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Dimitri Le Meur

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Dimitri Le Meur. Sur l'étude dynamique des tranches paraboliques de l'espace des polynômes cubiques et leurs enrichissements. Analyse fonctionnelle [math.FA]. Université Paul Sabatier - Toulouse III, 2023. Français. NNT : 2023TOU30125 . tel-04356645

**HAL Id: tel-04356645**

**<https://theses.hal.science/tel-04356645>**

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# THÈSE

**En vue de l'obtention du  
DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE  
Délivré par l'Université Toulouse 3 - Paul Sabatier**

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**Présentée et soutenue par  
Dimitri LE MEUR**

Le 13 juillet 2023

**Sur l'étude dynamique des tranches paraboliques de l'espace des  
polynômes cubiques et leurs enrichissements**

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Ecole doctorale : **EDMITT - Ecole Doctorale Mathématiques, Informatique et  
Télécommunications de Toulouse**

Spécialité : **Mathématiques et Applications**

Unité de recherche :  
**IMT : Institut de Mathématiques de Toulouse**

Thèse dirigée par  
**Arnaud CHERITAT et Pascale ROESCH**

Jury

**M. Marco ABATE**, Rapporteur  
**M. Christian HENRIKSEN**, Rapporteur  
**Mme Anna Miriam BENINI**, Examinatrice  
**M. Arnaud CHERITAT**, Directeur de thèse  
**Mme Pascale ROESCH**, Co-directrice de thèse  
**Mme Jasmin RAISSY**, Présidente

## Abstract

Douady, Lavaurs, A. Epstein, Shishikura, Lanford III, Yampolsky and others have discovered a class of functions which is invariant by parabolic renormalisation. This class is characterized by covering properties, contains the polynomials of degree 2 and enabled among other things to progress in the study of their dynamics. To date, the extension of these results for all polynomials of degree 3 has not been achieved, nevertheless there exist some elements to develop such a theory.

In this thesis, we start the study of these elements. We propose to extend the definition of this class of functions to include the polynomials of degree 3. This extension separates naturally, in degree 3, in 3 types of classes.

Since the candidates to be invariant by parabolic renormalisation are defined by covering properties, we study in the fourth part the relations between a notion of covering equivalence of horn maps of certain parabolic points and a notion of conjugacy associated to these parabolic points.

In part 2, we recall how to parameterise the slices of polynomials of degree 3 which fix a point  $z_0$  with multiplier  $\rho$ . Then we study the external rays that can end at  $z_0$ , when  $\rho$  is a root of unity. We propose a combinatorial way of enumerating orbit portraits.

## Résumé

Douady, Lavaurs, A. Epstein, Shishikura, Lanford III, Yampolsky et d'autres ont mis à jour une classe de fonctions qui est invariante par renormalisation parabolique. Cette classe est caractérisée par des propriétés de revêtement, contient les polynômes de degré deux, et a permis notamment de faire des progrès dans l'étude de leur dynamique. A ce jour l'extension de ces résultats à tous les polynômes de degré 3 n'a pas été réalisée de façon satisfaisante, mais des pistes existent pour développer une telle théorie.

Dans cette thèse, nous commençons l'étude d'une telle piste. Nous proposons d'étendre la définition de cette classe de fonctions pour inclure les polynômes de degré 3. Cette extension se scinde naturellement, en degré 3, en trois types de classes.

Les candidats à être invariant par renormalisation parabolique étant définis par des propriétés de revêtement, nous étudions en quatrième partie les relations entre une notion d'équivalence de revêtement des applications de cornes de certains points paraboliques et une notion de conjugaison associée à ces points paraboliques.

En partie 2, nous rappelons comment paramétrer les tranches des polynômes de degré 3 qui fixent un point  $z_0$  avec multiplicateur  $\rho$ . Puis, nous étudions les rayons externes qui peuvent aboutir en  $z_0$ , quand  $\rho$  est une racine de l'unité. Nous proposons une manière combinatoire d'énumérer les portraits d'orbite.

*Monsieur Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels. Un philosophe tel que lui aurait dû savoir que le but unique de la Science, c'est l'honneur de l'esprit humain et que, sous ce titre, une question de nombres vaut bien une question de système du monde.*

*Gustav Jacobi, 1830.*

## Remerciements

Je tiens d'abord à exprimer toute ma gratitude envers Arnaud, mon directeur de thèse, pour son soutien infaillible, ses conseils avisés et son encadrement tout au long de mon travail de recherche. Je le remercie pour ses commentaires constructifs, son optimisme, son soutien et sa disponibilité à toute épreuve pour répondre à mes questions et résoudre mes préoccupations. Grâce à ses encouragements et sa direction, j'ai été non seulement en mesure de mener à bien ce projet, d'atteindre mes objectifs de recherche, mais aussi de compléter ma formation en tant que mathématicien. Arnaud est pour moi à la fois un mathématicien modèle, ainsi qu'un exemple en termes d'humanité et de générosité. Il a su rester juste et droit malgré les difficultés humaines qui se sont présentées. J'espère que nous aurons l'occasion de rester en contact après la thèse, voire peut-être de devenir amis une fois que le rapport directeur/doctorant ne sera plus présent.

Je remercie ensuite Pascale, ma codirectrice de thèse, pour son soutien. Sa contribution à mon travail a été unique, et a contribué à sa qualité. Je suis reconnaissant pour la liberté qu'elle m'a laissé, et je suis fier d'avoir travaillé avec elle.

Je souhaite également remercier les rapporteurs, Christian Henriksen et Marco Abate, pour leur temps, leur expertise et leur contribution à l'amélioration de mon travail. Leurs commentaires et leurs suggestions ont été précieux et ont permis d'améliorer considérablement ma thèse. Je remercie ensuite Jasmin Raissy et Anna Miriam Benini, qui ont bien voulu faire partie du jury.

J'exprime ma gratitude à Vincent Guedj, et Henri Guenancia, mes anciens encadrants de thèse. Je les remercie sincèrement d'avoir accepté mon encadrement, alors que ma demande s'est faite presque au dernier moment, et de m'avoir guidé de façon inappréciable lors de mon choix de mon nouveau sujet. Je remercie Henri d'avoir accepté d'être mon garant pour mon appartement. Je remercie mes anciens collègues, Dat et Benoît, avec lesquels j'ai beaucoup appris.

Je suis reconnaissant à Adrien Douady, mon grand-père de thèse, par lequel je tire indirectement des enseignements, que ce soit par l'intermédiaire d'Arnaud, ou plus généralement par l'étude des systèmes dynamiques holomorphes dont il est l'un des pionniers.

Je remercie également John Hamal Hubbard, que j'ai eu l'occasion de rencontrer en personne, et qui par les notes d'Orsay coécrites avec Douady m'a motivé à prendre la décision de changer de domaine pour faire de la dynamique holomorphe.

Je tiens à remercier sincèrement Xavier Buff et Jasmin Raissy pour leur aide précieuse pendant ma thèse. Leur expertise mathématique et leur disponibilité m'ont été d'une grande aide dans la réussite de mon travail.

Je tiens à exprimer ma profonde gratitude envers mes anciens professeurs, qui ont

joué un rôle déterminant dans ma scolarité et dans mon développement personnel et professionnel. Je suis particulièrement reconnaissant envers Mme Serfati, Julien Clark, Olivier Bouverot, Philippe Chateaux, qui m'ont enseigné les mathématiques respectivement au lycée et en classes préparatoires. Leurs connaissances, leur dévouement et leur passion pour l'enseignement ont été une source d'inspiration pour moi et m'ont aidé à développer mes compétences académiques et personnelles. Je suis extrêmement reconnaissant pour leur aide et leur soutien, et je leur suis redevable pour leur contribution à ma formation et à ma réussite professionnelle. Merci infiniment à tous mes anciens professeurs.

Je remercie Raphaël Cerf de m'avoir guidé malgré l'urgence vers mon premier sujet de thèse, Emmanuel Paul de m'avoir accompagné lors de mon premier changement de thèse, et Bertrand Monthubert de m'avoir soutenu pour l'après-thèse.

Je remercie Agnès Manoretonil, Anne Hélène Cunningem et Xavier Passot de m'avoir apporté un soutien inestimable pour mon déménagement à Toulouse, et pour avoir conservé ensuite une bonne amitié. Je remercie tout particulièrement Anne Hélène qui m'a hébergé durant quelques mois le temps que je trouve un appartement.

Je suis également reconnaissant envers mes collègues de recherche, d'une part Runze, Valentin, Tu, ceux qui travaillent dans le même domaine que moi, et d'autre Sara, Florian, Elena, Bruno, William, Maxence, Jean-Marc, Sonny, Bruno, Dominique, Maximo et plus récemment Yekta, Mohammad, Baalu, qui m'ont apporté leur soutien, leur aide et leur encouragement tout au long de ma recherche. Leur soutien a été essentiel pour moi et m'a aidé à surmonter les défis que j'ai rencontrés.

Je tiens à exprimer ma profonde gratitude envers ma famille, David, Jacqueline, Noël, Eric, Zaga, Alexandre, mes amis, Laura, Camille, Anne-Laure, Sion, Soukayna, Titouan, Flavien, Timothée, Philippe, Iris, ainsi que les responsables de l'institut de mathématiques Laurent Manivel et Marie Hélène Vignal. Votre soutien et votre présence à mes côtés ont été d'une grande aide et m'ont permis de garder le moral dans les moments difficiles.

Je voulais vous remercier pour votre générosité, votre bienveillance et votre amitié. Je suis fier et chanceux d'avoir des personnes aussi exceptionnelles dans ma vie. Encore une fois, merci infiniment pour tout ce que vous avez fait pour moi.

Je remercie sincèrement toutes les personnes qui ont contribué à ma thèse, qu'elles soient nommées ici ou non. Sans leur aide, leur soutien et leur encouragement, je n'aurais pas été en mesure de mener à bien ce travail de recherche.

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# 1 Introduction

Le but de notre thèse est de donner des éléments préliminaires à l'étude des perturbations et les enrichissements d'une tranche parabolique du lieu de connexité des polynômes de degré 3. Ce lieu de connexité est un objet analogue à l'ensemble de Mandelbrot. On s'attardera dans cette introduction à contextualiser ce but, à donner quelques conjectures, et à présenter les théorèmes dont la démonstration fera l'objet des parties suivantes.

## 1.1 Quelques propriétés et conjectures sur l'ensemble de Mandelbrot

On rappelle quelques propriétés et conjectures sur l'ensemble de Mandelbrot, ce qui permettra de mieux guider l'étude du lieu de connexité des polynômes de degré 3.

- L'ensemble de Mandelbrot  $M$  est connexe. Ce résultat doit être attribué à Douady, Hubbard [DHL<sup>+</sup>84] ainsi qu'à Sibony, non publié.
- Conjecture (MLC) :  $M$  est localement connexe. Cette conjecture est mentionnée dans [DHL<sup>+</sup>84].

Il existe un modèle topologico-combinatoire explicite de l'ensemble de Mandelbrot qui est égal à l'ensemble de Mandelbrot si et seulement si ce dernier est localement connexe. Si l'ensemble de Mandelbrot n'est pas localement connexe, il se surjecte continûment dans ce modèle topologico-combinatoire. Voir [Thu09].

On partitionne l'espace en composantes stables (i.e composantes où l'ensemble de Julia admet un mouvement holomorphe). On montre que  $\mathbb{C} \setminus \partial M$  est l'union de ces composantes, l'union des composantes bornées étant  $\mathring{M}$ . On note  $M'$  l'union des composantes hyperboliques, formée par les paramètres admettant un cycle attractif (ce qui permet par définition de définir explicitement des équations de  $M'$ ).

- Conjecture (HG2) :  $\mathring{M} = M'$ . Il est connu que (MLC) implique (HG2), voir [DHL<sup>+</sup>84].
- L'ensemble de Mandelbrot est localement connexe en tout point  $c$  tel que  $P_c$  admet un point fixe indifférent. Ce résultat est attribué à Yoccoz par Hubbard dans [Hub93].
- $M$  est localement connexe en tout point  $c$  tel que  $P_c$  n'est pas infiniment renormalisable. Voir le même article [Hub93].
- $\mathring{M} \cap \mathbb{R} = M' \cap \mathbb{R}$  (Graczyk et Swiatek [GS97], et indépendamment Lyubich [Lyu97])
- Pour  $c \in \mathring{M} \setminus M'$  (ce qui conjecturalement n'existe pas),  $K_c$  est d'intérieur vide, de mesure de Lebesgue strictement positive et supporte une forme de Beltrami invariante (Mane, Sad, Sullivan [MnSS83]).

- Tout rayon externe d'argument rationnel aboutit. La situation est différente selon le cas dénominateur pair ou impair. Dans le cas pair, on a un point de Misurewicz (0 tombe en temps fini dans un cycle répulsif pour  $P_c$ ). Dans le cas impair,  $P_c$  admet un cycle indifférent rationnel. Voir [DHL<sup>+</sup>84] ainsi que [Pom86].

## 1.2 Généralités sur le lieu de connexité des polynômes de degré 3

Avant de s'intéresser à ses tranches paraboliques, donnons quelques résultats généraux valables sur le lieu de connexité  $\mathcal{C}_3$  des polynômes de degré 3.

- $\mathcal{C}_3$  est connexe, voir [Lav].
- $\mathcal{C}_3$  n'est pas localement connexe, par la thèse de Lavours [Lav], exploitant le phénomène d'implosion parabolique.
- $\mathcal{C}_3$  contient des copies de l'ensemble de Mandelbrot de façon dense sur la frontière, dans chaque tranche. De façon plus générale, il existe une universalité de l'ensemble de Mandelbrot, qui apparaît autour de chaque paramètre parabolique d'un espace de paramètres raisonnable ; il s'agit de l'article fameux de MacMullen, apparaissant dans le livre de Tan Lei [Lei00].

Contrairement à l'ensemble de Mandelbrot, on travaille dans un sous-ensemble de  $\mathbb{C}^2$  et non  $\mathbb{C}$  ; ceci n'est pas une différence anodine.

Par un travail de Milnor [Mil12] et [Mil92], les composantes hyperboliques de  $\mathcal{C}_3$  sont classifiées en quatre types de composantes, à savoir A, B, C, D.

- A : Adjacent, points critiques dans la même composante connexe du même bassin attractif.
- B : Bitransitif. Points critiques dans deux composantes connexes du bassin différents, envoyés l'un sur l'autre par des itérées de l'application  $f$ .
- C : Capture. Points critiques dans deux composantes connexes du bassin différents, l'un étant envoyé sur l'autre par un itéré de  $f$ , mais pas l'autre sur le premier.
- D : Disjoint. Points critiques dans deux bassins attractifs différents.

La pertinence de la décomposition repose sur ce fait : le nom de la composante permet de la caractériser à isomorphisme holomorphe près.

Voici les outils d'analyse complexe dont on dispose en dimension complexe 1 mais pas dans les dimensions supérieures :

- Théorème de représentation conforme de Riemann, topologie du plan (par exemple les compacts pleins du plan connexes et localement connexes sont les quotients du disque par des laminations, ceci étant donné par l'étude des rayons externes, voir [Dou93]). Rayons externes, puzzles, laminations.



- Théorème de représentation conforme mesurable. Voir Ahlfors, Bers [AB60] ainsi que Morrey pour une démonstration sans dépendance analytique [Mor38].
- Théorème de Montel (en dimension supérieure, il existe des analogues, mais plus faibles).

Ceci motive le fait que, pour étudier cet ensemble, on étudie différentes tranches et leur mouvements. On peut espérer par ces moyens obtenir des résultats qu'une approche globale ne permettrait pas. On partitionne le lieu de connexité en tranches avec point fixe/périodique attractif ou indifférent de multiplicateur fixé (remarquer que dans le cas où tout les cycles sont répulsifs, on est en dehors du lieu de connexité). Ceci permet d'utiliser dans chaque tranche des méthodes connues d'étude de lieu de bifurcation dans l'espace des paramètres, qui reposent sur la (ou les) position du (ou des) point critique.

Dans l'ensemble de Mandelbrot, on déplace un point du plan des paramètres, et on regarde comment se meut l'ensemble de Julia. Dans le lieu de connexité de degré 3, il y a deux mouvements possibles :

- On prend un point mobile sur la tranche (ou plus généralement sur le lieu de connexité, en s'autorisant à changer de tranche), on regarde comment l'ensemble de Julia se meut.
- On change de tranche, on regarde comment la tranche se meut.

### 1.3 Généralités d'analyse complexe

**Lemme 1.3.1** (Lemme 1/4 de Koebe). *L'image d'une fonction holomorphe injective  $f : \mathbb{D} \rightarrow \mathbb{C}$  contient un disque dont le centre est  $f(0)$  et dont le rayon est  $|f'(0)|/4$ .*

### 1.4 Portraits d'orbite

On présente ici le théorème 1.4.1 qui sera démontré en section 2.

Soit  $f$  une fonction holomorphe définie sur  $\mathbb{C}$ . Un portrait d'orbite se définit informellement par la forme, à transformation bicontinue près préservant les angles à l'infini, des rayons externes qui aboutissent sur une orbite de  $f$ . De façon équivalente, il est donné par les angles à l'infini et la manière dont ils doivent se recoller entre eux. Comme on le démontrera, le nombre de cycles dans la dynamique de la fonction  $f$  sur les rayons est limité si on travaille sur un certain type d'orbite, dit orbite effective; on démontrera également que dans ce cadre, les cycles ont tous la même taille.

On définit pour  $\lambda \in \mathbb{C}$  l'ensemble  $\text{Per}_1(\lambda)$  comme étant le lieu des polynômes  $P$  de degré 3 admettant un point parabolique fixe de multiplicateur  $\lambda$ , On se restreindra à la tranche parabolique  $\text{Per}_1(\lambda)$ , où  $\lambda = e^{2i\pi \frac{p}{q}}$ ,  $p \in \mathbb{Z}$ ,  $q \in \mathbb{N}^*$  des polynômes de degré 3, où  $\mathbb{N}^*$  dénote l'ensemble des entiers naturels strictement positifs (nous utiliserons cette convention française même dans les parties en anglais). L'existence et l'inexistence d'un certain portrait d'orbite dans le plan dynamique sont stables tant que le paramètre évite certains rayons du plan des paramètres, nommés rayons caractéristiques du portrait

d'orbite. Ceci implique topologiquement que les rayons caractéristiques en question vont séparer le plan des paramètres en plusieurs composantes connexes (la démonstration est non constructive). En travaillant un petit peu plus on démontre qu'ils coaboutissent.

Plus précisément, le paramètre appartient au rayon caractéristique lorsque le portrait d'orbites se brise dans le plan dynamique : un des rayons du portrait se brise sur un point critique de la coordonnée de Botcher (c'est à dire sur un point (pré-)critique de  $f$ ). Ou les rayons aboutissent en un cycle parabolique non persistant (et se séparent en un cycle attractif entouré par un cycle répulsif dont la taille est multiple de celle du cycle attractif).

Pour motiver l'étude des portraits d'orbites, notons encore que l'étude de l'aboutissement et du coaboutissement des rayons externes est quelque chose d'essentiel pour étudier les ensembles de Julia et le lieu de connexité : ceci permet de les voir comme quotient du disque par une certaine géolamination. La question de l'existence des portraits d'orbites donne des informations sur cette géolamination.

On introduit les portraits d'orbites marqués, aboutissant sur le point fixe marqué 0 du polynôme  $P^{\lambda,a} = \lambda z + az^2 + z^3$ . On introduira en section 2 les portraits d'orbites marqués formels. On démontrera que les portraits d'orbite marqués sont des portraits d'orbites marqués formels, et on conjecture que la réciproque est vraie. Voici le théorème que l'on démontre, où  $p' \in \mathbb{N}$  est un inverse de  $p$  modulo  $q$  :

**Théorème 1.4.1.** *Les portraits d'orbites marqués formels de nombre de rotation  $t = \frac{1}{q}$  sont les singletons  $\{A\}$  de  $P(\mathbb{R}/\mathbb{Z})$  avec  $A = \{\Theta_k^t\}$  ou  $A = \{\Theta_k^t, \Theta_{k+1}^t\}$ , et avec  $\Theta_k^t$  orbite de  $\theta_k^t = \frac{1}{3^q-1} + \frac{\sum_{s=0}^{k-1} 3^s}{3^q-1}$  où  $k \in \llbracket 0, q \rrbracket$  (convention somme nulle si  $k = 0$ ). La notation  $\llbracket 0, q \rrbracket$  signifie l'ensemble des entiers compris entre 0 et  $q$  inclus.*

*Les portraits d'orbites marqués formels de nombre de rotation  $t = \frac{p}{q}$ , où  $p$  premier avec  $q$  sont les singletons  $\{A\}$  de  $P(\mathbb{R}/\mathbb{Z})$  avec  $A = \{\Theta_k^t\}$  ou  $A = \{\Theta_k^t, \Theta_{k+1}^t\}$  et  $\Theta_k^t$  orbite de  $\theta_k^t = \frac{\sum_{s=0}^{p-1} 3^{(s+1)p'-1} + \sum_{s=0}^{k-1} 3^{(s+1)p'-1}}{3^q-1}$*

## 1.5 Applications de cornes de fonctions holomorphes localement pseudo-conjuguées sur leur bassin parabolique immédiat

Certains résultats sur les polynômes de degré deux ([IS08], [BC05], [Che19], [AC18], [CC15], [Che13]) sont obtenus par renormalisation presque parabolique, étudiée dans [IS08]. Il s'agit d'une perturbation de la renormalisation parabolique, cette dernière consistant à associer à une application  $f$  son application de cornes étendue.

Les travaux de Shishikura [Shi00] et indépendamment Landford, Yampolski [LY14] définissent une classe invariante par renormalisation sous certaines hypothèses. Plus précisément :

- Shishikura suppose que le bassin parabolique immédiat contient un unique point critique.
- Landford et Yampolski supposent que le bassin parabolique immédiat est localement connexe en zéro.

Notre objectif est d'étendre les travaux de Shishikura et Landford, Yamposlki, en levant les hypothèses mentionnées. À plus long terme, nous espérons étendre [IS08] à des classes permettant une application aux polynômes de degré supérieur, dont par exemple le degré trois. Pour cela, on prévoit de perturber des classes stables qu'il reste à définir. Une proposition conjecturale est donnée en fin de section 3. La section 4 est l'un des outils nécessaires à la preuve de la stabilité de cette classe par renormalisation parabolique.

Plus précisément, on démontre en section 4 le théorème suivant. Les définitions et notations seront introduites durant cette section 4.

**Théorème 1.5.1.** *Soient  $f_1, f_2$  deux fonctions holomorphes d'un voisinage ouvert de  $0 \in \mathbb{C}$  vers  $\mathbb{C}$ , avec le développement limité en  $0 : f_i(z) = z + a_i z^2 + o(z^2)$  où  $a_i \neq 0$ .*

*On suppose que  $f_1, f_2$  sont localement semi-conjugués en  $0$  sur leurs bassins immédiats  $B_0^{f_i}$ . Alors il existe  $\sigma \in \mathbb{C}$  et une paire de fonctions holomorphes  $\psi = (\psi^+, \psi^-)$ , où  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ , est telle que :*

$$\mathbf{h}_1^\pm = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm$$

*et telle que  $\psi^\pm$  a une singularité effaçable en  $\pm i\infty$ .*

*Réciproquement, s'il existe  $\sigma \in \mathbb{C}$  et une paire de fonctions holomorphes  $\psi = (\psi^+, \psi^-)$ , où  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ , avec des singularités effaçables en  $\pm i\infty$  tels que :*

$$\mathbf{h}_1^\pm = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm$$

*alors  $f_1, f_2$  sont localement semi-conjugués en  $0$  sur leurs bassins paraboliques immédiats.*

*De plus, les applications  $\psi^\pm$  ont en  $\pm i\infty$  un développement limité de la forme  $\psi^\pm(w) = w + \rho^\pm + o(1)$ .*

Voici un complément de ce théorème :

**Proposition 1.5.2.** *Le théorème reste valable en remplaçant chaque occurrence de "semi-conjugaison" par "pseudo-conjugaison", et chaque occurrence de "paire de fonctions holomorphes  $\psi = (\psi^+, \psi^-)$ , où  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ " par "paire de biholomorphismes  $\psi = (\psi^+, \psi^-)$ , où  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ ".*

*En ce cas,  $\mathbf{h}_1^\pm, \mathbf{h}_2^\pm$  sont équivalents en tant que revêtement ramifiés analytiques via biholomorphismes entre  $\mathcal{D}_1^\pm$  et  $\mathcal{D}_2^\pm$  (plus précisément via  $\psi^\pm$ ), et une translation de  $\mathbb{C}/\mathbb{Z}$  (plus précisément via  $T_\sigma$ ).*

*Plus explicitement. Soient  $f_1, f_2$  des fonctions holomorphes d'un voisinage ouvert de  $0 \in \mathbb{C}$  dans  $\mathbb{C}$ , avec développement limité en  $0 : f_i(z) = z + a_i z^2 + o(z^2)$  où  $a_i \neq 0$ .*

*Supposons  $f_1, f_2$  localement pseudo-conjugués en  $0$  sur  $B_0^{f_i}$ . Alors il existe  $\sigma \in \mathbb{C}$  et une paire de biholomorphismes  $\psi = (\psi^+, \psi^-)$ , où  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ , tels que :*

$$\mathbf{h}_1^\pm = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm$$

*et tels que l'application holomorphe  $\psi^\pm$  ait une singularité effaçable en  $\pm i\infty$ .*

*Réciproquement, s'il existe  $\sigma \in \mathbb{C}$  et une paire de biholomorphismes  $\psi = (\psi^+, \psi^-)$ , où  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ , avec des singularités effaçables en  $\pm i\infty$  tels que :*

$$\mathbf{h}_1^\pm = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm$$

alors  $f_1, f_2$  sont localement pseudo-conjugués en 0 sur leurs bassins immédiats.

De plus, les applications  $\psi^\pm$  ont en  $\pm i\infty$  un développement limité de la forme  $\psi^\pm(w) = w + \rho^\pm + o(1)$ .

## 2 Polynômes de degré 3

Dans cette section, nous rappelons, après avoir défini le lieu de connexité  $\mathcal{C}_3$  ainsi que les tranches paraboliques  $\text{Per}_1(\lambda)$ , comment paramétriser le complémentaire de ce lieu de connexité  $\mathcal{C}_3$  dans chaque tranche  $\text{Per}_1(\lambda)$ . Cette paramétrisation s'inspire des travaux de Faught [Fau92] et de Roesch [Roe07]. Puis nous étudions les portraits d'orbite de l'origine lorsque  $\lambda$  est une racine  $q$ -ième de l'unité. Blokh et Oversteegen ont énuméré les portraits d'orbites et démontré que ce que nous appelons les portraits d'orbites formels sont réalisés par des portraits d'orbite, en exploitant les laminations dans [BOT22]. Zhang fournit une autre approche vis-à-vis de ce problème dans [Zha23]. On propose ici de donner une nouvelle manière d'énumérer les portraits d'orbite formels, par des arguments combinatoires, mais cependant sans prouver leur réalisabilité.

### 2.1 Conventions

#### 2.1.1 Notations

Pour  $R > 0$  on notera  $\mathbb{D}_R$  le disque de rayon  $R$  inclus dans  $\mathbb{C}$ .

Pour  $n \in \mathbb{N}^*$ ,  $\mathcal{M}_3$  désigne l'espace des modules des polynômes complexes à une variable de degré 3. Cet espace est par définition l'ensemble des polynômes de degré 3 modulo conjugaison par les automorphismes de  $\mathbb{C}$ , c'est à dire modulo conjugaison par les bijections  $\mathbb{C}$ -affines. Pour  $P$  polynôme, on notera  $[P]$  l'image de  $P$  par l'application canonique. Pour  $n \in \mathbb{N}$ ,  $\lambda \in \mathbb{C}$ ,  $\text{Per}_n(\lambda)$  désigne le lieu des  $[P]$  de  $\mathcal{M}_3$  tels que  $P$  ait un cycle de taille exactement  $n$  et de multiplicateur  $\lambda$  (propriété indépendante du choix du représentant de  $[P]$ ).

On note  $\mathcal{C}_3$  le lieu de connexité des polynômes de degré 3, i.e le lieu des points tels que l'ensemble de Julia est connexe, ce qui équivaut à ce que toutes les orbites critiques soient bornées. On note  $\mathcal{C}_3(\lambda) = \mathcal{C}_3 \cap \text{Per}_1(\lambda)$ .

Paramétrisation de  $\text{Per}_1(\lambda)$  :  $P_a^\lambda(z) = P^{\lambda,a}(z) = P^\lambda(a, z) = \lambda z + az^2 + z^3$ .

$c_1^{\lambda,a} = c_1^\lambda(a)$ ,  $c_2^{\lambda,a} = c_2^\lambda(a)$  : points critiques de  $P_a^\lambda$  (choisis arbitrairement, sans souci de continuité).

$c_B^{\lambda,a} = c_B^\lambda(a)$ ,  $c_\infty^{\lambda,a} = c_\infty^\lambda(a)$  : point critique borné, resp. dans le bassin de l'infini de  $P_a^\lambda$  quand celui-ci est unique.

$\mathcal{A}_\infty^{\lambda,a} = \mathcal{A}_\infty^\lambda(a)$  : bassin de l'infini de  $P_a^\lambda$ .

$\phi_\infty^{\lambda,a}$  : coordonnée de Böttcher à l'infini de  $P_a^\lambda$ .

$\mathcal{K}_a^\lambda = \mathcal{K}^{\lambda,a} = \mathcal{K}^\lambda(a)$  : ensemble de Julia rempli de  $P_a^\lambda$ .

$\mathcal{J}_a^\lambda = \mathcal{J}^{\lambda,a} = \mathcal{J}^\lambda(a)$  : ensemble de Julia de  $P_a^\lambda$ .

$\mathcal{F}_a^\lambda = \mathcal{F}^{\lambda,a} = \mathcal{F}^\lambda(a)$  : ensemble de Fatou de  $P_a^\lambda$ .

$G_a^\lambda = G^{\lambda,a}$  : fonction de Green de  $K_a^\lambda$ .

$\Upsilon_\infty^\lambda$  : isomorphisme, dont on démontre l'existence lors de la proposition 2.2.1, de  $\text{Per}_1(\lambda) \setminus \mathbb{C}_3$  vers  $\mathbb{C} \setminus 3\overline{\mathbb{D}}$  et tangente à l'identité à l'infini.

### 2.1.2 Définitions : espaces de paramètres, espace de module

Il y a plusieurs espaces sur lesquels travailler :  $\mathcal{M}_3$ , constitué des classes de conjugaison des polynômes de degré 3 par les bijections  $\mathbb{C}$ -affines, dit l'espace des modules, qui possède plusieurs choix de paramétrages, c'est à dire des  $P_a$  recouvrant au moins une fois chaque classe de conjugaison. On appelle l'ensemble  $\mathcal{P}$  constitué par ces  $a$ , muni de l'application dite de paramétrage  $\mathcal{P} \rightarrow \mathcal{M}$  un espace de paramètres. Par abus, on dira souvent que l'ensemble  $\mathcal{P}$  est un espace de paramètres, quand il n'y aura pas d'ambiguïté concernant l'application de paramétrage. N'importe quel revêtement ramifié holomorphe de  $\mathbb{C}$  à valeurs dans l'espace des modules est un espace de paramètres. Travailler avec un espace de paramètres permet de calculer explicitement les éléments dynamiques (point critique, valeur critique, etc. ) ainsi que les applications holomorphes (coordonnées de Böttcher, coordonnées de Fatou). L'espace des modules s'obtient comme quotient d'un espace de paramètres lorsque ce dernier est un revêtement ramifié. Dans ce cas, les applications holomorphes ayant pour espace de départ l'espace des paramètres peuvent être quotientées en des applications holomorphes ayant pour espace de départ l'espace des modules ssi elles vérifient la (ou les) condition d'invariance en question : ceci donne un outil pratique pour construire les applications holomorphes provenant de l'espace des modules.

### 2.1.3 Paramétrisation des polynômes de $\text{Per}_1(\lambda)$

Soit  $\lambda$  un nombre complexe. On définit la paramétrisation suivante de  $\text{Per}_1(\lambda)$  :  $P_a^\lambda(z) = \lambda z + az^2 + z^3$ .

Dans cette paramétrisation, on calcule les différents éléments dynamiquement pertinents. Lorsqu'il n'y aura pas plus de précision,  $\sqrt{Z}$  désignera une détermination fixée de la racine carrée de  $Z$  définie pour tout  $Z \in \mathbb{C}$ , sans souci de continuité.

$P_a$  admet 0 comme point fixe parabolique de multiplicateur  $\lambda$ , et  $x_\pm^{\lambda,a} = \frac{-a \pm \sqrt{a^2 - 4(\lambda - 1)}}{2}$  comme point fixe de multiplicateur  $m_\pm^{\lambda,a} = 1 - ax_\pm = 1 + \frac{a^2 \mp a\sqrt{a^2 - 4\lambda}}{2}$ . Remarquer qu'en le cas  $\lambda = 1$ ,  $\{x_+^{1,a}, x_-^{1,a}\} = \{0, -a\}$  si bien que 0 est un point fixe (au moins) double de multiplicateur 1, le point fixe restant étant de multiplicateur  $1 + a^2$ .

Les points critiques sont :  $c_+^\lambda(a) = \frac{-a + \sqrt{a^2 - 3\lambda}}{3}$ ,  $c_-^\lambda(a) = \frac{-a - \sqrt{a^2 - 3\lambda}}{3}$  (potentiellement égaux, auquel cas il s'agit d'un point critique de multiplicité 2)

Les valeurs critiques sont :  $v_+^\lambda(a) = P_a^\lambda(c_+^\lambda)$ ,  $v_-^\lambda(a) = P_a^\lambda(c_-^\lambda)$

Les points cocritiques (unique point qui a même image que  $c_\pm^\lambda(a)$  par  $P_a^\lambda$ ) sont :

$$\begin{cases} co_+^\lambda(a) = -2c_+^\lambda(a) - a = \frac{-a - 2\sqrt{a^2 - 3\lambda}}{3} \\ co_-^\lambda(a) = -2c_-^\lambda(a) - a = \frac{-a + 2\sqrt{a^2 - 3\lambda}}{3} \end{cases}$$

**Remarque 2.1.1.** Soient  $\lambda_0, a_0 \in \mathbb{C}$ . Dans le cadre où les deux points critiques  $c_\pm^{\lambda_0, a_0}$  sont distincts, on a  $a_0^2 - 3\lambda_0 \neq 0$ . Ceci implique qu'une détermination ponctuelle de la

fonction  $(a, \lambda) \mapsto \sqrt{a^2 - 3\lambda}$  en le point  $(a_0, \lambda_0)$  peut être prolongée holomorphiquement sur un petit voisinage de  $(\lambda_0, a_0)$ . Pour montrer cela, prendre par exemple une demi-droite  $D$  émergeant de 0 et ne rencontrant pas  $\lambda_0^2 - 3a_0$ , et prendre la détermination de  $\sqrt{a^2 - 3\lambda}$  sur  $\mathbb{C} \setminus D$  étendant la détermination ponctuelle. Ainsi, un point critique  $c_{\pm}^{\lambda_0, a_0}$  défini ponctuellement peut être suivi holomorphiquement sur un voisinage de  $(\lambda_0, a_0)$  en exploitant la formule  $c_{\pm}^{\lambda, a} = \frac{-a \pm \sqrt{a^2 - 3\lambda}}{3}$ .

**Remarque 2.1.2.** Il n'existe pas de détermination holomorphe des deux points critiques  $c_{\pm}^{\lambda}$  sur  $\mathbb{C}$  tout entier, et de même pour les autres éléments associés. L'équation  $a^2 - 3\lambda = 0$  correspond dynamiquement au fait que les deux points critiques soient égaux. A  $\lambda$  fixé et  $a$  variable, il y a échange de détermination des deux points critiques suivis holomorphiquement lorsque  $a$  parcourt un lacet ne rencontrant pas  $a^2 - 3\lambda = 0$  et dont la somme des indices autour des deux points de  $a^2 - 3\lambda = 0$  (à savoir  $a = \pm\sqrt{3\lambda}$ , où l'on compte par convention le point deux fois si  $\lambda = 0$ ) ne vaut pas 0 modulo 2.

**Remarque 2.1.3.** L'application  $\mathbb{C}^2 \rightarrow \mathcal{M}_3$   
 $(\lambda, a) \mapsto P_a^{\lambda}$  est un revêtement ramifié. Elle induit, en passant au quotient, un isomorphisme d'orbifolds :

$$\begin{cases} X \rightarrow \mathcal{M}_3 \\ (\lambda, a) \mapsto P_a^{\lambda} \end{cases} \quad (1)$$

où  $X$  est  $\mathbb{C}^2$  quotienté par les relations  $(\lambda, a) = (\alpha(P_a^{\lambda})'(\beta), \alpha(a + 3\beta))$  où  $\alpha$  parcourt  $\{1, -1\}$  et  $\beta$  parcourt les points fixes de  $P_a^{\lambda}$ ; plus précisément,  $(\lambda', a')$  est identifié à  $(\lambda, a)$  si et seulement si l'une des six équations suivantes est vérifiée :

$$\begin{cases} (\lambda', a') = (\lambda, a) \\ (\lambda', a') = (-\lambda, -a) \\ (\lambda', a') = \left(1 + \frac{a^2 - a\sqrt{a^2 - 4(\lambda - 1)}}{2}, \frac{-a + 3\sqrt{a^2 - 4(\lambda - 1)}}{2}\right) \\ (\lambda', a') = \left(1 + \frac{a^2 + a\sqrt{a^2 - 4(\lambda - 1)}}{2}, \frac{-a - 3\sqrt{a^2 - 4(\lambda - 1)}}{2}\right) \\ (\lambda', a') = \left(-1 - \frac{a^2 - a\sqrt{a^2 - 4(\lambda - 1)}}{2}, \frac{a - 3\sqrt{a^2 - 4(\lambda - 1)}}{2}\right) \\ (\lambda', a') = \left(-1 - \frac{a^2 + a\sqrt{a^2 - 4(\lambda - 1)}}{2}, \frac{a + 3\sqrt{a^2 - 4(\lambda - 1)}}{2}\right) \end{cases}$$

Soit  $\lambda \in \mathbb{C}$ . L'application  $\mathbb{C} \rightarrow \text{Per}_1(\lambda)$   
 $a \mapsto P_a^{\lambda}$  est un revêtement ramifié. Elle induit, en passant au quotient, un isomorphisme de variétés complexes.

$$\begin{cases} Y \rightarrow \text{Per}_1(\lambda) \\ a \mapsto P_a^{\lambda} \end{cases} \quad (2)$$

où  $Y$  est  $\mathbb{C}$  quotienté par les relations  $a = \alpha(a + 3\beta)$ , où  $\alpha$  parcourt  $\{1, -1\}$  et  $\beta$  parcourt les points fixes de  $P_a^\lambda$  qui satisfont  $\alpha(P_a^\lambda)'(\beta) = \lambda$ . Remarquer que si  $\lambda = 1$ , les relations se réduisent à  $a = -a$ .

*Démonstration.* Soient  $a, \lambda, a', \lambda'$  des nombres complexes, et  $P = P_a^\lambda$ ,  $Q = P_{a'}^{\lambda'}$ . On suppose que  $P$  et  $Q$  sont conjugués via automorphisme complexe, c'est à dire qu'il existe  $\alpha \in \mathbb{C}^*$ ,  $\beta \in \mathbb{C}$  tels que  $\frac{P(\alpha z + \beta) - \beta}{\alpha} = Q$ .

Par égalité des coefficients en  $z^3$ ,  $\frac{\alpha^3}{\alpha} = 1$ , donc  $\alpha = \pm 1$ .

Par évaluation en  $z = 0$ ,  $P(\beta) = \beta$ .

Par dérivation puis évaluation en  $z = 0$ ,  $\frac{P'(\beta)}{\alpha} = \lambda'$ , c'est à dire  $\lambda' = \alpha P'(\beta)$  (car  $\alpha^2 = 1$ ).

Par égalité des coefficients en  $z^2$ ,  $\frac{3\alpha^2\beta + a\alpha^2}{\alpha} = a'$ , c'est à dire  $a' = \alpha(a + 3\beta)$ .

Réciproquement, cette série de 4 égalités implique l'égalité initiale  $\frac{P(\alpha z + \beta) - \beta}{\alpha} = Q$ . En conclusion,  $Q$  est conjugué à  $P$  si et seulement si il existe un  $\alpha \in \{1, -1\}$  et un point fixe  $\beta$  de  $P$  tel que  $a' = \alpha(a + 3\beta)$ ,  $\lambda' = \alpha P'(\beta)$ , auquel cas la conjugaison est donnée par  $z \mapsto \alpha z + \beta$ . Remarquer que les trois valeurs possibles de  $\beta$  sont donc 0, et  $x_\pm^{\lambda, a}$ .  $\square$

Soit  $\lambda \in \mathbb{C}$ . Pour  $a$  au voisinage de l'infini, on choisit les deux déterminations suivantes des points critiques :

$$\begin{cases} c_1^\lambda(a) = \frac{-a(1 + \sqrt{1 - \frac{3\lambda}{a^2}})}{3} \\ c_2^\lambda(a) = \frac{-a(1 - \sqrt{1 - \frac{3\lambda}{a^2}})}{3} \end{cases}$$

où les racines carrées sont calculées via la détermination principale. Remarquer que ces quantités constituent bien les points critiques de  $P_a^\lambda$ .

**Proposition 2.1.4.** (*Détermination holomorphe des deux points critiques au voisinage de l'infini.*)

Les fonctions  $c_1^\lambda, c_2^\lambda$  sont holomorphes au voisinage de l'infini et :

$$\begin{cases} c_1^\lambda(a) \sim_{a \rightarrow \infty} -\frac{2a}{3} \\ c_2^\lambda(a) \sim_{a \rightarrow \infty} -\frac{\lambda}{2a} \end{cases}$$

Ceci permet d'en déduire les équivalents analogues pour les autres paramètres dynamiques :

$$\begin{cases} co_1^\lambda(a) \sim_{a \rightarrow \infty} \frac{a}{3} \\ co_2^\lambda(a) \sim_{a \rightarrow \infty} -a \\ v_1^\lambda(a) = \left(-\frac{2a}{3}\right)^3 + \left(-\frac{2a}{3}\right)^2 a + O(a^2) \sim_{a \rightarrow \infty} \frac{4a^3}{27} \\ v_2^\lambda(a) \sim_{a \rightarrow \infty} -\frac{\lambda^2}{2a} \end{cases}$$

*Démonstration.* Un développement limité permet d'obtenir les équivalents suivants lorsque  $a$  tend vers l'infini :

$$\begin{cases} \frac{-a(1+\sqrt{1-\frac{3\lambda}{a^2}})}{3} \sim_{a \rightarrow \infty} -\frac{2a}{3} \\ \frac{-a(1-\sqrt{1-\frac{3\lambda}{a^2}})}{3} \sim_{a \rightarrow \infty} -\frac{\lambda}{2a} \end{cases}$$

Des calculs aisés permettent d'obtenir les autres équivalents. □

#### 2.1.4 Compacité de $\text{Per}_1(\lambda) \cap \mathcal{C}_3$

**Proposition 2.1.5.** *Soit  $P$  un polynôme unitaire. Alors son bassin de l'infini contient  $\mathbb{C} \setminus \mathbb{D}_R$  où  $R$  est la somme des modules des coefficients de  $P$ . En particulier, soient  $\lambda \in \mathbb{C}$ ,  $a \in \mathbb{C}$ . Alors le bassin de l'infini de  $P_a^\lambda$ , que l'on notera  $A_\infty^\lambda(a)$ , contient  $\mathbb{C} \setminus \overline{\mathbb{D}_R}$  où  $R = 1 + |a| + |\lambda|$  (somme des modules des coefficients de  $P_a^\lambda$ ). Ceci revient bien entendu à dire que  $\mathcal{K}_P$  (resp.  $\mathcal{K}_a^\lambda$ ) ensemble de Julia rempli de  $P$  (resp.  $P_a^\lambda$ ) est inclus dans  $\overline{\mathbb{D}_R}$ .*

*Démonstration.* Voir la démonstration dans les notes d'Orsay de Douady-Hubbard, exposé 8 [DHL<sup>+</sup>84]. □

**Proposition 2.1.6.** *Soit  $P$  un polynôme unitaire, tel que l'ensemble de Julia associé  $\mathcal{K}_P$  soit connexe. Alors le bassin de l'infini  $\mathcal{A}_P$  contient  $\mathbb{C} \setminus \overline{\mathbb{D}_4}$ ; ceci est équivalent à  $\mathcal{K}_P$  inclus dans  $\overline{\mathbb{D}_4}$ . En particulier, pour  $(\lambda, a)$  dans  $\mathcal{C}_3$ , le bassin à l'infini de  $P_a^\lambda$ ,  $A_\infty^{\lambda, a}$ , contient  $\mathbb{C} \setminus \overline{\mathbb{D}_4}$  et son ensemble de Julia  $\mathcal{K}_a^\lambda$  est inclus dans  $\overline{\mathbb{D}_4}$ .*

*Démonstration.* C'est une conséquence immédiate du fait que le rayon de capacité du bassin de l'infini est égal à 1 et du lemme 1/4 de Koebe. □

**Remarque 2.1.7.** Le contrôle uniforme des bornes de l'ensemble de Julia rempli est néanmoins faux hors du lieu de connexité  $\mathcal{C}_3$ . On vérifiera dans la suite de la section que le point cocritique  $co_2^{\lambda, a}$ , qui est équivalent à  $-a$  lorsque  $a \rightarrow \infty$ , appartient à l'ensemble de Julia rempli pour  $|\lambda| \leq 1$ . Ainsi, le contrôle de la proposition 2.1.5 est précis en ce sens : on ne peut pas remplacer  $|a|$  par  $(1 - \epsilon)|a|$ , où  $\epsilon > 0$

**Proposition 2.1.8.**  $\mathcal{C}_3$  est compact. En particulier,  $\mathcal{C}_3 \cap \text{Per}_1(\lambda)$  est compact.

*Démonstration.* La fermeture de  $\mathcal{C}_3$  résulte de l'ouverture de son complémentaire, qui provient de la stabilité des bassins à l'infini (car super-attractifs). Il reste donc à démontrer le caractère borné de  $\mathcal{C}_3$ . Soit  $(\lambda, a) \in \mathcal{C}_3$ . Alors par contrôle uniforme de la taille de l'ensemble de Julia rempli, proposition 2.1.6, les deux points critiques  $c_\pm^{\lambda, a}$  sont dans  $\overline{\mathbb{D}_4}$ . Or, la moyenne des deux points critiques est  $-a$ , donc  $a \in \overline{\mathbb{D}_4}$ . Ensuite, leur différence  $\frac{4\sqrt{a^2-3\lambda}}{3}$  est dans  $\overline{\mathbb{D}_8}$ , ce qui implique le caractère borné de  $\lambda$  (avec borne indépendante de  $a$ ). □

Dans le cadre où  $|\lambda| \leq 1$ , on peut raffiner le résultat d'holomorphicité des points critiques, en distinguant un point critique qui est dans le bassin de l'infini, avec un autre qui peut être naturellement associé au point fixe 0.



Dans un premier temps, si  $|\lambda| < 1$  ou  $\lambda = \exp(\frac{2i\pi p}{q})$ , pour  $a \in \text{Per}_1(\lambda) \setminus \mathcal{C}_3$ , on peut noter  $c_\infty^\lambda(a)$  l'unique point critique dans le bassin de l'infini, et  $c_B^\lambda(a)$  l'autre point critique, dans le bassin de 0, d'orbite bornée (on le notera aussi  $c_A^\lambda(a)$ ,  $c_P^\lambda(a)$ , si on sait que le bassin est attractif, parabolique). Ceci implique que les deux points critiques sont distincts.

**Remarque 2.1.9.** Les applications  $c_\infty^\lambda$ ,  $c_B^\lambda$  ainsi définies sur  $\text{Per}_1(\lambda) \setminus \mathcal{C}_3$  sont holomorphes.

*Démonstration.* Il suffit de vérifier que  $c_\infty^\lambda$  est holomorphe. Les points critiques  $c_\pm^\lambda(a_0) = \frac{-a_0 \pm \sqrt{a_0^2 - 3\lambda}}{3}$  sont distincts ; par la remarque 2.1.1, notons  $\bar{c}$  un prolongement holomorphe local de  $c_\infty^\lambda(a_0)$  au voisinage de  $a_0$ . Par stabilité du bassin de l'infini, l'ensemble  $\{(z, a) \in \mathbb{C}^2, z \in \mathcal{A}_\infty^{\lambda, a}\}$  est ouvert, si bien que  $\bar{c}(a)$  demeure dans le bassin de l'infini pour  $a$  proche de  $a_0$ . Il s'ensuit que  $c_\infty^\lambda$  coïncide avec  $\bar{c}$  au voisinage de  $a_0$ , ce qui conclut.  $\square$

Dans un second temps, en supposant seulement  $|\lambda| \leq 1$ , on démontre aussi l'existence de deux déterminations holomorphes des points critiques. On démontre pour cela le point suivant pour  $a \in \text{Per}_1(\lambda) \setminus \mathcal{C}_3$  (remarquer qu'il s'agit d'une condition ouverte en le couple  $(a, \lambda)$  par stabilité du bassin de l'infini, superattractif) :

**Remarque 2.1.10.** Soit  $\lambda$  nombre complexe de module 1 non racine de l'unité.  $P_a^\lambda$  admet un point critique borné, qui peut être associé naturellement au point fixe 0 : c'est l'unique point critique  $c_B^\lambda(a)$  de  $P_a^\lambda$  tel que, pour une perturbation assez petite de  $\lambda$  en  $\lambda'$  tel que  $|\lambda'| < 1$  ou  $\lambda' = \exp(\frac{2i\pi p}{q})$ ,  $|c_B^\lambda(a) - c_B^{\lambda'}(a)|$  est aussi proche que désiré de 0. On rappelle que dans le cas  $|\lambda'| < 1$  ou  $\lambda' = \exp(\frac{2i\pi p}{q})$ ,  $c_B^{\lambda'}(a)$  est un point critique de  $P_a^{\lambda'}$  appartenant au bassin de 0, qu'il soit attractif ou parabolique.

*Démonstration.* Pour l'existence d'un point critique d'orbite bornée, raisonnons par l'absurde. Si ce n'est pas le cas,  $P_a^\lambda$  n'admet que des points critiques d'orbites non bornées. Par le caractère super-attractif de l'infini, tous les points critiques sont attirés par l'infini. Par stabilité du bassin d'un point super-attractif, il existe  $\lambda'$ ,  $|\lambda'| < 1$  proche de  $\lambda$  tel que tous les points critiques de  $P_a^{\lambda'}$  sont attirés par l'infini, ce qui est absurde puisque le bassin attractif de 0 doit contenir un point critique. L'unicité du point critique borné est immédiate, puisque l'autre point critique doit être dans le bassin de l'infini. Enfin, pour la caractérisation de ce point via perturbation, utiliser la stabilité du bassin d'un point super-attractif.  $\square$

Ainsi, on peut encore noter  $c_\infty^\lambda(a)$  l'unique point critique dans le bassin de l'infini, et  $c_B^\lambda(a)$  l'autre point critique, d'orbite bornée. On le notera aussi  $c_A^\lambda(a)$ ,  $c_P^\lambda(a)$ ,  $c_S^\lambda(a)$ ,  $c_C^\lambda(a)$  si on sait que le point fixe 0 est attractif, parabolique, Siegel ou Cremer.

Ceci permet de démontrer le point suivant, par une preuve identique au cas précédent :

**Remarque 2.1.11.** Sous les mêmes hypothèses que la remarque précédente, les applications  $c_\infty^\lambda$ ,  $c_B^\lambda$  ainsi définies sur  $\text{Per}_1(\lambda) \setminus \mathcal{C}_3$  sont holomorphes.

**Remarque 2.1.12.** Soit  $\lambda \in \overline{\mathbb{D}}$  fixé. Pour  $a$  proche de l'infini,  $c_\infty^\lambda(a)$  est le point critique de module maximal,  $c_B^\lambda(a)$  le point critique de module minimal. On peut alors réappliquer les équivalents de la proposition 2.1.4, en remplaçant 1 par  $\infty$ , 2 par  $B$ . Ceci donne l'holomorphie de ces fonctions sur  $(\text{Per}_1(\lambda) \cup \{\infty\}) \setminus \mathcal{C}_3$

*Démonstration.* Comme déjà vu lors de la démonstration de la proposition 2.1.8, le point critique de  $P_a^\lambda$  de module maximal est dans le bassin de l'infini pour  $a$  assez grand.  $\square$

**Remarque 2.1.13.** L'holomorphie donnée par la précédente remarque, cumulée avec la remarque 2.1.2, implique que  $\pm\sqrt{3\lambda}$  appartiennent tous deux à une même composante connexe de  $\text{Per}_1(\lambda) \cap \mathcal{C}_3$  : il n'y a pas d'échange de détermination des deux points critiques paramétrés holomorphiquement par  $a$  parcourant un lacet au sein de  $\text{Per}_1(\lambda) \setminus \mathcal{C}_3$ . On démontrera bientôt qu'en fait,  $\text{Per}_1(\lambda) \cap \mathcal{C}_3$  est connexe.

