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Valeurs aux entiers négatifs de fonctions multizêta de  
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Applications à certaines fonctions zêta de Witten  
rattachées à des algèbres de Lie semi-simples

Simon Rutard

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**THÈSE de DOCTORAT DE L'UNIVERSITÉ JEAN  
MONNET SAINT-ÉTIENNE**

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**Simon Rutard**

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multizêta de type Hurwitz généralisées et de  
leurs dérivées. Applications à certaines  
fonctions zêta de Witten rattachées à des  
algèbres de Lie semi-simples**

---

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# Valeurs aux entiers négatifs de fonctions multizêta de type Hurwitz généralisées et de leurs dérivées. Applications à certaines fonctions zêta de Witten rattachées à des algèbres de Lie semi-simples

## Résumé

Le but de cette thèse est de fournir des formules explicites pour les valeurs directionnelles et les valeurs des dérivées directionnelles de fonctions multizêta de Hurwitz généralisées en des multi-entiers négatifs. On exprimera en particulier les valeurs directionnelles de ces fonctions multizêta en fonction de polynômes de Bernoulli. De plus, on exprimera les valeurs des dérivées directionnelles de ces fonctions multizêta en fonction de polynômes de Bernoulli, de valeurs aux entiers négatifs de la dérivée suivant la variable  $s$  de la fonction zêta d'Hurwitz, et de valeurs aux entiers négatifs de la dérivée de certaines fonctions zêta de type Barnes généralisé.

Comme première application, on déterminera des formules pour les valeurs en  $s = 0$  de la fonction zêta de Witten  $\zeta_{\mathfrak{g}_2}$  et de sa dérivée, ainsi que des valeurs en  $s = 0$  de la fonction zêta de Witten  $\zeta_{s_0(5)}$  et de sa dérivée. On trouvera en particulier une formule asymptotique explicite du nombre de représentations de l'algèbre de Lie  $\mathfrak{g}_2$  en combinant nos résultats avec un théorème de type Meinardus généralisé obtenu par Bridges, Brindle, Bringmann et Franke.

Comme seconde application, on déterminera des formules explicites des fonctions multigamma de type Shintani via des fonctions multigamma de type Barnes.

**Mots-clés :** Fonctions multizêta de type Hurwitz généralisées, Fonctions zêta de Witten, Valeurs et valeurs des dérivées aux entiers négatifs des fonctions multizêta, Représentations des algèbres de Lie semi-simples

**Special values of generalized multiple Hurwitz zeta functions and their derivatives. Applications towards some Witten zeta functions attached to semi-simple Lie algebras**

## Abstract

The goal of this thesis is to provide explicit formulas for directional values and directional derivative values of generalized Hurwitz multizeta functions at non positive integers. We will express the directional values of these multizeta functions in terms of Bernoulli polynomials. Furthermore, we will express the directional derivative values of these multizeta functions in terms of Bernoulli polynomials, of values at nonpositive integer values of the derivative with respect to the variable  $s$  of the Hurwitz zeta function, and of values at nonpositive integers of the derivative of some generalized Barnes zeta functions.

As a first application, we will determine formulas for the values at  $s = 0$  of the Witten zeta function  $\zeta_{\mathfrak{g}_2}$  and its derivative, as well as values at  $s = 0$  of the Witten zeta function  $\zeta_{s_0(5)}$  and its derivative. In particular, we will find an explicit asymptotic formula for the number of representations of the exceptional Lie algebra  $\mathfrak{g}_2$  by combining our result with a generalized Meinardus-type theorem proved by Bridges, Brindle, Bringmann, and Franke.

As a second application, we will determine explicit formulas for Shintani's multigamma functions in terms of Barnes' multigamma functions.

**Keywords:** Generalized multiple zeta functions of Hurwitz type, Witten zeta functions, Values and derivative values at nonpositive integers of multiple zeta functions, Representations of semi-simple Lie algebra



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Thèse de doctorat en Mathématiques  
**Simon Rutard**

Sous la direction de Driss Essouabri

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12 Janvier 2024



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# Chapitre 1

## Introduction

### 1.1 Notations

**Notation.** Soit  $A, B$  des ensembles. On notera l'inclusion large  $A \subset B$  (ce que certains auteurs notent  $A \subseteq B$ ), et l'inclusion stricte  $A \subsetneq B$ .

**Notation.** Soit  $A$  un ensemble, et  $B \subset A$  un sous ensemble de  $A$ , on notera  $B^c = A \setminus B$ .  
On note  $|A| \in \mathbb{N} \cup \{+\infty\}$  le cardinal de  $A$ .

**Notation.** On notera par une lettre en gras pour désigner les multi-indices de la forme  $\mathbf{x} = (x_a)_{a \in A} \in \mathbb{C}^A$  avec  $A$  un ensemble fini. Soit  $B \subset A$ , on notera  $|\mathbf{x}|_B := \sum_{b \in B} x_b$ . Si  $B = A$ , on notera directement  $|\mathbf{x}| := |\mathbf{x}|_A$ .

**Notation.** Soit  $1 \leq k \leq n$  des entiers. On notera  $\mathbf{e}_k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{C}^n$  le vecteur avec un 1 à la  $k$ -ième composante, et des 0 ailleurs.

