



# Combinatorics and computations: Cartan matrices of monoids & minimal elements of Shi arrangements

Balthazar Charles

## ► To cite this version:

Balthazar Charles. Combinatorics and computations: Cartan matrices of monoids & minimal elements of Shi arrangements. Combinatorics [math.CO]. Université Paris-Saclay, 2023. English. NNT : 2023UPASG063 . tel-04356646

**HAL Id: tel-04356646**

**<https://theses.hal.science/tel-04356646>**

Submitted on 20 Dec 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Combinatorics and Computations:  
Cartan matrices of monoids  
& Minimal elements of Shi arrangements.  
*Combinatoire et Calcul : Matrice de Cartan des monoïdes  
& Éléments minimaux des arrangements de Shi*

**Thèse de doctorat de l'université Paris-Saclay**

École doctorale n° 580, Sciences et Technologies de l'Information de la  
Communication (STIC)

Spécialité de doctorat: Informatique mathématique  
Graduate School : Informatique et Sciences du Numérique,  
Réfèrent : Faculté des Sciences d'Orsay

Thèse préparée dans l'unité de recherche **LISN (Université Paris-Saclay,  
CNRS)**, sous la direction de **Nicolas THIÉRY**.

**Thèse soutenue à Paris-Saclay, le 18 octobre 2023, par**

**Balthazar CHARLES**

**Composition du jury**

Membres du jury avec voix délibérative

**Vincent PILAUD**

Directeur de recherche, CNRS, LIX

**Matthew DYER**

Professeur, University of Notre-Dame

**James D. MITCHELL**

Professeur, University of Saint-Andrews

**Lila BOUKHATEM**

Maitresse de conférence, Université Paris-Saclay

**Susanna FISHEL**

Professeure Associée, Arizona State University

**Victor REINER**

Professeur, University of Minnesota

**Benjamin STEINBERG**

Professeur, City University of New-York

Président

Rapporteur & Examineur

Rapporteur & Examineur

Examinatrice

Examinatrice

Examineur

Examineur

**Titre :** Combinatoire et Calcul : Matrice de Cartan des monoïdes & Éléments minimaux des arrangements de Shi.

**Mots clés :** Algorithmique et combinatoire des monoïdes, Groupes de Coxeter, Théorie des représentations combinatoire et effective, Arrangements d'hyperplans, Algèbre effective, Exploration Informatique.

**Résumé :** Cette thèse présente le résultat de recherches sur deux thèmes combinatoires distincts: le calcul effectif des matrices de Cartan en théorie des représentations des monoïdes et l'exploration des propriétés des éléments minimaux dans les arrangements de Shi des groupes de Coxeter. Bien que disparates, ces deux domaines de recherche partagent l'utilisation de méthodes combinatoires et d'exploration informatique, soit en tant que fin en soi pour le premier domaine, soit comme aide à la recherche pour le second.

Dans la première partie de la thèse, nous développons des méthodes pour le calcul effectif des tables de caractères et des matrices de Cartan dans la théorie des représentations des monoïdes. À cette fin, nous présentons un algorithme basé sur nos résultats pour le calcul efficace des points fixes sous une action similaire à une conjugaison, dans le but de mettre en œuvre la formule de [Thiéry '12] pour la matrice de Cartan. Après une introduction largement auto-contenue aux notions nécessaires, nous présentons nos résultats sur le comptage des points fixes, ainsi qu'une nouvelle formule pour la table de caractères des monoïdes finis. Nous évaluons les performances des algorithmes résultants en termes de temps d'exécution et d'utilisation mémoire. Nous observons qu'ils sont plus efficaces

par plusieurs ordres de grandeur que les algorithmes non spécialisés pour les monoïdes. Nous espérons que l'implémentation (publique) résultant de ces travaux contribuera à la communauté des représentations des monoïdes en permettant des calculs auparavant difficiles.

La deuxième partie de la thèse se concentre sur les propriétés des éléments minimaux dans les arrangements de Shi. Les arrangements de Shi ont été introduits dans [Shi '87] et sont l'objet de la Conjecture 2 dans [Dyer, Hohlweg '14]. Initialement motivés par cette conjecture, après une introduction aux notions nécessaires, nous présentons deux résultats. Premièrement, une démonstration directe dans le cas des groupes de rang 3. Deuxièmement, dans le cas particulier des groupes de Weyl, nous donnons une description des éléments minimaux des régions de Shi en étendant une bijection issue de [Athanasiadis, Linusson '99] et [Armstrong, Reiner, Rhoades '15] entre les fonctions de parking et les régions de Shi permettant d'effectuer le calcul pratique des éléments minimaux. Comme application, à partir des propriétés de ce calcul, nous donnons une démonstration de la conjecture pour les groupes de Weyl indépendante de leur classification. Ces résultats révèlent une interaction intrigante entre les partitions non-croisées et non-embrassées dans le cas des groupes de Weyl classiques.

**Title:** Combinatorics and Computations: Cartan matrices of monoids & Minimal elements of Shi arrangements.

**Keywords:** Algorithmic and combinatorics of monoids, Coxeter groups, Combinatorial Representation Theory, Hyperplanes arrangements, Computational Algebra, Computer Exploration.

**Abstract:** This thesis presents an investigation into two distinct combinatorial subjects: the effective computation of Cartan matrices in monoid representation theory and the exploration of properties of minimal elements in Shi arrangements of Coxeter groups. Although disparate, both of these research focuses share a commonality in the utilization of combinatorial methods and computer exploration either as an end in itself for the former or as an aid to research for the latter.

In the first part of the dissertation, we develop methods for the effective computation of character tables and Cartan matrices in monoid representation theory. To this end, we present an algorithm based on our results for the efficient computations of fixed points under a conjugacy-like action, with the goal to implement Thiéry's formula for the Cartan matrix from [Thiéry '12]. After a largely self-contained introduction to the necessary background, we present our results for fixed-point counting, as well as a new formula for the character tables of finite monoids. We evaluate the performance of the resulting algorithms in terms of execution time and memory usage and find that they are more efficient than

algorithms not specialized for monoids by orders of magnitude. We hope that the resulting (public) implementation will contribute to the monoid representation community by allowing previously impractical computations.

The second part of the thesis focuses on the properties of minimal elements in Shi arrangements. The Shi arrangements were introduced in [Shi '87] and are the object of Conjecture 2 from [Dyer, Hohlweg '14]. Originally motivated by this conjecture, we present two results. Firstly, a direct proof in the case of rank 3 groups. Secondly, in the special case of Weyl groups, we give a description of the minimal elements of the Shi regions by extending a bijection from [Athanasiadis, Linusson '99] and [Armstrong, Reiner, Rhoades '15] between parking functions and Shi regions. This allows for the effective computation of the minimal elements. From the properties of this computation, we provide a type-free proof of the conjecture in Weyl groups as an application. These results reveal an intriguing interplay between the non-nesting and non-crossing worlds in the case of classical Weyl groups.



# Remerciements

Merci à Nicolas Thiéry pour avoir été mon directeur de thèse, pour les nombreuses opportunités qu'il m'a offertes, tant dans la recherche et dans l'enseignement qu'humainement, par les rencontres qu'il m'a permis de faire. Merci aussi pour la grande liberté qu'il m'a laissée au cours de ces 4 ans et la compréhension qu'il a manifestée dans les moments difficiles.

Merci aux rapporteurs, Pr. Matthew J. Dyer et Pr. James D. Mitchell, pour leur relecture attentive et leurs conseils avisés qui ont grandement contribué à la qualité de ce manuscrit.

Merci aux membres du jury, Dr. Lila Boukhatem, Pr. Susanna Fishel, Dr. Vincent Pilaud, Pr. Vic Reiner, Pr. Benjamin Steinberg, auxquels se joignent mes rapporteurs et mon directeur de thèse pour avoir accepté de donner leur temps et d'accorder leur attention à mes travaux.

Merci aux membres des équipes pédagogiques avec lesquels j'ai eu le plaisir de travailler : Thibault Balabonski, Joël Falcou, Giancarlo Fissore, Thomas Lavergne, François Pirot, Fanny Pouyet, Viviane Pons, Adeline Pierrot et, à nouveau, Nicolas Thiéry. Merci également aux étudiants pour lesquels enseigner a été un plaisir.

Merci au reste de l'équipe GALaC que je n'ai pas encore mentionné, et en particulier Nathalie Aubrun, Benjamin Héllouin de Méribus et Florent Hivert pour toute l'aide qu'ils ont pu m'apporter.

Merci à tous ceux qui m'ont permis de discuter de mes sujets de recherches, et en particulier merci à Antoine Abrams, Dr. Nathan Chapelier-Laget, Pr. Priyavrat Deshpande, Krishna Menon et au Pr. Pascal Weyl.

Merci à Daniel Tamayo-Jimenez, Hugo Mlodecki et Agustín Borgna pour avoir été de grands amis et collègues de thèse, dont le soutien m'a été essentiel durant la pandémie. Merci aux co-doctorants et amis que j'ai appris à connaître en revenant au LRI après les confinements : Doriann Albertin, Noémie Cartier, Loïc Le Mogne, Chiara Mantovani, Eva Philippe, Germain Poullot

et autres habitués de DGeCo qui ont revivifié mon goût pour la recherche. Merci également à Pierre Béaur, Nicolás Bitar, Valentin Dardilhac et Jonas Sénizergues pour les bons moments passés aux LRI.

Merci aux amis qui ont passé un moment chez nous, à Massy : Itzel Andrea Castro González, Clément Chenevière, Justine Falque, Léo Gayral, Quentin Gendre, Pauline Hubert.

Merci à Maria Abad Aldonza, Jorge Bretonnes Santamarina, Carlo Buccisano, Lisa Ding, Sarah Fajon, Rémi Goudey, Maud Le Bouedec, David Llerena, Sofía Lopez Ortiz, Katherine Morales pour les jeux, les soirées, les repas...

Merci à Maxime Plançon et Gunjin Udval pour leur amitié.

Merci à toute ma famille pour son soutien sous toutes ses formes. Merci Philippe et Christelle, Guillaume, Aurore pour leurs nombreuses invitations durant ma thèse et avant. Merci à Grand-mère Régine pour son accueil, non seulement de moi, mais aussi de mes amis. Merci à Ma Tante Françoise, à Manolo, à Geneviève. Merci à Hippolyte et Esther, Catherine et Clément. Merci à Papa et Maman pour plus de choses que je ne pourrais en énumérer.

Enfin, plus de gratitude que je ne pourrais le dire en quelques mots va à Monica pour tout et le reste.

# Contents

General Introduction . . . . .	9
French summary . . . . .	19
<b>I Computational monoid representation theory</b>	<b>29</b>
Introduction . . . . .	31
<b>1 On the structure of finite monoids</b>	<b>33</b>
1.1 Generalities . . . . .	33
1.2 Green structure of finite monoids and Schützenberger groups .	40
1.2.1 Green Structure . . . . .	40
1.2.2 Schützenberger groups . . . . .	44
1.3 Regularity and multiplication . . . . .	46
<b>2 Representation theory</b>	<b>49</b>
2.1 Modules . . . . .	49
2.1.1 Basic definitions . . . . .	49
2.1.2 Building blocks . . . . .	53
2.1.3 From projective to simple, and back. . . . .	59
2.2 Characters . . . . .	64
2.3 Cartan Matrix . . . . .	66
2.4 Clifford-Munn-Ponizovskii . . . . .	68



<b>3</b>	<b>Computing representation invariants</b>	<b>71</b>
3.1	A formula for the character table . . . . .	71
3.2	Fixed points counting . . . . .	76
3.3	Algorithms . . . . .	79
3.3.1	Computational hypotheses . . . . .	80
3.3.2	Fixed points . . . . .	82
3.3.3	Computing the radical . . . . .	84
3.3.4	Performance . . . . .	86
<b>II</b>	<b>Minimal elements of Shi arrangements</b>	<b>101</b>
	Introduction . . . . .	103
<b>4</b>	<b>Preliminaries on Coxeter theory</b>	<b>105</b>
4.1	Coxeter groups . . . . .	105
4.2	Geometric realization . . . . .	108
4.3	Inversion sets . . . . .	114
4.4	Projective picture . . . . .	117
4.5	(Af)Finiteness and Weyl groups . . . . .	120
4.6	Dual objects: Tits cone, Coxeter complex . . . . .	127
4.7	Small roots and Shi arrangement . . . . .	131
4.7.1	Arbitrary Coxeter group . . . . .	131
4.7.2	Weyl group . . . . .	133
<b>5</b>	<b>Minimal elements of Shi regions in Weyl groups</b>	<b>137</b>
5.1	Structure of the Shi relations . . . . .	138
5.2	Describing the minimal elements . . . . .	142
5.2.1	The Shi encoding . . . . .	142
5.2.2	The parking function bijection . . . . .	143
5.2.3	Minimal elements of Shi regions . . . . .	148
5.3	Specialization to classical groups . . . . .	154

<b>6</b>	<b>Low elements</b>	<b>161</b>
6.1	In rank three . . . . .	161
6.1.1	Wiggling hyperplanes . . . . .	162
6.1.2	$G_{bip}(w)$ is acyclic . . . . .	163
6.1.3	The sources of $G_{bip}(w)$ lie in $\Gamma_w$ . . . . .	166
6.2	In affine Weyl groups . . . . .	167
6.2.1	The dominant region . . . . .	169
6.2.2	General case . . . . .	170



# List of Figures

1.1	The fry pan diagram. . . . .	35
1.2	Green's relations in $\mathcal{T}_3$ . . . . .	43
1.3	Green's Lemma . . . . .	44
1.4	Location Theorem . . . . .	47
1.5	Illustration of Proposition 1.3.7. . . . .	48
3.1	Cartan Matrix of $\mathcal{T}_7$ . . . . .	92
3.2	Time and memory usage: transformation monoids . . . . .	94
3.3	Time and memory usage: partition monoids . . . . .	95
3.4	Time and memory usage: partial permutation monoids . . . . .	96
3.5	Cartan matrix of $\mathcal{T}_{12}$ . . . . .	98
4.1	Coxeter graph of $\tilde{G}_2$ . . . . .	106
4.2	The inversion set of $sts$ in $D_4$ . . . . .	111
4.3	The projective picture for the dihedral group $D_4$ . . . . .	118
4.4	Projective picture of the root system of $\langle S,  S  = 5 \mid \forall s, t \in S, (st)^7 \rangle$ . . . . .	118
4.5	Inversion polytope of $s_4s_1s_3s_2s_4$ in $\tilde{A}_3$ . . . . .	120
4.6	The irreducible crystallographic root systems of rank 2. . . . .	122
4.7	Affine arrangement and affine roots in $\tilde{A}_1$ . . . . .	126
4.8	Duality between projective picture and Coxeter complex. . . . .	129
5.1	The positive roots of the rank 2 crystallographic root systems. . . . .	137
5.2	"A skirt of a skirt gives a pyramid". . . . .	141

5.3	The Hasse diagram on the Shi relations for $A_5$ . . . . .	142
5.4	Repeated application of Proposition 5.2.9 . . . . .	153
5.5	Illustration of Proposition 5.3.2. . . . .	155
5.6	Examples of root posets of type $B_n$ and $C_n$ . . . . .	157
5.7	Root poset of type $D_4$ . . . . .	158
5.8	Idea of proof for Proposition 5.3.4. . . . .	159
6.1	Wiggling hyperplanes. . . . .	163
6.2	A removal order follows the arrows. . . . .	164
6.3	The 4 possible “bad” cases. . . . .	172

# General introduction

## Context

Algebraic combinatorics, as a field, is concerned with counting using algebraic tools and, in turn, using counting to prove results about these algebraic objects. For instance, being able to count the number of intervals of a partially ordered set (poset) is an interesting question in itself, and in turn, being able to do it can be used to prove that two posets not are isomorphic.

This thesis, although it is split into two independent parts, is entirely contained within the field of algebraic combinatorics. This manifests most obviously in the first part when we want to count fixed points, and in the second, when we count arcs, before using this counting argument to describe certain geometric objects.

Another uniting feature of the two subjects discussed here is computation. In the first part, the results we obtain were motivated by the “need” to algorithmically compute efficiently certain algebraic invariants. Thus, we give special attention to presenting effective algorithms and measuring their performance. In the second part, many of the results we obtain were first guessed from computer experiments, and although it does not remain in the version of the proof presented hereafter, at least one result was initially separated into a formal proof in the “classical” cases supplemented by a computer proof in the “exceptional” cases. Because this research is either based or aimed at algorithmic exploration, we would like to include the publicly available code<sup>1</sup> as a contribution of this PhD thesis of the same level, if narrower scope, as this manuscript.

---

<sup>1</sup>[github.com/ZoltanCoccyx/monoid-character-table](https://github.com/ZoltanCoccyx/monoid-character-table)

## Monoid representation theory

The work of this thesis on finite monoid representation theory is in the direct continuation of the work from Thiéry [Thi12]. This makes it a part of a renewed dynamic in the domain starting in the 90s but reaching full intensity over the last two decades, initially motivated by questions revolving around the random generation of “typical” combinatorial objects. Since then, it has taken on a life of its own, with notably the publication of the book *Representation Theory of Finite Monoids* by B. Steinberg [Ste16] and the creation of computer algebra tools, mainly [Mit+23] by Mitchell & *al.*

Despite this newfound interest, the roots of the field are quite a bit older, with the first significant results coming from the 40s and 50s, with Clifford [Cli42], Munn [Mun57], and Ponizovskii [Pon58]. A major result of this impetus is what is now called the Clifford-Munn-Ponizovskii Theorem that describes the irreducible representations of finite monoids in terms of the representation of certain subgroups. The study of the *full transformation monoid* was tackled in [HZ57]. The character theory of semigroups was further developed by the independent works from McAlister [McA72] and Rhodes and Zalcstein [RZ91] (which first appeared in the early 70s, although it was only formally published in 1991). However, monoid algebras not being semisimple (in general) has proven, at first, to be an obstacle to their study and to the applicability of the theory to other domains. With the development of a more powerful theory of representation on non-semisimple algebras (for which we mostly reference the very complete [CR66] as well as the more approachable [Ass97], [Eti+09]), the field was, in the 90s, ripe for a new blossom.

Arising from the work of Putcha [Put88] (among others) on linear algebraic monoids, questions about the representation theory of matrix monoids over a finite field began to find some answers ([OP91], [Kov92], [Put99]). The study of Markov chains, in the research from Diaconis and Brown ([BD98], [Bro00]), beginning in the late 90s-early 2000s was another strong motivator for the development of the theory, as it provided applications for probability theorists ([Bro04], [Sal09]) and combinatorialists ([Sch06], [Sal07], [HT09], [HST13]). Out of this movement powered by applications, also emerged the general study of finite monoid representation theory, with for instance [GMS09], [Thi12], [Ste16], [Ste22].

As pointed out by Steinberg in the introduction of [Ste16], “examples have motivated much of the theory”. In turn, by providing techniques to compute examples explicitly, we hope to contribute to the development of the theory. With the recent apparition of mature computer algebra tools for semigroup theory, we believe that the time is ripe for a new wave of combinatorial results based on the computer exploration of large classes of monoids.

## Minimal elements of Shi arrangements

This part of the thesis is situated in the larger context of Coxeter group theory. This theory is multifaceted, with possible angles of attack purely from group theory and language theory, but also from geometry and combinatorics. We are mainly concerned with questions regarding geometric aspects of Coxeter groups, and more specifically, with the questions about root system, the associate Coxeter hyperplanes arrangement and the special subarrangement called the Shi arrangement.

Coxeter groups were first introduced by Coxeter in [Cox34] as an abstract generalization of groups generated by Euclidean orthogonal reflections. The finite groups among them were classified, still by Coxeter, in [Cox35]. Among those finite groups are the *Weyl groups*, whose root systems classify important objects such as semisimple Lie groups [Che48]. This has been a major motivation for the study of Coxeter groups, which have given rise to a rich literature, with the basic theory collected for instance in [Hum90] and [BB05]. Two interesting developments of the theory have been the discovery of automata that recognize the *language of reduced words* of a general Coxeter group by [BH93], as well as the notion of *Shi arrangement*, introduced by Shi in [Shi87a] only for Weyl groups to tackle Kazhdan-Lusztig theory questions. Surprisingly, the two are intimately related, as noted in [BB05, p. 124].

Although not originally part of the thesis project, the resurgence of this subject was kick-started by discussions with Dr. Nathan Chapelier-Laget who, at the time, studied questions related to [DH16, Conjecture 2] from Dyer & Hohlweg, which has since been proven in [Dye+23]. Firstly, we present related inquiries about the Shi arrangement in the case of Weyl groups. In this special situation, the arrangement is highly structured (see [Fis19] for a survey), and several bijections with interesting combinatorial (parking function, Catalan objects) objects with Shi regions (or special subsets of regions) are known. In this particular context, we examine a bijection from Athanasiadis & Linusson (in [AL99]), extended by Armstrong, Reiner & Rhoades (in [ARR15]). This part of the thesis is “at home” in the context of hyperplane arrangements, as one of the possible perspectives of this work would be to extend the description of the minimal element to other arrangements. Secondly, we return closer to the original motivating conjecture. We begin with a proof in the case of rank 3 general Coxeter groups. The original idea for it was originally conceived during a research internship under the supervision of Pr. Hohlweg at LaCIM in 2017. However, the original proof was flawed to the point of being inapplicable. Under the impulsion of Nathan Chapelier-Laget, the proofs were rewritten during this PhD thesis, with notably the introduction of the bipodality graph. We then follow up by using the previously obtained description of the minimal elements to give a new answer to the question in the case Weyl groups.



## Contributions

### Computational monoid representation theory

The aim of our research in this area was to effectively use Thiéry's formula for the Cartan matrix of a monoid (see Proposition 2.3.2), that links the Cartan invariant matrix of a monoid algebra to the character table of the monoid, and the number of fixed points in the monoid under simultaneous left and right translations. Computationally, these objects are hard to obtain. To the best of our knowledge, we provide the first example of an algorithm for computing the character table of a general finite monoid (over  $\mathbb{C}$ ). Our performance tests indicate that it can reasonably be applied to monoids of cardinality up to  $10^5$  (and pushed to  $10^6$  in some special cases of interest). This algorithm is based on the following formula that we prove in the text:

**Proposition** (3.1.6 in the text). *We have the formula for the lines of the character table of the monoid  $M$  corresponding to simple modules of apex an idempotent  $e$ , denoted  $X_e(M)$ :*

$$X_e(M) = {}^t X(G_e)^{-1} \cdot \left( \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{\mathbf{S}}(m, g) - \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{N_e(\mathbf{S})}(m, g) \right)_{g \in C_{G_e}, m \in C_M}$$

where the dot is the matrix product and  $X(G_e)$  is the character table of the maximal group at  $e$ .

In addition, we partially reduce the question of counting the fixed points we are interested in to the group theoretic problem of computing the cardinality of a centralizer. This is done by the following proposition:

**Proposition** (3.2.4 in the text). *Let  $H$  be an  $\mathcal{H}$ -class,  $a \in H$ ,  $\Gamma'(H)$  be the right Schützenberger group of  $H$  and  $h \in {}_M \text{Stab}(\mathcal{R}(H))$ ,  $k \in \text{Stab}_M(\mathcal{L}(H))$ . Then*

$$|\text{Fix}_H(h, k)| = \begin{cases} |C_{\Gamma'(H)}(\times_H k)| & \text{if } \tau_a(h \times_H)^{-1} \in \overline{\times_H k} \\ 0 & \text{otherwise} \end{cases}$$

where  $\overline{\times_H k}$  is the conjugacy class of  $\times_H k$  in  $\Gamma'(H)$  and  $C_{\Gamma'(H)}(\times_H k)$  is the centralizer in  $\Gamma'(H)$  of  $\times_H k$ .

Based on this reduction, we describe an algorithm counting the number of fixed points under  $s \mapsto hsk$  (for relevant  $h, k$ ) that vastly outperforms the naive algorithm in monoid with “not too simple  $\mathcal{J}$ -structure”, allowing to tackle monoids of cardinality up to  $10^8$ . In the case where the monoid is  $\mathcal{J}$ -trivial (or close to be), our method, although worse in practice, is only slower by a (experimentally) constant factor.

## Minimal elements of Shi arrangements

In order to understand the relations describing the minimal elements of the Shi regions, we introduce the notion of *skirt* of a root.

**Definition** (5.1.2 in the text). The *skirt* of a positive root  $\gamma$  is the set

$$\text{Sk } \gamma = \{\alpha \in \Phi^+ \mid \gamma - \alpha \in \Phi^+\}.$$

The relation “being in the skirt of” is *almost* a total order (Propositions 5.1.3 and 5.1.4). This, together with the results from Athanasiadis & Linusson (see Theorem 5.2.4) Armstrong, Reiner & Rhoades (see Theorem 5.2.7) allows to locally propagate information about which hyperplanes bound a given Shi region, giving the following result in the *dominant chamber*.

**Proposition** (5.2.9 in the text). *Let  $A$  be an antichain of the root poset. Then the following vector is a Shi vector:*

$$\forall a \in \Phi^+, v_a = \max(\{\mathbb{1}_{a \in A}\} \cup \{v_b + v_{a-b} \mid b \in \text{Sk}(a)\}).$$

We can then transport this information from the dominant chamber to every other chamber to obtain a general description of the minimal elements of Shi regions in terms of their *Shi vectors* that encode their *inversion set*. Moreover, in the case of classical groups, our result gives a nice combinatorial description of the coefficients of its Shi vector as the number of non-crossing arcs in a non-nesting arc diagram associated to the Shi region. This puts those minimal elements at the intersection of non-crossing and non-nesting combinatorics in the classical cases. Because we get that combinatorial data on what is essentially a set of roots, we can deduce some necessary condition on a root to be a “vertex” of the inversion set of an element. We use this condition to provide a new answer, independent of [CLH22] and [Dye+23], to [DH16, Conjecture 2]. As this question can be formulated in terms of the vertices of the inversion sets, we are able to deduce the case of the affine groups as a consequence of the description of minimal elements we provide. This constitutes the first part of the following result.

**Proposition** (6.1.1 and 6.2.4 in the text). *The map associating to a low element the set of its small inversions is a surjection from the low element onto the small inversion sets in affine Weyl groups and rank 3 Coxeter groups.*

The case of rank 3 groups is an independent proof, since superseded by a complete general proof in [Dye+23]. We include it, however, as we think that there are interesting questions surrounding the proof, in particular regarding the notion of *bipodality graph*.

## Content and plan of the thesis

### Chapter I.1 – On the structure of finite monoids

In this chapter, we begin by describing the general theory of monoids, with the general definitions of monoid and semigroups, substructures and quotients, and special elements (in particular idempotents) (§1.1). It turns out that the associativity of the product in a monoid is actually a strong condition, that endows monoids with a *Green's classes structures*, which we discuss in a second section (§1.2). The major result of this last section is *Green's Lemma* (Lemma 1.2.7) that expresses the constrained relations between the different types of Green's classes (§1.2.1). We also discuss the notion of *Schützenberger groups*, which are groups that arise naturally as the monoid acts on parts of itself (§1.2.2). Finally, we examine the structure of multiplication in a single  $\mathcal{J}$ -class with the *Location Theorem* 1.3.2 (§1.3).

### Chapter I.2 – Representation theory of monoids

In this second chapter of background material, we describe the representation theory of finite monoids. We begin by presenting the language of modules over an algebra, with a particular attention given to simple and projective modules as they are central to the definition of the two invariants we are after (§2.1): the character table, discussed in the following section (§2.2) and the Cartan matrix, presented in §2.3 with the formula from Thiéry that motivates our study (Proposition 2.3.2). Then, in the last section of this chapter (§2.4) we state the *Clifford-Munn-Ponizovskii Theorem*, which plays a vital role in the proofs of the following chapter.

### Chapter I.3 – Computing representation invariants

Here, we present the results of this part of the thesis. At first, we prove a formula to compute the character table of a monoid (Proposition 3.1.6) in relation to group character tables, fixed points, and characters of the *radical* of (some) regular Green's  $\mathcal{L}$ -classes. We conclude by briefly discussing the modular case (§3.1). Then, we describe how to compute the number of fixed points in a monoid under the transformation  $s \mapsto hsk$ , and relate it to a group theoretic question on the Schützenberger group of a given  $\mathcal{J}$ -class (§3.2). In a final section (§3.3), we consider the practical applicability of the previous results. We provide explicit algorithms for counting the fixed points of interest and an equation system for the radical that is “optimized” using consequences of Green's Lemma.

## Chapter II.4 – Preliminaries on Coxeter theory

In this chapter, we give general background on Coxeter groups (§4.1) and related geometric objects: their geometric realization and root system (§4.2), inversion sets (§4.3) and projective picture (§4.4). As we use a geometric point of view, we reproduce certain classical proofs in that language as a way to warm up to the geometric considerations of the next chapters. Before this, we discuss the classification of finite Coxeter groups (Proposition 4.5.3), crystallographic root systems (Proposition 4.5.5) and affine Weyl groups (Proposition 4.5.6). Finally, we go to the dual point of view, with the Tits cone and Coxeter arrangement (§4.6) and the main object of our study: the Shi arrangement (§4.7). We discuss it in all generality in §4.7.1 and specialize it in affine Weyl groups (§4.7.2).

## Chapter II.5 – Minimal elements of Shi regions in Weyl groups

This chapter presents the results of our investigation into the minimal element of Shi regions in Weyl groups. First, as these minimal elements are constrained by the *Shi relations*, we study their structure in §5.1 to give ourselves some tools for the next section. It is in this section that we introduce the notion of *skirt* of a root (Definition 5.1.2) and investigate its properties. We then describe these minimal elements: we do so through their *Shi encoding* (§5.2.1), by extending a bijection with parking functions (§5.2.2) to give in Proposition 5.2.12 a description of those elements (§5.2.3). In the remaining section (§5.3), we discuss the specialization of the previous result to classical Weyl groups. As it turns out, the Shi encoding that counts the number of hyperplanes behind which is located a region can be computed in terms of non-crossing non-nesting arcs (Proposition 5.3.3).

## Chapter II.6 – Low elements

We go back, in this final chapter, to the motivation for our Coxeter groups theory research. The first section presents a complete proof of the results from [Cha21] for rank 3 groups. The result is obtained in three steps: we first consider the interaction between the primal (root system) and dual (Tits cone), to be able to recognize *descents* of an element (§6.1.1). We then prove that the *bipodality graph* is acyclic (§6.1.2) and that its sources lie in a convenient subset (§6.1.3). In a second time, the final section is focused on Weyl groups: from the combinatorial description of the Shi encoding of minimal element, we get a meaningful enough description of the inversion sets to get a necessary

condition for roots to be “vertices” of it. We proceed by showing this result in the dominant chamber (§6.2.1) before transporting it to other chambers (§6.2.2).

## Reading advice

The intended audience of this document, somewhat egotistically, is the first year PhD student I was. As such, it is written to be accessible to a “generalist” master student in algebra, with an interest in effective algebra. In a less self-centered way, as the two subjects of this thesis are somewhat disjointed, we hope to have made it accessible to readers that may be used to one of the parts, but not necessarily the other. That being said, the two parts are entirely independent, and the reader may skip one of them without losing any information.

With that in mind, we assume familiarity with basic notions of group theory, linear, bilinear and tensorial algebra, as well as group representation theory presented in a “classical” fashion. We will not use major results without citing them, even in these familiar fields, but we will be using them (especially group theory) as motivating examples and guides as to what to expect. Conversely, we do not suppose familiarity with semigroup and semigroup theory, general algebra representation theory (and in particular the notion of projective modules), monoid representation theory, and Coxeter group theory. Thus, Chapter 1, 2 and 4 aim to provide this familiarity, and the specialist reader should probably skip them.

On the contrary, we hope that the non-specialist reader will find a mostly self-contained introduction to the notions we use. We put special emphasis on the basic definitions and facts of each field. When it comes to more general or powerful results, we have tried not to stray too far away from the main text, and to provide references to the literature, provided it would not impact the understanding of the important points of the thesis. Perhaps the best example of this is the emphasis put on projective modules. We provide proofs of the needed elementary results, specialized to our context, with the added intention for those results to be used as “sanity checks” for the verification of the programs accompanying this document. However, with Proposition 2.1.55, we only provide a reference as this is a somewhat higher level result, and the proof would not be very enlightening in our context.

Should this approach seem too cavalier, we recommend to the reader to consult the following works for a more thorough introduction to the subjects:

- *Mathematical Foundations of Automata Theory* [Pin] by Pin for general semigroup theory,

- *Representation theory of finite groups: a guidebook* [Cra] by Craven for a beginner-friendly introduction to representation theory of finite groups,
- *Introduction to representation theory* [Eti+09] by Etingof et al. for representation theory of finite groups as well, but also for a gentle introduction to algebra representation theory,
- for the reader already familiar with the language of module and categories, *Representation Theory of Finite Monoids* [Ste16] treats the subject at a more advanced level, but with more completeness and precision than we ever could,
- *Reflection groups and Coxeter groups* [Hum90] by Humphreys for an introduction to Coxeter groups, and in particular many related geometric constructions such as the Tits cone and the root system,
- *Combinatorics of Coxeter Groups* [BB05] by Björner & Brenti for a combinatorial, group theory and language theory based approach to Coxeter groups.

These books have been a precious help during the preparation and redaction of this thesis, and we often refer to them.

Finally, we want to bring the attention of the reader to the examples. We have tried to provide as many relevant examples as possible, and some of them are referenced in proofs and later results. Moreover, and especially in the part regarding Coxeter groups, the examples of explicit geometric constructions provide a good intuition for the more abstract notions.



# French summary

## Contexte

La combinatoire algébrique s'intéresse à l'utilisation d'objets algébriques pour faire du comptage et à l'utilisation du comptage pour prouver des résultats sur ces mêmes objets. Par exemple, être capable de compter le nombre d'intervalles d'un ensemble partiellement ordonné (poset) est une question intéressante en soi, et à son tour, être capable de le faire peut permettre de réfuter que deux posets sont isomorphes.

Cette thèse, divisée en deux parties indépendantes, s'inscrit entièrement dans le domaine de la combinatoire algébrique. Cela se manifeste de manière la plus évidente dans la première partie lorsque l'on veut compter des points fixes, et dans la seconde, lorsque l'on compte des arcs, avant d'utiliser la façon même de les compter pour décider si un point d'un certain polytope peut en être un sommet.

L'autre caractéristique commune aux deux parties présentées ici est le calcul. Dans la première partie, les résultats théoriques présentés ont été motivés par le « besoin » de calculer algorithmiquement certains invariants algébriques. Nous accordons donc une attention particulière à la présentation d'algorithmes pratiques et efficaces ainsi qu'à la mesure de leur performance. Dans la seconde partie, l'intuition pour la plupart des résultats est issue d'expériences sur ordinateur. D'autre part, bien que nous en donnions ici une preuve entièrement théorique, un de nos résultats principaux (Proposition 5.2.13) était initialement séparé en une preuve formelle dans les cas « classiques » complétée par des calculs informatiques explicites dans les cas « exceptionnels ». Parce que cette recherche est basée sur, ou vise à, l'exploration algorithmique, nous aimerions inclure le code disponible publiquement<sup>2</sup> comme un produit de cette thèse de doctorat du même niveau que ce manuscrit.

---

<sup>2</sup>[github.com/ZoltanCoccyx/monoid-character-table](https://github.com/ZoltanCoccyx/monoid-character-table)



## Théorie des représentations des monoïdes

Les travaux de cette thèse sur la théorie des représentations des monoïdes finis s'inscrivent dans la droite ligne des travaux de Thiéry [Thi12]. Ils font donc partie d'une dynamique dans le domaine, renouvelée dans les années 1990, initialement impulsée par des questions tournant autour de la génération aléatoire d'objets combinatoires « typiques ». Depuis, le domaine a pris son essor, avec notamment la publication du livre *Representation Theory of Finite Monoids* de B. Steinberg [Ste16] et la création d'outils de calcul formel, principalement [Mit+23] de Mitchell & *al.*

Les racines du domaine sont plus anciennes, les premiers résultats significatifs datant des années 1940 et 50, avec Clifford [Cli42], Munn [Mun57], et Ponizovskii [Pon58]. Le théorème qui porte aujourd'hui leurs noms décrit les représentations irréductibles des monoïdes finis en termes de représentation de certains sous-groupes et est un des fruits majeurs de cette première période de développement. À la suite de ces études initiales, les représentations du monoïde de toutes les transformations ont été abordées dans [HZ57], puis la théorie des caractères a été développée par les travaux indépendants de McAlister [McA72] et de Rhodes et Zalcstein [RZ91] (qui est apparu pour la première fois au début des années 1970, mais n'a été formellement publié qu'en 1991). Cependant, le fait que les algèbres de monoïdes ne soient pas semisimples (en général) s'est avéré, à partir des années 1970 et jusqu'aux années 1990, un obstacle à leur étude et à l'applicabilité de la théorie à d'autres domaines. C'est avec le développement d'une théorie plus puissante des représentations des algèbres non semisimples (pour laquelle nous faisons référence au très complet [CR66] ainsi qu'aux plus abordables [Ass97], [Eti+09]), que le domaine est devenu, dans les années 1990, mûr pour une nouvelle floraison.

Suite aux travaux de Putcha [Put88] (entre autres) sur les monoïdes linéaires algébriques, les questions relatives à la théorie des représentations des monoïdes matriciels sur un corps fini ont commencé à trouver des réponses ([OP91], [Kov92], [Put99]). L'étude des chaînes de Markov, dans les recherches de Diaconis et Brown ([BD98], [Bro00]), à partir de la fin des années 1990 et du début des années 2000, a été une autre motivation importante pour le développement de la théorie, car elle a fourni des applications aux théoriciens des probabilités ([Bro04], [Sal09]) et aux combinatoriciens ([Sch06], [Sal07], [HT09], [HST13]). De ce mouvement, alimenté par les applications, est également issue nombre de travaux portants sur l'étude générale de la théorie des représentations des monoïdes finis, avec, par exemple, [GMS09], [Thi12], [Ste16], [Ste22].

Comme le souligne Steinberg dans l'introduction de [Ste16], « les exemples ont motivé une grande partie de la théorie ». À notre tour, en fournissant des techniques pour calculer explicitement des exemples, nous espérons contribuer

au développement de la théorie. Avec l'apparition récente d'outils matures de calcul formel pour la théorie des semigroupes, nous pensons que l'on peut s'attendre à une nouvelle vague de résultats combinatoires fondés sur l'exploration informatique de grandes familles de monoïdes.

## Éléments minimaux des arrangements de Shi

Cette partie de la thèse se situe dans le contexte plus large de la théorie des groupes de Coxeter. Cette théorie est multiforme, avec des angles d'attaque possibles purement en théorie des groupes et en théorie des langages, mais également en géométrie et en combinatoire. Nous nous intéressons principalement aux questions concernant les aspects géométriques des groupes de Coxeter, et plus particulièrement aux questions quant au système de racines, l'arrangement des hyperplans de Coxeter associés et un sous-arrangement spécial appelé l'arrangement de Shi.

Les groupes de Coxeter ont été introduits pour la première fois par Coxeter en [Cox34] comme généralisation abstraite des groupes générés par des réflexions euclidiennes orthogonales. Les groupes finis parmi eux ont été classés, toujours par Coxeter, dans [Cox35]. Parmi ces groupes finis se trouvent les *groupes de Weyl* dont le système racine classifie des objets importants tels que les groupes de Lie semisimples [Che48]. Cela a été une motivation majeure pour l'étude des groupes de Coxeter qui ont engendré une riche littérature, la théorie de base étant rassemblée, par exemple, dans [Hum90] et [BB05]. Deux développements intéressants de la théorie ont été la découverte d'automates qui reconnaissent le *langage des mots réduits* d'un groupe de Coxeter général par [BH93], ainsi que la notion d'*arrangement de Shi*, introduite par Shi dans [Shi87a] uniquement pour les groupes de Weyl afin d'aborder les questions de la théorie de Kazhdan-Lusztig. De manière surprenante, les deux sont intimement liés, comme indiqué dans [BB05, p. 124].

Bien que n'étant pas originellement prévu comme partie de cette thèse, la résurgence de ce sujet a été lancée par des discussions avec Nathan Chapelier-Laget qui, à l'époque, étudiait des questions connexes à la [DH16, Conjecture 2] de Dyer & Hohlweg, qui a depuis été prouvée. D'abord, nous présentons des questions relatives à l'arrangement de Shi dans le cas des groupes de Weyl. Dans cette situation particulière, de nombreux résultats sont connus (voir [Fis19] pour un aperçu), et l'arrangement est très structuré. Plusieurs bijections d'objets combinatoires intéressants (fonction de parking, objets catalans) avec des régions de Shi (ou des sous-ensembles spéciaux de régions) sont disponibles. Nous examinons en particulier une bijection d'Athanasiadis & Linusson (dans [AL99]) et étendue par Armstrong, Reiner & Rhoades (dans [ARR15]). Cette partie de la thèse est « chez elle » dans le contexte des arrangements

d'hyperplans, car l'une des perspectives possibles de ce travail serait d'étendre la description de l'élément minimal à d'autres arrangements. Dans un second temps, nous revenons plus près de la motivation initiale par la conjecture. Nous commençons par donner une preuve dans le cas des groupes de Coxeter généraux de rang 3. L'idée originelle a été conçue lors d'un stage de recherche sous la supervision du Pr. Hohlweg au LaCIM en 2017. Cependant, cette version de la preuve était imparfaite au point d'être inapplicable. Sous l'impulsion de Nathan Chapelier-Laget, les preuves ont été réécrites durant cette thèse, avec notamment l'introduction du graphe de bipodalité. Nous poursuivons en utilisant la description des éléments minimaux obtenue précédemment pour donner une nouvelle réponse à la question dans le cas des groupes de Weyl.

## Contribution

### Théorie des représentations des monoïdes

Le but de nos recherches dans ce domaine était d'utiliser efficacement la formule de Thiéry pour la matrice de Cartan d'un monoïde (voir la proposition 2.3.2), qui relie la matrice des invariants de Cartan d'une algèbre de monoïde à la table des caractères du monoïde d'une part et au nombre de points fixes dans le monoïde sous des translations simultanées par la gauche et par la droite d'autre part. D'un point de vue informatique, ces objets sont difficiles à obtenir. À notre connaissance, nous fournissons le premier exemple d'un algorithme calculant la table de caractères d'un monoïde fini général (sur  $\mathbb{C}$ ). Nos tests de performance montrent qu'il peut raisonnablement être appliqué à des monoïdes de cardinalité allant jusqu'à  $10^5$  (et poussé jusqu'à  $10^6$  dans certains cas particuliers d'intérêt). Cet algorithme est basé sur la formule suivante que nous prouvons dans le texte :

**Proposition** (3.1.6 dans le texte). *Les lignes de la table de caractères d'un monoïde  $M$  correspondant aux modules simples d'apex un idempotent  $e$ , notée  $X_e(M)$  sont données par la formule :*

$$X_e(M) = {}^t X(G_e)^{-1} \cdot \left( \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{\mathbf{S}}(m, g) - \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{N_e(\mathbf{S})}(m, g) \right)_{g \in C_{G_e}, m \in C_M}$$

où le point est le produit matriciel et  $X(G_e)$  est la table de caractères de groupe maximal à l'idempotent  $e$ .

Deuxièmement, nous réduisons partiellement la question du comptage des points fixes qui nous intéressent au problème de la théorie des groupes qui consiste à calculer la cardinalité d'un centralisateur. Cela se fait par la proposition suivante :

**Proposition** (3.2.4 dans le texte). *Soient  $H$  un  $\mathcal{H}$ -classe,  $a \in H$ ,  $\Gamma'(H)$  le groupe de Schützenberger à droite de  $H$  et finalement  $h \in {}_M\text{Stab}(\mathcal{R}(H))$ ,  $k \in \text{Stab}_M(\mathcal{L}(H))$ . Alors :*

$$|\text{Fix}_H(h, k)| = \begin{cases} |C_{\Gamma'(H)}(\times_H k)| & \text{si } \tau_a(h \times_H)^{-1} \in \overline{\times_H k} \\ 0 & \text{sinon} \end{cases}$$

où  $\overline{\times_H k}$  est la classe de conjugaison de  $\times_H k$  dans  $\Gamma'(H)$  et  $C_{\Gamma'(H)}(\times_H k)$  est le centralisateur de  $\times_H k$  dans  $\Gamma'(H)$ .

À partir de cette réduction, nous décrivons un algorithme comptant le nombre de points fixes sous  $s \mapsto hsk$  (pour  $h, k$  pertinents) qui surpasse l'algorithme naïf dans les monoïdes avec une « structure  $\mathcal{J}$  pas trop simple », permettant de s'attaquer à des monoïdes de cardinalité allant jusqu'à  $10^8$ . Si le monoïde est  $\mathcal{J}$ -trivial (ou proche de l'être), notre méthode, bien que pire en pratique, est seulement (expérimentalement) plus lente par un facteur constant.

## Éléments minimaux des arrangements de Shi

Afin de comprendre les relations décrivant les éléments minimaux des régions Shi, nous introduisons la notion de *jupe* d'une racine.

**Définition** (5.1.2 dans le texte). La *jupe* d'une racine positive  $\gamma$  est l'ensemble

$$\text{Sk } \gamma = \{\alpha \in \Phi^+ \mid \gamma - \alpha \in \Phi^+\}.$$

La relation « être dans la jupe de » est *presque* un ordre total (Propositions 5.1.3 et 5.1.4). Ceci, associé aux résultats d'Athanasiadis & Linusson (voir le Théorème 5.2.4) et Armstrong, Reiner & Rhoades (voir le théorème 5.2.7), permet de propager localement des informations sur les hyperplans qui délimitent une région de Shi donnée, ce qui donne le résultat suivant dans la *chambre dominante*.

**Proposition** (5.2.9 dans le texte). *Soit  $A$  une antichaine du poset des racines. Le vecteur suivant est un vecteur de Shi :*

$$\forall a \in \Phi^+, v_a = \max(\{\mathbb{1}_{a \in A}\} \cup \{v_b + v_{a-b} \mid b \in \text{Sk}(a)\}).$$

Nous pouvons ensuite transporter cette information de la chambre dominante à toutes les autres chambres pour obtenir une description générale des éléments minimaux des régions de Shi en termes de leurs *vecteurs de Shi* qui encodent leurs *ensembles d'inversions*. De plus, dans le cas des groupes classiques,

cela donne une jolie description combinatoire des coefficients de son vecteur de Shi comme nombre d'arcs non croisés dans un diagramme d'arcs non imbriqués associé à la région de Shi. Comme nous obtenons ces données combinatoires sur ce qui est essentiellement un ensemble de racines, nous sommes en mesure de déduire une condition nécessaire pour qu'une racine soit un « sommet » de l'ensemble d'inversions d'un élément. Puisque la question [DH16, Conjecture 2] est liée à ces sommets, nous sommes en mesure de déduire le cas des groupes affines dans le résultat suivant.

**Proposition** (6.1.1 et 6.2.4 dans le texte). *L'application associant à un élément bas l'ensemble de ses petites inversions est une surjection des éléments bas sur les petits ensembles d'inversions, dans les groupes de Weyl affines, ainsi que dans les groupes de Coxeter de rang 3.*

Le cas des groupes de rang 3 est une preuve indépendante, depuis (largement) dépassée par une preuve générale complète dans [Dye+23]. Nous l'incluons malgré tout, car nous pensons qu'il y a des questions intéressantes autour de la preuve, en particulier concernant la notion de *graphe de bipodalité*.

## Contenu et plan de la thèse

### Chapitre I.1 – Sur la structure des monoïdes finis

Dans ce chapitre, nous commençons par décrire la théorie générale des monoïdes, avec les définitions générales sur les monoïdes et semigroupes, les sous-structures et quotients, et les éléments spéciaux (en particulier les idempotents) (§1.1). Il s'avère que l'associativité du produit dans un monoïde est une condition forte, qui dote les monoïdes d'une *structure des classes de Green*, que nous examinons dans une seconde section (§1.2). Le principal résultat présenté ici est le *lemme de Green* (Lemme 1.2.7) qui exprime les relations contraintes entre les différents types de classes de Green (§1.2.1). Nous discutons également de la notion de *groupe de Schützenberger*, qui sont des groupes qui apparaissent naturellement lorsque le monoïde agit sur des parties de lui-même (§1.2.2). Enfin, nous examinons la structure de la multiplication dans une seule  $\mathcal{J}$ -classe avec le *Théorème de localisation* 1.3.2 (§1.3).

### Chapitre I.2 – Théorie des représentations

Dans ce second chapitre, nous décrivons la théorie des représentations des monoïdes finis. Tout d'abord, nous présentons le langage des modules sur une algèbre, en accordant une attention particulière aux modules simples et projectifs, car ils sont essentiels à la définition des deux invariants que nous cherchons

(§2.1). Le premier, la table des caractères, est discuté dans la section suivante (§2.2). La seconde, la matrice de Cartan, est présentée dans §2.3 avec la formule de Thiéry qui motive notre étude (Proposition 2.3.2). Dans la dernière section de ce chapitre (§2.4), nous énonçons le *Théorème de Clifford-Munn-Ponizovskii*, qui joue un rôle essentiel dans les preuves du chapitre suivant.

## Chapitre I.3 – Calculs des invariants

Nous présentons ici nos résultats. Tout d'abord, nous prouvons une formule pour calculer la table de caractères d'un monoïde (Proposition 3.1.6) en relation avec les tables de caractères des groupes, les points fixes et les caractères du *radical* de (certaines) classes  $\mathcal{L}$  de Green régulières. Nous concluons en discutant brièvement le cas modulaire (§3.1). Ensuite, nous décrivons comment calculer le nombre de points fixes d'un monoïde sous la transformation  $s \mapsto hsk$ , ce que nous relierons à une question de théorie des groupes sur le groupe de Schützenberger d'une classe  $\mathcal{J}$  donnée. Dans la dernière section (§3.3), nous considérons l'applicabilité pratique des résultats précédents. Nous fournissons des algorithmes explicites pour compter les points fixes qui nous intéressent et un système d'équations pour le radical « optimisé » grâce aux conséquences du lemme de Green.

## Chapitre II.4 – Préliminaires sur la théorie de Coxeter

Dans ce chapitre, nous donnons des résultats généraux sur les groupes de Coxeter (§4.1) et les objets géométriques associés : leur réalisation géométrique et leur système de racines (§4.2), les ensembles d'inversions (§4.3) et l'image projective (§4.4). Comme nous utilisons un point de vue géométrique, nous reproduisons certaines preuves classiques dans ce langage afin de nous préparer aux considérations géométriques des chapitres suivants. Avant cela, nous discutons de la classification des groupes de Coxeter finis (Proposition 4.5.3), des systèmes de racines cristallographiques (Proposition 4.5.5) et des groupes de Weyl affines (Proposition 4.5.6). Enfin, nous passons au point de vue dual avec le cône de Tits et l'arrangement de Coxeter (§4.6) et l'objet principal de notre étude : l'arrangement de Shi (§4.7). Nous le discutons dans toute sa généralité dans §4.7.1 et le spécialisons pour les groupes de Weyl affines (§4.7.2).

## Chapitre II.5 – Éléments minimaux des régions de Shi dans les groupes de Weyl

Ce chapitre présente les résultats de nos recherches sur les éléments minimaux des régions de Shi dans les groupes de Weyl. Tout d’abord, comme ces éléments minimaux sont contraints par les *relations de Shi*, nous étudions la structure de ces relations dans §5.1 afin de nous donner des outils pour la section suivante. C’est dans cette section que nous introduisons la notion de *jupe* d’une racine (Définition 5.1.2) et que nous étudions ses propriétés. Nous décrivons ensuite ces éléments minimaux : nous le faisons par le biais de leur codage de *Shi* (§5.2.1), en étendant une bijection avec les fonctions de parking (§5.2.2) pour donner dans la Proposition 5.2.12 une description de ces éléments (§5.2.3). Dans la section finale (§5.3), nous discutons de la spécialisation du résultat précédent aux groupes de Weyl classiques. Il s’avère que le codage de Shi qui compte le nombre d’hyperplans derrière lesquels se trouve une région peut être calculé en termes d’arcs non imbriqués non croisés (Proposition 5.3.3).

## Chapitre II.6 – Éléments bas

Dans ce dernier chapitre, nous revenons sur la motivation de notre recherche sur la théorie des groupes de Coxeter. Dans une première section, nous fournissons une preuve complète des résultats de [Cha21] pour les groupes de rang 3. Le résultat est obtenu en trois étapes : nous considérons d’abord l’interaction entre le primal (système de racines) et le dual (cône de Tits), afin de pouvoir reconnaître les *descentes* d’un élément (§6.1.1). Nous prouvons ensuite que le *graphique de bipodalité* est acyclique (§6.1.2) et que ses sources se trouvent dans un sous-ensemble commode (§6.1.3). Dans un second temps, nous retournons vers les groupes de Weyl : à partir de la description combinatoire du codage de Shi de l’élément minimal, nous obtenons une description suffisamment significative des ensembles d’inversions pour obtenir une condition nécessaire pour que les racines soient des « sommets » du groupe. Nous démontrons ce résultat dans la chambre dominante (§6.2.1) avant de le transporter dans d’autres chambres (§6.2.2).

## Conseils de lecture

Le public visé par ce document, de manière quelque peu égoïste, est l’étudiant en première année de doctorat que j’étais. En tant que tel, il est écrit pour être accessible à un étudiant en master d’algèbre « généraliste », avec un intérêt pour l’algèbre effective. De manière moins égocentrique, les deux sujets de

cette thèse étant quelque peu disjointes, nous espérons l'avoir rendue abordable à des lecteurs qui pourraient être habitués à l'une des parties, mais pas nécessairement à l'autre. Cela dit, les deux parties sont totalement indépendantes et le lecteur peut sauter l'une d'entre elles sans perdre d'informations.

Dans cette optique, nous supposons que le lecteur est familiarisé avec les notions de base de la théorie des groupes, de l'algèbre linéaire, bilinéaire et tensorielle, ainsi qu'avec la théorie des représentations des groupes présentée de manière « classique ». Nous n'utiliserons pas de résultats majeurs sans les citer, même dans ces domaines familiers, mais nous les utiliserons (en particulier la théorie des groupes) comme exemples motivants et comme guides pour savoir à quoi s'attendre. Inversement, nous ne supposons pas une familiarité avec les semigroupes et la théorie des semigroupes, la théorie des représentations des algèbres générales (et particulièrement la notion de modules projectifs), la théorie des représentations des monoïdes et la théorie des groupes de Coxeter. Les chapitres 1, 2 et 4 visent donc à fournir cette familiarité, et le lecteur spécialisé devrait probablement les sauter.

Au contraire, nous espérons que le lecteur non spécialiste y trouvera une introduction en grande partie autocontenue aux notions que nous utilisons. À cet effet, nous avons mis l'accent sur les définitions et les faits de base de chaque domaine. Lorsqu'il s'agit de résultats plus généraux ou plus puissants, nous avons essayé de ne pas trop nous éloigner du texte principal et de fournir des références à la littérature, à condition que cela n'affecte pas la compréhension des points importants de la thèse. Le meilleur exemple est peut-être l'accent mis sur les modules projectifs. Nous fournissons la preuve des résultats élémentaires nécessaires, spécialisés dans notre contexte, avec l'intention supplémentaire que ces résultats soient utilisés comme « sanity checks » pour la vérification des programmes accompagnant ces documents. Cependant, pour la proposition 2.1.55, nous ne fournissons qu'une référence, car il s'agit d'un résultat d'un niveau un peu plus élevé et dont la preuve serait peu éclairante dans notre contexte.

Si cette approche semble trop cavalière, nous recommandons au lecteur de consulter les ouvrages suivants pour une introduction plus approfondie à ces sujets :

- *Mathematical Foundations of Automata Theory* [Pin] de Pin pour la théorie générale des semigroupes,
- *Representation theory of finite groups : a guidebook* [Cra] par Craven pour une introduction délicate à la théorie des représentations des groupes finis,
- *Introduction to representation theory* [Eti+09] par Etingof et al. pour la



théorie des représentations des groupes finis, mais également pour une introduction douce à la théorie des représentations des algèbres,

- pour le lecteur déjà familiarisé avec le langage des modules et des catégories, *Representation Theory of Finite Monoids* [Ste16] traite le sujet à un niveau plus avancé, mais avec plus d'exhaustivité et de précision que nous n'aurions pu le faire,
- *Reflection groups and Coxeter groups* [Hum90] de Humphreys pour une introduction aux groupes de Coxeter, et en particulier à de nombreuses constructions géométriques connexes telles que le cône de Tits et le système de racines,
- *Combinatorics of Coxeter Groups* [BB05] par Björner & Brenti pour une approche combinatoire, de la théorie des groupes et de la théorie du langage des groupes de Coxeter.

Ces ouvrages ont été d'une aide précieuse lors de la préparation et de la rédaction de cette thèse et nous nous référons souvent à la plupart d'entre eux.

Enfin, nous souhaitons attirer l'attention du lecteur sur les exemples. Nous avons essayé de fournir autant d'exemples pertinents que possible, et certains d'entre eux sont référencés dans les preuves et les résultats ultérieurs. De plus, et surtout dans la partie concernant les groupes de Coxeter, les exemples de constructions géométriques explicites fournissent une bonne intuition pour les notions plus abstraites.

# Part I

## Computational monoid representation theory



## Introduction

The last two decades have seen the development of a new dynamic around the study of monoid representation theory. This is due to applications to certain types of discrete Markov chains and especially Markov chains used to generate combinatorial objects randomly, first uncovered in the seminal article of Brown [Bro00]. This has lead to an exploration of the combinatorial properties of monoid representations, for instance in [Ayy+14], [AST16] or [Thi12]. In this last article [Thi12], Thiéry gives a formula for the Cartan matrix of a finite monoid of  $M$  in terms of the number of fixed points and the character table of  $M$ . More precisely, it involves computing the cardinality of the set  $\{s \in M \mid hsk = s\}$  for any  $h, k \in M$ . In this paper, we set out to use this formula to effectively compute the Cartan matrix of the algebra of  $M$  over a perfect field  $\mathbf{k}$  of null characteristic.

Two difficulties have to be overcome in the pursuit of this goal. Firstly, the cardinality of many interesting families of monoids tends to increase very quickly. For instance, the cardinality of the *full transformation monoid*  $\mathcal{T}_n$  of all functions from  $\llbracket 1, n \rrbracket$  to  $\llbracket 1, n \rrbracket$  is  $n^n$ , making the naive computation of  $|\{s \in M \mid hsk = s\}|$  impractical even for small  $n$ . To remedy this, we provide an algorithm to efficiently compute this statistic. We obtain, and later utilize in a computer implementation using the algebra system GAP, the following result.

**Proposition** (3.2.4 in the text). *Let  $H$  be an  $\mathcal{H}$ -class,  $a \in H$  and  $h \in {}_M \text{Stab}(\mathcal{R}(H))$ ,  $k \in \text{Stab}_M(\mathcal{L}(H))$ . Then*

$$|\text{Fix}_H(h, k)| = \begin{cases} |C_{\Gamma'(H)}(\times_H k)| & \text{if } \tau_a(h \times_H)^{-1} \in \overline{\times_H k} \\ 0 & \text{otherwise} \end{cases}$$

where  $\overline{\times_H k}$  is the conjugacy class of  $\times_H k$  in  $\Gamma'(H)$  and  $C_{\Gamma'(H)}(\times_H k)$  is the centralizer in  $\Gamma'(H)$  of  $\times_H k$ .

This allows us to tackle this difficulty by relying on existing efficient group theoretic algorithms. Secondly, to use the formula, one has to compute the character table of the monoid. The Clifford-Munn-Ponizovskii Theorem (such as presented in [GMS09]) gives an explicit description of the simple  $\mathbf{k}M$ -modules and technically makes the computation of the character table possible, provided that we know how to compute the simple modules associated to certain groups. However, we note that although results on character tables are known in the case of some interesting families of monoids (most notable *inverse monoids* [Ste08]), no algorithms are available to compute the character table of an arbitrary finite monoid. This is no surprise, as an approach based directly on the Clifford-Munn-Ponizovskii Theorem, although explicit, is rather

convoluted and inefficient for several reasons. Firstly, we would need to explicitly construct irreducible representations of groups. Secondly, we would have to transform them into monoid representations, which necessitates computing several quotients of representations. Finally, we still would have to compute individually each character. This makes this approach computationally very costly.

In contrast, we prove the following formula.

**Proposition** (3.1.6 in the text). *We have the formula for the lines of the character table of the monoid  $M$  corresponding to simple modules of apex  $e$ , denoted  $X_e(M)$ :*

$$X_e(M) = {}^t X(G_e)^{-1} \cdot \left( \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{\mathbf{S}}(m, g) - \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{N_e(\mathbf{S})}(m, g) \right)_{g \in C_{G_e}, m \in C_M}$$

where the dot is the matrix product and  $X(G_e)$  is the character table of the maximal group at  $e$ .

This puts into relation the character table of the monoid with that of so-called *maximal subgroups at idempotents* on one hand, and modules whose character computation can be sped up using the *Green structure* of the monoid on the other hand. This has several advantages. Firstly, this formula is relatively “group representation agnostic” as it only requires knowing the character table and not an explicit construction of the representations. Secondly, the representations that we do compute are relatively easy to construct and well-behaved with respect to the action of the monoid. This allows us to use a combinatorial approach to the computation of their characters to develop faster algorithms.