### 2.1.5 Coordonnées de Böttcher, fonctions de Green

On note pour  $a \in \mathbb{C}$ ,  $\mathcal{A}_\infty^\lambda(a) = \mathcal{A}_\infty^{\lambda,a}$  le bassin de l'infini de  $P_a^\lambda$ .

Soit  $\phi_\infty^{\lambda,a} : \Omega_\infty^{\lambda,a} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}_R}$  une coordonnée de Böttcher définie sur un voisinage de l'infini, tangente à l'identité en l'infini. Son inverse  $(\phi_\infty^{\lambda,a})^{-1}$  a une unique extension radiale maximale  $(\phi_\infty^{\lambda,a})^{-1} : X_\infty^{\lambda,a} \rightarrow \tilde{\mathcal{A}}_\infty^{\lambda,a}$  où  $X_\infty^{\lambda,a} \subset \mathbb{C} \setminus \overline{\mathbb{D}}$  et  $\tilde{\mathcal{A}}_\infty^{\lambda,a} = (\phi_\infty^{\lambda,a})^{-1}(X_\infty^{\lambda,a}) \subset \mathcal{A}_\infty^{\lambda,a}$ . Cette extension s'arrête en un point  $w$  avec  $|w| > 1$  ssi  $(\phi_\infty^{\lambda,a})^{-1}$  s'étend continûment à  $w$  et  $(\phi_\infty^{\lambda,a})^{-1}(w)$  est un point (pré)critique pour  $P_a^\lambda$ . Remarquer que  $X_\infty^{\lambda,a}$  est ouvert, car l'ensemble des points (pré)critiques de  $P_a^\lambda$  (et donc l'union des segments joignant chacun d'entre eux à 0) est borné et n'a pas de point d'accumulation hors de  $\overline{\mathbb{D}}$ .

$\phi_\infty^{\lambda,a}$  est l'unique fonction holomorphe tangente à l'identité en l'infini vérifiant  $\phi_\infty^{\lambda,a}(P_a(z)) = \phi_\infty^{\lambda,a}(z)^3$ . Ceci permet d'écrire  $\phi_\infty^{\lambda,a}$  au voisinage de l'infini comme un produit infini. Pour une formule générale, voir le chapitre VIII des notes d'Orsay de Douady et de Hubbard [DHL<sup>+</sup>84] (attention, il y a une coquille dans cette référence : le produit infini doit commencer par  $n = 0$  comme on le vérifie dans la preuve ci-dessous). Notons  $f(z) = P_a^\lambda(z)/z^3$  et pour  $n \in \mathbb{N}$ ,  $z_n = (P_a^\lambda)^{\circ n}(z)$ . Notons  $F(z) = z \prod_{n=0}^{\infty} f(z_n)^{\frac{1}{3^{n+1}}} = z \prod_{n=0}^{\infty} (1 + \frac{a}{z_n} + \frac{\lambda}{z_n^2})^{\frac{1}{3^{n+1}}}$

**Remarque 2.1.14.**  $F$  est défini sur un voisinage de l'infini, et constitue bien (une restriction de) la coordonnée de Böttcher de  $P_a^\lambda$  en l'infini.

*Démonstration.* Pour  $z$  dans un voisinage de l'infini suffisamment petit,  $|\frac{a}{z_n} + \frac{\lambda}{z_n^2}| < 1$  pour tout  $n \in \mathbb{N}$  et  $|\frac{a}{z_n} + \frac{\lambda}{z_n^2}|$  est majoré uniformément en  $z$  par une suite géométrique tendant vers 0 (on pourrait en réalité être beaucoup plus précis) et la famille  $(\frac{a}{z_n} + \frac{\lambda}{z_n^2})_{n \in \mathbb{N}^*}$  est donc sommable uniformément en  $z$ . Ceci permet non seulement de définir sans ambiguïté les racines cubiques itérées des  $(1 + \frac{a}{z_n} + \frac{\lambda}{z_n^2})$ , via la détermination principale, mais aussi de dire que le produit au sein de  $F$  converge. Quand  $z$  est dans un voisinage assez petit de l'infini, on peut en fait majorer  $\sum_{n \in \mathbb{N}^*} |\frac{a}{z_n} + \frac{\lambda}{z_n^2}|$  par un  $\epsilon > 0$  arbitraire fixé, ce qui montre que le produit converge vers 1 quand  $z$  tend vers l'infini, et donc que  $F(z)$  est équivalente à  $z$  en l'infini. Il ne reste plus qu'à vérifier que  $F(P_a^\lambda(z)) = F(z)^3$ . On a :

$$\begin{cases} F(P_a(z)) = z_1 \prod_{n=0}^{\infty} f(z_{n+1})^{\frac{1}{3^{n+1}}} \\ F(z)^3 = z^3 \prod_{n=0}^{\infty} f(z_n)^{\frac{1}{3^n}} \end{cases}$$

On obtient donc en prenant le quotient des deux termes :

$$\frac{F(P_a(z))}{F(z)^3} = \frac{z_1}{z^3 f(z_0)} = 1$$

Bien sûr, si le produit commençait à  $n = 1$ , le quotient ne vaudrait plus 1 à cause de deux nouveaux termes apparaissant et en se compensant pas.  $\square$

**Remarque 2.1.15.** Le produit permet de définir la fonction  $|F|$  dans tout le bassin d'attraction  $\mathcal{A}_{\infty}^{\lambda,a}$  en notant que la majoration par une suite géométrique tendant vers 0 est encore valable sur tout compact inclus dans  $\mathcal{A}_{\infty}^{\lambda,a}$ . Néanmoins,  $F$  n'est pas définie à cause d'un problème de détermination de l'argument : on ne contrôle plus la taille des premiers termes.  $|F|$  étant analytique réelle, elle est égale à  $|\phi_{\infty}^{\lambda,a}|$  (domaine de définition compris).

On note pour  $a \in \mathbb{C}$   $G_a^{\lambda}$  la fonction de Green associée à  $K_a^{\lambda}$ . On a  $G_a^{\lambda}(z) = \log |\phi_{\infty}^{\lambda,a}(z)|$  pour  $z \in \mathcal{A}_{\infty}^{\lambda,a}$  et  $G_a^{\lambda}(z) = 0$  pour  $z \in K_a^{\lambda}$ . En particulier, pour  $z \in \mathbb{C}$ ,  $G_a^{\lambda}(P_a^{\lambda}(z)) = 3G_a^{\lambda}(z)$ .

**Remarque 2.1.16.** On a  $G_a^{\lambda}(z) = \lim_{n \rightarrow \infty} \frac{1}{3^n} \log \left( |(P_a^{\lambda})^n(z)| \right)$  avec convergence uniforme sur tout ensemble de la forme  $\{z \in \mathbb{C}, G_a^{\lambda}(z) \geq \delta\}$  où  $\delta > 0$ . Ceci se traduit en disant que pour  $\epsilon > 0$ , pour tout  $\delta > 0$ , il existe  $N \in \mathbb{N}$  tel que pour tout entier  $n \geq N$  et pour tout  $z \in \mathbb{C}$  vérifiant  $G_a^{\lambda}(z) \geq \delta$ ,  $(1 - \epsilon)^{3^n} \exp \left( G_a^{\lambda}(z) \right)^{3^n} \leq |(P_a^{\lambda})^n(z)| \leq (1 + \epsilon)^{3^n} \exp \left( G_a^{\lambda}(z) \right)^{3^n}$ . En ce sens,  $G_a^{\lambda}(z)$  est un indicateur de la vitesse de convergence de  $(P_a^{\lambda})^n(z)$  vers l'infini quand  $n$  tend vers l'infini.

## 2.2 Paramétrage de l'extérieur du lieu de connexité

**Proposition 2.2.1.** On pose pour  $a \in \text{Per}_1(\lambda) \setminus \mathcal{C}_3$   $\Phi_{\infty}^{\lambda}(a) = \phi_{\infty}^{\lambda,a}(co_{\infty}^{\lambda}(a))$ , où l'on rappelle que  $co_{\infty}^{\lambda}(a)$  est le point cocritique associé à  $c_{\infty}^{\lambda}(a)$ .  $\Phi_{\infty}^{\lambda}$  est un isomorphisme de  $\text{Per}_1(\lambda) \setminus \mathcal{C}_3$  vers  $\mathbb{C} \setminus \overline{\mathbb{D}}$ . Plus précisément,  $\Phi(a)$  équivaut à  $\frac{a}{3}$  en l'infini, et donc  $\Upsilon_{\infty}^{\lambda} = 3\Phi_{\infty}^{\lambda}$  est l'unique isomorphisme de  $\text{Per}_1(\lambda) \setminus \mathcal{C}_3$  vers  $\mathbb{C} \setminus 3\overline{\mathbb{D}}$  tangent à l'identité en l'infini. Ainsi, le rayon de capacité de  $\text{Per}_1(\lambda) \setminus \mathcal{C}_3$  est  $\frac{1}{3}$ .

*Démonstration.* Il est immédiat que  $\Phi_{\infty}^{\lambda}$  est à valeurs dans  $\mathbb{C} \setminus \overline{\mathbb{D}}$ , car  $G_a^{\lambda}(co_{\infty}^{\lambda}(a)) = G_a^{\lambda}(c_{\infty}^{\lambda}(a)) > 0$  par construction. Par les propositions 2.1.4 et 2.1.5, on rappelle que  $v_{\infty}^{\lambda}(a) \sim_{a \rightarrow \infty} \frac{4a^3}{27}$  et que  $\mathcal{A}_{\infty}^{\lambda} \subset \mathbb{C} \setminus \mathbb{D}_{R_a^{\lambda}}$  où  $R_a^{\lambda} = 1 + |a| + |\lambda|$ . Ceci implique que  $\lim_{a \rightarrow \infty} G_a^{\lambda}(v_{\infty}^{\lambda}(a)) = \infty$ , ce qui implique  $\lim_{a \rightarrow \infty} G_a^{\lambda}(co_{\infty}^{\lambda}(a)) = \infty$ .

On écrit :

$$\frac{\Phi_{\infty}^{\lambda}(a)}{co_{\infty}^{\lambda}(a)} = \prod_{n=0}^{\infty} \left( 1 + \frac{a}{P^n(co_{\infty}^{\lambda}(a))} + \frac{\lambda}{(P^n(co_{\infty}^{\lambda}(a)))^2} \right)^{\frac{1}{3^{n+1}}}$$

Pour montrer la détermination et la convergence du produit, exploitons la remarque 2.1.16. Par le fait que  $\lim_{a \rightarrow \infty} G_a^\lambda(\text{co}_\infty^\lambda(a)) = \infty$ , tous les facteurs du produit tendent uniformément vers 1 quand  $a$  tend vers l'infini, ce qui permet de définir au voisinage de l'infini les itérés des racines cubiques via détermination principale. L'uniforme minoration de la fonction de Green par une quantité strictement positive au voisinage de l'infini suffit pour montrer que le produit converge uniformément sur ce voisinage. Reexploitant  $\lim_{a \rightarrow \infty} G_a^\lambda(\text{co}_\infty^\lambda(a)) = \infty$ , le produit tend vers 1 quand  $a$  tend vers l'infini, donc  $\Phi_\infty^\lambda(a) \sim_{a \rightarrow \infty} \text{co}_\infty^\lambda(a) \sim_{a \rightarrow \infty} \frac{a}{3}$ . Cet argument montre que  $\Phi_\infty^\lambda$  est holomorphe au voisinage de l'infini, et que  $\Phi_\infty^\lambda$  s'étend holomorphiquement à l'infini via  $\Phi_\infty^\lambda(\infty) = \infty$ .

Il faut vérifier que  $\Phi_\infty^\lambda$  est holomorphe sur  $\text{Per}_1(\lambda) \setminus \mathcal{C}_3$ . Notons  $\mathcal{U} = \{(a, z) \in \mathbb{C}^2, G_a(z) > G_a(\text{co}_\infty^\lambda(a))\}$ , ouvert par continuité du paramétrage des fonctions de Green. L'application  $(a, z) \mapsto \phi_a(z)$  est holomorphe sur  $\mathcal{U}$  car  $\phi_a(z)$  est une détermination de  $\phi_a(P_a^{\circ n}(z))^{\frac{1}{3^n}}$ . Elle est aussi holomorphe au voisinage de  $(a, \text{co}_\infty^\lambda(a))$  par le même argument. D'où l'holomorphie de  $\Phi_\infty^\lambda$ .

$\Phi_\infty$  est donc une application holomorphe de  $\text{Per}_1(\lambda) \setminus \mathcal{C}_3 \cup \{\infty\}$  vers  $\mathbb{C} \setminus \overline{\mathbb{D}} \cup \{\infty\}$ . Elle est propre par continuité du paramétrage des fonctions de Green. C'est donc un revêtement ramifié, dont le degré est déterminé par le nombre de préimages de  $\infty$  comptées avec multiplicité. La seule préimage de  $\infty$  est  $\infty$ . Par l'équivalent de  $\Phi_\infty^\lambda$  en l'infini,  $\infty$  est une préimage de  $\infty$  de multiplicité 1. On en conclut donc que  $\Phi_\infty$  est un isomorphisme. La tangence à l'identité à l'infini nous montre que  $\Phi_\infty$  est l'application d'uniformisation de  $\text{Per}_1(\lambda) \setminus \mathcal{C}_3$ . □

**Conséquence 2.2.2.**  $\mathcal{C}_3 \cap \text{Per}_1(\lambda)$  est connexe.

*Démonstration.* Il s'agit du complémentaire d'un ouvert simplement connexe de  $\mathbb{C} \cup \{\infty\}$ . □

## 2.3 Orbites périodiques, rayons externes

Cette section reprend la définition des portraits d'orbite de l'article [Mil00]. Elle a pour objectif de donner une condition nécessaire pour être un portrait d'orbite marqué. Blokh et Oversteegen ont démontré un résultat plus précis dans [BOT22], mais on apporte ici une autre méthode.

### 2.3.1 Brisure des rayons externes dans le plan dynamique

On fixe  $\lambda \in \mathbb{C}$ . On note  $R_t^a := R_t^{\mathcal{K}^{\lambda, a}}$  le rayon externe dynamique de  $\mathcal{K}^{\lambda, a}$  d'angle  $t$  compté en nombre de tours. On notera  $R_t^{\mathcal{C}_{3, a}}$  le rayon externe paramétrique de  $\mathcal{C}_{3, a}$  d'angle  $t$ .

On se pose la question suivante : quelle condition nécessaire et suffisante donner sur le paramètre  $a \in \mathbb{C}$  pour que le rayon externe dynamique d'angle  $\alpha$  de  $P^{\lambda, a}$  se brise ?

Notons que si  $a \in \mathcal{C}_{3, a}$ , aucun rayon de  $P^{\lambda, a}$  ne se brise puisque l'ensemble de Julia rempli  $\mathcal{K}^{\lambda, a}$  est connexe. Si  $a \notin \mathcal{C}_{3, a}$ , on rappelle que  $P^{\lambda, a}$  admet deux points critiques, l'un  $c_P^{\lambda, a}$  dans le bassin parabolique et l'autre  $c_\infty^{\lambda, a}$  dans le bassin de l'infini. Un rayon

dynamique externe se brise si et seulement si il rencontre le point critique du bassin de l'infini, ou une de ses préimages. La paramétrisation du complémentaire du lieu de connexité ayant justement été pensée à partir du point cocritique du bassin de l'infini, on comprend qu'elle permet de traiter le problème.

On commence par calculer les rayons dynamiques externes de  $\mathcal{K}^{\lambda,a}$  qui se brisent en  $c_\infty^{\lambda,a}$  puis on calcule ensuite leur préimages. Noter en effet que les rayons qui se brisent sont exactement constitués par les préimages itérées (en un sens non strict) des rayons qui se brisent en  $c_\infty^{\lambda,a}$ .

Soit  $a$  dans le complémentaire de  $\mathcal{C}_{3,a}$ . On écrit :

$$\begin{aligned} a \in R_\theta^{\mathcal{C}_{3,a}} &\Leftrightarrow \arg(\Phi_\infty^{\lambda,a}(a)) = \theta \\ &\Leftrightarrow \arg(\phi_\infty^{\lambda,a}(co_\infty^{\lambda,a})) = \theta \\ &\Leftrightarrow co_\infty^{\lambda,a} \in R_\theta^a \\ &\Leftrightarrow c_\infty^{\lambda,a} \in R_{\theta+\frac{1}{3}}^a \cap R_{\theta+\frac{2}{3}}^a \end{aligned}$$

Ainsi,  $a \in R_\theta^{\mathcal{C}_{3,a}}$  si et seulement si  $R_{\theta+\frac{1}{3}}^a$  et  $R_{\theta+\frac{2}{3}}^a$  se brisent en  $c_\infty^{\lambda,a}$ .

Par passage aux préimages, dans le cadre  $a \in R_\theta^{\mathcal{C}_{3,a}}$ , les rayons dynamiques qui se brisent sont exactement les rayons d'angles appartenant à l'ensemble :

$$A_\theta = \left\{ \alpha \in \mathbb{R}/\mathbb{Z}, \exists n \in \mathbb{N} \ 3^n \alpha = \theta + \frac{1}{3} \text{ ou } 3^n \alpha = \theta + \frac{2}{3} \right\}$$

Soit  $\alpha \in \mathbb{R}/\mathbb{Z}$ . La question initiale de la section revient à savoir quand  $\alpha \in A_\theta$ . Remarquer que ce problème dépend seulement de l'angle du rayon paramétrique externe dans lequel  $a$  appartient, à savoir  $\theta$ . On pose donc l'ensemble de ces angles :

$$\begin{aligned} \Theta_\alpha &= \{ \theta \in \mathbb{R}/\mathbb{Z}, \alpha \in A_\theta \} \\ &= \{ \theta \in \mathbb{R}/\mathbb{Z}, \forall a \in R_\theta^{\mathcal{C}_{3,a}}, \text{ le rayon externe dynamique } R_\alpha^a \text{ se brise} \} \\ &= \{ \theta \in \mathbb{R}/\mathbb{Z}, \exists a \in R_\theta^{\mathcal{C}_{3,a}}, \text{ le rayon externe dynamique } R_\alpha^a \text{ se brise} \} \end{aligned}$$

Par définition de  $A_\theta$ , on a  $\Theta_\alpha = \{3^n \alpha + \frac{1}{3}, n \in \mathbb{N}\} \cup \{3^n \alpha + \frac{2}{3}, n \in \mathbb{N}\}$ .

En particulier, pour les angles  $\alpha$  fixes par multiplication par 3, à savoir 0 et  $\frac{1}{2}$  :

$$\Theta_0 = \left\{ \frac{1}{3}, \frac{2}{3} \right\}, \quad \Theta_{\frac{1}{2}} = \left\{ \frac{1}{6}, \frac{5}{6} \right\}$$

### 2.3.2 Portraits d'orbites

Lorsqu'on écrira  $\mathcal{O} = \{z_1, \dots, z_p\}$ , où  $\mathcal{O}$  est une orbite périodique pour une application cubique de la forme  $P^{\lambda,a}$ , on sous-entendra que  $P^{\lambda,a}(z_i) = z_{i+1}$  où les indices sont définis modulo  $p$ .

Pour trois angles comptés en nombre de tours  $\theta_1, \theta_2, \theta_3$  dans  $\mathbb{R}/\mathbb{Z}$ , on notera  $\theta_1 < \theta_2 < \theta_3$  si les trois angles sont distincts et apparaissent dans l'ordre cyclique. Même notation pour trois éléments de  $\mathbb{Z}/n\mathbb{Z}$ .

On note  $\sigma_3 : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$  l'application  $\theta \mapsto 3\theta$ .

Voici une définition due à Milnor, cf. l'article [Mil00] :

**Définition 2.3.1.** On dit qu'une orbite  $\mathcal{O}$  d'une application  $P^{\lambda,a}$  est effective si elle vérifie les deux conditions suivantes :

- elle est périodique
- il existe au moins un angle rationnel  $t \in \mathbb{Q}/\mathbb{Z}$  tel que le rayon dynamique d'angle  $t$  aboutisse en un point de  $\mathcal{O}$ .

**Remarque 2.3.2.** Noter qu'une orbite effective est nécessairement répulsive ou parabolique par le lemme de l'escargot. Ceci est démontré dans l'exposé VIII, partie II.2 de [DHL<sup>+</sup>84].

Si  $(\lambda, a) \in \mathcal{C}_3$ , il est possible de démontrer que toute orbite périodique répulsive ou parabolique est effective. Il s'agit du *théorème d'aboutissement de Douady-Hubbard*.

Néanmoins, ceci n'est plus vrai en général lorsque  $(\lambda, a)$  n'appartient pas au lieu de connexité, puisque les rayons externes peuvent se briser et non aboutir.

**Définition 2.3.3.** On nomme l'orbite  $\mathcal{O} = \{0\}$  du point parabolique marqué de l'application  $P^{\lambda,a}$  l'orbite marquée. On appelle les autres orbites effectives les orbites effectives libres, ou pour abrégé, orbites libres.

**Définition 2.3.4.** Soit  $\mathcal{O} = \{z_1, \dots, z_p\}$  une orbite effective. On note pour tout  $i \in \llbracket 1, p \rrbracket$   $A_i$  l'ensemble consistant en tous les angles de rayons dynamiques aboutissant en  $z_i$  ; remarquer que  $A_i \subset \mathbb{R}/\mathbb{Z}$ . On appelle l'ensemble  $\{A_i, i \in \llbracket 1, p \rrbracket\}$  le portrait d'orbite associé à  $\mathcal{O}$ , noté  $\mathcal{P}(\mathcal{O})$ . On appelle un sous-ensemble  $\mathcal{P}$  de l'ensemble des parties de  $\mathbb{R}/\mathbb{Z}$  un portrait d'orbite s'il existe une orbite effective  $\mathcal{O}$  dont il est le portrait.

On nommera portraits libres les portraits d'une orbite libre, et portraits marqués les portraits de l'orbite marquée.

On commence par démontrer des propriétés élémentaires des portraits d'orbites.

**Lemme 2.3.5.** *Si un rayon dynamique  $R_t^a$  aboutit en un point  $z \in K^{\lambda,a}$ , alors le rayon image  $P^{\lambda,a}(R_t^a) = R_{3t}^a$  aboutit au point image  $f(z)$ . De plus, si trois rayons ou plus  $R_{t_1}^a, \dots, R_{t_k}^a$  aboutissent en un même point  $z$  non critique, alors l'ordre cyclique des angles  $t_i$  dans  $\mathbb{R}/\mathbb{Z}$  est le même que l'ordre cyclique des angles triplés  $3t_i$  dans  $\mathbb{R}/\mathbb{Z}$  ; en particulier, la restriction du triplement à  $\{t_1, \dots, t_p\}$  est injective.*

*Démonstration.* La démonstration est identique au lemme 2.1 de [Mil00] . □

**Remarque 2.3.6.** Un rayon externe dynamique ne peut aboutir sur un point critique périodique. En effet, dans ce cadre, il aboutirait à l'intérieur du bassin superattractif de son orbite : absurde. Dans le cadre d'un portrait d'orbites, les rayons externes dynamiques correspondants aux angles du portrait n'aboutissent donc jamais sur un point critique, et le lemme peut être appliqué sans restriction.

**Lemme 2.3.7.** Soit  $\mathcal{O} = \{z_1, \dots, z_p\}$  une orbite effective, on note  $\mathcal{P} = \{A_1, \dots, A_p\}$  le portrait d'orbite associé.

1. Chaque  $A_j$  est un sous-ensemble fini de  $\mathbb{Q}/\mathbb{Z}$ .
2. Pour chaque  $j$  modulo  $p$ , l'application de triplement  $t \mapsto 3t \pmod{\mathbb{Z}}$  envoie  $A_j$  bijectivement sur  $A_{j+1}$  en préservant l'ordre cyclique dans  $\mathbb{R}/\mathbb{Z}$ . On appellera valence le cardinal commun des  $A_k$ , et on le notera  $v$ .
3. Tous les angles de  $A_1 \cup \dots \cup A_p$  sont périodiques via triplement, avec une période commune  $r$  multiple de  $p$ , que l'on appellera période des rayons.
4. Les ensembles  $A_1, \dots, A_p$  sont deux à deux séparés, c'est à dire pour tous  $i \neq j$  les ensembles  $A_i$  et  $A_j$  sont contenus dans des intervalles disjoints de  $\mathbb{R}/\mathbb{Z}$ .

**Remarque 2.3.8.** L'introduction de la notion d'orbite effective, qui pourrait paraître artificielle, est motivée par le premier point du lemme.

*Démonstration.* Puisqu'un certain  $A_i$  contient un élément de  $\mathbb{Q}/\mathbb{Z}$ , il s'ensuit par stabilité des rayons par image, lemme 2.3.5, qu'un certain  $A_j$  contient un élément  $t_0 \in \mathbb{Q}/\mathbb{Z}$  périodique par triplement. Notons  $n \in \mathbb{N}^*$  la période de façon que  $3^n t_0 = t_0$ . Par le même lemme, l'application  $\eta : t \mapsto 3^n t$  envoie  $A_j$  injectivement dans lui-même, préservant l'ordre cyclique dans  $\mathbb{R}/\mathbb{Z}$  et fixant  $t_0$ . Montrons que  $\eta$  fixe tous les éléments de  $A_j$ . Si ce n'est pas le cas, soit  $t \in A_j$  non fixe par  $\eta$ . On suppose par exemple  $t_0 = \eta(t_0) < t < \eta(t)$  (le cas  $\eta(t) < t < t_0 = \eta(t_0)$  se traitant de façon symétrique), ce qui peut aussi s'écrire  $t_0 = \eta(t_0) < t < \eta(t) < t_0 = \eta(t_0)$  (noter que par injectivité  $\eta(t) \neq \eta(t_0) = t_0$ ). Puisque  $\eta$  préserve l'ordre cyclique, il s'ensuit par récurrence que :

$$t_0 < t < \eta(t) < \eta^2(t) < \dots < t_0$$

Par conséquent, les images successives de  $t$  convergeraient vers un point fixe de  $\eta$ . Ceci est impossible car chacun des points fixes de  $\eta$  est répulsif. C'est pourquoi  $\eta$  fixe tous les points de  $A_j$ . L'ensemble  $A_j$  est donc fini et inclus dans  $\mathbb{Q}/\mathbb{Z}$ , car inclus dans l'ensemble des points fixes de  $\eta$ , égal à  $\frac{1}{3^n-1}\mathbb{Z}/\mathbb{Z}$ . Remarquer ensuite que par le lemme 2.3.5 de préservation cyclique des rayons,  $A_i$  s'envoie injectivement par triplement dans  $A_{i+1}$  qui s'envoie lui-même injectivement sur  $A_i$  par itération du triplement, tout ceci avec préservation de l'ordre cyclique dans  $\mathbb{R}/\mathbb{Z}$ ; ceci démontre (1) et (2).

Pour (3). Remarquons que par (2) tous les angles du portrait sont périodiques par triplement. Soient  $t_1, t_2$  deux angles du portrait, soient  $r_1, r_2$  leur périodes par triplement, on souhaite montrer que  $r_1 = r_2$ . Leur orbite par triplement rencontre nécessairement  $A_1$ , si bien qu'on peut supposer qu'en fait  $t_1$  et  $t_2$  sont tous deux dans  $A_1$ . On peut aussi supposer que ces deux orbites sont disjointes, sinon il n'y a rien à démontrer. Notons  $\alpha$  l'application de premier retour dans  $A_1$ ,  $\alpha : t \mapsto 3^p t$ . Notons  $r'_s = \frac{r_s}{p}$  la période de  $t_s$  selon  $\alpha$ ,  $s \in \llbracket 1, 2 \rrbracket$ .

Notons  $\mathcal{O}(t_1), \mathcal{O}(t_2)$  les orbites de  $t_1, t_2$  par  $\alpha$ .

**Lemme 2.3.9.** *Il existe  $a' \in \mathbb{N}^*$  inversible modulo  $r_1$  tel que la famille  $(\alpha^i(t_1))_{0 \leq i \leq r_1' - 1}$  vérifie :*

$$t_1 < \alpha^{a'}(t_1) < \alpha^{2a'}(t_1) < \dots < \alpha^{(r_1' - 1)a'}(t_1) < t_1$$

*Démonstration.* Comme l'orbite de  $t_1$  est finie, il existe  $a'$  tel que  $t_1 < \alpha^{a'}(t_1) < \alpha^{ka'}(t_1) < t_1$  pour tout  $k \in \mathbb{N}$  distinct de 0 et 1 modulo  $r_1'$  :  $\alpha^{a'}(t_1)$ . Ces inégalités sont équivalentes à dire que  $\alpha^{a'}(t_1)$  est l'élément de  $\mathcal{O}(t_1)$  le plus proche de  $t_1$  sur le cercle trigonométrique, le cercle étant parcouru dans le sens trigonométrique.

On exploite le fait que  $\alpha$  préserve l'ordre trigonométrique : en appliquant  $\alpha^{ka'}$  aux inégalités précédentes, on obtient  $\alpha^{ka'}(t_1) < \alpha^{(k+1)a'} < \alpha^{la'} < \alpha^{ka'}(t_1)$  pour tout  $l \in \mathbb{N}$  distinct de  $k$  et de  $k + 1$ . Ceci revient encore à dire que  $\alpha^{(k+1)a'}$  est l'élément de  $\mathcal{O}(t_1)$  le plus proche de  $\alpha^{ka'}(t_1)$  sur le cercle trigonométrique, le cercle étant parcouru dans le sens trigonométrique.

Il s'ensuit que  $t_1, \alpha^{a'}(t_1), \dots, \alpha^{(r_1' - 1)a'}(t_1)$  sont dans l'ordre trigonométrique, c'est à dire que  $t_1 < \alpha^{a'}(t_1) < \alpha^{2a'}(t_1) < \dots < t_1$ , ce qu'il fallait démontrer.  $\square$

On note pour  $i \in \mathbb{Z}/r_1'\mathbb{Z}$ ,  $I_i = \{t \in \mathcal{O}(t_2), \alpha^{a'i}(t_2) < t < \alpha^{a'(i+1)}(t_2)\}$ ; remarquer que ces ensembles finis sont disjoints et que pour  $k \in \mathbb{N}$ ,  $\alpha^{a'k}(I_i) \subset I_{i+k}$  par préservation de l'ordre cyclique de  $\alpha$  sur  $A_1$ . Le caractère disjoint des orbites de  $t_1$  et  $t_2$  fait qu'il existe (un unique)  $i \in \mathbb{Z}/r_1'\mathbb{Z}$  tel que  $t_2 \in I_i$ . Alors pour tout  $k \in \mathbb{N}$ ,  $\alpha^{a'k}(t_2) \in I_{i+k}$ . Par conséquent,  $\alpha^{a'k}(t_2) = t_2$  n'est possible que si  $k$  est divisible par  $r_2'$ , ce qui démontre que  $r_2'$  divise  $r_1'$ . Comme  $t_1$  et  $t_2$  jouent des rôles symétriques, on a également  $r_2'$  divise  $r_1'$ , d'où l'égalité des périodes  $r_1' = r_2'$  et donc  $r_1 = r_2$ .

Pour (4), cela découle du fait que les rayons externes dynamiques correspondant aux portrait d'orbite sont disjoints (sauf en leur point d'aboutissement appartenant à l'orbite).  $\square$

**Définition 2.3.10.** Soit  $\mathcal{P} = \{A_1, \dots, A_p\}$  un portrait d'orbites. On appelle arc du portrait  $\mathcal{P}$  en  $A_i \in \mathcal{P}$ , ou arc de  $A_i$ , toute composante connexe du complémentaire de  $A_i \in \mathcal{P}$ . On appelle arc du portrait  $\mathcal{P}$  tout arc d'un certain  $A_i \in \mathcal{P}$ .

**Exemple 2.3.11.** *On tire un exemple de [Mil00], donnant des portraits d'orbites pour des polynômes de degré 2.*

$\mathcal{P} = \left\{ \left\{ \frac{11}{63}, \frac{44}{63}, \frac{50}{63} \right\}, \left\{ \frac{22}{63}, \frac{25}{63}, \frac{37}{63} \right\} \right\}$  est le portrait d'orbite de  $z^2 + \left(\frac{1}{4}e^{\frac{2\pi i}{3}} - 1\right)$  en son orbite parabolique de période 2. Les arcs du portrait sont donnés par les intervalles  $] \frac{11}{63}, \frac{44}{63} [$ ,  $] \frac{44}{63}, \frac{50}{63} [$ ,  $] \frac{50}{63}, \frac{11}{63} [$  ainsi que  $] \frac{22}{63}, \frac{25}{63} [$ ,  $] \frac{25}{63}, \frac{37}{63} [$ ,  $] \frac{37}{63}, \frac{22}{63} [$ .

**Définition 2.3.12.** Soit  $\mathcal{O}$  une orbite effective de valence  $v \geq 2$  et période  $p \geq 1$ , soit  $z_i \in \mathcal{O}$ . Les  $v$  rayons aboutissant en  $z_i$  partagent le plan dynamique  $\mathbb{C}$  en  $v$  ouverts, dénommés secteurs basés en  $z_i$ . On appelle secteur de  $\mathcal{O}$  un secteur de  $\mathcal{O}$  basé en un certain  $z_i \in \mathcal{O}$ .

**Remarque 2.3.13.** Soit  $\mathcal{O} = \{z_1, \dots, z_n\}$  une orbite effective d'une application  $f = P^{\lambda, a}$ , et  $\mathcal{P} = \{A_1, \dots, A_p\}$  le portrait associé. Soit  $i \in \llbracket 1, p \rrbracket$ . Il y a une bijection naturelle



entre les secteurs  $S$  basés en  $z_i$  de  $\mathcal{O}$  et les arcs  $I$  en  $A_i$  de  $\mathcal{P}$  donnée par :

$$S \rightarrow \{t \in \mathbb{R}/\mathbb{Z}, R_t^a \subset S\} = \{t \in \mathbb{R}/\mathbb{Z}, R_t^a \cap S \neq \emptyset\}$$

Remarquer que les rayons  $R_t^a$  où  $t \in I$  n'aboutissent pas en  $z_i$ , contrairement aux rayons correspondants à  $t \in \partial I$ .

**Définition 2.3.14.** Soit  $\mathcal{O}$  une orbite effective, et  $\mathcal{P}$  le portrait associé. On appelle largeur angulaire d'un secteur de  $\mathcal{O}$  la longueur de l'arc de  $\mathcal{P}$  associé.

## 2.4 Portraits marqués

Dans cette section, on pose  $q \in \mathbb{N}$ ,  $q \geq 2$ ,  $p \in (\mathbb{Z}/q\mathbb{Z})^*$  de façon que le multiplicateur du point fixe parabolique marqué soit  $\lambda = \exp(2i\pi\frac{p}{q})$ . Noter que dans le cas  $q = 1$ , il n'y a qu'un portrait d'orbite, donné par  $\{\{0\}\}$ , c'est pourquoi on étudie les portraits d'orbites seulement pour  $q \geq 2$ .

On notera  $p' \in (\mathbb{Z}/q\mathbb{Z})^*$  l'inverse de  $p$ . On notera  $r$  l'application de translation par  $\frac{p}{q}$  dans  $\mathbb{R}/\mathbb{Z}$ , et pour  $k \in \mathbb{N}$  on note  $U^k$  l'union des orbites de  $0, \frac{1}{kq}, \dots, \frac{k-1}{kq}$  par cette application. On note pour  $k \in \mathbb{N}^*$ ,  $r^{(k)}$  l'application que  $r$  induit sur  $U^k$ . Par convention, à chaque fois que l'on mentionnera un  $q$ -uplet, les indices seront dans  $\mathbb{Z}/q\mathbb{Z}$ .

**Proposition 2.4.1.** Soit  $\mathcal{P} = \{A\}$  un portrait d'orbite marqué associé à une application de la forme  $P = P^{\lambda, a}$ ,  $a \in \mathbb{C}$ . Alors  $A$  est stable par multiplication par 3, et il existe  $k \in \mathbb{N}^*$  tel que cette application de multiplication sur  $A$  est conjuguée par une application strictement croissante à l'application  $r^{(k)}$ . Ceci signifie que  $A$  est de cardinal  $qk$  et est égal à l'union de  $k$  orbites de  $\sigma_3$ ,  $\Theta_i = \{\theta_0^i, \dots, \theta_{q-1}^i\}$  avec  $i \in [0, k-1]$ ,  $\theta_{i+1}^i = 3\theta_i^i$  pour  $i \in \mathbb{Z}/q\mathbb{Z}$  et

$$\begin{aligned} \theta_0^0 &< \theta_0^1 < \theta_0^2 < \dots < \theta_0^{k-1} < \\ \theta_{p'}^0 &< \theta_{p'}^1 < \theta_{p'}^2 < \dots < \theta_{p'}^{k-1} < \\ &\vdots \\ \theta_{p'(q-1)}^0 &< \dots < \theta_{p'(q-1)}^k < \\ &\theta_0^0 \end{aligned}$$

On appelle  $\frac{p}{q}$  le nombre de rotation de l'orbite  $\Theta$ , et nombre de rotation du portrait marqué, défini modulo 1.

*Démonstration.* On démontre le lemme suivant :

**Lemme 2.4.2.** Tout rayon externe convergeant vers 0 converge tangentiellement à un axe répulsif.

*Démonstration.* Le rayon externe doit être disjoint du bassin attractif, et donc des pétales attractifs. Comme il existe des pétales attractifs larges (voir définition 4.1.3), un tel rayon externe doit converger vers 0 en étant dans un interpétale, ce qui impose de converger tangentiellement à un axe répulsif.  $\square$

On utilise également que pour tout axe répulsif, il y a au moins un rayon externe convergeant vers ce dernier (voir notes d'Orsay [DHL<sup>+</sup>84] exposé IX.II.2 proposition 5).

Enfin, on utilise le lemme 2.3.5 et 2.3.7.

□

**Remarque 2.4.3.** Le choix des orbites définissant  $U^k$  est indifférent tant que ces orbites sont disjointes. Les orbites  $\Theta_i$  peuvent être aussi caractérisées de la façon suivante : ce sont des orbites de nombre de rotation  $\frac{p}{q}$  entrelacées dans l'ordre donné par les indices  $i = 0, 1, \dots, k-1$ , c'est à dire qu'elles peuvent s'écrire  $\Theta_i = \{\theta_0^i, \dots, \theta_{q-1}^i\}$ , avec

$$\begin{aligned} \theta_0^0 &< \theta_0^1 < \theta_0^2 < \dots < \theta_0^{k-1} < \\ \theta_1^0 &< \theta_1^1 < \theta_1^2 < \dots < \theta_1^{k-1} < \\ &\vdots \\ \theta_{q-1}^0 &< \dots < \theta_{q-1}^k < \\ &\theta_0^0 \end{aligned}$$

l'indexation étant a priori dans un ordre indifférent à celui donné par les itérations de  $\sigma_3$ . A posteriori, le fait que  $\theta_0^i < \theta_1^i < \dots < \theta_{q-1}^i < \theta_0^{i+1}$  et que  $\sigma_3$  soit de nombre de rotation  $p/q$  sur  $\Theta_i$  implique nécessairement que les termes adjacents sont obtenus les uns à partir des autres par application de  $\sigma_3^{p'}$ . D'où la caractérisation annoncée.

Pour abrégé, on notera  $\Theta_0 < \Theta_1 < \dots < \Theta_{k-1} < \Theta_0$  pour signifier que les  $\Theta_i$  sont entrelacés dans cet ordre. Noter enfin que pour  $k = 2$ , le cas qui nous intéressera, la question de l'ordre ne se pose pas.

**Définition 2.4.4.** Soit  $\mathcal{P}$  un sous-ensemble quelconque de  $P(\mathbb{R}/\mathbb{Z})$ . On dira que  $\mathcal{P}$  est un portrait d'orbite marqué formel s'il s'écrit sous la forme  $\mathcal{P} = \{A\}$  et vérifie la conclusion de la proposition 2.4.1. On appelle toujours  $\frac{p}{q}$  le nombre de rotation du portrait marqué formel.

Tout portrait d'orbite marqué est un portrait d'orbite marqué formel. La réciproque indiquée ci-dessous découle du théorème principal de [BOT22] :

**Théorème 2.4.5.** *Tout portrait d'orbite formel marqué  $\mathcal{P}$  est réalisé par un portrait d'orbite, c'est à dire qu'il existe un polynôme  $P^{\lambda,a} : \mathbb{C} \rightarrow \mathbb{C}$  dont le portrait d'orbite marqué est égal à  $\mathcal{P}$ .*

Notre but est de caractériser complètement les portraits d'orbite marqués formels.

**Définition 2.4.6.** On note  $\mathcal{C}$  l'ensemble des cycles de  $\sigma_3$  d'angle de rotation  $\frac{p}{q}$ .

L'étude des portraits d'orbites marqués formels se ramène à l'étude des cycles  $\mathcal{C}$  et de la relation d'entrelacement entre ces cycles.

Soit  $\Theta \in \mathcal{C}$ .

**Remarque 2.4.7.** Chaque élément de  $\Theta$  est fixe par  $\theta \mapsto 3^q\theta$ . L'orbite  $\Theta$  est donc un sous-ensemble de  $\{\frac{x}{3^q-1}, x \in \llbracket 0, 3^q - 1 \rrbracket\} \subset \mathbb{R}/\mathbb{Z}$ . Chaque élément de  $\theta \in \Theta$  peut alors être représenté de façon unique par un mot  $w = a_0 \cdots a_{q-1}$  de taille  $q$  dans l'alphabet  $\{0, 1, 2\}$  distinct de  $2 \cdots 2$ , par l'écriture en base 3 :  $\theta = \sum_{i \in \mathbb{Z}/q\mathbb{Z}} \frac{a_i 3^{q-1-i}}{3^q-1}$ . L'angle  $\theta$  est défini modulo 1, avec la convention suivante pour  $i \in \mathbb{Z}/q\mathbb{Z}$  :  $\frac{3^i}{3^q-1} = \frac{3^s}{3^q-1}$  pour tout  $s \in \mathbb{N}$  tel que  $s = i \pmod q$  (la valeur modulo 1 de  $\frac{3^s}{3^q-1}$  étant indépendante du  $s$  choisi puisque en fixant  $s_1, s_2 \in \mathbb{N}$  tels que  $s_1 < s_2$  et  $s_1 - s_2$  divisible par  $q$  on obtient  $3^{s_1} - 3^{s_2} = 3^{s_1}(1 - 3^{s_2-s_1})$  divisible par  $3^q - 1$ ). On appelle  $w$  le mot associé à  $\theta$ .

Par convention, les mots seront désignés par les lettres  $u, v, w$  et les lettres par  $a, b, c$ . On note  $\mathcal{W}$  l'ensemble des mots de taille  $q$  de l'alphabet  $\{0, 1, 2\}$ . On note  $W = W(\Theta)$  l'ensemble des mots associés aux éléments de  $\Theta$ . On définit dans  $\mathcal{W}$  l'application de permutation cyclique  $\phi$  qui à un mot  $w = a_0 \cdots a_{q-1} \in W$  associe le mot :

$$\phi(w) = a_1 \cdots a_q$$

où l'on rappelle que les indices sont définis modulo  $q$ . Remarquer que si  $w$  est le mot associé à  $\theta$ ,  $\phi(w)$  est le mot associé à  $3\theta$ , ce qui implique que  $\phi$  induit une application  $\phi_W$  de  $W$  dans  $W$ , qui est conjuguée à l'application de multiplication par 3 de  $\Theta$  dans  $\Theta$ . On munit ces mots de l'ordre lexicographique, de telle façon que pour  $\theta_0, \theta_1, \theta_2 \in A$  dont  $w_0, w_1, w_2 \in W$  sont les mots associés,  $\theta_0 < \theta_1 < \theta_2$  équivaut à  $w_k < w_{k+1} < w_{k+2}$  où  $w_k$  est le mot minimal pour l'ordre lexicographique parmi  $w_0, w_1, w_2$  et l'addition des indices doit être comprise modulo 3. Voici une conséquence de ce fait :

**Proposition 2.4.8.** *Soit  $\theta_0, \dots, \theta_{n-1} \in A$  dont  $w_0, \dots, w_{n-1}$  les mots associés. Alors en notant  $k$  tel que  $w_k$  minimal pour l'ordre lexicographique parmi  $w_0, \dots, w_{n-1}$ , on a  $\theta_0 < \dots < \theta_{n-1} < \theta_0$  si et seulement si  $w_k < w_{k+1} < \dots < w_{k+n-1}$  où les indices sont compris modulo  $n$ .*

*Démonstration.* Dire que  $\theta_0 < \dots < \theta_{n-1} < \theta_0$  revient à dire que  $\theta_k < \dots < \theta_{k-1} < \theta_k$ , ce qui équivaut encore à  $\theta_k < \theta_{k+i} < \theta_{k+i+1}$  pour  $i = 1, \dots, n-2$ . Ceci permet d'appliquer le fait précédent pour obtenir  $w_k < w_{k+i} < w_{k+i+1}$  pour  $i = 1, \dots, n-2$  c'est à dire  $w_k < w_{k+1} < \dots < w_{k-1}$ .  $\square$

L'étude de  $\mathcal{C}$  se ramène donc à l'étude des sous-ensembles  $W = \{w_0, \dots, w_{q-1}\}$  de  $\mathcal{W}$  tels que  $w_0 < w_{p'} < \dots < w_{p'(q-1)}$  et  $w_{k+1} = \phi(w_k)$ . Notons  $\mathfrak{X}$  l'ensemble de ces  $W$ .

**Proposition 2.4.9.** *Les éléments  $W$  de  $\mathfrak{X}$  sont caractérisés par la propriété suivante :  $W$  est l'orbite par  $\phi$  d'un élément  $w$  de la forme  $a_0 \cdots a_{q-1}$ , avec  $a_0 \leq a_{p'} \leq \dots \leq a_{(q-2)p'} \leq a_{(q-1)p'}$  et  $a_{q-1-p'} < a_{q-1}$ , avec unicité d'un tel élément  $w$  dans l'orbite de  $W$ .*

**Exemple 2.4.10.** *Si  $p = 2$  et  $q = 7$  (donc  $p' = 4$ ), la liste exhaustive des éléments de  $\mathfrak{X}$  est :*

$$\left\{ \begin{array}{l} w^0 = 0001001 \\ w^1 = 0002001 \\ w^2 = 0002002 \\ w^3 = 0012002 \end{array} \right. \quad \left\{ \begin{array}{l} w^4 = 0012012 \\ w^5 = 0112012 \\ w^6 = 0112112 \\ w^7 = 1112112 \end{array} \right.$$

Ces mots ont été construits à partir d'un calcul préalable de leurs lettres  $a_0, a_4, a_1, a_5, a_2, a_6, a_3$  dans cet ordre (incrémentés de  $p' = 4$  modulo 7) : lorsque les lettres sont dans cet ordre, on doit ajouter 0, 1 ou 2 pour passer d'une lettre à la suivante, à l'exception du passage de  $a_2$  à  $a_6$  où on ne peut ajouter que 1 ou 2. La somme des ajouts ne doit pas dépasser 2, puisque la dernière lettre  $a_3$  doit être égale à 1 ou 2. Ceci permet d'énumérer toutes les possibilités.

L'usage des applications  $\alpha_p$  et  $\alpha_{p'}$  (voir lemme 2.4.11) est nécessaire en vue de faire une démonstration générale, claire et rigoureuse concernant l'énumération des  $w^k$ .

*Démonstration.* Soit  $w$  de la forme  $a_0 \cdots a_{q-1}$ , où  $a_0 \leq a_{p'} \leq \cdots \leq a_{(q-2)p'} \leq a_{(q-1)p'}$  et  $a_{q-1-p'} < a_{q-1}$ . Remarquer qu'on a l'inégalité  $a_0 < a_{(q-1)p'}$  et l'égalité modulo  $q$  :  $(q-1)p' = q - p'$ , si bien que  $a_{(q-1)p'} = a_{q-p'} > a_0 = a_q$ . On appelle l'inégalité  $a_{q-p'} > a_q$  l'inégalité renversée puisque pour tout  $i \in \mathbb{Z}/p\mathbb{Z} \setminus \{q - p'\}$  on a  $a_i \leq a_{i+p'}$  avec l'unique exception pour  $i = q - p'$  pour lequel on a  $a_i > a_{i+p'}$ .

Montrons que l'orbite de  $w$  par  $\phi$  est un élément de  $\mathfrak{X}$ . Soit  $k \in \mathbb{Z}/q\mathbb{Z} \setminus \{q - p'\}$ . Comme  $a_k \leq a_{k+p'}$ ,  $a_{k+1} \leq a_{k+1+p'}$ ,  $\dots$ ,  $a_{q-1-p'} < a_{q-1}$  (remarquer que la comparaison renversée  $a_{q-p'} > a_q$  n'est pas dans la liste) il s'ensuit que  $\phi^k(w) < \phi^{k+p'}(w)$  par définition de l'ordre lexicographique. Donc l'orbite de  $w$  appartient à  $\mathfrak{X}$ .

Soit  $W = \{w_0, \dots, w_{q-1}\} \in \mathfrak{X}$  avec  $w_{k+p'} = \phi(w_k)$  pour  $k \in \mathbb{Z}/q\mathbb{Z}$  et  $w_0$  choisi de façon que  $w_0 < w_{p'} < \cdots < w_{p'(q-1)}$ . On écrit  $w_0 = a_0 \cdots a_{q-1}$ , ainsi  $w_k = a_k \cdots a_{k+q-1}$  pour  $k \in \mathbb{Z}/q\mathbb{Z}$ . Pour  $k \in \mathbb{Z}/q\mathbb{Z} \setminus \{q - 2p'\}$ ,  $w_k \leq w_{k+p'}$  et donc  $a_k \leq a_{k+p'}$  (comparaison des premières lettres). Ces inégalités impliquent que  $a_0$  est minimal parmi les  $a_k$ ,  $k \in \mathbb{Z}/q\mathbb{Z}$ , et  $a_{q-p'}$  maximal. Comme  $w_{q-1-p'} \leq w_{q-1}$ , on doit avoir  $a_{q-1-p'} < a_{q-1}$ . En effet, dans le cas contraire, on a  $a_{q-1-p'} = a_{q-1}$ , égalité des premières lettres de  $w_{q-1-p'}$ ,  $w_{q-1}$ , donc par comparaison des deuxièmes lettres  $a_{q-p'} \leq a_q = a_0$ ; ceci implique donc l'égalité de toutes les lettres de  $w_0$ , ce qui contredit les inégalités strictes  $w_0 < \cdots < w_{q-1}$ . D'où la caractérisation annoncée.  $\square$

**Lemme 2.4.11.** Notons pour  $l \in (\mathbb{Z}/q\mathbb{Z})^*$  l'application  $\alpha_l$  de  $\mathcal{W}$  vers  $\mathcal{W}$  définie par :

$$\alpha_l(a_0 \cdots a_{q-1}) = a_0 a_l \cdots a_{l(q-1)}$$

pour  $w = a_0 \cdots a_{q-1} \in \mathcal{W}$ .

Alors :

1.  $\alpha_1 = Id_{\mathcal{W}}$  et pour  $l_1, l_2 \in (\mathbb{Z}/q\mathbb{Z})^*$ ,  $\alpha_{l_1} \circ \alpha_{l_2} = \alpha_{l_1 l_2}$
2.  $\phi \circ \alpha_l = \alpha_l \circ \phi^l$

3. Par la bijection entre  $W$  et  $\Theta$ , la restriction de  $\alpha_p$  à  $W$  est conjuguée à l'application :

$$\theta = \sum_{i \in \mathbb{Z}/q\mathbb{Z}} \frac{a_i 3^{q-1-i}}{3^q - 1} \mapsto \sum_{i \in \mathbb{Z}/q\mathbb{Z}} \frac{a_i 3^{q-1-p'i}}{3^q - 1}$$

Ceci peut se réécrire encore :

$$\theta = \sum_{i \in \mathbb{Z}/q\mathbb{Z}} \frac{b_i 3^i}{3^q - 1} \mapsto \sum_{i \in \mathbb{Z}/q\mathbb{Z}} \frac{b_i 3^{(i+1)p'-1}}{3^q - 1}$$

On rappelle que la convention de la remarque 2.4.7, permet de définir les sommes indépendamment du choix du représentant.

