q'}$$

then there is a compact embedding from $W^{2,q}(\Omega)$ into $L^{q'}(\Omega)$. Reordering the latter inequality we get

$$q > \frac{2d}{d+2}.$$

This condition is sufficient to ensure the existence of an extraction such that $\boldsymbol{u}_n$ converges strongly in $L^{q'}([0,T]\times\Omega)$ toward some limit $\boldsymbol{u} \in L^q([0,T]\times\Omega)$. In particular by divergence free of the velocity field, we preserve the L^q norm of the datum, and we have

$$(\rho, \boldsymbol{u}) \in L^\infty(0,T;L^q(\Omega)) \times L^\infty(0,T;W^{2,q}(\Omega)).$$

Since T is arbitrary, and since the uniform norm of ρ is preserved along time, one can extend such a solution for any time. □

2.6 Well-posedness of the Stokes-transport equation for Sobolev data

The aim of this section is to prove the well-posedness of the Stokes-transport on the domain of interest of the present paper, namely $\Omega = \mathbb{T}\times(0,1)$. The proof is also valid on any regular enough bounded domain of $\mathbf{R}^d$ with $d = 2$ or 3. The author included this Section in [DGL23].

Theorem 2.6.1. *Let Ω satisfy either*

1. $\Omega = \mathbb{T}\times(0,1)$;
2. *Ω is a simply connected compact subdomain of $\mathbf{R}^d, d = 2,3$ asumed C^{m+2}.*

Let $m > 3$ and $\rho_0 \in H^m(\Omega)$. The system

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho &= 0 \\ -\Delta \boldsymbol{u} + \nabla p &= -\rho \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} &= 0 \\ \boldsymbol{u}|_{\partial\Omega} &= \boldsymbol{0} \\ \rho|_{t=0} &= 0, \end{cases} \tag{2.47}$$

has a unique global solution in the following class of regularity

$$(\rho, \boldsymbol{u}) \in C(\mathbf{R}_+; H^m(\Omega)) \times C(\mathbf{R}_+; H^{m+2}(\Omega)).$$

Moreover, the solution obeys the following energy estimate

$$\|\rho(t)\|_{H^m} \leq \|\rho_0\|_{H^m} \exp\left(C \int_0^t (\|\nabla \boldsymbol{u}(s)\|_{L^\infty} + \|\nabla \rho(s)\|_{L^\infty}) \, \mathrm{d}s \right). \tag{2.48}$$

This result requires rather classical techniques. It also relies on the previous Sections, constituting [Leb22], including in particular the well-posedness in the weak sense of the system (2.47).

A priori estimate. Formally, the energy estimate for any derivative of order m writes

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\partial^m \rho\|_{L^2}^2 = - \int_\Omega [\partial^m, \boldsymbol{u} \cdot \nabla] \rho \partial^m \rho,$$

due to the divergence free condition satisfied by $\boldsymbol{u}$. We apply the tame estimate (3.20), together with the continuous Sobolev embedding of $H^m(\Omega)$ in $L^\infty(\Omega)$ and the Stokes equation regularization estimate to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\partial^m \rho\|_{L^2}^2 \lesssim (\|\nabla \boldsymbol{u}\|_{L^\infty} + \|\nabla \rho\|_{L^\infty}) \|\rho\|_{H^m}^2.$$

One therefore obtains the same inequality with the complete H^m norm on the left-hand-side, and the estimate (2.48) follows. This energy estimate tells us that ρ remains in $H^m(\Omega)$ as long as $\|\nabla \boldsymbol{u}\|_{L^\infty}$ and $\|\nabla \rho\|_{L^\infty}$ are integrable in time. Regarding the properties we know from [Leb22] about the solutions of this equation it is enough to prove that the solution exists globally, and is unique. Let us recall from [Gal11, Theorem IV.6.1] and [Leb22, Section 2.1] that the source term and the solution of the Stokes equation satisfy for all time

$$\|\boldsymbol{u}\|_{H^m} \lesssim \|\rho\|_{H^{m-2}}, \qquad \|\boldsymbol{u}\|_{W^{1,\infty}} \lesssim \|\rho\|_{L^\infty}. \tag{2.49}$$

Also, the uniform norm of ρ is constant by incompressibility condition. We also observe that

$$\|\nabla \rho\|_{L^\infty} \leq \|\nabla \rho_0\|_{L^\infty} \exp\left(C \int_0^t \|\nabla \boldsymbol{u}(s)\|_{L^\infty} \, \mathrm{d}s \right) \leq \|\nabla \rho_0\|_{L^\infty} \exp(C \|\rho_0\|_{L^\infty} t). \tag{2.50}$$

All these consideration put together lead to

$$\|\rho\|_{H^m} \leq \|\rho_0\|_{H^m} \exp\left(C\|\rho_0\|_{L^\infty} t + \frac{\|\nabla \rho_0\|_{L^\infty}}{\|\rho_0\|_{L^\infty}} (\exp(C\|\rho_0\|_{L^\infty} t) - 1) \right).$$

This suggests that it is enough for ρ_0 to belong to $W^{1,\infty}(\Omega)$ for the solution to exist globally in time, which is satisfied here since $\rho_0 \in H^m(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for m large enough to ensure this embedding.

Proof. An iterative scheme allows to formalise the previous considerations. Let $\rho^0 : t \mapsto \rho_0$ which belongs to $C(\mathbf{R}_+; H^m(\Omega))$. If ρ^N belongs to $C(\mathbf{R}_+; H^m(\Omega))$, which is true for $N = 0$, we know that the Stokes system

$$\begin{cases} -\Delta \boldsymbol{u}^N + \nabla p^N &= -\rho^N \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u}^N &= 0 \\ \boldsymbol{u}^N|_{\partial\Omega} &= \mathbf{0} \end{cases}$$

admits for any time a unique solution $\boldsymbol{u}^N(t) \in H^{m+2}(\Omega)$ obeying inequalities (2.49). By linearity of the problem, $\boldsymbol{u}^N$ in $H^{m+2}(\Omega)$ inherits the continuity of ρ^N in $H^m(\Omega)$. Then since $\boldsymbol{u}^N$ belongs in particular to $C(\mathbf{R}_+; H^{m+2}(\Omega))$, the transport equation

$$\begin{cases} \partial_t \rho^{N+1} + \boldsymbol{u}^N \cdot \nabla \rho^{N+1} &= 0 \\ \rho^{N+1}|_{t=0} &= \rho_0, \end{cases}$$

has a unique strong solution $\rho^{N+1} \in C(\mathbf{R}_+; H^k(\Omega))$. This concludes the definition of a sequences $(\rho^N)_N$ and $(\boldsymbol{u}^N)_N$. We thereafter show that for any $T > 0$ the sequence $(\rho^N)_N$ is bounded in $L^\infty(0,T; H^m(\Omega))$ and equicontinuous in $C(0,T; H^{m-1}(\Omega))$, so it converges in $C(0,T; H^{m-1}(\Omega)$ to a solution of the original system up to an extraction. Since this is true for any $T > 0$ and by uniqueness of the weak solution ensured by [Leb22, Theorem 1.1 & 1.2], we get the well-posedness of the system and Proposition is proven.

Boundedness. Let us show that we have for any $N \in \mathbf{N}$,

$$\|\rho^N\|_{H^m} \leq \|\rho_0\|_{H^m} \exp\left(C\|\rho_0\|_{L^\infty} t + \frac{\|\nabla \rho_0\|_{L^\infty}}{\|\rho_0\|_{L^\infty}} (\exp(C\|\rho_0\|_{L^\infty} t) - 1) \right) =: B_{\rho_0}(t). \tag{2.51}$$

This inequality is immediately satisfied for $N = 0$ since ρ^0 is constant in time and equal to ρ_0. Let $N \in \mathbf{N}$ such that (2.51) is satisfied. Then the tame estimate (3.20) provides here

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\rho^{N+1}(t)\|_{H^m}^2 \lesssim \|\nabla \boldsymbol{u}^N\|_{L^\infty} \|\rho^{N+1}\|_{H^m}^2 + \|\nabla \rho^{N+1}\|_{L^\infty} \|\boldsymbol{u}^N\|_{H^m} \|\rho^{N+1}\|_{H^m}.$$

The considerations (2.49) and (2.50) applied to ρ^N, ρ^{N+1} and $\boldsymbol{u}^N$ conduct here to

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\rho^{N+1}\|_{H^m} \lesssim \|\rho_0\|_{L^\infty} \|\rho^{N+1}\|_{H^m} + \|\nabla \rho_0\|_{L^\infty} \exp(C\|\rho_0\|_{L^\infty} t) \|\rho^{N+1}\|_{H^m}.$$

From here, we use Grönwall lemma to estimate $\|\rho^{N+1}\|_{H^m}$,

$$\|\rho^{N+1}\|_{H^m} \leq \exp(C\|\rho_0\|_{L^\infty} t) \left(\|\rho_0\|_{H^m} + C\|\nabla \rho_0\|_{L^\infty} \int_0^t \|\rho^N(s)\|_{H^m} \, \mathrm{d}s \right), \tag{2.52}$$

then according to the assumption on ρ^N, observe

$$\begin{aligned}
C\|\nabla\rho_0\|_{L^\infty}&\int_0^t \|\rho^N(s)\|_{H^m}\,\mathrm{d}s\\
&\leq C\|\rho_0\|_{H^m}\|\nabla\rho_0\|_{L^\infty}\int_0^t \exp\left(C\|\rho_0\|_{L^\infty}s+\frac{\|\nabla\rho_0\|_{L^\infty}}{\|\rho_0\|_{L^\infty}}\left(e^{C\|\rho_0\|_{L^\infty}s}-1\right)\right)\mathrm{d}s\\
&\leq C\|\rho_0\|_{H^m}\|\nabla\rho_0\|_{L^\infty}\int_0^{e^{C\|\rho_0\|_{L^\infty}t}-1}\exp\left(\frac{\|\nabla\rho_0\|_{L^\infty}}{\|\rho_0\|_{L^\infty}}r\right)\frac{\mathrm{d}r}{C\|\rho_0\|_{L^\infty}}\\
&= \|\rho_0\|_{H^m}\left(\exp\left(\frac{\|\nabla\rho_0\|_{L^\infty}}{\|\rho_0\|_{L^\infty}}\left(\exp(C\|\rho_0\|_{L^\infty}t)-1\right)\right)-1\right).
\end{aligned}$$

The latter bound substituted in (2.52) yields exactly the result (2.51). Therefore, for any $T>0$ the sequence $(\rho^N)_N$ is uniformly bounded in $L^\infty(0,T;H^m(\Omega))$.

Equicontinuity. We find a uniform bound on $(\partial_t\rho^N)_N$ in $H^{m-1}(\Omega)$ to show the equicontinuity of the sequence in $C(0,T;H^{m-1}(\Omega))$. This bound, uniform in $N\in\mathbf{N}$ and $t\in[0,T]$ is obtained thanks to the tame estimate, the bounds (2.49) and the uniform bound (2.51) on ρ^N,

$$\begin{aligned}
\|\partial_t\rho^N\|_{H^{m-1}} &= \|\boldsymbol{u}^{N-1}\cdot\nabla\rho^N\|_{H^{m-1}}\\
&\lesssim \|\boldsymbol{u}^{N-1}\|_{L^\infty}\|\nabla\rho^N\|_{H^{m-1}}+\|\rho^N\|_{L^\infty}\|\boldsymbol{u}^{N-1}\|_{H^{m-1}}\\
&\lesssim \|\rho_0\|_{L^\infty}\|\rho^N\|_{H^m}+\|\rho_0\|_{L^\infty}\|\rho^{N-1}\|_{H^m}\\
&\lesssim \|\rho_0\|_{L^\infty}\|\rho_0\|_{H^m}\exp\left(C\|\rho_0\|_{L^\infty}t+\frac{\|\nabla\rho_0\|_{L^\infty}}{\|\rho_0\|_{L^\infty}}\left(\exp(C\rho_0\|_{L^\infty}t)-1\right)\right).
\end{aligned}$$

Regularity. Let us show that the limit ρ belongs to $L^\infty(0,T;H^m(\Omega))$. For any $t\in[0,T]$, $(\rho^N(t))_N$ is uniformly bounded in $H^m(\Omega)$ with respect to N and t, hence according to Banach–Alaoglu theorem, for any t the sequence is weakly compact in $H^m(\Omega)$. Hence up to an extraction, $\rho^N(t)$ converges weakly toward a $\bar\rho(t)\in H^m(\Omega)$, and this limit satisfies

$$\|\bar\rho(t)\|_{H^m}\leq\liminf_N\|\rho^N(t)\|_{H^m},\tag{2.53}$$

where the right-hand-side is uniformly bounded thanks to (2.51). As ρ^N already converges weakly in $H^m(\Omega)$, we can identify $\bar\rho$ and ρ, which then belongs to $L^\infty(0,T;H^m(\Omega))$. Finally, to reach the regularity $C(0,T;H^m(\Omega))$. The [BF12, Lemma II.5.6] tells us that since in particular $\rho\in L^\infty(0,T;H^m(\Omega))\cap C^0_{\mathrm{w}}(0,T;H^{m-1}(\Omega))$ then $\rho\in C^0_{\mathrm{w}}(0,T;H^m(\Omega))$. Hence it is enough to show that $t\mapsto\|\rho(t)\|_{H^m}$ is continuous to prove the strong continuity of ρ in $H^m(\Omega)$. By weak continuity, we have

$$\|\rho_0\|_{H^m}\leq\liminf_{t\searrow0}\|\rho(t)\|_{H^m}.$$

Also we have by weak convergence

$$\|\rho(t)\|_{H^m}\leq\liminf_{N\to\infty}\|\rho^N(t)\|_{H^m}\leq B_{\rho_0}(t),$$

which proves by clamping that $t\mapsto\|\rho(t)\|_{H^m}$ is continuous at $t=0$. This can be performed for any $t\in[0,T]$, hence the continuity. Finally, $\boldsymbol{u}\in C^0(0,T;H^{m+2}(\Omega))$ by (2.49) and linearity of the Stokes equation. □

2.7 The hydrostatic Stokes-transport equation

Let us recall that the Stokes equations models the steady state of a viscous and compressible fluid submitted to some external persistent force, described by its velocity $\boldsymbol{u}$ and its pressure p. When considering fluids to the scale of an ocean or of the atmosphere, is it common to perform the following approximation, so called *hydrostatic approximation*, consisting in neglegting the vertical acceleration of the fluid,

$$\partial_z p = -\rho.$$

The hydrostatic approximation is relevant for fluid occupying a thin domain, with typical width smaller than typical length. For instance, an ocean has an height average of order $\sim$ 1 - 10 km and horizontal scale of order $\sim$ 1000 - 10 000 km. This assumption in generally considered in Boussinesq systems frameworks and more complete oceanography models, [LTW92a; LTW92b; CT07], but Stokes equation alone has also been studied in [Zia95]. In the present Section, we study the well-posedness of the coupling between the transport equation and this modified Stokes system on $\Omega = \mathbf{T} \times (0,1)$, as a reduced model of the previous systems mentionned. The equation is

$$\left\{ \begin{array}{rcl} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho &=& 0 \\ -\Delta u_1 + \partial_x p &=& 0 \\ \partial_z p &=& -\rho \\ \operatorname{div} \boldsymbol{u} &=& 0 \\ u_2 = \partial_z u_1 &=& 0 \text{ on } \partial\Omega \\ \langle u_1 \rangle_\Omega &=& 0 \\ \rho|_{t=0} &=& \rho_0. \end{array} \right. \tag{2.54}$$

The extra boundary and average conditions on $\boldsymbol{u} = (u_1, u_2)$ are discussed in the following Subsection 2.7.1, in the mean course of a self-contained study of the Stokes system with hydrostatic approximation. We demonstrate the well-posedness of (2.54) in Subsection 2.7.2, as stated in the following result.

Theorem 2.7.1. *Let $m \geq 3$. For any $\rho_0 \in H^m(\Omega)$ there exists a time $T(\|\rho_0\|_{H^3}) > 0$ such that the hydrostatic Stokes-transport system (2.54) is locally well-posed, with solution*

$$(\rho, \boldsymbol{u}) \in C(0,T; H^m(\Omega)) \times C(0,T; H^m(\Omega)),$$

obeying moreover

$$\forall t \in [0,T), \quad \|\rho(t)\|_{H^m} \leq \frac{\|\rho_0\|_{H^m}}{1 - C\|\rho_0\|_{H^3} t}.$$

2.7.1 A hydrostatic Stokes system

We consider the Stokes system endowed with the hydrostatic approximation, on the spatial domain $\Omega = \mathbf{T} \times (0,1)$,

$$\left\{ \begin{array}{rcl} -\Delta u_1 + \partial_x p &=& 0 \\ \partial_z p &=& -\rho \\ \operatorname{div} \boldsymbol{u} &=& 0, \end{array} \right. \tag{2.55}$$

which requires extra boundary conditions on $\boldsymbol{u} = (u_1, u_2)$, which are given and discussed further in (2.56). It is important to stress how the classical Stokes system (2.3) and the present one differ. In the former, the velocity $\boldsymbol{u}$ depends globally on the forcing datum and benefits from the elliptic regularisation effect. In the latter, only the horizontal velocity satisfies a Laplace equation, with a datum already depending on derivatives of ρ, as we will see further. The vertical velocity is

recovered thanks to the divergence free assumption only, which implies a loss of regularity with respect to the horizontal one. In the end we obtain that the velocity and density have basically the same regularity, and only local well-posedness of the system is proven here. To simplify the analysis, we endow this system with the following slip boundary condition, together with a condition on the average of Ω of the horizontal compound, which we justify in the course of the proof,

$$u_2 = 0, \qquad \partial_{\boldsymbol{n}} u_1 = 0, \qquad \text{on } \partial\Omega, \qquad \langle u \rangle_\Omega = 0. \tag{2.56}$$

In view of the following result we define the following anisotropic Sobolev space having horizontal derivatives of order less than $k \in \mathbf{N}$ in H^m, for given $m, k \in \mathbf{N}$, as the subset of functions $f \in H^m(\Omega)$ satisfying

$$\|f\|_{H^k_x H^m} := \sum_{\ell=0}^{k} \|\partial_x^\ell f\|_{H^m} < \infty.$$

Lemma 2.7.2. *Let us set $m \in \mathbf{N}$ and $\rho \in H^m(\Omega)$ such that $\partial_x^2 \rho \in H^m(\Omega)$. Then the system (2.55)-(2.56) has a unique solution $\boldsymbol{u}$, which obeys*

$$\|\boldsymbol{u}\|_{H^{m+2}} \lesssim \|\rho\|_{H^2_x H^m} \tag{2.57}$$

To prove this Lemma we first proceed to an a priori analysis. Let us assume having a triplet $(\rho, \boldsymbol{u}, p)$ satisfying (2.55)-(2.56), as regular as necessary to proceed to the following computations. Let us integrate the system (2.55) with respect to $z \in (0,1)$, to get

$$\begin{cases} -\partial_x^2 U_1(x) - [\partial_z u_1(x,z')]^1_{z'=0} + \partial_x P(x) = 0 \\ \qquad\qquad\qquad\qquad u_2(x,1) - u_2(x,0) = -\partial_x U_1(x), \end{cases} \tag{2.58}$$

where we set

$$\forall x \in \mathbf{T}, \quad U_1(x) := \int_0^1 u_1(x,\zeta)\,\mathrm{d}\zeta, \quad P(x) := \int_0^1 p(x,\zeta)\,\mathrm{d}\zeta.$$

Due to the boundary conditions (2.56) we infer from (2.58) that

$$\forall x \in \mathbf{T}, \quad \partial_x U_1(x) = \partial_x^2 U_1(x) = \partial_x P(x) = 0.$$

In particular P does not depend on x and we deduce from the hydrostatic assumption that, up to an additive constant,

$$p(x,z) = -\int_0^z \rho(x,\zeta)\,\mathrm{d}\zeta.$$

Whence the following explicit source term for the Neumann problem satisfied by u_1,

$$\begin{cases} \Delta u_1(x,z) &= \partial_x \int_0^z \rho(x,\zeta)\,\mathrm{d}\zeta \\ \partial_z u_1|_{\partial\Omega} &= 0 \\ \langle u_1 \rangle_\Omega &= 0, \end{cases} \tag{2.59}$$

where the solution u_1 is defined up to an additive constant, so we will choose the average condition $\langle u_1 \rangle_\Omega := \int_\Omega u_1 = 0$. We denote $H^1(\Omega)\backslash\mathbf{R}$ the set of H^1 functions satisfying this condition. Classically, the solution is defined as the unique element $u_1 \in H^1(\Omega)/\mathbf{R}$ satisfying the variational formulation of the problem, for a generic source term f,

$$\forall \varphi \in H^1(\Omega)/\mathbf{R}, \quad \int_\Omega \nabla u_1 \cdot \nabla \varphi = \int_\Omega f\varphi,$$

since Poincaré–Wirtinger inequality and the zero average condition ensure the left bilinear form to be coercive on this functional space. Moreover we obtain the estimate

$$\|u_1\|_{H^1} \lesssim \|\partial_x \rho\|_{H^{-1}} \lesssim \|\rho\|_{L^2}.$$

The solution also obeys the elliptic regularisation, for any $m \in \mathbf{N}$, provided $\partial_x \rho \in H^m(\Omega)$, see [Bré11] for classical results on this topic,

$$\|u_1\|_{H^{m+2}} \lesssim \|\partial_x \rho\|_{H^m}.$$

Notice that the Neumann problem (2.59) is also satisfied for any horizontal derivative of the pair $(\partial_x^k u_1, \partial_x^{k+1}\rho)$. Together with the previous estimates, we have for any $k, m \in \mathbf{N}$,

$$\|\partial_x^k u_1\|_{H^1} \lesssim \|\partial_x^k \rho\|_{L^2}, \qquad \|\partial_x^k u_1\|_{H^{m+2}} \lesssim \|\partial_x^{k+1}\rho\|_{H^m}.$$

Now, given u_1, we deduce the following estimate on u_2, thanks to the divergence-free condition,

$$\begin{aligned} \|\partial_z u_2\|_{L^2} &= \|\partial_x u_1\|_{L^2} \lesssim \|\rho\|_{L^2} \\ \|\partial_z u_2\|_{H^1} &= \|\partial_x u_1\|_{H^1} \lesssim \|\partial_x \rho\|_{L^2}. \end{aligned}$$

Integrating the divergence equation with respect to z and recalling the boundary condition (2.56), we get

$$u_2(x,z) = -\int_0^z \partial_x u_1(x,\zeta)\,\mathrm{d}\zeta.$$

We observe that horizontal derivatives of u_2 do not share the same regularisation effect as u_1, and require more horizontal regularity on the datum ρ to ensure

$$\|\partial_x^2 u_2\|_{L^2} \lesssim \|\partial_x^3 u_1\|_{L^2} \lesssim \|\partial_x^2 u_1\|_{H^1} \lesssim \|\partial_x^2 \rho\|_{L^2}.$$

Let us gather these estimates in order to highlight how $\boldsymbol{u}$ depends on ρ,

$$\begin{aligned} \|\boldsymbol{u}\|_{L^2} + \|u_1\|_{H^1} + \|\partial_z u_2\|_{L^2} &\lesssim \|\rho\|_{L^2} \\ \|u_1\|_{H^2} + \|u_2\|_{H^1} &\lesssim \|\partial_x \rho\|_{L^2} \\ \|u_2\|_{H^2} &\lesssim \|\partial_x^2 \rho\|_{L^2}. \end{aligned} \tag{2.60}$$

This observation illustrates the non-isotropy of the phenomenon, contrary to the classical Stokes equation for which the velocity field regularity does not depend on the compounds and directions, recalling the estimate $\|\boldsymbol{u}\|_{H^2} \lesssim \|\rho\|_{L^2}$. Basically, horizontal regularity is required on ρ to compensate the loss of derivatives on u_2. These controls are consistent with the formal Fourier relation between ρ and $\boldsymbol{u}$, for any $\boldsymbol{\xi} = (\xi_x, \xi_z)$,

$$\widehat{u}_1(\boldsymbol{\xi}) = \frac{\xi_x}{\xi_z |\boldsymbol{\xi}|^2} \widehat{\rho}(\boldsymbol{\xi}), \quad \widehat{u}_2(\boldsymbol{\xi}) = \frac{\xi_x^2}{\xi_z^2 |\boldsymbol{\xi}|^2} \widehat{\rho}(\boldsymbol{\xi}).$$

The regularity of u_2 is therefore limiting the regularity of $\boldsymbol{u}$, and we have

$$\|u_2\|_{H^{m+2}} \lesssim \|\partial_x u_1\|_{H^{m+2}} \lesssim \|\rho\|_{H^2_x H^m},$$

which concludes the proof of Lemma 2.7.2 and the announced estimate (2.57).

2.7.2 Well-posedness of the hydrostatic Stokes-transport equation

According to the above discussion, we will conduct our analysis using that the solution of the hydrostatic Stokes equation satisfies

$$\|\boldsymbol{u}\|_{H^m} \lesssim \|\rho\|_{H^m}. \tag{2.61}$$

Although this estimate is not as optimal as (2.57), it is not clear how to use the anisotropic regularisation highlighted in (2.61). In the same spirit as the proof of Theorem 2.6.1, we compute the a priori estimate of ρ in H^m for any $m \geq 3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{H^m}^2 \lesssim \|[\partial^m, \boldsymbol{u}\cdot\nabla]\rho\|_{L^2}\|\rho\|_{H^m},$$

where the tame estimate introduced in (3.20) and inequality (2.61) provide here

$$\begin{aligned}\|[\partial^m, \boldsymbol{u}\cdot\nabla]\rho\|_{L^2} &\lesssim \|\boldsymbol{u}\|_{H^m}\|\nabla\rho\|_{L^\infty} + \|\nabla\boldsymbol{u}\|_{L^\infty}\|\rho\|_{H^m}\\ &\lesssim (\|\nabla\rho\|_{L^\infty} + \|\nabla\boldsymbol{u}\|_{L^\infty})\|\rho\|_{H^m}.\end{aligned}$$

Therefore, as long as ρ lies in H^m, we have the same a priori estimate as for the regular Stokes-transport equation, see Theorem 2.6.1:

$$\|\rho\|_{H^m} \leq \|\rho_0\|_{H^m}\exp\left(C\int_0^t (\|\nabla\boldsymbol{u}(s)\|_{L^\infty} + \|\nabla\rho(s)\|_{L^\infty})\,\mathrm{d}\tau\right).$$

The difference here lies in the fact that we do not have a uniform bound on the Lipschitz norm of $\boldsymbol{u}$, which ensures for the original system to have a finite growth speed, allowing global existence of the solution. Instead, we have here, combining (2.61) and the embedding $H^3 \hookrightarrow W^{1,\infty}$,

$$\|\nabla\boldsymbol{u}\|_{L^\infty} + \|\nabla\rho\|_{L^\infty} \lesssim \|\rho\|_{H^3}.$$

Therefore, if the H^m norm of the solution were to explode in finite time, there should be a regularity break at the H^3 level first. Including this latter inequality in the energy estimate, we obtain for $m \geq 3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{H^m} \lesssim \|\rho\|_{H^m}^2,$$

whence Grönwall lemma ensures that, at least for $0 \leq t < T(\|\rho_0\|_{H^3}) := (C\|\rho_0\|_{H^3})^{-1}$,

$$\|\rho(t)\|_{H^m} \leq \frac{\|\rho_0\|_{H^m}}{1 - C\|\rho_0\|_{H^3}t}. \tag{2.62}$$

Proof of Theorem 2.7.1. **Existence.** We follow the scheme of proof of Theorem 2.6.1 and obtain existence of a solution by a compactness argument. Let $\rho^N \in C(0,T;H^m(\Omega))$ such that

$$\|\rho^N(t)\|_{H^m} \lesssim \frac{\|\rho_0\|_{H^m}}{1 - C\|\rho_0\|_{H^3}t}, \tag{2.63}$$

where we consider $0 \leq t \leq T_*$ for some $0 < T_* < 1/C\|\rho_0\|_{H^3}$. Therefore, Lemma 2.7.2 ensures existence and uniqueness of $\boldsymbol{u}^N \in C(0,T_*;H^m(\Omega))$ satisfying for all $0 \leq t \leq T_*$ the hydrostatic

Stokes equation,

$$\left\{\begin{array}{rcl} -\Delta u_1^N + \partial_x p^N &=& 0 \\ \partial_z p^N &=& -\rho^N \\ \operatorname{div} \boldsymbol{u}^N &=& 0, \end{array}\right.$$

supplemented by conditions (2.56), obeying

$$\|\boldsymbol{u}^N\|_{H^m} \lesssim \|\rho^N\|_{H^m}.$$

In particular estimate (2.63) ensures for any $0 \leq t \leq T_*$ that

$$\|\boldsymbol{u}^N(t)\|_{W^{1,\infty}} \lesssim \|\boldsymbol{u}^N(t)\|_{H^3} \lesssim \frac{\|\rho_0\|_{H^3}}{1 - C\|\rho_0\|_{H^3} T_*},$$

whence $\boldsymbol{u}^N$ is globally Lipschitz in space uniformly in time. In particular the characteristics associated to $\boldsymbol{u}^N$ are well-defined and the transport problem

$$\left\{\begin{array}{rcl} \partial_t \rho^{N+1} + \boldsymbol{u}^N \cdot \nabla \rho^{N+1} &=& 0 \\ \rho^{N+1}_{|t=0} &=& \rho_0, \end{array}\right.$$

has a unique solution $\rho^{N+1} \in C(0, T_*; H^m(\Omega))$. Earlier a priori energy computation yields here

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{N+1}\|_{H^m}^2 &\lesssim (\|\nabla \rho^{N+1}\|_{L^\infty}\|\boldsymbol{u}^N\|_{H^m} + \|\nabla \boldsymbol{u}^N\|_{L^\infty}\|\rho^{N+1}\|_{H^m})\|\rho^{N+1}\|_{H^m} \\ &\lesssim (\|\rho^{N+1}\|_{H^3}\|\rho^N\|_{H^m} + \|\rho^N\|_{H^3}\|\rho^{N+1}\|_{H^m})\|\rho^{N+1}\|_{H^m}. \end{aligned} \tag{2.64}$$

We need to obtain estimate (2.63) for ρ^{N+1} in H^3 before concluding about $H^m, m \geq 3$. This is ensured by a Grönwall lemma after plugging assumption (2.63) in (2.64) for $m = 3$, which yields the following

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{N+1}\|_{H^3}^2 \lesssim \|\rho^N\|_{H^3}^2\|\rho^{N+1}\|_{H^3} + \|\rho^N\|_{H^3}\|\rho^{N+1}\|_{H^3}^2.$$

So, for any $m \geq 3$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{N+1}\|_{H^m}^2 \lesssim \frac{\|\rho\|_{H^3}}{1 - C\|\rho_0\|_{H^3} t}\|\rho^{N+1}\|_{H^m}^2,$$

implying

$$\|\rho^{N+1}\|_{H^m} \leq \frac{\|\rho_0\|_{H^m}}{1 - C\|\rho_0\|_{H^3} t}.$$

Initialising the sequence by $\rho^0(t) := \rho_0$ for all $t \in \mathbf{R}_+$, constant in time, therefore in $C(0, T; H^m(\Omega))$ and satisfying (2.63), we obtain sequences $(\rho^N)_N, (\boldsymbol{u}^N)_N$ uniformly bounded with respect to N. The sequence $(\rho^N)_N$ is moreover equicontinuous in $C(0, T_*; H^{m-2}(\Omega))$ since

$$\begin{aligned} \|\partial_t \rho^N\|_{H^2} &= \|\boldsymbol{u}^{N-1} \cdot \nabla \rho^N\|_{H^2} \\ &\lesssim \|\boldsymbol{u}^{N-1}\|_{L^\infty}\|\rho^N\|_{H^m} + \|\boldsymbol{u}^{N-1}\|_{H^{m-1}}\|\nabla \rho^N\|_{L^\infty} \\ &\lesssim \|\rho^{N-1}\|_{H^3}\|\rho^N\|_{H^m} + \|\rho^{N-1}\|_{H^{m-1}}\|\rho^N\|_{H^3} \\ &\lesssim \frac{\|\rho_0\|_{H^3}\|\rho_0\|_{H^m}}{(1 - C\|\rho_0\|_{H^3} T_*)^2}. \end{aligned}$$

From here, Aubin–Lions lemma and the same method as in the proof of Theorem 2.6.1 ensures

existence of a solution

$$(\rho, \boldsymbol{u}) \in C(0, T_*; H^m(\Omega)) \times C(0, T_*; H^{m+2}(\Omega)),$$

obeying estimate (2.62).

Uniqueness. Let us consider two solutions $(\rho^i, \boldsymbol{u}^i), i = 1, 2$ to (2.54), with same initial datum ρ_0. The stability estimate for the transport equation from Proposition 2.2.6 still holds here,

$$\|\rho^1 - \rho^2\|_{L^\infty(0,T_*,H^{-1})} \lesssim \|\rho_0\|_{L^\infty} e^{CT_*\|\nabla \boldsymbol{u}^1\|_{L^\infty}} \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{L^\infty(0,T_*;L^2)},$$

but does not allow to ensure uniqueness of the solution, since we do not have anymore the control $\|\boldsymbol{u}\|_{H^1} \lesssim \|\rho\|_{H^{-1}}$ on the Stokes solution to close the estimate. Bounding the density difference in L^2 instead of H^{-1}, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^1 - \rho^2\|_{L^2}^2 &= -\int_\Omega (\boldsymbol{u}^1 \cdot \nabla\rho^1 - \boldsymbol{u}^2 \cdot \nabla\rho^2)(\rho^1 - \rho^2) \\
&= -\int_\Omega \boldsymbol{u}^1 \cdot \nabla(\rho^1 - \rho^2)(\rho^1 - \rho^2) + (\boldsymbol{u}^1 - \boldsymbol{u}^2) \cdot \nabla\rho^2(\rho^1 - \rho^2) \\
&\leq \|\nabla\rho^1\|_{L^\infty} \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{L^2} \|\rho^1 - \rho^2\|_{L^2},
\end{aligned}$$

where the first term of the integrale does not contribute by divergence free of $\boldsymbol{u}^1$. We know from inequality (2.63) that $\|\nabla\rho^1\|_{L^\infty}$ is bounded on $[0, T)$. From (2.60) we obtain at best

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^1 - \rho^2\|_{L^2} \lesssim \frac{\|\rho_0\|_{H^3}}{1 - C\|\rho_0\|_{H^3} t} \|\rho^1 - \rho^2\|_{H^1},$$

which still does not allow to conclude. Therefore we perform the same computations in H^1. For first order derivatives of the solutions, inserting some relevant terms, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial(\rho^1 - \rho^2)\|_{L^2}^2 &= -\int_\Omega \partial\left(\boldsymbol{u}^1 \cdot \nabla\rho^1 - \boldsymbol{u}^2 \cdot \nabla\rho^2\right) \partial(\rho^1 - \rho^2) \\
&= -\int_\Omega \partial\left(\boldsymbol{u}^1 \cdot \nabla(\rho^1 - \rho^2) + (\boldsymbol{u}^1 - \boldsymbol{u}^2) \cdot \nabla\rho^2\right) \partial(\rho^1 - \rho^2) \\
&= -\int_\Omega \partial\boldsymbol{u}^1 \cdot \nabla(\rho^1 - \rho^2)\partial(\rho^1 - \rho^2) + \tfrac{1}{2}\boldsymbol{u}^1 \cdot \nabla|\partial(\rho^1 - \rho^2)|^2 \\
&\quad - \int_\Omega \partial(\boldsymbol{u}^1 - \boldsymbol{u}^2)\nabla\rho^2\partial(\rho^1 - \rho^2) + (\boldsymbol{u}^1 - \boldsymbol{u}^2)\nabla\partial\rho^2\partial(\rho^1 - \rho^2) \\
&\leq \|\partial\boldsymbol{u}^1\|_{L^\infty}\|\rho^1 - \rho^2\|_{H^1}^2 + 0 \\
&\quad + \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{H^1}\|\nabla\rho^2\|_{L^\infty}\|\rho^1 - \rho^2\|_{H^1} + \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{L^4}\|\rho^2\|_{W^{2,4}}\|\rho^1 - \rho^2\|_{H^1}.
\end{aligned}$$

Note that derivatives of order two on the difference $\rho^1 - \rho^2$ vanish due to the divergence free of $\boldsymbol{u}^1$. Using the embeddings $H^1 \hookrightarrow L^4$, $H^3 \hookrightarrow W^{2,4}$, the estimate (2.61), and adding the previous computation on the L^2 norm, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^1 - \rho^2\|_{H^1}^2 \lesssim \frac{\|\rho_0\|_{H^3}}{1 - C\|\rho_0\|_{H^3} t} \|\rho^1 - \rho^2\|_{H^1}^2,$$

whence $\rho^1 = \rho^2$ since $\rho^1|_{t=0} = \rho^2|_{t=0} = \rho_0$. □

Chapter 3

Long-time behaviour for the Stokes-transport equation

This chapter is dedicated to the study of the long-time behaviour of the solutions to the Stokes-transport equation in the periodic channel $\mathbf{T} \times (0,1)$. The author resumes the paper [DGL23] written together with Dalibard and Guillod, from Section 3.1 to Section 3.5. This work is devoted to the demonstration of a stability result for the Stokes-transport equation for data $\rho_0 \in H^6(\Omega)$ close enough to linear profiles, and to the formation of density boundary layers. The paper introduction is reported in Section 3.1 and completed according to the supplemented following Sections. In particular, Section 3.6 contains a study of the long-time properties of the solutions for general data $\rho_0 \in L^\infty(\Omega)$, and we discuss the natural conjecture of the density convergence toward its vertical rearrangement. A previous work of Kiselev & Yao [KY23] on the instability of the incompressible porous media equation for data in $H^{2^-}(\Omega)$ is adapted to the Stokes-transport equation in Section 3.7. Finally, properties and eigenfunction basis of the biharmonic operator with Dirichlet boundary conditions, at the heart of the long-time analysis of [DGL23], are compiled in Section 3.8.

3.1 Introduction

The Stokes-transport equation

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = 0 \\ -\Delta \boldsymbol{u} + \nabla p = -\rho \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} = 0 \\ \rho|_{t=0} = \rho_0 \end{cases}$$

models the evolution of an incompressible inhomogeneous fluid with density ρ, velocity and pressure fields $(\boldsymbol{u}, p)$. Since the density can be shifted by a constant, we assume for simplicity that the initial density ρ_0 is non-negative. This equation will be studied in a two-dimensional periodic strip, namely $\Omega = \mathbb{T} \times (0,1)$ with variables $(x,z) \in \Omega$ and with Dirichlet boundary condition of the velocity field:

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \partial\Omega.$$

It consists in a coupling of the transport equation for the density of the fluid, with a velocity field satisfying for all time the Stokes equation with gravity forcing $-\rho \boldsymbol{e}_z$ where $\boldsymbol{e}_z$ is the unitary

vertical vector. This equation has been studied in particular in the works of [Höf18a] and [Mec20] showing that (3.1) is a model obtained as the homogenization limit of inertialess particles in a fluid satisfying Stokes equation. A more recent paper by [Gra22] shows that this system is obtained as a formal limit where the Prandtl number is infinite. In this paper, the domain is chosen as $\Omega = \mathbb{T}\times(0,1)$ since it describe a physically meaningful situation which include Dirichlet boundary conditions.

Well-posedness. The well-posedness of this system has been shown in [Leb22] in bounded two-dimensional and three-dimensional domains as well as in the infinite strip $\mathbb{R}\times(0,1)$ with $\rho \in L^\infty(\mathbf{R}_+; L^\infty)$ and $\boldsymbol{u} \in L^\infty(\mathbf{R}_+; W^{1,\infty})$; the well-posedness in $\Omega = \mathbb{T}\times(0,1)$ being a direct consequence. Well-posedness in Sobolev spaces are required for our results. Since this result does not seem to appear in the literature, we provide a concise proof of the global well-posedness of this problem in Chapter 2, Section 2.6. More precisely, for any $\rho_0 \in H^m$ with $m \geq 3$, there exists a unique strong solution $(\rho, \boldsymbol{u})$ of (3.1) with $\rho \in C(\mathbf{R}_+; H^m(\Omega))$ and $\boldsymbol{u} \in C(\mathbf{R}_+; H^{m+2}(\Omega))$.

Well-posedness in other domains and spaces has also been proven, see for example the recent results [MS22; Inv23].

Steady states. Before going further let us observe that the stationary states, *i.e.* states such that $\partial_t \rho = 0$, of this system are precisely the *stratified* density profiles, which mean in this paper density profiles depending only on the vertical variable z. Indeed, for such a map $\Theta = \Theta(z)$,

$$(\rho, \boldsymbol{u}, p) = \left(\Theta, \mathbf{0}, -\int^z \Theta(z')\,\mathrm{d}z'\right).$$

is a solution of (3.1). To show the converse, let us introduce the potential energy associated to a density profile ρ,

$$\mathrm{E}(\rho) := \int_\Omega z\rho\,\mathrm{d}x\,\mathrm{d}z.$$

so the energy balance is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(\rho) = \int_\Omega z\partial_t\rho = -\int_\Omega z\boldsymbol{u}\cdot\nabla\rho = \int_\Omega u_2\rho = -\|\nabla\boldsymbol{u}\|_{L^2}^2, \tag{3.2}$$

where the divergence-free and the Dirichlet boundary conditions on $\boldsymbol{u}$ are used in the integration by parts. The last equality is simply the basic estimate of the Stokes equation. The potential energy dissipates exactly through the viscosity effects. From this observation we see that the whole evolution is non-reversible; the fluid only rearranges in states of lower potential energy. Moreover, a stationary state is exactly a state for which $\boldsymbol{u} = \mathbf{0}$, therefore it means that the density ρ and the pressure p must satisfy

$$\nabla p = -\rho\boldsymbol{e}_z,$$

so that the pressure is independent of the x variable, implying ρ to depend only on the z variable. These considerations are detailed in Section 3.6.2

Asymptotic properties for general initial data For initial data ρ_0 belonging to $L^\infty(\Omega)$, we do not know if the solution converges in general. We state the sufficient condition $\boldsymbol{u} \in L^1(\mathbf{R}_+; L^2(\Omega))$ ensuring the strong convergence of ρ in $L^2(\Omega)$, and notice that this condition is not ensured in general since we only have a priori $\boldsymbol{u} \in L^2(\mathbf{R}_+; H^1(\Omega))$. Nevertheless, we are

able to ensure that the ω-set associated to ρ_0, defined as the set of accumulation points of the trajectory $t \mapsto \rho(t)$, is never empty. The solution converges if and only if this set reduces to a singleton. We do not know in general if this is the case, or if the rearrangement ρ_0^* belongs to the ω-set of ρ_0 in general. Finally, we provide the following criterion. If the potential energy of the system converges toward the potential energy of the initial datum $E(\rho_0^*)$, then ρ_∞ is unique and coincides with ρ_0^*. These results are discussed and proved in Section 3.6.

Main stability result. The aim of the paper [DGL23], corresponding to Sections 3.1 to 3.5, is to study the long-time asymptotic behaviour of perturbation of stratified initial data with lighted fluid on top and lower fluid on the bottom, *i.e.* such that the profile Θ depends only on z and is strictly decreasing. For simplicity and for the rest of this chapter, we consider perturbations of the affine profile $\Theta(z) = 1 - z$. Our main stability result for perturbations vanishing on the boundary is the following:

Theorem 3.1.1. *There exists $\varepsilon_0 > 0$ small such that for any $\rho_0 \in H^6(\Omega)$ satisfying $\|\rho_0 - \Theta\|_{H^6} \leq \varepsilon_0$ and $\rho_0 - \Theta \in H^2_0(\Omega)$, the solution ρ of (3.1) satisfies:*

$$\|\rho - \rho_\infty\|_{L^2(\Omega)} \lesssim \frac{\varepsilon_0}{1+t}, \qquad \|\rho - \rho_\infty\|_{H^4(\Omega)} \lesssim \varepsilon_0,$$

where ρ_∞ is given by the decreasing vertical rearrangement of ρ_0:

$$\rho_\infty(z) := \int_0^\infty \mathbf{1}_{0 \leq z \leq |\{\rho_0 > \lambda\}|} \, \mathrm{d}\lambda.$$

Note that the condition on $H^2_0(\Omega)$ is equivalent to the following requirements, discussed in the following paragraphs:

$$\rho_0|_{\partial\Omega} = \Theta|_{\partial\Omega}, \qquad \partial_{\boldsymbol{n}}\rho_0|_{\partial\Omega} = \partial_{\boldsymbol{n}}\Theta|_{\partial\Omega}.$$

This theorem will be proven in Section 3.2.

Remarks on the main stability result. Since the set of steady-states is not discrete, it is expected that Θ is not asymptotically stable, and that the long-time behaviour is given by a slightly modified density profile. In general, this asymptotic profile is constructed thought the entire nonlinear dynamics is a very non-explicit way. However the transport equation is remarkable since it preserves the measure of the level sets. This property combined with the fact that the asymptotic profile is strictly decreasing (as a perturbation of Θ) allows us to identify the asymptotic profile as the decreasing vertical rearrangement of ρ_0, which can be computed directly from ρ_0 without dependence on the full non-linear dynamics. See Subsection 3.2.4 for details.

This result proves the stability of the particular state $\Theta(z) = 1 - z$. The result generalizes and our proofs adapt to the case of stratified Θ satisfying

$$\sup_{(0,1)} \partial_z \Theta < 0, \qquad \|\partial_z^2 \Theta\|_{H^s} \lesssim -\sup_{(0,1)} \partial_z \Theta,$$

for s large enough. This conditions mean that Θ is strictly decreasing and presents sufficiently mild variations. This remark is detailed at the end of Subsection 3.2.2. We note that without monotony assumption, lighter fluid might be below heavy one, and physically instabilities are expected to develop. Some weak convergence up to extraction toward a stationary state could

be proven, but the limit might be a non-trivial ω-limit set in general. Anyway, this is not clear if convergence to the rearranging steady-state is expected to hold.

In [GGS22] the interface problem for (3.1) is considered also in the domain $\Omega = \mathbb{T} \times (0,1)$. The interface problem treats the case where the density is equal to two different constant below and above an interface $\Gamma(t) \subset \Omega$. The question is about the regularity of the interface, the well-posedness for L^∞ densities being established in [Leb22]. More precisely, the authors prove local well-posedness for the interface in $C^{1,\gamma}$ for $0 < \gamma < 1$ as well as the global well-posedness and decay of small perturbation in $H^3(\mathbb{T})$ of the flat interface with lighter fluid on top. The proof is very different than our as it uses a contour dynamics equation, but the spirit of the stability result is pretty similar.

Finally, an interesting question is about the optimality of the $(1+t)^{-1}$ decay in (3.1). The dynamics of the equation preserve the fact that the perturbation and its normal derivative are vanishing on $\partial\Omega$ *i.e.* $\rho - \Theta \in H^2_0(\Omega)$. For higher normal derivatives this property is not preserved, and this is the main reason why the time-decay is limited. This is one of the main motivation to study boundary layers formations in this system, together with the possibility to allow non-vanishing perturbations on $\partial\Omega$.

Nonlinear instability for low regularity perturbation The results of [KY23] about the instability of the incompressible porous media equation being essentially geometric, they can be adapted to the Stokes-transport system, which is done in Section 3.7. The result is the same: there exists smooth perturbation small in $H^{2-}(\Omega)$-norm such that $\limsup_{t\to\infty} \|\rho(t) - \Theta\|_{H^s(\Omega)} = \infty$ for any $s > 1$. Therefore, this enlightens the existence of a regularity threshold between $H^2(\Omega)$ and $H^6(\Omega)$ between stability and instability.

Linear asymptotic expansion for non-vanishing perturbation on $\partial\Omega$. The previous result is only valid under the assumption that the perturbation and its normal derivative are vanishing on $\partial\Omega$ *i.e.* when $\rho_0 - \Theta \in H^2_0(\Omega)$. If the perturbation does not vanish on the boundary, this question is non trivial even for the linearized equations around $\Theta = 1 - z$:

$$\begin{cases} \partial_t \theta - \boldsymbol{u} \cdot \boldsymbol{e}_z = 0 \\ -\Delta \boldsymbol{u} + \nabla p = -\theta \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} = 0 \\ \theta|_{t=0} = \theta_0 \end{cases} \tag{3.3}$$

It turns out that θ vanishes as $t \to \infty$ but with a much slower rate than in the case when $\theta_0 = \partial_{\boldsymbol{n}} \theta_0 = 0$ on $\partial\Omega$. This is due to the formation of boundary layers of typical size $t^{-1/4}$ as $t \to \infty$, in the vicinity of $z = 0$ and $z = 1$. More precisely, we will prove the following result in Section 3.3:

Theorem 3.1.2. *Let $\theta_0 \in H^s(\Omega)$ for some s sufficiently large. Then the solution of (3.3) satisfies:*

$$\theta = \bar{\theta}_0 + \theta^{\mathrm{BL}} + O(t^{-1}) \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad t \to \infty,$$

where $\bar{\theta}_0(z) = \frac{1}{2\pi} \int_0^{2\pi} \theta_0(x,z)\,\mathrm{d}x$ is the horizontal average of the initial data and θ^{BL} is the boundary layer part whose leading terms are:

$$\theta^{\mathrm{BL}} = \Theta^0_{\mathrm{top}}(x, t^{1/4}(1-z)) + \Theta^0_{\mathrm{bot}}(x, t^{1/4} z) + l.o.t$$

with $\Theta^0_{\rm top}$ *and* $\Theta^1_{\rm bot}$ *decaying exponentially as* $Z = t^{1/4}z \to \infty$. *Note that* $\|\theta^{\rm BL}\|_{L^2(\Omega)} \lesssim (1+t)^{-1/8}$.

Non-linear asymptotic expansion. For the nonlinear problem, more complex boundary layers seems to develop when the perturbation is non-zero on $\partial\Omega$. To avoid such complications, we go back to the case where $\rho_0 - \Theta \in H^2_0(\Omega)$. The previous results will then allow us to derive uniform bounds in $H^8(\Omega)$, modulo some boundary layer terms:

Theorem 3.1.3. *There exists* $\varepsilon_0 > 0$ *small such that for any* $\rho_0 \in H^{14}(\Omega)$ *satisfying* $\|\rho_0 - \Theta\|_{H^{14}} \le \varepsilon_0$ *and* $\rho_0 - \Theta \in H^2_0(\Omega)$, *then the solution* ρ *of* (3.1) *satisfies:*

$$\rho = \rho_\infty + \rho_{\rm BL} + O(t^{-2}) \quad in \quad L^2(\Omega) \quad as \quad t \to \infty,$$

where $\rho_{\rm BL}$ *is the boundary layer part:*

$$\rho_{\rm BL} = \frac{1}{t}\Theta_{\rm top}(x, t^{1/4}(1-z)) + \frac{1}{t}\Theta_{\rm bot}(x, t^{1/4}z) + l.o.t.$$

with $\Theta_{\rm top}$ *and* $\Theta_{\rm bot}$ *decaying exponentially as* $Z = t^{1/4}z \to \infty$.

We note that $\|\rho_{\rm BL}\|_{L^2(\Omega)} \lesssim (1+t)^{-9/8}$ so this result strongly suggests that the optimal decay of $\rho - \rho_\infty$ is like $t^{-9/8}$ in $L^2(\Omega)$, which is close to the rate t^{-1} obtained in Theorem 3.1.1. Indeed, it would be very surprising but not excluded that the non-linear dynamics drive the system to the case where these boundary layers terms always vanish.

It seems there is no major obstacles in obtaining a similar non-linear result when the perturbation do not vanish on the boundary, except technicalities due to the fact that boundary layer equations will be non-linear. The present result proved in Section 3.4 requires only linear boundary layer and is already quite technical.

Comparison with incompressible porous media. We compare the results and properties of the Stokes-transport equation and of the incompressible porous medium equation, namely (3.1) where the Stokes equation is replaced by Darcy law,

$$\begin{cases} \partial_t \rho + \boldsymbol{u}\cdot\nabla\rho = 0 \\ \qquad \boldsymbol{u} + \nabla p = -\rho \boldsymbol{e}_z \\ \qquad \operatorname{div}\boldsymbol{u} = 0 \\ \qquad \rho|_{t=0} = \rho_0. \end{cases}$$

This equation has been intensively studied and we only cite comparable results. The question of well-posedness is way more difficult than for the Stokes-transport equation. In particular global well-posedness in Sobolev spaces as stated in Theorem 2.6.1 seems to remain an open question. Local in time well-posedness has been shown in [CGO07; Xue09; YH14; CVW15] whereas some ill-posedness through non-uniqueness in some spaces has been shown in [CFG11; Shv11; IV15].

Concerning classical global solutions, the only known results has been proven for initial data close enough in Sobolev space to the stratified initial data $\Theta(z) = 1 - z$ by [Elg17] in $\mathbb{R}^2$ and $\mathbb{T}^2$ and later improved in [CCL19a] to the domain $\mathbb{T}\times(0,1)$. More precisely, these results prove that the profile $\Theta(z) = 1 - z$ is asymptotically stable under small perturbation in H^m for some m.

In $\mathbb{T}\times(0,1)$, the boundary conditions used by [CCL19a] somehow ensure that everything works more or less as in the periodic or whole space case, in particular integration by parts of

high order derivatives are possible to obtain uniform bounds in high enough Sobolev spaces. By using similar boundary conditions for Stokes-transport in $\mathbb{T}^2 \times (0,1)$, the results of [CCL19a] can be adapted in a straightforward way. In our situation, the presence of the Dirichlet boundary condition is the major obstacle, in particular because uniform bounds in high Sobolev spaces are no more valid. This is due to the presence of boundary layers as explained above and more details are provided below in the Scheme of the proof.

In the previous results of [Elg17; CCL19a], the existence of the limiting profile was obtained through a fixed point argument. One contribution of the present paper is to precisely identify the long-time asymptotic profile as the decreasing vertical rearrangement of ρ_0.

Biharmonic operator and adapted basis During the analysis of the stability problem, the Stokes equation reduces to a bilaplacian equation. Depending on the boundary condition for the velocity required in the Stokes system, the bilaplacian equation is also endowed with different boundary conditions. It appears that considering slip boundary conditions, also called Neumann boundary conditions, leads to similar analysis and stability result as in the work [CCL19a], with the same proof. On the contrary, the no-slip assumption, also called Dirichlet condition, breaks the aforementioned proof at different steps, which we detailed all along our work. We provide a selfcontained proof of the classical estimates on the solutions of the bilaplacian equation. Moreover, it seemed convenient to study closer this bilaplacian operator endowed with Dirichlet boundary conditions, and to obtain an explicit eigenfunction basis to adapt the Fourier analysis of Castro, Córdoba & Lear. This basis is described in Section 3.8. The structure of the eigenfunctions, based on the Fourier basis with addititional terms rectifying the trace, is relatable to the boundary layer formation stated in Theorems 3.1.2 and 3.1.3.

Scheme of the proof

Here we explain the main steps and difficulties of the proof of Theorem 3.1.1. The proofs of Theorems 3.1.2 and 3.1.3 are way more technical, but in some sense the general strategy is more straightforward, therefore the explanations are postponed in the beginning of Sections 3.3 and 3.4 respectively.

Rewriting of the equation. Since perturbations of $\Theta(z) = 1 - z$ are considered, it is natural to introduce the perturbation θ as

$$\rho = \Theta + \theta$$

with initial perturbation $\theta_0 = \rho_0 - \Theta$. Substituting this expression in (3.1) and recalling that stratified states do not contribute to the velocity field in the Stokes equation, we obtain the following equation of the perturbation θ:

$$\begin{cases} \partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = u_2 \\ -\Delta \boldsymbol{u} + \nabla p = -\theta \boldsymbol{e}_z \\ \operatorname{div} \boldsymbol{u} = 0 \\ \theta|_{t=0} = \theta_0. \end{cases} \tag{3.4}$$

The Stokes equation can be simplified by introducing the stream function of the divergence-free velocity field $\boldsymbol{u}$ through $\boldsymbol{u} = \nabla^\perp \psi = (-\partial_z \psi, \partial_x \psi)$. Substituting it in the Stokes equation

and considering the rotational of this equation, we get

$$\begin{cases} \partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = u_2 \\ \Delta^2 \psi = \partial_x \theta \\ \boldsymbol{u} = \nabla^\perp \psi \\ \psi|_{\partial\Omega} = \partial_{\boldsymbol{n}} \psi|_{\partial\Omega} = 0. \end{cases}$$

Notice that this writing is consistent with the previous observation that any z dependant perturbation of the density does not affect the velocity field.

Once the steady states of (3.1) are identified as the stratified density profiles, *i.e.* functions depending only on z, it is natural to decompose the perturbation $\theta(t,x,z)$ as its horizontal average $\bar{\theta}(t,z)$ and as its complementary $\theta'(t,x,z)$ with zero horizontal average, following [Elg17] and others:

$$\theta(t,x,z) = \bar{\theta}(t,z) + \theta'(t,x,z) \qquad \bar{\theta}(t,z) = \frac{1}{2\pi}\int_0^{2\pi} \theta(t,x,z)\,\mathrm{d}x.$$

We note that contrary to [Elg17; CCL19a], the average is denoted by $\bar{\theta}$ and not its complement as this seems a more natural notation for the average.

This decomposition is actually orthogonal in any Sobolev space H^m and one can project the transport equation onto the two appropriates complementary subspaces, leading to

$$\begin{cases} \partial_t \theta' + (\boldsymbol{u} \cdot \nabla \theta')' = (1 - \partial_z \bar{\theta}) u_2 & \theta'|_{t=0} = \theta'_0 \\ \partial_t \bar{\theta} + \overline{\boldsymbol{u} \cdot \nabla \theta'} = 0 & \bar{\theta}|_{t=0} = \bar{\theta}_0 \\ \Delta^2 \psi = \partial_x \theta' & \psi|_{\partial\Omega} = 0 \\ \boldsymbol{u} = \nabla^\perp \psi & \partial_{\boldsymbol{n}} \psi|_{\partial\Omega} = 0. \end{cases} \tag{3.5}$$

Although more complicated at first sight, this equation allows us to distinguish the evolution of θ' and of the average perturbation $\bar{\theta}$; this is needed since the whole perturbation cannot be expected to decay in Sobolev spaces due to its pure transport, only the average-free part θ'.

Toy problem on the torus. In order to intuit the decay of $\|\theta'\|_{L^2(\Omega)}$ and to highlight the specific difficulties of our work, we will first explain the strategy in the case when the problem is set on the torus. More precisely, we consider the following linear problem for θ' on the torus $\mathbb{T}^2$:

$$\begin{cases} \partial_t \theta' = (1 - G)\partial_x \psi + S \\ \Delta^2 \psi = \partial_x \theta' \\ \theta'|_{t=0} = \theta'_0. \end{cases} \tag{3.6}$$

where G is a given small function of t and z, whose finality is to be replaced by $\partial_z \bar{\theta}$. A source term is needed but we omit it here for simplicity. Note that (3.6) differs from our original system through the periodic boundary conditions on ψ in the vertical variable. The choice of periodic boundary conditions simplifies the analysis in several ways, which we will detail below.

The first step is to consider (3.6) for $G = 0$. In this case the system has constant coefficients and since it is set within the torus, its solution is explicit and can be represented in terms of

Fourier modes:

$$\theta'(t,x,z) = \sum_{\substack{\boldsymbol{k}\in\mathbf{Z}^2\\ k_x\neq 0}} \widehat{\theta}_{\boldsymbol{k}}(0)\exp\left(-\frac{k_x^2}{|\boldsymbol{k}|^4}t\right)\exp(i\boldsymbol{k}\cdot(x,z)),$$

where the summation is done on $\boldsymbol{k} = (k_x, k_z) \in \mathbf{Z}^2$ except $k_x \neq 0$ since θ' has vanishing horizontal averages. In particular, all H^s norms of the solution of (3.6) for $G = S = 0$ are decreasing (provided $\theta_0 \in H^s(\mathbb{T}^2)$) and furthermore, for any $\alpha \in \mathbf{N}$, $s \in \mathbf{R}$, and $t > 0$,

$$\begin{aligned}\|\Delta^{s/2}\theta'(t)\|_{L^2}^2 &= \sum_{\substack{\boldsymbol{k}\in\mathbf{Z}^2\\ k_x\neq 0}} |\boldsymbol{k}|^{2s}\exp\left(-2\frac{k_x^2}{|\boldsymbol{k}|^4}t\right)|\widehat{\theta}_{\boldsymbol{k}}(0)|^2\\ &\leq \sum_{\substack{\boldsymbol{k}\in\mathbf{Z}^2\\ k_x\neq 0}} |\boldsymbol{k}|^{2s+4\alpha}t^{-\alpha}|k_x|^{-2\alpha}|\widehat{\theta}_{\boldsymbol{k}}(0)|^2\\ &\leq t^{-\alpha}\|\Delta^{s/2+\alpha}\partial_x^{-\alpha}\theta_0'\|_{L^2}^2,\end{aligned}$$

where $\partial_x^{-1}f$ denotes the anti-derivative of f with null horizontal average. Hence if the initial datum is sufficiently smooth, θ' decays algebraically and the rate of decay can be as large as desired, the cost being the regularity required on θ_0'.