In Chapter 1, we give some background on finite monoids (§1.1) and on their structure, mainly in terms of Green’s classes and Schützenberger groups (§1.2). We also give some attention to the structure of multiplication inside a regular  $\mathcal{J}$ -class and the role played by idempotents (§1.3). In Chapter 2, we recall the necessary notions on modules in general (§2.1) and the case of monoid representation in particular (§2.4). We also introduce the little of character theory that we need (§2.2) and discuss Thiéry’s formula for the Cartan matrix (§2.3). Finally, in the Chapter 3, we present our results. First, we derive a formula for the character table of a monoid as a function (mainly of) group character tables and number of fixed points in certain “nice” modules (§3.1). Secondly, we reduce the question of counting fixed points under a conjugacy-like action in the monoid to a group theoretic question (§3.2). In §3.3, we put these results together to provide effective algorithms for counting fixed points (§3.3.2) and computing the radical of an  $\mathcal{L}$ -class (§3.3.3). We also discuss the necessary computational hypotheses (§3.3.1) before measuring the effective performances of the computation of the bicharacter, the character table and the Cartan matrix (§3.3.4).

# Chapter 1

## On the structure of finite monoids

### 1.1 Generalities

**Definition 1.1.1.** A set  $S$  (the *ground set*) equipped with an internal composition law  $\cdot_S : S \times S \rightarrow S$  that is *associative* (meaning that for all  $s, t, u \in S$ ,  $(s \cdot_S t) \cdot_S u = s \cdot_S (t \cdot_S u)$ ) is called a *semigroup*. A semigroup equipped with an element  $1_S \in S$  such that for all  $s \in S$ ,  $1_S \cdot_S s = s \cdot_S 1_S = s$  is called a *monoid* and  $1_S$  is called the *identity*. A semigroup is *commutative* if  $s \cdot_S t = t \cdot_S s$  for all  $s, t \in S$ .

We will always simply refer to the internal composition law as the *multiplication* of the monoid and (unless ambiguous) drop the  $\cdot_S$ , writing  $st$  for  $s \cdot_S t$  as the multiplication is always assumed to be known in context. In the same fashion, we most often simply write  $1$  for  $1_S$ .

We can always transform a semigroup  $S$  into a monoid denoted  $S^1$  with ground set  $S \sqcup \{1\}$  and by extending the composition law by  $1s = s1 = s$  for all  $s \in S^1$ . Note that in the case where  $S$  is already a monoid,  $S^1$  is distinct from  $S$ : we have artificially added a new identity.

- Example 1.1.2.**
1. Any group is a monoid (and consequently, a semigroup).
  2.  $(\mathbb{N}, +, 0)$  is a monoid.
  3.  $(\mathbb{N}, \min)$  is a semigroup,  $(\mathbb{N} \cup \{\infty\}, \min, \infty)$  is a monoid.
  4. The set of  $n \times n$  matrices over a ring  $R$  equipped with matrix multiplication is a semigroup. If  $R$  is unitary, the identity matrix makes that set a monoid.

Past these few examples, in the totality of this thesis, we assume that we are working only with finite semigroups and monoids.

**Special elements** A first immediate property of monoids is that the identity element is necessarily unique, as if  $e, e'$  are two identity elements  $e = ee' = e'$ . However, the identity is not necessarily the only element of a semigroup  $S$  with special properties. Let us examine some special cases.

- A *zero* element, most often denoted  $0$  or  $0_S$  is such that for all  $s \in S$ ,  $0_S s = s 0_S = 0_S$ . Just like for the identity, there can be only one zero element in a semigroup  $S$ , and just like before, we may always add a zero, if necessary, as we did to define  $S^1$  from  $S$ .
- An idempotent element  $e \in S$  is such that  $e^2 = e$ . The identity element of a monoid is obviously an idempotent, but rarely the only one: since we assume our semigroups to be finite, there is a natural way presented in the following Proposition 1.1.3 to associate any element  $s$  of  $S$  to an idempotent denoted  $s^\omega$ .
- Inverses. There are diverse notions of inverse elements in semigroups. We will use firstly the *semigroup inverses*:  $s'$  is a semigroup inverse to  $s$  if  $s's' = s$  and  $ss' = s'$ . Secondly, the notion of *group inverse with respect to an idempotent  $e$* :  $s'$  is a group inverse of  $s$  with respect to  $e$  if it is a semigroup inverse and  $ss' = s's = e$ . In the special case where  $1 = e$ , we simply talk about a group inverse.

Questions relating inverses and idempotents will be explored further once we dispose of the notion of Green's relations in Section 1.2.

**Proposition 1.1.3.** *Let  $s$  be an element of a finite semigroup  $S$ . The sequence  $(s^n)_{n \in \mathbb{N}}$  is ultimately periodic. Let  $i, p \in \mathbb{N}$  be minimal such that  $s^i = s^{i+p}$ . Then  $\{s^{i+k} \mid k \in \llbracket 0, p-1 \rrbracket\}$  is a cyclic group with identity  $s^\omega$  defined by  $s^\omega = s^{i+j}$ , where  $j \equiv -i \pmod{p}$ .*

*Proof.* Since  $S$  is finite, the sequence cannot be injective. Thus, there exist minimal positive integers  $i$  and  $p$  such that  $s^i = s^{i+p}$ : the sequence is periodic with period  $p$  starting at index  $i$ . The set  $\{s^{i+k} \mid k \in \llbracket 0, p-1 \rrbracket\}$  is a group isomorphic to  $\mathbb{Z}/p\mathbb{Z}$ : we have  $s^{i+k}s^{i+k'} = s^{i+k+i+k'} = s^{i+(i+k+k' \pmod{p})}$ , which obviously makes  $s^\omega$  as defined the identity element.  $\square$

The previous proposition is often illustrated by the so-called “fry pan diagram” of Figure 1.1.

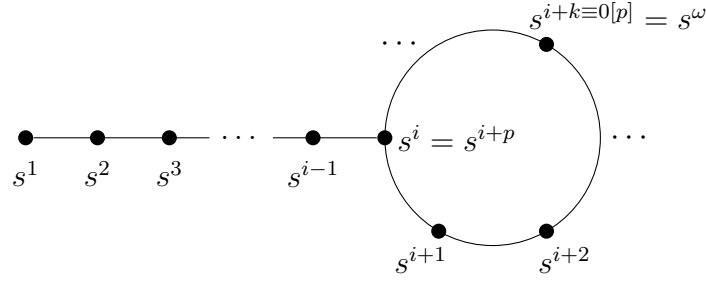


Figure 1.1: The fry pan diagram.

**Remark.** Semigroups for which any subsemigroup  $G$  that is a group is actually the trivial group  $\{1_G\}$  are called *aperiodic semigroups* because of this result.

Obviously,  $s^\omega$  is an idempotent, which gives the following property:

**Corollary 1.1.4.** *In a finite semigroup,  $S$  any element  $s$  has an idempotent power denoted  $s^\omega$ .*

**Definition 1.1.5.** An element  $s \in S$  is a *group element* if its index is 0 (the fry pan has no handle), or equivalently  $s = s^\omega s = s^{\omega+1}$ .

**Morphisms and substructures** As evidenced by the fact that a semigroup is just an easily added element away from being a monoid, the two notions are closely related. Although we will state most results only for monoids, they can be adapted to a semigroup  $S$  just by considering the monoid  $S^1$ . We could entirely ignore the notion of semigroup but it is useful to use that language. Indeed, unlike in the case of groups where the identity element is made truly special by the invertibility condition, in the case of monoids, natural maps that “should be” morphisms ignore the identity, making semigroups reappear.

**Definition 1.1.6.** A *semigroup morphism* from  $S_1$  to  $S_2$  is a map  $\varphi : S_1 \rightarrow S_2$  such that for any  $s, t \in S_1$ ,  $\varphi(st) = \varphi(s)\varphi(t)$ , *i.e.* if the following diagram is commutative<sup>1</sup>.

$$\begin{array}{ccc} S_1 \times S_1 & \xrightarrow{\cdot} & S_1 \\ \downarrow (\varphi, \varphi) & & \downarrow \varphi \\ S_2 \times S_2 & \xrightarrow{\cdot} & S_2 \end{array}$$

<sup>1</sup>Using commutative diagrams here is a bit excessive, but we do it to avoid them appearing for the first time with projective modules, which would make for a rougher introduction.



A *monoid morphism* from  $M_1$  to  $M_2$  is a semigroup morphism  $\varphi : M_1 \rightarrow M_2$  such that  $\varphi(1_{M_1}) = 1_{M_2}$ . An *isomorphism* is a bijective morphism.

We will sometimes also talk about *anti-morphisms* of these structure that reverse multiplication instead of preserving it:  $\varphi(st) = \varphi(t)\varphi(s)$ .

The remark that 1 is “less special” in monoids also extends to substructures:

**Definition 1.1.7.** A subset  $S'$  of a semigroup  $S$  is a subsemigroup of  $S$  if  $(S', \cdot_S)$  is a semigroup. A submonoid  $M'$  of a monoid  $M$  is a *subsemigroup* of  $M$  that contains the identity element.

**Example 1.1.8.** Consider a group  $G$  as a semigroup and add an identity element to get  $G^1$ : we naturally want to consider  $G$  inside  $G^1$ , but although  $G$  is a group (and consequently a monoid) it is only a subsemigroup of  $G^1$ .

We will also need the notion of product of semigroups, which is defined in the natural way.

**Definition 1.1.9.** Let  $I$  be a finite set and  $(S_i)_{i \in I}$  a family of semigroups. Then the set  $\prod_{i \in I} S_i$  equipped with the multiplication  $(s_i)_{i \in I} \cdot (t_i)_{i \in I} = (s_i t_i)_{i \in I}$  is the *product semigroup*. If all the  $S_i$  happen to be monoids, the resulting product semigroup equipped with the identity element  $(1_{S_i})_{i \in I}$  is the *product monoid*.

We will now focus our attention more on monoids, remarking on semigroups only where they naturally reappear.

**Definition 1.1.10.** An *ideal* (respectively *left ideal*, *right ideal*) of a monoid  $M$  is a subset  $I \subset M$  such that  $MIM = I$  (resp.  $MI = I, IM = M$ ).

Note that an ideal of a monoid is necessarily a subsemigroup, but the converse is obviously false. The notions of (left, right) ideal are well-behaved with respect to intersection: the intersection of any family of (left, right) ideals is still such an ideal. Similarly, it is well-behaved with respect to morphisms.

**Proposition 1.1.11.** Let  $\varphi : M_1 \rightarrow M_2$  be a monoid morphism, and  $I_1 \subset M_1, I_2 \subset M_2$  be ideals. Then  $\varphi^{-1}I_2$  is an ideal of  $M_1$ . If  $\varphi$  is surjective,  $\varphi(I_1)$  is an ideal of  $M_2$ . The same result holds for left and right ideals.

**Congruences and quotients** Another construction that will be of interest in the following sections of this thesis is the notion of quotient. There are two ways of defining a quotient, the first being with surjective morphisms.

**Definition 1.1.12.** A monoid  $M'$  is a *quotient* of a monoid  $M$  if there exists a surjective monoid morphism  $M \twoheadrightarrow M'$ .

The other way is through *monoid congruences*.

**Definition 1.1.13.** A *monoid congruence*  $\sim$  on a monoid  $M$  is an equivalence relation such that for all  $s, t, u, v \in M$ ,  $s \sim t \Rightarrow usv \sim utv$ .

Usually, a congruence on a structure is simply an equivalence relation that is compatible with the composition law in the sense that  $s \sim s'$  and  $t \sim t'$  implies  $st \sim s't'$  for all  $s, s', t, t' \in M$ , which we do not explicitly require here. However, it is implied by Definition 1.1.13.

**Proposition 1.1.14.** Let  $\sim$  be a monoid congruence on  $M$ , the quotient of  $M$  by  $\sim$ , denoted  $M/\sim$ , is itself a monoid and the projection map  $s \mapsto \bar{s}$  is a monoid morphism.

*Proof.* Multiplication is well-defined: if  $s \sim s'$  and  $t \sim t'$ ,  $(1)s(t) \sim (1)s'(t) = (1s')t(1) \sim (1s')t'(1) = s't'$ . Multiplication is associative: we have  $(\bar{s}\bar{t})\bar{u} = \overline{(st)u} = \overline{s(tu)} = \bar{s}(\bar{t}\bar{u})$ . The class  $\bar{1}$  is the identity of  $M/\sim$  since  $\bar{1}\bar{s} = \overline{1s} = \bar{s} = \overline{s1} = \bar{s}\bar{1}$ .  $\square$

Conversely, if we only had the property  $s \sim s'$  and  $t \sim t'$  implies  $st \sim s't'$  for all  $s, s', t, t' \in M$ , we trivially get back Definition 1.1.13 by  $s \sim t \Rightarrow us \sim ut \Rightarrow usv \sim utv$  because  $\sim$  is reflexive.

As expected, there is an isomorphism theorem (Theorem 1.1.15) stating that quotients in the sense of Definition 1.1.13 factor through the projection map of Proposition 1.1.14, where the congruence is the so-called *nuclear equivalence*, one of the two important examples of monoid congruences we will use in this thesis.

- The *kernel congruence* or *nuclear equivalence*. A morphism  $\varphi : M_1 \rightarrow M_2$  defines an equivalence relation  $\sim_\varphi$  by  $s \sim_\varphi t \Leftrightarrow \varphi(s) = \varphi(t)$ . This is obviously a congruence relation. Note that, unlike in a group, the fibers  $\{\varphi^{-1}\{t\} \mid t \in M_2\}$  are not simple translated copies of  $\varphi^{-1}\{1_{M_2}\}$  (indeed, this definition holds for semigroups where  $1_{M_2}$  does not necessarily exist). The idea of *kernel* of a monoid morphism thus needs to be slightly expanded as  $\ker \varphi = \{\varphi^{-1}\{\phi(s)\} \mid s \in M_1\}$ .

- The *Rees congruence*. Let  $I$  be an ideal of  $M$  and let  $\sim$  be the equivalence relation where  $i \sim i'$  for all  $i, i' \in I$  and  $s \sim t \Rightarrow s = t$  for all  $s, t \in M \setminus I$ . From the definition of an ideal, it is obvious that it is a congruence. The *Rees quotient* of  $M$  by  $I$  is the quotient  $M/\sim$ . A more explicit notation is almost always adopted in the literature by denoting the quotient by  $M/I$ , by  $0$  the class  $I$  and by  $s$  the class  $\{s\}$  for all  $s \in M$ .

The isomorphism theorem is as follows (see [Pin, Theorem II.3.2] for a proof).

**Theorem 1.1.15.** *Let  $\varphi : M \longrightarrow M'$ . Let  $\pi : M \longrightarrow M/\sim_\varphi$  be the canonical projection. Then there exists a unique injective morphism  $\tilde{\varphi} : M/\sim_\varphi \longrightarrow M'$  such that  $\varphi = \tilde{\varphi} \circ \pi$ , i.e. such that the following diagram commutes.*

$$\begin{array}{ccc} & & M' \\ & \nearrow \varphi & \uparrow \tilde{\varphi} \\ M & \xrightarrow{\pi} & M/\sim_\varphi \end{array}$$

Moreover,  $\tilde{\varphi}|_{\varphi(M)}$  is an isomorphism.

**Actions** Just as one can define a group action, monoid actions exist:

**Definition 1.1.16.** If  $M$  is a monoid and  $X$  is a set, a map  $M \times X \longrightarrow X$  is a *left (respectively right) monoid action* if:

- $\forall x \in X, 1 \cdot x = x$ .
- $\forall s, t \in M, x \in X, s \cdot (t \cdot x) = (st) \cdot x$  (resp.  $s \cdot (t \cdot x) = (ts) \cdot x$ ). That is, on the left, the following diagram commutes.

$$\begin{array}{ccc} M \times M \times X & \xrightarrow{(\text{Id}_M, \cdot)} & M \times X \\ (\cdot_M, \text{Id}_X) \downarrow & & \downarrow \cdot \\ M \times X & \xrightarrow{\cdot} & X \end{array}$$

It is often more convenient to write right action on the right, where the axioms become  $\forall x \in X, x \cdot 1 = x$  and  $\forall s, t \in M, x \in X, (x \cdot t) \cdot s = x \cdot (ts)$ . Still, written on the left, it is clear that an action on the right is reversing multiplication, which motivates the definition of *opposite semigroup*:

**Definition 1.1.17.** Given a semigroup  $(S, \cdot_S)$  the *opposite semigroup* (also-called *dual semigroup* in the literature) is the semigroup, denoted simply by  $S^{op}$ , with ground set  $S$  and multiplication  $\cdot_{S^{op}}$  defined by  $s \cdot_{S^{op}} t = t \cdot_S s$  for all  $s, t \in S$ . The *opposite monoid*  $M^{op}$  of a monoid  $M$  is the opposite semigroup of  $M$  equipped with the same identity element  $1_M$ .

Thus, a right action of a monoid  $M$  on a set  $X$  naturally corresponds to a left action of  $M^{op}$ , as the two commutative diagrams below summarize: the first one defines a right action, and the second is exactly the same given that the ground sets of  $M$  and  $M^{op}$  are the same, but it defines a left action of  $M^{op}$ .

$$\begin{array}{ccc}
 M \times M \times X & \xrightarrow{(\text{Id}_M, \cdot)} & M \times X \\
 (\cdot, \text{Id}_X) \downarrow & & \downarrow \cdot \\
 M \times X & \longrightarrow & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 M^{op} \times M^{op} \times X & \xrightarrow{(\text{Id}_{M^{op}}, \cdot)} & M^{op} \times X \\
 (\cdot_{op}, \text{Id}_X) \downarrow & & \downarrow \cdot \\
 M^{op} \times X & \longrightarrow & X
 \end{array}$$

We say that two actions  $\cdot_1 : M_1 \times X \rightarrow X$ ,  $\cdot_2 : M_2 \times X \rightarrow X$  *commute* if for all  $s_1 \in M_1, s_2 \in M_2, x \in X$ ,  $s_1 \cdot_1 (s_2 \cdot_2 x) = s_2 \cdot_2 (s_1 \cdot_1 x)$ . If  $\cdot_1$  happens to be a left action while  $\cdot_2$  is a right action, it is more conveniently expressed  $(s_1 \cdot_1 x) \cdot_2 s_2 = s_1 \cdot_1 (x \cdot_2 s_2)$ . Later, it will be convenient to see two commuting actions, one of  $M_1$  on the left and the other of  $M_2$  on the right, as a unique action on the left of  $M_1 \times M_2^{op}$ .

Note that if  $\varphi : M_1 \rightarrow M_2$  is an anti-morphism, then the same set map seen as  $\varphi : M_1 \rightarrow M_2^{op}$  is a morphism.

**Transformation semigroups** In the context of groups, one can define an action of a group  $G$  on a set  $X$  as a group morphism from  $G$  to the symmetric group on  $X$ ,  $\mathfrak{S}(X)$ . More broadly, a group acting on a set is often called a permutation group. The equivalent notion in terms of monoids is that of *transformation monoid*.

**Definition 1.1.18.** A *transformation monoid* on a set  $X$  is a subset of  $X^X$  that is a monoid for map composition (on the left side:  $fg = x \mapsto f \circ g(x) = f(g(x))$ ). The cardinality of  $X$  is called the *rank* of the transformation monoid. The full transformation monoid on  $X$ , denoted by  $\mathcal{T}_X$  is  $(X^X, \circ, \text{Id}_X)$ .

For a finite set  $X$  of cardinality  $n$ , up to assigning a number to each element of  $X$ ,  $\mathcal{T}_X$  is isomorphic to  $\mathcal{T}[[1, n]]$ , simply denoted  $\mathcal{T}_n$ , called the *full transformation monoid of rank  $n$* . Using that language, a left (respectively right) action of a monoid  $M$  on a set  $X$  is a monoid morphism  $\phi : M \rightarrow \mathcal{T}_X$  (resp. anti-morphism). The full transformation monoid is an important example we will often use in this text.

A monoid  $M$  naturally acts on itself by translation (either left or right). This makes the map  $s \mapsto (t \mapsto st)$  an injective monoid morphism from  $M$  to  $\mathcal{T}_M$  (and similarly  $s \mapsto (t \mapsto ts)$  embeds  $M$  into  $\mathcal{T}_M^{op}$ ). This is the monoidal equivalent of Cayley's Theorem for groups. Through this embedding, we may see any finite monoid as a transformation monoid, on which many relevant

computations are, if not easy, at least explicit (for instance, the nuclear equivalence class but also many others detailed in Section 3.3.1). As is already the case for the Cayley embedding for groups however, this is extremely inefficient. The cardinal of  $\mathcal{T}_n$  being  $n^n$  allows for effective computations of the kind we are after for monoids with only up to a dozen elements, possibly a few dozens taking advantage that  $\mathcal{T}_M$  is not entirely in the image of the embedding. However, despite the results in this thesis holding for all finite monoids, thinking of them in terms of transformation monoids gives a good intuition on the general results.

## 1.2 Green structure of finite monoids and Schützenberger groups

In this section, we first recall essential and elementary results on the Green structure of finite monoids and on Schützenberger groups. The informed reader may skip this first section, with the exception of the notation (1.2.6) that are used throughout this paper.

### 1.2.1 Green Structure

Although finite monoids have been considered to be much wilder objects than groups, it turns out that, with the right optics, they are actually highly structured by their internal multiplication. Consider the divisibility relation:  $x$  divides  $y$  if  $y = xz$  for some  $z$ . If  $x, y, z$  are taken in a group  $G$ , the relation is trivial. If, however, we take them in a general monoid  $M$ , left or right translation by an arbitrary element need not be surjective, making the question of  $x \in M$  being a left (or right) multiple of  $y \in M$  non-trivial. These questions of “divisibility” in a general monoid are studied under the name of Green structure, of which we give an overview necessary for our purpose in the subsection below.

**Definition 1.2.1** (Green’s relations). Let  $M$  be a finite monoid and  $s, t$  two of its elements. *Green’s relations* are:

- $s \leq_{\mathcal{L}} t \Leftrightarrow Ms \subset Mt$
- $s \leq_{\mathcal{R}} t \Leftrightarrow sM \subset tM$
- $s \leq_{\mathcal{J}} t \Leftrightarrow MsM \subset MtM$
- $s \leq_{\mathcal{H}} t \Leftrightarrow s \leq_{\mathcal{L}} t \text{ and } s \leq_{\mathcal{R}} t.$

For  $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}\}$ , Green's  $\mathcal{K}$ -equivalence relation is given by  $s\mathcal{K}t \Leftrightarrow s \leq_{\mathcal{K}} t$  and  $t \leq_{\mathcal{K}} s$ . Equivalently,  $s\mathcal{L}t \Leftrightarrow Ms = Mt$ ,  $s\mathcal{R}t \Leftrightarrow sM = tM$  and  $s\mathcal{J}t \Leftrightarrow MsM = MtM$ .

These preorders and equivalence relations are compatible with multiplication in various ways.

**Proposition 1.2.2.** *Let  $s, t, u, v$  be elements of a monoid  $M$ . Then:*

1.  $us \leq_{\mathcal{L}} s$ ,  $su \leq_{\mathcal{R}} s$  and  $usv \leq_{\mathcal{J}} s$ .
2.  $s \leq_{\mathcal{L}} t \Rightarrow su \leq_{\mathcal{L}} tu$  and  $s \leq_{\mathcal{R}} t \Rightarrow us \leq_{\mathcal{R}} ut$ .
3.  $s\mathcal{L}t \Rightarrow su\mathcal{L}tu$  and  $s\mathcal{R}t \Rightarrow us\mathcal{R}ut$ .
4. If  $s \leq_{\mathcal{L}} us$  then  $s\mathcal{L}us$ . If  $s \leq_{\mathcal{R}} su$  then  $s\mathcal{R}su$ .

*Proof.* Let us prove the first point in each case, as the others are obtained in the same way. (1)  $Mu \subset M \Rightarrow Mus \subset Ms$ . (2)  $Ms \subset Mt \Rightarrow Msu \subset Mt u$ . (3)  $Ms = Mt \Rightarrow Msu = Mt u$ . (4)  $s \leq_{\mathcal{L}} us \leq_{\mathcal{L}} s \Rightarrow s\mathcal{L}us$ .  $\square$

Green's relations are also compatible with each other in the following way.

**Proposition 1.2.3.** *For all  $s, t \in M$ ,  $s \leq_{\mathcal{J}} t \Rightarrow s \leq_{\mathcal{L}} t$  and  $s \leq_{\mathcal{R}} t$  firstly, and secondly  $s \leq_{\mathcal{L}} t$  or  $s \leq_{\mathcal{R}} t \Rightarrow s \leq_{\mathcal{H}} t$ . Consequently,  $s\mathcal{L}t$  or  $s\mathcal{R}t \Rightarrow s\mathcal{J}t$  firstly, and secondly  $s\mathcal{H}t \Rightarrow s\mathcal{R}t$  and  $s\mathcal{L}t$ .*

*Proof.* From the definition.  $\square$

Because of this, talking about “the  $\mathcal{L}$ -class  $L$  of some  $\mathcal{H}$ -class”, etc... is well-defined, and we shall denote it by  $\mathcal{L}(H)$ ,  $\mathcal{J}(L)$ ... There is a fifth Green's relation called the  $\mathcal{D}$ -relation defined by  $s\mathcal{D}t$  if and only if  $s\mathcal{R}u\mathcal{L}t$  for some  $u \in M$ . It happens to be that it is the same relation defined by  $s\mathcal{L}u\mathcal{R}t$  (see [Pin20, Proposition V.1.8]). However, we shall not need to make such differences, as in a finite monoid, which is the only case we are concerned with, the two relations  $\mathcal{J}$  and  $\mathcal{D}$  correspond. The following result is reproduced from [Pin, Theorem V.1.9].

**Proposition 1.2.4.** *For  $s$  and  $t$  in a finite monoid  $M$ ,  $s\mathcal{J}t$  if and only if there exist  $u \in M$  such that  $s\mathcal{L}u\mathcal{R}t$  if and only if there exists  $v \in M$  such that  $s\mathcal{R}v\mathcal{L}t$ .*

*Proof.* We will only prove the first equivalence, as the other proof is simply obtained by duality. If  $s\mathcal{L}u\mathcal{R}t$  then  $s\mathcal{J}u\mathcal{J}t$  so  $s\mathcal{J}t$ . Now, if  $s\mathcal{J}t$ , by definition, there are  $x, y, x', y' \in M$  such that  $xsy = t$  and  $x'ty' = s$  so  $x'xsyy' = s$  and further  $(x'x)^n s (yy')^n = s$ . From Corollary 1.1.4, we may choose  $n$  such that  $(x'x)^n$  and  $(yy')^n$  are both idempotent. Let  $u = (x'x)^n, v = (yy')^n$ . We have  $us = uuv = usv = s$  so  $s\mathcal{L}us$  and further  $s\mathcal{L}xs$ . Similarly,  $s\mathcal{R}sy$ . From the first relation, we get  $sy\mathcal{L}xsy = t$ :  $s\mathcal{R}sy\mathcal{L}t$  which by definition is  $s\mathcal{D}t$ .  $\square$

Some of the following results are stated in terms of  $\mathcal{D}$ -classes in the literature. Since in the context of finite monoid we are interested in, the  $\mathcal{J}$  and  $\mathcal{D}$ -classes coincide, we shall only use the terms  $\mathcal{J}$ -relation,  $\mathcal{J}$ -class, etc.

**Example 1.2.5** (Green's relations in  $\mathcal{T}_n$ ). Let  $a, b$  be two elements of  $M$ .

- $a\mathcal{L}b$  if and only if they have the same *kernel* or nuclear equivalence  $\ker a = \{a^{-1}\{i\} \mid i \in \llbracket 1, n \rrbracket\}$ .
- $a\mathcal{R}b$  if and only if they have the same image,  $\text{Im}(a) = \text{Im}(b)$ .
- Since  $\mathcal{T}_n$  is finite,  $\mathcal{J}$  is generated by  $\mathcal{L}$  and  $\mathcal{R}$  so  $a\mathcal{J}b$  if and only if  $\text{Im } a$  and  $\text{Im } b$  (or equivalently  $\ker a$  and  $\ker b$ ) have the same cardinality.
- $a\mathcal{H}b$  if and only  $a$  and  $b$  have the same image and the same kernel.

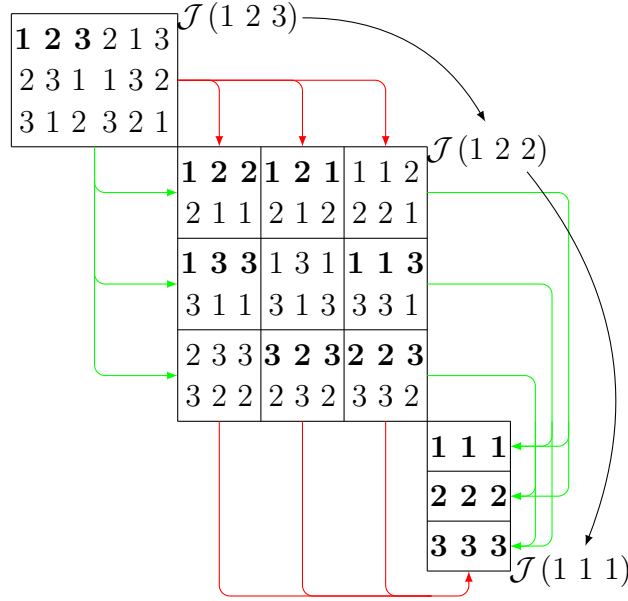
These conditions are necessary conditions in any transformation monoid. To get that they are sufficient, we use the fact that  $\mathfrak{S}_n \subset \mathcal{T}_n$  and that we can rearrange both image and kernel as we please.

These relations are illustrated in the case of the monoid  $\mathcal{T}_3$  in Figure 1.2.

The following notations will prove useful, as the study of Green's relations is, in part, the study of the maps given by left and right translations in the monoid.

**Notation 1.2.6.** Let  $h, k$  be elements of  $M$  and  $S$  be a subset of  $M$ . We denote by:

- $h \times_S$  the application from  $S$  to  $hS$  defined by  $s \mapsto hs$ ,
- $\times_S k$  the application from  $S$  to  $Sk$  defined by  $s \mapsto sk$ ,
- ${}_M \text{Stab}(S) = \{m \in M \mid mS = S\}$ ,
- $\text{Stab}_M(S) = \{m \in M \mid Sm = S\}$ ,


 Figure 1.2: Green's relations in  $\mathcal{T}_3$ .

Each block is a  $\mathcal{J}$ -class, each line is an  $\mathcal{R}$ -class, each column an  $\mathcal{L}$ -class and each case an  $\mathcal{H}$ -class. The red, green and black arrows represent the  $\mathcal{L}$ ,  $\mathcal{R}$  and  $\mathcal{J}$ -order respectively.

- $\text{Fix}_S(h, k) = \{s \in S \mid hsk = s\}$ .

Using these notations, let us recall Green's Lemma, which is one of the central elements of the theory of Green's relations, as it shows that the structure of the relations is actually heavily constrained, making their study practical.

**Lemma 1.2.7** (Green's Lemma). *Let  $a, a'$  be two elements in the same  $\mathcal{L}$ -class and let  $\lambda, \lambda'$  such that  $\lambda a = a'$  and  $\lambda' a' = a$ . Then  $\lambda \times_{\mathcal{R}(a)}$  and  $\lambda' \times_{\mathcal{R}(a')}$  are reciprocal bijections. Moreover, for any  $\mathcal{L}$ -class  $L$   $\lambda \times_{\mathcal{R}(a) \cap L}$  and  $\lambda' \times_{\mathcal{R}(a') \cap L}$  are reciprocal bijections. Similarly, if  $a, b$  are two elements in the same  $\mathcal{R}$ -class and  $\rho, \rho'$  are such that  $\rho a = b$  and  $\rho' b = a$ , then  $\times_{\mathcal{L}(a)} \rho$  and  $\times_{\mathcal{L}(b)} \rho'$  are reciprocal bijections. Moreover, for any  $\mathcal{R}$ -class  $R$   $\times_{\mathcal{L}(a) \cap R} \rho$  and  $\times_{\mathcal{L}(b) \cap R} \rho'$  are reciprocal bijections.*

An important consequence of Green's Lemma is that  $\mathcal{J}$ -classes can be neatly organized as *egg box pictures*<sup>2</sup>: the  $\mathcal{J}$ -class can be represented as a rectangular array with the  $\mathcal{L}$ -classes as columns, the  $\mathcal{R}$ -classes as rows and the  $\mathcal{H}$ -classes, the eggs, in the boxes, as can be seen in Figure 1.2. This level of organization is actually what allows for efficient computer representation of monoids and most of their algorithmic exploration.

<sup>2</sup>Terminology introduced in [CP61].



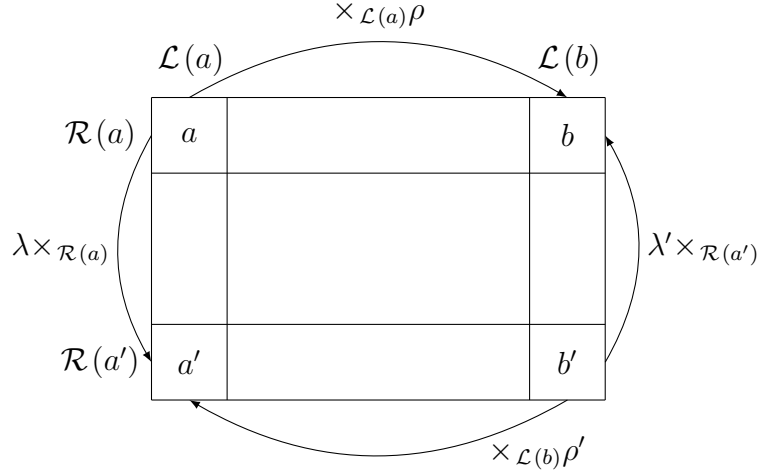


Figure 1.3: Green's Lemma

### 1.2.2 Schützenberger groups

The Green structure offers a second way of facilitating computer exploration of monoids through groups that arise as stabilizers of some Green's classes. These groups were introduced by Schützenberger in [Sch57]. This is a running theme of monoid theory: they allow for a number of monoid theoretic questions to be formulated in terms of groups for which we have at our disposal an array of efficient algorithms.

**Definition 1.2.8** (Schützenberger groups). Let  $H$  be an  $\mathcal{H}$ -class. The set  $\{h \times_H \mid h \in {}_M \text{Stab}(H)\}$  equipped with map composition is a subgroup of  $\mathfrak{S}(H)$  called the *left Schützenberger group* and denoted by  $\Gamma(H)$ .

Similarly,  $(\{\times_H k \mid k \in \text{Stab}_M(H)\}, \circ)$  is a subgroup of  $\mathfrak{S}(H)$  called the *right Schützenberger group* and denoted by  $\Gamma'(H)$ .

**Example 1.2.9.** Consider  $H = \mathcal{H}([1 \ 3 \ 1])$  (the elements of  $\mathcal{T}_n$  are given in function notation in all examples). We have :

$${}_M \text{Stab}(H) = \{[1 \ 2 \ 3], [3 \ 2 \ 1], [1 \ 3 \ 3], [3 \ 1 \ 1], [1 \ 1 \ 3], [3 \ 3 \ 1]\}$$

and subsequently,  $\Gamma(H) = \{[1 \ 2 \ 3] \times_H, [3 \ 3 \ 1] \times_H\}$ . Notice that, as elements of  $\Gamma(H)$ ,  $[3 \ 3 \ 1] \times_H = [3 \ 2 \ 1] \times_H$  and that the only important thing is the permutations induced by the elements of  ${}_M \text{Stab}(H)$  on  $\text{Im } H$ . Thus, in the case of transformation monoids, the left Schützenberger group of an  $\mathcal{H}$ -class  $H$  can be represented as a subgroup of  $\mathfrak{S}(\text{Im } H)$ . In the same way, the right Schützenberger groups can be represented as subgroups of  $\mathfrak{S}(\ker H)$ . This fact is used to represent the Schützenberger groups in Section 3.3.

Our precedent remark on exploiting Schützenberger groups to get efficient algorithms for computational monoid theoretic questions is seconded by the fact that Schützenberger groups do not contain any “superfluous information” in the following sense.

**Proposition 1.2.10.** *Let  $H$  be an  $\mathcal{H}$ -class. The natural actions of  $\Gamma(H)$  and  $\Gamma'(H)$  on  $H$  are free and transitive.*

We reproduce below a proof for Proposition 1.2.10 from [Sch57] for the purpose of showcasing the main argument. The argument itself is widely known, and we will use it multiple times in the remainder of this document.

*Proof.* If two elements  $h, h' \in H$  are in the same  $\mathcal{L}$ -class, there is some  $u \in M$  such that  $uh = h'$ . By Green’s Lemma, this means that  $u \in {}_M\text{Stab}(H)$ , so  $\Gamma(H)$  acts transitively on  $H$ . Suppose that  $uh = h$  for some  $u \in M$ . Since  $h, h'$  are also in the same  $\mathcal{R}$ -class, there exists some  $v$  such that  $h' = hv$ , so  $uh' = uhv = hv = h'$ : an element of  $\Gamma(H)$  either fixes all points in  $H$  or fixes none. The only element of  $\Gamma(H)$  that fixes all points (and, consequently, the only one that fixes any point) is the identity and thus the action is free. The same arguments apply for  $\Gamma'(H)$ .  $\square$

As pointed out in the same paper [Sch57], the left and right Schützenberger groups are essentially the same.

**Proposition 1.2.11.** *Given an  $\mathcal{H}$ -class  $H$ ,  $a \in H$  and  $g \in \Gamma(H)$ , we define  $\tau_a(g)$  as the unique element of  $\Gamma'(H)$  such that  $g \cdot a = a \cdot \tau_a(g)$ . Then  $\tau_a : \Gamma(H) \longrightarrow \Gamma'(H)$  is an anti-isomorphism<sup>3</sup>.*

*Proof.* We have, for a given  $a \in H$  and any  $h, g \in \Gamma(H)$ :

$$gh \cdot a = a \cdot \tau_a(gh) = g \cdot a \cdot \tau_a(h) = a \cdot \tau_a(g) \cdot \tau_a(h) = a \cdot (\tau_a(h)\tau_a(g))$$

where the inversion in the last equality comes from  $\Gamma'(H)$  having a right action on  $H$ . Because the action is transitive, we may cancel the  $a$  to obtain  $\tau_a(gh) = \tau_a(h)\tau_a(g)$ .  $\square$

From Green’s Lemma, it is straightforward that we have  ${}_M\text{Stab}(\mathcal{L}(H)) = {}_M\text{Stab}(H)$  and that the action of this stabilizer also induces a group that acts freely on the whole of  $\mathcal{L}(H)$ , stabilizing and acting transitively on each  $\mathcal{H}$ -class contained in  $\mathcal{L}(H)$ . The same phenomenon occurs from  $\text{Stab}(\mathcal{R}(H))_M$  acting on the right. This implies that the Schützenberger groups of any two  $\mathcal{H}$ -classes in the same  $\mathcal{J}$ -class are (anti-)isomorphic. However, we will not

<sup>3</sup>Note that some authors equip the right Schützenberger group with reversed composition, and thus obtain an isomorphism instead of an anti-isomorphism.

strictly need this result: for representation theory, we are satisfied with the weaker Proposition 1.3.7, while for fixed points counting in Section 3.2 we will in fact do our best to compute the Schützenberger groups of only one  $\mathcal{H}$ -class per  $\mathcal{J}$ -class.

### 1.3 Regularity and multiplication

As we have just seen, the Schützenberger groups are the objects that characterize the permutations of an  $\mathcal{H}$ -class induced by the action of the monoid. In this section, we want to understand the transformations of a  $\mathcal{J}$ -class induced by the action. We will then “zoom out” from the elements themselves to look at how the  $\mathcal{H}$ -classes are “moved around” by the action.

**Definition 1.3.1.** A  $\mathcal{J}$ -class is *regular* if it contains an idempotent.

Regular  $\mathcal{J}$ -classes play an important role in the representation theory of finite monoids because they are somewhat (but not entirely) stable by internal multiplication. Given this, they are responsible for the “mixing” that occurs in modules, whereas non-regular classes only “reduce” the space acted upon by the monoid, by making the multiplied element “fall down toward lower  $\mathcal{J}$ -classes”. As to what we mean by “somewhat stable”, another consequence of Green’s Lemma is the so-called “Location Theorem” from Clifford and Miller. We reproduce the proof from [Pin, Theorem 1.11]

**Theorem 1.3.2** (Location Theorem). *Let  $r, l$  be two elements in the same  $\mathcal{J}$ -class. We have:*

$$rl = \begin{cases} \gamma \in \mathcal{R}(r) \cap \mathcal{L}(l) & \text{if } \mathcal{L}(r) \cap \mathcal{R}(l) \text{ contains an idempotent,} \\ \gamma <_{\mathcal{J}} l, r & \text{otherwise.} \end{cases}$$

*Proof.* Note that  $rl \leq_{\mathcal{J}} r$  and  $rl \leq_{\mathcal{J}} l$ , so either we are in the second case or  $rl$  stays in the same  $\mathcal{J}$ -class. Suppose that  $rl \in \mathcal{R}(r) \cap \mathcal{L}(l)$  and let us show that  $\mathcal{L}(r) \cap \mathcal{R}(l)$  contains an idempotent  $e$ . By Green’s Lemma,  $\times_{\mathcal{L}(r)} l$  is a bijection from  $\mathcal{L}(r)$  to  $\mathcal{L}(l)$  preserving the  $\mathcal{H}$ -classes. In particular, there exists an element  $e \in \mathcal{R}(l) \cap \mathcal{L}(r)$  such that  $el = l$ . Since  $e\mathcal{R}l$ , we have  $e = lv$  for some  $v \in M$  and  $ee = elv = lv = e$  so  $e$  is idempotent. Suppose now that  $e \in \mathcal{R}(l) \cap \mathcal{L}(r)$  is an idempotent. We have  $el = l$  because  $e\mathcal{R}l$  means that there is some  $u \in M$  such that  $l = eu$  and so  $el = eeu = eu = l$ . Again because  $e\mathcal{R}l$ , we have  $l' \in M$  such that  $ll' = e$ . Setting  $\bar{l} = l'e$ , we get  $\bar{l}\bar{l} = ll'el = eel = l$ , so  $\bar{l}$  is a semigroup inverse of  $l$ . Similarly, there is a semigroup inverse  $\bar{r}$  of  $r$ . We have  $rl\bar{l} = rll'e = ree = r$  and in the same way  $\bar{r}rl = l$ . Clearly,  $rl \leq_{\mathcal{R}} r$  and this shows the converse  $r \leq_{\mathcal{R}} rl$  (and the same thing holds for  $\mathcal{L}$ ) so, by Proposition 1.2.2,  $rl \in \mathcal{R}(r) \cap \mathcal{L}(l)$ .  $\square$

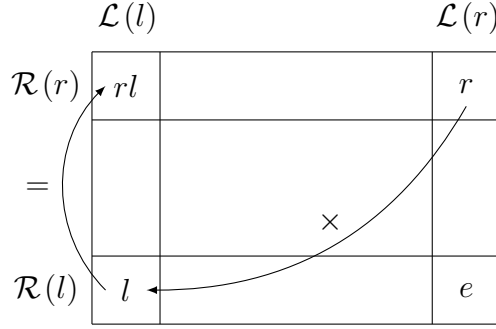


Figure 1.4: Location Theorem

Since there is an idempotent  $e$  in  $\mathcal{L}(r) \cap \mathcal{R}(l)$ ,  $rl$  stays in the same  $\mathcal{J}$ -class, in  $\mathcal{L}(l) \cap \mathcal{R}(r)$ .

Later, as, a corollary, that will allow us to identify regular  $\mathcal{J}$ -classes.

**Corollary 1.3.3.** *A  $\mathcal{J}$ -class  $J$  is regular if and only if  $JJ \cap J \neq \emptyset$ .*

*Proof.* If  $j \in JJ \cap J$ , then there is  $r, l \in J$  with  $rl = j$  such that  $\mathcal{R}(l) \cap \mathcal{L}(r)$  contains an idempotent and hence  $J$  is regular. The converse is obvious from the definition.  $\square$

It also permits to identify the groups doing the actual “mixing” we were referring to earlier: they are the  $\mathcal{H}$ -classes of the idempotents.

**Proposition 1.3.4.** *Let  $H$  be an  $\mathcal{H}$ -class. Then (1)  $H$  contains an idempotent if and only if (2) there exist  $r, l \in H$  such that  $rl \in H$  if and only if (3)  $H$  is a group.*

*Proof.* The first two points are equivalent by Theorem 1.3.2. Clearly (3) implies (1). Let us see that (1) implies (3) and let  $e \in H$  be an idempotent. From Theorem 1.3.2 and its proof,  $H$  is a monoid with identity element  $e$ . By Green’s Lemma, if  $h \in H$ , the maps  $\times_H h$  and  $h \times_H$  are permutations of  $H$  and in particular there exist left and right inverses of  $h$ , which must be equal.  $\square$

**Corollary 1.3.5.** *If  $S$  is a subsemigroup of a monoid  $M$  and  $S$  is a group,  $S$  is contained in an  $\mathcal{H}$ -class of  $M$ .*

This makes the special case where  $H$  is the  $\mathcal{H}$ -class of an idempotent interesting to note, and it will be important later:

**Definition 1.3.6.** Given an idempotent  $e$ , we denote  $G_e = \mathcal{H}(e)$  the *maximal subgroup at  $e$* .

While there may be several idempotents in a single  $\mathcal{J}$ -class (and in fact having exactly one is more or less the exception), the maximal subgroups at each idempotent in the same  $\mathcal{J}$ -class are isomorphic.

**Proposition 1.3.7.** *Let  $e, f$  be idempotents and suppose  $e \mathcal{J} f$ . Then  $G_e \cong G_f$ .*

*Proof.* By definition of  $e \mathcal{J} f$ , there are  $l, r \in M$  such that  $f = ler$ . Let  $l' = le, r' = er$ . From Green's Lemma,  $l' \in \mathcal{R}(f)$ , so in particular  $fl' = l'$  (because by definition  $l' = fu$  for some  $u$ ). Similarly,  $r'f = r'$ . We have  $l'r' = leer = f$  and conversely  $r'l' \in \mathcal{H}(e)$  by the Location Theorem. Since  $r'l'r'l' = r'fl' = r'l'$  is idempotent, it must be  $e$ . Let  $x, y \in \mathcal{H}(e)$ , we have  $l'xr'l'yr' = l'xeyr' = l'xyr'$ . So the map  $x \mapsto l'xr'$  is a bijective (from Green's Lemma) morphism between  $G_e$  and  $G_f$ .  $\square$

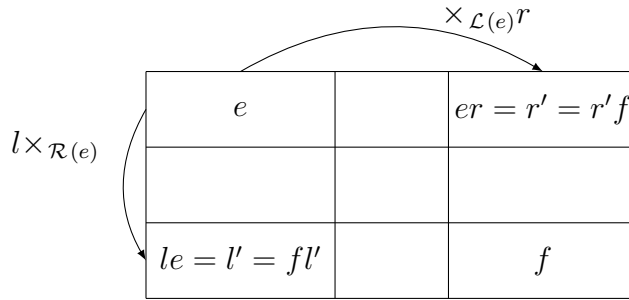


Figure 1.5: Illustration of Proposition 1.3.7.

**Proposition 1.3.8.** *Let  $e$  be an idempotent and  $H$  its  $\mathcal{H}$ -class. Then  $\Gamma_H$ ,  $G_e (= H)$  are canonically isomorphic and canonically anti-isomorphic to  $\Gamma'_H$ .*

*Proof.*  $G_e \subset {}_M \text{Stab}(H)$  naturally induce a map making it a subgroup of  $\Gamma(H)$  and that since  $G_e$  acts freely and transitively on  $H$  (Proposition 1.2.10), this map must be injective and surjective (and similarly for  $\Gamma'(H)$ ).  $\square$

Note that the Schützenberger groups are defined when  $H$  does not contain an idempotent, but in this case, we lose  $e$  as a natural anchor point that makes  $\Gamma_H$  and  $\Gamma'_H$  canonically anti-isomorphic.

**Example 1.3.9.** Consider, in Example 1.2.5,  $e = [1 \ 2 \ 2]$  and  $H = \mathcal{H}(e)$ .  $e$  is an idempotent, and, indeed,  $H = G_e$  is a group : setting  $t = [2 \ 1 \ 1]$ , we have  $e^2 = e, t^2 = e$  and  $et = te = t$ . As noted in Example 1.2.9:

$$\Gamma(H) = \mathfrak{S}(\{1, 2\}), \quad \Gamma'(H) = \mathfrak{S}(\{\{1\}, \{2, 3\}\}).$$

Note that the canonical isomorphism between  $\Gamma(H)$  and  $\mathcal{H}(e)$  is simply given by  $g \in \Gamma(H) \mapsto g \cdot e \in H$ .

# Chapter 2

## Representation theory

### 2.1 Modules

We will use the language of module and representation somewhat interchangeably in the rest of this document. In this chapter, we introduce these notions, see how they are related, and introduce the associated elementary notions: sum and products, simplicity, projective modules, etc. . .

#### 2.1.1 Basic definitions

Let us fix the language we use on elementary algebraic objects.

**Definition 2.1.1.** A *ring* is a triplet  $(R, (+_R, 0_R), (\cdot_R, 1_R))$  where  $R$  is the *ground set* and  $0_R, 1_R \in R$ , such that  $(R, +_R, 0_R)$  is a commutative group,  $(R, \cdot_R, 1_R)$  is a monoid and for all  $x, y, z \in R$ ,  $(a +_R b) \cdot_R c = a \cdot_R c + b \cdot_R c$  and  $a \cdot_R (b +_R c) = a \cdot_R b +_R a \cdot_R c$  (*distributivity*).

A *subring*  $R'$  is a subset of  $R$  such that  $(R', (+_R, 0_R), (\cdot_R, 1_R))$  is a ring.

The composition laws are clear from context most of the time, and similarly for the  $+_R$  *zero element*  $0_R$  and the  $\cdot_R$  *identity element*  $1_R$ . Thus, we shall refer to the ring by its ground set  $R$ , use  $+$ ,  $0$ ,  $1$  and denote  $x \cdot_R y$  simply by  $xy$  as we did for monoids.

**Definition 2.1.2.** A *ring morphism* between some rings  $R_1$  and  $R_2$  is a map  $\varphi : R_1 \rightarrow R_2$  such that for any  $r, s, t \in R$  we have

$$\varphi(r(s+t)) = \varphi(r)\varphi(s) + \varphi(r)\varphi(t) \text{ and } \varphi(1) = 1.$$

Note that we are requiring a ring to be *unitary*, that is, to have an identity element. This requirement is not present everywhere in the literature, but we adopt it for two main reasons. Firstly, the rings based on monoids we will construct later will naturally have an identity element. Secondly, as it is easy to add a neutral element to a semigroup to make it a monoid, it is easy to add an identity element to a non-unitary ring. Note that a field  $(\mathbf{k}, (+, 0), (\cdot, 1))$  is just a ring such that  $(\mathbf{k} \setminus \{0\}, \cdot, 1)$  is a commutative group. In the following, the bold letter  $\mathbf{k}$  is always a field.

**Definition 2.1.3.** A left (respectively right) *module* on a ring  $R$ , or  $R$ -module (resp. module- $R$ ) is a commutative group  $(V, +_V, 0_V)$  equipped with an action  $\cdot_l : R \times V \rightarrow V$  (resp.  $\cdot_r : V \times R \rightarrow V$ ) called *scalar multiplication* such that for all  $r, s \in R, u, v \in V$  we have

$$(r + s) \cdot_l (u + v) = r \cdot_l u + r \cdot_l v + s \cdot_l u + s \cdot_l v$$

(resp. exactly the reverse for  $\cdot_r$ ) and  $r \cdot_l (s \cdot_l u) = (rs) \cdot_l u$  (resp.  $(u \cdot_r r) \cdot_r s = u \cdot_r (rs)$ ). Moreover, we ask  $1 \cdot_l u = u$  (resp.  $u \cdot_r 1 = u$ ).

**Example 2.1.4.** For any ring  $R$ , the trivial group  $\{0\}$  is an  $R$ -module.

**Example 2.1.5.** For any ring  $R$ ,  $R$  is both a left and a right module on  $R$ . More generally, a *free*  $R$ -module  $F$  is an  $R$ -module for which there exists a *basis*  $(e_i)_{i \in I}$  that generates  $F$  as a module (*i.e.*  $F = \sum_i Rb_i$ ) and is “ $R$ -free” (*i.e.*  $\sum r_i e_i = 0 \Rightarrow \forall i, r_i = 0$ , for all  $(r_i)_{i \in I} \in R^I$ ). Equivalently,  $F$  is isomorphic to  $R^I$ .

Most often we denote the action simply by  $\cdot$  or drop the dot altogether as long as it is clear which element is the scalar and which is in  $V$ . We also denote the module by its ground set. For the same reason, we will drop the distinction between  $0_V$  and  $0_R$  unless not made clear by context. Note that a vector space is just a module on a field or, conversely, a module is a vector space “except that the scalars do not have to be a field”. If  $V$  is both an  $R$ -module and a module- $R$ , we say that  $V$  is an  *$R$ -bimodule*. We denote by  $R - \text{mod}$  the collection of  $R$ -modules,  $\text{mod} - R$  that of modules- $R$  and  $R - \text{mod} - R$  that of  $R$ -“bimodules” (that is, simultaneous left and right modules on  $R$ ). In our context, the left and right actions on a bimodule will always commute.

**Definition 2.1.6.** A *morphism of  $R$ -module* between some  $R$ -modules  $V_1$  and  $V_2$  is a map  $\varphi : V_1 \rightarrow V_2$  such that for any  $r \in R, u, v \in V$  we have

$$\varphi(ru + v) = r\varphi(u) + \varphi(v).$$

We denote by  $\text{hom}_{\mathbf{k}M}(V_1, V_2)$  the set of all morphisms from  $V_1$  to  $V_2$ .

**Example 2.1.7.** The zero map  $V_1 \longrightarrow \{0\} \subset V_2 : x \longmapsto 0$  is always a morphism of  $R$ -module.

This is similarly defined for a right module on  $R$  and for an  $R$ -bimodule (and so is the next definition). Note that a module morphism  $\varphi$  is in particular a linear map. As such, its nuclear equivalence in the sense of monoid morphism is much more well-behaved than for a general monoid. Hence, for linear maps, and more generally groups, rings, module morphism, we readopt the general convention that  $\ker \varphi = \varphi^{-1}\{0\}$ . The “full” nuclear equivalence is recovered by taking the translated copies of this kernel.

**Definition 2.1.8.** An  $R$ -submodule (or *left submodule on  $R$* )  $V'$  of an  $R$ -module  $V$  is a subset of  $V$  such that  $(V', +_V, 0_V)$  is a module and such that for any  $r \in R$ ,  $rV' \subset V$ .

**Example 2.1.9.** A ring  $R$  is obviously a bimodule on itself, and its  $R$ -subbimodules are its ideals.

**Example 2.1.10.** If  $\varphi : V_1 \longrightarrow V_2$  is an  $R$ -module morphism, then  $\text{Im } \varphi$  is an  $R$ -submodule of  $V_2$  and  $\ker \varphi$  is an  $R$ -submodule of  $V_1$ .

**Definition 2.1.11.** An *algebra on a field  $\mathbf{k}$*  or  *$\mathbf{k}$ -algebra* is a ring that has a structure of  $\mathbf{k}$ -vector space. A subalgebra is a subring that is also a  $\mathbf{k}$ -subspace. A  $\mathbf{k}$ -algebra morphism is a  $\mathbf{k}$ -linear ring morphism.

This ensures that any module on a  $\mathbf{k}$ -algebra is actually not just a commutative group, but also a  $\mathbf{k}$ -vector space.

**Definition 2.1.12.** The *algebra of a monoid  $M$*  over a field  $\mathbf{k}$ , denoted by  $\mathbf{k}M$ , is the  $\mathbf{k}$ -vector space formally generated by  $M$ , equipped with the multiplication of  $M$ , extended by linearity.

**Example 2.1.13.** Consider  $\text{hom}_{\mathbf{k}}(V, \mathbf{k})$  for a  $\mathbf{k}M$ -module  $V$ , denoted by  $V^*$ . It is naturally a  $\mathbf{k}M^{op}$ -module, called the *dual module*, by  $m \cdot f(v) = f(m \cdot v)$  for  $m \in M^{op}$ ,  $f \in V^*$  and  $v \in V$ . As we work with finite dimensional modules, it is easily seen that  $(V^*)^* \cong V$ .

Monoid (and possibly group) algebras will be the only kind of algebras we will have to deal with from now on. Note that we have supposed earlier that all our rings are with identity: clearly  $1_M$  is the unity of  $\mathbf{k}M$ . Hence, we most often state the following results in terms of  $\mathbf{k}M$ -algebras and modules although they may have much broader generalizations. The dimension of a module on a  $\mathbf{k}$ -algebra is its dimension as a  $\mathbf{k}$ -vector space. In all of this document, we



only consider finite monoids, which leads to the monoid algebras being finite dimensional. Moreover, we only ever consider finite dimensional modules on those algebras. We are now ready to introduce the closely related notion of representation.

**Definition 2.1.14.** A *representation* of a monoid  $M$  over the field  $\mathbf{k}$  is a pair  $(\rho, V)$ , where  $V$  is a  $\mathbf{k}$ -vector space and  $\rho$  is a monoid morphism  $\rho : M \rightarrow \mathcal{L}(V)$  where  $\mathcal{L}(V)$  is the set of linear maps on  $V$  equipped with map composition and the identity map.

**Example 2.1.15.** An important example is the case where  $M$  acts on a set  $X$ . Considering  $V = \text{Span}_{\mathbf{k}}(X)$  and the morphism  $\rho : M \rightarrow \mathcal{L}(V)$  given by extending linearly the action of  $M$  on the basis  $X$  we obtain a *representation by transformation* of  $M$ .

**Example 2.1.16.** Another important and even more special example of representation by permutation is the case when  $X = M$  with  $M$  acting on itself by (say left) translation. The resulting representation,  $\mathbf{k}M$ , is called the *regular representation* of  $M$  over  $\mathbf{k}$ . This corresponds to considering  $\mathbf{k}M$  as a module over itself.

**Definition 2.1.17.** Given a monoid  $M$ , a *representation morphism*  $T$  between two  $M$ -representations  $(\rho_1, V_1)$  and  $(\rho_2, V_2)$  is a linear map in  $\mathcal{L}(V_1, V_2)$  such that  $T \circ \rho_1(m) = \rho_2(m) \circ T$  for all  $m \in M$ .

The algebra of a monoid is obtained by linearizing “upstream” while representations linearize “downstream”. Modules are linear both on the side of the structure that is acting, the monoid algebra, and the structure that is acted upon, the module itself. This makes modules a somewhat more practical language as, in some sense, this notion is more stable. However, in the remainder of this document, we will retain some terminology coming from the “pure representation view”. For now, let us state clearly in what sense  $\mathbf{k}M$ -modules and representations of  $M$  in a  $\mathbf{k}$ -vector space are the same thing.

**Proposition 2.1.18.** *The  $\mathbf{k}M$ -modules and representation of  $M$  over  $\mathbf{k}$  are in bijection.*

*Proof.* Let  $V$  be a  $\mathbf{k}M$ -module and  $\rho : M \rightarrow \mathcal{L}(V)$  by defined by  $\rho(m) = v \mapsto m \cdot v$ . For all  $v \in V$ , we have

$$\rho(mm')(v) = (mm') \cdot v = m \cdot (m' \cdot v) = (\rho(m) \circ \rho(m'))(v) \quad (*)$$

so  $\rho$  is a monoid morphism and hence  $(\rho, V)$  is a representation over  $\mathbf{k}$ . In some sense, the module “restricts” to a representation.

Let  $(\rho, V)$  be a representation of  $M$  over  $\mathbf{k}$ . Then the scalar multiplication of  $v \in V$  by  $\sum_{m \in M} c_m m \in \mathbf{k}M$  is  $\sum_{m \in M} c_m \rho(m)(v)$ . The same equation (\*) shows that this definition respects the axiom of scalar multiplication on the basis  $M$  of  $\mathbf{k}M$  and on all of  $\mathbf{k}M$  by linearity. In some sense, the representation “generates” a module by linearly extending the action.

Obviously, these two transformations are inverse of each other: the restriction to  $M \subset \mathbf{k}M$  of the module generated by a representation  $(\rho, V)$  over  $\mathbf{k}$  is exactly  $(\rho, V)$  and reciprocally.  $\square$

Although other ways to consider a given module as a representation (and reciprocally) may exist, we systematically mean the specific one from Proposition 2.1.18 all throughout this document.

**Proposition 2.1.19.** *Let  $V_1, V_2$  considered at the same time as  $\mathbf{k}M$ -modules and representations of  $M$  over  $\mathbf{k}$ . Then a map  $T \in \mathcal{L}(V_1, V_2)$  is a module morphism if and only if it is a representation morphism.*

*Proof.* By definition of the maps in Proposition 2.1.18, we have, if  $T$  is either a module or representation morphism, for any  $v \in V_1, m \in M$ :

$$\begin{array}{ccc} (T \circ \rho_1(m))(v) & = & (\rho_2(m) \circ T)(v) \\ \parallel & & \parallel \\ m \cdot_2 T(v) & = & T(m \cdot_1 v) \end{array}$$

where  $\cdot_i, \rho_i$  are the scalar multiplication and monoid morphism corresponding to  $V_i, i = 1, 2$ . The red equalities are true by definition, so if one of the blue ones is, so is the other.  $\square$

The reader may notice that this just expresses “the categories of  $\mathbf{k}M$ -modules and of representation of  $M$  over  $\mathbf{k}$  are isomorphic”. However, we try not to rely on categorical vocabulary too heavily in order to keep the results as close as possible to the situation at hand.

### 2.1.2 Building blocks

From now on, we will almost exclusively deal in terms of (left) modules, with mentions of the differences in terms of representation, mainly vocabulary wise. Let  $M$  be a finite monoid and  $\mathbf{k}$  be field, which we suppose algebraically closed from now on. We are now only interested in modules where the ring is  $\mathbf{k}M$ , for which the underlying Abelian groups are actually  $\mathbf{k}$ -vector spaces. Recall that we only consider modules that are finite dimensional as  $\mathbf{k}$ -vector spaces. In most cases, this is a strong assumption and most results presented below can

be proven under weaker hypotheses (most often, Noetherianity or Artinianity), which the references we provide do. From now on, we denote the trivial module  $\{0\}$  simply by  $0$ .