**Notation.** Soit  $A$  et  $B$  des ensembles finis, et  $\mathbf{x} = (\mathbf{x}_a)_{a \in A} \in (\mathbb{N}^B)^A$ , avec  $\mathbf{x}_a = (x_{a,b})_{b \in B}$  pour tout  $a \in A$ . On pose pour tout  $b \in B$ ,

$$\mathbf{x}(b) := \sum_{a \in A} x_{a,b}.$$

**Remarque 1.1.1.** Soit  $A$  et  $B$  des ensembles finis, et  $\mathbf{x} = (\mathbf{x}_a)_{a \in A} \in (\mathbb{N}^B)^A$ . Le multi-indice  $\mathbf{x}$  correspond à une application  $\mathbf{x} : A \times B \rightarrow \mathbb{N}, (a, b) \mapsto x_{a,b}$ . En particulier, si  $A$  ou  $B$  est vide,  $\mathbf{x}$  correspond à l'application vide. Dans ce cas, on a alors pour tout sous ensemble  $\mathcal{A} \subset A$ ,

$$|\mathbf{x}|_{\mathcal{A}} = 0,$$

et pour tout  $b \in B$ ,

$$\mathbf{x}(b) = 0.$$

**Notation.** Soit  $s \in \mathbb{C}$  un nombre complexe. On note  $s = \sigma + i\tau$  avec  $\sigma$  et  $\tau$  respectivement la partie réelle et la partie imaginaire de  $s$ .

**Notation.** Soit  $x \in \mathbb{R}$  un réel. On note  $H_x$  le demi-plan complexe ouvert d'abscisse  $x$ , et  $\overline{H}_x$  son adhérence :

$$H_x := \{s \in \mathbb{C} \mid \sigma > x\},$$

$$\overline{H}_x := \{s \in \mathbb{C} \mid \sigma \geq x\}.$$

**Notation.** Soit  $a \in \mathbb{C}$ ,  $r \in \mathbb{R}_+$ . On note  $\overline{D}_a(r)$  le disque fermé de centre  $a$  et de rayon  $r$  :

$$\overline{D}_a(r) := \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

**Notation.** Soit  $s \in \mathbb{C}$  un nombre complexe, et  $k \in \mathbb{N}$ . On note  $s_{\Delta,k} := (s, \dots, s) \in \mathbb{C}^k$ . Lorsqu'il n'y a pas d'ambiguïté, on notera simplement  $s_{\Delta}$ .

**Notation.** Soit  $n \in \mathbb{N}$  un entier. On appelle  $n$ -ième nombre harmonique le nombre rationnel suivant

$$h_n := \sum_{k=1}^n \frac{1}{k},$$

avec la convention  $h_0 = 0$ .

On notera  $\gamma = 0.5772156649\dots$  la constante d'Euler [OEI23a].

**Définition-Propriété 1.1.2.** [EMOT81, Chap.1] On note  $\Gamma$  la fonction gamma d'Euler définie sur  $H_0$  par l'intégrale

$$\forall s \in H_0, \quad \Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx.$$

La fonction  $\Gamma$  est holomorphe sur  $H_0$ , et admet un prolongement méromorphe sur  $\mathbb{C}$  avec des pôles simples aux entiers négatifs. On note  $\Gamma$  son prolongement. On note également  $\psi$  la fonction digamma définie sur  $H_0$  par la relation

$$\forall s \in H_0, \quad \psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}.$$

La fonction  $\psi$  est holomorphe sur  $H_0$ , et admet un prolongement méromorphe sur  $\mathbb{C}$ , avec des pôles simples aux entiers négatifs.

Rappelons que  $\Gamma$  admet l'équation fonctionnelle suivante :

$$\Gamma(s+1) = s\Gamma(s) \quad (s \notin -\mathbb{N}),$$

et au voisinage de  $s = 0$ , on a le développement asymptotique suivant

$$\Gamma(s) = \frac{1}{s} - \gamma + \frac{1}{2} \left( \gamma^2 + \frac{\pi^2}{6} \right) s + O(s^2).$$

Aux entiers positifs, on a

$$\Gamma(N+1) = N!.$$

Rappelons également que la fonction  $\frac{1}{\Gamma}$  est holomorphe sur  $\mathbb{C}$ , qu'elle s'annule en les entiers négatifs.

**Notation.** On pose les coefficients binomiaux et multinomiaux de la manière suivante :

Pour tout  $k \in \mathbb{Z}$ , et  $s \in \mathbb{C}$ , on pose

$$\binom{s}{k} = \begin{cases} \frac{s(s-1)\dots(s-k+1)}{k!} & \text{si } k \geq 0, \\ 0 & \text{sinon.} \end{cases}$$

Pour tout entiers  $\mathbf{k} = (k_1, \dots, k_P)$  tels que  $n = k_1 + \dots + k_P$ , on considère le coefficient multinomial :

$$\binom{n}{\mathbf{k}} = \begin{cases} \frac{n!}{k_1! \dots k_P!} & \text{si } k_p \geq 0 \text{ pour tout } p \in \llbracket 1, P \rrbracket, \\ 0 & \text{sinon.} \end{cases}$$

Dans ce manuscrit, on utilisera la détermination principale du logarithme, et on écrira pour tout complexe  $z \in \mathbb{C}$ ,  $z = |z|e^{i \arg(z)}$  avec  $\arg(z) \in ]-\pi, \pi]$ .