When G is nonzero and for any $s \geq 0$ even, applying $\Delta^{s/2}$ to the first equation of (3.6) and projecting on $\Delta^{s/2}\theta'$, we obtain, after several integrations by parts in the right-hand side,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta^{s/2}\theta'\|_{L^2}^2 &= -\int_{\mathbb{T}^2}\Delta^{s/2}((1-G)\psi)\Delta^{s/2+2}\psi\\ &= -\int_{\mathbb{T}^2}\Delta^{s/2+1}((1-G)\psi)\Delta^{s/2+1}\psi\\ &\leq -(1-\bar{C}\|G\|_{H^{s+2}})\|\Delta^{s/2+1}\psi\|_{L^2}^2.\end{aligned} \tag{3.7}$$

where $\bar{C}$ is a universal constant. As a consequence, if $\bar{C}\|G(t)\|_{H^{s+2}} < 1$, then the H^s norm of θ' is non-increasing, and whence uniformly bounded.

Then, the decay of $\|\theta'(t)\|_{L^2}$ is deduced by using the following Gagliardo–Nirenberg interpolation inequality:

$$\|\Delta^2\psi\|_{L^2}^2 \lesssim \frac{1}{K}\|\Delta\psi\|_{L^2}^2 + K^2\|\Delta^4\psi\|_{L^2}^2 \tag{3.8}$$

which is valid for any $K > 0$. More precisely, combining (3.7) with $s = 0$ and (3.8) lead to:

$$\frac{d}{dt}\|\theta'\|_{L^2}^2 \lesssim -\|\Delta\psi\|_{L^2}^2 \lesssim K^3\|\Delta^4\psi\|_{L^2}^2 - K\|\Delta^2\psi\|_{L^2}^2 \lesssim K^3\|\Delta^2\partial_x\theta'\|_{L^2}^2 - K\|\theta'\|_{L^2}^2$$

remaining that $\Delta^2\psi = \partial_x\theta'$. Taking $K \simeq (1+t)^{-1}$ we deduce:

$$\frac{d}{dt}\|\theta'\|_{L^2}^2 + \frac{3}{1+t}\|\theta'\|_{L^2}^2 \lesssim \frac{1}{(1+t)^3}\|\Delta^2\partial_x\theta'\|_{L^2}^2$$

and since $\partial_x\theta'$ is uniformly bounded in $H^4(\mathbb{T}^2)$ by $\|\partial_x\theta_0'\|_{H^4}$, this integrate into

$$\|\theta'\|_{L^2} \lesssim \frac{\|\partial_x\theta_0'\|_{H^4}}{1+t}. \tag{3.9}$$

Difficulties with Dirichlet boundary conditions. Let us now explain the main differences between (3.6) on $\mathbb{T}^2$ and the original system (3.5) on $\Omega = \mathbb{T}\times(0,1)$. The strategy will be identical, the first step in to deduce uniform bound for θ' in $H^4(\Omega)$ and then use interpolation inequalities together with the energy estimate to obtain the decay estimate (3.9), but their derivation will be substantially more involved.

Since the equation is no longer set on the torus, but rather in the domain $\Omega = \mathbb{T} \times (0,1)$ endowed with boundary conditions, we can no longer perform a (discrete) Fourier transform in the vertical variable. However it is possible to analyze explicitly the eigenfunctions of the operator

$$L : \theta \in L^2 \mapsto \partial_x \psi \in L^2, \quad \text{where } \Delta^2 \psi = \partial_x \theta, \quad \psi|_{\partial\Omega} = \partial_{\boldsymbol{n}} \psi|_{\partial\Omega} = 0.$$

and show that the eigenvalues $(\lambda_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbf{Z}^2}$ of the operator L behave asymptotically as $k_x^2/|\boldsymbol{k}|^4$ (see Proposition 3.8.3), so that the estimate (3.9) remains true. Details on the spectral analysis are presented in Section 3.8.

However, we are able to prove uniform $H^4(\Omega)$ bound for θ' directly from the equation without spectral analysis. More precisely, the estimate (3.7) remains valid for $s = 0$ since $\psi|_{\partial\Omega} = \partial_{\boldsymbol{n}}\psi|_{\partial\Omega}$. Higher order uniform estimates in $H^s(\Omega)$ fails in general due to non-vanishing terms on the boundary. The question is therefore when the integration by parts done in (3.7) can be performed. The traces of θ' and $\partial_{\boldsymbol{n}}\theta'$ being zero, the traces of $\Delta^2\psi$ and $\partial_{\boldsymbol{n}}\Delta^2\psi$ are also vanishing (see Subsection 3.2.1) so integrations by parts in (3.7) can be done for $s = 4$ provided $G = 0$. Therefore a uniform H^4 bound can be deduced when $G = 0$. When G is nonzero, some traces are not vanishing anymore. The strategy will be to treat them perturbativly, *i.e.* not performing integration by parts on $\Delta^2(G\psi)\Delta^3\psi$.

A similar interpolation argument in Lemma 3.2.7 allow us to then deduce the analogue of (3.9) *i.e.* that $\|\theta'\|_{L^2}$ is bounded by $(1+t)^{-1}$.

Bootstrap. The last step of the proof consists in bringing the previous linear analysis into the full nonlinear system. Intuitively, the strategy is the following: denote by $(0, T^*)$ the maximal time interval over which $\|\theta'\|_{L^2} \leq B(1+t)^{-1}$ and $\|\theta'\|_{H^4} \leq B$ are valid with a constant B. In fact more estimates need to be included in the boostrap argument for technical reasons, see (3.18). On this time interval, the quadratic terms can be treated perturbatively, provided $\|\theta_0\|_{H^4}$ is sufficiently small. Hence the bootstrap estimates hold with a constant which is better than B, and thus $T^* = \infty$. It follows that θ' converges towards zero in L^2, and that the time derivative of $\bar{\theta}$ is integrable. Hence $\bar{\theta}$ has a limit in L^2 as $t \to \infty$. This is the main part of the proof which is detailed in Subsection 3.2.3.

Identitification of the limit. Since θ' converges to zero in any H^m for $m < 4$ as $t \to \infty$ and $\bar{\theta}$ has a limit in L^2 as $t \to \infty$, the whole density $\rho = \Theta + \theta = \Theta + \bar{\theta} + \theta'$ converges to some limit ρ_∞ in L^2 and ρ_∞ depends only on z. Since θ is small compared to Θ, $\partial_z\theta$ is small, hence ρ_∞ is strictly decreasing with respect to z as is Θ. The transport of the density by the divergence-free field $\boldsymbol{u}$ ensures that the level sets of ρ are preserved by the time evolution, and by strong convergence this is also the case for the limit ρ_∞. According to rearrangement theory, ρ_∞ is therefore a rearrangement of ρ_0. One can show that there exists a unique decreasing vertical rearrangement of ρ_0, hence ρ_∞ is uniquely determined. This part of the proof is detailed in Subsection 3.2.4.

3.2 Long time stability of stratified profiles: proof of Theorem 3.1.1

This section is devoted to the proof of Theorem 3.1.1. The scheme of proof follows the steps highlighted in the introduction: we decompose θ into $\theta = \bar{\theta} + \theta'$, and we prove that θ' vanishes in L^2 with algebraic decay, while $\bar{\theta}$ converges in L^2 towards a profile $\bar{\theta}_\infty(z)$. To that end, we first study the linearized Stokes-transport system around a solution θ close to an affine profile. Thanks to a crucial interpolation inequality (see Lemma 3.2.7), which somehow replaces the spectral decomposition in the periodic setting, we quantify the L^2 decay of solutions of the linearized equation with a source term (see Proposition 3.2.6). We then use a bootstrap argument to show that the decay predicted by the linear analysis persists for the nonlinear evolution. This allows us to prove that $\theta'(t) \to 0$ and that $\bar{\theta} \to \bar{\theta}_\infty$ in $H^s(\Omega)$ as $t \to \infty$, for all $s < 4$. Eventually, we identify the asymptotic profile $\bar{\theta}_\infty$ in terms of the initial data.

The organization of this section is the following. We start in Subsection 3.2.1 with some preliminary remarks concerning the traces of θ and $\partial_{\boldsymbol{n}}\theta$. We then turn towards the analysis of the linearized system in Subsection 3.2.2. The bootstrap argument is presented in Subsection 3.2.3. Eventually, we prove in Subsection 3.2.4 that ρ_∞ is the rearrangement of the initial data ρ_0.

3.2.1 Vanishing traces for θ' and $\partial_{\boldsymbol{n}}\theta'$

We prove here the following preliminary result:

Lemma 3.2.1. *Let $\theta_0 \in H^m(\Omega)$ with $m \geq 3$, and let $\theta \in L^\infty_{loc}(\mathbf{R}_+, H^m)$ be the solution of (3.5). Assume that $\theta_0 = \partial_{\boldsymbol{n}}\theta_0 = 0$ on $\partial\Omega$, and that $\partial_z^2\bar{\theta}_0|_\Omega = 0$.*

Then for all $t \geq 0$,

$$\theta(t)|_\Omega = \partial_{\boldsymbol{n}}\theta(t)|_\Omega = 0, \quad \partial_z^2\bar{\theta}(t)|_\Omega = 0.$$

Remark 3.2.2. *If $\rho_0 \in H^m_0(\Omega)$ then the solution $\rho(t)$ of (3.1) belongs to $H^m_0(\Omega)$ for all times. Indeed, the solution of the transport equation writes*

$$\rho(t) = \rho_0(\boldsymbol{X}(t)^{-1}),$$

where $\boldsymbol{X} : \mathbf{R}_+ \times \Omega \to \Omega$ is the characteristic function associated to $\boldsymbol{u}$, defined as the solution of the ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t) = \boldsymbol{u}(t, \boldsymbol{X}(t)), \\ \boldsymbol{X}(0) = \mathrm{Id}_\Omega. \end{cases}$$

We recall that $\boldsymbol{X}(t)$ is a diffeomorphism of Ω for all time $t \in \mathbf{R}_+$. Since $\boldsymbol{u}(t) \in H^1_0(\Omega)$ due to the homogeneous Dirichlet condition, the boundary $\partial\Omega$ is stable for characteristic function at all time t, namely $\boldsymbol{X}(t)|_{\partial\Omega} = \mathrm{Id}_{\partial\Omega}$, so is its inverse, $\boldsymbol{X}(t)^{-1}|_{\partial\Omega}$. It follows that if $\rho_0 \in H^1_0(\Omega)$, $\rho(t)|_{\partial\Omega} = 0$ for all $t \geq 0$. The claim for higher values of m follows easily by induction.

Note that the assumptions of 3.2.1 are different since $\rho_0 = -z + \theta_0$ does not vanish on the boundary.

Proof. Formally,

$$\partial_t\theta|_{\partial\Omega} + \boldsymbol{u}|_{\partial\Omega} \cdot \nabla\theta'|_{\partial\Omega} = u_2|_{\partial\Omega},$$

where $\boldsymbol{u}|_{\partial\Omega} = \mathbf{0}$, hence $\partial_t\theta|\partial_\Omega = 0$ and the trace of θ is constant it time, equal to 0. Since horizontal derivatives preserve this property, we have even $\partial_x^\ell\theta|_{\partial\Omega} = 0$ for any ℓ. Now consider one vertical derivative applied to the equation,

$$\partial_t\partial_z\theta + \partial_z\boldsymbol{u}\cdot\nabla\theta + \boldsymbol{u}\cdot\nabla\partial_z\theta = \partial_z u_2$$

where

$$\partial_z\boldsymbol{u}|_{\partial\Omega}\cdot\nabla\theta|_{\partial\Omega} = \partial_z u_1|_{\partial\Omega}\partial_x\theta|_{\partial\Omega} + \partial_z u_2|_{\partial\Omega}\partial_z\theta|_{\partial\Omega},$$

and we know that $\partial_x\theta|_{\partial\Omega} = 0$ and we use the divergence free condition to observe $\partial_z u_2|_{\partial\Omega} = -\partial_x u_1|_{\partial\Omega} = 0$. In the end we get $\partial_z\theta|_{\partial\Omega} = 0$ for all time, hence $\theta \in H^2_0(\Omega)$. Trying to go further, applying the same ideas, we get

$$\partial_t\partial_z^2\theta|_{\partial\Omega} = \partial_z^2 u_2|_{\partial\Omega},$$

where we cannot ensure anything on $\partial_z^2 u_2$. Hence we cannot suppose $\partial_z^2\theta = 0$. Nevertheless, we get

$$\partial_t\partial_z^2\bar{\theta}|_{\partial\Omega} = \int_{\mathbb{T}}\partial_z^2 u_2|_{\partial\Omega} = -\int_{\mathbb{T}}\partial_x\partial_z u_1|_{\partial\Omega} = 0.$$

Note that for higher orders of derivation, we cannot infer any cancellation in general.

□

Definition 3.2.3. *In the rest of the paper, we will set*

$$G(t,z) = \partial_z\bar{\theta}(t,z).$$

The above Lemma ensures that if θ_0 satisfies the assumptions of Lemma 3.2.1, $G|_{\partial\Omega} = \partial_z G|_{\partial\Omega} = 0$.

3.2.2 Study of the linearized system

This subsection is concerned with the study of the linear system

$$\begin{cases} \partial_t\theta' = (1-G)\partial_x\psi + S \\ \Delta^2\psi = \partial_x\theta' \\ \psi|_{\partial\Omega} = \partial_{\boldsymbol{n}}\psi|_{\partial\Omega} = 0, \theta'|_{t=0} = \theta'_0. \end{cases} \tag{3.10}$$

which is satisfied by θ' and ψ in the first place, with $G = \partial_z\bar{\theta}$ and $S = -\left(\nabla^\perp\psi\cdot\nabla\theta'\right)'$. It will also be satisfied for various derivates of θ' and ψ with different S. The term G will always be $\partial_z\bar{\theta}$.

Our goal is to analyze the long time behaviour of θ', under suitable decay assumptions on S. For later purposes, we have decomposed our results into several separate statements, whose proof is postponed to the end of the section. The first one is a uniform L^2 bound on the solutions when the source term is time integrable:

Lemma 3.2.4 (Uniform L^2 bound on solutions of the linearized system)**.** *Let $G \in L^\infty(\mathbf{R}_+, H^2)$, $S \in L^\infty(\mathbf{R}_+, L^2)$, and $\theta'_0 \in L^2$. Let $\theta' \in L^\infty(\mathbf{R}_+, L^2)$ be the unique solution of (3.10).*

Assume that S can be decomposed as $S = S_\perp + S_\parallel$ satisfying for some $\sigma, \delta > 0$,

$$\int_\Omega S_\perp\theta' = 0, \qquad \|S_\parallel\|_{L^2} \lesssim \frac{\sigma}{(1+t)^{1+\delta}}. \tag{3.11}$$

There exists a universal constant $\gamma_0 \in (0,1)$ such that if

$$\|G\|_{H^2} \leq \gamma_0, \tag{3.12}$$

then

$$\|\theta'\|_{L^2} \leq \|\theta'_0\|_{L^2} + C_\delta \sigma.$$

Remark 3.2.5. *The term $S_\perp$ will often have the structure $S_\perp = \boldsymbol{u} \cdot \nabla \theta'$: indeed, provided $\boldsymbol{u}$ and θ' have sufficient regularity, the divergence free condition and the non-penetration condition ensure that*

$$\int_\Omega (\boldsymbol{u} \cdot \nabla \theta')\theta' = \frac{1}{2}\int_\Omega \boldsymbol{u} \cdot \nabla |\theta'|^2 = -\frac{1}{2}\int_\Omega \underbrace{\operatorname{div} \boldsymbol{u}}_{=0} |\theta'|^2 + \frac{1}{2}\int_{\partial\Omega} \underbrace{\boldsymbol{u} \cdot \boldsymbol{n}}_{=0} |\theta'|^2 = 0.$$

Our second result, which is at the core of Theorem 3.1.1, gives a quantitative algebraic decay on θ':

Proposition 3.2.6. *Assume that the hypotheses of Proposition 3.2.4 are satisfied. There exists a universal constant $\gamma_0 \in (0,1)$ such that the following result holds. Assume moreover that θ' and $\partial_{\boldsymbol{n}}\theta'$ vanish on the boundaries of Ω, and that S decomposes into $S = S_\perp + S_\parallel + S_\Delta$ with for some $\sigma, \delta > 0$,*

$$\int_\Omega S_\perp \theta' = 0, \qquad \|S_\parallel\|_{L_2} \leq \frac{\sigma}{(1+t)^{1+\delta}}, \qquad \|S_\Delta\|_{L^2} \lesssim \frac{\gamma_0^{1/2}\|\Delta\psi\|_{L^2}}{(1+t)^{1/2}}. \tag{3.13}$$

Assume that G satisfies (3.12), and that there exists $A, \alpha \geq 0$ such that

$$\|\Delta^2 \partial_x^{-2} \theta'\|_{L^2} \leq \frac{A}{(1+t)^\alpha}. \tag{3.14}$$

Then

$$\|\theta'\|_{L^2} \lesssim \frac{\|\theta'_0\|_{L^2} + A + \sigma}{(1+t)^{\min(1+\alpha,\delta)}}.$$

In order to prove this quantitative decay, we shall need to analyze the structure of the dissipation term

$$-\int_\Omega \partial_x \psi \theta' = \int_\Omega (\Delta\psi)^2.$$

In previous works for different but related models [CCL19a], at this stage, an explicit spectral decomposition of the solution was used, relying on Fourier series. Note that such a spectral decomposition is also available for the operator $\Delta^{-2}\partial_x^2$ (see Proposition 3.8.3). However, since we cannot interpolate for an arbitrary regularity, we choose here to use a different approach. We replace this spectral analysis with the following result, which can be seen as an interpolation Lemma. It is noteworthy that in spite of its deceitfully simple form (and proof), this Lemma provides the correct scaling for the solutions.

Lemma 3.2.7. *For any $\alpha \geq 0$, and for all $\psi \in H^{8+\alpha-2}(\Omega)$ satisfying*

$$\Delta^2\psi|_{\partial\Omega} = \partial_{\boldsymbol{n}}\Delta^2\psi|_{\partial\Omega} = 0\,,$$

we have for all $K > 0$

$$\|\partial_x^{\alpha-1}\Delta^2\psi\|_{L^2}^2 \lesssim \frac{1}{K}\|\partial_x^\alpha \Delta\psi\|_{L^2}^2 + K^2\|\partial_x^{\alpha-3}\Delta^4\psi\|_{L^2}^2.$$

Proof. Since $\Delta^2\psi$ and $\partial_{\boldsymbol{n}}\Delta^2\psi$ vanish on the boundary $\partial\Omega$, we have after two integrations by parts

$$\begin{aligned}\|\partial_x^{\alpha-1}\Delta^2\psi\|_{L^2}^2 &= -\int_\Omega \partial_x^\alpha\Delta^2\psi\partial_x^{\alpha-2}\Delta^2\psi = -\int_\Omega \nabla\partial_x^\alpha\Delta\psi\cdot\nabla\partial_x^{\alpha-2}\Delta^2\psi\\ &= \int_\Omega \partial_x^\alpha\Delta\psi\partial_x^{\alpha-2}\Delta^3\psi \le \|\partial_x^\alpha\Delta\psi\|_{L^2}\|\partial_x^{\alpha-2}\Delta^3\psi\|_{L^2}.\end{aligned}$$

On another hand, we have also by integrations by parts

$$\begin{aligned}\|\partial_x^{\alpha-2}\Delta^3\psi\|_{L^2}^2 &= -\int_\Omega \partial_x^{\alpha-1}\Delta^3\psi\partial_x^{\alpha-3}\Delta^3\psi = -\int_\Omega \nabla\partial_x^{\alpha-1}\Delta^2\psi\cdot\nabla\partial_x^{\alpha-3}\Delta^3\psi\\ &= \int_\Omega \partial_x^{\alpha-1}\Delta^2\psi\partial_x^{\alpha-3}\Delta^4\psi \le \|\partial_x^{\alpha-1}\Delta^2\psi\|_{L^2}\|\partial_x^{\alpha-3}\Delta^4\psi\|_{L^2},\end{aligned}$$

hence, together with the previous bound, we obtain

$$\|\partial_x^{\alpha-1}\Delta^2\psi\|_{L^2}^2 \le \|\partial_x^\alpha\Delta\psi\|_{L^2}^{\frac43}\|\partial_x^{\alpha-3}\Delta^2\psi\|_{H^4}^{\frac23} \lesssim \frac{1}{K}\|\partial_x^\alpha\Delta\psi\|_{L^2}^2 + K^2\|\partial_x^{\alpha-3}\Delta^4\psi\|_{L^2}^2,$$

where Young's inequality is used for the last step. □

Let us now turn towards the proof of Propositions 3.2.4 and 3.2.6.

Proof of Proposition 3.2.4. The energy estimate in (3.10) writes

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2}^2 = \int_\Omega (1-G)\partial_x\psi\theta' + \int_\Omega S_\parallel\theta'.$$

A few integrations by parts provide

$$\begin{aligned}\int_\Omega (1-G)\partial_x\psi\theta' &= -\int_\Omega (1-G)\psi\Delta^2\psi\\ &= -\int_\Omega \Delta((1-G)\psi)\Delta\psi\\ &\le -(1-C\|G\|_{H^2})\|\Delta\psi\|_{L^2}^2.\end{aligned} \tag{3.15}$$

Since ψ belongs in particular to $H_0^2(\Omega)$, the boundary integrals vanish. At this point we have

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2}^2 \le -(1-C\gamma_0)\|\Delta\psi\|_{L^2}^2 + \int_\Omega S_\parallel\theta'. \tag{3.16}$$

Therefore if γ_0 is small enough, in a universal way, the first right-hand side term is non-positive. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2} \lesssim \|S_\parallel\|_{L^2} \lesssim \frac{\sigma}{(1+t)^{1+\delta}}$$

and since $\delta>0$ this inequality integrates as

$$\|\theta'\|_{L^2} \le \|\theta_0'\|_{L^2} + C_\delta\sigma.$$

□

Proof of Proposition 3.2.6. Back to (3.16) and plugging the decomposition of S we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2}^2 &\lesssim -(1-C\gamma_0)\|\Delta\psi\|_{L^2}^2 + \left(\|S_\parallel\|_{L^2}+\|S_\Delta\|_{L^2}\right)\|\theta'\|_{L^2}\\ &\lesssim -(1-C\gamma_0)\|\Delta\psi\|_{L^2}^2 + \frac{\sigma^2}{(1+t)^{1+2\delta}} + \frac{1}{1+t}\|\theta'\|_{L^2}^2,\end{aligned}$$

where Young inequality allows in particular to gather the dissipative terms $\gamma_0\|\Delta\psi\|^2$. Hence if γ_0 is small enough, we have for some $c_0 \in (0,1)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2}^2 + c_0\|\Delta\psi\|_{L^2}^2 \leq \frac{\sigma^2}{(1+t)^{1+2\delta}} + \frac{1}{1+t}\|\theta'\|_{L^2}^2.$$

The interpolation Lemma 3.2.7 yields, since $\Delta^2\psi = \partial_x\theta' = 0$ and $\partial_{\boldsymbol{n}}\Delta^2\psi = \partial_{\boldsymbol{n}}\partial_x\theta' = 0$ on the boundary, and making the choice $K \simeq \kappa/c_0(1+t)^{-1}$ with $\kappa > 0$ arbitrary large independently of the data,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2}^2 + \frac{\kappa}{1+t}\|\theta'\|_{L^2} \lesssim \frac{\|\Delta^2\partial_x^{-2}\theta'\|_{L^2}^2}{(1+t)^3} + \frac{\sigma^2}{(1+t)^{1+2\delta}}.$$

Plugging assumption (3.14) provides

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2}^2 + \frac{\kappa}{1+t}\|\theta'\|_{L^2}^2 \lesssim \frac{(A+\sigma)^2}{(1+t)^{\min(3+2\alpha,1+2\delta)}}$$

which integrates into

$$\|\theta'\|_{L^2} \lesssim \frac{\|\theta_0'\|_{L^2}+A+\sigma}{(1+t)^{\min(1+\alpha,\delta)}}.$$

□

Stability for more general stationary profiles. This is a remark about generalising our stability result to some other stationary profiles than $1-z$. We find sufficient conditions on general stratified Θ ensuring so. Basically it is sufficient for Θ to be sufficiently close to an affine function. We illustrate how the computations adapt in this case. For instance, the linear evolution equation on θ' for a general background profile Θ writes

$$\partial_t\theta' = -\partial_z\Theta\partial_x\psi,$$

and the computation (3.15) writes in this case

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2}^2 \leq \int_\Omega \partial_z\Theta|\Delta\psi|^2 + \int_\Omega [\Delta,\partial_z\Theta]\psi\Delta\psi.$$

The approach in our work consists in obtaining a negative sign for the main dissipative term $\int_\Omega \partial_z\Theta|\Delta\psi|^2$. This property is ensured for the sufficient condition

$$\sup_{(0,1)} \partial_z\Theta < 0,$$

Now, a bound on the commutator provides

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2}^2 \leq \left(\sup_{(0,1)} \partial_z\Theta + C\|\partial_z^2\Theta\|_{H^1}\right)\|\Delta\psi\|_{L^2}^2.$$

To stick to the proof developped in this paper, meaning to obtain a negative right handside in the above, it is necessary to have small enough derivatives of $\partial_z\Theta$ regarding the minimal slope of Θ. For the full system we obtain, instead of (3.15), an estimate of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta'\|_{L^2}^2 \leq \left(\sup_{(0,1)} \partial_z\Theta + C(\|G\|_{H^2} + \|\partial_z^2\Theta\|_{H^1})\right)\|\Delta\psi\|_{L^2}^2,$$

and the smallness condition (3.12) on G becomes

$$\|G\|_{H^2} + \|\partial_z^2\Theta\|_{H^1} \lesssim -\sup_{(0,1)} \partial_z\Theta.$$

This condition might appear at different steps of our demonstration, and require having higher order derivates of Θ small in L^2. By adjusting this assumption, the system remains stable in the sense of Theorem 3.1.1. Our demonstration only tackles the linear case for sake of simplicity.

3.2.3 Bootstrap argument

The purpose of this paragraph is to prove, thanks to a bootstrap argument, that under the assumptions of Theorem 3.1.1, the solution θ' of (3.5) enjoys the same decay rates as the ones predicted by the linear analysis (see Proposition 3.2.6). More precisely, we shall prove the following result:

Proposition 3.2.8. *Let $\theta_0 \in H^6(\Omega)$ such that $\theta_0|_{\partial\Omega} = \partial_{\boldsymbol{n}}\theta_0|_{\partial\Omega} = 0$, $\partial_z^2\bar{\theta}_0|_{\partial\Omega} = 0$.*
There exists $\varepsilon_0 > 0$ such that if $\|\theta_0\|_{H^6} \leq \varepsilon \leq \varepsilon_0$ the solution of (3.5) satifies

$$\|\partial_x^3\theta'(t)\|_{L^2} \lesssim \frac{\varepsilon}{1+t}, \quad \|\partial_x\theta'(t)\|_{H^4} \lesssim \varepsilon, \quad \|\bar{\theta}(t)\|_{H^5} \lesssim \varepsilon, \quad \forall t > 0. \tag{3.17}$$

Remark 3.2.9. *The interplay between horizontal derivatives of θ and the considered regularities is consistent with the operator $\Delta^{-2}\partial_x^2$ from the linearized system*

$$\partial_t\theta' = \partial_x\psi = \Delta^{-2}\partial_x^2\theta'.$$

Note that Δ^{-2} denotes the operator solving the bilaplacian $\Delta^2\psi = f$ equation endowed with the boundary condition $\psi = \partial_z\psi = 0$ on $\partial\Omega$.

A demonstration of the decays validity is provided in the rest of this Section. Remarks motivating the necessity of the bootstrap hypothesis and the method in general are included along the demonstration. We also develop our understanding of the obstacle to the iteration of this method to higher regularity on the perturbation.

Bootstrap assumption and general argument. Let $0 < B < 1$ and $C_0 > 1$ constants to adjust. Let $\theta_0 \in H_0^2 \cap H^6$ such that $\|\theta_0\|_{H^6} \leq B/C_0$. In particular $\|\partial_x^3\theta_0'\|_{L^2} \leq B/C_0$ and $\|\partial_x\theta_0'\|_{H^4} \leq B/C_0$. If C_0 is large enough, independently of the data, we also have

$$\|\psi_0\|_{H^4} \leq C\|\partial_x\theta'\|_{L^2} \leq CB/C_0,$$

and therefore

$$\|\partial_t\partial_x\theta'|_{t=0}\|_{L^2} \le \|1-\partial_z\bar\theta_0\|_{L^\infty}\|\partial_x^2\psi_0\|_{L^2} + \|\partial_x(\boldsymbol{u}_0\cdot\nabla\theta_0')\|_{L^2} \le C(B/C_0)^2 \le B.$$

By continuity of the Sobolev norms of θ, ensured by 2.6.1, there exists a maximal time $T^* \in \mathbf{R}_+ \cup \{+\infty\}$ such that the following inequalities are satisfied on $[0,T^*)$:

$$\begin{aligned} &\|\partial_x^3\theta'\|_{L^2} \le \frac{B}{1+t}, && \|\partial_x\theta'\|_{H^4} \le B,\\ &\|G\|_{H^2} \le B, && \|\partial_t\partial_x\theta'\|_{L^2} \le \frac{B}{(1+t)^2}. \end{aligned} \tag{3.18}$$

We recall that these decay rates follow the behaviour of the linearized system, see Proposition 3.2.6.

Let us assume by contradiction that $T^* < +\infty$. We show below by a bootstrap argument that hypothesis (3.18), combined with Propositions 3.2.4 and 3.2.6, actually leads to an improvement of the inequalities, satisfied with some new constant $0 < \underline{B} < B$ which contradicts with the maximality of T^*. Whence $T^* = +\infty$ and inequalities (3.18) hold for all time.

Preliminary bounds. All along the proof we require estimates on θ' and ψ derivated from the bootstrap hypothesis (3.18). For the sake of readability, we introduce the following notation:

$$\underbrace{\|f\|\|g\|}_{r+r'} \qquad \text{when} \qquad \|f\| \lesssim \frac{B}{(1+t)^r} \text{ and } \|g\| \lesssim \frac{B}{(1+t)^{r'}}.$$

First, from an integration by parts

$$\|\partial_x^2\theta'\|_{H^2}^2 = \int_\Omega \partial_x^2\Delta\theta'\partial_x^2\Delta\theta' = -\int_\Omega \partial_x^3\theta'\partial_x\Delta^2\theta' \le \|\partial_x^3\theta'\|_{L^2}\|\partial_x\theta'\|_{H^4}, \tag{3.19}$$

we deduce, by assumption (3.18), for all $t\in(0,T^*)$

$$\|\partial_x^2\theta'\|_{H^2} \lesssim \|\partial_x^3\theta'\|_{L^2}^{1/2}\|\partial_x\theta'\|_{H^4}^{1/2} \lesssim \frac{B}{(1+t)^{1/2}}.$$

We get by interpolation for any $0\le m\le 4$, for all $t\in(0,T^*)$

$$\|\partial_x\theta'\|_{H^m} \lesssim \|\partial_x\theta'\|_{L^2}^{1-m/4}\|\partial_x\theta'\|_{H^4}^{m/4} \lesssim \frac{B}{(1+t)^{1-m/4}}.$$

We will frequently use *Agmon's inequality* in dimension 2, namely

$$\forall f\in H_0^1\cap H^2(\Omega), \qquad \|f\|_{L^\infty} \lesssim \|f\|_{L^2}^{1/2}\|f\|_{H^2}^{1/2},$$

together with the following direct consequence

$$\forall f\in H_0^2\cap H^4(\Omega), \qquad \|\nabla f\|_{L^\infty} \lesssim \|f\|_{L^2}^{1/2}\|f\|_{H^4}^{1/2}.$$

We infer in particular, for all $t \in (0, T^*)$

$$\|\partial_x^2 \theta'\|_{L^\infty} \lesssim \|\partial_x^2 \theta'\|_{L^2}^{1/2} \|\partial_x^2 \theta'\|_{H^2}^{1/2} \lesssim \frac{B}{(1+t)^{3/4}},$$
$$\|\nabla \partial_x \theta'\|_{L^\infty} \lesssim \|\partial_x \theta'\|_{L^2}^{1/2} \|\partial_x \theta'\|_{H^4}^{1/2} \lesssim \frac{B}{(1+t)^{1/2}}.$$

We also need estimates on ψ. Any Sobolev norm of order larger than 4 inheritates decays from θ' thanks to Proposition 3.8.1, providing, for $t \in (0, T^*)$,

$$\|\partial_x^2 \psi\|_{H^4} \lesssim \|\partial_x^3 \theta'\|_{L^2} \lesssim \frac{B}{1+t},$$
$$\|\partial_x \psi\|_{H^6} \lesssim \|\partial_x^2 \theta'\|_{H^2} \lesssim \frac{B}{(1+t)^{1/2}}.$$

We also need higher order decays on $\partial_x \psi$ in $L^2(\Omega)$. We access this quantity thanks to the control of $\partial_t \theta'$ by rewriting

$$\partial_x \psi = \frac{\partial_t \theta' + (\boldsymbol{u} \cdot \nabla \theta')'}{1 - G}.$$

We know that $\|G\|_{L^\infty} \lesssim \|G\|_{H^2} \leq B$ so it is enough to ask B to be smaller than 1 to ensure that the inverse of $(1 - G)$ is well-defined. It alows to estimate $\partial_x \psi$ and $\partial_x^2 \psi$ in L^2. We illustrate the computation for $\partial_x \psi$ since the same reasoning applies for $\partial_x^2 \psi$ with a few extra terms.

$$\begin{aligned}\|\partial_x \psi\|_{L^2} &\lesssim \|\partial_t \theta'\|_{L^2} + \|\boldsymbol{u} \cdot \nabla \theta'\|_{L^2} \\ &\lesssim \|\partial_t \theta'\|_{L^2} + \|\partial_z \psi\|_{L^\infty} \|\partial_x \theta'\|_{L^2} + \|\partial_x \psi\|_{L^2} \|\partial_z \theta'\|_{L^\infty} \\ &\lesssim \underbrace{\|\partial_t \theta'\|_{L^2}}_{2} + \underbrace{\|\partial_x \theta'\|_{L^2}^2}_{2\times 2} + \|\partial_x \psi\|_{L^2} \frac{B}{(1+t)^{1/2}}.\end{aligned}$$

Hence for $B > 0$ small enough once again we get

$$\|\partial_x^2 \psi\|_{L^2} \lesssim \frac{B}{(1+t)^2}.$$

By interpolation and Agmon inequalities we deduce, in the same fashion as above, the following decay estimates, with the latest valid for $0 \leq m \leq 4$,

$$\|\partial_x^2 \psi\|_{H^2} \lesssim \frac{B}{(1+t)^{3/2}}, \qquad \|\partial_x^2 \psi\|_{L^\infty} \lesssim \frac{B}{(1+t)^{7/4}}, \qquad \|\nabla \partial_x^2 \psi\|_{L^\infty} \lesssim \frac{B}{(1+t)^{3/2}},$$
$$\|\nabla^2 \psi\|_{L^\infty} \lesssim \frac{B}{(1+t)^{5/4}}, \quad \|\partial_x \psi\|_{H^m} \lesssim \|\partial_x \psi\|_{L^2}^{1-m/4} \|\partial_x \psi\|_{H^4}^{m/4} \lesssim \frac{B}{(1+t)^{2-m/4}}.$$

H^6 bound on the solution and H^4 bound on G. In our nonlinear bootstrap argument, we shall need some high Sobolev bound on the solution. In order to lighten the proof of the bootstrap as much as possible, we isolate in the present paragraph this technical step.

Lemma 3.2.10. *Let $\theta_0 \in H^6(\Omega)$ such that $\theta_0|_{\partial\Omega} = \partial_{\boldsymbol{n}} \theta_0|_{\partial\Omega} = 0$, $\partial_z^2 \bar{\theta}_0|_{\partial\Omega} = 0$. Let T^* be the*

maximal time on which the assumptions (3.18) *are satisfied. Then for all* $t \in (0, T^*)$,

$$\|\theta'(t)\|_{H^6} \lesssim B(1+t)^{1/2}, \qquad \|G(t)\|_{H^4} \leq \frac{B}{C_0} + CB^2, \qquad \|\partial_t G(t)\|_{L^\infty} \lesssim \frac{B^2}{(1+t)^2}.$$

Proof. We cannot estimate θ' in H^6 directly from its evolution equation since it requires an assumption on $G = \partial_z \bar{\theta} \in H^6(\Omega)$, therefore on θ in $H^7(\Omega)$. To get around this, we directly perform an estimate from the whole perturbed evolution equation, namely

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = u_2$$

for any derivative of order 6 (and less) as follows,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^6\theta\|_{L^2}^2 + \int_\Omega [\partial^6, \boldsymbol{u}\cdot\nabla]\theta\partial^6\theta = \int_\Omega \partial^6\partial_x\psi\partial^6\theta,$$

where the commutator comes from the incompressibility assumption. Hence we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^6} \lesssim \|\partial_x\psi\|_{H^6} + \|[\partial^6, \boldsymbol{u}\cdot\nabla]\theta'\|_{L^2}.$$

On the one hand, the first term is dealt with thanks to the bilaplacian regularization,

$$\|\partial_x\psi\|_{H^6} \lesssim \|\partial_x^2\theta'\|_{H^2} \lesssim \frac{B}{(1+t)^{1/2}}.$$

Notice that this 1/2-algebraic decay, issued from the linear system, is critical to prove the 1/2-algebraic growth control of θ in $H^6(\Omega)$. Any smaller decay would not be sufficient to get so. Since the bilaplacian regularization cannot provide better, that $\|\partial_x^3\theta'\|_{L^2}$ decays as $(1+t)^{-1}$ and $\|\partial_x\theta'\|_{H^4} \leq \underline{B}$. This can be seen by interpolation or with the computation (3.19).

Concerning the nonlinear term, we rely on the following *tame estimate*, valid for any $m \in \mathbf{N}$,

$$\forall f, g \in H^m \cap L^\infty(\Omega), \qquad \|fg\|_{H^m} \lesssim \|f\|_{L^\infty}\|g\|_{H^m} + \|f\|_{H^m}\|g\|_{L^\infty}, \tag{3.20}$$

which leads to

$$\forall f \in H^m \cap W^{1,\infty}(\Omega), g \in H^{m-1} \cap L^\infty(\Omega), \quad \|[D^m, f]g\|_{L^2} \lesssim \|\nabla f\|_{L^\infty}\|g\|_{H^{m-1}} + \|f\|_{H^m}\|g\|_{L^\infty}. \tag{3.21}$$

Hence we decompose the nonlinear commutator into

$$[\partial^6, \boldsymbol{u}\cdot\nabla]\theta = -[\partial^6, \partial_z\psi\partial_x]\theta + [\partial^6, \partial_x\psi\partial_z]\theta.$$

Each part estimates thanks to (3.21) as follows

$$\begin{aligned}
\|[\partial^6, \partial_z\psi]\partial_x\theta\|_{L^2} &\lesssim \|\nabla\partial_z\psi\|_{L^\infty}\|\partial_x\theta\|_{H^5} + \|\partial_z\psi\|_{H^6}\|\partial_x\theta\|_{L^\infty} \\
&\lesssim \underbrace{\|\nabla^2\psi\|_{L^\infty}}_{5/4}\|\theta\|_{H^6} + \underbrace{\|\partial_x\theta'\|_{H^3}\|\partial_x\theta'\|_{L^\infty}}_{1/4+3/4} \\
&\lesssim \frac{B}{(1+t)^{5/4}}\|\theta\|_{H^6} + \frac{B^2}{1+t},
\end{aligned}$$

and

$$\begin{aligned}\|[\partial^6,\partial_x\psi]\partial_z\theta'\|_{L^2} &\lesssim \|\nabla\partial_x\psi\|_{L^\infty}\|\partial_z\theta\|_{H^5}+\|\partial_x\psi\|_{H^6}\|\partial_z\theta\|_{L^\infty}\\ &\lesssim \underbrace{\|\nabla\partial_x\psi\|_{L^\infty}}_{3/2}\|\theta\|_{H^6}+\underbrace{\|\partial_x^2\theta'\|_{H^2}}_{1/2}\|\nabla\theta\|_{L^\infty}\\ &\lesssim \frac{B}{(1+t)^{7/4}}\|\theta\|_{H^6}+\frac{B^2}{1+t}+\frac{B^2}{(1+t)^{1/2}}\end{aligned}$$

where we observed in particular that

$$\|\nabla\theta\|_{L^\infty}\le\|\nabla\theta'\|_{L^\infty}+\|G\|_{L^\infty}\lesssim\frac{B}{(1+t)^{1/2}}+B.$$

In the end, gathering and summing up all these bounds provides

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^6}\lesssim\frac{B}{(1+t)^{5/4}}\|\theta\|_{H^6}+\frac{B^2}{(1+t)^{1/2}},$$

and we get, for some universal constant C,

$$\|\theta\|_{H^6}\le\|\theta_0\|_{H^6}+CB(1+t)^{1/2}\lesssim B(1+t)^{1/2}.$$

Eventually, let us prove decaying bounds on $\partial_t G$ and uniform bounds on G. We recall that $G=\partial_z\bar\theta$ depends only on the variables t and z. From the evolution equation and one integration by parts we observe

$$\partial_t\bar\theta=-\overline{\boldsymbol{u}\cdot\nabla\theta'}=\int_{\mathbb{T}}\partial_z\psi\partial_x\theta'-\partial_x\psi\partial_z\theta'=-\partial_z\int_{\mathbb{T}}\partial_x\psi\theta',$$

so we can write

$$\partial_t G=-\partial_z^2\int_{\mathbb{T}}\partial_x\psi\theta'.$$

The same arguments as above lead to

$$\|\partial_t G\|_{L^2(0,1)}\lesssim\underbrace{\|\partial_x\psi\|_{L^\infty}\|\theta'\|_{H^2}}_{7/4+1/2}+\underbrace{\|\partial_x\psi\|_{H^2}\|\theta'\|_{L^\infty}}_{3/2+3/4}\lesssim\frac{B^2}{(1+t)^{2+1/4}}.$$

Using the H^6 estimate, we also have

$$\|\partial_t G\|_{H^4(0,1)}\lesssim\underbrace{\|\partial_x\psi\|_{L^\infty}\|\theta'\|_{H^6}}_{7/4-1/2}+\underbrace{\|\partial_x\psi\|_{H^6}\|\theta'\|_{L^\infty}}_{1/2+3/4}\lesssim\frac{B^2}{(1+t)^{1+1/4}}.$$

Since the right-hand side of the above inequality is time-integrable, we infer that

$$\|G\|_{H^4(0,1)}\le\|\partial_z\theta_0\|_{H^4}+\int_0^t\|\partial_t G\|_{H^4(0,1)}\le B/C_0+CB^2.$$

Hence $\|G\|_{H^4(0,1)}$ is as small as necessary for $B>0$ small enough. Moreover, for all $t\in(0,T^*)$,

$$\|\partial_t G\|_{L^\infty}\lesssim \underbrace{\|\partial_t G\|_{L^2}^{3/4}\|\partial_t G\|_{H^4}^{1/4}}_{3/4\times 9/4+1/4\times 5/4}\lesssim \frac{B^2}{(1+t)^2}.$$

□

Remark 3.2.11. *It would be tempting to proceed to the same computations as in (3.15) in order to get a signed term in our estimate, which would allow to simply forget about its contribution, as for lower order derivatives. Doing so requires to control the boundary integrals, which do not vanish* a priori *in this case,*

$$\begin{aligned}\int_\Omega D^6\partial_x\psi D^6\theta &= -\int_\Omega D^6\psi\Delta^2 D^6\psi\\ &= -\|\Delta D^6\psi\|_{L^2}^2+\int_{\partial\Omega}(\partial_{\boldsymbol{n}}D^6\psi\Delta D^6\psi - D^6\psi\partial_{\boldsymbol{n}}D^6\Delta\psi).\end{aligned}$$

For instance, trying to bound the integral comprising the higher order of $z-$derivatives on ψ provides at best

$$\left|\int_{\partial\Omega}D^6\psi\partial_z\Delta D^6\psi\right|\lesssim\|\partial_x\psi\|_{H^7}\|\partial_x^{-1}\psi\|_{H^{10}}\lesssim B^2(1+t)^{1/4}.$$

This estimate ensures no better growth control than $\|\theta'\|_{H^6}\lesssim(1+t)^{3/4}$, which is not enough to close the bootstrap and get the control by $(1+t)^{1/2}$.

Improvements of the bootstrap bounds. We now improve the uniform bound on θ' and $\partial_x\theta'$ in $H^4(\Omega)$, relying on the linear analysis from Subsection 3.2.2. Since $\|\theta'\|_{H^4}\leq\|\partial_x\theta'\|_{H^4}$, it is enough to treat $\partial_x\theta'$. Also we have according to Proposition 3.8.1 the inequality

$$\|\partial_x\theta'\|_{H^4}\lesssim\|\Delta^2\partial_x\theta'\|_{L^2}$$

since $\partial_x\theta'$ belongs in particular to $H^2_0\cap H^4(\Omega)$ as detailed in Subsection 3.2.1, so it is enough to deal with $\partial_x\Delta^2\theta'$ in $L^2(\Omega)$.

Lemma 3.2.12. *As long as the bootstrap hypothesis (3.18) holds we have*

$$\|\partial_x\Delta^2\theta'\|_{L^2}\leq\|\partial_x\Delta^2\theta_0'\|_{L^2}+CB^2. \tag{3.22}$$

Proof. In view of the application of Proposition 3.2.6 to $\Delta^2\partial_x\theta'$, we observe that its evolution is governed by the equation

$$\partial_t\Delta^2\partial_x\theta' = (1-G)\partial_x\Delta^2\partial_x\psi-[\Delta^2\partial_x,G]\partial_x\psi+\Delta^2\partial_x(\boldsymbol{u}\cdot\nabla\theta')',$$

which is of the form (3.10) with $\Delta^2\partial_x\psi=\partial_x^2\theta'$ and $\partial_z\Delta^2\partial_x\psi=\partial_z\partial_x^2\theta'$ vanishing on the boundary $\partial\Omega$ and

$$S=\underbrace{-[\Delta^2\partial_x,G]\partial_x\psi-[\Delta^2\partial_x,\boldsymbol{u}\cdot\nabla]\theta'}_{S_\parallel}+\underbrace{\boldsymbol{u}\cdot\nabla\Delta^2\partial_x\theta'}_{S_\perp}.$$

We already know that $\|G\|_{H^2}$ satisfies the smallness assumption (3.18) for $B>0$ small enough. We show that $S_\parallel$ presents an algebraic decay strictly larger than 1, as in (3.11). To do so we apply the tame estimates (3.20) to the two commutator terms, using the assumption (3.18).

Let us emphasize that we need to be thorough by substituting $\boldsymbol{u} = \nabla^\perp \psi$ such that the transport operator writes

$$\boldsymbol{u}\cdot\nabla\cdot = -\partial_z\psi\partial_x\cdot + \partial_x\psi\partial_z\cdot .$$

Hence the nonlinear term only presents formally a vertical derivative of order 1. This makes a difference in the estimates and allows to reach more optimal decay rates.

On the one hand we get for the perturbation due to G,

$$\|[\Delta^2, G]\partial_x^2\psi\|_{L^2} \leq \|G\|_{H^4}\underbrace{\|\partial_x^2\psi\|_{L^\infty}}_{7/4-3/32} + \|\nabla G\|_{L^\infty}\underbrace{\|\partial_x^2\psi\|_{H^3}}_{5/4-2/32} \lesssim \frac{B^2}{(1+t)^{1+1/8}},$$

where the algebraic decay exponent is not optimal but enough and easier to read than $1+5/32$. Note that we used here the uniform H^4 bound on G from Lemma 3.2.10. On the other hand the contribution of $[\Delta^2\partial_x, \boldsymbol{u}\cdot\nabla]\theta'$ splits into four terms as follows

$$\begin{aligned}[\Delta^2\partial_x, \boldsymbol{u}\cdot\nabla]\theta' = &-\Delta^2(\partial_x\partial_z\psi\partial_x\theta') + \Delta^2(\partial_x^2\psi\partial_z\theta')\\ &- [\Delta^2, \partial_z\psi]\partial_x\theta' + [\Delta^2, \partial_x\psi]\partial_z\theta'.\end{aligned}$$

Each term estimates accordingly. The limiting decay comes from

$$\begin{aligned}\|\Delta^2(\partial_x^2\psi\partial_z\theta')\|_{L^2} &\lesssim \|\partial_x^2\psi\|_{H^4}\|\partial_z\theta'\|_{L^\infty} + \|\partial_x^2\psi\|_{L^\infty}\|\partial_z\theta'\|_{H^4}\\ &\lesssim \underbrace{\|\partial_x^3\theta'\|_{L^2}\|\nabla\theta'\|_{L^\infty}}_{1+1/2} + \underbrace{\|\partial_x^2\psi\|_{L^\infty}\|\theta'\|_{H^5}}_{7/4-1/4} \lesssim \frac{B^2}{(1+t)^{1+1/4}}.\end{aligned}$$

The other term admit the bound $B^2(1+t)^{-1/2}$, such as

$$\begin{aligned}\|\Delta^2(\partial_x\partial_z\psi\partial_x\theta')\|_{L^2} &\lesssim \|\partial_x\partial_z\psi\|_{H^4}\|\partial_x\theta'\|_{L^\infty} + \|\partial_x\partial_z\psi\|_{L^\infty}\|\partial_x\theta'\|_{H^4}\\ &\lesssim \underbrace{\|\partial_x\psi\|_{H^5}\|\partial_x\theta'\|_{L^\infty}}_{3/4+3/4} + \underbrace{\|\nabla\partial_x\psi\|_{L^\infty}\|\partial_x\theta'\|_{H^4}}_{3/2+0} \lesssim \frac{B^2}{(1+t)^{1+1/2}}.\end{aligned}$$

Gathering these estimates provides

$$\|S_\parallel\|_{L^2} \lesssim \frac{B^2}{(1+t)^{1+1/8}},$$

and Lemma 3.2.4 applies, ensuring

$$\|\Delta^2\partial_x\theta'\|_{L^2} \leq \|\Delta^2\partial_x\theta_0'\|_{L^2} + CB^2.$$

□

Lemma 3.2.13. *As long as the bootstrap hypothesis (3.18) holds, we have*

$$\|\partial_x^3\theta'\|_{L^2} \lesssim \frac{\|\theta_0'\|_{H^5} + B^2}{1+t}.$$

Proof. Note that $\partial_x^3\theta'$ satisfies (3.10) with the source term

$$S = S_\parallel = -\partial_x^3(\boldsymbol{u}\cdot\nabla\theta').$$

We can bound the whole term $S = S_{\|}$ as follows,

$$\begin{aligned}\|S\|_{L^2} &\leq \|\partial_x^3(\partial_z\psi\partial_x\theta')\|_{L^2} + \|\partial_x^3(\partial_x\psi\partial_z\theta')\|_{L^2}\\ &\lesssim \|\partial_x^3\partial_z\psi\|_{L^2}\|\partial_x\theta'\|_{L^\infty} + \|\partial_z\psi\|_{L^\infty}\|\partial_x^4\theta'\|_{L^2}\\ &\quad + \|\partial_x^4\psi\|_{L^2}\|\partial_z\theta'\|_{L^\infty} + \|\partial_x\psi\|_{L^\infty}\|\partial_x^3\partial_z\theta'\|_{L^2}\\ &\lesssim \underbrace{\|\partial_x^2\psi\|_{H^2}\|\partial_x\theta'\|_{L^\infty}}_{2+3/4>2} + \underbrace{\|\nabla\psi\|_{L^\infty}\|\partial_x^2\theta'\|_{H^2}}_{3/2+1/2=2}\\ &\quad + \underbrace{\|\partial_x^2\psi\|_{H^2}\|\nabla\theta'\|_{L^\infty}}_{3/2+3/4>2} + \underbrace{\|\partial_x\psi\|_{L^\infty}\|\partial_x^2\theta'\|_{H^2}}_{7/4+1/2>2} \lesssim \frac{B^2}{(1+t)^2}.\end{aligned}$$

Assumption (3.13) is satisfied with $\delta = 1$. Besides, the norm of $(\Delta^2\partial_x^{-2})\partial_x^3\theta' = \Delta^2\partial_x\theta'$ is bounded according to (3.22), so assumption (3.14) is satisfied with $A = \|\Delta^2\partial_x\theta_0'\|_{L^2} + CB^2$ and $\alpha = 0$. Moreover, the traces of $\partial_x^3\theta'$ and $\partial_{\boldsymbol{n}}\partial_x^3\theta'$ vanish as a direct consequence of 3.2.1. Therefore $\min(1+\alpha,\delta) = 1$ and Proposition 3.2.6 provides

$$\|\partial_x^3\theta'\|_{L^2} \leq \left(\|\partial_x^3\theta_0'\|_{L^2} + C\|\Delta^2\partial_x\theta'\|_{L^\infty((0,t),L^2)} + B^2\right)\frac{1}{1+t}.$$

Using (3.22), we obtain the desired estimate. □

Lemma 3.2.14. *Under assumptions (3.18) we have*

$$\|\partial_t\partial_x\theta'\|_{L^2} \leq \frac{\|\partial_x\theta_0'\|_{H^4} + CB^2}{(1+t)^2}.$$

Proof. Taking $\partial_t\partial_x\theta' \rightsquigarrow \theta'$, equation (3.10) is satisfied with

$$\partial_t\partial_x\psi \rightsquigarrow \psi, \quad S = -\partial_t G\partial_x^2\psi - \partial_t\partial_x(\boldsymbol{u}\cdot\nabla\theta').$$

Note that $\partial_t\partial_x\theta'$ and $\partial_{\boldsymbol{n}}\partial_t\partial_x\theta'$ vanish on the boundary, from Lemma 3.2.1. We have to bound $(\Delta^2\partial_x^{-2})\partial_t\partial_x\theta'$ in $L^2(\Omega)$. This is done by applying the convenient operator to the equation

$$\Delta^2\partial_x^{-1}\partial_t\theta' = \Delta^2((1-G)\partial_x^{-1}\psi) - \Delta^2\partial_x^{-1}(\boldsymbol{u}\cdot\nabla\theta')),$$

the norm of which bounds as

$$\|\Delta^2\partial_x^{-1}\partial_t\theta'\|_{L^2} \leq (1+\|G\|_{H^4})\|\Delta^2\partial_x^{-1}\psi\|_{L^2} + \|\Delta^2\partial_x^{-1}(\boldsymbol{u}\cdot\nabla\theta')\|_{L^2}$$

since

$$\begin{aligned}\|\boldsymbol{u}\cdot\nabla\theta'\|_{H^4} &\lesssim \|\boldsymbol{u}\|_{H^4}\|\nabla\theta'\|_{L^\infty} + \|\boldsymbol{u}\|_{L^\infty}\|\theta'\|_{H^5}\\ &\lesssim \underbrace{\|\psi\|_{H^5}\|\nabla\theta'\|_{L^\infty}}_{3/4+1/2} + \underbrace{\|\nabla\psi\|_{L^\infty}\|\theta'\|_{H^5}}_{3/2-1/4} \lesssim \frac{B^2}{1+t}.\end{aligned}$$

satisfying assumption (3.14) with $\alpha = 1$.

Let us set $S_\perp = \boldsymbol{u}\cdot\nabla\partial_t\partial_x\theta'$, indeed orthogonal to $\partial_t\partial_x\theta'$ in $L^2(\Omega)$. Therefore we have to consider

$$S_{\|} + S_\Delta = -\partial_t G\partial_x\psi - \partial_x\boldsymbol{u}\cdot\nabla\partial_t\theta' - \partial_t\partial_x\boldsymbol{u}\cdot\nabla\theta' - \partial_t\boldsymbol{u}\cdot\nabla\partial_x\theta',$$

and find how to decompose such that

$$\|S_{\|}\|_{L^2} \lesssim \frac{B^2}{(1+t)^3}, \qquad \|S_{\Delta}\|_{L^2} \lesssim \frac{\gamma_0^{1/2}\|\Delta\partial_t\partial_x\psi\|_{L^2}}{(1+t)^{1/2}}.$$

The first term bounds directly as follows,

$$\|\partial_t G\partial_x^2\psi\|_{L^2} \leq \underbrace{\|\partial_t G\|_{L^\infty}\|\partial_x^2\psi\|_{L^2}}_{2+2} \lesssim \frac{B^2}{(1+t)^4}.$$

The second requires for instance a bound on $\partial_t\theta'$ in $H^1(\Omega)$, obtained directly from the evolution equation,

$$\begin{aligned}\|\partial_t\theta'\|_{H^1} &\lesssim (1+\gamma_0)\|\partial_x\psi\|_{H^1} + \|\boldsymbol{u}\cdot\nabla\theta'\|_{H^1}\\ &\lesssim (1+\gamma_0)\|\partial_x\psi\|_{H^1} + \|\boldsymbol{u}\|_{H^1}\|\nabla\theta'\|_{L^\infty} + \|\boldsymbol{u}\|_{L^\infty}\|\nabla\theta'\|_{H^1}\\ &\lesssim (1+\gamma_0)\underbrace{\|\partial_x\psi\|_{H^1}}_{7/4} + \underbrace{\|\psi\|_{H^2}\|\nabla\theta'\|_{L^\infty}}_{3/2+1/2} + \underbrace{\|\nabla\psi\|_{L^\infty}\|\theta'\|_{H^2}}_{3/2+1/2} \lesssim \frac{B}{(1+t)^{7/4}}.\end{aligned}$$

Hence

$$\|\partial_x\boldsymbol{u}\cdot\nabla\partial_t\theta'\|_{L^2} \lesssim \|\partial_x\boldsymbol{u}\|_{L^\infty}\|\partial_t\theta'\|_{H^1} \lesssim \underbrace{\|\nabla\partial_x\psi\|_{L^\infty}\|\partial_t\theta'\|_{H^1}}_{3/2+7/4} \lesssim \frac{B^2}{(1+t)^3},$$

and $S_{\|} := -\partial_t G\partial_x^2\psi - \partial_x\boldsymbol{u}\cdot\nabla\partial_t\theta'$ satisfies the assumption (3.13) with $\sigma = CB^2$ and $\delta = 2$. Pursuing our computations,

$$\|\partial_t\partial_x\boldsymbol{u}\cdot\nabla\theta'\|_{L^2} \lesssim \|\Delta\partial_t\partial_x\psi\|_{L^2}\underbrace{\|\nabla\theta'\|_{L^\infty}}_{1/2} \lesssim \frac{B\|\Delta\partial_t\partial_x\psi\|_{L^2}}{(1+t)^{1/2}},$$

and the same consideration applies for

$$\|\partial_t\boldsymbol{u}\cdot\nabla\partial_x\theta'\|_{L^2} \lesssim \|\nabla\partial_t\psi\|_{L^2}\underbrace{\|\nabla\partial_x\theta'\|_{L^\infty}}_{1/2} \lesssim \frac{B\|\Delta\partial_t\partial_x\psi\|_{L^2}}{(1+t)^{1/2}}.$$

For $B > 0$ small enough we have $B \leq \gamma_0^{1/2}$, so that

$$S_{\Delta} = \partial_t\partial_x\boldsymbol{u}\cdot\nabla\theta' - \partial_t\boldsymbol{u}\cdot\nabla\partial_x\theta'$$

indeed satisfies assumption (3.13). Finally Proposition 3.2.6 applies with $\min(1+\alpha,\delta) = 2$ and we obtain

$$\|\partial_t\partial_x\theta'\|_{L^2} \lesssim \frac{B/C_0 + CB^2}{(1+t)^2}.$$

□

Conclusion. Let us close the bootstrap argument. Assuming $\|\theta_0\|_{H^6} \leq B/C_0$, we had, by continuity in time of the Sobolev norms of θ ensured by Theorem 2.6.1, existence of a maximal

time $T^* \in \mathbf{R}_+ \cup \{+\infty\}$ such that (3.18) is satisfied for any $t \in [0, T^*)$, reported here

$$\begin{aligned} &\|\partial_x^3 \theta'\|_{L^2} \leq \frac{B}{1+t}, && \|\partial_x \theta'\|_{H^4} \leq B, \\ &\|G\|_{H^2} \leq B, && \|\partial_t \partial_x \theta'\|_{L^2} \leq \frac{B}{(1+t)^2}. \end{aligned} \tag{3.23}$$

These decay estimate induces, as shown in Lemmas 3.2.10, 3.2.12, 3.2.13 and 3.2.14 that (3.23) holds for another constant $\underline{B}$ defined as

$$\underline{B} = \frac{CB}{C_0} + CB^2,$$

where $C > 0$ is universal. By choosing B small and C_0 large enough, we have $\underline{B} < B$ and inequalities (3.23) are strictly satisfied for any $t \in [0, T^*)$. Therefore T^* must be $+\infty$, otherwise the continuity of $t \mapsto \|\theta(t)\|_{H^6}$ would imply the existence of a larger validity time interval for (3.23). In the end, these bounds are valid for all time, and setting $\varepsilon_0 = B/C_0$ closes demonstration of Proposition 3.2.8 .