### Modules: building tools and building blocks

**Definition 2.1.20.** Let  $V_1, V_2$  be  $\mathbf{k}M$ -modules. Then the *direct sum*  $V_1 \oplus V_2$  is the  $\mathbf{k}$ -vector space  $V_1 \oplus V_2$  equipped with the scalar multiplication defined by  $r \cdot (v_1 \oplus v_2) = (r \cdot_1 v_1) \oplus (r \cdot_2 v_2)$  with  $r_i \in \mathbf{k}M_i$  for  $i = 1, 2$  and  $(v_1, v_2) \in V_1 \oplus V_2$ .

**Definition 2.1.21.** Let  $V_1$  be in  $\mathbf{k}M_1 - \text{mod}$  and  $V_2$  be in  $\mathbf{k}M_2 - \text{mod}$ . Then the *tensor product* of  $V_1$  and  $V_2$  is the  $\mathbf{k}$ -vector space  $V_1 \otimes V_2$  equipped with scalar multiplication defined by  $(r_1, r_2) \cdot (v_1 \otimes v_2) = (r_1 \cdot_1 v_1) \otimes (r_2 \cdot_2 v_2)$  and extended linearly to  $V_1 \otimes V_2$ .

Note that we are tensoring over the field  $\mathbf{k}$  here, as we do systematically whenever the tensor product is unlabeled. However, we will need to tensor over an algebra.

**Definition 2.1.22.** Let  $V$  be a  $\text{mod} - R$  and  $W$  be an  $R - \text{mod}$ . Then  $V \otimes_R W$  is defined as  $V \otimes W$  quotiented by the ideal

$$\langle v \cdot r \otimes w - v \otimes r \cdot w \mid v \in V, w \in W, a \in R \rangle.$$

In particular, the structures of right and left  $\mathbf{k}M$ -modules disappear in the quotient over  $\mathbf{k}M$ , and, *a priori*, only a vector space remains. However, if  $V$  happens to be a left module over a ring  $R_l$  and  $W$  is a right module over  $R_r$ , then  $V \otimes_{\mathbf{k}M} W$  is in  $R_l - \text{mod} - R_r$ . This leads to a special case that we will use later:

**Proposition 2.1.23.** *If  $V$  is a right module over the ring  $\mathbf{k}M$  and  $I \subset \mathbf{k}M$  is a 2-sided ideal, then  $V \otimes_{\mathbf{k}M} \mathbf{k}M/I \cong V/VI$ . In particular  $V \otimes_{\mathbf{k}M} \mathbf{k}M = V$ .*

*Proof.* We recall that our rings (and in particular  $\mathbf{k}M$ ) are unital. Consider the two following maps:

$$\pi : \begin{cases} V \otimes_{\mathbf{k}M} \mathbf{k}M/I \longrightarrow V/VI \\ v \otimes r + I \longmapsto v \cdot r \end{cases} \quad \iota : \begin{cases} V/VI \longrightarrow V \otimes_{\mathbf{k}M} \mathbf{k}M/I \\ v \longmapsto v \otimes 1 + I \end{cases}.$$

These two maps are linear and inverses of each other.  $\square$

**Definition 2.1.24.** If  $\varphi : V_1 \longrightarrow V_2$  is a surjective  $\mathbf{k}M$ -module morphism,  $V_2$  is called a *quotient*. Equivalently, if  $V'_1$  is a submodule of  $V_1$ , then let  $V_2 = V_1/V'_1$  as an Abelian group quotient. Then  $V_2$  is naturally equipped with a structure of  $\mathbf{k}M$ -module and is called the quotient of  $V_1$  by  $V'_1$ .

**Definition 2.1.25.** Let  $V \neq 0$  be a  $\mathbf{k}M$ -module. Then  $V$  is *indecomposable* if whenever  $V \cong V_1 \oplus V_2$  then  $V_1 = 0$  or  $V_2 = 0$ .

**Proposition 2.1.26.** Let  $V \neq 0$  be a  $\mathbf{k}M$ -module. Then the following are equivalent:

1. for any submodule  $V'$  of  $V$ , either  $V' = 0$  or  $V' = V$ .
2. for all  $v \in V \setminus \{0\}$ ,  $\mathbf{k}Mv = V$ .

In that case, we say that  $V$  is a simple module. As a representation, we use the term *irreducible* instead.

*Proof.* Suppose (1) and let  $v \in V', v \neq 0$ . Then  $\mathbf{k}Mv$  is a nonzero submodule of  $V$ , so  $\mathbf{k}Mv = V$ . Suppose (2) and let  $0 \subsetneq V' \subset V$ . Then for any  $v \in V', v \neq 0$ ,  $\mathbf{k}Mv = V$ . Since  $\mathbf{k}Mv \subset V'$ , we have  $V' = V$ .  $\square$

**Lemma 2.1.27** (Schur's Lemma). Let  $S_1, S_2$  be simple  $\mathbf{k}M$ -modules. Then  $\text{hom}_{\mathbf{k}M}(S_1, S_2) = 0$  if  $S_1$  and  $S_2$  is not isomorphic and  $\text{hom}_{\mathbf{k}M}(S_1, S_2) \cong \mathbf{k}$  otherwise.

*Proof.* Let  $f \in \text{hom}_{\mathbf{k}M}(S_1, S_2)$ . Since  $\ker f$  is a submodule of  $S_1$  and  $\text{Im } f$  is a submodule of  $S_2$ ,  $f$  is injective if and only if it is surjective, if and only if it is non-zero. Thus,  $\text{hom}_{\mathbf{k}M}(S_1, S_2) \neq 0$  if and only if  $S_1 \cong S_2$ . Let  $f \in \text{hom}_{\mathbf{k}M}(S_1, S_1)$ . Because  $\mathbf{k}$  is algebraically closed,  $f$  has an eigenvalue  $\lambda$ , such that  $\ker(f - \lambda \text{Id}_{S_1}) \neq 0$ : this implies  $f = \lambda \text{Id}_{S_1}$ .  $\square$

**Proposition 2.1.28.** Let  $V$  be a  $\mathbf{k}M$ -module. Then the following are equivalent:

1. for any submodule  $V' \subset V$ , there exists  $V'' \subset V$  such that  $V = V' \oplus V''$ .
2. for any direct sum of simple submodules  $V' \subset V$ , there exists  $V'' \subset V$  such that  $V = V' \oplus V''$ .
3.  $V$  is a direct sum of simple modules.

In that case, we say that  $V$  is semisimple. As a representation, we use the term *completely reducible* instead.

*Proof.* Clearly (1) implies (2). Also, (2) applied recursively implies (3) (the recursion terminates as the modules are finite dimensional). Suppose (3) and let  $V' \subsetneq V$  be a maximal dimensional submodule of  $V$  for which (1) does not hold. By (3), there must be a simple submodule  $S \subset V$  with  $S \cap V' = 0$  (by simplicity of  $S$ ). So  $V' + S = V' \oplus S$ . Since  $V'$  is maximal with the property of not being a summand, there exists a submodule  $V''$  such that  $V'' \oplus (V' \oplus S) = V$ . This is absurd as we supposed that  $V'$  was not a direct summand, so (1) must hold.  $\square$

**Example 2.1.29.** Consider  $\text{Irr}_{\mathbf{k}M}$  a set of representative of the isomorphism classes of simple  $\mathbf{k}M$  modules (which is necessarily finite). Then

$$\bigoplus_{V \in \text{Irr}_{\mathbf{k}M}} V \otimes V^*$$

is a semisimple  $\mathbf{k}M$ -module.

It is quite straightforward that semi-simplicity is stable by direct sum, submodule, and quotient. Jacobson's density Theorem 2.1.47 ensures that is also stable by tensor product. Moreover, let  $f : V_1 \rightarrow V_2$  be a morphism of  $\mathbf{k}M$ -modules. If  $f$  is injective and  $V_2$  is semisimple, so is  $V_1$ . If  $f$  is surjective and  $V_1$  semisimple, so is  $V_2$ . Obviously, simple modules are in particular semisimple and simple modules are necessarily indecomposable. Sadly, for general (and even finite) monoids, the notions of indecomposable and simple modules do not coincide.

**Example 2.1.30.** Consider  $M = (\{0, 1\}, \times, 1)$  acting on itself by multiplication. Then the module  $\mathbf{k}M$  (where the scalar multiplication is purely formal) is not irreducible as  $\mathbf{k}0$  is a non-trivial submodule, but it is indecomposable because the image of the scalar multiplication by  $0$  is  $\mathbf{k}0$ .

However, simple and indecomposable modules are still elementary building blocks, as we have the two following theorems. The first one is the Krull-Schmidt Theorem (see [Jac09, Section 3.4] for a proof), that affirms that although when decomposing a module, we may not get to simple modules, at least the decomposition obtained is essentially unique.

**Theorem 2.1.31** (Krull-Schmidt). *Any finite dimensional  $\mathbf{k}M$ -module has a unique (up to reordering of the summands and isomorphism) decomposition as a direct sum into indecomposable modules.*

The second one is the Jordan-Hölder Theorem (see [Jac09, Section 3.3] for a proof), which affirms that although we may not access them by direct sum decomposition, when decomposing into simple modules via quotients, we get an essentially unique assemblage of simple modules.

**Theorem 2.1.32** (Jordan-Hölder). *Let  $V$  be any finite dimensional  $\mathbf{k}M$ -module. There exists  $n \in \mathbb{N}$  and a strictly increasing sequence of submodule of  $V$ ,  $0 = V_0 \subset V_1 \subset V_{n-1} \subset V_n = V$ , called a composition series, such that  $V_i/V_{i-1}$  is simple for all  $i \in \llbracket 1, n \rrbracket$ . These quotients are called composition factors and for a given simple module  $S$ , its multiplicity as a composition factor of  $V$  is independent of the choice of composition series and denoted  $[V : S]$ .*

**Projective modules** In particular, the Krull-Schmidt Theorem invites us to consider modules that are well-behaved with respect to direct sums.

**Definition 2.1.33.** A module  $P$  is *projective* if for any  $\mathbf{k}M$ -modules  $V, W$  and any morphism  $g : P \rightarrow V$  and surjective morphism  $f : W \rightarrow V$ , there exist a morphism  $h : P \rightarrow W$  such that  $g = f \circ h$  i.e. such that the following diagram commutes.

$$\begin{array}{ccc} & & W \\ & \nearrow h & \downarrow f \\ P & \xrightarrow{g} & V \end{array}$$

**Proposition 2.1.34.** If  $Q = \bigoplus_{i \in I} P_i$  is a (finite) direct sum of  $\mathbf{k}M$ -modules,  $Q$  is projective if and only if every  $P_i$  is.

*Proof.* We denote by  $\iota_i$  the canonical injection of  $P_i$  in  $Q$  and  $\pi_i$  the canonical projection of  $Q$  on  $P_i$ . Let  $V, W$  be  $\mathbf{k}M$ -modules and  $f : W \rightarrow V$  be any surjective morphism. Suppose  $Q$  is projective and let  $u : P_i \rightarrow V$  be any morphism. We have the following commutative diagram:

$$\begin{array}{ccccc} & & & & W \\ & & & \nearrow v & \downarrow f \\ Q & \xrightarrow{\pi_i} & P_i & \xrightarrow{u} & V \\ & \nwarrow \iota_i & & & \end{array}$$

with  $w$  existence assured by  $Q$  being projective and  $v$  defined as  $w \circ \iota_i$ . We must verify  $f \circ v = u$  but  $f \circ v = (f \circ w) \circ \iota_i = u \circ (\pi_i \circ \iota_i) = u$  because  $\pi_i \circ \iota_i = \text{Id}_{P_i}$ .

Suppose now that  $P_i$  is projective for all  $i \in I$ , and let  $u : Q \rightarrow V$  be any morphism. For all  $i \in I$ , we have the following commutative diagram:

$$\begin{array}{ccccc} & & & & W \\ & & & \nearrow v & \downarrow f \\ P_i & \xleftarrow{\iota_i} & Q & \xrightarrow{u} & V \end{array}$$

where  $v_i$  exists because  $P_i$  is projective and  $v = (v_i)_{i \in I}$ . We have  $f \circ v \circ \iota_i = u \circ \iota_i$  for all  $i \in I$  which implies  $f \circ v = u$ :  $Q$  is projective.  $\square$

**Proposition 2.1.35.** A free  $\mathbf{k}M$ -module is projective.

*Proof.* Let  $F$  be a free  $\mathbf{k}M$ -module with  $\mathbf{k}M$ -basis  $(e_i)_{i \in I}$ . Let  $V, W$  be  $\mathbf{k}M$ -modules,  $f : W \rightarrow V$  be a surjective morphism and  $u : F \rightarrow V$  be any morphism. By surjectivity of  $f$ , for all  $i \in I$ , there exists  $w_i \in W$  such that

$u(e_i) = f(w_i)$ . We define  $h(e_i) = w_i$  for all  $i \in I$ . Since  $F$  is free,  $h$  can be uniquely extended into a  $\mathbf{k}M$ -module morphism  $h$  such that  $fh = u$ :  $F$  is projective.  $\square$

As a consequence of this, any direct summand of  $\mathbf{k}M$  is projective. But the converse is also true!

**Proposition 2.1.36.** *A module is projective if and only if it is a summand of a free module.*

*Proof.* We took care of the “only if” in the previous proposition. Suppose that  $P$  is projective. Then it is a quotient of a free module  $F$ , meaning there exists a short exact sequence

$$0 \longrightarrow \ker f \xrightarrow{\iota} F \xrightarrow{f} P \longrightarrow 0$$

where  $\iota$  is the inclusion. By projectivity of  $P$ , we can transform this into:

$$\begin{array}{ccccccc} & & & & P & & \\ & & & g \swarrow & \downarrow \text{Id}_P & & \\ 0 & \longrightarrow & \ker f & \xrightarrow{\iota} & F & \xrightarrow{f} & P \longrightarrow 0 \end{array}$$

This implies that  $F = \text{Im } g \oplus \ker f \cong P \oplus \ker f$ . Indeed, let  $g(x) \in \ker f$ . Then  $f(g(x)) = 0$  so  $0 = (g \circ f \circ g)(x) = g(x)$ , so  $\text{Im } g \cap \ker f = 0$ . Since we have finite dimensional modules and  $\dim F = \dim \ker f + \dim \text{Im } g$ , this concludes:  $P$  is (isomorphic to) a summand of  $F$ .  $\square$

Projective modules interest us mainly because, as we have seen, they behave especially cleanly with regard to direct sum. We will see that this implies that there is a finite number of projective indecomposable modules in Proposition 2.1.37. The second reason is that the set of projective indecomposable modules is closely related to the set of simple modules, as we shall see in the next paragraph.

**Proposition 2.1.37.** *The projective indecomposable  $\mathbf{k}M$ -modules are precisely the indecomposable summands of  $\mathbf{k}M$ .*

*Proof.* Let  $\mathbf{k}M = \bigoplus_{i \in I} P_i$  the decomposition in indecomposable modules given by the Krull-Schmidt Theorem. From Proposition 2.1.36, the  $P_i$  are projective modules. Conversely, still by Proposition 2.1.36, if  $P$  is projective indecomposable, then there exists some  $Q \in \mathbf{k}\text{-mod}$  and  $n > 0$  such that  $P \oplus Q = (\mathbf{k}M)^n$ . But  $(\mathbf{k}M)^n = \bigoplus_{i \in I} P_i^n$ , so by unicity  $P$  is one of the  $P_i$ .  $\square$

In particular, since  $\mathbf{k}M$  is finite dimensional, there is a finite number of non-isomorphic projective indecomposable modules.

### 2.1.3 From projective to simple, and back.

The objective of this section is to show that the projective indecomposable  $\mathbf{k}M$ -modules are in natural bijection with the simple  $\mathbf{k}M$ -modules. We follow [CR66, Section 5.4] as well as [Ass97, Chapter VII and VIII].

#### Jacobson Radical.

To get there, we need to discuss the notion of maximal submodule and Jacobson radical. This will be immediately useful toward the cited goal, but also later when we will need some properties of the radical to derive a formula for the Character table in Section 3.1.

**Definition 2.1.38.** A submodule  $V'$  of a  $\mathbf{k}M$ -module  $V$  is *maximal* if  $V' \neq V$  and if it is maximal for inclusion.

**Proposition 2.1.39.** A submodule  $V'$  of a  $\mathbf{k}M$ -module  $V$  is maximal if and only if  $M/M'$  is simple.

*Proof.* Suppose  $M/M'$  is simple and let  $s \in M \setminus M'$ . Then  $s + M' \in M/M'$  is not the class of 0. By simplicity,  $s + M'$  generates  $M/M'$ , so  $\{s\} \cup M'$  generates  $M$ :  $M'$  is maximal. Conversely, if  $M'$  is maximal, for any  $s \in M \setminus M'$ ,  $\{s\} \cup M'$  generates  $M$ , or equivalently  $s + M'$  generates  $M/M'$ : any non-zero element of  $M/M'$  is generator,  $M/M'$  is simple.  $\square$

**Definition 2.1.40.** The *radical* of a  $\mathbf{k}M$ -module  $V$ , denoted by  $\text{rad}_{\mathbf{k}M}(V)$ , is the set:

$$\text{rad}_{\mathbf{k}M}(V) = \bigcap_{V' \subset V, \text{ maximal}} V'.$$

Because of Proposition 2.1.39, it is obvious that  $\text{rad}_{\mathbf{k}M}(V)$  could also be defined as the intersection of the kernel of all morphisms with a simple module as image:

$$\text{rad}_{\mathbf{k}M}(V) = \bigcap_{S \text{ simple}} \bigcap_{\varphi \in \text{hom}_{\mathbf{k}M}(V, S)} \ker \varphi.$$

For now, we consistently deal with modules over  $\mathbf{k}M$  and we momentarily drop  $\text{rad}_{\mathbf{k}M}(V)$  for  $\text{rad}(V)$  for the sake of lighter notation. However, in Section 3.1, we will simultaneously deal with modules on different algebras for which the heavier notation will be necessary.

**Proposition 2.1.41.** Let  $V$  be  $\mathbf{k}M$ -module. Then  $\text{rad}(V)$  is the smallest (for inclusion) submodule such that  $V/\text{rad } V$  is semisimple.



*Proof.* First, suppose that  $V/V' = \bigoplus_{i \in I} S_i$  is semisimple (with  $S_i$  simple for all  $i$ ), and let  $\pi$  be the canonical projection of  $V$  onto  $V/V'$  and  $\pi_i$  be that of  $V/V'$  onto  $S_i$ . Then  $\ker \pi = \bigcap_{i \in I} \ker \pi_i \pi \subset \text{rad}(V)$  because  $\pi_i \pi : V \rightarrow S_i$  is surjective and thus its kernel is a maximal submodule.

Secondly, the maximal submodules of  $Q = V/\text{rad } V$  lift to maximal submodules of  $V$ , and they intersect at 0, so this quotient has 0 as radical. This implies that it is semisimple. Indeed, because  $Q$  is finite dimensional, we may choose a minimal dimensional element among the finite intersections of maximal submodules, say  $Q' = \bigoplus_{i \in I} Q_i$  where  $Q_i$  is maximal in  $Q$  for all  $i$ . Let  $Q''$  be any maximal submodule. Then by minimality of  $Q'$ ,  $Q'' \cap Q' = Q'$ . This is true for any  $Q''$ , so  $\text{rad}(Q) \cap Q' = Q' = 0$ . Let  $\pi_i$  be the canonical projection of  $Q$  onto  $Q/Q_i$ . Then  $f = (\pi_i)_{i \in I} : Q \rightarrow \bigoplus_{i \in I} Q/Q_i$  is injective, so  $Q$  is semisimple.  $\square$

**Corollary 2.1.42.** *In particular, a  $\mathbf{k}M$ -module is semisimple if and only if its radical is 0.*

Since a ring is a bimodule on itself, we have a slight ambiguity when it comes to apply the previous definition this is dissipated by the next proposition (see [Ass97, Section VII-3]).

**Proposition 2.1.43.** *The following definitions of the Jacobson radical of  $\mathbf{k}M$ , denoted by  $\text{rad}(\mathbf{k}M)$  are equivalent:*

1. *The intersection of two-sided maximal ideals.*
2. *The intersection of left maximal ideals.*
3. *The intersection of right maximal ideals.*
4. *The set  $\{x \in \mathbf{k}M \mid xS = 0 \text{ for all simple } \mathbf{k}M - \text{mod } S\}$ .*

Note that as submodules of an algebra are its ideals, this is essentially the same definition as before, and it only gives the reassurance that the side on which the algebra is acting on itself does not matter. In our case, where the algebra is finite dimensional, we have a convenient description of the radical of a module in terms of the Jacobson radical of the algebra (see [Ass97, VII-Theorem 4.6]).

**Proposition 2.1.44.** *Let  $V$  be a finite dimensional  $\mathbf{k}M$  module. Then*

$$\text{rad}_{\mathbf{k}M}(V) = \text{rad}(\mathbf{k}M)V.$$

**Remark.** Because of this, we say that an algebra (in our case  $\mathbf{k}M$ ) is semisimple if all its modules are semisimple and, equivalently, if its radical is 0.

**Example 2.1.45.** If  $M$  is a group and  $k$  is a field whose characteristic does not divide  $|G|$ , then  $\mathbf{k}G$  is a semisimple algebra (this is called *Maschke's Theorem*, see [Eti+09, Theorem 4.1.1]). This case is called *ordinary* while the “bad case”, where  $\text{char}(\mathbf{k})$  does divide  $|G|$ , is called the *modular* case.

**Definition 2.1.46.** The *top* (or *cosocle*) of a module  $V$  over  $\mathbf{k}M$ , denoted by  $\text{top}_{\mathbf{k}M}(V)$  or  $\text{top}(V)$  when clear in context, is  $V/\text{rad}_{\mathbf{k}M}(V)$ . The top of the algebra  $\mathbf{k}M$  is its top as a module on itself.

Since clearly  $\text{top}(\mathbf{k}M)$  is itself an algebra, and given Proposition 2.1.43.(4), simple  $\mathbf{k}M$ -modules are in bijection with simple  $\text{top}(\mathbf{k}M)$ -modules because the radical acts as zero on any simple module. The question of describing the decomposition of  $\text{top}(\mathbf{k}M)$  into simple modules arises naturally. To answer this question, we use the following result from [Jac45], for which a proof can be found in [Eti+09, Theorem 3.2.2].

**Theorem 2.1.47** (Jacobson's density Theorem). *Let  $V$  be a simple  $\mathbf{k}M$ -module. The map  $\mathbf{k}M \rightarrow \mathcal{L}(V)$  given by  $a \mapsto (v \mapsto a \cdot v)$  is surjective.*

**Proposition 2.1.48.** *Consider a set of representative of the isomorphism classes of simple  $\mathbf{k}M$  modules denoted  $\text{Irr}_{\mathbf{k}M}$ . Then*

$$\text{top}(\mathbf{k}M) \cong \bigoplus_{V \in \text{Irr}_{\mathbf{k}M}} V \otimes V^*$$

*is a semisimple  $\mathbf{k}$ -module.*

*Proof.* As noted in Example 2.1.29, this is a semisimple  $\mathbf{k}M$ -module and thus also a semisimple  $\text{top}(\mathbf{k}M)$ -module. Because  $\mathcal{L}(V) \cong V \otimes V^*$  as a  $\text{top}(\mathbf{k}M)$ -module, from Theorem 2.1.47, the direct sum is a quotient of  $\mathbf{k}M$ . Because of Proposition 2.1.41, it suffices to see that its kernel is contained in  $\text{rad}(\mathbf{k}M)$ . But from Proposition 2.1.43.(4), if  $x \notin \text{rad}(\mathbf{k}M)$  then  $xV \neq 0$  for some  $V \in \text{Irr}_{\mathbf{k}M}$  so  $x$  does not act as 0 on one of the summands.  $\square$

In the case where  $\mathbf{k}M$  is already semisimple,  $\text{top}(\mathbf{k}M) = \mathbf{k}M$  and this result is called the *Wedderburn-Artin Theorem*.

### The bijection between simple and projective indecomposable.

To get to the bijection between projective indecomposable and simple modules, we now can apply the above definition toward describing the radical of the indecomposable projective modules.

**Proposition 2.1.49.** *Let  $P$  be a projective indecomposable  $\mathbf{k}M$ -module. Then  $P$  admits one unique maximal submodule.*

*Proof.* The collection of strict submodules is non-empty (as 0 is one) and because  $P$  is finite dimensional, we can just choose a submodule of maximal dimension. Let us show that this choice is unique. Suppose that  $M, M'$  are two distinct maximal submodules of  $P$ . We have the following commutative diagram.

$$\begin{array}{ccccc}
 & & v & \cdots & P \\
 & & \swarrow & & \downarrow \pi' \\
 M & \xrightarrow{\iota} & P & \xrightarrow{\pi'} & P/M' \\
 & \searrow & \swarrow \pi'\iota & & \\
 & & & & 
 \end{array}$$

The canonical projection on  $P/M'$  is  $\pi'$  and  $\iota : M \rightarrow P$  is the inclusion. The morphism  $\pi'\iota$  has a submodule of the simple module  $P/M'$  as image. Therefore,  $\pi'\iota(M) = 0$  or  $P/M'$ . The first possibility implies that  $M \subset \ker \pi' = M'$  and by maximality,  $M = M'$ . Thus,  $\pi'\iota$  must be surjective. Since  $P$  is projective, there exists  $v : P \rightarrow M$  making the diagram commute. Consider  $\iota v : P \rightarrow P$ : since  $P$  is finite dimensional, for some  $n > 0$ , we have  $P = \text{Im}(\iota v)^n \oplus \ker(\iota v)^n$ . But  $P$  is indecomposable so either  $P = \ker(\iota v)^n$ , in which case we read on the diagram that  $0 \neq \pi'(P) = \pi'(\iota v)^n(P) = 0$ : this is absurd. The remaining possibility is that  $P = \text{Im}(\iota v)^n$ , but it would mean that  $\iota$  is surjective, which is impossible. In conclusion,  $M$  and  $M'$  cannot be distinct.  $\square$

**Corollary 2.1.50.** *If  $P$  is a projective indecomposable  $\mathbf{k}M$ -module,  $P/\text{rad}(P)$  is a simple  $\mathbf{k}M$ -module.*

This map is the bijection between projective and simple modules that we are after. Let us first show that it is an injection.

**Proposition 2.1.51.** *If  $V_1, V_2$  are  $\mathbf{k}M$ -modules and  $f : V_1 \rightarrow V_2$  is a morphism,  $f(\text{rad } V_1) \subset \text{rad } V_2$ .*

*Proof.* Let  $S$  be a simple  $\mathbf{k}M$ -module and  $g : V_2 \rightarrow S$  be a morphism. By definition of the radical,  $g \circ f : V_1 \rightarrow S$  vanishes on  $\text{rad } V_1$ , so  $g$  vanishes on  $f(\text{rad } V_1)$  for any  $g$  with a simple module as image:  $f(\text{rad } V_1) \subset \text{rad } V_2$ .  $\square$

This shows that a morphism  $f : V_2 \rightarrow V_2$  factors to a morphism  $\tilde{f} : V_1/\text{rad } V_1 \rightarrow V_2/\text{rad } V_2$  defined by  $\tilde{f}(v + \text{rad } V_1) = f(v) + \text{rad } V_2$ . In other words, the following diagram is commutative:

$$\begin{array}{ccc}
V_1 & \xrightarrow{f} & V_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
V_1/\text{rad } V_1 & \xrightarrow{\tilde{f}} & V_2/\text{rad } V_2
\end{array}$$

where  $\pi_i$  is the canonical projection of  $V_i$  onto  $V_i/\text{rad } V_i$ ,  $i = 1, 2$ .

**Lemma 2.1.52** (Nakayama's Lemma). *Let  $f : V_1 \rightarrow V_2$  be a morphism between two finite dimensional  $\mathbf{k}M$ -modules. Then  $f$  is surjective if and only if the induced morphism  $\tilde{f} : V_1/\text{rad } V_1 \rightarrow V_2/\text{rad } V_2$  is surjective.*

*Proof.* Using the notations on the diagram, if  $f$  is surjective, since  $\pi_2$  is too,  $\tilde{f}$  must be. Conversely, suppose  $\tilde{f}$  is surjective. This implies that  $V_2 = f(V_1) + \text{rad}(V_2)$ . If  $f(V_1) \neq V_2$  we choose  $f(V_1) \subset W \subsetneq V_2$  a maximal submodule (we can, since  $V_2$  is finite dimensional). Then  $V_2 = f(V_1) + \text{rad } V_2 \subset W + \text{rad } V_2 \subset W \subsetneq V_2$ . This is absurd, so  $f(V_1) = V_2$ .  $\square$

**Proposition 2.1.53.** *Let  $P_1, P_2$  be projective, finite dimensional,  $\mathbf{k}M$ -modules. Then  $P_1, P_2$  are isomorphic if and only if  $P_1/\text{rad } P_1$  and  $P_2/\text{rad } P_2$  are isomorphic.*

*Proof.* If  $P_1$  and  $P_2$  are isomorphic, clearly, so are their semisimple quotient. Conversely, suppose that  $f : P_1/\text{rad } P_1 \rightarrow P_2/\text{rad } P_2$  is an isomorphism. We have the following commutative diagram.

$$\begin{array}{ccc}
P_1 & \xrightarrow{\quad v \quad} & P_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
P_1/\text{rad } P_1 & \xrightarrow{f} & P_2/\text{rad } P_2
\end{array}$$

In particular  $\tilde{f}\pi_1$  is surjective, so  $f$  exists because  $P_2$  is a projective module. Since  $f$  is surjective, so is  $v$  (by Lemma 2.1.52). By considering  $f^{-1}$ , we get  $w : P_2 \rightarrow P_1$  surjective.  $P_1, P_2$  thus have the same dimension, and  $v$  is in fact an isomorphism.  $\square$

We now know that any projective indecomposable module corresponds to a unique (up to isomorphism) simple module. It remains to see why this map is surjective. A complete demonstration of this fact would necessitate quite a long introduction to the general theory of algebra representations, and thus we admit the following result. We refer to [CR66, Section 54] for a detailed proof.

**Definition 2.1.54.** Let  $A$  be an algebra (which in our case, by supposition, is unitary). Like in semigroups, an idempotent  $e \in A$  is an element such that  $e^2 = e$ . Two idempotents  $e, f$  are *orthogonal* if  $ef = fe = 0$ . An idempotent  $e$  is *primitive* if whenever  $e = e_1 + e_2$  with  $e_1, e_2$  idempotent,  $e_1 = 0$  or  $e_2 = 0$ . A set  $\{e_1, \dots, e_n\}$  of primitive orthogonal idempotents form a *decomposition of unity* if  $1 = \sum e_i$ .

**Proposition 2.1.55.** • *There exists  $\{e_1, \dots, e_n\}$  a decomposition of unity of  $\mathbf{k}M$  in primitive orthogonal idempotents.*

- *The projective indecomposable  $\mathbf{k}M$ -modules are the  $\mathbf{k}Me_i$ .*
- $\text{rad}(\mathbf{k}Me_i) = \text{rad}(\mathbf{k}M)e_i$ .
- *A  $\mathbf{k}M$ -module  $V$  has a composition factor isomorphic to  $\mathbf{k}Me_i/\text{rad}(\mathbf{k}M)e_i$  if and only if  $e_iV \neq 0$ .*

**Proposition 2.1.56.** *The map  $P \mapsto P/\text{rad } P$  is a bijection between isomorphism classes of indecomposable projective modules.*

*Proof.* Corollary 2.1.50 gives the injectivity. Let  $S$  be a simple  $\mathbf{k}M$ -module, and  $\sum e_i$  be a decomposition of unity into primitive orthogonal idempotents. Then  $1S = S = \sum e_i S$ . So for some  $e_i$ ,  $e_i S = S$ . On the other hand, if  $P_i = e_i \mathbf{k}M$ , then  $S_i = P_i/\text{rad } P_i$  simple and clearly  $e_i S_i = S_i$ . So  $S \cong S_i$ : we have the surjectivity.  $\square$

## 2.2 Characters

One of the major features of finite group representation theory is the fact that all the information on a representation can be summarized in its *character*. This (partially) carries over to monoid representation theory. In this section, we give the elementary results of character theory needed for our purpose: mainly how to compute with them and what the character table is. We have in mind the goal of explicitly computing these objects, and, in particular, we would like to know *where* to compute a character. Note that, until now, we have remained agnostic regarding the characteristic of the field  $\mathbf{k}$ , only supposing it algebraically closed. This agnosticism has to stop in this section, where we will have different results for the 0 and positive characteristics.

**Definition 2.2.1.** If  $V$  is a finite dimension  $\mathbf{k}M$ -module, its *character* is the map from  $M$  to  $\mathbf{k}$  defined by  $\chi_{\mathbf{k}M}^V : m \mapsto \text{Tr}(v \mapsto m \cdot v)$ .

We recall the following well-known facts about characters. Proofs for fact 2 and 3 are respectively (ii) and (iii) of [Ste16, Proposition 7.12]<sup>1</sup>.

**Proposition 2.2.2.** 1. Let  $V$  be a  $\mathbf{k}M$ -module. We have  $\chi_{\mathbf{k}M}^V = \chi_{\mathbf{k}M^{op}}^{V^*}$ .

2. Consider the short exact sequence of  $\mathbf{k}M$ -modules :

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0.$$

$$\text{Then } \chi_{\mathbf{k}M}^{B/A} = \chi_{\mathbf{k}M}^B - \chi_{\mathbf{k}M}^A.$$

3. Consider  $M, M'$  two finite monoids,  $V$  a  $\mathbf{k}M$ -module and  $W$  a  $\mathbf{k}M'^{op}$ -module. Then  $\chi_{\mathbf{k}M \otimes \mathbf{k}M'}^{V \otimes W} = \chi_{\mathbf{k}M}^V \chi_{\mathbf{k}M'}^W$ .

The previous properties are simply extensions of similar properties on groups, and their proof is similar. From groups, we also keep in the case of monoids the linear independence of irreducible characters (see [MQS15] for the general case):

**Proposition 2.2.3.** The irreducible characters  $\{\chi_{\mathbf{k}M}^S \mid S \text{ is a simple } \mathbf{k}M\text{-mod}\}$  are linearly independent as  $\mathbf{k}$  valued functions.

This, together with the second point in the Proposition 2.2.2, has a nice consequence. As we are interested in finite dimensional modules over finite monoids, those modules have a composition series. Say that a  $\mathbf{k}M$ -module  $V$ , has  $S$  as a composition factor with multiplicity  $[V : S]$  for any simple  $\mathbf{k}M$ -module  $S$ . Then:

$$\chi_{\mathbf{k}M}^V = \sum_S [V : S] \chi_{\mathbf{k}M}^S.$$

In that way, since characters of the simple modules are linearly independent, the character of a module can be seen as a record of its composition factors.

The question of where to compute characters is worth asking: in the case of groups, one needs only to compute the character for a transversal of conjugacy classes to get its value everywhere.

**Definition 2.2.4.** We say that two elements  $m, m'$  in  $M$  are in the same *generalized conjugacy class* or *character equivalency class* if for every  $\mathbf{k}M$ -module  $V$ ,  $\chi_M^V(m) = \chi_M^V(m')$ . We denote by  $C_M$  the set of generalized conjugacy classes.

The description in the case of monoids in 0 characteristic was described for the first time by McAlister in [McA72]. The case of the positive characteristic

---

<sup>1</sup>Note that for Fact 3, our reference deals only with the case  $M = M'$ , but the proof is the same.

is elucidated by Masuda, Quooos and Steinberg in [MQS15]. More modestly, we only want to be able to compute a transversal of these character equivalency classes. The particular result we are interested in is [MQS15, Corollary 2.16], stated in the case of an algebraically closed field.

**Definition 2.2.5.** Let  $p$  be a prime number or 0. An element  $m \in M$  is  $p$ -regular if it is a group element and either  $p = 0$  or  $p$  and the order of  $m$  are relatively prime.

**Proposition 2.2.6** ([MQS15]). *Let  $\mathbf{k}$  be a field of any characteristic and  $\mathcal{E} = \{e_1, \dots, e_n\}$  be idempotent representatives of the regular  $\mathcal{J}$ -classes of  $M$ . For each  $e_i$ , let  $\mathcal{C}_i = \{c_{i,1}, \dots, c_{i,m_i}\}$  be representatives of the conjugacy classes of the  $p$ -regular elements of  $G_{e_i}$ . Then the set  $\mathcal{C}_M = \bigcup_{e_i \in \mathcal{E}} \mathcal{C}_i$  is a set of representatives of character equivalency classes of  $M$ .*

We can now recall the definition of the character table of a monoid.

**Definition 2.2.7.** Let  $\text{Irr}_M$  be the set of isomorphism classes of simple  $\mathbf{k}M$ -modules and  $\mathcal{C}_M$  as in definition 2.2.4. The *character table* of  $M$  over  $\mathbf{k}$  is the (square) matrix defined by :

$$X(M) = (\chi_{\mathbf{k}M}^V(m))_{V \in \text{Irr}_M, m \in \mathcal{C}_M}.$$

Moreover, anticipating on Definition 2.4.2, if  $e \in M$  is an idempotent, we define  $X_e(M)$  as the matrix obtained by extracting from  $X(M)$  only the rows corresponding to simple modules with apex  $e$ .

Stated simply, the (matrix inverse of the) character table takes the character of a representation and gives back the vector of multiplicities of the simple modules as composition factors.

## 2.3 Cartan Matrix

We have just seen that projective indecomposable modules and simple modules are in correspondence, which allows for the definition of the *Cartan matrix* of the algebra  $\mathbf{k}M$ .

**Definition 2.3.1** (Cartan matrix). The *Cartan (invariants) matrix* of  $\mathbf{k}M$  is the integer matrix  $C(\mathbf{k}M)$  with columns indexed by the set  $\{P_i\}_{i \in I}$  of projective modules and rows indexed by the set  $\{S_i\}_{i \in I}$  of simple modules defined by:

$$C(\mathbf{k}M)_{i,j} = ([P_j : S_i]) \text{ for } i, j \in I.$$

We have seen that it is a square matrix, with positive entries. It can be seen as a measure of how “not semisimple” the algebra of the monoid is: if every indecomposable is also simple, the Cartan Matrix is the identity matrix. Conversely, in a loose sense, the more non-diagonal entries are non-zero, and the higher the coefficients are, the less semisimple the algebra is. The objective of this section is to prove the following result due to Thiéry [Thi12].

**Proposition 2.3.2** (Thiéry [Thi12]). *Let  $\{S_1, \dots, S_n\}$  be a set of representatives of the isomorphism classes of simple  $\mathbf{k}M$ -modules. Denote by  $[\mathbf{k}M : S_i \otimes S_j^*]$  the multiplicity of  $S_i \otimes S_j^*$  as a composition factor of  $\mathbf{k}M$ . The Cartan matrix of  $\mathbf{k}M$  is equal to:*

$$C(\mathbf{k}M) = ([\mathbf{k}M : S_i \otimes S_j^*])_{i,j}$$

Note that this makes formal sense: recall that we assumed that  $\mathbf{k}$  is algebraically closed and that all modules are finite dimensional. Then we have the following result (see [Eti+09, Proposition 3.10.2]):

**Proposition 2.3.3.** *The  $S_i \otimes S_j^*$  are the simple  $\mathbf{k}M \otimes \mathbf{k}M^{op}$ -modules where  $S_i, S_j$  run over the simple  $\mathbf{k}M$ -modules.*

In other words, the Cartan matrix is a recording of the multiplicities of the composition factors of  $\mathbf{k}M$  as a  $\mathbf{k}M \otimes \mathbf{k}M^{op}$  module. But so is its character! The difference being that the character of  $\mathbf{k}M$  as it is computed is expressed in the basis of the character equivalency classes of  $\mathbf{k}M \otimes \mathbf{k}M^{op}$  while the Cartan matrix is expressed directly in the basis of the simple modules. Since the basis change between the two is precisely given by the character table, we have Thiéry’s Formula for the Cartan matrix.

**Corollary 2.3.4** (Thiéry [Thi12]). *The Cartan matrix is given by the formula:*

$$C(\mathbf{k}M) = {}^t X_M^{-1} B X_M^{-1}$$

where  $B = (|\{s \in M \mid msm' = s\}|)_{m,m' \in C_M}$

*Proof.* It follows from the previous remark and the fact that  $B$  is indeed the character of  $\mathbf{k}M$  as a  $\mathbf{k}M \otimes \mathbf{k}M^{op}$ -module: for any  $(m, m') \in M \times M^{op}$ ,  $v \mapsto mvm'$  is a transformation of the set  $M$  as a basis of  $M$ . As a consequence, the  $s^{\text{th}}$  line of the matrix of  $v \mapsto mvm'$  contributes 1 to the trace if  $s = msm'$  and 0 otherwise.  $\square$

We will now present a proof of Proposition 2.3.2 following [Ste16, Section 7.5]. The first and main ingredient is [Ste16, Proposition 7.27], itself a reformulation in our context of Proposition 11.3 and its proof in [SY11].



**Theorem 2.3.5.** *Let  $e_1, \dots, e_n$  be a decomposition of unity in  $\mathbf{k}M$  into primitive orthogonal idempotents. Then  $\{e_i \otimes e_j \mid i, j \in \llbracket 1, n \rrbracket\}$  is a decomposition of the unity of  $\mathbf{k}M \otimes \mathbf{k}M^{op}$  into primitive orthogonal idempotents.*

*Moreover, if for any  $k \in \llbracket 1, n \rrbracket$ ,  $\mathbf{k}Me/\text{rad}(\mathbf{k}M)e$  is the simple  $\mathbf{k}M$ -module corresponding to an idempotent  $e$ , then the simple  $\mathbf{k}M \otimes \mathbf{k}M^{op}$ -module corresponding to  $e_i \otimes e_j$  is  $S_i \otimes S_j^*$ .*

Note that Propositions 2.3.3 and Theorem 2.3.5 are essentially the same, the latter putting more emphasis on the role of the idempotents. This result being stated, we just need the classical lemma that follows to finally give a proof of Proposition 2.3.2. A proof can be found in [Ass97, Proposition VIII-1].

**Lemma 2.3.6.** *Let  $e \in \mathbf{k}M$  an idempotent and  $V \in \mathbf{k} - \text{mod}$ . Then*

$$\text{hom}_{\mathbf{k}M}(\mathbf{k}Me, V) \cong eV.$$

*Proof of Proposition 2.3.2.* Let us denote  $\mathbf{k}M^\circ$  for  $\mathbf{k}M \otimes \mathbf{k}M^{op}$ . Let  $e_1, \dots, e_n$  be a decomposition of the unity of  $\mathbf{k}M$  into primitive indecomposable idempotents. For  $k \in \llbracket 1, n \rrbracket$ ,  $P_k = \mathbf{k}Me_k$  and  $S_k = P_k/\text{rad}(P_k)$ . We have  $[P_j : S_i] = [\mathbf{k}Me_j : S_i] = \dim e_i \mathbf{k}Me_j$  by Lemma 2.3.6. But  $e_i \mathbf{k}Me_j = (e_i \otimes e_j) \mathbf{k}M$  as a  $\mathbf{k}M^\circ$ -module. In the end,

$$[P_j : S_i] = \dim(e_i \otimes e_j) \mathbf{k}M = [\mathbf{k}M : S_i \otimes S_j^*]$$

because from Theorem 2.3.5 the simple module associated to  $e_i \otimes e_j$  is  $S_i \otimes S_j^*$ .  $\square$

## 2.4 Clifford-Munn-Ponizovskii

We recall that  $\mathbf{k}$  is an algebraically closed field (in particular, it is perfect) and  $M$  remains a finite monoid. In this section, we deal with monoid representation theory, with the goal in mind to compute the character table of  $M$ . Using the Clifford-Munn-Ponizovskii Theorem, this can largely be reduced to group representation theory. Stated differently, the representation theory of a monoid  $M$  is an extension of the representation theory of certain groups embedded in  $M$ . The groups in question are precisely the groups of Definition 1.3.6.

We have previously stated that the representation theory of monoids is an extension of the representation theory of some subgroups. This mainly expressed using the two following functors.

**Definition 2.4.1.** Let  $e \in M$  be an idempotent,  $\mathcal{L}(e)$  its  $\mathcal{L}$ -class,  $G_e$  the associated maximal subgroup. We define the two following maps :

$$\begin{aligned} \text{Ind}_{G_e}^M : & \begin{cases} \mathbf{k}G_e\text{-mod} \longrightarrow \mathbf{k}M\text{-mod} \\ V \longmapsto \mathbf{k}\mathcal{L}(e) \otimes_{\mathbf{k}G_e} V \end{cases} \\ N_e : & \begin{cases} \mathbf{k}M\text{-mod} \longrightarrow \mathbf{k}M\text{-mod} \\ V \longmapsto \{v \in V \mid eMv = 0\} \end{cases} . \end{aligned}$$

Note that if we choose an element  $a \in M$  and denote by  $L$  its  $\mathcal{L}$ -class and  $H$  its  $\mathcal{H}$ -class, we can equip  $\mathbf{k}L$  with a  $\mathbf{k}M\text{-mod} - \mathbf{k}\Gamma'(H)$  structure.  $\mathbf{k}L$  is already a  $\text{mod} - \mathbf{k}\Gamma'(H)$  by definition of  $\Gamma'(H)$ . Since  $H$  contains an idempotent  $e$ ,  $\Gamma'(H) \cong \mathcal{H}(e) = G_e$ . Moreover, this isomorphism is canonical. We can also make it into a  $\mathbf{k}M\text{-mod}$  by setting, for every  $m \in M$  and  $l \in L$ :

$$m \cdot l = \begin{cases} ml & \text{if } ml \in L \\ 0 & \text{otherwise} \end{cases} .$$

This is well-defined, as  $ml \notin L$  implies that  $l >_{\mathcal{L}} ml$  and so for every  $m' \in M$ ,  $l >_{\mathcal{L}} m'ml \notin L$ . This makes  $\mathbf{k}\mathcal{L}(e) \otimes_{\mathbf{k}G_e} V$  a well-defined  $\mathbf{k}M$ -module.

We are almost ready to state the Clifford-Munn-Ponizovskii Theorem, which is the central piece connecting group and monoid representation theory. We will need the notion of the *apex* of a  $\mathbf{k}M$ -module. A complete proof module theoretic proof of the Clifford-Munn-Ponizovskii Theorem can be found in [Ste16, Chapter 5]. We follow the exposition given in [GMS09, Theorem 7].

**Definition 2.4.2.** If  $V$  is a  $\mathbf{k}M\text{-mod}$ , we denote its annihilator in  $M$  by  $\text{Ann}_M(V) = \{m \in M \mid mV = 0\}$ . This is clearly a two-sided ideal of  $M$  and, as such, is a union of  $\mathcal{J}$ -classes. Let  $V$  be a  $\mathbf{k}M\text{-mod}$ . A regular  $\mathcal{J}$ -class  $J$  is said to be the *apex* of  $V$  if  $\text{Ann}_M(V) = I_J$  where  $I_J = \{s \in M \mid J \not\leq_{\mathcal{L}} \mathcal{J}(s)\}$ . If  $e \in J$  is an idempotent, we also say that  $V$  has apex  $e$ .

Note in terms of idempotents, the definition allows for a module to have more than one apex, but in that case, they must both lie in the same  $\mathcal{J}$ -class.

**Proposition 2.4.3.** *Let  $V$  be a finite dimensional simple  $\mathbf{k}M$ -module. Then  $V$  has an apex.*

*Proof.* Because  $\mathbf{k}M$  is unitary,  $\mathbf{k}MV \neq 0$ . Let  $J$  be a  $\leq_{\mathcal{J}}$ -minimal  $\mathcal{J}$ -class such that  $JV \neq 0$ . The ideal  $I = MJM$  contains  $J$  and for any  $m \in I \setminus J$ ,  $m \leq_{\mathcal{J}} J$ . By minimality  $\mathbf{k}IV = \mathbf{k}JV \neq 0$ . By simplicity, this implies that  $\mathbf{k}JV = V$ . From this, we get that  $\text{Ann}_M(V) \subset I_J$  : by definition of being  $\leq_{\mathcal{J}}$  above, if  $j \in J \leq_{\mathcal{J}} m \in M$  and  $mV = 0$  there exists  $u, v \in M$  such that  $j = umv$  and so  $jV = 0$ . Hence, the first inclusion. Moreover,  $I_J J \subset$

$I \setminus J \subset \text{Ann}_M(V)$  because elements of  $I_J$  are not above  $J$ , they fall into a  $\mathcal{J}$ -class lower than  $J$  once multiplied by an element of  $J$ . We indeed have  $\text{Ann}_M(V) = I_J$ . We now must show that  $J$  is regular, that is that  $JJ \cap J \neq \emptyset$  (Corollary 1.3.3). But if that was the case, then  $JJ \subset I \setminus J \subset \text{Ann}_M(V)$  which is absurd.  $\square$

**Theorem 2.4.4** (Clifford-Munn-Ponizovskii). *Let  $M$  be a finite monoid,  $e \in M$  an idempotent and  $\mathbf{k}$  be a field.*

1. *There is a bijection between isomorphism classes of simple  $\mathbf{k}M$ -modules with apex  $e$  and isomorphism classes of simple  $\mathbf{k}G_e$ -modules given by :*

$$V \mapsto V^\# = \text{Ind}_{G_e}^M(V) / \text{rad}(\text{Ind}_{G_e}^M(V)).$$

*The reciprocal bijection is given by  $S \mapsto eS$ .*

2.  $\text{rad}(\text{Ind}_{G_e}^M(V)) = N_e(\text{Ind}_{G_e}^M(V))$ .
3. *Every composition factor of  $\text{Ind}_{G_e}^M(V)$ , except for  $V^\#$ , has an apex strictly  $\mathcal{J}$ -greater than  $e$ . Moreover,  $V^\#$  has apex  $e$  and is a factor of multiplicity one.*

The proof relies on the following more general result from Green [Gre06, Proposition 6.2], and we present a formulation taken from [GMS09, Lemma 6].

**Lemma 2.4.5.** *Let  $A$  be an algebra and  $e \in A$  an idempotent. Then:*

1. *If  $V$  is a simple  $A$ -module, then either  $eV = 0$  or  $eV$  is a simple  $eAe$ -module.*
2. *If  $V$  is a simple  $eAe$ -module, then  $Ae \otimes_{eAe} V$  has a unique maximal  $A$ -submodule  $N = \{w \in Ae \otimes_{eAe} V \mid eAw = 0\}$ .*
3.  $V^\# = Ae \otimes_{eAe} V / N$  *is the unique simple  $A$ -module such that  $eV^\# \cong V$ .*

*Proof of Theorem 2.4.4.* Let  $e \in M$  be an idempotent,  $J = \mathcal{J}(e)$  and  $V$  be a simple  $\mathbf{k}M$ -module with apex  $e$ . Clearly  $V$  is also a simple  $A_J = \mathbf{k}M/\mathbf{k}I_J$ -module as elements in  $I_J$  act as 0. Now  $eA_Je = \mathbf{k}eMe/\mathbf{k}eI_Je = \mathbf{k}G_e$ . In the same way, as  $\mathbf{k}I_J$  acts as 0,  $Ae \otimes_{eAe} V \cong A_Je \otimes_{\mathbf{k}G_e} V$ . By definition of the  $\mathcal{J}$  order,  $A_Je \leq_{\mathcal{J}} e$  so  $A_Je \subset \mathbf{k}\mathcal{J}(e)$ . From the Location Theorem 1.3.2, we have in fact  $A_Je = \mathbf{k}\mathcal{L}(e)$ . The Clifford-Munn-Ponizovskii Theorem is obtained by applying Lemma 2.4.5 replacing  $A$  by  $\mathbf{k}M$  and  $Ae \otimes_{eAe} V$  by  $\mathbf{k}\mathcal{L}(e) \otimes_{\mathbf{k}G_e} V$ .  $\square$

Since we have seen that  $G_e \cong G_f$  whenever  $e\mathcal{J}f$ , the simple  $\mathbf{k}M$ -modules are indexed by the simple  $\mathbf{k}G_e$ -modules where  $e$  runs over a family of idempotent representatives of the regular  $\mathcal{J}$ -classes. Note also that this theorem is independent of the characteristic of the field  $\mathbf{k}$ .

## Computing representation invariants

### 3.1 A formula for the character table

In this section, we will often keep notations between results. Thus, we fix once and for all a finite monoid  $M$ ,  $e \in M$  an idempotent,  $G_e$  the maximal subgroup at  $e$ , and  $\text{Irr}_e$  a set of representatives of the isomorphism classes of simple  $\mathbf{k}G_e$ -modules. The field  $\mathbf{k}$  is supposed algebraically closed, but no hypothesis is made on its characteristic.

The previous discussion, and in particular the Clifford-Munn-Ponizovskii Theorem, allows us the following description of  $\mathbf{k}\mathcal{L}(e)$  for an idempotent  $e$ .

**Proposition 3.1.1.** *With the previously defined notations, we have:*

$$\text{top}_{\mathbf{k}G_e^{\text{op}}}(\mathbf{k}\mathcal{L}(e)) \cong \bigoplus_{V \in \text{Irr}_e} \text{Ind}_{G_e}^M(V) \otimes_{\mathbf{k}} V^*$$

where  $V^*$  is the dual of  $V$  and  $\text{top}_{\mathbf{k}G_e^{\text{op}}}(\mathbf{k}\mathcal{L}(e))$  is taken to mean the top of  $\mathbf{k}\mathcal{L}(e)$  as a right module over  $\mathbf{k}G_e$ .

*Proof.* By definition,  $\text{Ind}_{G_e}^M(V) = \mathcal{L}(e) \otimes_{\mathbf{k}G_e} V$ . Now, since direct sum and tensor product over a ring with identity commute :

$$\bigoplus_{V \in \text{Irr}_e} \text{Ind}_{G_e}^M(V) \otimes V^* = \bigoplus_{V \in \text{Irr}_e} \mathbf{k}\mathcal{L}(e) \otimes_{G_e} V \otimes V^* = \mathbf{k}\mathcal{L}(e) \otimes_{G_e} \left( \bigoplus_{V \in \text{Irr}_e} V \otimes V^* \right)$$

Because of Proposition 2.1.48,  $\bigoplus_{V \in \text{Irr}_e} V \otimes V^* = \text{top}(\mathbf{k}G_e)$  so :

$$\begin{aligned} \bigoplus_{V \in \text{Irr}_e} \text{Ind}_{G_e}^M(V) \otimes V^* &= \mathbf{k}\mathcal{L}(e) \otimes_{\mathbf{k}G_e} \mathbf{k}G_e / \text{rad}(\mathbf{k}G_e) \\ &= \mathbf{k}\mathcal{L}(e) / \mathbf{k}\mathcal{L}(e) \cdot \text{rad}_{\mathbf{k}G_e^{\text{op}}} \mathbf{k}\mathcal{L}(e) \end{aligned}$$

because of Proposition 2.1.23. This is equal to  $\text{top}_{\mathbf{k}G_e^{op}}(\mathbf{k}\mathcal{L}(e))$  by Proposition 2.1.44.  $\square$

In the ordinary case, this puts in relation three kinds of modules : the simple  $\mathbf{k}G_e$ -modules, which are well-understood,  $\mathbf{k}\mathcal{L}(e)$  which is understood as well, because it is a combinatorial module, and finally the  $\text{Ind}_{G_e}^M(V)$  which contain, in a sense, the simple  $\mathbf{k}M$ -module that we are after. In the modular case, however, not only are the simple  $\mathbf{k}G_e$ -modules not entirely understood even in (*a priori*) sympathetic cases like  $\mathfrak{S}_n$ , but the decomposition gets complicated further by the quotient by  $\text{rad}_{\mathbf{k}G_e^{op}}(\mathbf{k}\mathcal{L}(e))$ . Still, we want to proceed and apply the Clifford-Munn-Ponizovskii Theorem, according to which we need to remove the radical of each  $\text{Ind}_{G_e}^M(V)$  factor to reach the simple  $\mathbf{k}M$ -modules. Proposition 3.1.4 puts the radical in a form similar to Theorem 2.4.4 while Proposition 3.1.5 does exactly this. Lemma 3.1.2 and its corollary are technical results on radicals used in the proof of Proposition 3.1.4.

**Lemma 3.1.2.** *Let  $A, B$  be two finite dimensional algebras over a perfect field  $\mathbf{k}$ . Then:*

$$\text{rad}(A \otimes B) = \text{rad}(A) \otimes B + A \otimes \text{rad}(B).$$

A proof of this fact can be found in the proof of [Eti+09, Proposition 3.10.2]<sup>1</sup>. From Lemma 3.1.2, we get the following Corollary by recalling Proposition 2.1.44: if  $V$  is an  $A$ -module and  $A$  is finite dimensional,  $\text{rad}_A(V) = \text{rad}(A) \cdot V$ .

**Corollary 3.1.3.** *Let  $A, B$  be two finite dimensional unitary algebras over a perfect field. If  $V_A \otimes V_B$  is an  $A - \text{mod} - B$  (or equivalently an  $A \otimes B^{op} - \text{mod}$ ), then*

$$\text{rad}_{A \otimes B^{op}}(V_A \otimes V_B) = \text{rad}_A(V_A) \otimes B + A \otimes \text{rad}_B(V_B).$$

This allows us to identify the radical of  $\text{top}_{\mathbf{k}G_e^{op}}(\mathbf{k}\mathcal{L}(e))$  as a  $\mathbf{k}M \otimes \mathbf{k}G_e^{op}$ -module. We denote  $\text{top}_{\mathbf{k}G_e^{op}}(\mathbf{k}\mathcal{L}(e))$  by  $\mathbf{S}$  from now on.

**Proposition 3.1.4.** *With the previously defined notations, we have:*

$$\text{rad}_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}(\mathbf{S}) = N_e(\mathbf{S}).$$

*Proof.* Using Lemma 3.1.2, for  $V$  a simple  $G_e$ -module, we have that :

$$\begin{aligned} & \text{rad}_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}(\text{Ind}_{G_e}^M(V) \otimes_{\mathbf{k}} V^*) \\ &= \text{rad}_{\mathbf{k}M} \text{Ind}_{G_e}^M(V) \otimes V^* + \text{Ind}_{G_e}^M(V) \otimes \text{rad}_{\mathbf{k}G_e^{op}}(V^*) \\ &\stackrel{(1)}{=} \text{rad}_{\mathbf{k}M} \text{Ind}_{G_e}^M(V) \otimes V^* \\ &\stackrel{(2)}{=} N_e(\text{Ind}_{G_e}^M(V)) \otimes V^* \end{aligned}$$

<sup>1</sup>We would like to warmly thank Pr. Pierre-Guy Plamondon for providing us with a proof of this fact when we could not find a reference for it. It seems to be folklore in the algebra representation community, to the point that it is rarely written down.

where equality (1) comes from the simplicity of  $V^*$  as a  $\mathbf{k}G_e^{op}$ -module and (2) is the second point of Theorem 2.4.4.

Since radical and direct sums commute, from Proposition 3.1.1, we know that:

$$\text{rad}_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}(\mathbf{S}) = \bigoplus_{V \in \text{Irr}_e} N_e(\text{Ind}_{G_e}^M(V)) \otimes V^*.$$

It remains to be seen why

$$\bigoplus_{V \in \text{Irr}_e} N_e(\text{Ind}_{G_e}^M(V)) \otimes V^* = N_e(\mathbf{S}).$$

It is clear the direct sum on the left is a subset of the set on the right. For the other inclusion, we see that if  $V, V'$  are  $\mathbf{k}M$ -modules,  $N_e(V \oplus V') = N_e(V) \oplus N_e(V')$ . Given the Proposition 3.1.1, it is enough to show that  $N_e(\text{Ind}_{G_e}^M(V)) \otimes V^* = N_e(\text{Ind}_{G_e}^M(V) \otimes V^*)$ . Let  $x \in \text{Ind}_{G_e}^M(V) \otimes V^*$  be such that for every  $m \in M$ ,  $emx = 0$ .  $x$  can be written as  $\sum_i (\sum_j x_{i,j} b_j) \otimes b'_i$  where  $\{b_j\}_j$  is a basis of  $\text{Ind}_{G_e}^M(V)$  and  $\{b'_i\}_i$  is a basis of  $V^*$ . For every  $m \in M$ , we have :

$$em \cdot x = em \cdot \sum_i \left( \sum_j x_{i,j} b_j \right) \otimes b'_i = \sum_i \left( em \cdot \sum_j x_{i,j} b_j \right) \otimes b'_i = 0$$

that is, for every  $b'_i$  we get  $em \cdot \sum_j x_{i,j} b_j = 0$  so  $\sum_j x_{i,j} b_j \in N_e(\text{Ind}_{G_e}^M(V))$  which means  $x \in N_e(\text{Ind}_{G_e}^M(V)) \otimes V^*$ .  $\square$

**Proposition 3.1.5.** *With the previously defined notations, we have:*

$$\text{top}_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}(\mathbf{S}) \cong \bigoplus_{V \in \text{Irr}_e} V^\# \otimes V^*.$$

*Proof.* From Proposition 3.1.4, we have a decomposition of  $\text{rad}_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}(\mathbf{S})$  as a direct sum adapted to the decomposition of  $\mathbf{S}$  as  $\bigoplus_{V \in \text{Irr}_e} \text{Ind}_{G_e}^M(V) \otimes_{\mathbf{k}} V^*$ . So:

$$\text{top}_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}(\mathbf{S}) \cong \bigoplus_{V \in \text{Irr}_e} (\text{Ind}_{G_e}^M(V) \otimes_{\mathbf{k}} V^*) / (N_e(\text{Ind}_{G_e}^M(V)) \otimes V^*).$$

From Theorem 2.4.4, we know that

$$0 \longrightarrow N_e(\text{Ind}_{G_e}^M(V)) \longrightarrow \text{Ind}_{G_e}^M(V) \longrightarrow V^\# \longrightarrow 0$$

is a short exact sequence. Since  $N_e(\text{Ind}_{G_e}^M(V)) \otimes V^*$  a submodule of  $\text{Ind}_{G_e}^M(V) \otimes V^*$  and because tensor product over a field is exact, we have a short exact sequence :

$$0 \longrightarrow N_e(\text{Ind}_{G_e}^M(V)) \otimes V^* \longrightarrow \text{Ind}_{G_e}^M(V) \otimes V^* \longrightarrow V^\# \otimes V^* \longrightarrow 0$$

which proves the result.  $\square$

Finally, we can apply the language of characters to Proposition 3.1.5, which yields a formula for computing the character table of  $M$  over  $\mathbf{k}$  given the character tables of the groups  $G_e$  over  $\mathbf{k}$ .