*Démonstration.* Les deux premiers points résultent d'un calcul immédiat. Le troisième point résulte de la formule  $\sum_{i \in \mathbb{Z}/q\mathbb{Z}} \frac{a_i 3^{q-1-p'i}}{3^q - 1} = \sum_{i \in \mathbb{Z}/q\mathbb{Z}} \frac{a_{pi} 3^{q-1-i}}{3^q - 1}$ . Sa retraduction est immédiate.  $\square$

Notons  $\tilde{w}^k = 0^{q-p}1^p + 0^{q-k}1^k$ , où l'addition entre mots est définie comme l'addition lettre à lettre (remarquer qu'elle n'excède jamais 2),  $k \in \llbracket 0, q \rrbracket$  et avec la convention  $p \in \llbracket 1, q-1 \rrbracket$ . Voici une conséquence du lemme 2.4.11 :

**Conséquence 2.4.12.** *Les éléments de  $\mathfrak{X}$  sont au nombre de  $q+1$ . Ce sont les orbites par  $\phi$  des éléments  $w^k = \alpha_p(\tilde{w}^k)$ .*

**Remarque 2.4.13.** Dans le cas  $p=1$ , les éléments  $W$  de  $\mathfrak{X}$  sont caractérisés par la propriété suivante :  $W$  est l'orbite par  $\phi$  d'un élément  $w$  de la forme  $a_0 \cdots a_{q-1}$ , avec  $a_0 \leq \cdots \leq a_{q-2} < a_{q-1}$ . Dans ce cadre, l'ensemble  $\mathfrak{X}$  se comprend aisément sans changement de variable : il est constitué par l'orbite de  $w^0 = 0^{q-1}1$ , et celle des éléments  $w^k = 0^{q-k}1^{k-1}2$  où  $k \in \llbracket 1, q \rrbracket$ .

*Démonstration.* Quoique le cas  $p=1$  soit clair, le cas général mérite vérification.

Soit  $W = \{w_0, \dots, w_{q-1}\}$  un élément de  $\mathcal{W}$ ,  $w_0$  minimal pour l'ordre lexicographique et  $w_{k+1} = \phi(w_k)$  pour  $k \in \mathbb{Z}/q\mathbb{Z}$ . On écrit  $w_0 = a_0 \cdots a_{q-1}$ . Dire que  $W \in \mathfrak{X}$  revient à dire  $a_0 \leq \cdots \leq a_{(q-1)p'}$  et  $a_{q-1-p} < a_{q-1}$ . Retraduisons cette condition sur le mot  $\tilde{w} = \alpha_{p'}(w_0) = a_0 a_{p'} \cdots a_{p'(q-1)} = b_0 \cdots b_{q-1}$ . La première partie de la condition se traduit ainsi :  $b_0 \leq b_1 \leq \cdots \leq b_{q-1}$ . La deuxième partie se traduit par l'inégalité  $b_s < b_{s+1}$  où  $s$  est l'unique entier dans  $\llbracket 0, q-1 \rrbracket$  tel que  $sp' = q-1-p'$  modulo  $q$ , c'est à dire en multipliant par  $p$ ,  $s = pq - p - 1 = q - p - 1$  modulo  $q$ . Donc  $s = q - p - 1$ , cet entier appartenant bien à l'intervalle  $\llbracket 0, q-1 \rrbracket$  avec la convention  $p \in \llbracket 1, q-1 \rrbracket$ .

Ainsi, la condition sur les  $b_k$  se retraduit ainsi :  $(b_k)$  est croissante à valeur dans  $\{0, 1, 2\}$  et  $b_s + 1 \leq b_{s+1}$ . Si il n'y a pas de deuxième incrément de 1,  $\tilde{w}$  doit être égal à  $\tilde{w}^0 = 0^{q-p}1^p$ . Sinon, il y a un deuxième incrément de 1 (qui peut se regrouper avec le premier incrément de 1 en un incrément de 2), auquel cas  $\tilde{w}$  est égal à l'un des  $\tilde{w}^k = 0^{q-p}1^p + 0^{q-k}1^k$ ,  $k \in \llbracket 0, q \rrbracket$ . Réciproquement, chacun des  $\tilde{w}^k = b_0^k \cdots b_{q-1}^k$ ,  $k \in \llbracket 0, q \rrbracket$  vérifient les conditions retraduites. Il s'ensuit que les éléments de  $\mathfrak{X}$  sont au nombre de  $q+1$ , donnés par les orbites par  $\phi$  des éléments  $w^k = \alpha_p(\tilde{w}^k)$  où  $k \in \llbracket 0, q \rrbracket$ .  $\square$

**Remarque 2.4.14.** Notons désormais  $w^k = \alpha_p(\tilde{w}^k)$  les mots ainsi obtenus.

Soit  $\delta_{i,j}$  le symbole de Kronecker, et pour  $i \in \mathbb{Z}/q\mathbb{Z}$ ,  $e_i$  le mot  $\delta_{0,i} \cdots \delta_{q-1,i}$ .

On a  $\phi(e_i) = e_{i-1}$  et  $\alpha_p(e_i) = e_{p'i}$ . Donc pour  $k \in \llbracket 0, q \rrbracket$ ,  $\alpha_p(0^{q-k}1^k) = \sum_{s=1}^k e_{-sp'}$ .  
Noter que le dernier terme de la somme est  $e_{-1} = e_{q-1}$  lorsque  $k = p$ . On obtient l'expression pour  $w^k$  :

$$w^k = \sum_{s=1}^p e_{-sp'} + \sum_{s=1}^k e_{-sp'}$$

**Exemple 2.4.15.** Si  $p = 2$  et  $q = 7$  (donc  $p' = 4$ ) on a  $e_3 = 0001000$ ,  $\phi(e_3) = 0010000 = e_2$  et  $\alpha_p(e_3) = e_{4 \times 3} = e_5 = 0000010$ .

On obtient également :

$$\begin{cases} w^0 &= e_{-p'} + e_{-2p'} = 0001000 + 0000001 = 0001001 \\ w^1 &= e_{-p'} + e_{-2p'} + e_{-p'} = 0002001 \\ w^2 &= e_{-p'} + e_{-2p'} + e_{-p'} + e_{-2p'} = 0002002 \\ w^3 &= e_{-p'} + e_{-2p'} + e_{-p'} + e_{-2p'} + e_{-3p'} = 0012002 \\ w^4 &= 0012012 \\ w^5 &= 0112012 \\ w^6 &= 0112112 \\ w^7 &= 1112112 \end{cases}$$

**Remarque 2.4.16.** Notons  $\tilde{\gamma}_k = \sum_{\substack{i \in \mathbb{Z}/q\mathbb{Z} \\ 0 \leq i \leq k-1}} 3^i / 3^q - 1$ , ainsi que  $\gamma_k = \sum_{\substack{i \in \mathbb{Z}/q\mathbb{Z} \\ 0 \leq i \leq k-1}} 3^{p'(i+1)-1} / 3^q - 1$

pour  $k \in \llbracket 0, q \rrbracket$  (on rappelle que les sommes sont réglées par la convention de la remarque 2.4.7). L'angle associé à  $\tilde{w}^k$  est  $\tilde{\theta}^k = \tilde{\gamma}_p + \tilde{\gamma}_k$ , et donc l'angle associé à  $w^k$  est  $\theta^k = \gamma_p + \gamma_k$ .  
On note désormais cet angle  $\theta^k$ .

On a donc démontré la :

**Proposition 2.4.17.** Les éléments de  $\mathcal{C}$  sont formés par les orbites  $\Theta_k$  d'un  $\theta^k$  où  $k \in \llbracket 0, q \rrbracket$ .

**Exemple 2.4.18.** Comme dans les exemples précédents, on suppose  $p = 2$ ,  $q = 7$ .  
Notons  $W_k$  l'ensemble des mots associés à  $\Theta_k$ . On écrit l'orbite de  $W_k$  suivant l'ordre lexicographique  $<$  :  $W_k = \{w^k, \phi^{p'}(w^k), \dots, \phi^{q-p'}(w^k)\}$  avec  $w^k < \phi^{p'}(w^k) < \dots < \phi^{q-p'}(w^k)$  (on rappelle que  $w^k$  est l'élément minimal pour l'ordre lexicographique de son orbite par  $\phi$ ).

$$\left\{ \begin{array}{l} W_0 = \{0001001, 0010001, 0010010, 0100010, 0100100, 1000100, 1001000\} \\ W_1 = \{0002001, 0010002, 0020010, 0100020, 0200100, 1000200, 2001000\} \\ W_2 = \{0002002, 0020002, 0020020, 0200020, 0200200, 2000200, 2002000\} \\ W_3 = \{0012002, 0020012, 0120020, 0200120, 1200200, 2001200, 2002001\} \\ W_4 = \{0012012, 0120012, 0120120, 1200120, 1201200, 2001201, 2012001\} \\ W_5 = \{0112012, 0120112, 1120120, 1201120, 1201201, 2011201, 2012011\} \\ W_6 = \{0112112, 1120112, 1121120, 1201121, 1211201, 2011211, 2112011\} \\ W_7 = \{1112112, 1121112, 1121121, 1211121, 1211211, 2111211, 2112111\} \end{array} \right.$$

Ceci permet d'écrire  $\Theta_k$ , qui est rappelons-le défini modulo 1 (noter que  $2186 = 3^q - 1$ ) :

$$\left\{ \begin{array}{l} \Theta_0 = \frac{1}{2186} \{28, 82, 84, 246, 252, 738, 756\} \\ \Theta_1 = \frac{1}{2186} \{55, 83, 165, 249, 495, 747, 1485\} \\ \Theta_2 = \frac{1}{2186} \{56, 164, 168, 492, 504, 1476, 1512\} \\ \Theta_3 = \frac{1}{2186} \{137, 167, 411, 501, 1233, 1503, 1513\} \\ \Theta_4 = \frac{1}{2186} \{140, 410, 420, 1230, 1260, 1504, 1594\} \\ \Theta_5 = \frac{1}{2186} \{383, 419, 1149, 1257, 1261, 1585, 1597\} \\ \Theta_6 = \frac{1}{2186} \{392, 1148, 1176, 1258, 1342, 1588, 1840\} \\ \Theta_7 = \frac{1}{2186} \{1121, 1175, 1177, 1339, 1345, 1831, 1849\} \end{array} \right.$$

Il reste à étudier la relation d'entrelacement sur ces cycles. On démontre d'abord la condition nécessaire suivante pour l'entrelacement :

**Lemme 2.4.19.** *Soient  $E_1, E_2$  deux sous-ensembles de  $\mathbb{R}/\mathbb{Z}$  finis. Soit  $I$  un intervalle ouvert de  $\mathbb{R}/\mathbb{Z}$ . Pour que  $E_1, E_2$  soient entrelacés, il faut que la différence des cardinaux de  $E_1 \cap I, E_2 \cap I$  n'excède pas 1 en valeur absolue.*

*Démonstration.* Si cela n'était pas le cas, on aurait par le lemme des tiroirs deux angles successifs de  $(E_1 \cup E_2) \cap I$  (et donc de  $E_1 \cup E_2$ ) appartenant à un même  $E_i$ , ce qui contredit l'entrelacement.  $\square$

Cette condition nécessaire va pouvoir exploitée dans notre cadre. En effet :

**Lemme 2.4.20.** *Soit  $I = ]0, \frac{1}{2}[$ . Alors  $|\Theta_k \cap I| = q - k$  (noter que  $\Theta_k$  est disjoint de  $\{0, \frac{1}{2}\}$ ).*

Ce lemme découle de la propriété technique suivante sur les mots  $w^k$  :

**Lemme 2.4.21.** *Dans le mot  $w^k$  :*

- Si  $k \leq p$ ,  $w^k$  contient exactement  $q - p$  fois 0,  $p - k$  fois 1,  $k$  fois 2. De plus, tous les 2 sont précédés (cycliquement) par un mot de la forme  $02^s$  où  $s \in \mathbb{N}$ .

- Si  $k \geq p$ ,  $w^k$  contient exactement  $q - k$  fois 0,  $k - p$  fois 1,  $p$  fois 2. Tous les 1 apparaissant sont suivis (cycliquement) par un mot de la forme  $1^s 2$  où  $s \in \mathbb{N}$ .

Donc, en considérant une permutation cyclique  $v^k = \phi^s(w^k)$  de  $w^k$ , on a l'équivalence (noter que  $v^k \neq 1 \cdots 1$ ) :

- Si  $k \leq p$ ,  $v^k < 1 \cdots 1$  si et seulement si  $v^k$  commence par 0 ou 1.
- Si  $k \geq p$ ,  $v^k < 1 \cdots 1$  si et seulement si  $v^k$  commence par 0.

Dans tous les cas, le nombre de permutations cycliques de  $w^k$  inférieures à  $1 \cdots 1$  est exactement égal à  $q - k$ .

**Exemple 2.4.22.** On reprend l'exemple 2.4.14, où l'on rappelle que  $p = 2$ ,  $p' = 4$ ,  $q = 7$ , colonne de gauche, et un autre exemple, colonne de droite, avec  $p = 5$ ,  $p' = 3$ ,  $q = 7$ .

$$\left\{ \begin{array}{l} w^0 = 0001001 \\ w^1 = 0002001 \\ w^2 = 0002002 \\ w^3 = 0012002 \\ w^4 = 0012012 \\ w^5 = 0112012 \\ w^6 = 0112112 \\ w^7 = 1112112 \end{array} \right. \quad \left\{ \begin{array}{l} w^0 = 0110111 \\ w^1 = 0110211 \\ w^2 = 0210211 \\ w^3 = 0210221 \\ w^4 = 0220221 \\ w^5 = 0220222 \\ w^6 = 0221222 \\ w^7 = 1221222 \end{array} \right.$$

On constate, lors du passage de  $w^k$  à  $w^{k+1}$ , une première phase, où les 1 sont remplacés successivement par des 2 ( $k < p$ ), et une deuxième phase, où les 0 sont remplacés successivement par des 1 ( $k \geq p$ ). Lors de la première phase, les 2 apparaissent toujours immédiatement à droite d'un 0 ou d'un 2. Lors de la deuxième phase, les 1 apparaissent toujours immédiatement à gauche d'un 1 ou d'un 2. C'est ce qu'il s'agit de démontrer dans le cas général.

Un point très important pour l'étude des deux phases sera que le dernier 1 à être transformé en 2 est le 1 en position  $q - 1$  (il y aura toujours un 1 en position  $q - 1$  lors de la phase  $k < p$ , peu important les valeurs de  $p, q$ ), ceci permettra de tirer profit d'une division euclidienne bien choisie dans chaque phase.

*Démonstration.* Le décompte du nombre de 0, 1, 2 découle immédiatement de la conséquence 2.4.12. Noter que dans le cas  $k = p$ , le mot  $w^k$  ne contient que des 0 et des 2, et donc il n'y a rien de plus à démontrer. On suppose désormais  $k \neq p$ . Notons  $I_0, I_1, I_2$  l'ensemble des indices de  $w^k$  en lesquels apparaissent respectivement les nombres 0, 1, 2. On rappelle que les indices de  $w^k$  sont considérés modulo  $q$ .

Supposons  $k < p$ . On a grâce à l'expression de  $w^k$  en fonction des  $e_k$  (voir remarque



2.4.14 ) :

$$\begin{aligned} I_2 &= \{-p', -2p', \dots, -kp'\} \\ I_1 &= \{-(k+1)p', \dots, -pp'\} \text{ (noter que } -pp' = -1 \text{ )} \\ I_0 &= \{-1 - p', \dots, -1 - (q-p)p'\} \text{ (noter que } -1 - (q-p)p' = 0 \text{ )} \end{aligned}$$

où les indices sont considérés modulo  $q$ .

Notons  $\phi$  la translation de 1 dans  $\mathbb{Z}/q\mathbb{Z}$ , et  $\tau = \phi^{-1}$  la translation de  $-1$ . Soit  $B = \{-p', -2p', \dots, -(q-p)p'\} = \phi(I_0)$ . On note aussi pour  $l \in \mathbb{N} \cup \{-1\}$ ,  $B_l = \{-p', -2p', \dots, -(l+1)(q-p)p'\}$  (noter que  $B_{-1} = \emptyset$  et  $B_0 = B$ ).

**Lemme 2.4.23.** *Pour  $l \in \mathbb{N} \cup \{-1\}$ ,  $B_l = \cup_{s=1}^{l+1} \phi^s(I_0)$ .*

*Démonstration.* On a :

$$\begin{aligned} \phi(I_0) &= \{-p', -2p', \dots, -(q-p)p'\} \text{ (noter que } -(q-p)p' = 1 \text{ )} \\ \phi^2(I_0) &= \{1 - p', 1 - 2p', \dots, 1 - (q-p)p'\} \text{ (noter que } 1 - (q-p)p' = 2 \text{ )} \\ &\vdots \\ \phi^l(I_0) &= \{l - p', l - 2p', \dots, l - (q-p)p'\} \text{ (noter que } l - (q-p)p' = l + 1 \text{ )} \\ \phi^{l+1}(I_0) &= \{l + 1 - p', l + 1 - 2p', \dots, l + 1 - (q-p)p'\} \end{aligned}$$

Ceci permet de dire que  $\cup_{i=1}^{l+1} \phi^i(I_0)$  est l'ensemble des éléments d'une suite arithmétique finie de  $\mathbb{Z}/q\mathbb{Z}$  de cardinal  $(l+1)(q-p)$  de raison  $-p'$  et de premier terme  $q-p'$ . Un tel ensemble est par définition égal à  $B_l$ .  $\square$

Montrer que tous les 2 sont précédés (cycliquement) par un mot de la forme  $02^s$  où  $s \in \mathbb{N}$  revient exactement à montrer que  $\tau(I_2) \subset I_0 \cup I_2$ . Si  $k \leq q-p$ , on a immédiatement  $I_2 \subset B$  et donc  $\tau(I_2) \subset I_0$ . Dans le cas général, on a par division euclidienne de  $k$  par  $q-p$ , en écrivant  $k = l(q-p) + r$  où  $0 \leq r < q-p$  :  $B_{l-1} \subset I_2 \subset B_l$  (ce jeu d'inclusions doit se comprendre ainsi :  $I_2$  est une union successive de translatés à droite de  $I_0$ , avec le dernier d'entre eux incomplet). On a alors  $\tau(I_2) \subset \tau(B_l) \subset I_0 \cup B_{l-1} \subset I_0 \cup I_2$ , ce qui conclut.

On suppose  $k > p$ . On reconstitue un raisonnement semblable au cas  $k < p$ . On a grâce à l'expression de  $w^k$  en fonction des  $e_k$  (voir remarque 2.4.14 ) :

$$\begin{aligned} I_2 &= \{-p', -2p', \dots, -pp'\} \text{ (noter que } -pp' = -1 \text{ )} \\ I_1 &= \{-1 - p', \dots, -1 - (k-p)p'\} \text{ (noter que } -1 - (k-p)p' = -kp' \text{ )} \\ I_0 &= \{-(k+1)p', \dots, -qp'\} \text{ (noter que } -qp' = 0 \text{ )} \end{aligned}$$

Dire que tous les 1 de  $w^k$  sont suivis (cycliquement) par un mot de la forme  $1^s 2$  revient à dire que  $\phi(I_1) \subset I_1 \cup I_2$ . Notons  $B = \{-1 - p', -1 - 2p', \dots, -1 - pp'\} = \tau(I_2)$ , ainsi que pour  $l \in \mathbb{N} \cup \{-1\}$ ,  $B_l = \{-1 - p', -1 - 2p', \dots, -1 - (l+1)pp'\}$ . Remarquer que  $B_{-1} = \emptyset$  et  $B_0 = B$ .

**Lemme 2.4.24.** Pour  $l \in \mathbb{N} \cup \{-1\}$ ,  $B_l = \cup_{s=1}^{l+1} \tau^s(I_2)$ .

*Démonstration.* La preuve est analogue à celle du lemme 2.4.23.  $\square$

Si  $k - p \leq p$  on a  $I_1 \subset B$ , et donc  $\phi(I_1) \subset I_2$ . Dans le cas général, par division euclidienne de  $k - p$  par  $p$ ,  $k - p = lp + r$  où  $0 \leq r < p$ , on a  $B_{l-1} \subset I_1 \subset B_l$  ce qui permet de conclure comme lors du raisonnement pour  $k < p$ .

Il reste à vérifier la comparaison d'un élément  $v^k$  de la forme  $\phi^s(w^k)$  avec le mot  $1 \cdots 1$ . Si  $v^k$  commence par 0 ou 2, la comparaison avec  $1 \cdots 1$  est claire. Il reste le cas où  $v^k$  commence par 1.

Supposons  $k < q$ . Dire que  $v^k > 1 \cdots 1$  revient à dire que  $v^k$  commence par un mot de la forme  $1^s 2$  où  $s \in \mathbb{N}^*$ . Ceci est impossible car les 2 de  $v^k$  ne peuvent pas être immédiatement précédés par un 1. Donc  $v^k < 1 \cdots 1$ .

Supposons  $k > q$ . Alors  $v^k$  commence par un mot de la forme  $1^s 2$  où  $s \in \mathbb{N}^*$ , ce qui implique  $v^k > 1 \cdots 1$ .  $\square$

Comme un cycle ne peut être entrelacé avec lui-même, ceci démontre que les cycles  $\Theta_{k_1}, \Theta_{k_2}$  sont entrelacés seulement si  $|k_2 - k_1| = 1$ . Cette condition nécessaire s'avère en réalité suffisante.

**Proposition 2.4.25.**  $\Theta_{k_1}$  et  $\Theta_{k_2}$  sont entrelacés si et seulement si  $|k_2 - k_1| = 1$ .

**Exemple 2.4.26.** Voir exemple 2.4.18, où  $p = 2, q = 7$ . Le lecteur patient peut vérifier à la main dans ces exemples que  $\Theta_k$  et  $\Theta_{k+1}$  sont bien entrelacés. La proposition démontre ceci dans le cas général.

*Démonstration.* Il reste seulement à montrer que pour  $k \in \llbracket 0, q - 1 \rrbracket$ ,  $\Theta_k$  et  $\Theta_{k+1}$  sont entrelacés. A cause du fait que  $w^k, w^{k+1}$  sont minimaux pour l'ordre lexicographique dans  $W_k, W_{k+1}$  (on rappelle que  $W_k$  est l'ensemble de mots associés à  $\Theta_k$ ), et qu'il est clair que  $w^k < w^{k+1}$ , ceci ne laisse pas de choix pour l'entrelacement, il faut montrer que :

$$w^k < w^{k+1} < \phi^{p'}(w^k) < \phi^{p'}(w^{k+1}) < \cdots < \phi^{q-p'}(w^k) < \phi^{q-p'}(w^{k+1})$$

Comme  $w^{k+1}$  est identique à  $w^k$ , à substitution près ou bien d'un 1 par un 2 ou bien d'un 0 par un 1, on a pour tout  $s \in \mathbb{Z}$ ,  $\phi^s(w^k) < \phi^s(w^{k+1})$ .

Il reste à démontrer que  $\phi^s(w^{k+1}) < \phi^{s+p'}(w^k)$  pour  $s \in \llbracket 0, q - 1 \rrbracket \setminus \{q - p'\}$ . Cela repose sur le même type d'idée. Pour comprendre quels chiffres sont substitués, on utilise l'expression commode des  $w^k$  à l'aide des  $e_k$ , remarque 2.4.14, pour calculer la différence, définie lettre à lettre :

$$\begin{aligned} \phi^{p'}(w^k) - w^{k+1} &= \sum_{s=2}^{p+1} e_{-sp'} + \sum_{s=2}^{k+1} e_{-sp'} - \sum_{s=1}^p e_{-sp'} - \sum_{s=1}^{k+1} e_{-sp'} \\ &= e_{-1-p'} - 2e_{-p'} \end{aligned}$$

Si bien que, pour  $s \in \llbracket 0, q - 1 \rrbracket$  :

$$\phi^{s+p'}(w^k) - \phi^s(w^{k+1}) = e_{-1-p'-s} - 2e_{-p'-s}$$

Ainsi,  $\phi^{s+p'}(w^k)$  est obtenu à partir de  $\phi^s(w^{k+1})$  en ajoutant 1 à la lettre d'indice  $-1-p'-s$ , et en retirant 2 à la lettre d'indice  $-p'-s$ , c'est à dire l'indice suivant dans l'ordre cyclique. L'inégalité  $\phi^s(w^{k+1}) < \phi^{s+p'}(w^k)$  est donc toujours vraie, à l'exception du cas  $s = q - p'$ , où l'indice suivant mentionné est le premier indice. L'entrelacement est donc montré.  $\square$

**Théorème 2.4.27.** *Les portraits d'orbites marqués formels de nombre de rotation  $t = \frac{1}{q}$  sont les singletons  $\{A\}$  de  $P(\mathbb{R}/\mathbb{Z})$  avec  $A = \{\Theta_k^t\}$  ou  $A = \{\Theta_k^t, \Theta_{k+1}^t\}$ , et avec  $\Theta_k^t$  orbite de  $\theta_k^t = \frac{1}{3^q-1} + \frac{\sum_{s=0}^{k-1} 3^s}{3^q-1}$  où  $k \in \llbracket 0, q \rrbracket$  (convention somme nulle si  $k = 0$ ).*

*Les portraits d'orbites marqués formels de nombre de rotation  $t = \frac{p}{q}$ , où  $p$  premier avec  $q$  sont les singletons  $\{A\}$  de  $P(\mathbb{R}/\mathbb{Z})$  avec  $A = \{\Theta_k^t\}$  ou  $A = \{\Theta_k^t, \Theta_{k+1}^t\}$  et  $\Theta_k^t$  orbite de  $\theta_k^t = \frac{\sum_{s=0}^{p-1} 3^{(s+1)p'-1} + \sum_{s=0}^{k-1} 3^{(s+1)p'-1}}{3^q-1}$*

**Exemple 2.4.28.** *Voir l'exemple 2.4.18, où  $p = 2, q = 7$ .*

*Démonstration.* Les deux points découlent de la conséquence 2.4.12, de la remarque 2.4.16 et de la proposition 2.4.25.  $\square$

### 3 General properties of coverings

#### 3.1 Blaschke products and parabolic Blaschke products

Let  $f$  be an holomorphic function with parabolic point of multiplier  $\lambda = \exp(2i\pi\frac{p}{q})$ . We introduce some classes which enables us to study the covering properties of  $f^q$  over its immediate parabolic basin.

**Definition 3.1.1.** Let  $\mathbf{B}$  be the set of the Blaschke products  $\phi : \mathbb{D} \rightarrow \mathbb{D}$ . For  $n \in \mathbb{N}^*$ , define  $\mathbf{B}_n$  the set of Blaschke products  $\phi : \mathbb{D} \rightarrow \mathbb{D}$  of degree  $n$ . More generally, define for all tuple of integers  $(n_0, \dots, n_{s-1}) \in (\mathbb{N}^*)^s$  the following class :

$$\mathbf{B}_{n_0, \dots, n_{s-1}} = \{\phi_{s-1} \circ \dots \circ \phi_0, \forall i \in \llbracket 0, s-1 \rrbracket, \phi_i \in \mathbf{B}_{n_i}\}$$

Let  $\mathbf{B}^P$ , resp.  $\mathbf{B}_{n_0, \dots, n_{s-1}}^P$  the subset of  $\mathbf{B}$ , resp.  $\mathbf{B}_{n_0, \dots, n_{s-1}}$  consisting of elements with the following Taylor expansion of order 3 at 1 :  $f(z) = (z-1) - (z-1)^3 + O((z-1)^4)$ .

Finally, let  $\tilde{\mathbf{B}}^P$ , resp.  $\tilde{\mathbf{B}}_{n_0, \dots, n_{s-1}}^P$  be the subset of  $\mathbf{B}$  resp.  $\mathbf{B}_{n_0, \dots, n_{s-1}}$  consisting of elements with a parabolic point at 1 with multiplier 1 with two petals.

**Remark 3.1.2.** Let  $\mathbf{n} = (n_0, \dots, n_{s-1}) \in (\mathbb{N}^*)^s$ , and  $\tilde{\mathbf{n}} = (m_0, \dots, m_{t-1})$ ,  $t \leq s$ , the family obtained by removing the occurrences of 1 in the family  $\mathbf{n}$  and preserving the order of the terms, then  $B_{\mathbf{n}} = B_{\tilde{\mathbf{n}}}$ .

In a similar way, if  $\mathbf{n} = (n_0, \dots, n_{s-1})$  and  $\mathbf{I} = (I_0, \dots, I_{t-1})$  is a partition of  $\llbracket 0, s-1 \rrbracket$  in successive disjoint segments, and if we denote  $n_I = \prod_{i \in I} n_i$  for all segment  $I \subset \llbracket 0, s-1 \rrbracket$  one gets  $B_{\mathbf{n}} \subset B_{\tilde{\mathbf{n}}}$  with  $\tilde{\mathbf{n}} = (n_{I_0}, \dots, n_{I_{t-1}})$ .

**Remark 3.1.3.** The functions  $f \in \mathbf{B}$  conjugate to an element of  $\mathbf{B}^P$  with a conjugacy of which the continuous extension to the closed disk fixes 1 are exactly the elements of  $\tilde{\mathbf{B}}^P$ . Furthermore, the elements of  $\tilde{\mathbf{B}}^P$  have an attracting axis orthogonal to  $\partial\mathbb{D}$ .

*Proof.* Let  $f \in \mathbf{B}$ . Saying that  $f \in \tilde{\mathbf{B}}^P$  is equivalent to saying that the function  $f$  admits a Taylor expansion at 1 of the form:

$$f(z) = z - a(z-1)^3 + O((z-1)^4)$$

with  $a \in \mathbb{C}^*$ .

Notice that the Julia set of a holomorphic function with parabolic point 1 intersects all punctured sector neighborhood of a repelling axis; since the Julia set of a Blaschke product is included in the unit circle, this excludes that the repelling axis of  $f$  being non vertical. Thus the repelling axis of  $f$  is vertical, which implies that  $a \in \mathbb{R}_+^*$  (we use the french convention :  $0 \in \mathbb{R}_+$ ).

It remains to be shown that  $f$  is equal up to a conjugacy to a map  $g$  satisfying the following Taylor expansion at 1:

$$g(z) = z - (z-1)^3 + O((z-1)^4)$$

Let  $\alpha : \mathbb{D} \rightarrow \mathbb{H}$  the following biholomorphism, sending 0 on  $i$ , and 1 on 0 (after taking the continuous extension):

$$\begin{aligned} \alpha &: \mathbb{D} \rightarrow \mathbb{H} \\ z &\mapsto i \frac{-z+1}{z+1} \end{aligned}$$

The map  $f$  induces a function  $\tilde{f} = \alpha \circ f \circ \alpha^{-1}$  from  $\mathbb{H}$  to  $\mathbb{H}$ . We use the following lemma, where the value of  $a'$  is obtained by an explicit calculation of the Taylor expansion at 0 of  $\tilde{f} = \alpha \circ f \circ \alpha^{-1}$  (we used the formal calculation software Sage):

**Lemma 3.1.4.** *Let  $f : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic map, and  $\tilde{f} = \alpha \circ f \circ \alpha^{-1}$ . Then  $f$  admits a Taylor expansion at 1 of the form  $f(z) = z - a(z-1)^3 + O((z-1)^4)$  if and only if  $\tilde{f}$  admits a Taylor expansion at 0 of the form  $\tilde{f}(w) = w + a'w^3 + O(w^4)$ . Furthermore, in this setting,  $a' = 4a$ .*

The homotheties  $h_\lambda : z \mapsto \lambda z$  where  $\lambda > 0$  are automorphisms of  $\mathbb{H}$ . Up to conjugating  $\tilde{f}$  by such an automorphism  $h_{\lambda_0}$  (it is essential that  $\alpha' > 0$ ), one gets a function  $\tilde{g} = h_{\lambda_0} \circ \tilde{f} \circ h_{\lambda_0}$ , with the following Taylor expansion:  $\tilde{g}(w) = w + 4w^3 + O(w^4)$ . This enables us to set  $g = \alpha^{-1} \circ \tilde{g} \circ \alpha$  and to finish the proof:  $g$  has the desired Taylor expansion  $g(z) = z - (z-1)^3 + O((z-1)^4)$  by the lemma, and  $g$  is conjugate to  $f$ , by successive conjugacies.  $\square$

**Definition 3.1.5.** Let  $f : M_1 \rightarrow N_1$ ,  $g : M_2 \rightarrow N_2$  where  $M_1, N_1, M_2, N_2$  are complex Riemann surfaces and  $f, g$  analytic ramified coverings. The maps  $f, g$  are said to be

equivalent as analytic ramified coverings (which will be abbreviated as follows:  $f, g$  equivalent) if there exists biholomorphisms  $\phi : M_1 \rightarrow M_2, \psi : N_1 \rightarrow N_2$  such that the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & N_1 \\ \phi \downarrow & & \downarrow \psi \\ M_2 & \xrightarrow{g} & N_2 \end{array}$$

Denote  $\sim$  this equivalence relation.

**Remark 3.1.6.** There exists an other terminology, which is more standard (an equivalence of analytic ramified covering does not change the base), but the equivalence of analytic ramified covering which appears naturally in the parabolic renormalization phenomena is the one defined previously. According to this standard terminology, it would be necessary to suppose  $N_1 = N_2$  and  $\psi = id$ , this will not be done here.

**Remark 3.1.7.** We will define other equivalence relations, which will also be noted  $\sim$ . There will be no ambiguity insofar as these equivalence relations will apply on disjoint sets; we will thus talk without ambiguity of the equivalence of two elements,  $a \sim b$ , and of the quotient sets  $A/\sim$ .

**Proposition 3.1.8.** *Let  $f \in \mathbf{B}_{n_0, \dots, n_{s-1}}$ . Then there exists  $g \in \mathbf{B}_{n_0, \dots, n_{s-1}}^P$  equivalent to  $f$ . More precisely, we show that there exists  $g_1 \in \tilde{\mathbf{B}}_{n_0, \dots, n_{s-1}}^P$  and  $\phi$  a disk automorphism such that  $f = \phi \circ g_1$ . We then use remark 3.1.3, which affirms that  $g_1$  is conjugate to  $g \in \mathbf{B}_{n_0, \dots, n_{s-1}}^P$  by an homothety centered at  $1 \in \partial\mathbb{D}$ .*

*Proof.* The sets  $\mathbf{B}_{n_0, \dots, n_{s-1}}$  are invariant by equivalence of analytic ramified covering. In order to show the result, it is sufficient to take  $f \in \mathbf{B}$ , and to determine a  $g \in \mathbf{B}^P$  which is equivalent to  $f$ .

Let  $\tilde{f} = \alpha \circ f \circ \alpha^{-1}$ . The proof goes as follows: postcompose  $\tilde{f}$  by several automorphisms of  $\mathbb{H}$  until you get a new function  $\tilde{g}_1$  with a Taylor expansion with the desired form. This allows to show, by taking  $g_1 := \alpha^{-1} \circ \tilde{g}_1 \circ \alpha$ , that  $g_1 \in \tilde{\mathbf{B}}^P$  and to finish the proof. Indeed,  $g_1$  will be equal to  $f$  modulo postcomposition by an automorphism of  $\mathbb{D}$ .

Up to postcomposing  $\tilde{f}$  by an automorphism of  $\mathbb{H}$ , we may suppose that  $\tilde{f}$  sends 0 to 0 after taking its continuous extension.

The function  $\tilde{f}$  has now a Taylor expansion at 0 of the form:

$$\tilde{f}(w) = aw + bw^2 + dw^3 + O(w^4)$$

with  $a \in \mathbb{R}_+^*$ ,  $b, c \in \mathbb{R}$ . Indeed,  $a$  must be positive real since  $\tilde{f}$  stabilises  $\mathbb{H}$ . The number  $a$  is nonzero since  $f$  cannot have, after holomorphic extension, any critical point in  $\partial\mathbb{H}$  (a Blaschke product cannot have any critical point in  $\partial\mathbb{D}$ ). The numbers  $b, c$  must be real since  $\tilde{f}$  induces after holomorphic extension and restriction a map from  $\partial\mathbb{H}$  to  $\partial\mathbb{H}$ .

To obtain the result, it remains, even if it means postcomposing  $\tilde{f}$  again, to get a Taylor expansion with  $a = 1, b = 0$ . Indeed, lemma 3.1.4 enables us to get the Taylor expansion of  $f := \alpha^{-1} \circ \tilde{f} \circ \alpha$  corresponding to the definition of  $f \in \tilde{\mathbf{B}}^P$ .

The set of automorphisms of  $\mathbb{H}$  preserving 0 is equal to the semi-direct product formed by the two following automorphism subgroups (le sous-groupe distingué étant celui des translations):

$$\begin{array}{ll} h_\lambda : & w \mapsto \lambda w & \text{homotheties} \\ \tau_\mu : & w \mapsto \frac{1}{\frac{1}{w} + \mu} = \frac{w}{1 + \mu w} = w - \mu w^2 + \mu^2 w^3 + O(w^4) & \text{translations} \end{array}$$

This set is a real Lie group of dimension 2. Since we can freely post-compose  $\tilde{f}$  by such maps, we expect that we can control the value of 2 real parameters in the Taylor expansion of  $\tilde{f}$ , which would be sufficient to finish the proof.

Postcompose  $\tilde{f}$  by an homothety to get  $a = 1$ . Then, postcompose by a translation to get a Taylor expansion with  $a = 1$ ,  $b = 0$ . This finishes the proof.  $\square$

**Proposition 3.1.9.** *Let  $f, g \in \tilde{\mathbf{B}}^P$ . Then  $f, g$  are equivalent via biholomorphisms  $\phi, \psi$  where the continuous extension of  $\phi$  over  $\overline{\mathbb{D}}$  fixes 1 if and only if  $f, g$  are conjugate by  $\phi$ .*

*Proof.* Let  $f, g \in \tilde{\mathbf{B}}^P$ . Suppose that  $f, g$  are equivalent. We show that  $f, g$  are conjugate (note that the converse implication is automatic).

Let  $\tilde{f} = \alpha \circ f \circ \alpha^{-1}, \tilde{g} = \alpha \circ g \circ \alpha^{-1}$ ; these maps are equivalent. Let  $\tilde{\phi} = \alpha \circ \phi \circ \alpha^{-1}, \tilde{\psi} = \alpha \circ \psi \circ \alpha^{-1}$  automorphisms of  $\mathbb{H}$  completing the following commutative diagram:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\tilde{f}} & \mathbb{H} \\ \tilde{\phi} \downarrow & & \downarrow \tilde{\psi} \\ \mathbb{H} & \xrightarrow{\tilde{g}} & \mathbb{H} \end{array}$$

$\tilde{f}$  and  $\tilde{g}$  both satisfy the same Taylor expansion of order 2 at 0 :

$$\tilde{f}(w) = \tilde{g}(w) + O(w^3) = w + O(w^3)$$

Since  $\tilde{\phi}$  sends 0 on 0, this implies that  $\tilde{\psi}$  sends 0 on 0.

Write the Taylor expansions of  $\tilde{\phi}$  and  $\tilde{\psi}$  at 0 :

$$\begin{aligned} \tilde{\phi}(w) &= b_1 w + c_1 w^2 + O(w^3) \\ \tilde{\psi}(w) &= b_2 w + c_2 w^2 + O(w^3) \end{aligned}$$

We identify the obtained Taylor expansions in the equality  $\tilde{\psi} \circ \tilde{f} = \tilde{g} \circ \tilde{\phi}$ , this allows us to get  $b_1 = b_2$  and  $c_1 = c_2$ .

Since  $\tilde{\phi}$  and  $\tilde{\psi}$  both stabilise 0, each of them is equal to an element of the form  $h_\lambda \circ \tau_\mu$ , with  $\lambda \in \mathbb{R}_+^*, \mu \in \mathbb{R}$ , satisfying the following Taylor expansion:

$$h_\lambda \circ \tau_\mu(w) = \lambda w + \lambda \mu w^2 + O(w^3)$$

Thus, the first two terms of the Taylor expansion of  $h_\lambda \circ \tau_\mu$  characterises completely the map  $h_\lambda \circ \tau_\mu$ , since they allow to compute  $\lambda$  and  $\mu$  (note that  $\lambda \neq 0$ ). Thus  $\phi = \psi$  and  $\tilde{f}, \tilde{g}$  are conjugate.  $\square$

**Remark 3.1.10.** To complement the proposition. If one suppose furthermore that  $f, g \in \mathbf{B}^P$ , one can say that the conjugacy  $\phi$  will be a translation by exploiting the Taylor expansion of order 3.

We write:

$$\tilde{f}(w) = \tilde{g}(w) + O(w^4) = w + 4w^3 + O(w^4)$$

And:

$$\tilde{\phi}(w) = bw + cw^2 + dw^3 + O(w^4)$$

Then:

$$\begin{aligned} \tilde{\phi} \circ \tilde{f}(w) &= b(w + 4w^3) + c(w + 4w^3)^2 + d(w + 4w^3)^3 + O(w^4) \\ &= bw + cw^2 + (4b + d)w^3 + O(w^4) \end{aligned}$$

$$\begin{aligned} \tilde{g} \circ \tilde{\phi}(w) &= bw + cw^2 + dw^3 + 4(bw + cw^2 + dw^3)^3 + O(w^4) \\ &= bw + cw^2 + (d + 4b^3)w^3 + O(w^4) \end{aligned}$$

We get  $4b = 4b^3$ , therefore  $b = -1, 0, 1$ . By writing  $\tilde{\phi} = h_\lambda \circ \tau_\mu$  with  $\lambda \in \mathbb{R}_+^*$ ,  $\mu \in \mathbb{R}$ , the condition on  $b$  is equivalent to  $\lambda = 1$  (since  $\lambda = b \in \mathbb{R}_+^*$ ). Finally,  $\tilde{\phi} = \tau_\mu$  is a translation.

More precisely, if  $f \in \mathbf{B}^P$ ,  $g \in \mathbf{B}$  are conjugate,  $g = \phi \circ f \circ \phi^{-1}$  with  $\phi$  automorphism of  $\mathbb{D}$ , then  $g \in \mathbf{B}^P$  if and only if  $\phi$  is a translation.

**Proposition 3.1.11.**

$$\tilde{\mathbf{B}}_{n_0, \dots, n_{s-1}}^P = \{\phi_{s-1} \circ \dots \circ \phi_0, \forall i \in \llbracket 0, s-1 \rrbracket, \phi_i \in \tilde{\mathbf{B}}_{n_i}^P\}$$

*Proof.* Let  $\phi \in \tilde{\mathbf{B}}_{n_0, \dots, n_{s-1}}$ , and  $(\phi_{s-1}, \dots, \phi_0)$  be an adapted decomposition. We use proposition 3.1.8 to show by induction the existence of a family of automorphisms  $\chi_0, \dots, \chi_{s-2}$  of  $\mathbb{D}$  such that:

$$\begin{aligned} \chi_0 \circ \phi_0 &\in \tilde{\mathbf{B}}^P \\ \forall 1 \leq i \leq s-2, \quad \chi_i \circ \phi_i \circ \chi_{i-1}^{-1} &\in \tilde{\mathbf{B}}^P \end{aligned}$$

Let us denote for  $0 \leq i \leq s-1$ ,  $\psi_i = \chi_i \circ \phi_i \circ \chi_{i-1}$  (we noted  $\chi_{-1} = \chi_{s-1} = id$ ). We have  $\phi = \psi_{s-1} \circ \dots \circ \psi_0$  and  $\psi_i \in \tilde{\mathbf{B}}^P$  for  $i \in \llbracket 0, s-2 \rrbracket$ . By identifying the Taylor expansions of the equality  $\phi = \psi_{s-1} \circ \dots \circ \psi_0$ , the fact that  $\phi, \psi_0, \dots, \psi_{s-2} \in \tilde{\mathbf{B}}^P$  implies that  $\psi_{s-1} \in \tilde{\mathbf{B}}^P$ . This allows us to finish the proof, since the precompositions and postcompositions by automorphisms do not modify the degree.  $\square$

**Remark 3.1.12.** Notice that for  $s \geq 2$ :

$$\mathbf{B}_{n_0, \dots, n_{s-1}}^P \neq \{\phi_{s-1} \circ \dots \circ \phi_0, \forall i \in \llbracket 0, s-1 \rrbracket, \phi_i \in \mathbf{B}_{n_i}^P\}$$

Indeed, for  $s \geq 2$  the two sets are disjoint, the elements  $f$  of the right set having a Taylor expansion at 1 of the form  $f(z) = (z-1) - s(z-1)^2 + O((z-1)^3)$ .

**Theorem 3.1.13.** *The elements of  $\mathbf{B}$  up to equivalence are characterised by the relative position of the critical points.*

*More precisely, let  $f, g \in \mathbf{B}$ . If  $f$  and  $g$  are equivalent, then by noting  $\phi, \psi$  some automorphisms of  $\mathbb{D}$  such that the following diagram  $D(\phi, \psi)$  commutes:*

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{f} & \mathbb{D} \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{D} & \xrightarrow{g} & \mathbb{D} \end{array}$$

*we have  $\phi_*C(f) = C(g)$  where  $C(f)$  denotes the divisor of critical points of  $f$ ,  $C(g)$  the divisor of critical points of  $g$ .*

*Conversely, if an automorphism  $\phi$  of  $\mathbb{D}$  satisfies  $\phi_*C(f) = C(g)$  then there exists an automorphism  $\psi$  of  $\mathbb{D}$  such that the diagram  $D(\phi, \psi)$  commutes.*

*We will denote by  $\sim$  the equivalence relation over the divisors of  $\mathbb{D}$ , given by  $D \sim D'$  if and only if there exists an automorphism  $\phi : \mathbb{D} \rightarrow \mathbb{D}$  such that  $D' = \phi_*D$ ; if  $D \sim D'$  we will say that  $D$  and  $D'$  have the same relative position. Then, the maps  $f$  and  $g$  are equivalent if and only if  $C(f)$  and  $C(g)$  have the same relative position.*

*Proof.* This statement is shown, in a slightly different but equivalent form, in [Zak98].  $\square$

### 3.2 Covering decompositions

**Definition 3.2.1.** A tuple of maps  $\mathbf{f} = (f_{s-1}, \dots, f_0)$  is a *ramified covering decomposition* (or more simply a *decomposition*) if there exists Riemann surfaces  $S_0, \dots, S_s$  such that the map  $f_i : S_i \rightarrow S_{i+1}$  is a ramified covering for all  $i \in \llbracket 0, s-1 \rrbracket$ . In this case, we say that  $\mathbf{f}$  passes through the surfaces  $S_0, \dots, S_s$ . If  $f$  is such that  $f = f_{s-1} \circ \dots \circ f_0$ , we say that  $\mathbf{f}$  is a decomposition of  $f$ . We call  $f_i$  the factors of  $\mathbf{f}$ .

We note for  $i, j \in \llbracket 0, s \rrbracket$ ,  $i \leq j$ ,  $\mathbf{f}_{i,j} = f_{j-1} \circ \dots \circ f_i$ , with the convention  $\mathbf{f}_{i,i} = \text{id}_{S_i}$  (in all cases,  $\mathbf{f}_{i,j} : S_i \rightarrow S_j$ ). We call  $\mathbf{f}_{i,j}$  the *partial compositions* of  $\mathbf{f}$  (even for  $i = 0$  and  $j = s$ ), and  $f = \mathbf{f}_{0,s}$  the *composition* of  $\mathbf{f}$ .

**Definition 3.2.2.** Two decomposition tuples  $\mathbf{f} = (f_{s-1}, \dots, f_0)$  and  $\mathbf{g} = (g_{s'-1}, \dots, g_0)$  passing through the surfaces  $S_0, \dots, S_s$ , resp.  $S'_0, \dots, S'_t$  are said to be *equivalent as covering decomposition* (or more simply *equivalent*) if  $s = t$  and if there exists  $\chi_0, \dots, \chi_s$  isomorphisms satisfying the following commutative diagram:

$$\begin{array}{ccccccc} S_0 & \xrightarrow{f_0} & S_1 & \xrightarrow{f_1} & \dots & \xrightarrow{f_{s-1}} & S_s \\ \chi_0 \downarrow & & \chi_1 \downarrow & & & & \chi_s \downarrow \\ S'_0 & \xrightarrow{g_0} & S'_1 & \xrightarrow{g_1} & \dots & \xrightarrow{g_{s-1}} & S'_s \end{array}$$

**Proposition 3.2.3.** *Let  $f, g$  be two coverings. The following conditions are equivalent:*

- *The covering  $f$  and  $g$  are equivalent.*



- $f$  and  $g$  have the same decompositions modulo decomposition equivalence.
- For all  $\mathbf{f}$  decomposition of  $f$ , there exists  $\mathbf{g}$  decomposition of  $g$  equivalent to  $\mathbf{f}$ .
- There exists  $\mathbf{f}, \mathbf{g}$  decompositions of  $f, g$  which are equivalent.

*Proof.* Note that the second property implies the third, the third implies the fourth (use that  $\mathbf{f} = (f)$  is a decomposition of  $f$ ) and the fourth implies the first. It remains to show that the first property implies the second.

Assume that  $f, g$  are equivalent. Thus there exists  $\phi, \psi$  such that the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & N_1 \\ \phi \downarrow & & \downarrow \psi \\ M_2 & \xrightarrow{g} & N_2 \end{array}$$

Let  $\mathbf{f} = (f_{s-1}, \dots, f_0)$  be a decomposition of  $f$  passing through the surfaces  $S_0 = M_1, S_1, \dots, S_{s-1}, S_s = N_1$ . Then the decomposition  $\mathbf{g} = (g_{s-1}, \dots, g_0) = (\psi \circ f_{s-1}, f_{s-2}, \dots, f_0 \circ \phi^{-1})$  of  $g$  is equivalent to  $\mathbf{f}$  by the diagram :

$$\begin{array}{ccccccc} M_1 & \xrightarrow{f_0} & S_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{s-2}} & S_{s-1} & \xrightarrow{f_{s-1}} & N_1 \\ \phi \downarrow & & id \downarrow & & & & id \downarrow & & \psi \downarrow \\ M_2 & \xrightarrow{g_0} & S_1 & \xrightarrow{g_1} & \cdots & \xrightarrow{g_{s-2}} & S_{s-1} & \xrightarrow{g_{s-1}} & N_2 \end{array}$$

Symmetrically, for  $\mathbf{g}$  decomposition of  $g$ , there exists  $\mathbf{f}$  decomposition of  $f$  equivalent to  $\mathbf{g}$ . Hence  $f$  and  $g$  have the same decompositions modulo decomposition equivalence. This ends the proof.  $\square$

**Definition 3.2.4.** A covering  $f$  is said *irreducible* if  $f$  is not a biholomorphism, and for all covering decomposition  $f = f_1 \circ f_2$  of  $f$ ,  $f_1$  or  $f_2$  is a biholomorphism.

**Remark 3.2.5.** For a covering to be irreducible, it is sufficient that its degree is finite and equal to a prime number.

An immediate induction over the degree implies that:

**Proposition 3.2.6.** *Every covering  $f$  of finite degree  $\geq 2$  admits a decomposition  $\mathbf{f} = (f_{s-1}, \dots, f_0)$  with  $f_0, \dots, f_{s-1}$  irreducible.*

**Remark 3.2.7.** The decomposition into irreducible factors is not unique, even if we consider it modulo permutation of the terms and modulo equivalence term by term. Counter-example:  $z(z+1)^3 \circ z^3 = z^3 \circ z(z^3+1)$  are two decompositions into irreducible factors, and the factors are not equivalent up to permuting the order of the terms.

**Definition 3.2.8.** Let  $f \in \mathbf{B}_{n_0, \dots, n_{s-1}}$ . A decomposition  $\mathbf{f} = (f_{s-1}, \dots, f_0)$  of  $f$  is *adapted to the multidegree*  $(n_0, \dots, n_{s-1})$  of  $f$  (or more simply *adapted to  $f$* , or *adapted*, when the multidegree can be implied) if  $f_i \in \mathbf{B}_{n_i}$  for all  $i \in \llbracket 0, s-1 \rrbracket$ .

**Proposition 3.2.9.** *Let  $f, g$  be two elements of  $\mathbf{B}_{n_0, \dots, n_{s-1}}$ . Then proposition 3.2.3 is valid by replacing each occurrence of "decomposition" by "adapted decomposition".*

*Proof.* The same proof as in proposition 3.2.3 can be applied, the compositions by automorphisms preserving the degree.  $\square$

### 3.3 Stability of the classes $\Sigma^0$ by parabolic renormalization

Section 3 and theorem 1.5.1, shown in section 4 are motivated by the conjecture mentioned at the end of this subsection.