Par convention, on dira qu'une somme sur l'ensemble vide est nulle, et qu'un produit sur l'ensemble vide vaut 1.

## 1.2 Tour d'horizon sur les fonctions zêta et multizêta

On s'intéressera dans cette thèse à une classe de fonctions multizêta définie par la série de Dirichlet

$$Z(\mathbf{s}, \mathbf{s}') = \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n} + \mathbf{d}))^{-s'_q},$$

avec  $P \geq 1$ ,  $Q \geq 1$ ,  $\mathbf{d} \in H_0^P$ , et  $l_q$  des formes linéaires de rang  $P$ , dépendants de chaque variable  $n_p$  pour  $1 \leq p \leq P$ . On étudiera en particulier un domaine de convergence, son prolongement méromorphe, puis on établira des relations entre les valeurs directionnelles aux entiers négatifs de  $Z$  et aussi les valeurs spéciales de sa dérivée selon cette même direction.

Dans [Kom08], Komori obtient une nouvelle représentation intégrale des fonctions double zêta de type Mordell-Tornheim. Il déduit de ce résultat une formule des valeurs aux entiers négatifs de cette fonction zêta. Par la suite, via une méthode similaire et en utilisant des nombres de Bernoulli généralisés, il prouvera une formule pour les valeurs aux entiers négatifs des fonctions multizêta plus générales dans [Kom10]. Cependant, ces nombres de Bernoulli généralisés sont souvent difficiles à calculer. Pour des fonctions multizêta de type Euler-Zagier, des formules explicites ont été obtenus par Onozuka [Ono13], et pour une classe plus générale de fonction multizêta de type Euler-Zagier dans [EM20]. Ces formules s'expriment à l'aide de nombres de Bernoulli classiques.

Les travaux exposés ici permettent d'obtenir des relations explicites en fonction de polynômes de Bernoulli classiques pour les valeurs directionnelles aux entiers négatifs de  $Z(\mathbf{s}, \mathbf{s}')$ . De plus, on obtient également des valeurs explicites pour la dérivée directionnelle en les entiers négatifs de  $Z(\mathbf{s}, \mathbf{s}')$ . On obtient ces expressions à l'aide d'une expansion de Crandall (c.f. [Cra12, §9.2]). Dans [BD18], cette stratégie a déjà fourni des approximations pour  $\zeta^{AV}(-N)$  et  $\partial_s \zeta^{AV}(-N)$ , avec  $\zeta^{AV}$  la fonction zêta de type Apostol-Vu sur la diagonale :

$$\zeta^{AV}(s) = \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s}.$$

Dans [BD18], Borwein et Tomkins étudient une fonction zêta de type Tornheim :

$$\zeta_P^T(s) := \sum_{n_1, \dots, n_P \geq 1} \frac{1}{n_1^s \dots n_P^s (n_1 + \dots + n_P)^s},$$

et calculent les valeurs aux entiers négatifs de  $\zeta_P^T(s)$ , et de la dérivée  $\partial_s \zeta_P^T(s)$ . Dans [Ono21], Onodera étudie également cette classe de fonction zêta, et calcule également les valeurs de la dérivée seconde en  $s$  de  $\zeta_P^T(s)$  en les entiers négatifs.

On pourra déduire de nos travaux des formules pour les valeurs spéciales des deux fonctions zêta de Witten suivantes, ainsi que de leurs dérivées :

$$\zeta_{\mathfrak{so}(5)}(s) := 6^s \sum_{n_1, n_2=1}^{+\infty} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s}$$

$$\zeta_{\mathfrak{g}_2}(s) := 120^s \sum_{n_1, n_2=1}^{+\infty} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_2)^s (2n_1 + 3n_2)^s}$$

De même, on pourra en déduire des formules pour les valeurs spéciales de certaines fonctions zêta de type Shintani, ainsi que de leurs dérivées

$$\zeta^{Sh}(s, \mathbf{d} | \mathbf{c}_1, \dots, \mathbf{c}_Q) := \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{q=1}^Q \left( d_q + \sum_{p=1}^P c_{q,p} n_p \right)^{-s},$$

avec  $P, Q \geq 1$  des entiers, et  $d_q \in H_0$  et  $c_{q,p} \in H_0$  pour tout  $1 \leq p \leq P$  et  $1 \leq q \leq Q$ .

### 1.2.1 Quelques résultats sur les valeurs spéciales de certaines fonctions zêta

On souhaite donner ici un bref récapitulatif des résultats sur les valeurs spéciales de la fonction zêta de Riemann, sur la fonction zêta de Hurwitz, sur les fonctions zêta de Barnes, sur les fonctions zêta d'Euler-Zagier, sur certaines fonctions zêta de Witten et sur les fonctions zêta de Shintani. On mentionnera également leur domaine d'holomorphie et de méromorphie. On donnera en particulier des résultats sur les valeurs aux entiers négatifs pour certaines de ces fonctions.

Toutes les fonctions zêta mentionnées dans le paragraphe précédent ont un domaine de convergence de la forme  $H_{\sigma_0} = \{s = \sigma + i\tau \in \mathbb{C} \mid \sigma > \sigma_0\}$  avec  $\sigma_0 \in \mathbb{R}$ , et elles sont holomorphes sur leur demi-plan de convergence. Pour donner un sens aux valeurs aux entiers  $s = N \in \mathbb{Z}$  ne se situant pas dans le demi-plan de convergence de ces fonctions, on doit au préalable discuter de leur prolongement méromorphe, et de montrer que ce prolongement est régulier en l'entier que l'on considère.