**Proposition 3.1.6.** *With the previously defined notations, we have the formula for  $X_e(M)$  :*

$$X_e(M) = {}^t X(G_e)^{-1} \cdot \left( \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{\mathbf{S}}(m, g) - \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{N_e(\mathbf{S})}(m, g) \right)_{g \in C_{G_e}, m \in C_M}$$

where the dot is the matrix product.

*Proof.* First, we have, because of Proposition 2.2.2.(2), we have:

$$\chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{\mathbf{S}/N_e(\mathbf{S})} = \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{\mathbf{S}}(m, g) - \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{N_e(\mathbf{S})}(m, g)$$

Then, from Proposition 3.1.5, we know that:

$$\begin{aligned} \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{\mathbf{S}/N_e(\mathbf{S})}(m, g) &= \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{\bigoplus_{V \in \text{Irr}_e} V^\# \otimes V^*}(m, g) \\ &= \sum_{V \in \text{Irr}_e} \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{V^\# \otimes V^*}(m, g) \\ &= \sum_{V \in \text{Irr}_e} \chi_{\mathbf{k}M}^{V^\#}(m) \chi_{\mathbf{k}G_e}^V(g) \end{aligned}$$

This last sum is clearly the dot product between the column of  $X(G_e)$  indexed by  $g$  and the column of  $X_e(M)$  indexed by  $m$ . That is, the coefficient in position  $(g, m)$  of  $\chi_{\mathbf{k}M \otimes \mathbf{k}G_e}^{\text{top } \mathbf{S}}$  is equal to the coefficient in position  $(g, m)$  of  ${}^t X(G_e) \cdot X_e(M)$ , which, together with Proposition 2.2.2-(ii), proves the equality.  $\square$

Let us state it in the particularly simple ordinary case where  $\mathbf{S} = \mathbf{k}\mathcal{L}(e)$ .

**Corollary 3.1.7.** *We suppose that  $\text{char } \mathbf{k} \nmid |G_e|$ . With the previously defined notations, we have the formula for  $X_e(M)$  :*

$$X_e(M) = {}^t X(G_e)^{-1} \cdot \left( \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{\mathbf{k}\mathcal{L}(e)}(m, g) - \chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{N_e(\mathbf{k}\mathcal{L}(e))}(m, g) \right)_{g \in C_{G_e}, m \in C_M}$$

where the dot is the matrix product.

This is nice because this formula gives the character table of the monoid as a function of the character tables of groups, the bicharacter of the  $\mathcal{L}$ -classes and the bicharacter of their radical (and nothing else). The first is generally well-understood. The second can efficiently be computed from the result of Section 3.2. The third, as Section 3.3.4 reveals, is our bottleneck. Still, we make it reasonably efficient using Green structure results in Section 3.3.3. In the modular case, the situation is not nearly as favorable. Modular

representation theory of groups is much more complicated than in the ordinary case. For instance, the computer algebra system GAP, that we use to apply this formula later in this manuscript, does not have an algorithm to compute the character table of any finite group in the modular case, despite being at or near the state of the art in terms of computational group theory. Despite this, we still can separate the “monoid” computations from the “characteristic” computations in the radical.

**Proposition 3.1.8.** *We have:*

$$\text{top}_{\mathbf{k}M \otimes \mathbf{k}G_e^{\text{op}}}(\mathbf{S}) = \mathbf{k}\mathcal{L}(e) / N_e(\mathbf{k}\mathcal{L}(e)) + \text{rad}_{\mathbf{k}G_e^{\text{op}}}(\mathbf{k}\mathcal{L}(e)).$$

*Proof.* It is easy to see that

$$N_e(\mathbf{S}) = \{v \in \mathbf{k}\mathcal{L}(e) \mid Mev \subset \text{rad}_{\mathbf{k}G_e^{\text{op}}}(\mathbf{k}\mathcal{L}(e))\} / \text{rad}_{\mathbf{k}G_e^{\text{op}}}(\mathbf{k}\mathcal{L}(e)).$$

For every  $\mathcal{H}$ -class  $H \subset \mathcal{L}(e)$ , let  $l_H \in M$  such that  $l_H \mathcal{H}(e) = H$  and denote by  $\mathfrak{L}$  the set of those chosen  $l_H$ . We have the decompositions adapted to the  $\mathbf{k}G_e$ -right scalar multiplication:

$$\mathbf{k}\mathcal{L}(e) = \bigoplus_{l \in \mathfrak{L}} l \cdot G_e, \quad \text{rad}_{\mathbf{k}G_e^{\text{op}}}(\mathbf{k}\mathcal{L}(e)) = \bigoplus_{l \in \mathfrak{L}} l \cdot \text{rad}(\mathbf{k}G_e).$$

Note that from Green’s Lemma, for any  $m \in M$ :

$$\text{Im}(me \times_{\mathbf{k}\mathcal{L}(e)}) = \bigoplus_{H \subset \text{Im}(me \times_{\mathbf{k}\mathcal{L}(e)})} \mathbf{k}H$$

where the direct sum is taken over  $\mathcal{H}$ -classes. In particular, since the  $\mathbf{k}M$  and  $\mathbf{k}G_e^{\text{op}}$  actions commute, this induces a surjection between  $\mathbf{k}G_e$ -radicals:

$$me \times : \bigoplus_{\substack{l \in \mathfrak{L} \\ mel \cdot \mathbf{k}G_e = \mathbf{k}H}} l \cdot \text{rad}(\mathbf{k}G_e) \mapsto \mathbf{k}H \cdot \text{rad}(\mathbf{k}G_e)$$

whenever  $H$  is in the image of  $me \times$ . Given  $v \in \mathbf{k}\mathcal{L}(e)$ , suppose that  $mev = r \in \text{rad}_{\mathbf{k}G_e^{\text{op}}}(\mathbf{k}\mathcal{L}(e))$ . We can decompose  $v$  and  $r$  along the previous direct sum:

$$me \cdot \sum_{l \in \mathfrak{L}} v_l = \sum_{l \in \mathfrak{L}} r_l \implies \forall l \in \mathfrak{L}, r_l = me \cdot \sum_{\substack{l' \in \mathfrak{L} \\ me(l \cdot \mathbf{k}G_e) = l' \cdot \mathbf{k}G_e}} v_{l'}.$$

Because we have a surjection, for all  $l \in \mathfrak{L}$  let  $r'_l \in \text{rad}_{\mathbf{k}G_e}(\mathcal{L}(e))$  such that  $mer'_l = r_l$  and let  $r' = \sum_{l \in \mathfrak{L}} r'_l$ . Then  $v = (v - r') + r'$  where, by design,  $v - r' \in N_e(\mathbf{k}\mathcal{L}(e))$  and  $r' \in \text{rad}_{\mathbf{k}G_e}(\mathcal{L}(e))$ . So,

$$N_e(\mathbf{S}) = N_e(\mathbf{k}\mathcal{L}(e)) + \text{rad}_{\mathbf{k}G_e^{\text{op}}}(\mathbf{k}\mathcal{L}(e)) / \text{rad}_{\mathbf{k}G_e^{\text{op}}}(\mathbf{k}\mathcal{L}(e)).$$



Replacing in  $\text{top}_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}(\mathbf{S})$ , we get that:

$$\text{top}_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}(\mathbf{S}) = \mathbf{k}\mathcal{L}(e) /_{N_e(\mathbf{k}\mathcal{L}(e)) + \text{rad}_{\mathbf{k}G_e^{op}}(\mathbf{k}\mathcal{L}(e))}.$$

□

Thus, even in the modular case, we can reuse the computations described later for the bicharacter of  $\mathbf{k}\mathcal{L}(e)$  and the equations for  $N_e(\mathbf{k}\mathcal{L}(e))$ . Supposing we dispose of a “black box” algorithm giving the radical of  $\mathbf{k}G_e$ , the computations of the bicharacter  $\text{rad}_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}(\mathbf{S})$  should not be much more inefficient than that of  $N_e(\mathbf{k}\mathcal{L}(e))$  in the ordinary case, as it necessitates the same linear algebra operations. That being said, in Section 3.3 where we present effective algorithms, we focus our attention on the ordinary case, and the modular case remains to be tackled in practice.

## 3.2 Fixed points counting

Consider the problem of counting the number of elements of the set  $\text{Fix}_G(h, k)$  where  $G$  is a finite group and  $h, k \in G$ . If  $\text{Fix}_G(h, k)$  is non-empty, it contains an element  $\gamma$  such that  $h\gamma k = \gamma$ , or equivalently  $h = \gamma k^{-1} \gamma^{-1}$ . So for any  $g \in \text{Fix}_G(h, k)$  we have:

$$h g k = g \Leftrightarrow \gamma k^{-1} \gamma^{-1} g k = g \Leftrightarrow \gamma^{-1} g k = k \gamma^{-1} g.$$

This means that  $g \in \gamma C_G(k)$  where  $C_G(k)$  is the centralizer of  $k$  in  $G$ . Because the other inclusion is obvious, we get a description of  $|\text{Fix}_G(h, k)|$ : either  $h$  and  $k^{-1}$  are conjugates in which case there are  $|C_G(k)|$  fixed points, or there are none, and there are no fixed points. In the case of a monoid, this reasoning mostly breaks: we crucially used the invertibility property, which monoids lack. The Schützenberger groups seem to be ideal candidates to get back some of this invertibility. In this section, we clarify the role of the Schützenberger groups for counting fixed points, how to give meaning to “ $h \times_H$  and  $\times_H k$  are in the same conjugacy class”, and how to factorize our previous remark over all the  $\mathcal{H}$ -classes of the same  $\mathcal{J}$ -class.

As the bijections between  $\mathcal{L}$  (and  $\mathcal{R}$ ) classes will play a major role in the remainder of this section, we introduce the following notations.

**Notation 3.2.1.** Given  $R, R'$  two  $\mathcal{R}$ -classes in the same  $\mathcal{J}$ -class, we say that  $(\lambda, \lambda')$  is a *left Green pair* with respect to  $(R, R')$  if:

- $\lambda R = R'$  and  $\lambda' R' = R$ .
- $(\lambda \lambda') \times_R = \text{Id}_R$  and  $(\lambda' \lambda) \times_{R'} = \text{Id}_{R'}$

Similarly, given two  $\mathcal{L}$ -classes  $L, L'$  in the same  $\mathcal{J}$ -classes,  $(\rho, \rho')$  is a *right Green pair* with respect to  $(L, L')$  if:

- $L\rho = L'$  and  $L'\rho' = L$ .
- $\times_L(\rho\rho') = \text{Id}_L$  and  $\times_{L'}(\rho'\rho) = \text{Id}_{L'}$

Using Green pairs, one can transport the problem of counting fixed points in an arbitrary  $\mathcal{H}$ -class to a reference  $\mathcal{H}$ -class.

**Proposition 3.2.2.** *Let  $H_1, H_2 \subset J$  be two  $\mathcal{H}$ -classes contained in the same  $\mathcal{J}$ -class. Let  $\lambda, \lambda', \rho, \rho'$  such that:*

- $(\lambda, \lambda')$  is a left Green pair with respect to  $(\mathcal{R}(H_1), \mathcal{R}(H_2))$ ,
- $(\rho, \rho')$  is a right Green pair with respect to  $(\mathcal{L}(H_1), \mathcal{L}(H_2))$ .

Finally, let  $(h, k) \in {}_M \text{Stab}(\mathcal{R}(H_2)) \times \text{Stab}_M(\mathcal{L}(H_2))$  and define  $(h', k') = (\lambda'h\lambda, \rho k\rho')$ . Then the maps  $x \mapsto \lambda'x\rho'$  and  $x \mapsto \lambda x\rho$  are reciprocal bijections between the sets  $\text{Fix}_{H_2}(h, k) = \{a \in H_2 \mid hak = a\}$  and  $\text{Fix}_{H_1}(h', k') = \{a \in H_1 \mid h'ak' = a\}$ .

*Proof.* First, notice that Green's Lemma gives us the existence of  $\lambda, \lambda', \rho, \rho'$  respecting the hypothesis we demand, and also gives that  $x \mapsto \lambda'x\rho'$  and  $x \mapsto \lambda x\rho$  are reciprocal bijections between  $H_1$  and  $H_2$ . Let  $a_1$  be an element of  $H_1$  and denote by  $a_2 = \lambda a_2 \rho$ . Then:

$$ha_2k = a_2 \Leftrightarrow \lambda'ha_2k\rho' = \lambda'a_2\rho' \Leftrightarrow (\lambda'h\lambda)a_1(\rho k\rho') = a_1 \Leftrightarrow h'a_1k' = a_1$$

so these bijections restrict to  $\text{Fix}_{H_1}(h', k')$  and  $\text{Fix}_{H_2}(h, k)$ .  $\square$

Keeping in mind our computational goals, transporting the problem of counting fixed points from  $H_2$  to  $H_1$  is helpful, as for the price of 4 monoid multiplications, we can use a lot of precomputations specific to a particular  $\mathcal{H}$ -class, avoiding the repetition of multiple similar computations for each  $\mathcal{H}$ -class.

The question is now to determine the fixed points in a single  $\mathcal{H}$ -class, using our previous remark on conjugacy. Let us first clarify the idea of elements of the left and right Schützenberger groups being in the same conjugacy class.

**Proposition 3.2.3.** *Given an  $\mathcal{H}$ -class  $H$ ,  $a \in H$  and  $g \in \Gamma(H)$ , we define  $\tau_a(g)$  as in Proposition 1.2.11. Then,  $\tau_a$  gives rise to a bijection between the conjugacy classes of  $\Gamma(H)$  and  $\Gamma'(H)$  that is independent of the choice of  $a$ .*

*Proof.* We want to check that for  $a \in H, g \in \Gamma(H)$ , the conjugacy class of  $\tau_a(g)$  is defined independently of  $a$ . Take any  $a, b \in H$ . By definition of  $\Gamma(H)$ , there exist some  $h \in \Gamma(H)$  such that  $b = h \cdot a$ . So :

$$b \cdot \tau_a(g) = (h \cdot a) \cdot \tau_a(g) = h \cdot (g \cdot a) = hgh^{-1} \cdot (h \cdot a) = b \cdot \tau_b(hgh^{-1}).$$

Since  $\Gamma'(H)$  acts freely, this means that  $\tau_a(g) = \tau_b(h)\tau_b(g)\tau_b(h)^{-1}$  and thus  $\tau_a(g)$  is conjugated with  $\tau_b(g)$ , which proves that the conjugacy class of  $\tau_a(g)$  is indeed defined independently of  $a$ . Finally, as  $\tau_a$  is a group morphism, the images of two conjugated elements are conjugated, meaning that  $\tau_a$  does indeed induce bijection between the conjugacy classes of the left and right Schützenberger groups, independently of the choice of  $a$ .  $\square$

In the next proposition, we formalize the idea of searching for the fixed points as some centralizer, but in the context of a monoid.

**Proposition 3.2.4.** *Let  $H$  be an  $\mathcal{H}$ -class,  $a \in H$  and  $(h, k) \in {}_M \text{Stab}(\mathcal{R}(H)) \times \text{Stab}_M(\mathcal{L}(H))$ . Then*

$$|\text{Fix}_H(h, k)| = \begin{cases} |C_{\Gamma'(H)}(\times_H k)| & \text{if } \tau_a(h \times_H)^{-1} \in \overline{\times_H k} \\ 0 & \text{otherwise} \end{cases}$$

where  $\overline{\times_H k}$  is the conjugacy class of  $\times_H k$  in  $\Gamma'(H)$  and  $C_{\Gamma'(H)}(\times_H k)$  is the centralizer in  $\Gamma'(H)$  of  $\times_H k$ .

*Proof.* For simplicity, we commit an abuse of notation by denoting  $h \times_H$  as  $h$  and  $\times_H k$  as  $k$ . Let  $a$  be any element of  $H$ .

$$\begin{aligned} \text{Fix}_H(h, k) &= \{b \in H \mid hbk = b\} \\ &= \{a \cdot g \mid g \in \Gamma'(H) \text{ and } ha \cdot gk = a \cdot g\} \\ &= \{a \cdot g \mid g \in \Gamma'(H) \text{ and } a \cdot \tau_a(h)gk = a \cdot g\} \\ &= \{a \cdot g \mid g \in \Gamma'(H) \text{ and } \tau_a(h)gk = g\}. \end{aligned}$$

The last equality comes from the fact that  $\Gamma'(H)$  acts freely, so we can simplify the  $a$ . Suppose that  $\text{Fix}_H(h, k)$  is non-empty and let  $\gamma \in \Gamma'(H)$  such that  $\tau_a(h)\gamma k = \gamma$ . Then, for any  $g \in \Gamma'(H)$  :

$$\tau_a(h)gk = g \Leftrightarrow g^{-1}\tau_a(h)gk = e \Leftrightarrow g^{-1}\gamma k^{-1}\gamma^{-1}gk = e \Leftrightarrow [\gamma^{-1}g, k] = e$$

where  $[\cdot, \cdot]$  is the commutation bracket. This means that

$$\text{Fix}_H(h, k) = \{a \cdot g \mid g \in \gamma C_{\Gamma'(H)}(k)\}.$$

Note that because, again,  $\Gamma'(H)$  acts freely,  $\text{Fix}_H(h, k)$  has the same cardinality as  $C_{\Gamma'(H)}(k)$  and that, from Proposition 3.2.3, this is independent of the choice of  $a$ , which proves the result.  $\square$

**Example 3.2.5.** Consider  $a = [1\ 2\ 2\ 3] \in \mathcal{T}_4$  and  $H = \mathcal{H}(a)$ . We have  $\text{Im } a = \{1, 2, 3\}$  and  $\ker a = \{\{1\}, \{2, 3\}, \{4\}\}$ . Notice that  $H$  is not a group, since  $a^2 = [1\ 2\ 2\ 2] \notin H$ . Considering the Schützenberger groups as symmetric groups on the image and kernel common to all elements of  $H$  as in Example 1.2.9, we have  $\Gamma(H) = \mathfrak{S}(\text{Im } a)$  and  $\Gamma'(H) = \mathfrak{S}(\ker a)$ . Let us first check for fixed points under the action of  $h = [1\ 2\ 3\ 4]$  on the left and  $k = [2\ 1\ 1\ 4]$  on the right. Seen as an element of  $\Gamma(H)$ ,  $h$  corresponds to  $\text{Id}_{\text{Im } a}$ , and  $k$  corresponds to  $(\{2, 3\} \{1\})$  in  $\Gamma'(H)$ . Since we have  $\tau_a(h) = \text{Id}_{\ker a}$ , it follows that  $|\text{Fix}_H(h, k)| = 0$ . If we now take  $h$  to be  $[1\ 3\ 2\ 4]$ , the corresponding element in  $\Gamma(H)$  is  $(2\ 3)$  and  $\tau_a(h) = (\{2, 3\} \{4\})$ . Since  $(\{2, 3\} \{4\})$  and  $(\{2, 3\} \{1\})$  are conjugated in  $\mathfrak{S}(\ker a)$ . Their centralizers have cardinal 2 and one can indeed check that  $[2\ 3\ 3\ 1]$  and  $[3\ 2\ 2\ 1]$  are the only fixed points in  $H$ .

Putting together the previous results, we get the following corollary on the cardinality of  $\text{Fix}_J(h, k)$ .

**Corollary 3.2.6.** *Let  $J$  be a  $\mathcal{J}$ -class,  $a$  be any element of  $J$  and denote by  $H_0 = \mathcal{H}(a)$ ,  $R_0 = \mathcal{R}(a)$ ,  $L_0 = \mathcal{L}(a)$ . Let  $h, k$  be any elements of  $M$ . We denote by:*

- $(\lambda_R, \lambda'_R)$  a left Green pair with respect to  $(R_0, R)$  for each  $\mathcal{R}$ -class  $R \subset J$ , and  $(\rho_L, \rho'_L)$  a right Green pair with respect to  $(L_0, L)$  for each  $\mathcal{L}$ -class  $L \subset J$ ,
- $S_{\mathcal{R}}(h) = \{R \subset J \mid R \text{ is a } \mathcal{R}\text{-class and } hR = R\}$ ,
- $S_{\mathcal{L}}(k) = \{L \subset J \mid L \text{ is an } \mathcal{L}\text{-class and } Lk = L\}$

Denoting the set of conjugacy classes of  $\Gamma'(H_0)$  as  $C$ , we further define two vectors:

- $r_J(h) = (|C_{\Gamma'(H)}(g)| \cdot |\{R \in S_{\mathcal{R}}(h) \mid \tau_a(\lambda'_R h \lambda_R) \in \bar{g}\}|)_{\bar{g} \in C}$ ,
- $l_J(k) = (|\{L \in S_{\mathcal{L}}(k) \mid \rho'_L k \rho_L \in \bar{g}\}|)_{\bar{g} \in C}$ .

Then  $\text{Fix}_J(h, k)$  has cardinality the dot product of  $r_J(h)$  with  $l_J(k)$ .

### 3.3 Algorithms

We suppose here that we are *not* in the modular case. In particular, we do not discuss here the questions of  $p$ -regular elements and  $\mathbf{k}G_e$ -radicals.

**Question 3.3.1.** Implement a version of the algorithms presented below in the modular case.

### 3.3.1 Computational hypotheses

In this section, we discuss the computational hypotheses necessary for the algorithms in the next section. This section is based on the work [Eas+19] in which East, Egri-Nagy, Mitchell and Péresse provide efficient algorithms for all basic computational questions on finite semigroups (which include monoids). Although we limit our scope, for this section, to the case of transformation monoids, the methods described in [Eas+19] allow the algorithms described below to be applied to other interesting classes of monoids. In Section 3.3.4, we do just that and provide timing and measure for other types of monoids. Moreover, the algorithm of this section can theoretically be applied to any finite monoid using a Cayley embedding in a full transformation monoid. In general, however, this is very inefficient and not feasible in practice, even with our non-naive algorithms.

Following the authors of [Eas+19], we make the following fundamental assumptions that we can compute:

- Assumption I : a product of two elements of the monoid.
- Assumption II : the image and kernel of a transformation (note that we do not explicitly use this assumption, but that it is necessary for the algorithms of [Eas+19] that we do use).
- Assumption III : Green pairs.
- Assumption IV : Given  $h \in {}_M \text{Stab}(H)$  compute the corresponding element in  $\Gamma(H)$  (understood as a permutation group of the image common to all elements of  $H$  as seen in Example 1.2.9), and similarly on the right.

Not only do we directly need to be able to do these computations for our own algorithm, but they are also prerequisite for the algorithms from [Eas+19]. As such, we refer to the top of Section 5.2 of [Eas+19] on how to realize these computations in the case of transformation monoids.

We, again, refer to [Eas+19] for the specific algorithms meeting our computational prerequisites.

- Computing the Schützenberger groups: [Eas+19, Algorithm 4]
- Checking membership of an element in a Green class: [Eas+19, Algorithms 7 & 8].

- Finding idempotents: [Eas+19, Algorithm 10]. This algorithm also allows for finding the regular  $\mathcal{J}$ -classes.
- Decomposing the monoid in  $\mathcal{R}$ ,  $\mathcal{L}$  and  $\mathcal{J}$ -classes : [Eas+19, Algorithm 11] and its discussion. Note that by storing this decomposition, we can, given an element of the monoid, find the classes that contain it.
- Obtaining a representative of a Green class: this is given by the data structure representing the Green's classes described at the top of [Eas+19, Section 5.4].

Finally, we require the following points that, although they are not described in [Eas+19], are easily obtained from it.

- Computing a set  $C_M$  of character equivalency representatives: given Proposition 2.2.6, this can be done in four steps:
  1. compute a set  $\mathcal{E}$  of idempotent representatives of the regular  $\mathcal{J}$ -classes,
  2. compute  $\Gamma(\mathcal{H}(e))$  for each  $e \in \mathcal{E}$ ,
  3. compute a set  $C_e$  of representatives of the conjugacy classes of  $\Gamma(\mathcal{H}(e))$  for each  $e \in \mathcal{E}$ , using for instance the procedure described in [Hul00],
  4. for each  $e \in \mathcal{E}$  and  $c \in C_e$  compute the corresponding element of  $\mathcal{H}(e)$  as in Example 1.3.9.
- Computing  $\tau_a$  as in Proposition 3.2.3: in the case of transformation monoids, given  $g \in \Gamma(H)$ ,  $\tau_a(g)$  is simply, seen as an element of  $\mathfrak{S}(\ker a)$ :

$$a^{-1}\{i\} \mapsto (g \cdot a)^{-1}\{g \cdot a(i)\},$$

which can be computed in  $O(n)$ .

- Testing that two elements  $g, g'$  in  $\Gamma'(\mathcal{H}(a))$  are conjugated :  $\Gamma'(H)$  is represented as a subgroup of  $\mathfrak{S}(\ker a)$  and known procedures, such as the one described in [But94], can be used.
- Computing the cardinality of a conjugacy class of a Schützenberger group: for instance, the computer algebra system GAP uses the method described in [Hul00].

### 3.3.2 Fixed points

We are now ready to present the algorithm for fixed-point counting, keeping in mind that we want first to use the formula from Section 3.1 to compute the character table of the monoid and further to compute its Cartan matrix. In the cases we are interested in, we use the formalism of character computing, since, as stated in the Lemma below, computing the characters of so-called *combinatorial modules* is actually counting fixed points, exactly for the same reason as in Corollary 2.3.4.

**Lemma 3.3.2.** *Let  $M, M'$  be two finite monoids and  $(V, B)$  a finite dimensional  $\mathbf{k}M - \text{mod} - \mathbf{k}M'$  space equipped with a basis  $B$ . If the actions of  $M, M'$  on  $(V, B)$  are combinatorial, meaning for any  $(m, b, m') \in M \times B \times M', mbm' \in (B \cup \{0\})$ , then:*

$$\chi_{\mathbf{k}M \otimes \mathbf{k}M'^{\text{op}}}^V = |\{b \in B \mid mbm' = b\}|.$$

*Proof.* In the basis  $B$ , the matrix of the linear map  $x \mapsto mxm'$  is a  $\{0, 1\}$ -matrix, with for every  $b \in B$  exactly one 1 in the  $b$ -th column, that 1 being on the  $b$ -th row if  $mbm' = b$ . Thus, the trace counts the number of fixed points.  $\square$

Note that we have already defined a structure of combinatorial  $\mathbf{k}M - \text{mod} - \mathbf{k}G_e$  on  $\mathbf{k}\mathcal{L}(e)$  for any idempotent  $e$ . In the same way,  $\mathbf{k}J$  for a  $\mathcal{J}$ -class  $J$ , can be equipped with a structure of  $\mathbf{k}M - \text{mod} - \mathbf{k}M$  by setting for every  $(m, j) \in M \times J$ :

$$m \cdot j = \begin{cases} mj & \text{if } mj \in J \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad j \cdot m = \begin{cases} jm & \text{if } jm \in J \\ 0 & \text{otherwise} \end{cases}.$$

As before, this is well-defined: firstly because the actions on the left and on the right commute (because the monoid's law is associative by assumption) and secondly because either  $m \leq_{\mathcal{L}} m'$  or  $m \leq_{\mathcal{R}} m'$  imply  $m \leq_{\mathcal{J}} m'$ . Thus, if  $ml$  or  $lm$  has “fallen to 0”, it can’t “climb back up” to  $J$ .

This structure makes  $(\mathbf{k}J, J)$  into a combinatorial module and we may apply our fixed points counting methods to compute its character.

**Algorithm 3.3.3** (Computing the bicharacter of a  $\mathcal{J}$ -class). Keeping the assumptions and notations of the previous Paragraph 3.3.1, we get from Corollary 3.2.6 an algorithm to compute the bicharacter of  $\mathbf{k}J$  as a  $\mathbf{k}M - \text{mod} - \mathbf{k}M$ :

- Input : A  $\mathcal{J}$ -class  $J$ , a set of representatives of the character equivalency classes  $C_M$ .
- Output : A matrix  $(|\{m \in J \mid hmk = m\}|)_{(h,k) \in C_M^2}$

1. Preparations:
  - (a) Choose  $a \in J$  and define  $H = \mathcal{H}(a)$ .
  - (b) Compute Green pairs  $(\lambda_R, \lambda'_R)$  (respectively  $(\rho_L, \rho'_L)$ ) for  $(\mathcal{R}(a), R)$  (resp.  $(\mathcal{L}(a), L)$ ) for all  $\mathcal{R}$ -class  $R \subset J$  (resp.  $\mathcal{L}$ -class  $L \subset J$ ).
  - (c) Compute the set  $C$  of conjugacy classes of  $\Gamma'(H)$ .
2. For each character equivalency representative  $h \in C_M$ , initialize  $r_J(h)$  and  $l_J(h)$  to both be  $(0)_{\bar{g} \in C}$ .
  - (a) For each  $\mathcal{R}$ -class  $R \subset J$ , test if  $h\lambda_R a \in R$ . If so, denoting by  $\bar{g}$  the conjugacy class of  $\tau_a((\lambda'_R h \lambda_R) \times_H)$  in  $\Gamma'(H)$ , increment  $r_J(h)$  by  $|C_{\Gamma'(H)}(\bar{g})|$  at position  $\bar{g}$ .
  - (b) For each  $\mathcal{L}$ -class  $L \subset J$ , test if  $a\rho_R h \in L$ . If so, denoting by  $\bar{g}$  the conjugacy class of  $\times_H(\rho'_L h \rho_L)$  in  $\Gamma'(H)$ , increment  $r_J(h)$  by 1 at position  $\bar{g}$ .
3. Compute the matrix  $\chi = (r_J(h) \cdot l_J(k))_{(h,k) \in C_M^2}$  using the previously computed vectors and return  $\chi$ .

**Example 3.3.4.** Recall that an aperiodic monoid is a monoid where all  $\mathcal{H}$ -classes are singletons. Let us apply the algorithm we just described in the case of a  $\mathcal{J}$ -class  $J$  with trivial  $\mathcal{H}$ -classes. Several simplifications occur: first, we don't need to check for the conjugacy class, as there is only one. Secondly, the conjugacy class has cardinality one. Consider the vectors  $r_J = (|S_{\mathcal{R}}(h)|)_{h \in C_M}$  and  $l_J = (|S_{\mathcal{L}}(h)|)_{h \in C_M}$  with  $S_{\mathcal{L}}(h)$  and  $S_{\mathcal{R}}(h)$  defined as in Corollary 3.2.6. The bicharacter is simply the matrix product of  $r_J^T$  with  $l_J$ . The particular case of this algorithm for aperiodic monoid is described in [Thi12, Section 1]

**Algorithm 3.3.5** (Computing the bicharacter of  $\mathbf{k}M$ ). If we consider  $(\mathbf{k}M, M)$  as a combinatorial  $\mathbf{k}M - \text{mod } -\mathbf{k}M$ , we immediately have that:

$$\chi_{\mathbf{k}M \otimes \mathbf{k}M'^{op}}^{\mathbf{k}M} = \sum_{J \in \mathcal{J}} \chi_{\mathbf{k}M \otimes \mathbf{k}M'^{op}}^{\mathbf{k}J}$$

and we can therefore compute the bicharacter of the whole monoid  $M$ : we first compute a set  $C_M$  of representatives of the character equivalency classes and we then iterate Algorithm 3.3.3 over all  $\mathcal{J}$ -classes and sum the results.

The final useful example is the case of counting fixed points in a single regular  $\mathcal{L}$ -class, for the purpose of computing the character table of the monoid.



**Algorithm 3.3.6** (Computing the bicharacter of an  $\mathcal{L}$ -class). Let  $e$  be an idempotent and let  $L = \mathcal{L}(e)$ . In this example  $\mathbf{k}L$  is still a combinatorial module, but it has the particularity, compared with the other two examples, that the monoids on the left and right are not the same. However, as the maximal subgroup at  $e$ ,  $G_e$ , is a subsemigroup of  $M$ , the same results apply at no extra costs.

We can simply adapt Algorithm 3.3.3. Since an element of  $C_{G_e}$  acts “as itself” on the right, we don’t need to keep track of the action of the right with a vector  $r_L$ , as we did previously.

1. Initialize  $\chi$  to  $(0)_{(h,k) \in C_M \times C}$
2. For each  $h \in C_M$ , for each  $\mathcal{H}$ -class  $H$ , test if  $h\lambda_H a \in H$ . If so, denoting by  $k$  the conjugacy class of  $\lambda'_H h\lambda_H$  in  $G_e$ , increment  $\chi$  by  $|C_{G_e}(h)|$  at position  $(h, k)$ .
3. Return  $\chi$

### 3.3.3 Computing the radical

We are now almost in position to use the formula of Proposition 3.1.6: the character tables of the groups are supposed to be given, as we dispose of efficient group algorithms in the literature to compute them. From Algorithm 3.3.6 we now know how to efficiently compute the bicharacter of  $\mathbf{k}\mathcal{L}(e)$  as a  $\mathbf{k}M \otimes \mathbf{k}G_e^{op}$ -module for some idempotent  $e \in M$ . It remains to compute the bicharacter of  $N_e(\mathbf{k}\mathcal{L}(e))$  as a  $\mathbf{k}M \otimes \mathbf{k}G_e^{op}$ -module, which we discuss now.

Let  $L = \mathcal{L}(e)$ . Recall that, by definition,  $N_e(\mathbf{k}L) = \{x \in \mathbf{k}L \mid eMx = 0\}$ . Taking  $L$  as a basis for  $\mathbf{k}L$ , we can form a matrix with rows indexed by  $M \times L$  and columns indexed by  $L$ , with the coefficient at  $((m, l), l') = 1$  if  $eml = l'$  and 0 otherwise. Computing the kernel of this matrix yields a basis for  $N_e(\mathbf{k}L)$  but is extremely inefficient as the number of rows is many times the cardinality of the monoid.

Notice first that for any  $m \in M$ ,  $em \leq_{\mathcal{R}} e$  so we can consider only the elements of  $M$  that are  $\mathcal{R}$ -smaller than  $e$ . Conversely, recall that the structure of  $\mathbf{k}M$ -module on  $M$  is defined by  $m \cdot l = ml$  if  $ml \in L$  and 0 otherwise and that this latter case happens if  $ml \leq_{\mathcal{L}} l$ . Since if  $m \leq_{\mathcal{L}} l$  implies  $ml \leq_{\mathcal{L}} l$ , we have that the  $(m, l)$ -th row of the matrix is null and that we may omit it. This shows that we need only to consider the element of  $M$  that are not  $\mathcal{L}$ -below  $e$ . Together with the previous point, this means that the similarly defined matrix but whose rows are only indexed by  $\mathcal{R}(e) \times L$  has the same kernel. This is good news, as we may now exploit the structure of the  $\mathcal{J}$ -class given

by Green's Lemma and in particular the Location Theorem to further reduce the dimension of this matrix. Let us first choose convenient Green pairs.

**Lemma 3.3.7.** *Let  $e \in M$  be an idempotent,  $R = \mathcal{R}(e)$  its  $\mathcal{R}$ -class and  $R'$  another  $\mathcal{R}$ -class of  $\mathcal{J}(e)$ . Let  $(\lambda, \lambda')$  be a left Green pair for  $(R, R')$ . Then  $(\lambda e, e\lambda')$  is a left Green pair for  $(R, R')$ . Similarly, if  $L = \mathcal{L}(e)$ ,  $L'$  is an  $\mathcal{L}$ -class of  $\mathcal{J}(e)$  and  $(\rho, \rho')$  is a right Green pair for  $(L, L')$ , then  $(e\rho, \rho'e)$  is a right Green pair for  $(L, L')$*

*Proof.* Let  $g$  be any element of  $\mathcal{H}(e)$  and  $g' = \lambda g$ . Since  $e$  is idempotent,  $\mathcal{H}(e)$  is a group with identity  $e$  so we have  $e\lambda'\lambda eg = e\lambda'\lambda g = eg = g$  and  $\lambda ee\lambda'g' = \lambda eg = \lambda g = g'$  which, from Green's Lemma, make  $(\lambda e, e\lambda')$  a left Green pair for  $(L, L')$ . A similar argument applies to the second part of the proposition.  $\square$

**Remark.** This lemma means that for a regular  $\mathcal{J}$ -class  $J$  and for any two  $\mathcal{L}$ -class (or  $\mathcal{R}$ -class) it contains, we may choose a corresponding Green pair among the elements of those two classes.

**Proposition 3.3.8.** *Let  $e \in M$  be an idempotent and  $H = \mathcal{H}(e)$ ,  $L = \mathcal{L}(e)$ ,  $R = \mathcal{R}(e)$  and  $J = \mathcal{J}(e)$ . For each  $\mathcal{R}$ -class  $R' \subset J$ , we choose a left Green pair  $(l, l') \in J^2$ . We denote by  $\mathfrak{L}$  the set of all  $l$  for the chosen left Green pairs. We define  $\mathfrak{R}$  similarly. Then  $N_e$  is the set of solutions of :*

$$\forall r \in \mathfrak{R}, \forall g \in H, \quad \sum_{l \in \mathfrak{L}} \mathbb{1}_H(rl) x_{l(rl)^{-1}g} = 0$$

*Proof.* Consider an element  $a \in R$ . It can be written in a unique way as  $gr$ , with  $g \in H$  and  $r \in \mathfrak{R}$  corresponding to  $\mathcal{L}(a)$ . Similarly, an element  $b$  in  $L$  has a unique decomposition as  $l\gamma$ ,  $l \in \mathfrak{L}$ ,  $\gamma \in H$ . For an element  $x \in \mathbf{k}L$ , we note:

$$x = \sum_{l \in \mathfrak{L}, \gamma \in H} x_{l\gamma} l\gamma$$

its decomposition over the basis  $L$ .

We want to find the equations that describe  $\ker(gr \times_L)$  (where  $gr \times_L$  is the linear map on  $\mathbf{k}L$  obtained by extending the monoid's multiplication by linearity). From the Location Theorem, we get that  $\text{Im}(gr \times_L) \subset \mathbf{k}H$ . For  $k \in H$ , denote by  $f_{k,gr}$  the  $k$ -th coordinate function of  $gr \times_L$ . Because  $gr \times_L$  acts combinatorially on  $\mathbf{k}L$ , we have :

$$f_{k,gr}(x) = \sum_{l \in \mathfrak{L}, \gamma \in H} \mathbb{1}_{\{k\}}(grl\gamma) x_{l\gamma}$$

Note that  $x_{l\gamma}$  appears in the sum if and only if  $grl\gamma = k$ . From the Location Theorem, and because we choose  $l \in L, r \in R$ , we have  $grl\gamma = k$  if and only if  $rl \in H$  and  $\gamma = (rl)^{-1}g^{-1}k$  and thus the equation becomes:

$$f_{k,gr}(x) = \sum_{l \in \mathfrak{L}} \mathbb{1}_H(rl) x_{l(rl)^{-1}g^{-1}k}.$$

For  $x$  to be in  $\ker(gr \times_L)$ ,  $x$  must cancel simultaneously  $f_{k,gr}$  for all  $k \in H$ . We now have a set of equations for  $\ker(gr \times_L)$ , and we can deduce that the set of equations

$$\forall r \in \mathfrak{R}, \forall g, k \in H, \quad f_{k,gr}(x) = \sum_{l \in \mathfrak{L}, \gamma \in H} \mathbb{1}_H(rl) x_{l(rl)^{-1}g^{-1}k} = 0$$

describes  $N_e(\mathbf{k}L)$ . However, the equation system is redundant, as the equation  $f_{k,gr}(x) = 0$  is the same for all pairs  $(g, gk')$  with  $k' \in H$ . Removing the duplicate equations gives the system announced in the proposition.  $\square$

**Example 3.3.9** ( $N_e$  in the case of an aperiodic monoid). As in Example 3.3.4, let us consider the case of a  $\mathcal{J}$ -class with trivial  $\mathcal{H}$ -classes. In this case, we have  $L = \mathfrak{L}, R = \mathfrak{R}$  and  $H = \{1_H\}$ , so the equations become:

$$\forall r \in R, \quad \sum_{l \in M} \mathbb{1}_H(rl) x_l.$$

Again from the Location Theorem, we have that  $\mathbb{1}_H(rl) = 1$  if and only if there is an idempotent in  $\mathcal{L}(r) \cap \mathcal{R}(l)$ . So if we form a matrix  $A$  with rows indexed by  $L$  and columns indexed by  $R$ , and with coefficients 1 at  $(\mathcal{L}(r), \mathcal{R}(l))$  if  $\mathcal{L}(r) \cap \mathcal{R}(l)$  contains an idempotent and 0 otherwise, the above equations become:

$$(x_l)_{l \in L}^T A = 0,$$

that is, in the case of an  $\mathcal{H}$ -trivial  $\mathcal{J}$ -class,  $N_e(\mathbf{k}L)$  is the left kernel of the egg box picture seen as a  $\{0, 1\}$ -matrix.

Note that given this set of equations, we can compute the character  $\chi_{\mathbf{k}M \otimes \mathbf{k}G_e^{op}}^{N_e(\mathbf{k}\mathcal{L}(e))}$  from the formula in Proposition 3.1.6 using classical linear algebra algorithms to find a basis of  $N_e(\mathbf{k}\mathcal{L}(e))$  and then computing the value of the character at any  $(m, g) \in C_M \times C_{G_e}$  by iterating over the basis vectors, applying  $(m, g)$  as a linear map and computing the relevant coefficient in the image vector.

### 3.3.4 Performance

To begin with, in this section, we discuss the challenges and choices we have been led to make to measure the performance of our algorithms. In a second

part, we present the experimental results as well as, when possible, their complexity analysis. We discuss performance for the computation of the number of fixed points, the character table and finally the Cartan matrix. At the end of this section, in Figures 3.2, 3.3 and 3.4, we give experimental measures of the complexity of our algorithms as a function of cardinality.

### Challenges, experimental choices and methodology

Monoids being as diverse as they are, a meaningful analysis of the time and space complexities of the above algorithms is difficult, as many relevant metrics (such as the number of Green's classes of a given type) cannot be straightforwardly computed. Moreover, in some sense these metrics vary a lot: two transformation monoid acting on the same number of points with the same number of generators can have vastly different Green structures. Although we can provide some time complexities in terms of number of  $\mathcal{L}$ ,  $\mathcal{R}$ -classes and cardinality of  $\mathcal{H}$ -classes (as we do below), the real test of viability is to see if the algorithm effectively terminates in practice. Thus, we provide timings and memory usage measures for the computation of the three main objects of our discussion: the bicharacter, the character table and the Cartan matrix.

The performance measures provided for these new algorithms (as well as the computation results presented hereafter) all come from an implementation using the computer algebra system GAP. All performance measures are realized on a laptop equipped with an Intel Core i7-10850H @ 2.7GHz (on one core) and 16 GB memory. Our specific implementation, as well as the test cases used and the raw data, are publicly available on our git repository<sup>2</sup>. As the following section will show, these algorithms are, on our test machine, memory limited. We have made some test computations on a machine equipped with 128 GB of RAM. Even then, due to the rapid explosion of memory requirements, we cannot compute the character table of  $\mathcal{T}_8$  using these methods, and it seems that monoids containing  $\mathfrak{S}_{12}$  generally fail to reach the end of the bicharacter computation. Since in the cases where this more powerful machine enables more computation by bypassing memory limitations, computation time can reach the hour range, we choose to focus on the smaller scale tests allowed by our smaller machine.

We consider three families of monoids implemented by the package GAP Semigroups [Mit+23]: transformation monoids, which we have introduced before, partition monoids, and partial permutation monoids.

**Definition 3.3.10.** A *partition*  $P$  of a set  $S$  is a collection of subsets of  $S$  such that  $\bigcup P = S$  and for all  $p, p' \in P$ ,  $p \cap p' = \emptyset$ . The elements of  $P$  are called its *blocks*. Given  $s \in S$ ,  $P(s)$  is the unique block of  $P$  containing  $s$ .

<sup>2</sup>[github.com/ZoltanCoccyx/monoid-character-table](https://github.com/ZoltanCoccyx/monoid-character-table)

Let  $P, Q$  be two set partitions of the set  $S = \llbracket -n, -1 \rrbracket \cup \llbracket 1, n \rrbracket$ . The *partition product* of  $P$  by  $Q$  is the partition  $PQ$  where  $i, j \in S$  are in the same block if:

- $i, j > 0$  and are in the same  $P$ -block.
- $i, j < 0$  and are in the same  $Q$ -block.
- $i > 0$  and  $j < 0$  and there exists  $k \in \llbracket -n, -1 \rrbracket$  such that  $k \in P(i)$  and  $k \in Q(j)$ .
- $i$  and  $j$  are related in the transitive closure of the relation given by the previous point.

This defines **the partition monoid**  $\mathcal{P}_n$  (with identity  $\{\{i, -i\} \mid i \in \llbracket 1, n \rrbracket\}$ ). A partition monoid (of rank  $n$ ) is a submonoid of  $\mathcal{P}_n$ .

**Definition 3.3.11.** A *partial permutation* of  $\llbracket 1, n \rrbracket$  is an injective partial function from  $\llbracket 1, n \rrbracket$  to itself. Equipped with the identity function and the partial map composition, this defines the *inverse symmetric monoid*  $\mathcal{I}_n$ . A partial permutation monoid (of rank  $n$ ) is a submonoid of  $\mathcal{I}_n$ .

We test our functions on the families  $\mathcal{T}_n, \mathcal{P}_n$  and  $\mathcal{I}_n$ , with numeric values provided for our canonical example  $\mathcal{T}_n$  in Tables 3.2, 3.3 and 3.4. However, many computer algebra systems, including GAP, are smart enough to detect that the Schützenberger groups are actually symmetric groups and thus use some non-general algorithms that could not be used on a typical finite monoid. This has the potential to falsify our measures (and indeed probably does, given Figures 3.2-(a), 3.3-(a) and 3.4-(a)). To mitigate this issue, we provide timings for randomly chosen finite monoids. The question of picking a “generic” monoid is entirely outside the scope of this paper. We simply choose a monoid  $R(m, n)$  of rank  $m$  with  $n$  generators by picking uniformly  $n$  elements of  $\mathcal{T}_m, \mathcal{P}_m$  or  $\mathcal{I}_m$ . The set of (rank, number of generators) pairs used is given in Table 3.1. For reasons discussed hereafter, we have chosen to have “enough” generators to have non-trivial  $\mathcal{J}$ -structure. Experimentally, we note that the resources in time and memory used by two transformation monoids acting of the same rank with the same number of generators can differ by up to an order of magnitude. Thus, for each pair  $(m, n)$ , we measure performance on 10 randomly chosen test cases. To experimentally evaluate the complexity as a function of cardinality, we do a linear regression of the logarithm of the (time and memory) measures against the logarithm of the cardinality. In the case of transformation monoids and partition monoid, the data follows a relatively tight distribution (see Figures 3.2 and 3.3). To mitigate threshold effects, we weight the sample values linearly (and not logarithmically) so that the higher values have more importance.

By contrast, the randomly generated partial permutation monoids give more dispersed samples. Given this dispersion, considering only (or with a heavy weight) the highest values makes for an overestimation of necessary resources in most cases. Thus, in that case, we do the linear regression with equal weights on all samples. This does not dramatically change the measured complexity (we refer to the actual code as the final arbiter), but we advise caution when using these figures. For ease of discussion, we will call *structured monoids* the non-random ones present in our tests.

Type	Bicharacter	Character table and Cartan matrix
Transformation	(4,3), (5,3), (5,4), (6,5), (7,6), (8,8), (9,8)	(4,3), (5,3), (5,4), (6,5), (7,6)
Partition	(4,3), (4,6), (5,10), (5,15), (6,18), (7,20), (8,20), (9,25), (10,30), (12,40)	(4,3), (4,6), (5,10), (5,15), (6,18), (7,20), (8,20), (9,25), (10,30), (12,40)
Partial Permutation	(3,6), (3,9), (4,8), (4,12), (5,15), (6,20), (7,21), (7,25), (7,30), (8,25), (8,30)	(3,6), (3,9), (4,8), (4,12), (5,15), (6,20), (7,21), (7,25), (7,30)

Table 3.1: Rank and number of generators for random monoids

### Performances and experimental results.

In the case of Algorithm 3.3.3, we can give some analysis of the time complexity in terms of the Green structure of the particular  $\mathcal{J}$ -class Algorithm 3.3.3 is applied to.

**Proposition 3.3.12.** *Consider a  $\mathcal{J}$ -class  $J$  containing  $n_L$   $\mathcal{L}$ -classes,  $n_R$   $\mathcal{R}$ -classes, containing an  $\mathcal{H}$ -class  $H$  with  $n_C$  conjugacy classes in  $\Gamma'(H)$ , and let  $C_M$  be a set of representatives of the character equivalence classes, as before. Then, the Algorithm 3.3.3 does:*

- $n_C$  computations of conjugacy classes of  $\Gamma'(H)$  cardinality (if we precompute those cardinalities to be able to do a lookup in step 2-a of Algorithm 3.3.3, instead of computing it on the fly),

- $O(|C_M|(n_L+n_R))$  monoid multiplications, Green class membership tests and conjugacy class of  $\Gamma'(H)$  membership tests,
- $O(|C_M|n_R)$  computations of  $\tau_a$ ,
- $O(|C_M|n_L)$  conjugacy class of  $\Gamma'(H)$  cardinality lookups,
- $n_C|C_M|^2$  integer multiplications.

*Proof.* This simply results from an inspection of Algorithm 3.3.3, with the caveat that we precompute the cardinalities of the conjugacy classes of  $\Gamma'(H)$ .  $\square$

In the case of Algorithm 3.3.5 applied to an arbitrary monoid, such an analysis is mostly meaningless because we would need to express the number of Green's classes and understand their breakdown. However, because of the vast variety of possible monoids, there is no meaningful way to do it from simple parameters such as the rank or the number of generators.

Note that we do not provide a cumulative formula for the complexity of Algorithm 3.3.3 as, for instance, the complexity of a conjugacy class membership test heavily depends on the algorithm used by the computer algebra system, that can itself vary depending on the characteristics of the Schützenberger groups. This makes the task of providing a meaningful evaluation of the global complexity of the algorithm quite difficult, mainly because expressing the complexity of those “elementary” operations of monoid multiplications, membership testing, etc. . . in terms of the same parameters is not straightforward. However, we can at least compare this to the naive algorithm of testing if every element of  $J$  is a fixed point which demands  $O(n_L n_R |H|^2 |C_M|^2)$  monoid multiplications: as long as the complexity of the more complex operations of Green class or conjugacy class membership testing remains limited in terms of monoid multiplications, our complexity is better. For instance, in the case of the monoid  $\mathcal{T}_n$ , all the required operations can be done on  $O(n)$ , making Algorithm 3.3.3 (and, in turn, Algorithm 3.3.5) more efficient than the naive algorithm, as can be seen in Table 3.2, with a sublinear (with respect to cardinality) measured complexity (Figure 3.2).

As shown in Table 3.3, the computation of the character table takes much longer. This is because, to compute the radical of  $\mathbf{k}\mathcal{L}(e)$  for an idempotent  $e$ , we must solve a linear system of size  $|\mathcal{R}(e)| \times |\mathcal{L}(e)|$ , which necessitates  $O(|\mathcal{R}(e)|^2 |\mathcal{L}(e)|)$  arithmetic operations. In the case of the full transformation semigroup  $\mathcal{T}_n$ , if  $e$  has  $k$  elements in its image,  $|\mathcal{L}(e)| = k! \times \binom{n}{k}$ , while  $|\mathcal{R}(e)| = k! \times S(n, k)$  where  $S(n, k)$  is a Stirling number of the second kind, which gives  $|\mathcal{R}(e)| \sim k^n$ . The size of that linear system becomes rapidly intractable. Moreover, once we have a basis of  $N_e(\mathbf{k}L)$  of cardinality  $d$ , we

Monoid	Cardinality	Coefficients	Naive	Ours
$\mathcal{T}_3$	27	$6^2$	29 ms	8 ms
$\mathcal{T}_4$	256	$11^2$	92 ms	32 ms
$\mathcal{T}_5$	3125	$18^2$	1.44 s	84 ms
$\mathcal{T}_6$	46656	$29^2$	53.0 s	0.30 s
$\mathcal{T}_7$	823543	$44^2$	>30 min	1.54 s
$\mathcal{T}_8$	16777216	$66^2$	...	8.65 s
$\mathcal{T}_9$	387420489	$96^2$	...	58.2 s

Table 3.2: Computation time of the regular representation bicharacter.

still have to compute the  $C_M^2$  character values in  $O(d^2)$  operations each. Experiments indicate that the computation time of the character tables of the maximal subgroups is small in comparison to all radical related computations. As can be seen in Figures 3.2-(b, c), 3.3-(b, c) and 3.4-(b, c) the limiting factor is memory (the test on random monoids fails for the random transformation monoids of the form  $R(9, 8)$  by exceeding the 16 GB memory capacity of our testing machine). Although computation requirements are close to linear in the cardinality, the cardinality tends to be more than exponential in the rank, limiting these methods to small ranks.

Monoid	Cardinality	Coefficients	Ours
$\mathcal{T}_3$	27	$6^2$	80 ms
$\mathcal{T}_4$	256	$11^2$	182 ms
$\mathcal{T}_5$	3125	$18^2$	1.30 s
$\mathcal{T}_6$	46656	$29^2$	17.6 s
$\mathcal{T}_7$	823543	$44^2$	5.93 min

Table 3.3: Computation time of the character table.

Finally, for the computation of the Cartan Matrix, the previous timings show that the vast majority of the computation time is spent computing the character table of the monoid. As the computation of the combinatorial bicharacter is more than a hundred times faster than the computation of the character table, this is a clear invitation to improve in particular the computation of the character of the radical of the  $\mathcal{L}$ -classes. In Table 3.4, we show some timings for that computation, and a comparison with Sage's generalist algorithm (based on the Peirce decomposition of the monoid algebra) for the computation of the Cartan Matrix: despite its limitations, our specialized algorithm allows for the handling of larger objects. Indeed, our algorithm has near linear performance with respect to cardinality, while Sage's has roughly cubic complexity.

Again, memory fails before time for  $\mathcal{T}_8$  and onward. For the transformation monoids of the form  $R(9, 8)$ , using the regression, we can predict a computation



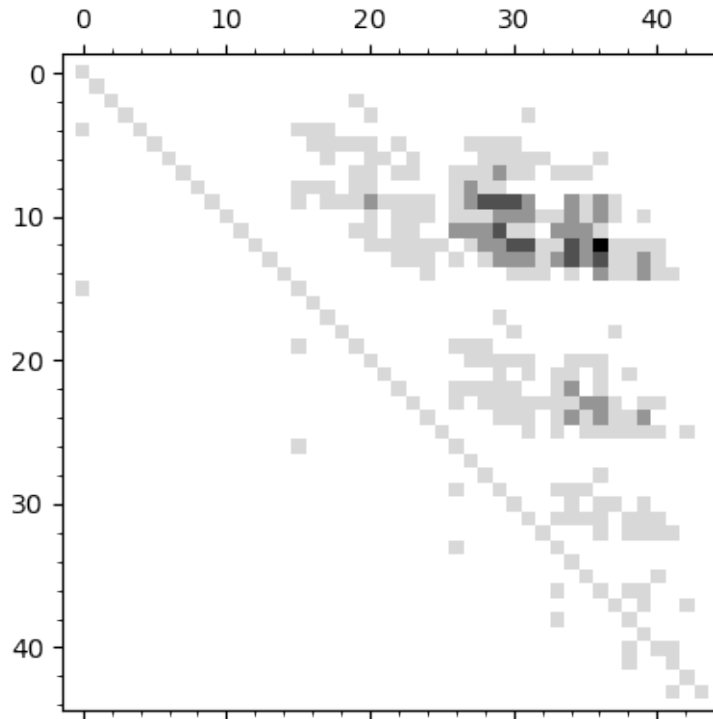
Monoid	Coefficients	Sage's	Ours
$\mathcal{T}_3$	$6^2$	575 ms	42 ms
$\mathcal{T}_4$	$11^2$	5.23 min	146 ms
$\mathcal{T}_5$	$18^2$	>2h	1.29 s
$\mathcal{T}_6$	$29^2$	...	17.7 s
$\mathcal{T}_7$	$44^2$	...	5.48 min

Table 3.4: Computation time of the Cartan matrix.

In the case  $\mathcal{T}_5$ , Sage's algorithm was interrupted before the end of the computation.

time of around 6 hours on our testing machine if it was not memory limited.

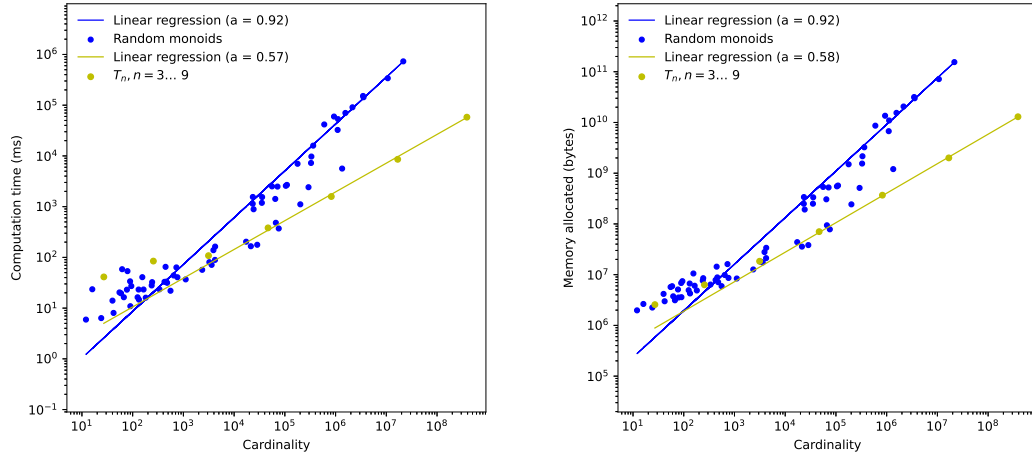
An example of a Cartan matrix obtained using our Algorithms and Thiéry's formula is pictured in Figure 3.1.

Figure 3.1: Cartan Matrix of  $\mathcal{T}_7$ 

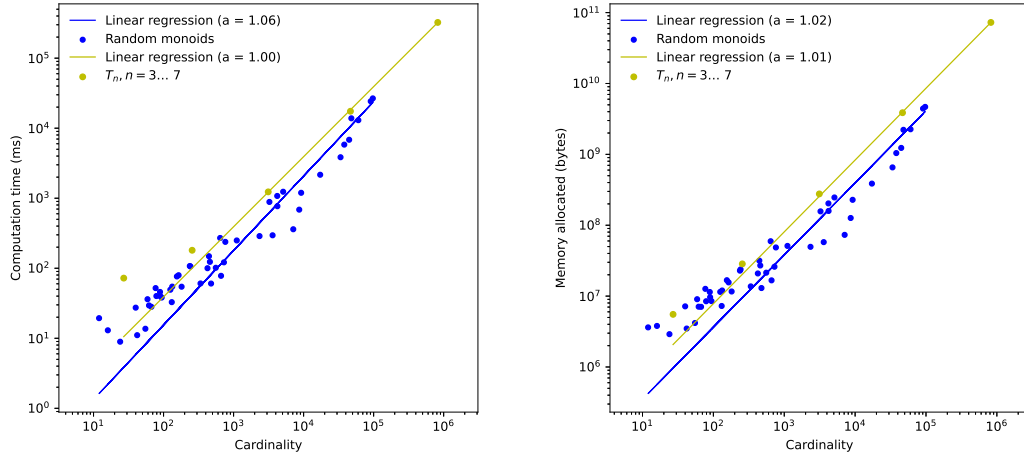
For legibility, the entries are represented as gray values. The entries are integers from 0 (in white) to 4 (the single black pixel).

Generally, our algorithms achieve (at worst) near linear measured complexity. The fixed point counting is most efficient in the structured monoids, where there are few, big  $\mathcal{J}$ -class with big Schützenberger groups. In the contrary,

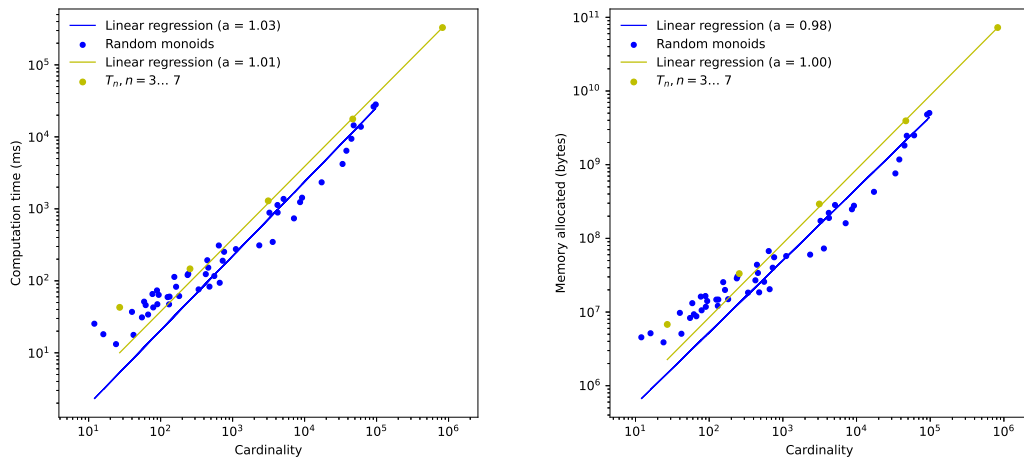
on “sparse” monoid with many small  $\mathcal{J}$ -class, the efficiency of the algorithm drops. This is most apparent in Figure 3.4, where the blue clusters are (roughly) semigroups of the same rank with the same number of generators. The partial permutation semigroups generated that way tends to have either (at our scale) low tens *or* multiple thousands  $\mathcal{J}$ -classes (without intermediate values). The latter kind tends to be much more expensive to deal with, despite implementing “lazy” fixed point counting on small  $\mathcal{J}$ -classes and  $\mathcal{J}$ -classes with trivial Schützenberger groups to avoid costly computations. The situation balances out (or even inverses itself, see Figure 3.3) when it comes to computing the character table and the Cartan matrix. This seems to come from the fact that bigger  $\mathcal{J}$ -classes mean higher dimensional radical and bigger Schützenberger groups mean less sparse basis vectors in the radical. Still, the measured complexity remains near linear in our experiments.