**Definition 3.3.1.** Let  $\mathbf{P}$  the set of the functions  $f$  satisfying the following :

- $f : U \rightarrow \mathbb{C}$  is holomorphic on an open neighborhood  $U$  of  $0 \in \mathbb{C}$
- $f$  has a parabolic point at 0 with exactly one cycle  $B_f^0 = U_0 \cup \dots \cup U_{q-1}$  of connected components of the immediate basin, with  $f(U_i) = U_{i+1}$  (the index  $i$  is defined modulo  $q$ ).
- Each  $U_k$  is simply connected.

Let  $f \in \mathbf{P}$ . Set for  $X \subset U$ ,  $f_X = (f : X \rightarrow f(X))$ , and  $\mathbf{g} = (f_{U_{q-1}}, \dots, f_{U_0})$  the natural decomposition of  $f^q : U_0 \rightarrow U_0$ ; notice that  $\mathbf{g}$  depends on the arbitrary choice of  $U_0$ .

Let  $\Sigma^0$  the set of  $f \in \mathbf{P}$  such that for a certain choice of  $U_0$  (or equivalently, for all choice of  $U_0$ ),  $\mathbf{g}$  is equivalent to an element of  $\mathbf{B} \times \dots \times \mathbf{B}$ . Denote by  $\Sigma_{n_0, \dots, n_{q-1}}^0$  the set of  $f \in \mathbf{P}$  such that for a certain choice of  $U_0$ ,  $\mathbf{g}$  is equivalent to an element of  $\mathbf{B}_{n_{q-1}} \times \dots \times \mathbf{B}_{n_0}$ . Notice that the set  $\Sigma_{n_0, \dots, n_{q-1}}^0$  is the same when permuting cyclically the  $n_i$ .

**Remark 3.3.2.** Let  $P$  be a polynomial of degree  $n$  with a parabolic point at 0 such that the parabolic basin contains exactly one cycle of connected components  $B_f = U_0 \cup \dots \cup U_{q-1}$ , where  $U_{k+1} = f(U_k)$  for all  $k \in \mathbb{Z}/q\mathbb{Z}$ . Then  $P \in \Sigma^0$ . More precisely, if  $m_k$  is the number of critical points counted with multiplicity of  $P$  in  $U_k$ , then  $P \in \Sigma_{m_0+1, \dots, m_{q-1}+1}^0$ . An analogue remark is possible when  $P$  is a polynomial of degree  $n$  with any parabolic point at 0.

*Proof.* Notice that the  $U_i$  are simply connected since  $P$  is a polynomial. The map  $P : U_i \rightarrow U_{i+1}$  is proper of degree  $m_i + 1$ . This map is hence equivalent as covering to a Blaschke product of degree  $m_i + 1$  over the unit disk. This shows that  $P \in \Sigma_{m_0+1, \dots, m_{q-1}+1}^0$ .  $\square$

In the following proposition, we are interested by the polynomials of degree 3, and thus by the classes  $\Sigma_2^0, \Sigma_{2,2}^0, \Sigma_3^0$ .

**Proposition 3.3.3.**

- $\Sigma_2^0/\sim$  is reduced to one element represented by the parabolic Blaschke product  $B_2 : z \mapsto \frac{z^2 + \frac{1}{3}}{\frac{1}{3}z^2 + 1}$
- $\Sigma_3^0/\sim$  is an real analytic manifold of dimension 1, the chart being given by the hyperbolic distance between the two critical points.
- $\Sigma_{2,2}^0/\sim$  is a real analytic manifold of dimension 1, the chart being given by the hyperbolic distance between the critical point  $\phi_0$  (independent on the choice of the decomposition  $\phi = \phi_1 \circ \phi_0$ ) and the two preimages by  $\phi_0$  of the critical point of  $\phi_1$  for  $\Sigma_{2,2}^0/\sim$ .

*Proof.* We think that the result is already known in litterature. Since we do not have a full reference on this topic, we give here a proof.

For the first point. An element  $f \in \Sigma_2^0$  is equivalent to a Blaschke product  $B \in \mathbf{B}_2$ . This Blaschke product  $B$  admits an unique critical point and an unique critical value. By pre-composing and post-composing by an automorphism of  $\mathbb{D}$ , one may get that the critical point is 0 and the critical value is 0. This implies that  $B$  can be written  $z \mapsto \lambda z^2$ . Post-composing by a rotation,  $B$  can be written  $z \mapsto z^2$ . Hence there is a unique element in  $\Sigma_2^0/\sim$ . Since  $B_2 \in \Sigma_2^0$ , this concludes the proof.

For the second point. Let  $f \in \Sigma_3^0$ . The map  $f$  is equivalent to a Blaschke product  $B \in \mathbf{B}_3$ . The map  $B$  admits two critical points counted with multiplicity, and two critical values counted with multiplicity. By post-composing and pre-composing by automorphisms of  $\mathbb{D}$ , one can get that the critical point is 0 and its image is 0. By post-composing and pre-composing again by automorphisms of  $\mathbb{D}$ , one can get furthermore that the second critical point and the associated critical value are on the nonnegative real axis. We have shown that  $\Sigma_3^0/\sim = S/\sim$  where  $S$  is the set of Blaschke products of degree 3 with 0 as fixed critical point, and a critical point real nonnegative with the critical value associated real nonnegative. It remains to show that two elements of  $S$  of which the critical points are at distinct hyperbolic distances give two elements of  $S/\sim$ , i.e. are not equivalent as coverings. This must be the case since an automorphism of the unit disk preserves the hyperbolic distance.

For the third point. Let  $f \in \Sigma_{2,2}^0$ . The map  $f$  is equivalent to a Blaschke product  $B \in \mathbf{B}_{2,2}$ . We write  $B = \phi_1 \circ \phi_0$  where  $\phi_0, \phi_1 \in \mathbf{B}_2$ . By pre-composing  $\phi_0$  by an automorphism  $\alpha$  of the disk and post-composing it by an automorphism  $\beta$  of the disk (it is implied that we pre-compose  $\phi_1$  by  $\beta^{-1}$  in such a way that we have the same  $B = \phi_1 \circ \phi_0$  up to pre-composition and post-composition by automorphisms of  $\mathbb{D}$ ) we get that 0 is the critical point of  $\phi_0$  and is fixed by this map. This implies that  $\phi_0 : z \mapsto \lambda z^2$  where  $\lambda \in \mathbb{C}$ ,  $|\lambda| = 1$ . Modify  $\beta$  in such a way that the critical point  $c_1$  of  $\phi_1$  is on the real nonnegative axis. Then, modify  $\alpha$  in such a way that  $\lambda = 1$ ; this implies that the pre-images of  $c_1$  by  $\phi_0$  are  $\pm\sqrt{c_1}$  where  $\sqrt{\cdot}$  is the usual square root of the nonnegative real numbers. Notice that  $B$  admits three critical points, namely 0,  $\pm\sqrt{c_1}$ , and the critical point of  $\phi_0$  is independent on the choice of the decomposition  $B = \phi_1 \circ \phi_0$ , since it is the critical point of  $B$  which is midway between the two others in terms of hyperbolic geometry. Post-compose  $B$  by an automorphism  $\gamma$  of the disk to get that, the critical

point 0 is fixed by  $B$ , and that the two others critical points  $\pm\sqrt{c_1}$  map to the same nonnegative real number. This implies that  $\Sigma_{2,2}^0/\sim$  is equal to  $S/\sim$ , where  $S$  is the set of  $B \in \mathbf{B}_{2,2}$  such that  $B$  admits a fixed critical point at 0 and two critical points at  $\pm a$  where  $a \in [0, 1[$ , and such that  $B(a) = B(-a) \in [0, 1[$ . It remains to show that two elements of  $S$  corresponding to two distinct numbers  $a \in [0, 1[$  must give two distinct elements of  $S/\sim$ . This is true since  $a$  is equal to the hyperbolic distance between the central critical point of  $B$  and the two others : pre-composing or post-composing  $B$  by automorphisms of the disk must preserve this hyperbolic distance.  $\square$

**Definition 3.3.4.** For all  $f \in \Sigma^0$ , denote by  $\mathfrak{R}(f)$  the normalized horn map  $\mathbf{h}^+$  of  $f$ , analytically extended  $+i\infty$ .

Notice that  $\mathfrak{R}(f)$  admits a parabolic point at  $+i\infty$ . We identify in the following paragraphs  $\mathbb{C}/\mathbb{Z} \cup \{\pm i\infty\}$  and  $\mathbb{C} \cup \{\infty\}$  via  $z \mapsto \exp(2i\pi z)$ . Before giving the conjecture, we introduce the following definition (**S** stands for singular):

**Definition 3.3.5.** Denote by **S** the set of functions  $f$  satisfying the following properties:

- $f : U \rightarrow \mathbb{C}$  is holomorphic on an open neighborhood of  $0 \in \mathbb{C}$
- $f$  admits a parabolic point at 0 with two cycles  $U_0 \cup \dots \cup U_{q-1}$ ,  $V_0 \cup \dots \cup V_{q-1}$  of connected components of the immediate basin, with  $f(U_i) = U_{i+1}$  and  $f(V_i) = V_{i+1}$  (the index  $i$  is defined modulo  $q$ ).
- Each of the  $U_k$  and  $V_k$  are simply connected.
- $f^q : U_0 \rightarrow U_0$  and  $f^q : V_0 \rightarrow V_0$  are conjugate to a possibly different Blaschke product of degree 2.

We conjecture that:

**Conjecture 3.3.6.** *The elements of  $\Sigma_3^0$ ,  $\Sigma_{2,2}^0$ ,  $\Sigma_2^0$  are, excepted a finite number of equivalence classes, renormalizable, and their parabolic renormalization obeys the following rules:*

$$\begin{aligned} \mathfrak{R}(\Sigma_3^0) &\subset \Sigma_3^0 \cup \Sigma_{2,2}^0 \cup \Sigma_2^0 \cup \mathbf{S} \\ \mathfrak{R}(\Sigma_{2,2}^0) &\subset \Sigma_{2,2}^0 \cup \Sigma_2^0 \cup \mathbf{S} \\ \mathfrak{R}(\Sigma_2^0) &\subset \Sigma_2^0 \end{aligned}$$

Theorem 1.5.1, which is shown in section 4, and more precisely its complement given by proposition 1.5.2, would be a crucial element of the proof.

Here is an idea of how to demonstrate it. In the spirit of [Shi00] and [LY14], we may show that  $\mathfrak{R}(\Sigma_2^0) \subset \Sigma_2^0$  thanks to the following lemma applied to  $f = \mathfrak{R}(f_0)$ , where  $f_0 \in \Sigma_2^0$ :

**Lemma 3.3.7.** *Let  $f : \mathbb{D} \rightarrow \mathbb{D}$  be an holomorphic function, of which the set of singular values is equal to  $\{0\}$ . Then  $f$  is equivalent as a covering to  $z \mapsto z^d$  or  $\exp \circ \phi$  where  $\phi$  is a biholomorphism from  $\mathbb{D}$  to the left half-plane.*

If the singular value in question is a critical value, which is the case for  $f = \mathfrak{R}(f_0)$ , then  $\deg(f) < \infty$ . This is not true if  $f$  has two critical values in  $\mathbb{D}$ , and hence when  $f = \mathfrak{R}(f_0)$  where  $f_0 \in \Sigma_3^0$  or  $f_0 \in \Sigma_{2,2}^0$ . In this case, we hope to use the covering properties of  $f$  to continue the proof.

More precisely, let  $f_0 \in \Sigma_3^0$  resp.  $f_0 \in \Sigma_{2,2}^0$ . There exists a Blaschke product  $B_0 \in \mathbf{B}_3$  resp.  $B_0 \in \mathbf{B}_{2,2}$  such that  $f_0$  is conjugate to  $B_0$ . This implies by proposition 1.5.2 that  $f = \mathfrak{R}(f_0)$  and  $\mathfrak{R}(B_0)$  are equivalent as coverings: we reduced the problem to the specific study of coverings properties of  $\mathfrak{R}(B_0)$ .

Let  $v_1, v_2$  be the two critical values of  $f = \mathfrak{R}(f_0) : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$ . By the preceding paragraph, the study of the parabolic chessboard of  $f_0$ , which is equal by definition to  $f^{-1}\left((v_1 + \mathbb{R}/\mathbb{Z}) \cup (v_2 + \mathbb{R}/\mathbb{Z})\right)$ , reduces to the study of the parabolic chessboard of  $B_0$ . The idea of the proof would be to use the parabolic chessboard of  $B_0$  to show that the parabolic basin  $B_f^0$  of  $f$  contains a limited number of critical points. Set  $C = \pi^{-1}(B_f^0)$ , where  $\pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$  is the canonical projection and let  $\tilde{f}$  be a lift of  $f$  mapping  $C$  in  $C$ . Notice that if a curve  $\gamma$  is included in  $C$ , then  $C$  is disjoint of the  $\gamma + k$ ,  $k \in \mathbb{Z}$ ,  $k \neq 0$  and of their pre-images by  $f$ . By constructing sufficiently many disjoint curves of  $C$ , we hope to enclose  $C$  in some space containing a limited number of critical points of  $\tilde{f}$ . This must imply that  $C$  contains a limited number of critical points of  $\tilde{f}$  and thus that  $B_f^0$  contains a limited number of critical points of  $f$ .

## 4 Horn maps of holomorphic functions locally pseudo-conjugate on their local parabolic basin

**Plan of the section:** In part 4.1, we recall a definition of Fatou coordinate and petals which is not fully compliant with [Mil06], and their classical consequences. In part 4.2, we define the Fatou coordinate extensions and the attracting, repelling cylinders, isomorphic via Fatou coordinates to  $B^f/f$  and  $P_R/f$  (these objects will be defined in this part 4.2). In part 4.3, we define the horn map, seen as a map from the repelling cylinder  $\mathcal{C}_R$  to the attracting cylinder  $\mathcal{C}_A$ , using the natural map  $P_R/f \rightarrow B^f/f$ . In part 4.4, we define the pseudo-conjugacies and we prove some properties, which will allow in part 4.5 to show the main theorem (theorem 1.5.1). We give in part 4.6 the definition of generalized pseudo-conjugacy for parabolic points with several petal cycles, and the expected generalization of the theorem in this setting.

### 4.1 Fatou coordinates, petals

Let  $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$  a holomorphic function defined on an open neighborhood  $U$  of 0. We assume that  $f$  has a parabolic point 0 with multiplier 1 with one petal, in other words

$f$  has the Taylor expansion:

$$f(z) = z + az^2 + O(z^3)$$

with  $a \in \mathbb{C}^*$ . The attracting axis of  $f$  has argument  $-\arg(a) + \pi$ , and the repelling axis of  $f$  has argument  $-\arg(a)$ .

Here is a definition of attracting petal that comes from [LY14] with slight modifications. Among others, the condition  $f(P_A) \setminus \{0\} \subset P_A$  is replaced by  $f(P_A) \subset P_A$ . This modified definition will prove to be useful to show that a local conjugation of the immediate basin, not assumed continuous at 0, sends an attracting petal on an attracting petal, see proposition 4.4.27. This is not a standard definition, it differs for instance from the more classical definition of Milnor [Mil06]. Nevertheless, a petal in our sense is a petal in the sense of Milnor. The classical existence results are hence still valid concerning our definition.

**Definition 4.1.1.** An attracting petal  $P_A$  of  $f$  is an open subset of  $\mathbb{C}$  with a holomorphic function  $\Phi_A : P_A \rightarrow \mathbb{C}$  named Fatou coordinate such that:

- $P_A$  is connected, simply connected (i.e.  $P_A$  is homeomorphic to  $\mathbb{D}$ ).
- $f(P_A) \subset P_A$ .
- $\forall z \in P_A, \lim_{n \rightarrow \infty} f^n(z) = 0$ .
- Conversely, if  $f^n(z)$  converges to 0, it may be stationary at 0, or else  $f^n(z)$  must belong to  $P_A$  for  $n$  big enough.
- We have the following conjugacy:  $\forall z \in P_A, \Phi_A \circ f(z) = T_1 \circ \Phi_A(z)$ , with  $T_1(z) = z + 1$  for all  $z \in \mathbb{C}$ .
- $\Phi_A$  is injective on  $P_A$ .

We call (petal) the following proposition, for a better comparison with the definitions of  $\alpha$ -petal, large petal, very large petal which will be soon defined.

**Proposition 4.1.2** (petal). *An attracting petal  $P_A$  satisfies the following properties:*

- for all  $w \in \mathbb{C}$ , there exists  $n \in \mathbb{N}$  such that  $w + n \in \Phi_A(P_A)$
- for any compact subset  $K$  of  $\mathbb{C}$ , there exists  $n \in \mathbb{N}$  such that  $K + n \subset \Phi_A(P_A)$ .

*Proof.* The first point is already shown in [Mil06] for a specific petal  $P_A^1$ . Let  $P_A^2$  be a general petal. The isomorphism induced by the identity  $P_A^1/f_1 \rightarrow P_A^2/f_2$  induces an isomorphism  $\mathbb{C}/\mathbb{Z} = \Phi_A^1(P_A^1)/T_1 \rightarrow \Phi_A^2(P_A^2)/T_1$  where  $T_1 : z \mapsto z + 1$  (the quotients  $P_A/f$  are introduced in section 4.2). Since  $\mathbb{C}/\mathbb{Z}$  is parabolic,  $\Phi_A^2(P_A^2)/T_1$  must also be parabolic. This implies that  $\Phi_A^2(P_A^2) = \mathbb{C}/\mathbb{Z}$ .

The set  $\Phi_A(P_A)$  is open, so the map which from  $z$  associates the least integer  $\tau(z)$  such that  $z + \tau(z) \in \Phi_A(P_A)$  is upper semi-continuous. This implies that  $\tau(z)$  admits a maximum  $n$  in the compact set  $K$ . As  $\Phi_A(P_A)$  is forward invariant by the translation  $T_1$ , we have  $K + n \subset \Phi_A(P_A)$ .  $\square$

**Definition 4.1.3.** Let  $\alpha \in ]0, \pi[$ , and  $P_A$  an attracting petal. We say that  $P_A$  is an attracting  $\alpha$ -petal, resp. large petal, very large, if:

- ( $\alpha$ -petal) The image of  $P_A$  by  $\Phi_A$  equals a sector, more precisely:

$$\exists w_0 \in \mathbb{C}, \exists R > 0, \{w \in \mathbb{C}, |w - w_0| > R \text{ and } |\arg(w)| < \alpha\} = \Phi_A(P_A)$$

- (large) For all  $\alpha > \pi/2$ ,  $P_A$  contains an  $\alpha$ -petal.
- (very large) The image of  $P_A$  by  $\Phi_A$  contains an upper half-plane, a lower half-plane, and a right half-plane.

To define repelling petals of  $f$ : it is an attracting petal for a local inverse  $f^{-1}$  of  $f$  which sends 0 on 0. But concerning the Fatou coordinate, we post-compose it by  $z \mapsto -z$ , in such a way that it conjugates  $f$  (and not  $f^{-1}$ ) to  $T_1$ .

**Lemma 4.1.4.** [equivalent] Let  $w_0 \in \mathbb{C}$  and  $\beta \in ]0, \pi[$ . Let a sector  $S = \{w \in \mathbb{C}, |w - w_0| > R \text{ and } |\arg(w)| < \beta\}$  be included in  $\Phi_A(P_A)$ , and  $P'_A = \Phi_A^{-1}(S)$ . We have  $\Phi_A(z) \sim -\frac{1}{az}$  when  $z \rightarrow 0$ ,  $z \in P'_A$ .

*Proof.* This is a classical result presented in [Ché08]. □

**Consequence 4.1.5.** Recall that  $f(z) = z + az^2 + O(z^3)$  where  $a \in \mathbb{C}^*$ . The fact that an attracting petal is large is equivalent to the fact that it contains an  $\alpha'$ -sector for all  $0 < \alpha' < \pi$ , in the following sense: there exists  $R > 0$  such that  $\{z \in \mathbb{C}, 0 < |z| < R, |\arg(-az)| < \alpha'\} \subset P_A$ . More geometrically, the petal  $P_A$  contains a punctured sector of width  $2\alpha'$  and small radius centered on the attracting axis.

The fact that an attracting petal is an  $\alpha$ -petal implies that it contains an  $\alpha'$ -sector for all  $0 < \alpha' < \alpha$ , in the same sense as before.

**Proposition 4.1.6.** Let  $P_A$  be an attracting petal containing a  $\frac{\pi}{2}$ -petal,  $D$  a fundamental domain of  $P_A$  (i.e. a set containing exactly once each grand orbit of  $f$  restricted to  $P_A$ ) inverse image of a strip  $\{z \in \mathbb{C}, r \leq \Re(z) < r + 1\}$  by  $\Phi_A$ , and  $P_R$  a large repelling petal. Let  $M > 0$ . Then there exists  $M' > 0$  satisfying the following property: if  $z \in D$  is such that  $\Im(\Phi_A(z)) > M'$ , then  $z \in P_R$  and  $\Im(\Phi_R(z)) > M$ . There also exists  $M'$  satisfying the property: if  $z \in D$  is such that  $\Im(\Phi_A(z)) < -M'$ , then  $z \in P_R$  and  $\Im(\Phi_R(z)) < -M$ .

The property is also available if one permutes simultaneously each occurrence of repelling by attracting and vice versa. In particular, two large petals, one of which is attracting and the other repelling, always share a non-empty intersection.

**Remark 4.1.7.** The constant  $M'$  depends on the choice of  $P_A$ ,  $P_R$ , and the choice of  $D$ .

*Proof.* Let  $w \in \mathbb{C}$  such that  $r \leq w < r + 1$ . The equivalent of  $\Phi_A$  shows that:

$$\Phi_A^{-1}(w) \sim -\frac{1}{aw}$$

when  $|\Im(w)| \rightarrow +\infty$ .

When  $\Im(w)$  is large enough (resp. large enough with negative sign), then we have  $|\Phi_A^{-1}(w)|$  small and  $\arg(\Phi_A^{-1}(w))$  close of  $-\arg(a) - \pi/2$  (resp. close of  $-\arg(a) + \pi/2$ ).

Recall that the attracting axis admits as argument  $-\arg(a) + \pi$ , the repelling axis as argument  $-\arg(a)$ . The quantity  $\Phi_A^{-1}(w)$  is close to 0, in a little cone containing a germ of the line  $X$ , where  $X$  is the line passing through 0 perpendiculary to both attracting and repelling axis. Since  $P_R$  is large, we may apply remark 4.1.5 to deduce that  $\Phi_A^{-1}(w)$  must belongs to  $P_R$ .

Now, if  $z \in D$  is such that the quantity  $\Im(\Phi_A(z))$  is big, resp. big with negative sign, we may apply the preceding argument to deduce that  $z \in P_R$ . Since  $z$  is close to 0 and avoids a sector centered on the attracting axis and a sector centered on the repelling axis, we have the equivalents  $\Phi_A(z) \sim -\frac{1}{az} \sim \Phi_R(z)$ , hence  $\Im(\Phi_R(z))$  must be big (resp. big with negative sign) when  $\Im(\Phi_A(z))$  is big (resp. big with negative sign). This finishes the proof.

The reasoning obtained inversing the role of attracting and repelling petals is identical.  $\square$

**Proposition 4.1.8.**  *$f$  admits attracting  $\alpha$ -petals, large petals and very large petals.*

*Proof.* Although it is not expressed in the same terminology, it is shown in [Mil06] that  $f$  admits attracting  $\alpha$ -petals, where  $0 < \alpha < \pi$ . It is sufficient to prove the existence of a very large attracting petal, since it is a particular case of a large petal.

*Existence of very large attracting petals:* Let  $P_A$  a attracting  $\pi/2$ -petal of  $f$ . By definition,  $D = \Phi_A(P_A)$  is a right half-plane:  $D = \{z \in \mathbb{C}, a < \Re(z)\}$ . Let  $B = \{z \in \mathbb{C}, a < \Re(z) \leq a + 1\}$ . For  $M > 0$  we note  $\Delta = \{z \in \mathbb{C}, |\Im(z)| \leq M\}$ ,  $P_A^M = \Phi_A^{-1}(D \setminus \Delta)$  and  $X^M = \Phi_A^{-1}(B \setminus \Delta)$ . If  $M$  is big enough,  $X^M$  is included in a large repelling petal  $P_R$  fixed by proposition 4.1.6. We note  $f^{-1} : P_R \rightarrow f^{-1}(P_R)$  a local inverse of  $f$  at 0.

Let  $X_{-n}^M = f^{-n}(X^M)$  for all  $n \in \mathbb{N}$ , and

$$\begin{cases} P_A^L = P_A \cup \bigcup_{n \in \mathbb{N}^*} X_{-n}^M \\ D^L = D \cup (\mathbb{C} \setminus \Delta) = D \cup \bigcup_{n \in \mathbb{N}^*} ((B - n) \setminus \Delta) \end{cases}$$

Let  $n \in \mathbb{N}^*$ . Note that  $X_{-n}^M$  is disjoint of  $P_A$ . Indeed,  $\Phi_A(f^n(X_{-n}^M)) \subset B$  and  $\Phi_A(f^n(P_A)) \subset D + n$ , and  $B, D + n$  are disjoint. By a similar reasoning,  $X_{-n}^M$  and  $X_{-m}^M$  are also disjoint.

Let for  $z \in P_A^L$ ,  $\Phi_A(z) = \Phi_A(f^n(z)) - n$  where  $n$  is such that  $f^n(z) \in P_A$  (the value of the expression does not depend on the integer  $n$  chosen). Note that the map  $\Phi_A$  is a biholomorphism from  $P_A^L$  to  $D^L$ . We check this point. It is holomorphic since  $n$  can be chosen bounded when  $z$  is in a compact subset of  $B^f \supset P_A^L$ . It is surjective by construction; it remains to show injectivity. The map  $\Phi_A$  is injective on  $P_A$ , and on  $X_{-(n+1)}^M$  for  $n \in \mathbb{N}^*$ . These sets are a partition of  $P_A^L$  and have disjoint images by  $\Phi_A$ , this finally implies that  $\Phi_A$  is injective.

It remains to check that  $P_A^L$  has the desired properties.

- $P_A^L$  is homeomorphic via  $\Phi_A$  to  $D^L$ , which is homeomorphic to the unit disk.



- $f(P_A^L) \subset P_A^L$ , because  $f(P_A) \subset P_A$  and  $f(X_{-n}^M) = X_{-n+1}^M \subset P_A^L$  for  $n \in \mathbb{N}^*$ .
- $\forall z \in P_A^L$ ,  $\lim_{n \rightarrow \infty} f^n(z) = 0$ . This property is indeed valid if  $z \in P_A$ , and if  $z \in X_{-n}^M$ , then  $f^n(z) \in P_A$ .
- Conversely, if  $f^n(z)$  converges to 0, either it is stationary at 0, or  $f^n(z)$  belongs to  $P_A^L$  for  $n$  big enough. It results from  $P_A^L \supset P_A$ .
- $\Phi_A$  is injective on  $P_A^L$
- (very large) The image of  $P_A^L$  by  $\Phi_A$  contains an upper half-plane, a lower half-plane, and a right half-plane.
- We have the following conjugacy:  $\forall z \in P_A, \Phi_A \circ f(z) = T_1 \circ \Phi_A(z)$ , with  $T_1(z) = z + 1$  for all  $z \in \mathbb{C}$ .

□

**Proposition 4.1.9** (Uniqueness of Fatou coordinates). *Let  $P_A^1, P_A^2$  be two petals of  $f$ , and  $\Phi_A^1, \Phi_A^2$  their Fatou coordinates. Then  $\Phi_A^1, \Phi_A^2$  differ by a constant on  $P_A^1 \cap P_A^2$ .*

*Proof.* We work on  $P = P_A^1 \cap P_A^2$ . This set is not necessarily homeomorphic to the unit disk. However  $P$  satisfies all the other points of the definition of petals, definition 4.1.1, with Fatou coordinates restricted to  $P$ , namely  $\Phi_{A|P}^1, \Phi_{A|P}^2$ . Note that  $P$  still satisfies proposition (petal). Let  $w \in \mathbb{C}$ , there exists  $n \in \mathbb{N}$  such that  $w + n \in \Phi_A^1(P_A^1)$ . Let  $z = \Phi_A^1(w + n) \in P_A^1$ . There exists  $m \in \mathbb{N}$  such that  $f^m(z) \in P_A^2$ . Since  $P_A^1$  is forward invariant under the action of  $f$ , we have  $f^m(z) \in P_A^1 \cap P_A^2 = P$ , this enables us to say that  $w + n + m \in \Phi_A^1(P)$ .

Let  $P/f$ , the quotient set of  $P$  obtained from the equivalence relation whose equivalence classes are the grand orbits of  $f|_P$ .

Since  $\Phi_A^i$  conjugates  $f$  to the translation  $T_1$  and  $\Phi_A^i(P)$  is forward invariant by  $T_1$ , we have  $P/f \sim \Phi_A^i(P)/\mathbb{Z}$ . By proposition (petal), for  $z \in \mathbb{C}$ , there exists  $n \in \mathbb{N}$  such that  $z + n \in \Phi_A^1(P)$ . Thus  $P/f \sim \mathbb{C}/\mathbb{Z}$ .

The function  $\Phi_A^1 \circ (\Phi_A^2)^{-1}|_{\Phi_A^1(P)} : \Phi_A^1(P) \rightarrow \Phi_A^2(P)$  commutes with the translation  $T_1$ , since  $T_1 \circ \Phi_A^1 = \Phi_A^1 \circ f|_{P_A^1}$  and  $f \circ (\Phi_A^2)^{-1} = (\Phi_A^2)^{-1} \circ T_1|_{\Phi_A^2(P_A^2)}$ . The map  $\Phi_A^1 \circ (\Phi_A^2)^{-1}$  induces therefore a holomorphic function

$$\alpha : \Phi_A^2(P)/\mathbb{Z} = \mathbb{C}/\mathbb{Z} \rightarrow \Phi_A^1(P)/\mathbb{Z} = \mathbb{C}/\mathbb{Z}$$

Similarly for  $\Phi_A^2 \circ (\Phi_A^1)^{-1}$ , which induces a map  $\beta : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$ . The equalities

$$\text{id}_{\Phi_A^2(P)} = \left( \Phi_A^2 \circ (\Phi_A^1)^{-1} \right) \circ \left( \Phi_A^1 \circ (\Phi_A^2)^{-1} \right)$$

$$\text{id}_{\Phi_A^1(P)} = \left( \Phi_A^1 \circ (\Phi_A^2)^{-1} \right) \circ \left( \Phi_A^2 \circ (\Phi_A^1)^{-1} \right)$$

become after quotient  $\beta \circ \alpha = \alpha \circ \beta = \text{id}_{\mathbb{C}/\mathbb{Z}}$ . The map  $\alpha$  is a cylinder automorphism. Therefore  $\alpha(w) = w + \sigma$  or  $\alpha(w) = -w + \sigma$ , where  $\sigma \in \mathbb{C}/\mathbb{Z}$ . Let  $X := \Phi_A^1(P) = \bigcup_{i \in I} X_i$ ,

where the  $X_i$  are the connected components of  $X$ , with the set  $I$  at most countable. Even though  $X$  might be not connected, this set is forward invariant by  $T_1 : w \mapsto w + 1$ . Lifting the map  $\alpha$  to a map from  $\mathbb{C}$  to  $\mathbb{C}$  we distinguish two cases.

- $\Phi_A^2 \circ (\Phi_A^1)^{-1}(w) = w + \tilde{\sigma}_i$  for all  $w \in X_i$ , where  $\tilde{\sigma}_i \in \mathbb{C}$  and  $\tilde{\sigma}_i \bmod \mathbb{Z} = \sigma$ . Let  $w \in X_i, w' \in X_j$  and  $K$  a connected compact subset of  $\mathbb{C}$  containing  $\{w, w'\}$ . By proposition (petal), there exists  $n \in \mathbb{N}$  such that  $K + n \subset X$ . For such an integer  $n$ , the complex numbers  $w + n, w' + n$  belongs to a connected component  $X_k$  of  $X$ . So  $\Phi_A^2 \circ (\Phi_A^1)^{-1}(w + n) = w + n + \tilde{\sigma}_k = \Phi_A^2 \circ (\Phi_A^1)^{-1}(w) + n = w + n + \tilde{\sigma}_i$ , that is why  $\tilde{\sigma}_i = \tilde{\sigma}_k$ . In the same way,  $\tilde{\sigma}_j = \tilde{\sigma}_k$ , hence  $\tilde{\sigma}_i = \tilde{\sigma}_j$  for all  $i, j \in I$ . By noting  $\tilde{\sigma}$  the common value of  $\tilde{\sigma}_i$  we get  $\Phi_A^2(w) = \Phi_A^1(w) + \tilde{\sigma}$  for all  $w \in P$ . Which was to be proven.
- $\Phi_A^2 \circ (\Phi_A^1)^{-1}(w) = -w + \tilde{\sigma}_i$  for all  $w \in X_i$ . We apply the preceding reasoning by choosing an integer  $n \geq 1$ . We get  $\Phi_A^2 \circ (\Phi_A^1)^{-1}(w + n) = -w - n + \tilde{\sigma}_k = \Phi_A^2 \circ (\Phi_A^1)^{-1}(w) + n = -w + n + \tilde{\sigma}_i$ , so  $\tilde{\sigma}_i = -2n + \tilde{\sigma}_k$ , and in the same way  $\tilde{\sigma}_k = -2n + \tilde{\sigma}_k$ . We get the following contradiction:  $\tilde{\sigma}_i = \tilde{\sigma}_j$  for all  $i, j \in I$ , but  $\tilde{\sigma}_i \neq \tilde{\sigma}_k$ . This case is thus impossible.

Which completes the proof. □

**Proposition 4.1.10.** *We assume that  $f(z) = z + az^2 + bz^3 + o(z^3)$  where  $a \in \mathbb{C}^*, b \in \mathbb{C}$ . Let  $\gamma = 1 - \frac{b}{a^2}$  be the iterative residue of  $f$ . This quantity  $\gamma$  is the residue at 0 of the meromorphic form  $\frac{dz}{f(z)-z} + \frac{dz}{z}$ .*

*Let  $P_A, P_R$  be attracting, repelling petals of  $f$ , and  $\Phi_A, \Phi_R$  be Fatou coordinates defined on these petals. We have the following asymptotic expansion, when  $z \mapsto 0$  avoiding a sector containing the repelling resp. attracting axis:*

$$\begin{aligned}\Phi_A(z) &= \frac{-1}{az} - \gamma \log \frac{-1}{az} + \text{cst} + O(z) \\ \Phi_R(z) &= \frac{-1}{az} - \gamma \log \frac{1}{az} + \text{cst} + O(z)\end{aligned}$$

*where  $\log$  denotes the principal branch of the logarithm.*

*Elements of proof.* This is a classical property presented in [Mil06]. Nevertheless, we give some elements of proof, from a more general demonstration presented in the first chapter of [Ché08]. We will not detail the precision of the approximations. We consider only the attracting Fatou coordinate (for the repelling coordinate, the reasoning is done with the local inverse branch  $f^{-1}$  of  $f$ ). We consider the two following vector fields:  $\frac{f(z)-z}{dz}, \frac{\log(f'(z))(f(z)-z)}{(f'(z)-1)dz}$ . The second one is well defined in a neighborhood of  $z = 0$ , via the principal branch of the logarithm (since  $f'(0) = 1$ ) and the analytic continuation at 0.

The flow associated to these vector fields will constitute approximations of the Fatou coordinate. We will not give a proof concerning the precision of the approximation. Nevertheless, let motivate the choice of these vector fields. Where  $z$  is close of the

parabolic point, the distance between  $f(z)$  and  $z$  is insignificant with respect to the distance between  $z$  and the parabolic point 0. Thus the quantity  $f(z) - z$  must be almost constant along an integral curve of the vector field  $\frac{f(z)-z}{dz}$  between  $t = 0$  and  $t = 1$  close to the parabolic point, which suggests that its flow is a good candidate to be an approximation of the Fatou coordinate. Note that  $\frac{\log(f'(z))}{f'(z)-1}$  is close to 1: it is a corrective factor which enables us to precise the approximation. The introduction of this corrective term, which is an idea of Buff, is motivated by the following fact. When we perturb  $f$  to get a function  $f_n$  close to  $f$ , the new function  $f_n$  has two fixed points which are in general distinct in a neighborhood of 0. The vector field  $\frac{f_n(z)-z}{dz}$  has for eigenvalues at these fixed points  $f'_n(z) - 1$ , so its flow admits for multiplier at these points  $\exp(f'_n(z) - 1)$ . Furthermore, the fixed points of  $f_n$  has multiplier  $f'_n(z)$ . The corrective factor ensures that the vector field  $\frac{\log(f'(z))(f(z)-z)}{(f'(z)-1)dz}$  has for eigenvalue  $\ln(f'_n(z))$ , so the associated flow has for multiplier  $f'_n(z)$  at these fixed points.

The flow of a vector field is obtained by integrating the differential form associated to this vector field, which is the inverse of this vector field (the complex dimension is 1). The anti-derivatives of the differential forms  $\frac{dz}{f(z)-z}$ ,  $\frac{f'(z)-1}{\log(f'(z))(f(z)-z)}$  will constitute good approximations of Fatou coordinates. The expansion:

$$\frac{f'(z) - 1}{\log(f'(z))(f(z) - z)} = \frac{1}{f(z) - z} + \frac{1}{z} + O(1)$$

suggests that an anti-derivative of the differential form  $\frac{dz}{f(z)-z} + \frac{dz}{z}$  will give a better approximation of  $f$  than an antiderivative of  $\frac{dz}{f(z)-z}$ .

We have:

$$\begin{aligned} \frac{1}{f(z) - z} &= \frac{1}{az} \frac{1}{1 + \frac{b}{a}z} + O(1) \\ &= \frac{1}{az^2} \left(1 - \frac{b}{a}z\right) + O(1) \\ &= \frac{1}{az^2} - \frac{b}{a^2z} + O(1) \end{aligned}$$

This shows that  $\gamma = 1 - \frac{b}{a^2}$  is the residue at 0 of the meromorphic form  $\frac{dz}{f(z)-z} + \frac{dz}{z}$ . We obtain by integrating:

$$\int \frac{dz}{f(z) - z} + \frac{dz}{z} = \frac{-1}{az} - \gamma \log \frac{-1}{az} + \text{cst} + O(z)$$

Where  $\log$  is the principal branch of the logarithm, and  $z$  close to 0 and avoiding a sector containing the repelling axis. This corresponds to the terms appearing in the asymptotic expansion of the attracting Fatou coordinate.  $\square$

**Definition 4.1.11.** By proposition 4.1.9, there exists a unique attracting Fatou coordinate where the constant of its asymptotic expansion is 0, and the same holds for the repelling Fatou coordinate. We call these Fatou coordinates the *normalized Fatou coordinates* of  $f$ .

**Remark 4.1.12.** Assume that  $\Phi_A, \Phi_R$  are normalized. Let  $S \subset \mathbb{C}$  be an open nonempty cone, symmetrical with respect to 0, avoiding the attracting and repelling axis. See figure 1. Let  $S^+$  be the connected component of  $S$  included in  $-\frac{1}{a}\mathbb{H}$ ,  $S^-$  the component included in  $\frac{1}{a}\mathbb{H}$ . The sets  $S^+, S^-$  could also be defined as the connected components of  $S$  satisfying the following property: each neighborhood of 0 intersected with  $S^+$  is mapped by  $\Phi_A \bmod \mathbb{Z}$  (or by  $\Phi_R \bmod \mathbb{Z}$ , this is immaterial) onto a punctured neighborhood of  $+i\infty$ , and each neighborhood of 0 intersected with  $S^-$  is mapped onto a punctured neighborhood of  $-i\infty$ .

By computing the difference between the two asymptotic expansions, one gets:

$$\Phi_A(z) = \Phi_R(z) + \epsilon i\pi\gamma + O(z) \quad (3)$$

with  $\epsilon = \mp 1$ , where  $z \rightarrow 0$ ,  $z \in S^\pm$ .

The maps  $\Phi_A, \Phi_R$  share indeed the same asymptotic expansion up to the logarithmic term; the logarithmic terms are equal up to a constant depending on whether  $z$  belongs to  $-\frac{1}{a}\mathbb{H}$  or to  $\frac{1}{a}\mathbb{H}$ .

**Remark 4.1.13.** More generally, the Fatou coordinates admits a complete asymptotic expansion of the form:

$$\begin{aligned} \Phi_A(z) &= \frac{-1}{az} - \gamma \log \frac{-1}{az} + \sum_{k=0}^n b_k z^k + o(z^n) \\ \Phi_R(z) &= \frac{-1}{az} - \gamma \log \frac{1}{az} + \sum_{k=0}^n b_k z^k + o(z^n) \end{aligned}$$

where only the terms in log differs. This is shown for instance in [LY14].

Equation (3) is also true by writing  $O(z^n)$  (and more precisely  $O(\exp(\pm \frac{2i\pi}{az}))$  when  $z \in S^\pm$ ) instead of  $O(z)$ , with arbitrary  $n \in \mathbb{N}$ . The neglected term is nonzero in general, simply negligible with respect to  $z^n$  for all  $n \in \mathbb{N}$ .

The series  $\sum b_n z^n$  admits generically a convergence radius of zero.

## 4.2 Extension of Fatou coordinates and attracting, repelling cylinders

The results of this section are already known, although the presentation might be original.

Let  $f : U \rightarrow \mathbb{C}$  a holomorphic function, where  $U \subset \mathbb{C}$  is an open neighborhood of 0, with a simple parabolic point at 0.

**Definition 4.2.1.** Define the *extended attracting Fatou coordinate* of  $f$  in the following way. For  $z \in B^f$ , there exists  $n \in \mathbb{N}$  such that  $f^n(z) \in P_A$ . The quantity  $w := \Phi_A(f^n(z)) - n$  does not depend on the chosen integer  $n$ . Let  $\Phi_A^{\text{ext}}(z) = w$  be the extended attracting Fatou coordinate.

**Remark 4.2.2.** Let  $P_A^1, P_A^2$  be two attracting petals of  $f$ , whose Fatou coordinates  $\Phi_A^{P_A^1}, \Phi_A^{P_A^2}$  coincides on their intersection. Then these two Fatou coordinates gives the

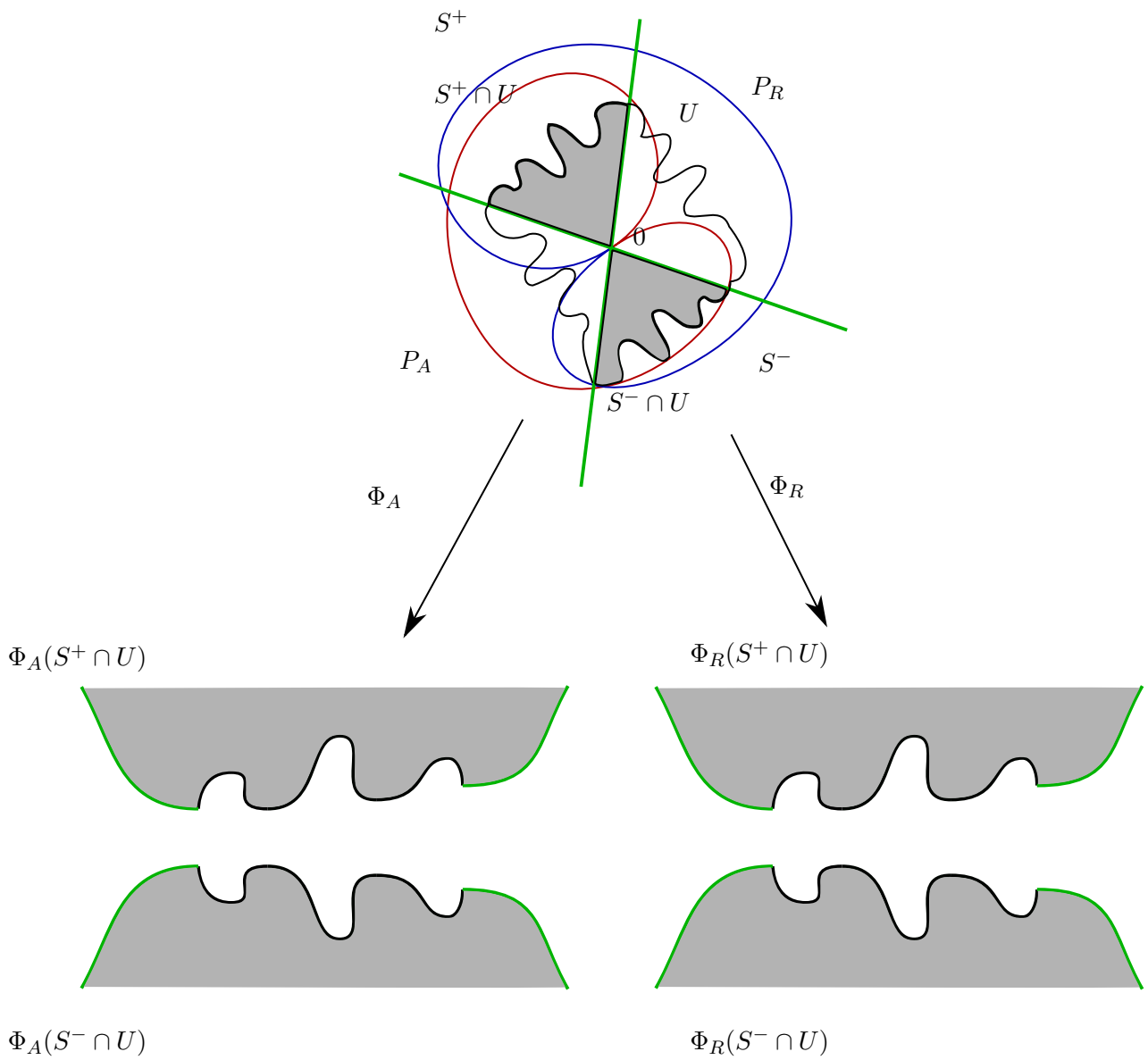


Figure 1: Taylor expansion of  $\Phi_A - \Phi_R$  over  $S^+$  and  $S^-$

*The cone  $S^+ \cup S^-$  is delimited by green lines. The set  $U$  is a neighborhood of 0. The grey parts represent  $S^+ \cap U$ ,  $S^- \cap U$  and their images by  $\Phi_A$  and  $\Phi_R$ .*

same extended Fatou coordinate. This comes from the fact that  $\Phi_A^{\text{ext}}(z)$  equals to  $\Phi_A(f^n(z)) - n$  for all  $n$  such that  $f^n(z) \in P_A^1 \cap P_A^2$ . In this sense, the extended Fatou coordinate does not depend on the choice of the petals. It is a holomorphic function on  $B^f$ . Two extended Fatou coordinates differs by a constant (post-composition by a translation).

**Definition 4.2.3.** Define the extended repelling Fatou parametrisation of  $f$  in the following way. Let  $P_R$  be a repelling petal of  $f$ . Let  $w \in \mathbb{C}$ . By proposition (petal), there exists  $n \in \mathbb{N}$  such that  $w - n \in \Phi_R(P_R)$ . Then  $z = f^n(\Phi_R^{-1}(w - n))$  (if this quantity does exist) is independent of the chosen integer  $n$ . We note  $\Psi_R^{\text{ext}}(w) = z$ .

**Remark 4.2.4.** In the same way as the preceding paragraph, the extended repelling Fatou parametrisation does not depend on the chosen repelling petal. If  $f$  is defined on  $\mathbb{C}$ , the holomorphic map  $\Psi_R^{\text{ext}}$  is defined on  $\mathbb{C}$ . Else,  $\Psi_R^{\text{ext}}$  is holomorphic on its domain of definition, which is open. In all other cases, two extended Fatou parametrisations differ by pre-composition by a translation.

We have  $\text{Im}(\Psi_R^{\text{ext}}) = \cup_{n \in \mathbb{N}} f^n(P_R)$ , independent of the chosen petal  $P_R$ . If  $f$  is defined on  $\mathbb{C}$ , the fact that  $\Psi_R^{\text{ext}}$  is a nonconstant holomorphic function on  $\mathbb{C}$  (since it is injective on the sets of the form  $\Phi_R(P_R)$ ) implies that  $\text{Im}(\Psi_R^{\text{ext}})$  avoids at most one point of  $\mathbb{C}$ .

**Definition 4.2.5.** Let  $E, F$  be two sets (whose intersection will be generally nonempty), and  $\phi : E \rightarrow F$ . We call grand orbit of  $\phi$  a nonempty subset  $X$  of  $E$  which is minimal for the inclusion satisfying the following properties:

- $\phi(X) \cap E \subset X$
- $\phi^{-1}(X) \subset X$

**Remark 4.2.6.** The grand orbits of  $\phi$  might be characterized as the equivalence classes of the following equivalence relation:

$$x \sim y \Leftrightarrow \exists n, m \in \mathbb{N}, \phi^n(x) = \phi^m(y) \in E$$

where it is implied that the intermediate iterates are also in  $E$ . We denote by  $GO(z)$  the grand orbit of  $z$ , which is by definition the equivalence class of  $z$ .

The set  $E$  is not supposed forward invariant by  $\phi$ . In this setting, it is important to suppose that  $\phi^n(x) = \phi^m(y)$  belongs to  $E$ .

Let  $f : \text{dom}(f) \rightarrow \text{range}(f)$  be a holomorphic function, nowhere locally constant. Let  $A$  be an open subset of the domain of definition of  $f$ . Let  $A/f$  be the topological space obtained by quotienting  $A$  by the equivalence relation la relation whose equivalence classes are the grand orbits of  $f|_A$ . When no confusion can arise, we name grand orbits of  $A$  the elements of this set (instead of the grand orbits of  $f$  restricted to  $A$ ). We note  $\pi_A : A \rightarrow A/f$ . By definition a subset  $V$  of  $A/f$  is an open subset of  $A/f$  if and only if  $\pi_A^{-1}(V)$  is an open subset of  $A$ . If  $U$  is a saturated set of  $A$ , then  $U$  is open if and only  $\pi_A(U)$  is open. Since  $f$  is open and continuous, then  $\pi_A$  is open.

If  $A \subset B$ , the inclusion of  $A$  in  $B$  induces a non necessarily injective map  $i : A/f \rightarrow B/f$ , which associates to a grand orbit of  $f$  restricted to  $A$  the unique grand orbit of  $f$  restricted to  $B$  which contains it.

**Proposition 4.2.7.** *The map  $i$  is open and continuous. Furthermore, it is:*

- *a surjection if and only if every grand orbit of  $f$  in  $B$  intersects  $A$ .*
- *an injection if and only if every grand orbit of  $f$  in  $B$  intersects at most one grand orbit in  $A$ . This last condition is automatically verified if  $f(A) \cap B \subset A$ . To satisfy  $f(A) \cap B \subset A$ , it is sufficient that  $A$  is forward invariant by  $f$ .*
- *a homeomorphism if and only if  $i$  is bijective. In this case,  $D \subset A$  is a fundamental domain of  $A$  (i.e. it intersects exactly once each grand orbit of  $A$ ) if and only if it is a fundamental domain of  $B$ .*

*Proof.* We check the continuity of  $i$ . Let  $\pi_B(U)$  be an open subset of  $B/f$ , where  $U$  is a saturated open subset of  $B$ . Notice that  $U \cap A$  is a saturated open subset of  $A$ , and that  $i^{-1}(\pi_B(U)) = \pi_A(U \cap A)$ , that is why this set is open and  $i$  is continuous. We check that  $i$  is open. Let  $\pi_A(U)$  be an open subset of  $A/f$  where  $U$  is a saturated open subset of  $A$ . Notice that  $i(\pi_A(U)) = \pi_B(U)$  is open, this implies the openness of  $i$ .

The map  $i$  is open and continuous,  $i$  is a homeomorphism if and only if it is bijective. The necessary and sufficient conditions of surjectivity, injectivity are clear.

We check that the injectivity of  $i$  is satisfied as soon as  $f(A) \cap B \subset A$ . Let two grand orbits of  $A$  be included in the same grand orbit of  $B$ . Let  $z_1 \in A$  be an element of the first grand orbit, and  $z_2 \in A$  an element of the second one. There exists  $n_1, n_2 \in \mathbb{N}$  such that  $f_B^{n_1}(z_1) = f_B^{n_2}(z_2) \in B$ , where it is implied that the intermediate iterates are all in  $B$ . Since  $f(A) \cap B \subset A$  and  $z_1, z_2 \in A$ , this implies that the intermediate iterates are also in  $A$ , and finally that  $f_A^{n_1}(z_1) = f_A^{n_2}(z_2) \in A$ , hence  $z_1, z_2$  are in the same grand orbit of  $A$ . Hence  $i$  is injective.  $\square$

**Proposition 4.2.8.** *Let  $P_A$  an attracting petal of  $f$ . Then the induced map by the inclusion  $P_A \subset B^f$ ,  $i_1 : P_A/f \rightarrow B^f/f$  is a homeomorphism.*

*Let  $P_R^1, P_R^2$  two repelling petals of  $f$ . Then  $P_R^1/f$  and  $P_R^2/f$  are homeomorphic.*

**Remark 4.2.9.** During the proof, the two sets  $P_R^1, P_R^2$  both appear to be homeomorphic to  $(P_R^1 \cap P_R^2)/f$  via the induced maps from the inclusions. Note that a grand orbit of  $P_R^1$  intersects exactly one grand orbit of  $P_R^2$  and conversely, the homeomorphism constructed between  $P_R^1/f$  and  $P_R^2/f$  is formed by the association of these two grand orbits.