On se propose tout d'abord de rappeler la formule d'Euler-Maclaurin.

**Théorème 1.2.1** (Formule d'Euler-Maclaurin). [Ten15] Soit  $k \in \mathbb{N}$  un entier, et  $f : [a, b + 1] \rightarrow \mathbb{C}$  une fonction de classe  $C^{k+1}$  sur  $[a, b]$ . Alors, on a

$$\begin{aligned} \sum_{n=a}^{b-1} f(n) &= \int_a^b f(t) dt + \sum_{n=0}^k \frac{(-1)^{n+1} B_{n+1}}{(n+1)!} (f^{(n)}(b) - f^{(n)}(a)) \\ &\quad + \frac{(-1)^k}{(k+1)!} \int_a^b b_{k+1}(x) f^{(k+1)}(x) dx, \end{aligned}$$

où  $B_n$  désigne le  $n$ -ième nombre de Bernoulli, et  $b_n(t)$  désigne le  $n$ -ième polynôme de Bernoulli périodisé.

On rappelle également l'expression de la fonction zêta de Riemann.

**Définition 1.2.2.** Soit  $s \in \mathbb{C}$ . On appelle fonction zêta de Riemann la fonction suivante

$$\forall s \in H_1, \quad \zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

On verra dans la proposition suivante que l'on peut prolonger cette fonction sur  $\mathbb{C}$  :

**Proposition 1.2.3.** La fonction zêta de Riemann est holomorphe sur  $H_1$ , elle se prolonge méromorphiquement sur  $\mathbb{C}$  avec un seul pôle en  $s = 1$ , qui est simple, de résidu 1.

On peut démontrer le prolongement méromorphe à l'aide de la formule d'Euler-Maclaurin 1.2.1, et cette formule donne au passage les valeurs de la fonction zêta de Riemann aux entiers négatifs en fonction des nombres de Bernoulli :

$$\forall N \in \mathbb{N}, \quad \zeta(-N) = (-1)^N \frac{B_{N+1}}{N+1}.$$

En particulier, on en déduit que  $\zeta(-2N) = 0$  pour tout entier  $N \in \mathbb{N}^*$ .

A l'aide de l'équation fonctionnelle de la fonction zêta de Riemann, on peut également relier les valeurs de  $\zeta$  aux entiers positifs avec les valeurs de sa dérivée aux entiers négatifs.

**Théorème 1.2.4.** Soit  $s \in \mathbb{C} \setminus \{0, 1\}$ , on a la relation fonctionnelle

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s).$$

On en déduit en particulier que, pour tout  $n \in \mathbb{N}^*$ ,

$$\zeta'(-2N) = \frac{(-1)^N (2N)!}{2^{2N+1} \pi^{2N}} \zeta(2N+1).$$

Pour des petites valeurs, on a également ces résultats-ci :

$$\begin{aligned} \zeta'(0) &= -\frac{1}{2} \ln(2\pi), \\ \zeta'(-1) &= \frac{1}{12} - \ln(A), \end{aligned}$$

avec  $A = 1.28242712\dots$  la constante de Glaisher-Kinkelin définie dans [OEI23b].

**Définition 1.2.5.** Soit  $d \in H_0$  un complexe, on appelle fonction zêta d'Hurwitz la fonction zêta

$$\forall s \in H_0, \quad \zeta(s, d) := \sum_{n=0}^{+\infty} \frac{1}{(n+d)^s}.$$

Observons qu'en  $d = 1$ , on retrouve la fonction zêta de Riemann. Cette fonction zêta admet un prolongement méromorphe sur  $\mathbb{C}$ .

**Proposition 1.2.6.** La fonction zêta de Hurwitz se prolonge méromorphiquement sur  $\mathbb{C}$ , avec un unique pôle en  $s = 1$ , qui est simple. De plus, le résidu en ce pôle vaut 1.

*Démonstration.* Soit  $n \in \mathbb{N}^*$  un entier. Via la formule d'Euler-Maclaurin, on obtient pour  $\sigma > 1$  que

$$\begin{aligned} \zeta(s, d) &= \frac{d^{1-s}}{1-s} + \sum_{i=0}^n \binom{-s}{i} \frac{(-1)^i B_{i+1}}{i+1} d^{-s-i} \\ &\quad + (-1)^n \binom{-s}{n+1} \int_0^{+\infty} b_{n+1}(x) (x+d)^{-s-n-1} dx. \end{aligned} \quad (1.1)$$

L'intégrale présente dans la formule précédente converge absolument pour tout  $\sigma > -n$ . Ainsi, cette formule permet de prolonger méromorphiquement la fonction zêta de Hurwitz pour  $\sigma > -n$ , avec un pôle unique en  $s = 1$ , d'ordre 1, et de résidu 1.  $\square$