(a) Bicharacter

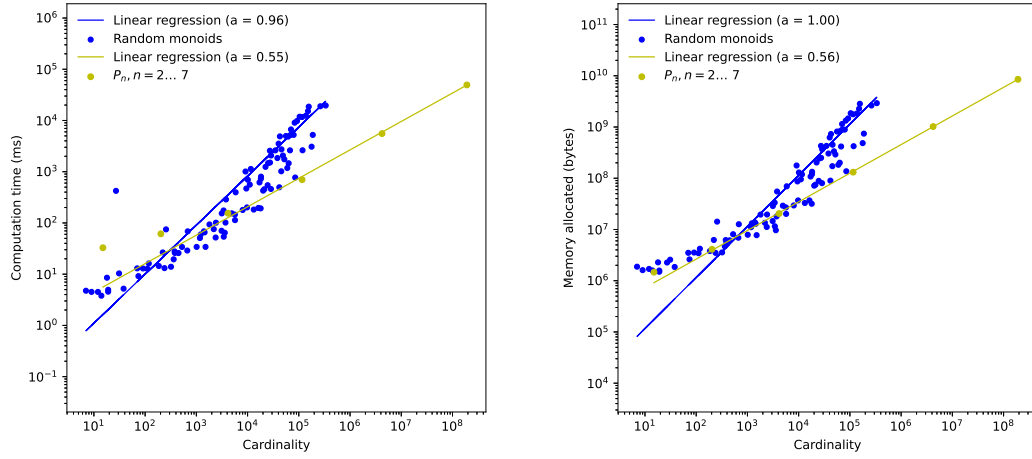


(b) Character table

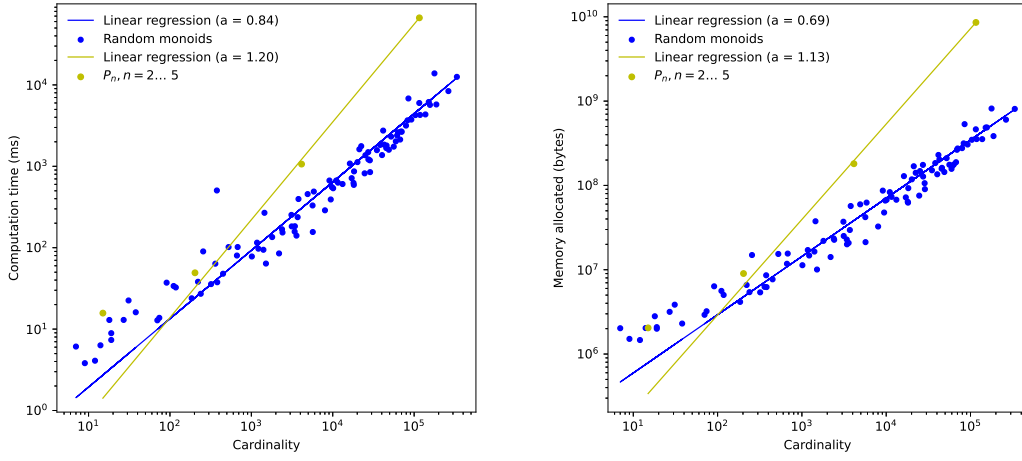


(c) Cartan Matrix

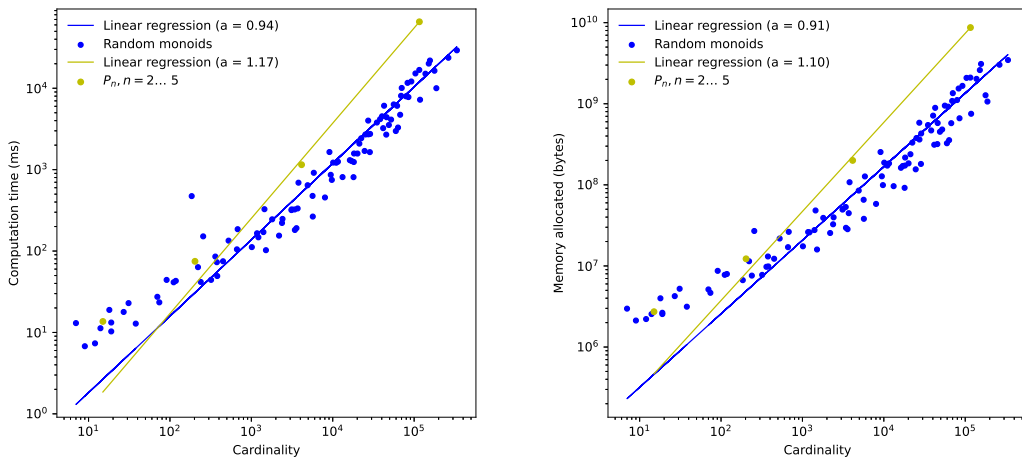
Figure 3.2: Time and memory usage: transformation monoids



(a) Bicharacter

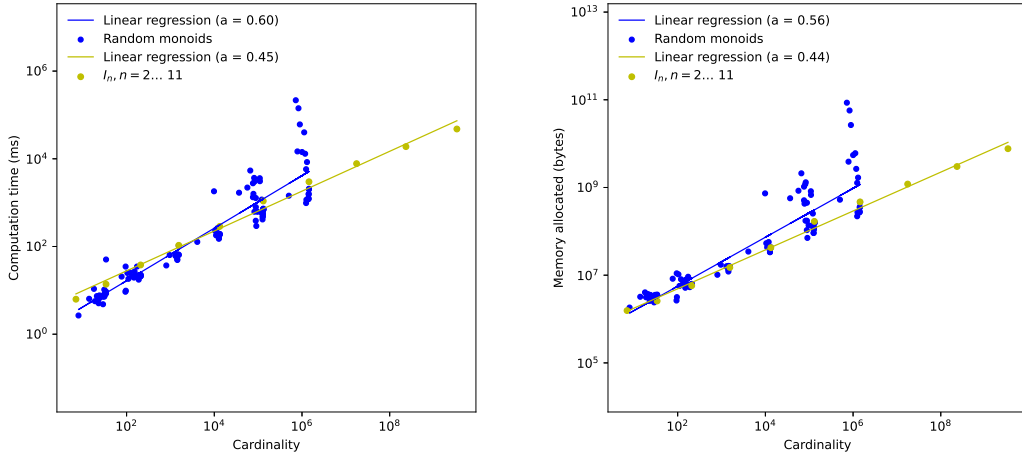


(b) Character table

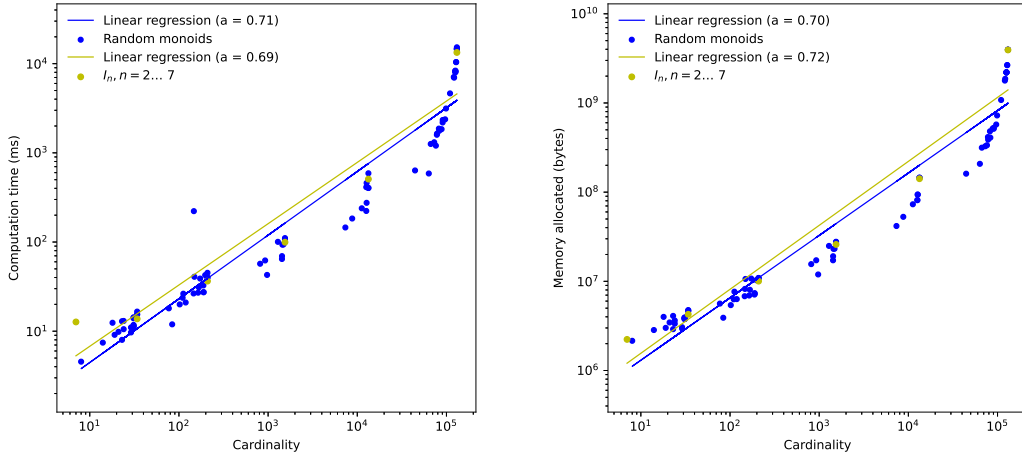


(c) Cartan Matrix

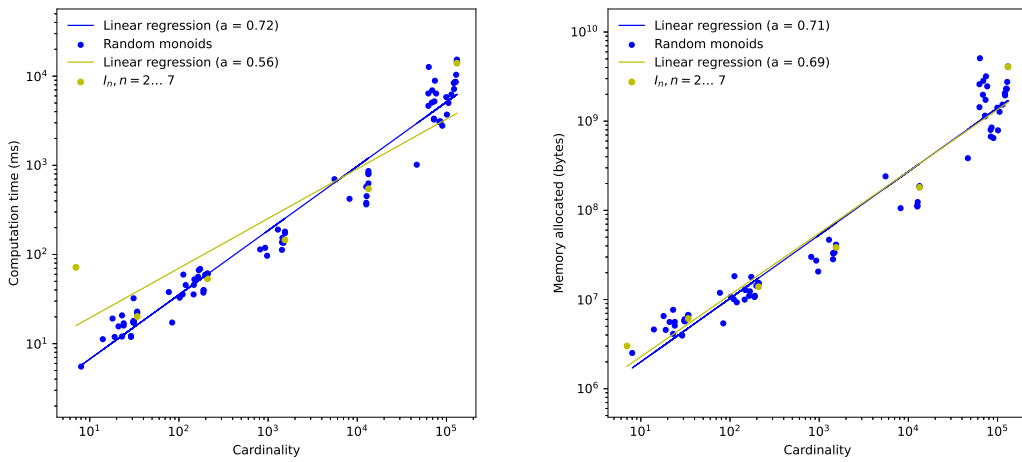
Figure 3.3: Time and memory usage: partition monoids



(a) Bicharacter



(b) Character table



(c) Cartan Matrix

Figure 3.4: Time and memory usage: partial permutation monoids

## Bonus: a formula for $\mathbb{C}\mathcal{T}_n$

For convenience, we suppose that  $\mathcal{T}_n = \llbracket 0, n-1 \rrbracket^{\llbracket 0, n-1 \rrbracket}$  here, and in the same way that  $\mathfrak{S}_n$  permutes  $\llbracket 0, n-1 \rrbracket$  instead of  $\llbracket 1, n \rrbracket$ .

As mentioned in Example 1.2.5, the  $\mathcal{J}$ -classes of  $\mathcal{T}_n$  are ordered by the cardinality of the image (of any element in the  $\mathcal{J}$ -class). It is clear that  $J_l$ , the  $\mathcal{J}$ -class with image cardinality  $l$ , has  $\mathfrak{S}_l$  as Schützenberger group and (since  $J_k$  is regular since  $e_l = i \mapsto \min(i, l-1) \in J_l$  is idempotent) as maximal subgroup at its idempotents (up to relabeling). It is well-known that the conjugacy classes of  $\mathfrak{S}_l$  are indexed by the *integer partitions* of  $l$ , that is, the set of decreasing strictly positive integer sequence that sum to  $l$ . Thus, the character equivalence classes of  $\mathcal{T}_n$  are naturally labeled by integer partitions up to  $n$ . To fix the ideas, given an integer partition  $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$ , let  $l = \sum \lambda$ ,  $\Lambda = (\sum_{i=1}^j \lambda_i)_{j \in \llbracket 1, r \rrbracket}$  and

$$c_\lambda = e_l \circ [(0 \ 1 \dots \lambda_1 - 1)(\lambda_1 \dots \lambda_2 - 1) \dots (\lambda_j \dots \lambda_{j+1} - 1) \dots (\lambda_{r-1} \dots l - 1)]$$

presented as product of cycles.

**Proposition 3.3.13.** *Let  $\lambda, \mu$  be integers partitions with  $\sum \lambda, \sum \mu < n$ . Let  $h = c_\lambda, k = c_\mu$ . Then:*

$$|\{s \in \mathcal{T}_n \mid hsk = s\}| = \prod_{\mu_i} \sum_{\lambda_j \mid \mu_i} \lambda_i$$

*Proof.* Consider the case where  $\lambda = (\lambda_1)$  and  $\mu = (\mu_1)$  and suppose that  $hsk = s$  for some  $s$ . Note that it is necessary that  $\text{Im } s \subset \text{Im } h$  and that  $\ker s$  is coarser than  $\ker h$ . Thus, we can concentrate only on the “injective support” of  $k$ , meaning  $\llbracket 0, \mu_1 - 1 \rrbracket$ . For all  $i \in \llbracket 0, \mu_1 - 1 \rrbracket$ , we have

$$\begin{aligned} s(i) &= hsk(i) = hs(i+1 \pmod{\mu_1}) \\ \Rightarrow s(i+1 \pmod{\mu_1}) &= h^{-1}(s(i)) = s(i) - 1 \pmod{\lambda_1}. \end{aligned}$$

Note that we commit an abuse of notation in that  $h$  is not invertible as a function. However, it is invertible as an element of the maximal subgroup at  $e_{n-\lambda_1}$  where it behaves like a cycle. This implies that  $\lambda_1 \mid \mu_1$ , as otherwise  $s$  would be multivalued. Conversely, if  $\lambda_1 \mid \mu_1$  and

$$\exists k, \forall i, s(i) = \begin{cases} (k-i) \pmod{\lambda_i} & \text{if } i \leq \mu_i \\ (k-1) \pmod{\lambda_i} & \text{otherwise} \end{cases}$$

then  $hsk = s$ . There are clearly  $\lambda_i$  such functions.

Consider now the case where  $\lambda$  and  $\mu$  are general. The same argument (up to relabelling) shows that for each part  $\mu_i$ , any  $\lambda_j$  such that  $\lambda_j \mid \mu_i$  can

be associated to  $\mu_i$ , hence the sum  $\sum_{\lambda_j|\mu_i} \lambda_i$ . Since that association is independent between the parts of  $\mu$ , (a  $\lambda_j$  can be reused), we get the announced product.  $\square$

As it happens, from [Put96], we have an explicit formula for the character table of  $\mathcal{T}_n$ . We need a bit of notation. A partition  $\lambda = (\lambda_1, \dots, \lambda_r)$  is abbreviated by  $\lambda = \left( \lambda_i^{|\{j \mid \lambda_j = \lambda_i\}|} \right)_{\lambda_i \in \{\lambda_j \mid j\}}$ . We say that  $\lambda \subseteq \mu$  when  $\mu - \lambda$  (completed by some zeros if need be) is component-wise non-negative. The result is denoted  $\mu \setminus \lambda$  and is a *horizontal strip* if its entries are at most 1.

**Proposition 3.3.14.** *Let  $D$  be the block diagonal matrix of the group character tables  $\text{diag}(X(\mathfrak{S}_k)_{k \in [1, n]})$  and  $U$  be the matrix defined by:*

$$U_{\lambda, \mu} = \begin{cases} 1 & \text{if } \lambda = (1^r), \mu = (m - r + 1, 1^{r-1}) \text{ with } m \geq r \\ 1 & \text{if } \lambda \neq (1^r), \lambda \subseteq \mu \text{ and } \mu \setminus \lambda \text{ is a horizontal strip} \\ 0 & \text{otherwise} \end{cases}$$

where  $\lambda, \mu$  are partitions with  $\sum \lambda, \sum \mu \leq n$ . Then  $X(M) = UD$ .

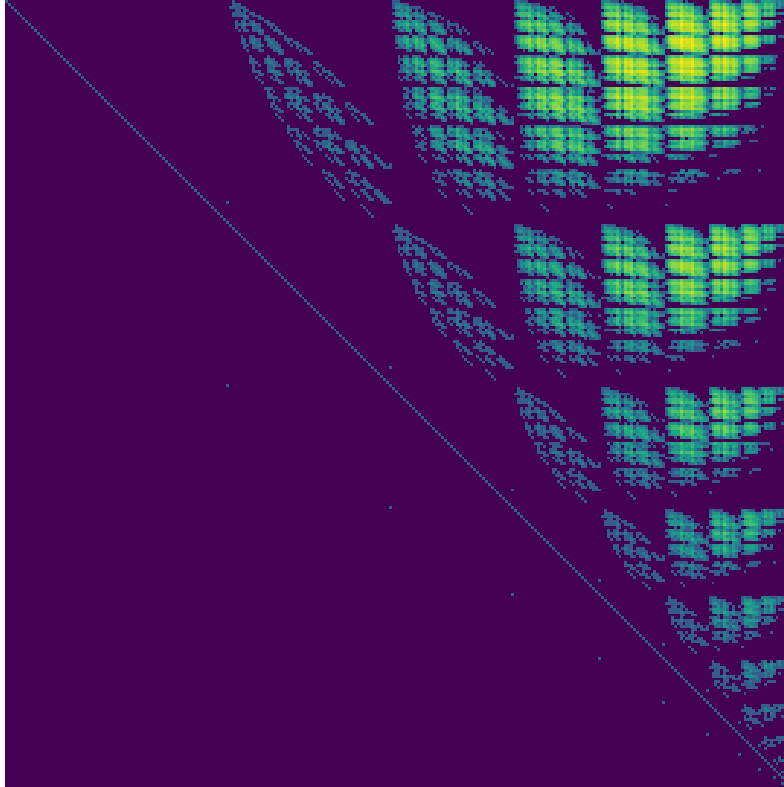


Figure 3.5: Cartan matrix of  $\mathcal{T}_{12}$

Together with known formulas for the character tables of the symmetric groups (directly in terms of tableaux or in terms of basis change on symmetric functions), this allows to compute bigger examples for the Cartan matrix of  $\mathbb{C}\mathcal{T}_n$ .

**Question 3.3.15.** Explain the fern-like structure.





## Part II

# Minimal elements of Shi arrangements



## Introduction

The Shi arrangement associated to a crystallographic group  $W$  was introduced in 1987 par J.-Y. Shi in [Shi87a]. It has many interesting properties, and in particular the regions of the Shi arrangement correspond to the states of the so-called canonical automaton recognizing the language of reduced words of  $W$  introduced by B. Brink and R. B. Howlett in [BH93]. Motivated by the search of minimal automata ([HNW16]) and similar questions in braid groups ([DH16]) a push has been made recently by Chapelier-Laget, Dyer, Fishel, Hohlweg... to understand the minimal elements of the Shi regions. In particular, in [DH16, Conjecture 2] (which as since been solved in [Dye+23]), Hohlweg and Dyer ask, in a sense, to retrieve geometric information on the *inversion sets* of those minimal elements that the conjecture implies do exist (which was not know for Shi arrangements general Coxeter groups at the time).

In this part of the thesis, we present our work around understanding these minimal elements. Chronologically, it begins with the case of rank 3 Coxeter groups, discussed in Section 6.1. It should be noted that although this section is a result of this thesis, it is only partly so, as the original research on that subject was originally done during a master's research internship with Pr. Hohlweg at LaCIM, Montréal, in 2017. However, the proofs we present, although they originate in spirit in that internship, were first produced in their correct form during the preparation of the extended abstract [Cha21]. This document extends, and improves on, the sketch of proofs from that extended abstract.

The rest of this part of the thesis originates in a working group launched by Dr. Chapelier-Laget during the summer 2022, with the objective of algorithmically computing those minimal elements in type  $A_n$ . This working group was the major motivation for the investigations presented here on the special case of Weyl groups. Considering the structure of the Shi relations lead to [AL99], in which Athanasiadis and Linusson give a bijection between parking functions and Shi regions in type  $A_n$ . This bijection is “natural” in the sense that it encodes in the parking function the set of defining inequalities of the Shi region. Citing the same article as a motivation, Armstrong Reiner and Rhoades provide a generalization of it to all crystallographic groups as [ARR15, Proposition 10.4]. In the extended abstract [Cha22], the author describes how to compute the Shi vector coefficients of the minimal element of a Shi regions using this bijection, in the case of the classical Weyl groups. We expand here on this extended abstract in three main ways. Most obviously, we provide full proofs of the results that were announced. Secondly, we do so in a (mostly) type-free fashion that allows us to expand the results to exceptional groups. Finally, and this was the original motivation for this work in the first

place, we provide as an application of our descriptive result a proof of [DH16, Conjecture 2] in affine Weyl groups.

In Chapter 4 we present our setting, our notations and some facts that we will use in the following chapters. We go over basic definitions on Coxeter groups (§4.1), before introducing their associated geometric representation (§4.2) and the notion of inversion sets (§4.3). All of these notions will be well known to the informed reader, and may be skipped, but we do take the time to go about them in details as a way to build familiarity for some of the geometric considerations made in the latter sections. We then present the projective picture of a root system (§4.4), before recalling the classification of finite and affine Coxeter groups and some related facts (§, 4.5). We then discuss dual notions to the root systems and inversion sets: the Coxeter arrangement and Tits cone (§4.6). In the final stretch of this introductory chapter we introduce the object of our study: the Shi arrangement, both in a general setting (§4.7.1) and specialized to affine Weyl groups (§4.7.2). In Chapter 5, we describe the minimal elements of the regions of the Shi arrangement in Weyl groups. We begin by analyzing the Shi relations constraining the possible inversions sets in a Weyl group (§5.1). We then give the description itself in Section 5.2.3. This description gives the *Shi encoding* (§5.2.1) of the minimal elements by extending the bijections of the Shi regions with parking functions from Athanasiadis & Linusson and Armstrong, Reiner & Rhoades (§5.2.2). Up to that point, this discussion is mostly type-free, but in Section 5.3, we come back to the case of classical Weyl groups, where the previous results have an interpretation in terms of non-crossing arcs counting in a non-nesting partition. This puts, somewhat intriguingly, these minimal elements at the intersection of non-nesting and non-crossing combinatorics. In the final Chapter 6, we come back to the motivating question from Dyer & Hohlweg [DH16], first in the case of rank 3 Coxeter groups (§4.7.1) and then in Weyl groups using our description of minimal element from the previous chapter (§6.2.2).

# Chapter 4

## Preliminaries on Coxeter theory

In these preliminaries, we go over well known results on Coxeter groups, for which we refer the classical (and excellent) references *Combinatorics of Coxeter groups* by A. Björner and F. Brenti [BB05] and *Reflection groups and Coxeter groups* by J. Humphreys [Hum90]. However, since much of the original work of this part of the thesis concerns itself with the notion of *inversion sets*, we have taken the approach of presenting proofs using this notion whenever doing so gives a better understanding of the behavior of inversion sets, where some canonical proofs may be of a more combinatorial or group theoretic nature. The goal of this approach is to make the considerations of the later sections seem more natural, and not to present “original proof” as, again, they can all be found in a more or less explicitly “inversion set related” form in [BB05] or [Hum90].

### 4.1 Coxeter groups

**Definition 4.1.1.** (Coxeter system) Let  $S$  be a finite set. For each  $s, t \in S$  we choose  $m_{s,t} \in \mathbb{N}_{>0} \cup \{\infty\}$  under the constraints that  $m_{s,t} = 1 \Leftrightarrow s = t$  and  $m_{s,t} = m_{t,s}$  for every  $s, t \in S$ . The associated *Coxeter group*  $W$  is the group with presentation  $\langle S \mid (st)^{m_{s,t}} = 1 \rangle$  where  $(st)^\infty = 1$  means that no relation is imposed.  $(W, S)$  is called a *Coxeter system* of which  $|S|$  is the *rank*. Given two distinct  $s, t \in S$ , the relation  $(st)^{m_{s,t}} = (ts)^{m_{s,t}} = 1$  is called the *braid relation* between  $s$  and  $t$ .

Note that, because the generators have order 2, the braid relation  $(st)^{m_{s,t}} = 1$  can also be written as

$$\underbrace{stst \cdots}_{m_{s,t}} = \underbrace{tsts \cdots}_{m_{s,t}}.$$

**Example 4.1.2.** Consider a *dihedral group*, that is, a Coxeter group of rank 2:  $W = \langle s, t \mid ss, tt, (st)^{m_{s,t}}, (ts)^{m_{s,t}} \rangle$ . Because of the braid relation, any element of  $W$  can be written as an alternating product of  $s$  and  $t$  of length at most  $m_{s,t}$ . Indeed, if we have a word for some element  $w \in W$ , we can remove any  $ss$  or  $tt$  from that word until we obtain an alternating expression for  $w$ . If the alternating word is longer than  $m_{s,t}$ , for example  $w = tstststs$ , and  $m_{s,t} = 3$ , then  $tstststs = tststtst = ts$ .

From now on, we fix a Coxeter system  $(W, S)$  of finite rank. It is usual to represent a Coxeter group using a graph with vertices the elements of  $S$ , with an edge between two distinct generators  $s, t$  labeled by  $m_{s,t}$  if  $m_{s,t} > 3$ , an unlabeled edge if  $m_{s,t} = 3$  and no edge if  $m_{s,t} = 2$ . This convention comes from the fact that many interesting Coxeter groups have few braids of length more than 3.

**Example 4.1.3.** Consider the Coxeter group given by the relation matrix

$$m = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 6 \\ 2 & 6 & 1 \end{pmatrix}.$$

Let us suppose that its rows and columns are indexed (in order) by the set  $S = \{r, s, t\}$ : the ones on the diagonal indicate that the generators have order two (since  $(ss)^1 = 1$  for instance), that  $r$  and  $t$  commute (since  $rtrt = 1 \Leftrightarrow rt = tr$ ), that the pair  $(r, s)$  is *simply braided* (meaning that the braid relation has length 3:  $rsr = srs$ ) and finally that  $ststst = tststs$ . The Coxeter graph encoding these relations is represented below.

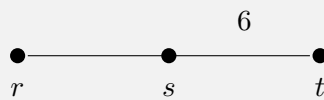


Figure 4.1: Coxeter graph of  $\tilde{G}_2$ .

Thus, a Coxeter group is the set of classes of words on the alphabet  $S$  under the relations  $(st)^{m_{s,t}} = 1$ . A *word* for an element  $w \in W$  is a product of generators equal to  $w$ . A *reduced word* for  $w$  is a word for  $w$  of minimal length. The *length of  $w$*  is the length of any reduced word for  $w$  and we note it as  $|w|$ . In particular, the class containing the empty word corresponds to the identity element which has length 0.

**Proposition 4.1.4.** For  $s \in S$  and  $w \in W$ , either  $|sw| = |w| + 1$  or  $|sw| = |w| - 1$ .

*Proof.* Denote  $|w|$  by  $n$  and let  $s_1 \dots s_n$  be a reduced word for  $w$ . Either  $ss_1 \dots s_n$  is reduced, and obviously  $|sw| = n+1$ , or it is not. Since the relations defining the group preserve the parity of the length, so  $|sw|$  has length at most  $|w| - 1$ . In this case, we take a reduced expression  $s'_1 \dots s'_k$  with  $k < n$  for  $sw$ . If  $k < n - 1$ ,  $ss'_1 \dots s'_k$  is an expression for  $w$  of length strictly less than  $n$ . This is absurd by definition of  $|w|$ , so  $|sw| = |w| \pm 1$ .  $\square$

The similar result on the right is also true.

**Definition 4.1.5** (Left weak order). If  $s \in S, w \in W$  are such that  $|sw| = |w| - 1$ ,  $s$  is called a *left descent* of  $w$  and we say that  $w$  is a *left cover* of  $sw$ , denoted by  $sw \triangleleft_L w$ . The *left weak order* is the transitive reflexive closure of the left cover relation and is denoted by  $\leq_L$ .

The concept of right descent, as well as the right weak order with  $\triangleleft_R$  and  $\leq_R$  are similarly defined. We will denote by  $D_L(w)$  and  $D_R(w)$  the sets of left and right descents of  $w$ . It is easy to see that  $s \in D_L(w)$  if and only if there is some reduced word for  $w$  beginning by  $s$ : if there is, then obviously  $|sw| = |w| - 1$  and  $s$  is a descent. Conversely, let  $s_1 \dots s_{|w|-1}$  be a reduced expression for  $sw$ , then  $w = ss_1 \dots s_{|w|-1}$  is a reduced word. Further, two elements  $u, w \in W$  verify  $u \leq_L w$  if and only if there exist a reduced expression for  $w$  whose suffix of size  $|u|$  is a reduced expression for  $u$  (and similarly for prefixes on the right).

**Definition 4.1.6** (Parabolic subgroups). Consider  $J \subset S$ . The subgroup of  $W$  generated by  $J$ , denoted by  $W_J$ , is called the *standard parabolic subgroup* generated by  $J$ . More generally, a subgroup of  $W$  is *parabolic* if it is conjugated to  $W_J$  for some  $J$ .

It is easy to show that if the Coxeter graph associated to  $(W, S)$  is not connected and has connected components  $J_1, \dots, J_n$ , then  $W$  is isomorphic to the direct product of standard parabolic subgroups  $W_{J_1} \times \dots \times W_{J_n}$ . If the Coxeter graph of  $(W, S)$  is connected, the Coxeter system (and by extension, the group) is called *irreducible*. From now on, we only consider irreducible Coxeter systems. Because of this direct product decomposition, this is not an important restriction in our proofs.

Let us choose a  $J \subset S$  and given a  $w \in W$ , if  $w$  has any element  $s$  of  $J$  as a left descent, replace  $w$  by  $sw$  and repeat. Eventually, this process terminates, as the length of the resulting element of each step is strictly decreasing. Say it stabilizes on some element  $w^J$ . By construction,  $D_L(w^J) \cap J = \emptyset$ , and  $w_J = w(w^J)^{-1} \in W_J$ . In fact, this decomposition of  $w$  as  $w_J w^J$  is independent of the choice of descent removed at each step, making the decomposition unique. One can find a proof of this fact in [BB05, Proposition 2.4.4], and we will later provide another – geometric – one.



**Proposition 4.1.7.** *Given  $J \subset S$  and  $w \in W$ , we denote  $W^J = \{\omega \in W \mid D_L(\omega) \cap J = \emptyset\}$ . Then there is a unique factorization of  $w$  as  $w_J w^J$  with  $w_J \in W_J$  and  $w^J \in W^J$ .*

As usual, the corresponding result with right descents is also true.

## 4.2 Geometric realization

The notions of Coxeter groups and reflections groups are often conflated, because given  $(W, S)$  we may obtain a realization of  $W$  as a group generated by orthogonal reflections for some well-chosen definition of “orthogonal”. Although we may obtain many interesting properties of Coxeter groups only using elementary proofs from the definition, this geometric detour provides nice ways to prove these combinatorial results, as well as a view of the kind of proof technics used later.

**Proposition 4.2.1.** *Let  $V$  be the  $\mathbb{R}$  vector space with base  $\Delta = \{e_s \mid s \in S\}$  and  $B$  the bilinear form on  $V$  defined by*

$$B(e_s, e_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{s,t}}\right) & \text{if } m_{s,t} \neq \infty \\ b_{s,t} & \text{with } b_{s,t} \in ]-\infty, -1] \text{ if } m_{s,t} = \infty \end{cases}.$$

*Then  $W$  can be embedded in the orthogonal group for  $B$  by extending the map  $s \mapsto (x \mapsto x - 2B(e_s, x)e_s)$  from  $S$  to  $W$ .*

$V$  equipped with  $B$  and  $\Delta$  is called a *geometric representation* of  $W$ . If we choose  $b_{s,t}$  to be equal to  $-1$  whenever  $m_{s,t} = \infty$ , we get the *classical* geometric representation of  $W$ . We will see that this distinction is useful, as the hypothesis  $b_{s,t} = -1$  is not stable by restriction to subgroups of  $W$  acting on subspaces of  $V$ .

Notice that this embeds  $W$  as a subgroup of

$$\mathcal{O}_B(V) = \{f \in GL(V) \mid B(f(x), f(y)) = B(x, y)\},$$

the linear maps that are orthogonal with respect to the form  $B$ . Indeed,  $\forall s \in S, x, y \in V$ :

$$\begin{aligned} B(s(x), y) &= B(x - 2B(x, e_s)e_s, y) \\ &= B(x, y) - 2B(e_s, x)B(e_s, y) \\ &= B(x, y - 2B(e_s, y)e_s) = B(x, s(y)) \end{aligned}$$

so the image of this map is contained in  $\mathcal{O}_B(V)$ .

This result is well known, and can be found in [Hum90, Section 5.3]. However, for the sake of having a “complete geometric and inversion set based” version of the results we need, we reproduce the proofs.

**Example 4.2.2.** Let us consider the simplest non-trivial example: the case of dihedral groups  $(W, \{s, t\})$ . Given that the generators are of order 2, we only need to consider the alternating words beginning with  $s$  or  $t$ .

For now, we abuse the notation by denoting by  $s$  and  $t$  the linear transformations previously defined. Consider first the case where  $m_{s,t} < \infty$ . Observe that  $r(e_r) = -e_r$  for  $r \in \{s, t\}$ . The transformations being linear, we only need to find the values of:

$$(st)^n(e_s), \quad t(st)^n(e_s), \quad (ts)^n(e_t), \quad s(ts)^n(e_t)$$

for  $n > 0$  to entirely describe the actions of the group elements, since  $(ts)^n(e_s)$  will just be  $-t(st)^{n-1}(e_s)$ . We prove that:

$$(st)^n(e_s) = \left(2 \sum_{k=0}^n \cos \frac{2k\pi}{m_{s,t}} - 1\right) e_s + 2 \sum_{k=1}^n \cos \frac{(2k-1)\pi}{m_{s,t}} e_t.$$

This is true when  $n = 0$ . For brevity, let us denote  $\cos(k\pi/m_{s,t})$  by  $\lambda_k$ ,  $\sum_{k=0}^n \lambda_{2k}$  by  $\Lambda_{0,n}$  and  $\sum_{k=1}^n \lambda_{2k-1}$  by  $\Lambda_{1,n}$ . Using cosine linearization formulas for  $2\lambda_1\lambda_k = \lambda_{k-1} + \lambda_{k+1}$ , we observe that

$$2\lambda_1\Lambda_{0,n} = 2\Lambda_{1,n+1} + \lambda_1 - \lambda_{2n+1} \text{ and } 2\lambda_1\Lambda_{1,n} = 2\Lambda_{0,n} - (1 + \lambda_{2n}).$$

Consider now  $t(st)^n(e_s)$ . We have:

$$\begin{aligned} t(st)^n(e_s) &= t((2\Lambda_{0,n} - 1)e_s + 2\Lambda_{1,n}e_t) \\ &= (2\Lambda_{0,n} - 1)e_s + (2\lambda_1(2\Lambda_{0,n} - 1) - 2\Lambda_{1,n})e_t \\ &= (2\Lambda_{0,n} - 1)e_s + (4\Lambda_{1,n+1} - 2\Lambda_{1,n} + 2\lambda_1 - 2\lambda_{2n+1} - 2\lambda_1)e_t \\ &= (2\Lambda_{0,n} - 1)e_s + 2\Lambda_{1,n+1}e_t \\ (st)^{n+1}(e_s) &= s((2\Lambda_{0,n} - 1)e_s + 2\Lambda_{1,n+1}e_t) \\ &= (2\lambda_1 \cdot 2\Lambda_{1,n+1} - (2\Lambda_{0,n} - 1))e_s + 2\Lambda_{1,n+1}e_t \\ &= (4\Lambda_{0,n+1} - 2\Lambda_{0,n} + 1 - 2 - 2\lambda_{2n})e_s + 2\Lambda_{1,n+1}e_t \\ &= (2\Lambda_{0,n+1} - 1)e_s + 2\Lambda_{1,n+1}e_t \end{aligned}$$

This gives us the wanted expression for  $(st)^n(e_s)$  with that of  $t(st)^n$ . Reversing the roles of  $s$  and  $t$ , we get the remaining cases. Because this is essentially a geometric sum, we can compute it, giving:

$$\Lambda_{0,n} = \frac{\sin((n+1)\pi/m_{s,t})}{\sin(\pi/m_{s,t})} \cos \frac{n\pi}{m_{s,t}}, \quad \Lambda_{1,n} = \frac{\sin(n\pi/m_{s,t})}{\sin(\pi/m_{s,t})} \cos \frac{n\pi}{m_{s,t}}$$

which simplifies to:

$$\Lambda_{0,n} = \frac{1}{2} + \frac{\sin((2n+1)\pi/m_{s,t})}{\sin(\pi/m_{s,t})}, \quad \Lambda_{1,n} = \frac{1}{2} \frac{\sin(2n\pi/m_{s,t})}{\sin(\pi/m_{s,t})}.$$

This gives  $(st)^n = \text{Id}$  if and only if  $m_{s,t}$  divides  $n$ , that is  $st$  (and, in the same way  $ts$ ) have order  $n$ , so the map given on the generators can be extended to a (surjective) morphism. In the dihedral case, it is easy to see that it is also injective, and so is an isomorphism.

If  $m_{s,t} = 0$ ,  $B(e_s, e_t) \leq -1$ , it is easy to show by induction that we have  $(st)^n(e_s) = a_s e_s + a_t e_t$  with  $a_s, a_t \geq n$ , meaning that  $st$  is of infinite order. Consequently, the map can still be extended to an isomorphism.

The interpretation is that the generators can be thought of as  $B$ -orthogonal reflections, and that the form  $B$  encodes the angle between the fixed hyperplanes of  $s$  and  $t$  as  $\pi/m_{s,t}$  such that  $st$  can be thought of as the rotation of angle  $2\pi/m_{s,t}$  around the intersection of the fixed hyperplanes. This interpretation somewhat fails when  $m_{s,t} = \infty$ , but remains a good source of intuition. In fact, one can think of  $B$  as the “dot product” that gives the correct “angles” between the generators to force those relations to hold. However, in general,  $B$  is not a dot product, and in fact its isotropic cone will play an important role later.

**Definition 4.2.3.** The *isotropic cone* of  $B$ , denoted by  $Q$ , is the set of “self  $B$ -orthogonal vectors”  $\{v \in V \mid B(v, v) = 0\}$ .

The previous example shows that the linear maps associated to the generators defined in Proposition 4.2.1 verify the same relations as the elements of  $S$ , and that the map defined on the generators does indeed extend to a surjective morphism for any number of generators. Proposition 4.2.1 announces that this is actually an isomorphism, and we shall shortly provide a proof. However, we first need to introduce the notion of *root system*.

**Definition 4.2.4.** Given  $(W, S)$ , we define the quadratic space  $(V, B)$  as before. The *root system* (of  $W$  in  $V$ ), denoted by  $\Phi$  is the set  $\{w(e_s) \mid w \in W, s \in S\}$ . Its elements are called *roots*. The set  $\Delta = \{e_s \mid s \in S\}$  is called the set of *simple roots*.

The following properties can immediately be checked, keeping in mind that  $W$  acts on  $V$  as a subgroup of  $\mathcal{O}_B(V)$ .

**Proposition 4.2.5.** *Let  $\Phi$  be the root system of  $W$  in  $V$ . Then:*

1.  $\forall \phi \in \Phi, B(\phi, \phi) = 1$ .

2.  $\forall \phi \in \Phi, \mathbb{R}\phi \cap \Phi = \{\pm\phi\}$ .
3.  $\forall \phi \in \Phi, x \mapsto x - 2B(x, \phi)\phi \in \mathcal{O}_B(V)$  and it stabilizes  $\Phi$ .
4. Suppose for an instant that  $W$  is not irreducible and let  $W = W_1 \times W_2$  with  $W_1, W_2$  non-trivial standard parabolic subgroups. Then  $\Phi_W = \Phi_{W_1} \sqcup \Phi_{W_2}$  and  $\Phi_{W_1} \perp \Phi_{W_2}$ .

From the root system, we are ready to define the *inversion sets*, which will be at the center of attention for most of this part of the thesis.

**Definition 4.2.6** (Positive roots and inversion sets). A root  $\phi \in \Phi$  is *positive* if all of its coordinates in the basis  $\Delta$  are non-negative. The set of positive roots is denoted by  $\Phi^+$ , and we denote  $-\Phi^+$  by  $\Phi^-$ . The *inversion set* of  $w$  is defined as  $N(w) = \Phi^+ \cap w^{-1}(\Phi^-)$ .

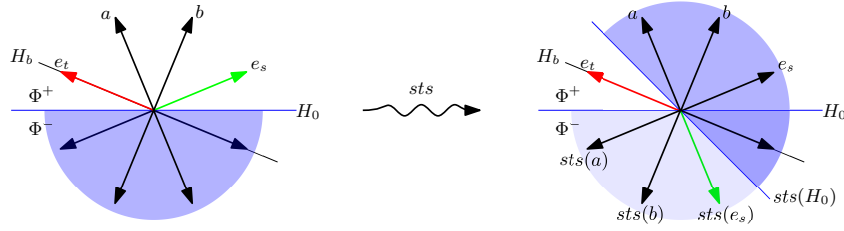


Figure 4.2: The inversion set of  $sts$  (the orthogonal reflection associated with  $b$ ) in the dihedral group  $D_4$ :  $N(sts) = \{e_s, a, b\}$ .

**Notation 4.2.7.** For any  $A \subset \Phi$ , we will denote  $\text{Span}_{\mathbb{R}} A$  by  $V_A$ ,  $B$  restricted to  $V_A$  by  $B_A$ ,  $\text{Span}_{\Phi}(A)$ ,  $\text{Span}_{\Phi^+}(A)$ ,  $\text{Span}_{\Phi^-}(A)$  will respectively be  $\Phi \cap V_A$ ,  $\Phi^+ \cap V_A$  and  $\Phi^- \cap V_A$ . When more convenient, we will also use the notations  $\Phi_A$ ,  $\Phi_A^+$  and  $\Phi_A^-$  for these three last sets.

It is clear from the definition that  $\Phi^+ \sqcup \Phi^- \subset \Phi$ . This inclusion is in fact an equality, as the next result will make apparent.

**Proposition 4.2.8.** Let  $w \in W$  and  $s \in S$ . Then  $s \in D_R(w) \Leftrightarrow e_s \in N(w)$ .

**Example 4.2.9.** Let us again examine the dihedral case  $(W, \{s, t\})$  first. In Example 4.2.2, we actually computed the root system. If  $m_{s,t}$

is finite, the root system is the set

$$\Phi = \left\{ \begin{array}{l} (st)^n(e_s) = (2\Lambda_{0,n} - 1)e_s + 2\Lambda_{1,n}e_s, \\ (ts)^n(e_t) = (2\Lambda_{0,n} - 1)e_t + 2\Lambda_{1,n}e_s, \\ t(st)^n(e_s) = (2\Lambda_{0,n} - 1)e_s + 2\Lambda_{1,n+1}e_t, \\ s(ts)^n(e_t) = (2\Lambda_{0,n} - 1)e_t + 2\Lambda_{1,n+1}e_s \end{array} \middle| n \in \llbracket 0, m_{s,t} - 1 \rrbracket \right\}$$

where

$$\Lambda_{0,n} = \frac{1}{2} + \frac{\sin((2n+1)\pi/m_{s,t})}{\sin(\pi/m_{s,t})}, \quad \Lambda_{1,n} = \frac{1}{2} \frac{\sin(2n\pi/m_{s,t})}{\sin(\pi/m_{s,t})}.$$

It is easy to see that both coefficients are either both non-negative or non-positive depending on whether  $n \leq m_{s,t}/2$ , so  $\Phi = \Phi^+ \sqcup \Phi^-$  in that case. Let  $w$  be some element of  $W$ . If  $s \notin D_R(w)$ , then  $w = (st)^n$  or  $w = t(st)^n$  with  $n < m_{s,t}$  (if  $m_{s,t}$  is even and  $w = (st)^{m_{s,t}}$ , then  $t \in D_R(w)$ ) and we have just seen that  $w(e_s) \in \Phi^+$ . Conversely, if  $s \in D_R(w)$ , then  $w$  can be written as  $(st)^n s$  or  $(ts)^n$  with  $n < m_{s,t}/2$  and, as before,  $ws(e_s) \in \Phi^+$ . So  $w(e_s) = wss(e_s) = ws(-e_s) = -ws(e_s) \in \Phi^-$ . If  $m_{s,t}$  is infinite, the reasoning is essentially the same, if not simpler, as since the roots escape to infinity we do not need to worry about them “coming around” as the condition  $n \leq m_{s,t}/2$  is always true.

From this example, we can get the general case. We reproduce the proof from [Hum90, Sections 5.3, 5.4]

*Proof of Proposition 4.2.8.* Suppose that if for some  $s \in S, w \in W, s \in D_R(w)$  and  $e_s \in N(w)$ , then  $s \notin D_R(ws)$  and  $ws(e_s) = -w(e_s) \in \Phi^+$  because  $w(e_s) \in \Phi^-$ , so  $e_s \notin N(ws)$ . Thus, we only need to prove one implication.

We will show by recursion on  $|w|$  that if  $s \notin D_R(w)$ , then  $w(e_s) \in \Phi^+$  i.e.  $e_s \notin N(w)$ . This is obvious if  $|w| = 0$  as  $w = 1$  and  $N(1) = \emptyset$ . If  $|w| > 0$ , then  $w$  has some right descent  $t$ . Consider  $J = \{s, t\}$  and  $w = w^J w_J$  a factorization in the fashion of Proposition 4.1.7. We do not need to know that this factorization is unique at this point, only that  $|w^J| < |w|$  and  $D_R(w) \cap J = \emptyset$ . In particular, by recursion, this means that  $w^J(e_s), w^J(e_t) \in \Phi^+$ . Recall how we constructed  $w_J$ : by removing rights descents in  $\{s, t\}$  from  $w$  and accumulating them. Because  $t \in D_R(w)$  and  $s \notin D_R(w)$ ,  $w_J$  can only be written as an alternating word of  $s$  and  $t$ , ending in  $t$  with  $|w_J| < m_{s,t}$ . But from Example 4.2.9, we know that this implies  $w_J(e_s) = a_s e_s + a_t e_t \in \Phi^+$  (and in particular  $a_s, a_t \geq 0$ ). Finally,

$$w(e_s) = w^J w_J(e_s) = a_s \underbrace{w^J(e_s)}_{\in \Phi^+} + a_t \underbrace{w^J(e_t)}_{\in \Phi^+} \in \Phi^+$$

so  $s \notin D_R(w) \Rightarrow e_s \notin N(w)$ , and we conclude by recursion.  $\square$

**Corollary 4.2.10** (of Proposition 4.2.8). *The action of  $W$  on  $V$  (and on  $\Phi$ ) is faithful.*

*Proof.* If  $w \in W \setminus \{1\}$ ,  $D_R(w)$  contains some  $s \in S$ . Then  $w(e_s) \neq e_s$  so  $w$  does not act trivially.  $\square$

Because of this, from now on, we will conflate the elements of the Coxeter groups as word quotiented by relations with their associated orthogonal map acting on  $V$ .

**Corollary 4.2.11** (of Proposition 4.2.8). *We have  $\Phi = \Phi^+ \sqcup \Phi^-$ .*

*Proof.* For any  $\phi \in \Phi$ , there is some  $w \in W, s \in S$  such that  $\phi = w(e_s)$ . If  $s \in D_R(w)$ , then  $e_s \in N(w)$  i.e.  $w(e_s) = \phi \in \Phi^-$ . If  $s \notin D_R(w)$  then  $s \in D_R(ws)$  so  $ws(e_s) = -w(e_s) = -\phi \in \Phi^-$  so  $\phi \in \Phi^+$ .  $\square$

Finally, we have a very useful and desirable (although non-trivial) property of root systems: they restrict well in the following sense, uncovered independently by Deodhar in [Deo89] and Dyer [Dye90].

**Proposition 4.2.12.** *Let  $A \subset \Phi$  and  $V_A, B_A, \Phi_A^+$  as defined in Notation 4.2.7. Let  $\Delta_A$  be the set of roots of  $\Phi_A^+$  generating the extreme rays of  $\text{cone}(\Phi_A^+)$  and for any  $\delta \in \Delta_A$ , denote  $s_\delta$  the  $B_A$ -orthogonal reflection of  $V_A$  with respect to  $\delta$ . Then if  $W_A = \langle s_\delta \mid \delta \in \Delta \rangle$ ,  $(W, \{s_\delta \mid \delta \in \Delta_A\})$  is a Coxeter system.*

The result proven in [Deo89], [Dye90] is in fact richer, as they actually take as  $W_A$  the subgroup generated by the reflections associated to the roots in  $A$ , while we are only interested in sets  $A$  that are “maximal in a given subspace”. Note however that if  $(V, B)$  is a classical realization of a group  $W$ , for some  $A \subset \Phi$ ,  $(V_A, B_A)$  may not be a classical representation, which is why we paid special attention to the case  $b_{s,t} < -1$ : all the results in this section apply to reflection subgroups of Coxeter groups.

This result also tells us that given the set of positive roots, the set of simple roots is uniquely determined. However, if instead of  $\Phi^+$  we only have  $\Phi$ , not necessarily given in the basis  $\Delta$  of simple roots, we may have multiple choices for  $\Delta$ . From now on, when the choice of simple roots is non-obvious, we will use the terminology of *based root system* from [HLR14] by specifying the pair  $(\Phi, \Delta)$ . To sum up Proposition 4.2.12 “based root systems restrict naturally”: given a subspace  $V_A$  generated by a set  $A$ , we get a based subsystem  $(\Phi_A, \Delta_A)$ .

### 4.3 Inversion sets

Proposition 4.2.8 allows for a description of our “most central” object in this presentation: the inversion set.

**Proposition 4.3.1.** *Let  $w \in W$  and suppose that  $s_1 \cdots s_n$  is a reduced word for  $w$ . Then  $N(w) = \{e_n, s_n(e_{s_{n-1}}), s_n s_{n-1}(e_{s_{n-2}}), \dots, w^{-1}s_1(e_{s_1})\}$ . Moreover,  $|N(w)| = |w|$ .*

*Proof.* It is easy to see that  $\{e_s\} \cup sN(w) \subset N(ws)$  if  $s \notin D_R(w)$ : by Proposition 4.2.8,  $e_s \in N(ws)$  and if  $\phi \in N(w)$ ,  $ws(s(\phi)) = w(\phi) \in \Phi^-$ , so  $s(\phi) \in N(ws)$ . For the other inclusion, observe that  $s$  exchanges  $e_s$  and  $-e_s$  and stabilizes  $\Phi^+ \setminus \{e_s\}$ . Because  $s$  acts bijectively, it means that  $s$  permutes the set  $\Phi^+ \setminus \{e_s\}$ . Suppose  $\phi \in N(ws) \setminus \{e_s\}$  so  $s(\phi) \in \Phi^+ \setminus \{e_s\}$ . Because  $ws(\phi) \in \Phi^- \Leftrightarrow w(s(\phi)) \in \Phi^-$ ,  $s(\phi) \in N(w)$ . Because  $s$  is an involution,  $\phi$  is the image by  $s$  of a root in  $N(w)$ :  $N(ws) \subset \{e_s\} \cup sN(w)$ . By applying repetitively this result to the reduced decomposition of  $w$ , for the initial inversion set  $N(1) = \emptyset$ , we obtain the proposed formula. This also proves (via a recursion) that  $|N(w)| = |w|$ . It is obvious if  $w = 1$ :  $|N(1)| = |\emptyset| = 0 = |w|$ . Because  $s$  permutes  $\Phi^+ \setminus \{e_s\}$ ,  $|N(ws)| = |\{e_s\} \sqcup sN(w)| = 1 + |N(w)| = 1 + |w| = |ws|$ .  $\square$

This essentially says that the inversion sets contains all the information we might want on the elements.

**Corollary 4.3.2** (of Proposition 4.3.1). *The map  $N$  is an increasing injection (for inclusion of inversion sets) on  $(W, \leq_L)$ .*

*Proof.* Recalling that  $v \leq_L w$  if and only if some reduced word for  $v$  is a suffix of a reduced word for  $w$ , it is clear that  $N$  is increasing from the description of  $N(w)$  in Proposition 4.3.1.  $N$  is injective on elements on length 0 (in fact,  $N^{-1}\{\emptyset\} = \{1\}$  from Proposition 4.2.8). Suppose that  $N$  is injective for all  $w$  of length at most  $n$ , and consider  $w_1, w_2 \in W \setminus \{1\}$  of length at most  $n+1$  such that  $N(w_1) = N(w_2)$ . From Proposition 4.2.8,  $w_1$  and  $w_2$  have some common right descent  $s$ . Then, as in Proposition 4.3.1, it is easy to prove that

$$N(w_1s) = s(N(w_1) \setminus \{e_s\}) = s(N(w_2) \setminus \{e_s\}) = N(w_2s).$$

Since  $w_1s$  and  $w_2s$  have length at most  $n$ ,  $w_1s = w_2s$  so  $w_1 = w_2$ .  $\square$

As things stand now, the definition of inversion sets we have adopted seems to favor the left weak order, and indeed, to have similar results on the left, we would need to define a *right inversion set*  $N_L = w \mapsto N(w^{-1})$ . However, the left weak order is still “readable” in  $N$  (which we will not call the *left inversion set* for brevity, although it morally is).

**Corollary 4.3.3** (of Proposition 4.3.1). *The sets  $\Gamma_w = \{\gamma \in N(w) \mid \exists w' N(w') = N(w) \setminus \{\gamma\}\}$  and  $\{-w^{-1}(e_s) \mid s \in D_L(w)\}$  are equal and in bijection with  $D_L(w)$ .*

*Proof.* It is clear that  $D_L(w)$  is in bijection with  $\Gamma_w$ : consider  $\{w' \mid \exists \gamma \in N(w), N(w') = N(w) \setminus \{\gamma\}\}$ : it is the set of element covered by  $w$  in the left weak order, and each of these covers is uniquely labeled by a left descent of  $w$ . Next,  $\{-w^{-1}(e_s) \mid s \in D_L(w)\} \subset \Gamma_w$ : for  $s \in D_L(w)$ , choose a reduced word for  $w$  beginning by  $s$ . From Proposition 4.3.1,  $N(w) = N(ws) \cup \{w^{-1}s(e_s)\}$ , then  $w^{-1}s(e_s) = -w^{-1}(e_s)$  corresponds to  $\gamma$  and  $ws$  to  $w'$ . The other inclusion is the same thing, taking  $s$  as  $w'w^{-1}$ .  $\square$

We will call the set  $\Gamma_w$  the *left geometric descents* of  $w$ , omitting the “left” if clear in context, as we will later be much more interested in the set  $\Gamma_w$  than in  $\{e_s \mid s \in D_R(w)\}$  the *right geometric descents* of  $w$ . If the inversion sets contain, in some sense, all the information on  $W$ , we might want to be able to recognize whether a subset of  $\Phi^+$  is an inversion set. Fortunately, there is an elementary characterization.

**Proposition 4.3.4.** *Let  $A \subset \Phi^+$ .  $A$  is an inversion if and only if  $A$  is finite and there is  $f \in V^*$  such that  $\forall \phi \in A, f(\phi) < 0$  and  $\forall \phi \in \Phi^+ \setminus A, f(\phi) > 0$  (we say that  $A$  is a separable subset of  $\Phi^+$ ).*

*Proof.* Given that  $|N(w)| = |w|$ , the finiteness condition is obvious. Suppose that  $A = N(w)$  for some  $w$ . Then since  $\Phi^+ \subset \text{cone}(\Delta)$  and  $\Phi = \Phi^+ \sqcup \Phi^-$ , the form  $v \mapsto \langle v \mid \mathbf{1} \rangle$ , where  $\mathbf{1} = (1)_{s \in S}$  and  $\langle \cdot \mid \cdot \rangle$  is the usual dot product, is such that  $f(\phi) < 0$  if  $\phi \in \Phi^-$  and  $f(\phi) > 0$  if  $\phi \in \Phi^+$ . Consider  $f \circ w$ : by definition of  $N$ ,  $\phi \in N(w) \Leftrightarrow \phi \in \Phi^+ \cap w^{-1}\Phi^- \Leftrightarrow f \circ w(\phi) < 0$ :  $f \circ w$  separates  $A$  from  $\Phi^+ \setminus A$ . Conversely, if we have  $f$  that separates a finite set  $A$  from  $\Phi^+ \setminus A$ , then  $f^{-1}\{0\}$  crosses  $\text{cone}(\Delta)$  so  $f^{-1}\mathbb{R}_+^*$  must contain some  $e_s$ . Then  $f \circ s$  separates  $s(A \setminus \{e_s\})$ . By iterating this process, since  $A$  is finite, we obtain that  $f \circ w$  for some  $w$  separate  $\emptyset$  from  $\Phi^+$  and that  $f$  separates  $N(w)$  from its complement, so  $A = N(w)$ .  $\square$

**Corollary 4.3.5.** *Let  $A \subset N(w)$  for some  $w$ . Then  $\text{cone}(A) \cap \Phi^+ \subset N(w)$ .*

*Proof.* Let  $f$  a separating form for  $N(w)$ . For all  $\phi \in A$ ,  $f(\phi) < 0$  so any positive linear combination  $v$  of elements of  $A$  verifies  $f(v) < 0$ .  $\square$

With those properties of inversion sets, we can go back to prove properties of the weak order in terms of inversion sets. For instance, take the following simple proposition.



**Proposition 4.3.6.** *If  $W$  is finite, then it contains a unique maximal element for the (left or right) weak order, denoted by  $w_0$ .*

*Proof.* If  $W$  is finite, so is  $\Phi^+$ .  $\Phi^+$  is clearly separable by  $v \mapsto -\langle v, \mathbf{1} \rangle$ , so it is the inversion set of some element  $w_0$ . Because  $N(w_0)$  is the unique maximal element of the inversion sets for inclusion,  $w_0$  is the unique maximal element of  $W$  for  $\leq_R$ . The proof on the left is the same.  $\square$

This could have been proven without any explicit geometric construction (as is done for instance in [BB05, Chapters 1 & 3]), but the inversion set proof is particularly simple. Let us review a few well-known results about the weak order, beginning with the proof of Proposition 4.1.7 on parabolic decompositions.