*Proof.* Let us use proposition 4.2.7. Each grand orbit of  $B^f$  intersects  $P_A$ , this implies that  $i_1$  is surjective. The petal  $P_A$  being forward invariant by  $f$ , this implies that  $i_1$  is injective. Hence  $i_1$  is a homeomorphism.

The inclusion  $P_R^1 \cap P_R^2 \subset P_R^1$  induces a homeomorphism  $i_2 : (P_R^1 \cap P_R^2)/f \rightarrow P_R^1/f$ . Indeed, each grand orbit of  $P_R^1$  intersects  $P_R^1 \cap P_R^2$ , so  $i_2$  is surjective. Let  $f^{-1}$  be the inverse of  $f : P_R^1 \rightarrow f(P_R^1)$ . The grand orbits of  $f$  on  $P_R^1$ , resp.  $P_R^1 \cap P_R^2$  coincides

with those of  $f^{-1}$ . The set  $P_R^1 \cap P_R^2$  is forward invariant by  $f^{-1}$ , this implies that  $i_2$  is injective. Hence  $i_2$  is a homeomorphism.  $\square$

**Remark 4.2.10.** We will consider with a convenient abuse that the constructed homeomorphisms of the proposition are identifications: we will hence write  $P_A/f = B^f/f$ , and  $P_R^1/f = P_R^2/f$ . More generally, we will identify  $A/f = B/f$  when the inclusions  $A \cap B \subset A, B$  both induce homeomorphisms from  $(A \cap B)/f$  to  $A/f, B/f$ ; the identified orbits of  $A$  and  $B$  (the identified elements of  $A/f, B/f$ ) are those which intersect together.

**Proposition 4.2.11.** *Let  $P_A$  be an attracting petal of  $f$ . Passing to the quotient, the map  $\Phi_A : P_A \rightarrow \mathbb{C}$  becomes a homeomorphism  $\tilde{\Phi}_A : P_A/f \rightarrow \mathbb{C}/\mathbb{Z}$ .*

*Let  $P_R$  be a repelling petal of  $f$ . Passing to the quotient, the map  $\Phi_{R|P_R} : P_R \rightarrow \mathbb{C}$  becomes a homeomorphism  $\tilde{\Phi}_R : P_R/f \rightarrow \mathbb{C}/\mathbb{Z}$ .*

*Proof.* Passing to the quotient, the map  $\Phi_A$  becomes a homeomorphism  $\tilde{\Phi}_A : P_A/f \rightarrow \mathbb{C}/\mathbb{Z}$  for the following reasons:

- $\Phi_A$  conjugates  $f$  to the translation, so it induces a map  $\tilde{\Phi}_A : P_A/f \rightarrow \mathbb{C}/\mathbb{Z}$ .
- Proposition (petal) implies that  $\tilde{\Phi}_A : P_A/f \rightarrow \mathbb{C}/\mathbb{Z}$  is surjective.
- Recall that  $\Phi_A$  is injective on  $P_A$ . Let  $z, z' \in P_A$  be such that  $\Phi_A(z) = \Phi_A(z') \pmod{\mathbb{Z}}$ . If  $\Phi_A(z) = \Phi_A(z') + n$  where  $n \in \mathbb{N}$ , we get  $\Phi_A(z) = \Phi_A(f^n(z'))$  and  $z = f^n(z')$ . If the integer  $n$  is negative, note that  $\Phi_A(z) + (-n) = \Phi_A(z')$  which enables us to apply the same argument. Hence  $\Phi_A(z) = \Phi_A(z') \pmod{\mathbb{Z}}$  implies  $z = z' \pmod{f}$ . Hence  $\tilde{\Phi}_A : P_A/f \rightarrow \mathbb{C}/\mathbb{Z}$  is injective.
- $\Phi_A$  is a nonconstant holomorphic function, hence it is open. So  $\tilde{\Phi}_A$  is open.

$\tilde{\Phi}_A : P_A/f \rightarrow \mathbb{C}/\mathbb{Z}$  is an open continuous bijection, that is why it is a homeomorphism.

The reasoning for  $\tilde{\Phi}_R$  is identical.  $\square$

**Proposition-Definition 4.2.12.** *We consider the space of the grand orbits of  $f$  on  $B^f : B^f/f$ . The map  $\Phi_A^{\text{ext}} : B^f \rightarrow \mathbb{C}$  goes to quotient in a homeomorphism  $\bar{\Phi}_A : B^f/f \rightarrow \mathbb{C}/\mathbb{Z}$ . By transport of structure, this gives to  $B^f/f$  a Riemann surface structure. Let  $\mathcal{C}_A = \mathbb{C}/\mathbb{Z}$  be the codomain of  $\bar{\Phi}_A$ , which we label the attracting cylinder of  $f$ . Via the identification  $B^f/f = P_A/f$ , we have  $\bar{\Phi}_A = \tilde{\Phi}_A$  (in particular  $\bar{\Phi}_A$  does not depend on the chosen petal  $P_A$ , but only on the normalisation).*

*Proof.* The map  $\Phi_A^{\text{ext}}$  goes to quotient since it semi-conjugates  $f$  to  $T_1$ . The fact that  $\bar{\Phi}_A$  is a homeomorphism results from  $\bar{\Phi}_A = \tilde{\Phi}_A$ .  $\square$

**Proposition-Definition 4.2.13.** *Let  $P_R$  be a repelling petal of  $f$ . The topological space  $P_R/f$  is independent of the chosen repelling petal. Furthermore, the map  $\bar{\Phi}_R : P_R/f \rightarrow \mathbb{C}/\mathbb{Z}$  is independent of the chosen repelling petal  $P_R$ . By transport of structure, this gives*



to  $P_R/f$  a Riemann surface structure. We denote by  $\mathcal{C}_R = \mathbb{C}/\mathbb{Z}$  the codomain of  $\bar{\Phi}_R$ , which we name repelling cylinder of  $f$ . We also denote by  $\bar{\Psi}_R = \bar{\Phi}_R^{-1} : \mathcal{C}_R \rightarrow P_R/f$  the map independent of the choice of the petal that  $\Psi_R : \Phi_R(P_R) \rightarrow P_R$  induces by quotient by  $\mathbb{Z}$  on the domain and by  $f$  on the codomain.

*Proof.* The space  $P_R/f$  is independent of the repelling petal by proposition 4.2.8. If we note  $P_R^1, P_R^2$  two repelling petals of  $f$  and  $\Phi_R^1, \Phi_R^2$  their respective Fatou coordinates, chosen in such a way that they coincide on  $P_R^1 \cap P_R^2$ , we have via the identification  $P_R^1/f = P_R^2/f = (P_R^1 \cap P_R^2)/f$  the equality  $\tilde{\Phi}_R^1 = \tilde{\Phi}_R^2$ . This finishes the proof.  $\square$

**Remark 4.2.14.** Let  $D_A, D_R$  be fundamental domains of  $f$  restricted respectively to an attracting petal  $P_A$ , to a repelling petal  $P_R$  (for instance the inverse images by the Fatou coordinates of a semi-open strip of width 1). The cylinder  $\mathcal{C}_A$  is homeomorphic to  $B^f/f$ , which is in natural bijection with  $D_A$  (each grand orbit has a unique element in  $D_A$ ), by the same process  $\mathcal{C}_R$  is in natural bijection with  $D_R$ .

If  $D_A, D_R$  are inverse images by  $\Phi_A, \Phi_R$  of two strips of the form  $\{z \in \mathbb{C}, a \leq \Re(z) < a + 1\}$  (we call  $\overline{D_A} \setminus \{0\}, \overline{D_R} \setminus \{0\}$  crescents), one may also notice that  $B^f/f = (\overline{D_A} \setminus \{0\})/f$ , and  $P_R/f = (\overline{D_R} \setminus \{0\})/f$ , where the quotient consists exactly to glue the inverse images of the extremal vertical lines of the strips by the map  $f$ . This phenomenon enables us to see  $B^f/f$  and  $P_R/f$  as topological cylinders via gluing the two boundaries of a crescent.

### 4.3 Horn maps

In this section, we remind the elements of a standard theory, with a presentation which might be original. We suppose in this section and those that follow that the Fatou coordinates  $\Phi_A, \Phi_R$  are normalized (see definition 4.1.11).

Let  $P_R$  be a repelling petal. The grand orbits of  $P_R \cap B^f$  are equal to grand orbits of  $P_R$ , in such a way that by 4.2.7, the inclusion  $P_R \cap B^f \subset P_R$  induces an injection of  $(P_R \cap B^f)/f$  in  $P_R/f$ ; we assimilate the first set to a subset of the second one. These sets and this inclusion do not depend on the choice of  $P_R$ .

On the other hand, the map  $i : (P_R \cap B^f)/f \rightarrow B^f/f$  induced by the inclusion  $P_R \cap B^f \subset B^f$  is in general noninjective. This map  $i$  and its domain do not depend on the chosen repelling petal.

**Definition 4.3.1.** Set

$$\mathbf{h} = \bar{\Phi}_A \circ i \circ \bar{\Psi}_R : \bar{\Psi}_R^{-1}((P_R \cap B^f)/f) \subset \mathcal{C}_R \rightarrow \mathcal{C}_A$$

where  $\bar{\Phi}_A, \bar{\Psi}_R$  are defined in proposition-definition 4.2.12 and 4.2.13. We name this map the horn map associated to  $f$ . We note

$$\mathcal{D} = \mathcal{D}_{\mathbf{h}} = \bar{\Psi}_R^{-1}((P_R \cap B^f)/f)$$

the definition domain of  $\mathbf{h}$ . Note that:

$$\bar{\Psi}_R^{-1}((P_R \cap B^f)/f) = (\Psi_R^{\text{ext}})^{-1}(P_R \cap B^f)/\mathbb{Z} = \bar{\Phi}_R((P_R \cap B^f)/f) = \Phi_R(P_R \cap B^f)/\mathbb{Z}$$

The horn map is said to be normalized if both attracting and repelling Fatou coordinates are normalized.

**Remark 4.3.2.** Let  $\Phi_A$  be the normalized extended attracting Fatou coordinate, and  $\Psi_R^{\text{ext}}$  be the normalized extended repelling Fatou parametrisation. Then

$$\mathbf{h}(w) = \Phi_A(\Psi_R^{\text{ext}}(\tilde{w})) \bmod \mathbb{Z}$$

for all  $\tilde{w} \in (\Psi_R^{\text{ext}})^{-1}(B^f)$  such that  $w = \tilde{w} \bmod \mathbb{Z}$ .

**Remark 4.3.3.** Let  $D_A, D_R$  be fundamental domains of  $f$  restricted to an attracting petal  $P_A$ , resp. restricted to a repelling petal  $P_R$ . We associate a point  $z \in \mathcal{C}_R$  to the corresponding element of  $D_R$ , then we associate, if it does exist, the unique element of  $D_A$  which belongs to the same grand orbit for  $f$  (this element exists if and only if  $\bar{\Psi}_R(z) \in (P_R \cap B^f)/f$ , this explains the definition domain of  $\mathbf{h}$ ), then we finally associate the corresponding element of  $\mathcal{C}_A$ .

In a certain sense,  $\mathbf{h}$  may be assimilated to an iterate with nonconstant number of iterations of  $f$ . More precisely, we associate  $z_0 \in D_R \cap B^f$  first to  $f^n(z_0) \in P_A$  where  $n \in \mathbb{N}$ , then by applying the right number of times  $f$  or  $f^{-1} : f(P_A) \rightarrow P_A$  we get  $(f|_{P_A})^k \circ f^n(z_0) \in D_A$  where  $k \in \mathbb{Z}$ , the obtained result is independent of  $n, k$  chosen as soon as we respect the conditions  $f^n(z_0) \in P_A$  and  $(f|_{P_A})^k \circ f^n(z_0) \in D_A$ . notice that even with  $n$  minimal, it is still possible that  $k$  is negative. The issue of the negativity of  $k$  might be suppressed locally: there exists a neighborhood  $V$  of  $z_0$  and  $n \in \mathbb{N}$  such that  $\overline{f^n(V)} \subset P_A$ ; provided we choose  $V$  small enough and consider another fundamental domain  $D'_A$  of  $P_A$ , we can assume that  $\overline{f^n(V)} \subset D'_A$ . The map  $f^n : z \in V \mapsto f^n(z) \in D'_A$ , which gives after quotient  $\mathbf{h}|_{\bar{\Phi}_R(V)} : \bar{\Phi}_R(V) \rightarrow \bar{\Phi}_A^{-1}(D'_A) = \mathbb{C}/\mathbb{Z}$ , is the first passing map for  $f$  to  $D'_A$  from a  $z \in V \subset D_R$ .

**Remark 4.3.4.** The definition domain  $\mathbf{h}$  contains  $\Phi_R(P_A \cap P_R)/\mathbb{Z}$ , where  $P_A, P_R$  are attracting, repelling petals of  $f$ . Let  $D_R$  be a fundamental domain of  $f$  restricted to a repelling petal  $P_R$ . The definition domain of  $\mathbf{h}$  is exactly equal to  $\bar{\Phi}_R((B^f \cap D_R)/f) = \Phi_R(B^f \cap D_R)/\mathbb{Z}$ .

Recall that  $\mathcal{D}$  is the domain of  $\mathbf{h}$ , and that  $\mathcal{D}^\pm$  is the connected component of  $\mathcal{D}$  containing a neighborhood of  $\pm i\infty$  (which exists by proposition 4.3.6).

**Proposition-Definition 4.3.5.** Let  $\pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$  be the canonical projection. Set

$$\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_{\mathbf{h}} = \pi^{-1}(\mathcal{D})$$

The extended Fatou parametrisation  $\Psi_R^{\text{ext}}$  is defined at least on  $\tilde{\mathcal{D}}$  (and in particular on  $\tilde{\mathcal{D}}^\pm$ ) and  $\Psi_R^{\text{ext}}(\tilde{\mathcal{D}}) \subset B^f$ .

More precisely, for  $z$  in the definition domain of  $\Psi_R^{\text{ext}}$ ,  $z \in \tilde{\mathcal{D}}$  if and only if  $\Psi_R^{\text{ext}}(z) \in B^f$ . This enables us to set  $h = \Phi_A \circ \Psi_R^{\text{ext}}$ , the lifted horn map, defined on  $\tilde{\mathcal{D}}$  ( $\Phi_A$  denotes the extended attracting Fatou coordinate).

*Proof.* Let  $P_R$  be a repelling petal of  $f$ . Recall that by definition 4.3.1 of the horn map, we have  $\mathcal{D} = \Phi_R(P_R \cap B^f) \bmod \mathbb{Z}$ , in such a way that  $\tilde{\mathcal{D}} = \Phi_R(P_R \cap B^f) + \mathbb{N}$  given that  $\Phi_R(P_R \cap B^f)$  is already forward invariant by  $T_{-1}$ . The extended Fatou parametrisation  $\Psi_R^{\text{ext}}$  is defined at least on  $\Phi_R(P_R \cap B^f)$ , with values on this set in  $B^f$ , it indeed coincides with  $\Phi_R^{-1}$ . Since  $B^f$  is included in the definition domain of  $f$  and forward invariant by  $f$ , this implies by definition 4.2.1 of the extended repelling Fatou parametrisation that  $\Psi_R^{\text{ext}}$  is also defined on  $\Phi_R(P_R \cap B^f) + \mathbb{N}$  (use the formula  $\Psi_R^{\text{ext}}(w+n) = f^n(\Psi_R^{\text{ext}}(w))$ ), well defined for  $w \in \Phi_R(P_R \cap B^f)$ . Thus the definition domain of  $\Psi_R^{\text{ext}}$  contains  $\tilde{\mathcal{D}}$  and  $\Psi_R^{\text{ext}}(\tilde{\mathcal{D}}) \subset B^f$ .

Conversely, let  $w \in \text{Dom}(\Psi_R^{\text{ext}})$  be such that  $z = \Psi_R^{\text{ext}}(w) \in B^f$ . Since  $w \in \text{Dom}(\Psi_R^{\text{ext}})$ , there exists  $z_0 \in P_R$  and  $n \in \mathbb{N}$  such that  $f^n(z_0) = z$ . Since  $f^{-1}(B^f) \subset B^f$ , we also have  $z_0 \in B^f$ . Note  $w_0 = \Phi_R(z_0) = w - n$ . We have

$$w \bmod \mathbb{Z} = w_0 \bmod \mathbb{Z} = \Phi_R(z_0) \bmod \mathbb{Z} \in \bar{\Phi}_R(B^f \cap P_R) = \mathcal{D}$$

hence  $w \in \tilde{\mathcal{D}}$ , which completes the proof.  $\square$

**Proposition 4.3.6.** *The definition domain of  $\mathbf{h}$  contains a punctured neighborhood  $V_+$  of  $+i\infty$ , and a punctured neighborhood  $V_-$  of  $-i\infty$ . More precisely, if  $P_A, P_R$  are attracting, repelling large petals of  $f$ , take the following punctured neighborhoods (which are non connected in general) :*

$$V_+ = \bar{\Phi}_A \left( P_A \cap P_R \cap \left(-\frac{1}{a}\mathbb{H}\right) \right)$$

$$V_- = \bar{\Phi}_A \left( P_A \cap P_R \cap \left(\frac{1}{a}\mathbb{H}\right) \right)$$

. We denote  $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$  the Poincare half-plane. For  $\lambda \in \mathbb{C}^*$ , the set  $\lambda\mathbb{H}$  is the image of  $\mathbb{H}$  by the map  $z \mapsto \lambda z$ .

Let  $\mathcal{D}_{\mathbf{h}}^+$  and  $\mathcal{D}_{\mathbf{h}}^-$  be the connected components of  $\mathcal{D}_{\mathbf{h}}$ , the definition domain of  $\mathbf{h}$  appearing in definition 4.3.1, punctured neighborhood of  $+i\infty$  and  $-i\infty$  respectively (these connected components might possibly be equal).

*Proof.* Let  $P_A, P_R$  be  $\alpha$ -petals of  $f$ , where  $\frac{\pi}{2} < \alpha < \pi$ .

When  $z$ , belonging to the intersection  $P_A \cap P_R$ , tends to 0:

$$\Phi_A(z) \sim -\frac{1}{az} \sim \Phi_R(z)$$

Let  $0 < \delta < \alpha - \frac{\pi}{2}$ . Then, by remark 4.1.5, there exists  $r > 0$  such that

$$\left\{ z \in \mathbb{C}, 0 < |z| < r, \pi/2 - \delta < \arg\left(\frac{z}{-a}\right) < \pi/2 + \delta \right\} \subset P_A \cap P_R$$

By the preceding equivalent, and applying a winding number argument, we have an inclusion of the form  $\Phi_R(P_A \cap P_R \cap \left(\frac{-1}{a}\mathbb{H}\right)) \supset \{x + iy \in \mathbb{C}, 0 \leq x \leq 1, y > R\}$  where  $R > 0$  is large enough.

The set :

$$V_+ = \overline{\Phi}_R(P_A \cap P_R \cap (-\frac{1}{a}\mathbb{H}))$$

is then a punctured neighborhood of  $+i\infty$ .

The proof is analogous for  $V_-$ .  $\square$

**Remark 4.3.7.** One may replace  $-\frac{1}{a}\mathbb{H}$  by any nonempty open sector centered at 0 contained in this set. The new set formed  $V_+$  will also be a neighborhood of  $+i\infty$ . The same holds for  $V_-$  by replacing  $\frac{1}{a}\mathbb{H}$  by an nonempty open sector centered at 0 contained in this set.

**Proposition 4.3.8.** *Set  $\tilde{\mathcal{D}}^\pm = \pi^{-1}(\mathcal{D}^\pm)$ . Then we have  $\tilde{\mathcal{D}}^\pm \subset \text{Dom}(\Psi_R^{\text{ext}})$  and  $\Psi_R^{\text{ext}}(\tilde{\mathcal{D}}^\pm) \subset B_0^f$ , where  $B_0^f$  is the immediate basin of  $f$  at 0.*

*Proof.* By proposition-definition 4.3.5, we have already the inclusions  $\tilde{\mathcal{D}}^\pm \subset \text{Dom}(\Psi_R^{\text{ext}})$  and  $\Psi_R^{\text{ext}}(\tilde{\mathcal{D}}^\pm) \subset B^f$ . The set  $\Psi_R^{\text{ext}}(\tilde{\mathcal{D}}^\pm)$  is connected. Moreover, let  $G$  be a left half plane such that  $P_R = \Psi_R^{\text{ext}}(G)$  is a repelling petal and let  $P_A$  be a large attracting petal. By proposition 4.1.6,  $P_R \cap P_A \neq \emptyset$ , hence  $P_R$  intersects  $B_0^f$ . The desired inclusion is proved.  $\square$

**Proposition 4.3.9.** *The map  $\mathbf{h}$  has a removable singularity at  $+i\infty$ . More precisely,  $\mathbf{h}$  admits the following expansion:*

$$\mathbf{h}(w) = w - i\pi\gamma + O\left(\exp(-2i\pi|\text{Im}(w)|)\right)$$

*Likewise,  $\mathbf{h}$  has a removable singularity at  $-i\infty$ . More precisely,  $\mathbf{h}$  admits the following expansion:*

$$\mathbf{h}(w) = w + i\pi\gamma + O\left(\exp(-2i\pi|\text{Im}(w)|)\right)$$

*The expansion of  $\mathbf{h}$  implies in particular:*

- For all  $M > 0$ , there exists  $M' > 0$  such that  $\Im(w) > M'$  implies  $\Im(\mathbf{h}(w)) > M$ .
- For all  $M > 0$ , there exists  $M' > 0$  such that  $\Im(w) < -M'$  implies  $\Im(\mathbf{h}(w)) < -M$ .

*These property are summarized in the following way: if  $w$  has a high imaginary part (positively, negatively) then  $\mathbf{h}(w)$  also have a high imaginary part (positively, negatively).*

*Proof.* This is a consequence of the asymptotic expansion of Fatou coordinates, more precisely of remark 4.1.12. Recall that the maps  $\Phi_A, \Phi_R$  are normalized; this remark (and consequently the expansion of  $\mathbf{h}$ ) would not be true without this assumption: there would be an additional constant term.

Let  $S, S^\pm$  as in remark 4.1.12. Since  $\Phi_A(z) = \Phi_R(z) + \epsilon\pi\gamma + O(z)$  when  $z \rightarrow 0$ ,  $z \in S^\pm$ , with  $\epsilon = \mp 1$ , we get  $\Phi_A \circ \Psi_R^{\text{ext}}(w) = w + \epsilon\pi\gamma + O(w)$  when  $w \mapsto \pm i\infty$ .

The result is thus obtained by passing the domain and the codomain to quotient by  $\mathbb{Z}$ :  $\lim_{w \rightarrow \pm i\infty} \mathbf{h}(w) = \pm i\infty$  and the singularity of  $\mathbf{h}$  at  $\pm i\infty$  is removable. Since a neighborhood of  $\pm i\infty$  in  $\mathbb{C}/\mathbb{Z}$  is isomorphic to a neighborhood of 0 in  $\mathbb{C}$  via the map  $z \mapsto \exp(\pm 2i\pi w)$ , the neglected term  $O(w)$  must in reality be a  $O\left(\exp(\pm 2i\pi w)\right) = O\left(\exp(-2i\pi|\operatorname{Im}(w)|)\right)$ .  $\square$

**Consequence 4.3.10.** *Let  $P_A, P_R$  be attracting, repelling petals of  $f$  included in a neighborhood  $W$  of 0 where  $f$  admits a local inverse branch  $f^{-1}$ . We note  $\Phi_A, \Phi_R$  Fatou coordinates on these petals. Then for  $z \in P_A \cap P_R$ ,  $\Im(\Phi_A(z))$  is high (resp. low) if and only if  $\Im(\Phi_R(z))$  is high (resp. low).*

*Nevertheless for  $z \in B^f \cap P_R$  we have in general only one implication: if  $\Im(\Phi_R(z))$  is high (resp. low) then  $\Im(\Phi_A^{\text{ext}}(z))$  is high (resp. low), where  $\Phi_A^{\text{ext}}$  is the extended Fatou coordinate.*

*Proof.* To show the equivalence, we use the horn maps  $\mathbf{h}, \mathbf{h}'$  of  $f$  and  $f^{-1}$ . The formulas, when  $z \in P_A \cap P_R$ :

$$\begin{aligned}\Phi_A(z) \bmod \mathbb{Z} &= \mathbf{h}(\Phi_R(z) \bmod \mathbb{Z}) \\ -\Phi_R(z) \bmod \mathbb{Z} &= \mathbf{h}'(-\Phi_A(z) \bmod \mathbb{Z})\end{aligned}$$

enable us easily to finish the proof by applying the expansions of  $\mathbf{h}$  and  $\mathbf{h}'$ .

To show the implication, we use the fact that the first formula is still available when  $z \in B^f \cap P_R$ , by replacing  $\Phi_A$  by  $\Phi_A^{\text{ext}}$ . Note nonetheless that the second formula would be only available for  $z \in B^{f^{-1}} \cap P_A$  where  $P_A$  is an attracting petal of  $f$  (and consequently a repelling petal of  $f^{-1}$ ), and this is not enough to show the converse statement.

This converse statement is false. Let us give a counter-example. Note  $f : z \mapsto z + z^2$  the cauliflower map. An attracting petal  $P_A$  admits iterated inverse images  $P'$  by  $f$  which accumulate densely over the boundary of  $B^f$ , with Euclidean diameters converging to 0. Since a repelling petal  $P_R$  contains in its interior boundary elements of  $B^f$ , it must contain the closure of one of the  $P'$  totally. That is why  $\Im(\Phi_R)$  is bounded on  $P'$ , although it is not the case for  $\Im(\Phi_A)$ . This finishes the proof: the forementioned converse statement is false.  $\square$

**Remark 4.3.11.** We have shown that the converse statement of the consequence 4.3.10 is false, i.e. for  $z \in B^f \cap P_R$ , if  $\Im(\Phi_A^{\text{ext}}(z))$  is high (resp. low), it is not necessary that  $\Im(\Phi_R(z))$  is high (resp. low), where  $\Phi_A^{\text{ext}}$  is the extended Fatou coordinate. This implies that the converse statement of proposition 4.3.9 is also false : for  $z \in \mathcal{D}$ , if  $\mathbf{h}(w)$  has a high imaginary part (positively, resp. negatively), it is not necessary that  $w$  also have a high imaginary part (positively, resp. negatively).

We show some properties concerning the very large attracting petals:

**Proposition 4.3.12.** *Let  $f$  be a holomorphic function admitting an expansion at 0 of the form  $f(z) = z + az^2 + o(z^2)$  where  $a \in \mathbb{C}^*$ . Let  $P_A$  be a very large attracting petal of  $f$ , and  $P_R$  be a repelling petal  $f$ . Then there exists an upper half-plane  $H$  and a lower*

half-plane  $B$  such that  $\Phi_R^{-1}(H \cup B) = \Phi_R^{-1}((H \cup B) \cap \Phi_R(P_R)) \subset P_A$ . The same holds by interchanging the roles of  $P_A, P_R$  (this amounts to applying the result to  $f^{-1}$ ).

*Proof.* Let  $P_A$  be a very large attracting petal. Let  $H', B'$  be upper, lower half-planes such that  $\Phi_A(P_A) \supset H' \cup B'$ . Proposition 4.3.9 shows that there exists two half-planes  $H, B$  such that  $\Phi_A(\Psi_R^{\text{ext}}(H \cup B)) \subset H' \cup B'$ , so  $\Psi_R^{\text{ext}}(H \cup B) \subset P_A$ .  $\square$

**Proposition 4.3.13.** *Let  $f$  be a holomorphic function admitting an expansion at 0 of the form  $f(z) = z + az^2 + o(z^2)$  where  $a \in \mathbb{C}^*$ .*

*Let  $P_A$  be a very large attracting petal, contained in a small neighborhood  $f(W)$  of 0 where  $f$  admits a local inverse branch  $f^{-1} : f(W) \rightarrow W$ . Let  $P_R$  be a repelling petal containing an  $\alpha$ -petal, where  $\pi/2 < \alpha < \pi$  (it is sufficient that  $P_R$  is large). Then there exists an upper half plane  $H$ , a lower half-plane  $B$  both included in  $\Phi_A(P_A)$ , and a left half-plane  $G$  such that  $L = L_{AR} := \Phi_A^{-1}((H \cup B) \cap G)$  is included in  $P_R$ .*

*Let  $P_R$  be a very large repelling petal, contained in a small neighborhood of 0 where  $f$  admits a local inverse branch  $f^{-1}$ . By applying the preceding statement (formed by the first two paragraphs of the present proposition) to  $f^{-1}$ , we get the following statement: let  $P_A$  be an attracting petal containing an  $\alpha$ -petal, where  $\pi/2 < \alpha < \pi$ . Then there exists an upper half-plane  $H$ , a lower half-plane  $B$  both included in  $\Phi_R(P_R)$ , and a right half-plane  $D$  such that  $L = L_{RA} := \Phi_R^{-1}((H \cup B) \cap D)$  is included in  $P_A$ .*

*In both cases, we call  $L$  a lepal pair of  $f$ .*

See figure 2, next page.

*Proof.* Without loss of generality we suppose (up to taking a conjugate of  $f$ ) that  $f(z) = z + z^2 + o(z^2)$  at 0. We only show the statement for the lepal pair  $L_{AR}$ , the other statement admits a similar proof. Let  $R > 0$  be such that  $S_{R, \alpha'} := \{z \in \mathbb{C}, |z| > R \text{ and } \pi - \alpha' < |\arg(z)| < \alpha'\} \subset \Phi_A(P_A)$ , where  $\alpha' \in ]\pi/2, \alpha[$ . We also note  $\alpha'' \in ]\alpha', \alpha[$ , which will be used later. Set  $P'_A = \Phi_A^{-1}(S_{R, \alpha'}) \subset P_A$ .

The Fatou coordinate  $\Phi_A$  admits within  $P'_A$  the equivalent:

$$\Phi_A(z) \sim -\frac{1}{z}$$

Let  $\epsilon > 0$ . The Fatou coordinate equivalent implies that there exists a neighborhood  $V = V_\epsilon$  of infinity in  $S_{R, \alpha'} = \Phi_A(P'_A)$  such that  $\Phi_A^{-1}(V)$  is included in a cone  $\{z \in \mathbb{C}, |z| < \epsilon \text{ and } \pi - \alpha'' < |\arg(z)| < \alpha''\}$ . We choose  $\epsilon > 0$  in such a way that this last cone is included in a repelling  $\alpha$ -petal  $P_R$  of  $f$ . The existence of such an  $\epsilon$  is shown likewise by applying the equivalent of  $\Phi_R$  instead of the one of  $\Phi_A$ . This finally implies that for a neighborhood  $V$  of infinity in  $S_{R, \alpha'} = \Phi_A^{-1}(P'_A)$ , the image of  $\Phi_A(V)$  is included in a repelling  $\alpha$ -petal  $P_R$  of  $f$ . Let  $H$  be an upper half-plane, chosen in such a way that  $H'_0 := H \cap \{z \in \mathbb{C}, a - 1 \leq \Re(z) < a\} \subset V$  for some  $a \in \mathbb{R}$ .

Set for  $n \in \mathbb{N}$ ,  $H'_n = H \cap \{z \in \mathbb{C}, a - n - 1 \leq \Re(z) < a - n\}$ . We show by induction that for all  $n \in \mathbb{N}$ ,  $\Phi_A^{-1}(H'_n) \subset P_R$ . This property is indeed true for  $n = 0$ , since  $H'_0 \subset V$ . Now, let  $n \in \mathbb{N}$  be such that  $\Phi_A^{-1}(H'_n) \subset P_R$ . So  $f(\Phi_A^{-1}(H'_{n+1})) = \Phi_A^{-1}(H'_n) \subset P_R$ . The

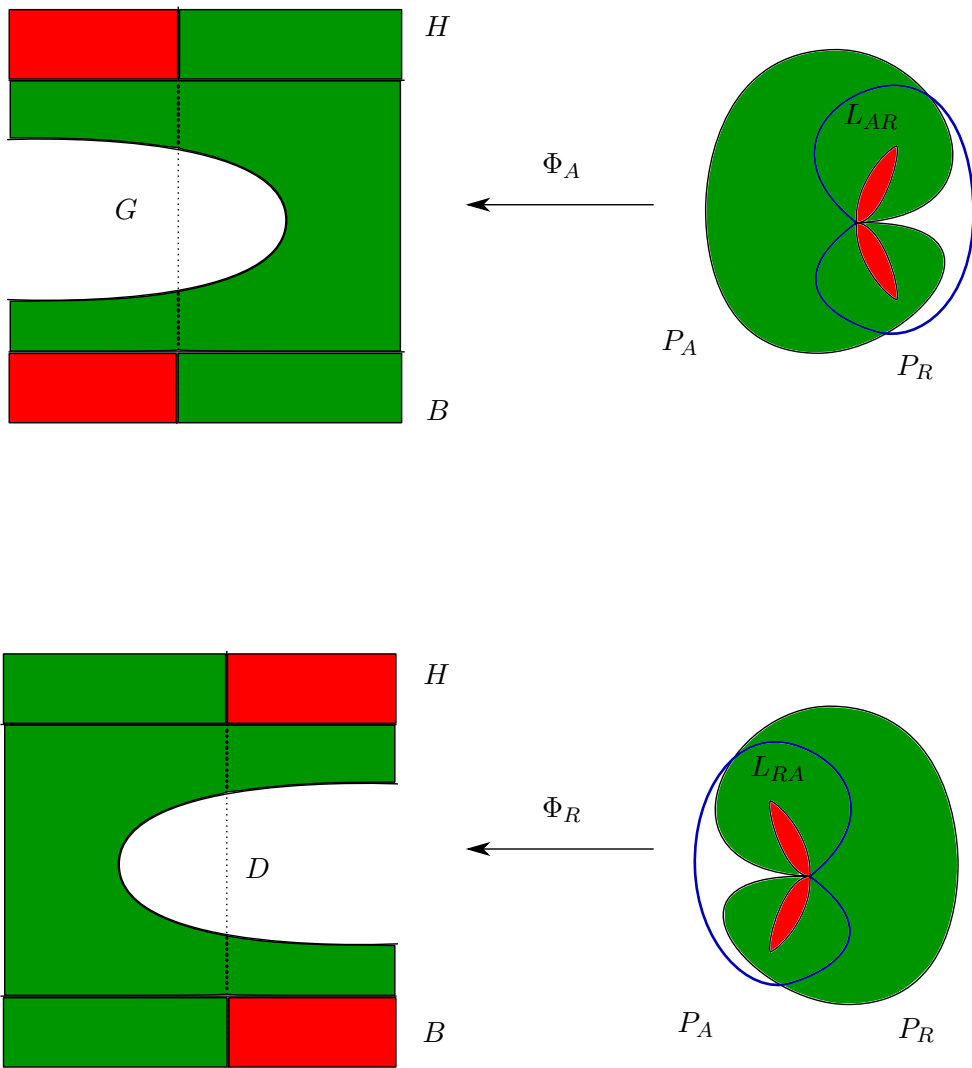


Figure 2: Lepal pairs of  $f$

very large attracting petal  $P_A$  is included in  $W$ . This implies that  $\Phi_A^{-1}(H'_{n+1})$  is also included in  $W$ , and then  $\Phi_A^{-1}(H'_{n+1}) = f^{-1}(\Phi_A^{-1}(H'_n))$  where  $f^{-1} : f(W) \rightarrow W$  is the local inverse branch of  $f$ . Hence  $\Phi_A^{-1}(H'_{n+1}) \subset P_R$ .

This finally implies, denoting  $G' = \{z \in \mathbb{C}, \Re(z) < a\}$  that we have  $\Phi_A^{-1}(H \cap G') \subset P_R$ . Likewise, there exists a lower half-plane  $B$  and a left half-plane  $G''$  such that  $\Phi_A^{-1}(B \cap G'') \subset P_R$ . This finishes the proof by setting  $G = G \cap G''$ .  $\square$

## 4.4 Definition of local pseudo-conjugacy and immediate consequences

### 4.4.1 Intermediate statement

We show proposition 4.4.2. It will be useful in the proofs of proposition 4.4.17 and proposition 4.4.27, which are also treated in this section.

**Notation 4.4.1.** Let  $\mathcal{S}$  be the set of (infinite) punctured closed sectors at 0 of  $\mathbb{C} \setminus \mathbb{R}_+$ . Note for  $A \subset \mathbb{C}$  and  $r > 0$ :  $A[r] = A \cap r\mathbb{D}$

**Proposition 4.4.2.** Let  $U, V$  be two simply connected open sets of  $\mathbb{C}$  such that  $\lambda U \rightarrow \mathbb{C} \setminus \mathbb{R}_+$ ,  $\lambda V \rightarrow \mathbb{C} \setminus \mathbb{R}_+$ , the limits are taken in the Caratheodory sense, when  $\lambda \rightarrow +\infty$ . This implies that 0 belongs to the boundary of  $U$  and  $V$ .

Let  $X = \{x_n, n \in \mathbb{N}\}$  be a subset of  $U$  such that  $(x_n)$  converges in the Euclidean sense to 0 and the sequence of hyperbolic distances in  $U$ ,  $(d_U(x_n, x_{n+1}))$ , converges to 0. We also assume that  $X$  is included in a sector  $S_X \in \mathcal{S}$ .

Let  $\phi : U \rightarrow V$  be a holomorphic function. Suppose that there exists  $\lambda_0 \in \mathbb{R}_+^*$  such that  $\phi(z) \sim \lambda_0 z$  when  $z \rightarrow 0$ ,  $z \in X$  (this amounts to saying that  $\phi(x_n) \sim \lambda_0 x_n$  when  $n \rightarrow \infty$ ). Then:

- for all sector  $S \in \mathcal{S}$  we have  $\phi(z) \sim \lambda_0 z$  when  $z \rightarrow 0$ ,  $z \in S$
- for all sector  $S' \in \mathcal{S}$ , and all sector  $S$  such that  $S' \cap \partial\mathbb{D} \subset \text{int}(S)$  there exists  $r, r' > 0$  such that  $S[r] \subset U$ ,  $S'[r'] \subset V$ , and  $\phi(S[r]) \supset S'[r']$ . The set  $\text{int}(S)$  denotes the interior of  $S$ .

**Remark 4.4.3.** One may replace  $X$  by any set which contains such a sequence  $(x_n)$ , for instance any curve included in a sector  $S_X \in \mathcal{S}$  converging towards the origin, or  $X = S_X \in \mathcal{S}$ .

**Remark 4.4.4.** The proposition is still valid with the following modifications:

- replace  $\mathbb{C} \setminus \mathbb{R}_+$  by  $\mathbb{C} \setminus S_0$ , where  $S_0 = \{0\} \cup \{z \in \mathbb{C}^*, \alpha_0 \leq \arg(z) \leq \beta_0\}$  nonpunctured closed sector at the origin
- replace the definition of  $\mathcal{S}$  by: the set of punctured closed sectors at the origin such that  $S \cap S_0 = \emptyset$ .

The given proof of the proposition readapts in this more general case with slight adaptations.



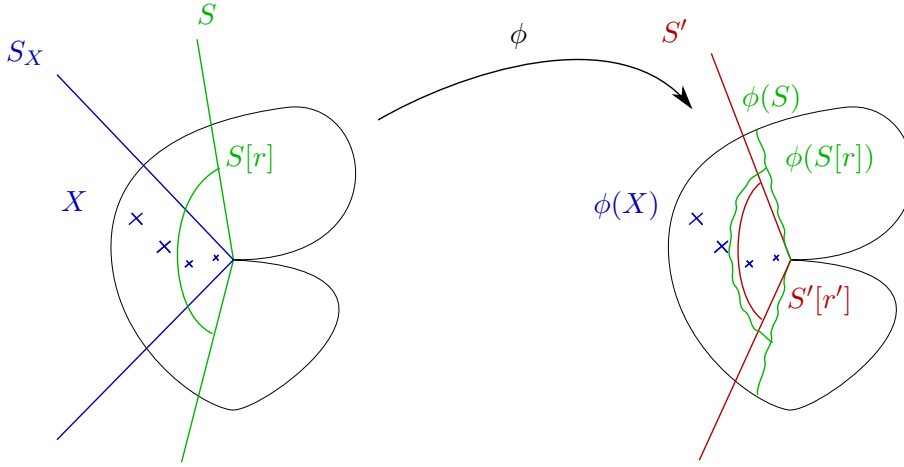


Figure 3: Intermediate proposition

Discussion about the technical hypothesis done on the sequence  $(x_n)$ :

**Lemma 4.4.5.** *The hypothesis “ $d_U(x_n, x_{n+1})$  converges to 0” is equivalent (if all the other hypothesis of the proposition are assumed) to “ $\frac{x_{n+1}}{x_n}$  converges to 1”.*

*Proof.* Note that  $d_U(x_n, x_{n+1}) = d_{\frac{1}{|x_n|}U}(\frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|})$ . Let us show that  $\frac{x_n}{|x_n|}$  and  $\frac{x_{n+1}}{|x_n|}$  are for  $n$  big enough in a fixed compact set  $K$  of  $\mathbb{C} \setminus \mathbb{R}_+$  as soon as  $\frac{x_{n+1}}{x_n}$  converges to 1 or  $d_U(x_n, x_{n+1})$  converges to 0. This will allow in both cases to compare the hyperbolic metric of  $\frac{1}{|x_n|}U$  and the one of  $\mathbb{C} \setminus \mathbb{R}_+$ .

- Suppose that  $(\frac{x_{n+1}}{x_n})$  converges to 1. Then choose for  $K$  any  $h$ -neighborhood of  $S_X \cap \partial\mathbb{D}$  where  $0 < h < 1$ . The element  $\frac{x_n}{|x_n|}$  is in  $S_X \cap \partial\mathbb{D}$  by definition of  $S_X$ , thus the element  $\frac{x_{n+1}}{|x_n|} = \frac{x_n}{|x_n|} \frac{x_{n+1}}{x_n}$  is in  $K$  for  $n$  big enough.
- Suppose that  $d_U(x_n, x_{n+1}) = d_{\frac{1}{|x_n|}U}(\frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|})$  converges to 0. Then choose for  $K$  any  $h'$ -hyperbolic neighborhood of  $S_X \cap \partial\mathbb{D}$  for the hyperbolic metric of  $\mathbb{C} \setminus \mathbb{R}_+$  where  $h' > 0$ . The element  $\frac{x_{n+1}}{|x_n|}$  is in  $K$ . Using the Caratheodory convergence of  $(\frac{1}{|x_n|}U)$  to  $\mathbb{C} \setminus \mathbb{R}_+$ ,  $K$  is included in  $\frac{1}{|x_n|}U$  for  $n$  big enough and the Poincaré coefficient of the hyperbolic metric of  $\frac{1}{|x_n|}U$  converges uniformly over  $K$  to the Poincaré coefficient of the hyperbolic metric of  $\mathbb{C} \setminus \mathbb{R}_+$ . Thus for  $n$  big enough, the hyperbolic ball of  $\frac{1}{|x_n|}U$  of center  $\frac{x_n}{|x_n|}$  and radius  $h'/2$  is included in  $K$ . Since  $d_{\frac{1}{|x_n|}U}(\frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_n|})$  converges to 0, this implies that  $\frac{x_{n+1}}{|x_n|} \in K$  for  $n$  big enough.

This implies in both cases, by Caratheodory convergence of  $\frac{1}{|x_n|}U$  to  $\mathbb{C} \setminus \mathbb{R}_+$  (and thus

the uniform convergence of the Poincaré coefficient over the compact set  $K \ni \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_{n+1}|}$ :

$$A_n d_{\mathbb{C} \setminus \mathbb{R}_+} \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_{n+1}|} \right) \leq d_{\frac{1}{|x_n|}U} \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_{n+1}|} \right) \leq B_n d_{\mathbb{C} \setminus \mathbb{R}_+} \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_{n+1}|} \right)$$

where  $\lim A_n = \lim B_n = 1$ . That is why in both cases,  $d_U(x_n, x_{n+1}) = d_{\frac{1}{|x_n|}U} \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_{n+1}|} \right)$  is equivalent to  $d_{\mathbb{C} \setminus \mathbb{R}_+} \left( \frac{x_n}{|x_n|}, \frac{x_{n+1}}{|x_{n+1}|} \right) = d_{\mathbb{C} \setminus \mathbb{R}_+}(x_n, x_{n+1})$ .

We write:

$$d_{\mathbb{C} \setminus \mathbb{R}_+}(x_n, x_{n+1}) = d_{\mathbb{C} \setminus \mathbb{R}_+} \left( \alpha_n, \frac{x_{n+1}}{|x_{n+1}|} \right)$$

where for all  $n \in \mathbb{N}$ ,  $\alpha_n = \frac{x_n}{|x_n|} \in S_X \cap \partial\mathbb{D}$ .

Since  $\alpha_n$  belongs to a fixed compact subset of  $\mathbb{C} \setminus \mathbb{R}_+$ , the property  $d_{\mathbb{C} \setminus \mathbb{R}_+} \left( \alpha_n, \frac{x_{n+1}}{|x_{n+1}|} \right)$  converges to 0 is equivalent to say that the Euclidean distance  $|\alpha_n - \frac{x_{n+1}}{|x_{n+1}|}| = \left| \frac{x_n}{|x_n|} - \frac{x_{n+1}}{|x_{n+1}|} \right|$  converges to 0, i.e.  $|x_{n+1} - x_n| = o(x_n)$ , which is equivalent to  $x_{n+1} \sim x_n$ . This finishes the proof.  $\square$

Let us show proposition 4.4.2.

*Proof.* Denote for  $\lambda \geq 1$ ,  $\phi_\lambda : \lambda U \rightarrow \lambda V$  defined by  $\phi_\lambda(z) = \lambda \phi\left(\frac{z}{\lambda}\right)$ . We show that  $(\phi_\lambda)_{\lambda \geq 1}$  is normal on  $G = \text{int}(S[2r])$  where  $S \in \mathcal{S}$  and  $r > 0$  is chosen such that  $S[2r] \subset U$ .

Set  $Y = \phi(X)$ . Notice that  $\lambda Y = \phi_\lambda(\lambda X)$ . Since  $\lambda V \rightarrow \mathbb{C} \setminus \mathbb{R}_+$  when  $\lambda \rightarrow \infty$ , we have that for all  $z \in \mathbb{C} \setminus \mathbb{R}_+$ , there exists  $a > 0$  such that for all  $\lambda \geq a$ ,  $z \in \lambda V$ . We also have that for all  $z_0 \in \mathbb{R}_+$ ,  $\epsilon > 0$ , there exists  $a > 0$  such that for  $\lambda \geq a$ ,  $B(z_0, \epsilon) \cap (\lambda V)^c \neq \emptyset$ . Taking the values  $z_0 = 1$ ,  $\epsilon = \frac{1}{2}$ , we get for  $\lambda \geq a$  that there exists  $z(\lambda) \in B(1, \frac{1}{2}) \cap (\lambda V)^c$ . Note that the function  $z$  is not necessarily continuous. The family  $(\phi_\lambda)_{\lambda \geq a}$  is normal since  $\phi_\lambda(\lambda U)$  avoids three points,  $0, z(\lambda), \infty$  where  $z(\lambda)$  belongs to a fixed compact set avoiding  $0, \infty$ .

Let  $\psi$  be an accumulation point of  $(\phi_\lambda)$  for the uniform local convergence. Let  $(\lambda_n)$  be such that  $(\phi_{\lambda_n})$  converges to  $\psi$ . Let  $F$  be an accumulation point of the sequence  $\lambda_n X$  for the Hausdorff metric. Note that  $F$  is a closed set included in  $S_X$ . Up to extraction of a subsequence, we may assume that  $F$  is the limit of the sequence  $\lambda_n X$ .

**Lemma 4.4.6.** *For all  $r > 0$ , the set  $F$  contains an element  $z$  of modulus  $r$ . In particular,  $F$  admits accumulation points.*

*Proof.* Let  $\epsilon > 0$ . Let  $A$  be the annulus delimited by circles of center 0 and respective radii  $r, r + \epsilon$ .

Since  $\lim_{n \rightarrow +\infty} \frac{x_{n+1}}{x_n} = 1$ , there exists  $k_0 \geq 0$  such that for  $k \geq k_0$  we have  $\frac{r}{r+\epsilon} \leq \frac{x_{k+1}}{x_k} \leq \frac{r+\epsilon}{r}$ . Let  $N \in \mathbb{N}$  be such that for all  $n \geq N$  we have  $|\lambda_n x_{k_0}| \geq r$ . Then  $\lambda_n X \cap A$  is nonempty for all  $n \geq N$ . When  $n \rightarrow \infty$ , this set converges in the Hausdorff sense to a set included in  $F \cap A$ . This set must be nonempty as a limit of nonempty sets included in a compact subset of  $\mathbb{C}$ . The property  $F \cap A$  nonempty must be true for all  $\epsilon > 0$ . The set  $F$  is closed, it thus intersects  $r\partial\mathbb{D}$ . This finishes the proof.  $\square$

Let  $z \in F$ . There exists  $z'_n \in \lambda_n X$  such that  $|z - z'_n|$  tends to 0 when  $n$  tends to infinity. Note that the  $z'_n$  belong to a compact neighborhood  $K$  of  $z$ . We have:

$$\phi_{\lambda_n}(z'_n) = \lambda_n \phi\left(\frac{z'_n}{\lambda_n}\right) \xrightarrow[n \rightarrow \infty]{} \lambda_0 z$$

where  $\lambda_0$  is given in the hypothesis of proposition 4.4.2. The elements of the sequence  $(z'_n)$  belongs to a fixed compact set  $K$ , and  $\phi_{\lambda_n}$  converges uniformly to  $\psi$  over  $K$ . This gives the existence of the limit:

$$\lim_{n \rightarrow \infty} \phi_{\lambda_n}(z'_n) = \psi(z) = \lambda_0 z$$

Hence  $\psi(z) = \lambda_0 z$  for all  $z \in F$ . By the preceding lemma and the principle of isolated zeros, we must have  $\psi(z) = \lambda_0 z$  for all  $z \in \mathbb{C} \setminus \mathbb{R}_+$ . Since the family  $(\phi_\lambda)$  is normal and has only one accumulation point, it follows that  $(\phi_\lambda)$  converges to  $\lambda_0 \text{id}$ .

Let  $K$  be a compact subset of  $U$ . Set  $\hat{K} = \cup_{\lambda \geq 1} \frac{1}{\lambda} K$ . We show that, when  $z \in \hat{K}$  and  $z \rightarrow 0$ , we have the equivalent  $\phi(z) \sim \lambda_0 z$ . Let  $z \in \hat{K}$ . We write  $z = \frac{1}{\lambda} z_0$  where  $z_0 \in K$ . The element  $z$  converges to 0 if and only if  $\lambda$  converges to infinity (although the decomposition  $z = \frac{1}{\lambda} z_0$  is not unique). We have  $\phi(z) = \frac{1}{\lambda} \phi_\lambda(\lambda z) = \frac{1}{\lambda} \phi_\lambda(z_0) \sim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \lambda_0 z_0 = \lambda_0 z$ . We note that the equivalent is uniform in  $z_0 \in K$  when  $\lambda \rightarrow \infty$  by uniformity of the limit of  $\phi_\lambda$  on the compact set  $K$ . This gives an equivalent when  $z \rightarrow 0$ ,  $z \in \hat{K}$ .

Note that for all  $S \in \mathcal{S}$ , we have  $S \cap \overline{\mathbb{D}} = \hat{K}$ , with  $K = S \cap \partial \mathbb{D}$ . In particular:

$$\phi(z) \underset{z \in S}{\sim}_{z \rightarrow 0} \lambda_0 z$$

This finishes the proof of the first part of the proposition.

Concerning the proof of the second part, let us assume, even if it means to post-compose by a homothety centered at 0, that  $\lambda_0 = 1$ . The result to be shown is a direct consequence of the following lemma, applied to  $g = \ln_S \circ \phi \circ \exp|_B$ , where  $\ln_S : S \rightarrow \mathbb{C}$  is a holomorphic branch of the logarithm on the sector  $S$ , of image  $B$ . The hypothesis  $\phi(z) \underset{z \in S}{\sim}_{z \rightarrow 0} z$  is equivalent to the Taylor expansion  $g(z) = z + o(1)$  when  $\text{Re}(z) \rightarrow -\infty$ .