On a également des résultats sur les valeurs aux entiers négatifs de la fonction zêta de Hurwitz (voir [Apo76, Théorème 12.13]) :

$$\forall N \in \mathbb{N}, \quad \zeta(-N, x) = -\frac{b_{N+1}(x)}{n+1},$$

avec  $b_{N+1}(x)$  le  $N+1$ -ième polynôme de Bernoulli. De même, il était connu par Hurwitz (voir [Ber85, Formule (3)]) que

$$(\partial_s \zeta(s, d))|_{s=0} = \ln(\Gamma(d)) + \zeta'(0) = \ln(\Gamma(d)) - \frac{1}{2} \ln(2\pi). \quad (1.2)$$

Observons également que, pour tout  $s \neq 1$ , et tout  $d \in H_0$ , on a  $\zeta(s, d) = d^{-s} + \zeta(s, d+1)$ . En dérivant selon  $s$ , on trouve alors

$$\partial_s \zeta(s, d)|_{s=-N} = -d^N \ln(d) + \partial_s \zeta(s, d+1)|_{s=-N}. \quad (1.3)$$

Par un jeu de réécriture de la série de Dirichlet  $\zeta\left(s, \frac{1}{2}\right)$ , on obtient que  $\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s)$ . Ainsi, en dérivant des deux côtés par rapport à  $s$ , on trouve

$$\partial_s \zeta\left(s, \frac{1}{2}\right)|_{s=-1} = -\frac{\ln(2)}{6} - \frac{1}{2} \zeta'(-1). \quad (1.4)$$

Des travaux de Miller et Adamchik (voir [MA98]) ont établi des relations explicites entre les valeurs des dérivées de la fonction zêta d'Hurwitz suivant  $s$ , en les entiers négatifs, avec un coefficient rationnel  $d$ , et des valeurs spéciales de logarithme, de fonctions polygamma, et de la fonction zêta de Riemann.

La fonction zêta d'Hurwitz admet une relation fonctionnelle appelée formule d'Hurwitz. Avant de l'exprimer, on a besoin d'introduire la fonction zêta de Lerch :

**Définition 1.2.7.** Soit  $s \in \mathbb{C}$  tel que  $\sigma > 1$ ,  $d \in H_0$ , et  $z \in \mathbb{C}$  tel que  $|z| \leq 1$ . On pose

$$\phi(z, s, d) := \sum_{n \geq 0} \frac{z^n}{(n+d)^s}.$$

Cette fonction zêta admet un prolongement méromorphe sur  $\mathbb{C}$ . Elle est même holomorphe pour tout  $z \neq 1$ , et elle admet un unique pôle, qui est simple, en  $s = 1$  lorsque  $z = 1$  [EMOT81, §1.11].

**Théorème 1.2.8** (Formule d'Hurwitz). [Apo76, §12.7] Soit  $d \in ]0, 1[$  un réel, et  $s \in \mathbb{C}$  un complexe tel que, soit  $d \in ]0, 1[$  et  $\operatorname{Re}(s) > 0$ , soit  $d = 1$  et  $\operatorname{Re}(s) > 1$ . Alors,

$$\zeta(1-s, d) = \frac{\Gamma(s)}{(2\pi)^s} \left( e^{-i\pi s/2} \phi(e^{2i\pi d}, s, 1) + e^{i\pi s/2} \phi(e^{-2i\pi d}, s, 1) \right).$$

Remarquons au passage que le cas  $d = 1$  de la formule précédente correspond à l'équation fonctionnelle de la fonction zêta de Riemann.

On mentionne maintenant quelques résultats sur les fonctions zêta de Barnes. On utilisera certains de ces résultats plus tard dans le manuscrit.

**Définition 1.2.9.** Soit  $P \in \mathbb{N}$  un entier, et  $(c_1, \dots, c_P)$  une famille de complexes dans  $H_0$ , et  $d \in H_0$ . On appelle fonction zêta de Barnes la fonction zêta de la forme

$$\forall s \in H_P, \quad \zeta^B(s, d|c_1, \dots, c_P) := \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{(\sum_{p=1}^P c_p n_p + d)^s}.$$

Il était connu de Barnes que cette fonction zêta était méromorphe sur  $\mathbb{C}$ , avec des pôles simples en certains entiers positifs. Plus précisément :

**Proposition 1.2.10.** [Bar04] La fonction zêta de Barnes  $s \mapsto \zeta^B(s, d|c_1, \dots, c_P)$  admet un prolongement méromorphe sur  $\mathbb{C}$ , avec comme seules singularités des pôles simples en les  $s = 1, \dots, P$ .

La valeur de la dérivée en  $s = 0$  de ces fonctions a été étudiée par Barnes en développant la théorie des fonctions multigamma (voir [Bar01] et [Bar04]). Par la suite, Vardi obtient une relation explicite entre la valeur en  $\frac{1}{2}$  de la fonction multigamma de dimension  $n$ , et le déterminant régularisé du Laplacien de la  $n$ -sphère dans [Var88]. Il fournit également une formule explicite de la valeur en  $\frac{1}{2}$  de la fonction multigamma de dimension  $n$  en fonction des valeurs aux entiers négatifs de la dérivée de la fonction zêta de Riemann.

Quelques-uns de nos résultats s'exprimeront en fonction de valeurs spéciales de ces fonctions là. On aura en particulier besoin d'un résultat démontré par Sakane et Aoki dans [SA22]. On pose tout d'abord

$$C_{P,x}(t) := (t-x+P-1)(t-x+P-2)\dots(t-x+1) \in (\mathbb{Z}[x])[t] \quad (P \geq 2),$$

et  $C_{1,x}(t) = 1$ . Remarquons au passage que  $C_{2,x}(t) = t-x+1$ .