*Proof of Proposition 4.1.7.* Let  $w \in W$ . Although we have already seen the existence of the decomposition by successive factorization of elements of  $J$ , we can construct  $w_J$  directly from the inversion sets. Consider  $N_J(w) = \Phi_J \cap N(w)$ . Suppose that a form  $f$  separates  $N(w)$  from its complement, then  $f|_{V_J}$  separates  $N_J(w)$  from  $\Phi_J^+ \setminus N_J(w)$ . Since  $N_J(w)$  is also finite, it is the inversion set of some element  $w_J$  in  $W_J$ . Then  $w^J = ww_J^{-1}$  is in  $W^J$ , that is, it has no right descent in  $J$ :  $w_J$  stabilizes  $V_J$  and is a bijection, so in particular  $N(w^J) = w_J(N(w) \setminus N(w_J))$  does not intersect  $V_J$  because  $(N(w) \setminus N(w_J)) \cap V_J = \emptyset$ . But  $V_J$  contains the simple roots in  $\Delta_J = \{e_s \mid s \in J\}$  so  $N(w_J)$  does not: by Proposition 4.2.8, they are not descents. Similarly, if we have another decomposition  $w = v^J v_J$  with  $v^J \in W^J, v_J \in W_J$ , either  $N(v_J) = N(w_J)$  so it is the same decomposition, or  $N(v_J) \subset N(w_J)$  (since we have taken  $w_J$  such that  $N(w_J) = \Phi_J^+ \cap N(w)$  and that  $v_J$  is a suffix of  $w$ , it cannot contain other roots). But in that case:

$$\begin{aligned} N(v^J) &= v_J(N(w) \setminus N(v_J)) \\ &= v_J(N(w) \setminus N(w_J) \sqcup N(w_J) \setminus N(v_J)) \\ &= v_J(N(w) \setminus N(w_J)) \sqcup v_J(N(w_J) \setminus N(v_J)) \end{aligned}$$

The set  $v_J(N(w_J) \setminus N(v_J))$  is the (by supposition non-empty) inversion set of  $w_J v_J^{-1}$ , so it contains a simple root in  $\Delta_J$ : again from Proposition 4.2.8, this means that  $w v_J^{-1} = v^J$  has a descent in  $J$ , so  $v^J \notin W^J$ . This is absurd, which proves the uniqueness of the parabolic decomposition of  $w$ .  $\square$

**Proposition 4.3.7.** *Let  $w \in W$  and suppose  $J \subset D_R(w)$ . Then  $W_J$  is finite and in the decomposition  $w = w^J w_J$ ,  $w_J$  is the maximal element of  $W_J$ .*

*Proof.* By supposition,  $N(w)$  contains  $\Delta_J$ . By separability,  $\Phi_J^+ \subset N(w)$ , so  $W_J$  is finite and  $N(w_J) = \Phi_J^+$  that is  $w_J$  is the maximal element of  $W_J$ .  $\square$

**Proposition 4.3.8** (Subword property). *Suppose that  $v, w \in W$  with  $v \leq_R w$ . Then any reduced expression  $w_1 \cdots w_n$  contains a reduced expression of  $v$  as a subword, that is, there is a reduced expression  $v = v_1 \cdots v_k$  and a sequence  $1 \leq i_1 < \cdots < i_k \leq n$  such that  $v_j = v_{i_j}$  for all  $1 \leq j \leq k$ .*

*Proof.* This is obviously true if  $v = 1$ . Suppose it is true for  $v$  of length at most  $k - 1$  and consider the sequence

$$N(w_n) \subset N(w_{n-1}w_n) \subset N(w_{n-2}w_{n-1}w_n) \subset \cdots \subset N(w).$$

Since  $v \leq_R w$ ,  $N(v) \subset N(w)$ , there is some  $j \in \llbracket 1, n-1 \rrbracket$  such that  $N(v) \not\subset N(w_{n-j+1} \cdots w_n)$  and  $N(v) \subset N(w_{n-j} \cdots w_n)$ . Denote  $w_{n-j}$  by  $s$ . Because the sequence is increasing (by exactly one root at each step), this means that the root that was just added,  $-(w_{n-j} \cdots w_n)e_s$ , lies in  $N(v)$ . So  $s \in D_L(w)$  and  $sv \leq_R w_{n-j+1} \cdots w_n$  because  $N(sv) \subset N(w_{n-j+1} \cdots w_n)$ . By recursion, there is a sequence  $n-j+1 \leq i_2 < \cdots < i_k \leq n$  such that the subword of  $w_{i_2} \cdots w_{i_k} = sv$ . By taking  $i_1 = n-j$ , we have found a subword for  $v$  in  $w$ .  $\square$

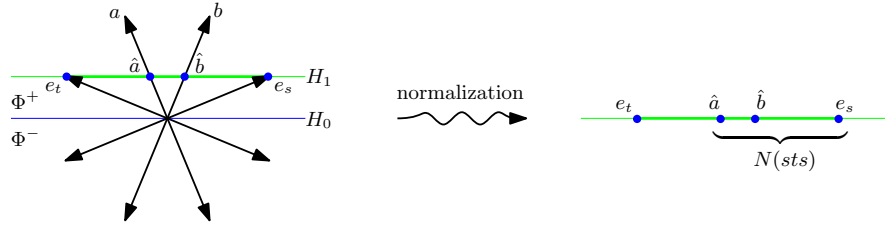
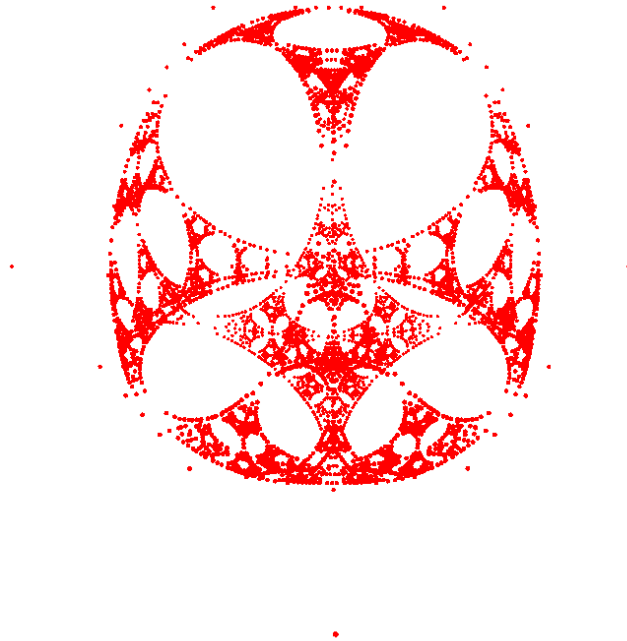
Note that this is also true for the left weak order, so this is not an equivalence. In fact, the relation “being a reduced subword of some reduced expression” gives rise to an interesting order relation called the *Bruhat order* on  $W$ . In this thesis, we consider only the weak orders.

## 4.4 Projective picture

From now on, we set  $(V, B)$  to be the classical geometric representation of  $(W, S)$  and  $(\Phi, \Delta)$  the corresponding based root system.

In the case of infinite Coxeter groups,  $\Phi$  can prove challenging to picture: there are infinitely many roots, they are of unbounded norm... Following [HLR14], we may obtain a handy depiction of the root system: the *projective picture*. To a root  $\rho$ , we associate a *normalized root*  $\hat{\rho} = \mathbb{R}\rho \cap H_1$ , where  $H_1$  is the affine hyperplane generated by  $\Delta$ . The normalized roots form the set  $\hat{\Phi} = \{\hat{\rho} \mid \rho \in \Phi\}$ , as shown in Figure 4.3. We call *projective picture* the set  $\hat{\Phi}$  seen as embedded in  $H_1$ . Since  $\Phi^+$  is naturally in bijection with  $\hat{\Phi}$ , we will identify  $a \in \Phi^+$  with  $\hat{a}$ .

**Remark.** Since  $\Phi^+ \subset \text{cone}(\Delta)$  and  $\Delta \subset H_1$ , we have  $\hat{\Phi} \subset \text{conv}(\Delta)$ , the convex hull of  $\Delta$ . Because of this, nothing happens at infinity and the projective picture is essentially the image of  $\Phi$  by the canonical projection in the projective space  $\mathbb{P}(V)$ . This is also why  $H_1$  is chosen: the choice of cutting hyperplane does not matter because the underlying object is the same, and the case  $H_1$  is especially simple. We chose to see the projective picture as a subset

Figure 4.3: The projective picture for the dihedral group  $D_4$ .Figure 4.4: Projective picture of the root system of  $\langle S, |S| = 5 \mid \forall s, t \in S, (st)^7 \rangle$ .

of the affine hyperplane  $H_1$ , but one could rewrite everything as occurring in the standard open sets  $\mathbb{P}(\mathbb{V})$ .

More than a simple tool to draw pictures of rank 3 or 4 root systems, the projective picture has several advantages. For instance, it is a subset of a compact set, meaning that if we have an infinite group (and thus an infinite root system),  $\widehat{\Phi}$  has accumulation points. These points, as indicated in [HLR14], are on the isotropic cone:

**Proposition 4.4.1.** *Let  $(\rho_n)_{n \in \mathbb{N}}$  such that  $\rho_n / \|\rho_n\| \rightarrow_{n \rightarrow \infty} l$  for some  $l$ .*

Suppose that  $l \neq \rho_n$  for all  $n$ . Then  $l \in Q$ .

*Proof.* Let us denote  $v/\|v\|$  by  $\tilde{v}$ . Since for all  $v \in V$ ,  $\|v\| \Rightarrow B(v, v) = 0$  and for all  $\phi \in \Phi$ ,  $B(\phi, \phi) = 1$ , the map  $v \mapsto \tilde{v}$  is continuous on the closure of  $\Phi$ .  $B$  is continuous too so  $B(\tilde{\rho}_n, \tilde{\rho}_n) \xrightarrow{n \rightarrow \infty} B(l, l)$ , which means  $B(l, l)$  is positive, because  $B(\rho_n, \rho_n) = 1$  for all  $n$ . Conversely, the following maps are all continuous for the usual topologies:

$$\left\{ \begin{array}{l} V \longrightarrow \mathcal{L}(V) \\ v \longmapsto (s_v : x \mapsto x - 2B(v, x)v) \end{array} \right\}, \quad \left\{ \begin{array}{l} \mathcal{L}(V) \longrightarrow V^* \\ s \longmapsto f \circ s \quad \text{where } f = \langle \cdot | \mathbf{1} \rangle \end{array} \right\}.$$

So the map  $v \mapsto f \circ s_v(l)$  is continuous as well and  $f \circ s_{\tilde{\rho}_n}(l) \xrightarrow{n \rightarrow \infty} f \circ s_l(l)$ . Notice that for any  $w \in W$ ,  $f \circ w$  is a form separating  $N(w)$  from its complement, so  $f_n = f \circ s_{\tilde{\rho}_n}$  defines an inversion set  $A$ , which is in particular finite. So  $f^{-1}\mathbb{R}_+^*$  cannot contain  $l$  (otherwise an infinity of  $\tilde{\rho}_n$  would also be contained in the inversion set). So  $f \circ s_l(l) \geq 0$  and consequently  $B(l, l) \leq 0$ . Finally,  $B(l, l) = 0$ , that is  $l \in Q$ .  $\square$

This can be observed in Figure 4.7, where the roots diverge, but tend to a limit direction given by the isotropic cone which is, in that case, a line.

**Notation 4.4.2.** Let us denote by  $\lim \Phi^+$  the set of (unitary) limit directions of  $\Phi^+$ , that is the set:

$$\{l \in V \mid \exists (\phi_n)_{n \in \mathbb{N}}, \forall n \ \phi_n/\|\phi_n\| \neq l, \phi_n/\|\phi_n\| \rightarrow_n l\}$$

This circumscribes the treatment of many topological issues to a neighborhood of  $Q_1 = Q \cap H_1$ . In our proofs in Section 6.1, it will also allow us to simplify the formalism: cones become convex hulls, extreme rays become points, etc. This allows us to translate the inversion set into an “even more” geometric object.

**Definition 4.4.3.** The *inversion polytope* of an element  $w$  in  $W$  is the convex hull of  $N(w)$  in  $H_1$ . It is denoted by  $\mathcal{P}_w$ . The set of its vertices is denoted by  $N^1(w)$ .

Recall that  $N(w)$  is separable. This too translates in the projective picture: there exists some affine hyperplane  $H$  of  $H_1$  strictly separating the inversion polytope  $\mathcal{P}_w$  from its complement in  $\hat{\Phi}$ . This imposes that  $N(w)$  is convex in the sense that  $\mathcal{P}_w \cap \hat{\Phi} = N(w)$  (see [HL16, Lemma 2.10]):  $\mathcal{P}_w$  and  $N(w)$  hold the same information.

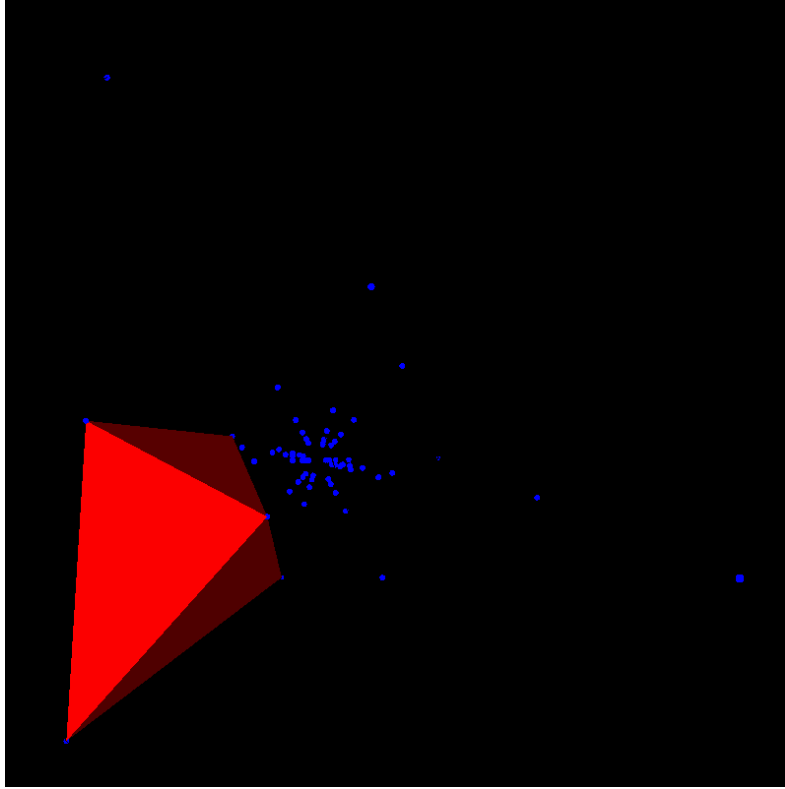


Figure 4.5: Inversion polytope of  $s_4s_1s_3s_2s_4$  in  $\tilde{A}_3$ .

## 4.5 (Af)Finiteness and Weyl groups

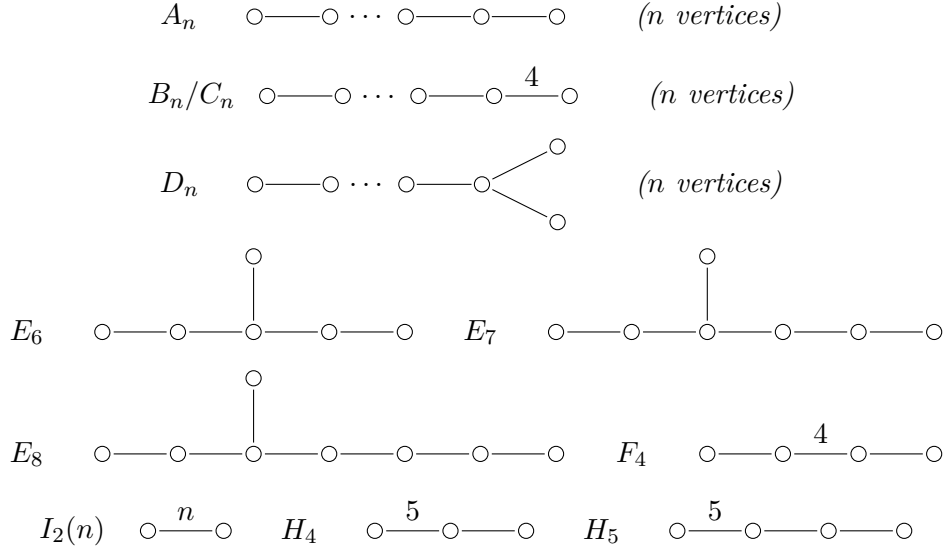
A consequence of Proposition 4.4.1 is that if  $W$  is infinite, then  $B$  cannot be a definite form. Indeed, if  $W$  is infinite, so must be  $\Phi$ . Suppose that  $\Phi$  is infinite and bounded: it must have an accumulation point  $l$ , which lies on the isotropic cone, meaning  $B(l, l) = 0$ . But  $\Phi \subset (v \mapsto B(v, v))^{-1}\{1\}$  and  $B$  is continuous, so  $B(l, l) = 1$ . This is absurd, so  $\Phi$  is unbounded. Since the roots are “ $B$ -unitary”, the “unit” sphere is also unbounded and  $B$  cannot be definite. In fact, the converse is also true (see [Hum90, Theorem 6.4]).

**Proposition 4.5.1.**  *$W$  is finite if and only if  $B$  is positive definite.*

**Corollary 4.5.2.** *Let  $A \subset \Phi$ . Using the notations above,  $H_A$  is finite if and only if  $\Phi_A$  is finite, if and only if the restriction of  $B$  to  $\text{Span}(A)$  is positive definite.*

With this condition on  $B$  being definite in mind, it becomes relatively easy to classify the finite irreducible Coxeter groups (see for instance [Hum90, Section 2.7]).

**Proposition 4.5.3.** *The complete classification of irreducible finite Coxeter groups is given by the following Coxeter diagrams.*



Note that for  $n = 1$ , we have  $A_1 = B_1 = C_1$ . For  $n = 2$ ,  $D_2$  and  $I_2(2)$  are not irreducible, so  $D_n$  and  $I_2(n)$  are in the list for  $n > 2$ .

Note that, in general, if we construct an orthogonal reflection with respect to a given vector, the reflection “forgets” the norm of that vector. It so happens that in the way we have defined root systems until now, all roots are  $B$ -unitary, so this information is not actually lost. However, this is not a hard requirement as, until now, we have essentially asked the root system to be a stable set of directions, them being given by  $B$ -unitary vectors being a nice but non-essential property *when it comes to encoding the Coxeter group*  $W$ . Imagine that, instead of constructing the root systems from the abstract Coxeter systems, we try first to define root systems and then to get Coxeter groups out of them, we lose an advantage: previously, we crafted an ad-hoc way of measuring angles by defining  $B$ . Now, we are given a Euclidean space  $(V, \langle \cdot | \cdot \rangle)$  which is more rigid in terms of defining angles, but we may play on the length of the roots. The natural things to ask of a root system  $\Phi$  would be the immediate properties that we got earlier:

1. For all  $\phi \in \Phi$ , let  $s_\phi = x \mapsto x - 2 \frac{\langle x | \phi \rangle}{\langle \phi | \phi \rangle} \phi$ . Then  $s_\phi(\Phi) = \Phi$ .
2. For all  $\phi \in \Phi$ ,  $\mathbb{R}\phi \cap \Phi = \{\pm\phi\}$ .

This would make for a rather insufficient definition, as objects respecting only these requirements can get quite wild. Let us add two more requirements:

3. **Irreducibility** If  $U \subset V$  is a subspace such that  $(U \cap \Phi) \sqcup (U^\perp \cap \Phi)$  then  $U = \{0\}$  or  $U = V$ .

4. **Essentiality**  $\text{Span}_{\mathbb{R}}(\Phi) = V$ .

Still, this is insufficient as this does not guaranty properties we would like to have, like the fact that  $\Phi$ , as a consequence of Proposition 4.4.1, is a discrete set. We might ask for  $\Phi$  to be finite. This is indeed a good way to define what a root system is in the sense that it captures all the finite groups, and we can hardly hope for more because, from Proposition 4.5.1, infinite groups are inaccessible due to  $\langle \cdot | \cdot \rangle$  being definite. However, let us be more restrictive and ask more of  $\Phi$ .

5. **Integrality** For all  $\phi, \psi \in \Phi$ ,  $2\langle \phi | \psi \rangle / \langle \phi | \phi \rangle \in \mathbb{Z}$ .

If  $\Phi$  satisfies these 5 conditions, it is called an (essential irreducible) *crystallographic root system*. There are not many such root systems: for instance, in dimension (or *rank*) 2 there are only 3 (see Figure 4.6).

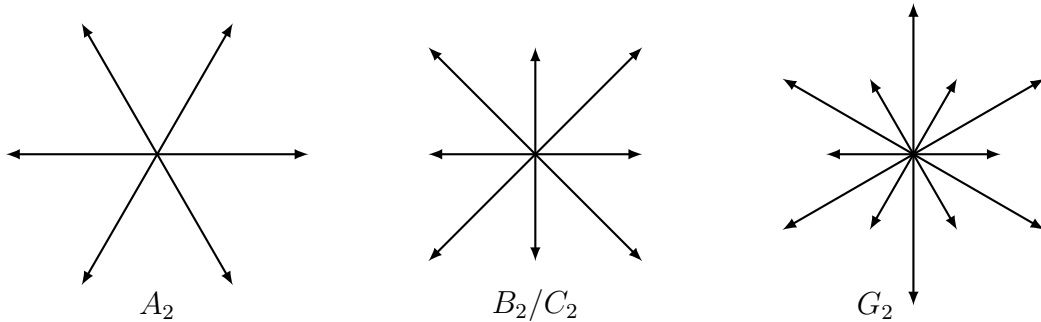


Figure 4.6: The irreducible crystallographic root systems of rank 2.

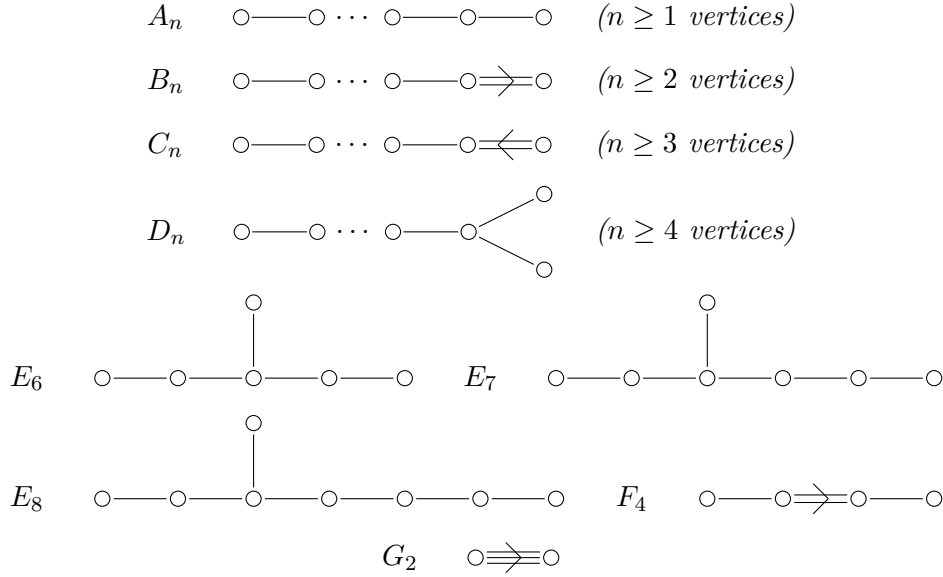
Given  $\Phi$  an irreducible essential crystallographic root system in  $V$ , we choose  $f \in V^*$  such that  $f^{-1}\{0\} \cap \Phi = \emptyset$  and say that the positive roots  $\Phi^+$  are  $f^{-1}\mathbb{R}^+ \cap \Phi$ . The simple roots  $\Delta$  are the positive roots generating the extreme rays of  $\text{cone}(\Phi^+)$ . Up to an isometry on  $V$ , the specific choice of  $f$  has no influence.

**Definition 4.5.4.** Let  $\Phi$  be crystallographic root system. The *Weyl group* associated to  $\Phi$  is the group generated by the orthogonal reflections through the roots.

We will see in an instant (Proposition 4.5.5) that the irreducible root systems can be naturally labeled by  $\{A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2\}$ . If  $\Phi$  is of these types  $T$ , the Weyl group is often also denoted by  $T$ . We have the same

properties as before:  $W$  is actually generated by the reflections through the simple roots,  $\Phi = \Phi^+ \sqcup \Phi^-$ ,  $\Phi = W\Delta$ ,  $W$  acts faithfully on  $\Phi$ ,  $(W, \{s_\delta \mid \delta \in \Delta\})$  is a Coxeter system... Because of this, in the same way that finite Coxeter groups are classified by their Coxeter diagrams, the irreducible crystallographic root systems are classified by their *Dynkin diagrams*: the vertices are the simple roots, with, between any two simple roots  $\delta, \delta'$  no edge if  $\langle \delta \mid \delta' \rangle = 0$ , a simple undirected edge if  $\langle \delta \mid \delta' \rangle = -1/2$  (they generate an  $A_2$  root subsystem), a double directed edge from the longest root to the shortest if  $\langle \delta \mid \delta' \rangle = -1$  ( $B_2/C_2$  subsystem) and a triple directed edge from longest to shortest if  $\langle \delta \mid \delta' \rangle = -3/2$  ( $G_2$  subsystem). We obtain the following classification of the crystallographic root systems (see [Hum90, Section 2.9]).

**Proposition 4.5.5.** *The complete classification of irreducible crystallographic root system is given by the following Dynkin diagrams.*

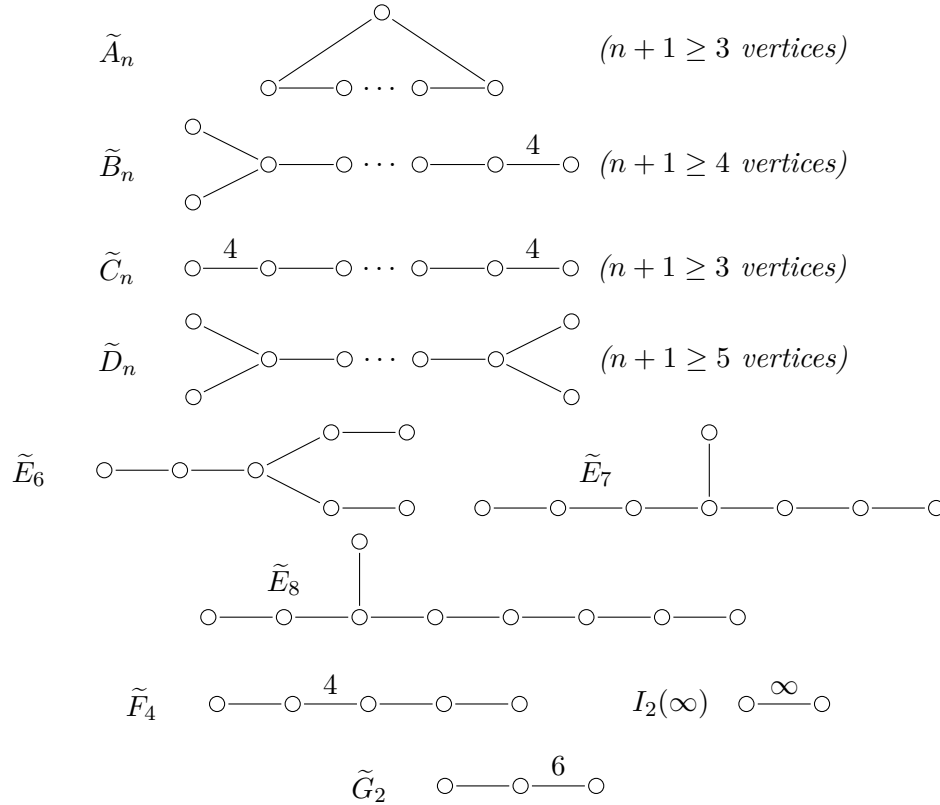


As before, the number of vertices requirement is there to ensure uniqueness in the classification, but we can otherwise consider the root systems  $A_1 = B_1 = C_1$ ,  $B_2 = C_2$ ,  $A_3 = D_3$ .

Note that even though the  $B_n$  and  $C_n$  root systems are not isometric, their associated Weyl groups are isomorphic: as said before, the reflection group “forgets” about the length of the roots and the Coxeter diagrams of the Weyl groups associated to the root system  $B_n$  and  $C_n$  merge into the  $B_n/C_n$  diagram of Proposition 4.5.3. However, this distinction is significant: if we recall Proposition 4.5.1, we may ask what (irreducible) groups correspond to, for instance, a positive semi-definite bilinear form?

**Proposition 4.5.6.** *The irreducible Coxeter systems such that  $B$  is positive semi-definite are classified by the following Coxeter diagrams.*





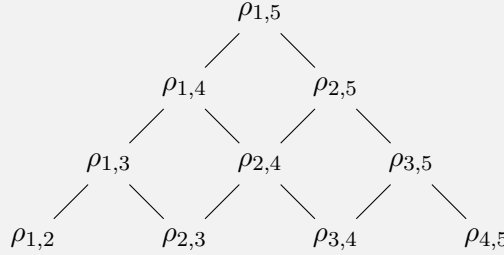
Same remark as before on the number of vertices. However, we will often write  $I_2(\infty)$  as  $\tilde{A}_1$ .

The groups associated with these diagrams are called (irreducible) *affine Coxeter groups* or (irreducible) *affine Weyl groups*. Visibly, the irreducible crystallographic root systems are in one-to-one correspondence with the irreducible affine Coxeter groups (with  $I_2(\infty)$  corresponding to  $A_1$ ). All these Coxeter diagrams can be obtained from the Coxeter diagrams of the Weyl groups by adding a vertex and an edge. The fact that  $B_n/C_n$  can be extended in two ways seems to point to the Weyl groups “remembering” the lengths of the crystallographic root systems. Let us present a well-known construction of the affine Weyl groups that explains this and that we adapt here from [Hum90, Section 6.5].

Let  $\Phi$  be an essential irreducible crystallographic root system and  $V$  its ambient space. Recall the integrality condition: since  $\Phi = W\Delta$ ,  $\Phi \in \mathbb{Z}\Delta$  and in particular  $\Phi^+ \in \mathbb{N}\Delta$ .

**Definition 4.5.7.** The *root poset* is  $(\Phi^+, \leq)$  where  $\alpha \leq \beta$  if and only if  $\beta - \alpha \in \mathbb{N}\Delta$ .

**In type A :** In type  $A_n$ , we can choose the positive roots as the set  $\{\rho_{i,j} = e_i - e_j \mid 1 \leq i < j \leq n+1\}$ . In the root poset, we have  $e_i - e_l \geq e_j - e_k$  if and only if  $i \leq j < k \leq l$ . The Hasse diagram of  $A_4$  is as follows. In this case, the highest root is  $\alpha_0 = \rho_{1,5}$ .



As in the previous example, the root poset of type  $A_n$  has a unique maximal element. This is general: if  $\Phi$  is irreducible, the root poset has a unique maximal element, called the *highest root* (see [Hum90, Section 2.9 (3)]), denoted by  $\alpha_0$ .

Let  $\delta$  be a symbol and consider the space  $V \oplus \mathbb{R}\delta$  where the dot product of  $V$  is extended by  $\langle \delta | \delta \rangle = 1$  and  $\delta \in V^\perp$ . We also define a bilinear form  $B$  by  $B|_{V \times V} = \langle \cdot | \cdot \rangle$ , and  $(V \oplus \mathbb{R}\delta)^{\perp B} = \mathbb{R}\delta$ .

**Definition 4.5.8.** The *affine root system*  $\Phi_a$  associated to  $\Phi$  is the set  $\Phi + \mathbb{Z}\delta$ . The positive roots are  $(\Phi^+ + \mathbb{N}\delta) \sqcup (\Phi^- + \mathbb{N}^*\delta)$ . The simple roots are  $\Delta \sqcup \{-\alpha_0 + \delta\}$ .

As before, we get a group by considering the  $B$ -orthogonal reflections associated to the roots. This is not the only way to obtain that group, however, and we shall prefer the following way:

**Definition 4.5.9.** Let  $\Phi$  be an irreducible crystallographic root system. Consider the *affine Weyl group* associated to  $\Phi$  is the group generated by the affine reflections for all  $\alpha \in \Phi, k \in \mathbb{Z}$ :

$$s_{\alpha,k} : \begin{cases} V \longrightarrow V \\ x \longmapsto x - 2(\langle \alpha | x \rangle - k) \frac{\alpha}{\langle \alpha | \alpha \rangle} \end{cases}.$$

If  $\Phi$  is of type  $T \in \{A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2\}$ , the Weyl group will be denoted by  $\tilde{T}$ .

The reader may question why in this case only, the reflection group does not arise in terms of reflection with respect to roots, but to (affine) hyperplanes. We refer to [Hum90, Chapter 4] for more details, as we shall not develop further the relation between these two points of view: put simply, the affine

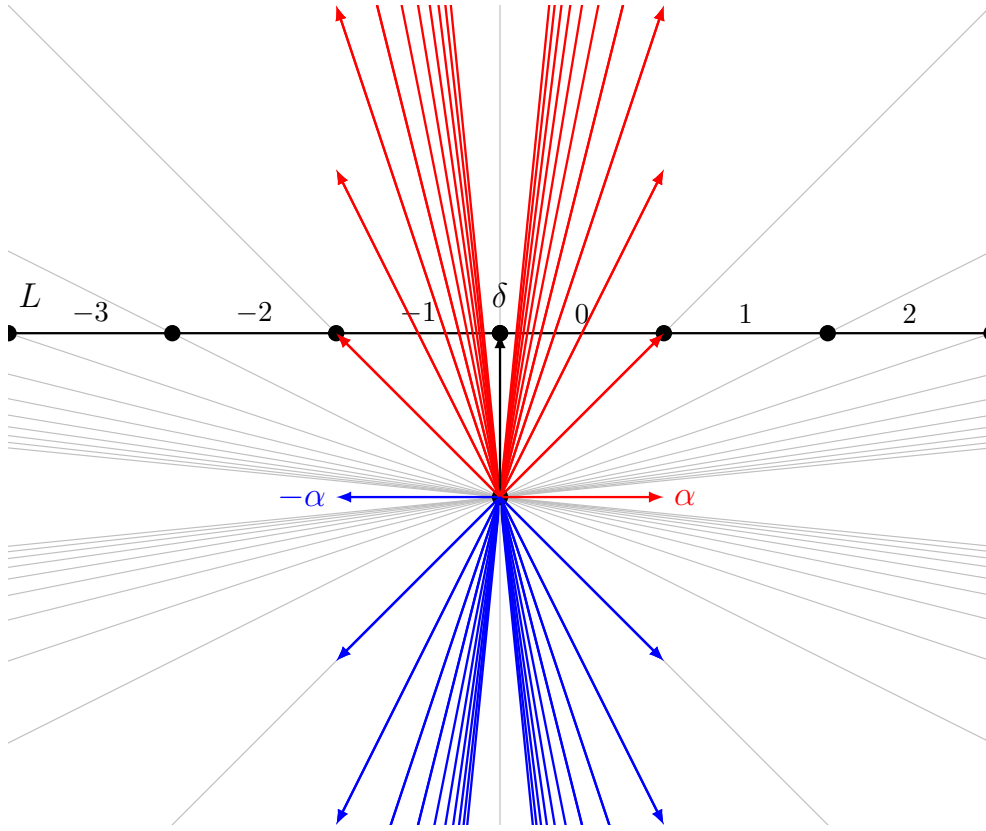


Figure 4.7: Affine arrangement and affine roots in  $\tilde{A}_1$ .

The dots are the affine hyperplane in the black line  $H_1$ . The red vectors are  $\Phi_a^+$  and the blue ones are  $\Phi_a^-$ . Their direction approaches that of  $\delta$  as their associated affine hyperplane (meaning, here, their intersection with the line  $L$ ) goes to infinity.

hyperplanes in  $V$  can be seen as a cut of the dual of the affine root system as illustrated in Figure 4.7.

The same phenomenon can also be observed in Figure 4.8 in the case of  $\tilde{A}_2$ . We shall only use the fact, readily observable in the figures, that there is a one-to-one correspondence between positive roots and affine hyperplanes. To fix the notations in this correspondence, let:

$$H_{\alpha+k\delta} = \{x + d\delta \in V \oplus \mathbb{R}\delta \mid \langle x + d\delta \mid \alpha + k\delta \rangle = 0\}.$$

If we restrict ourselves to  $d = 1$ , we get:

$$H_{\alpha+k\delta} \cap \{d = 1\} = \{x \in V \mid \langle x \mid \alpha \rangle + k\langle \delta \mid \delta \rangle = 0\} + \delta$$

so the root  $\alpha + k\delta$  corresponds to the affine hyperplane  $\langle x \mid \alpha \rangle = -k$ .

If we set  $H_{\alpha,k} = \{x \in V \mid \langle x \mid \alpha \rangle = k\}$ , the root  $\alpha + k\delta$  naturally corresponds to  $H_{\alpha,-k}$ . Because of this, the set  $\Phi_a^+ = (\Phi^+ + \mathbb{N}\delta) \sqcup (\Phi^- + \mathbb{N}^*\delta)$  is a natural

indexing set for the affine hyperplanes in  $V$  generating  $\widetilde{W}$ . For ease of use reasons that will appear in Section 4.7, we will prefer to manipulate the sets  $H_{\alpha,k}^+ = \{x \in V \mid \langle x \mid \alpha \rangle > k\}$  and  $H_{\alpha,k}^- = \{x \in V \mid \langle x \mid \alpha \rangle < k\}$ . However, we get that  $H_{\alpha+k\delta}^+ = H_{\alpha,-k}^-$  if  $\alpha \in \Phi^-$ . To avoid this slightly unpleasant issue, we chose a different indexing set: note that  $H_{\alpha+k\delta} = H_{-\alpha-k\delta}$  so  $\Phi^+ + \mathbb{Z}\delta$  is also an indexing set. Thus, we get the following correspondence between the  $\Phi_a^+$  and  $\Phi^+ \times \mathbb{Z}$  based indexing:

$$\phi \mapsto \begin{cases} H_{-\alpha,k} & \text{if } \phi = \alpha + k\delta \in \Phi^- + \mathbb{N}^*\delta \\ H_{\alpha,-k} & \text{if } \phi = \alpha + k\delta \in \Phi^+ + \mathbb{N}\delta \end{cases}.$$

That way, given a root  $\alpha \in \Phi$ , the sets  $H_{\alpha,k}^+$  are (strictly) decreasing for inclusion.

To come back to the question of the length of the roots, it is somewhat clear that in this construction  $\tilde{B}_n$  and  $\tilde{C}_n$  “remembers” whether it was constructed from the  $B_n$  and  $C_n$  because the length of the roots is reflected in the spacing in the arrangement of affine hyperplanes used to construct the Weyl group.

## 4.6 Dual objects: Tits cone, Coxeter complex

We present here the notions of *Tits cone* and *Coxeter complex*, which can be understood as dual objects to the root system and the inversion sets. This dual point of view will be useful in stating simply the proofs in §6.1.1 where we examine the interactions of inversion sets and moving separation hyperplanes. We refer to [AB08] for more details. Recall that we have fixed  $(W, S)$  of finite rank, which means that  $V$  is finite dimensional. As such, we commit an abuse of notation by conflating the dot product and  $\langle \cdot \mid \cdot \rangle : V \times V^* \rightarrow \mathbb{R}$  the duality bracket. We do not, however, identify  $V$  with  $V^*$ : the objects described in this section live in a space separate from  $V$ .

For  $v \in V$ , we set  $H_v = \{f \in V^* \mid \langle v \mid f \rangle = 0\}$ ,  $H_v^+ = \{f \in V^* \mid \langle v \mid f \rangle > 0\}$  and  $H_v^- = -H_v^+$ . Note that unlike what we did in the case of affine groups, those are hyperplanes in the *dual space*. In the affine case, because the bilinear form with respect to which we defined the reflections was so close to an actual dot product, affine roots and their orthogonal hyperplanes could meaningfully be represented in the same space. In the general case however, the bilinear form is not necessarily positive definite, and we must work in the dual space.

**Definition 4.6.1.** The union of the  $H_\phi$  taken over the roots is called the *Coxeter arrangement*:

$$\mathcal{A} = \bigcup_{\phi \in \Phi} H_\phi.$$

The connected components of the complement of the Coxeter arrangement in  $V^*$  are called the *alcoves*.

Recall from Proposition 4.3.4 the definition of a separating form. Let  $f$  be an element contained in an alcove  $A$ : by definition, it does not lie in any  $H_\phi$  and thus is a separating form for the set of roots  $\Phi^+ \cap f^{-1}\mathbb{R}_-$ . Moreover, this set is independent of the choice of  $f$  in  $A$ , and we may denote by  $N(A) = \Phi^+ \cap f^{-1}\mathbb{R}_-$  for any  $f \in A$ . By duality to the question of which subsets of the roots correspond to an inversion set, we can ask which alcoves  $A$  correspond to inversion sets  $N(A)$ . We have by construction the separability, so according to Proposition 4.3.4 we only lack finiteness.

Beginning with the identity element, let the set  $\mathcal{C} = \bigcap_{\delta \in \Delta} H_\delta^+ = \bigcap_{\phi \in \Phi^+} H_\phi^+$  be the *fundamental alcove* and  $\bar{\mathcal{C}}$  be its closure for the usual topology. It is immediate that  $f \in V^*$  separates  $N(1) = \emptyset$  if and only if  $f \in \mathcal{C}$  and that  $\mathcal{C}$  is non-empty (as it contains  $\langle \cdot | \mathbf{1} \rangle$ ). The *Tits cone of  $W$*  is  $U = \bigcup_{w \in W} w(\bar{\mathcal{C}})$ , where  $w$  acts on the right on  $V^*$  by  $w(f) = f \circ w$ . Note that this goes against the usual convention (used for instance in [Hum90]) of acting on  $V^*$  by duality on the left, but has the advantage that it helps us more immediately understand the elements of the Weyl alcoves as forms separating the inversion set of their corresponding element from their complement. This change of convention does not essentially change the classical results on the Tits cone.

The map  $w \mapsto w\mathcal{C}$  is injective as  $w\mathcal{C}$  is non-empty (it contains  $\langle w(\cdot) | \mathbf{1} \rangle$ ) and all the forms in  $w\mathcal{C}$  are separating forms for  $N(w)$ : since  $N$  is injective, so is  $w \mapsto w\mathcal{C}$ . The resulting alcove lies in  $H_\phi^-$  if  $\phi \in N(w)$  and  $H_\phi^+$  otherwise. In that sense, an alcove  $w\mathcal{C}$  is the dual of the inversion set  $N(w)$ . Let us just formally state the characterization of the inversion set in terms of the alcove

**Proposition 4.6.2.** *Let  $w \in W$ . Then for all  $\Phi \in \Phi^+$ ,  $\phi \in N(w)$  if and only if  $\mathcal{C}$  and  $w\mathcal{C}$  lie on different sides of  $H_\phi$ .*

*Proof.* This is immediate given that  $\mathcal{C}$  is in  $H_\phi^+$  for all  $\phi \in \Phi^+$  and that  $\phi \in N(w)$  if and only if  $w\mathcal{C} \subset H_\phi^-$ .  $\square$

Given this interpretation of the inversion set as separating hyperplanes between two alcoves in the dual space, we can also say that the Tits cone is the union of (the closure of) all alcoves that are separated from  $\mathcal{C}$  by finitely many hyperplanes  $H_\phi, \phi \in \Phi^+$ .

As the inversion set of an element  $w$  can be read as the set of hyperplanes separating the Weyl alcove  $w\mathcal{C}$  from the Weyl alcove  $\mathcal{C}$  of the identity element, logically, we could expect that the descents  $\Gamma_w$ , being the first roots that can be removed from  $N(w)$ , the associated hyperplanes should be in some sense the “closest” to  $w\mathcal{C}$ . As it turns out, this can be formalized. Fix a root  $\phi \in \Phi^+$

and a Weyl alcove  $w(\mathcal{C})$ . We say that  $H_\phi$  is a *wall of  $w(\mathcal{C})$*  if  $H_\phi \cap \overline{w\mathcal{C}}$  spans  $H_\phi$  (or, equivalently, if  $H_\phi \cap \overline{w\mathcal{C}}$  contains a relative open set of  $H_\phi$ ). Walls give us yet another way to recognize inversions and descents.

**Proposition 4.6.3.** *For any  $\phi \in \Phi$ ,  $\phi \in \Gamma_w$  if and only if  $\phi \in N(w)$  and  $H_\phi$  is a wall of  $w(\mathcal{C})$ .*

*Proof.* Seen in the geometric light, this is obvious: a descent  $\phi$  is a root that is an inversion such that  $N(w) \setminus \{\phi\} = N(w')$  for some  $w'$ . Since removing a root  $\phi$  from  $N(w)$  corresponds to crossing  $H_\phi$ , we may cross only  $H_\phi$  from  $w\mathcal{C}$  if and only if  $H_\phi \cap w\overline{\mathcal{C}}$  is a proper face of  $\overline{\mathcal{C}}$  of dimension  $\dim V - 1$ : that is, a wall.  $\square$

Because for any non-zero linear form  $f \in V^*$ , for all  $\lambda \in \mathbb{R}_+^*$ ,  $f$  and  $\lambda f$  define the same half spaces and thus potentially the same inversion sets, we consider the quotient  $(U \setminus \{0\})/\mathbb{R}_+^*$ , called the *Coxeter complex* associated to  $\Phi$ . We may choose, for instance, a set of representatives of the elements of the Coxeter complex as the intersection of the Tits cone with the unit sphere of  $V^*$ .

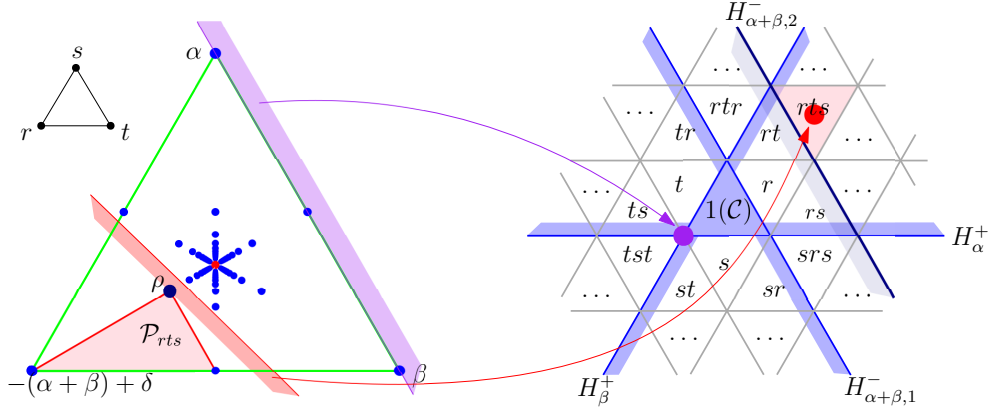


Figure 4.8: On the left, the projective picture. On the right, the Coxeter complex.

Weyl alcoves are labeled by their associated element. Half-spaces in the primal are sent to points of the same color in the Coxeter complex.

Conversely, the descent  $\rho$  of  $rts$  is sent to the wall  $H_\rho$  of  $rts(\mathcal{C})$  with  $rts(\mathcal{C}) \subset H_\rho^-$ .

The map  $f \mapsto f^{-1}\mathbb{R}_+$  is a bicontinuous bijection from  $(V^* \setminus \{0\})/\mathbb{R}_+$  to the set of closed half spaces of  $V$ . In particular, if we restrict the map to the Coxeter complex, this means that a small perturbation of a point in the dual, translates to a small perturbation of the corresponding hyperplane in the primal. In fact, in the same way that in the projective picture we get a

nice discrete topology on the normalized roots as long as we stay away from the isotropic cone, in the dual space, the hyperplanes corresponding to roots remain “separated” from each other as long as we stay in the Tits cone. Several formalizations of this remark exist, but the one that will be most useful to us is reproduced here from [Hum90, Theorem 5.13].

**Proposition 4.6.4.** *The Tits cone  $U$  is convex. Any line segment with extremities in  $U$  meets finitely many alcoves and finitely many hyperplanes  $H_\phi$  for  $\phi \in \Phi^+$ .*

**Remark.** A coarse (and false, but useful) intuition for the Tits cone is as the dual cone of the isotropic cone. In fact, it is the (strict) dual cone of the limit directions of the positive root system *i.e.* the set  $\{f \in V^* \mid f(l) > 0 \forall l \in \lim \Phi^+\}$  (see [Dye12]). Let us see in the finite, affine and general cases what this means.

- In the case of a finite group, any alcove of the Coxeter arrangement is a finite number of hyperplanes away from  $\mathcal{C}$ , as there is only a finite number of hyperplanes. This is indeed the dual cone of the limit directions of the positive roots, that is, the empty set.
- In the case of an affine set, the positive roots have exactly one limit direction: the imaginary root  $\delta$ . The dual cone is the (open) half-space  $H_\delta^+$ .
- In the other cases, given two roots  $\alpha, \beta$  such that  $B(\alpha, \beta) < -1$ ,  $\Phi_{\alpha, \beta}$  has two accumulation points and thus, the Tits cone is bounded along the direction of the line  $(\langle \alpha \mid \cdot \rangle, \langle \beta \mid \cdot \rangle)$ .

We want to bring the attention of the reader on the fact that in the affine case, we can realize the Coxeter complex as the totality of the affine space  $H_{\delta,1} = \{v + \delta \mid v \in V\}$ , for instance. In that sense, the picture of Figure 4.7 is not exceptional: the Tits cone is the dual cone of  $\delta$ , so it contains  $H_{\delta,1}$  and conversely, every alcove in the Tits cone encounters  $H_{\delta,1}$ .

**Notation 4.6.5.** In the case of a general Coxeter group, we will call the connected components of the complement of the Coxeter arrangement *alcoves*, as we have done in this section. However, in the case of affine Weyl groups, the Coxeter arrangement of a given  $\widetilde{W}$  also contains the Coxeter arrangement of the finite group  $\widetilde{W}$ . To distinguish, when we talk about alcoves we mean *of the affine group*, and we call *chambers* the connected components when considering only the arrangement of the finite group.

## 4.7 Small roots and Shi arrangement

### 4.7.1 Arbitrary Coxeter group

Let  $(W, S)$  be an arbitrary Coxeter system of finite rank.

**Definition 4.7.1.** Let  $\alpha, \beta \in \Phi^+$ , distinct. We say that  $\alpha$  *dominates*  $\beta$  if for any  $w \in W$ ,  $\alpha \in N(w) \Rightarrow \beta \in N(w)$ . The statistic  $\text{dp}_\infty(\alpha) = |\{\beta \in \Phi^+ \mid \alpha \text{ dominates } \beta\}|$  is called the *infinite depth* of  $\alpha$ . We denote  $\Sigma_n$  the set of roots with infinite depth at most  $n$ .

The elements of  $\Sigma_0$  (resp.  $\Sigma_n$ ) are called *small roots* (resp. *n-small*) and were introduced in [BH93] by Brink & Howlett in connection with constructing an automaton recognizing the language of reduced words of  $W$ .

**Example 4.7.2.** The simple roots are always small roots as for any  $s$ ,  $\{e_s\}$  is an inversion set so  $e_s$  dominates no one. More generally, if  $W$  is finite, all roots are small because all separable sets of roots are inversion sets. Given two distinct positive roots  $\alpha, \beta$ , any separating linear form for the pair that does not vanish on any other root (which can be assured) gives an inversion set in which one is and not the other.

A major result is that the sets  $\Sigma_n$  are finite, as proven by Brink & Howlett in [BH93] for the case  $n = 0$  and Fu in [Fu12] for arbitrary  $n$ .

**Proposition 4.7.3** (Brink & Howlett, Fu). *For all  $n$ ,  $\Sigma_n$  is finite.*

Without going into the details, it follows from Proposition 4.3.1 that if  $s \notin D_R(w)$ ,  $\Sigma_n(ws) = \{e_s\} \cup (s\Sigma_n(w) \cap \Sigma_n)$ .

**Definition 4.7.4.** For any  $w$  in let  $\Sigma_n(w) = \Sigma_n \cap N(w)$  be the *n-small inversion set* of  $w$ . The set of all *n-small inversion set* is denoted by  $\Lambda_n$ .

Note that since  $\Delta \subset \Sigma_n$ , if two alcoves  $w\mathcal{C}, w'\mathcal{C}$  are in the same Shi region then  $\Sigma_n(w) = \Sigma_n(w')$  and in particular  $D_R(w) = D_R(w')$ . Thus, the free monoid on  $S$  acts on the finite set  $\Lambda_n \cup \{\mathbf{reject}\}$  by  $s\lambda = \mathbf{reject}$  if  $e_s \in \lambda$ ,  $s\lambda = \{e_s\} \cup (s\lambda \cap \Sigma_n)$  otherwise and  $s\mathbf{reject} = \mathbf{reject}$ . This gives a finite automaton where a word is accepted if and only if it doesn't end on  $\mathbf{reject}$ , which happens if and only if it is reduced.

**Proposition 4.7.5** (Brink, Howlett [BH93]). *The language of reduced words of a Coxeter group is recognized by a finite automaton.*



This is a very interesting and important property in language theoretic terms. It is especially notable that although the result itself has no obvious connection to geometry, its proof is dependent on the definition of the small roots, which are geometric in nature. As before, looking at the dual of the  $n$ -small root gets us another interesting and well-studied object: the  $n$ -Shi arrangement.

**Definition 4.7.6.** The  $n$ -Shi arrangement is the union of the hyperplanes

$$\mathcal{A}_n = \bigcup_{\phi \in \Sigma_n} H_\phi.$$

The Shi arrangements are subarrangements of the Coxeter arrangement, so for any region  $R$  of  $\mathcal{A}_n$ , and any alcove  $A$ , either  $A \subset R$  or  $A \cap R = \emptyset$  so the Shi regions are (a topological closure away from) a union of alcoves. However, when looking only at the accessible alcoves, meaning the Tits cone, the situation is not as clean. For example, the fundamental alcove  $\mathcal{C}$  is a region of the Shi arrangement (because the simple roots are small) and so is  $-\mathcal{C}$ . However, unless  $W$  is finite,  $-\mathcal{C}$  is not in the Tits cone (it would correspond to the element that inverses every root). In this thesis, we are only concerned about Shi regions that meet the Tits cone.

The  $n$ -small inversions sets are naturally in bijection with the regions of the  $n$ -Shi arrangement that intersect the Tits cone: for any such region  $R$ , let  $w\mathcal{C} \subset R$ . By definition of the  $n$ -Shi arrangement  $R = \bigcap_{\phi \in \Sigma_n} H_\phi^{\varepsilon_\phi}$  with  $\varepsilon \in \{+, -\}^{\Sigma_n}$  so by Proposition 4.6.2,  $\Sigma_n(w) = \{\phi \in \Sigma_n \mid \varepsilon_\phi = -\}$ . Conversely, if  $\lambda \in \Lambda_n$ , by definition  $\lambda = \Sigma_n(w)$  for some  $w$ , and  $\lambda$  corresponds to the Shi region containing  $w\mathcal{C}$ . In general, the intersection of a Shi region with the Tits cone contains an infinite number of alcoves in the Tits cone and the  $n$ -small inversion set map cannot be injective. Is there a natural notion of a privileged element of the group having a given  $n$ -small inversion set?

**Definition 4.7.7** (From [DH16]). An element  $w \in W$  is  $n$ -low if the inversion set can be recovered from the  $n$ -small inversion set by

$$N(w) = \text{cone}(\Sigma_n(w)) \cap \Phi^+.$$

The set of  $n$ -low elements is denoted by  $L_n$ .

If  $\lambda \in \Lambda_n$  is the  $n$ -small inversion set of some low element  $w$ , it is natural to say that  $w$  should be the privileged representation for  $\lambda$  as from Corollary 4.3.5,  $\text{cone}(\Sigma_n(w)) \cap \Phi^+ \subset N(w)$  so the equality case defining a low element is asking for the low element to be minimal in its Shi region in terms of inversion set inclusion – or equivalently, in the left weak order. It is clear that the map  $\Sigma_n : L_n \longrightarrow \Lambda_n$  is injective as a low element is determined by its small inversion set.

**Conjecture 4.7.8** (Dyer & Hohlweg, [DH16] Conjecture 2). *The map  $\Sigma_n : L_n \rightarrow \Lambda_n$  is surjective.*

Since this conjecture was proposed, several special cases were solved (mostly for  $n = 0$ ): the rank 3 case in [Cha21] (reproduced below), the case of affine Weyl groups in [CLH22], another proof of the case of affine Weyl groups in [Cha23] (also reproduced below) until a recent paper [Dye+23] from Dyer, Hohlweg, Fishel and Mark settled the question by proving the conjecture for any group and any  $n$ .

An important tool for our proof of the conjecture in rank 3 as well as the general proof from Dyer, Hohlweg, Fishel and Mark [Dye+23] is the notion of *bipodality* of a subset of the positive roots.

**Definition 4.7.9.** For any maximal dihedral subgroup  $H$  and  $\alpha, \beta \in \Phi_H^+$ , if  $\alpha \notin \Delta_H$  and  $\beta \in \Delta_H$ , we say that  $(\alpha, \beta)$  is an *arrow* from  $\alpha$  to  $\beta$ , and we write  $\alpha \rightarrow \beta$ .

In the projective picture, deciding if any two distinct roots  $(\alpha, \beta)$  form an arrow can be interpreted as drawing the line  $d = (\alpha, \beta)$ , considering all the normalized roots lying on  $d$  and answering the question “is  $\alpha$  between other roots while  $\beta$  is not?”.

**Definition 4.7.10** (Dyer & Hohlweg [DH16]). A set  $A \subset \Phi^+$  is *bipodal* if whenever  $\alpha \rightarrow \beta$  and  $\alpha \in A$  then  $\beta \in A$ .

In [DH16], the authors first prove that  $\Sigma_0$  is bipodal, which is then generalized by Dyer in [Dye22]. We refer to [Dye22, Theorem 1.6] for the proof of the general case.

**Theorem 4.7.11** (Dyer & Hohlweg, Dyer). *For all  $n$ , the set  $\Sigma_n$  of  $n$  small roots is bipodal.*

## 4.7.2 Weyl group

In the case of affine Weyl groups, the  $n$ -Shi arrangement takes a special, simpler form, because infinite depth is essentially given by the  $\delta$  coordinate of a root. In this section  $(W, S)$  is a crystallographic group with root system  $\Phi$  in the Euclidean space  $V$  with associated affine root system  $\Phi_a \in V \oplus \mathbb{R}\delta$  and affine Weyl group  $\widetilde{W}$ .

**Proposition 4.7.12.** *Let  $\phi \in \Phi_a^+$ . Then*

$$\text{dp}_\infty(\phi) = \begin{cases} k & \text{if } \phi = \alpha + k\delta, \quad (\alpha, k) \in \Phi^+ \times \mathbb{N} \\ k-1 & \text{if } \phi = -\alpha + k\delta, \quad (\alpha, k) \in \Phi^+ \times \mathbb{N}^* \end{cases}.$$

*Proof.* Note that because of the choice of indexing for the hyperplanes at the end of Section 4.5,  $\alpha + k\delta$ , where  $(\alpha, k) \in \Phi^+ \times \mathbb{N}$ , is an inversion of  $w$  if and only if  $w\mathcal{C} \subset H_{\alpha, -k}^-$ . But, conversely,  $-\alpha + k\delta$  for  $(\alpha, k) \in \Phi^+ \times \mathbb{N}^*$  is an inversion if and only if  $w\mathcal{C} \subset H_{\alpha, k}^+$ . Recall that the sets  $H_{\alpha, k}^+$  and  $H_{\alpha, -k}^-$  are decreasing for inclusion when  $k$  increases. Thus, in the first case  $w\mathcal{C} \subset H_{\alpha, -k}^- \Rightarrow w\mathcal{C} \subset H_{\alpha, -k'}^-$  for all  $k' \in \llbracket 0, k-1 \rrbracket$ , so  $\text{dp}_\infty(\alpha + k\delta) \geq k$  because  $\alpha + k\delta$  dominates all those  $\alpha + k'\delta$ . Similarly, in the second case  $w\mathcal{C} \subset H_{\alpha, -k}^- \Rightarrow w\mathcal{C} \subset H_{\alpha, -k'}^-$  for  $k' \in \llbracket 1, k-1 \rrbracket$  so  $\text{dp}_\infty(\alpha + k\delta) \geq k-1$ .

Let us show that the roots  $\{\alpha + k'\delta \mid k' \in \llbracket 0, k \rrbracket\}$  are the only ones dominated by  $\alpha + k\delta$ . Firstly, no root of the form  $-\alpha + d\delta$  is dominated: if  $\alpha + k\delta$  and  $-\alpha + d\delta$  are in the same inversion set, from Corollary 4.3.5, so are every positive linear combination of the two. In that case, the inversion set would contain an infinite number of roots (of the form  $\alpha + (k + md)\delta, m \in \mathbb{N}$  for instance). Secondly, let  $\beta \in \Phi^+$  be distinct from  $\alpha$ . Since  $\alpha$  and  $\beta$  are not co-linear, for any  $k' \in \mathbb{N}$  the affine hyperplanes  $H_{\alpha, k}$  and  $H_{\beta, k'}$  intersect. Because  $\widetilde{W}$  is affine, the Coxeter complex contains that intersection, meaning that the region  $H_{\alpha, k}^+ \cap H_{\beta, k'}^-$  contains accessible alcoves and any alcove in it corresponds to an element inverting  $\alpha + k\delta$  but not  $\beta + k'\delta$  (and similarly for the other combinations  $(\alpha + k\delta, -\beta + (k+1)\delta), (-\alpha + (k+1)\delta, \beta + k\delta), (-\alpha + (k+1)\delta, -\beta + (k+1)\delta)$ ).  $\square$

**Definition 4.7.13.** We denote by  $\mathcal{A}_n$  the  $n$ -Shi arrangement defined by:

$$\mathcal{A}_n = \bigcup_{\alpha \in \Phi^+} \bigcup_{k \in \llbracket 1-n, n \rrbracket} H_{\alpha, k}.$$

It is clear from Proposition 4.7.12 that this is equivalent, in the context of Weyl groups, to the previous definition of the Shi arrangement. As before, the Shi arrangement  $\mathcal{A}_n$  being a sub-arrangement of  $\mathcal{A}$  which defines the alcoves, for a Shi region  $R$  we have  $\bigcup_{w\mathcal{C} \cap R \neq \emptyset} \overline{w\mathcal{C}} \subset \overline{R}$ . Unlike before however, this inclusion is in fact an equality because, again, the Coxeter complex is the whole affine space. Said more loosely, a Shi region is a union of alcoves. Using this fact, for  $w \in \widetilde{W}$  we (abusively) write that  $w \in R$  to mean  $w\mathcal{C} \subset R$ .

We will mostly be interested in the case of  $\mathcal{A}_1$  for which we have a handy description of the regions. Setting, for a real number  $x$ ,  $\text{sign}(x) = -$  if  $x < 0$ ,  $\text{sign}(x) = 0$  if  $x = 0$  and  $\text{sign}(x) = +$  if  $x > 0$ , we can define  $\text{sign}(w)$  for  $w \in \widetilde{W}$  as  $(\text{sign}(k(w, \alpha)))_{\alpha \in \Phi^+}$ . The following is straightforward (see [Shi87b] for a discussion and a precise proof):

**Proposition 4.7.14.** *Let  $R$  be a Shi region. The value of  $\text{sign}(w)$  is constant for all  $w \in R$ . The sign type of  $R$ , denoted by  $\text{sign}(R)$ , is defined as  $\text{sign}(w)$  for any  $w \in R$ . The map  $R \mapsto \text{sign}(R)$  is an injection of the set of Shi regions in  $\{-, 0, +\}^{\Phi^+}$ .*

**In type  $A_2$ :** A consequence of Theorem 5.2.1 is that not all sign types are possible. In type  $A_2$  for instance, we have 3 roots  $\alpha = e_1 - e_2, \beta = e_2 - e_3, \alpha + \beta = e_1 - e_3$ . Giving the signs in the order  $(\alpha, \alpha + \beta, \beta)$  (as we present sign triplets everywhere, with the sum in the middle), the  $A_2$  sign types are the following:

$$\left\{ \begin{array}{cccccc} (+, +, +), & (-, -, -), & (+, +, -), & (-, +, +), & (-, -, +), & (+, -, -), \\ (+, +, 0), & (0, +, +), & (+, 0, -), & (-, 0, +), & (-, -, 0), & (0, -, -), \\ & (0, +, 0), & (0, 0, -), & (-, 0, 0), & (0, 0, 0) & \end{array} \right\}.$$

Notice that replacing all the zeros of a possible  $A_2$  sign type with pluses gives one of the possible  $A_2$  sign types of the first row above.