**Lemma 4.4.7.** *Let  $B = \{z \in \mathbb{C}, \text{Re}(z) \leq a \text{ and } b \leq \text{Im}(z) \leq c\}$  be a half strip of the plane  $\mathbb{C}$ . Let  $0 < h < \frac{c-b}{2}$  and  $g : B \rightarrow \mathbb{C}$  be a continuous functions such that  $|g(z) - z| \leq h$  for all  $z \in B$ . Then the strip  $B' = \{z \in \mathbb{C}, \text{Re}(z) \leq a - h \text{ and } b + h \leq \text{Im}(z) \leq c - h\}$  is included in  $g(B)$ .*

See figure 4.

*Proof.* Denote by  $X$  the open  $h$ -neighborhood of  $\partial B$ . We write  $\mathbb{C} \setminus X = C_0 \sqcup C_1$  the partition in connected components, where  $C_0$  is contained in  $B$ . Notice that  $C_0 = B'$ .

Set  $g(\infty) = \infty$ , in such a way that  $g$  defines a continuous function of  $B \cup \{\infty\}$  to  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . The closed loops of  $\hat{\mathbb{C}}$  given by  $\gamma = \partial B \cup \{\infty\}$  and  $\gamma' = g \circ \gamma = g(\partial B) \cup \{\infty\}$  are homotopic in  $X \cup \{\infty\}$ . The homotopy is indeed given by:

$$\phi(s, t) = (1 - t)\gamma(s) + t\gamma'(s)$$

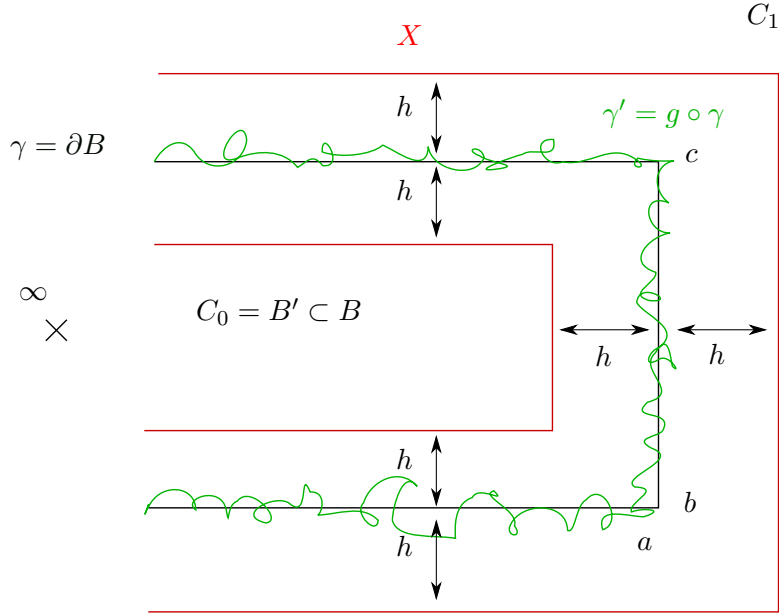


Figure 4: Topological lemma

Since  $\gamma$  is contractible in  $B$ , by direct image by  $g$ , the curve  $\gamma' = g(\partial B) \cup \{\infty\}$  is contractible in  $g(B) \cup \{\infty\}$ . Since  $\gamma$  is homotopic to  $\gamma'$  in  $X$ , this implies that  $\gamma$  is contractible in  $g(B) \cup X \cup \{\infty\}$ . This is possible only if  $g(B)$  contains  $C_0$  or  $C_1$  (see figure 4). Indeed, if  $g(B)$  avoids one point  $z_0$  of  $C_0$  and one point  $z_1$  on  $C_1$ , the homotopy class of  $\gamma$  in  $\mathbb{C} \cup \{\infty\} \setminus \{z_0, z_1\}$  would be nontrivial, thus  $\gamma$  cannot be contractible in  $g(B) \cup X \cup \{\infty\} \subset \mathbb{C} \cup \{\infty\} \setminus \{z_0, z_1\}$ , which is absurd. The set  $g(B)$  is included in  $B \cup X$  so it cannot contain  $C_1$ . It follows that  $g(B)$  contains  $C_0$ . This finishes the proof.  $\square$

This finishes the proof of the second part of the proposition.  $\square$

#### 4.4.2 Local semi-conjugacy on immediate basins

For all open set  $U_1$  of  $\mathbb{C}$  containing 0, denote by  $U_1^0$  the connected component of  $U_1 \cap B_0^{f_1}$  containing a germ of the attracting axis of  $f_1$ .

**Definition 4.4.8.** A holomorphic map  $\phi$  is said to be a local semi-conjugacy of  $f_1, f_2$  on their immediate basins if  $\phi : U_1^0 \rightarrow B_0^{f_2}$  where  $U_1$  open neighborhood of 0 in  $\mathbb{C}$ , where  $\phi$  semi-conjugacy, in other words:

$$\phi \circ f_1(z) = f_2 \circ \phi(z) \text{ for } z \in U_1^0 \text{ such that } f_1(z) \in U_1^0.$$

**Remark 4.4.9.** We require neither that  $\phi$  admits a continuous extension at 0, nor that  $\phi(U_1^0)$  is included in a small neighborhood of 0.

To motivate the choice of the set  $U_1^0$ , let us give some properties of it.

**Remark 4.4.10.** The attracting petals are connected and contain a germ of the attracting axis. Thus the set  $U_i^0$  might be understood as the connected component of  $B_0^{f_i} \cap U_i$

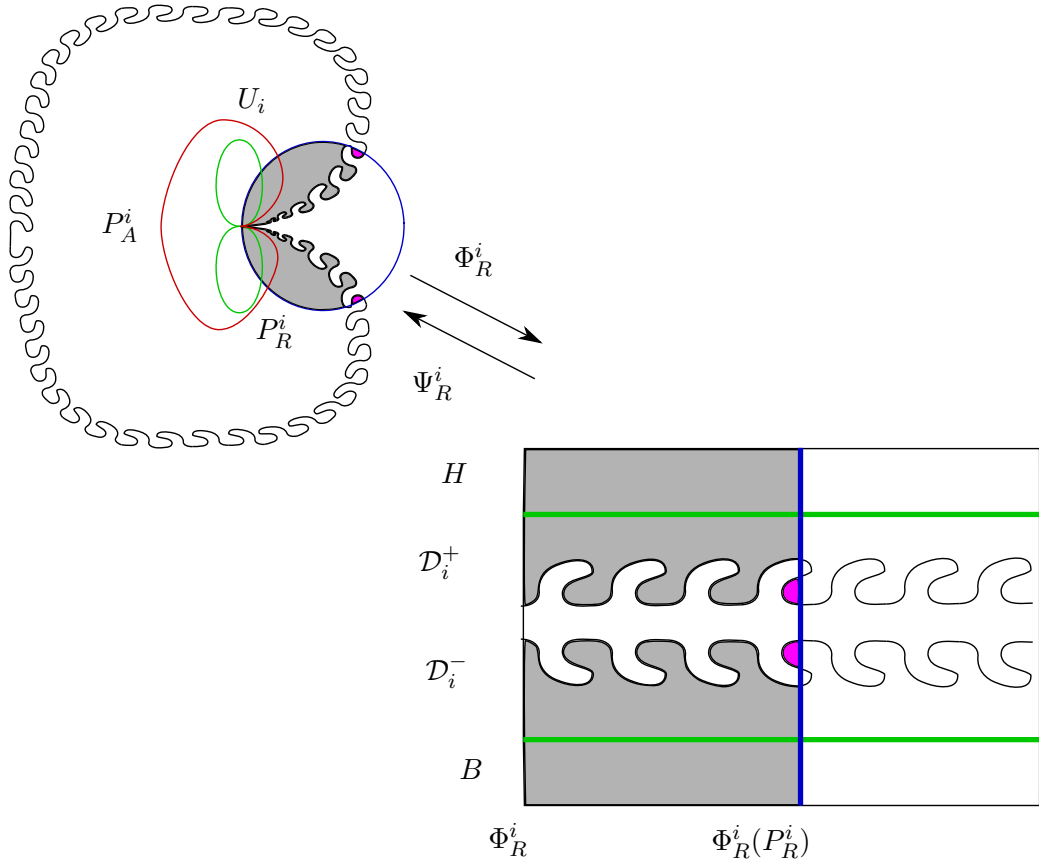


Figure 5: Some connected components of  $(\Phi_R^i)^{-1}(\tilde{\mathcal{D}}_i^\pm \cap \Phi_R^i(P_R^i))$  included in  $U_i^0$

On the figure,  $U_i = P_A^i \cup P_R^i \cup \{0\}$  and the set  $\Phi_R^i(P_R^i)$  is a left half-plane. The grey sets represent the connected components  $C$  of  $(\Phi_R^i)^{-1}(\tilde{\mathcal{D}}_i^\pm \cap \Phi_R^i(P_R^i))$  such that  $\Phi_R^i(C)$  intersects  $H \cup B$ , and their image by  $\Phi_R^i$ . The fuchsia sets represents the other connected components  $C$  of  $(\Phi_R^i)^{-1}(\tilde{\mathcal{D}}_i^\pm \cap \Phi_R^i(P_R^i))$  and their images by  $\Phi_R^i$ . One can see that the grey components are included in  $U_i^0$ .

containing a given attracting petal (resp. all the attracting petals) of  $f_i$  included in  $U_i$ . In particular,  $U_i^0$  contains all attracting petals of  $f_i$  included in  $U_i$ .

**Remark 4.4.11.** Let  $P_A^i$  be a very large attracting petal of  $f_i$  included in  $U_i$  containing  $\Psi_R^{i,\text{ext}}(H \cup B)$  where  $H, B$  are upper, lower half planes such that  $H \cup B = \{z \in \mathbb{C}, |\text{Im}(z)| > M\}$ . Let  $P_R^i$  be a repelling petal included in  $U_i$ . Then the open set  $U_i^0$  contains all the connected components  $C$  of  $(\Phi_R^i)^{-1}(\tilde{\mathcal{D}}_i^\pm \cap \Phi_R^i(P_R^i))$  such that  $|\text{Im}(\Phi_R^i)|$  is non majorized by  $M$ , where  $P_R^i$  is a repelling petal included in  $U_i$ . Recall that  $\mathcal{D}_i^\pm$  is the connected component of the domain  $\mathcal{D}_i$  of  $\mathbf{h}_i$  containing a neighborhood of  $\pm i\infty$ , which exists by proposition 4.3.6, and that  $\tilde{\mathcal{D}}_i^\pm = \pi^{-1}(\mathcal{D}_i^\pm)$ , where  $\pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$  is the canonical projection. These sets  $C$  are connected, included in  $U_i \cap B_0^{f_i}$  and intersect  $P_A^i$  by proposition 4.3.12, that is why they must be included in  $U_i^0$ . See figure 5.

This remark implies the following proposition:

**Proposition 4.4.12.** *Let  $P_R^i$  be a repelling petal of  $f_i$ . Then*

$$\Phi_R^i(U_i^0 \cap P_R^i) \bmod \mathbb{Z} \supset \mathcal{D}_i^+ \cup \mathcal{D}_i^-$$

*Proof.* We consider  $z \in \mathcal{D}_i^\pm$  and a path  $\gamma$  joining  $z$  with an element  $z_0 \in \mathcal{D}_i^\pm$  such that  $|\operatorname{Im}(z_0)| > M$ . We lift this path by  $\pi$  to obtain a path  $\tilde{\gamma}$  of  $\tilde{\mathcal{D}}_i^+ \cup \tilde{\mathcal{D}}_i^-$  joining  $\tilde{z}, \tilde{z}_0 \in \mathbb{C}$  such that  $\pi(\tilde{z}) = z$  and  $\pi(\tilde{z}_0) = z_0$ . The set  $\tilde{\gamma}$  is compact, so by proposition (petal) there exists  $n \in \mathbb{N}$  such that  $\tilde{\gamma} - n$  is included in  $\Phi_R^i(P_R)$ . This shows that  $\tilde{z} - n$  is in a connected component of  $\tilde{\mathcal{D}}_i^\pm \cap \Phi_R^i(P_R)$  whose imaginary part has an absolute value non majorized by  $M$ . Thus  $\tilde{z} - n \in \Phi_R^i(U_i^0 \cap P_R^i)$ . Hence  $z \in \Phi_R^i(U_i^0 \cap P_R^i) \bmod \mathbb{Z}$ .  $\square$

If  $U_i$  is small enough, we have the converse inclusion. More precisely, let  $\alpha \in ]\pi/2, \pi[$  and let  $P_A^i, P_R^i$  be attracting, repelling  $\alpha$ -petals of  $f_i$  such that  $P_A^i \cap P_R^i$  has two connected components  $C_+^i, C_-^i$ . This is possible by the Leau-Fatou flower theorem. In this case, we must have (even if it means permuting  $C_+^i$  and  $C_-^i$ ):

$$\Phi_R^i(C_\pm^i) \bmod \mathbb{Z} \subset \mathcal{D}_i^\pm$$

Indeed,  $\Phi_R(C_\pm^i)$  is connected, included in  $\mathcal{D}_i$  and intersects  $\mathcal{D}_i^\pm$ . In particular :

$$\Phi_R^i(P_A^i \cap P_R^i) \bmod \mathbb{Z} \subset \mathcal{D}_i^+ \cup \mathcal{D}_i^-$$

**Proposition 4.4.13.** *With these notations, suppose that  $U_i \subset P_A^i \cup P_R^i \cup \{0\}$ . Then:*

$$\Phi_R^i(U_i^0 \cap P_R^i) \bmod \mathbb{Z} \subset \mathcal{D}_i^+ \cup \mathcal{D}_i^-$$

*Proof.* It is sufficient to show this inclusion when  $U_i = P_A^i \cup P_R^i \cup \{0\}$ , since  $U_i \subset U_i'$  implies  $U_i^0 \subset U_i'^0$ . Without loss of generality, suppose  $i = 1$ . We set

$$(P_R^1)^0 = (\Phi_R^1)^{-1}(\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-)$$

and

$$V_1^0 = P_A^1 \cup (P_R^1)^0$$

We show the following lemma:

**Lemma 4.4.14.**  $U_1^0 \subset V_1^0$ .

If this lemma is shown, we obtain by applying  $\Phi_R^1$  the desired inclusion, which finishes the proof. Indeed,  $\Phi_R^1(U_1^0) = \Phi_R^1(U_1^0 \cap P_R^1)$  and  $\Phi_R^1(V_1^0) = \Phi_R^1(P_A^1 \cap P_R^1) \cup \Phi_R^1((P_R^1)^0) \subset \mathcal{D}_1^+ \cup \mathcal{D}_1^-$ . Let us show this lemma.

*Proof.* Since  $U_1 = P_A^1 \cup P_R^1 \cup \{0\}$ , we have  $U_1 \cap B_0^{f_1} = P_A^1 \cup (P_R^1 \cap B_0^{f_1})$ . The set  $U_1^0$  is the connected component of  $U_1 \cap B_0^{f_1}$  containing  $P_A^1$ . It is then the union of  $P_A^1$  with the connected components  $C$  of  $P_R^1 \cap B_0^{f_1}$  intersecting  $P_A^1$ .

To show the lemma, it is thus sufficient to show that each connected component  $C$  of  $P_R^1 \cap B_0^{f_1}$  intersecting  $P_A^1$  (and thus intersecting  $P_A^1 \cap P_R^1 = C_+^1 \cup C_-^1$ ) is contained in

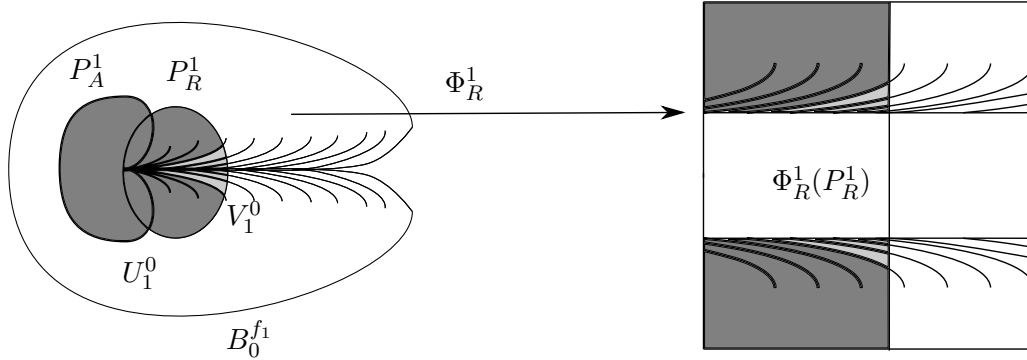


Figure 6: Inclusion of  $U_1^0$  in  $V_1^0$

The attracting petal is still represented by a cardioid, and the repelling petal by a circle. The set  $\Phi_R^1(P_R^1)$  is represented by a left half plane. The dark grey region at the left represents  $U_1^0$ , while the union of the dark grey and the light grey is equal to  $V_1^0$ . The colored regions at the right represents the images by  $\Phi_R^1$  of the zones colored at the left.

$(P_R^1)^0$ . The nonempty set  $(C_+^1 \cup C_-^1) \cap C$  is included in the definition domain of  $\Phi_R^1$ , this implies that  $C' = \Phi_R^1(C) \bmod \mathbb{Z}$  (connected set included in  $\mathcal{D}_1$ , the definition domain of  $\mathbf{h}_1$ ), intersects

$$\Phi_R^1(C_+^1 \cup C_-^1) \bmod \mathbb{Z} \subset \mathcal{D}_1^+ \cup \mathcal{D}_1^-$$

Thus  $C' \subset \mathcal{D}_1^+ \cup \mathcal{D}_1^-$ . Hence  $\Phi_R^1(C) \subset \tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-$ , and  $C \subset (P_R^1)^0$ , this finishes the proof.  $\square$

$\square$

**Remark 4.4.15.** We do not know if proposition 4.4.13 is still true for general  $U_i$ , which would be equivalent to say that:

$$\Phi_R^i(B_0^{f_i} \cap P_R^i) \bmod \mathbb{Z} \subset \mathcal{D}_i^+ \cup \mathcal{D}_i^-$$

**Remark 4.4.16.** The equality valid for small enough  $U_i$

$$\Phi_R^i(U_i^0 \cap P_R^i) \bmod \mathbb{Z} = \mathcal{D}_i^+ \cup \mathcal{D}_i^-$$

is the motivation of the definition of  $U_i^0$ : restricting  $f_i$  to  $U_i^0$  corresponds to restricting its horn map to  $\mathcal{D}_i^+ \cup \mathcal{D}_i^-$ . See definition 4.3.1 of the horn map  $\mathbf{h}$  of a function  $f$ , especially its definition domain  $\mathcal{D}_{\mathbf{h}}$ , which depends on the definition domain of  $f$ , since the immediate basin of  $f|_U$  is smaller than the immediate basin of  $f$ .

The map  $\phi$  induces a map

$$\bar{\phi} : B^{f_1}/f_1 \rightarrow B^{f_2}/f_2$$

in the following way: for  $\bar{z} = z \bmod f_1 \in B^{f_1}/f_1$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $f_1^n(z) \in U_1^0$ . The element  $\phi(f_1^n(z)) \bmod f_2 \in B^{f_2}/f_2$  is independent of the representative  $z$  of  $\bar{z}$  and of the chosen integer  $n \geq N$ .

**Proposition 4.4.17.** *Let  $\phi$  be a local semi-conjugacy. Let  $P_A^1$  be an attracting petal included in  $U_1$ , with a Fatou coordinate denoted  $\Phi_A^1$ . Then  $\bar{\phi} : B^{f_1}/f_1 \rightarrow B^{f_2}/f_2$  is a biholomorphism preserving each end of the cylinders, where we recall that the ends of the cylinder  $B^{f_i}/f_i$  are given by the biholomorphism induced by the Fatou coordinate  $\bar{\Phi}_A^i : B^{f_i}/f_i \rightarrow \mathbb{C}/\mathbb{Z}$ . This amounts to saying that the map  $\alpha = \bar{\Phi}_A^2 \circ \bar{\phi} \circ (\bar{\Phi}_A^1)^{-1} : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$  is a translation.*

*Proof.* Let  $z \in P_A^1$ . We note

$$X = \{f_1^n(z), n \in \mathbb{N}\} \subset P_A^1$$

and:

$$Y = \phi(X) = \{f_2^n(\phi(z)), n \in \mathbb{N}\} \subset B^{f_2}$$

We notice that, by the equivalent of the inverses of Fatou coordinates,  $f_1^n(z) = (\Phi_A^1)^{-1}(\Phi_A^1(z) + n) \sim -\frac{C_1}{n}$  and  $f_2^n(\phi(z)) = (\Phi_A^2)^{-1}(\Phi_A^2(\phi(z)) + n) \sim -\frac{C_2}{n}$  where  $C_1, C_2 > 0$  (recall that the  $z^2$  coefficient of  $f_i$  is supposed positive real), in such a way that  $\phi(z) \sim \frac{C_2}{C_1}z$  when  $z \in X$ ,  $z \rightarrow 0$ . Also note that  $\lim_{n \rightarrow \infty} \frac{f_1^{n+1}(z)}{f_1^n(z)} = 1$ , thus by lemma 4.4.5  $d_{U_1^0}(f_1^n(z), f_1^{n+1}(z))$  converges to 0. We may apply proposition 4.4.2 with  $U = U_1^0 \supset P_A^1$ ,  $V = B^{f_2}$ ,  $\lambda_0 = \frac{C_2}{C_1}$ . Hence for all sector  $S \in \mathcal{S}$  (recall that  $\mathcal{S}$  is the set of infinite punctured closed sectors at 0 of  $\mathbb{C} \setminus \mathbb{R}_+$ ):

$$\phi(z) \underset{z \in S}{\sim} \frac{C_2}{C_1}z$$

Let  $P_A^2$  be a large attracting petal of  $f_2$  and denote  $\tilde{\alpha} = \Phi_A^2 \circ \phi \circ (\Phi_A^1)^{-1}$ . Let  $S \subset \Phi_A^1(P_A^1)$  be a nonempty open sector containing no horizontal half-line such that  $\phi((\Phi_A^1)^{-1}(S)) \subset P_A^2$ . This last inclusion is possible by the equivalents of  $\phi$  and  $(\Phi_A^1)^{-1}$ . Notice that  $\text{Im}(z)$  is unbounded on  $S$ .

The map  $\tilde{\alpha} = \Phi_A^2 \circ \phi \circ (\Phi_A^1)^{-1} : S \rightarrow \mathbb{C}$  satisfies, by exploiting the Fatou coordinates equivalents and their inverse, and the equivalent of  $\phi$  given by proposition 4.4.2:

$$\tilde{\alpha}(w) \sim \left(-\frac{1}{C_2}\right) \frac{C_2}{C_1} (-C_1)w = w$$

where the equivalent is taken when  $w \rightarrow \infty$ ,  $w \in S$ .

In particular, since  $S$  contains no horizontal ray,  $\text{Im}(\tilde{\alpha}(w)) \underset{w \in S}{\rightarrow \text{Im}(w) \rightarrow \pm\infty} \pm\infty$ .



Notice that  $S \bmod \mathbb{Z}$  is a neighborhood of  $\pm i\infty$ , the sign depending on the choice of  $S$ . Thus the map  $\alpha = \overline{\Phi}_A^2 \circ \overline{\phi} \circ (\overline{\Phi}_A^1)^{-1} : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$  satisfies :

$$\operatorname{Im}(\alpha(w)) \rightarrow_{w \rightarrow \pm i\infty} \pm i\infty$$

The singularities at  $\pm i\infty$  are thus removable. Via the biholomorphism  $\mathbb{C}/\mathbb{Z} \cup \{\pm i\infty\} \rightarrow \hat{\mathbb{C}}$  given by  $w \mapsto \exp(2i\pi w)$ , the map  $\alpha$  induces a map  $\hat{\alpha}$  from  $\hat{\mathbb{C}}$  to itself admitting  $\{0\}$  and  $\{\infty\}$  as completely invariant sets (i.e. invariant by direct and inverse images). The map  $\hat{\alpha}$  is thus of the form  $z \mapsto Cz^n$  with  $C \in \mathbb{C}^*$  and  $n \in \mathbb{N}^*$ . Hence the map  $\alpha$  must be of the form  $z \mapsto nz + \sigma$ . Since  $\alpha$  commutes with  $z \mapsto z + 1$ , we must have  $n = 1$ , thus  $\alpha$  is a translation. This allows us to finish the proof:  $\overline{\phi}$  is a biholomorphism of the cylinder preserving the ends.  $\square$

Let  $\phi$  be a local semi-conjugacy from  $f_1$  to  $f_2$ . Let  $\Phi_A^1, \Phi_A^2$  be Fatou coordinates of  $f_1, f_2$ , associated to the attracting petals  $P_A^1, P_A^2$  included in  $U_1, U_2$ . The map  $\alpha = \overline{\Phi}_A^2 \circ \overline{\phi} \circ (\overline{\Phi}_A^1)^{-1} : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$  is a biholomorphism preserving each end of the cylinder, in such a way that it is a translation  $T_\sigma$ . We call  $\sigma$  the *phase shift* of  $\phi$  with respect to the Fatou coordinates, and denote it  $\sigma(\phi)$ .

**Proposition 4.4.18.** *The map  $\tilde{\alpha} = \Phi_A^{2, \text{ext}} \circ \phi \circ (\Phi_A^1)^{-1} : \Phi_A^1(P_A^1) \rightarrow \mathbb{C}$  is a translation  $T_{\tilde{\sigma}}$ . We call  $\tilde{\sigma}$  the lifted phase shift of  $\phi$  with respect to the attracting Fatou coordinates, and denote it  $\tilde{\sigma}(\phi)$ . We have  $\tilde{\sigma} \bmod \mathbb{Z} = \sigma$ . This phase shift depends only on the normalizations of Fatou coordinates, and not on their definition domains.*

*Proof.* In a composition of functions, it is equivalent to go to quotient at each step and to go to quotient only at the beginning and at the end. Then, the map  $\tilde{\alpha} : \Phi_A^1(P_A^1) \rightarrow \mathbb{C}$  is a lift of  $\alpha : \Phi_A^1(P_A^1)/\mathbb{Z} = \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$ . That is why  $\tilde{\alpha}$  is a translation  $T_{\tilde{\sigma}}$  where  $\tilde{\sigma} \bmod \mathbb{Z} = \sigma$ . It is clear that  $\tilde{\sigma}$  neither depends on the choice of the definition domain of  $\Phi_A^2$ , nor the one of  $\Phi_A^1$ .  $\square$

**Corollary 4.4.19.**  *$\phi$  is injective on  $P_A^1$ .*

*Proof.* The injectivity of  $\phi$  on  $P_A^1$  follow from the equality  $T_{\tilde{\sigma}|_{\Phi_A^1(P_A^1)}} = \Phi_A^{2, \text{ext}} \circ \phi \circ (\Phi_A^1)^{-1}$  and of the injectivity of  $T_{\tilde{\sigma}|_{\Phi_A^1(P_A^1)}}$ .  $\square$

**Remark 4.4.20.** The post-composition of  $\phi$  by positive powers of  $f_2$  leaves the definition domain invariant. It is not the case for pre-composition by integer powers of  $f_1$ .

**Proposition 4.4.21.** *Let  $\phi$  be a local semi-conjugacy of  $f_1, f_2$ . For  $m \in \mathbb{N}$ , the map  $f_2^m \circ \phi$  is also a local semi-conjugacy. Furthermore:*

$$\tilde{\sigma}(f_2^m \circ \phi) = \tilde{\sigma}(\phi) + m$$

*Let  $m \in \mathbb{N}$  and  $W_1, W_1'$  be open neighborhoods of 0 such that  $f_1^m : W_1 \rightarrow W_1'$  is a biholomorphism. We note  $f_1^{-m} : W_1' \rightarrow W_1$  the inverse of this biholomorphism. Set  $V_1 = f_1^{-m}(U_1 \cap W_1')$  the direct image of  $U_1 \cap W_1'$  by the biholomorphism  $f_1^{-m}$  and*

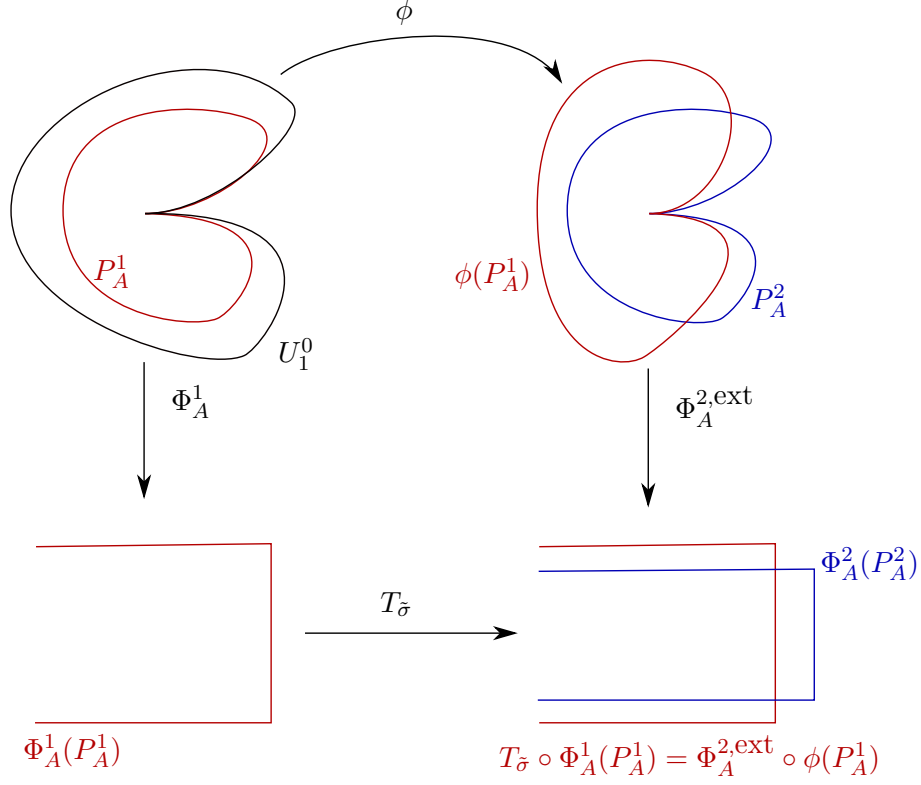


Figure 7: Petal inclusion

$V_1' = f_1^m(U_1 \cap W_1)$ . The sets  $V_1, V_1'$  are an open neighborhoods of 0. Let finally  $V_1^0$  resp.  $V_1'^0$  be the connected component of  $V_1 \cap B^{f_i}$  resp.  $V_1' \cap B^{f_i}$  containing a germ of the attracting axis. The maps  $\phi \circ f_1^m|_{V_1^0}$ ,  $\phi \circ f_1^{-m}|_{V_1'^0}$  are also a local semi-conjugacies. Furthermore:

$$\begin{aligned}\tilde{\sigma}(\phi \circ f_1^m|_{V_1^0}) &= \tilde{\sigma}(\phi) + m \\ \tilde{\sigma}(\phi \circ f_1^{-m}|_{V_1'^0}) &= \tilde{\sigma}(\phi) - m\end{aligned}$$

*Proof.* It is clear that the indicated maps are also semi-conjugacies, since  $f_2^m$  commutes with  $f_2$ , and  $f_1^{\pm m}$  commutes with  $f_1$ . For the equalities with the lifted phase shift, use  $\Phi_A^2 \circ f_2^m = T_m \circ \Phi_A^2$  and  $f_1^{\pm m} \circ (\Phi_A^1)^{-1} = (\Phi_A^1)^{-1} \circ T_{\pm m}$  where  $T_a = z \mapsto z + a$ .  $\square$

**Proposition 4.4.22.** Let  $P_A^1, P_A^2$  be attracting petals of  $f_1, f_2$ . Then:

- $\phi(P_A^1) \subset P_A^2$  is equivalent to  $T_{\tilde{\sigma}}(\Phi_A^1(P_A^1)) \subset \Phi_A^2(P_A^2)$
- $\phi(P_A^1) \supset P_A^2$  is equivalent to  $T_{\tilde{\sigma}}(\Phi_A^1(P_A^1)) \supset \Phi_A^2(P_A^2)$

In particular, there exists a very large attracting petal  $P_A^1 \subset U_1$  such that  $\phi(P_A^1) \subset U_2$ .

See figure 7.

*Proof.* We know that  $T_{\tilde{\sigma}|_{\Phi_A^1(P_A^1)}} = \Phi_A^{2,\text{ext}} \circ \phi \circ (\Phi_A^1)^{-1}$ . So:

$$T_{\tilde{\sigma}}(\Phi_A^1(P_A^1)) = \Phi_A^{2,\text{ext}}(\phi(P_A^1))$$

and  $\Phi_A^{2,\text{ext}}$  induces a biholomorphism from  $\phi(P_A^1)$  to  $\Phi_A^{2,\text{ext}}(\phi(P_A^1))$ . Furthermore:

**Lemma 4.4.23.** *Let  $X, Y$  be two topological spaces,  $U, V$  two non-disjoint open connected subsets of  $X$ . Let  $\Phi : X \rightarrow Y$  be a map such that  $\Phi : V \rightarrow \Phi(V)$  is open and proper. Then  $U \subset V$  is equivalent to  $\Phi(U) \subset \Phi(V)$ .*

*Proof.* The direct implication is clear. For the other implication, we use contraposition. Since  $U$  is connected and intersects  $V$ , the fact that  $U$  is not included in  $V$  implies that  $U$  intersects the boundary of  $V$ . Let  $z \in U \cap \partial V$ . Then  $\Phi(z) \in \Phi(U)$  and  $\Phi(z) \in \partial\Phi(V)$  since  $\Phi : V \rightarrow \Phi(V)$  is proper, so  $\Phi(z) \notin \Phi(V)$  since  $\Phi(V)$  is open.  $\square$

We apply the preceding lemma with  $U = \phi(P_A^1)$ ,  $V = P_A^2$  and  $\Phi = \Phi_A^{2,\text{ext}}$ . We apply then the same lemma, permuting the role of  $U$  and  $V$ . Since  $U$  contains a forward orbit of  $f_2$ , it is not disjoint from  $V$ . We also note that  $\Phi_A^{2,\text{ext}} : U \rightarrow \Phi_A^{2,\text{ext}}(U)$  and  $\Phi_A^{2,\text{ext}} : V \rightarrow \Phi_A^2(V)$  are biholomorphisms (since, for the first one,  $\Phi_A^{2,\text{ext}} \circ \phi \circ (\Phi_A^1|_{\phi(P_A^1)})^{-1}$  is a translation, and for the second one the extended Fatou coordinates coincides with Fatou coordinates on petals), hence open and proper.

We thus obtain:

- $\phi(P_A^1) \subset P_A^2$  if and only if  $T_\sigma(\Phi_A^1(P_A^1)) \subset \Phi_A^2(P_A^2)$
- $\phi(P_A^1) \supset P_A^2$  if and only if  $T_\sigma(\Phi_A^1(P_A^1)) \supset \Phi_A^2(P_A^2)$

To finish the proof, let  $P_A^2 \subset U_2$  be a very large attracting petal for  $f_2$ . There exists a very large attracting petal  $P_A^1$  for  $f_1$  such that  $\Phi_A^1(P_A^1) \subset T_{\tilde{\sigma}}(\Phi_A^2(P_A^2))$ , that is why  $\phi(P_A^1) \subset U_2$ .  $\square$

**Proposition 4.4.24.** *If  $P_A^1, P_A^2$  are chosen such that  $\phi(P_A^1) \subset P_A^2$ , we have for  $z \in P_A^1$ :*

$$\phi(z) = (\Phi_A^2)^{-1} \circ T_{\tilde{\sigma}} \circ \Phi_A^1(z)$$

*Proof.* This follows from the equality:

$$T_{\tilde{\sigma}|_{\Phi_A^1(P_A^1)}} = \Phi_A^{2,\text{ext}} \circ \phi \circ (\Phi_A^1)^{-1} = \Phi_A^2 \circ \phi \circ (\Phi_A^1)^{-1}$$

$\square$

Let  $\phi$  be a local semi-conjugacy between  $f_1, f_2$ . The expression of  $\phi$  on attracting petals, cf. proposition 4.4.24, implies by uniqueness of analytic extension that  $\phi$  is uniquely determined by  $\tilde{\sigma}$ . More precisely:

**Proposition 4.4.25** (Uniqueness of the local semi-conjugacy). *Let  $\Phi_A^1, \Phi_A^2$  be Fatou coordinate of  $f_1, f_2$ . Let  $\phi$  and  $\psi$  be local semi-conjugacies of  $f_1, f_2$  admitting the same lifted phase shift. Denote  $U_1$  the open set corresponding to  $\phi$  and  $V_1$  the one corresponding to  $\psi$ . Set  $W_1 = U_1 \cap V_1$ , and let  $W_1^0$  be the connected component of  $W_1$  containing a germ of the attracting axis of  $f_1$ . Then:*

$$\phi|_{W_1^0} = \psi|_{W_1^0}$$

**Corollary 4.4.26.** *Let  $\phi$  and  $\psi$  be two local semi-conjugacies such that  $\bar{\phi} = \bar{\psi}$  (recall that  $\bar{\phi}, \bar{\psi} : U_1^0/f_1 \rightarrow B^{f_2}/f_2$  are the quotient maps induced by  $\phi, \psi$ ), i.e. having the same phase shift ; recall that the phase shift is defined modulo  $\mathbb{Z}$ . Let  $U_1$  be the open set corresponding to  $\phi$  and  $V_1$  the one corresponding to  $\psi$ . Set  $W_i = U_1 \cap V_1$  and let  $W_1^0$  be the connected component of  $W_1$  containing a germ of the attracting axis of  $f_1$ . Then there exists  $m \in \mathbb{N}$  such that:*

$$f_2^m \circ \phi|_{W_1^0} = \psi|_{W_1^0} \quad \text{or} \quad \phi|_{W_1^0} = f_2^m \circ \psi|_{W_1^0}$$

*Proof.* We use the preceding proposition 4.4.25 and the lifted phase shift calculus given by proposition 4.4.21.  $\square$

**Proposition 4.4.27.** *Let  $\phi$  be a local semi-conjugacy of  $f_1, f_2$  on their immediate basin. Let  $P_A^1$  be an attracting petal (resp. large, very large petal or  $\alpha$ -petal) of  $f_1$  included in  $U_1$ , we note  $P_A^2 = \phi(P_A^1)$ . Suppose that  $P_A^1$  contains an  $\alpha$ -petal, where  $0 < \alpha < \pi$ . Then  $P_A^2$  is an attracting petal (resp. large, very large petal or  $\alpha$ -petal) of  $f_2$  for the map  $\Phi_A^2 = \Phi_A^1 \circ (\phi_{P_A^1})^{-1}$ .*

*Furthermore, let  $U_2$  be an open neighborhood of 0. If  $\Phi_A^1(P_A^1)$  is included in  $H \cup B \cup D$  where  $H, B, D$  are upper, lower, right half-plane small enough in the sense of inclusion, we have  $\phi(P_A^1) \subset U_2$ .*

*Proof.* Note that the attracting petal  $P_A^1$  is included in  $U_1^0$ . Set  $G = \Phi_A^1 \circ (\phi_{P_A^1})^{-1}$ . We show that  $Y_A^2 := \phi(P_A^1)$  is an attracting petal of  $f_2$  with Fatou coordinate  $G$ .

- $Y_A^2 = \phi(P_A^1)$  is homeomorphic to  $P_A^1$ , since  $\phi_{P_A^1} = (\phi : P_A^1 \rightarrow \phi(P_A^1))$  is a biholomorphism by corollary 4.4.19. So  $Y_A^2$  is homeomorphic to the unit disk.
- The local semi-conjugacy  $\phi$  and the stability of  $P_A^1$  by  $f_1$  show that  $f_2(Y_A^2) \subset Y_A^2$ .
- The inclusion of  $Y_A^2$  in the immediate parabolic bassin of  $f_2$  shows that:  $\forall z \in Y_A^2, \lim_{n \rightarrow \infty} f_2^n(z) = 0$ .
- Since  $G = \Phi_A^1 \circ (\phi_{P_A^1})^{-1}$ , the map  $G$  is injective.
- By exploiting the conjugacy  $\phi_{P_A^1}$  between  $f_1 : P_A^1 \rightarrow f_1(P_A^1)$  and  $f_2 : Y_A^2 \rightarrow f_2(Y_A^2)$ , we show that for all  $z \in B^{f_2}$ ,  $f_2^n(z)$  belongs to  $Y_A^2$  for  $n$  big enough. Let  $z \in B^{f_2}$ . Recall that  $(f_2^n(z))$  converges to 0 tangentially to the attracting axis, and  $U_2^0$  is the connected component of  $U_2 \cap B_0^{f_2}$  containing a germ of the attracting axis. There exists  $n_0 \in \mathbb{N}$  such that for  $n \geq n_0$ ,  $f_2^n(z) \in \phi(P_A^1) = Y_A^2$  (since by proposition 4.4.2,  $\phi(P_A^1)$  contains a set of the form  $S'[r']$ ; it follows from the fact that  $P_A^1$  contains a set of the form  $S[r]$ ).
- The condition large, very large,  $\alpha$ -petal, for  $Y_A^2$  (if  $P_A^1$  is a petal of this type) follows from  $G(Y_A^2) = (\Phi_A^1 \circ (\phi_{P_A^1})^{-1})(Y_A^2) = \Phi_A^1(P_A^1)$ .

- We have the following conjugacy:  $\forall z \in Y_A^2, G \circ f_2(z) = T_1 \circ G(z)$ , with  $T_1(z) = z+1$  for all  $z \in \mathbb{C}$ .

Concerning the proof of the second point, denote by  $\Phi_A^2$  the Fatou coordinate associated to  $P_A^2 = \phi(P_A^1)$ . Suppose that  $P_A^1$  very large such that  $\Phi_A^1(P_A^1) = H_1 \cup B_1 \cup D_1$  where  $H_1, B_1, D_1$  upper, lower, right half planes. We have that  $P_A^2$  is also very large, with  $\Phi_A^2(P_A^2) = \Phi_A^1(P_A^1)$ . Let  $Q_A^2$  be a very large attracting petal included in  $U_2$  with Fatou coordinate  $G$  such that  $G^{\text{ext}} = \Phi_A^{2,\text{ext}}$  (such a  $G$  exists by uniqueness of Fatou coordinates and uniqueness of holomorphic extension) and  $\Phi_A^{2,\text{ext}}(Q_A^2)$  can be written  $H_2 \cup B_2 \cup D_2$  with  $H_2, B_2, D_2$  upper, lower and right half planes (note that  $\Phi_A^{2,\text{ext}}$  is injective on  $Q_A^2$ ). We note  $H = H_1 \cap H_2, B = B_1 \cap B_2, D = D_1 \cap D_2$ . Set  $P_A'^1 = (\Phi_A^1)^{-1}(H \cup B \cup D)$  and  $P_A'^2 = (\Phi_A^2)^{-1}(H \cup B \cup D)$ . We have  $\Phi_A^2(P_A^2) \subset \Phi_A^{2,\text{ext}}(Q_A^2)$  and  $P_A^2 \cap Q_A^2 \neq \emptyset$  since two attracting petals must intersect. By lemma 4.4.23 applied with  $\Phi = \Phi_A^{2,\text{ext}}$  we get  $P_A'^2 = \phi(P_A'^1) \subset U_2$ . More generally, for all petal  $P_A''^1$  included in  $P_A'^1$  (this amounts to saying that  $\Phi_A^1(P_A''^1) \subset H \cup B \cup D$  by proposition 4.4.22), we have  $P_A''^2 := \phi(P_A''^1) \subset P_A'^2 \subset U_2$ .  $\square$

We give an essential corollary to prove proposition 4.5.1:

**Corollary 4.4.28.** *Let  $\phi$  be a local semi-conjugacy of  $f_1, f_2$  on their immediate basin. The petals of  $f_1$  are assumed to be included in  $U_1$ .*

*Let  $P_R^1$  be a repelling petal of  $f_1$  and  $P_R^2$  be a repelling petal of  $f_2$ . Denote  $\Phi_R^1, \Phi_R^2$  their Fatou coordinates. Then there exists  $M > 0$  satisfying the following property. Let  $z_{-n} \in B_0^{f_1}$  be such that for all  $n \in \mathbb{N}^*$ ,  $f_1(z_{-n}) = z_{-n+1}$ . Suppose that  $z_{-n} \in P_R^1$  for all  $n \in \mathbb{N}$  and  $|\Im(\Phi_R^1(z_0))| > M$ . Note  $z'_{-n} = \phi(z_{-n})$  ( $z_{-n}$  is in  $U_1^0$  by proposition 4.4.12). Then  $z'_{-n}$  belongs to  $P_R^2$  for  $n$  big enough.*

*Proof.* See figure 8. It can be assumed that  $P_R^1$  is a  $\frac{\pi}{2}$ -petal as small as desired, and that  $P_R^2$  is large. Indeed, by proposition-definition 4.2.13, two repelling petals  $P_R, Q_R$  of a function  $f$  have the same quotient by  $f$ , so a backwards orbit is eventually in  $P_R$  if and only if it is eventually in  $Q_R$ . Let  $P_A^1$  be a very large attracting petal of  $f_1$  with Fatou coordinate  $\Phi_A^1$ , small enough for the inclusion. Then, by proposition 4.4.27, the set  $P_A^2 := \phi(P_A^1)$  is a very large attracting petal of  $f_2$  with Fatou coordinate  $\Phi_A^2 = \Phi_A^1 \circ \phi$ . Up to taking a petal  $P_A^1$  small enough,  $P_A^2$  satisfies the hypothesis of proposition 4.3.13: it is included in a small enough neighborhood 0, this follows from  $\Phi_A^2(P_A^2) = T_{\tilde{\sigma}(\phi)}(\Phi_A^1(P_A^1))$ .

Let  $M' > 0$  be big enough, and  $a \in \mathbb{R}$  be big enough negatively in such a way that the half planes  $H = \{z \in \mathbb{C}, \Im(z) > M'\}$ ,  $B = \{z \in \mathbb{C}, \Im(z) < -M'\}$ , and  $G = \{z \in \mathbb{C}, \Re(z) < a\}$  satisfies  $L_{AR}^2 := (\Phi_A^2)^{-1}((H \cup B) \cap G) \subset P_R^2$  (see proposition 4.3.13). Set  $X := (\Phi_A^1)^{-1}((H \cup B) \cap G) = \phi^{-1}(L_{AR}^2)$ . Note that the set  $X$  is a lepal pair AR for  $f_1$ .

Even if it means choosing the left half plane  $\Phi_R^1(P_R^1) = G$  small enough for the inclusion, by proposition 4.3.13, there exists  $M > 0$  such that  $z \in P_R^1$  and  $|\Im(\Phi_R^1(z))| >$

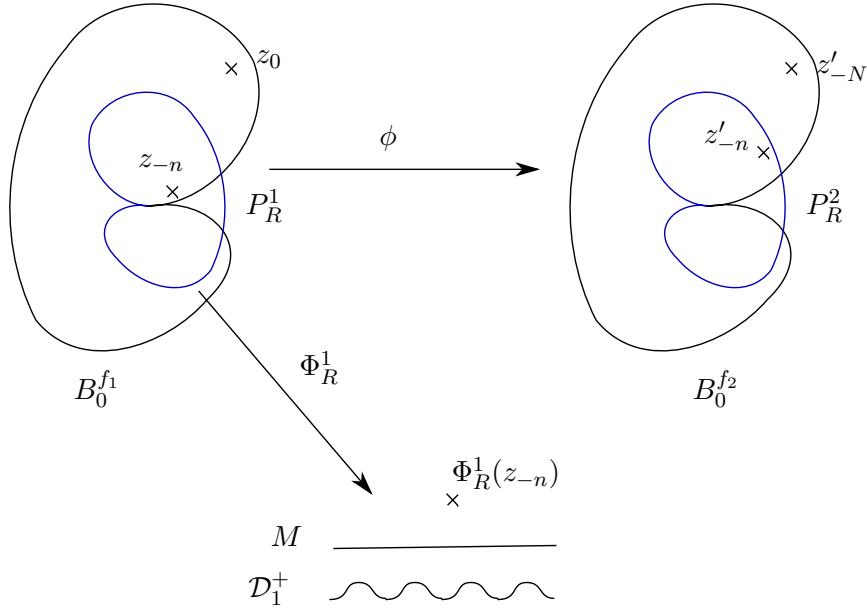


Figure 8: A local semi-conjugacy preserves some type of backward orbits

$M$  implies  $z \in P_A^1$  ( $z$  belongs to a lepal pair  $L_{RA}^1$ ). Up to taking  $P_A^1, P_R^1$  small enough and  $M$  big enough, we can apply consequence 4.3.10, we get that for all  $z \in P_R^1$  such that  $|\Im(\Phi_R^1(z))| > M$ , we have  $|\Im(\Phi_A^1(z))| > M'$ . Let  $(z_{-n})$  be a backward orbit of  $f_1$  such that for all  $n \in \mathbb{N}$ ,  $z_{-n} \in P_R^1$  and  $|\Im(\Phi_R^1(z_0))| > M$ . This implies that  $|\Im(\Phi_A^1(z_0))| > M'$ . Denote for  $n \in \mathbb{N}$ ,  $z'_{-n} = \phi(z_{-n})$ . Since  $z_n \in X$  for all big enough integer  $n$ , we must have for such integers  $z'_{-n} \in L_{AR}^2 \subset P_R^2$ . This finishes the proof.  $\square$

#### 4.4.3 Local pseudo-conjugacies

**Definition 4.4.29.** A pair of maps  $(\phi, \phi')$  is said to be a pair of local semi-conjugacies of  $f_1, f_2$  if:

- $\phi$  is a local semi-conjugacy of  $f_1, f_2$
- $\phi'$  is a local semi-conjugacy of  $f_2, f_1$

**Definition 4.4.30.** Let  $(\phi, \phi')$  be a pair of local semi-conjugacies. The pair  $(\phi, \phi')$  is said to be a local pseudo-conjugacy of  $f_1, f_2$  if  $\bar{\phi} : B^{f_1}/f_1 \rightarrow B^{f_2}/f_2$  (we already know by proposition 4.4.17 that  $\bar{\phi}$  is a biholomorphism) has for inverse  $\bar{\phi}' : B^{f_2}/f_2 \rightarrow B^{f_1}/f_1$ .

The term pseudo-conjugacy is motivated by the fact that the maps  $\phi, \phi'$  are mutually inverse only after quotient.

**Remark 4.4.31.** The fact that  $\bar{\phi}' \circ \bar{\phi} = \text{id}$  may be reformulated in the following way. For all  $z \in U_1^0$ , there exists  $n_0 \in \mathbb{Z}$  and  $N \geq \max(0, -n_0)$  such that for all  $n \geq N$  we

have the three following conditions (the first two are automatically true for big enough  $n$ ):

1.  $f_1^n(z) \in U_1^0$
2.  $\phi(f_1^n(z)) \in U_2^0$
3.  $\phi' \circ \phi(f_1^n(z)) = f_1^{n+n_0}(z)$

*Proof.* Let  $N_0$  such that for all  $n \geq N_0$  the points 1. and 2. are true. Notice that for  $n \geq N_0$  we have by definition of  $\bar{\phi}$  and  $\bar{\phi}'$ :

$$\bar{\phi}' \circ \bar{\phi}(z \bmod f_1) = \phi' \circ \phi(f_1^n(z)) \bmod f_1 \quad (4)$$

If  $\bar{\phi}' \circ \bar{\phi} = \text{id}$ , we must have  $z \bmod f_1 = \phi' \circ \phi(f_1^{N_0}(z)) \bmod f_1$ . Hence there exists  $a, b \in \mathbb{N}$  such that  $f^a(z) = f^b(\phi' \circ \phi(f_1^{N_0}(z))) = \phi' \circ \phi(f_1^{N_0+b}(z))$ . Let  $N = N_0 + b$ . We have  $\phi' \circ \phi(f_1^N(z)) = f^a(z)$ , hence for all  $n \geq N$  we get  $\phi' \circ \phi(f_1^n(z)) = f^{n+a-N}(z)$ . Thus we take  $n_0 = a - N$  and this finishes the proof since  $N \geq -n_0$ .