**Théorème 1.2.11.** [SA22, Théorème 4] Soit  $d \in H_0$ , et  $\mathbf{c} = (c_1, \dots, c_P) \in \mathbb{Q}^P$  des rationnels. On écrit  $c_1 = \frac{a_1}{b_1}$ , ...,  $c_P = \frac{a_P}{b_P}$  avec  $a_p, b_p \geq 1$  des entiers premiers entre eux pour tout  $p$ . On pose  $x(\mathbf{c}) := \frac{\operatorname{ppcm}(a_1, \dots, a_P)}{\operatorname{pgcd}(b_1, \dots, b_P)}$ , et  $\beta_1 := \frac{x(\mathbf{c})}{c_1}$ , ...,  $\beta_P := \frac{x(\mathbf{c})}{c_P}$ , alors on a

$$\zeta^B(s, d|c_1, \dots, c_P) = \frac{x(\mathbf{c})^{-s}}{(P-1)!} \sum_{v_1=0}^{\beta_1-1} \dots \sum_{v_{P-1}=0}^{\beta_{P-1}-1} \sum_{k=0}^{P-1} C_{P, \frac{d+c_1 v_1 + \dots + c_P v_P}{x(\mathbf{c})}}^{(k)}(0) \zeta \left( s-k, \frac{d+c_1 v_1 + \dots + c_P v_P}{x(\mathbf{c})} \right)$$

On démontrera une généralisation du théorème précédent dans le Théorème B, en utilisant une stratégie similaire à celle utilisée par Aoki et Sakane dans [SA22].



**Exemple 1.2.12.** *On trouve*

$$\zeta^B(s, d|1, 1) = (1-d)\zeta(s, d) + \zeta(s-1, d),$$

$$\zeta^B(s, d|1, 2) = 2^{-s} \left[ \left(1 - \frac{d}{2}\right) \zeta\left(s, \frac{d}{2}\right) + \left(1 - \frac{d+1}{2}\right) \zeta\left(s, \frac{d+1}{2}\right) + \zeta\left(s-1, \frac{d}{2}\right) + \zeta\left(s-1, \frac{d+1}{2}\right) \right],$$

$$\zeta^B(s, d|1, 3) = 3^{-s} \left[ \left(1 - \frac{d+2}{3}\right) \zeta\left(s, \frac{d+2}{3}\right) + \left(1 - \frac{d+1}{3}\right) \zeta\left(s, \frac{d+1}{3}\right) + \left(1 - \frac{d}{3}\right) \zeta\left(s, \frac{d}{3}\right) \right. \\ \left. + \zeta\left(s-1, \frac{d+2}{3}\right) + \zeta\left(s-1, \frac{d+1}{3}\right) + \zeta\left(s-1, \frac{d}{3}\right) \right],$$

$$\zeta^B(s, d|2, 3) = 6^{-s} \left[ \left(1 - \frac{d+7}{6}\right) \zeta\left(s, \frac{d+7}{6}\right) + \left(1 - \frac{d+5}{6}\right) \zeta\left(s, \frac{d+5}{6}\right) + \left(1 - \frac{d+4}{6}\right) \zeta\left(s, \frac{d+4}{6}\right) \right. \\ \left. + \left(1 - \frac{d+3}{6}\right) \zeta\left(s, \frac{d+3}{6}\right) + \left(1 - \frac{d+2}{6}\right) \zeta\left(s, \frac{d+2}{6}\right) + \left(1 - \frac{d}{6}\right) \zeta\left(s, \frac{d}{6}\right) \right. \\ \left. + \zeta\left(s-1, \frac{d+7}{6}\right) + \zeta\left(s-1, \frac{d+5}{6}\right) + \zeta\left(s-1, \frac{d+4}{6}\right) + \zeta\left(s-1, \frac{d+3}{6}\right) + \zeta\left(s-1, \frac{d+2}{6}\right) \right. \\ \left. + \zeta\left(s-1, \frac{d}{6}\right) \right].$$

On obtient au passage les valeurs spéciales  $\zeta^B(0, 2|1, 1) = \frac{5}{12}$ ,  $\zeta^B(0, 3|1, 2) = \frac{11}{24}$ ,  $\zeta^B(0, 4|1, 3) = \frac{19}{36}$ , et  $\zeta^B(0, 5|2, 3) = \frac{31}{72}$ . On peut également dériver les formules obtenues précédemment pour  $\zeta^B(s, d|c_1, c_2)$ . On obtient alors

$$(\zeta^B)'(0, 2|1, 1) = -\zeta'(0, 2) + \zeta'(-1, 2),$$

$$(\zeta^B)'(0, 3|1, 2) = -\frac{11 \ln(2)}{24} - \frac{1}{2} \zeta'\left(0, \frac{3}{2}\right) - \zeta'(0, 2) + \zeta'\left(-1, \frac{3}{2}\right) + \zeta'(-1, 2),$$

$$(\zeta^B)'(0, 4|1, 3) = -\frac{19 \ln(3)}{36} - \zeta'(0, 2) - \frac{2}{3} \zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3} \zeta'\left(0, \frac{4}{3}\right) + \zeta'(-1, 2) + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right),$$