Remark 3.2.15 (Obstacle to a generalisation at any order)**.** *Motivated by the fact that the following perturbated subproblem is stable under horizontal derivation*

$$\begin{cases} \partial_t \partial_x^\ell \theta' = (1-G)\partial_x^{\ell+1}\psi \\ \Delta^2 \partial_x^{\ell-1}\psi = \partial_x^\ell \theta', \\ \partial_x^{\ell-1}\psi|_{\partial\Omega} = \partial_{\boldsymbol{n}} \partial_x^{\ell-1}\psi|_{\partial\Omega} = 0, \end{cases}$$

we could expect to propagate arbitrary high horizontal regularity on θ'. Nevertheless, our proof relies on the control

$$\|\theta\|_{H^6} \lesssim (1+t)^{1/2},$$

that we can obtain thanks to the classical divergence free condition on $\boldsymbol{u}$ cancelling the extra-derivative term. Trying to do the same on $\partial_x^\alpha \theta$, driven by

$$\partial_t \partial_x^\alpha \theta + \sum_{\beta=0}^{\alpha-1} C_{\alpha,\beta} \partial_x^{\alpha-\beta} \boldsymbol{u} \cdot \nabla \partial_x^\beta \theta + \boldsymbol{u} \cdot \nabla \partial_x^\alpha \theta = \partial_x^{\alpha+1}\psi,$$

so that the following estimation does not close, even though one of its term does not contribute, just as in the initial equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha \theta\|_{H^6}^2 + \sum_{\beta=0}^{\alpha-1} C_{\alpha,\beta} \int_\Omega D^6(\partial_x^{\alpha-\beta}\boldsymbol{u} \cdot \nabla \partial_x^\beta \theta) D^6 \partial_x^\alpha \theta + \frac{1}{2} \underbrace{\int_\Omega \boldsymbol{u} \cdot \nabla |D^6 \partial_x^\alpha \theta|^2}_{=0} \leq \|\partial_x^\alpha \psi\|_{H^6} \|\partial_x^\alpha \theta\|_{H^6},$$

since crossed derivatives integrands do no lead to a vanishing integral, such as

$$\int_\Omega \partial_x^\alpha \boldsymbol{u} \cdot \nabla D^6 \theta D^6 \partial_x^\alpha \theta.$$

3.2.4 Convergence as $t \to \infty$ and identification of the asymptotic profile

Regarding the asymptotic behaviour of the density for the Stokes-transport system without any assumption on the type of initial data, we can only say that if ρ converges toward some ρ_∞ in H^{-1},

this limit profile is stratified. Indeed,, the energy balance (3.2) ensures that $\boldsymbol{u} \in L^2(\mathbf{R}_+; H^1(\Omega))$, and since $\boldsymbol{u}$ is also $\mathrm{Lip}(\mathbf{R}_+; H^1(\Omega))$ by linearity of the Stokes system, we infer that $\|\boldsymbol{u}(t)\|_{H^1} \to 0$ as $t \to \infty$, but without any information about its decay rate. At least we have

$$\|\nabla p + \rho \boldsymbol{e}_z\|_{H^{-1}} \lesssim \|\boldsymbol{u}\|_{H^1} \underset{t\to\infty}{\longrightarrow} 0.$$

The H^{-1} convergence of ρ leads to the existence of p_∞ such that

$$\nabla p_\infty = -\rho_\infty \boldsymbol{e}_z,$$

so observing that $\partial_x p_\infty = 0$ ensures that p_∞ and ρ_∞ are both independent of the horizontal coordinate x.

In the context of a small perturbation of the stationary profile $\Theta(z) = 1 - z$ we obtained explicit decay rates for Sobolev norm of $\boldsymbol{u}$. We show that these decays are enough to ensure the strong convergence of ρ toward a limit profile ρ_∞. Moreover, the smallness of the perturbation θ does not affect the vertical monotonicity of the whole density ρ, from which we deduce that ρ_∞ is exactly the vertical rearrangement of ρ_0.

Proposition 3.2.16. *Under the assumptions of Theorem 3.1.1, the whole density ρ converges in $H^m, m < 4$, toward its vertical decreasing rearrangement.*

Proof. The proof is divided in the following steps:

Convergence. It is enough to show that $\partial_t \rho$ belongs to $L^1(\mathbf{R}_+; H^m)$ for $m < 4$, which implies the strong convergence of $\rho(t)$ in H^m and existence of a limit ρ_∞. Let us estimate $\partial_t \rho$ in H^m for any $0 \le m \le 4$, a range of indices for which the following bounds are valid,

$$\begin{aligned}\|\partial_t \rho\|_{H^m} &= \|\boldsymbol{u} \cdot \nabla \rho\|_{H^m} \le \|\partial_z \psi \partial_x \rho\|_{H^m} + \|\partial_x \psi \partial_z \rho\|_{H^m} \\ &\lesssim \|\nabla \psi\|_{L^\infty} \|\partial_x \rho\|_{H^m} + \|\psi\|_{H^{m+1}} \|\partial_x \rho\|_{L^\infty} + \|\partial_x \psi\|_{L^\infty} \|\rho\|_{H^{m+1}} + \|\partial_x \psi\|_{H^m} \|\partial_z \rho\|_{L^\infty}.\end{aligned}$$

Recalling that $\partial_z \rho = -1 + G + \partial_z \theta'$ is bounded in $H^5(\Omega)$, that $\partial_x \rho = \partial_x \theta'$ decays as $(1+t)^{1-m/4}$ for $m \le 4$, and the previous decay estimates, we find

$$\|\partial_t \rho\|_{H^m} \lesssim \frac{\|\rho_0\|_{H^6}^2}{(1+t)^{2-m/4}},$$

which is integrable for any $m < 4$, hence the convergence.

Stratified limit. Since ρ converges, so do $\theta' = (\rho - \Theta)'$ and $\bar{\theta} = \overline{\rho - \Theta}$. We obtained in (3.17) that θ' vanishes in H^m for $m < 4$ The limit ρ_∞ is stratified since θ' vanishes in $H^m(\Omega)$ for $m < 4$. Hence ρ_∞ writes as the sum of Θ and the limit $\bar{\theta}_\infty$ of $\bar{\theta}$. In view of (3.17) this limit satisfies in particular $\|\partial_z \bar{\theta}_\infty\|_{L^\infty} \le C\varepsilon$. At least for $\varepsilon > 0$ such that $C\varepsilon < 1 = -\partial_z \Theta$, we know that $\sup_{(0,1)} \partial_z \rho_\infty < 0$, which means that ρ_∞ is strictly decreasing with respect to z.

Rearrangement. The divergence free character of the velocity field $\boldsymbol{u}$ ensures that all L^q norms and the cumulative distribution function of $\rho(t)$ are preserved along time, in the sense

$$\forall \lambda \ge 0, \qquad |\{\rho(t) > \lambda\}| = |\{\rho_0 > \lambda\}|. \tag{3.24}$$

This property transfers to the limit state ρ_∞ by L^q strong convergence of ρ. According to rearrangement theory such as developed in [LL01, Chapter 3] for instance, we say that two

maps are rearrangement of each other if they have the same level sets, in the sense of (3.24). Adapting slightly the construction of [LL01], we know there exists a unique vertical decreasing rearrangement of $\rho_0 : \Omega \to \mathbf{R}_+$, which can be defined as

$$\rho_0^*(z) := \int_0^\infty \mathbf{1}_{0\leq z\leq |\{\rho_0>\lambda\}|}\, \mathrm{d}\lambda.$$

In the end, we know that ρ_∞ is a decreasing rearrangement of ρ_0, therefore it is ρ_0^* by uniqueness. □

Notice that we actually have $\|\partial_z\theta\|_{L^\infty} \lesssim \varepsilon$ for all time, therefore the total density as a strictly negative vertical derivative, for all $x \in \mathbb{T}$ and for all time $t \in \mathbf{R}_+$, since

$$\partial_z\rho(t,x,\cdot) = -1 + \partial_z\theta(t,x,z),$$

and the density reordering is essentially horizontal. This is a rare case in which we can describe the asymptotic profile. This intuition of having heavy fluids sinking under the lighter ones prompts to wonder if, at least in a weak sense, the density profile should always converge toward the vertical rearrangement of the initial datum, unless it is already stratified. This question remains open, either for the Stokes-transport equation as for the incompressible porous media.

3.3 Formation of linear boundary layers for large times: proof of Theorem 3.1.2

The purpose of this section is to prove Theorem 3.1.2. We consider the linear problem

$$\begin{cases} \partial_t\theta' = \partial_x\psi \quad \text{in } (0,+\infty)\times\Omega, \\ \Delta^2\psi = \partial_x\theta \quad \text{in } \Omega, \quad \psi|_{\partial\Omega} = \partial_{\boldsymbol{n}}\psi|_{\partial\Omega} = 0, \\ \theta'(t=0) = \theta_0', \end{cases} \tag{3.25}$$

with $\theta_0' \in H^4(\Omega)$ arbitrary. The difference with the linear analysis of Subsection 3.2.2, and in particular with Proposition 3.2.6, lies in the fact that we do not assume that θ_0' and $\partial_{\boldsymbol{n}}\theta_0'$ vanish on the boundary. As a consequence, boundary layers are created as $t \to \infty$ close to $z = 0$ and $z = 1$, and the purpose of this section is precisely to describe the mechanism driving the apparition of boundary layers. We will therefore decompose θ' as the sum of an interior term decaying like t^{-1} in L^2, and some boundary layer terms which lift the traces of θ' and $\partial_z\theta'$ on the boundary. This will lead us to Theorem 3.1.2. We will then return to our nonlinear system (3.5) in Section 3.4.

The organization of this section is the following. We start with some preliminary observations explaining the reason why boundary layers are formed, and present our Ansatz. We then construct the boundary layer part of the solution, denoted by θ^{BL}, and we establish some properties. Eventually, we prove that $\theta' - \theta^{\mathrm{BL}}$ satisfies the assumptions of Proposition 3.2.6, and we conclude.

3.3.1 Preliminary observations and Ansatz

We start with rather simple observations:

- First, it follows from the equation that $\partial_t \theta'|_{\partial\Omega} = \partial_t \partial_z \theta'|_{\partial\Omega} = 0$. Therefore, for all $t \geq 0$,

$$\theta'_{|\partial\Omega}(t) = \theta'_{0|\partial\Omega} \quad \partial_z \theta'_{|\partial\Omega}(t) = \partial_z \theta'_{0|\partial\Omega}.$$

- Since $(b_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbf{Z}\times\mathbf{N}^*}$ is a hilbertian basis in $L^2(\Omega)$, we can always write

$$\theta'(t) = \sum_{\boldsymbol{k}} \widehat{\theta'_{\boldsymbol{k}}}(t) b_{\boldsymbol{k}},$$

with $(\widehat{\theta'_{\boldsymbol{k}}})_{\boldsymbol{k}\in\mathbf{Z}\times\mathbf{N}^*} \in \ell^2$. Taking the scalar product of (3.25) with $b_{\boldsymbol{k}}$, we obtain, for any $\boldsymbol{k} \in \mathbf{Z}\times\mathbf{N}^*$, $t > 0$,

$$\frac{d}{dt}\widehat{\theta'_{\boldsymbol{k}}}(t) = -\lambda_{\boldsymbol{k}} \widehat{\theta'_{\boldsymbol{k}}}(t),$$

and thus $\widehat{\theta'_{\boldsymbol{k}}}(t) = \exp(-\lambda_{\boldsymbol{k}} t)\widehat{\theta'_{\boldsymbol{k}}}(t=0)$. It then follows from Lebesgue's dominated convergence theorem that $\theta(t) \to 0$ in L^2 as $t \to \infty$

Therefore $\theta(t)$ vanishes in L^2 while keeping a constant - and non-zero - value on the boundary. As a consequence, it is reasonable to expect that boundary layers are formed in the vicinity of $z = 0$ and $z = 1$ as $t \to \infty$. Hence we take an Ansatz of the form

$$\begin{aligned}\theta'(t) \simeq \theta^{\rm int} &+ \Theta^0_{\rm bot}\left(x, (1+t)^\alpha z\right) + \Theta^0_{\rm top}(x, (1+t)^\alpha (1-z)) \\ &+ (1+t)^{-\alpha}\Theta^1_{\rm bot}\left(x, (1+t)^\alpha z\right) + (1+t)^{-\alpha}\Theta^1_{\rm top}(x, (1+t)^\alpha(1-z)) + \text{l.o.t.}\end{aligned}$$

for some $\alpha > 0$ to be determined, where

$$\begin{gathered}\theta^{\rm int}|_{\partial\Omega} = \partial_z \theta^{\rm int}|_{\partial\Omega} = 0, \\ \Theta^j_{\rm bot}(x, Z) \to 0 \quad \text{and} \quad \Theta^j_{\rm top}(x, Z) \to 0 \quad \text{as} \quad Z \to \infty.\end{gathered}$$

The role of $\Theta^0_{\rm top}$ (resp. of $\Theta^0_{\rm bot}$) is to lift the trace of θ'_0 at the top boundary $z = 1$ (resp. at the bottom boundary $z = 0$). Whence we take

$$\begin{aligned}&\Theta^0_{\rm top}(x, Z=0) = \theta'_0(x, z=1), && \partial_Z \Theta^0_{\rm top}(x, Z=0) = 0, \\ &\Theta^0_{\rm bot}(x, Z=0) = \theta'_0(x, z=0), && \partial_Z \Theta^0_{\rm bot}(x, Z=0) = 0.\end{aligned}$$

In a similar way, the next order boundary layer terms $\Theta^1_{\rm top}$ and $\Theta^1_{\rm bot}$ lift the traces of $\partial_{\boldsymbol{n}}\theta'_0$ on $\partial\Omega$, *i.e.*

$$\begin{aligned}&\Theta^1_{\rm top}(x, Z=0) = 0, && \partial_Z \Theta^0_{\rm top}(x, Z=0) = -\partial_z \theta'_0(x, z=1), \\ &\Theta^0_{\rm bot}(x, Z=0) = 0, && \partial_Z \Theta^0_{\rm bot}(x, Z=0) = \partial_z \theta'_0(x, z=0).\end{aligned}$$

Similarly, we assume that

$$\begin{aligned}\psi(t) \simeq \psi^{\rm int} &+ (1+t)^{-4\alpha}\Psi^0_{\rm bot}\left(x, (1+t)^\alpha z\right) + (1+t)^{-4\alpha}\Psi^0_{\rm top}(x, (1+t)^\alpha(1-z)) \\ &+ (1+t)^{-5\alpha}\Psi^1_{\rm bot}\left(x, (1+t)^\alpha z\right) + (1+t)^{-5\alpha}\Psi^1_{\rm bot}(x, (1+t)^\alpha(1-z)) + \text{l.o.t.}\end{aligned}$$

where

$$\partial_Z^4 \Psi^j_a = \partial_x \Theta^j_a, \quad \text{with } \Psi^j_a = \partial_Z \Psi^j_a = 0 \text{ on } Z = 0, \text{ and } \Psi^j_a \to 0 \text{ as } Z \to \infty, \quad a \in \{\text{top}, \text{bot}\}.$$

Plugging these Ansatz into equation (3.25), we find that at main order

$$\alpha(1+t)^{-1}Z\partial_Z\Theta_a^0 = (1+t)^{-4\alpha}\partial_x\Psi_a^0.$$

Consequently, identifying the powers of $(1+t)$, we take $\alpha = 1/4$. Hence the equation for Ψ_a^0, $a \in \{\text{top}, \text{bot}\}$ becomes

$$\begin{cases} \frac{1}{4}Z\partial_Z^5\Psi_a^0 = \partial_x^2\Psi_a^0 \quad \text{in } \mathbb{T}\times(0,+\infty), \\ \Psi^0_{a|Z=0} = \partial_Z\Psi^0_{a|Z=0} = 0, \\ \partial_Z^4\Psi^0_{a|Z=0} = \gamma_a^0(x), \quad \partial_Z^5\Psi^0_{a|Z=0} = 0, \\ \lim_{Z\to\infty}\Psi_a^0(x,Z) = 0, \end{cases} \tag{3.26}$$

where $\gamma^0_{\text{bot}}(x) = \partial_x\theta_0'(x, z=0)$, $\gamma^0_{\text{top}}(x) = \partial_x\theta_0'(x, z=1)$. Note that the above boundary conditions are redundant: indeed, if $\partial_Z\Psi^0_{a|Z=0} = 0$, then it follows from the equation (after one differentiation with respect to Z) that $\partial_Z^5\Psi^0_{a|Z=0} = 0$. Hence in the following paragraph we will drop the condition $\partial_Z^5\Psi^0_{a|Z=0} = 0$.

In a similar fashion, the equation for Ψ_a^1, $a \in \{\text{top}, \text{bot}\}$ is

$$\begin{cases} -\partial_Z^4\Psi_a^1 + Z\partial_Z^5\Psi_a^1 = 4\partial_x^2\Psi_a^1 \quad \text{in } \mathbb{T}\times(0,+\infty) \\ \Psi^1_{a|Z=0} = \partial_Z\Psi^1_{a|Z=0} = 0, \\ \partial_Z^4\Psi^1_{a|Z=0} = 0, \quad \partial_Z^5\Psi^1_{a|Z=0} = \gamma_a^1(x), \\ \lim_{Z\to\infty}\Psi_a^1(x,Z) = 0, \end{cases} \tag{3.27}$$

where $\gamma^1_{\text{bot}}(x) = \partial_x\partial_z\theta_0'(x, z=0)$, $\gamma^1_{\text{top}}(x) = -\partial_x\partial_z\theta_0'(x, z=1)$. Once again, the condition $\partial_Z^4\Psi^1_{a|Z=0} = 0$ is redundant and is automatically satisfied when one takes the trace of the equation at $Z = 0$, using the other boundary conditions. We now turn towards the well-posedness of (3.26) and (3.27).

3.3.2 Construction of the main profiles

The well-posedness of equations (3.26) and (3.27) stems from the following result:

Lemma 3.3.1. *Let $m \geq m_0 > 0$ and let $S \in C([0,+\infty))$, $\delta > 0$ such that*

$$\|S\|^2 := \int_0^1 \frac{S(Z)^2}{Z^2}\,dZ + \int_0^\infty S(Z)^2 e^{\delta Z^{4/5}}\,dZ < +\infty$$

Consider the ODE

$$Z\Psi^{(5)}(Z) = -m\Psi(Z) + S(z) \quad \text{in } (0,+\infty), \quad \lim_{Z\to\infty}\Psi(Z) = 0, \tag{3.28}$$

endowed with one of the four following boundary conditions:

1. $\Psi(0) = \Psi'(0) = \Psi^{(4)}(0) = 0$;
2. *or* $\Psi(0) = \Psi^{(3)}(0) = \Psi^{(4)}(0) = 0$;
3. *or* $\Psi(0) = \Psi''(0) = \Psi^{(3)}(0) = 0$;
4. *or* $\Psi(0) = \Psi'(0) = \Psi''(0) = 0$.

Then there exists a constant $c > 0$ depending only on m_0 and δ such that equation (3.28) endowed with one of the four previous conditions has a unique solution $\Psi \in H^5_{loc}(\mathbf{R}_+)$ such that for all $k \in \{0, \cdots, 5\}$,

$$\int_0^\infty |\Psi^{(k)}(Z)|^2 \exp(cZ^{4/5})\, dZ \leq C\|S\|^2 < +\infty.$$

As a consequence, for $k \leq 4$, there exists a constant C such that

$$\left|\Psi^{(k)}(Z)\right| \leq C\|S\| \exp\left(-\frac{c}{4}Z^{4/5}\right) \quad \forall Z > 0.$$

The proof of Lemma 3.3.1 is postponed to the Appendix, and relies on the use of the Lax-Milgram Lemma in weighted Sobolev spaces. As a corollary, we have the following result:

Corollary 3.3.2. *For all $j \in \{0, 1\}$, there exists a unique solution $\chi_j \in C^\infty(0, +\infty)$ of the ODE*

$$Z\chi_j^{(5)} - j\chi_j^{(4)} + 4\chi_j = 0 \quad \text{on } (0, +\infty),$$

endowed with the following boundary conditions:

- $\chi_0(0) = \chi_0'(0) = 0$, $\chi_0^{(4)}(0) = 1$;
- $\chi_1'(0) = \chi_1^{(4)}(0) = 0$, $\chi_1^{(5)}(0) = 1$,

and such that for $j = 0, 1$, $0 \leq k \leq 4$,

$$\int_0^\infty |\chi_j^{(k)}(Z)| \exp(cZ^{4/5})\, dZ < +\infty.$$

Furthermore, $\chi_0^{(5)}(0) = \chi_1(0) = 0$.

Proof. Let us start with χ_0. Let $\eta \in C_c^\infty(\mathbf{R})$ such that $\eta \equiv 1$ in a neighborhood of zero. Then $\chi_0 - Z^4\eta$ satisfies (3.28) with the boundary conditions (i) and with a C^∞ and compactly supported source term. Hence the result follows from Lemma 3.3.1. The C^∞ regularity of χ_0 follows easily from the ODE (3.28) and from an induction argument. Differentiating the ODE and taking the trace at $Z = 0$, we obtain $\chi_0^{(5)}(0) = -4\chi_0'(0) = 0$.

Concerning χ_1, we first consider the solution of the ODE

$$Z\phi^{(5)} + 4\phi = 0 \quad \text{on } (0, +\infty),$$
$$\phi(0) = \phi^{(3)}(0) = 0,\ \phi^{(4)}(0) = 1,\ \phi(+\infty) = 0.$$

The existence, uniqueness, and exponential decay of ϕ follows from Lemma 3.3.1 and from a lifting argument. We then set $\chi_1(Z) = -\int_Z^\infty \phi$, and we observe that $\partial_Z(Z\chi_1^{(5)} - \chi_1^{(4)} + 4\chi_1) = 0$. As a consequence, $Z\chi_1^{(5)}(Z) - \chi_1^{(4)}(Z) + 4\chi_1(Z) = \text{cst.} = 0$ on $(0, +\infty)$, thanks to the decay properties of ϕ at infinity. Hence the existence, uniqueness and decay of χ_1 follow. Taking the trace of the equation at $Z = 0$, we find that $\chi_1(0) = 0$.

□

Let us now explain how we construct Ψ_a^j for $a \in \{\text{top}, \text{bot}\}$ and $j = 0, 1$. Taking the Fourier

transform of (3.26) with respect to x and dropping the index a, we infer that $\widehat{\Psi^0_k}$ satisfies

$$\frac{1}{4}Z\partial_Z^5\widehat{\Psi^0_k} = -k^2\widehat{\Psi^0_k},$$
$$\widehat{\partial_Z^4\Psi^0_{k|Z=0}} = \widehat{\gamma^0_{a,k}}, \quad \widehat{\Psi^0_{k|Z=0}} = \widehat{\partial_Z\Psi^0_{k|Z=0}} = 0.$$

Considering the function χ_0 defined in 3.3.2, it is then easily checked that

$$\widehat{\Psi^0_k} = |k|^{-2}\widehat{\gamma^0_k}\,\chi_0(|k|^{1/2}Z)$$

is a solution of the problem. We infer that

$$\Psi^0_a := \sum_{k\in\mathbf{Z}\setminus\{0\}} |k|^{-2}\widehat{\gamma^0_{a,k}}\,\chi_0(|k|^{1/2}Z)e^{ikx} \tag{3.29}$$

is a solution of (3.26). In a similar way,

$$\Psi^1_a := \sum_{k\in\mathbf{Z}\setminus\{0\}} |k|^{-5/2}\widehat{\gamma^1_{a,k}}\,\chi_1(|k|^{1/2}Z)e^{ikx} \tag{3.30}$$

satisfies (3.27).

As a consequence, we have the following estimates, which follow easily from the formulas (3.29) and (3.30):

Corollary 3.3.3. *Let $\gamma^0_a \in L^2(\mathbb{T})$, $\gamma^1_a \in L^2(\mathbb{T})$. Then equation (3.26) (resp. equation (3.27)) has a unique solution $\Psi^0_a \in H^{9/4}_x L^2_Z \cap L^2_x H^{9/2}_Z$ (resp. $\Psi^1_a \in H^{11/4}_x L^2_Z \cap L^2_x H^{11/2}_Z$). Furthermore, for all $m \in \mathbf{N}$,*

$$\|\Psi^0_a\|_{H^m_x L^2_z} \lesssim \|\gamma^0_a\|_{H^{m-\frac{9}{4}}} \lesssim \|\theta_0'\|_{H^{m-\frac{1}{4}}(\Omega)}, \quad \|\Psi^0_a\|_{L^2_x H^m_z} \lesssim \|\gamma^0_a\|_{H^{\frac{m}{2}-\frac{9}{4}}} \lesssim \|\theta_0'\|_{H^{\frac{m}{2}-\frac{1}{4}}(\Omega)},$$
$$\|\Psi^1_a\|_{H^m_x L^2_z} \lesssim \|\gamma^1_a\|_{H^{m-\frac{11}{4}}} \lesssim \|\theta_0'\|_{H^{m+\frac{1}{4}}(\Omega)}, \quad \|\Psi^1_a\|_{L^2_x H^m_z} \lesssim \|\gamma^1_a\|_{H^{\frac{m}{2}-\frac{11}{4}}} \lesssim \|\theta_0'\|_{H^{\frac{m}{2}+\frac{1}{4}}(\Omega)}.$$

Additionally, the profiles Ψ^0_a and Ψ^1_a have exponential decay: for any $Z_0 \geq 1$, for any $m \in \mathbf{N}$,

$$\|\Psi^0_a\|_{H^m(\mathbb{T}\times(Z_0,+\infty))} \lesssim \|\theta_0'\|_{H^1(\Omega)}\exp(-\bar{c}Z_0^{4/5}),$$
$$\|\Psi^1_a\|_{H^m(\mathbb{T}\times(Z_0,+\infty))} \lesssim \|\theta_0'\|_{H^2(\Omega)}\exp(-\bar{c}Z_0^{4/5}).$$

3.3.3 Construction of an approximate solution

The idea is now to find a decomposition of θ' as $\theta' = \theta^{\mathrm{BL}} + \theta^{\mathrm{int}}$, where θ^{BL} is a solution of

$$\partial_t\theta^{\mathrm{BL}} = \partial_x^2\Delta^{-2}\theta^{\mathrm{BL}} + S_r,$$

with a source term S_r such that, for some $\delta > 0$,

$$S_r(t) = O((1+t)^{-2}) \text{ in } L^2(\Omega), \quad S_r(t) = O((1+t)^{-1-\delta}) \text{ in } H^4(\Omega),$$
$$\partial_t S_r(t) = O((1+t)^{-3}) \text{ in } L^2(\Omega),$$

and a boundary layer profile θ^{BL} such that $\theta^{\mathrm{BL}}|_{\partial\Omega}(t=0) = \theta_0'|_{\partial\Omega}$, $\partial_n\theta^{\mathrm{BL}}|_{\partial\Omega} = \partial_{\boldsymbol{n}}\theta'|_{\partial\Omega}$. Note that the operator Δ^{-2} is endowed with homogeneous conditions for the trace and the normal

derivative on the boundary of $\partial\Omega$. In the assumptions on the source term S above, we omitted the x derivatives. Actually, we will construct a source term S such that $\partial_x^k S$ satisfies the above assumptions for some $k \geq 3$. Note that in this case the assumptions of Proposition 3.2.6 are satisfied.

As a consequence, the interior part θ^{int} satisfies

$$\partial_t \theta^{\mathrm{int}} = \partial_x^2 \Delta^{-2} \theta^{\mathrm{int}} - S_r$$

and the trace of θ^{int} vanishes on $\partial\Omega$, together with its normal derivative. Thus we may apply Proposition 3.2.6, and we obtain $\|\theta^{\mathrm{int}}\|_{L^2} = O((1+t)^{-1})$, which will complete the proof of Theorem 3.1.2.

The main order part of θ^{BL} will be given by the profiles Θ_a^j, $j = 0, 1$, $a \in \{\mathrm{top}, \mathrm{bot}\}$ constructed in 3.3.3. However, a few adjustments must be made in order to have a suitable decomposition:

- First, the profiles Θ_a^j must be truncated away from $z = 0$ and $z = 1$, so that their (exponentially small) trace does not pollute the opposite boundary. Since Θ_a^j have exponential decay, this introduces a remainder of order $\exp(-ct^{1/5})$, which will be included in S_r. More precisely, the error terms generated by this truncation will be dealt with thanks to the following Lemma, whose proof is left to the reader:

  Lemma 3.3.4. *Let $\Psi \in L^2(\mathbb{T} \times (0, +\infty))$ such that there exists $c, C > 0$ such that*

  $$\int_{\mathbb{T}} \int_0^{\infty} |\Psi(x, Z)|^2 \exp(cZ^{4/5})\, dZ \leq C < +\infty.$$

  Let $\zeta \in L^\infty(0,1)$ such that Supp $\zeta \subset (1/4, 1)$. *Then*

  $$\left\| \Psi(x, (1+t)^{1/4} z) \zeta(z) \right\|_{L^2(\mathbb{T})} \lesssim C \|\zeta\|_\infty \exp(-c'(1+t)^{1/5}),$$

  where c' depends only on c.

- More importantly, the main order profiles (Θ_a^j, Ψ_a^j) do not satisfy exactly

  $$\Delta^2 \left((1+t)^{-1} \Psi_{\mathrm{bot}}^j (x, (1+t)^{1/4} z) \right) = \Theta_{\mathrm{bot}}^j (x, (1+t)^{1/4} z).$$

  Indeed, when constructing Ψ_a^0, we only kept the main order terms in Δ^2, *i.e.* the z derivatives. It turns out that the term $2\partial_x^2 \partial_z^2$ in the bilaplacian generates an error term in the equation which is not $O((1+t)^{-2})$. As a consequence, we introduce lower order correctors, whose purpose is precisely to cancel this error term. We emphasize that the construction of such additional correctors is quite classical in multiscale problems. In order to determine the order at which the expansion can be stopped, we will rely on the following Lemma, whose proof is postponed to the end of this section:

  Lemma 3.3.5. *Let $f \in H^4(\mathbb{T}, L^2(\mathbf{R}_+))$ such that there exist constants $c, C > 0$ such that*

  $$|\partial_x^k f(x, Z)| \leq C \exp(-cZ^{4/5}) \quad \forall k \in \{0, \cdots, 4\},\ \forall (x, Z) \in \mathbb{T} \times \mathbf{R}_+.$$

  Then

  $$\left\| \Delta^{-2} \left(f(x, (1+t)^{1/4} z) \chi(z) \right) \right\|_{L^2} \lesssim \frac{1}{(1+t)^{3/4}}.$$

Furthermore, if

$$\int_0^\infty Z^2 f(x,Z)\,\mathrm{d}Z = \int_0^\infty Z^3 f(x,Z)\,\mathrm{d}Z = 0 \quad \forall x \in \mathbb{T},$$

this estimate becomes

$$\left\| \Delta^{-2} \left(f(x,(1+t)^{1/4}z)\chi(z) \right) \right\|_{L^2} \lesssim \frac{1}{1+t}.$$

With the two above Lemmas in mind, we define θ^{BL} in the following way. Let $\chi \in C_c^\infty(\mathbf{R})$ be a cut-off function such that $\chi \equiv 1$ on $(-1/4, 1/4)$, and Supp $\chi \subset (-1/2, 1/2)$. We look for θ^{BL} in the form

$$\begin{aligned} \theta^{\mathrm{BL}}(t,x,z) &:= \sum_{j=0}^{4} (1+t)^{-j/4} \Theta^j_{\mathrm{bot}}(x,(1+t)^{1/4}z)\chi(z) \\ &\quad + \sum_{j=0}^{4} (1+t)^{-j/4} \Theta^j_{\mathrm{top}}(x,(1+t)^{1/4}(1-z))\chi(z-1) \\ &=: \theta^{\mathrm{BL}}_{\mathrm{bot}} + \theta^{\mathrm{BL}}_{\mathrm{top}} \end{aligned}$$

and

$$\begin{aligned} \psi^{\mathrm{BL}}(t,x,z) &:= \sum_{j=0}^{4} (1+t)^{-1-\frac{j}{4}} \Psi^j_{\mathrm{bot}}(x,(1+t)^{1/4}z)\chi(z) \\ &\quad + \sum_{j=0}^{4} (1+t)^{-1-\frac{j}{4}} \Psi^j_{\mathrm{top}}(x,(1+t)^{1/4}(1-z))\chi(z-1) \\ &=: \psi^{\mathrm{BL}}_{\mathrm{bot}} + \psi^{\mathrm{BL}}_{\mathrm{top}}. \end{aligned}$$

The profiles Θ^j_a, Ψ^j_a for $j = 0, 1$ and $a \in \{\mathrm{bot}, \mathrm{top}\}$ were defined in the previous paragraph, and we now proceed to define Θ^j_a, Ψ^j_a for $j \geq 2$. The reason why we stop the expansion at $j = 4$ follows from Lemma 3.3.5, as we will see shortly.

We focus on the part near $z = 0$, since the part near $z = 1$ works identically. Setting $Z = (1+t)^{1/4}z$, we have

$$\frac{\partial}{\partial t}\theta^{\mathrm{BL}}_{\mathrm{bot}} = (1+t)^{-1} \sum_{j=0}^{4} (1+t)^{-j/4} \left[-\frac{j}{4}\Theta^j_{\mathrm{bot}}(x,Z) + \frac{1}{4} Z \partial_Z \Theta^j_{\mathrm{bot}}(x,Z) \right] \chi(z)$$

For $j = 0, 1$, the bracketed term in the right-hand side is simply $\partial_x \Psi^j_{\mathrm{bot}}(x,(1+t)^{1/4}z)$. Similarly, we choose Ψ^j_a for $j = 2, 3, 4$ and $a \in \{\mathrm{bot}, \mathrm{top}\}$ so that

$$\partial_x \Psi^j_a = -\frac{j}{4}\Theta^j_a + \frac{1}{4} Z \partial_Z \Theta^j_a. \tag{3.31}$$

With this choice, we have

$$\partial_t \theta^{\mathrm{BL}} = \partial_x \psi^{\mathrm{BL}}.$$

There remains to choose Θ^j_a so that $\partial_x \psi^{\mathrm{BL}} = \Delta^{-2}\partial_x^2 \theta^{\mathrm{BL}} + O((1+t)^{-2})$ in L^2. To that end, we

observe that

$$\begin{aligned}\Delta^2\psi^{\mathrm{BL}}_{\mathrm{bot}} &= \sum_{j=0}^{4}(1+t)^{-\frac{j}{4}}\partial_Z^4\Psi^j_{\mathrm{bot}}(x,Z)\chi(z)+2\sum_{j=0}^{4}(1+t)^{-\frac12-\frac{j}{4}}\partial_x^2\partial_Z^2\Psi^j_{\mathrm{bot}}(x,Z)\chi(z)\\ &+\sum_{j=0}^{4}(1+t)^{-1-\frac{j}{4}}\partial_x^4\Psi^j_{\mathrm{bot}}(x,Z)\chi(z)\\ &+\sum_{j=0}^{4}\sum_{k=0}^{3}\binom{k}{4}(1+t)^{-1+\frac{k-j}{4}}\partial_Z^k\Psi^j_{\mathrm{bot}}(x,Z)\chi^{(4-k)}(z)\\ &+2\sum_{j=0}^{4}\sum_{k=0}^{1}\binom{k}{2}(1+t)^{-1+\frac{k-j}{4}}\partial_x^2\partial_Z^k\Psi^j_{\mathrm{bot}}(x,Z)\chi^{(2-k)}(z).\end{aligned}$$

The last two terms are handled by Lemma 3.3.4 (anticipating that Ψ^j_a will have exponential decay for $j=2,3,4$).

We obtain

$$\begin{aligned}\Delta^2\psi^{\mathrm{BL}}_{\mathrm{bot}} &= \partial_x\theta^{\mathrm{BL}}_{\mathrm{bot}}+O(\exp(-c(1+t)^{1/5}))\\ &+(1+t)^{-1/2}\left[-\partial_x\Theta^2_{\mathrm{bot}}+\partial_Z^4\Psi^2_{\mathrm{bot}}+2\partial_x^2\partial_Z^2\Psi^0_{\mathrm{bot}}\right](x,Z)\chi(z)\\ &+(1+t)^{-3/4}\left[-\partial_x\Theta^3_{\mathrm{bot}}+\partial_Z^4\Psi^3_{\mathrm{bot}}+2\partial_x^2\partial_Z^2\Psi^1_{\mathrm{bot}}\right](x,Z)\chi(z)\\ &+(1+t)^{-1}\left[-\partial_x\Theta^4_{\mathrm{bot}}+\partial_Z^4\Psi^4_{\mathrm{bot}}+2\partial_x^2\partial_Z^2\Psi^2_{\mathrm{bot}}+\partial_x^4\Psi^0\right](x,Z)\chi(z)\\ &+\sum_{j\geq 5}(1+t)^{-j/4}\Phi^j_{\mathrm{bot}}(x,Z)\chi(z),\end{aligned}$$

for some functions Φ^j_{bot} depending on the profiles Ψ^j_{bot} (for instance $\Phi^5=2\partial_x^2\partial_Z^2\Psi^3_{\mathrm{bot}}+\partial_x^4\Psi^1_{\mathrm{bot}}$). Thanks to Lemma 3.3.5, the inverse bilaplacian of the last term has a size of order $(1+t)^{-2}$ in L^2. Hence it will be included in the remainder S_r. Note that the reason why we need to stop the expansion in θ^{BL} at $j=4$ is dictated by the above formula and by Lemma 3.3.5. If we stop the expansion for a lower j, then the remainder may be greater than $(1+t)^{-2}$ in L^2.

Therefore we focus on the terms of order $(1+t)^{-j/4}$ with $j=2,3,4$. We treat the cases $j=2$ and $j=3$ simultaneously, and we will focus on the case $j=4$ later.

- **Construction of Ψ^j_a for $j=2,3$:**

Remembering (3.31), we choose Θ^j_a and Ψ^j_a for $a\in\{\mathrm{bot},\mathrm{top}\}$ and $j=2,3$ so that

$$\begin{aligned}\partial_x\Psi^j_a &= -\frac{j}{4}\Theta^j_a+\frac14 Z\partial_Z\Theta^j_a,\\ -\partial_x\Theta^j_a+\partial_Z^4\Psi^j_a+2\partial_x^2\partial_Z^2\Psi^{j-2}_a &= 0,\\ \lim_{Z\to\infty}\Psi^j_a &= 0,\\ \Psi^j_a(Z=0)=\partial_Z\Psi^j_a(Z=0) &= 0,\\ \Theta^j_a(Z=0)=\partial_Z\Theta^j_a(Z=0) &= 0.\end{aligned}$$

As before, we note that the boundary conditions at $Z=0$ are redundant. Eliminating Θ^j_a from

the equation, we find that Ψ_a^j satisfies

$$\begin{aligned} Z\partial_Z^5\Psi_a^j - j\partial_Z^4\Psi_a^j &= 4\partial_x^2\Psi_a^j + S_a^j, \\ \Psi_a^j(Z=0) = \partial_Z\Psi_a^j(Z=0) &= 0, \\ \partial_Z^4\Psi_a^j = -2\partial_x^2\partial_Z^2\Psi_a^{j-2} = -\frac{1}{j}S_a^j \quad &\text{at } Z=0, \\ \partial_Z^5\Psi_a^j = -2\partial_x^2\partial_Z^3\Psi_a^{j-2} = -\frac{1}{j-1}\partial_Z S_a^j \quad &\text{at } Z=0, \\ \lim_{Z\to\infty}\Psi_a^j &= 0, \end{aligned} \tag{3.32}$$

where $S_a^j = -2(Z\partial_Z - j)\partial_x^2\partial_Z^2\Psi_a^{j-2}$. Therefore $\partial_Z^j S_a^j = -2Z\partial_Z^{j+3}\partial_x^2\Psi_a^{j-2} = -8\partial_x^4\partial_Z^{j-2}\Psi_a^{j-2}$.

As a consequence, we find that $\partial_Z^j\Psi_a^j$ satisfies

$$\begin{aligned} Z\partial_Z^5\partial_Z^j\Psi_a &= 4\partial_x^2\partial_Z^j\Psi_a - 8\partial_x^4\partial_Z^{j-2}\Psi_a^{j-2}, \\ \partial_Z^j\Psi_a = 0 \quad &\text{at } Z=0, \\ \partial_Z^4\Psi_a^j = -\frac{1}{j}S_a^j, \quad \partial_Z^5\Psi_a^j = -\frac{1}{j-1}\partial_Z S_a^j \quad &\text{at } Z=0, \\ \lim_{Z\to\infty}\partial_Z^j\Psi_a^j &= 0, \end{aligned}$$

Note that the boundary condition $\partial_Z^j\widetilde{\Psi_a^j}(Z=0)=0$ follows from the identity

$$\partial_x\partial_Z^j\Psi_a^j = \frac{1}{4}Z\partial_Z^{j+1}\Theta_a^j.$$

Taking the Fourier transform with respect to x, we observe that $\widehat{\partial_Z^j\Psi_a}(k)$ satisfies (3.28) with nonhomogeneous boundary conditions of type (iii) (for $j=2$) or (iv) (for $j=3$). Using the Fourier representations (3.29) and (3.30) for Ψ^0 and Ψ^1, we anticipate that Ψ_a^2 and Ψ_a^3 can be written as

$$\begin{aligned} \Psi_a^2(x,Z) &= \sum_{k\in\mathbf{Z}\setminus\{0\}} |k|^{-1}\widehat{\gamma_{a,k}^0}\,\chi_2(|k|^{1/2}Z)e^{ikx}, \\ \Psi_a^3(x,Z) &= \sum_{k\in\mathbf{Z}\setminus\{0\}} |k|^{-3/2}\widehat{\gamma_{a,k}^1}\,\chi_3(|k|^{1/2}Z)e^{ikx}. \end{aligned} \tag{3.33}$$

with $\chi_2,\chi_3\in C^\infty((0,+\infty))$ decaying like $\exp(-\bar{c}Z^{4/5})$. The precise construction of χ_2 and χ_3 will be performed below. We obtain the following result:

Lemma 3.3.6. *Let $a\in\{\mathrm{top},\mathrm{bot}\}$ and $\gamma_a^0,\gamma_a^1\in L^2(\mathbb{T})$. Consider the solutions Ψ_a^0,Ψ_a^1 of (3.26), (3.27) given by Corollary 3.3.3.*

Then there exist unique solutions $\Psi_a^2\in H_x^{5/4}L_Z^2\cap L_x^2H_Z^{5/2}$, $\Psi_a^3\in H_x^{7/4}L_Z^2\cap L_x^2H_Z^{7/2}$ of (3.32). Furthermore, for any $m\in\mathbf{N}$,

$$\begin{aligned} \|\Psi_a^2\|_{H_x^mL_z^2} &\lesssim \|\theta_0'\|_{H^{m+\frac{3}{4}}(\Omega)}, \quad \|\Psi_a^2\|_{L_x^2H_z^m} \lesssim \|\theta_0'\|_{H^{\frac{m}{2}+\frac{3}{4}}(\Omega)}, \\ \|\Psi_a^1\|_{H_x^mL_z^2} &\lesssim \|\theta_0'\|_{H^{m+\frac{5}{4}}(\Omega)}, \quad \|\Psi_a^1\|_{L_x^2H_z^m} \lesssim \|\theta_0'\|_{H^{\frac{m}{2}+\frac{5}{4}}(\Omega)}. \end{aligned}$$

Additionally, the profiles Ψ_a^2 *and* Ψ_a^3 *have exponential decay: for any* $Z_0 \geq 1$*, for any* $m \in \mathbf{N}$*,*

$$\|\Psi_a^2\|_{H^m(\mathbb{T}\times(Z_0,+\infty))} \lesssim \|\theta_0'\|_{H^1(\Omega)} \exp(-\bar{c}Z_0^{4/5}),$$
$$\|\Psi_a^3\|_{H^m(\mathbb{T}\times(Z_0,+\infty))} \lesssim \|\theta_0'\|_{H^2(\Omega)} \exp(-\bar{c}Z_0^{4/5}).$$

Proof. In view of (3.33), it is sufficient to construct χ_2 and χ_3. We first construct the solution of

$$Z\partial_Z^5\phi_j(Z) = -4\phi_j(Z) - 8\partial_Z^{j-2}\chi_{j-2},$$
$$\phi^j(0) = 0, \quad \partial_Z^{4-j}\phi_j(0) = -2\chi_{j-2}''(0), \quad \partial_Z^{5-j}\phi_j(0) = -2\chi_{j-2}^{(3)}(0),$$
$$\lim_{Z\to\infty} \phi_j(Z) = 0.$$

Note that after a suitable lifting, ϕ_j satisfies (3.28) with the boundary conditions (iii) from Lemma 3.3.1 (for $j = 2$) or (iv) (for $j = 3$). Hence the existence and uniqueness of ϕ_j (and its exponential decay) follow from Lemma 3.3.1. Now, define χ_j as

$$\partial_Z^j\chi_j = \phi_j, \quad \partial_Z^k\chi_j(+\infty) = 0 \text{ for } 0 \leq k \leq j-1.$$

It follows that χ_j decays like $\exp(-\bar{c}Z^{4/5})$. Furthermore, by construction

$$\partial_Z^j\left[Z\partial_Z^5\chi_j - j\partial_Z^4\chi_j + 4\chi_j + 2(Z\partial_Z - j)\chi_{j-2}''\right] = 0.$$

It follows that $Z\partial_Z^5\chi_j - j\partial_Z^4\chi_j + 4\chi_j + 2(Z\partial_Z - j)\chi_{j-2}''$ is a polynomial of order at most $j-1$, which has exponential decay at infinity. We infer that

$$Z\partial_Z^5\chi_j - j\partial_Z^4\chi_j + 4\chi_j + 2(Z\partial_Z - j)\chi_{j-2}'' = 0.$$

Taking the trace of the above identity at $Z = 0$, we infer that $\chi_j(0) = 0$. In a similar way, we also find that $\chi_j'(0) = 0$. Now, defining Ψ_a^j by (3.33), we obtain that Ψ_a^j satisfies (3.32). The Sobolev estimates are then a consequence of the Fourier representation formula.

□

- **Construction of Ψ_a^4:**

The definition of Ψ_a^4 and Θ_a^4 is similar. We have

$$Z\partial_Z^5\Psi_a^4 - 4\partial_Z^4\Psi_a^4 = 4\partial_x^2\Psi_a^4 + S_a^4,$$
$$\Psi_a^4(Z=0) = \partial_Z\Psi_a^4(Z=0) = 0,$$
$$\partial_Z^4\Psi_a^4 = -\frac{1}{4}S_a^4 \quad \text{at } Z = 0,$$
$$\partial_Z^5\Psi_a^4 = -\frac{1}{3}\partial_Z S_a^4 \quad \text{at } Z = 0,$$
$$\lim_{Z\to\infty} \Psi_a^4 = 0,$$

where

$$S_a^4 = -(Z\partial_Z - 4)(2\partial_x^2\partial_Z^2\Psi_a^2 + \partial_x^4\Psi_a^0).$$

Therefore the Fourier transform of $\partial_z^4\Psi_a^4$, after a suitable lifting, is a solution of (3.28). The main difference with the construction of Ψ_a^j for $j \leq 3$ lies in the fact that $\partial_Z^4\Psi_a^4$ is not fully determined. Indeed, we lack a boundary condition on $\partial_Z^k\Psi_a^4$ for some $k \geq 6$. Once again, this phenomenon (a high order corrector is under-determined) is quite common in multiscale problems. In fact it

turns out that Ψ_a^4 could be determined in a unique fashion if we were looking for a higher order expansion (see Remark 3.3.7). In this case, we should choose $\Psi^4_{\rm bot}$ so that $\partial_Z^4\Theta^4_{{\rm bot}|Z=0}$ lifts the trace of $\Delta^2\theta'|_{z=0}$. In the present case, since we merely wish to close the first order expansion, we simply further require that $\partial_Z^8\Psi^4_{a|Z=0}=0$, so that the lifted Fourier transform of $\partial_Z^4\Psi_a^4$ satisfies the boundary conditions (i) of Lemma 3.3.1. We conclude that Ψ_a^4 is well-defined and satisfies the same estimates as Ψ_a^j for $j\leq 3$. The details of the proof are left to the reader.

3.3.4 Estimate of the remainder and conclusion

At this stage, we have constructed $\theta^{\rm BL}$ such that

$$\theta^{\rm BL}|_{\partial\Omega}=\theta'_{\partial\Omega}=\theta'_{0|\partial\Omega},$$
$$\partial_{\boldsymbol{n}}\theta^{\rm BL}|_{\partial\Omega}=\partial_{\boldsymbol{n}}\theta'_{\partial\Omega}=\partial_{\boldsymbol{n}}\theta'_{0|\partial\Omega},$$

and

$$\partial_t\theta^{\rm BL}=\Delta^{-2}\partial_x^2\theta^{\rm BL}+\Delta^{-2}\partial_x T_r+O(\exp(-c(1+t)^{1/5})\quad\text{in } L^2,$$

where $T_r=T_{\rm top}+T_{\rm bot}$ and

$$T_{\rm bot}:=\left[\sum_{j\geq 5}(1+t)^{-j/4}\Phi^j_{\rm bot}(x,(1+t)^{1/4}z)\chi(z)\right],$$

with a similar expression for $T_{\rm top}$. According to Lemma 3.3.5,

$$\|\Delta^{-2}\partial_x T_r\|_{L^2}\lesssim\|\theta'_0\|_{H^s}(1+t)^{-2},\quad \|\partial_t\Delta^{-2}\partial_x T_r\|_{L^2}\lesssim\|\theta'_0\|_{H^s}(1+t)^{-3}.$$

Furthermore,

$$\|\Delta^{-2}\partial_x T_r\|_{H^4}=\|\partial_x T_r\|_{L^2}\lesssim\|\theta'_0\|_{H^s}(1+t)^{-5/4}.$$

Therefore $\theta^{\rm int}=\theta'-\theta^{\rm BL}$ solves

$$\partial_t\theta^{\rm int}=\partial_x^2\Delta^{-2}\theta^{\rm int}-\Delta^{-2}\partial_x T_r+O(\exp(-c(1+t)^{1/5}),$$

and $\theta^{\rm int}=\partial_{\boldsymbol{n}}\theta^{\rm int}=0$ on $\partial\Omega$. According to Proposition 3.2.6, $\|\theta^{\rm int}\|_{L^2}=O((1+t)^{-1})$, which completes the proof of Theorem 3.1.2.

Remark 3.3.7 (Construction of an approximation at any order)**.** *Since $\Delta^2\theta'$ solves the same equation as θ', one can easily iterate this construction. More precisely, if $\theta'_0\in H^{4k}$ it can be proved that there exist sequences of profiles $(\Theta^j_{bot},\Theta^j_{top})_{0\leq j\leq 4k}$ such that the following result holds:*

$$\theta'(t,x)=\sum_{j=1}^{4k}(1+t)^{-j/4}\left[\Theta^j_{bot}(x,(1+t)^{1/4}z)\chi(z)+\Theta^j_{top}(x,(1+t)^{1/4}(1-z))\chi(z-1)\right]+\theta^j_{\rm rem}(t),$$

and

$$\|\theta^j_{\rm rem}(t)\|_{L^2}\lesssim\frac{1}{(1+t)^k},\quad \|\theta^j_{\rm rem}(t)\|_{H^{4k}}\lesssim 1.$$

For instance, the role of Θ^{4j}_{bot} is to lift the trace of $\Delta^{2j}\theta'$ at $z=0$, the one of Θ^{4j+1}_{top} to lift the one of $\partial_z\Delta^{2j}\theta'$ at $z=1$, etc.

The details of the construction are very similar to the ones of the profiles Θ^j_a for $0\leq j\leq 3$

above and are left to the reader.

Proof of Lemma 3.3.5. We first define a function f_1 such that

$$\partial_Z^4 f_1 = f,$$

and $\partial_Z^k f_1(+\infty) = 0$ for $0 \leq k \leq 3$. Note that the exponential decay assumption on f ensures that f_1 exists, and $f_1 \in W^{4,\infty} \cap H^4$. Moreover, for $0 \leq m1, m_2 \leq 4$,

$$|\partial_x^{m_1} \partial_Z^{m_2} f_1(x,Z)| \leq C \exp(-cZ^{1/5}),$$

with possibly different constants C and c. Setting $Z = (1+t)^{1/4} z$, we infer that

$$\begin{aligned} \Delta^2 \left((1+t)^{-1} f_1(x,Z)\chi(z) \right) &= f(x,Z)\chi(z) \\ &\quad +2(1+t)^{-1/2} \partial_x^2 \partial_Z^2 f_1(x,Z)\chi(z) \\ &\quad +(1+t)^{-1} \partial_x^4 f_1(x,Z)\chi(z) + O(e^{-ct^{1/5}}) \text{ in } L^2, \end{aligned}$$

where the term $O(e^{-ct^{1/5}})$ stems from the commutator involving derivatives of χ. Note that $\partial_x^2 \partial_Z^2 f_1$ satisfies the same decay assumptions as f, and therefore we can lift it by another corrector f_2 such that

$$\partial_Z^4 f_2 = -2\partial_x^2 \partial_Z^2 f_1,$$

i.e. $\partial_Z^2 f_2 = -2\partial_x^2 f_1$. Therefore

$$\Delta^2 \left(\left((1+t)^{-1} f_1(x,Z) + (1+t)^{-3/2} f_2(x,Z) \right) \chi(z) \right) = f(x,Z)\chi(z) + O((1+t)^{-1}) \quad \text{in } H^{-2}.$$

The only remaining issue lies in the fact that f_1, f_2 and their normal derivatives do not vanish on the boundary. Hence we set $a_i(x) = f_i(x,0)$, $b_i(x) = \partial_Z f_i(x,0)$, and we add a corrector

$$f_3(t,x,z) := -\sum_{i=1,2} (1+t)^{-\frac{i-1}{2}} (a_i(x) + z(1+t)^{1/4} b_i(x))\chi(z).$$

Now

$$\begin{aligned} \Delta^2 \left(\left((1+t)^{-1} f_1(x,Z) + (1+t)^{-3/2} f_2(x,Z) \right) \chi(z) + (1+t)^{-1} f_3 \right) \\ = f(x,Z)\chi(z) + O((1+t)^{-3/4}) \quad \text{in } H^{-2}, \end{aligned}$$

and for $k = 0, 1$

$$\partial_z^k \left(\left((1+t)^{-1} f_1(x,Z) + (1+t)^{-3/2} f_2(x,Z) \right) \chi(z) + (1+t)^{-1} f_3 \right) |_{\partial\Omega} = 0.$$

It follows that

$$\begin{aligned} \left((1+t)^{-1} f_1(x,Z) + (1+t)^{-3/2} f_2(x,Z) \right) \chi(z) + (1+t)^{-1} f_3 \\ = \Delta^{-2}(f(x,Z)\chi(z)) + O((1+t)^{-3/4}) \quad \text{in } H^2. \end{aligned}$$

Let us now prove that when $\int_0^\infty Z^2 f(\cdot,Z)\,\mathrm{d}Z = \int_0^\infty Z^3 f(\cdot,Z)\,\mathrm{d}Z = 0$, we gain an additional

factor $(1+t)^{-1/4}$. It can be easily checked that

$$f_{1|Z=0} = \frac{1}{6}\int_0^\infty Z^3 f(\cdot, Z)\,\mathrm{d}Z = 0, \quad \partial_Z f_{1|Z=0} = -\frac{1}{2}\int_0^\infty Z^2 f(\cdot, Z)\,\mathrm{d}Z = 0.$$

Hence, with the notation above, $a_1 = b_1 = 0$ and therefore $f_3 = O((1+t)^{-1/4})$. With the same arguments, we infer that

$$\Big((1+t)^{-1} f_1(x,Z) + (1+t)^{-3/2} f_2(x,Z)\Big)\chi(z) + (1+t)^{-1} f_3 \\ = \Delta^{-2}(f(x,Z)\chi(z)) + O((1+t)^{-1}) \quad \text{in } H^2.$$

□

3.4 Nonlinear boundary layers: proof of Theorem 3.1.3

We now go back to the long time analysis of (3.5) when $\theta_0' = \partial_{\boldsymbol{n}}\theta_0' = 0$ on $\partial\Omega$. We recall (see Theorem 3.1.1) that in this case, $\theta'(t)$ converges towards zero in H^s for all $s < 4$ as $t \to \infty$. This result follows from a uniform bound on the H^4 norm of the solution and from an interpolation inequality applied to the dissipation term.

A natural question is to investigate whether the algebraic decay rate provided by 3.1.1 can be improved, possibly at the cost of a stronger regularity requirement on the initial data. In other words, if we assume that $\theta_0' \in H^s$ with s large, can we prove a uniform H^s bound on a solution, and thereby a higher decay estimate on θ'?

As explained in the Introduction, such a result does not follow immediately from an induction argument. Indeed, the traces of $\Delta^2\theta'$ and of $\partial_z\Delta^2\theta'$ do not vanish on the boundary (even when the traces of $\Delta^2\theta_0'$ and of $\partial_z\Delta^2\theta_0'$ do), and therefore we cannot apply Proposition 3.2.6 to $\Delta^2\theta'$.

However, it turns out that we can use (a variant of) the linear analysis of Section 3.3 to analyze the long time behaviour of $\Delta^2\theta'$, and more generally, of $\Delta^{2k}\theta'$ for $k \geq 1$. In other words, in this case, there are boundary layers in the vicinity of the boundary, but they are driven by a linear mechanism. Theorem 3.1.3 will follow.

To that end, the strategy is to consider the equation satisfied by $\Delta^2\theta'$. As we have seen previously, the structure of the equation is overall the same. The main difference lies in the fact that the traces of $\Delta^2\theta'$ and $\partial_z\Delta^2\theta'$ do not vanish on the boundary. However, following the methodology of the previous section, we may lift them thanks to a corrector which remains linear at main order. Modifying slightly our bootstrap argument in order to account for these boundary layers, we eventually prove Theorem 3.1.3.

3.4.1 General strategy

Following the same strategy as in Section 3.3, we look for an Ansatz for θ' as a sum of a boundary layer part θ^{BL}, whose role is to lift the trace of $\Delta^2\theta'$ and $\partial_z\Delta^2\theta'$ on the boundary, and an interior part θ^{int}, which vanishes at a high order on the boundary, and for which we will therefore be able to prove better decay estimates. Let us give a few additional details on these two parts:

- As in the previous secion, the boundary layer term will be defined as an asymptotic expansion in powers of $(1+t)^{-1/4}$, and the size of the boundary layers will also be $(1+t)^{1/4}$. The different terms of the expansion will be constructed recursively: the main order terms will lift the traces of $\Delta^2\theta'$ and $\partial_z\Delta^2\theta'$ (or rather, their limits as $t \to \infty$), and the next order

terms will correct error terms generated by the first order ones. The precise construction of the boundary layer is the purpose of Subsection 3.4.4 below.

- Thanks to the design of the boundary layer, the remaining part $\theta^{\rm int}$ is such that

$$\theta^{\rm int} = \partial_z \theta^{\rm int} = \Delta^2 \theta^{\rm int} = \partial_z \Delta^2 \theta^{\rm int} = 0 \quad \text{on } \partial\Omega.$$

As a consequence, $\Delta^2\theta^{\rm int}$ satisfies assumptions that are similar to those of Proposition 3.2.6, and it is reasonable to expect that $\|\Delta^2\theta^{\rm int}\|_{L^2} = O((1+t)^{-1})$. Using once again Proposition 3.2.6, we then infer that $\|\theta^{\rm int}\|_{L^2} = O((1+t)^2)$. We will use a bootstrap argument to propagate these bounds; the corresponding argument is described in Subsection 3.4.5.

Before constructing $\theta^{\rm BL}$ and proving the decay estimates on $\theta^{\rm int}$, some preliminary (and somewhat technical) steps are in order. The traces of $\Delta^2\theta'$ and $\partial_z\Delta^2\theta'$ need to be decomposed as an asymptotic expansion in powers of $(1+t)^{-1/4}$, in order to identify the relevant boundary conditions for the terms in the expansion of $\theta^{\rm BL}$. This is performed in 3.4.5 below, whose proof involves some high regularity bounds on θ. As a consequence, the organisation of the rest of this section is the following. In 3.4.2, we prove some quantitative H^{10} and H^{12} bounds on θ' under our bootstrap assumption. In Subsection 3.4.3, we provide a decomposition of $\Delta^2\theta'$ and $\partial_z\Delta^2\theta'$ under the bootstrap assumption. The main results of each section are given in the beginning of the section. The reader wishing to avoid the technicalities may jump to Subsection 3.4.4, in which we construct the boundary layer, using the decomposition of Subsection 3.4.3. Eventually, we close the bootstrap argument in Subsection 3.4.5.