Let us show the converse implication. Let  $z \in U_1^0$ . The equality  $\phi' \circ \phi(f_1^n(z)) = f_1^{n+n_0}(z)$  for all integer  $n$  bigger than  $N$  implies that  $\bar{\phi}' \circ \bar{\phi}(z \bmod f_1) = z \bmod f_1$ . This is valid for all  $z \in U_1^0$ , hence  $\bar{\phi}' \circ \bar{\phi} = \text{id}$ .  $\square$

We will not use this property, but one can show that  $n_0$  may be chosen independently of  $z$ . Note that the first  $N$  terms of the orbit  $z, f_1(z), \dots, f_1^{N-1}(z)$  might go out of  $U_1^0$ . The same for  $\phi(z), f_2(\phi(z)), \dots, f_2^{N-1}(\phi(z))$  and  $U_2^0$ . This prevents in general to write the equality  $f_1^n \circ \phi' \circ \phi(z) = f_1^{n+n_0}(z)$ . Nevertheless, under the assumption that  $\phi, \phi'$  can be extended into maps defined on the whole basin  $\phi : B_0^{f_1} \rightarrow B_0^{f_2}, \phi' : B_0^{f_2} \rightarrow B_0^{f_2}$ , it is possible to write  $f_1^n \circ \phi' \circ \phi(z) = f_1^{n+n_0}(z)$  without any restriction. This last equality has the advantage to let only a finite number of possibilities (depending on  $n$ ) for the value of  $\phi' \circ \phi(z)$ , under the condition that  $f : B^{f_1} \rightarrow B^{f_1}$  is a ramified covering of finite degree.

The fact that  $\bar{\phi} \circ \bar{\phi}' = \text{id}$  can be reformulated in a symmetrical way.

Recall that  $\sigma(\phi), \sigma(\phi')$ , the phase shifts of  $\phi, \phi'$ , are defined in  $\mathbb{C}/\mathbb{Z}$ , and  $\tilde{\sigma}(\phi), \tilde{\sigma}(\phi')$ , the lifted phase shifts, are defined in  $\mathbb{C}$  by proposition 4.4.18.

**Remark 4.4.32.** Let  $(\phi, \phi')$  be a pair of local semi-conjugacies. The pair  $(\phi, \phi')$  is a pseudo-conjugacy if and only if  $\sigma(\phi) + \sigma(\phi') = 0$ . Note that  $\sigma(\phi)$  and  $\sigma(\phi')$  depends on the normalisation of the Fatou coordinates, but that by proposition 4.4.35,  $\sigma(\phi) + \sigma(\phi')$  is independant of the normalisation.

**Definition 4.4.33.** A local pseudo-conjugacy is said to be *synchronous* if  $\tilde{\sigma}(\phi) + \tilde{\sigma}(\phi') = 0$ .

**Example 4.4.34.** A semi-conjugacy  $\phi$  inducing a biholomorphism  $\phi : U_1^0 \rightarrow U_2^0$  is a particular case of synchronous local pseudo-conjugacy, with  $\phi' = \phi^{-1}$ . In this case, the map  $\phi$  is said to be a local conjugacy of  $f_1, f_2$  on their immediate basins.

**Proposition 4.4.35.** *The value  $\tilde{\sigma} + \tilde{\sigma}'$  does not depend on the choice of the normalization of Fatou coordinates.*

*Proof.* We have by definition of  $\sigma, \sigma'$  :

$$\begin{aligned} T_{\tilde{\sigma}|\Phi_A^1(P_A^1)} &= \Phi_A^2 \circ \phi \circ (\Phi_A^1)^{-1} \\ T_{\tilde{\sigma}'|\Phi_A^2(P_A^2)} &= \Phi_A^1 \circ \phi' \circ (\Phi_A^2)^{-1} \end{aligned}$$

choose  $P_A^1, P_A^2$  such that  $\phi(P_A^1) \subset P_A^2$ . This is possible by proposition 4.4.22. By composition, we get that  $T_{\tilde{\sigma}+\tilde{\sigma}'|\Phi_A^1(P_A^1)} = \Phi_A^1 \circ \phi' \circ \phi \circ (\Phi_A^1)^{-1}$ . This map does not depend on the normalization of Fatou coordinates. More precisely, if we had chosen an other Fatou coordinate  $T_\alpha \circ \Phi_A^1$ , the resulting function would be the same (up to restriction) since  $T_{\tilde{\sigma}+\tilde{\sigma}'}$  commutes with  $T_\alpha$ . This is equivalent to say that  $\tilde{\sigma} + \tilde{\sigma}'$  is independent of the petal choice.  $\square$

There is a consequence of proposition 4.4.21:

**Proposition 4.4.36.** *The existence of a local pseudo-conjugacy between  $f_1, f_2$  is equivalent to the existence of a synchronous local pseudo-conjugacy.*

*Proof.* Let  $(\phi, \phi')$  be a local pseudo-conjugacy. Since  $\sigma(\phi') = -\sigma(\phi)$ , we obtain  $\tilde{\sigma}(\phi') = -\tilde{\sigma}(\phi)$  up to post-composing or pre-composing  $\phi, \phi'$  by the adapted integer powers of  $f_1, f_2$ .  $\square$

**Proposition 4.4.37.** *Let  $(\phi, \phi')$  a synchronous pseudo-conjugacy. Let  $P_A^1, P_A^2$  be attracting petals of  $f_1, f_2$  included in  $U_1, U_2$ . Then  $\phi(P_A^1) = P_A^2$  is equivalent to  $P_A^1 = \phi'(P_A^2)$ .*

*Proof.* This is an immediate consequence of proposition 4.4.22.  $\square$

**Proposition 4.4.38.** *A pseudo-conjugacy is synchronous if and only if there exists an attracting petal  $P_A^1$  such that  $\phi : P_A^1 \rightarrow \phi(P_A^1)$  is a biholomorphism with inverse  $\phi' : \phi(P_A^1) \rightarrow P_A^1$ , where it is implied that  $P_A^1 \subset U_1$  and  $\phi(P_A^1) \subset U_2$ . This amounts to saying that for all attracting petal  $P_A^1$  such that  $P_A^1 \subset U_1$  and  $\phi(P_A^1) \subset U_2$ , the map  $\phi : P_A^1 \rightarrow \phi(P_A^1)$  is a biholomorphism of inverse  $\phi' : \phi(P_A^1) \rightarrow P_A^1$ .*

*Proof.* Let  $(\phi, \phi')$  be a pseudo-conjugacy. By proposition 4.4.22, there exists a petal  $P_A^1 \subset U_1^0$  such that  $\phi(P_A^1) \subset U_2^0$ . Denote  $P_A^2 := \phi(P_A^1)$ , which is an attracting petal for  $f_2$  by proposition 4.4.27. Suppose that  $\phi'(P_A^2) \subset P_A^1$ .

By proposition 4.4.24, we have:

$$\forall z \in P_A^1, \phi' \circ \phi(z) = (\Phi_A^1)^{-1} \circ T_{\tilde{\sigma}+\tilde{\sigma}'} \circ \Phi_A^1(z) \quad (5)$$

$$\forall z \in P_A^2, \phi \circ \phi'(z) = (\Phi_A^2)^{-1} \circ T_{\tilde{\sigma}+\tilde{\sigma}'} \circ \Phi_A^2(z) \quad (6)$$

Now,  $P_A^1$  is any petal such that  $P_A^1 \subset U_1^0$  and  $\phi(P_A^1) \subset U_2^0$  and  $P_A^2 := P_A^1$ . It is sufficient to show the following equivalence :  $\phi$  is synchronous if and only if  $\phi : P_A^1 \rightarrow \phi(P_A^1)$  is a biholomorphism with inverse  $\phi' : \phi(P_A^1) \rightarrow P_A^1$ . Indeed, since this equivalence



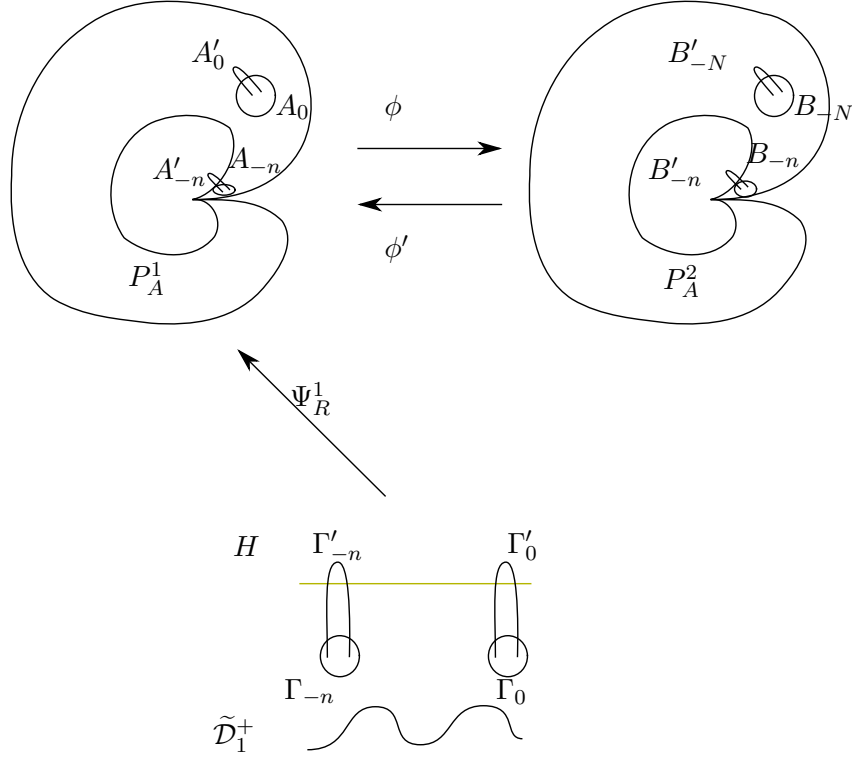


Figure 9:  $\phi$  is pseudo-inversible

does not depend on the choice of the petal  $P_A^1$  such that  $\phi(P_A^1) \subset U_1^0$ , this will finish the proof.

Suppose that  $\phi : P_A^1 \rightarrow P_A^2$  is a biholomorphism of inverse  $\phi' : P_A^2 \rightarrow P_A^1$ . We can use equation (5) since  $\phi'(P_A^2) = P_A^1$ . By this equation, the pair of maps  $(\phi, \phi')$  is synchronous. Conversely, suppose that  $(\phi, \phi')$  is synchronous. Then by proposition 4.4.37,  $P_A^1 = \phi'(P_A^2)$ , so we can use equations (5) and (6). Thus we get that  $\phi : P_A^1 \rightarrow P_A^2$  is a biholomorphism of inverse  $\phi' : P_A^2 \rightarrow P_A^1$ .  $\square$

Let  $\phi$  be a local semi-conjugacy on the immediate basins. We now show a lemma which will be called the lemma of pseudo invertibility of  $\phi$ . Before invoquing the lemma, let us introduce some notations.

Let  $W$  a neighborhood of 0 in which  $f_1$  admits a local inverse branch  $f_1^{-1} : W \rightarrow f_1^{-1}(W)$ . Suppose that  $P_R^1 \subset U_1 \subset W$  and that  $\Phi_R^1(P_R^1)$  contains a left half-plane  $G$ . Let  $H$  an upper half-plane included in  $\tilde{\mathcal{D}}_1^+$  and  $B$  a lower half-plane included in  $\tilde{\mathcal{D}}_1^-$ . Let  $X = C_1 \cup C_2$  where  $C_1$  is the connected component of  $(\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-) \cap \Phi_R^1(P_R^1)$  containing  $G \cap H$ , and  $C_2$  the connected component containing  $G \cap B$ . Notice that  $X$  is forward invariant by the left translation  $T_{-1}$ , since  $G, B, H$  and  $\tilde{\mathcal{D}}^\pm, \Phi_R^1(P_R^1)$  are forward invariant by  $T_{-1}$ . Let  $Y := (\Phi_R^1)^{-1}(X)$ ; notice that  $Y \subset U_1^0 \cap P_R^1$  is forward invariant by the local inverse branch  $f_1^{-1}$  of  $f_1$ , that is why we introduced  $Y$ .

Let  $A_0$  be a set of the form  $f_1^{n_0}(A_{-n_0})$  where  $n_0$  is a possibly big natural integer and

$A_{-n_0}$  a connected open set relatively compact in  $Y \subset U_1^0 \cap P_R^1$ . Note that  $A_{-n_0} \subset U_1^0 \subset B_0^{f_1}$ , in such a way that  $A_{-n_0}$  is included in the definition domain of  $f_1^n$  for all  $n \in \mathbb{N}$ , and in particular for  $n = n_0$ .

Such a set  $A_0$  may be characterized by the following property. There exists a backward orbit  $(A_{-n})$  for  $f_1$  such that:

- for all  $n \in \mathbb{N}$ ,  $f_1(A_{-n-1}) = A_{-n}$  (backward orbit definition)
- $A_{-n}$  is a connected open set relatively compact in  $Y$  for  $n$  big enough, let us say  $n \geq N$

Let us show this equivalence.

*Proof.* If there exists a backward orbit  $(A_{-n})$  for  $f_1$  as mentionned before, take  $n_0 = N$ . Thus  $A_0 = f_1^{n_0}(A_{-n_0})$  and  $A_{-n_0}$  is a connected open set relatively compact in  $U_1^0 \cap P_R^1$ .

If  $A_0$  is a set of the form  $f_1^{n_0}(A_{-n_0})$  with  $A_{-n_0}$  a connected open set relatively compact in  $Y \subset U_1^0 \cap P_R^1$ , take for  $n \in \mathbb{N}$ :

$$A_{-n} = f_1^{n_0-n}(A_{n_0})$$

where for  $n \in \mathbb{N}^*$ ,  $f_1^{-n}$  is the local inverse branch restricted to  $P_R^1$  of  $f_1^n$ . For all  $n \in \mathbb{N}$  we have  $f_1(A_{-n-1}) = A_{-n}$ .

Notice that  $f_1^{-1}$  from  $Y \subset P_R^1$  to its image, included in  $Y$ , is open and continuous. Hence if  $A_{-n}$  is a connected open set relatively compact in  $Y$ , then  $A_{-n-1}$  satisfies the same properties. Since  $A_{-n_0}$  satisfy this property, the property is also true for all integer  $n$  bigger than  $n_0$ . Thus we take  $N = n_0$ , and this finishes the proof.  $\square$

**Remark 4.4.39.** For  $n \geq N$ , the map  $f_1 : A_{-n-1} \rightarrow A_{-n}$  defines a biholomorphism. More precisely,  $f_1^{-1} : A_{-n} \rightarrow A_{-n-1}$  is a restriction of the local inverse branch of  $f_1$  defined on  $P_R^1$ , this implies that  $A_{-n-1}$  is defined uniquely from  $A_{-n}$ . Notice that given  $A_0$ , the backward orbit  $(A_{-n})$  is not unique, since  $f_1 : A_{-n-1} \rightarrow A_{-n}$  is in general only surjective, in such a way that there is several definitions of  $A_{-n-1}$  from  $A_{-n}$ .

Let for  $n \geq N$ ,  $B_{-n} = \phi(A_{-n})$ . We prove that the sequence  $(B_{-n})$  satisfies the following properties:

- for all  $n \geq N$ ,  $f_2(B_{-n-1}) = B_{-n}$  (backward orbit definition)
- $B_{-n}$  is a connected open set compactly contained in  $P_R^2$  for  $n$  big enough
- $B_{-n}$  is compactly contained in  $U_2^0$  for  $n$  big enough

*Proof.* We prove successively the three points.

- Since  $\phi$  is a semi-conjugacy, it is clear that  $(B_{-n})$  is also a backward orbit.

- $B_{-n}$  is connected as the direct image of a connected set by a continuous map; it is open as the direct image of an open set by a nonconstant holomorphic map. The orbit  $(A_{-n})$  is compactly contained in an orbit  $(A'_{-n})$  satisfying the same properties. To build such an orbit  $(A'_{-n})$ , denote  $A'_{-N}$  an open neighborhood of  $\overline{A_{-N}}$ . Then apply  $f_1$  to build  $A'_{-n}$  for  $n < N$  and apply the local inverse branch of  $f_1$  on  $P_R^1$  to build  $A'_{-n}$  for  $n > N$ . Suppose without loss of generality that  $N = 0$ . Consider the map  $\omega$  which at  $z \in A'_0$  associates the least integer  $n \in \mathbb{N}$  such that  $\phi(f_1^{-n}(z)) \in P_R^2$ : such an integer exists by the construction of  $\psi : \mathcal{D}_1^+ \rightarrow \mathcal{D}_2^+$ . Since the set  $P_R^2$  is open and the maps  $\phi, f_1$  are continuous, the map  $\omega$  is upper semi-continuous. Since the set  $A_0$  is relatively compact in  $A'_0$ , the map  $\omega$  is majorized over  $A_0$ . Thus there exists  $N' \in \mathbb{N}$  such that for all  $z \in A_0$ ,  $\phi(f_1^{-n}(z)) \in P_R^2$ , namely such that  $B_{-n} \subset P_R^2$ . Even if it means choosing  $P_R^2$  small enough, one may suppose that  $P_R^2 \subset U_2$ .
- See figure 9. Since  $A_0 \subset P_R^1 \cap U_1^0$ , the set  $A_0$  is of the form  $A_0 = \Psi_R^1(\Gamma_0)$  where  $\Gamma_0 = \Phi_R^1(A_0)$  open set included in  $\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-$  by the inclusion  $\Phi_R^1(P_R^1 \cap U_1^0) \bmod \mathbb{Z} \subset \mathcal{D}_1^+ \cup \mathcal{D}_1^-$  stated in proposition 4.4.13. Let  $H, B$  be upper, lower half-planes such that  $\Psi_R^{1,\text{ext}}(H \cup B)$  is included in an very large attracting petal  $P_A^1$ . There exists a connected open set  $\Gamma'_0$  relatively compact in  $\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-$  containing  $\Gamma_0$  and intersecting  $H \cup B$ . To prove this, join  $\Gamma_0$  with  $H \cup B$  by a path included in  $\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-$ , and consider an open neighborhood of the path which is relatively compact in  $\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-$ . By noting for all  $n \in \mathbb{N}$ ,  $A'_{-n} = \Psi_R^1(\Gamma'_0 - n)$ , it follows that  $A'_{-n}$  is connected, that it intersects  $P_A^1$ , and that for  $n$  big enough it is relatively compact in  $U_1$ , and thus in  $U_1^0$ . Its image  $B'_{-n}$  by  $\phi$  is then a connected set relatively compact in  $U_2$  intersecting  $P_A^2$ , it is thus relatively compact in  $U_2^0$ . It follows that  $B_{-n}$  is relatively compact in  $U_2^0$ . □

This enables us to state the lemma of pseudo-invertibility of  $\phi$ :

**Lemma 4.4.40.** *For all  $n$  big enough, the map  $\phi : A_{-n} \rightarrow B_{-n}$  is a biholomorphism of inverse  $\phi' : B_{-n} \rightarrow A_{-n}$ .*

*Proof.* Take back the notations of the last point of the preceding proof, see figure 9. Even if it means removing the first terms of the orbit  $(A_{-n})$ , one may suppose that  $A'_{-n}$  is relatively compact in  $U_1^0$  for all  $n \in \mathbb{N}$ . Write  $B'_{-n} = \phi(A'_{-n})$ . We apply the second point of the preceding proof to  $(A'_{-n})$ : there exists  $N' \in \mathbb{N}$  such that for all  $n \geq N'$ ,  $B'_{-n} \subset P_R^2 \subset U_2$ . There exists also  $N'' \in \mathbb{N}$  such that for all  $n \geq N''$ ,  $\phi'(B'_{-n}) \subset P_R^1$ . Thus for  $n \geq \max(N', N'')$ ,  $\phi' \circ \phi$  is defined on the connected set  $A'_{-n}$  and  $\phi \circ \phi'$  is defined on the connected set  $B'_{-n}$ . Since  $\phi$  is a bijection from  $P_A^1$  to  $P_A^2 := \phi(P_A^1)$  of inverse  $\phi' : P_A^2 \rightarrow P_A^1$ , we obtain that  $\phi' \circ \phi$  equals the identity over  $P_A^1 \cap A_{-n} \neq \emptyset$ , and thus equals identity over the connected set  $A_{-n}$ . Likewise  $\phi' \circ \phi$  equals the identity over the connected set  $B_{-n}$ . Thus  $\phi : A_{-n} \rightarrow B_{-n}$  is a biholomorphism of inverse  $\phi' : B_{-n} \rightarrow A_{-n}$ . □

The reader may wisely consult figure 10 by replacing mentally the paths  $\gamma, \gamma'$  by open sets  $\Gamma, \Gamma'$  relatively compact, and the paths  $\alpha_{-n}, \beta_{-n}$  by open sets  $A_{-n}, B_{-n}$  relatively compact.

Notice that the pseudo-invertibility lemma may be reformulated in this way.

**Proposition 4.4.41.** *There exists open sets  $V \subset \tilde{\mathcal{D}}_1^+$ ,  $W \subset \tilde{\mathcal{D}}_2^+$  forward invariant by the translation  $T_{-1} : z \mapsto z - 1$  such that  $V \bmod \mathbb{Z} = \mathcal{D}_1^+$ ,  $W \bmod \mathbb{Z} = \mathcal{D}_2^+$  and  $\phi : \Psi_R^{1, \text{ext}}(V) \rightarrow \Psi_R^{2, \text{ext}}(W)$  is a biholomorphism of inverse  $\phi' : \Psi_R^{2, \text{ext}}(W) \rightarrow \Psi_R^{1, \text{ext}}(V)$ . The same holds by replacing  $+$  with  $-$ . The pseudo-invertibility is therefore retranslated in a concrete invertibility on a fixed domain.*

*Proof.* The pseudo-invertibility lemma allows to say that for all open set  $V'$  relatively compact in  $\tilde{\mathcal{D}}_1^+$ , there exists an open set  $W'$  relatively compact in  $\tilde{\mathcal{D}}_2^+$  such that  $\phi : \Psi_R^{1, \text{ext}}(V' - n) \rightarrow \Psi_R^{2, \text{ext}}(W' - n)$  is a biholomorphism of inverse  $\phi'$  for all  $n$  big enough, say  $n \geq n_{V'}$  where  $n_{V'}$  is chosen minimal. The roles of  $V'$  and  $W'$  are symmetrical, therefore we may also denote the integer corresponding to  $W'$ :  $m_{W'}$ . Writing  $V = \bigcup_{n \geq n_{V'}} V' - n$  and  $W = \bigcup_{m \geq m_{W'}} W' - m$ , where the unions are taken under all the open sets  $V', W'$  relatively compact in  $\tilde{\mathcal{D}}_1^+, \tilde{\mathcal{D}}_2^+$ , we get the result.  $\square$

Here is a last variant where the invertibility domain is bigger (we add an attracting petal).

**Proposition 4.4.42.** *Let  $P_A^1, P_A^2 := \phi(P_A^1)$  be attracting petals included in  $U_1^0, U_2^0$ . There exists open sets  $A \subset U_1^0$  and  $B \subset U_2^0$  such that  $\phi : A \rightarrow B$  biholomorphism of inverse  $\phi' : B \rightarrow A$  satisfying the following properties.*

- $A$  and  $B$  contains the attracting petals  $P_A^1, P_A^2$
- $A \cap P_R^1$  resp.  $B \cap P_R^2$  is invariant by the local inverse branch of  $f_1$  resp.  $f_2$  defined over  $P_R^1$ , resp.  $P_R^2$
- $(A \cap P_R^1)/f_1 = (U_1^0 \cap P_R^1)/f_1$  and  $B/f_2 = (U_2^0 \cap P_R^2)/f_2$ .

**Remark 4.4.43.** The last proposition gives a characterisation of being a pseudo-conjugacy:  $\phi$  is a pseudo-conjugacy if and only if it satisfies this property, with invertibility on domains  $A, B$  before quotient.

*Proof.* Notice that the last point is equivalent to  $\Phi_R^1(A) \bmod \mathbb{Z} = \mathcal{D}_1^+ \cup \mathcal{D}_1^-$  and  $\Phi_R^2(B) \bmod \mathbb{Z} = \mathcal{D}_2^+ \cup \mathcal{D}_2^-$  by the equality  $\Phi_R^i(U_i^0 \cap P_R^i) \bmod \mathbb{Z} = \mathcal{D}_i^+ \cup \mathcal{D}_i^-$  shown in propositions 4.4.12 and 4.4.13.

Consider  $A = P_A^1 \cup \Psi_R^{1, \text{ext}}(V^+) \cup \Psi_R^{1, \text{ext}}(V^-)$  where  $V^+, V^-$  are the open sets built by the preceding proposition, and in the same way  $B = P_A^2 \cup \Psi_R^{1, \text{ext}}(W^+) \cup \Psi_R^{1, \text{ext}}(W^-)$ . The three points are shown easily. This finishes the proof.  $\square$

#### 4.5 Proof of the theorem 1.5.1

We first show the direct implication of the theorem 1.5.1. Recall that the sets  $\mathcal{D}_1, \mathcal{D}_2$  are the domains of  $\mathbf{h}_1, \mathbf{h}_2$ , and that  $\mathcal{D}_1^\pm, \mathcal{D}_2^\pm$  are the connected components of these domains containing a punctured neighborhood of  $\pm i\infty$  as in proposition 4.3.6. The maps  $\mathbf{h}_i^\pm$  are the horn maps of  $\mathbf{h}_i$  restricted to  $\mathcal{D}_i^\pm$ . Here is the direct implication of the theorem 1.5.1:

**Proposition 4.5.1.** *Let  $f_1, f_2$  be two holomorphic maps with parabolic point at 0, with Taylor expansion at 0 of the form  $f_i(z) = z + a_i z^2 + O(z^3)$  with  $a_i \neq 0$ . Assume that  $f_1, f_2$  are locally semi-conjugate at 0 on  $B_0^{f_i}$ . Denote  $\phi$  a local semi-conjugacy (see definition 4.4.8).*

*Then there exists  $\sigma \in \mathbb{C}$ , and a pair of holomorphic maps  $\psi = (\psi^+, \psi^-)$ , where  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$  such that:*

$$\mathbf{h}_1^\pm = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm$$

where  $T_\sigma$  is the translation of the cylinder  $\mathbb{C}/\mathbb{Z}$  given by the formula  $z \mapsto z + \sigma$ .

Besides, the constructed maps  $\psi^\pm$  admits at  $\pm i\infty$  removable singularities, and an expansion of the form:

$$\psi^\pm(w) = w + \rho^\pm + o(1)$$

**Complement 4.5.2.** *We add the following assumption:  $\phi$  is a local pseudo-conjugacy, i.e. there exists a semi-conjugacy  $\phi' : U_2^0 \rightarrow B_0^{f_1}$  such that the induced maps  $\bar{\phi} : B_0^{f_1}/f_1 \rightarrow B_0^{f_2}/f_2$  and  $\bar{\phi}' : B_0^{f_2}/f_2 \rightarrow B_0^{f_1}/f_1$  are mutually inverses (see definition 4.4.30). Then the horn maps  $\mathbf{h}_1^\pm, \mathbf{h}_2^\pm$  of  $f_1, f_2$  are equivalent as ramified analytic coverings via biholomorphisms from  $\mathcal{D}_1^\pm$  to  $\mathcal{D}_2^\pm$  and translation of  $\mathbb{C}/\mathbb{Z}$ . More precisely, there exists  $\sigma \in \mathbb{C}$ , and a pair of biholomorphisms  $\psi = (\psi^+, \psi^-)$ , where  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$  such that:*

$$\mathbf{h}_1^\pm = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm$$

where  $T_\sigma$  is the translation of the cylinder  $\mathbb{C}/\mathbb{Z}$  given by the formula  $z \mapsto z + \sigma$ .

*Proof.* We simultaneously prove the proposition and the complement. Let  $\phi$  be a local semi-conjugacy, which might be completed in a local pseudo-conjugacy  $(\phi, \phi')$ . We will clearly identify the parts of the proof devoted to the complement, in which the local pseudo-conjugacy  $(\phi, \phi')$  will be exploited. We give first the main elements of the proof, and we will give in the following parts the details.

We will build  $T_\sigma$  and  $\psi$  satisfying the relation  $T_\sigma \circ \mathbf{h}_1^\pm = \mathbf{h}_2^\pm \circ \psi^\pm$  instead of the announced equality (to get the exact relation of the proposition, replace  $\sigma$  by  $-\sigma$ ). The map  $\phi$  will induce a map  $T_\sigma$  from  $\mathcal{C}_A^{f_1}$  to  $\mathcal{C}_A^{f_2}$ , and a partial map  $\psi$  from  $\mathcal{C}_R^{f_1}$  to  $\mathcal{C}_R^{f_2}$  that can be restricted to the maps  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ . These maps  $T_\sigma$  and  $\psi^\pm$  will enable us to show the desired equality  $T_\sigma \circ \mathbf{h}_1^\pm = \mathbf{h}_2^\pm \circ \psi^\pm$ .

Concerning the proof of the complement, we will follow the same construction with  $\phi'$ : it induces the map  $T_{-\sigma}$  from  $\mathcal{C}_A^{f_2}$  to  $\mathcal{C}_A^{f_1}$  and maps  $\psi'^\pm$  from  $\mathcal{D}_2^\pm$  to  $\mathcal{D}_1^\pm$  (thus  $T_{-\sigma} \circ \mathbf{h}_2^\pm = \mathbf{h}_1^\pm \circ \psi'^\pm$ ). We will prove that the map  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$  is a biholomorphism of inverse  $\psi'^\pm$ . We will see that the maps  $T_\sigma$  and  $\psi^\pm$  built after quotient remain identical when we

post-compose, pre-compose  $\phi$  (or  $\phi'$ ) by powers of  $f_1, f_2$ . Thus, if  $\phi$  is a local pseudo-conjugacy, we may suppose without loss of generality that the pair  $(\phi, \phi')$  is synchronous.

During the proof, we will assume that all the petals of  $f_1$ , resp.  $f_2$  are included in  $U_1$ , resp.  $U_2$ , this will enable us to apply remark 4.4.10 and proposition 4.4.12 concerning attracting and repelling petals. We will use in a remark (which is useful for a better understanding, but not for the construction itself) proposition 4.4.13, which is valid only when the open sets  $U_1, U_2$  are supposed small enough.

**Construction of  $T_\sigma$ :**

The quotient map  $T_\sigma : \mathcal{C}_A^{f_1} \rightarrow \mathcal{C}_A^{f_2}$  induced by  $\phi : P_A^1 \rightarrow B^{f_2}$  is built in proposition 4.4.17.

**Construction of  $\psi$ :**

The construction of the induced map on the repelling petals is harder. We give in a first paragraph the main elements of the construction; we will give the details in the subsequent paragraphs.

Summary of the construction: First, set  $\hat{z} = \bar{\Psi}_R^1(w) = \{z_{-n}, n \in \mathbb{N}\} \in P_R^1/f_1$ . Recall that  $\bar{\Psi}_R^1 : \mathbb{C}/\mathbb{Z} \rightarrow P_R^1/f_1$  is the map induced by the Fatou parametrisation  $\Psi_R^1 : \Phi_R^1(P_R^1) \rightarrow P_R^1$  after quotient by  $\mathbb{Z}$  on the domain and by  $f_1$  on the codomain. The element  $\hat{z}$  is a backward orbit for  $f_1$ . When  $\phi(\hat{z})$  belongs eventually to  $P_R^2$ , choose the following element of  $\mathcal{C}_R^{f_2}$ :

$$\psi(w) = \Phi_R^2(\phi(\hat{z}_N)) \bmod \mathbb{Z}$$

where  $\hat{z}_N := \{z_{-n}, n \geq N\}$  and  $N$  is chosen big enough such that  $\phi$  is defined on  $\hat{z}_N$  and the backward orbit  $\phi(\hat{z}_N)$  is completely included in  $P_R^2$ .

We check in the next paragraphs that for all initial choice of  $w \in \mathcal{D}_1^+ \cup \mathcal{D}_1^-$ , the backward orbit  $\phi(\hat{z}_N)$  is contained in  $P_R^2$  for  $N$  big enough. This point follows first from the corollary 4.4.28 when  $|\Im(w)|$  big; we then show that it is valid for all  $w \in \mathcal{D}_1^\pm$ , by exploiting the arc connectedness of this set and by controlling the hyperbolic length of the paths formed at each step of the construction.

This will allow to quotient the map  $\phi$ , to get a map  $\psi^\pm$  from  $\mathcal{D}_1^\pm$  to  $\mathcal{D}_2^\pm$ .

Why  $\psi(w) = \Phi_R^2(\phi(\hat{z}_N)) \bmod \mathbb{Z}$  is the natural choice : The map  $\phi$  induces by quotient a partial map  $\psi$  from  $\mathcal{C}_R^{f_1}$  to  $\mathcal{C}_R^{f_2}$ . We want to define an element of  $\mathcal{C}_R^{f_2}$  from an element  $w \in \mathcal{C}_R^{f_1}$ . Set  $\hat{z} = \bar{\Psi}_R^1(w) = \{z_{-n}, n \in \mathbb{N}\} \in P_R^1/f_1$ . It is possible to apply  $\phi$  at the elements of  $\hat{z}_N$  for  $N$  big enough if and only if  $\hat{z}_N$  is included in  $U_1^0$  for  $N$  big enough. To satisfy this property, it is sufficient that  $w \in \mathcal{D}_1^+ \cup \mathcal{D}_1^-$  by proposition 4.4.12.

**Remark 4.5.3.** If the open sets  $U_1, U_2$  are small enough to apply proposition 4.4.13,  $\hat{z}_N$  is included for  $N$  big enough in  $U_1^0$  if and only if  $w \in \mathcal{D}_1^+ \cup \mathcal{D}_1^-$ . In this setting, there is no reasonable definition of  $\psi$  on a bigger set than  $\mathcal{D}_1^+ \cup \mathcal{D}_1^-$ .

Consider the backward orbit  $\phi(\hat{z}_N)$  for  $f_2$ ;  $N$  is first chosen such that  $\phi$  is defined on  $\hat{z}_N$ . To get at least one element of  $\mathcal{C}_R^{f_2}$ , the backward orbit  $\phi(\hat{z}_N)$  must intersect a

repelling petal  $P_R^2$ . However, such an intersection with  $P_R^2$  is a priori not ensured, since the map  $\phi$  is not supposed continuous at 0. Also, note that the backward orbit  $\phi(\hat{z}_N)$  for  $f_2$  might enter several times in the repelling petal  $P_R^2$ , in such a way that the intersection of the orbit and the repelling petal may in general define distinct elements in  $P_R^2/f_2$  at each step. Each of these values is a candidate for the value of  $\bar{\Phi}_R^2$  applied on each of these orbits of  $P_R^2$ . Doing a choice independent of the petal  $P_R^1$  is equivalent to doing a choice independent of the backward orbit  $\hat{z}_N$  of  $f_1$ , where  $N \in \mathbb{N}$ . The identification  $P_R^1/f_1 = P_R^1/f_1$  remove indeed the first terms of the backward orbit, the removed initial segment may be chosen arbitrarily long. For this purpose, choose  $P_R^1 = f_1^{-N}(P_R^1)$  to remove the first  $N$  elements of the orbit.

Let us now begin the construction.

Construction of  $\psi^+(w)$  when  $\Im(w)$  is big: An element  $w \in \mathcal{D}_1$  such that  $\Im(w)$  is big, using proposition 4.3.9, is send by the horn map  $\mathbf{h}_1$  on an element  $\mathbf{h}_1(w)$  such that  $\Im(\mathbf{h}_1(w))$  is big. Let  $P_A^1$  be a very large attracting petal of  $f_1$  and  $P_R^2$  be a large repelling petal of  $f_2$ . With the aim of applying the consequence 4.3.10, we suppose that  $P_A^2 := \phi(P_A^1)$ ,  $P_R^2$  are included in a neighborhood  $W_2$  of 0 in which  $f_2$  admits a local inverse branch  $f_2^{-1}$ . It is possible if  $P_A^1$  is small enough, see proposition 4.4.27 with  $U_2$  of the proposition equal to  $W_2$  here ; note that  $P_A^2$  is very large since  $\Phi_A^2(P_A^2) = \Phi_A^1(P_A^1)$  (recall that  $\Phi_A^1 = \Phi_A^2 \circ \phi_{P_A^1}$  with  $\phi_{P_A^1} : P_A^1 \rightarrow P_A^2$  biholomorphism). Let also  $P_R^1$  be a repelling petal of  $f_1$ .

Let  $w \in \mathcal{C}_R^1$ , with a high imaginary part. Let  $z \in P_R^1$  be a pre-image of  $w$  by  $\Phi_R^1 \bmod \mathbb{Z}$ . Note  $z_{-n} = f_1^{-n}(z)$  where  $f_1^{-1} : f(P_R^1) \rightarrow P_R^1$ , and  $z'_{-n} = \phi(z_{-n})$  ( $z_{-n}$  belongs to  $U_1^0$  by proposition 4.4.12). If we choose  $w$  with big enough imaginary part,  $z'_{-n}$  will necessarily belong to  $P_R^2$  for  $n$  big enough,  $n \geq N$ , by corollary 4.4.28. This enables us to denote by  $\psi(w)$  the common value of the  $\Phi_R^2(z'_{-n}) \bmod \mathbb{Z} \in \mathcal{C}_R^2$  when  $n \geq N$ . Note that by consequence 4.3.10, since  $\Phi_R^1(z_{-n})$  has a high imaginary part (independent of  $n$ ),  $z_{-n}$  must be in  $P_A^1$  and  $\Phi_A^1(z_{-n}) = \Phi_A^2(z'_{-n})$  also have a high imaginary part (which is independent modulo  $\mathbb{Z}$  of  $n \geq N$ ). Thus, exploiting this consequence 4.3.10 again,  $\psi(w) = \Phi_R^2(z'_{-n}) \bmod \mathbb{Z}$  must have a high imaginary part for all  $n \geq N$ .

This finally implies that if  $w$  has a big enough imaginary part, we must have  $w \in \mathcal{D}_1^+$  and  $\psi(w) \in \mathcal{D}_2^+$  ; define for all such  $w$   $\psi^+(w) = w$ . Then  $\lim_{w \rightarrow +i\infty} \psi^+(w) = +i\infty$ , and  $\psi^+$  admits a continuous extension at  $+i\infty$ . This will imply, when it will be shown that  $\psi^+$  is holomorphic, that its singularity at  $+i\infty$  is removable.

Construction of  $\psi^+ : \mathcal{D}_1^+ \rightarrow \mathcal{D}_2^+$ . See figure 10.

Let  $w \in \mathcal{D}_1^+$ . Note  $w_0 \in \mathcal{D}_1^+$  with high imaginary part as in the preceding paragraph, in such a way that  $\psi^+(w_0) := \psi(w_0) \in \mathcal{D}_2^+$ . There exists a path  $\gamma$  included in  $\mathcal{D}_1^+$  such that  $\gamma(0) = w_0$ ,  $\gamma(1) = w$ . Recall that  $\mathcal{D}_1^+ \subset \mathcal{C}_R^1 = \mathbb{C}/\mathbb{Z}$ . Note  $\pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$  the canonical projection. Note  $\hat{\gamma}_0$  a lift of  $\gamma$  by  $\pi$ , and for  $n \in \mathbb{N}$ ,  $\hat{\gamma}_{-n} = \hat{\gamma}_0 - n$ . We choose  $\hat{\gamma}_0$  sufficiently to the left to guarantee the existence of a repelling  $\pi/2$ -petal  $P_R^1$  and a Fatou coordinate  $\Phi_R^1$  such that the left half plane  $G := \Phi_R^1(P_R^1)$  contains  $\hat{\gamma}_0$ . Then  $G$  contains all the  $\hat{\gamma}_{-n}$ . Notice that  $\mathcal{D}_1^+$  contains a upper half-plane  $\text{Im}(z) > M$  where

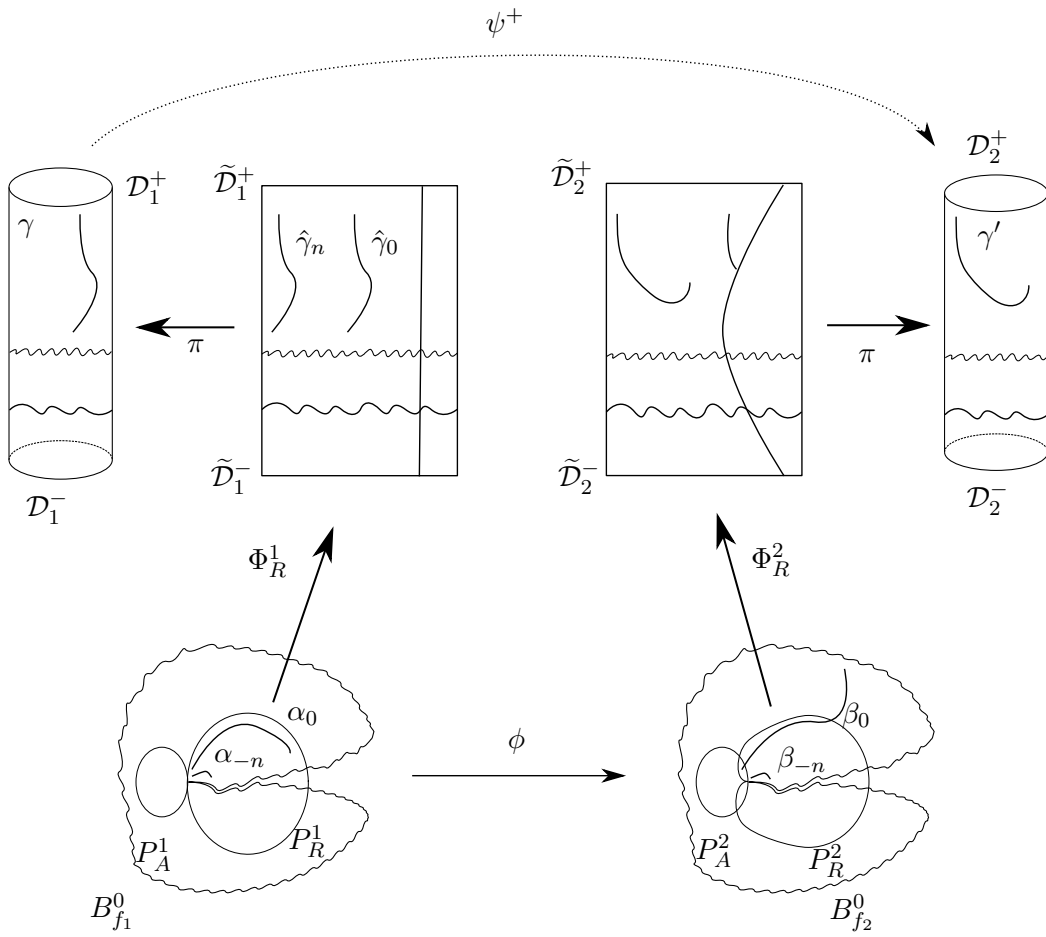


Figure 10: Construction of the image of a path  $\gamma$  by  $\psi^+ : \mathcal{D}_1^+ \rightarrow \mathcal{D}_2^+$

$M \in \mathbb{R}$ . We choose  $w_0$  satisfying furthermore  $\text{Im}(z) > M$  (which is possible), hence we have for  $n \in \mathbb{N}$ ,  $\hat{\gamma}_{-n} \subset X$  where  $X$  is the connected component of  $\pi^{-1}(\mathcal{D}_1^+) \cap G$  whose imaginary part is nonmajorized. We have by proposition 4.4.12,  $\Psi_R^1(X) \subset U_1^0$ . The map  $\Psi_R^1 : X \rightarrow \Psi_R^1(X)$  is a biholomorphism. Note  $\alpha_{-n} = \Psi_R^1 \circ \gamma_{-n}$  and  $\beta_{-n} = \phi \circ \alpha_{-n}$  for all  $n \in \mathbb{N}$ .

We give a general notation: for all hyperbolic Riemann surface  $S$  and  $\gamma$  included in  $S$ , denote by  $L_S(\gamma)$  the hyperbolic length of the path  $\gamma$ .

We have for  $n \in \mathbb{N}$ :

$$L_{\phi(U_1^0)}(\beta_{-n}) \leq L_{U_1^0}(\alpha_{-n}) \leq L_{\Psi_R^1(X)}(\alpha_{-n}) = L_X(\hat{\gamma}_{-n}) \leq L_X(\hat{\gamma}_0)$$

we use at each equality the fact that the biholomorphisms (more generally the nonramified holomorphic coverings, although it is not used here) preserve the hyperbolic length, and at each inequality that the holomorphic maps (especially the inclusions) reduce non-strictly the hyperbolic length. The successive inequalities, equalities, are justified by the



following inclusions, holomorphic maps:

$$\begin{aligned}
\phi &: U_1^0 \rightarrow \phi(U_1^0) \\
\Psi_R^1(X) &\subset U_1^0 \\
\Psi_R^1 : X &\rightarrow \Psi_R^1(X) \quad (\text{biholomorphism}) \\
T_{-n} &: X \rightarrow X
\end{aligned}$$

Notice that  $T_{-n}$  sends  $X \rightarrow X$ . Indeed, let  $M$  such that the upper half plane  $H$  of equation  $\text{Im}(z) > M$  is included in  $\tilde{\mathcal{D}}_1^+$ . The set  $X$  is the union of the connected components of  $\tilde{\mathcal{D}}_1^+ \cap G$  intersecting  $H$ . Since  $T_{-n}$  sends  $\tilde{\mathcal{D}}_1^+$  in  $\tilde{\mathcal{D}}_1^+$  and  $H$  in  $H$ , it must send a connected component  $C$  of  $\tilde{\mathcal{D}}_1^+ \cap G$  intersecting  $H$  in a connected component  $C$  of  $\tilde{\mathcal{D}}_1^+ \cap G$  intersecting  $H$ .

The sequence of hyperbolic lengths  $(L_{\phi(U_1^0)}(\beta_{-n}))$  is thus majorized.

Note for  $n \in \mathbb{N}$ ,  $z_0^{-n} = \alpha_{-n}(0)$  and  $z_1^{-n} = \alpha_{-n}(1)$ . The sequence  $(z_0^{-n})$  constitutes a backward orbit of  $f_1$  each term of which is a pre-image of  $w_0$  by  $\Phi_R^1 \bmod \mathbb{Z}$ . In a similar way,  $(z_1^{-n})$  is a backward orbit each term of which is a pre-image of  $w$  by  $\Phi_R^1 \bmod \mathbb{Z}$ . Recall that the two orbits are included in the repelling petal  $P_R^1 = \Psi_R^1(G)$ .

Note for  $n \in \mathbb{N}$ ,  $z_0'^{-n} = \phi(z_0^{-n}) = \beta_{-n}(0)$  and  $z_1'^{-n} = \phi(z_1^{-n}) = \beta_{-n}(1)$ . The element  $w_0$  has a high imaginary part. By the first paragraph, the element  $z_0'^{-n}$  must belong to  $P_R^2$  for  $n$  big enough, where  $P_R^2$  is an unspecified repelling petal. Thus  $(z_0'^{-n})$  converges to 0. Since  $(\beta_{-n})$  is a sequence of paths of  $B_0^{f_2}$  with one extremity converging in the Euclidean sense to  $0 \in \partial B_{f_2}^0$  and having a bounded hyperbolic length in  $\phi(U_1^0) \subset B_{f_2}^0$ , this implies that  $(\beta_{-n})$  has a Euclidean diameter converging to 0. Thus  $(\beta_{-n})$  converges uniformly to 0 in the Euclidean sense.

Choose  $P_R^2$  as a *large* repelling petal of  $f_2$  included in  $U_2$ . Let  $P_A^2$  be an attracting petal of  $f_2$ . We choose  $P_A^2$  as the inverse image of a right half plane. By Taylor expansion of Fatou coordinates (proposition 4.1.10), we must have that  $P_A^2 \cup P_R^2 \cup \{0\}$  is a neighborhood of 0. Note for  $n \in \mathbb{N}$ ,  $F_n = \{t \in [0, 1], \beta_{-n}(t) \in \overline{f_2(P_A^2)}\}$ . The sequence  $(F_n)$  is a sequence of compacta of  $[0, 1]$ , decreasing for the inclusion, this is shown by the stability of  $\overline{f_2(P_A^2)}$  by  $f_2$  and the fact that  $f_2 \circ \beta_{-n-1} = \beta_{-n}$ . Note that their intersection is empty. If we had a  $t$  in their intersection, then the equality  $\Phi_A^2(\beta_{-n}(t)) = \Phi_A^2(\beta_0(t)) - n$  valid for all  $n \in \mathbb{N}$  would give a contradiction: the image of  $\Phi_A^2$ , defined on  $P_A^2$ , is supposed to be a right half plane. The sequence  $(F_n)$  is a sequence of compacta decreasing for the inclusion, and its intersection is empty, thus there exists an integer  $n$  such that  $F_n = \emptyset$ . Since  $(\beta_{-n})$  converges uniformly in the Euclidean sense to 0, this finally proves that  $\beta_{-n}$  is included in  $P_R^2$  for  $n$  big enough,  $n \geq N$  (recall that  $P_A^2 \cup P_R^2 \cup \{0\}$  is a neighborhood of 0). Since  $P_R^2 \subset U_2$ , we get  $\beta_{-n} \subset U_2^0 \cap P_R^2$  if  $\beta_{-n}(0) \in U_2^0$ .

Let us show that  $\beta_{-n}(0) \in U_2^0$ . The point  $\beta_{-n}(0)$  corresponds to the point  $z'_{-n}$  in the construction of  $\psi^+(w)$  when  $\text{Im}(w)$  is big. It has been shown in this part that  $z'_{-n} \in P_A^2$ . Since we know that for  $n$  big enough,  $z'_{-n} \in U_2$ , it follows that  $z'_{-n} \in U_2^0$ .

Let us define a path  $\gamma'$  as follows. For  $t \in [0, 1]$ ,  $\gamma'(t)$  is equal to the common value

of the  $\Phi_R^2(\beta_{-n}(t)) \bmod \mathbb{Z}$  when  $n \geq N$ . Finally, set

$$\psi^+(w) = \gamma'(1)$$

The path  $\gamma' = (\Phi_R^2 \bmod \mathbb{Z}) \circ \beta_{-N}$  links  $\psi^+(w_0)$  to  $\psi^+(w)$  and is included in  $\Phi_R^2(U_2^0 \cap P_R^2) \bmod \mathbb{Z} \subset \mathcal{D}_2$ . Since  $\psi^+(w_0) \in \mathcal{D}_2^+$ , this shows that  $\psi^+(w) \in \mathcal{D}_2^+$ .

The quantity  $\psi^+(w)$  depends a priori of the choice of the path  $\gamma$  made at the beginning.

The map  $\psi^+$  is well-defined from  $\mathcal{D}_1^+$  to  $\mathcal{D}_2^+$ : The element  $\psi^+(w)$  may be expressed without any paths. This will prove that it is independent of the chosen path. Let  $\Phi_R^1 : P_R^1 \rightarrow \Phi_R^1(P_R^1)$  and  $\Phi_R^2 : P_R^2 \rightarrow \Phi_R^2(P_R^2)$  be normalized Fatou coordinates, with arbitrary repelling petals  $P_R^1, P_R^2$ . Let for  $w \in \mathcal{D}_1^+$ ,  $\hat{z} = \bar{\Psi}_R^1(w) = \{z_{-n}, n \in \mathbb{N}\}$  the grand orbit of  $P_R^1/f_1$  (hence a backward orbit) associated to  $w \in \mathcal{C}_R^{f_1}$ . We have for all  $w \in \mathcal{D}_1^+$  and  $N$  big enough:

$$\psi^+(w) = \Phi_R^2(\phi(\hat{z}_N)) \bmod \mathbb{Z} \quad (7)$$

We recall that  $\hat{z}_N = \{z_{-n}, n \geq N\}$ . The integer  $N$  is chosen big enough such that the backward orbit  $\hat{z}_N$  is completely included in  $P_R^2$ . Let  $N(w)$  be the minimal integer such that  $\hat{z}_N$  is included in  $P_R^2$ .

Since the petal is open, the integer  $N(w)$  can only decrease locally, this corresponds to an upper semi-continuity of  $N(w)$  as a function of  $w$ .

In this paragraph, the notations are inspired by those of figure 10, replacing letters by capital letters and paths by relatively compact open sets. Given an relatively compact open set  $\Gamma \Subset \mathcal{D}_1^+$ , one can get a uniform upper bound  $N$  for  $N(w)$  on  $\bar{\Gamma}$ . This allows us to get an expression of  $\psi^+$  as a composition of holomorphic maps in  $\Gamma$ . More generally, set  $\hat{\Gamma}_0 = \{w \in \mathbb{C}, a \leq \operatorname{Re}(z) \leq a+1 \text{ and } w \bmod \mathbb{Z} \in \Gamma\}$  where  $a \in \mathbb{R}$  is chosen such that  $\hat{\Gamma}_0 \Subset \Phi_R^1(P_R^1)$  and for  $n \in \mathbb{N}$ ,  $\hat{\Gamma}_{-n} = \hat{\Gamma}_0 - n$ . We have  $A_{-n} := \Psi_R^1(\hat{\Gamma}_{-n}) \Subset P_R^1$  and there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $B_{-n} := \phi(\Psi_R^1(\hat{\Gamma}_{-n})) \Subset P_R^2$ . By equation (7), we get for all  $n \geq N$  and all  $w \in \hat{\Gamma}_{-n}$ :

$$\psi^+(w \bmod \mathbb{Z}) = \Phi_R^2\left(\phi(\Psi_R^1(w - n))\right) \bmod \mathbb{Z}$$

Hence  $\psi^+$  is holomorphic on  $\Gamma$ . This holds for all  $\Gamma \Subset \mathcal{D}_1^+$ . That is why  $\psi^+$  is holomorphic on its domain  $\mathcal{D}_1^+$ .