Given the sign type  $\text{sign}(R)$  of some Shi region  $R$ , it seems natural to ask to exhibit an element  $w \in \widetilde{W}$  such that  $\text{sign}(w) = \text{sign}(R)$ . As announced in the introduction, we can actually do slightly better than exhibit any element: in [Shi87b, Proposition 7.2], Shi proves that every Shi region has a *minimal element*, in the sense that its *Shi encoding* is minimal. Given  $w \in \widetilde{W}$  and its corresponding alcove  $w\mathcal{C}$ , define:

$$k(w, \alpha) = \max\{i \in \mathbb{Z} \mid w\mathcal{C} \subset H_{\alpha, i}^+\}.$$

**Proposition 4.7.15.** *Let  $R$  be a Shi region in the type  $\widetilde{W}$  Shi arrangement. There exists a unique minimal element  $\min(R) \in R$  in the sense that for every  $w \in R$  and every  $\alpha \in \Phi^+$ ,  $|k(\min(R), \alpha)| \leq |k(w, \alpha)|$ .*

**Remark.** The minimal element can be defined in other equivalent ways: it is also the unique element such that  $\sum_{\alpha \in \Phi^+} |k(w, \alpha)|$  is minimized, or the unique minimal element of  $R$  for the *left weak order* of  $\widetilde{W}$ . We refer to [Shi87b, Section 7] for details. For our application, the version of Proposition 5.2.13 is the most convenient.



# Chapter 5

## Minimal elements of Shi regions in Weyl groups

In this chapter,  $W$  is an irreducible Weyl group with crystallographic essential root system  $\Phi$  in an ambient Euclidean space  $V$ , positive roots  $\Phi^+$  and negative roots  $\Phi^-$ . If  $v$  is a vector indexed by  $\Phi^+$ , we abuse the notation by setting  $v_{-a} = v_a$ <sup>1</sup> for any  $a \in \Phi^+$ . If  $A$  is a subset of  $V$ , we have the objects  $V_A, B_A, \text{Span}_\Phi(A), \Phi_A, \Phi_A^+, \Phi_A^-$  as defined in Notation 4.2.7.

It is obvious that, since  $\Phi$  is crystallographic, for any  $A \subset \Phi$ ,  $\Phi_A$  is too. As pointed out in Proposition 4.5.5, the irreducible crystallographic root systems are fully classified, but we shall not need the classification for the most important part of this chapter, except for two points:

- The list of rank 2 crystallographic root systems. Fortunately, there are very few, as presented in Figure 5.1 below.
- That only  $G_2$  has  $G_2$  as a rank two subsystem.

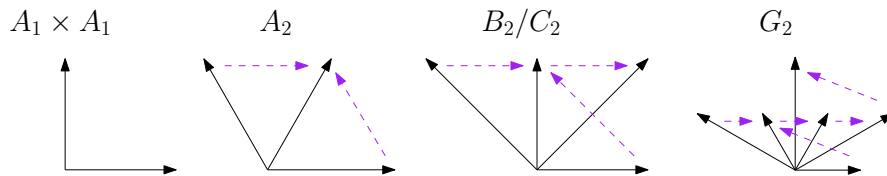


Figure 5.1: The positive roots of the rank 2 crystallographic root systems.  
The Hasse diagram of the root poset is represented in purple.

<sup>1</sup>This choice, which will reveal itself to be slightly non-standard, is – as will be explained later – intentional.

From Figure 5.1, we can make the two following observations that will be useful in Section 5.1.

- Observation 5.0.1.** 1. In all crystallographic root systems of rank 2, if  $a, b$  are positive roots, if  $a - b \notin \Phi$  then  $\langle a | b \rangle \leq 0$ .
2. Moreover, if we assume that  $b \geq a$ , then  $\langle a | b \rangle = 0$ .

Given the finite group  $W$ , we denote  $\widetilde{W}$  the corresponding affine Weyl group, as before.

## 5.1 Structure of the Shi relations

We begin by considering the notion of *Shi vectors*, following [Shi87a]. We do so somewhat independently of the original context of hyperplane arrangement, in the sense that we are (for now) only interested in understanding their defining relations. For now, Shi vectors will only be  $\Phi^+$ -indexed integer vectors satisfying some relations called the *Shi relations*. The objective of this section is to understand those relations: we will see that, in some sense, they are almost totally ordered, giving them a rigid structure, which we will use in Sections 5.2 and 6.2.

**Definition 5.1.1.** A vector  $v \in \mathbb{Z}^{\Phi^+}$  is a *Shi vector* if  $\forall \alpha, \beta, \gamma \in \Phi^+, \alpha = \beta + \gamma$  implies that  $\exists \varepsilon_{\beta, \gamma} \in \{0, 1\}$  such that  $v_\alpha = v_\beta + v_\gamma + \varepsilon_{\beta, \gamma}$ . This relation is called a *Shi relation* (with top  $\alpha$ ).

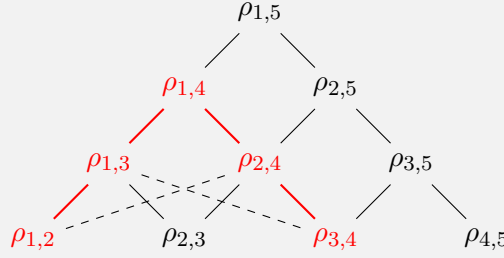
**Remark.** We want to bring the attention of the reader to the fact that in Shi's original formulation given as reference, the condition on  $\alpha, \beta, \gamma$  in the previous definition is instead  $\alpha^\vee + \beta^\vee = \gamma^\vee$  where for some non-zero vector  $v \in V$ ,  $v^\vee$  is defined as  $2v / \langle v | v \rangle$ . However, this is because the  $H_{\alpha, k}$  are instead defined as  $\{x \in V \mid \langle \alpha^\vee | x \rangle = 0\}$ . This is a minor difference, as  $\{\alpha^\vee \mid \alpha \in \Phi\}$  is a crystallographic root system whenever  $\Phi$  is: thus when we study for instance the Shi relations of type  $B_n$  for our definition, we are actually studying the Shi relations of type  $C_n$  in Shi's original sense. We adopt this difference in convention for the sake of lighter notation.

The definition of the Shi relations naturally leads to consider the following set:

**Definition 5.1.2.** The *skirt* of a positive root  $\gamma$  is the set

$$\text{Sk } \gamma = \{\alpha \in \Phi^+ \mid \gamma - \alpha \in \Phi^+\}.$$

**In type  $A_n$  :** In the Hasse diagram of a type  $A$  root poset, the skirt of a root  $\rho_{i,j}$  is the set  $\{\rho_{i,k} \mid i < k < j\} \cup \{\rho_{k,j} \mid i < k < j\}$ . We represent below the skirt of the root  $\rho_{1,4}$  in  $A_4$ . The dashed lines show the Shi relations with top  $\rho_{1,4}$ .

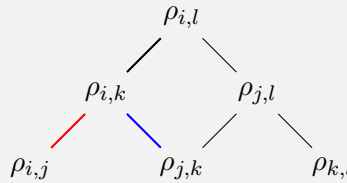


The name “skirt” comes from the shape of this set in the case of type  $A$ .

It is obvious that  $a \in \text{Sk}(b)$  implies  $a \leq b$  in the root poset. In the remainder of this section, we want to establish two features of the skirt of a root that are immediately apparent in type  $A$  from the picture above: the relation “being in the skirt of” is *almost* transitive (Proposition 5.1.3) and any two elements of the skirt are *almost* comparable (Proposition 5.1.4). The following proposition formalizes this “quasi-transitivity”. It should be noted<sup>2</sup> that it is equivalent to the case  $n = 3$  of [FP18, Note 3.4].

**Proposition 5.1.3.** *Let  $a, b, c$  be positive roots such that  $b \in \text{Sk}(a)$  and  $c \in \text{Sk}(b)$ . Then  $c \in \text{Sk}(a)$  or  $b - c \in \text{Sk}(a)$ .*

**In type  $A$  :** This proposition is the transposition in any Weyl group of the obvious equivalent in type  $A$ : if  $\rho_{i,k} \in \text{Sk}(\rho_{i,l})$  and  $a \in \text{Sk}(\rho_{i,k})$ , then either  $a = \rho_{i,j} \in \text{Sk}(\rho_{i,l})$  for some  $j$  (in red below) or  $a = \rho_{j,k}$  for some  $j$  and  $\rho_{i,k} - a = \rho_{i,j} \in \text{Sk}(a)$  (in blue below), and similarly for the “other side” of the skirt.



*Proof.* Let us show that  $c \notin \text{Sk}(a)$  implies  $b - c \in \text{Sk}(a)$ , that is, by definition of the skirt,  $a - (b - c) \in \Phi^+$ . Denote  $\text{Span}_{\Phi}(\{a, c\})$  by  $\Phi_1$  :  $\Phi_1$  must be of

<sup>2</sup>The author thanks Pr. Matthew Dyer for pointing it out.



one of the types  $A_1 \times A_1, A_2, B_2/C_2$  or  $G_2$ . From Observation 5.0.1-2,  $a$  and  $c$  are orthogonal.

Denote now  $\text{Span}_\Phi(\{a - b, c\})$  by  $\Phi_2$ . We want to show that  $(a - b) + c = a - (b - c)$  is a positive root. Note that it is enough to prove it is a root, since  $a - b$  and  $c$  are already positive, so their sum is positive. We have several possible cases:

- $\Phi_2 = G_2$ . Since only  $G_2$  has a root subsystem of type  $G_2$ , it means  $\Phi = \Phi_1 = \Phi_2$ . Reasoning by contradiction, we suppose that  $a - (b - c) \notin \Phi^+$ . This means that  $a$  is orthogonal to  $c$  and to  $b - c$ . Since  $b - c$  and  $c$  are distinct and no two positive roots are positively collinear, this means that  $c = \lambda(b - c)$ ,  $\lambda < 0$ . This is a contradiction because  $c$  and  $b - c$  are both positive roots.

We now assume that  $\Phi_2$  is *not* of type  $G_2$ .

- $\langle a - b | c \rangle > 0$  which since  $\langle a | c \rangle = 0$  implies  $\langle b | c \rangle < 0$ . Notice in Figure 5.1 (Observation 5.0.1-2) that given  $b \geq c$ , this cannot happen.
- $\langle a - b | c \rangle < 0$ . In that case, it is easy to check that in all possible configurations for  $\Phi_2$  where two positive roots form an obtuse angle, their sum is still a positive root.
- $\langle a - b | c \rangle = 0$ . This means that  $a - b, a$  and  $b$  are of the same length, because they are of the same length as  $c$ . Thus,  $\Phi_3 = \{\pm a, \pm b, \pm(a - b)\}$  is a root subsystem of type  $A_2$  and in particular  $\langle a | b \rangle = \|a\|^2/2$ . Since  $\Phi$  is stable by the linear reflection orthogonal to the roots, we have:

$$(b - c) - 2 \frac{\langle b - c | a \rangle}{\|a\|^2} a = (b - c) - 2 \frac{\langle b | a \rangle}{\|a\|^2} a = b - c - a = -(a - (b - c)).$$

We conclude by recalling that  $\Phi$  is also stable by  $x \mapsto -x$ .  $\square$

We will use several times the following consequence of Proposition 5.1.3: if we have  $b \in \text{Sk}(a), c \in \text{Sk}(b)$ , then  $d = b - c$  and  $e = a - b$  are positive roots by definition of the skirt. From Proposition 5.1.3, either  $c$  or  $d$  are in  $\text{Sk}(a)$ . Suppose, for instance, that  $d \in \text{Sk}(a)$ . Then  $f = a - d$  is a positive root. We have  $a = e + (c + d) = f + d$ , so  $f = e + c$ . We illustrate this fact in Figure 5.2. Roughly speaking, if we add all the skirt complements to  $a, b$  and  $c$ , and consider the subposet of the root poset they form, we get a height two pyramid.

**Proposition 5.1.4.** *Let  $a$  be a positive root and  $b, c \in \text{Sk}(a)$ . Denote  $a - b = d$ ,  $a - c = e$  and suppose that  $\{b, d\} \neq \{c, e\}$ . Then  $b \in \text{Sk}(c)$  or  $c \in \text{Sk}(b)$  or  $d \in \text{Sk}(c)$  or  $c \in \text{Sk}(d)$ .*

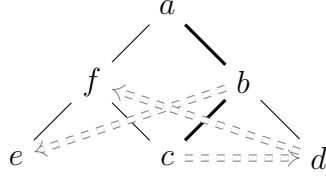


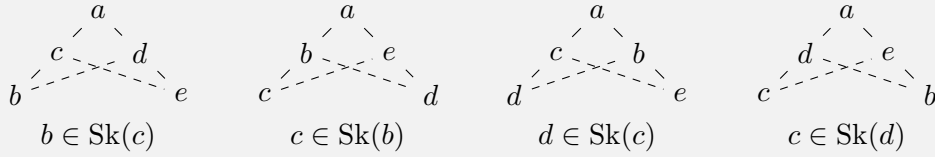
Figure 5.2: “A skirt of a skirt gives a pyramid”:  $c \in \text{Sk}(b)$  implies that  $d$  is a positive root,  $b \in \text{Sk}(a)$  implies that  $e$  is a positive root, and later, from Proposition 5.1.3 if for instance  $d \in \text{Sk}(a)$  then  $f \in \text{Sk}(a)$  and  $f = e + c$ .

*Proof.* We reason by contradiction. Suppose that  $b \notin \text{Sk}(c)$  and  $c \notin \text{Sk}(b)$  and  $d \notin \text{Sk}(c)$  and  $c \notin \text{Sk}(d)$ . Note that because  $a = b + d = c + e$ ,  $b \in \text{Sk}(c)$  is equivalent to  $c - b = d - e \in \Phi^+$ , that is, to  $e \in \text{Sk}(d)$ . So by our supposition, we also have  $e \notin \text{Sk}(d)$  and  $d \notin \text{Sk}(e)$  and  $e \notin \text{Sk}(b)$  and  $b \notin \text{Sk}(e)$ . By definition of the skirt,  $b \notin \text{Sk}(c)$  and  $c \notin \text{Sk}(b)$  means that  $b - c, c - b \notin \Phi^+$ , that is,  $b - c \notin \Phi$ . By Observation 5.0.1-1,  $\langle b | c \rangle \leq 0$ . Similarly, the dot products  $\langle d | c \rangle, \langle b | e \rangle, \langle d | e \rangle$  are all negative. We have:

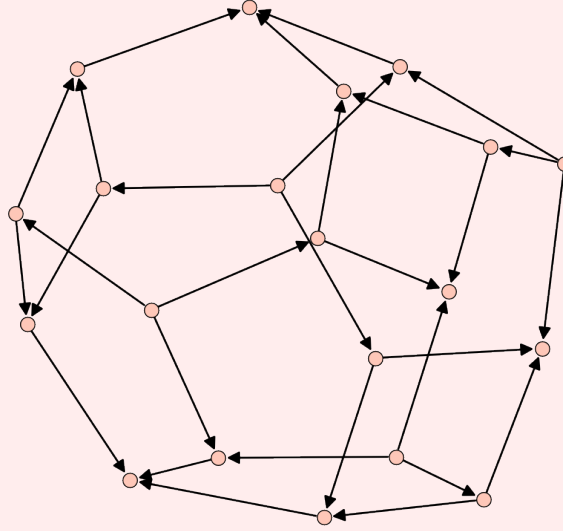
$$0 < \|a\|^2 = \langle b + d | b + d \rangle = \langle b + d | c + e \rangle = \langle b | c \rangle + \langle b | e \rangle + \langle d | c \rangle + \langle d | e \rangle \leq 0.$$

This is a contradiction.  $\square$

**In type  $A_n$  :** Again, this is simply the translation of an obvious type  $A$  fact: if  $a = \rho_{i,l}, b = \rho_{i,j}, c = \rho_{i,k}, d = \rho_{j,l}$  and  $e = \rho_{k,l}$  such that  $a = b + d = c + e$ , then  $b \in \text{Sk}(c)$  and  $e \in \text{Sk}(d)$ . The 4 ways alternative in the proposition simply comes from the fact that, in a Weyl group, there is not, as far as we know, an obvious way to define left and right “sides” of the skirt, and we find ourselves having to check 4 cases as pictured.



**Question 5.1.5.** Consider the set  $\{(\alpha, \beta, \gamma) \in \Phi^+ \mid \alpha + \beta = \gamma\}$  ordered by the transitive reflexive closure of  $(\alpha_1, \beta_1, \gamma_1) \leq (\alpha_2, \beta_2, \gamma_2)$  if and only if  $\gamma_1 \in \{\alpha_2, \beta_2\}$ . The Hasse diagram of this order in type  $A_n$  seems to be the 1-skeleton of some  $(n - 2)$ -dimensional polytope. Is it? For  $B_3$  and  $D_4$ , it already is not a polytope. Can this still be described in terms of quotient of the type  $A_n$  case? What about exceptional groups?

Figure 5.3: The Hasse diagram on the Shi relations for  $A_5$ .

## 5.2 Describing the minimal elements

The goal of this section is to describe the minimal elements of the Shi regions in affine Weyl groups. We first describe the *Shi encoding* of an element of the affine group, before getting to the Shi arrangement itself. We then present the bijective results from [AL99] (in type  $A_n$ ) and [ARR15] (type-free) which we finally use for our description of the minimal elements.

### 5.2.1 The Shi encoding

Consider the collection of affine half-spaces associated to the affine roots as in Section 4.6:

$$H_{\alpha,k}^+ = \{x \in V \mid \langle \alpha \mid x \rangle > k\}, \quad H_{\alpha,k}^- = \{x \in V \mid \langle \alpha \mid x \rangle < k\}$$

for  $\alpha \in \Phi^+$  and  $k \in \mathbb{Z}$ .

We have already introduced the notion of Shi encoding to express in what sense Proposition 4.7.15 affirms the existence of a minimal element in each affine Shi region. Let us recall it here. Given  $w \in \widetilde{W}$  and its corresponding alcove  $w\mathcal{C}$ , define:

$$K(w) = (k(w, \alpha))_{\alpha \in \Phi^+} \text{ where } k(w, \alpha) = \max\{i \in \mathbb{Z} \mid w\mathcal{C} \subset H_{\alpha,i}^+\}.$$

We are interested in the elements of  $\mathbb{Z}^{\Phi^+}$  of the form  $K(w)$ . The following Theorem from [Shi99] shows that these elements are exactly the Shi vectors as in Definition 5.1.1.

**Theorem 5.2.1.** *[Shi99, Theorem 1.1] Consider  $v \in \mathbb{Z}^{\Phi^+}$ . The map  $w \mapsto K(w)$  is a bijection between  $\widetilde{W}$  and the set of Shi vectors.*

Theorem 5.2.1 already tells us that the Shi encoding is a bijection between Shi vectors and elements of the Affine group  $\widetilde{W}$ . Nonetheless, let us recall a reason why the Shi encoding is an injection: the Shi encoding  $K(w)$  *compresses the information* of the inversion set  $N(w)$ .

We have a description of the inversion set of  $w$  in terms of  $K(w)$ . Indeed, recall (Proposition 4.6.2) that a root  $\rho \in \Phi^+$  is in the inversion set of  $w$  if and only if  $w\mathcal{C}$  and  $\mathcal{C}$  lie on opposite sides of the hyperplane  $H_\rho$ . The absolute value of  $k(w, \alpha)$  counts the number of hyperplanes separating  $w\mathcal{C}$  from  $\mathcal{C}$  in the direction of  $\alpha$  while the sign indicates whether the hyperplanes should be counted in the positive or negative direction. More formally put, the description of  $N(w)$  from  $K(w)$  is as follows.

**Proposition 5.2.2.** *Let  $w \in W_a$  and  $v$  the Shi vector of  $w$ . Then:*

$$N(w) = \bigsqcup_{a \in \Phi^+, v_a < 0} \{a + k\delta \mid k \in \llbracket 0, -v_a + 1 \rrbracket\} \sqcup \bigsqcup_{a \in \Phi^+, v_a > 0} \{-a + k\delta \mid k \in \llbracket 1, v_a \rrbracket\}.$$

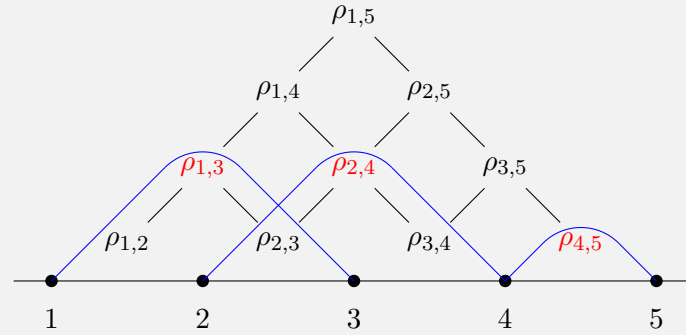
Another property of the inversions set that we would like to recall is that it is *separable*. This fact is not strictly necessary for our purposes, but will be useful in understanding the spirit of the proofs in Section 6.2.

### 5.2.2 The parking function bijection

One of the first bijections between parking functions and Shi (in type  $A_n$ ) regions is given by Pak & Stanley in [Sta96, Section 5]. Shortly after, Athanasiadis & Linusson in [AL99] gave a related bijection (still in type  $A_n$ ). This result was later generalized by Armstrong, Reiner & Rhoades [ARR15] to all Weyl groups. In this section, we present that result and its implications in terms of sign types. We begin by recalling the Athanasiadis-Linusson bijection before stating and proving the Armstrong-Reiner-Rhoades result. Before all of this however, we need a type-free notion of non-nesting partition given by Postnikov [Rei97, Remark 2].

**Definition 5.2.3.** A *non-nesting partition* of type  $W$  is an antichain of the type  $W$  root poset.

**In type  $A_n$ :** This corresponds to the usual definition of a non-nesting partition in type  $A_n$ , where a non-nesting partition is defined as a partition  $P = \{P_1, \dots, P_k\}$  of  $\llbracket 1, n+1 \rrbracket$  such that if  $a, d \in P_i$  and  $b, c \in P_j$  with  $a < b < c < d$ , then  $i = j$ . This is equivalent to the following formulation: we place the numbers 1 through  $n+1$  on an axis and for every  $P_i \in P$  for every pair  $a < b$  of “consecutive elements” of  $P_i$  (in the sense that  $a < c < b \Rightarrow c \notin P_i$ ), we draw an arc above the axis between  $a$  and  $b$ . Then  $P$  is non-nesting if and only if no two arcs nest in each other. We give below an example to illustrate this definition of non-nesting corresponds to that of Definition 5.2.3 for the partition  $\{\{1, 3\}, \{2, 4, 5\}\}$  with the corresponding antichain in red and the corresponding arcs in blue.



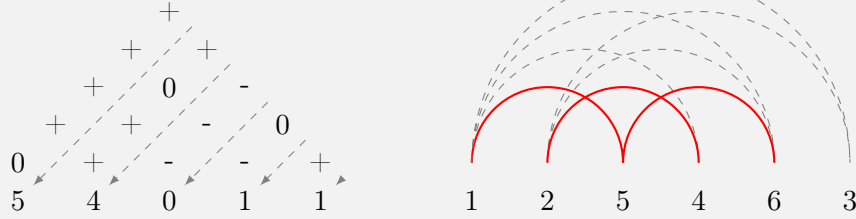
Let us now understand the result in type  $A_n$ .

**Theorem 5.2.4** (Athanasiadis & Linusson). *Let  $R$  be a type  $A_n$  Shi region with sign type  $s$ . Consider the following procedure :*

- Define the vector  $I$  by  $I_i = \#\{j \in \llbracket i+1, n+1 \rrbracket \mid s_{i,j} \in \{0, +\}\}$  for each  $i \in \llbracket 1, n \rrbracket$ .
- Define  $\pi$  to be the unique permutation in  $\mathfrak{S}_{n+1}$  such that for every  $i \in \llbracket 1, n+1 \rrbracket$ , the number of values greater than 1 appearing to the right of  $i$  is  $I_i$ .
- For every pair  $i, j, 1 \leq i < j \leq n+1$  such that  $s_{i,j} = +$ , draw an arc between  $i$  and  $j$ .
- Remove any arc that contains another.

This procedure gives a bijection between pairs  $(\pi, P)$  where  $\pi \in A_n$  and  $P$  is a non-nesting partition with sorted blocks on one hand and type  $A_n$  Shi regions on the other hand.

**In type  $A_5$ ,** We work out below an example of the procedure.



The Athanasiadis-Linusson procedure clearly gives a parking function in the sense of Definition 5.2.6: the fact that the roots in the non-nesting partition are not inversion of the permutation corresponds to the fact that values increase along arcs.

**Remark.** The mouthful “ $(\pi, P)$  where  $\pi \in A_n$  and  $P$  is a non-nesting partition with sorted blocks” characterization of the image of this procedure is chosen for coherence with the definition used in other types. However, in [AL99], the authors go further and show how this corresponds to the usual parking functions (*i.e.*, the elements of  $\{p \in \llbracket 1, n+1 \rrbracket^{\llbracket 1, n+1 \rrbracket} \mid |p^{-1}(\llbracket 1, i \rrbracket)| \geq i\}$ ). From  $(\pi, P)$ , we can define a parking function  $p$  in the following way: given  $P = (P_1, \dots, P_k)$  (ordered by minimal element) an element  $i \in P_j$  is sent to  $p(i) = \pi^{-1}(\min(P_j))$ . We invite the reader to consult [AL99] for details, but loosely, this must be a parking function because for every  $i$  the union of the blocks with minimal element lesser than  $i$  has at least cardinality  $i$ . It is a bijection because it is reversible: given a parking function  $p$ , the blocks of the partition are recovered as the fibers  $P = \{p^{-1}\{i\} \mid i \in \llbracket 1, n+1 \rrbracket\}$ . The permutation  $\pi$  is recovered from the blocks: take the block  $P_1$  containing 1, write it in order with 1 in position 1, and draw the arcs between its consecutive elements. Then take  $P_2$  containing the smallest element not in  $P_1$  and write it in order, with its arcs, such that no nesting happens, putting  $\min(P_2)$  in position  $p(P_2)$ . Note that in the “in-construction” permutation, the position  $p(P_2)$  is unambiguously defined as  $p$  is a parking function and so at least  $|p^{-1}(\llbracket 1, \min(P_2) - 1 \rrbracket)| \geq \min(P_2) - 1$  values have been inserted. Continue this way for the rest of the blocks of  $P$ .

The philosophy of Athanasiadis & Linusson’s bijection is to encode a Shi region by two things: an element  $w \in W$  and the set of its *floors*, in the following sense.

**Definition 5.2.5.** A hyperplane  $H_{\alpha,1}$  is a *floor* of a Shi region  $R$  if its intersection with  $\overline{R}$  is a facet of  $\overline{R}$  and  $R$  and  $\mathcal{C}$  lie on different sides of  $H_{\alpha,1}$ . Equivalently,  $H_{\alpha,1}$  is a floor of  $R$ , if the inequality  $\langle x | \alpha \rangle$  for all  $x \in R$  is irredundant.

The element  $w$  allows to situate  $R$  with respect to the hyperplanes  $H_{\alpha,0}$ , while the floors give the missing information about the (relevant)  $H_{\alpha,1}$ . The fact that these floors form a non-nesting partition is not a “type  $A_n$  miracle” and the objective of this paragraph is to state a result from Armstrong, Reiner and Rhoades [ARR15] that generalizes this labeling of Shi regions in a type-free fashion. With the Definition 5.2.3 of non-nesting partition, we are finally ready to give a low-tech version of the definition of parking function from [ARR15].

**Definition 5.2.6.** A parking function of type  $W$  is a pair  $(w, A)$  with  $w \in W$  and  $A$  a non nesting partition of type  $W$  such that  $\forall a \in A, a \notin N(w)$ .

The type-free equivalent of the Athanasiadis-Linusson result is as follows:

**Theorem 5.2.7** (Armstrong, Reiner, Rhoades). *Let  $R$  be a region of the Shi arrangement of type  $W$  and  $w$  be the unique element of  $W$  such that  $R \subset wC$ . Then  $w^{-1}fl(R)$  is an antichain of the root poset. Moreover, the map  $\mu_{ARR} : R \mapsto (w, w^{-1}fl(R))$  is a bijection between Shi regions and parking functions.*

This proposition appears as [ARR15, Proposition 10.4], although in a slightly different form. Firstly, we borrowed the authors’ definition of (non-nesting) parking functions in a somewhat stripped down form more suitable for our needs: where we use an element in  $w$ , they use an entire coset of which our  $w$  is a representative, privileged by the fact that it does not invert any element of the antichain. Secondly, the original proof is done for the *ceilings* of a region (with the same definition as floors except for the fact that  $R$  and  $C$  lie on the same side of  $H_{\alpha,1}$ ) (In that sense, this makes the result from Armstrong, Reiner and Rhoades more closely resemble the Pak-Stanley bijection from [Sta96, Section 5]). We encourage the interested reader to consult the original work, keeping in mind these two changes. Still, for convenience, we translate below the proof from [ARR15] in the language we have adopted for this article. More modestly, we begin by proving that this map is injective. The surjectivity will come from a later result in this paper. We choose to separate this proof in two parts to avoid a cardinality argument that would perhaps necessitate further translation of results from [ARR15] (specially Proposition 2.11, in the case of the cardinality argument).

*Proof of injectivity.* Firstly, the map is well-defined: we have to check that  $w^{-1}fl(R) \subset \Phi^+$  and that it is an antichain. The first point is clear: since  $w$  is orthogonal as a linear map:

$$wH_{\alpha,1} = \{w(x) \mid \langle x \mid \alpha \rangle = 1\} = \{w(x) \mid \langle w(x) \mid w(\alpha) \rangle = 1\} = H_{w(\alpha),1},$$

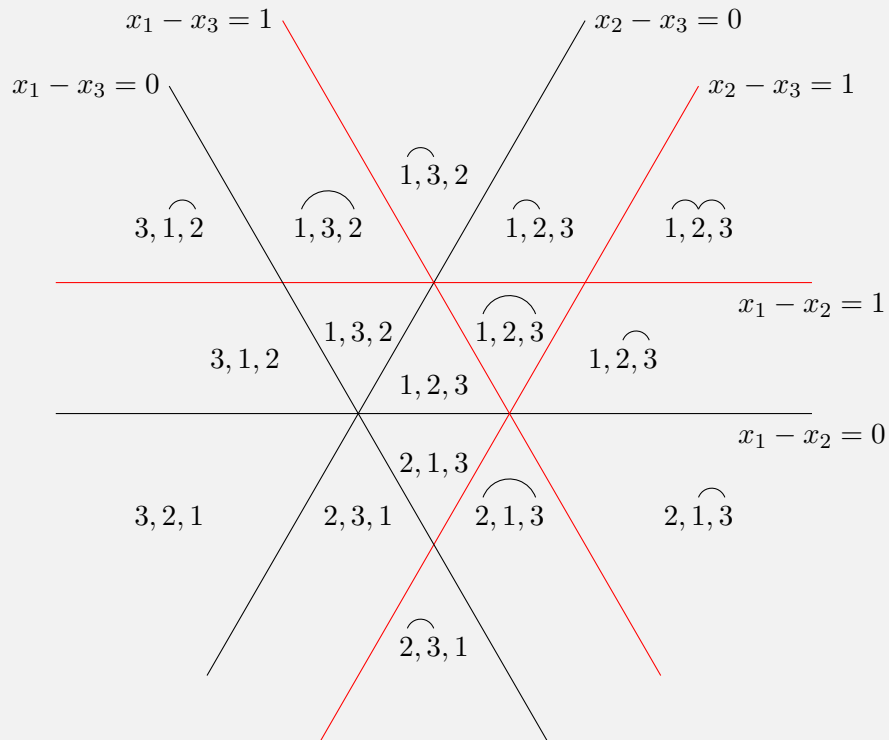
the hyperplane  $H_{a,1}$  of the Shi arrangement intersects  $wC$  if and only if  $a = w(b)$  for some  $b \in \Phi^+$ , so  $w^{-1}fl(R) \in \Phi^+$ . We now prove that  $w^{-1}fl(R)$

is an antichain. Suppose that  $a = w^{-1}(\alpha), b = w^{-1}(\beta)$ , with  $\alpha, \beta \in fl(R)$  and  $a < b$  in the root poset, that is  $b = a + c$  with  $c \in \mathbb{N}\Delta$  non-zero. By definition of  $\mathcal{C}$ , for any  $x$  in  $\mathcal{C}$  and  $\rho \in \Delta$ ,  $\langle x | \rho \rangle > 0$ . Thus, for all  $x \in \mathcal{C}$ ,  $\langle x | b \rangle = \langle x | a + c \rangle > \langle x | a \rangle$ . Again, because  $w$  is orthogonal, it follows that for all  $x \in w\mathcal{C}$ ,  $\langle x | \beta \rangle > \langle x | \alpha \rangle$ : if  $\alpha$  is a floor of  $R$ , in particular  $\langle x | \alpha \rangle > 1$  implies  $\langle x | \beta \rangle > 1$ , meaning that this inequality is redundant and that  $\beta$  is not a floor. This proves that  $w^{-1}fl(R)$  is an antichain.

Secondly,  $\mu_{ARR}$  is injective. This is easy to see, as  $(w, A) = \mu_{ARR}(R)$  records the position of  $R$  with respect to all the hyperplanes in the Shi arrangement. More precisely,  $w$  records the position of  $R$  with respect to the  $H_{\alpha,0}$  hyperplanes, while  $A$  gives the floors of  $R$  (and further the position relative to the  $H_{\alpha,1}$ :  $R$  is in  $H_{\alpha,1}^+$  if and only if  $w^{-1}(\alpha) \geq \beta$  in the root poset for some  $\alpha \in A$ ) which together with  $w$  determines entirely  $R$ .  $\square$

Another way to think of the proof that  $A$  is an antichain in Theorem 5.2.7, is that we showed that the set  $\{w_i^{-1}(r) \mid \text{sign}(v_r) = +\}$  is an upper ideal of the root poset and  $A$  is simply the set of its minimal elements.

**In type  $A_2$** , the bijection is represented in the following picture.



The determination of the sign type of a Shi region in terms of its parking function can be made explicit.



**Proposition 5.2.8.** *Consider a region  $R$  labeled by a parking function  $(w, A)$ . Then the sign type  $s$  of  $R$  is, for any root  $r$ :*

$$s_r = \begin{cases} - & \text{if } r \in N(w) \\ 0 & \text{if } r \notin N(w) \text{ and } \forall a \in A, w^{-1}(r) \not\geq a \\ + & \text{if } r \notin N(w) \text{ and } \exists a \in A, w^{-1}(r) \geq a \end{cases}$$

*Conversely, if the sign type of  $R$  is  $s$ , then  $R$  is labeled by the parking function  $(w, A)$  where  $w$  is the unique element of  $W$  with  $r \in N(w) \Leftrightarrow s_r = -$  and  $A$  is the set of minimal elements of  $\{w^{-1}(r) \in \Phi^+ \mid s_r = +\}$ .*

*Proof.* This is the same proof as that of Theorem 5.2.7 (specifically the last paragraph), encoding the inclusion in the half spaces in the signs.  $\square$

### 5.2.3 Minimal elements of Shi regions

The objective of this section is to give a description of the minimal element of a Shi region  $R$  (or rather, of the Shi vector of its minimal element) in terms of the parking functions of  $R$  as given by Proposition 5.2.7. We proceed in two steps: we deal with the case of the Shi regions contained in the fundamental chamber (that is, the chambers associated to the identity of the finite group, also-called *dominant region* in the literature) in the Propositions 5.2.9 and 5.2.11. Then, in the Propositions 5.2.12 and 5.2.13, we transport the result to Shi regions contained in other chambers.

**Proposition 5.2.9.** *Let  $A$  be an antichain of the root poset. Then the following vector is a Shi vector:*

$$\forall a \in \Phi^+, v_a = \max(\{\mathbf{1}_{a \in A}\} \cup \{v_b + v_{a-b} \mid b \in \text{Sk}(a)\}).$$

*Proof.* We prove by induction on the height of  $a$  that all the Shi relation associated to triplets  $(b, a, a - b)$  for  $b \in \text{Sk}(a)$  are verified. If  $a$  has height 1, this is obviously true, as there are no relations to check. If  $a$  has height  $k > 1$ , then consider two relations  $a = b + d = c + e$  for  $b, c, d, e \in \Phi^+$ . From Proposition 5.1.4, we may suppose without loss of generality that  $c \in \text{Sk}(b)$  which is equivalent to  $d \in \text{Sk}(e)$ . Let us denote the positive root  $b - c = e - d$  by  $f$ . By induction, we have  $v_b = v_c + v_f + \varepsilon_{c,f}$  and  $v_e = v_d + v_f + \varepsilon_{d,f}$  with  $\varepsilon_{c,f}$  and  $\varepsilon_{d,f}$  in  $\{0, 1\}$ . Thus :

$$(v_b - \varepsilon_{c,f}) + v_d = v_c + v_f + v_e - v_f - \varepsilon_{d,f} = v_c + (v_e - \varepsilon_{d,f})$$

so  $v_b + v_d$  and  $v_c + v_e$  cannot differ by more than 1. Let  $b \in \text{Sk}(a)$  such that  $M = v_b + v_{a-b}$  is maximized. By comparing all relations to the one associated to  $a = b + (a - b)$ , we get that  $\{v_c + v_{a-c} \mid c \in \text{Sk}(a)\} \subset \{M, M - 1\}$ .

We now have two possibilities: either  $v_a = M$  and thus all Shi relations with top  $a$  are verified with  $\varepsilon_{c,a-c} = M - (v_c + v_{a-c})$  for any  $c \in \text{Sk}(a)$ , or  $v_a = \mathbb{1}_{a \in A}$ , we have  $\{v_b + v_{a-b} \mid b \in \text{Sk}(a)\} \subset \{\mathbb{1}_{a \in A}, \mathbb{1}_{a \in A} - 1\}$  and again taking  $\varepsilon_{b,a-b} = \mathbb{1}_{a \in A} - (v_b + v_{a-b})$  yields an  $\varepsilon_{b,a-b} \in \{0, 1\}$ .  $\square$

The method of proof is taken from [ACLR21, Lemma 24] where this proof appears in type  $A_n$ . In other types, it is not obvious that two Shi relations with the same top are “comparable” in the sense needed for the central algebraic manipulation of the proof. However, it turns out that the proof from [ACLR21, Lemma 24] still works with the help of Proposition 5.1.3.

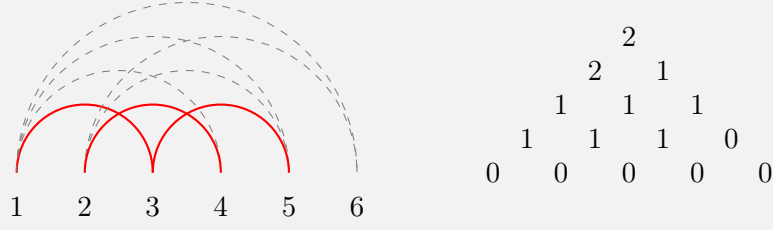
**Question 5.2.10.** The expression from Proposition 5.2.9 very much looks like a tropical polynomial. Is there a tropical interpretation of the minimal elements of Shi regions?

**Proposition 5.2.11.** *Let  $R$  be the Shi region labeled by the parking function  $(e, A)$ . Then the Shi vector of the minimal element of  $R$  is given by Proposition 5.2.9.*

*Proof.* Firstly, keeping the notation of Proposition 5.2.9,  $v$  is a Shi vector. Secondly,  $v$  has the sign type  $s$  encoded by  $(e, A)$ : a trivial induction shows that for a positive root  $r$ ,  $v_r > 0 \Leftrightarrow \exists a \in A, r \geq a$ , which proves that  $s_r = + \Leftrightarrow v_r > 0$ . Secondly,  $v_r$  is the minimal vector of that sign type. We show this by proving by induction on the height of a root  $a$  that the value of  $a$  is minimal given the sign type and the Shi relations. If  $a$  is of height 1,  $v_a = \mathbb{1}_{a \in A} \in \{0, 1\}$ : both cases minimize the value of  $v_a$  under the constraint of the sign type. If  $a$  is of height  $k > 1$ , either  $v_a = \mathbb{1}_{a \in A}$  and the argument is the same, or  $v_a = \max\{v_b + v_{a-b} \mid b \in \text{Sk}(a)\}$ , so there exists  $b \in \text{Sk}(a)$  such that  $v_a = v_b + v_{a-b}$  and thus  $v_a$  is minimal given the Shi relations.  $\square$

We now have a complete description of the Shi vectors for the region contained in the fundamental chamber, and they are extremely simple to compute: said loosely, we can get the minimal element by doing the minimum.

**In type  $A_5$ ,** if we forget the permutation in our previous example, we can compute the minimal element of the region  $(e, \{e_1 - e_3, e_2 - e_4, e_3 - e_5\})$



More generally it is easy to prove that in type  $A_n$ , using the Athanasiadis-Linusson bijection, Proposition 5.2.9 can be translated to “ $v_{e_i-e_j}$  is equal to the maximum number of non-crossing arcs in the non nesting partition between position  $i$  and  $j$ ”.

We now want to transport this result to the other chambers.

**Proposition 5.2.12.** *Let  $R$  be a Shi region labeled by the parking function  $(w, A)$  and  $v$  be a Shi vector of the region labeled by  $(e, A)$ . Then the following vector is a Shi vector of an alcove in  $R^3$ :*

$$\forall a \in \Phi^+, v'_{w(a)} = \begin{cases} v_a & \text{if } w(a) \in \Phi^+ \\ -(v_a + 1) & \text{if } w(a) \in \Phi^- \end{cases}.$$

*Proof.* First, comparing with Proposition 5.2.12, it is clear that  $v'$  does lie in  $R$ . If  $a$  is a root, we pose  $|a| = a$  if  $a \in \Phi^+$  and  $|a| = -a$  otherwise. Consider a relation  $a = b + c$  and denote  $|w(a)|, |w(b)|, |w(c)|$  by  $\alpha, \beta, \gamma$ , respectively. Because  $w$  is linear, the relation  $a = b + c$  is sent to  $w(a) = w(b) + w(c)$ . Depending on whether  $a, b$  and  $c$  lie in  $N(w)$  or not, we can rearrange this to obtain a relation between the positive roots  $\alpha, \beta, \gamma$ , as specified in Table 5.1. We define  $s_a$  (and similarly,  $s_b, s_c$ ), by  $s_a = +$  if  $w(a) \in \Phi^+$  and  $s_a = -$  otherwise.

$(s_b, s_a, s_c)$	Root relation	Shi relation						$\varepsilon'$
$(+, +, +)$	$\alpha = \beta + \gamma$	$v_a$	=	$v_b$	+	$v_c$	+	$\varepsilon$
$(-, -, -)$	$\alpha = \beta + \gamma$	$-(v_a + 1)$	=	$-(v_b + 1)$	+	$-(v_c + 1)$	+	$1 - \varepsilon$
$(+, +, -)$	$\beta = \alpha + \gamma$	$v_b$	=	$v_a$	+	$-(v_c + 1)$	+	$1 - \varepsilon$
$(-, -, +)$	$\beta = \alpha + \gamma$	$-(v_b + 1)$	=	$-(v_a + 1)$	+	$v_c$	+	$\varepsilon$
$(+, -, -)$	$\gamma = \alpha + \beta$	$-(v_c + 1)$	=	$-(v_a + 1)$	+	$v_b$	+	$\varepsilon$
$(-, +, +)$	$\gamma = \alpha + \beta$	$v_c$	=	$v_a$	+	$-(v_c + 1)$	+	$1 - \varepsilon$

Table 5.1: The map  $v \mapsto v'$  preserves the Shi relations.

Since the sign 0 is a special case of  $+$  in the sense that it indicates a non inversion, there are only 6 cases to check, which we do in Table 5.1. In each

<sup>3</sup>Recall that we took the convention that  $v_{w(a)} = v_{-w(a)}$  in the case where  $a \in N(w)$ .

☐

**Proposition 5.2.13.** *Let  $R$  be a Shi region labeled by the parking function  $(w, A)$ , and let  $v$  be the Shi vector of the minimal element of the  $(e, A)$  region given by Proposition 5.2.11. Then,  $v'$  as described in Proposition 5.2.12 is the Shi vector of the minimal element of  $R$ .*

*Proof.* Firstly, from Proposition 5.2.12  $v'$  is a Shi vector of the region  $R$ . Secondly,  $v'$  is minimal. We show by induction on the height of a root  $a$  that  $|v'_{w(a)}|$  is minimal given the sign requirement and the Shi relations. Let  $a$  be a root of height 1. Since  $\text{Sk}(a) = \emptyset$ ,  $v_a = \mathbb{1}_{a \in A}$ . If  $a \in A$ ,  $a \notin N(w)$  and  $v'_{w_a} = 1$  which is the minimal possible value satisfying the sign requirement. If  $a \notin A$ ,  $v_a = 0$  so  $v'_{w(a)} \in \{-1, 0\}$ : in both case, it is the minimal absolute value with the appropriate sign. Now let  $a$  be a root of height  $k > 1$ . If  $a \in A$ , again  $v'_{w(a)} = 1$  and the absolute value is minimal. If  $a \notin A$ , from Proposition 5.2.11, there exists  $b, c \in \Phi^+$  such that  $v_a = v_b + v_c$ . Consider the last column of Table 5.1: this corresponds to setting  $\varepsilon = 0$ . By induction, we can suppose that  $|v'_{w(b)}|$  and  $|v'_{w(c)}|$  are minimal. We can conclude by noticing then that the value taken by  $\varepsilon'$  always minimizes the absolute value of  $v'_{w(a)}$ .  $\square$

$$\begin{array}{ccccc} v & & & 2 & \\ & & & 2 & 1 \\ & & 1 & 1 & 1 \\ & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{ccccc} v' & & & 2 & \\ & & & 1 & 1 \\ & & 1 & 0 & -1 \\ & 2 & 1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 1 \end{array}$$

For instance, since  $(3, 6)$  is sent by the permutation to  $(5, 3)$ , in position  $e_3 - e_5$ , we have  $-2$  because there was a  $1$  in position  $(3, 6)$  in  $v$ . Similarly,

$(1, 6)$  is sent to  $(1, 3)$ , so the 2 in position  $(1, 6)$  in  $v$  is “transported” to position  $(1, 3)$  in  $v'$ .

As before, it is easy to prove that in type  $A_n$ , Proposition 5.2.12 can be translated to “ $v'_{e_i - e_j}$  is the maximal number of non-crossing arcs in the non-nesting partition between positions  $w^{-1}(i)$  and  $w^{-1}(j)$  if  $w^{-1}(i) < w^{-1}(j)$  and the opposite of that number minus 1 if  $w^{-1}(i) > w^{-1}(j)$ ”.

As a corollary of this result and of Theorem 5.2.1, we can now get to the proof of surjectivity in 5.2.7 that we had postponed.

*Theorem 5.2.7, proof of surjectivity.* Note that Propositions 5.2.9 and 5.2.12 can be applied to any parking function (not necessarily one that labels a Shi region) and yield a minimal element of some region  $R$ . This transformation is clearly injective: let  $(w_1, A_1), (w_2, A_2)$  with the same image  $v$ . We first have  $N(w_1) = \{r \mid \text{sign}(v_r) = -\} = N(w_2)$  which implies  $w_1 = w_2$ . Since  $A_i$  is, by construction, the set of minimal elements of upper ideal  $\{w_i^{-1}(r) \mid \text{sign}(v_r) = +\}$  for  $i = 1, 2$ , we also have  $A_1 = A_2$ .

In conclusion, we have an injection of Shi regions in the parking functions by the first part of the proof of Theorem 5.2.7, an injection of the parking functions in the minimal elements and one from minimal elements to Shi regions by Theorem 5.2.1. These sets are finite, so these injections are also surjections.  $\square$

Note that as a consequence of the results of this section, we actually get a new proof of Proposition 4.7.15 as we have just computed that minimal element.

**Question 5.2.14.** Let  $\Phi_0^+ = \Phi^+ \cup \{0\}$  and consider the commutative magma  $(\Phi_0^+, +)$  where for all  $\phi \in \Phi_0^+, \phi + 0 = 0$  and for all  $\alpha, \beta \in \Phi^+, \alpha + \beta = 0$  if  $\alpha + \beta \notin \Phi^+$  and is the regular sum otherwise.

Consider now the fully parenthesized expressions in this magma that evaluate to some  $\gamma \in \Phi^+$ . Since the magma is commutative, we can quotient this set by the application of the commutativity rule. We construct the directed graph with these classes as vertices, and where we put an edge in two cases. Let  $c_1, c_2$  be two classes of expressions. Firstly, there is an edge from  $c_1$  to  $c_2$  if  $c_1$  can be obtained from  $c_2$  by a single application of the associativity rule. Secondly, there is an edge  $c_1$  can be obtained from  $c_2$  by the evaluation of one of the innermost parentheses.

This comes out naturally of repetitively applying Proposition 5.2.9. As partial evaluation is also a natural and independent idea, and because,

in type  $A_n$  at least, this is so closely related to the associahedron, we expect that this graph is already known. What are its properties? Can they be used to obtain a simpler, non-recursive computation of the Shi vectors of minimal elements in any Weyl group?

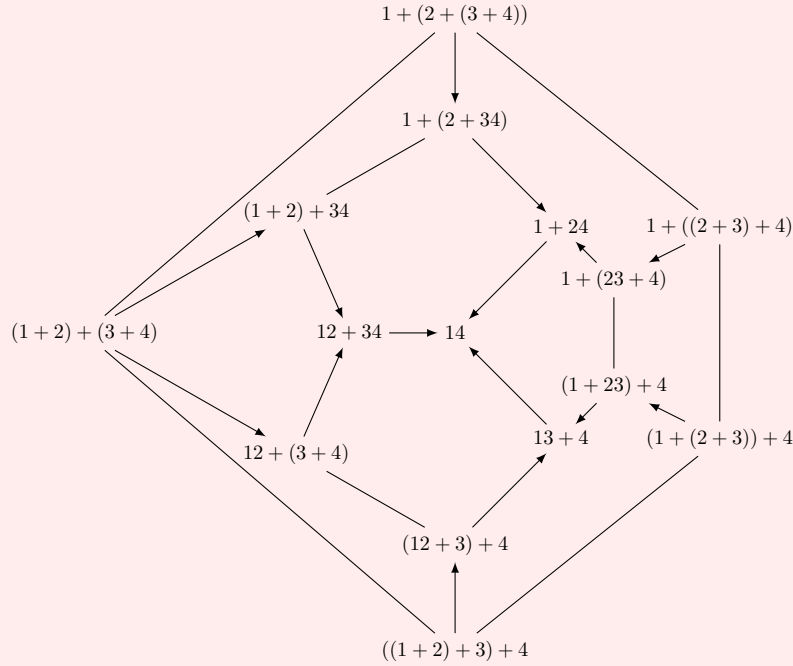
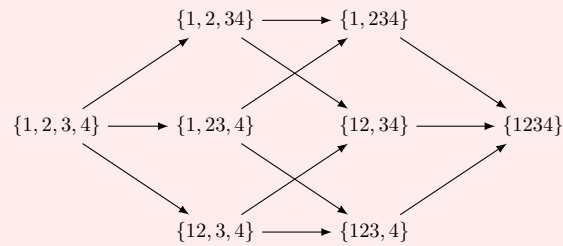


Figure 5.4: Repeated application of Proposition 5.2.9  
For clarity,  $e_i - e_{i+1}$  is represented by  $i$  and  $e_i - e_{j+1}$  by  $ij$ .

As is visible in the picture, it turns out that when we apply Proposition 5.2.9 recursively, we generate, in some sense, too many possible sums. Indeed, the value of the sums depends only on the multiset of roots that appears in it. This inspires a second question.

**Question 5.2.15.** Consider the same set of expressions quotiented by commutativity as before, and identify all the vertices that use the same multiset of roots, removing all loops and multi-edges that could have been created. In the previous case, we obtain the following graph.



In type  $A_n$ , we can easily see that this gives the boolean lattice. In other groups however, the situation is less clear. In general, do we obtain the same graph just by contracting undirected edges? In general, given a root in  $\Phi^+$ , what is the set of vertices of this graph?

### 5.3 Specialization to classical groups

All along this paper, we have given examples in type  $A_n$  of our results as motivation for our methods or explanation for our proofs. Most of the intuition in type  $A_n$  comes from the existence of the easily understandable arc diagram. In turn, they exist because type  $A_n$  groups can be seen as permutation groups and because there is a notion of “pictorially non-nesting” arcs. Of course, type  $A_n$  groups are not the only ones with these properties, and we briefly expose what this translates to in the case of the classical groups. In this section, we will use the classification of the crystallographic root systems (see Proposition 4.5.5). This section is adapted and expanded from the extended abstract [Cha22].

**Definition 5.3.1.** The root systems described in Table 5.2 are called the *classical root systems*.

Type	$\Phi^+$	$\Delta$
$A_n$	$e_i - e_j$ for $i, j \in \llbracket 1, n \rrbracket, i < j$	$e_i - e_{i+1}$ for $i \in \llbracket 1, n \rrbracket$
$B_n$	$e_i \pm e_j$ for $i, j \in \llbracket 1, n \rrbracket, i < j$ , $e_i$ for $i \in \llbracket 1, n \rrbracket$	$e_n, e_i - e_{i+1}$ for $i \in \llbracket 1, n-1 \rrbracket$
$C_n$	$e_i \pm e_j$ for $i, j \in \llbracket 1, n \rrbracket, i < j$ , $2e_i$ for $i \in \llbracket 1, n \rrbracket$	$2e_n, e_i - e_{i+1}$ for $i \in \llbracket 1, n-1 \rrbracket$
$D_n$	$e_i \pm e_j$ for $i, j \in \llbracket 1, n \rrbracket, i < j$	$e_{n-1} + e_n, e_i - e_{i+1}$ for $i \in \llbracket 1, n-1 \rrbracket$

Table 5.2: Classical root systems

It is easy to see that these sets of vectors indeed constitute roots systems of types  $A_n, B_n, C_n, D_n$ , and the interested reader can consult [Hum90, Section 2.10] for more details about their construction.

Before proceeding to extend the non-nesting arc diagrams to other classical types, let us reprove Proposition 5.2.9 in a more pictorial fashion in type  $A_n$ . Recall that in type  $A_n$ , we have “pictorial” notions of a non-nesting partition (if we have  $a < b < c < d$  with  $a, d$  in a block  $B$  of the partition and  $c, b$  in a block  $B'$ , then  $B = B'$ ) and of a non-crossing partition (if we have  $a < b < c < d$  with  $a, c$  in a block  $B$  of the partition and  $b, d$  in a block  $B'$ , then  $B = B'$ ).

**Proposition 5.3.2.** *Let  $(e, A)$  be a parking function presented in diagram form, that is, the identity permutation with an arc between  $i$  and  $j$  for all*

$e_i - e_j \in A$ . Define  $v_{e_i - e_j}$  to be the maximum number of non-crossing arcs that can be chosen in  $A$  between  $i$  and  $j$ . Then  $v \in \mathbb{Z}^{\Phi^+}$  is a Shi vector.

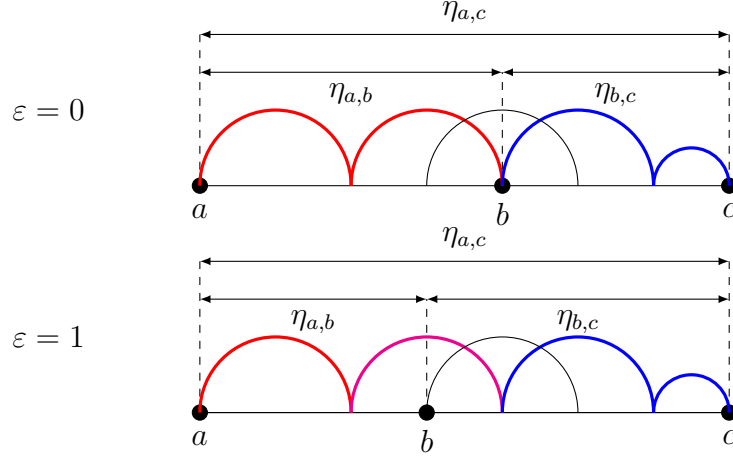


Figure 5.5: Illustration of Proposition 5.3.2: the arcs of  $S_{a,b}$  are represented in bold. Note that we do not need to choose the same set of non-nesting non-crossing arcs to compute  $v_{a,b}$ ,  $v_{b,c}$  and  $v_{a,c}$ .

*Proof.* Let denote  $v_{e_i - e_j}$  by  $v_{i,j}$  for short. For  $i, j \in \llbracket 1, n \rrbracket$ , let  $S_{i,j}$  be a set of non-nesting, non-crossing arcs between  $i$  and  $j$  of maximal cardinality. It is clear that  $v_{a,c} \geq v_{a,b} + v_{b,c}$  since  $S_{a,b} \cup S_{b,c}$  is a set of non-crossing, non-nesting arcs. Conversely, in  $S_{a,c}$ , by definition of  $v$ , at most  $v_{a,b}$  arcs are between  $a$  and  $b$  and similarly for  $b$  and  $c$ . Any arc in  $S_{a,b}$  that is not between  $a$  and  $b$  or  $b$  and  $c$  must therefore be of the form  $(d, e)$  with  $d < b < e$  and two such arcs must be either nesting or crossing. Thus,  $S_{a,b}$  contains at most  $v_{a,b} + v_{b,c} + 1$  elements: the Shi relations are verified,  $v$  is a Shi vector.  $\square$

Figure 5.5 can actually give a good intuition of what could be expected in other types: most importantly, it is easy to predict on the picture that in the dominant region, whenever  $e_i - e_j$  is not in the antichain (that is, there is no arc with extremities  $(i, j)$ ), we can find  $k \in \llbracket i + 1, j - 1 \rrbracket$  such that  $v_{i,j} = v_{i,k} + v_{k,j}$ . Because  $e_i - e_j \notin A$ , it means that there is at least one with an extremity strictly between  $i$  and  $j$ . Let  $k$  be the right extremity of the leftmost arc:  $k$  does the trick! This is a simple visual proof of Proposition 5.2.9 in type  $A_n$ . The same trick (with two non-crossing arcs this time) also gives a visual interpretation to the proof of Proposition 6.2.3.

The proof for why Proposition 5.3.2 allows to compute the minimal elements is exactly the same as in Proposition 5.2.11 (for minimality in the dominant region), Proposition 5.2.12 (for transporting a Shi vector from the



dominant region to another) and 5.2.13 (for sending a minimal element to another<sup>4</sup>). Thus, as a corollary of these four results, and given that for the permutation realization of the type  $A_n$  groups, a root  $e_i - e_j$  is an inversion of an element  $w$  if and only if  $j$  appears before  $i$  in the function notation of  $w$ , we get:

**Corollary 5.3.3.** *Let  $R$  be a Shi region of type  $A_n$  labeled by a parking function  $(w, A)$ . Consider the arc diagram obtained by writing the permutation  $w$  and then drawing an arc between positions  $i$  and  $j$  for all  $e_i - e_j$  in  $A$ . For all  $1 \leq i < j \leq n + 1$ , let  $v_{i,j}$  be the maximal number of non-crossing arcs that can be chosen between the values  $i$  and  $j$ . Then the vector defined by:*

$$v'_{i,j} = \begin{cases} v_{i,j} & \text{if } i \text{ appears before } j \text{ in } w \\ -(v_{i,j} + 1) & \text{if } j \text{ appears before } i \text{ in } w \end{cases}$$

*is the Shi encoding of the minimal element of  $R$ .*

To get the pictures we are after in other classical groups, we just need an equivalent of Proposition 5.3.2 in types  $B_n, C_n, D_n$ . Thus, we need to check mainly two things:

1. In type  $A_n$ , we used the fact that a relation  $\alpha + \beta = \gamma$  corresponds to a triple of integers in  $\llbracket 1, n \rrbracket$  to which we can apply Proposition 5.3.2. This implicitly used the realization of  $A_n$  as a permutation group. As  $B_n, C_n, D_n$  can also be realized as permutation groups of  $\llbracket -n, n \rrbracket$ , this extends to all classical groups, using the correspondence between roots and pairs of integers given in Table 5.3.

Root	$e_i - e_j$	$e_i + e_j$	$2e_i$	$e_i$
Extremities	$i$ to $j$ and $-j$ to $-i$	$i$ to $-j$ and $j$ to $-i$	$i$ to $-i$	$i$ to 0 and 0 to $-i$

Table 5.3: Roots and corresponding arcs in classical types.

It is easy to check that all the Shi relations between roots on which Theorem 5.2.1 applies correspond to suitable integer triples such that sums of roots correspond to “telescoping identities” in the encoding above. For instance, in type  $C_n$ , the Shi relation  $e_i - e_j + e_i + e_j = 2e_i$  can correspond to two triples  $(i, j, -i)$  ( $e_i - e_j$  is encoded by  $(i, j)$ ,  $e_i + e_j$  by  $(j, -i)$  and  $2e_i$  by  $(i, -i)$  with the two  $j$  corresponding to a cancellation in the sum) or  $(-i, -j, i)$  (similarly with the opposite signs): because the encodings of any two roots in the relation share a point, we can apply the argument of Proposition 5.3.2.

<sup>4</sup>Notably, in Proposition 5.2.13, we proceed by induction on the height of a root  $|w^{-1}(a)|$  which translates nicely in the permutation group setting as the difference in position of two values.

2. Proposition 5.3.2 applies only for the “pictorial” notion on non-nesting arcs, meaning for two arcs  $[a, b], [c, d]$  with  $a < b, c < d$  and  $a < c$  we don’t have  $a < c < d < b$ . Thus, to use Theorem 5.2.7, we need to get a “pictorial” representation of the “antichain” definition of non-nesting. We discuss this point below. In particular, this requires a slightly different proof of Proposition 5.3.2 in the case  $D_n$ .