Let us now introduce the bootstrap assumption that will be used throughout this section. As mentioned before, we shall decompose θ' as $\theta' = \theta^{\rm BL} + \theta^{\rm int}$. The term $\theta^{\rm BL}$ will take the form

$$\theta^{\rm BL} = \frac{1}{1+t}\Theta_{\rm top}(x,(1+t)^{1/4}(1-z)) + \frac{1}{1+t}\Theta_{\rm bot}(x,(1+t)^{1/4}z) + \text{l.o.t.}$$

with boundary layer profiles $\Theta_{\rm top}, \Theta_{\rm bot}$ such that

$$\|\Theta_a\|_{H^9(\mathbb{T}\times\mathbf{R}_+)} \leq B$$

for some constant $B > 0$, while

$$\begin{aligned} \sup_{t\in[0,T]} (1+t)^2 \|\partial_x^4 \theta^{\rm int}(t)\|_{L^2} + \|\Delta^4\theta^{\rm int}(t)\|_{L^2} \leq B, \\ \sup_{t\in[0,T]} (1+t)^3 \|\partial_t\partial_x^4 \theta^{\rm int}(t)\|_{L^2} + (1+t)^3\|\partial_x^5\psi^{\rm int}\|_{L^2} \leq B, \end{aligned} \tag{3.34}$$

where $\psi^{\rm int} = \Delta^{-2}\partial_x\theta^{\rm int}$.

As a consequence, our bootstrap assumptions on θ' read as follows:

$$\begin{aligned} \forall t\in(0,T), \quad \forall k\in\{4,\cdots,8\}, \quad \|\partial_x^k\theta'\|_{L^2} \leq B(1+t)^{-9/8} + B(1+t)^{\frac{k-8}{2}}, \\ \forall t\in(0,T), \quad \forall k\in\{0,\cdots,8\}, \quad \|\partial_z^k\theta'\|_{L^2} \leq B(1+t)^{-\frac{9}{8}+\frac{k}{4}}, \\ \forall t\in(0,T), \quad \|\partial_x^5\psi(t)\|_{L^2} \leq B(1+t)^{-17/8}. \end{aligned} \tag{3.35}$$

Note that these assumptions imply in particular that

$$\|\psi\|_{W^{3,\infty}} \lesssim B(1+t)^{-9/8} \quad \forall t\in[0,T]. \tag{3.36}$$

Indeed, by the Agmon inequality,

$$\|\psi\|_{W^{3,\infty}} \lesssim \|\psi\|_{H^3}^{1/2}\|\psi\|_{H^5}^{1/2} \lesssim \|\psi\|_{L^2}^{1/5}\|\psi\|_{H^5}^{4/5}.$$

By (3.35), $\|\psi\|_{L^2} \lesssim \|\partial_x^5\psi\|_{L^2} \lesssim B(1+t)^{-17/8}$, while

$$\|\partial_z^5\psi\|_{L^2} \lesssim \|\partial_z\partial_x\theta'\|_{L^2} \lesssim \|\partial_x^2\theta'\|_{L^2}^{1/2}\|\partial_z^2\theta'\|_{L^2}^{1/2} \lesssim B(1+t)^{-7/8}.$$

Estimate (3.36) follows.

3.4.2 High regularity bounds under the bootstrap assumption

The purpose of this subsection is to prove the following bounds:

Lemma 3.4.1. *Let $\theta = \theta' + \bar\theta$ be a solution of (3.5), and assume that $\theta_0 \in H^{14}$. Let $T > 0$ be such that the bounds (3.35) hold on $(0,T)$ for some constant $B \in (0,1)$. Assume furthermore that $\|\theta_0\|_{H^{14}} \le B$. Then for all $t \in [0,T]$,*

$$\|\bar\theta(t)\|_{H^6} \lesssim B, \quad \|\theta\|_{H^{14}} \lesssim B(1+t)^{5/2}, \quad \|\partial_x^8\Delta^2\theta'\|_{L^2} \lesssim B,$$
$$\|\partial_x^{10}\theta'\|_{L^2} \lesssim B(1+t)^{-1}, \quad \|\partial_x^{10}\psi\|_{L^2} \lesssim B(1+t)^{-2}.$$

Proof. First, recalling that

$$\partial_t\bar\theta = -\overline{\nabla^\perp\psi\cdot\nabla\theta'}$$

and using the bootstrap assumptions (3.35), we infer that

$$\|\partial_t\partial_z^6\bar\theta\|_{L^2} \lesssim B^2(1+t)^{-5/4},$$

and thus $\|\bar\theta(t)\|_{H^6} \lesssim \|\theta_0\|_{H^6} + B^2 \lesssim B$. A similar argument also shows that $\|\bar\theta(t)\|_{H^7} \lesssim B + B^2\ln(1+t)$.

Let us then compute the equation satisfied by $D^{10}\theta$, where $D = \partial_x$ or ∂_z. We have

$$\partial_t D^{10}\theta + u\cdot\nabla D^{10}\theta = D^{10}\partial_x\psi - \left[D^{10}, u\cdot\nabla\right]\theta.$$

Mutliplying by $D^{10}\theta$ and integrating by parts, we obtain

$$\frac{d}{dt}\|D^{10}\theta\|_{L^2} \le 2\|D^{10}\partial_x\psi\|_{L^2} + 2\left\|\left[D^{10}, u\cdot\nabla\right]\theta\right\|_{L^2}.$$

Using the bootstrap assumptions (3.35), we have

$$\|D^{10}\partial_x\psi\|_{L^2} \lesssim B(1+t)^{3/8} \quad \forall t\in[0,T].$$

As for the commutator term, using the Leibniz formula, we write

$$\left[D^{10}, u\cdot\nabla\right]\theta = \sum_{k=0}^{9}\binom{10}{k}\nabla^\perp D^{10-k}\psi\cdot\nabla D^k\theta.$$

For $k \ge 8$, we simply write

$$\left\|\nabla^\perp D^{10-k}\psi\cdot\nabla D^k\theta\right\|_{L^2} \lesssim \|\psi\|_{W^{3,\infty}}\|\theta\|_{H^{10}} \lesssim B(1+t)^{-9/8}\|\theta\|_{H^{10}}.$$

For $k \leq 7$, the bootstrap assumptions together with the preliminary bounds on $\bar{\theta}$ ensure that

$$\left\|\nabla^{\perp} D^{10-k}\psi \cdot \nabla D^k \theta\right\|_{L^2} \lesssim B^2(1+t)^{3/8} + B(1+t)^{-9/8}\|\theta\|_{H^{10}}.$$

Therefore, assuming that $B < 1$, we obtain

$$\frac{d}{dt}\|D^{10}\theta\|_{L^2} \lesssim B(1+t)^{3/8} + B(1+t)^{-9/8}\|\theta\|_{H^{10}}.$$

Of course the same computation can be performed for any derivative of the type $\partial_x^k \partial_z^{10-k}\theta$, and we infer

$$\frac{d}{dt}\|\theta\|_{H^{10}} \lesssim B(1+t)^{3/8} + B(1+t)^{-9/8}\|\theta\|_{H^{10}}.$$

The Gronwall Lemma then ensures that

$$\|\theta(t)\|_{H^{10}} \lesssim \|\theta_0\|_{H^{10}} + B(1+t)^{11/8} \lesssim B(1+t)^{11/8}.$$

We then use the same strategy to estimate $\|\partial_x^2 \Delta^4 \theta\|_{L^2}$. The linear term in the right-hand side is now

$$\partial_x^3 \Delta^4 \psi = \partial_x^4 \Delta^2 \theta' = O(1) \quad \text{in } L^2.$$

The only difference in the treatment of the commutator term lies in the bound of terms of the form $\partial_z^2\psi \partial_x^3 \partial_z^7 \theta'$. For those, we use our first estimate on $\|\theta\|_{H^{10}}$ together with the bootstrap assumptions, and we obtain

$$\|\partial_z^2\psi \partial_x^3 \partial_z^7 \theta'\|_{L^2} \lesssim \|\partial_z^2\psi\|_{\infty}\|\theta'\|_{H^{10}} \lesssim B^2(1+t)^{-\frac{11}{8}+\frac{11}{8}} \lesssim B^2.$$

It follows that

$$\frac{d}{dt}\|\partial_x^2\Delta^4\theta\|_{L^2} \lesssim B + B(1+t)^{-9/8}\|\partial_x^2\Delta^4\theta\|_{L^2},$$

and therefore $\|\partial_x^2\Delta^4\theta\|_{L^2} \lesssim B(1+t)$. The next step is to prove that $\sup_{t\in[0,T]} \|\partial_x^6\Delta^2\theta'\|_{L^2} \lesssim B$. To that end, we check that $\partial_x^6\Delta^2\theta'$ satisfies the assumptions of Proposition 3.2.4. The source term is $S = u\cdot\nabla\partial_x^6\Delta^2\theta' + \left[\partial_x^6\Delta^2, u\cdot\nabla\right]\theta$. Classically, the first term is orthogonal to θ'. It is therefore sufficient to bound the commutator. The terms involving $\bar{\theta}$ can be treated as perturbations of the dissipation term $\|\partial_x^7\Delta\theta'\|_{L^2}^2$, and therefore we focus on $\left[\partial_x^6\Delta^2, u\cdot\nabla\right]\theta'$. First, note that

$$\left\|(\nabla^{\perp}\partial_x^6\Delta^2\psi)\cdot\nabla\theta'\right\|_{L^2} \leq \|\nabla\theta'\|_{\infty}\|\nabla\partial_x^7\theta'\|_{L^2} \lesssim B(1+t)^{-3/4}\|\partial_x^7\Delta\theta'\|_{L^2}$$

The other terms can be estimated thanks to the bootstrap assumptions together with the preliminary bounds on $\|\theta\|_{H^{10}}$ and $\|\partial_x^2\Delta^4\theta\|_{L^2}$. We obtain

$$\left\|\left[\partial_x^6\Delta^2, u\cdot\nabla\right]\theta\right\|_{L^2} \lesssim B(1+t)^{-1-\delta}\|\partial_x^6\Delta^2\theta'\|_{L^2} + B^2(1+t)^{-1-\delta} + B(1+t)^{-\frac{1}{2}-\delta}\|\partial_x^7\Delta\theta'\|_{L^2}$$

for some $\delta > 0$. The details are left to the reader. Using a Cauchy-Schwarz inequality, it follows that

$$\frac{d}{dt}\|\partial_x^6\Delta^2\theta'\|_{L^2}^2 + c\|\partial_x^7\Delta\theta'\|_{L^2}^2 \lesssim B^2(1+t)^{-1-\delta} + B(1+t)^{-1-\delta}\|\partial_x^6\Delta^2\theta'\|_{L^2}^2.$$

The Gronwall Lemma then implies that $\sup_{t\in[0,T]} \|\partial_x^6\Delta^2\theta'\|_{L^2} \lesssim \|\theta_0\|_{H^{10}} + B^2 \lesssim B$.

We then follow the same strategy to obtain bounds on $\|\theta\|_{H^{12}}$, $\|\partial_x^4\Delta^4\theta\|_{L^2}$ and $\|\partial_x^8\Delta^2\theta\|_{L^2}$.

We have

$$\partial_t \|\theta\|_{H^{12}} \lesssim \|\partial_x^2 \theta'\|_{H^8} + \sum_{k\leq 11, D\in\{\partial_x,\partial_z\}} \left\|\nabla^{12-k}\psi\cdot\nabla D^{12}\theta\right\|_{L^2}.$$

The first term in the right-hand side is bounded by $B(1+t)$. We estimate the quadratic term thanks to the bootstrap assumptions and our prelimilary bounds on derivatives up to order 10. We obtain $\|\theta(t)\|_{H^{12}} \lesssim B(1+t)^2$. We then write

$$\partial_t \partial_x^4 \Delta^4 \theta' + u\cdot\nabla\partial_x^4\Delta^4\theta' = \partial_x^6\Delta^2\theta' - \left[\partial_x^4\Delta^4, u\cdot\nabla\right]\theta.$$

The first term in the right-hand side is bounded by CB. We then check that the nonlinear term can be treated perturbatively, using the bounds on θ' obtained so far, and we infer that $\|\partial_x^4\Delta^4\theta'(t)\|_{L^2} \lesssim B(1+t)$. Once again, we then use Proposition 3.2.4 in order to prove that $\|\partial_x^8\Delta^2\theta'(t)\|_{L^2}\lesssim B$, and that

$$\frac{d}{dt}\|\theta(t)\|_{H^{14}} \lesssim B(1+t)^{-1-\delta}\|\theta(t)\|_{H^{14}} + B(1+t)^{3/2}.$$

The computations are very similar to the ones above and left to the reader, and lead to the estimate of $\|\theta(t)\|_{H^{14}}$.

The last step is to prove additional decay on $\|\partial_x^{10}\theta'\|_{L^2}$ and $\|\partial_x^{11}\psi\|_{L^2}$. Setting $S = -\partial_x^{10}(u\cdot\nabla\theta')$, we can decompose S into $S = S_\parallel + S_\perp + S_\Delta$, with $S_\perp = u\cdot\nabla\partial_x^{10}\theta'$, and

$$\|S_\parallel\|_{L^2} \lesssim B^2(1+t)^{-2}, \quad \|S_\Delta\|_{L^2}\lesssim B(1+t)^{-1/2}\|\partial_x^{10}\Delta\psi\|_{L^2}.$$

Hence for B sufficiently small, S satisfies the assumptions of 3.2.6, and we obtain

$$\|\partial_x^{10}\theta'(t)\|_{L^2}\lesssim B(1+t)^{-1}.$$

Differentiating the equation on $\partial_x^9\theta$ with respect to time, we get

$$\partial_t\partial_t\partial_x^9\theta' = (1-G)\partial_t\partial_x^{10}\psi - \partial_t\partial_x^9(u\cdot\nabla)\theta' - \partial_t G\partial_x^{10}\psi.$$

Estimating the norm of each term in the right-hand side and using Proposition 3.2.6, we obtain, for some large fixed constant κ,

$$\frac{d}{dt}\|\partial_t\partial_x^9\theta'\|_{L^2}^2 + \frac{\kappa}{1+t}\|\partial_t\partial_x^9\theta'\|_{L^2}^2 \lesssim \frac{\|\partial_t\partial_x^7\Delta^2\theta'\|_{L^2}^2}{(1+t)^3} + B^2\frac{1}{(1+t)^5}.$$

Writing

$$\partial_t\partial_x^7\Delta^2\theta' = \partial_x^9\theta' - \partial_x^7\Delta^2(u\cdot\nabla\theta),$$

we find that $\|\partial_t\partial_x^7\Delta^2\theta'\|_{L^2}\lesssim B(1+t)^{-1}$, and thus $\|\partial_t\partial_x^9\theta'\|_{L^2}\lesssim B(1+t)^{-2}$. Going back to the equation on $\partial_x^9\theta'$, we find eventually that

$$\partial_x^{10}\psi = \partial_t\partial_x^9\theta' + \partial_x^9(\nabla^\perp\psi\cdot\nabla\theta) = O((1+t)^{-2}) \quad \text{in } L^2.$$

This concludes the proof of the Lemma.

□

Let us now prove a useful result concerning the trace of $\partial_z^3\theta'$:

Corollary 3.4.2. *Under the assumptions of Lemma 3.4.1,*

$$\|\partial_z^3\theta'|_{z=0}\|_{H^{33/4}(\mathbb{T})} \lesssim B(1+t)^{-1/8}.$$

Proof. Using Theorem 3.1 in [LM12, Chapter 1],

$$\|\partial_z^3\theta'|_{z=0}\|_{H^s(\mathbb{T})} \lesssim \|\theta'\|^{1/8}_{H^\beta_x L^2_z}\|\partial_z^4\theta'\|^{7/8}_{H^\gamma_x L^2_z},$$

where $\frac{1}{8}\beta + \frac{7}{8}\gamma = s$. Taking $\beta = 10$ and $\gamma = 8$ and using the bounds of Lemma 3.4.1, we obtain the desired result. □

3.4.3 Decomposition of the traces of $\Delta^2\theta'$ and $\partial_z\Delta^2\theta'$

The first result of this section concerns the long time behaviour of $\Delta^2\theta'|_{\partial\Omega}$ and $\partial_z\Delta^2\theta'|_{\partial\Omega}$:

Lemma 3.4.3 (Long-time behaviour of $\partial_z^k\Delta^2\theta'|_{\partial\Omega}$ and of $\partial_z^{2+k}G(t)|_{\partial\Omega}$, $k = 0, 1$)**.** *For $k = 0, 1$, let*

$$\gamma^k_{\rm top}(t,x) := \partial_z^k\Delta^2\theta'(t,x,z=1), \quad \gamma^k_{\rm bot}(t,x) := \partial_z^k\Delta^2\theta'(t,x,z=0).$$

Assume that $\theta_0 \in H^{14}(\Omega)$ and $\theta_0' = \partial_z\theta_0' = 0$ on $\partial\Omega$. Let $T > 0$, $B \in (0,1)$ such that the bootstrap assumptions (3.35) hold on $[0,T]$. Assume furthermore that $\|\theta_0\|_{H^{14}} \le B$. Then there exists universal constant $B_0, \delta > 0$ and functions $\gamma^0_{a,T} \in H^s(\mathbb{T})$, $\gamma^1_{a,T} \in H^{s-1}$ for all $s < 8 + \frac{5}{8}$, such that if $B \le B_0$,

$$\|\gamma^0_{a,T}\|_{H^s(\mathbb{T})} \lesssim_s \|\theta_0\|_{H^{14}} + B^2 \quad \text{and} \quad \|\gamma^0_a(t) - \gamma^0_{a,T}\|_{H^s(\mathbb{T})} \lesssim_s B^2\frac{1}{(1+t)^\delta} \quad \forall t \in [0,T],$$

$$\|\gamma^1_{a,T}\|_{H^{s-1}(\mathbb{T})} \lesssim_s \|\theta_0\|_{H^{14}} + B^2 \quad \text{and} \quad \|\gamma^0_a(t) - \gamma^0_{a,T}\|_{H^{s-1}(\mathbb{T})} \lesssim_s B^2\frac{1}{(1+t)^\delta} \quad \forall t \in [0,T].$$

In a similar fashion, for $k = 2, 3$

$$\left|\partial_t\partial_z^k G(t)|_{\partial\Omega}\right| \lesssim B^2(1+t)^{-3+\frac{k+3}{4}},$$

and therefore there exists $g^k_{a,\infty} \in \mathbf{R}$ such that, for $k = 2, 3$,

$$|g^k_{a,\infty}| \lesssim \|\theta_0\|_{H^{14}} + B^2,$$
$$\partial_z^k G(t,0) \to g^k_{\rm bot,\infty}, \quad \partial_z^k G(t,1) \to g^k_{\rm top,\infty} \quad \text{as } t \to \infty.$$

The proof of Lemma 3.4.3 is postponed to the end of this section.

The second intermediate result of this section pushes further the decomposition of $\gamma^k_a(t)$. It holds under additional structural assumptions on $\theta^{\rm BL}$ and $\theta_{\rm rem} = \theta' - \theta^{\rm BL}$. More precisely, let us assume that

$$\begin{aligned}
\theta^{\rm BL}(t,x,z) &= \sum_{j=0}^4 (1+t)^{-1-\frac{j}{4}}\left(\Theta^j_{\rm bot}(x,(1+t)^{1/4}z) + \Theta^j_{\rm top}(x,(1+t)^{1/4}(1-z))\right) + \theta_c,\\
\psi^{\rm BL}(t,x,z) &= \sum_{j=0}^4 (1+t)^{-2-\frac{j}{4}}\left(\Psi^j_{\rm bot}(x,(1+t)^{1/4}z) + \Psi^j_{\rm top}(x,(1+t)^{1/4}(1-z))\right) + \psi_c
\end{aligned} \tag{3.37}$$

where

$$\begin{aligned}\|\Theta_a^j\|_{H^8(\mathbb{T}\times\mathbf{R}_+)} &\lesssim \|\theta_0\|_{H^{14}} + B^2, \quad \|\Psi_a^j\|_{H^{11}(\mathbb{T}\times\mathbf{R}_+)} \lesssim \|\theta_0\|_{H^{14}} + B^2,\\ \|\theta_c\|_{H_x^m H_z^k} &\lesssim (\|\theta_0\|_{H^{14}} + B^2)(1+t)^{-2-\frac{1}{8}+\frac{k}{4}} \text{ for } 0 \le k+m \le 8,\\ \|\psi_c\|_{H_x^m H_z^k} &\lesssim (\|\theta_0\|_{H^{14}} + B^2)(1+t)^{-3-\frac{1}{8}+\frac{k}{4}} \text{ for } 0 \le k+m \le 11.\end{aligned} \tag{3.38}$$

In the course of the proof, we shall also need the following assumption:

$$\partial_Z^2 \Theta_a^j|_{Z=0} \in H^7(\mathbb{T}), \quad \partial_Z^3 \Theta_a^j|_{Z=0} \in H^{15/2}(\mathbb{T}). \tag{3.39}$$

Remark 3.4.4. *Note that we have slightly changed our notation: we now set $\theta_{rem} = \theta' - \theta^{BL}$. This is linked to our bootstrap assumptions, and to the fact that the traces of $\Delta^2\theta_{rem}$ and $\partial_z\Delta^2\theta_{rem}$ do not vanish. We refer to the next subsection (see Lemma 3.4.10) for more details.*

Lemma 3.4.5 (Decomposition of γ_a^k)**.** *Assume that $\theta_0 \in H^{14}(\Omega)$ and $\theta_0' = \partial_z\theta_0' = 0$ on $\partial\Omega$. Let $T > 0$, $\|\theta_0\|_{H^{14}(\Omega)} < B < 1$ such that the bootstrap assumptions (3.35) hold on $[0,T]$. Assume furthermore that θ^{BL}, ψ^{BL} can be decomposed as (3.37), where the profiles Θ_a^j, Ψ_a^j and θ_c, ψ_c satisfy (3.38) and (3.39), and that $\theta_{rem} = \theta' - \theta^{BL}$ satisfies (3.34).*

Then for $j = 0, 1$, $a \in \{\text{top}, \text{bot}\}$, there exists $\gamma_{a,k}^j \in L^2(\mathbb{T})$, $\Gamma_{a,T}^j \in W^{1,\infty}((0,T); L^2(\mathbb{T}))$ such that for all $t \in [0,T]$,

$$\begin{aligned}\gamma_a^0(t) =& \gamma_{a,T}^0 + \gamma_{a,2}^0(1+t)^{-1/2} + \gamma_{a,3}^0(1+t)^{-3/4}\\ &+ \Gamma_{a,T}^0(t) - \gamma_{a,2}^0(1+T)^{-1/2} - \gamma_{a,3}^0(1+T)^{-3/4},\\ \gamma_a^1(t) =& \gamma_{a,T}^1 + \gamma_{a,1}^1(1+t)^{-1/4} + \gamma_{a,2}^1(1+t)^{-1/2}\\ &+ \Gamma_{a,T}^1(t) - \gamma_{a,1}^1(1+T)^{-1/4} + \gamma_{a,2}^1(1+T)^{-1/2},\end{aligned}$$

where $\|\gamma_{a,k}^j\|_{L^2(\mathbb{T})} \lesssim B^2$ and for all $t \in [0,T]$, for $j = 0, 1$, $l = 0, 1, 2$,

$$\left\|\partial_t^l \Gamma_{a,T}^j(t)\right\|_{L^2(\mathbb{T})} \lesssim B^2(1+t)^{-1-l+\frac{j}{4}}, \left\|\Gamma_{a,T}^j(t)\right\|_{H^4(\mathbb{T})} \lesssim B^2(1+t)^{-\frac{23}{24}+\frac{j}{4}}.$$

Furthermore, $\gamma_{a,k}^j$ can be written explicitely in terms of Ψ_a^0, Ψ_a^1 and Θ_a^0.

Remark 3.4.6. *Note that in the above decomposition, the coefficients $\gamma_{a,k}^j$ do not depend on time.*

Remark 3.4.7. *As mentioned in Lemma 3.4.5, $\gamma_{a,k}^j$ can be written explicitely in terms of the main order boundary layer profiles. More precisely, taking $\eta_{\text{bot}} = 1$ and $\eta_{\text{top}} = -1$,*

$$\begin{aligned}\gamma_{a,2}^0 &= 12 g_{a,\infty}^2 \partial_x \partial_Z^2 \Psi_{a|Z=0}^0,\\ \gamma_{a,3}^0 &= 8 g_{a,\infty}^2 \partial_x \partial_Z^2 \Psi_{a|Z=0}^1 - \frac{4\eta_a}{3}\left[\partial_Z^4 \left\{\Psi_a^0, \Theta_a^0\right\}_{x,Z}\right]' |_{z=0},\\ \gamma_{a,1}^1 &= 40 \eta_a g_{a,\infty}^2 \partial_x \partial_Z^3 \Psi_{a|Z=0}^0\\ \gamma_{a,2}^1 &= 20\left(\eta_a g_{a,\infty}^2 \partial_x \partial_Z^3 \Psi_{a|Z=0}^1 + g_{a,\infty}^3 \partial_x \partial_Z^2 \Psi_{a|Z=0}^0\right)\\ &\quad + 2\left[\partial_Z^5 \left\{\Psi_a^0, \Theta_a^0\right\}_{x,Z}\right]' |_{z=0}.\end{aligned} \tag{3.40}$$

where $\{\cdot,\cdot\}_{x,Z}$ denotes the Poisson bracket

$$\{f, g\}_{x,Z} = \partial_x f \partial_Z G - \partial_Z f \partial_x g.$$

Let us now prove the two previous Lemmas:

Proof of Lemma 3.4.3. We have

$$\begin{aligned}\frac{\partial}{\partial t}\Delta^2\theta' &= (1-G)\partial_x^2\theta' - 4\partial_z G\partial_z\partial_x^3\psi - 2\partial_z^2 G\partial_x^3\psi - \sum_{k=1}^{4}\binom{4}{k}\partial_z^k G\partial_x\partial_z^{4-k}\psi \\ &\quad +\Delta^2(\partial_z\psi\partial_x\theta')' - \Delta^2(\partial_x\psi\partial_z\theta')'.\end{aligned} \tag{3.41}$$

We now take the trace of the above equation at $z=0$, recalling that $G(z=0)=\partial_z G(z=0)=0$, $\psi|_{z=0}=\partial_z\psi|_{z=0}=0$, and $\theta'|_{z=0}=\partial_z\theta'|_{z=0}=0$. We obtain

$$\begin{aligned}\frac{d}{dt}\gamma^0_{\rm bot} =& -6\partial_z^2 G|_{z=0}\partial_x\partial_z^2\psi|_{z=0} \\ &+6\left(\partial_z^3\psi|_{z=0}\partial_x\partial_z^2\theta'|_{z=0}\right)' + 4\left(\partial_z^2\psi|_{z=0}\partial_x\partial_z^3\theta'|_{z=0}\right)' \\ &-6\left(\partial_x\partial_z^2\psi|_{z=0}\partial_z^3\theta'|_{z=0}\right)' - 4\left(\partial_x\partial_z^3\psi|_{z=0}\partial_z^2\theta'|_{z=0}\right)'.\end{aligned} \tag{3.42}$$

We then estimate each term in the right-hand side using Lemma 3.4.1. Note that $\partial_z^2 G|_{z=0}$ is bounded in $L^\infty(\mathbf{R}_+\times(0,1))$. We focus on the first term, which has the smallest decay. Using Theorem 3.1 in Chapter 1 of [LM12], we infer that for any $s>0$,

$$\|\partial_x\partial_z^2\psi|_{z=0}\|_{H^s} \leq \|\partial_z^2\psi|_{z=0}\|_{H^{s+1}} \lesssim \|\psi\|^{3/8}_{L^2_z H^\beta_x}\|\partial_z^4\psi\|^{5/8}_{L^2_x H^\gamma_x},$$

where β,γ are such that $\frac{3}{8}\beta+\frac{5}{8}\gamma = s+1$. In view of Lemma 3.4.1, we take $\beta=10$. As for the other term, we have, if $\gamma\in(9,11)$

$$\|\partial_z^4\psi\|_{L^2_x H^\gamma_x} \lesssim \|\theta\|_{H^{\gamma+1}_x L^2_z} \lesssim \|\partial_x^{10}\theta\|^{\frac{11-\gamma}{2}}_{L^2}\|\partial_x^{12}\theta\|^{\frac{\gamma-9}{2}}_{L^2} \lesssim B(1+t)^{-\frac{11-\gamma}{2}+2\frac{\gamma-9}{2}} \lesssim B(1+t)^{\frac{3\gamma-29}{2}}.$$

We infer that

$$\|\partial_x\partial_z^2\psi|_{z=0}\|_{H^s} \lesssim B(1+t)^{-\frac{3}{4}+\frac{5}{16}(3\gamma-29)}.$$

We then choose s so that the exponent of $(1+t)$ in the right-hand side is strictly less than -1, which leads to $\gamma<47/5$, and thus $s<69/8$.

The quadratic terms, involving traces of derivatives of ψ and of θ', have a higher decay. Let us estimate for instance $\partial_z^3\psi\partial_x\partial_z^2\theta'$ at $z=0$. We have, for any $s>1/2$,

$$\begin{aligned}\|\partial_z^3\psi|_{z=0}\partial_x\partial_z^2\theta'|_{z=0}\|_{H^s(\mathbb{T})} \lesssim& \|\partial_z^3\psi|_{z=0}\|_{L^\infty(\mathbb{T})}\|\partial_x\partial_z^2\theta'|_{z=0}\|_{H^s(\mathbb{T})} \\ &+\|\partial_z^3\psi|_{z=0}\|_{H^s(\mathbb{T})}\|\partial_x\partial_z^2\theta'|_{z=0}\|_{L^\infty(\mathbb{T})}.\end{aligned}$$

It follows from the bootstrap assumptions (3.35) that

$$\begin{aligned}\|\partial_z^3\psi\|_\infty \lesssim \|\partial_z^4\partial_x\psi\|_{L^2} \lesssim \|\partial_x^2\theta'\|_{L^2} \lesssim B(1+t)^{-9/8}, \\ \|\partial_x\partial_z^2\theta'\|_{L^\infty(\mathbb{T})} \lesssim B(1+t)^{-1/2}.\end{aligned} \tag{3.43}$$

Using once again Theorem 3.1 in Chapter 1 of [LM12], we have

$$\begin{aligned}\|\partial_x\partial_z^2\theta'|_{z=0}\|_{H^{19/2}(\mathbb{T})} &\lesssim \|\theta'\|^{3/8}_{H^{12}_x L^2_z}\|\partial_z^4\theta'\|^{5/8}_{H^8_x L^2_z} \\ &\lesssim \|\partial_x^8\Delta^2\theta'\|_{L^2} \lesssim B,\end{aligned}$$

while

$$\|\partial_z^3\psi|_{z=0}\|_{H^s(\mathbb{T})} \lesssim \|\psi\|_{L^2_z H^\beta_x}^{1/8}\|\partial_z^4\psi\|_{L^2_z H^\gamma_x}^{7/8} \lesssim \|\psi\|_{L^2_z H^\beta_x}^{1/8}\|\theta'\|_{L^2_z H^{\gamma+1}_x}^{7/8},$$

with $\frac{1}{8}\beta + \frac{7}{8}\gamma = s$. Taking $\beta = 10$ and $\gamma = 9$, we obtain, for some $s > 9$,

$$\|\partial_z^3\psi|_{z=0}\partial_x\partial_z^2\theta'|_{z=0}\|_{H^s(\mathbb{T})} \lesssim B^2(1+t)^{-9/8}.$$

The other terms are treated in a similar fashion. The estimate on γ_a^0 follows.

The estimate for γ_a^1 follows from a similar argument. Taking the vertical derivative of (3.41), we have

$$\begin{aligned}\frac{\partial}{\partial t}\partial_z\Delta^2\theta' &= (1-G)\partial_z\partial_x^2\theta' - \sum_{k=1}^{3}\binom{3}{k}\partial_z^k G\partial_z^{3-k}\partial_x^3\psi - \sum_{k=1}^{5}\binom{5}{k}\partial_z^k G\partial_x\partial_z^{5-k}\psi\\ &\quad +\partial_z\Delta^2(\partial_z\psi\partial_x\theta')' - \Delta^2(\partial_x\psi\partial_z\theta')'.\end{aligned}$$

Taking the trace of the above equation at $z = 0$, we obtain

$$\begin{aligned}\frac{d}{dt}\gamma^1_{\mathrm{bot}} =& -10\partial_z^2 G|_{z=0}\partial_x\partial_z^3\psi|_{z=0} - 10\partial_z^3 G|_{z=0}\partial_x\partial_z^2\psi|_{z=0}\\ &+ 6\partial_x^2\left(\partial_z^2\psi|_{z=0}\partial_x\partial_z^2\theta'|_{z=0} - \partial_x\partial_z^2\psi|_{z=0}\partial_z^2\theta'|_{z=0}\right)\\ &+ 10\left(\partial_z^3\psi|_{z=0}\partial_x\partial_z^3\theta'_{z=0} + \partial_z^4\psi_{z=0}\partial_x\partial_z^2\theta'|_{z=0}\right)' + 5\left(\partial_z^2\psi|_{z=0}\partial_x\partial_z^4\theta'_{z=0}\right)'\\ &- 10\left(\partial_x\partial_z^3\psi|_{z=0}\partial_z^3\theta'|_{z=0} + \partial_x\partial_z^4\psi_{z=0}\partial_z^2\theta'|_{z=0}\right)' - 5\left(\partial_x\partial_z^2\psi|_{z=0}\partial_z^4\theta'|_{z=0}\right)'.\end{aligned}$$

The highest order term is the first one. We recall that ∂_z^2 and $\partial_z^3 G$ are uniformly bounded in L^∞, and that the trace of $\partial_z^3\psi$ in H^s is evaluated thanks to (3.43). Following the same argument as above, we find that

$$\|\partial_x\partial_z^3\psi|_{z=0}\|_{H^s} \lesssim B(1+t)^{-\frac{1}{4}+\frac{7(3\gamma-29)}{16}},$$

where γ is such that $\frac{1}{8}10 + \frac{7}{8}\gamma = s+1$. Hence the exponent of $(1+t)$ is strictly less than -1 if and only if $s < 8 + \frac{5}{24}$. Once again, the quadratic terms have a higher decay and can be handled as perturbations. The trace of $\partial_z^4\theta'|_{z=0}$ can estimated thanks to γ^0_{bot}. We then obtain the desired estimate for γ_a^1.

Let us now address the convergence of $\partial_z^k G(t)|_{\partial\Omega}$ as $t \to \infty$. We recall that

$$\begin{aligned}\partial_t\partial_z^k G(t, z=0) =& -\partial_z^{k+1}\overline{\boldsymbol{u}\cdot\nabla\theta'}_{z=0}\\ =&\partial_z^{k+1}\overline{\partial_z\psi\partial_x\theta' - \partial_x\psi\partial_z\theta'}|_{z=0}.\end{aligned}$$

Since $\psi|_{z=0} = \partial_z\psi|_{z=0} = 0$, and $\theta'|_{z=0} = \partial_z\theta'|_{z=0} = 0$, we have

$$\partial_t\partial_z^2 G(t, z=0) = 3\overline{\partial_z^2\psi|_{z=0}\partial_x\partial_z^2\theta'|_{z=0} - \partial_x\partial_z^2\psi|_{z=0}\partial_z^2\theta'|_{z=0}}.$$

As above,

$$\begin{aligned}\|\partial_z^2\psi|_{z=0}\partial_x\partial_z^2\theta'|_{z=0}\|_{L^1(\mathbb{T})} \le& \|\partial_z^2\psi|_{z=0}\|_{L^2(\mathbb{T})}\|\partial_x\partial_z^2\theta'|_{z=0}\|_{L^2(\mathbb{T})}\\ \lesssim& \|\partial_z^2\psi\|_{H^{1/2}(\Omega)}\|\partial_x\partial_z^2\theta'\|_{H^{1/2}(\Omega)}\\ \lesssim& \|\partial_x^2\theta'_0\|_{H^4(\Omega)}(1+t)^{-7/4}.\end{aligned}$$

The estimate on $\partial_t\partial_z^3 G(t, z=0)$ is similar and left to the reader.

□

We now turn towards the decomposition of γ_a^0 and γ_a^1 for $a \in \{\mathrm{top}, \mathrm{bot}\}$:

Proof of Lemma 3.4.5. We focus on $a = \mathrm{bot}$ by symmetry, and we start with the decomposition of γ^0_{bot}. Let us go back to (3.42). The main term in the right-hand side is $-6\partial_z^2 G|_{z=0}\partial_x\partial_z^2\psi|_{z=0}$. Following Lemma 3.4.3 and using the decomposition $\theta' = \theta^{\mathrm{BL}} + \theta^{\mathrm{int}}$, we write

$$\begin{aligned}\partial_z^2 G|_{z=0}\partial_x\partial_z^2\psi|_{z=0} \;=\;& (1+t)^{-3/2} g^2_{\mathrm{bot},\infty}\partial_x\partial_Z^2\Psi^0_{\mathrm{bot}|Z=0} + (1+t)^{-7/4} g^2_{\mathrm{bot},\infty}\partial_x\partial_Z^2\Psi^1_{\mathrm{bot}|Z=0}\\ &+ \sum_{j=2,3}(1+t)^{-\frac{3}{2}-\frac{j}{4}}\partial_z^2 G|_{z=0}\partial_x\partial_Z^2\Psi^j_{\mathrm{bot}|Z=0}\\ &+ \sum_{j=0,1}(1+t)^{-\frac{3}{2}-\frac{j}{4}}\left(\partial_z^2 G|_{z=0} - g^2_{\mathrm{bot},\infty}\right)\partial_x\partial_Z^2\Psi^j_{\mathrm{bot}|Z=0}\\ &+\partial_z^2 G|_{z=0}\partial_x\partial_z^2\psi^{\mathrm{int}}|_{z=0}.\end{aligned}$$

The assumptions of the Lemma ensures that for all $t \in [0,T]$,

$$\left|\sum_{j=2,3}(1+t)^{-\frac{3}{2}-\frac{j}{4}}\partial_z^2 G|_{z=0}\partial_x\partial_Z^2\Psi^j_{\mathrm{bot}|Z=0}\right| \lesssim B\|\partial_z^2 G\|_\infty (1+t)^{-2},$$

$$\left\|\partial_z^2 G|_{z=0}\partial_x\partial_z^2\psi^{\mathrm{int}}|_{z=0}\right\|_{L^2(\mathbb{T})} \lesssim \|\partial_z^2 G\|_\infty B(1+t)^{-3+\frac{1}{2}+\frac{1}{8}} \lesssim B\|\partial_z^2 G\|_\infty (1+t)^{-2}.$$

Recalling that $\|\partial_z^2 G\|_\infty \lesssim \|\theta_0\|_{H^s} \lesssim B$, the two terms in the right-hand side are bounded by $B^2(1+t)^{-2}$. Furthermore, Lemma 3.4.3 ensures that

$$\left|\partial_z^2 G|_{z=0} - g^2_{\mathrm{bot},\infty}\right| \lesssim B(1+t)^{-3/4},$$

and therefore

$$\left\|\sum_{j=0,1}(1+t)^{-\frac{3}{2}-\frac{j}{4}}\left(\partial_z^2 G|_{z=0} - g^2_{\mathrm{bot},\infty}\right)\partial_x\partial_Z^2\Psi^j_{\mathrm{bot}|Z=0}\right\|_{L^2(\mathbb{T})} \lesssim B^2(1+t)^{-9/4}.$$

We now address the quadratic terms in (3.42), namely

$$\mathcal{B}(\psi,\theta') := 6\left\{\partial_z^2\psi|_{z=0}, \partial_z^2\theta'\right\}' + 4(\partial_z^2\psi\partial_x\partial_z^3\theta' - \partial_x\partial_z^3\psi\partial_z^2\theta')'|_{z=0}.$$

Decomposing ψ and θ' into their boundary layer and their interior part, we find that the main order quadratic term is

$$\begin{aligned}&(1+t)^{-7/4}\left[6\partial_Z^3\Psi^0_{\mathrm{bot}}\partial_x\partial_Z^2\Theta^0_{\mathrm{bot}} + 4\partial_Z^2\Psi^0_{\mathrm{bot}}\partial_x\partial_Z^3\Theta^0_{\mathrm{bot}} - 6\partial_x\partial_Z^2\Psi^0_{\mathrm{bot}}\partial_Z^3\Theta^0_{\mathrm{bot}} - 4\partial_x\partial_Z^3\Psi^0_{\mathrm{bot}}\partial_Z^2\Theta^0_{\mathrm{bot}}\right]'|_{z=0}\\ &=: (1+t)^{-7/4}\gamma^0_{\mathrm{bot},NL},\end{aligned}$$

while all the other terms are bounded in $L^2(\mathbb{T})$ by $CB^2(1+t)^{-2}$.

Therefore, we set

$$\gamma^0_{a,2} := 12 g^2_{a,\infty}\partial_x\partial_Z^2\Psi^0_{a|Z=0}, \quad \gamma^0_{a,3} := 8 g^2_{a,\infty}\partial_x\partial_Z^2\Psi^1_{a|Z=0} - \frac{4}{3}\gamma^0_{a,NL}.$$

Now, define $\Gamma^0_{\mathrm{bot},T}$ by

$$\begin{aligned}\Gamma^0_{\mathrm{bot},T}(t) &= 6\int_t^T \sum_{j=2,3}(1+s)^{-\frac{3}{2}-\frac{j}{4}}\partial_z^2 G(s)|_{z=0}\partial_x\partial_Z^2\Psi^j_{\mathrm{bot}|Z=0}\,\mathrm{d}s\\ &\quad+6\int_t^T\sum_{j=0,1}(1+s)^{-\frac{3}{2}-\frac{j}{4}}\left(\partial_z^2G(s)|_{z=0}-g^2_{\mathrm{bot},\infty}\right)\partial_x\partial_Z^2\Psi^j_{\mathrm{bot}|Z=0}\,\mathrm{d}s\\ &\quad+\int_t^T\mathcal{B}\left(\sum_{j=1}^3(1+s)^{-2-\frac{j}{4}}\Psi^j_{\mathrm{bot}}(x,(1+s)^{1/4})+\psi_{\mathrm{rem}},\theta'(s)\right)\mathrm{d}s\\ &\quad+\int_t^T\mathcal{B}\left((1+s)^{-2}\Psi^0_{\mathrm{bot}}(x,(1+s)^{1/4}),\sum_{j=1}^3(1+s)^{-1-\frac{j}{4}}\Theta^j_{\mathrm{bot}}(x,(1+s)^{1/4})+\theta_{\mathrm{rem}}\right)\mathrm{d}s.\end{aligned}$$

Note that the assumptions (3.34) on θ^{int} ensure that

$$\|\partial_z^3\theta_{\mathrm{rem}}|_{z=0}\|_{L^2}\lesssim\|\theta_{\mathrm{rem}}\|_{L^2}^{1/8}\|\partial_z^4\theta_{\mathrm{rem}}\|_{L^2}^{7/8}\lesssim B(1+t)^{-1/4}.$$

Recalling Corollary 3.4.2 and using the assumption $\partial_Z^3\Theta^j_a|_{Z=0}\in H^{15/2}(\mathbb{T})$ (see (3.39)), we also infer that

$$\|\partial_z^3\theta_{\mathrm{rem}}|_{z=0}\|_{H^{15/2}}\lesssim B(1+t)^{-1/8}.$$

Interpolating between these two estimates, we find in particular that

$$\|\partial_z^3\theta_{\mathrm{rem}}|_{z=0}\|_{H^5}\lesssim B(1+t)-11/24.$$

The above estimates ensure that for $k=0,1$

$$\begin{aligned}\left\|\partial_t^k\Gamma^0_{\mathrm{bot},T}(t)\right\|_{L^2(\mathbb{T})}&\lesssim B^2(1+t)^{-k-1},\\ \left\|\Gamma^0_{\mathrm{bot},T}(t)\right\|_{H^4(\mathbb{T})}&\lesssim B^2(1+t)^{-23/24}.\end{aligned}$$

Therefore we obtain the decomposition announced in the Lemma for γ^0_a.

The decomposition of γ^1_a follows from similar arguments and is left to the reader. □

3.4.4 Iterative construction of the boundary layer profile

Let us now turn towards the construction of the boundary layer profile, and more generally, of an approximate solution. The purpose of this subsection is to prove the two following Lemmas. Our first result, which is truly the core of the construction, is valid under the bootstrap assumption (3.35) on θ':

Lemma 3.4.8. *Let $\theta_0\in H^{14}(\Omega)$ such that $\|\theta_0\|_{H^{14}}\le B<1$. Let $\theta=\theta'+\bar\theta$ be a solution of (3.5), and assume that the bounds (3.35) hold on $(0,T)$. Let $\gamma^0_{a,T}$, $\gamma^1_{a,T}$ be defined by Lemma 3.4.3.*

Then there exist profiles $\Theta^j_a\in H^8(\mathbb{T}\times R_+)$, $\Psi^j_a\in H^9(\mathbb{T}\times\mathbf{R}_+)$, $j\in\{0,\cdots,4\}$ and $\theta_c\in H^9(\Omega)$, depending only on $\gamma^0_{a,T},\gamma^1_{a,T},g^2_{a,\infty}$ and $g^3_{a,\infty}$, such that, defining θ^{BL} by (3.37), the following properties hold:

1. *Bounds on the profiles: Θ^j_a, Ψ^j_a, θ^{lin}_c, $\psi^{lin}_c=\Delta^{-2}\partial_x\theta_c$ satisfy (3.38);*

2. *Traces at the top and bottom: at $z = 0$,*

$$\begin{aligned}\Delta^2\theta^{BL}|_{z=0} &= \gamma^0_{\mathrm{bot},T} + \gamma^0_{\mathrm{bot},2}(1+t)^{-1/2} + \gamma^0_{\mathrm{bot},3}(1+t)^{-3/4}\\ &\quad -\gamma^0_{\mathrm{bot},2}(1+T)^{-1/2} - \gamma^0_{\mathrm{bot},3}(1+T)^{-3/4},\\ \partial_z\Delta^2\theta^{BL}|_{z=0} &= \gamma^1_{\mathrm{bot},T} + \gamma^1_{\mathrm{bot},1}(1+t)^{-1/4} + \gamma^1_{\mathrm{bot},2}(1+t)^{-1/2}\\ &\quad -\gamma^1_{\mathrm{bot},1}(1+T)^{-1/4} - \gamma^1_{\mathrm{bot},2}(1+t)^{-1/2},\end{aligned}$$

where $\gamma^j_{a,k}$ are defined in (3.40). Similar formulas hold at $z = 1$.

3. *Evolution equation: θ^{BL} satisfies*

$$\partial_t\theta^{BL} = (1-G)\partial_x^2\Delta^{-2}\theta^{BL} - \left(\nabla^\perp\Delta^{-2}\partial_x\theta^{BL}\cdot\nabla\theta^{BL}\right)' + R^{BL},$$

and the remainder R^{BL} is such that for all $k, m \geq 0$ with $k + m \leq 8$,

$$\|R^{BL}\|_{H^m_x H^k_z} \lesssim B^2(1+t)^{-3+\frac{k}{4}-\frac{1}{8}}.$$

Remark 3.4.9. *Actually, all profiles Θ^j_a, Ψ^j_a, and therefore θ^{BL}, ψ^{BL} depend on T through $\gamma^0_{a,T}, \gamma^1_{a,T}$. However, in order not to burden unecessarily the notation, we will omit this dependency.*

Once the boundary layer part is constructed, under an additional bootstrap assumption on the remainder, we can define a nonlinear corrector:

Lemma 3.4.10. *Let $\theta_0 \in H^{14}(\Omega)$ such that $\|\theta_0\|_{H^{14}} \leq B < 1$. Let $\theta = \theta' + \bar\theta$ be a solution of (3.5), and assume that the bounds (3.35) hold on $(0, T)$.*

Let θ^{BL}, ψ^{BL} be given by Lemma 3.4.8, and let $\theta_{rem} = \theta' - \theta^{BL}$. Assume that (3.34) holds on $(0, T)$, and define $\Gamma^j_{a,T}$ as in Lemma 3.4.5.

Then there exists $\theta^{NL}_c \in H^8(\Omega)$ such that

$$\begin{aligned}\Delta^2\sigma^{NL}_{lift}|_{z=0} &= \Gamma^0_{\mathrm{bot},T}, \quad \partial_z\Delta^2\sigma^{NL}_{lift}|_{z=0} = \Gamma^1_{\mathrm{bot},T},\\ \Delta^2\sigma^{NL}_{lift}|_{z=1} &= \Gamma^0_{\mathrm{top},T}, \quad \partial_z\Delta^2\sigma^{NL}_{lift}|_{z=1} = \Gamma^1_{\mathrm{top},T}\end{aligned}$$

and for all $k, m \geq 0$ with $k + m \leq 8$

$$\|\sigma^{NL}_{lift}\|_{H^m_x H^k_z} \lesssim B^2(1+t)^{-2+\frac{k}{4}-\frac{1}{8}}.$$

As a consequence, setting $\theta^{\mathrm{app}} := \theta^{BL} + \sigma^{NL}_{lift}$, we have

$$\Delta^2\theta^{\mathrm{app}} = \Delta^2\theta', \quad \partial_z\Delta^2\theta^{\mathrm{app}} = \partial_z\Delta^2\theta' \quad \text{on } \partial\Omega,$$

and θ^{app} is a solution of

$$\partial_t\theta^{\mathrm{app}} = (1-G)\partial_x^2\Delta^{-2}\theta^{\mathrm{app}} - \left(\nabla^\perp\Delta^{-2}\partial_x\theta^{\mathrm{app}}\cdot\nabla\theta^{\mathrm{app}}\right)' + S_{\mathrm{rem}},$$

and the remainder S_{rem} is such that for all $k, m \geq 0$ with $k + m \leq 8$,

$$\|S_{\mathrm{rem}}\|_{H^m_x H^k_z} \lesssim B^2(1+t)^{-3+\frac{k}{4}-\frac{1}{8}}.$$

The main part of this section will be devoted to the proof of Lemma 3.4.8. The strategy will be very similar to the one of Section 3.3, and we will often refer the reader to the computations therein. We begin with the construction of the profiles Θ_a^j, Ψ_a^j. To that end, we plug the Ansatz (3.37) into equation (3.5) and identify the powers of $1+t$ in the vicinity of $z=0$ or $z=1$. Note that for $z \ll 1$, setting $Z = (1+t)^{1/4} z$ and using Lemma 3.4.3,

$$\begin{aligned} G(t,z) &= \frac{1}{2}\partial_z^2 G(t,0) z^2 + \frac{1}{6}\partial_z^3 G(t,0) z^3 + O(z^4) \\ &= (1+t)^{-1/2} \frac{g^2_{\mathrm{bot},\infty}}{2} Z^2 + (1+t)^{-3/4} \frac{g^3_{\mathrm{bot},\infty}}{6} Z^3 \\ &\quad + O((1+t)^{-1} Z^4 + (1+t)^{-5/4}(Z^2+Z^3)). \end{aligned} \tag{3.44}$$

A similar expansion holds in the vicinity of $z=1$. Furthermore, in the vicinity of $z=0$, setting $S = -(\nabla^\perp \psi \cdot \nabla \theta')'$,

$$S = \sum_{0 \le i,j \le 4} (1+t)^{-3-\frac{i+j-1}{4}} \left(\partial_Z \Psi^i_{\mathrm{bot}} \partial_x \Theta^j_{\mathrm{bot}} - \partial_x \Psi^i_{\mathrm{bot}} \partial_Z \Theta^j_{\mathrm{bot}} \right)' + O((1+t)^{-15/4}). \tag{3.45}$$

Following the computations of the previous section and identifying the coefficient of $(1+t)^{-2-\frac{j}{4}}$, we obtain (compare with (3.31))

$$-\left(1+\frac{j}{4}\right)\Theta_a^j + \frac{1}{4} Z \partial_Z \Theta_a^j = \partial_x \Psi_a^j + S_a^j, \tag{3.46}$$

where the source terms S_a^j are defined by

$$\begin{aligned} &S_a^0 = S_a^1 = 0, \\ &S_a^2 = -\frac{g^2_{a,\infty}}{2} Z^2 \partial_x \Psi_a^0, \\ &S_a^3 = -\frac{g^2_{a,\infty}}{2} Z^2 \partial_x \Psi_a^1 - \eta_a \frac{g^3_{a,\infty}}{6} Z^3 \partial_x \Psi_a^0 + \eta_a \left(\partial_Z \Psi_a^0 \partial_x \Theta_a^0 - \partial_x \Psi_a^0 \partial_Z \Theta_a^0 \right)', \end{aligned} \tag{3.47}$$

with $\eta_{\mathrm{bot}} = 1, \eta_{\mathrm{top}} = -1$.

Let us now proceed to define recursively the profiles Θ_a^j, Ψ_a^j.

Main order boundary layer terms: Θ_a^0 and Θ_a^1. The roles of the boundary layer profiles Θ_a^j for $j = 0, 1$ is to correct the traces of $\Delta^2 \theta'$ and $\partial_z \Delta^2 \theta'$ on $\partial\Omega$ *at main order*, *i.e.* $\gamma^j_{a,T}$ (see Lemma 3.4.5). Choosing $S_a^j = 0$ and Ψ_a^j such that $\partial_Z^4 \Psi_a^j = \partial_x \Theta_a^j$ and recalling (3.46), we are led to

$$\begin{cases} Z \partial_Z^5 \Theta_a^0 = 4 \partial_x^2 \Theta_a^0 \quad \text{in } \mathbb{T} \times (0, +\infty) \\ \partial_Z^4 \Theta^0_{a|Z=0} = \gamma^0_{a,T}, \quad \partial_Z^5 \Theta^0_{a|Z=0} = 0, \\ \Theta^0_{a|Z=0} = 0, \quad \partial_Z \Theta^0_{a|Z=0} = 0, \quad \lim_{Z \to \infty} \Theta_a^0 = 0, \end{cases}$$

and

$$\begin{cases} Z \partial_Z^6 \Theta_a^1 = 4 \partial_x^2 \partial_Z \Theta_a^1 \quad \text{in } \mathbb{T} \times (0, +\infty) \\ \partial_Z^4 \Theta^1_{a|Z=0} = 0, \quad \partial_Z^5 \Theta^1_{a|Z=0} = \eta_a \gamma^1_{a,T}, \\ \Theta^1_{a|Z=0} = 0, \quad \partial_Z \Theta^1_{a|Z=0} = 0, \quad \lim_{Z \to \infty} \Theta_a^1 = 0. \end{cases}$$

Note that these systems are identical to (3.26) and (3.27) respectively. As a consequence, as in the previous section, we find that

$$\begin{aligned}\Theta_a^0(x,Z) &= \sum_{k\in\mathbf{Z}\setminus\{0\}} |k|^{-2}\widehat{\gamma_{a,T}^0}(k)\chi_0(|k|^{1/2}Z)e^{ikx},\\ \Psi_a^0(x,Z) &= \sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{1}{ik|k|^2}\widehat{\gamma_{a,T}^0}(k)\left[\frac{1}{4}|k|^{1/2}Z\chi_0'(|k|^{1/2}Z)-\chi_0(|k|^{1/2}Z)\right]e^{ikx},\end{aligned} \tag{3.48}$$

where χ_0 is defined in Corollary 3.3.2. Since $\|\gamma_{a,T}^0\|_{H^s}\lesssim\|\theta_0\|_{H^{14}}+B^2$ for all $s<69/8$, it follows that

$$\begin{aligned}\|\Theta_a^0\|_{H_x^{10+\frac{3}{4}}L_Z^2}+\|\Theta_a^0\|_{L_x^2H_Z^{20+\frac{3}{2}}}&\lesssim\|\theta_0\|_{H^{14}}+B^2,\\ \|\Psi_a^0\|_{H_x^{11+\frac{3}{4}}L_Z^2}+\|\Psi_a^0\|_{L_x^2H_Z^{22+\frac{3}{2}}}&\lesssim\|\theta_0\|_{H^{14}}+B^2.\end{aligned}$$

In a similar fashion, recalling the definition of χ_1 from Corollary 3.3.2,

$$\begin{aligned}\Theta_a^1(x,Z) &= \eta_a\sum_{k\in\mathbf{Z}\setminus\{0\}} |k|^{-5/2}\widehat{\gamma_{a,T}^1}(k)\chi_1(|k|^{1/2}Z)e^{ikx},\\ \Psi_a^1(x,Z) &= \eta_a\sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{1}{ik|k|^{5/2}}\widehat{\gamma_{a,T}^1}(k)\left[\frac{1}{4}|k|^{1/2}Z\chi_1'(|k|^{1/2}Z)-\chi_1(|k|^{1/2}Z)\right]e^{ikx}.\end{aligned} \tag{3.49}$$

Since $\|\gamma_{a,T}^1\|_{H^{15/2}}\lesssim\|\theta_0\|_{H^{14}}+B^2$, we also have

$$\begin{aligned}\|\Theta_a^1\|_{H_x^{10+\frac{1}{4}}L_Z^2}+\|\Theta_a^1\|_{L_x^2H_Z^{20+\frac{1}{2}}}&\lesssim\|\theta_0\|_{H^{14}}+B^2,\\ \|\Psi_a^1\|_{H_x^{11+\frac{1}{4}}L_Z^2}+\|\Psi_a^1\|_{L_x^2H_Z^{22+\frac{1}{2}}}&\lesssim\|\theta_0\|_{H^{14}}+B^2.\end{aligned}$$

Let us now define the boundary terms $\gamma_{a,j}^0$ and $\gamma_{a,j}^1$ by (3.40). It follows from the above expressions for Ψ_a^j and Θ_a^j and from the boundary conditions $\chi_0(0)=\chi_0'(0)=0$ that

$$\begin{aligned}\|\gamma_{a,2}^0\|_{H^9(\mathbb{T})}&\lesssim B^2,\quad \|\gamma_{a,3}^0\|_{H^{15/2}(\mathbb{T})}\lesssim B^2,\\ \|\gamma_{a,1}^1\|_{H^{17/2}(T)}&\lesssim B^2,\quad \|\gamma_{a,2}^1\|_{H^7(\mathbb{T})}\lesssim B^2.\end{aligned} \tag{3.50}$$

In a similar fashion, defining the source terms S_a^2, S_a^3 by (3.47), we have

$$\begin{aligned}\|S_a^2\|_{H_x^{11+\frac{1}{4}}L_Z^2}+\|S_a^2\|_{L_x^2H_Z^{22+\frac{1}{2}}}&\lesssim B^2,\\ \|S_a^3\|_{H_x^{9+\frac{3}{4}}L_Z^2}+\|S_a^3\|_{L_x^2H_Z^{18+\frac{3}{2}}}&\lesssim B^2.\end{aligned} \tag{3.51}$$

Note however that because of the quadratic term $\{\Psi_a^0,\Theta_a^0\}_{x,Z}$, S_a^3 does not have the same self-similar structure as Ψ_a^j,Θ_a^j for $j=0,1$, which is also shared by S_a^2.