$$(\zeta^B)'(0, 5|2, 3) = -\frac{31 \ln(6)}{72} - \zeta'(0, 2) - \frac{2}{3} \zeta'\left(0, \frac{5}{3}\right) - \frac{1}{2} \zeta'\left(0, \frac{3}{2}\right) - \frac{1}{3} \zeta'\left(0, \frac{4}{3}\right) - \frac{1}{6} \zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6} \zeta'\left(0, \frac{5}{6}\right) \\ + \zeta'(-1, 2) + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{3}{2}\right) + \zeta'\left(-1, \frac{4}{3}\right) + \zeta'\left(-1, \frac{7}{6}\right) + \zeta'\left(-1, \frac{5}{6}\right),$$

en notant  $(\zeta^B)' = \partial_s \zeta^B$ , et  $\zeta'(s, d) = \partial_s \zeta(s, d)$ .

Shintani a introduit des fonctions zêta lui permettant d'étudier des fonctions zêta de Dedekind rattachées à un corps de nombre totalement réel (voir [Shi76b] [Shi76a], [Shi77a], [Shi77b], [Shi77c], [Shi80]). Ces fonctions généralisent en particulier les fonctions zêta de Barnes :

**Définition 1.2.13.** *Soit  $P, Q \in \mathbb{N}$  des entiers,  $\mathbf{c} = (c_{q,p})_{1 \leq p \leq P, 1 \leq q \leq Q}$  et  $(d_q)_{1 \leq q \leq Q}$  deux familles de complexes dans  $H_0$ . On appelle fonction zêta de Shintani la fonction zêta suivante*

$$\zeta^{Sh}(s, \mathbf{d}|\mathbf{c}) = \sum_{n_1, \dots, n_P \geq 0} \prod_{q=1}^Q (c_{q,1}n_1 + \dots + c_{q,P}n_P + d_q)^{-s}.$$

Observons que, pour  $Q = 1$ , on retrouve la définition de la fonction zêta de Barnes. Cette fonction zêta est holomorphe sur  $H_{P/Q}$ , et admet un prolongement méromorphe sur  $\mathbb{C}$ . Shintani a prouvé que cette fonction possède les mêmes pôles que ceux de la fonction  $\frac{\Gamma(Ps - Q)}{\Gamma(s)}$ . Il a également prouvé certaines formules aux entiers négatifs de ces fonctions. Une partie de notre travail permet en particulier d'étudier les valeurs de certaines de ces fonctions aux entiers négatifs, ainsi que les valeurs de leurs dérivées.

On déduit de ce qui précède que les fonctions zêta de Barnes et de Shintani sont régulières en les entiers négatifs. La valeur de leurs dérivées par rapport à la variable  $s$  en  $s = 0$  permettent de définir des fonctions multigamma, introduites par Barnes dans [Bar04], puis généralisées par Friedmann et Ruijsenaars.

**Définition 1.2.14.** Soit  $P \in \mathbb{N}^*$  un entier, et  $(c_1, \dots, c_P)$  une famille de complexes dans  $H_0$ , et  $d \in H_0$ . On appelle fonction multigamma de Barnes la fonction

$$\Gamma_P(d|c_1, \dots, c_P) := \exp((\partial_s \zeta^B(s, d|c_1, \dots, c_P))|_{s=0}).$$

**Exemple 1.2.15.** Soit  $d \in H_0$ , on a

$$\begin{aligned} \Gamma_0(d) &= \frac{1}{d} \\ \Gamma_1(d|1) &= \frac{1}{\sqrt{2\pi}} \Gamma(d), \end{aligned}$$

où  $\Gamma$  désigne la fonction gamma d'Euler.

Barnes obtient aussi une relation fonctionnelle

$$\Gamma_P(d|c_1, \dots, c_P) = \Gamma_{P-1}(d|c_1, \dots, c_{P-1}) \Gamma_P(d + c_P|c_1, \dots, c_P),$$

qui généralise la relation fonctionnelle de la fonction gamma d'Euler.

Via les fonctions zêta de type Shintani, Friedman et Ruijsenaars ont introduit dans [FR04] des fonctions multigamma de type Shintani, via une construction similaire à celle utilisée par Barnes.

**Définition 1.2.16.** Soit  $P, Q \in \mathbb{N}^*$  des entiers,  $\mathbf{c} = (c_{q,p})_{1 \leq p \leq P, 1 \leq q \leq Q}$  et  $(d_q)_{1 \leq q \leq Q}$  deux familles de complexes dans  $H_0$ . On appelle fonction multigamma de Shintani la fonction :

$$\Gamma_P(\mathbf{d}|\mathbf{c}_1, \dots, \mathbf{c}_P) := \exp((\partial_s \zeta^{Sh}(s, \mathbf{d}|\mathbf{c}_1, \dots, \mathbf{c}_P))|_{s=0}),$$

en notant  $\mathbf{c}_q = (c_{q,p})_{p \in [1, P]}$ .