**Type  $B_n, C_n$ :** The groups  $B_n$  and  $C_n$  are isomorphic to the permutation group:

$$\{w \in \mathfrak{S}_{[-n, n]} \mid \forall i \in [1, n], w(i) = -\pi(-i)\}.$$

Note that although  $B_n$  and  $C_n$  are isomorphic as groups, they require a slightly different treatment as their root systems and thus their Shi arrangements are not the same. We say by convention that the interval  $[-n, n]$  written in the following order is sorted.

$$1 \ 2 \ \dots \ n \ 0 \ -n \ \dots \ -2 \ -1$$

Notice that the encoding given in Table 5.3 puts in correspondence a root with one or two arcs: the length of each arc (omitting 0 in type  $C_n$ ) corresponds to the height of the root (see Figure 5.6 below for an example of type  $B_n$  and  $C_n$  root posets).

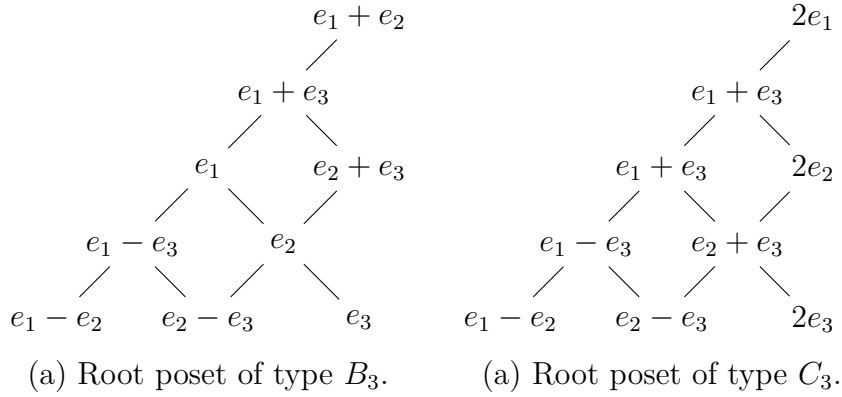


Figure 5.6: Examples of root posets of type  $B_n$  and  $C_n$

It is known (see for instance [Ath98]) that, using these conventions, a non-nesting partition in the “antichain” sense corresponds to a non-nesting partition in the “pictorial” sense as we have used in Proposition 5.3.2. Consequently, the Proposition 5.3.2 and its consequences also holds for type  $B_n$  and  $C_n$ . In particular if we take any root  $a$  and group element  $w$ ,  $a \in N(w)$  if and only if the extremities of the arc(s) corresponding to  $a$  are inverted when the permutation  $w$  is written

$$w(1) \ w(2) \ \dots \ w(n) \ 0 \ w(-n) \ \dots \ w(-2) \ w(-1),$$

and we also have the Corollary 5.3.3 for type  $B_n$  and  $C_n$ .

**Type  $D_n$ :** The group  $D_n$  is isomorphic to the permutation group:

$$\{w \in B_n \mid \{i \in \llbracket 1, n \rrbracket \mid w(i) < 0\} \text{ has even cardinality}\}.$$

Given such a permutation  $W$ , we will write it as the sequence of its values in the following order:

$$\begin{array}{ccccccccccc} & & & & & w(n) & & & & & \\ w(1) & w(2) & \dots & w(n-1) & & & w(1-n) & \dots & w(-2) & w(-1) & \\ & & & & & w(-n) & & & & & \end{array}$$

Again, by convention, we say that the identity permutation written in this format is sorted, meaning  $n$  and  $-n$  are incomparable, both greater than  $n-1$  and both lesser than  $-n+1$ . We do this to “solve” an issue noted in [Ath98]: if we simply ordered the elements of  $\llbracket -n, n \rrbracket$  as we did in case  $B_n, C_n$ , the antichain  $\{e_{n-1} - e_n, e_{n-1} + e_n\}$  in type  $D_n$  would correspond to two nesting arcs (see Figure 5.7).

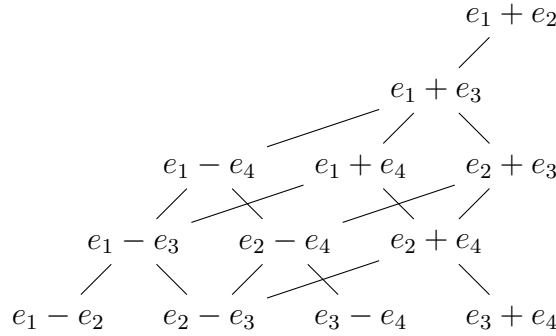


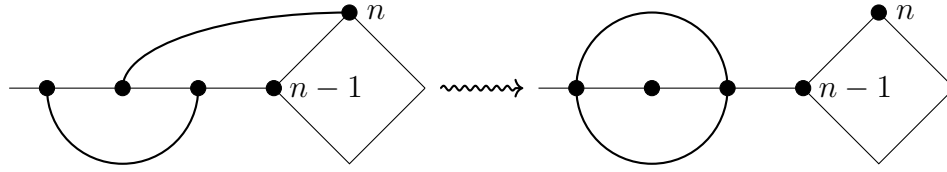
Figure 5.7: Root poset of type  $D_4$ .

Note that  $\{e_3 - e_4, e_3 + e_4\}$  is an antichain in that poset. However, the pair  $(3, 4)$  is “inside” the pair  $(3, -4)$ , so if we were to choose the order  $3 < 4 < -4$  as we did in types  $B_n, C_n$ , this antichain would correspond to pictorially nesting arcs, hence why we make  $n$  and  $-n$  incomparable.

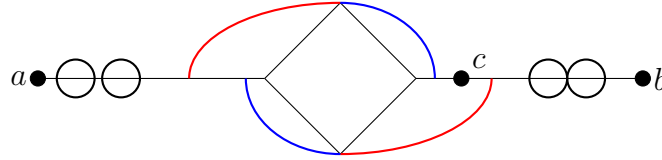
Making  $n$  and  $-n$  incomparable preserves the correspondence between the “antichain” definition of non-nesting and the “pictorial” definition. This is necessary to use an argument similar to that of Proposition 5.3.2. However, the question of how to count arcs becomes slightly more difficult.

**Proposition 5.3.4.** *Let  $(e, A)$  be a parking function of type  $D_n$  in diagram form. Let  $a, b \in \llbracket -n, n \rrbracket \setminus \{0\}$  with  $a \neq -b$ . We define  $v_{a,b}^+$  (resp.  $v_{a,b}^-$ ) as the maximal number of non-nesting, non-crossing arcs between  $a$  and  $b$ , excluding arcs connected to  $-n$  (resp. to  $n$ ). Define  $v_{a,b} = \max(v_{a,b}^+, v_{a,b}^-)$ . Then  $v$  is a type  $D_n$  Shi vector.*

*Proof.* We just have to show that the Shi relations are respected. We interpret  $A$  as a set of arcs and again, we note  $S_{a,b}^+$  a set of non-crossing arcs of  $A$  ignoring  $-n$  with maximal cardinality and similarly  $S_{a,b}^-$ . Suppose that  $A$  does not contain an arc with extremity  $n$  (or  $-n$ , this is equivalent from Table 5.3). Then  $S_{a,b}^+$  and  $S_{a,b}^-$  can be chosen to be the same, and this is just Proposition 5.3.2 for all  $a < c < b$  with  $-a \neq c \neq -b$ . Similarly, if  $-n, n \notin \llbracket a, b \rrbracket$ , the Shi relations are verified for all  $c$  between  $a$  and  $b$ . Let us now deal with the difficult case: suppose that  $A$  does contain an arc with extremity  $n$  and that  $a < n, -n < b$ . We are going to construct the sets  $S_{a,b}^\pm$  to be as similar as possible: we choose the non-crossing arcs by adding the leftmost possible arcs between  $a$  and  $\pm n$  and the rightmost between  $\pm n$  and  $b$ . It is obvious that this strategy yields maximal sets for  $S_{a,b}^+$  and  $S_{a,b}^-$ . Moreover, it has the advantage that if an arc  $(i, j)$  with  $i, j \neq \pm n$  is in one of the sets, then it is in the other (see Figure 5.8a).



(a) Except in  $\pm n$ , a symmetric choice of arcs to compute  $v_{a,b}^\pm$  can be made, thus...



(b) ... allowing to reason on this kind of picture. Different cases corresponding to the positions of  $a, b, c$  must be checked.

Figure 5.8: Idea of proof for Proposition 5.3.4.

With  $S_{a,b}^\pm$  chosen this way, we can ignore the arcs that do not have  $\pm n$  as an extremity, as they contribute the same thing in both sets. We can reason on the case where  $A \subset \{e_i - e_n, e_j + e_n\}$  for some fixed integers  $i, j$ . Because there are at most two non-crossing arcs, the two only possible cases of failure to respect the Shi relations would correspond to having some  $c \in \llbracket a, b \rrbracket$  such that  $v_{a,b} = 2$  and  $v_{a,c} = v_{c,b} = 0$  or  $v_{a,c} + v_{c,b} > 2$ . Both are impossible. For the first case, it is obvious from the picture that if  $v_{a,b} = 2$ , then at least one of  $v_{a,c}^-, v_{a,c}^+, v_{c,b}^-, v_{c,b}^+$  is non-zero. For the second case it is also obvious from the picture that  $v_{a,c}, v_{c,b} \leq 2$  and that if one is 2, the other must be 0.  $\square$

As in the previous types, this gives an equivalent of Corollary 5.3.3 considering the correspondence between roots being inversions of an element and

pairs of values appearing in reverse order in the permutation.

**Question 5.3.5.** As Question 5.2.14 touches upon, we do not have, in general, a non-recursive way of computing the Shi coefficients. However, we have just seen that, in classical types, the Shi coefficients can be computed by counting non-crossing arcs in a non-nesting partition. In our type-free proof, this non-crossing phenomenon vanishes. Where has it gone? Is there a uniform way, not only for exceptional groups, but that also circumvents the special constructions we just presented in classical types, to compute directly the Shi coefficients? Can this description be obtained in terms of non-nesting arcs in non-crossing partitions? If so, is there an interpretation of it compatible with the notion of non-crossing partition that is defined for all Coxeter groups?

# Chapter 6

## Low elements

### 6.1 In rank three

**Proposition 6.1.1.** *In rank 3,  $\Sigma_n : L_n \longrightarrow \Lambda_n$  is surjective.*

The key observation is that the notion of bipodality allows to propagate information along the arrows of Definition 4.7.9. We package this in a graph.

**Definition 6.1.2.** Let  $w \in W$ . The *bipodality graph* of  $w$ , denoted by  $G_{bip}(w)$  is the directed graph with:

- vertices set the vertices of  $\mathcal{P}_w$ ,
- edges set the edges of  $\mathcal{P}_w$  that are arrows in the sense of Definition 4.7.9.

*Strategy.* Recall that an element  $w$  is  $n$ -low if and only if the vertices of  $\mathcal{P}_w$  are in  $\Sigma_n$ . Thus, we can prove this result by exhibiting for any small inversion set  $\lambda \in \Lambda$  an element  $w$  such that  $N^1(w) \subset \Sigma_n$  and  $\Sigma_n(w) = \lambda$ . Our strategy is as follows:

1. Choose  $w$  with  $\Sigma_n(w) = \lambda$  such that its descents are  $n$ -small, *i.e.*  $\Gamma_w \subset \Sigma_n$  (Lemma 6.1.5).
2. Prove that every vertex on the bipodality graph  $G_{bip}(w)$  is accessible from  $\Gamma_w$  in two steps:
  - a. show that  $G_{bip}(w)$  is acyclic (Proposition 6.1.7),
  - b. show that its sources lie in  $\Gamma_w$  (Proposition 6.1.10).

By bipodality, since there is a path of arrows from the sources, which have controlled infinite depth from step 1, to any vertex of  $\mathcal{P}_w$ , this imposes by Theorem 4.7.11 that  $N^1(w) \subset \Sigma_n$ , proving that  $w$  is an  $n$ -low element.  $\square$

The most difficult point is to show that the sources form a subset of the descents. We present here a proof in rank 3, and it is the only part of the proof that does not easily extend to higher ranks. We begin with some remarks on the interaction between moving separations hyperplane and inversion polytopes in §6.1.1. This gives us some tools to prove that  $G_{bip}(w)$  is acyclic in §6.1.2 and to localize the sources of the graph in §6.1.3.

### 6.1.1 Wiggling hyperplanes

As can be seen in Figure 4.8, the set of separation hyperplanes  $H$  for  $N(w)$  is homeomorphic to  $w\mathcal{C}/\mathbb{R}^+$ . This shows, for instance, that if  $H, H'$  are two separation hyperplanes for  $N(w)$ , we can continuously move one to the other without crossing a root. In this spirit, we examine in this section some interactions between roots and moving hyperplanes, beginning with a relaxation of the separability condition.

**Definition 6.1.3.** Let  $w$  be an element of  $W$  and  $\mathcal{P}_w$  its inversion polytope. We say that an affine hyperplane  $H \subset H_1$  is a *weak separation hyperplane* for  $\mathcal{P}_w$  if  $\mathcal{P}_w$  is on one side of  $H$ , while its complement  $\widehat{\Phi} \setminus \mathcal{P}_w$  is strictly on the other side and  $H$  does not intersect the normalized isotropic cone.

Stated in terms of separating forms in the Tits cone, this means that we authorize the form to be in the closure  $\overline{w\mathcal{C}}$  as long as it stays in the topological interior of the Tits cone.

For finite subsets of  $\widehat{\Phi}$ , this is equivalent to the regular separability condition: a weak separation hyperplane  $H$  does not cross the isotropic cone, so it is at a strictly positive distance from the roots it does not contain. This gives us a root free envelope (fig. 6.1a) in which we may translate  $H$  to make it into a strict separation hyperplane (fig. 6.1b). Because the set  $\widehat{\Phi}$  is bounded, we may actually exit this envelope, as long as we do it “far enough”: we can tilt  $H$  to remove the vertices of  $\text{conv}(H \cap \widehat{\Phi})$  (fig. 6.1c).

Again, stated in the Tits cone, this is loosely saying that since a small open set around the weak separating form intersects finitely many hyperplanes, we may cleanly back up in the open set  $w\mathcal{C}$  before crossing one of its walls. This gives us indications on the descents of an element:

**Lemma 6.1.4.** *Let  $w$  be an element of  $W$ , and  $\mathcal{P}_w$  its inversion polytope. We suppose that  $H$  is a weak separation hyperplane for  $\mathcal{P}_w$  containing a face  $F$  of  $\mathcal{P}_w$ . Then the vertices of  $F$  are in  $\Gamma_w$ .*

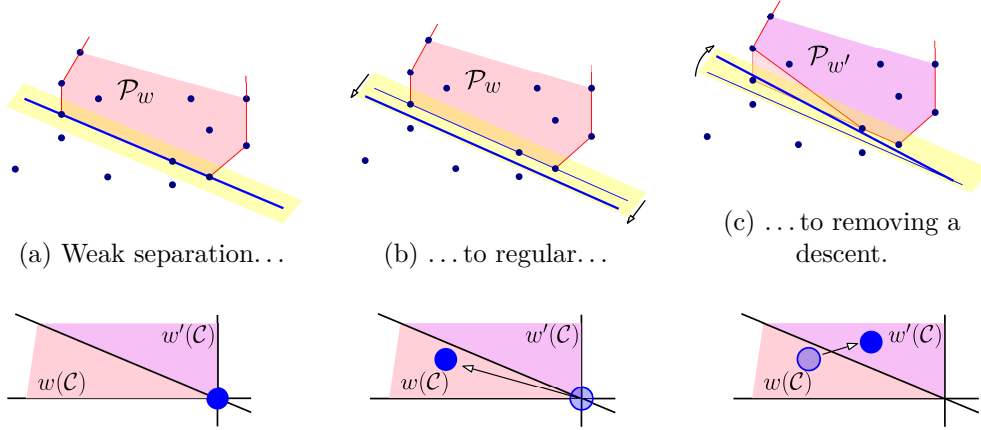


Figure 6.1: Wiggling hyperplanes : movements of the separation hyperplane on the top row (primal) correspond to a movement of the associated point of the Coxeter complex in the bottom row (dual). The root-free envelope is figured in yellow.

*Proof.* Let  $f$  be a weak separation form for  $\mathcal{P}_w$  with  $f(F) = 0$ . Since  $f \in \overline{w\mathcal{C}}$ , for all  $\phi \in F$ ,  $H_\phi$  intersects  $\overline{w\mathcal{C}}$ . We have to show which of these roots are walls. Again by definition of weak separation  $f^{-1}\{0\} \cap \widehat{\Phi} = F \cap \widehat{\Phi}$ , so  $\Phi_F^+$  is the positive root system of a maximal reflection subgroup of  $W$  with simple roots  $\Delta_F$  corresponding to the vertices of  $F$ . The roots in  $\Phi_F^+$  are inversions of  $w$ , so

$$w\mathcal{C} \subset \bigcap_{\phi \in \Phi_F^+} H_\phi^- = \bigcap_{\delta \in \Delta_F} H_\delta^-.$$

In other words, among the roots contained in  $F$ , only the vertices can possibly be descents of  $w$ . Conversely, considering a small open set  $A$  around  $f$ , we meet only the hyperplanes corresponding to  $\Phi_F^+$ . So  $w\mathcal{C} \cap A = \bigcap_{\delta \in \Delta_F} H_\delta^- \cap A$ . Thus, every  $H_\delta, \delta \in \Delta_F$  is a wall of  $w\mathcal{C}$ . Indeed, if  $H_\delta$  is not a wall, then

$$H_\delta \cap (\overline{w\mathcal{C}} \cap A) = \bigcap_{\delta' \in \Delta'} H_{\delta'} \cap A$$

for  $\Delta' \subset \Delta \setminus \{\delta\}$ : dually,  $\delta \in \text{conv}(\Delta')$  so  $\delta$  is not a vertex: absurd.  $\square$

This lemma expresses a more general idea: because small perturbations of a point in the Coxeter complex correspond to small perturbations of a separation hyperplane, *we can choose to enter an alcove through one of its walls*.

### 6.1.2 $G_{bip}(w)$ is acyclic

Let  $\lambda \in \Lambda_n$  be a small inversion set, let us show that we can choose  $w \in W$  with  $\Sigma_n(w) = \lambda$  and  $\Gamma_w \subset \Sigma_n$ , as per the first point of our strategy. Indeed,



if it is not the case, then remove  $\gamma \in \Gamma_w \setminus \Sigma_n$ . Notice that the resulting set of roots is still the inversion set of some  $w'$ , and that the  $n$ -small inversion set is unchanged. Repeat. Because the initial  $w$  is of finite length, this eventually terminates. We obtained the following lemma:

**Lemma 6.1.5.** *Let  $\lambda \in \Lambda_n$  be an  $n$ -small inversion set. Then there exists some  $w \in W$  such that  $\Sigma_n(w) = \lambda$  and  $\Gamma_w \subset \Sigma_n$ .*

We now want to show that every vertex of  $\mathcal{P}_w$  is accessible from  $\Gamma_w$ . It is enough to prove that  $G_{bip}(w)$  is acyclic and that the sources of  $G_{bip}(w)$  (meaning the vertices connected only to outward edges) are a subset of  $\Gamma_w$ . Indeed, to find a path from  $\Gamma_w$  to a vertex  $v$  we just have to follow the following process: if  $v$  is a source, stop. Else,  $v$  must be connected by an inward edge to  $v'$ , replace  $v$  by  $v'$  and iterate. Since there is no cycle, this terminates on a source, which is in  $\Gamma_w$ .

We now fix the  $w$  obtained from Lemma 6.1.5 for the remainder of this section. Our next step is to show that the bipodality graph is acyclic. Let us first prove the following useful lemma.

**Lemma 6.1.6.** *We say that a permutation  $R = (\rho_1, \rho_2, \dots, \rho_{|w|})$  of  $N(w)$  is a removal order if for all  $i \in \{1, \dots, |w|\}$ ,  $N(w) \setminus \{\rho_j \mid i \leq j\} = N(w_i)$  for some  $w_i$ . Let  $[a, b]$  be an edge of  $\mathcal{P}_w$  with  $a \rightarrow b$ . Then the roots on  $[a, b]$  are removed from  $a$  to  $b$ .*

Note that Proposition 4.3.4 makes obvious that such removal orders are in bijection with reduced words of  $w$ . However, we are more preoccupied by the geometric aspect of moving the separation hyperplane through the roots than with the direct interpretation of it in terms of reduced words, although this is in itself a very interesting question.

*Proof.* Notations are defined in Figure 6.2. Suppose we remove  $b$  first. Since the remaining roots form an inversion set, it means that a separation hyperplane  $H$  must cut the line  $(a, b)$  between  $a$  and  $b$  and between  $\alpha$  and  $a$ : this is absurd.  $\square$

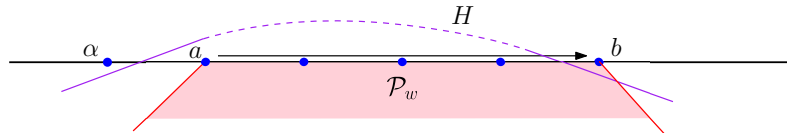


Figure 6.2: A removal order follows the arrows.  $\alpha$  exist by definition of  $[a, b]$  being an arrow from  $a$  to  $b$ .

**Remark.** Note that this argument has two consequences. Firstly, an edge  $[a, b]$  of  $\mathcal{P}_w$  is either an arrow or *complete*, meaning  $(a, b) \cap \hat{\Phi} = [a, b] \cap \hat{\Phi}$ . Secondly, it means that any removal order determines an orientation on the edges of  $\mathcal{P}_w$ . Indeed, even if  $[a, b]$  is complete, if  $a$  is removed first, the remaining roots on  $[a, b]$  must be removed from  $a$  to  $b$ . This allows us to define a (directed) *removal graph*  $G_R(w)$  formed of the vertices and edges of  $\mathcal{P}_w$  oriented by  $R$ : if  $[a, b]$  is an edge of  $\mathcal{P}_w$ , we note  $a \rightarrow_R b$  the corresponding edge in  $G_R(w)$ .

We can now restate Lemma 6.1.6 as: for any order  $R$ ,  $G_{bip}(w)$  is a subgraph of  $G_R(w)$ . This will help us to reach our goal for this paragraph:

**Proposition 6.1.7.** *The bipodality graph is acyclic.*

*Proof.* Because  $G_{bip}(w)$  is a subgraph of  $G_R(w)$  for any removal order  $R$ , it is enough to show that  $G_R(w)$  is acyclic for some  $R$ . Let  $H$  be a separation hyperplane. In the spirit of Lemma 6.1.4, we further assume that  $H$  is generic: it is not parallel to any line through two roots in  $\mathcal{P}_w$ . We continuously translate  $H$  through  $\mathcal{P}_w$ , following its orthogonal line. Because  $H$  is generic, the roots will traverse  $H$  one at a time: except at these times,  $H$  is a separation hyperplane for the remaining roots. Hence, this defines a removal order. Since  $a \rightarrow_R b$  is only possible if  $H$  visits  $a$  before  $b$ , the related removal graph  $G_R(w)$  is acyclic.  $\square$

We now know that the bipodality graph is acyclic and thus that it induces a poset. We also know that it is, in some sense, compatible with all factorizations into generators.

**Question 6.1.8.** Is it the case that  $\alpha \rightarrow \beta$  in the bipodality graph if and only if  $\alpha \rightarrow_R \beta$  for all removal orders  $R$ ? In other words, is bipodality the largest suborder on vertices of  $\mathcal{P}_w$  common to all removal orders?

An interesting first step to take in answering this question is to investigate whether the bipodality graph is “maximal”.

**Question 6.1.9.** In [Dye+23], the authors define the *short inversion poset*, which is essentially the bipodality graph, but where all the diagonals that are arrows are edges of the graph generating the poset, instead of just the edges of the inversion polytope. Is it the case that the short inversion poset and the poset generated by the bipodality graph coincide?

If the short inversion poset has more relations, it is a better candidate for the previous question.

### 6.1.3 The sources of $G_{bip}(w)$ lie in $\Gamma_w$

The goal of this section is to prove the following proposition:

**Proposition 6.1.10.** *Assume that  $|S| = 3$ . Then every source of the bipodality graph on  $\mathcal{P}_w$  is a descent.*

Let  $v$  be a source: either it is only connected to outward arrows (case 1), or it is connected to at least one complete edge (case 2). Case 1 is easy: such a source in  $G_{bip}(w)$  must be a source in any removal graph. In particular, removal graphs obtained by pushing hyperplanes through polytopes (as in Proposition 6.1.7) only have one source, which is the first root they meet and thus must be a descent.

Case 2 is harder. First, notice that a simple root cannot be a source because it can always be removed last, which means it is a sink in some removal graph. The only remaining possibility is dealt with in the following lemma.

**Lemma 6.1.11.** *We suppose that  $(W, S)$  is a Coxeter system of rank 3. Let  $\mathcal{P}_w$  be the inversion polygon of an element  $w \in W$ . Let  $[a, b]$  be a complete edge of  $\mathcal{P}_w$ . If  $a \notin \Delta$ , then  $a \in \Gamma_w$ .*

*Proof.* Notice first that because  $[a, b]$  is complete and contains a finite number of roots, the line  $d = (a, b)$  does not intersect the isotropic cone ( $B_d$  is definite, Proposition 4.5.1). If we can prove that  $d$  is a weak separation line for  $\mathcal{P}_w$ , from Lemma 6.1.4 we get that  $a$  must be a descent. Let  $f$  be an affine form corresponding to  $d$  such that  $P_w \subset f^{-1}\mathbb{R}_-$ . Let  $g$  be a separating affine form for  $w$ .

- Suppose that  $f^{-1}\mathbb{R}_- \cap \widehat{\Phi}$  is finite and let  $w'$  be the corresponding element with strong separating form  $f'$ . By Proposition 4.6.4, the segment  $\{(1-t)g + tf' \mid t \in [0, 1]\}$  (considered oriented from  $g$  to  $f'$ ) crosses a finite number of hyperplanes associated to roots. If the segment crosses no hyperplane, then  $g$  and  $f'$  are in the same alcove and thus  $f$  is indeed a weak separation hyperplane for  $\mathcal{P}_w$  so  $\{a, b\} \in \Gamma_{w'}$  and we have the result. Otherwise, up to a small perturbation of  $f'$  or  $g$  we may assume that the oriented segment crosses one unique last hyperplane associated to a root  $c$  at some  $t_c \in ]0, 1[$ . This last root must necessarily be distinct from  $a$  and  $b$  as  $f'(a), f'(b), g(a), g(b)$  are strictly negative, and consequently so are  $((1-t)g + tf')(a)$  and  $((1-t)g + tf')(b)$  for all  $t$ . Then  $\{a, b\} \subset \Gamma_{w'}$  by construction. But we also have a third distinct element  $c \in \Gamma_{w'}$  because for  $\varepsilon > 0$  small enough  $f_c = (1 - t_c + \varepsilon)g + (t_c - \varepsilon)f'$  is in the Tits cone (because it is convex), vanishes on no roots and  $f_c^{-1}\mathbb{R}_- \cap \widehat{\Phi} = \widehat{N(w')} \setminus \{\hat{c}\}$ . Since  $a \notin \Delta$ , the line  $(a, b)$  has roots of  $\Delta$

on either side of it, so  $w'$  cannot be the maximal element because the maximal element inverts all roots. But only the maximal element has  $|\Gamma_{w'}| = 3$ : this is absurd.

- Suppose that  $f^{-1}\mathbb{R}_+^* \cap \widehat{\Phi}$  is finite and let  $w'$  be the corresponding element with strong separating form  $f'$ . In exactly the same fashion, we deduce that either  $W$  is finite and  $w' = w_0 w^{-1}$ , or that  $w'$  is a non-neutral element that is covered by three distinct elements, which is also absurd.

Since  $d = (a, b)$  does not cross the isotropic cone, one of these situations (possibly both in finite groups) must occur, which proves the result.  $\square$

## 6.2 In affine Weyl groups

In this section, we use the main result from the previous section, Proposition 5.2.13, to prove that in the case of an irreducible affine Weyl group, low elements and small inversion sets are in bijection. As before, we split the proof by considering only the dominant region and later transporting the result to other chambers.

We present the result in the language of Shi arrangements to remain as coherent as possible with our previous considerations. As exposed in Proposition 4.7.12, the 0-small roots of  $\Phi_a$  form the set  $\Sigma_0 = \Phi^+ \sqcup (\Phi^- + \delta)$ . Those are the roots associated with the hyperplanes  $\bigcup_{\alpha \in \Phi^+} H_{\alpha,0} \cap \bigcup_{\alpha \in \Phi^+} H_{\alpha,1}$  in the (1-)Shi arrangement.

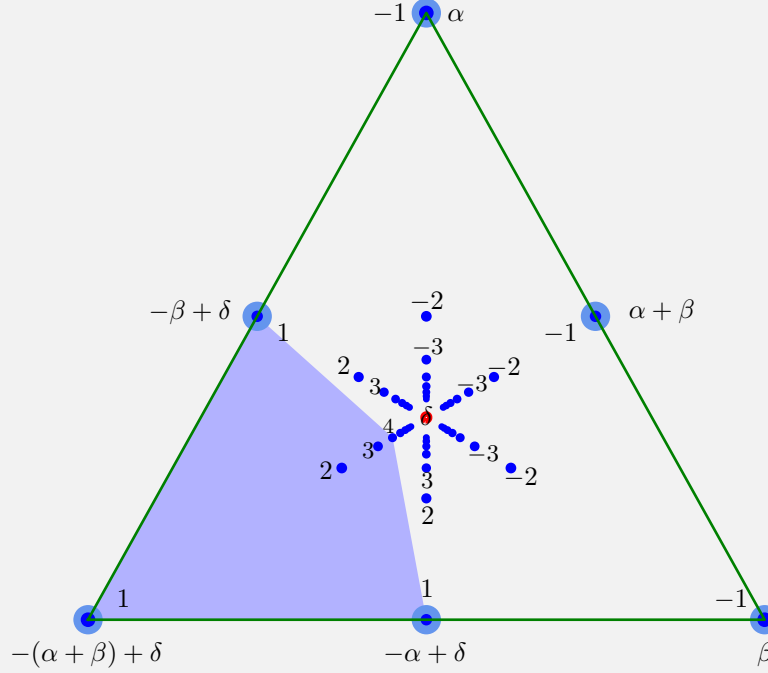
We set out to prove the following result (the case with 1-low elements and 1-small roots of [DH16, Conjecture 2]):

**Proposition 6.2.1.** *The map  $\Sigma_0 : w \mapsto N(w) \cap \Sigma_0$  from  $L_0$  to  $\Lambda_0$  is surjective in irreducible affine Weyl groups.*

Recall that, because a root  $a \in N(w)$  if and only if the alcove  $w\mathcal{C}$  is separated from  $\mathcal{C}$  by the hyperplane associated to  $a$ , small inversion sets are naturally in bijection with the Shi regions as defined by Definition 4.7.13. The map taking  $w$  to  $N(w) \cap \Sigma_0$  is known to be injective from  $L_0$  to  $\Lambda_0$  [DH16, Proposition 3.26]. Until recently (and particularly [CLH22] for the case of Weyl groups), not much was known about the surjectivity of this map.

Let us look at an example to understand the geometric meaning of the question.

In the picture, we construct what [HLR14] calls a “projective picture”. We took the positive roots in type  $\tilde{A}_2$  and took the intersection of the rays they generate with the affine hyperplane with equation  $x_\delta = 1$ . We note  $\alpha, \beta$  the simple roots of  $A_2$ .



In the picture, we have labeled the roots  $\alpha + k\delta$  by  $k$  if  $\alpha \in \Phi^-$  and  $-(k + 1)$  otherwise. That way, if we consider an inversion set  $N(w)$  with “innermost” root  $\alpha + k\delta$  in the  $\{\alpha, \delta\}$  plane, our label gives the coefficient  $k(w, \alpha)$ . The small roots are overlined in blue.

As appears in the figure above, the roots that share an  $\alpha \in \Phi^+$  are aligned (or rather, coplanar). Thus, the only roots that could be candidate to be extreme rays of  $\text{cone}(N(w))$  are the “exterior” roots of  $N(w) \cap \mathbb{R}\{\alpha, \delta\}$ . This gives an easy reduction for the proof, which we formalize below.

**Proposition 6.2.2.** *Let  $v$  be the minimal element of a Shi region  $R$ . Define  $r_a$  by:*

$$r_a = \begin{cases} a - (v_a + 1)\delta & \text{if } v_a < 0 \\ -a + v_a\delta & \text{if } v_a \geq 0 \end{cases}.$$

*Then  $v$  corresponds to a low element if and only if for all positive roots  $a$ ,  $r_a \in N^1(w) \Rightarrow v_a = \pm 1$ .*

*Proof.* From Proposition 5.2.2, if  $v_a = 0$ ,  $r_a \notin N(w)$  so  $r_a \notin N^1(w)$ . If  $v_a > 0$ , then  $N(w)$  contains no root lying in  $a + \mathbb{N}\delta$ . If  $r = -a + k\delta, k > 0$ ,  $r \in \text{cone}(r_a, -a + \delta)$ , so  $r \in N^1(w) \Rightarrow r \in \{-a + \delta, r_a\}$ .  $-a + \delta$  is a small root and  $r_a$  is a small root if and only if  $v_a = 1$ . Similarly, if  $v_a < 0$ , only

$\{a, a - (v_a + 1)\delta\}$  can be in  $N^1(w)$ , and  $a - (v_a + 1)\delta$  is small if and only if  $v_a = -1$ . Since, by definition,  $w$  is low if and only if  $N^1(w)$  contains only small roots, we can conclude.  $\square$

Translated in a less formal language,  $r_a$  is the innermost positive root of  $N(w)$  in the plane  $\{\alpha, \delta\}$  and Proposition 6.2.2 says that testing if an element is low is reducible to testing if the innermost roots generating extreme rays are small. Note that only in the special case  $v_a = 0$ ,  $r_a$  is not a positive root and  $r_a \notin N(w)$ . We will need to give special attention to that 0 case in our proofs.

Let us examine these innermost roots on our example. The convex hull represented in blue would have  $(1, 4, 1)$  as Shi encoding. Obviously, from Theorem 5.2.1, this does not correspond to an element of  $\widetilde{W}$ , because  $(1, 4, 1)$  is not a Shi vector: the 4 “sticks out” too far. Recall that inversion sets are separable: correspondingly, in the picture, the root on the  $\{\alpha + \beta, \delta\}$  plane sticks out too far for a separation hyperplane to exist. In that sense, Theorem 5.2.1 is a rigidity result on the position of the roots.

Our proof harnesses this fact: in the dominant region, the use of  $\varepsilon = 1$  in the relation  $v_a = v_b + v_c + \varepsilon$  means that the roots  $r_a, r_b$  and  $r_c$  are not co-planar:  $r_a$  “sticks out”. This allows us to prove the result in the dominant region in Proposition 6.2.3. Transposing that idea in other chambers, questions of sign arise that we deal with in Proposition 6.2.4.

### 6.2.1 The dominant region

The spirit of the following proof can be summarized as: for  $v_a > 0$ , the root  $r_a$  generates an extreme ray of  $\text{cone}(N(w))$  only if  $\varepsilon_{b,a-b} = 1$  for all  $b \in \text{Sk}(a)$ , and this only happens if  $a \in A$ , in which case  $v_a = 1$ .

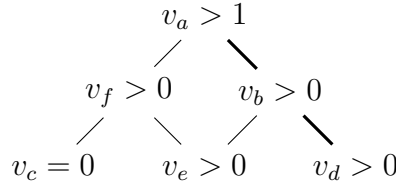
**Proposition 6.2.3.** *Let  $m$  be the minimal element of the Shi region labeled by  $(e, A)$ . Then  $m$  is a low element.*

*Proof.* We want to prove that for any positive root  $a$ ,  $v_a > 1$  implies that  $-a + v_a\delta \notin N^1(m)$ . Note that if we have  $a \in \Phi^+$  with  $v_a > 1$  and  $b, c \in \Phi^+$  such that  $v_b$  and  $v_c$  are non-zero and  $v_a = v_b + v_c$ , then  $r_b, r_c \in N(m)$  and

$$r_a = -a + v_a\delta = -(b + c) + (v_b + v_c)\delta = -b + v_b\delta - c + v_c\delta = r_b + r_c$$

so  $r_a \in \text{cone}(r_b, r_c)$  and  $r_a \notin N^1(m)$ . Let us prove, by induction on the height of a root  $a$ , that if  $v_a > 1$  then there exist  $b, c \in \text{Sk}(a)$ , with  $c = a - b$ ,  $v_a = v_b + v_c$  and  $v_b, v_c$  non-zero. If  $a$  has height 1,  $v_a$  can only be 0 or 1, depending on  $a$  belonging to  $A$  or not, so it is true at height 1. If  $a$  has height  $k > 1$  and  $v_a > 1$ , from Proposition 5.2.9, there exists  $b, c \in \text{Sk}(a)$  such that

$v_a = v_b + v_c$ . If  $v_b, v_c$  are non-zero, there is nothing to prove. Otherwise, we may suppose that  $v_a = v_b$ . By induction, there exists  $d, e \in \text{Sk}(b)$  with  $v_b = v_d + v_e$  and  $v_d, v_e$  non-zero. From Proposition 5.1.3, we may suppose, for instance, that  $d \in \text{Sk}(a)$  and denote  $a - d$  by  $f$ . We have  $f = e + c$  so  $v_f \geq v_e + v_c = v_e \geq 0$ .

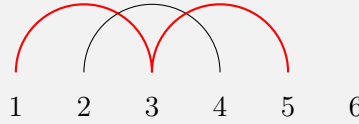


Further, we have:

$$v_a = v_b + v_c = v_d + v_e + v_c \leq v_d + v_f.$$

But from Proposition 5.2.11,  $a = d + f$  implies that  $v_d + v_f \leq v_a$ , so  $v_a = v_d + v_f$  and  $v_d, v_f$  are non-zero, and we can conclude by induction.  $\square$

**In type  $A_n$**  The proof translates something that is easy to see on the non-nesting arc diagrams of type  $A_n$ .



Imagine we want to compute  $v_{e_1 - e_6}$ . By our previous remarks, we just have to choose a set  $S$  of non-crossing arcs of maximal cardinality, which we have represented in red. The fact that  $v_{e_1 - e_6} > 1$  implies that there is some  $1 < j < 6$  such that  $v_{e_1 - e_j} + v_{e_j - e_6} = v_{e_1 - e_6}$  and  $v_{e_1 - e_j}, v_{e_j - e_6}$  are non-zero is visible in the picture: just pick  $j$  to be the right extremity of the leftmost arc in  $S$ , here, 3.

### 6.2.2 General case

We now transport the result to any chamber. As we reintroduce signs, notice a sign triplet  $(s_a, s_{a+b}, s_b)$  can only be one of the sign types of  $A_2$  (given on 135). We refer to that fact as the *sign rules*. For instance, if we have  $s_a = -$  and  $s_b = -$  then the sign rules allow us to deduce that  $s_{a+b} = -$ . Conversely,  $s_a = -$  and  $s_b = 0$  we could have  $s_{a+b} = -$  or 0, and the sign rules do not help us decide with it is. Recall our convention that for any vector  $v$  indexed by  $\Phi^+$  we allow ourselves to denote  $v_a$  by  $v_{-a}$  for  $a \in \Phi^+$ .

**Proposition 6.2.4.** *The minimal element of the Shi region labeled by  $(w, A)$  is a low element.*

*Proof.* Let  $v$  be the Shi vector of the minimal element  $m$  of the region  $(e, A)$ ,  $v'$  that of the minimal element  $m'$  of the region  $(w, A)$ , and for any  $a \in \Phi^+$ , let us denote

$$r_a = -a + v_a \delta \text{ and } r'_a = \begin{cases} a - (v'_a + 1)\delta & \text{if } v_a < 0 \\ -a + v'_a \delta & \text{if } v_a \geq 0 \end{cases}.$$

We want to show that for any  $a \in \Phi^+$ ,  $r'_a \in N^1(m') \Rightarrow v'_a = \pm 1$ . We will prove the contrapositive. Note that:

$$w(r_a) = w(-a + v_a \delta) = -w(a) + v_a \delta = \begin{cases} |w(a)| - (v'_{w(a)} + 1)\delta & \text{if } w(a) \in \Phi^- \\ -w(a) + v'_{w(a)} \delta & \text{if } w(a) \in \Phi^+ \end{cases}$$

that is,  $w(r_a) = r_{w(a)}$ . So if  $r_a, r_b, r_c$  belong to  $N(m)$ , then  $r_a = r_b + r_c$  implies  $r'_{w(a)} = r'_{w(b)} + r'_{w(c)}$ . Thus,  $v_a > 1$  implies both  $|v'_{w(a)}| > 1$  and  $r'_{w(a)} \notin N^1(m')$  because from the proof of Proposition 6.2.3 we can choose  $b, c$  with  $v_b, v_c$  non-zero. If  $v_a = 0$ , then  $v'_{w(a)} \in \{-1, 0\}$  so in that case the contrapositive is still verified. Finally, if  $v_a = 1$ , we have several cases: if  $w(a) \in \Phi^+$ , which, in particular, happens whenever  $a \in A$ , then  $v'_{w(a)} = 1$  and, again, the contrapositive is verified. Conversely, if  $w(a) \in \Phi^-$ , then  $a \notin A$  and we have  $b, c \in \text{Sk}(a)$  such that  $v_a = v_b + v_c$ . Let us assume that we have  $v_a = v_b = 1$ . Again, we must examine cases depending on whether  $a, b, c$  belong in  $N(w)$  or not. Denote by  $\alpha, \beta, \gamma$  the roots  $|w(a)|, |w(b)|, |w(c)|$ , respectively.

**Flat<sup>-</sup> case** If  $a, b, c \in N(w)$ , then  $(v'_\alpha, v'_\beta, v'_\gamma) = (-2, -2, -1)$ . This means that:

$$r'_\alpha = \alpha - (-2 + 1)\delta = \alpha + \delta = \beta + \delta + \gamma = r'_\beta + r'_\gamma.$$

In that case,  $r'_\alpha$  is not in  $N^1(m')$ .

**Flat<sup>+</sup> case** If  $a, c \in N(w)$  and  $b \notin N(w)$  then  $(v'_\alpha, v'_\beta, v'_\gamma) = (-2, 1, -1)$ , and we have

$$r'_\alpha = \alpha + \delta = -\beta + \delta + \gamma = r'_\beta + r'_\gamma,$$

meaning again that  $r'_\alpha \notin N^1(m')$ .

**Bad case** If  $a, b \in N(w)$  and  $c \notin N(w)$ , then  $(v'_\alpha, v'_\beta, v'_\gamma) = (-2, -2, 0)$ . We prove by induction on the height of  $a$  that there exists  $d, e$  such that  $a = d + e$ ,  $v_a = v_d + v_e$ , and  $a, d, e \in N(w)$  or  $a, e \in N(w)$  and  $d \notin N(w)$ , which gets us back to the two previous cases. If  $a$  has height 1, this is true, as in this case  $v_a = 1 \Rightarrow a \in A \Rightarrow a \notin N(w)$ .



If  $a$  has height  $k > 1$ , by supposition,  $a, b \in N(w)$  and  $c \notin N(w)$ . By the induction hypothesis, there exist  $d, f \in \Phi^+$  such that  $b = f + d$ , satisfying the hypothesis. We may suppose from Proposition 5.1.3 that  $d \in \text{Sk}(a)$ . We examine the situation in Figure 6.3, depending on whether  $v_d = 0$  or 1 and if  $b, d, f$  are in the  $\text{flat}^+$  or  $\text{flat}^-$  case.

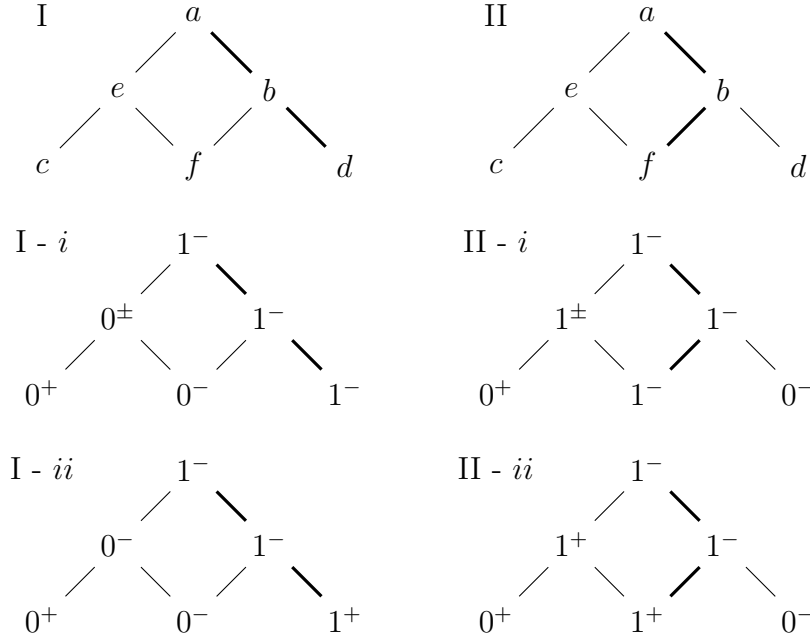


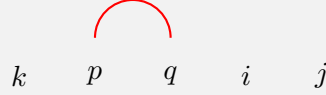
Figure 6.3: The 4 possibles “bad” cases.

We give the value of the coefficient in  $v$  and indicate a root is in  $N(w)$  by a “-” exponent, is not in  $N(w)$  by a “+” and put a  $\pm$  when it cannot be determined by the sign rules.

Case I:  $v_d = 1$ . Then either, case I -  $i$ ,  $d \in N(w)$  that is,  $b, d, f$  are  $\text{flat}^-$ . In that case, sign rules are not enough to determine if  $e \in N(w)$ . However, we can suppose that we chose  $b$  of minimal height among the elements that give a  $(-2, -2, 0)$  triplet, which forces  $e \in N(w)$ : we get back in the  $\text{flat}^-$ . Or, case I -  $ii$ ,  $d \notin N(w)$ , that is  $b, d, f$  are  $\text{flat}^+$ , and the sign rules give  $e \in N(w)$ : this is the  $\text{flat}^+$  case. Case II:  $v_d = 0$ . Then either, case II -  $i$ ,  $f \in N(w)$  ( $\text{flat}^-$ ) and although the sign rules do not determine if  $e \in N(w)$ , both possibilities are  $\text{flat}$ . Or, case II -  $ii$ ,  $f \notin N(w)$  ( $\text{flat}^+$ ) implies  $e \notin N(w)$ .

□

**In type  $A_n$**  Again, this proof has a nice interpretation in type  $A_n$ . Let  $1 \leq i < j < k \leq n+1$  and take  $a = e_i - e_k$ ,  $b = e_j - e_k$  and  $c = e_i - e_j$  and suppose  $a, b \in N(w)$  and  $c \notin N(w)$  so  $i, j, k$  appear in the order  $k, i, j$  in the permutation. Since  $v_a = 1$ , there must be some arc between  $k$  and  $i$  in the arc diagram, with extremities  $p < q$ . In the picture that gives:



Depending on how  $p, q$  and  $i, j, k$  are interlaced, it is easy to check we can always find  $d \in \{|e_p - e_k|, |e_q - e_k|\}$  and  $e = a - d$  such that  $v'_d, v'_e$  are non-zero. For instance, if  $i < j < k < p < q$ , we are in case I- $i$ , with  $d = e_k - e_q$ ,  $v'_d = 1$  and  $e = e_j - e_q$ ,  $v'_e = -1$ .



# Bibliography

- [AB08] P. Abramenko and K. Brown. *Buildings: theory and applications*. Vol. 248. Springer Science & Business Media, 2008. DOI: 10.1007/978-0-387-78835-7.
- [ACLR21] Antoine Abram, Nathan Chapelier-Laget, and Christophe Reutenauer. “An Order on Circular Permutations”. In: *Electronic Journal of Combinatorics* 28 (3 2021). DOI: 10.37236/9982.
- [AL99] Christos A. Athanasiadis and Svante Linusson. “A simple bijection for the regions of the Shi arrangement of hyperplanes”. In: *Discrete mathematics* 204.1-3 (1999), pp. 27–39. DOI: 10.1016/S0012-365X(98)00365-3.
- [ARR15] Drew Armstrong, Victor Reiner, and Brendon Rhoades. “Parking spaces”. In: *Advances in Mathematics* 269 (2015), pp. 647–706. DOI: 10.1016/j.aim.2014.10.012.
- [Ass97] Ibrahim Assem. *Algèbres et modules: cours et exercices*. Masson Paris, 1997.
- [AST16] Arvind Ayyer, Anne Schilling, and Nicolas M. Thiéry. “Spectral gap for random-to-random shuffling on linear extensions”. In: *Experimental Mathematics* (July 2016). arXiv:1412.7488, pp. 1–9. DOI: <http://dx.doi.org/10.1080/10586458.2015.1107868>.
- [Ath98] Christos A Athanasiadis. “On non-crossing and non-nesting partitions for classical reflection groups”. In: *The Electronic Journal of Combinatorics* 5.1 (1998), R42. DOI: 10.37236/1380.

- [Ayy+14] Arvind Ayyer et al. “Markov chains, R-trivial monoids and representation theory”. In: *International Journal of Algebra and Computation* 25 (1n02 2014). arXiv:1401.4250, pp. 169–231. DOI: 10.1142/S0218196715400081. URL: <http://dx.doi.org/10.1142/S0218196715400081>.
- [BB05] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*. Vol. 231. Graduate Texts in Mathematics. Springer, New York, 2005, pp. xiv+363.
- [BD98] Kenneth S Brown and Persi Diaconis. “Random walks and hyperplane arrangements”. In: *Annals of Probability* (1998), pp. 1813–1854.
- [BH93] B. Brink and R. Howlett. “A finiteness property and an automatic structure for Coxeter groups”. In: *Mathematische Annalen* 296.1 (1993), pp. 179–190. DOI: 10.1007/BF01445101.
- [Bro00] Kenneth S Brown. “Semigroups, rings, and Markov chains”. In: *Journal of Theoretical Probability* 13 (2000), pp. 871–938.
- [Bro04] Kenneth S Brown. “Semigroup and ring theoretical methods in probability”. In: *Representations of finite dimensional algebras and related topics in Lie theory and geometry* 40 (2004), pp. 3–26.
- [But94] Greg Butler. “An inductive schema for computing conjugacy classes in permutation groups”. In: *mathematics of computation* 62.205 (1994), pp. 363–383.
- [Cha21] Balthazar Charles. “Low elements and small inversion sets are in bijection in rank 3 Coxeter groups”. In: *Séminaire Lotharingien de Combinatoire* (2021). URL: <http://arxiv.org/abs/2104.03040>.
- [Cha22] Balthazar Charles. “A description of the minimal elements of Shi regions in classical Weyl Groups”. In: *arXiv preprint arXiv:2201.10233* (2022).
- [Cha23] Balthazar Charles. “Minimal elements of Shi regions in affine Weyl groups”. In: *preparation* (2023+).
- [Che48] Claude Chevalley. “Theorie des Groupes - Sur la classification des algèbres de Lie simples et de leurs representations”. In: *Comptes Rendus Hebdomadaires des Séances de l’Académie des Sciences* 227.22 (1948), pp. 1136–1138.
- [CL20] Nathan Chapelier-Laget. “Shi variety corresponding to an affine Weyl group”. In: *arXiv preprint arXiv:2010.04310* (2020).

- [CLH22] Nathan Chapelier-Laget and Christophe Hohlweg. “Shi arrangements and low elements in affine Coxeter groups”. In: *arXiv preprint arXiv:2201.06491* (2022).
- [Cli42] AH Clifford. “Matrix representations of completely simple semigroups”. In: *American Journal of Mathematics* 64.1 (1942), pp. 327–342.
- [Cox34] Harold SM Coxeter. “Discrete groups generated by reflections”. In: *Annals of Mathematics* (1934), pp. 588–621.
- [Cox35] Harold SM Coxeter. “The Complete Enumeration of Finite Groups of the Form  $R \wr_2 (R \wr_j)_{kij=1}$ ”. In: *Journal of the London Mathematical Society* 1.1 (1935), pp. 21–25.
- [CP61] Alfred H Clifford and Gordon B Preston. “The algebraic theory of semigroups, vol. 1”. In: *AMS surveys* 7 (1961), p. 1967.
- [CR66] Charles W Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. Vol. 356. American Mathematical Soc., 1966.
- [Cra] David A Craven. *Representation theory of finite groups: a guidebook*. Springer.
- [Deo89] Vinay V Deodhar. “A note on subgroups generated by reflections in Coxeter groups”. In: *Archiv der Mathematik* 53.6 (1989), pp. 543–546.
- [DH16] M. Dyer and C. Hohlweg. “Small roots, low elements, and the weak order in Coxeter groups”. In: *Advances in Mathematics* 301 (2016), pp. 739–784. DOI: 10.1016/j.aim.2016.06.022.
- [Dye12] Matthew Dyer. “Imaginary cone and reflection subgroups of Coxeter groups”. In: *arXiv preprint arXiv:1210.5206* (2012).
- [Dye22] MJ Dyer. “n-low elements and maximal rank k reflection subgroups of Coxeter groups”. In: *Journal of Algebra* 607 (2022), pp. 139–180.
- [Dye+23] Matthew Dyer et al. “Shi arrangements and low elements in Coxeter groups”. In: *arXiv preprint arXiv:2303.16569* (2023).
- [Dye90] Matthew Dyer. “Reflection subgroups of Coxeter systems”. In: *Journal of Algebra* 135.1 (1990), pp. 57–73.
- [Eas+19] James East et al. “Computing finite semigroups”. In: *Journal of Symbolic Computation* 92 (2019), pp. 110–155.
- [Eti+09] Pavel Etingof et al. *Introduction to representation theory*. 2009.

- [Fis19] Susanna Fishel. “A survey of the Shi arrangement”. In: *Recent trends in algebraic combinatorics*. Springer, 2019, pp. 75–113.
- [FP18] Timothy Ferdinands and Annette Pilkington. “A note on sums of roots”. In: *Rocky Mountain Journal of Mathematics* 48.3 (2018), pp. 819–829. DOI: 10.1216/RMJ-2018-48-3-819. URL: <https://doi.org/10.1216/RMJ-2018-48-3-819>.
- [Fu12] Xiang Fu. “The dominance hierarchy in root systems of Coxeter groups”. In: *Journal of Algebra* 366 (2012), pp. 187–204.
- [GMS09] Olexandr Ganyushkin, Volodymyr Mazorchuk, and Benjamin Steinberg. “On the irreducible representations of a finite semigroup”. In: *Proceedings of the American Mathematical Society* 137.11 (2009), pp. 3585–3592.
- [Gre06] James A Green. “Polynomial representations of  $GL_n$ ”. In: *Algebra Carbondale 1980: Lie Algebras, Group Theory, and Partially Ordered Algebraic Structures Proceedings of the Southern Illinois Algebra Conference, Carbondale, April 18 and 19, 1980*. Springer, 2006, pp. 124–140.
- [HL16] C. Hohlweg and J.-P. Labbé. “On inversion sets and the weak order in Coxeter groups”. In: *European Journal of Combinatorics* 55 (2016), pp. 1–19. DOI: 10.1016/j.ejc.2016.01.002.
- [HLR14] Christophe Hohlweg, Jean-Philippe Labbé, and Vivien Ripoll. “Asymptotical behaviour of roots of infinite Coxeter groups”. In: *Canadian Journal of Mathematics* 66.2 (2014), pp. 323–353.
- [HNW16] C. Hohlweg, P. Nadeau, and N. Williams. “Automata, reduced words and Garside shadows in Coxeter groups”. In: *Journal of Algebra* 457 (2016), pp. 431–456. DOI: 10.1016/j.jalgebra.2016.04.006.
- [HST13] Florent Hivert, Anne Schilling, and Nicolas Thiéry. “The biHecke monoid of a finite Coxeter group and its representations”. In: *Algebra & Number Theory* 7.3 (2013), pp. 595–671.
- [HT09] Florent Hivert and Nicolas M Thiéry. “The Hecke group algebra of a Coxeter group and its representation theory”. In: *Journal of Algebra* 321.8 (2009), pp. 2230–2258.

- [Hul00] Alexander Hulpke. “Conjugacy classes in finite permutation groups via homomorphic images”. In: *Mathematics of computation* 69.232 (2000), pp. 1633–1651.
- [Hum90] J. Humphreys. *Reflection groups and Coxeter groups*. Vol. 29. Cambridge university press, 1990. DOI: 10.1017/CB09780511623646.
- [HZ57] Edwin Hewitt and Herbert S Zuckerman. “The irreducible representations of a semigroup related to the symmetric group”. In: *Illinois Journal of Mathematics* 1.2 (1957), pp. 188–213.
- [Jac09] Nathan Jacobson. *Basic algebra II*. Dover Editions, 2009.
- [Jac45] Nathan Jacobson. “Structure theory of simple rings without finiteness assumptions”. In: *Transactions of the American Mathematical Society* 57.2 (1945), pp. 228–245.
- [Kov92] LG Kovács. “Semigroup algebras of the full matrix semigroup over a finite field”. In: *Proceedings of the American Mathematical Society* 116.4 (1992), pp. 911–919.
- [McA72] Donald B McAlister. “Characters of finite semigroups”. In: *Journal of Algebra* 22.1 (1972), pp. 183–200.
- [Mit+23] J. D. Mitchell et al. *Semigroups - GAP package, Version 5.2.1*. 2023. DOI: 10.5281/zenodo.592893. URL: <http://dx.doi.org/10.5281/zenodo.592893>.
- [MQS15] Ariane M. Masuda, Luciane Quoos, and Benjamin Steinberg. “Character theory of monoids over an arbitrary field”. In: *J. Algebra* 431 (2015), pp. 107–126. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2015.02.017. URL: <https://doi.org/10.1016/j.jalgebra.2015.02.017>.
- [Mun57] WD Munn. “Matrix representations of semigroups”. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 53. 1. Cambridge University Press. 1957, pp. 5–12.
- [OP91] Jan Okniński and Mohan S Putcha. “Complex representations of matrix semigroups”. In: *Transactions of the American Mathematical Society* 323.2 (1991), pp. 563–581.
- [Pin] Jean-Éric Pin. *Mathematical Foundations of Automata Theory*. URL: <https://www.irif.fr/~jep/PDF/MPRI/MPRI.pdf>.
- [Pin20] Jean-Éric Pin. “Mathematical Foundations of Automata Theory”. In: *Cours du MPRI* (2020).
- [Pon58] Iosif Solomonovich Ponizovskii. “On irreducible matrix representations of finite semigroups”. In: *Uspekhi Matematicheskikh Nauk* 13.6 (1958), pp. 139–144.



- [Put88] Mohan S Putcha. *Linear algebraic monoids*. Vol. 133. Cambridge University Press, 1988.
- [Put96] Mohan S Putcha. “Complex representations of finite monoids”. In: *Proceedings of the London Mathematical Society* 3.3 (1996), pp. 623–641.
- [Put99] Mohan S Putcha. “Hecke algebras and semisimplicity of monoid algebras”. In: *Journal of Algebra* 218.2 (1999), pp. 488–508.
- [Rei97] Victor Reiner. “Non-crossing partitions for classical reflection groups”. In: *Discrete Mathematics* 177.1-3 (1997), pp. 195–222. DOI: 10.1016/S0012-365X(96)00365-2.
- [RZ91] John Rhodes and Yechezkel Zalcstein. “Elementary representation and character theory of finite semigroups and its application”. In: *Monoids and semigroups with applications (Berkeley, CA, 1989)* (1991), pp. 334–367.
- [Sal07] Franco V Saliola. “The quiver of the semigroup algebra of a left regular band”. In: *International Journal of Algebra and Computation* 17.08 (2007), pp. 1593–1610.
- [Sal09] Franco V Saliola. “The face semigroup algebra of a hyperplane arrangement”. In: *Canadian Journal of Mathematics* 61.4 (2009), pp. 904–929.
- [Sch06] Manfred Schocker. “The module structure of the Solomon–Tits algebra of the symmetric group”. In: *Journal of Algebra* 301.2 (2006), pp. 554–586.
- [Sch57] Marcel-Paul Schutzenberger. “ $\bar{D}$ -représentation des demi-groupes”. In: *Comptes-Rendus de l’Académie des Sciences* 224 (1957), pp. 1994–1996.
- [Shi87a] Jian Yi Shi. “Alcoves corresponding to an affine Weyl group”. In: *J. London Math. Soc. (2)* 35.1 (1987), pp. 42–55. DOI: 10.1112/jlms/s2-35.1.42.
- [Shi87b] Jian Yi Shi. “Sign types corresponding to an affine Weyl group”. In: *J. London Math. Soc. (2)* 35.1 (1987), pp. 56–74. DOI: 10.1112/jlms/s2-35.1.56.
- [Shi99] Jian Yi Shi. “On two presentations of the affine Weyl groups of classical types”. In: *J. Algebra* 221.1 (1999), pp. 360–383. DOI: 10.1006/jabr.1999.8000.
- [Sta96] Richard P Stanley. “Hyperplane arrangements, interval orders, and trees.” In: *Proceedings of the National Academy of Sciences* 93.6 (1996), pp. 2620–2625.

- [Ste08] Benjamin Steinberg. “Möbius functions and semigroup representation theory II: Character formulas and multiplicities”. In: *Advances in Mathematics* 217.4 (2008), pp. 1521–1557.
- [Ste16] Benjamin Steinberg. *Representation theory of finite monoids*. Springer, 2016.
- [Ste22] Benjamin Steinberg. “Extensions of Theorems of Gaschütz, Žmud’ and Rhodes on Faithful Representations”. In: *Algebras and Representation Theory* (2022), pp. 1–27.
- [SY11] Andrzej Skowroński and Kunio Yamagata. *Frobenius algebras*. Vol. 12. European Mathematical Society, 2011.
- [Thi12] Nicolas M. Thiéry. “Cartan invariant matrices for finite monoids: description and computation using characters”. In: *DMTCS Proceedings* 01 (2012). FPSAC’12 Nagoya, 12 pages [math.RT], pp. 887–898. URL: <http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/viewArticle/dmAR0178>.