Correctors $\Theta_{c,a}^0$ and $\Theta_{c,a}^1$. We recall that the coefficients $\gamma_{a,k}^j$ are defined by (3.40), and are estimated in (3.50) above. The terms $\gamma_{a,2}^0(1+T)^{-1/2}$ and $\gamma_{a,3}^0(1+T)^{-3/4}$ in Lemma 3.4.5 are constant in time, but smaller (for $T\gg1$) than $\gamma_{a,T}^0$. Hence they give rise to a profile $\Theta_{c,a}^0$ whose construction is very similar to the one of Θ_a^0, but whose size is much smaller. More precisely, we

set

$$\Theta^0_{c,a}(x,Z) = \sum_{k\in\mathbf{Z}\setminus\{0\}} |k|^{-2}\left[\widehat{\gamma^0_{a,2}}(k)(1+T)^{-1/2} + +\widehat{\gamma^0_{a,3}}(k)(1+T)^{-3/4}\right]\chi_0(|k|^{1/2}Z)e^{ikx},$$
$$\Theta^1_{c,a}(x,Z) = \eta_a \sum_{k\in\mathbf{Z}\setminus\{0\}} |k|^{-5/2}\left[\widehat{\gamma^1_{a,1}}(k)(1+T)^{-1/4} + \widehat{\gamma^1_{a,2}}(1+T)^{-1/2}\right]\chi_1(|k|^{1/2}Z)e^{ikx}.$$

Remembering (3.50), we have, for $j=0,1$

$$\|\Theta^j_{c,a}\|_{H^{9+\frac{3}{4}}{}_xL^2_Z} + \|\Theta^j_{c,a}\|_{L^2_xH^{18+\frac{3}{2}}_Z} \lesssim B^2(1+T)^{-\frac{1}{2}+\frac{j}{4}},$$

and

$$\|\partial_Z^2\Theta^j_{c,a}|_{Z=0}\|_{H^{17/2}(\mathbb{T})} \lesssim B^2(1+T)^{-\frac{1}{2}+\frac{j}{4}}.$$

Analogously to Ψ^0_a and Ψ^1_a, we also define

$$\Psi^0_{c,a} = \sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{1}{ik|k|^2}\left[\widehat{\gamma^0_{a,2}}(k)(1+T)^{-1/2} + +\widehat{\gamma^0_{a,3}}(k)(1+T)^{-3/4}\right]\left(\frac{1}{4}\xi\chi_0'(\xi) - \chi_0(\xi)\right)|_{\xi=|k|^{1/2}Z}e^{ikx},$$
$$\Psi^1_{c,a} = \eta_a\sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{1}{ik|k|^{5/2}}\left[\widehat{\gamma^1_{a,1}}(k)(1+T)^{-1/4} + \widehat{\gamma^1_{a,2}}(1+T)^{-1/2}\right]\left(\frac{1}{4}\xi\chi_1'(\xi) - \chi_1(\xi)\right)|_{\xi=|k|^{1/2}Z}e^{ikx},$$

so that $\partial_Z^4\Psi^j_{c,a} = \partial_x\Theta^j_{c,a}$, and we have

$$\|\Psi^j_{c,a}\|_{H^{10+\frac{3}{4}}{}_xL^2_Z} + \|\Psi^j_{c,a}\|_{L^2_xH^{20+\frac{3}{2}}_Z} \lesssim B^2(1+T)^{-\frac{1}{2}+\frac{j}{4}}.$$

Lower order boundary layer terms: Θ^2_a, Θ^3_a and $\Theta^2_{c,a}$. We recall that Θ^j_a, Ψ^j_a must satisfy (3.46), where the source term S^j_a is given by (3.47). Note that since Ψ^0_a, Ψ^1_a and Θ^0_a have been constructed in the previous step, the source terms S^2_a and S^3_a are defined unequivocally and have exponential decay. Moreover, following Lemma 3.4.5 and noting that

$$\Delta^2\theta^{\mathrm{BL}}|_{z=0} = \sum_{j=0}^{3}(1+t)^{-j/4}\partial_Z^4\Theta^j_{\mathrm{bot}|Z=0} + 2\sum_{j=0}^{3}(1+t)^{-\frac{1}{2}-\frac{j}{4}}\partial_x^2\partial_Z^2\Theta^j_{\mathrm{bot}|Z=0} + O((1+t)^{-1}),$$

we enforce the following boundary conditions:

$$\begin{aligned}
&\partial_Z^4\Theta^2_{a|Z=0} = \gamma^0_{a,2} - 2\partial_x^2\partial_Z^2\Theta^0_{a|Z=0}, \quad \partial_Z^5\Theta^2_{a|Z=0} = \eta_a\left(\gamma^1_{a,1} - 2\partial_x^2\partial_Z^3\Theta^0_{a|Z=0}\right),\\
&\partial_Z^4\Theta^3_{a|Z=0} = \gamma^0_{a,3} - 2\partial_x^2\partial_Z^2\Theta^1_{a|Z=0}, \quad \partial_Z^5\Theta^3_{a|Z=0} = \eta_a\left(\gamma^1_{a,2} - 2\partial_x^2\partial_Z^3\Theta^1_{a|Z=0}\right),
\end{aligned} \tag{3.52}$$

where the coefficients $\gamma^j_{a,k}$ are defined in (3.40) and estimated in (3.50). There remains to specify the relationship between Ψ^j_a and Θ^j_a. In order that $\Delta^2\psi^{\mathrm{BL}} = \partial_x\theta^{\mathrm{BL}}$ at main order, following the computations of the previous section, we take, for $j=2,3$

$$\partial_Z^4\Psi^j_a + 2\partial_x^2\partial_Z^2\Psi^{j-2}_a = \partial_x\Theta^j_a.$$

Eliminating Ψ_a^2 from the equation on Θ_a^2, we find that the system satisfied by Θ_a^2 is

$$\begin{cases} Z\partial_Z^5\Theta_a^2 - 2\partial_Z^4\Theta = 4\partial_x^2\Theta_a^2 - 2g_{\mathrm{bot},\infty}^2\partial_Z^4(Z^2\partial_x\Psi_a^0) - 8\partial_x^3\partial_Z^2\Psi_a^0, \\ \Theta_{a|Z=0}^2 = \partial_Z\Theta_{a|Z=0}^2 = 0, \\ \Theta_a^2(Z) \to 0 \quad \text{as } Z \to \infty, \end{cases} \tag{3.53}$$

together with (3.52). Note that $2\partial_x\partial_Z^2\Psi_{a|Z=0}^0 = -\partial_Z^2\Theta_{a|Z=0}^0$ and $4\partial_x\partial_Z^3\Psi_{a|Z=0}^0 = \partial_Z^3\Theta_{a|Z=0}^0$, so that the boundary conditions are redundant. In other words, taking the trace of (3.53) at $Z=0$, we find $\partial_Z^4\Theta_{a|Z=0}^2 = \gamma_{a,2}^0 - 2\partial_x^2\partial_Z^2\Theta_{a|Z=0}^0$. Differentiating twice more with respect to Z, we find that the Fourier transform of $\partial_Z^2\Theta_a^2$, after a suitable lifting, satisfies an equation of the form (3.28) with boundary conditions of the type (iii) from Lemma 3.3.1. Using the explicit Fourier representation of Ψ_a^0 and Θ_a^0 (3.48), we find that

$$\|\Theta_a^2\|_{H_x^{9+\frac{3}{4}}L_Z^2} + \|\Theta_a^2\|_{L_x^2H_Z^{19+\frac{3}{2}}} \lesssim B, \quad \|\Psi_a^2\|_{H_x^{10+\frac{3}{4}}L_Z^2} + \|\Psi_a^2\|_{L_x^2H_Z^{20+\frac{3}{2}}} \lesssim B.$$

In a similar fashion, Θ_a^3 satisfies the system

$$\begin{cases} Z\partial_Z^5\Theta_a^3 - 3\partial_Z^4\Theta_a^3 = 4\partial_x^2\Theta_a^3 + 4\partial_Z^4S_a^3 + 6\partial_x^2\partial_Z^2\Theta_a^1 - 2Z\partial_x^2\partial_Z^3\Theta_a^1, \\ \Theta_{a|Z=0}^3 = \partial_Z\Theta_{a|Z=0}^3 = 0, \\ \Theta_a^3(Z) \to 0 \quad \text{as } Z \to \infty, \end{cases}$$

together with (3.52). Once again, we find that the lifted Fourier transform of $\partial_Z^3\Theta_a^3$ satisfies an equation of the form (3.28) with boundary conditions of the type (iv) from Lemma 3.3.1. Using the explicit Fourier representation of Θ_a^1 (3.49) together with the estimates on S_a^3 (3.51), we find that

$$\|\Theta_a^3\|_{H_x^{9+\frac{1}{4}}L_Z^2} + \|\Theta_a^3\|_{L_x^2H_Z^{19+\frac{1}{2}}} \lesssim \|\theta_0\|_{H^{14}} + B^2, \quad \|\Psi_a^3\|_{H_x^{10+\frac{1}{4}}L_Z^2} + \|\Psi_a^3\|_{L_x^2H_Z^{20+\frac{1}{2}}} \lesssim \|\theta_0\|_{H^{14}} + B^2.$$

Note that the Fourier representation of Θ_a^2 and of the linear part of Θ_a^3 also ensure that for $j = 2, 3$,

$$\|\partial_Z^2\Theta_a^j|_{Z=0}\|_{H^8(\mathbb{T})} + \|\partial_Z^3\Theta_a^j|_{Z=0}\|_{H^{15/2}(\mathbb{T})} \lesssim \|\theta_0\|_{H^{14}} + B^2. \tag{3.54}$$

Eventually, we define $\Theta_{c,a}^2$ analogously to Θ_a^2 so that

$$\begin{cases} Z\partial_Z^5\Theta_{c,a}^2 - 2\partial_Z^4\Theta_{c,a}^2 = 4\partial_x^2\Theta_{c,a}^2 - 8\partial_x^3\partial_Z^2\Psi_{c,a}^0, \\ \Theta_{c,a}^2|_{Z=0} = \partial_Z\Theta_{c,a}^2|_{Z=0} = \partial_Z^2\Theta_{c,a}^2|_{Z=0} = 0, \\ \partial_Z^j\Theta_{c,a}^2|_{Z=0} = -2(-1)^j\partial_x^2\partial_Z^{j-2}\Theta_{c,a}^0|_{Z=0} \quad \forall j \in \{4,5\}, \\ \Theta_{c,a}^2(x,Z) \to 0 \quad \text{as } Z \to \infty. \end{cases}$$

Once again, note that the boundary conditions are redundant. We also define $\Psi_{c,a}^2$ by $\partial_Z^4\Psi_{c,a}^2 = \partial_x\Theta_{c,a}^2 - 2\partial_x^2\partial_Z^2\Psi_{c,a}^2$, with homogeneous boundary conditions at $Z=0$. We obtain

$$\|\Theta_{c,a}^2\|_{H_x^{8+\frac{3}{4}}L_Z^2} + \|\Theta_{c,a}^2\|_{L_x^2H_Z^{16+\frac{3}{2}}} + \|Z\Theta_{c,a}^2\|_{H^{9+\frac{1}{4}}{}_xL_Z^2} \lesssim B^2(1+T)^{-\frac{1}{2}},$$
$$\|\Psi_{c,a}^2\|_{H_x^{9+\frac{3}{4}}L_Z^2} + \|\Psi_{c,a}^2\|_{L_x^2H_Z^{18+\frac{3}{2}}} \lesssim B^2(1+T)^{-\frac{1}{2}}.$$

Boundary layer corrector Θ_a^4. As in the previous section, we need to define a higher order boundary layer corrector Θ_a^4, whose role is to ensure that

$$\left\|\partial_x^2\Delta^{-2}\theta^{\mathrm{BL}} - \partial_x\psi^{\mathrm{BL}}\right\|_{L^2} \lesssim B(1+t)^{-3}.$$

To that end, we choose Θ_a^4, Ψ_a^4 so that

$$\begin{aligned}\partial_Z^4\Psi_a^4 + 2\partial_x^2\partial_Z^2\Psi_a^2 + \partial_x^4\Psi_a^0 &= \partial_x\Theta_a^4,\\ Z\partial_Z\Theta_a^4 - 8\Theta_a^4 &= 4\partial_x\Psi_a^4.\end{aligned}$$

Eliminating Ψ_a^4 from the equation, we find

$$Z\partial_Z^5\Theta_a^4 - 4\partial_Z^4\Theta_a^4 = 4\partial_x^2\Theta_a^4 - 8\partial_x^3\partial_Z^2\Psi_a^2 - 4\partial_x^5\Psi_a^2.$$

We enforce the following boundary conditions (which are redundant):

$$\Theta_a^4|_{Z=0} = \partial_Z\Theta_a^4|_{Z=0} = 0,\ \partial_Z^4\Theta_a^4|_{Z=0} = \frac{1}{2}\partial_x^3\partial_Z^2\Psi_a^2|_{Z=0},\quad \partial_Z^5\Theta_a^4|_{Z=0} = \frac{2}{3}\partial_x^3\partial_Z^3\Psi_a^2|_{Z=0},$$

together with a decay assumption at infinity. Looking at the equation satisfied by the Fourier transform and applying Lemma 3.3.1, we infer that there exists a (non unique) solution Θ_a^4 of this equation such that

$$\|\Theta_a^4\|_{H_x^{8+\frac{3}{4}}L_Z^2} + \|\Theta_a^4\|_{L_x^2H_Z^{16+\frac{3}{2}}} + \|Z\Theta_a^4\|_{H_x^{9+\frac{1}{4}}L_Z^2} \lesssim \|\theta_0\|_{H^{14}} + B^2.$$

As in the previous section, non-uniqueness comes from the fact that the Fourier transform of $\partial_Z^4\Theta_a^4$ satisfies an ODE of the form (3.28), with boundary conditions at $Z=0$ for $\partial_Z^4\Theta_a^4$ and $\partial_Z^5\Theta_a^4$. However, the boundary conditions above do not prescribe any condition on $\partial_Z^k\Theta_a^4$ for any $k \geq 6$. We lift this indetermination by requiring (somewhat arbitrarily) that $\partial_Z^8\Theta_a^4|_{Z=0} = 0$. The solution thus obtained satisfies the previous Sobolev estimates, and its trace satisfies

$$\|\partial_Z^2\Theta_a^4|_{Z=0}\|_{H^{15/2}(\mathbb{T})} + \|\partial_Z^3\Theta_a^4|_{Z=0}\|_{H^7(\mathbb{T})} \lesssim \|\theta_0\|_{H^{14}} + B^2. \tag{3.55}$$

Lift of the remaining traces of order B. At this stage, we have defined Θ_a^j, Ψ_a^j for $0 \leq j \leq 4$ together with $\Theta_{c,a}^j$, $\Psi_{c,a}^j$ for $0 \leq j \leq 2$. Let $\chi \in C_c^\infty(\mathbf{R})$ be a cut-off function such that $\chi \equiv 1$ on $(-1/4, 1/4)$ and $\mathrm{Supp}\chi \subset (-1/2, 1/2)$. Setting $\Theta_{c,a}^j = \Psi_{c,a}^j = 0$ for $j \geq 3$, the main boundary layer term is given by

$$\begin{aligned}\theta_{\mathrm{main}}^{\mathrm{BL}} &:= \sum_{j=0}^4 (1+t)^{-1-\frac{j}{4}}\left(\Theta_{\mathrm{bot}}^j + \Theta_{c,\mathrm{bot}}^j\right)(x,(1+t)^{1/4}z)\chi(z)\\ &\quad + \sum_{j=0}^4 (1+t)^{-1-\frac{j}{4}}\left(\Theta_{\mathrm{top}}^j + \Theta_{c,\mathrm{top}}^j\right)(x,(1+t)^{1/4}(1-z))\chi(1-z),\\ \psi_{\mathrm{main}}^{\mathrm{BL}} &:= \sum_{j=0}^4 (1+t)^{-2-\frac{j}{4}}\left(\Psi_{\mathrm{bot}}^j + \Psi_{c,\mathrm{bot}}^j\right)(x,(1+t)^{1/4}z)\chi(z)\\ &\quad + \sum_{j=0}^4 (1+t)^{-2-\frac{j}{4}}\left(\Psi_{\mathrm{top}}^j + \Psi_{c,\mathrm{top}}^j\right)(x,(1+t)^{1/4}(1-z))\chi(1-z),\end{aligned}$$

By construction, we have

$$\begin{aligned}
\Delta^2\theta^{\rm BL}_{\rm main}|_{z=0} &= \gamma^0_{{\rm bot},T} + \gamma^0_{{\rm bot},2}(1+t)^{-1/2} + \gamma^0_{{\rm bot},3}(1+t)^{-3/4} \\
&\quad -\gamma^0_{{\rm bot},2}(1+T)^{-1/2} - \gamma^0_{{\rm bot},3}(1+T)^{-3/4} \\
&\quad +2(1+t)^{-5/4}\partial_x^2\partial_Z^2\Theta^3_{\rm bot}|_{Z=0} + 2(1+t)^{-3/2}\partial_x^2\partial_Z^2\Theta^4_{\rm bot}|_{Z=0}, \\
\partial_z\Delta^2\theta^{\rm BL}_{\rm main}|_{z=0} &= \gamma^1_{{\rm bot},T} + \gamma^1_{{\rm bot},1}(1+t)^{-1/4} + \gamma^1_{{\rm bot},2}(1+t)^{-1/2} \\
&\quad -\gamma^1_{{\rm bot},1}(1+T)^{-1/4} - \gamma^1_{{\rm bot},2}(1+t)^{-1/2} \\
&\quad +2(1+t)^{-1}\partial_x^2\partial_Z^3\Theta^3_{\rm bot}|_{Z=0} + 2(1+t)^{-5/4}\partial_x^2\partial_Z^3\Theta^4_{\rm bot}|_{Z=0}.
\end{aligned}$$

Similar formulas hold at $z=1$. We first lift the remaining traces thanks to a corrector $\sigma^{\rm lin}_{\rm lift}$ which we define in Fourier in the following way. Let $\zeta_4, \zeta_5 \in C^\infty_c(\mathbf{R})$ such that $\zeta_j(Z) = Z^j/j!$ in a neighborhood of zero and such that ${\rm Supp}\,\zeta_j \subset (-1/4, 1/4)$. In order to apply the last estimate of Lemma 3.3.5, we further choose ζ_j so that

$$\int_0^\infty Z^k\zeta_j(Z)\,{\rm d}Z = 0 \quad \forall k \in \{2,3\}. \tag{3.56}$$

We then take

$$\begin{aligned}
\widehat{\sigma^{\rm lin}_{\rm lift}}(t,k,z) &= 2\sum_{l\geq 3, j=0,1} (1+t)^{-\frac{3}{2}-\frac{j+l}{4}}|k|^{-2-j}\widehat{\partial_Z^{2+j}\Theta^l_{\rm bot}}(k)|_{Z=0}\zeta_{4+j}(|k|z(1+t)^{1/4}) \\
&\quad +2\sum_{l\geq 3, j=0,1} (1+t)^{-\frac{3}{2}-\frac{j+l}{4}}|k|^{-2-j}\widehat{\partial_Z^{2+j}\Theta^l_{\rm top}}(k)|_{Z=0}\zeta_{4+j}(|k|(1-z)(1+t)^{1/4})
\end{aligned}$$

The estimates on the traces Θ^j_a for $j \geq 2$ (see (3.54), (3.55)) ensure that for all $k, m \geq 0$ such that $k+m \leq 10$,

$$\begin{aligned}
\|\sigma^{\rm lin}_{\rm lift}\|_{H^m_x H^k_z} &\lesssim (\|\theta_0\|_{H^{14}} + B^2)(1+t)^{-2-\frac{1}{8}+\frac{k}{4}}, \\
\|\partial_t\sigma^{\rm lin}_{\rm lift}\|_{H^m_x H^k_z} &\lesssim (\|\theta_0\|_{H^{14}} + B^2)(1+t)^{-3-\frac{1}{8}+\frac{k}{4}}.
\end{aligned} \tag{3.57}$$

We define an associated corrector $\phi^{\rm lin}_{\rm lift} = \Delta^{-2}\partial_x\sigma^{\rm lin}_{\rm lift}$. According to Lemma 3.3.5 and using (3.56), we have, for all $k, m \geq 0$ such that $k+m \leq 13$,

$$\begin{aligned}
\|\phi^{\rm lin}_{\rm lift}\|_{H^m_x H^k_z} &\lesssim (\|\theta_0\|_{H^{14}} + B^2)(1+t)^{-3-\frac{1}{8}+\frac{k}{4}} \\
\|\partial_t\phi^{\rm lin}_{\rm lift}\|_{H^m_x H^k_z} &\lesssim (\|\theta_0\|_{H^{14}} + B^2)(1+t)^{-4-\frac{1}{8}+\frac{k}{4}}.
\end{aligned} \tag{3.58}$$

Evaluation of the remainder. Let us now focus on the different remainder terms in the equation satisfied by $\theta^{\rm BL}_{\rm main}$, in view of defining one last linear corrector.

- *Remainder stemming from the nonlinear term:* Using Lemma 3.3.4 together with the esti-

mates on Θ_a^j, we have, setting $Z_{\rm bot} = (1+t)^{1/4}z$, $Z_{\rm top} = (1+t)^{1/4}(1-z)$

$$\begin{aligned}
&\nabla^\perp \psi^{\rm BL}_{\rm main} \cdot \nabla \theta^{\rm BL}_{\rm main} \\
= &\sum_{0\le j,k\le 4} (1+t)^{-3-\frac{k+j-1}{4}} \left\{ \Psi^j_{\rm bot} + \Psi^j_{c,\rm bot}, \Theta^k_{\rm bot} + \Theta^k_{c,\rm bot} \right\}_{x,Z}(x, Z_{\rm bot})\chi(z) \\
&- \sum_{0\le j,k\le 4} (1+t)^{-3-\frac{k+j-1}{4}} \{\Psi^j_{\rm top} + \Psi^j_{c,\rm top}, \Theta^k_{\rm top} + \Theta^k_{c,\rm top}\}_{x,Z}(x, Z_{\rm top})\chi(1-z) \\
&+ O(\exp(-c(1+t)^{1/5})) \quad \text{in } H^9(\Omega).
\end{aligned}$$

In the above expansion, we put aside the terms corresponding to $k = j = 0$, which are part of S_a^3 and are lifted by Θ_a^3. If $j + k \ge 1$, we claim that for all $s \le 8$,

$$\left\| \{\Psi^j_{\rm bot} + \Psi^j_{c,\rm bot}, \Theta^k_{\rm bot} + \Theta^k_{c,\rm top}\}_{x,Z}(x, (1+t)^{1/4}z)\chi(z) \right\|_{H^s} \lesssim B^2 (1+t)^{\frac{s}{4}-\frac{1}{8}},$$

and the same estimate holds for the top boundary layer. For $s \le 7$, the above estimate follows simply from the estimates on Ψ_a^j, Θ_a^j. For $s = 8$, there is a small difficuly rising from the fact that Θ_a^4 does not belong to $H^9_x L^2_Z$. For this term, we write

$$\partial_Z \Psi_a^j \partial_x \Theta_a^4 = \frac{\partial_Z \Psi_a^j}{Z} Z \partial_x \Theta_a^4.$$

Since $\partial_Z \Psi_a^j$ vanishes at $Z = 0$, both terms belong to $H^8(\mathbb{T} \times \mathbf{R}_+)$, and the H^8 estimate follows. We infer that

$$\begin{aligned}
\nabla^\perp \psi^{\rm BL}_{\rm main} \cdot \nabla \theta^{\rm BL}_{\rm main} = &(1+t)^{-11/4} \{\Psi^0_{\rm bot}, \Theta^0_{\rm bot}\}_{x,Z}(x, (1+t)^{1/4}z)\chi(z) \\
&- (1+t)^{-11/4} \{\Psi^0_{\rm top}, \Theta^0_{\rm top}\}_{x,Z}(x, (1+t)^{1/4}(1-z))\chi(1-z) \\
&+ R_{NL},
\end{aligned}$$

where for all $s \in [0, 8]$,

$$\|R_{NL}\|_{H^s} \lesssim B^2 (1+t)^{-3+\frac{s}{4}-\frac{1}{8}}.$$

- *Remainder stemming from the Taylor expansion of G:* As explained in the construction of Θ_a^2, Θ_a^3, when defining the boundary layer term, we replaced G by its Taylor expansion in the vicinity of $z = 0$ and $z = 1$. Recalling (3.44), we have, in the vicinity of $z = 0$, setting $Z = (1+t)^{1/4}z$,

$$\begin{aligned}
G\partial_x \psi^{\rm BL}_{\rm main} &= \frac{1}{2(1+t)^{1/2}} g^2_{\rm bot,\infty} Z^2 \partial_x \psi^{\rm BL}_{\rm main} + \frac{1}{6(1+t)^{3/4}} g^3_{\rm bot,\infty} Z^3 \partial_x \psi^{\rm BL}_{\rm main} \\
&\quad + O((1+t)^{-1}(Z^2 + Z^4)\partial_x \psi^{\rm BL}_{\rm main}) \\
&= \frac{1}{2(1+t)^{5/2}} g^2_{\rm bot,\infty} Z^2 \partial_x \Psi^0_{\rm bot}(x, Z)\chi(z) \\
&\quad + (1+t)^{-11/4} \left(\frac{g^2_{\rm bot,\infty}}{2} Z^2 \partial_x \Psi^1_{\rm bot}(x, Z) + \frac{g^3_{\rm bot,\infty}}{6} Z^3 \partial_x \Psi^0_{\rm bot}(x, Z) \right) \chi(z) \\
&\quad + R_G,
\end{aligned}$$

where the first two terms enter the definition of $\Theta^2_{\rm bot}$ and $\Theta^3_{\rm bot}$ respectively, and the re-

mainder term R_G satisfies

$$\|R_G\|_{H^s} \lesssim B^2(1+t)^{-3-\frac{1}{8}+\frac{s}{4}} \quad \forall s \in \{0,\cdots,8\}.$$

- *Remainder stemming from $\psi^{BL} - \Delta^{-2}\partial_x\theta^{BL}$:* We now address the fact that $\Delta^2\psi^{\rm BL}_{\rm main}$ is not equal to $\partial_x\theta^{\rm BL}_{\rm main}$. More precisely, using the definition of Ψ^j_a, we have, in $\Omega\cap\{z\le 1/2\}$,

$$\begin{aligned}\Delta^2\psi^{\rm BL}_{\rm main} - \partial_x\theta^{\rm BL}_{\rm main} \quad = \quad & 2\sum_{j=3,4}(1+t)^{-2-\frac{j-2}{4}}\partial_x^2\partial_Z^2\Psi^j_{\rm bot}(x,(1+t)^{1/4}z)\chi(z)\\ &+\sum_{j\ge 1}(1+t)^{-2-\frac{j}{4}}\partial_x^4\Psi^j_{\rm bot}(x,(1+t)^{1/4}z)\chi(z)\\ &+2\sum_{j=1,2}(1+t)^{-2-\frac{j-2}{4}}\partial_x^2\partial_Z^2\Psi^j_{c,\rm bot}(x,(1+t)^{1/4}z)\chi(z)\\ &+\sum_{j\ge 0}(1+t)^{-2-\frac{j}{4}}\partial_x^4\Psi^j_{c,\rm bot}(x,(1+t)^{1/4}z)\chi(z)\\ &+O(\exp(-c(1+t)^{1/5})) \quad \text{in } H^8(\Omega).\end{aligned}$$

A similar expression holds in $\Omega\cap\{z\ge 1/2\}$, replacing bot with top and z with $1-z$. The exponentially small remainder comes from the commutator of the bilaplacian with multiplication by χ (see Lemma 3.3.4), and from the estimates on Ψ^j_a, $\Psi^j_{c,a}$. It follows that

$$\begin{aligned}\partial_x\psi^{\rm BL}_{\rm main} - \Delta^{-2}\partial_x^2\theta^{\rm BL}_{\rm main} =: R_{\Delta^2},\\ \text{with } \|R_{\Delta^2}\|_{H^s} \lesssim (\|\theta_0\|_{H^{14}}+B^2)(1+t)^{-3-\frac{1}{8}+\frac{s}{4}} \quad \forall s\in\{0,\cdots,9\}.\end{aligned}$$

Note that the decay of this remainder is similar to the one of R_{NL} and R_G, but its order of magnitude is B. Hence we call it a "linear" remainder, and it cannot be included in a bootstrap argument. Therefore we will lift it thanks to another (linear) corrector.

- *Remainder stemming from σ^{lin}_{lift}:* Recalling (3.57), (3.58), we have, setting $R_{c,\rm lin} = \partial_t\sigma^{\rm lin}_{\rm lift} - \Delta^{-2}\partial_x^2\sigma^{\rm lin}_{\rm lift}$,

$$\begin{aligned}\|R_{c,\rm lin}\|_{H^m_x H^k_z} \lesssim (\|\theta_0\|_{H^{14}}+B^2)(1+t)^{-3+\frac{k}{4}-\frac{1}{8}},\\ \left\|G\Delta^{-2}\partial_x^2\sigma^{\rm lin}_{\rm lift}\right\|_{H^m_x H^k_z} \lesssim B^2(1+t)^{-3+\frac{k}{4}-\frac{1}{8}}.\end{aligned}$$

Once again, $R_{c,\rm lin}$ is a linear remainder, and must be lifted before the bootstrap argument of the next subsection.

In the remainders above, all terms of order $B^2(1+t)^{-3}$ in L^2 will be included in the remainder for the interior part (see 3.4.5), while the terms of order $B(1+t)^{-3}$ will be lifted thanks to a linear corrector $\sigma^{\rm rem}$, which we now construct.

Definition of $\sigma^{\rm rem}$. Let $\sigma^{\rm rem}$ be the solution of

$$\begin{aligned}\partial_t\sigma^{\rm rem} = \partial_x^2\Delta^{-2}\sigma^{\rm rem} - R_{\Delta^2} - R_{c,\rm lin},\\ \sigma^{\rm rem}(t=0) = 0.\end{aligned}$$

Note that $\partial_t \sigma^{\rm rem}|_{\partial\Omega} = \partial_t \partial_{\boldsymbol{n}} \sigma^{\rm rem}|_{\partial\Omega} = 0$, and therefore $\sigma^{\rm rem}|_{\partial\Omega} = \partial_{\boldsymbol{n}}\sigma^{\rm rem}|_{\partial\Omega} = 0$ for all $t > 0$. Applying Δ^2 to the above equation and taking the trace at $z = 0$, we have

$$\begin{aligned}
\partial_t \Delta^2 \sigma^{\rm rem}|_{z=0} &= -\Delta^2 \left(R_{\Delta^2} + R_{c,\rm lin}\right)|_{z=0} \\
&= -\partial_x \Delta^2 \psi^{\rm BL}_{\rm main}|_{z=0} - \partial_t \partial_z^4 \sigma^{\rm lin}_{\rm lift}|_{z=0} \\
&= -2\sum_{j\geq 3} (1+t)^{-2-\frac{j-2}{4}} \partial_x^3 \partial_Z^2 \Psi^j_{\rm bot}|_{Z=0} + 2\sum_{j\geq 3} \frac{2+j}{4} (1+t)^{-2-\frac{j-2}{4}} \partial_x^2 \partial_Z^2 \Theta^j_{\rm bot}|_{Z=0} \\
&= 0.
\end{aligned}$$

Hence $\Delta^2 \sigma^{\rm rem}|_{z=0} = 0$ for all $t \in (0,T)$. In a similar way, $\partial_z \Delta^2 \sigma^{\rm rem}|_{z=0} = 0$ for all $t \in (0,T)$, and the same properties hold at $z = 1$. Applying Proposition 3.2.6, (3.2.4), we infer that

$$\begin{gathered}
\|\Delta^4 \sigma^{\rm rem}\|_{L^2} \lesssim \|\theta_0\|_{H^{14}} + B^2, \quad \|\partial_x^2 \Delta^2 \sigma^{\rm rem}\|_{L^2} \lesssim (\|\theta_0\|_{H^{14}} + B^2)(1+t)^{-1}, \\
\|\partial_x^4 \sigma^{\rm rem}\|_{L^2} \lesssim (\|\theta_0\|_{H^{14}} + B^2)(1+t)^{-2}, \\
\|\partial_t \partial_x^4 \sigma^{\rm rem}\|_{L^2} \lesssim (\|\theta_0\|_{H^{14}} + B^2)(1+t)^{-3}, \quad \|\partial_x^6 \Delta^{-2} \sigma^{\rm rem}\|_{L^2} \lesssim (\|\theta_0\|_{H^{14}} + B^2)(1+t)^{-3}.
\end{gathered}$$

Furthermore, looking at the expressions of R_{Δ^2} and $R_{c,\rm lin}$ and recalling the estimates on Ψ^j_a, we find that

$$\begin{gathered}
\|\Delta^4 \sigma^{\rm rem}\|_{H^{3/4}_x L^2_z} + \|\Delta^4 \sigma^{\rm rem}\|_{L^2_x H^1_z} \lesssim \|\theta_0\|_{H^{14}} + B^2, \\
\|z\Delta^4 \sigma^{\rm rem}\|_{H^1_x L^2_z(z\leq 1/2)} + \|(1-z)\Delta^4 \sigma^{\rm rem}\|_{H^1_x L^2_z(z\geq 1/2)} \lesssim \|\theta_0\|_{H^{14}} + B^2.
\end{gathered}$$

Conclusion. Let

$$\theta^{\rm BL} := \theta^{\rm BL}_{\rm main} + \sigma^{\rm lift}_{\rm lin} + \sigma^{\rm rem}.$$

Then $\theta^{\rm BL}$ satisfies the boundary conditions stated in Lemma 3.4.8. Most of the remainder terms have already been evaluated. There only remains to evaluate the quadratic terms involving $\sigma^{\rm lift}_{\rm lin}$ and $\sigma^{\rm rem}$. We have for instance, for all $s \leq 7$,

$$\left\|\nabla^\perp \psi^{\rm BL}_{\rm main} \cdot \nabla \sigma^{\rm rem}\right\|_{H^s} \lesssim B^2 (1+t)^{-2+\frac{1+s}{4}} (1+t)^{-2+\frac{1}{4}} \lesssim B^2 (1+t)^{-\frac{7}{2}+\frac{s}{4}},$$

and for the H^8 estimate, we write as before, for $z \leq 1/2$,

$$\partial_x \psi^{\rm BL}_{\rm main} \partial_z \sigma^{\rm rem} = \frac{\partial_x \psi^{\rm BL}_{\rm main}}{z} z \partial_z \sigma^{\rm rem}.$$

Both terms in the right-hand side belong to H^8, and we infer

$$\left\|\nabla^\perp \psi^{\rm BL}_{\rm main} \cdot \nabla \sigma^{\rm rem}\right\|_{H^8} \lesssim B^2 (1+t)^{-5/4}.$$

The statement of 3.4.8 follows. □

Proof of Lemma 3.4.10. Assume that $\theta_{\rm rem} = \theta' - \theta^{\rm BL}$ satisfies (3.34), and define $\Gamma^j_{a,T}$ as in Lemma 3.4.5. According to Lemma 3.4.5,

$$\begin{aligned}
\|\Gamma^j_{a,T}(t)\|_{L^2(\mathbb{T})} &\lesssim B^2 (1+t)^{-1+\frac{j}{4}}, \\
\|\partial_t \Gamma^j_{a,T}(t)\|_{L^2(\mathbb{T})} &\lesssim B^2 (1+t)^{-2+\frac{j}{4}}, \\
\|\Gamma^j_{a,T}(t)\|_{H^4(\mathbb{T})} &\lesssim B^2 (1+t)^{-\frac{23}{24}+\frac{j}{4}}.
\end{aligned}$$

We now lift these traces thanks to a corrector $\sigma^{\rm NL}_{\rm lift}$, whose definition is similar to the one of $\sigma^{\rm lin}_{\rm lift}$, namely

$$\begin{aligned}\widehat{\sigma^{\rm NL}_{\rm lift}}(t,k,z) &= \sum_{j=0,1}(1+t)^{-1-\frac{j}{4}}|k|^{-4-j}\widehat{\Gamma^j_{{\rm bot},T}}(t,k)\zeta_{4+j}(|k|z(1+t)^{1/4})\\ &\quad+\sum_{j=0,1}(1+t)^{-1-\frac{j}{4}}|k|^{-4-j}\widehat{\Gamma^j_{{\rm top},T}}(t,k)\zeta_{4+j}(|k|(1-z)(1+t)^{1/4})\end{aligned}$$

where we recall that $\zeta_j\in C^\infty_c(\mathbf{R})$, $\zeta(Z)=Z^j/j!$ in a neighborhhod of zero, and ζ_j satisfies (3.56). It follows from the above estimates and from the formulas that for $l=0,1$,

$$\begin{aligned}\|\partial_t^l\sigma^{\rm NL}_{\rm lift}\|_{H^m_xH^k_z}&\lesssim B^2(1+t)^{-2-l+\frac{k}{4}-\frac18}\quad\text{if } k+m\le 9/2,\\ \|\sigma^{\rm NL}_{\rm lift}\|_{H^{17/2}(\Omega)}&\lesssim B^2(1+t)^{-1/12},\\ \|z\sigma^{\rm NL}_{\rm lift}\|_{H^{19/2}(\Omega\cap\{z\le1/2\})}+\|(1-z)\sigma^{\rm NL}_{\rm lift}\|_{H^{19/2}(\Omega\cap\{z\ge1/2\})}&\lesssim B^2(1+t)^{-1/12}.\end{aligned}$$

Furthermore, the function $\sigma^{\rm NL}_{\rm lift}$ has been designed so that

$$\begin{aligned}\Delta^2\sigma^{\rm NL}_{\rm lift}|_{z=0}=\Gamma^0_{\rm bot}(t),\quad \partial_z\Delta^2\sigma^{\rm NL}_{\rm lift}|_{z=0}=\Gamma^1_{\rm bot}(t),\\ \Delta^2\sigma^{\rm NL}_{\rm lift}|_{z=1}=\Gamma^0_{\rm top}(t),\quad \partial_z\Delta^2\sigma^{\rm NL}_{\rm lift}|_{z=1}=\Gamma^1_{\rm top}(t).\end{aligned}$$

Furthermore, according to Lemma 3.3.5 and using (3.56), we have, for all $k,m\ge0$ such that $k+m\le12$,

$$\begin{aligned}\|\Delta^{-2}\sigma^{\rm NL}_{\rm lift}\|_{H^m_xH^k_z}&\lesssim B^2(1+t)^{-3-\frac18+\frac{k}{4}}\\ \|\partial_t\Delta^{-2}\sigma^{\rm NL}_{\rm lift}\|_{H^m_xH^k_z}&\lesssim B^2(1+t)^{-4-\frac18+\frac{k}{4}}.\end{aligned}$$

The statement of Lemma 3.4.10 follows immediately from these estimates and from Lemmas 3.4.5 and 3.4.8.

□

3.4.5 Boostrap argument for $\theta^{\rm int}$

In this subsection, we complete the proof of Theorem 3.1.3 thanks to a bootstrap argument. We start with an initial data $\theta_0\in H^{14}(\Omega)$, with $\|\theta_0\|_{H^s(\Omega)}\le B$ and $\theta_0=\partial_{\boldsymbol{n}}\theta_0=0$ on $\partial\Omega$. We assume that $B\le B_0<1$, where B_0 is a small universal constant, so that Theorem 3.1.1 holds.

Let $\bar C\ge2$ be a universal constant to be determined. We define

$$T_1=\sup\left\{T>0,\ \text{(3.35) holds on } (0,T) \text{ with } B=\bar C\|\theta_0\|_{H^{14}}\right\}.$$

By continuity, $T_1>0$. For any $T\in(0,T_1)$, we define an associated boundary layer profile $\theta^{\rm BL}_T$ (see Lemma 3.4.8 and Remark 3.4.9). We recall that there exists a universal constant C_1 such that for all $m,k\ge0$ with $k+m\le8$, forall all $T\in(0,T_1)$,

$$\|\theta^{\rm BL}_T\|_{H^m_xH^k_z}\le C_1\bar C\|\theta_0\|_{H^{14}}(1+t)^{-1+\frac{k}{4}-\frac18}.$$

We then introduce a new time

$$T_2=\sup\left\{T\in(0,T_1),\ \theta_{\rm rem}=\theta'-\theta^{\rm BL}_T \text{ satisfies (3.34) on } (0,T) \text{ with } B=2(1+C_1)\bar C\|\theta_0\|_{H^{14}}\right\}. \tag{3.59}$$

On $(0, T_2)$, according to Lemma 3.4.10, we construct an approximate solution θ^{app}. We now set $\theta^{\mathrm{int}} = \theta' - \theta^{\mathrm{app}} = \theta_{\mathrm{rem}} - \sigma^{\mathrm{NL}}_{\mathrm{lift}}$. Note that we can always choose $\|\theta_0\|_{H^{14}}$ small enough so that for all $t \in (0, T_2)$, for $0 \leq k + m \leq 8$,

$$\|\sigma^{\mathrm{NL}}_{\mathrm{lift}}\|_{H^m_x H^k_z} \leq \bar{C}\|\theta_0\|_{H^{14}}(1+t)^{-2+\frac{k}{4}-\frac{1}{8}}.$$

Consequently, θ^{int} satisfies (3.34) with $B = (3 + 2C_1)\bar{C}\|\theta_0\|_{H^{14}}$ on $(0, T_2)$.

Our goal is now to prove that $T_1 = T_2 = +\infty$ for a suitable choice of $\bar{C}$, provided $\|\theta_0'\|_{H^{14}}$ is sufficiently small. To that end, we check that $\Delta^2\theta^{\mathrm{int}}$ satisfies the assumptions of Proposition 3.2.6.

By construction (see 3.4.10),

$$\theta^{\mathrm{int}} = \partial_{\boldsymbol{n}}\theta^{\mathrm{int}} = \Delta^2\theta^{\mathrm{int}} = \partial_{\boldsymbol{n}}\Delta^2\theta^{\mathrm{int}} = 0 \quad \text{on } \partial\Omega.$$

Furthermore, defining the quadratic form

$$Q(f, g) = -\left(\nabla^\perp\Delta^{-2}\partial_x f \cdot \nabla g\right)',$$

we have, recalling Lemma 3.4.10,

$$\partial_t\theta^{\mathrm{int}} = (1 - G)\partial_x^2\Delta^{-2}\theta^{\mathrm{int}} + S^1_{\mathrm{rem}}, \tag{3.60}$$

where

$$S^1_{\mathrm{rem}} = -S^0_{\mathrm{rem}}Q(\theta^{\mathrm{app}} + \theta^{\mathrm{int}}, \theta^{\mathrm{int}}) + Q(\theta^{\mathrm{int}}, \theta^{\mathrm{app}}).$$

We claim that we have the following estimates on S^1_{rem}:

Lemma 3.4.11 (Estimates on S^1_{rem})**.** *Let T_2 be defined by (3.59).*

- *L^2 and H^4 estimates: for $0 \leq s \leq 4$, for all $t \in [0, T^*)$,*

$$\|S^1_{\mathrm{rem}}(t)\|_{H^s} \lesssim B^2\frac{1}{(1+t)^{3-\frac{s}{4}}};$$

- *H^8 estimate: there exists $S^2_{\mathrm{rem}}, S^2_\perp \in L^\infty([0, T^*), L^2(\Omega))$ such that $\Delta^4 S^0_{\mathrm{rem}}(t) = S^2_{\mathrm{rem}} + S^2_\perp$, with*

$$\|S^2_{\mathrm{rem}}(t)\|_{L^2} \lesssim B^2\frac{1}{(1+t)^{9/8}} \quad \forall t \in [0, T^*) \quad \text{and} \quad \int_\Omega S^2_\perp(t)\Delta^4\theta_{rem}(t) = 0.$$

- *Estimates on the time derivative: for all $t \in [0, T^*)$,*

$$\|\partial_t S^0_{\mathrm{rem}}(t)\|_{L^2} \lesssim B^2\frac{1}{(1+t)^4}.$$

Furthermore, there exists $S^3_{\mathrm{rem}}, S^3_\perp \in L^\infty([0, T^*), L^2(\Omega))$ such that $\Delta^2\partial_t S^0_{\mathrm{rem}}(t) = S^3_{\mathrm{rem}} + S^3_\perp$, with

$$\|S^3_{\mathrm{rem}}(t)\|_{L^2} \lesssim B^2\frac{1}{(1+t)^3} \quad \forall t \in [0, T^*) \quad \text{and} \quad \int_\Omega S^3_\perp(t)\Delta^2\partial_t\theta_{rem}(t) = 0.$$

Proof. We estimate each term separately. The estimates on $S^0_{\rm rem}$ have already been proved in the previous paragraph. Therefore we focus on the quadratic terms. It follows from the estimates of Lemmas 3.4.8, 3.4.10 and from the definition of $\theta^{\rm app}$ that for $s \in \{0, \cdots, 7\}$,

$$\big\|Q(\theta^{\rm app} + \theta^{\rm int}, \theta^{\rm int})\big\|_{H^s} \lesssim B^2(1+t)^{-4+\frac{s+1}{4}}.$$

For $s = 8$, the situation is slightly different, because $\Delta^4 Q(\theta^{\rm app} + \theta^{\rm int}, \theta^{\rm int})$ involves derivatives of order 9 of $\theta^{\rm int}$, for which we have no estimate. Therefore we decompose $\Delta^4 Q(\theta^{\rm app} + \theta^{\rm int}, \theta^{\rm int})$ into two parts, writing

$$\begin{aligned}\Delta^4 Q(\theta^{\rm app} + \theta^{\rm int}, \theta^{\rm int}) = &- \big(\nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int})\big) \cdot \nabla \Delta^4 \theta^{\rm int} \\ &- \partial_z^8 \overline{\nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int}) \cdot \theta^{\rm int}} \\ &- \big[\Delta^4, \nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int}) \cdot \nabla\big] \theta^{\rm int}.\end{aligned}$$

It can be easily checked that the term $\big[\Delta^4, \nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int}) \cdot \nabla\big] \theta^{\rm int}$ can be evaluated as above, and we have

$$\big\|\big[\Delta^4, \nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int}) \cdot \nabla\big] \theta^{\rm int}\big\|_{L^2} \lesssim B^2(1+t)^{-2+\frac{9}{4}}(1+t)^{-2+\frac{1}{4}} \lesssim B^2(1+t)^{-3/2}.$$

Furthermore, since $\langle \Delta^4 \theta^{\rm int}(t, \cdot, z)\rangle = 0$ for all t, z,

$$\int_\Omega \partial_z^8 \overline{\nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int}) \cdot \theta^{\rm int}} \Delta^4 \theta^{\rm int} = 0.$$

Eventually, integrating by parts the remaining term,

$$\begin{aligned}-\int_\Omega \big(\big(\nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int})\big) \cdot \nabla \Delta^4 \theta^{\rm int}\big) \Delta^4 \theta^{\rm int} \\ = \frac{1}{2} \int_\Omega \nabla \cdot \big(\nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int})\big) |\Delta^4 \theta^{\rm int}|^2 = 0.\end{aligned}$$

Therefore, setting

$$\begin{aligned}S^2_\perp = &- \nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int}) \cdot \nabla \Delta^4 \theta^{\rm int} - \partial_z^8 \overline{\nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int}) \cdot \theta^{\rm int}}, \\ S^2_{\rm rem} = &\, \Delta^4 Q(\theta^{\rm int}, \theta^{\rm app}) - \big[\Delta^4, \nabla^\perp \Delta^{-2} \partial_x (\theta^{\rm app} + \theta^{\rm int}) \cdot \nabla\big] \theta^{\rm int},\end{aligned}$$

we obtain the desired H^8 estimates.

We now need to estimate the time derivative of $S^1_{\rm rem}$ in L^2 and H^4. Note that the definition of T_2 (see (3.59)) ensures that

$$\|\partial_t \theta^{\rm int}(t)\|_{L^2} \le B(1+t)^{-3}, \quad \|\partial_t \theta^{\rm int}(t)\|_{H^4} \le B(1+t)^{-2}.$$

Setting $\psi^{\rm int} = \Delta^{-2} \partial_x \theta^{\rm int}$, it follows that

$$\|\partial_t \psi^{\rm int}\|_{H^3} \lesssim B(1+t)^{-3}, \quad \|\partial_t \psi^{\rm int}\|_{H^7} \lesssim B(1+t)^{-2}.$$

From there, differentiating with respect to time $S^1_{\rm rem}$, we obtain the desired estimate in L^2. The only problematic term is $\partial_t \partial_x \psi^{\rm int} \partial_z \theta^{\rm app}$, which we decompose as

$$\partial_t \partial_x \psi^{\rm int} \partial_z \theta^{\rm app} \chi(z) + \partial_t \partial_x \psi^{\rm int} \partial_z \theta^{\rm app} (1 - \chi(z)),$$

with $\chi \in C_c^\infty(\mathbf{R})$ such that $\chi \equiv 1$ in a neighbourhood of zero and $\chi(z) = 0$ for $|z| \geq 1/2$. Let us consider the first term. Recalling that $\psi^{\rm int}(z=0) = 0$, we write, using the Hardy inequality,

$$\begin{aligned}\|\partial_t\partial_x\psi^{\rm int}\partial_z\theta^{\rm app}\chi(z)\|_{L^2} &\leq \left\|\frac{1}{z}\partial_t\partial_x\psi^{\rm int}\right\|_{L^2}\|z\partial_z\theta^{\rm app}\chi(z)\|_{L^\infty}\\ &\lesssim \left\|\partial_t\partial_x\partial_z\psi^{\rm int}\right\|_{L^2}\|z\partial_z\theta^{\rm app}\chi(z)\|_{L^\infty}\\ &\lesssim B(1+t)^{-3}\times B(1+t)^{-1} \lesssim B^2(1+t)^{-4}.\end{aligned}$$

The term involving $(1-\chi(z))$ is treated similarly, exchanging the roles of $z=0$ and $z=1$.

The decomposition of $\Delta^2\partial_t S^1_{\rm rem}$ goes along the same lines as the one of $\Delta^4 S^1_{\rm rem}$ and is left to the reader.

□

Conclusion. We apply the operator Δ^2 to equation (3.60). We recall that by construction, $\Delta^2\theta^{\rm int} = \partial_{\boldsymbol{n}}\Delta^2\theta^{\rm int} = 0$ on $\partial\Omega$. We obtain

$$\partial_t\Delta^2\theta^{\rm int} = (1-G)\partial_x\theta^{\rm int} + \Delta^2 S^1_{\rm rem} - \sum_{k=1}^{4}\binom{4}{k}\partial_z^k G\partial_z^{4-k}\partial_x\psi^{\rm int} - 2\sum_{k=1}^{2}\binom{2}{k}\partial_z^k G\partial_z^{2-k}\partial_x^2\psi^{\rm int}.$$

Let us now check that the assumptions of Proposition 3.2.6 are satisfied. The decay assumptions on $\Delta^2 S^1_{\rm rem}$ follow from Lemma 3.4.11. Therefore it suffices to check that the decay of the two commutator terms (involving derivatives of G) satisfy the desired bounds. We focus on the first term, which involves the largest number of z-derivatives. By definition of T_2 and according to the bounds of G (see Lemma 3.4.1), we have, for all $t \in (0,T_2)$, for $k \in \{1,\cdots,4\}$,

$$\begin{aligned}\left\|\partial_z^k G\partial_z^{4-k}\partial_x\psi^{\rm int}(t)\right\|_{L^2} &\lesssim B^2(1+t)^{-2},\\ \left\|\partial_z^k G\partial_z^{4-k}\partial_x\psi^{\rm int}(t)\right\|_{H^4} &\lesssim B^2(1+t)^{\frac{9}{8}-3+\frac{1}{4}} + B^2(1+t)^{-5/4} \lesssim B^2(1+t)^{-5/4}.\end{aligned}$$

Furthermore, for $k \in \{1,\cdots,4\}$,

$$\left\|\partial_t\partial_z^k G\partial_z^{4-k}\partial_x\psi^{\rm int}(t)\right\|_{L^2} \lesssim B^3(1+t)^{\frac{k+3}{4}-3-\frac{1}{8}-3+\frac{4-k}{4}} \lesssim B^2(1+t)^{-\frac{17}{4}-\frac{1}{8}}.$$

According to Proposition 3.2.6, there exists a universal constant C_2 such that for all $t \in (0,T^*)$, setting $B = (3+2C_1)\bar{C}\|\theta_0\|_{H^{14}}$ (see (3.59)),

$$\begin{aligned}\|\Delta^2\theta^{\rm int}(t)\|_{L^2} &\leq C_2\left(\|\theta_{\rm rem}(t=0)\|_{H^8} + B^2\right)(1+t)^{-1},\\ \|\Delta^2\theta^{\rm int}(t)\|_{H^4} &\leq C_2\left(\|\theta_{\rm rem}(t=0)\|_{H^8} + B^2\right),\\ \|\partial_x\theta^{\rm int}(t)\|_{L^2} &\leq C_2\left(\|\theta_{\rm rem}(t=0)\|_{H^8} + B^2\right)(1+t)^{-2}.\end{aligned} \tag{3.61}$$

There remains to bound $\partial_t\theta^{\rm int}$ and $\psi^{\rm int}$ in L^2. To that end, we differentiate (3.60) with respect to time, and we obtain

$$\partial_t\partial_t\theta^{\rm int} = (1-G)\partial_x\partial_t\psi^{\rm int} + \partial_t S^1_{\rm rem} - \partial_t G\partial_x\psi^{\rm int}.$$

The source term $\partial_t S^1_{\rm rem}$ is evaluated in Lemma 3.4.11. As for the commutator term, we have

$$\left\|\partial_t G\partial_x\psi^{\rm int}\right\|_{L^2} \leq \|\partial_t G\|_{L^2}\|\partial_x\psi^{\rm int}\|_\infty \lesssim B^3(1+t)^{\frac{3}{8}-3-2} \lesssim B^3(1+t)^{-4}.$$

Using Proposition 3.2.6, we find that for any $t \in (0, T_2)$,

$$\|\partial_t \theta^{\text{int}}\|_{L^2} \leq C_2 \left(\|\theta^{\text{int}}(t=0)\|_{H^8} + B^2\right) \frac{1}{(1+t)^3}.$$

Using equation (3.60),

$$\|\partial_x \psi^{\text{int}}(t)\|_{L^2} \leq C_2 \left(\|\theta^{\text{int}}(t=0)\|_{H^8} + B^2\right) \frac{1}{(1+t)^3} \quad \forall t \in (0, T_2).$$

Grouping these estimates with the ones on $\sigma_{\text{lift}}^{\text{NL}}$ from Lemma 3.4.10, we infer that up to a change of the constant C_2, for any $t \in (0, T_2)$

$$\begin{aligned}
\|\theta_{\text{rem}}(t)\|_{H^s} &\leq C_2 \left(\|\theta_0\|_{H^8} + B^2\right)(1+t)^{-2+\frac{s}{4}} \quad \forall s \in \{0, \cdots, 8\}, \\
\|\psi_{\text{rem}}(t)\|_{H^s} &\leq C_2 \left(\|\theta_0\|_{H^8} + B^2\right)(1+t)^{-3+\frac{s}{4}} \quad \forall s \in \{0, \cdots, 8\}, \\
\|\partial_t \theta_{\text{rem}}(t)\|_{H^s} &\leq C_2 \left(\|\theta_0\|_{H^8} + B^2\right)(1+t)^{-3+\frac{s}{4}} \quad \forall s \in \{0, \cdots, 4\}.
\end{aligned}$$

We now recall that$B = (3 + 2C_1)\bar{C}\|\theta_0\|_{H^{14}}$for some constant $\bar{C}$ that remains to be chosen. We want to pick $\bar{C}$ so that

$$C_2 \left(\|\|\theta_0\|_{H^8} + (3+2C_1)^2 \bar{C}^2 \|\theta_0\|_{H^{14}}^2\right) \leq (1+C_1)\bar{C}\|\theta_0\|_{H^{14}}.$$

It is sufficient to take C_b such that $2C_2 \leq (1+C_1)\bar{C}$, and $\|\theta_0\|_{H^{14}}$ sufficiently small. We then infer that the bounds within (3.59) are satisfied with B replaced by $B/2$. It follows that $T_2 = T_1$. From there, recalling the estimates on θ^{BL}, we deduce that there exists a universal constant C_3 such that for all $t \in (0, T_1)$, for all $k \in \{4, \cdots, 8\}$,

$$\begin{aligned}
\|\partial_x^k \theta'\|_{L^2} &\leq \|\partial_x^k \theta^{\text{BL}}\|_{L^2} + \|\partial_x^k \theta_{\text{rem}}\|_{L^2} \\
&\leq C_3 \left(\|\theta_0\|_{H^{14}} + B^2\right)\left((1+t)^{-9/8} + (1+t)^{\frac{k-8}{2}}\right).
\end{aligned}$$

Similar estimates hold for $\partial_z^k \theta'$ and $\partial_x^5 \psi$ in L^2. Hence we further choose the constant $\bar{C}$ so that

$$2C_3 \left(\|\theta_0\|_{H^{14}} + B^2\right) \leq \bar{C}\|\theta_0\|_{H^{14}}$$

provided $\|\theta_0\|_{H^{14}}$ is sufficiently small. We conclude that $T_1 = +\infty$. Theorem 3.1.3 follows.

3.5 Appendix: Proof of Lemma 3.3.1

The proof of the Lemma relies on energy estimates in weighted Sobolev spaces, with weights that grow like $\exp(cZ^{4/5})$ for $Z \gg 1$. Unfortunately, we have not been able to treat all four cases for the boundary conditions simultaneously, but we will treat (i) and (iii) (resp. (ii) and (iv)) together. Note that when equation (3.28) is multiplied (formally) by Ψw or by $-\Psi' w$, there are many commutator terms when we integrate by parts the fifth order derivative. The main idea is that if the weight is adequately chosen, all these commutators can be absorbed in the main order terms, which will be designed to have a positive sign. Hence we start with the following result, which will allow us to control the commutators:

Lemma 3.5.1. *Let $\Psi \in C_c^\infty([0, +\infty))$ such that $\Psi(0) = 0$, and let $r > 0$. Let $W \in C^\infty([0, +\infty))$ such that $W(Z) = \exp(Z^{4/5})$ for $Z \geq 1$, and $W \geq 1$, $W \equiv 1$ in a neighbourhood of zero.*

Then for $k \in \{1,2\}$, there exists a constant C_k, independent of r, such that

$$\begin{aligned}&\left|\int_0^\infty |\Psi^{(k)}(Z)|^2 \frac{|W^{(k)}(rZ)|^2}{W(rZ)}\,dZ\right| \\ \leq & C_k r^{-\frac{2}{3}(3-k)}\left[\int_0^\infty |\Psi^{(3)}(Z)|^2 W(rZ)\,dZ + \int_0^\infty \Psi^2(Z)\left(\frac{W'(rZ)}{Z}+\frac{W(rZ)}{Z^2}\right)\,dZ\right].\end{aligned}$$

Proof. For $k = 0, \cdots, 3$, let us consider weights $\omega_k \in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}_+)$ such that

$$\begin{aligned}\forall k \in \{0,\cdots,3\}, \quad \forall Z \geq 1, \omega_k(Z) = e^{-1} Z^{-\frac{2}{5}(3-k)} \exp(Z^{4/5}),\\ \forall Z \in (0,1), \quad \omega_1(Z) = \omega_3(Z) = 1, \quad \omega_0(Z) = Z^{-2}, \quad \omega_2(Z) = Z^2.\end{aligned}$$

Note that the weights ω_k satisfy the following assumptions:

- For $k \in \{1,2\}$, $\omega_k \leq \sqrt{\omega_{k-1}\omega_{k+1}}$;
- For $k \in \{1,2\}$, $|\omega_k'| \leq C_k\sqrt{\omega_k\omega_{k-1}}$ for some constant C_k;
- $\omega_2(0) = 0$.

Let us now introduce, for $k = 0, \cdots, 3$

$$I_k := \int_0^\infty |\Psi^{(k)}(Z)|^2 \omega_k(rZ)\,dZ.$$

Then by definition of W, ω_0, ω_3, there exists a constant C (independent of $r > 0$) such that

$$r^2 I_0 + I_3 \leq C\int_0^\infty |\Psi^{(3)}(Z)|^2 W(rZ)\,dZ + \int_0^\infty \Psi^2(Z)\left(\frac{W'(rZ)}{Z}+\frac{W(rZ)}{Z^2}\right)\,dZ.$$

Let us set $E = r^2 I_0 + I_3$. For $k = 1, 2$, integrating by parts and using the conditions $\Psi(0) = \omega_2(0) = 0$, we have

$$I_k = -\int_0^\infty \Psi^{(k-1)}(Z)\Psi^{(k+1)}(Z)\omega_k(rZ)\,dZ - r\int_0^\infty \Psi^{(k-1)}(Z)\Psi^{(k)}(Z)\omega_k'(rZ)\,dZ.$$

Using the properties of ω_k, we deduce that there exist constants C_k such that

$$I_k \leq C_k\left(\sqrt{I_{k-1}I_{k+1}} + r\sqrt{I_k I_{k-1}}\right).$$

Since $r^2 I_0 \leq E$, we deduce first that $I_1 \lesssim E + r^{-1}\sqrt{I_2 E}$, and plugging this inequality into the bound on I_2, we find

$$I_1 \lesssim r^{-4/3}E, \quad I_2 \lesssim r^{-2/3}E.$$

The result the follows easily by noticing that

$$\left|\int_0^\infty |\Psi^{(k)}(Z)|^2 \frac{|W^{(k)}(rZ)|^2}{W(rZ)}\,dZ\right| \lesssim I_k.$$

□

We now turn towards the proof of Lemma 3.3.1. In both cases, we start with a formal *a priori* estimate, from which we deduce an appropriate notion of variational solution in a suitable Hilbert

space. Existence and uniqueness then follow in a straightforward manner from the Lax-Milgram Lemma.