The map  $\psi^+$  is a biholomorphism from  $\mathcal{D}_1^+$  to  $\mathcal{D}_2^+$ : this paragraph is only for the proof of the complement 4.5.2.

We take again the construction, in the other sense. More precisely, we define a map from  $\mathcal{D}_2^+$  to  $\mathcal{D}_1^+$ :

$$\psi'^+(w) = \Phi_R^1(\phi'(\hat{z}_N)) \bmod \mathbb{Z} \quad (8)$$

where  $\hat{z} = \bar{\Psi}_R^2(w) = \{z_{-n}, n \in \mathbb{N}\}$  and  $\hat{z}_N = \{z_{-n}, n \geq N\}$ .

Let  $\Gamma, \Gamma'$  be connected open relatively compact subsets of  $\mathcal{D}_1^+, \mathcal{D}_2^+$ . We apply the lemma 4.4.40 of pseudo-invertibility to show that  $\psi^+ : \Gamma \rightarrow \Gamma'$  is a biholomorphism of

inverse  $\psi' : \Gamma' \rightarrow \Gamma$ . This is independent of the choice of  $\Gamma$  and  $\Gamma'$ , hence  $\psi^+ : \mathcal{D}_1^+ \rightarrow \mathcal{D}_2^+$  is a biholomorphism of inverse  $\psi'^+ : \mathcal{D}_2^+ \rightarrow \mathcal{D}_1^+$ .

Construction (and possible biholomorphy) of  $\psi^-$  from  $\mathcal{D}_1^-$  to  $\mathcal{D}_2^-$ : A reasoning identical to the preceding paragraphs enables us to build  $\psi^-$  in the general case, and to show its biholomorphy when  $\phi$  is a pseudo-conjugacy.

**Conclusion:** Finally, it remains to show that  $T_\sigma \circ \mathbf{h}_1^\pm = \mathbf{h}_2^\pm \circ \psi^\pm$  (recall that  $\mathbf{h}_1^\pm$  and  $\psi^\pm$  are defined on the same domain  $\mathcal{D}_1^\pm$ ) and to establish the Taylor expansion of  $\psi^\pm$ .

Recall that  $T_\sigma = \bar{\Phi}_A^2 \circ \bar{\phi} \circ (\bar{\Phi}_A^1)^{-1}$  where  $\bar{\phi}$  is the map that  $\phi$  induces from  $B^{f_1}/f = P_A^1/f$  to  $B^{f_2}/f = P_A^2/f$  by quotient (it may also be thought as the map that  $\phi$  induces by direct image if  $P_A^2 = \phi(P_A^1)$ ). Recall also that  $\mathbf{h}_1^\pm = \bar{\Phi}_A^1 \circ i \circ \bar{\Psi}_R^1$  where  $i$  is the quotient of the inclusion map  $P_R^1 \cap B^{f_1} \rightarrow B^{f_1}$  from  $(P_R^1 \cap B^{f_1})/f_1$  to  $B^{f_1}/f_1 = P_A^1/f_1$ , see definition 4.3.1. We get:

$$T_\sigma \circ \mathbf{h}_1^\pm = \bar{\Phi}_A^2 \circ \bar{\phi} \circ (\bar{\Phi}_A^1)^{-1} \circ \bar{\Phi}_A^1 \circ i \circ \bar{\Psi}_R^1 = \bar{\Phi}_A^2 \circ \bar{\phi} \circ i \circ \bar{\Psi}_R^1 \quad (9)$$

Use the fact that in a composite function, passing to quotient at each step is equivalent to passing to quotient on the range and the target. Recall that  $\pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$  is the canonical projection, that  $\Phi_R^i$  is defined on  $P_R^i$ , and that  $\Phi_A^i$  is defined over  $P_A^i$ . Let  $P_R^1$  be a repelling petal containing a  $\pi/2$ -petal, and  $G$  be a left half plane such that  $\Phi_R^1(P_R^1) \supset G$ . Let  $H = \{z \in \mathbb{C}, \text{Im}(z) > M\} \subset \mathcal{D}_1^+$  be an upper half-plane such that  $\Psi_R^1(H \cap G) \subset P_A^1$ , where  $P_A^1$  is a very large attracting petal, and such that  $M$  satisfies the corollary 4.4.28. For all  $w \in H \cap G$ , there exists  $n(w) \in \mathbb{N}$  such that  $\phi(\Psi_R^1(w)) \in P_R^2$  by corollary 4.4.28. Let

$$X = \bigcup_{w \in H \cap G} \{w - n(w)\} \subset H \cap G$$

This enables us to calculate the composite functions before the quotient (note that  $\pi(X) = \pi(H)$ ):

- By equation (9), the composite  $(T_\sigma \circ \mathbf{h}_1^+)|_{\pi(H)}$  is equal to  $\Phi_A^2 \circ \phi \circ (\Psi_R^1)|_X$  after passing to quotient by  $\mathbb{Z}$  at domain and range.
- $(\mathbf{h}_2^+ \circ \psi^+)|_{\pi(H)}$  is equal to  $\Phi_A^2 \circ \Psi_R^{2,\text{ext}} \circ \Phi_R^2 \circ \phi \circ (\Psi_R^1)|_X = \Phi_A^2 \circ \phi \circ (\Psi_R^1)|_X$  after passing to quotient by  $\mathbb{Z}$  at domain and range. Note that the term  $\Phi_R^2 \circ \phi \circ (\Psi_R^1)|_X$  corresponds to  $\psi^+$ , and the term  $\Phi_A^2 \circ \Psi_R^{2,\text{ext}}$  to  $\mathbf{h}_2^+$ .

We thus obtain that  $(T_\sigma \circ \mathbf{h}_1^+)|_{\pi(H)}$  and  $(\mathbf{h}_2^+ \circ \psi^+)|_{\pi(H)}$  are both equal to  $\Phi_A^2 \circ \phi \circ (\Psi_R^1)|_X$  after passing to quotient by  $\mathbb{Z}$  at domain and range. This implies by uniqueness of analytic extension ( $\pi(H)$  is open in the connected set  $\mathcal{D}_1^+$ ) that  $T_\sigma \circ \mathbf{h}_1^+ = \mathbf{h}_2^+ \circ \psi^+$ . We would show symmetrically that  $T_\sigma \circ \mathbf{h}_1^- = \mathbf{h}_2^- \circ \psi^-$ .

The singularity of  $\psi^\pm$  at  $\pm i\infty$  being removable, we already know that the Taylor expansion of  $\psi^\pm$  exists and is of the form  $\psi^\pm(w) = dw + c + o(1)$ . We have to show that  $d = 1$ . We may write locally at  $\pm i\infty$ :

$$\psi^\pm = (\mathbf{h}_2^\pm)^{-1} \circ T_{-\sigma} \circ \mathbf{h}_1^\pm$$

where  $(\mathbf{h}_2^\pm)^{-1}$  is the local inverse branch of  $\mathbf{h}_2^\pm$  at  $\pm i\infty$ . We finish the proof by inversion and composition of Taylor expansions (the  $\mathbf{h}_i^\pm$  are in  $\pm i\infty$  approximated to  $o(1)$  by translations, and an inverse or composite of translations is a translation).  $\square$

**Remark 4.5.4.** Suppose that  $\phi : U_1 \cap B_0^{f_1} \rightarrow U_2 \cap B_0^{f_2}$  defines a biholomorphism of inverse  $\phi^{-1} = \phi'$ , where  $U_1, U_2$  neighborhood of 0, conjugating  $f_1|_{U_1 \cap B_0^{f_1}}, f_2|_{U_2 \cap B_0^{f_2}}$  (to simplify, one might suppose  $U_1 = U_2 = \mathbb{C}$ , in such a way that  $\phi$  is a global biholomorphism on immediate basins); it is a more restrictive hypothesis than being a local conjugacy on immediate basins, see example 4.4.34. Note that the domains of definition are not supposed connected. In this setting, one might define  $\psi$  over a bigger set than  $\mathcal{D}_1^+ \cup \mathcal{D}_1^-$ . More precisely, we may rewrite the formula:

$$\psi(w) = \Phi_R^2(\phi(\hat{z}_N)) \bmod \mathbb{Z}$$

where  $\hat{z} = \overline{\Psi}_R^1(w) = \{z_{-n}, n \in \mathbb{N}\}$  and  $\hat{z}_N = \{z_{-n}, n \geq N\}$ . The formula is defined when  $w \in \mathcal{D}_1$  is such that there exists  $N \in \mathbb{N}$  such that  $\hat{z}_N \subset P_R^2$ . The condition  $w \in \mathcal{D}_1$  is equivalent to the backward orbit  $\hat{z} = \overline{\Psi}_R^1(w) \in P_R^1/f$  being included in  $U_1 \cap B_0^{f_1}$ . Recall that we choose  $P_R^1 \subset U_1$ . Since the smallest  $N(w)$  such that  $\hat{z}_N \subset P_R^2$  is an upper semi-continuous function of  $w$ , the domain of  $\psi$  is open. For the same reason, the map  $\psi$  is holomorphic, since it can be written locally as a composite of holomorphic functions.

The proof of proposition 4.5.1 defines  $\psi^+$  on the connected set  $\mathcal{D}_1^+$  from the definition of  $\psi^+$  at a single point  $w_0$  of this set, using paths. By following to the letter this proof, we may define  $\psi$  over a whole connected component of  $\mathcal{D}_1$  from the definition of  $\psi$  on a single point of this component; this shows that the domain of  $\psi$  must be a union of connected components of  $\mathcal{D}_1$ . Considering  $\psi'$  as in the proof in the case  $\phi$  is a pseudo-conjugacy (substituting  $w \in \mathcal{D}_2$  for the restriction  $w \in \mathcal{D}_2^+$ ), the arguments of the proof does not apply to show that  $\psi$  is a biholomorphism over its image of inverse  $\psi'$ , even though one might expect it is true.

We will not think anymore about this extension question in this article, our aim being to build a local classification (cf. theorem 1.5.1), which only needs to exploit the components  $\mathcal{D}_i^\pm$ .

Let us show now the converse statement of theorem 1.5.1:

**Proposition 4.5.5.** *Let  $f_1, f_2$  be two holomorphic maps with parabolic point at 0, with Taylor expansion at 0 of the form  $f_i(z) = z + a_i z^2 + O(z^3)$  with  $a_i \neq 0$ .*

*Suppose that there exists  $\sigma \in \mathbb{C}$  and a pair of holomorphic maps  $\psi = (\psi^+, \psi^-)$ , where  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ , with removable singularities at  $\pm i\infty$  such that:*

$$\mathbf{h}_1^\pm(w) = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm(w)$$

for all  $w \in \mathcal{D}_1^\pm$ .

Then there exists a local semi-conjugacy  $\phi$  over the immediate basins of  $f_1, f_2$  (see definition 4.4.8).

Moreover, the maps  $\psi^\pm$  admits an expansion at  $\pm i\infty$  of the form  $\psi^\pm(w) = w + \rho^\pm + o(1)$ .

**Complement 4.5.6.** *Same hypothesis as the preceding proposition. Suppose furthermore that  $\psi = (\psi^+, \psi^-)$  is a pair of biholomorphisms  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ .*

*Then there exists a local pseudo-conjugacy  $(\phi, \phi')$  over the immediate basins of  $f_1, f_2$  (see definition 4.4.30). We may choose this local pseudo-conjugacy synchronous.*

*Proof.* We show simultaneously the proposition and its complement. Let  $T_\sigma$  and  $\psi^\pm$  satisfy the hypothesis of the proposition. We will suppose that  $\psi^\pm$  are biholomorphisms during the paragraphs referring to the demonstration of the complement, paragraphs which will be explicitly mentionned.

Taylor expansions of  $\psi^\pm$  at  $\pm i\infty$ : The Taylor expansions of  $\psi^\pm$  are shown by applying the end of the proof of proposition 4.5.1, last paragraph. Denote by  $\rho^\pm \in \mathbb{C}$  the corresponding constant terms.

Choice of good lifts  $\tilde{\psi}^\pm$  of  $\psi^\pm$  by  $\pi$ : Denote by  $\pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$  the canonical projection and:

$$\begin{aligned}\tilde{\mathcal{D}}_i &= \pi^{-1}(\mathcal{D}_i) \\ \tilde{\mathcal{D}}_i^\pm &= \pi^{-1}(\mathcal{D}_i^\pm)\end{aligned}$$

for  $i = 1, 2$ . Let  $\tilde{\psi}^\pm : \tilde{\mathcal{D}}_1^\pm \rightarrow \tilde{\mathcal{D}}_2^\pm$  be a lift of  $\psi^\pm$  by  $\pi$ . Since  $\tilde{\psi}^\pm(w+1) = \tilde{\psi}^\pm(w) \bmod \mathbb{Z}$ , we get by continuity that there exists  $n \in \mathbb{Z}$  such that  $T_n \circ \tilde{\psi}^\pm = \tilde{\psi}^\pm \circ T_1$ . The Taylor expansion of  $\psi^\pm$  at  $\pm i\infty$  gives by continuity of the lift  $\tilde{\psi}$  an expansion of  $\tilde{\psi}^\pm$  at  $\pm i\infty$  over a vertical band  $D$  of width 1 (fundamental domain for the translation  $T_1$ )

$$\tilde{\psi}^\pm(w) = w + \rho^\pm + o(1)$$

even if it means adding fixed integers to  $\rho^\pm$ . This last Taylor expansion implies that  $n = 1$ . Hence  $\tilde{\psi}^\pm$  commutes with  $T_1$ .

Note that  $\mathbf{h}_1^\pm = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm$ . This implies:

$$\Phi_A^1 \circ \Psi_R^{1,\text{ext}}(w) = T_\sigma \circ \Phi_A^2 \circ \Psi_R^2 \circ \tilde{\psi}^\pm(w) \bmod \mathbb{Z}$$

for  $w \in \tilde{\mathcal{D}}_1^\pm$ , connected set.

Recall that  $T_\sigma$  and  $\Phi_A^2 \circ \Psi_R^2$  commute with  $T_1$ , this implies that  $T_\sigma \circ \Phi_A^2 \circ \Psi_R^2$  commutes with  $T_1$ . Chosing the right lifts of  $\psi^\pm$ , we get the equality

$$\Phi_A^1 \circ \Psi_R^1(w) = T_\sigma \circ \Phi_A^2 \circ \Psi_R^2 \circ \tilde{\psi}^\pm(w)$$

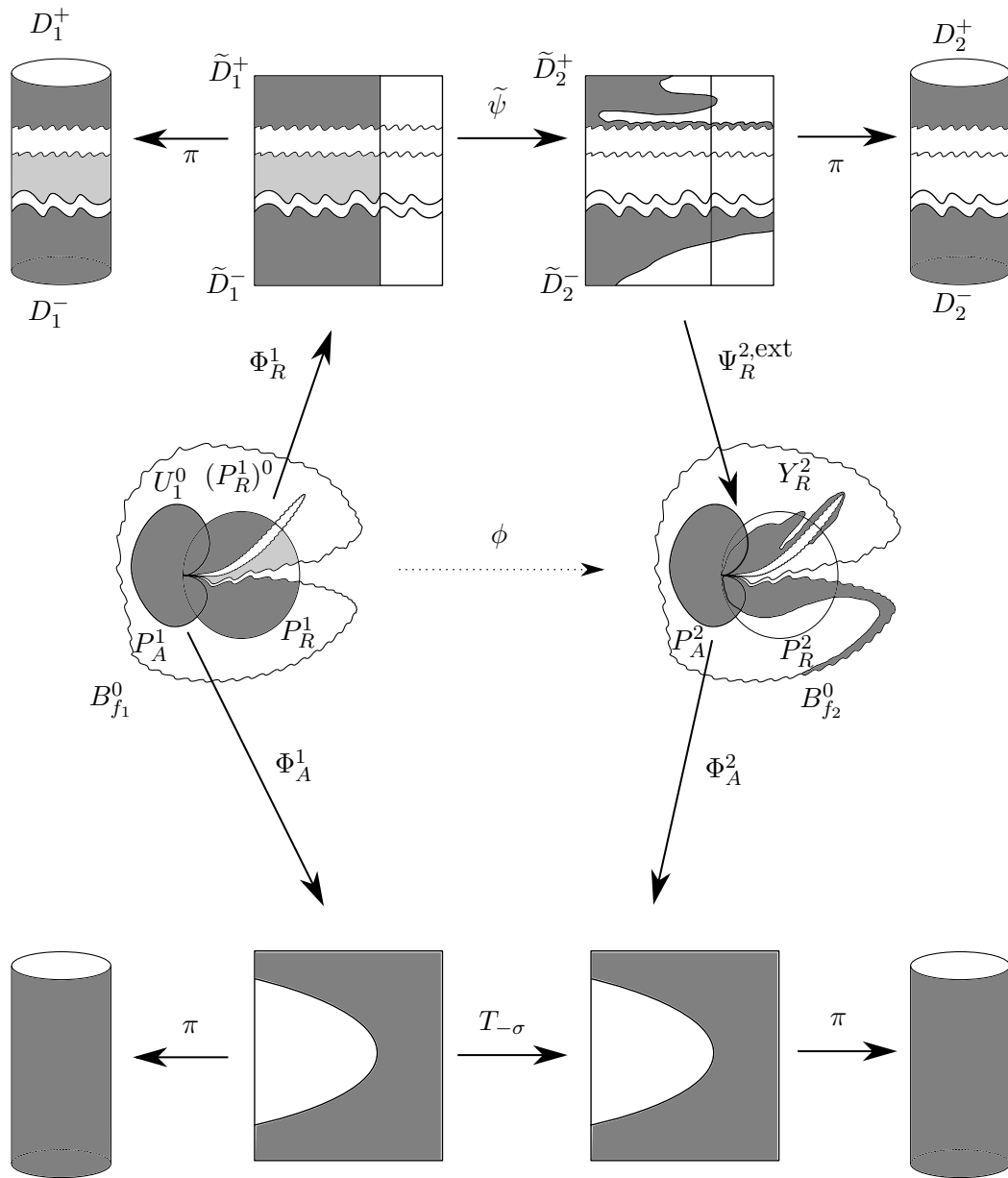


Figure 11

The attracting petals  $P_A^1$ ,  $P_A^2$  are represented by cardioids, the repelling petals  $P_R^1$ ,  $P_R^2$  by circles. We have chosen  $U_1 = P_A^1 \cup P_R^1 \cup \{0\}$ . The set  $U_1^0$  is represented in dark grey, in the central part at the left of the diagram. The set  $U_1 \cap B_{f_1}^0 \setminus U_1^0$  is represented in light grey. We represent also in dark grey, light grey their diverse images by the maps of the diagram. The map  $\phi$  will be build as the unique function completing this commutative diagram. Note that a priori,  $\phi(U_1^0)$  is not a set of the form  $U_2^0$ , and that  $\phi(P_R^1)$  is not a set of the form  $(P_R^2)^0$  where  $P_R^2$  repelling petal of  $f_2$ . Notice that  $Y_R^2$  is the direct image

without the modulo. Note that the reasoning is also valid when  $\mathcal{D}_1^+ = \mathcal{D}_1^-$  i.e.  $\psi^+ = \psi^-$ . We choose these lifts  $\tilde{\psi}^\pm$  for the rest of the argument. Note  $\tilde{\psi} = \tilde{\psi}^+ \cup \tilde{\psi}^-$ .

Construction of the petals  $P_A^1, P_A^2$ , and of the set  $Y_R^2$ : We choose the attracting, repelling petals  $P_A^1, P_R^1$  of  $f_1$ , such that

$$U_1 = P_A^1 \cup P_R^1 \cup \{0\}$$

is an open neighborhood of 0 included in a neighborhood  $W_1$  of 0 where  $f_1$  admits an inverse branch  $f_1^{-1} : W \rightarrow f_1^{-1}(W_1)$  sending 0 to 0. This allows to define  $\mathbf{h}'_1$  the horn map of  $f^{-1}$ . We choose  $P_A^1, P_R^1$  such that  $P_A^1 \cap P_R^1$  contains two connected components  $C^\pm$ . The component  $C^\pm$  is characterized by the fact that its image by  $\Phi_A^1 \bmod \mathbb{Z}$  (or equivalently by  $\Phi_R^1 \bmod \mathbb{Z}$ ) is a punctured neighborhood of  $\pm i\infty$ .

Write

$$(P_R^1)^0 = (\Phi_R^1)^{-1}(\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-) = (\Phi_R^1)^{-1}(\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^- \cap \Phi_R^1(P_R^1))$$

and

$$V_1^0 = P_A^1 \cup (P_R^1)^0$$

Let  $U_1^0$  be the connected component of  $U_1 \cap B_0^{f_1}$  containing a germ of the attracting axis.

Recall that by lemma 4.4.14, we have  $U_1^0 \subset V_1^0$  (the reader might look at the proof of the lemma again and see figure 6).

Notice that choosing  $P_A^1$  small enough, which is equivalent to choosing  $\Phi_A^1(P_A^1)$  small, there exists a petal  $P_A^2$  of  $f_2$  with a Fatou coordinate  $\Phi_A^2$  such that  $\Phi_A^2(P_A^2) = \Phi_A^1(P_A^1) - \sigma$ , i.e.

$$P_A^2 = (\Phi_A^2)^{-1}(\Phi_A^1(P_A^1) - \sigma)$$

the petal  $P_A^2$  may be arbitrarily small.

Let  $\Psi_R^{2,\text{ext}}$  be the extended Fatou parametrisation of  $f_2$ . By proposition-definition 4.3.8,  $\text{Im}(\tilde{\psi}) = \tilde{\mathcal{D}}_2^+ \cup \tilde{\mathcal{D}}_2^-$  is included in the domain of  $\Psi_R^2$  and  $\Psi_R^2(\text{Im}(\tilde{\psi})) \subset B_0^{f_2}$ . This allows to note the following open subset of the attracting immediate basin of  $f_2$ :

$$Y_R^2 = \Psi_R^{2,\text{ext}}(\tilde{\psi}(\Phi_R^1(V_1^0))) \subset B_0^{f_2}$$

**Remark 4.5.7.** Since  $\tilde{\psi}$  does not admit a continuous extension to the border of  $\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-$ , it seems not obvious that  $Y_R^2$  is included in a set of the form  $(P_R^2)^0 = (\Phi_R^2)^{-1}(\tilde{\mathcal{D}}_2^+ \cup \tilde{\mathcal{D}}_2^-)$  where  $P_R^2$  repelling petal of  $f_2$ . Figure 11 shows how a counter-example might look like. That is why we used the extended repelling parametrisation to define  $Y_R^2 = \Psi_R^{2,\text{ext}}(\tilde{\psi}(\Phi_R^1(V_1^0)))$ , since it seems difficult to use  $(\Phi_R^2)^{-1}$  where  $\Phi_R^2$  repelling Fatou coordinate of a well chosen repelling petal  $P_R^2$ .

The continuity problem of  $\tilde{\psi}$  at the border leads also to the following issue: it is not obvious that  $Y_R^2$  contains a set of the form  $(P_R^2)^0$ , where  $P_R^2$  repelling petal of  $f_2$ , see again the same figure to understand how a counter-example might look like.

Construction of  $\phi : V_1^0 \rightarrow P_A^2 \cup Y_R^2$ : The relation valid over  $(\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-) \cap \Phi_R^1(P_R^1)$

$$\Phi_A^1 \circ (\Phi_R^1)^{-1} = T_\sigma \circ \Phi_A^2 \circ (\Phi_R^2)^{-1} \circ \tilde{\psi}$$

enables us to show the following lemma:

**Lemma 4.5.8.**

$$(\Phi_A^2)^{-1} \circ T_{-\sigma} \circ \Phi_A^1(z) = \Psi_R^{2,\text{ext}} \circ \tilde{\psi} \circ \Phi_R^1(z)$$

for  $z \in P_A^1 \cap P_R^1$ .

*Proof.* The two members of the equality are indeed well defined by construction of  $P_A^1, P_R^1$  and  $P_A^2$ . More explicitly, the left member is defined over  $P_A^1$  since  $T_{-\sigma}(\Phi_A^1(P_A^1)) = P_A^2$ , and the right member is defined over  $P_A^1 \cap P_R^1$  since  $\Phi_R^1(P_A^1 \cap P_R^1) = \Phi_R^1(C^+) \cup \Phi_R^1(C^-) \subset \mathcal{D}_1^+ \cup \mathcal{D}_1^-$ . Recall we chose during the  $P_A^1, P_R^1$  construction the petals  $P_A^1, P_R^1$  such that  $P_A^1 \cap P_R^1$  contains two connected components  $C^+, C^-$ , and that the component  $C^\pm$  is characterized by the fact that its image by  $\Phi_A^1 \bmod \mathbb{Z}$  (or equivalently by  $\Phi_R^1 \bmod \mathbb{Z}$ ) is a punctured neighborhood of  $\pm i\infty$ .

Let  $P_R^2$  be a repelling petal for  $f_2$  with Fatou coordinate  $\Phi_R^2$ . Denote

$$\begin{cases} S_R &= \Phi_R^1(P_R^1) \\ S_R^0 &= S_R \cap (\tilde{\mathcal{D}}_1^+ \cup \tilde{\mathcal{D}}_1^-) \\ T_R &= \Phi_R^2(P_R^2) \\ T_R^0 &= T_R \cap (\tilde{\mathcal{D}}_2^+ \cup \tilde{\mathcal{D}}_2^-) \end{cases}$$

Let  $\tilde{\psi}^{-1}$  be a local inverse branch of  $\tilde{\psi}$  at  $\pm i\infty$ . Let  $W^\pm$  be a neighborhood of  $\pm i\infty$  in  $T_R^0$ . By continuity of  $\tilde{\psi}^{-1}$  at  $\pm i\infty$ , we choose  $W^\pm$  small enough such that,  $\tilde{\psi}^{-1}(W^\pm) \subset S_R \cap \mathcal{D}_1^\pm$ . We have immediately:

$$(\Phi_A^2)^{-1} \circ T_{-\sigma} \circ \Phi_A^1(z) = (\Phi_R^2)^{-1} \circ \tilde{\psi} \circ \Phi_R^1(z) \quad \left( = \Psi_R^{2,\text{ext}} \circ \tilde{\psi} \circ \Phi_R^1(z) \right)$$

for  $z \in P_A^1 \cap P_R^1 \cap \Phi_R^{-1}(\tilde{\psi}^{-1}(W^\pm))$ , open set of non-empty intersection with  $C^\pm$  since  $\tilde{\psi}^{-1}(W^\pm)$ . Hence by uniqueness of analytic extension, we have for  $z \in P_A^1 \cap P_R^1 = C^+ \cup C^-$ :

$$(\Phi_A^2)^{-1} \circ T_{-\sigma} \circ \Phi_A^1(z) = \Psi_R^{2,\text{ext}} \circ \tilde{\psi} \circ \Phi_R^1(z)$$

□

We may now denote the holomorphic map  $\phi : V_1^0 \rightarrow P_A^2 \cup Y_R^2$  defined by:

$$\phi(z) = \begin{cases} (\Phi_A^2)^{-1} \circ T_{-\sigma} \circ \Phi_A^1(z) & \text{if } z \in P_A^1 \\ \Psi_R^{2,\text{ext}} \circ \tilde{\psi} \circ \Phi_R^1(z) & \text{if } z \in (P_R^1)^0 \end{cases}$$

By construction of  $P_A^2$  and  $Y_R^2$ :



- $\phi$  induces a biholomorphism from  $P_A^1$  to  $P_A^2$
- $\phi$  induces a surjection from  $(P_R^1)^0$  to  $Y_R^2$ .

In particular  $\phi$  is surjective.

Note that  $\phi$  induces a conjugacy between  $(f_1)|_{P_A^1}$  and  $(f_2)|_{P_A^2}$ , and a semi-conjugacy between  $(f_1)|_{(P_R^1)^0}$  and  $(f_2)|_{Y_R^2}$  (for the semi-conjugacy, exploit that  $\tilde{\psi}$  commutes with  $T_1$ ). The map  $\phi$  then constitutes a semi-conjugacy between  $(f_1)|_{V_1^0}$  and  $(f_2)|_{P_A^2 \cup Y_R^2}$ .

The map  $\phi$  is then a local semi-conjugacy in the sense of definition 4.4.8.

Construction of  $\phi' : V_2^0 \rightarrow P_A^1 \cup Y_R^1$ : This paragraph is only for the proof of the complement. We take again the elements of the construction of  $\phi$ , in the opposite direction: we build in the same way  $V_2, V_2^0, (P_R^2)^0$  and  $Y_R^1$ .

Notice that  $P_A^1, P_R^1, P_A^2, P_R^2$  were already built in the preceding construction, that is why we will note the new petals  $Q_A^2, Q_R^2, Q_A^1, Q_A^1$ .

Let us check that we can take  $Q_A^i = P_A^i$  and  $Q_R^i = P_R^i$  with all the conditions satisfied on the petals. Recall the conditions.

1.  $P_A^1 \cup P_R^1$  is a punctured neighborhood of 0 containing two connected components, included in a neighborhood  $W_1$  of 0 where  $f_1$  admits an inverse branch  $f_1^{-1} : W_1 \rightarrow f_1^{-1}(W_1)$  sending 0 to 0.
2.  $\Phi_A^2(P_A^2) = \Phi_A^1(P_A^1) - \sigma$
3.  $Q_A^2 \cup Q_R^2$  is a punctured neighborhood of 0 containing two connected components, included in a neighborhood  $W_2$  of 0 where  $f_2$  admits an inverse branch  $f_2^{-1} : W_2 \rightarrow f_2^{-1}(W_2)$  sending 0 to 0.
4.  $\Phi_A^1(Q_A^1) = \Phi_A^2(Q_A^2) + \sigma$
5.  $Q_A^i = P_A^i$  and  $Q_R^i = P_R^i$

Let us take  $P_A^1, P_R^1, P_A^2$  such that the conditions 1 and 2 are true, as in the construction of  $\phi$ . Note that  $P_A^1$  can be taken arbitrarily small, this implies, by condition 1, that  $P_A^2$  can be taken arbitrary small. Take  $Q_A^2 := P_A^2$ . Since  $Q_A^2$  is arbitrarily small, it is possible to build  $Q_R^2$  such that condition 3 is true. Take  $Q_A^1 := P_A^1$ , this imply condition 4. Since  $\Phi_A^1(Q_A^1) = \Phi_A^1(P_A^1) = \Phi_A^2(P_A^2) + \sigma = \Phi_A^2(Q_A^2) + \sigma$  by condition 2. Finally, define  $Q_R^1 := P_R^1$  and  $P_R^2 := Q_R^2$  to have condition 5.

This enables us to build  $\phi' : V_2^0 \rightarrow P_A^1 \cup Y_R^1$ , the formula being the same for  $\phi$  with the roles of  $f_1, f_2$  inverted:

$$\phi'(z) = \begin{cases} (\Phi_A^1)^{-1} \circ T_\sigma \circ \Phi_A^2(z) & \text{if } z \in P_A^2 \\ \Psi_R^{1, \text{ext}} \circ \tilde{\psi}^{-1} \circ \Phi_R^2(z) & \text{if } z \in (P_R^2)^0 \end{cases}$$

Notice that the biholomorphism induced by  $\phi'$  from  $P_A^2$  to  $P_A^1$  turns out to be exactly the inverse of the biholomorphism induced by  $\phi$  from  $P_A^1$  to  $P_A^2$ . This enables us to say that  $(\phi, \phi')$  is a synchronous pseudo-conjugacy.  $\square$

**Remark 4.5.9.** There is no reason a priori for a local semi-conjugacy, even if it is a synchronous local pseudo-conjugacy, to be a local conjugacy over the immediate basins (see the definitions 4.4.8, 4.4.30). Recall that by the example 4.4.34, a local conjugacy over the immediate basins is a biholomorphism of the form  $\phi : U_1^0 \rightarrow U_2^0$  which conjugates  $f_1$  and  $f_2$ , it can be interpreted as a local pseudo-conjugacy  $(\phi, \phi')$  where  $\phi'$  is the inverse of  $\phi$ .

## 4.6 Parabolic point with several petals

### 4.6.1 Generalized definitions of semi-conjugacy, pseudo-conjugacy and reviewed statements

Let  $f_1, f_2$  be two holomorphic functions from an open neighborhood of  $0 \in \mathbb{C}$  to  $\mathbb{C}$  with a general parabolic point at 0:

$$f_i(z) = e^{2i\pi p_i/q_i} z + o(z)$$

where  $p_i \in \mathbb{Z}$ ,  $q_i \in \mathbb{N}^*$  and  $p_i \wedge q_i = 1$ .

Denote  $A_1, A_2$  attracting axes of  $f_1, f_2$ , and  $R_i^-, R_i^+$  the adjacent repelling axes of  $A_i$  such that  $R_i^-, A_i, R_i^+$  are in the trigonometrical order (if  $q_i = 1$  then  $R_i^- = R_i^+$ ). Let  $B^{f_1}, B^{f_2}$  be the parabolic basins of  $f_1, f_2$  and  $C^1, C^2$  the connected components of  $B^{f_1}, B^{f_2}$  containing a germ of  $A_1, A_2$ . Notice that  $q_i$  is the least positive integer such that  $f_i^{q_i}(C^i) = C^i$ .

For  $U_1, U_2$  open neighborhoods of  $0 \in \mathbb{C}$ , denote  $U_i^0$  the connected component of  $U_i \cap B^{f_i}$  containing a germ of the attracting axis  $A_i$ . For all attracting petal  $P_A^i$  of  $f_i$  associated to the axis  $A_i$  and included in  $U_i$ , we have  $P_A^i \subset U_i^0$ .

See figure 12 for the next two definitions.

**Definition 4.6.1.** A holomorphic map  $\phi$  is said to be a local semi-conjugacy of  $(f_1, A_1), (f_2, A_2)$  over their immediate basins if  $\phi : U_1^0 \rightarrow C^2$  where  $U_1, U_2$  open neighborhood of  $0 \in \mathbb{C}$ , and satisfies the property of semi-conjugacy:  $\phi \circ f_1^{q_1}(z) = f_2^{q_2} \circ \phi(z)$  for all  $z \in U_1^0$  such that  $f_1^{q_1}(z) \in U_1^0$ .

**Definition 4.6.2.** A pair of local semi-conjugacies  $(\phi, \phi')$  of  $(f_1, A_1), (f_2, A_2)$  is a local pseudo-conjugacy of  $(f_1, A_1), (f_2, A_2)$  over their immediate basins if  $\bar{\phi} : C^1/f_1^{q_1} \rightarrow C^2/f_2^{q_2}$  has for inverse  $\bar{\phi}' : C^2/f_2^{q_2} \rightarrow C^1/f_1^{q_1}$ .<sup>1</sup>

**Remark 4.6.3.** Suppose that  $f_1, f_2$  are pseudo-conjugate, with sets  $U_1, U_2$  small enough such that  $f_i$  admits a local inverse at 0:

$$f_i^{-1} : f_i(U_i) \rightarrow U_i$$

Write  $V_i = f_i(U_i)$ , and  $V_i^0$  the connected component of  $V_i \cap B^{f_i}$  containing a germ of the attracting axis  $Tf_i(A_i)$ , where  $Tf_i$  denotes the tangent map of  $f_i$  at 0. Saying

<sup>1</sup>It would be possible to show, as in the case of simple parabolic points, that the sets  $C^i/f_i^{q_i}$  are cylinders, and the map  $\bar{\phi}$  is a translation. This would enable us to define the concept of synchronous pseudo-conjugacy.

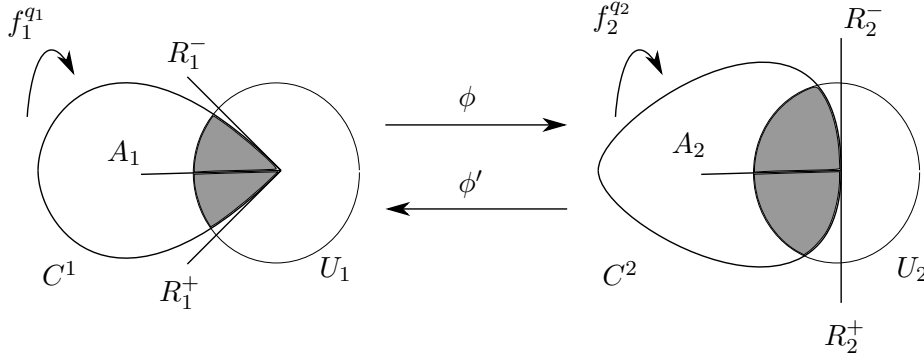


Figure 12: General definition of pseudo-conjugacy

The zones represented in gray are respectively  $U_1^0$  at the left and  $U_2^0$  at the right. Note these sets are not image the one by the other by  $\phi, \phi'$ . Notice also that the angles between the attracting axis and the adjacent repelling axis are not necessarily the same, which is equivalent to say that  $f_1$  and  $f_2$  do not necessarily have the same number of petals.

that  $(f_1, A_1)$  and  $(f_2, A_2)$  are pseudo-conjugate by  $(\phi, \phi')$  is equivalent to saying that  $(f_1, T f_1(A_1))$  and  $(f_2, T f_2(A_2))$  are pseudo-conjugate by  $(\phi \circ f_1^{-1}, \phi' \circ f_2^{-1})$ , where  $\phi \circ f_1^{-1}$  and  $\phi' \circ f_2^{-1}$  are defined over  $V_1^0, V_2^0$ .

Note  $\Phi_A^{\text{ext}}[A_i] : C^i \rightarrow \mathbb{C}$  an extended Fatou coordinate associated to the attracting axis  $A_i$ , and  $\Psi_R^{\text{ext}}[R_i^\pm]$  the extended repelling Fatou parametrisation associated to the repelling axis  $R_i^\pm$ . Note  $\mathcal{D}_i^\pm$  the connected component containing  $\pm i\infty$  of the domain of the map

$$\Phi_A^{\text{ext}}[A_i] \circ \Psi_R^{\text{ext}}[R_i^\mp]$$

after quotient at the domain and range by  $\mathbb{Z}$ . Denote  $\mathbf{h}_i^\pm$  this map  $\Phi_A[A_i]^{\text{ext}} \circ \Psi_R^{\text{ext}}[R_i^\mp]$  after quotient and restricted to  $\mathcal{D}_i^\pm$ . Call  $\mathbf{h}_i^\pm$  the *horn maps* of  $(f_i, A_i)$ . Notice that  $\mathbf{h}_i^+$  and  $\mathbf{h}_i^-$  both correspond to the same attracting axis  $A_i$ , but to distinct repelling axis  $R_i^-$  and  $R_i^+$ .

We believe that the proof of theorem 1.5.1 may be followed in this more general case, where the functions  $f_i$  have a general parabolic point at 0. Here is the statement we should obtain:

**Statement 4.6.4.** *Suppose that  $(f_1, A_1), (f_2, A_2)$  are locally pseudo-conjugate at 0 over  $B_0^{f_i}$ . Then  $\mathbf{h}_1^\pm, \mathbf{h}_2^\pm$  are equivalent as analytic ramified coverings via biholomorphism between  $\mathcal{D}_1^\pm$  and  $\mathcal{D}_2^\pm$ , and a translation of  $\mathbb{C}/\mathbb{Z}$ . More precisely, there exists  $\sigma \in \mathbb{C}$ , and a biholomorphism pair  $\psi = (\psi^+, \psi^-)$ , where  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ , such that:*

$$\mathbf{h}_1^\pm = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm$$

Furthermore, the built maps  $\psi^\pm$  admit at  $\pm i\infty$  removable singularities.

Conversely, if there exists  $\sigma \in \mathbb{C}$  and a pair of biholomorphisms  $\psi = (\psi^+, \psi^-)$ , where  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$ , with removable singularities at  $\pm i\infty$  such that:

$$\mathbf{h}_1^\pm = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm$$

then  $f_1, f_2$  are locally pseudo-conjugate at 0 on their immediate basins.

Furthermore, the maps  $\psi^\pm$  admits at  $\pm i\infty$  an expansion of the form  $\psi^\pm(w) = w + \rho^\pm + o(1)$ .

#### 4.6.2 A class of pseudo-conjugacies examples

We give here a last class of pseudo-conjugacies examples, and we apply the conjectural statement 4.6.4. Let  $A, B : \mathbb{D} \rightarrow \mathbb{D}$  be two Blaschke products. Suppose there exists two holomorphic functions  $f_1 : D_{f_1} \rightarrow \mathbb{C}$ ,  $f_2 : D_{f_2} \rightarrow \mathbb{C}$  satisfying the following properties. The maps  $f_1, f_2$  have a parabolic point at 0, and their immediate basins a cycle of two connected components  $B_0^{f_1} = C_0 \cup C_1$ ,  $B_0^{f_2} = C'_0 \cup C'_1$ . Notice that the  $\partial C_i, \partial C'_i$  are not necessarily locally connected. Suppose that we have the following commutative diagram:

$$\begin{array}{ccccc}
 C_0 & \xrightarrow{f_1} & C_1 & \xrightarrow{f_1} & C_0 \\
 \alpha_0 \downarrow & & \alpha_1 \downarrow & & \alpha_0 \downarrow \\
 \mathbb{D} & \xrightarrow{A} & \mathbb{D} & \xrightarrow{B} & \mathbb{D} \\
 \beta_0 \uparrow & & \beta_1 \uparrow & & \beta_0 \uparrow \\
 C'_0 & \xrightarrow{f_2} & C'_1 & \xrightarrow{f_2} & C'_0
 \end{array}$$

where  $\alpha_i : C_i \rightarrow \mathbb{D}$  and  $\beta_i : C'_i \rightarrow \mathbb{D}$  biholomorphisms. See figure 13.

Set  $g_1 = B \circ A$ ,  $g_2 = A \circ B$ . The functions  $g_1, g_2$  both have a parabolic point at  $z = 1$ , as composition of functions admitting fixed points at  $z = 1$  of which the product of the multipliers is 1. Let  $X = [0, 1[$  be their common attracting axis. Notice that  $g_1, g_2$  are semi-conjugate by  $A$ , and  $g_2, g_1$  are semi-conjugate by  $B$ . The pair of maps  $(A, B)$  constitutes a non-synchronous pseudo-conjugacy between  $(g_1, X)$  and  $(g_2, X)$ . The pseudo-conjugacy is non-synchronous since  $B$  is not a local inverse branch of  $A$  locally at  $z = 1$ ; indeed,  $B$  is not a global section of  $A$  and  $B$  is holomorphic.

Write  $X_1 = \alpha_1^{-1}(X)$ ,  $X_2 = \beta_2^{-1}(X)$ . The maps  $(f_1, X_1)$  and  $(f_2, X_2)$  are then pseudo-conjugate by  $(\beta_2^{-1} \circ A \circ \alpha_1, \alpha_1^{-1} \circ B \circ \beta_2)$ . If true, the statement 4.6.4 would be applied, by noting  $\mathbf{h}_1^\pm, \mathbf{h}_2^\pm$  the horn maps of  $(f_1, X_1)$  and  $(f_2, X_2)$ . There would exist  $\sigma \in \mathbb{C}$  and  $\psi^\pm : \mathcal{D}_1^\pm \rightarrow \mathcal{D}_2^\pm$  with removable singularities at  $\pm i\infty$  such that:

$$\mathbf{h}_1^\pm = T_\sigma \circ \mathbf{h}_2^\pm \circ \psi^\pm$$

Insofar as  $f_1$  and  $f_2$  exist, this enables us to build a class of pseudo-conjugacies examples. If  $\partial C_0$  is locally connected at 0 and  $\partial C'_1$  non locally connected at 0, this would allow to get a pseudo-conjugacy which is not continuous at 0.

This examples motivates a general question of existence. Let us fix some notations. Write  $\frac{p}{q} \in \mathbb{Q}$  where  $p \in \mathbb{Z}$ ,  $q \in \mathbb{N}^*$ ,  $p \wedge q = 1$ . Let  $\mathbf{P}$  be the set of functions  $f$  satisfying the following properties:

- $f : U \rightarrow \mathbb{C}$  is holomorphic on an open neighborhood of  $0 \in \mathbb{C}$

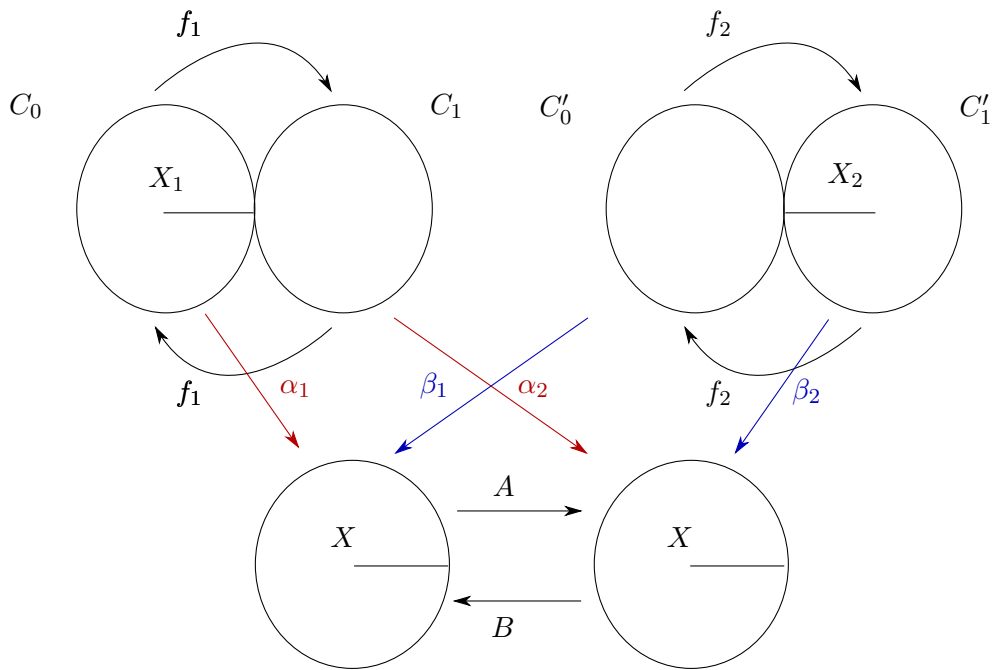


Figure 13: Pseudo-conjugate maps illustration

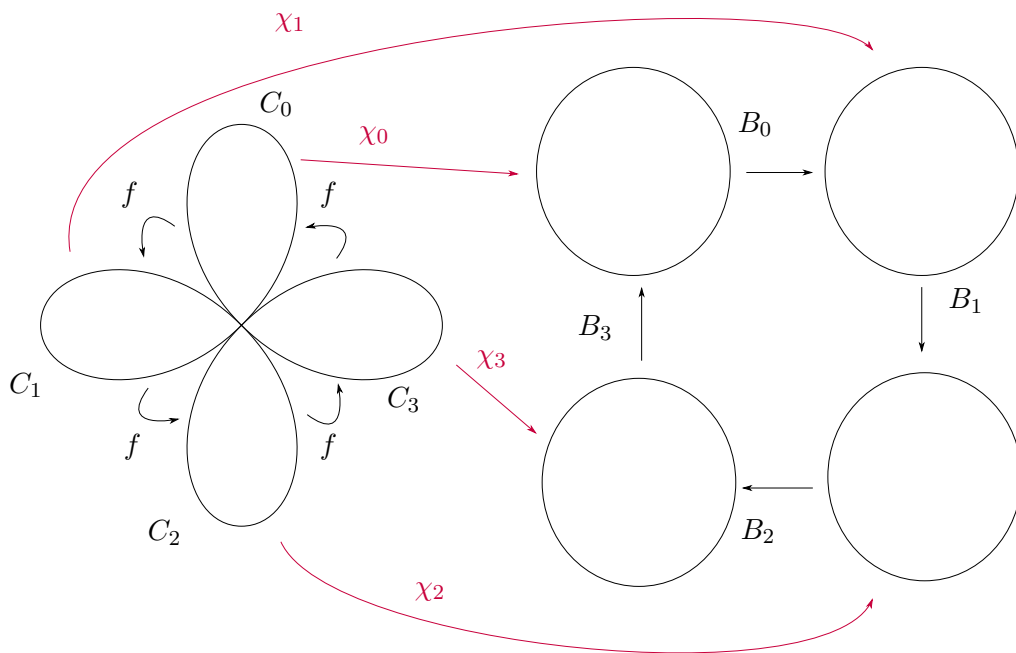


Figure 14: Blaschke products

- $f$  has a parabolic point at 0 of multiplier  $e^{2i\pi p/q}$  with a cycle of attracting axis, i.e. a single cycle  $B_0^f = C = C_0 \cup \dots \cup C_{q-1}$  of connected components of the immediate parabolic basin, with  $f(C_i) = C_{i+1}$  (the index  $i$  being defined modulo  $q$ ).
- Each of the  $C_i$  is simply connected.

**Question 4.6.5.** (see figure 14 for an illustration where  $q = 4$ ) Let  $B_0, \dots, B_{q-1}$  be Blaschke products from  $\mathbb{D}$  to  $\mathbb{D}$ . On what condition does there exist  $f \in \mathbf{P}$  and biholomorphisms  $\chi_i : C_i \rightarrow \mathbb{D}$  such that the following diagram commutes:

$$\begin{array}{ccccccc}
 C_0 & \xrightarrow{f} & C_1 & \xrightarrow{f} & \dots & \xrightarrow{f} & C_q \\
 \chi_0 \downarrow & & \chi_1 \downarrow & & & & \chi_q \downarrow \\
 \mathbb{D} & \xrightarrow{B_0} & \mathbb{D} & \xrightarrow{B_1} & \dots & \xrightarrow{B_{q-1}} & \mathbb{D}
 \end{array}$$

where we recall that  $C_q = C_0$  ?

Can it be further imposed that some specific  $\partial C_i$  are locally connected, and the others non locally connected ?

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## Abstract

Douady, Lavaurs, A. Epstein, Shishikura, Lanford III, Yampolsky and others have discovered a class of functions which is invariant by parabolic renormalisation. This class is characterized by covering properties, contains the polynomials of degree 2 and enabled among other things to progress in the study of their dynamics. To date, the extension of these results for all polynomials of degree 3 has not been achieved, nevertheless there exist some elements to develop such a theory.

In this thesis, we start the study of these elements. We propose to extend the definition of this class of functions to include the polynomials of degree 3. This extension separates naturally, in degree 3, in 3 types of classes.

The slice  $\text{Per}_1(\rho)$  is by definition the set of polynomials of degree 3 which fixes a point  $z_0$  with multiplier  $\rho$ ; the polynomials are here taken up to a conjugacy by an affine map where  $z_0$  is fixed. This slice is naturally parametrized by a complex number  $a$ . In part 2, we study the external rays which can land at the fixed parabolic point.

Since the candidates to be invariant by parabolic renormalisation are defined by covering properties, we study in the fourth part the relations between a notion of covering equivalence of horn maps of certain parabolic points and a notion of conjugacy associated to these parabolic points.

## Résumé

Douady, Lavaurs, A. Epstein, Shishikura, Lanford III, Yampolsky et d'autres ont mis à jour une classe de fonctions qui est invariante par renormalisation parabolique. Cette classe est caractérisée par des propriétés de revêtement, contient les polynômes de degré deux, et a permis notamment de faire des progrès dans l'étude de leur dynamique. A ce jour l'extension de ces résultats à tous les polynômes de degré 3 n'a pas été réalisée de façon satisfaisante, mais des pistes existent pour développer une telle théorie.

Dans cette thèse, nous commençons l'étude d'une telle piste. Nous proposons d'étendre la définition de cette classe de fonctions pour inclure les polynômes de degré 3. Cette extension se scinde naturellement, en degré 3, en trois types de classes.

La tranche  $\text{Per}_1(\rho)$  est par définition l'ensemble des polynômes de degré 3 qui fixent un point  $z_0$  avec multiplicateur  $\rho$ , polynômes pris à conjugaison près par une application affine où  $z_0$  est fixé. Il se trouve que cette tranche est naturellement paramétrée par un nombre complexe  $a$ . En partie 2, nous étudions les rayons externes qui peuvent aboutir au point fixe parabolique.

Les candidats à être invariant par renormalisation parabolique étant définis par des propriétés de revêtement, nous étudions en quatrième partie les relations entre une notion d'équivalence de revêtement des applications de cornes de certains points paraboliques et une notion de conjugaison associée à ces points paraboliques.