First case: *conditions (ii) and (iv):*

As explained above, we start with a formal *a priori* estimate. Let $w \in C^\infty(\mathbf{R}_+)$ be an arbitrary weight function, and multiply (3.28) by $(\Psi(Z)w(Z))'/Z$. On the one hand,

$$\int_0^\infty \Psi^{(5)}(Z)(\Psi w)'(Z)\,dZ = \int_0^\infty \Psi^{(3)}(Z)(\Psi w)^{(3)}(Z)\,dZ - \Psi^{(4)}(0)(\Psi w)'(0) + \Psi^{(3)}(0)(\Psi w)''(0).$$

Note that the two boundary terms vanish in cases (ii) and (iv). We obtain

$$\int_0^\infty \Psi^{(5)}(Z)(\Psi w)'(Z)\,dZ = \int_0^\infty (\Psi^{(3)}(Z)^2 w(Z) + \sum_{k=1}^{3} \binom{3}{k} \int_0^\infty \Psi^{(3)}(Z)\Psi^{(3-k)}(Z)w^{(k)}(Z)\,dZ.$$

On the other hand, since $\Psi(0) = 0$,

$$\int_0^\infty \Psi(Z)(\Psi w)'(Z)\frac{dZ}{Z} = \int_0^\infty (\Psi w)(Z)(\Psi w)'(Z)\frac{dZ}{Zw(Z)} = -\frac{1}{2}\int_0^\infty (\Psi w)^2(Z)\frac{d}{dZ}\left(\frac{1}{Zw(Z)}\right)dZ.$$

Choosing w such that $w' > 0$, the right-hand side has a positive sign. We then choose $w(Z) = W(rZ)$ for some $W \in C^\infty(\mathbf{R}_+)$ such that $W(\xi) = \exp(\xi^{4/5})$ for ξ large enough, $W(Z) = 1$ for $Z \le 1$, $W' \ge 0$, and $r > 0$ small enough. With this choice, the positive terms in the energy are bounded from below by

$$\int_0^\infty (\Psi^{(3)}(Z)^2 W(rZ)\,dZ + \int_0^\infty \Psi^2(Z)\left(\frac{W(rZ)}{Z^2} + r\frac{W'(rZ)}{Z}\right)dZ.$$

3.5.1 then implies that there exists an explicit constant $\delta > 0$ such that for $k = 1, 2, 3$,

$$\left|\int_0^\infty \Psi^{(3)}(Z)\Psi^{(3-k)}(Z)w^{(k)}(Z)\,dZ\right| \le r^\delta \int_0^\infty (\Psi^{(3)}(Z)^2 W(rZ)\,dZ + \int_0^\infty \Psi^2(Z)\left(\frac{W(rZ)}{Z^2} + r\frac{W'(rZ)}{Z}\right)dZ.$$

Therefore, for $r > 0$ sufficiently small, we obtain

$$\int_0^\infty (\Psi^{(3)}(Z)^2 W(rZ)\,dZ + \int_0^\infty \Psi^2(Z)\left(\frac{W(rZ)}{Z^2} + r\frac{W'(rZ)}{Z}\right)dZ \lesssim \int_0^1 \frac{S(Z)^2}{Z^2}\,dZ + \int_0^\infty S(Z)^2 W(rZ)\,dZ.$$

This leads us to the following formulation: let

$$\mathcal{H} := \{\Psi \in H^3(\mathbf{R}_+), \Psi(0) = 0, \int_0^\infty (\Psi^{(3)}(Z)^2 e^{(rZ)^{4/5}}\,dZ < +\infty \quad \int_0^\infty \Psi(Z)^2(Z^{-2} + Z^{-1/5})e^{(rZ)^{4/5}}\,dZ < +\infty\},$$

and let $\mathcal{H}_0 := \{\Psi \in \mathcal{H},\ \Psi'(0) = \Psi''(0) = 0\}$. We endow $\mathcal{H}$ and $\mathcal{H}_0$ with the norm

$$\|\Psi\|_{\mathcal{H}}^2 = \int_0^\infty (\Psi^{(3)}(Z)^2 W(rZ)\, dZ + \int_0^\infty \Psi^2(Z) \left(\frac{W(rZ)}{Z^2} + r\frac{W'(rZ)}{Z} \right) dZ,$$

where W is the previous weight. We say that $\Psi \in \mathcal{H}$ is a solution of (3.28)-(ii) (resp. $\Psi \in \mathcal{H}_0$ is a solution of (3.28)-(iv)) if and only if for all $\Phi \in \mathcal{H}$ (resp. $\Phi \in \mathcal{H}_0$),

$$\int_0^\infty \Psi^{(3)}(\Phi W(r\cdot))^{(3)} + \int_0^\infty \Psi(Z) \frac{(\Phi(Z)W(rZ))'}{Z}\, dZ = \int_0^\infty \frac{S(Z)}{Z}(\Phi(Z)W(rZ))'\, dZ.$$

Existence and uniqueness of solutions of (3.28)-(ii) (resp. of (3.28)-(iv)) in $\mathcal{H}$ (resp. $\mathcal{H}_0$) follow easily from the Lax-Milgram Lemma. Using the equation, we then infer that

$$\int_0^\infty (\Psi^{(5)}(Z))^2 e^{(rZ)^{4/5}}\, dZ < +\infty.$$

The result follows.

Second case: *conditions (i) and (iii):*

The estimates in the case of conditions (i) and (iii) are similar, but slightly less straightforward, since we shall need to combine two estimates.

We first mutliply (3.28) by $-\frac{\Psi^{(3)(Z)}(Z)w_1(Z)}{Z}$. We obtain on the one hand

$$-\int_0^\infty \Psi^{(5)}(Z)\Psi^{(3)}(Z)w_1(Z)\, dZ = \int_0^\infty (\Psi^{(4)}(Z))^2 w_1(Z) + \int_0^\infty \Psi^{(4)}(Z)\Psi^{(3)}(Z)w_1'(Z)\, dZ.$$

The first term gives a positive contribution to the energy, and the second one will be treated with the help of Lemma 3.5.1. On the other hand, we obtain for the zero-th order term, noticing that either $\Psi''(0) = 0$ or $(Z^{-1}\Psi(Z))|_{z=0} = \Psi'(0) = 0$,

$$-\int_0^\infty \frac{\Psi(Z)}{Z}\Psi^{(3)}(Z)w_1(Z)\, dZ = \int_0^\infty \Psi''(Z)\frac{d}{dZ}\left(\frac{\Psi(Z)}{Z}w_1(Z)\right) dZ.$$

Now, let us write Ψ'' as

$$\Psi''(Z) = \left(\frac{\Psi(Z)}{Z}Z\right)'' = 2\left(\frac{\Psi(Z)}{Z}\right)' + Z\left(\frac{\Psi(Z)}{Z}\right)''.$$

Performing integrations by parts and assuming that $w_1'(0) = 0$, we obtain

$$\begin{aligned}&\int_0^\infty \Psi''(Z)\frac{d}{dZ}\left(\frac{\Psi(Z)}{Z}w_1(Z)\right) dZ \\ =&\frac{3}{2}\int_0^\infty \left(\frac{d}{dZ}\frac{\Psi(Z)}{Z}\right)^2 (w_1 - Zw_1') + \frac{1}{2}\Psi(Z)^2 Z^{-1} w_1^{(3)}(Z)\, dZ.\end{aligned}$$

We choose w_2 so that $\partial_Z^3 w_2 > 0$, so that the last term has a positive sign. However, for $Z \gg 1$, $w_1 - Zw_1' < 0$, and therefore we add another term to the energy. More precisely, we now mutliply (3.28) by $-Z^{-1}((\Psi/Z)'w_2)'$, with a weight w_2 which vanishes identically in a neighborhood of

zero. We obtain

$$-\int_0^\infty \frac{\Psi(Z)}{Z}\left(\frac{d}{dZ}\frac{\Psi(Z)}{Z}w_2\right)' dZ = \int_0^\infty \left(\frac{d}{dZ}\frac{\Psi(Z)}{Z}\right)^2 w_2(Z)\, dZ.$$

We then take $w_i(Z) = W_i(rZ)$, with $0 < r \ll 1$ and W_1, W_2 such that $W_2 + \frac{3}{2}(W_1 - ZW_1') \gtrsim \exp(Z^{4/5})$. Our energy is then

$$\begin{aligned}
&\int_0^\infty (\Psi^{(4)}(Z))^2 W_1(rZ)\, dZ + \int_0^\infty \left(\frac{d}{dZ}\frac{\Psi(Z)}{Z}\right)^2 \left[W_2 + \frac{3}{2}(W_1 - ZW_1')\right](rZ)\, dZ \\
&+\frac{r^3}{2}\int_0^\infty \Psi(Z)^2 Z^{-1} W_1^{(3)}(rZ)\, dZ \\
\geq\ &\int_0^\infty (\Psi^{(4)}(Z))^2 \exp((rZ)^{4/5})\, dZ + \int_0^\infty \left(\frac{d}{dZ}\frac{\Psi(Z)}{Z}\right)^2 \exp((rZ)^{4/5})\, dZ \\
&+r^3\int_0^\infty \Psi(Z)^2 Z^{-1}(1+rZ)^{-3/5}\, dZ \exp((rZ)^{4/5})\, dZ.
\end{aligned}$$

The commutator terms, namely

$$r\int_0^\infty \Psi^{(4)}(Z)\Psi^{(3)}(Z)W_1'(rZ)\, dZ$$

$$\text{and} \quad \int_0^\infty \Psi^{(4)}(Z)\left(\frac{d}{dZ}\frac{\Psi(Z)}{Z}w_2\right)'' dZ$$

are treated perturbatively for r small enough, using 3.5.1. As before, we find that the energy is controlled by

$$\int_0^1 \frac{S(Z)^2}{Z^2}\, dZ + \int_0^\infty S(Z)^2 \exp{(rZ)^{4/5}}\, dZ.$$

We conclude by a Lax-Milgram type argument.

□

Acknowledgments. The authors thank Miguel Rodrigues for fruitful discussions especially for the identification of the limit profile. This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 637653, project BLOC) and by the French National Research Agency (Grant ANR-18-CE40-0027, project SingFlows). A.-L. Dalibard acknowledges the support of the Institut Universitaire de France. J. Guillod acknowledges the support of the Initiative d'Excellence (Idex) of Sorbonne University through the Emergence program.

3.6 Asymptotic behaviour for Lebesgue data

The aim of this section is to identify some long-time properties of the solutions to the Stokes-transport equation for data in L^∞ and other Lebesgue spaces, with no further assumption. In view of harmonizing the framework with the previous sections, and in order to circumvent unnecessary technical difficulties, we will work with the spatial domain $\Omega = \mathbf{T} \times (0,1)$. A monotonous Lyapunov functional is identified, as the potential energy of the system, which ensures that the density evolution is monotonous. We also remark that the backward Stokes-

transport system is globally well-defined. Nevertheless, no strong convergence of ρ is established in general, since the ω-limit is not necessarily trivial.

3.6.1 Backward well-posedness

Consider $(\rho, \boldsymbol{u}) \in L^\infty(\mathbf{R}_+; L^\infty(\Omega)) \times L^\infty(\mathbf{R}_+; W^{1,\infty})$ satisfying the Stokes-transport equation. Let $T > 0$ and set

$$\check{\rho}(t) = \rho(T-t), \quad \check{\boldsymbol{u}}(t) = -\boldsymbol{u}(T-t), \quad \check{p}(t) = -p(T-t), \quad t \in [0,T].$$

Then these satisfy the Stokes-transport equation with reversed gravity, modelling buoyancy for instance,

$$\left\{ \begin{array}{rcl} \partial_t \check{\rho} + \check{\boldsymbol{u}} \cdot \nabla \check{\rho} &=& 0 \\ -\Delta \check{\boldsymbol{u}} + \nabla \check{p} &=& \check{\rho} \boldsymbol{e}_z \\ \operatorname{div} \check{\boldsymbol{u}} &=& 0 \\ \check{\boldsymbol{u}}|_{\partial\Omega} &=& \mathbf{0} \\ \check{\rho}|_{t=0} &=& \rho(T). \end{array} \right. \tag{3.62}$$

The well-posedness proof provided in Section 2.2 is valid for this system, and there exists a unique global solution $(\check{\rho}, \check{\boldsymbol{u}})$ to (3.62) for the initial datum $\rho(T) \in L^\infty(\Omega)$. As a consequence of Theorem 2.1.1, there exists a unique global solution on $\mathbf{R}$.

Corollary 3.6.1. *For any $\rho_0 \in L^\infty(\Omega)$, the Stokes-transport equation has a unique weak solution $(\rho, \boldsymbol{u})$ in*

$$L^\infty(\mathbf{R}; L^\infty(\Omega)) \times L^\infty(\mathbf{R}; W^{1,\infty}(\Omega)).$$

3.6.2 Potential energy and stationary states

Since the Stokes-transport equation involves gravity, it is natural to consider the potential energy of the density profile solution $\rho(t)$, namely

$$\mathrm{E}(\rho(t)) := \int_\Omega z\rho(t,x,z)\,\mathrm{d}x\,\mathrm{d}z,$$

which might also be denoted $\mathrm{E}(t)$. Using the Stokes-transport equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) = \int_\Omega z\partial_t\rho = -\int_\Omega z\boldsymbol{u}\cdot\nabla\rho = \int_\Omega \operatorname{div}(z\boldsymbol{u})\rho = \int_\Omega u_2\rho = -\|\nabla\boldsymbol{u}\|_{L^2}^2,$$

which provides the classical energy balance between the potential energy and the dissipative effects due to the viscosity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) + \|\nabla\boldsymbol{u}(t)\|_{L^2}^2 = 0. \tag{3.63}$$

The potential energy is therefore a monotonous quantity, non-increasing, satisfying the definition of a *Lyapunov functional* in the sense of [AB20, Definition 1.1], for instance. The study of such other monotonous quantities shall be discussed in Subection 4.1.5 for a related free interface problem. This property, in particular, prevents the system to evolve periodically. Moreover, equation (3.63) allows us to identify the stationary profiles of the system. Indeed, a stationary density profile $\rho_0 \in L^2(\Omega)$ with associated solution ρ constant with respect to time, has a constant potential energy. Hence (3.63) requires that the velocity field must satisfy $\|\nabla\boldsymbol{u}\|_{L^2} = 0$ for all time, implying $\boldsymbol{u}$ to be identically zero. This latter holds by Poincaré inequality in bounded domains, see Subsection 2.2.1 or thanks to conditions such as $\boldsymbol{u}(\boldsymbol{x}) \to \mathbf{0}$ while $|\boldsymbol{x}| \to \infty$ in the

whole space, as in (1.4) for instance. In the case where $\boldsymbol{u} = \mathbf{0}$ in Stokes equation, from Theorem 2.2.1 there exists $p_0 \in H^1(\Omega)$ such that the hydrostatic equilibrium is satisfied, namely

$$\nabla p_0 = -\rho_0 \boldsymbol{e}_z.$$

In particular $\partial_x p_0 = 0$, hence p_0 depends only on the vertical coordinate, and so does ρ_0. Reciprocally, for any $\rho_0 = \rho_0(z)$ the following triplet satisfies the Stokes system

$$(\boldsymbol{u}, p, \rho)(t, x, z) = \left(\mathbf{0}, -\int_0^z \rho_0(z')\,\mathrm{d}z, \rho_0(z)\right).$$

This solution also satisfies the transport equation, since $\partial_t \rho_0 = 0$ and $\boldsymbol{u} = \mathbf{0}$. Hence it proves that the stationary states are exactly the density profiles depending only on the vertical coordinate. We will say that such functions are *stratified*.

It is tempting to conjecture that the solution of the equation converges toward a stationary state, and to wonder if this hypothetical asymptotic state can be described. In the following paragraphs we refine this conjecture, dealing in particular with the notion of rearrangement.

3.6.3 Rearrangement conjecture and weak convergence

Since the Stokes-transport equation modelises a fluid subject to gravity, intuitively we can expect the density to sediment and reorder vertically from bottom to top. Also, since the density is transported by a divergence free velocity field, all the measure of its level sets are preserved along time, and therefore $\rho(t)$ is for all time $t \geq 0$ a rearrangement of ρ_0 in the sense of Definition 3.6.2. In what follows we choose the spatial domain $\Omega = \mathbf{T} \times (0,1)$ where $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is the torus. This domain is mathematically convenient and physically relevant since the vertical coordinate plays a particular role in the system due to gravity. Nevertheless, most of the following is true in general in bounded domains, and some of it still hold in more general domains. Since the dynamic of the solution does not change if one add a constant to $\rho_0 \in L^\infty(\Omega)$, according to the previous paragraph, we consider once and for all, whithout loss of generality, positive initial data $\rho_0 \geq 0$.

Let us recall a definition and a few properties of the rearrangements. For further results on the topic, we refer to [LL01, Chapter 3]. Let us denote by $|\{f > \alpha\}|$ the Lebesgue measure of the $\alpha \in \mathbf{R}$ level set of f defined as $\{f > \alpha\} := \{\boldsymbol{x} \in \Omega : f(\boldsymbol{x}) > \alpha\}$.

Definition 3.6.2. *Two measurable maps $f, g : \Omega \to \mathbf{R}$ are* rearrangement *of each other if the measure of any of their level sets coincide, namely*

$$\forall \alpha \in \mathbf{R}, \qquad |\{f > \alpha\}| = |\{g > \alpha\}|.$$

From the previous discussion, we might be interested in a vertical rearrangement of a density profile. For maps defined on segments (resp. on the whole space), it is classical to consider their decreasing (resp. radially symmetric decreasing) rearrangement. Basically it consists in constructing the unique decreasing (resp. radially decreasing) map noted f^* sharing the same level sets as f, see [LL01, Section 3.3]. In our case we can adapt these considerations to define a *vertical decreasing rearrangement*, based on the idea of the decreasing rearrangement on an interval.

Definition 3.6.3. *Given a Borel set $A \subset \Omega$ we define its* vertical rearrangement *as*

$$A^{**} := \mathbf{T} \times (0, |A|),$$

with the convention $A^{**} = \emptyset$ *if* $|A| = 0$. *Given a measurable map* $f : \Omega \to \mathbf{R}$ *we define its* vertical deacreasing rearrangement *as*

$$f^*(x,z) := \int_{\mathbf{R}} \mathbf{1}_{\{f>\alpha\}^{**}}(z)\,\mathrm{d}\alpha. \tag{3.64}$$

Figure 3.1 illustrates such a rearrangement.

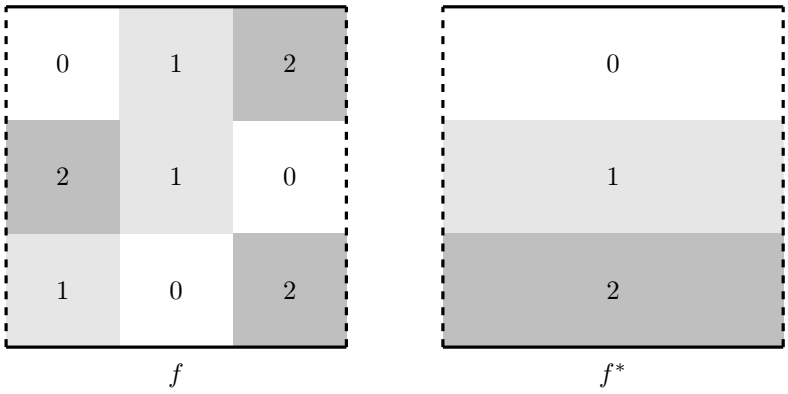

Figure 3.1: Vertical rearrangement of a piecewise constant map on $\Omega = \mathbf{T} \times (0,1)$.

Rearranged maps present a few properties, see [LL01, Section 3.3], of which we provide here a selection, adapted to the present vertical rearrangement definition. Most of its proof relies on the representation formula (3.64), which holds either in the segment or in the periodic strip case.

Properties 3.6.4. *For any positive measurable map* $f : \Omega \to \mathbf{R}$ *the followings hold:*

1. f^* *depends only on the vertical variable* z.
2. f^* *is nonnegative and nonincreasing.*
3. f^* *is a rearrangement of* f, *namely*
$$\forall \alpha \in \mathbf{R}, \qquad |\{f^* > \alpha\}| = |\{f > \alpha\}|.$$
4. *If* $f \in L^q(\Omega)$ *then* $f^* \in L^q(\Omega)$ *and* $\|f^*\|_{L^q} = \|f\|_{L^q}$.
5. *The vertical decreasing rearrangement is order preserving. If* $f \le g$ *then* $f^* \le g^*$. *Also, if* $f = f(z)$ *and is nonincreasing with respect to* z, *then* $f^* = f$.

The simplest rearrangement inequality, stated in [LL01, Theorem 3.4] also adapts here. We refer to this reference for its proof.

Theorem 3.6.5. *Let* f, g *be nonnegative functions on* Ω. *Then*

$$\int_\Omega fg \le \int_\Omega f^* g^*.$$

If f *is strictly decreasing, then there is equality if and only if* $g^* = g$.

Now, let us study the general properties of the Stokes-transport equation solutions. By divergence free of the velocity field $\boldsymbol{u}$, the density ρ shares for all time the same level sets as ρ_0, therefore $\rho(t)^* = \rho_0^*$ for all $t \geq 0$, all L^q norms are preserved for all time, and $\int_\Omega \rho$ is constant to $\int_\Omega \rho_0$. In particular for a given initial datum ρ_0, the potential energy is bounded from below. Indeed, consider $\Theta : (x, z) \in \Omega \mapsto 1 - z$, which is positive and strictly decreasing with respect to z. According to Theorem 3.6.5, we have the following equivalent inequalities

$$\begin{aligned} \int_\Omega \Theta\rho(t) &\leq \int_\Omega \Theta^* \rho(t)^* \\ \int_\Omega (1-z)\rho(t) &\leq (1-z)\rho_0^* \\ \int_\Omega z\rho_0^* &\leq \int_\Omega z\rho(t) = E(t), \end{aligned}$$

and there is equality if and only if $\rho(t) = \rho_0^*$. Since the potential energy is decreasing and bounded from below, it admits a finite limit that we denote $E_\infty := \lim_{t\to\infty} E(t)$. We also denote $E_0 = E(\rho_0)$. Thus, integrating the energy balance (3.63) with respect to time provides

$$\int_0^\infty \|\nabla \boldsymbol{u}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau = E_0 - E_\infty < \infty.$$

Consequently, $\boldsymbol{u}$ belongs to $L^2(\mathbf{R}_+; H^1(\Omega))$, since Poincaré inequality $\|\boldsymbol{u}\|_{H^1} \lesssim \|\nabla \boldsymbol{u}\|_{L^2}$ holds here.

Lemma 3.6.6. *For any $\rho_0 \in L^\infty(\Omega)$, the Stoke-transport equation velocity solution satisfies*

$$\boldsymbol{u} \in L^2(\mathbf{R}_+; H^1(\Omega)).$$

Let us identify some sufficient conditions on the velocity $\boldsymbol{u}$ which would ensure that ρ converges.

Proposition 3.6.7. *Let $\rho_0 \in L^\infty(\Omega)$ and $\boldsymbol{u} \in L^\infty_{\mathrm{loc}}(\mathbf{R}_+; W^{1,\infty}(\Omega))$ a divergence free velocity field and consider $\rho \in L^\infty(\mathbf{R}_+; L^\infty(\Omega))$ the solution of the associated transport equation*

$$\partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = 0, \quad \rho|_{t=0} = \rho_0.$$

If $\boldsymbol{u}$ belongs to $L^1(\mathbf{R}_+; L^2(\Omega))$ then $\rho(t)$ converges strongly in L^2 toward some $\rho_\infty \in L^\infty(\Omega)$.

Let us first establish the convergence in $H^{-1}(\Omega)$, then in $L^2(\Omega)$ with $\rho_0 \in H^1(\Omega)$, and finally prove Proposition 3.6.7. Let us start with the simple observation that if $\partial_t \rho$ belongs to $L^1(\mathbf{R}_+; X(\Omega))$ for some generic functional space $X(\Omega)$, then $\rho(t)$ converges in $X(\Omega)$ as t goes to infinity,

$$\rho(t) = \rho_0 + \int_0^t \partial_t \rho(\tau) \,\mathrm{d}\tau.$$

Considering $X = H^{-1}$, we get

$$\|\partial_t \rho\|_{H^{-1}} = \|\boldsymbol{u} \cdot \nabla \rho\|_{H^{-1}} \leq \|\boldsymbol{u}\rho\|_{L^2} \leq \|\rho\|_{L^\infty} \|\boldsymbol{u}\|_{L^2}.$$

Since $\|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty}$ and $\boldsymbol{u}$ is assumed to belong to $L^1(\mathbf{R}_+; L^2(\Omega))$, the strong convergence of ρ in $H^{-1}(\Omega)$ follows. Perfoming the same computation in $L^2(\Omega)$, we have a priori

$$\|\partial_t \rho\|_{L^2} \leq \|\boldsymbol{u}\|_{L^\infty} \|\nabla \rho\|_{L^2}.$$

Recall that $\rho(t) = \rho_0 \circ \boldsymbol{X}(-t)$ where $\boldsymbol{X}$ is the flow map associated to $\boldsymbol{u}$, see Subsection 2.2.2. As a consequence, we have $\nabla\rho(t) = \nabla\boldsymbol{X}(-t)\nabla\rho_0(\boldsymbol{X}(-t))$ and therefore

$$\|\nabla\rho(t)\|_{L^2} \leq \|\nabla\boldsymbol{X}(-t)\|_{L^\infty}\|\nabla\rho_0(\boldsymbol{X}(-t))\|_{L^2}.$$

Let us recall that $\boldsymbol{X}(-t)$ is a measure preserving diffeomorphism, so that $\|\nabla\rho_0(\boldsymbol{X}(-t))\|_{L^2} = \|\nabla\rho_0\|_{L^2}$ and from Lemma 2.2.3 (or [BCD11, Proposition 3.10]) we infer

$$\|\nabla\boldsymbol{X}(-t)\|_{L^\infty} \leq \exp\left(\int_0^t \|\nabla\boldsymbol{u}(\tau)\|_{L^\infty}\,\mathrm{d}\tau\right).$$

Combining these latter inequalities, we get

$$\|\partial_t\rho\|_{L^2} \leq \|\nabla\rho_0\|_{L^2}\|\boldsymbol{u}\|_{L^\infty}\exp\left(\int_0^t \|\nabla\boldsymbol{u}(\tau)\|_{L^\infty}\,\mathrm{d}\tau\right).$$

Hence if $\boldsymbol{u} \in L^1(\mathbf{R}_+; W^{1,\infty}(\Omega))$ then $\partial_t\rho \in L^1(\mathbf{R}_+; L^2(\Omega))$ and the strong convergence occurs.

Noneless, it is possible to take more advantage of the fact that ρ is the pushforward of ρ_0 by $\boldsymbol{X}$. Indeed, the flow $\boldsymbol{X}(t)$ satisfying

$$\partial_t\boldsymbol{X}(t) = \boldsymbol{u}(t, \boldsymbol{X}(t))$$

do converge in L^2 if $\boldsymbol{u} \in L^1(\mathbf{R}_+; L^2(\Omega))$, and we have

$$\|\boldsymbol{X}(t) - \boldsymbol{X}_\infty\|_{L^2} \leq \int_t^\infty \|\boldsymbol{u}(\tau)\|_{L^2}\,\mathrm{d}\tau, \tag{3.65}$$

where the remainder converges toward 0 as t goes to infinity. The limit X_∞ remains a measure preserving diffeomorphism. Indeed, Liouville formula and $\boldsymbol{u}$ being divergence free provide

$$\text{a.e. } \boldsymbol{x}, \quad |\det\nabla\boldsymbol{X}(t;\boldsymbol{x})| = \exp\left(\int_0^t \operatorname{div}\boldsymbol{u}(\tau, \boldsymbol{X}(\tau;\boldsymbol{x}))\,\mathrm{d}\tau\right) = 1,$$

which transfers to the limit $\boldsymbol{X}_\infty$. In passing, observe the following, detailed for instance in [BCD11, Section 3.2],

$$\partial_t\nabla\boldsymbol{X}(t) = \nabla\boldsymbol{X}(t)\nabla\boldsymbol{u}(t, \boldsymbol{X}(t)), \qquad \|\nabla\boldsymbol{X}(t)\|_{L^\infty} \leq \exp\left(\int_0^t \|\nabla\boldsymbol{u}(\tau)\|_{L^\infty}\,\mathrm{d}\tau\right).$$

Now let ρ_0 be in $L^2(\Omega)$, and define $\rho_\infty = \rho_0 \circ \boldsymbol{X}_\infty$. Let us show that $\rho(t)$ converges toward ρ_∞ in $L^2(\Omega)$. For any $\varepsilon > 0$, there exists $\varphi_0 \in C^\infty(\Omega)$ such that $\|\rho_0 - \varphi\|_{L^2} \leq \varepsilon$. Define $\varphi(t) := \varphi_0 \circ \boldsymbol{X}(-t)$ and $\varphi_\infty = \varphi_0 \circ \boldsymbol{X}_\infty$. Hence

$$\|\rho(t) - \rho_\infty\|_{L^2} \leq \|\rho(t) - \varphi(t)\|_{L^2} + \|\varphi(t) - \varphi_\infty\|_{L^2} + \|\varphi_\infty - \rho_\infty\|_{L^2}.$$

On the one hand we have for any $t \in [0, \infty]$,

$$\|\rho(t) - \varphi(t)\|_{L^2} = \|(\rho_0 - \varphi_0)(\boldsymbol{X}(-t))\|_{L^2} = \|\rho_0 - \varphi_0\|_{L^2} \leq \varepsilon. \tag{3.66}$$

On the other hand,

$$\|\varphi(t) - \varphi_\infty\|_{L^2} = \|\varphi_0(\boldsymbol{X}(-t)) - \varphi_0(\boldsymbol{X}_\infty)\| \leq \|\nabla\varphi_0\|_{L^\infty}\|\boldsymbol{X}(-t) - \boldsymbol{X}_\infty\|_{L^2}. \tag{3.67}$$

Plugging (3.66) and (3.67) combined with (3.65) in (3.66) yields

$$\|\rho(t)-\rho_\infty\|_{L^2} \leq 2\varepsilon + \|\nabla\varphi_0\|_{L^\infty}\int_t^\infty \|\boldsymbol{u}(\tau)\|_{L^2}\,\mathrm{d}\tau$$

Since $\boldsymbol{u}\in L^1(\mathbf{R}_+;L^2(\Omega))$, the remainder integral is smaller than ε for t large enough, depending on $\|\nabla\varphi_0\|_{L^\infty}$. In the end we proved the convergence $\rho(t)\to\rho_\infty$ in $L^2(\Omega)$ when $t\to\infty$ and the Proposition 3.6.7. Note that no convergence rate is available here.

Unfortunately, the velocity $\boldsymbol{u}$ solving ST belongs a priori to $L^2(\mathbf{R}_+;H^1(\Omega))$, which does not imply $\boldsymbol{u}\in L^1(\mathbf{R}_+;L^2(\Omega))$. In the case where ρ is a small perturbation of $\Theta(z)=1-z$, we showed the solution converges at infinity, using in this particular case that the velocity field stream function is integrable enough in time. For general data $\rho_0\in L^\infty(\Omega)$, it is not clear whether or not convergence occur. For now, we cannot exclude the possibility that the solution approaches successively several admissible sationary states, while having monotonous potential energy, without converging toward one in particular. We borrow the definition of ω-set to the dynamical system field, cut for the present problem, which precisely defines as the set of asymptotically approached states.

Definition 3.6.8. *We define the ω-set associated to $\rho_0\in L^\infty(\Omega)$ in a functional space $X(\Omega)$ as the intersection of all the accumulation points of the Stokes-transport equation solution ρ, denoted as follows*

$$\omega_X(\rho_0) := \bigcap_{t\geq 0} \overline{\{\rho(\tau):\tau\geq t\}}^X.$$

This notion allows to state some properties of ω-sets for ρ_0 in $L^\infty(\Omega)$ in a convenient way.

Proposition 3.6.9. *Let $\rho_0\in L^\infty(\Omega)$, and let ρ be the associated Stokes-transport equation solution. If ρ_0 is stratified, then $\rho(t)=\rho_0$ for all time. For any ρ_0 non-stratified, the followings hold.*

1. *The potential energy $E(t)$ is strictly deacreasing and converges toward some $E_\infty\geq E(\rho_0^*)$.*
2. *$\omega_{H^{-1}}(\rho_0)$ is nonempty.*
3. *For any $\rho_\infty\in\omega_{H^{-1}}(\rho_0)$, ρ_∞ is a stratified rearrangement of ρ_0, and $E(\rho_\infty)=E_\infty$.*
4. *If $E_\infty=E(\rho_0^*)$ then $\omega_{H^{-1}}(\rho_0)=\{\rho_0^*\}$ and the solution converges in the sense*

$$\rho(t)\xrightarrow[t\to\infty]{H^{-1}}\rho_0^*.$$

Proof. The monotonicity of the potential energy was shown in Subsection 3.6.2. If the potential energy is stationary from a certain time T, then the density profile $\rho(T)$ is stationary, and Corollary 3.6.1 proves that $\rho(t)=\rho(T)$ for all $t\in\mathbf{R}$. Hence the dichotomy between stationary profiles and strictly monotonous evolutions, which concludes item *1.* By linearity of the Stokes equation, we have

$$\|\partial_t\boldsymbol{u}\|_{H^1}\lesssim\|\partial_t\rho\|_{H^{-1}},$$

where the right-hand side estimates as follows in the first place,

$$\|\partial_t\rho\|_{H^{-1}}=\|\mathrm{div}(\rho\boldsymbol{u})\|_{H^{-1}}\lesssim\|\rho\boldsymbol{u}\|_{L^2}\lesssim\|\rho_0\|_{L^\infty}^2. \tag{3.68}$$

The combination of these two inequalities ensures that $\boldsymbol{u}$ belongs to $W^{1,\infty}(\mathbf{R}_+; H^1(\Omega))$. Together with the fact that $\boldsymbol{u} \in L^2(\mathbf{R}_+; H^1(\Omega))$, it implies

$$\boldsymbol{u}(t) \xrightarrow[t\to\infty]{H^1} \mathbf{0}. \tag{3.69}$$

In return we deduce from (3.68) that $\partial_t \rho$ also goes to 0 in H^{-1} as t goes to infinity. We also deduce by considering the H^{-1} norm on both side of the Stokes equation that

$$\|\nabla p + \rho \boldsymbol{e}_z\|_{H^{-1}} \leq \|\Delta \boldsymbol{u}\|_{H^{-1}} \lesssim \|\boldsymbol{u}\|_{H^1} \underset{t\to\infty}{\longrightarrow} 0.$$

These considerations have also been observed for the Boussinesq system with no diffusion on the density as for instance in [Doe+18, Theorem 1.2] or [AKZ23].

Now let us show that the only possible limits are stationary. Since $\|\rho(t)\|_{L^q} = \|\rho_0\|_{L^q}$ for all time $t \geq 0$ and any $1 \leq q \leq \infty$ we know by weak compactness in $L^q(\Omega)$ that there exists a time extraction $(t_n)_{n\in\mathbf{N}}$, *i.e.* a strictly increasing sequence of times such that $t_n \to \infty$ as $n \to \infty$, such that $(\rho(t_n))_{n\in\mathbf{N}}$ as a L^q-weak limit $\rho_\infty \in L^q(\Omega)$ for any $1 < q < \infty$, in the sense

$$\forall \varphi \in L^{q'}(\Omega), \qquad \int_\Omega \varphi \rho(t_n) \underset{n\to\infty}{\longrightarrow} \int_\Omega \varphi \rho_\infty.$$

Since $z \mapsto z$ belongs to any $L^{q'}(\Omega)$, and since the potential energy of the system converges, we obtain in particular that

$$\int_\Omega \rho_0 = \int_\Omega \rho(t_n) \underset{n\to\infty}{\longrightarrow} \int_\Omega \rho_\infty, \qquad \int_\Omega z\rho(t_n) \underset{n\to\infty}{\longrightarrow} \int_\Omega z\rho_\infty = E_\infty. \tag{3.70}$$

Moreover, $\rho(t_n)$ converges toward ρ_∞ in $H^{-1}(\Omega)$ strong, proving item *2.* Set $(\boldsymbol{u}_\infty, p_\infty)$ the solution to the Stokes system with source $-\rho_\infty \boldsymbol{e}_z$. From (3.69) we know that $\boldsymbol{u}_\infty = \mathbf{0}$. Hence

$$\nabla p_\infty = -\rho_\infty \boldsymbol{e}_z.$$

In particular $\partial_x p_\infty = 0$, therefore p_∞ depends only on z, and so does ρ_∞, which shows ρ_∞ is a stratified profile, which shows

By weak convergence, we also know that in any $L^q(\Omega), 1 < q < \infty$ we have

$$\|\rho_\infty\|_{L^q} \leq \liminf_{n\to\infty} \|\rho(t_n)\|_{L^q} = \|\rho_0\|_{L^q}, \tag{3.71}$$

and we get the inequality for $q = 1$ by passing to the limit. Using the identity

$$\forall 1 \leq q < \infty, \qquad \|f\|_{L^q}^q = q \int_0^\infty \alpha^{q-1} |\{|f| > \alpha\}| \, \mathrm{d}\alpha,$$

we infer from (3.71) that

$$\forall s \geq 0, \qquad \int_0^\infty \alpha^s \left(|\{|\rho_0| > \alpha\}| - |\{\rho_\infty| > \alpha\}| \right) \mathrm{d}\alpha \geq 0.$$

Hence we deduce that all the level set of ρ_∞ must be smaller than these of ρ_0 in the sense

$$\forall \alpha \geq 0, \qquad |\{|\tilde{\rho}_\infty| > \alpha\}| \leq |\{|\rho_0| > \alpha\}|. \tag{3.72}$$

Since we obtained preservation of the mass for the weak limit in (3.70), we have $\|\rho_\infty\|_{L^1} = \|\rho_0\|_{L^1}$ which is only possible together with (3.72) if equality (3.72) holds for all $\alpha \geq 0$. Therefore ρ_∞ is necessarily a rearrangement of ρ_0.

Since we have for any $\rho_\infty \in \omega_{H^{-1}}(\rho_0)$ the equality $E_\infty = E(\rho_\infty) \geq E(\rho_0^*)$ and ρ_∞ stratified, we know from Theorem 3.6.5 that $E_\infty = E(\rho_0^*)$ if and only if $\tilde{\rho}_\infty = \rho_0^*$, proving *4* and concluding the proof. □

Although any element ρ_∞ of the ω-set of ρ_0 satisfies $E(\rho_\infty) = E_\infty$, unless $E_\infty = E(\rho_0^*)$ we cannot ensure that ρ_∞ is unique. We do not know if the system can converge toward a rearrangement of ρ_0 different from the vertical rearrangement ρ_0, except in the trivial case where the initial datum is already stratified but not monotonous. Figure 3.2 illustrates two rearrangements of a given ρ_0, noted ρ_0^1 and ρ_0^2, stratified and satisfying $E(\rho_0^1) = E(\rho_0^2)$. Note that such rearrangements must necessarily satisfy $E(\rho_0^i) > E(\rho_0^*)$. We also do not know if the rearrangement of the initial datum belongs to the ω-set in general.

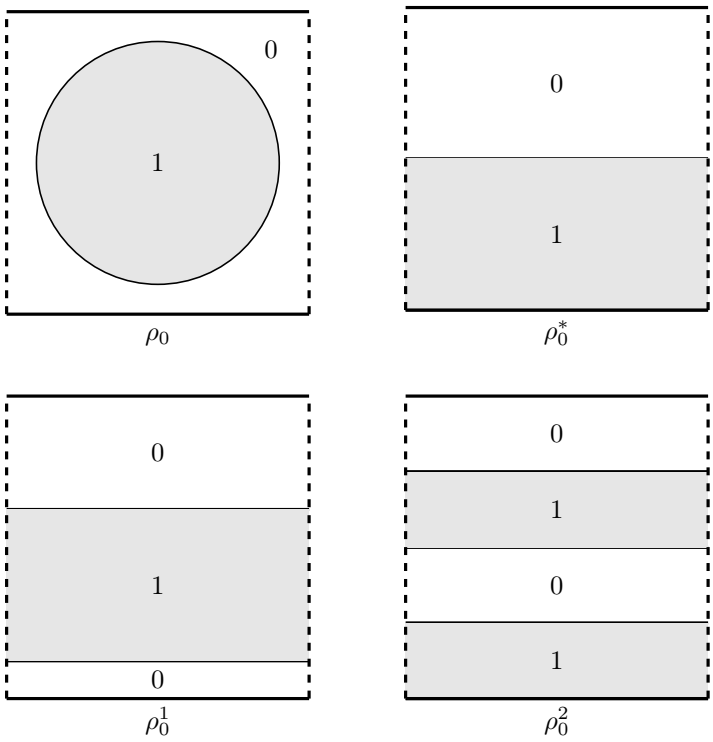

Figure 3.2: A function ρ_0, its vertical rearrangement ρ_0^*, and two other rearrangements with equal potential energy $E(\rho_0^1) = E(\rho_0^2)$.

3.7 Nonlinear instability for low regularity perturbation

In [KY23], Kiselev & Yao show that the Incompressible Porous Media equation admits a nonlinear instability around any stationary profile, in the following sense. Any stratified state $\Theta = \Theta(z)$ can be perturbed by some θ_0 small in $H^{2-\gamma}, \gamma \in (0,2)$ such that for any $s > 0$ the H^s-norm of the total solution $\rho = \Theta + \theta$ diverges. Their statement hold for spatial domain Ω being either

the whole plan $\mathbf{R}^2$, the torus $\mathbf{T}^2$ or the periodic strip $\mathbf{T}\times(0,1)$. Let us recall the stability result of [CCL19a] stating that $\Theta(z)=1-z$ is a stable state for IPM if perturbated by small data in $H^{10}(\Omega)$. Comparing both results we can wonder about the existence of a stability threshold $2\leq\alpha\leq 10$, for a given Θ, in the following sense:

1. For any $m<\alpha$ and any $\varepsilon>0$ there exists a perturbation $\|\theta_0\|_{H^m}\leq\varepsilon$ of Θ such that the solution $\rho=\Theta+\theta$ diverges in $H^m(\Omega)$;
2. For any $m>\alpha$ there exists $\varepsilon_m>0$ such that the solution Θ is stable for any perturbation satisfying $\|\theta_0\|_{H^m}\leq\varepsilon_m$.

These does not constitute an exhaustive dichotomy, and the actual existence of such a threshold remains unknown. The same question also remains open regarding the ST equation. Indeed, from Sections 3.1 to 3.5, corresponding to the paper [DGL23], we know that there exists $\varepsilon_6>0$ such that for any $\|\theta_0\|_{H^6}\leq\varepsilon_6$, the initial datum $\Theta+\theta_0$ associated solution remains close to $\Theta(z)=1-z$. Conversely, the aforementioned unstability result demonstrated by Kiselev & Yao does adapt to the ST equation. Therefore, wether a thereshold α should exists for the Stokes-transport equation, it shall satisfy $2\leq\alpha\leq 6$.

We report here the statement, [KY23, Theorem 1.5 and Corollary 4.2], written with notations consistent with the rest of the present manuscript.

Theorem 3.7.1 (Kiselev & Yao '21)**.** *Let $\Theta\in C^\infty(\Omega)$ be a stationary solution for (IPM). For any $\varepsilon>0$ and $2>\gamma>0$, there exists a perturbation $\theta_0\in C^\infty(\Omega)$ satisfying*

$$\|\theta_0\|_{H^{2-\gamma}}\leq\varepsilon,$$

such that the solution associated to $\rho_0=\Theta+\theta_0$, provided it exists globally in time, obeys

$$\limsup_{t\to\infty} t^{-s/2}\|\rho(t)-\Theta\|_{\dot H^s}=\infty,\qquad \text{for all } s>0.$$

We report and adapt the proof of Kiselev & Yao to the case of ST, for sake of completeness.

Theorem 3.7.2 (L.)**.** *Let $\Theta\in C^\infty(\Omega)$ be a stationary solution for (ST). For any $\varepsilon>0$ and $2>\gamma>0$, there exists a perturbation $\theta_0\in C^\infty(\Omega)$ satisfying*

$$\|\theta_0\|_{H^{2-\gamma}}\leq\varepsilon,$$

such that the solution associated to $\rho_0=\Theta+\theta_0$ obeys

$$\limsup_{t\to\infty} t^{-s/4}\|\rho(t)-\Theta\|_{\dot H^s}=\infty,\qquad \text{for all } s>0.$$

There are two main differences between the IPM and ST cases regarding these results. First, the global well-posedness of IPM remains an open question any Sobolev space $H^m(\Omega)$, and although global solutions exist for small perturbations of $\Theta(z)=1-z$, Theorem 3.7.1 is conditional to the global existence of a solution in general. On the contrary, the ST is well-posed in Sobolev spaces $H^m(\Omega)$, at least for $m\geq 3$. Second, the growth rates are different, which comes from different velocity regularities due to the respective regularisation effects of Darcy and Stokes equations.

The proof of this result relies on geometric considerations. It consists in showing that any initial profile satisfying a particular structure called "bubble" introduced in Definition 3.7.3 induces a growth of all the solution derivatives, as stated in Proposition 3.7.4. Then the construction of a small perturbation inducing a bubble type initial datum is given in Lemma 3.7.7.

Definition 3.7.3. *A density profile ρ is of* "bubble" type *if and only if there exists a closed curve $\Gamma_0 \subset \Omega$ enclosing a simply connected domain such that ρ is constant on Γ_0 and $\inf_{\Gamma_0} |\nabla \rho| > 0$.*

A typical case satisfying this assumption is any regular profile having a local extremum in the subdomain enclosed by Γ_0. See Figure 3.3 for an illustration.

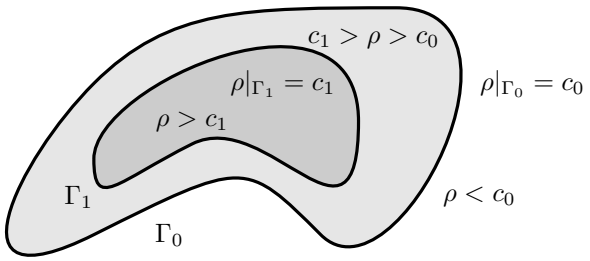

Figure 3.3: Illustration of a bubble type map.

Since the tendency of the density is to rearrange accordingly to the vertical axis, and since the transport of the density is pure and measure preserving, the bubble has to smash which translates as derivative growths. This is showed in the following result, also proved by Kiselev & Yao in [KY23, Proposition 4.1] and adapted here to the Stokes-transport case, for which a functional inequality and the rates are different.

Proposition 3.7.4. *If $\rho_0 \in C^\infty(\Omega)$ satisfies the "bubble" type definition 3.7.3, the solution with initial datum ρ_0 satisfies*

$$\int_0^\infty \|\partial_x \rho\|_{H^s}^{-4/s} \leq C_{s,\rho_0}, \qquad \forall s > 0,$$

implying

$$\limsup_{t\to\infty} t^{-s/4} \|\partial_x \rho\|_{H^s} = \infty, \qquad \forall s > 0.$$

We need the following lemma to prove this result.

Lemma 3.7.5. *Let $\Omega = \mathbf{T} \times (0,1)$ and let $s > 0$. For any ρ such that $\partial_x \rho \in H^s(\Omega)$ we have*

$$\|\partial_x \rho\|_{L^2} \lesssim \|\Delta \psi\|_{L^2}^{2s/(s+2)} \|\partial_x \rho\|_{H^s}^{4/(s+2)}, \tag{3.73}$$

where ψ is the solution to

$$\Delta^2 \psi = \partial_x \rho, \quad \psi|_{\partial\Omega} = \partial_{\boldsymbol{n}} \psi|_{\partial\Omega} = 0.$$

Proof. We plug $\Delta^2 \psi = \partial_x \rho$ and use the classical Sobolev inequalities to get

$$\|\partial_x \rho\|_{L^2}^2 = \|\Delta^2 \psi\|_{L^2} \lesssim \|\psi\|_{H^4} \lesssim \|\psi\|_{H^2}^{2s/(s+2)} \|\psi\|_{H^{s+4}}^{4/(s+2)}.$$

Then we use the norm equivalence $\|\psi\|_{H^2} \simeq \|\Delta \psi\|_{L^2}$ on $H_0^1 \cap H^2(\Omega)$ and the biharmonic equation regularization $\|\psi\|_{H^{s+4}} \lesssim \|\partial_x \rho\|_{H^s}$ from Proposition 3.8.1 to get the result. □

Remark 3.7.6. *Here the norm $\|\Delta \psi\|_{L^2}$ imitates the H^{-2}-norm of $\partial_x \rho$ since $\Delta \psi = \Delta(\Delta^{-2} \partial_x \rho)$. In the IPM case, the norm at play in the potential energy decay is $\|\nabla \psi\|_{L^2}$, and since $\Delta \psi = -\partial_x \rho$ one can identify $\|\nabla \psi\|_{L^2} \simeq \|\partial_x \rho\|_{\dot{H}^{-1}}$ by a proper choice of definition of $\dot{H}^{-1}$. This result is also generalised in Lemma 3.2.7.*

Proof of Proposition 3.7.4. Since $\inf_{\Gamma_0}|\nabla\rho_0| > 0$ and ρ_0 is smooth we have $|\nabla\rho_0|$ uniformly bounded from below in the viscinity of Γ_0, and there exists an other closed curve Γ_1 included in D_0 enclosing a simply connected subdomain $D_1 \subset D_0$ such that $\rho|_{\Gamma_1} =: c_1 \neq c_0 =: \rho|_{\Gamma_0}$. Let us recall the definition of the characteristic (or flow) map $\boldsymbol{X}$ associated to the velocity field $\boldsymbol{u}$, as the solution of the ordinary differential equation governed by $\boldsymbol{u}$, namely

$$\forall \boldsymbol{x} \in \Omega, \quad \begin{cases} \dfrac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t}(t;\boldsymbol{x}) = \boldsymbol{u}(t,\boldsymbol{X}(t;\boldsymbol{x})) \\ \boldsymbol{X}(0;\boldsymbol{x}) = \boldsymbol{x}, \end{cases}$$

For all time $t \geq 0$ the map $\boldsymbol{X}(t)$ is a measure preserving diffeomorphism of Ω onto itself because $\boldsymbol{u}$ is divergence free. In particular $|\boldsymbol{X}(t;D_i)| = |D_i|, i = 0,1$ and all inclusions are preserved, such as $\boldsymbol{X}(t;D_1) \subset \boldsymbol{X}(t;D_0)$ for all $t \geq 0$. Also, since ρ_0 is transported by $\boldsymbol{u}$, the density remains constant along the trajectories of the flow, let us note $\rho(t)|_{\boldsymbol{X}(t;\Gamma_i)} = \rho_0|_{\Gamma_i} = c_i$. Let us denote the projector $\Pi_z : (x,z) \in \Omega \mapsto z \in (0,1)$ onto the vertical axis and observe that

$$\Pi_z(\boldsymbol{X}(t;\Gamma_1)) \subset \Pi_z(\boldsymbol{X}(t;\Gamma_0)), \quad \forall t \geq 0.$$

Consequently, for any $z \in \Pi_z(\boldsymbol{X}(t;\Gamma_1))$ the horizontal interval $\mathbf{T}\times\{z\}$ intersects both $\boldsymbol{X}(t;\Gamma_i), i = 0,1$ and we can bound from below the integral of $\partial_x\rho$ as follows

$$\forall z \in \Pi_z(\boldsymbol{X}(t;\Gamma_1)), \quad \int_{\mathbf{T}} |\partial_x\rho(t,x,z)|\,\mathrm{d}x \geq |c_1 - c_0|.$$

Since $\boldsymbol{X}(t;\Gamma_1)$ encloses the simply connected domain $\boldsymbol{X}(t;D_1) \subset \Omega$ of constant measure $|D_1|$ and since Ω has sides lengths of size 1, we have necessarily $|\boldsymbol{X}(t;\Gamma_1)| \geq |D_1|$. Therefore we can uniformly bound from below the L^2 norm of $\partial_x\rho$ as follows

$$\|\partial_x\rho(t)\|_{L^2} \geq \|\partial_x\rho(t)\|_{L^1} \geq |c_1 - c_0|.$$

Now we infer from the interpolation inequality (3.73) that

$$\|\Delta\psi\|_{L^2}^2 \geq \|\partial_x\rho\|_{L^2}^{1+2/s}\|\partial_x\rho\|_{H^s}^{-4/s}.$$

Since we know from Lemma 3.6.6 that $\Delta\psi \in L^2(\mathbf{R}_+;L^2(\Omega))$, and since $\|\partial_x\rho\|_{L^2}$ is uniformly bounded from below, we get

$$C_{\rho_0} \geq |c_1 - c_0|^{1+2/s}\int_0^\infty \|\partial_x\rho\|_{H^s}^{-4/s}.$$

hence the integrability. By comparison with the non-integrability of $t \mapsto 1/t$ in any neighborhood of ∞ we infer the following equivalent growth assertions,

$$\limsup_{t\to\infty} t^{-1}\|\partial_x\rho\|_{H^s}^{4/s} = \infty \quad \Leftrightarrow \quad \limsup_{t\to\infty} t^{-s/4}\|\partial_x\rho\|_{H^s} = \infty.$$

□

Now let us show that any stratified datum can be pertubated by a small element of $H^{2-\gamma}(\Omega)$ for any given $\gamma \in (0,2)$ in a profile satisfying the bubble type definition.

Lemma 3.7.7. *For any $\Theta = \Theta(z) \in C^\infty(\Omega)$, for any $\varepsilon,\gamma > 0$ there exists a perturbation*

$\theta_0 \in C^\infty(\Omega)$ *satisfying*

$$\|\theta_0\|_{H^{2-\gamma}} \leq \varepsilon, \tag{3.74}$$

such that $\rho_0 := \Theta + \theta_0$ *satisfies the "bubble" type definition.*

Proof. Let $\varphi \in C_c^\infty(\mathbf{R}^2)$ such that $\operatorname{supp}\varphi \subset B_{\mathbf{R}^2}(\mathbf{0}, 1)$ and $\varphi(\mathbf{0}) = 1$. Set a point $\boldsymbol{x}_0 \in \Omega$ and

$$\theta_{0,\lambda} := 2\|\partial_z\Theta\|_{L^\infty}\lambda\varphi(\lambda^{-1}(\cdot - \boldsymbol{x}_0)),$$

with $\lambda > 0$ to adjust. This map $\theta_{0,\lambda}$ has support in $B(\boldsymbol{x}_0, \lambda) \subset \Omega$ for λ small enough. It is proportionnal to $\|\partial_z\Theta\|_{L^\infty}$ to compete with the gradient of the stationary profile we perturbate, and to λ in order to play we the scale of $\theta_{0,\lambda}$ in the due norms.

We can compare the value of the whole density $\rho_{0,\lambda} := \Theta + \theta_{0,\lambda}$ at point $\boldsymbol{x}_0$,

$$\rho_{0,\lambda}(\boldsymbol{x}_0) = \Theta(\boldsymbol{x}_0) + 2\lambda\|\partial_z\theta\|_{L^\infty}$$

and compare it to the supremum of the map on the boundary of $B(\boldsymbol{x}_0, \lambda)$ as, by the mean value theorem,

$$\rho_{0,\lambda}|_{\partial B(\boldsymbol{x}_0,\lambda)} = \theta|_{\partial B(\boldsymbol{x}_0,\lambda)} \leq \theta(\boldsymbol{x}_0) + \lambda\|\partial_z\Theta\|_{L^\infty}.$$

In particular

$$(\theta(\boldsymbol{x}_0) + \lambda\|\partial_z\Theta\|_{L^\infty}, \theta(\boldsymbol{x}_0) + 2\lambda\|\partial_z\Theta\|_{L^\infty}) \subset \rho_{0,\lambda}(\Omega)$$

so Sard lemma [Sar42] ensures that for almost every h in this interval the gradient $\nabla\rho_{0,\lambda}$ never vanishes on the level set $\{\rho_{0,\lambda} = h\}$. Finally, any connected component of such a level set is a curve Γ_0 enclosing a simply connected domain on which $\rho_{0,\lambda}|_{\Gamma_0}$ is constant with $\inf_{\Gamma_0}|\nabla\rho_{0,\lambda}| > 0$, therefore satisfying the "bubble" definition.

To show that $\theta_{0,\lambda}$ can be choosen arbitrarily small, notice that

$$\|\theta_{0,\lambda}\|_{L^2} = 2\|\partial_z\Theta\|_{L^\infty}\lambda^2\|\varphi\|_{L^2(\mathbf{R}^2)}, \qquad \|\partial^2\theta_{0,\lambda}\|_{L^2(\Omega)} = 2\|\partial_z\Theta\|_{L^\infty}\|\partial^2\varphi\|_{L^2(\mathbf{R}^2)}.$$

Hence by interpolating between $L^2(\Omega)$ and $H^2(\Omega)$ we obtain an estimate of the perturbation in any $H^{2-\gamma}(\Omega)$ space, for $\gamma \in (0, 2)$,

$$\|\theta_0\|_{H^{2-\gamma}} \lesssim \lambda^\gamma\|\partial_z\Theta\|_{L^\infty}\|\varphi\|_{L^2}^{\gamma/2}\|\partial^2\varphi\|_{L^2}^{1-\gamma/2},$$

and observe that assumption (3.74) is satisfied for λ small enough. □

3.8 Biharmonic operator and adapted basis

This section is devoted to the study of the biharmonic, or bilaplace, operator, appearing in the study of the Stokes equation. In particular, we detail the issues raised by different boundary conditions one imposes and its links with our stability Theorem 3.1.1. It motivated the study of boundary layers for the Stokes-transport equation, described in Theorems 3.1.2 and 3.1.3.

Let us recall how the bilaplace equation stems from the Stokes problem. Given a solution $\boldsymbol{u}$ of the Stokes equation with a generic source $\boldsymbol{f} = (f_1, f_2)$, namely

$$-\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \operatorname{div}\boldsymbol{u} = 0,$$

there exists a scalar field ψ such that $\boldsymbol{u} = \nabla^\perp\psi = (-\partial_z\psi, \partial_x\psi)$, and applying the rotational operator to the above yields

$$\Delta^2\psi = \partial_z f_1 - \partial_x f_2.$$

Regarding the Stokes-transport problem, $\boldsymbol{f} = -\rho \boldsymbol{e}_z$ and the above writes

$$\Delta^2 \psi = \partial_x \theta',$$

with the decomposition $\rho = \Theta + \bar{\theta} + \theta'$ introduced in Section 3.1. Conversely, if ψ solves the above then $\boldsymbol{u} = \nabla^\perp \psi$ satisfies the Stokes equation, with p recovered as described in Subsection 2.2.1 for instance. Now, the Stokes problem requires boundary condition on $\boldsymbol{u}$. Let us consider two classical boundary assumptions. The first one, the slip boundary condition, also called after Neumann, writes

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \partial_{\boldsymbol{n}} u_1 = 0, \quad \text{on } \partial\Omega.$$

The normal compound of the velocity equal to 0 on the boundary comes to assume the physical domain to be impermeable. The second means that the normal derivative of the tangential compound, denoted here u_1, vanishes on the boundary. The second one, the no-slip boundary condition, also called homogeneous Dirichlet boundary condition, writes

$$\boldsymbol{u} = \boldsymbol{0}, \quad \text{on } \partial\Omega.$$

It means that the fluid remains still for all time on the boundary, even in the tangential direction. Notice that since the Laplacian is an operator of order two, we need as much boundary conditions. Now, these conditions on $\boldsymbol{u}$ have their corresponences on ψ. As in this whole Chapter, we chose the periodic channel $\Omega = \mathbf{T} \times (-1, 1)$, vertically symmetric for computation purposes. In Subsection 3.8.1, we discuss the Neumann case and how the Stokes-transport stability can be dealt with under this assumption. We present the Dirichlet problem formulation in Subsection 3.8.2, and provide a self-contained proof of the regularisation effect of the operator on the bi-laplace equation solution, for sake of completeness. We also obtain an explicit description of an orthonormal eigenfunctions basis for the biharmonic operator with the Dirichlet boundary condition, in Subsection 3.8.3. Finally, we explain to what extend the Dirichlet framework differs from the previous, how it modifies the analysis of the Stokes-transport stability problem and why it leads to the formation of boundary layers.

3.8.1 The Neumann boundary conditions

Let us recall that the normal derivatives on $\Omega = \mathbf{T} \times (-1, 1)$ correspond to vertical derivatives up to a multiplicative sign. Substituting $\boldsymbol{u} = \nabla^\perp \psi$ in the slip boundary condition,

$$u_2 = \partial_z u_1 = 0 \text{ on } \partial\Omega,$$

leads to the following constraint on the stream function on the boundaries,

$$\psi = \partial_z^2 \psi = 0 \text{ on } \partial\Omega. \tag{3.75}$$

Indeed, since $u_2 = \partial_x \psi = 0$ on $\partial\Omega$, ψ is constant on each boundary compound, and one can choose $\psi|_{\partial\Omega} = 0$. Regarding the normal derivative condition, we get $\partial_z u_1 = -\partial_z^2 \psi = 0$. Since $\partial_x^2 \psi = 0$, it is equivalent to have $\partial_z^2 \psi = \Delta \psi = 0$ on $\partial\Omega$. We also call Neumann condition the equalities (3.75), as is customary for polyharmonic problems, see for instance [GGS10].

Endowed with Neumann boundary conditions, the stream formulation of the Stokes equation splits into two identical Poisson problem. Indeed, ψ is a solution of

$$\Delta^2 \psi = f, \qquad \psi = \Delta \psi = 0 \text{ on } \partial\Omega,$$

if and only if there exists Ψ such that

$$\begin{aligned} \Delta\psi &= \Psi, \qquad \psi = 0 \text{ on } \partial\Omega, \\ \Delta\Psi &= f, \qquad \Psi = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.76}$$

In other words, denoting Δ^{-1} the operator associating to f the solution of the Poisson problem with source term f, we find the solution ψ by applying twice this operator, whence $\psi = (\Delta^{-1})^2 f$.

Stability result for IPM and ST. In [CCL19a] in which Castro, Córdoba & Lear showed the stability of the profile $\Theta(z) = 1 - z$ for the incompressible porous media equation, the Darcy law rewrites as the following Poisson problem on the stream function

$$\Delta\psi = -\partial_x\theta', \quad \psi = 0 \text{ on } \partial\Omega. \tag{3.77}$$

They introduce a convenient hilbertian basis $(\omega_\lambda)_\lambda$, specified in Subsection 3.8.3 of the following $H^m(\Omega)$ subspace,

$$X^m(\Omega) := \{f \in H^m(\Omega) : \partial_{\boldsymbol{n}}^{2n} f = 0 \text{ on } \partial\Omega, 0 \leq n \leq (m-1)/2\}.$$

In particular, they show that an initial datum $\rho_0 \in X^m(\Omega)$ remain in $X^m(\Omega)$ for all time, and that ψ also belong to this functional space. For such functions, the energy estimates on ρ are basically the same for any derivative order. On the computation side, the integrations by parts required to set up a bootstrap argument vanish at any order, thanks to the conditions $\partial_z^{2n}\psi = \partial_z^{2n}\rho = 0$ for convinient n. This allows to obtain a decay estimate valid for any derivative order,

$$\|\nabla \boldsymbol{u}\|_{L^\infty} \lesssim \|\theta'\|_{H^3} \lesssim \frac{\|\rho_0\|_{H^{\kappa(\gamma)}}}{(1+t)^{\gamma/4}}, \tag{3.78}$$

where $\gamma, \kappa(\gamma)$ are related exponents, with γ that can be chosen arbitrary large. It appears that if $\|\nabla \boldsymbol{u}\|_{L^\infty}$ is integrable in time, then the perturbation θ of the background profile $\Theta(z) = 1 - z$ remains small. Since γ is arbitrary, the stability holds for small initial data regular enough. This is relatable to the a priori estimate, valid both for IPM and ST, also stated in Theorem 2.6.1,

$$\|\rho\|_{H^\kappa} \leq \|\rho_0\|_{H^\kappa} \exp\left(\int_0^t (\|\nabla \boldsymbol{u}(\tau)\|_{L^\infty} + \|\nabla\rho(\tau)\|_{L^\infty})\, \mathrm{d}\tau\right),$$

and to the convergence condition from Proposition 3.6.7, stating that if $\boldsymbol{u}$ is in $L^1(\mathbf{R}_+; W^{1,\infty}(\Omega))$ then ρ converges. Note that similar decays as (3.78) have been obtained in [Elg17], with a similar scheme of proof, in the whole plane $\mathbf{R}^2$ and in the torus, which are domains without boundaries.

When analysing the similar Stokes-tansport stability problem, we realised that the same functional space $X^m(\Omega)$ fits and that the same estimates can be computed, up to a different decay rate. This is related to the aforementioned splitting of the bilaplace problem into two Poisson problems, similar to the stream formulation (3.77) of IPM. The result for the stability for the Stokes-transport equation with slip boundary condition would state as for IPM and its proof would be a slight adaptation of [CCL19a].

The convenient orthonormal basis of $X^m(\Omega)$ considered in [CCL19a] describes as the renormalised following functions

$$\omega_{k,n}(x,z) = e^{ikx} \begin{cases} \cos(n\pi z/2) & n \text{ odd}, \\ \sin(n\pi z/2) & n \text{ even}, \end{cases} \qquad (k,n) \in \mathbf{Z} \times \mathbf{N},$$

on the strip $\mathbf{T}\times(-1,1)$, symmetric with respect to $\{z=0\}$ to simplify the expressions. This family corresponds to the product of the Fourier basis of $L^2(\mathbf{T})$ together with eigenfunctions for the operator $1-\partial_z^2$ on the functional space $H_0^1\cap H^2(\Omega)$. This family actually mimics the discrete Fourier basis on $\mathbf{T}^2$. Notice that any element of this basis is stable under derivatives of the following form,

$$\forall \alpha,\beta\in\mathbf{N},\quad \partial_x^\alpha\partial_z^{2\beta}\omega_{k,n}=(ik)^\alpha(-1)^\beta\left(\frac{n\pi}{2}\right)^{2\beta}\omega_{k,n}.$$

Let us say one word about the regularisation of the biharmonic operator for the Stokes equation with Neumann boundary conditions. Since it is well known that the operator Δ^{-1} is bounded $L^2\to H^2$ in bounded domains, we deduce rightaway the well-posedness of the bilaplacian problem with Neumann condition, with estimate

$$\forall m\in\mathbf{N},\forall f\in H^m(\Omega),\qquad \|(\Delta^{-1})^2 f\|_{H^{4+m}}\lesssim\|f\|_{H^m}.$$

In the following section we will obtain the same estimate for the Dirichlet condition case, and provide a self-contained proof of this result since it does not stem as directly.

3.8.2 The Dirichlet boundary conditions and bilaplacian estimates

In this framework, the Stokes equation with homogeneous Dirichlet boundary condition writes

$$-\Delta\boldsymbol{u}+\nabla p=-\rho\boldsymbol{e}_z,\quad \operatorname{div}\boldsymbol{u}=0,\quad \boldsymbol{u}=\mathbf{0}\text{ on }\partial\Omega,$$

whence its biharmonic formulation, obtained in the same fashion as in the previous case,

$$\Delta^2\psi=\partial_x\rho,\quad \psi=\partial_{\boldsymbol{n}}\psi=0\text{ on }\Omega. \tag{3.79}$$

We call *Dirichlet* conditions on ψ these equalities, at it is it customary when dealing with so called polyharmonic equations, see for instance [GGS10] on this topic.

When endowed with Dirichlet boundary conditions, there is no possible splitting in two elliptic problems, as in (3.76), since such a split would let the Poisson equations either over or underdetermined, see

$$\begin{array}{lll}\Delta\psi=\Psi, & \psi=\partial_z\psi=0 & \text{on }\partial\Omega,\\ \Delta\Psi=f, & \Psi=? & \text{on }\partial\Omega.\end{array}$$

In this case the solving operator will be denoted Δ^{-2}, with the understanding that it solves the biharmonic equation in the Dirichlet case. It differs from $(\Delta^{-1})^2$ where Δ^{-1} solves the Poisson equation. Note that for a non-trivial source f, a function ψ such that $\Delta^2\psi=f$ cannot satify the Neumann and the Dirichlet conditions at the same time, as the problem would be overdetermined with three distinc boundary conditions.

Well-posedness and estimates This paragraph is devoted to the proof of the following result.

Proposition 3.8.1. *Let $f\in H^m(\Omega)$ with $m\geq -2$. The problem*

$$\Delta^2\psi=f,\quad \psi=\partial_{\boldsymbol{n}}\psi=0\text{ on }\partial\Omega, \tag{3.80}$$

has a unique solution $\psi\in H_0^2(\Omega)$. Moreover, it belongs to $H_0^2\cap H^{4+m}(\Omega)$ and obeys

$$\|\psi\|_{H^{4+m}}\lesssim\|f\|_{H^m}.$$

We consider the variational formulation of the equation. Assuming all functions are regular enough to perform the following computations, we know that is ψ is a solution of (3.80), then for any test function φ such that $\varphi = \partial_{\boldsymbol{n}}\varphi = 0$ on $\partial\Omega$, we have

$$\int_\Omega \Delta^2\psi\varphi = \int_\Omega \Delta\psi\Delta\varphi + \int_{\partial\Omega} \partial_{\boldsymbol{n}}\Delta\psi\varphi \,\mathrm{d}\sigma - \int_{\partial\Omega} \Delta\psi\partial_{\boldsymbol{n}}\varphi \,\mathrm{d}\sigma = \int_\Omega \Delta\psi\Delta\varphi.$$

Therefor we call *weak* or *variational* solution any $\psi \in H^2_0(\Omega)$ such that

$$\int_\Omega \Delta\psi\Delta\varphi = \langle f, \varphi\rangle_{H^{-2},H^2_0}, \quad \forall\varphi \in H^2_0(\Omega),$$

where the right hand-side term corresponds to the duality bracket on $H^2_0(\Omega)$, for sake of generality. From the Poisson problem estimate, we know that

$$\forall\varphi \in H^2_0(\Omega), \qquad \|\varphi\|_{H^2} \lesssim \|\Delta\varphi\|_{L^2}. \tag{3.81}$$

Therefore, we observe that the bilinear form associated to the variational problem satisfies the coercivity assumption required to apply Lax–Milgram theorem, in $H^2_0(\Omega)$,

$$\forall\varphi \in H^2_0(\Omega), \qquad \|\varphi\|^2_{H^2} \lesssim \int_\Omega (\Delta\varphi)^2,$$

ensuring existence and uniqueness of a variational solution $\psi \in H^2_0(\Omega)$, satisfying moreover

$$\|\psi\|_{H^2} \lesssim \|f\|_{H^{-2}}.$$

At this point we showed that the operator $\Delta^{-2} : H^{-2}(\Omega) \to H^2_0(\Omega)$ is bounded. We now establish the regularisation effect on the solution.

To do so we apply the translation method, as performed in [Bré11, Section 9.6], for instance. Let us define the finite horizontal difference operator

$$\forall h \neq 0, \qquad D_h : \psi \mapsto \frac{\psi(\cdot + h\boldsymbol{e}_x) - \psi}{h},$$

where $D_h\psi$ inherits the regularity and homogeneous Dirichlet boundary condition of ψ, since the null trace is preserved under translation. From [Bré11, Section 9.6] we will use that for any $\psi \in L^2(\Omega)$ we have

$$\partial_x\psi \in L^2(\Omega) \quad \Rightarrow \quad \forall h \neq 0, \quad \|D_h\psi\|_{L^2} \leq \|\partial_x\psi\|_{L^2}, \tag{3.82}$$

and conversely, a characterisation of having a derivative in L^2,

$$(\exists C > 0, \forall h \neq 0, \quad \|D_h\psi\|_{L^2} \leq C) \quad \Rightarrow \quad \partial_x\psi \in L^2(\Omega). \tag{3.83}$$

Let us set $f \in L^2(\Omega)$. By linearity of the problem (3.79) we observe that $D_h\psi$ satisfies the same problem as ψ in the variational sense, since $\psi \in H^2_0(\Omega)$ implies $D_h\psi \in H^2_0(\Omega)$:

$$\Delta^2 D_h\psi = D_h f \qquad \Leftrightarrow \qquad \forall\varphi \in H^2_0(\Omega), \int_\Omega \Delta D_h\psi\Delta\varphi = \int_\Omega f D_{-h}\varphi \tag{3.84}$$

One can consider the test function $\varphi = D_h\psi$, use the norm equivalence (3.81) and inequality

(3.82) to get

$$\|D_h\psi\|_{H^2}^2 \lesssim \int_\Omega |\Delta D_{-h}\psi|^2 = \int_\Omega f D_{-h} D_h\psi \leq \|f\|_{L^2}\|D_h\psi\|_{H^2},$$

providing a uniform bound as in (3.83), ensuring $\partial_x\psi \in L^2(\Omega)$ and $\|\partial_x\psi\|_{L^2} \lesssim \|f\|_{L^2}$. Passing to the limit $h \to 0$ in (3.84) we obtain that $\partial_x\psi$ satisfies the same problem as ψ, in the sense

$$\Delta^2\partial_x\psi = \partial_x f \qquad \Leftrightarrow \qquad \forall\varphi \in H_0^2(\Omega), \quad \int_\Omega \Delta\partial_x\psi\Delta\varphi = -\int_\Omega f\partial_x\varphi.$$

Let us apply once again the finite difference operator, consider the test funciton $D_h\partial_x\psi$ to reach $\|D_h\partial_x\psi\|_{H^2} \lesssim \|f\|_{L^2}$ and deduce that $\partial_x^2\psi \in H_0^2(\Omega)$ with $\|\partial_x^2\psi\|_{H^2} \lesssim \|f\|_{L^2}$. Assuming $\partial_x^\ell f \in L^2(\Omega)$ we can even prove in the same fashion that

$$\forall k \in \mathbf{N}, \quad \|\partial_x^{k+2}\psi\|_{H^2} \lesssim \|\partial_x^k f\|_{L^2}.$$

Hence, we obtain that $\partial_z^4\psi$ is L^2 integrable by considering, in the distributional sense,

$$\partial_z^4\psi = f - (2\partial_z^2 + \partial_x^2)\partial_x^2\psi,$$

where the right hand-side lies in $L^2(\Omega)$ by assumption on f and since we proved that $\partial_x^2\psi$ belongs to $H^2(\Omega)$, so that

$$\|\partial_z^4\psi\|_{L^2} \lesssim \|f\|_{L^2} + \|\partial_x^2\psi\|_{H^2} \lesssim \|f\|_{L^2}.$$

From interpolation results such as [LM12, Chapter 1, Theorem 3.1], we have the following.

Lemma 3.8.2. *Let $\psi : \Omega \to \mathbf{R}$ and $k, \ell \in \mathbf{N}$ such that $\partial_x^k\psi, \partial_z^\ell\psi \in L^2(\Omega)$. Then, for any $\theta \in [0,1]$ we have*

$$\|\partial_x^{\theta k}\partial_z^{(1-\theta)\ell}\psi\|_{L^2} \lesssim \|\partial_x^k\psi\|_{L^2} + \|\partial_z^\ell\psi\|_{L^2}.$$

Since we proved the bound $\|\partial_x^4\psi\|_{L^2} + \|\partial_z^4\psi\|_{L^2} \lesssim \|f\|_{L^2}$, this lemma yields the expected inequality $\|\psi\|_{H^4} \lesssim \|f\|_{L^2}$. An induction provides the estimate for higher regularity data, with even order $m \in 2\mathbf{N}$ in a first time, namely for any $n \geq 0$,

$$\|\psi\|_{H^{4+2n}} \lesssim \|f\|_{H^{2n}}. \tag{3.85}$$

Assume this latter to be established for some $n \geq 0$. By the aforementioned arguments, the same holds for $\partial_x^2\psi$, namely

$$\|\partial_x^2\psi\|_{H^{4+2n}} \lesssim \|\partial_x^2 f\|_{H^{2n}}.$$

Therefore, in the distributional sense

$$\partial_z^{4+2(n+1)}\psi = \partial_z^{2(n+1)} f - \partial_z^{2(n+1)}(2\partial_z^2 + \partial_x^2)\partial_x^2\psi,$$

which yields

$$\|\partial_z^{4+2(n+1)}\psi\|_{L^2} \lesssim \|\partial_z^{2(n+1)} f\|_{L^2} + \|\partial_x^2\psi\|_{H^{4+2n}} \lesssim \|f\|_{H^{2(n+1)}}.$$

Then we recover the crossed derivatives bounds thanks to Lemma 3.8.2 and reach (3.85) with index $n+1$, thus the iteration. Since we proved (3.85) is valid for $n = 0$, the induction ensures that the property holds for any $m \in 2\mathbf{N}$, provided $f \in H^m(\Omega)$. The case m odd stems by interpolation.

3.8.3 Biharmonic eigenfunction basis for Dirichlet boundary conditions

In this Subsection we provide a description of the spectrum and eigenfunction basis associated to the bilaplace equation endowed with Dirichlet boundary condition. Moreover, we compare it to the basis adapted to the Neumann case, and comment its link with the boundary layers appearing in the stability results for the Stokes-transport equation.

Proposition 3.8.3 (Spectrum of the bilaplacian)**.** *The eigenvalues of the operator Δ^2 on H^2_0 in $\mathbb{T}\times(-1,1)$ are the union, for all $k\in\mathbf{Z}$, of strictly increasing sequences $(\lambda_{k,n})_{n\in\mathbf{N}}$ such that*

$$\lambda_{n,k}\simeq (n^2+k^2)^2$$

with associated (unnormalized) eigenfunctions:

$$b_{n,k}=e^{ikx}\begin{cases}\cos(\omega_{k,n}z)-\frac{\cos(\omega_{k,n})}{\cosh(r_{k,n})}\cosh(r_{k,n}z), & n\in 2\mathbf{N},\\ \sin(\omega_{k,n}z)-\frac{\sin(\omega_{k,n})}{\sinh(r_{k,n})}\sinh(r_{k,n}z), & n\in 2\mathbf{N}+1,\end{cases}$$

with $\omega_{k,n}=(k^2-\lambda_{k,n}^{1/2})^{1/2}$ and $r_{k,n}=(k^2+\lambda_{n,k}^{1/2})^{1/2}$. Note that to simplify the calculations, the domain was choosen to be $\mathbb{T}\times(-1,1)$ and not $\Omega=\mathbb{T}\times(0,1)$.

Let us determine the nontrivial eigenvalues and eigenfunctions (λ,b_λ) satisfying

$$\Delta^2 b_\lambda=\lambda b_\lambda,\qquad b_\lambda=\partial_z b_\lambda=0 \text{ on } \partial\Omega.$$

From Proposition 3.8.1 we know that these eigenfunctions have to belong to $H^2_0\cap C^\infty(\Omega)$. We apply horizontal Fourier transform to the above equation, for a given eigenvalue $\lambda>0$, and any horizontal mode $k\in\mathbf{Z}$ of b_λ must satisfy the following ordinary linear differential equation

$$(\partial_z^2-k^2)^2\,\widehat{b}_\lambda(k,\cdot)=\lambda\widehat{b}_\lambda(k,\cdot),\qquad \widehat{b}_\lambda(k,\pm1)=\partial_z\widehat{b}_\lambda(k,\pm1)=0. \tag{3.86}$$

A function b_λ therefore admits a Fourier mode k if and only if this linear differential equation has a nontrivial solution. The roots $\{\pm r,\pm\omega\}$ of the associated characteritic polynomial satisfy

$$(\pm r)^2=k^2+\lambda^{1/2},\quad (\pm\omega)^2=k^2-\lambda^{1/2}.$$

The only case with a multiple root is $\lambda=k^4$, does not admit nontrivial solution. This is detailed further for sake of completeness. Let us assume without loss of generality that $k^4\neq\lambda$. Then any solution to the differential equation has the form

$$z\mapsto A\mathrm{ch}(rz)+B\mathrm{sh}(rz)+C\mathrm{ch}(\omega z)+D\mathrm{sh}(\omega z).$$

Let us notice that such a map satisfy the boundary conditions if and only if its even and odd parts also satisfy these. Therefore, let us evaluate both parts and there derivatives at $z=\pm1$, to obtain the sufficient and necessary condition to the existence of nontrivial solution to (3.86),

$$\begin{cases} A\mathrm{ch}(r) + C\mathrm{ch}(\omega) = 0\\ Ar\mathrm{sh}(r) + C\omega\mathrm{sh}(\omega) = 0\end{cases}\quad\text{or}\quad\begin{cases} B\mathrm{sh}(r) + D\mathrm{sh}(\omega) = 0\\ Br\mathrm{ch}(r) + D\omega\mathrm{ch}(\omega) = 0.\end{cases}$$

This is satisfied if and only if one of these system has a trivial determinant, namely

$$r\mathrm{th}(r)+\omega\mathrm{th}(\omega)=0\quad\text{or}\quad\omega\mathrm{th}(r)=r\mathrm{th}(\omega). \tag{3.87}$$

Two cases present. If $k^2 > \lambda^{1/2}$, then ω is real. Therefore, the condition above cannot be satisfied, by positivity of $\sigma \mapsto \sigma \mathrm{th}(\sigma)$ and by monotonicity of $\sigma \mapsto \sigma^{-1}\mathrm{th}(\sigma)$, respectively. The other case is $k^2 < \lambda^{1/2}$, so let us redefine ω as $(\lambda^{1/2} - k^2)^{1/2}$ such that $i\omega$ is the actual characteristic polynomial root. Hence the above condition (3.87) rewrites as

$$r\mathrm{th}(r) + \omega \tan(\omega) = 0 \quad \text{or} \quad \omega \mathrm{th}(r) = r \tan(\omega), \tag{3.88}$$

where we ruled out the case $\omega \in \pi/2 + \pi\mathbf{Z}$ since such a root cannot satisfy (3.88). Let us detail and reorder the second equality as follows

$$\left(\frac{\lambda^{1/2} - k^2}{\lambda^{1/2} + k^2}\right)^{1/2} = \frac{\tan\left((\lambda^{1/2} - k^2)^{1/2}\right)}{\mathrm{th}\left((\lambda^{1/2} + k^2)^{1/2}\right)}.$$

For any $k \in \mathbf{Z}$ there exists an infinite number of solutions $\lambda > k^4$. Indeed, the left hand-side of the equality is continuous with respect to $\lambda > k^4$ and equivalent to 1 as λ goes to ∞. The right hand-side is piecewise continuous due to the tangente function. The intermediate value theorem for strictly monotonous maps ensures existence and uniqueness of a solution $\lambda_{k,n}$ for all $n \in \mathbf{N}$ satisfying

$$\left(\lambda_{k,n}^{1/2} - k^2\right)^{1/2} \in]n\pi, (n + 1/2)\pi[\,. \tag{3.89}$$

For such $\lambda = \lambda_{k,n}$, the differential equation (3.86) admits the following solution, up to a multiplicative constant,

$$z \mapsto \cos(\omega_{n,k} z) - \frac{\cos(\omega_{n,k})}{\mathrm{ch}(r_{n,k})}\mathrm{ch}(r_{n,k} z).$$

Similar considerations show that the first equality also admits a unique solution within each following interval, disjoint from the previous (3.89),

$$\left(\lambda_{k,n+1/2}^{1/2} - k^2\right)^{1/2} \in \left]\left(n + \tfrac{1}{2}\right)\pi, (n+1)\pi\right[,$$

with associated differential equation solutions proportional to

$$z \mapsto \sin\left(\omega_{k,n+1/2} z\right) + \frac{\sin\left(\omega_{k,n+1/2}\right)}{\mathrm{sh}\left(r_{k,n+1/2}\right)}\mathrm{sh}\left(r_{k,n+1/2} z\right).$$

Up to a slight reindexation, we obtained a description of the eigenvalues of Δ^2 on $H_0^2(\Omega)$ as the union, for all $k \in \mathbf{Z}$, of strictly increasing sequences $(\lambda_{n,k})_{n\in\mathbf{N}}$ such that

$$\lambda_{n,k} \simeq (n^2 + k^2)^2,$$

with associated (unnormalized) eigenfunctions

$$b_{n,k} = e^{ikx} \begin{cases} \cos(\omega_{k,n} z) - \frac{\cos(\omega_{k,n})}{\cosh(r_{k,n})}\cosh(r_{k,n} z), & n \in 2\mathbf{N}, \\ \sin(\omega_{k,n} z) - \frac{\sin(\omega_{k,n})}{\sinh(r_{k,n})}\sinh(r_{k,n} z), & n \in 2\mathbf{N} + 1, \end{cases} \qquad k \in \mathbf{Z},$$

with $r_{k,n} = (k^2 + \lambda_{k,n}^{1/2})^{1/2}$ and $\omega_{k,n} = (k^2 - \lambda_{k,n}^{1/2})^{1/2}$. We can check that $(b_{k,n})_{k,n}$ is indeed

orthonormal in $L^2(\Omega)$, since

$$\forall k, k' \in \mathbf{Z}, \quad \int_{\mathbf{T}} e^{ikx} e^{-ik'x} \, \mathrm{d}x = \delta_{k=k'},$$

and $(\partial_z^2 - k^2)^2$ is self-adjoint,

$$\begin{aligned} \forall k \in \mathbf{Z}, \forall n, n' \in \mathbf{N}, \quad \lambda_{k,n} \int_\Omega b_{k,n} \overline{b_{k,n'}} &= \int_\Omega (\partial_z^2 - k^2)^2 b_{k,n} \overline{b_{k,n'}} \\ &= \int_\Omega b_{k,n} (\partial_z^2 - k^2)^2 \overline{b_{k,n'}} \\ &= \lambda_{k,n'} \int_\Omega b_{k,n} \overline{b_{k,n'}}, \end{aligned}$$

with $\lambda_{k,n} \neq \lambda_{k,n'}$ when $n \neq n'$ according to (3.89), hence the orthogonality.

For completeness, we explain why we can rule out the case $\lambda^{1/2} = k^2$. In this situation, the roots of the characteristic polynomial are $r := \sqrt{2}|k|, -r$ and 0 with multiplicity 2. Hence the solutions of the associated differential equation are of the form

$$z \mapsto A + Bz + C\mathrm{ch}(rz) + D\mathrm{sh}(rz),$$

and we find the sufficient and necessary relation to satisfy the boundary condition

$$r = \mathrm{th}(r)$$

which is never satisfied for $k \neq 0$, and the case $k = \lambda = 0$ admits only the trivial solution 0.

Bases comparison and link with the Stokes-transport stability Both Neumann and Dirichlet bases functions are obtained as the product between an exponential e^{ikx} and a circular map in the z variable. In the Neumann case, the basis is the tensor products of independent x and z variable and 1d bases. In particular the z dependent factor was either a sine or a cosine, therefore stable under vertical derivatives of order 2, which relates to the space $X^m(\Omega)$ and its stability under such operations. In the Dirichlet case, the z dependent part is also a sine or cosine but supplemented with its hyperbolic avatar, this precisely to satisfy the boundary conditions, and the coefficients also depend on the horizontal mode k, which was not the case for Neumann. The consequences and differences with the Neumann basis are multiple. The map $\partial_z^2 b_{k,n}$ are no longer colinear to $\mathrm{b}_{k,n}$ and $\partial_z^{2n} b_{k,n}$ does not satisfy the Dirichlet boundary conditions in general. Therefore the integrations by part required in the proofs of [CCL19a] does not hold for any order, only for low derivtive orders, since boundary terms do not vanish anymore in general. In other words, $\Delta^2 \psi$ does not satisfy a Dirichlet biharmonic equation, contrary to ψ. This is why boundary layers are introduced in Theorem 3.1.2 and 3.1.3, to split the data in boundary parts, bearing the data traces, and in an interior part, satisfying the Dirichlet condition, on which the previous estimates available at low regularity level do hold.

Chapter 4

Numerical study of the graph interface evolution

The aim of this chapter is to provide a numerical analysis of the evolution of free interfaces derived from the Stokes-transport equation. We chose to generically consider the problem in the periodic strip domain $\Omega = \mathbf{T} \times (0,1)$. Since (ST) is well-posed for bounded initial data $\rho_0 \in L^\infty$, one can consider $\rho_0 = \mathbf{1}_{P_0}$ where P_0 is a connected subdomain of Ω. We call these *density patches*. Since the solution ρ is the pushforward of the initial datum by the flow $\boldsymbol{X}$ associated to the velocity field, the solution remains a patch for all time, satisfying $\rho(t) = \mathbf{1}_{\boldsymbol{X}(t;P_0)}$. We will focus on the case of an initial patch coinciding with the hypograph of a map $\eta_0 : \mathbf{T} \to (0,1)$, *i.e.* $P_0 = \{z < \eta_0(x)\} := \{(x,z) \in \Omega : z < \eta_0(x)\}$. Let us assume that the interface evolution can be described by a graph $\eta : (t,x) \in [0,T] \times \mathbf{T} \mapsto (0,1)$, such that

$$\rho(t,x,z) = \mathbf{1}_{\{z<\eta(t,x)\}}.$$

Such a function ρ satisfies (ST) on Ω if and only if the graph η satisfies the following system

$$\left\{\begin{array}{rcll} \partial_t \eta(t,x) + u_1(t,x,\eta(t,x))\partial_x \eta(t,x) &=& u_2(t,x,\eta(t,x)), & \text{for } (t,x,z) \in [0,T]\times\Omega, \\ -\Delta \boldsymbol{u} + \nabla p &=& -\mathbf{1}_{\{z<\eta(t,x)\}}\boldsymbol{e}_z, & \text{on } [0,T]\times\Omega, \\ \operatorname{div} \boldsymbol{u} &=& 0, & \text{on } [0,T]\times\Omega, \\ \boldsymbol{u} &=& \mathbf{0}, & \text{on } [0,T]\times\partial\Omega, \\ \eta|_{t=0} &=& \eta_0. & \end{array}\right. \tag{4.1}$$

Section 4.1 is devoted to theoretical considerations regarding the system (4.1). In particular we introduce the notations of this problem in Subsection 4.1.1. The system (4.1) is derived from (ST) in Subsection 4.1.2, where several formulations of the problem are discussed. In Subsection 4.1.3 the graph persistence in short time of the interface is proven. Subsection 4.1.4 is a discussion about the maximal regularity we can expect from the velocity field for patch data.

Section 4.2 consists in a numerical analysis of the evolution of graph interfaces for (4.1). In particular we repport in Subsection 4.2.1 the recent theoretical stability result from [GGS22] of Gancedo, Granero-Belinchón & Salguero for graph interfaces in the horizontally periodic plane $\mathbf{T} \times \mathbf{R}$, as a comparison point for our numerical observations. Subsection 4.2.2 introduces the numerical scheme we set up to run our simulations. In Subsection 4.2.3 we set up the main parameters of the scheme, to ensure enough precision for our analysis. Once this is done, we observe in Subsection 4.2.4 the evolution of several graphs and compare the observed convergence rates

with the result of [GGS22]. The influence of the boundaries is illustrated in Subsection 4.2.5. The maximum principle for the interface is briefly investigated in Subsection 4.2.6. After introducing the concept of Lyapunov functionals in Subsection 4.1.5, we disqualify several functionals thanks to examples for which these are not monotonous for the solutions of the system (4.1). We also speculate on the status of the entropy functional, for which we did not find counter-examples. This is done in Subsection 4.2.7.

4.1 Theoretical analysis of the graph interface problem

4.1.1 Notations

Let $\eta_0 : \mathbf{T} \times (0,1)$ be an initial map, parametrising the interface $\Sigma_0 := \{(x, \eta_0(x)) : x \in \mathbf{T}\}$. We note $\Sigma(t)$ its transport image at time t. If the interface corresponds to the graph of some η at time t, we have $\Sigma(t) = \{(x, \eta(t,x)) : x \in \mathbf{T}\}$. The image at time t of the initial hypograph subdomain $\Omega_0^- := \{z < \eta_0(x)\}$ will be denoted $\Omega^-(t)$, and the epigraph $\Omega^+(t)$ in the same fashion. The outward unitary normal to $\Omega^-(t)$ on $\Sigma(t)$ will be denoted $\boldsymbol{n}_{\Sigma(t)}$. We define $f|_\eta$ the trace on Σ when it is the graph of some η. We also use the notation $\langle \xi \rangle = \sqrt{1+\xi^2}$.

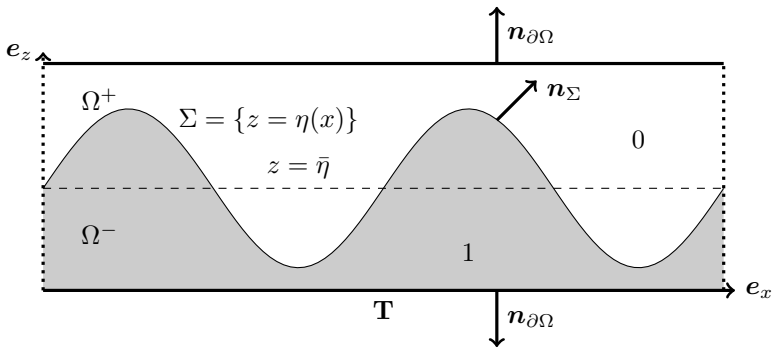

Figure 4.1: Illustration of the problem and notations.

4.1.2 Derivation of the free boundary problem

In this subsection we deduce the graph interface evolution equation (4.1) from (ST). Let us assume that there exists $T > 0$ such that the solution of (ST) with $\rho_0 = \mathbf{1}_{\{z<\eta_0(x)\}}$ writes for $t \in [0,T)$ as a patch associated to the hypograph of $x \to \eta(t,x)$, in the sense

$$\forall (t,x) \in [0,T) \times \mathbf{T}, \qquad \rho(t,x) = \mathbf{1}_{\{z<\eta(t,x)\}} = \mathbf{1}_{\Omega^-(t)}.$$

We will show in Subsection 4.1.3 that the interface indeed remains a graph for short time. We showed in Theorem 2.1.1 that ρ exists for all times and belongs to $L^\infty(\mathbf{R}_+; L^\infty(\Omega))$. In particular, it satisfies the transport equation in the weak (or distributional) sense

$$\forall \varphi \in C_c^\infty([0,T) \times \Omega), \quad \int_0^T \int_\Omega \rho(\partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi) = -\int_\Omega \rho_0 \varphi(0,\cdot). \tag{4.2}$$

Since ρ is the indicator function of the subdomain $\Omega^-(t)$, the weak formulation (4.2) also writes, for any given test function φ,

$$\int_0^T \int_{\Omega^-(t)} \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi = -\int_{\Omega_0^-} \varphi(0,\cdot). \tag{4.3}$$

Using the following identity, valid for $\eta \in C^1([0,T) \times \mathbf{T})$ for instance,

$$\forall x \in \mathbf{T}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^{\eta(t,x)} \varphi \,\mathrm{d}z\right] = \int_0^{\eta(t,x)} \partial_t \varphi \,\mathrm{d}z + \partial_t \eta(t,x) \varphi(t,x,\eta(t,x)),$$

and the graph character of the subdomain $\Omega^-(t)$, we get

$$\begin{aligned}
\int_0^T \int_{\Omega^-(t)} \partial_t \varphi &= \int_0^T \int_{\mathbf{T}} \int_0^{\eta(t,x)} \partial_t \varphi(t,x,z) \,\mathrm{d}z \,\mathrm{d}x \\
&= \int_0^T \int_{\mathbf{T}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^{\eta(t,x)} \varphi \,\mathrm{d}z\right] - \partial_t \eta(t,x) \varphi(t,x,\eta(t,x)) \right) \mathrm{d}x \,\mathrm{d}t \\
&= -\int_{\Omega_0^-} \varphi(0,\cdot) - \int_0^T \int_{\mathbf{T}} \partial_t \eta(t,x) \varphi(t,x,\eta(t,x)) \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned} \tag{4.4}$$

where we use for the last step that $\varphi(T,\cdot) = 0$.

Besides, the Green theorem ensures, since $\partial\Omega^-(t) = \{z = 0\} \cup \Sigma(t)$,

$$\int_{\Omega^-(t)} \boldsymbol{u} \cdot \nabla \varphi = -\int_{\Omega^-(t)} \operatorname{div} \boldsymbol{u} \, \varphi + \int_{\Sigma(t)} \boldsymbol{u} \cdot \boldsymbol{n}_{\Sigma(t)} \,\mathrm{d}\sigma(t) - \int_{\{z=0\}} u_2 \varphi \,\mathrm{d}x.$$

Since $\boldsymbol{u}$ is divergence-free and $u_2 = 0$ on the boundaries of Ω, only the integral on $\Sigma(t)$ remains. The outward normal vector and surface measure at time t write

$$\forall x \in \mathbf{T}, \quad \boldsymbol{n}_{\Sigma(t)}(x) = \langle \partial_x \eta(t,x) \rangle^{-1} \begin{pmatrix} -\partial_x \eta(t,x) \\ 1 \end{pmatrix}, \qquad \mathrm{d}\sigma(t) = \langle \partial_x \eta(t,x) \rangle \,\mathrm{d}x. \tag{4.5}$$

Recalling the notation $\boldsymbol{u} = (u_1, u_2)$, we obtain

$$\int_{\Omega^-(t)} \boldsymbol{u} \cdot \nabla \varphi = \int_{\Sigma(t)} \boldsymbol{u} \cdot \boldsymbol{n}_{\Sigma(t)} \,\mathrm{d}\sigma(t) = \int_{\mathbf{T}} (u_2|_\eta - u_1|_\eta \partial_x \eta) \varphi|_{z=\eta(t,x)}. \tag{4.6}$$

Substituting (4.4) and (4.6) in (4.3) we get

$$\forall \varphi \in C_{\mathrm{c}}^\infty([0,T) \times \Omega), \qquad \int_0^T \int_{\mathbf{T}} (\partial_t \eta + u_1|_\eta \partial_x \eta - u_2|_\eta) \varphi|_\eta = 0.$$

Let us set $\phi \in C_{\mathrm{c}}^\infty([0,T) \times \mathbf{T})$ and a smooth cutoff function $\chi_\varepsilon : (0,1) \to [0,1]$ compactly supported such that $\chi_\varepsilon(z) = 1$ for any $z \in [\varepsilon, 1-\varepsilon]$, for suitable $\varepsilon > 0$. For any $\varepsilon > 0$, $\varphi_\varepsilon := \phi \chi_\varepsilon$ is a test function on $[0,T) \times \Omega$, and we have

$$\int_0^T \int_\Omega (\partial_t \eta + u_1|_\eta \partial_x \eta - u_2)(t,x) \phi(t,x) \chi_\varepsilon(\eta(t,x)) \,\mathrm{d}x \,\mathrm{d}t = 0.$$

Assuming that the graph of $\eta(t)$ is at strict distance from the boundary of the domain $\partial\Omega$, then for $\varepsilon > 0$ small enough we have $\eta(t,x) \in [\varepsilon, 1-\varepsilon]$ for any $x \in \mathbf{T}$. We obtain that

$$\forall \phi \in C^\infty([0,T) \times \mathbf{T}), \quad \int_0^T \int_{\mathbf{T}} (\partial_t \eta + u_1|_\eta \partial_x \eta - u_2|_\eta)\phi = 0, \tag{4.7}$$

hence the evolution equation on η, reported in (4.1),

$$\partial_t \eta + u_1|_\eta \partial_x \eta = u_2|_\eta. \tag{4.8}$$

Other evolution formulations We obtain the following equivalent formulations of the equation (4.8).

$$\partial_t \eta - \langle \partial_x \eta \rangle \boldsymbol{u}|_\eta \cdot \boldsymbol{n}_\Sigma = 0, \tag{4.9}$$

$$\partial_t \eta + \partial_x \int_0^\eta u_1 = 0, \tag{4.10}$$

$$\partial_t \eta + \partial_x(\psi|_\eta) = 0, \qquad \Delta^2 \psi = \partial_x \mathbf{1}_{\{z<\eta\}}. \tag{4.11}$$

Equation (4.9) is obtained by identifying the components of the normal vector $\boldsymbol{n}$ defined in (4.5). It illustrate the fact that the interface is transported according to the normal component of the velocity field $\boldsymbol{u}$.

Both (4.10) and (4.11) stem from the incompressibility of the fluid. Integrating the divergence free condition and using that $u_2|_{z=0} = 0$ we get,

$$\begin{aligned} \forall x \in \mathbf{T}, \quad -u_2(x,\eta(x)) &= \int_0^{\eta(x)} \partial_x u_1(x,\zeta)\,\mathrm{d}z' \\ &= \partial_x \left[\int_0^{\eta(x)} u_1(x,\zeta)\,\mathrm{d}z' \right] - u_1|_\eta \partial_x \eta. \end{aligned}$$

Back to (4.7), the terms $u_1|_\eta \partial_x \eta$ compensate, providing (4.10)

$$\int_0^T \int_{\mathbf{T}} \left(\partial_t \eta + \partial_x \int_0^\eta u_1(x,z')\,\mathrm{d}z' \right) \phi = 0.$$

Since $\boldsymbol{u}$ is divergence free, there exists stream function ψ such that $\boldsymbol{u} = \nabla^\perp \psi$. In particular ψ is the vertical primitive of u_1. We saw in Section 3.8 that the Stokes equation also writes in terms of ψ. With $\rho = \mathbf{1}_{\{z<\eta(t,x)\}}$ we get

$$\Delta^2 \psi = \partial_x \mathbf{1}_{\{z<\eta(x)\}}, \quad \psi = \partial_{\boldsymbol{n}} \psi = 0 \text{ on } \partial\Omega,$$

where the source term belongs to $H^{-2}(\Omega)$, and is understood in the dual sense

$$\varphi \in H_0^2(\Omega), \quad \langle \partial_x \mathbf{1}_{\{z<\eta(x)\}}, \varphi \rangle := -\langle \mathbf{1}_{\{z<\eta(x)\}}, \partial_x \varphi \rangle = -\int_{\{z<\eta(x)\}} \partial_x \varphi.$$

Since $\psi = 0$ at $\{z = 0\}$, we find that $\int_0^\eta u_1 = \psi|_\eta$ and we recognise (4.11) from (4.10) by the

chain rule,

$$\begin{aligned}\partial_x(\psi|_\eta) &= \partial_x\psi|_\eta + \partial_z\psi|_\eta\partial_x\eta \\ &= -u_2|_\eta + u_1|_\eta\partial_x\eta.\end{aligned}$$

4.1.3 Graph persistence for short time

By an elementary geometric consideration, we prove that any Lipschitz graph interface $\Sigma_0 = \{(x, \eta_0(x)\}$ remains a graph in short time.

Lemma 4.1.1. *For any $\eta_0 : \mathbf{T} \to (0,1)$ such that $\|\partial_x\eta_0\|_{L^\infty} < \infty$, there exists $T > 0$ such that the interface $\Sigma(t)$ is the graph of some map $\eta(t) : \mathbf{T} \to (0,1)$ for any $t \in [0,T)$.*

Proof. We know from [Leb22] that if $\rho_0 \in L^\infty(\Omega)$ then the solution to ST satisfies in particular

$$\rho(t) = \rho_0 \circ \boldsymbol{X}(t)^{-1}, \qquad \|\boldsymbol{u}\|_{L^\infty(\mathbf{R}_+;W^{1,\infty})} \lesssim \|\rho_0\|_{L^\infty},$$

where $\boldsymbol{X} = (X_1, X_2) : \mathbf{R}_+ \times \mathbf{R}_+ \times \Omega \to \Omega$ is the flow map associated to $\boldsymbol{u}$, as defined in Subsection 2.2.2. We will note $\boldsymbol{X}(t) = \boldsymbol{X}(t;0) = \boldsymbol{X}(t;0,\cdot)$ and $\boldsymbol{X}(t;x,z) = \boldsymbol{X}(t;0,(x,z))$ for concision. The interface of the patch at time t writes as the image of the flow $\boldsymbol{X}$ of the initial graph $\Sigma_0 = \{(x,\eta_0(x)) : x \in \mathbf{T}\}$, as follows

$$\Sigma(t) = \{\boldsymbol{X}(t;x,\eta_0(x)) : x \in \mathbf{T}\}. \tag{4.12}$$

We know by definition of the flow that $\|\partial_t\boldsymbol{X}\|_{L^\infty} \leq \|\boldsymbol{u}\|_{L^\infty(\mathbf{R}_+;L^\infty)} \lesssim \|\rho_0\|_{L^\infty}$, ensuring the transport to occur at finite speed. In particular, since the latter is satisfied at $t = 0$, for any $x < y$ there is $X_1(t;x,\eta_0(x)) < X_1(t;y,\eta_0(y))$ for $t > 0$ small enough depending on (x,y) a priori. Conversely, $\Sigma(t)$ is a graph if and only if

$$\forall x < y, \quad X_1(t;x,\eta_0(x)) < X_1(t;y,\eta_0(y)).$$

This condition is in particular satisfied if the horizontal derivative of $X_1(t;\cdot,\eta_0(\cdot))$ remains strictly positive. We know that $\boldsymbol{X}(t) \in W^{1,\infty}(\Omega)$ for all time, so that

$$\partial_x[X_1(t;x,\eta_0(x))] = \partial_x X_1(t;x,\eta_0(x)) + \partial_x\eta_0(x)\partial_z X_1(t;x,\eta_0(x)).$$

Recalling the identity (2.13), see also [BCD11, Proposition 3.10] for more precise estimates on transport flows,

$$\|\nabla\boldsymbol{X}(t) - \mathrm{Id}_\Omega\|_{L^\infty} \lesssim e^{Ct\|\nabla\boldsymbol{u}\|_{L^\infty(0,t;L^\infty)}} - 1,$$

we extract

$$\|\partial_x X_1(t) - 1\|_{L^\infty} \lesssim e^{Ct\|\rho_0\|_{L^\infty}} - 1, \qquad \|\partial_z X_1(t)\|_{L^\infty} \lesssim e^{Ct\|\rho_0\|_{L^\infty}}.$$

Gathering these latter considerations yields

$$\|\partial_x[X_1(t,\cdot,\eta_0(\cdot)) - 1\|_{L^\infty} \lesssim (1 + \|\partial_x\eta_0\|_{L^\infty})(e^{Ct\|\rho_0\|_{L^\infty}} - 1).$$

In particular, $\partial_x[X_1(t,\cdot,\eta_0(\cdot)]$ remains strictly positive for small times, and the graph condition is satisfied. Moreover, is $\Sigma(t)$ the graph of some $\eta(t)$ if and only if

$$\forall y \in \mathbf{T}, \exists! x \in \mathbf{T}, \quad \begin{cases} X_1(t;y,\eta_0(y)) &= x \\ X_2(t;y,\eta_0(y)) &= \eta(t,x). \end{cases}$$

Since $X_1(t;\cdot,\eta_0(\cdot))$ is a diffeomorphism of $\mathbf{T}$, we can invert this system and infer

$$\forall x \in \mathbf{T}, \quad \eta(t,x) = X_2(t;\cdot,\eta_0(\cdot)) \circ (X_1(t;\cdot,\eta_0(\cdot))^{-1}(x). \tag{4.13}$$

□

4.1.4 Data regularity

We discuss here the maximal regularity propagation that we can expect for η and $\boldsymbol{u}$. We start by introducing some notations for Hölder spaces, and recall an embedding between Sobolev and Hölder spaces.

Definition 4.1.2. *Let $k \in \mathbf{N}$ and $\mu \in (0,1)$. The Hölder space $C^{k,\mu}(\Omega)$ is the subset of functions $f \in C^k(\Omega)$ such that the following semi-norm is finite,*

$$\|f\|_{\dot{C}^{k,\mu}} := \sup_{\boldsymbol{x}\neq\boldsymbol{y}} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|}.$$

This space is endowed with the norm

$$\|f\|_{C^{k,\mu}} := \max_{0\leq\ell\leq k} \|\nabla^\ell f\|_{L^\infty} + \|f\|_{\dot{C}^{k,\mu}}.$$

To relate the Sobolev and Hölder spaces, we point out the following continous embedding holds, as a consequence of Morrey's inequality, see [Bré11, Theorem 9.12],

$$\forall m \in \mathbf{N}^*, \forall q > 2, \quad W^{m,q}(\Omega) \hookrightarrow C^{m-1,1-2/q}(\Omega). \tag{4.14}$$

We also illustrate how the Hölder regularity is preserved by composition.

Lemma 4.1.3. *Let $I = (0,1)$ and let $f \in C^{1,\alpha}(I;I)$ and $g \in C^{1,\beta}(I;I)$ with $\alpha,\beta \in (0,1)$. Then $f \circ g \in C^{1,\min(\alpha,\beta)}$.*

This statement writes on segments for sake of clarity. The same result holds when considering bounded domains of $\mathbf{R}^2$ or $\mathbf{R}^3$ instead, as long as the composition between the two Hölder functions is well-defined.

Proof. Since in particular $f,g \in C^1$, so is $f \circ g$. The only thing to show is that $\|f\circ g\|_{\dot{C}^{1,\delta}}$ is indeed finite, and for which δ. For any $x,y \in I$ such that $x \neq y$, we have

$$\begin{aligned}
(f\circ g)'(x) - (f\circ g)'(y) &= g'(x)f'(g(x)) - g'(y)f'(g(y)) \\
&= g'(x)(f'(g(x)) - f'(g(y))) + (g'(x) - g'(y))f'(g(y)) \\
&\leq \|g'\|_{L^\infty}\|f'\|_{\dot{C}^{0,\alpha}}|g(x)-g(y)|^\alpha + \|f'\|_{L^\infty}\|g'\|_{\dot{C}^{0,\beta}}|x-y|^\beta \\
&\leq \|g'\|_{L^\infty}^{1+\alpha}\|f\|_{\dot{C}^{1,\alpha}}|x-y|^\alpha + \|f'\|_{L^\infty}\|g\|_{\dot{C}^{1,\beta}}|x-y|^\beta.
\end{aligned}$$

Then we have

$$\frac{|(f\circ g)'(x) - (f\circ g)'(y)|}{|x-y|^\delta} \leq \left(\|g'\|_{L^\infty}^{1+\alpha}\|f\|_{\dot{C}^{1,\alpha}} + \|f'\|_{L^\infty}\|g\|_{\dot{C}^{1,\beta}}\right)|x-y|^{\min(\alpha,\beta)-\delta},$$

so if $\delta \leq \min(\alpha,\beta)$ then the supremum on $x \neq y$ is finite. □

Velocity estimate In this paragraph we link the velocity regularity to the graph L^1 norm. From here we decompose any graph η into its horizontal average $\overline{\eta} = \frac{1}{2\pi}\int_{\mathbf{T}} \eta$ and its zero average complementary ζ,

$$\eta = \overline{\eta} + \zeta.$$

The following holds.

Lemma 4.1.4. *Let $\eta : \mathbf{T} \to (0,1)$ measurable. Then the solution to the Stokes equation with datum $\mathbf{1}_{\{z<\eta(x)\}}$ satisfies*

$$\forall q \in (2,\infty), \qquad \|\boldsymbol{u}\|_{C^{1,1-2/q}} \lesssim \|\boldsymbol{u}\|_{W^{2,q}(\Omega)} \lesssim \|\zeta\|_{L^1(\mathbf{T})}^{1/q}.$$

This latter estimate illustrates that the regularity of ζ does not affect $\boldsymbol{u}$. Only its L^1 norm does. We provide a proof of this lemma thereafter.

Since for any $c \in \mathbf{R}$, the couple $(\rho_c, \boldsymbol{u}_c) = (c, \mathbf{0})$ solves the Stokes equation, the folllowing holds by linearity of the system, for any $1 < q < \infty$,

$$\forall c \in \mathbf{R}, \quad \|\boldsymbol{u}\|_{W^{2,q}} \lesssim \|\rho - c\|_{L^q}.$$

Since additive constants in the source term do not contribute to the dynamic of the system, the following norms are relevant when estimating the Stokes equation solutions

$$\|\rho\|_{L^q/\mathbf{R}} := \inf_{c\in\mathbf{R}} \|\rho - c\|_{L^q},$$

In the same fashion, recalling estimate (2.7), we have

$$\|\boldsymbol{u}\|_{W^{1,\infty}} \lesssim \|\rho\|_{L^\infty/\mathbf{R}}.$$

In the case of a bounded density ρ and especially for patches $\mathbf{1}_P$ with $P \subset \Omega$ non-negligible, we have

$$\|\rho\|_{L^\infty/\mathbf{R}} = \sup_\Omega \rho - \inf_\Omega \rho, \qquad \|\mathbf{1}_P\|_{L^\infty/\mathbf{R}} = 1,$$

which shows that the density differences matter more than its actual values. In particular, there is no relation between the uniform norm of ρ and the regularity of the patch P or the graph η. Hence we focus on the L^q estimate for the Stokes equation for $q \in (2,\infty)$. The L^q norm of $\rho = \mathbf{1}_{\{z<\eta(x)\}}$ computes as follows,

$$\|\rho\|_{L^q}^q = \int_{\Omega^-} 1 = \int_{\mathbf{T}} \int_0^{\eta(x')} 1 \,\mathrm{d}z' \,\mathrm{d}x' = \|\eta\|_{L^1}.$$

Proof of Lemma 4.1.4. As discussed right above, the linearity of the Stokes equation allows to shed some parts of the source term non contributing to the velocity field. Here we consider the solution $(\rho_h, \boldsymbol{u}_h) = (\mathbf{1}_{\{z<h\}}, \mathbf{0})$ to the Stokes equation and as in the previous case, the linearity of the system yields for any $h \in [0,1]$,

$$\|\boldsymbol{u}\|_{W^{2,q}} \lesssim \|\mathbf{1}_{\{z<\eta(x)\}} - \mathbf{1}_{\{z<h\}}\|_{L^q} = \|\eta - h\|_{L^1(\mathbf{T})}^{1/q},$$

since the difference of indicator functions satisfies the following, illustrated by Figure 4.2,

$$\forall (x,z)\in\Omega,\quad \mathbf{1}_{\{z<\eta(x)\}}-\mathbf{1}_{\{z<h\}}=\begin{cases}1 & \text{if } h<z<\eta(x),\\ -1 & \text{if } \eta(x)<z<h,\\ 0 & \text{otherwise.}\end{cases} \tag{4.15}$$

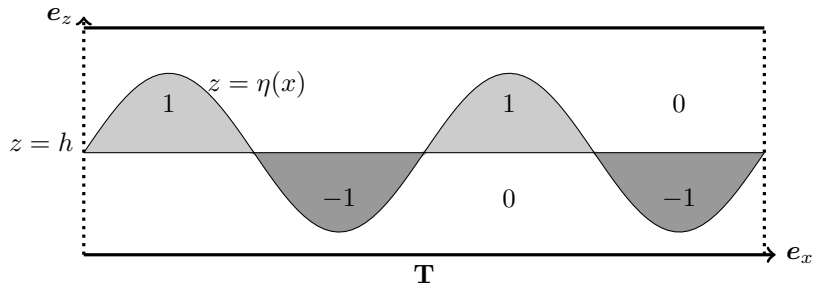

Figure 4.2: Representation of the difference (4.15).

Recalling $\bar\eta := \int_{\mathbf{T}}\eta$ the average of η, and define ζ such that $\eta=\bar\eta+\zeta$, with ζ has zero average. For $h=\bar\eta$ we get

$$\|\mathbf{1}_{\{z<\eta(x)\}}-\mathbf{1}_{\{z<\bar\eta\}}\|_{L^q}^q=\|\zeta\|_{L^1(\mathbf{T})}.$$

In the end, considering Morrey inequality (4.14), we have for any $2<q<\infty$,

$$\|\boldsymbol{u}\|_{C^{1,1-2/q}}\lesssim\|\boldsymbol{u}\|_{W^{2,q}}\lesssim\|\rho\|_{L^q}\lesssim\|\zeta\|_{L^1}^{1/q}.$$

□

Interface regularity We discuss here the regularity propagation of η. Lemma 4.1.1 ensures that η remains a Lipschitz graph for short time. From the interface description (4.1.1) we know that the interface, as a curve, has the regularity of the composition of the flow with the initial datum, namely $\boldsymbol{X}(t;0,\eta_0)$. Taking a look at the following flow estimate proof from [BCD11, Proposition 3.10],

$$|\nabla^2 X_i(t;s,\boldsymbol{x})|\le e^{\left|\int_s^t|\nabla\boldsymbol{u}(\tau,\boldsymbol{X}(\tau;s,\boldsymbol{x}))|\,\mathrm{d}\tau\right|}\left|\int_s^t|\nabla^2\boldsymbol{u}(\tau;\boldsymbol{X}(\tau;s,\boldsymbol{x}))e^{\left|\int_s^t|\nabla\boldsymbol{u}(\tau,\boldsymbol{X}(\tau;\sigma,\boldsymbol{x}))|\,\mathrm{d}\sigma\right|}\,\mathrm{d}\tau\right|,$$

we infer the following, thanks to the embedding (4.14),

$$\|\boldsymbol{X}(t;s)\|_{C^{1,1-2/q}}\lesssim\|\boldsymbol{X}(t;s)\|_{\dot W^{2,q}}\lesssim e^{C(t-s)\|\nabla\boldsymbol{u}\|_{L^\infty(t;s,L^\infty)}}\int_s^t\|\boldsymbol{u}(\tau)\|_{W^{2,q}}\,\mathrm{d}\tau.$$

In particular, $\boldsymbol{X}$ shares the spatial regularity of $\boldsymbol{u}$. We infer that for any s,t small enough, $\boldsymbol{X}(t;s,\cdot)$ belongs to $C^{1,\mu}(\Omega)$ for any $\mu\in(0,1)$. Considering $\eta_0\in C^{1,\delta}(\mathbf{T})$ for some $\delta\in(0,1)$, we obtain according to Lemma 4.1.3 that $X_2(t;\cdot,\eta_0(\cdot))\in C^{1,\delta}(\mathbf{T})$. From the explicit expression (4.13) of $\eta(t)$ as the composition of $X_2(t,\cdot,\eta_0(\cdot))$ and $(X_1(t,\cdot,\eta_0(\cdot)))^{-1}$, we get that $\eta(t)$ at best belong to the class $C^{1,\delta}$.

Notice that we are studying here the regularity of the map $x \to \eta(x)$ as a function $\mathbf{T} \to (0,1)$, which does not necessarily correspond to the regularity of a generic interface Σ, since this latter can be parametrised in different ways. In the case of a graph interface, Σ is at least as regular as η, but the converse does not hold. Think for instance of $x \mapsto \sqrt{x}$, continous on $\mathbf{R}_+$ but not Lipschitz, but whose graph is C^∞ as a curve. Further considerations and other works on patches are discussed in Subsection 1.4.2.

4.1.5 Lyapunov functionals and long-time behaviour

A Lyapunov functional is a monotonous quantity associated to the solution of a time dependent system. More precisely, following [AB20, Definition 1.1], we provide an adapted definition to our study.

Definition 4.1.5. *Consider a function $I : L^\infty(\Omega) \to [0,\infty)$. We say that I is a* Lyapunov functional *for the system (4.1) if for any solution $\eta : [0,T^*) \times \mathbf{T} \to (0,1)$, regular enough and with T^* its maximal time of existence, we have*

$$\forall t \in [0,T^*), \quad \frac{\mathrm{d}}{\mathrm{d}t} I(\eta(t)) \leq 0.$$

The Lyapunov functions allows us to study the long-time behaviour of solutions. In [AB20], Alazard & Bresch provide an extensive gathering of common Lyapunov functionals for several free surface equations arising in fluid mechanics, such as the Mean curvature equation, the Hele-Shaw equation or the Thin-film equation. We provide a selection of classical Lyapunov functionals, expressed for a generic solution $\eta : [0,T] \times \mathbf{T} \times \mathbf{R}$ to a free surface problem on a periodic 2-dimensional problem,

$$\int_{\mathbf{T}} \eta^2, \quad \int_{\mathbf{T}} |\partial_x \eta|^2, \quad \int_{\mathbf{T}} \sqrt{1+|\partial_x \eta|^2}, \quad \int_{\mathbf{T}} \eta \log \eta. \tag{4.16}$$

The first one is the L^2 norm and often corresponds to the energy of the system. The second is the H^1 norm. The third is the curve length. The latter is called Boltzmann's entropy. Other integrals involving other powers, derivatives or quantities, such as the curvature, can also be Lyapunov functionals. Note that all of these are convex functionals with respect to η.

In the case of the graph interface evolution for the Stokes-transport equation, we could only prove that $\eta \mapsto \|\eta\|_{L^2}$ is a Lyapunov functional for the system. This fact is related to the potential energy monotonicity established in Subsection 3.6.2. We numerically studied other Lyapunov functionals, in particular the ones mentionned above, and found counterexemples disqualifying most of them as monotonous quantities, including the curve length.

Lemma 4.1.6. *As long as $t \mapsto \eta(t)$ exists, its average is constant in time, in the sense*

$$\int_{\mathbf{T}} \eta(t,x)\,\mathrm{d}x = \int_{\mathbf{T}} \eta_0(x)\,\mathrm{d}x.$$

Proof. This is a direct consequence of the measure preserving flow,

$$\int_{\mathbf{T}} \eta(t,x)\,\mathrm{d}x = \int_{\Omega} \mathbf{1}_{\{z \leq \eta(t,x)\}} = |\Omega^-(t)| = |\Omega_0^-| = \int_{\mathbf{T}} \eta_0(x)\,\mathrm{d}x.$$

□

We recall the decomposition $\eta = \bar{\eta} + \zeta$ with $\bar{\eta} = \int_{\mathbf{T}} \eta(t,x)\,\mathrm{d}x = \bar{\eta_0}$ and ζ having zero average.

The potential energy of the system, as introduced in 3.6.2, is linked to the L^2 norm of the graph. Indeed,

$$E(t) = \int_\Omega z\rho(t) = \int_{\Omega^-(t)} z = \int_{\mathbf{T}} \int_0^{\eta(t,x)} z \,\mathrm{d}z \,\mathrm{d}x = \frac{1}{2}\int_{\mathbf{T}} \eta(t,x)^2 \,\mathrm{d}x = \frac{1}{2}\|\eta(t)\|^2_{L^2(\mathbf{T})}.$$

In particular, we deduce right away from (3.63) the following

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|^2_{L^2} = \frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|\nabla \boldsymbol{u}\|^2_{L^2}.$$

We can also recover this equation by considering directly the energy estimate for the evolution equation (4.8) of η. Notice that $\|\eta\|^2_{L^2} = \bar{\eta}^2 + \|\zeta\|^2_{L^2}$ since $\bar{\eta}$ and ζ are orthogonal in L^2 due to the zero average of ζ. Then $\eta \mapsto \|\zeta\|_{L^2}$ is also a Lyapunov functional. Note that no decay rate is given here. To sum up, we proved the following.

Lemma 4.1.7. *The potential energy, and equivalently the L^2 norm of the graph, is a Lyapunov functional. Moreover, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|^2_{L^2} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta\|^2_{L^2} = -\|\nabla \boldsymbol{u}\|^2_{L^2}.$$

We did not identify formally any other Lyapunov functional. To illustrate the situation, let us attempt to estimate η in $H^1(\mathbf{T})$. Derivate (4.8) to get

$$\partial_t\partial_x\eta + (\partial_x u_1|_\eta + \partial_z u_1|_\eta \partial_x\eta)\partial_x\eta + u_1|_\eta \partial_x^2\eta = \partial_x u_2|_\eta + \partial_z u_2|_\eta \partial_x\eta,$$

since

$$\partial_x(u_i|_\eta) = \partial_x(u_i(t,\cdot,\eta)) = \partial_x u_i|_\eta + \partial_x\eta\partial_z u_i|_\eta.$$

The fact that the evolution of $\partial_x\eta$ depends on $\partial_x^2\eta$ does not allow a priori to get an estimate in $H^1(\mathbf{T})$ without assuming more regularity on η.

Long-time behaviour From the discussion on possible asymptotic states, we infer that the only possible stationary states ρ_∞ must be stratified and satisfy in particular

$$\int_\Omega z\rho_0^* \le \int_\Omega z\rho_\infty \le \int_\Omega z\rho_0,$$

where ρ_0^* is the vertical rearrangement of ρ_0 defined in Subsection 3.6.3. In the case $\rho_0 = \mathbf{1}_{\{z<\eta_0(x)\}}$, its vertically decreasing rearrangement is explicit, and we have $\rho_0^* = \mathbf{1}_{\{z<\bar{\eta}\}}$. If the asymptotic state is the patch of some hypograph of η_∞, then η_∞ is therefore constant and the above implies $\eta_\infty = \bar{\eta}$ since $\int_\Omega z\mathbf{1}_{\{z<h\}} h^2/2$ for $h \in (0,1)$ and since $\|\eta\|_{L^2} = \|\eta_0\|_{L^2}$.

$$\bar{\eta} = \|\bar{\eta}\|^2_{L^2(\mathbf{T})} \le \eta_\infty = \|\eta_\infty\|^2_{L^2(\mathbf{T})} \le \|\eta\|_{L^2} = \|\eta_0\|_{L^2}.$$

Note that we can not exclude other scenarios to occur, see Subsection 3.6.3 on the rearrangement and Figure 3.2 illustrating possible asymptotic profiles.

4.2 Numerical study

4.2.1 Results of reference

To our knowledge, the only theoretical work on graph interfaces for the Stokes-transport equation with gravity is [GGS22]. We summarise this work here and we compare our numerical observation to these theoretical results in Section 4.2. Other works involving the interface for Stokes flows exist, but involve capillarity effects. The evolution of a free interface for the Incompressible Porous Medium equation has been much more studied, and is known as the Muskat problem. An overview of this broader literature is reported in Subsection 1.4.3.

Gancedo, Granero-Belinchón and Salguero address in [GGS22] the free interface problem for gravity Stokes flow, corresponding to the system (4.8) on the 2-dimensional horizontally periodic domain $\mathbf{T}\times\mathbf{R}$. In all generality they express the interface as a curve, and consider the particular case of a graph. We will stick to the graph point of view in the following. With our notations, once the constants are renormalised, the equation writes as follows.

$$\partial_t\eta = \boldsymbol{u}|_\eta\cdot\boldsymbol{n}_\Sigma\langle\partial_x\eta\rangle, \qquad \text{on } \mathbf{T}, \tag{4.17a}$$
$$-\Delta\boldsymbol{u}^\pm + \nabla p^\pm = -\rho^\pm\boldsymbol{e}_z, \qquad \text{in } \Omega^\pm(t), \tag{4.17b}$$
$$\operatorname{div}\boldsymbol{u}^\pm = 0, \qquad \text{in } \Omega^\pm(t), \tag{4.17c}$$
$$[\![\nabla\boldsymbol{u}^\pm + (\nabla\boldsymbol{u}^\pm)^\perp - 2p^\pm I]\!]\boldsymbol{n}_\Sigma = \mathbf{0}, \qquad \text{on } \Sigma(t), \tag{4.17d}$$
$$[\![\boldsymbol{u}]\!] = 0, \qquad \text{on } \Sigma(t), \tag{4.17e}$$
$$\eta|_{t=0} = \eta_0, \tag{4.17f}$$

where $\rho^+,\rho^-\in\mathbf{R}$ and $[\![f]\!] := f^+_{|\Sigma} - f^-_{|\Sigma}$ denotes the jump of the quantity f at the interface point. Here equation (4.17a) corresponds to the formulation (4.9) of the graph interface evolution, presented at the end of Subsection 4.1.2. The graph is assumed to have an average of 0, so η corresponds to ζ in this case. Equations (4.17b) and (4.17c) are restrictions of the Stokes equations on the subdomains $\Omega^\pm$, completed with the continuity of the stress tensor (4.17d) and velocity field (4.17e) at the interface. These conditions are equivalent to the Stokes equation considered on the whole domain. We know in particular that the velocity field and the Stokes stress tensor are continous for bounded source terms. The Stokes tensor is

$$\frac{\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\perp}{2} - pI,$$

such that the Stokes equation rewrites

$$-\operatorname{div}\left(\frac{\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\perp}{2} - pI\right) = -\rho\boldsymbol{e}_z.$$

We report here [GGS22, Theorem 1], which contains both qualitative and quantitative results, which we compare our numerical observations to.

Theorem 4.2.1 (Gancedo, Granero-Belinchón & Salguero, '22)**.** *Take* $3/2 < m < 2$. *There exists* $\varepsilon_0 > 0$ *such that for any* $\zeta_0\in H^3(\mathbf{T})$ *satisfying*

$$\|\zeta_0\|_{H^3} < \varepsilon_0,$$

there exists a unique global solution $\zeta \in L^\infty(\mathbf{R}_+; H^3(\mathbf{T}))$ *to* (4.17)*, which moreover obeys*

$$(1+t)^m \|\zeta\|_{L^2} + \|\zeta\|_{H^3} \lesssim \|\zeta_0\|_{H^3}.$$

The proof of this theorem relies on a contour dynamic formulation of this problem, which is further explained in Subsection 1.4.3. In the first place, Gancedo, Granero-Belinchón & Salguero prove that any generic curve in $C^{1,\mu}$ for $\mu \in (0,1)$ remains a curve at least for short time. Then, Theorem 4.2.1 means that the flat profile of height 0 is asymptotically stable for small enough perturbations in H^3. They also provided a second global existence theorem for solutions of (4.17) in another class of regularity, in which the solution decays exponentially, see [GGS22, Theorems 2 and 3].

Of course the evolution occurs in their case in the unbounded domain $\mathbf{T} \times \mathbf{R}$, and the decay rates might differ in the domain $\Omega = \mathbf{T} \times (0,1)$ in which we simulate the evolution of the curve. See also Subsection 4.2.5 on the boundaries influence.

4.2.2 Motivation and approach

The aim of this section is to motivate and describe the approach adopted for the implementation of the code simulating the evolution of the interface.

The code has been implemented using the FEniCS library, see [Aln+15]. This library presents ready to use finite elements methods and mesh generation algorithms. The elliptic equations to solve can be implemented through their variational formulation, and the finite elements can be easily specified.

As we did not know in the first place whether or not a graph break up could occur, in the sense that the interface would cease to be a graph, we chose to implement the whole density transport, instead of considering the interface evolution equation (4.1), whose solution could become singular in finite time. For simplicity, we consider the numerical domain $\Omega_{\text{num}} = (0,3) \times (0,1)$ with horizontal periodic conditions, to represent $\Omega = \mathbf{T} \times (0,1)$.

Since we want to simulate a patch evolution, we wish to preserve this structure. First we provide an initial graph curve, that we discretise, providing a sequence of segments. In order to respect the horizontal periodic boundary condition, the first and last points of this sequence should have same vertical coordinate. A mesh for the whole domain Ω_{num} is then generated, comprising the segments of the interface. We keep track of the subdomains $\Omega^\pm_{\text{num}}$ corresponding to $\Omega^\pm$. Then, the density is implemented as a valuation on the mesh cells, with value 1 on the cells belonging to Ω^-_{num} and 0 on Ω^+_{num}. This step is illustrated in Figure 4.3, applied to a sinusoid interface as in Figure 4.1.

The Stokes equation is solved by finite elements using P2-P1 lagrangian elements for $\boldsymbol{u}$ and p respectively, and periodic boundary conditions. The Pk lagrangian elements are basically piecewise polynomial of degree k. P2 and P1 are adapted for $\boldsymbol{u}$ and p respectively since the source term ρ is in L^∞, we expect $(\boldsymbol{u}, p) \in W^{2,q} \times W^{1,q}$, according to Theorem 2.2.1.

The set of vertices is then transported following the computed velocity field. The density function, depending only on the cells and not on the coordinates in the domain, remains an indicator function. The time step should be small enough in order to avoid cells overlaps that could occur due to this process.

Finally, the Stokes problem solving and the mesh displacement are repeated as long as the mesh quality is good enough. In case the mesh quality is degraded, the points on the interface are extracted, and a new mesh is created as above.

Observe in Figure 4.4 the evolution of the solution to the interface problem for the initial sinusoidal initial profile illustrated Figure 4.3, with the above protocol.

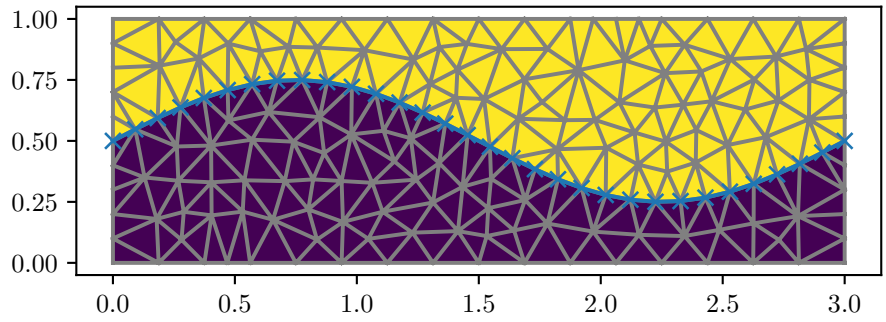

Figure 4.3: Sinusoidal interface and adapted mesh.

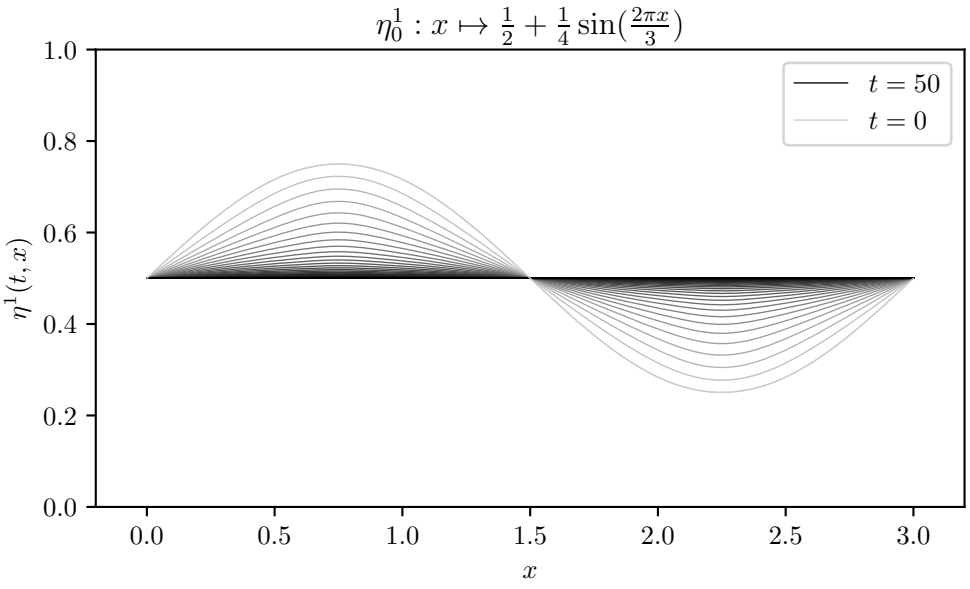

Figure 4.4: Graph evolution for a vertically centred sinus.

4.2.3 Parameters configuration

The aforementioned algorithm requires to set several parameters, among which the three following ones we will focus on:

- The interface resolution I_{res}. The interface is discretised in $3I_{\text{res}}$ points, hence the segments have typical length $1/I_{\text{res}}$.
- The mesh resolution M_{res}. The typical mesh edges length is $1/M_{\text{res}}$.
- The time step h.

For instance, Figure 4.3 has mesh resolution $I_{\text{res}} = 10$ and $M_{\text{res}} = 10$. Figure 4.4 is run with $I_{\text{res}} = 50, M_{\text{res}} = 50$ and $h = 0,05$. Our aim in this paragraph is to obtain a reference triplet $(I_{\text{res}}, M_{\text{res}}, h)$ for which the simulations are accurate enough for our purpose. The graph of reference is the sine from Figure 4.3, as it is smooth and symmetric with respect to the vertical coordinate, defined as

$$\eta_0^1 : x \in (0,3) \mapsto \frac{1}{2} + \frac{1}{4}\sin\left(\frac{2\pi x}{3}\right).$$

In the following, we will also consider $\zeta^i = \eta^i - \bar{\eta}^i$, the difference between η^i and its average $\bar{\eta}^i$. Now, we compare the influence of the ratio of the interface resolution and the mesh resolution, namely $I_{\text{res}}/M_{\text{res}}$. Since the problem relies entirely on the graph, we can expect the simulation to be more sensitive to the interface resolution than to the mesh resolution. In Figure 4.5, we plot the L^2 decay of ζ for several settings of I_{res} and M_{res} and a fixed time step $h = 0.1$. We observe that higher is I_{res} and higher is the ratio $I_{\text{res}}/M_{\text{res}}$, latter the numerical threshold is reached, occurring near $t = 10^2$ where the curve has a slope break.

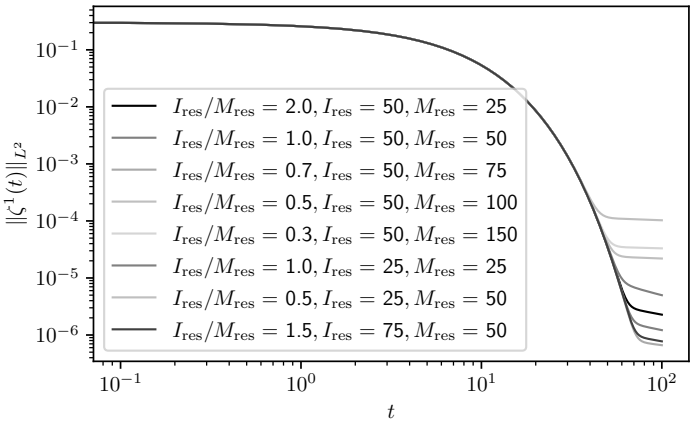

Figure 4.5: Decay of $\|\zeta^1\|_{L^2}$ for several I_{res} and M_{res}, with $h = 0.1$. Darkness increasing with $I_{\text{res}}/M_{\text{res}}$.

We run simulations for the smaller time step $h = 0.05$ and plot the evolution of the $\dot{H}^2$ norm of ζ, as this quantity should be more sensible to the different resolutions, see Figure 4.6. We observe that the red curve for which $I_{\text{res}} < M_{\text{res}}$ does not fit with the ones representing settings where $I_{\text{res}} \geq M_{\text{res}}$.

Finally, Figure 4.7 shows that there is no major difference in the evolution of $\|\zeta\|_{H^2}$ for $I_{\text{res}} \in \{25, 50, 75\}, M_{\text{res}} = 50$ and $h \in \{0.1, 0.05, 0.01, 0.005\}$. From now on we choose the generic set of parameters $I_{\text{res}} = 50, M_{\text{res}} = 50, h = 0,01$.

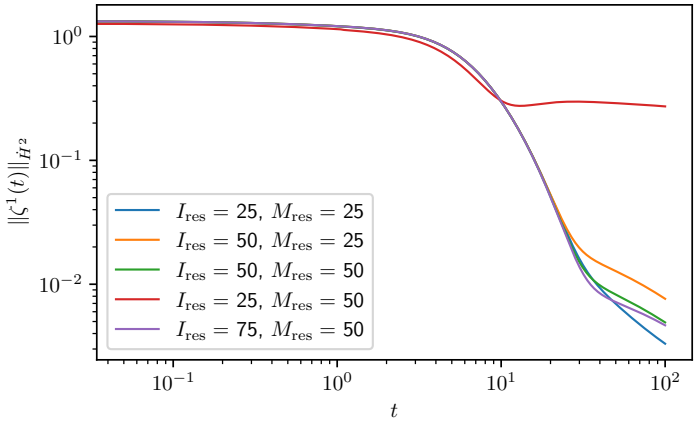

Figure 4.6: Decay of $\|\zeta^1\|_{\dot{H}^2}$ for $h = 0.05$.

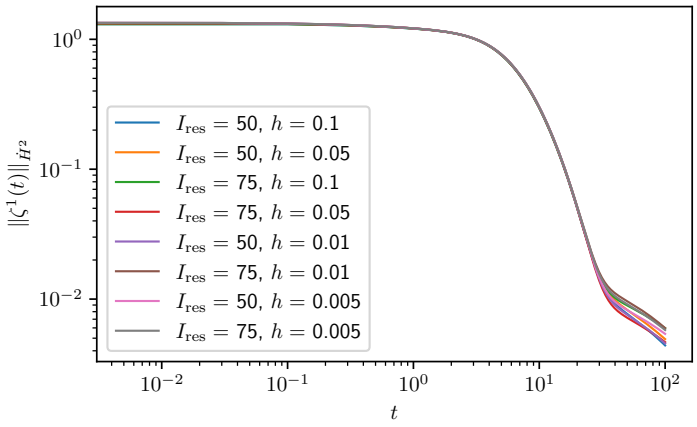

Figure 4.7: Decay of $\|\zeta^1\|_{\dot{H}^2}$ with $I_{\text{res}} = M_{\text{res}} = 50$ and several h.

4.2.4 Evolution rates

The aim of the present subsection is to consider regular initial data, and to compare the evolution of their quantities of interest. The case of graph break will be considered in Subsection 4.2.8. We consider four initial graphs, whose evolution is represented in Figure 4.8, with given expressions. From here, unless mentioned otherwise, the generic set of resolutions is $(I_{\text{res}}, M_{\text{res}}, h) = (50, 50, 0.01)$.

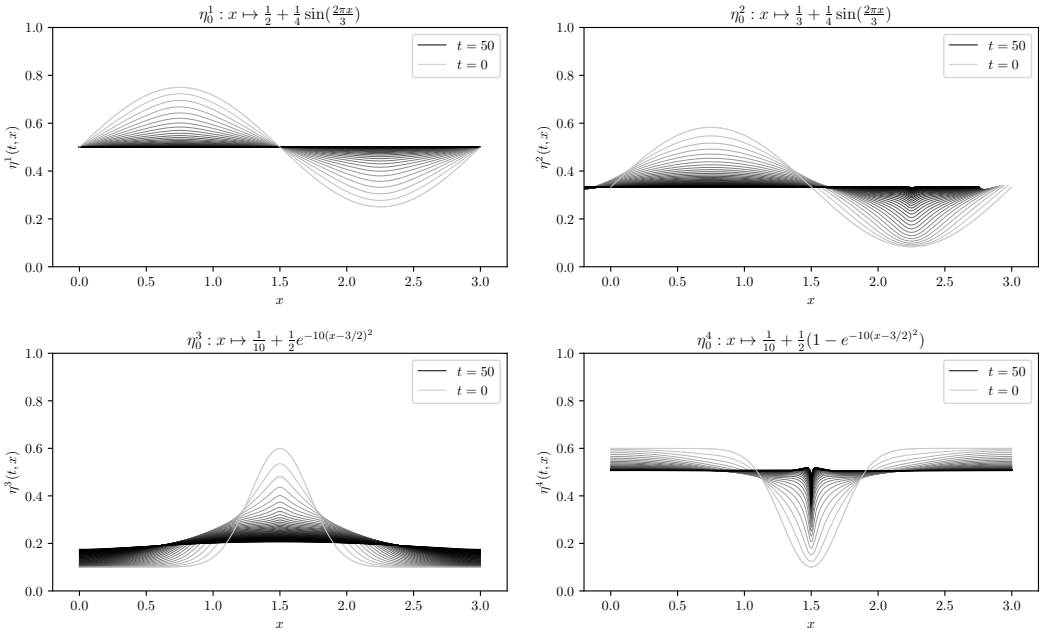

Figure 4.8: Graph interface evolution for several initial data.

We plot the evolution of the mass $\int_{\mathbf{T}} \eta$, the L^2, H^1 and H^2 norms of ζ and its length, for η_0^i with $i \in \{1, 2, 3, 4\}$, reported in Figure 4.9. Observe that the masses $\|\eta\|_{L^1}$ are well preserved, conformly to Lemma 4.1.6. The L^2 norm decays for all graphs, accordingly to Lemma 4.1.7. Interestingly, we observe that the $\dot{H}^1$ norm of η^4 is the only one not monotonous. This is surely linked to the corner formation occurring around $x = 1.5$. Even more insterestingly, the $\dot{H}^2$ norm of η^3 also grows before decaying. These behaviours are investigated and confirmed in Subsection 4.2.7, proving that these norms are not Lyapunov functional for the system. The $\dot{H}^3$ norm, supposedly bounded for data initially small, seems bounded except for η_0^4. Finally, the lengths of the curves are decreasing, except for η_0^4, which presents a hump. This also suggests that the length of η is not a Lyapunov function either, see Subsection 4.2.7.

According to Theorem 4.2.1, for data ζ_0 small enough in H^3, we have

$$\forall\, m \in \left(\frac{3}{2}, 2\right), \quad \|\zeta(t)\|_{L^2} \lesssim \frac{\|\zeta_0\|_{H^3}}{(1+t)^m}, \qquad \|\zeta(t)\|_{H^3} \lesssim \|\zeta_0\|_{H^3}. \tag{4.18}$$

By interpolation, we deduce that

$$\forall\, m' \in \left(1, \frac{4}{3}\right), \quad \|\zeta(t)\|_{H^1} \lesssim \frac{\|\zeta_0\|_{H^3}}{(1+t)^{m'}}, \qquad \forall\, m'' \in \left(\frac{1}{2}, \frac{2}{3}\right), \quad \|\zeta(t)\|_{H^2} \lesssim \frac{\|\zeta_0\|_{H^3}}{(1+t)^{m''}}. \tag{4.19}$$

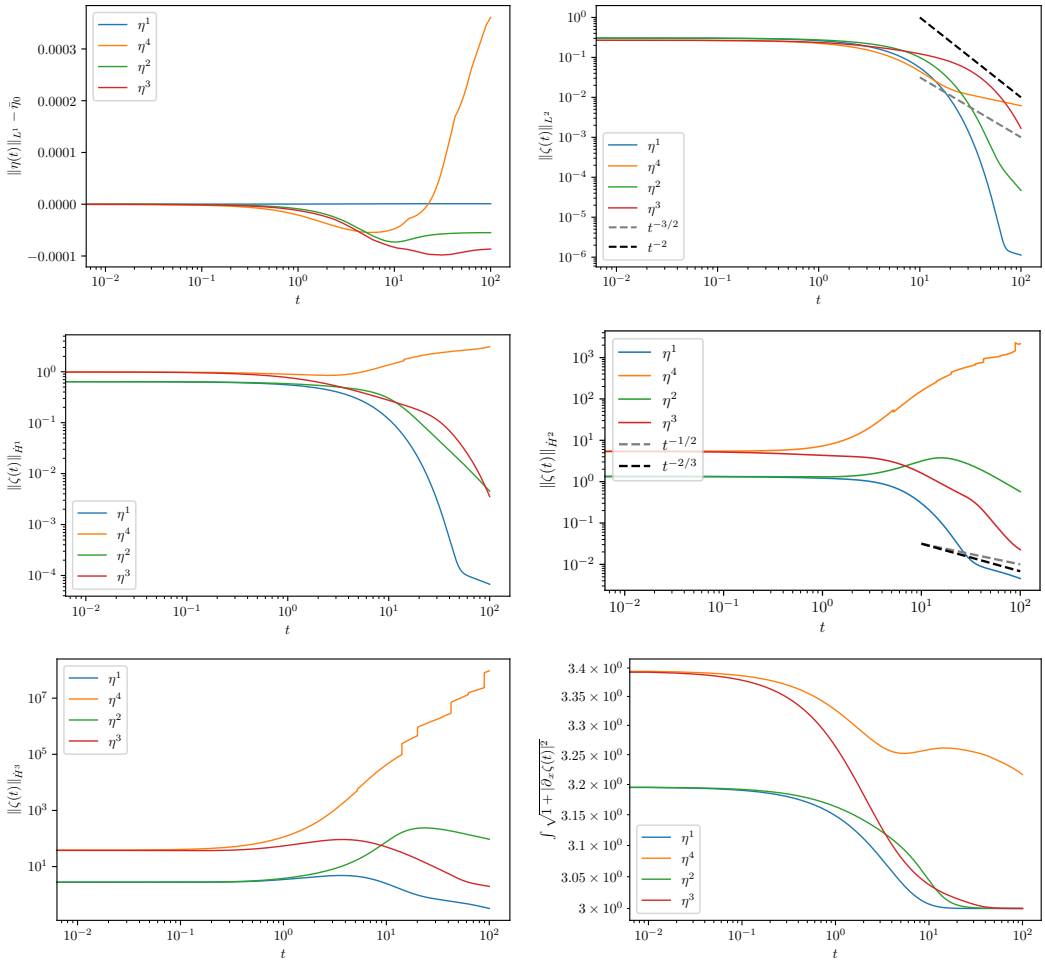

Figure 4.9: Several quantities evolution for η_i with $i \in \{1, 2, 3, 4\}$.

We observe that the decays in L^2, $\dot{H}^1$ and $\dot{H}^2$ for η^i occur for even faster rates than the maximal theoretical ones (4.18) and (4.19), available from [GGS22]. For η^1, the fastest to converge among our tests, the scheme seems to reach a numerical threshold, indicated by the slowdown before $t = 10^2$, which does not even respect the minimal decay rate (4.19) in $\dot{H}^1$, in particular. As mentioned before, the $\dot{H}^3$ norm may slightly increase for some time, then decrease again, accordingly to (4.18).

4.2.5 Boundaries influence

Observe that η_0^1 and η_0^2 are vertical translations of eachother, where η_0^1 is a vertically centred sinus, whereas η_0^2 is closer to the bottom of the domain. In $\mathbf{T} \times \mathbf{R}$, since the domain is invariant under vertical translation, the evolutions of η^1 and η^2 would be the same. However, in the confined channel $\Omega = \mathbf{T} \times (0, 1)$, the interfaces evolve differently. Their evolution is given in Figure 4.10. The curve of $\eta^1(t)$ is for all t centrally symmetric with respect to the point $(1.5, 0.5)$. The left part of $\eta^2(t)$ converges toward the flat state in the same way than $\eta^1(t)$, but the right part develops a small hump. We also observe that the Sobolev norms of η^2 locally increase, before decaying. Note that all the quantities decay faster for η^1 than for η^2, although both remain higher than the theoretical ones (4.18) and (4.19) obtained in [GGS22]. This explains by the Dirichlet condition $\boldsymbol{u}|_{\partial\Omega} = \mathbf{0}$, which makes the velocity decrease as we get closer to the boundary. Because the right hand-side of η^2 is closer to $\{z = 0\}$ than for η^1, the right hand-side extremum is slower than the rest of the interface, and this effect self-reinforces, causing the singularisation of the curve. This phenomenon might also explain the singularity formation we observe for η^4 in Figure 4.8. Indeed, η^3 and η^4 are similar, up to a vertical symmetry and a translation, and their evolutions differ essentially due to their distinct distances to the boundaries. Nevertheless, in all these case, the interfaces converge in L^2 toward the flat state.

These observations are relatable with the Muskat problem, whose infinite depth $\mathbf{R}^2$ and confined domain $\mathbf{R} \times (-1, 1)$ are compared in [GGI12], where Córdoba, Granero-Belinchón and Orive-Illera obtained numerical evidences that the confined problem is more singular than the infinite depth one, with initial data developping singularities in the former case, but becoming graphs in the latter.

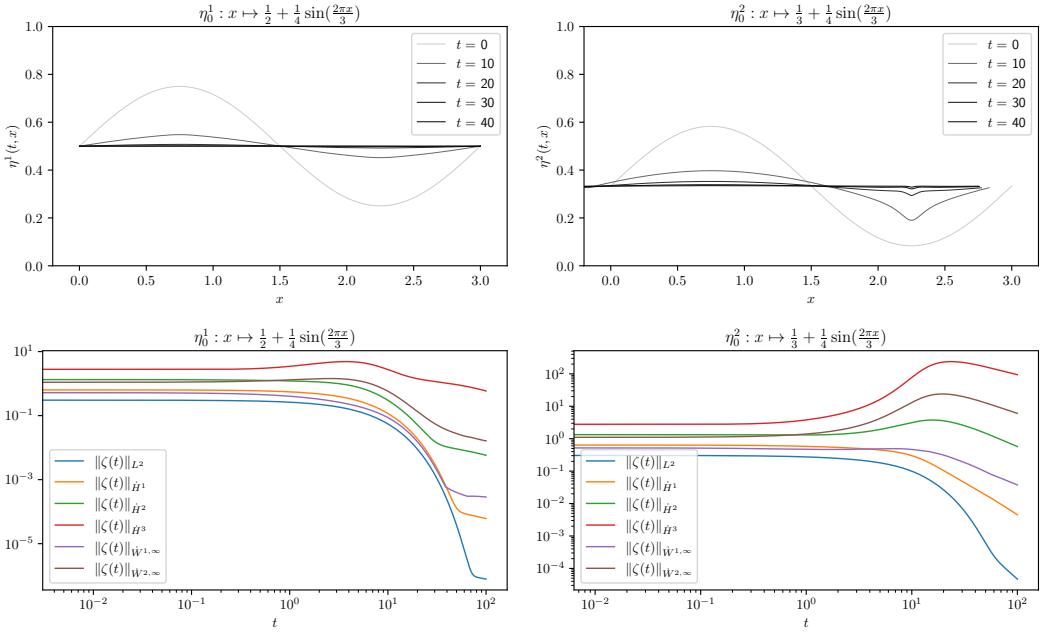

Figure 4.10: Evolution of η^1 and η^2 with their respective norms.

4.2.6 Maximum principle

In [CG09; GGI12], it is shown that the Muskat problem satisfies a maximum principle, in the sense that for any time t such that the solution exists, we have

$$\inf_x \eta_0 \leq \inf_x \eta(t) \leq \max_x \eta(t) \leq \max_x \eta_0.$$

This is true both for the Muskat problem considered in the domains $\mathbf{R}^2$, and in $\mathbf{R} \times (-1, 1)$ if $\|\partial_x \eta\|_{L^\infty} \leq 1$. We can wonder if such principle is satisfied for (4.1). To our knowledge, no theoretical answer to this question is available. See also Subsection 1.4.3 on the Muskat problem.

Considering in Figure 4.11 the evolution of the maximum and minimum for our four reference initial data, we observe that, except for one, these are monotonous, respectively decreasing and increasing, which is conform to the principle. The exception is the maximum of η^4, which we tested for higher resolutions in Figure 4.12. The curves seem to coincide for the different resolutions. Nevertheless, the behaviour modification around $t = 20$ is suspect and would require further investigation to ensure that the maximum indeed increases. Note that the condition $\|\partial_x \eta_0^4\|_{L^\infty} \leq 1$ on the datum, for which the maximum principle is ensured according to the aforementioned works, was not respected in the first place.

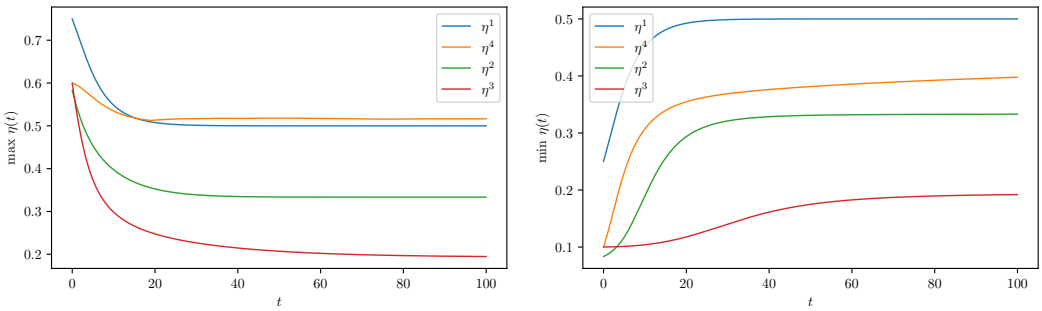

Figure 4.11: Evolution of the maximum and minimum of η^i for $i \in \{1, 2, 3, 4\}$.

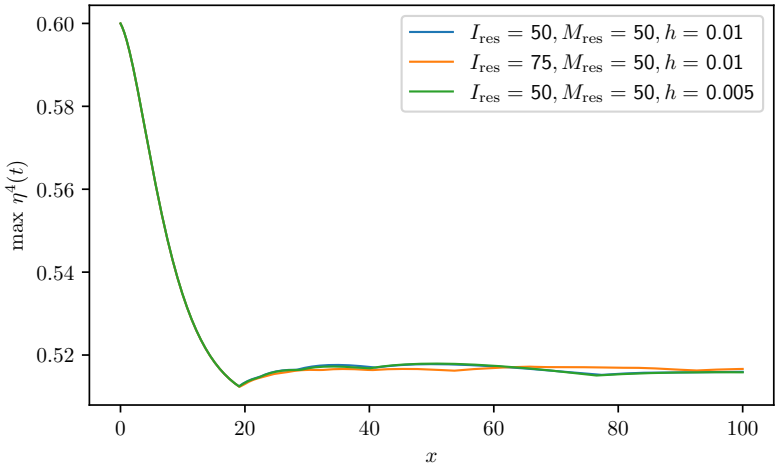

Figure 4.12: Evolution of the maximum of η^4 for several resolutions

4.2.7 Lyapunov functionals observations

The aim of this section is to numericaly observe the evolution of several quantities appearing classicaly as Lyapunov functional of free surface problems. As mentioned in Subsection 4.1.5, a Lyapunov functional defines as a monotonous quantity for the studied system. Among the functionals (4.16), only $\eta \mapsto \|\zeta\|_{L^2}$ is known to be a Lyapunov functional, according to Lemma 4.1.7. In all our figures, this quantity decays, see in particular Figure 4.9. In Subsection 4.2.4 and in Figure 4.9 we also have conterexamples of graph interfaces for which $\zeta \mapsto \|\partial_x \zeta\|_{L^2}$ and $\zeta \mapsto \|\partial_x^2 \zeta\|_{L^2}$ increase. Interestingly, the length of the curve is also excluded as the length of η^4 increases locally, see Figure 4.9. This behaviour confirms while increasing the space and time resolutions of the scheme in Figure 4.13. All the examples we tested also have monotonous

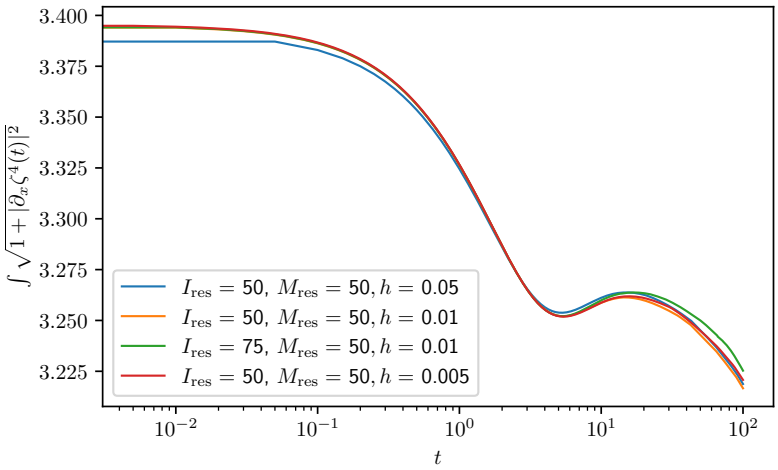

Figure 4.13: Evolution of the length of η^4 for several resolutions.

entropies $\eta \mapsto \int \eta \log(\eta)$, see Figure 4.14. At this point we can only suppose that it could be a Lyapunov functional. We sum up our conclusions in Table 4.1, with the functional proved to be Lyapunov ($\surd$), those for which we exhibited counter-examples ($\times$), and the one for which we could not (?).

	L^2-norm $\|\eta\|_{L^2}$	$\dot{H}^m$-norm, $m \in \{1,2,3\}$ $\|\eta\|_{\dot{H}^m}$	Length $\int_{\mathbf{T}} \sqrt{1+\lvert\partial_x \eta\rvert^2}$	Entropy $\int_{\mathbf{T}} \eta \log(\eta)$
Lyapunov	$\surd$	x	x	?

Table 4.1: Overview of functionals for the graph interface Stokes-transport problem in the periodic channel.

Note that we cannot exclude, from this study, that the observed behaviours, and in particular the non-monotonous functional counter-examples, would remain so in the unbounded domain $\mathbf{T} \times \mathbf{R}$.

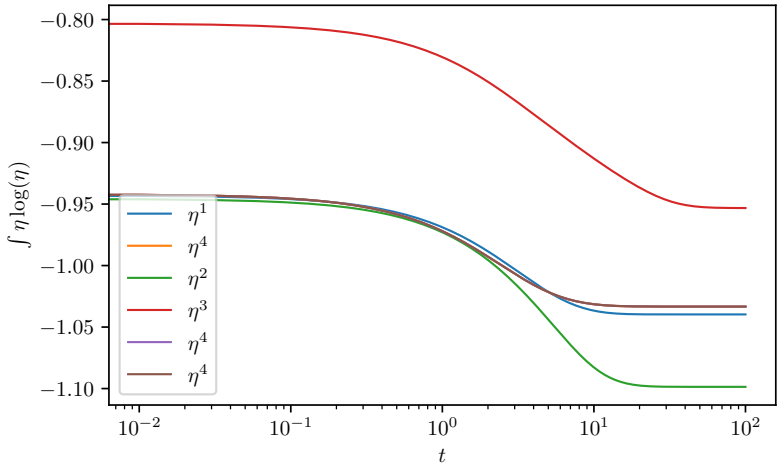

Figure 4.14: Evolution of the entropy of η^i.

4.2.8 Singularities and graph break

We found an example of regular graph such that the interface ceases to be a graph in finite time. Of course, the interface remains a curve, and our scheme, by construction and as described in Subsection 4.2.2, allows us to observe the evolution of the interface after the graph breaks. The initial datum defines as follows,

$$\eta_0^5 : x \mapsto \frac{1}{4} + 2e^{-100(x-1.5)^4} - 1.8e^{-200(x-1.5)^4}.$$

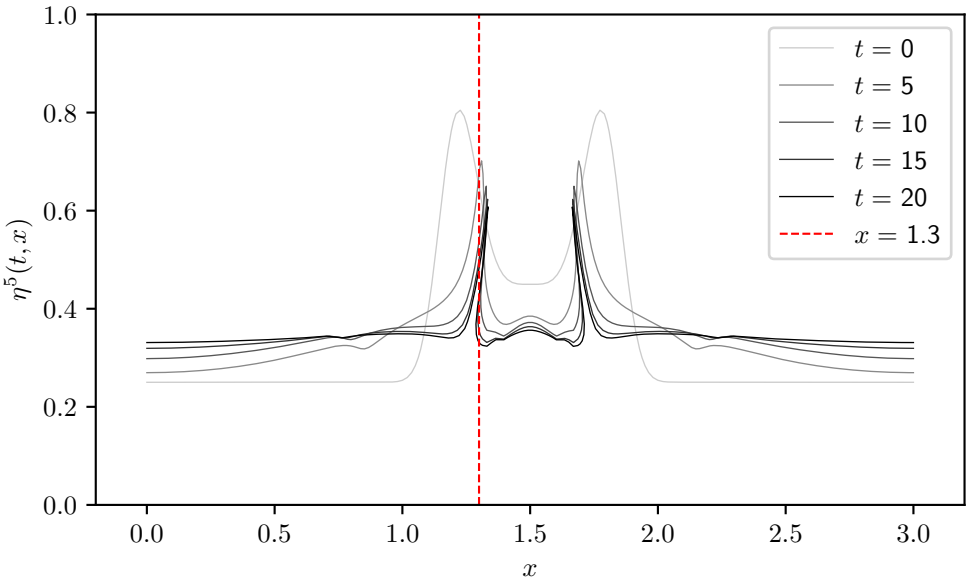

Figure 4.15: Evolution of η^5 with $I_{\text{res}} = M_{\text{res}} = 50$ and $h = 0.01$.

We observe that the interface is no longer a graph from a certain point prior to $t = 20$. The two humps seem to collapse on themselves. The first derivative of the graph also explodes in finite time, see Figure 4.16. We refined the resolutions to check that this behaviour is not due to some numerical approximation. All the obtained curves seem to coincide, which tends to confirm the graph break. In particular, we cannot expect to prove that (4.1) is globally well-posed for any initial graph, even for smooth ones.

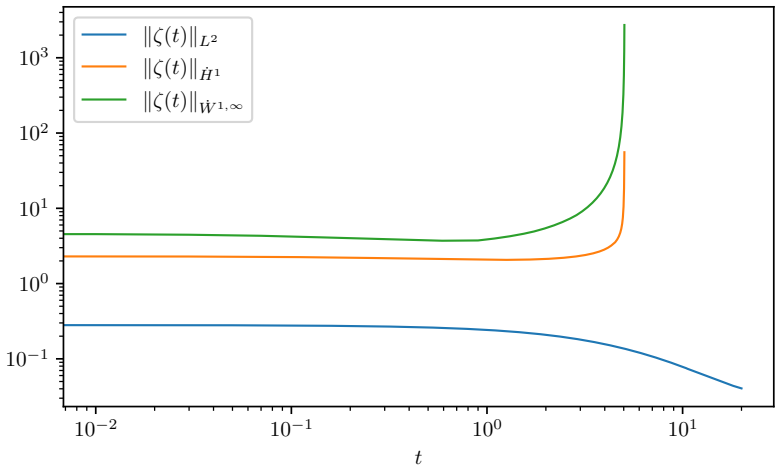

Figure 4.16: Evolution of η^5 norms on $[0, 20]$.

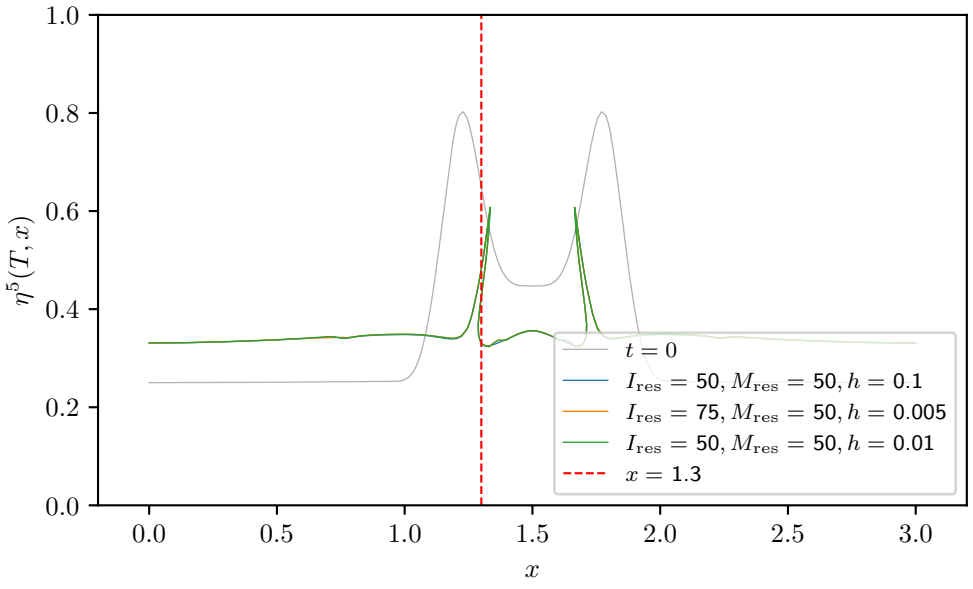

Figure 4.17: Plot of the interface of η^5 at time $T = 20$ for several resolutions.

Bibliography

[AB20] Thomas Alazard and Didier Bresch. "Functional inequalities and strong Lyapunov functionals for free surface flows in fluid dynamics". In: (2020). arXiv: 2004.03440.

[ABZ16] Thomas Alazard, Nicolas Burq, and Claude Zuily. "Cauchy theory for the gravity water waves system with non-localized initial data". In: *Annales de l'Institut Henri Poincare (C) Non Linear Analysis* 33.2 (Mar. 2016), pp. 337–395. DOI: https://doi.org/10.1016/j.anihpc.2014.10.004.

[AKZ23] Mustafa Sencer Aydın, Igor Kukavica, and Mohammed Ziane. *On Asymptotic Properties of the Boussinesq Equations*. 2023. arXiv: 2304.00481.

[Aln+15] Martin Alnæs et al. "The FEniCS project version 1.5". In: *Archive of numerical software* 3.100 (2015).

[AN21] Thomas Alazard and Quoc-Hung Nguyen. "On the Cauchy problem for the Muskat equation with non-Lipschitz initial data". In: *Communications in Partial Differential Equations* 46.11 (2021), pp. 2171–2212.

[AYM00] Stanislav Antontsev, Vadim Yurinsky, and Anvarbek Meirmanov. "A free-boundary problem for Stokes equations: Classical solutions". In: *Interfaces and Free Boundaries* 2.4 (2000), pp. 413–424.

[BCD11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*. Vol. 343. Springer, 2011.

[BD98] A Badea and J Duchon. "Capillary driven evolution of an interface between viscous fluids". In: *Nonlinear Analysis: Theory, Methods & Applications* 31.3-4 (1998), pp. 385–403.

[BF12] Franck Boyer and Pierre Fabrie. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*. Vol. 183. Springer Science & Business Media, 2012.

[BFL17] Animikh Biswas, Ciprian Foias, and Adam Larios. "On the attractor for the semi-dissipative Boussinesq equations". In: *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*. Vol. 34. 2. Elsevier. 2017, pp. 381–405.

[BG15] Hantaek Bae and Rafael Granero-Belinchón. "Global existence for some transport equations with nonlocal velocity". In: *Advances in Mathematics* 269 (2015), pp. 197–219.

[Bré11] Haim Brézis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer, 2011.

[CCL19a] Ángel Castro, Diego Córdoba, and Daniel Lear. "Global existence of quasi-stratified solutions for the confined IPM equation". In: *Archive for Rational Mechanics and Analysis* 232 (2019), pp. 437–471.

[CCL19b] Ángel Castro, Diego Córdoba, and Daniel Lear. "On the asymptotic stability of stratified solutions for the 2D Boussinesq equations with a velocity damping term". In: *Mathematical Models and Methods in Applied Sciences* 29.07 (2019), pp. 1227–1277.

[CFG11] Diego Córdoba, Daniel Faraco, and Francisco Gancedo. "Lack of uniqueness for weak solutions of the incompressible porous media equation". In: *Archive for Rational Mechanics and Analysis* 200.3 (2011), pp. 725–746.

[CG07] Diego Córdoba and Francisco Gancedo. "Contour dynamics of incompressible 3-D fluids in a porous medium with different densities". In: *Communications in Mathematical Physics* 273 (2007), pp. 445–471.

[CG09] Diego Córdoba and Francisco Gancedo. "A maximum principle for the Muskat problem for fluids with different densities". In: *Communications in Mathematical Physics* 286 (2009), pp. 681–696.

[CGO07] Diego Córdoba, Francisco Gancedo, and Rafael Orive. "Analytical behavior of two-dimensional incompressible flow in porous media". In: *Journal of Mathematical Physics* 48.6 (2007), p. 065206.

[CGS16] CH Arthur Cheng, Rafael Granero-Belinchón, and Steve Shkoller. "Well-posedness of the Muskat problem with H2 initial data". In: *Advances in Mathematics* 286 (2016), pp. 32–104.

[Che93] Jean-Yves Chemin. "Persistance de structures géométriques dans les fluides incompressibles bidimensionnels". In: *Annales scientifiques de l'Ecole normale supérieure*. Vol. 26. 4. 1993, pp. 517–542.

[Cob23] Dimitri Cobb. "On the Well-Posedness of a Fractional Stokes-Transport System". In: (2023). arXiv: 2301.10511.

[Con+12] Peter Constantin et al. "On the global existence for the Muskat problem". In: *Journal of the European Mathematical Society* 15.1 (2012), pp. 201–227.

[CT07] Chongsheng Cao and Edriss S Titi. "Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics". In: *Annals of Mathematics* (2007), pp. 245–267.

[CVW15] Peter Constantin, Vlad Vicol, and Jiahong Wu. "Analyticity of Lagrangian trajectories for well posed inviscid incompressible fluid models". In: *Advances in Mathematics* 285 (2015), pp. 352–393. ISSN: 0001-8708. DOI: 10.1016/j.aim.2015.05.019.

[DGL23] Anne-Laure Dalibard, Julien Guillod, and Antoine Leblond. *Long-time behavior of the Stokes-transport system in a channel*. 2023. arXiv: 2306.00780.

[Doe+18] Charles R Doering et al. "Long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion". In: *Physica D: Nonlinear Phenomena* 376 (2018), pp. 144–159.

[DP14] Anne-Laure Dalibard and Christophe Prange. "Well-posedness of the Stokes–Coriolis system in the half-space over a rough surface". In: *Analysis & PDE* 7.6 (2014), pp. 1253–1315.

[Elg17] Tarek M Elgindi. "On the asymptotic stability of stationary solutions of the inviscid incompressible porous medium equation". In: *Archive for Rational Mechanics and Analysis* 225 (2017), pp. 573–599.

[Gal11] Giovanni Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems*. Springer Science & Business Media, 2011.

[Gan17] Francisco Gancedo. "A survey for the Muskat problem and a new estimate". In: *SeMA Journal* 74.1 (2017), pp. 21–35.

[GGI12] Diego Córdoba Gazolaz, Rafael Granero-Belinchón, and Rafael Orive Illera. "The confined Muskat problem: differences with the deep water regime". In: *arXiv preprint arXiv:1209.1575* (2012).

[GGS10] Filippo Gazzola, Hans-Christoph Grunau, and Guido Sweers. *Polyharmonic boundary value problems: positivity preserving and nonlinear higher order elliptic equations in bounded domains*. Springer Science & Business Media, 2010.

[GGS22] Francisco Gancedo, Rafael Granero-Belinchón, and Elena Salguero. "Long time interface dynamics for gravity Stokes flow". arXiv:2211.03437. 2022.

[GL20] Rafael Granero-Belinchón and Omar Lazar. "Growth in the Muskat problem". In: *Mathematical Modelling of Natural Phenomena* 15 (2020), p. 7.

[GM10] David Gérard-Varet and Nader Masmoudi. "Relevance of the slip condition for fluid flows near an irregular boundary". In: *Communications in Mathematical Physics* 295.1 (2010), pp. 99–137.

[Gra22] Hezekiah Grayer II. "Dynamics of density patches in infinite Prandtl number convection". In: (2022). arXiv: 2207.09738.

[HKZ13] Weiwei Hu, Igor Kukavica, and Mohammed Ziane. "On the regularity for the Boussinesq equations in a bounded domain". In: *Journal of Mathematical Physics* 54.8 (2013), p. 081507.

[Höf18a] Richard M Höfer. "Sedimentation of inertialess particles in Stokes flows". In: *Communications in Mathematical Physics* 360 (2018), pp. 55–101.

[Höf18b] Richard M Höfer. "The Inertialess Limit of Particle Sedimentation Modeled by the Vlasov–Stokes Equations". In: *SIAM Journal on Mathematical Analysis* 50.5 (2018), pp. 5446–5476.

[HS21] Richard M. Höfer and Richard Schubert. "The influence of Einstein's effective viscosity on sedimentation at very small particle volume fraction". In: *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* (2021). ISSN: 0294-1449. DOI: https://doi.org/10.1016/j.anihpc.2021.02.001.

[HTZ02] Changbing Hu, R Temam, and M Ziane. "Regularity results for linear elliptic problems related to the primitive equations". In: *Chinese Annals of Mathematics* 23.02 (2002), pp. 277–292.

[Hu+16] Weiwei Hu et al. "Boussinesq equations with zero viscosity or zero diffusivity: A review". In: *Recent Progress in the Theory of the Euler and Navier-Stokes Equations, Part of London Mathematical Society Lecture Note Series* 430 (2016).

[Inv23] Marco Inversi. "Lagrangian solutions to the transport–Stokes system". In: (2023). arXiv: 2303.05797.

[IV15] Philip Isett and Vlad Vicol. "Hölder Continuous Solutions of Active Scalar Equations". In: *Annals of PDE* 1.1 (2015). DOI: 10.1007/s40818-015-0002-0.

[Jun+22] Johann H Jungclaus et al. "The ICON earth system model version 1.0". In: *Journal of Advances in Modeling Earth Systems* 14.4 (2022), e2021MS002813.

[Kor21] Peter Korn. "Global well-posedness of the ocean primitive equations with nonlinear thermodynamics". In: *Journal of Mathematical Fluid Mechanics* 23.3 (2021), p. 71.

[KPY22] Alexander Kiselev, Jaemin Park, and Yao Yao. "Small scale formation for the 2D Boussinesq equation". In: *arXiv preprint arXiv:2211.05070* (2022).

[KT23] Peter Korn and Edriss S Titi. "Global Well-Posedness of the Primitive Equations of Large-Scale Ocean Dynamics with the Gent-McWilliams-Redi Eddy Parametrization Model". In: (2023). arXiv: 2304.03242.

[KY23] Alexander Kiselev and Yao Yao. "Small scale formations in the incompressible porous media equation". In: *Archive for Rational Mechanics and Analysis* 247.1 (2023), p. 1.

[Leb22] Antoine Leblond. "Well-posedness of the Stokes-transport system in bounded domains and in the infinite strip". In: *Journal de Mathématiques Pures et Appliquées* 158 (2022), pp. 120–143. DOI: 10.1016/j.matpur.2021.10.006.

[LL01] Elliott H Lieb and Michael Loss. *Analysis*. Vol. 14. American Mathematical Soc., 2001.

[LM12] Jacques Louis Lions and Enrico Magenes. *Non-homogeneous boundary value problems and applications: Vol. 1*. Vol. 181. Springer Science & Business Media, 2012.

[LPZ11] Ming-Jun Lai, Ronghua Pan, and Kun Zhao. "Initial boundary value problem for two-dimensional viscous Boussinesq equations". In: *Archive for rational mechanics and analysis* 199 (2011), pp. 739–760.

[LS83] Olga A. Ladyzhenskaya and Vsevolod A. Solonnikov. "Determination of the solutions of boundary value problems for stationary stokes and Navier-Stokes equations having an unbounded Dirichlet integral". In: *Journal of Soviet Mathematics* 21.5 (Mar. 1983), pp. 728–761. DOI: 10.1007/bf01094437.

[LTW92a] J-L Lions, Roger Temam, and Shouhong Wang. "On the equations of the large-scale ocean". In: *Nonlinearity* 5.5 (1992), p. 1007.

[LTW92b] Jacques-Louis Lions, Roger Temam, and Shouhong Wang. "New formulations of the primitive equations of atmosphere and applications". In: *Nonlinearity* 5.2 (1992), p. 237.

[Ma23] Caochuan Ma. "Global well-posedness and optimal decay estimate for the incompressible porous medium equation near a nontrivial equilibrium". In: *Applied Mathematics and Computation* 440 (2023), p. 127680.

[Mec18] Amina Mecherbet. "Sedimentation of particles in Stokes flow". In: (2018). arXiv: 1806.07795.

[Mec20] Amina Mecherbet. "On the sedimentation of a droplet in Stokes flow". In: (2020). arXiv: 2007.02597.

[MP21] Bogdan–Vasile Matioc and Georg Prokert. "Two-phase Stokes flow by capillarity in full 2d space: an approach via hydrodynamic potentials". In: *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 151.6 (2021), pp. 1815–1845.

[MP22] Bogdan–Vasile Matioc and Georg Prokert. "Two-phase Stokes flow by capillarity in the plane: The case of different viscosities". In: *Nonlinear Differential Equations and Applications NoDEA* 29.5 (2022), p. 54.

[MS22] Amina Mecherbet and Franck Sueur. "A few remarks on the transport-Stokes system". In: (2022). arXiv: 2209.11637.

[Ped13] Joseph Pedlosky. *Geophysical fluid dynamics*. Springer Science & Business Media, 2013.

[San14] Filippo Santambrogio. *Introduction to optimal transport theory*. 2014.

[San15] Filippo Santambrogio. "Optimal transport for applied mathematicians". In: *Birkäuser, NY* 55.58-63 (2015), p. 94.

[Sar42] Arthur Sard. "The measure of the critical values of differentiable maps". In: (1942).

[Shv11] R. Shvydkoy. "Convex integration for a class of active scalar equations". In: *Journal of the American Mathematical Society* 24.4 (2011), pp. 1159–1174. DOI: 10.1090/s0894-0347-2011-00705-4.

[Szé12] László Székelyhidi Jr. "Relaxation of the incompressible porous media equation". In: *Annales scientifiques de l'École Normale Supérieure*. Vol. 45. 3. 2012, pp. 491–509.

[Tao+20] Lizheng Tao et al. "Stability near hydrostatic equilibrium to the 2D Boussinesq equations without thermal diffusion". In: *Archive for Rational Mechanics and Analysis* 237.2 (2020), pp. 585–630.

[Tem79] Roger Temam. *Navier-Stokes equations: theory and numerical analysis*. revised. North-Holland Publishing Company, 1979.

[Xue09] Liutang Xue. "On the well-posedness of incompressible flow in porous media with supercritical diffusion". In: *Applicable Analysis* 88.4 (2009), pp. 547–561.

[YH14] Wenxin Yu and Yigang He. "On the well-posedness of the incompressible porous media equation in Triebel-Lizorkin spaces". In: *Boundary Value Problems* 2014.1 (2014). DOI: 10.1186/1687-2770-2014-95.

[Yud63] Victor Iosifovich Yudovich. "Non-stationary flows of an ideal incompressible fluid". In: *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki* 3.6 (1963), pp. 1032–1066.

[Zia95] Mohammed Ziane. "Regularity results for Stokes type systems related to climatology". In: *Applied Mathematics Letters* 8.1 (1995), pp. 53–58.

Well-posedness and long-time behaviour of the Stokes-transport equation

Abstract

The Stokes-transport equation models an incompressible, viscous and inhomogeneous fluid, subject to gravity. It is a reduced model for oceanography and sedimentation. The density is transported by the velocity field, satisfying at any time the momentum balance between viscosity, pressure and gravity effects, namely the Stokes equation. In the first part, we establish the global well-posedness of this system in bounded domains and in the infinite channel, in the weak sense and for Lebesgue initial data. The unbounded channel case is solved in uniformly local Sobolev spaces, with solutions having infinite energy. These results are compared with previous works in the whole space and in the plane. In the second part, we focus on the long-time evolution of the solutions of the Stokes-transport equation in a periodic channel. We show that a class of monotonous stratified density profiles is stable for small and regular enough perturbations. We consider no-slip boundary conditions on the velocity field, which raises mathematical difficulties due to the presence of boundary effects. We obtain explicit algebraic convergence rates and show that the density rearranges vertically and monotonously, in line with the common intuition of sedimentation. We also give a refined description of the density profile, involving a boundary layer expansion in the vicinity of the boundaries. Besides, we extend a previous result obtained for a related problem, proving that any stationary profile is unstable in low regularity topologies. We also highlight properties compatible with the conjecture that the density always stratifies. In the last part, we undertake a numerical study of the evolution of graph density interfaces governed by the Stokes-transport equation. Several behaviours are observed, from the convergence toward the flat rest interface to the graph break. We compare our observations with existing theoretical results.

Keywords: incompressible viscous fluid, Stokes-transport, active scalar equation, global well-posedness, steady Stokes equation, transport equation, long-time behaviour, stability, rearrangement, boundary layers, numerical observation

---

Caractère bien posé et comportement en temps long de l'équation de Stokes-transport

Résumé

L'équation de Stokes-transport modélise un fluide incompressible, visqueux et inhomogène, soumis à la gravité. Il s'agit d'un modèle réduit d'océanographie et de sédimentation. La densité est transportée par le champ de vitesse du fluide, satisfaisant à tout instant l'équilibre entre les effets de viscosité, de pression et de gravité, d'après l'équation de Stokes. Dans la première partie, nous établissons le caractère bien posé de ce système dans les domaines bornés et dans un canal infini, au sens faible et pour des données intégrables. La cas du canal inclut des solutions d'énergie infinie, impliquant des espaces de fonctions uniformément localement Sobolev. Ces résultats sont comparés à des travaux antérieurs, dans l'espace et le plan. Dans la deuxième partie, nous nous concentrons sur le comportement en temps long des solutions de l'équation de Stokes-transport dans un canal périodique. Nous montrons qu'une classe de profils stratifiés est stable pour des perturbations assez petites et régulières. Nous supposons le fluide non-glissant aux bords, ce qui pose des problèmes particuliers dûs aux effets de bords induits. Nous obtenons des taux de convergence algébriques et montrons que la densité se réarrange verticalement et de façon monotone. Nous donnons également un développement de type couche limite du profil de densité à proximité des bords. En outre, nous prouvons, en adaptant un résultat antérieur, que tout profil stationnaire est instable pour des perturbations peu régulières. Nous mettons enfin en évidence des propriétés du système, compatibles avec la conjecture selon laquelle la densité tend toujours à se réordonner. Dans la dernière partie, nous menons une analyse numérique de l'évolution d'interfaces de densité de type graphe, gouvernée par l'équation de Stokes-transport. Plusieurs comportements sont observés, de la convergence vers l'équilibre plat à la rupture de graphe. Nous comparons nos observations à des résultats théoriques existants.

Mots clés : fluide visqueux incompressible, Stokes-transport, active scalar equation, caractère globalement bien-posé, équation de Stokes stationnaire, équation de transport, comportement en temps long, stabilité, réarrangement, couches limites, observation numérique

---

Laboratoire Jacques-Louis Lions
Sorbonne Université – Campus Pierre et Marie Curie – 4 place Jussieu – 75005 Paris – France