Dans [FR04], Friedman et Ruijsenaars étudient ces fonctions multigamma de Shintani pour en déduire des formules de type Raabe, généralisant la formule de Raabe suivante :

$$\int_0^1 \ln(\Gamma(x)) dx = \ln(\sqrt{2\pi}).$$

On discutera à présent d'une classe de fonctions zêta rattachées à des algèbres de Lie. Witten a introduit dans [Wit91] la série

$$\sum_{\rho} \frac{1}{\dim(\rho)^{2g}},$$

où  $g$  est un entier positif, et  $\rho$  parcourt les représentations irréductibles, à isomorphisme près, d'une algèbre de Lie  $\mathfrak{g}$ , et  $g \in \mathbb{N}^*$ . Witten démontre en particulier que

$$\sum_{\rho} \frac{1}{\dim(\rho)^{2g}} \in \pi^{2g} \mathbb{Q}.$$

Zagier introduira par la suite les fonctions zêta de Witten :

**Définition 1.2.17.** [Zag94] Soit  $\mathfrak{g}$  une algèbre de Lie semisimple. On appelle fonction zêta de Witten rattachée à  $\mathfrak{g}$  la fonction zêta

$$\zeta_{\mathfrak{g}}(s) := \sum_{\rho} \dim(\rho)^{-s},$$

où  $\rho$  parcourt les classes d'isomorphismes des représentations irréductibles de  $\mathfrak{g}$ .

On introduit ensuite  $r_{\mathfrak{g}}(n)$  le nombre de classes d'isomorphismes parmi les représentations de dimension  $n$ .

**Définition 1.2.18.** *Soit  $\mathfrak{g}$  une algèbre de Lie semi-simple. Notons  $r_{\mathfrak{g}}(n)$  le nombre de représentations de dimension  $n$  de  $\mathfrak{g}$ , à isomorphisme près.*

Il est bien connu qu'une représentation d'une algèbre de Lie semi-simple se décompose de manière unique en somme directe de représentations irréductibles. Ainsi pour une représentation  $\rho$  de  $\mathfrak{g}$ , on a en particulier qu'il existe des entiers  $k_1, \dots, k_m$  et des représentations irréductibles  $\phi_1, \dots, \phi_m$  tels que

$$\dim(\rho) = \sum_{i=1}^m k_i \dim(\phi_i).$$

Plus précisément, pour les algèbres de Lie  $\mathfrak{so}(5)$  et  $\mathfrak{g}_2$ , on a :

**Exemple 1.2.19.** [Hum72, §24.3]

1) L'ensemble des classes d'isomorphismes des représentations irréductibles de  $\mathfrak{sl}(3)$  est de la forme  $(\phi_{i,j})_{i,j \geq 1}$ , avec

$$\dim(\phi_{i,j}) = \frac{ij(i+j)}{2}.$$

On obtient que le nombre de représentations irréductibles de dimension  $n$  de  $\mathfrak{sl}(3)$  est

$$\left| \left\{ (i, j) \in \mathbb{N}^{*2} \mid \frac{ij(i+j)}{2} = n \right\} \right|,$$

et on obtient une expression du nombre de représentations  $r_{\mathfrak{sl}(3)}(n)$  de  $\mathfrak{sl}(3)$  :

$$\begin{aligned} r_{\mathfrak{sl}(3)}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\phi_{i,j}) = n \right\} \right|, \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)}{2} = n \right\} \right|. \end{aligned}$$

2) L'ensemble des classes d'isomorphismes des représentations irréductibles de  $\mathfrak{so}(5)$  est de la forme  $(\phi_{i,j})_{i,j \geq 1}$ , avec

$$\dim(\phi_{i,j}) = \frac{ij(i+j)(i+2j)}{3!}.$$

On obtient que le nombre de représentations irréductibles de dimension  $n$  de  $\mathfrak{so}(5)$  est

$$\left| \left\{ (i, j) \in \mathbb{N}^{*2} \mid \frac{ij(i+j)(i+2j)}{3!} = n \right\} \right|,$$

et on obtient une expression du nombre de représentations  $r_{\mathfrak{so}(5)}(n)$  de  $\mathfrak{so}(5)$  :

$$\begin{aligned} r_{\mathfrak{so}(5)}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\phi_{i,j}) = n \right\} \right| \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)}{3!} = n \right\} \right|. \end{aligned}$$

3) L'ensemble des classes d'isomorphismes des représentations irréductibles de  $\mathfrak{g}_2$  est de la forme  $(\phi_{i,j})_{i,j \geq 1}$ , avec

$$\dim(\phi_{i,j}) = \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!}.$$

On obtient que le nombre de représentations irréductibles de dimension  $n$  de  $\mathfrak{g}_2$  est

$$\left| \left\{ (i,j) \in \mathbb{N}^{*2} \mid \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|,$$

et on obtient une expression du nombre de représentations  $r_{\mathfrak{g}_2}(n)$  de  $\mathfrak{g}_2$  :

$$\begin{aligned} r_{\mathfrak{g}_2}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\phi_{i,j}) = n \right\} \right| \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|. \end{aligned}$$

Posons  $f_{\mathfrak{g}}(n)$  le nombre de représentations irréductibles de dimension  $n$  d'une algèbre de Lie semi-simple  $\mathfrak{g}$ . Alors par définition des fonctions zêta de Witten, on a

$$\zeta_{\mathfrak{g}}(s) = \sum_{n=1}^{+\infty} \frac{f_{\mathfrak{g}}(n)}{n^s}.$$

On verra dans le Chapitre 6 que ces fonctions zêta admettent une écriture simplifiée à l'aide de produits de formes linéaires, via le Théorème de Weyl 6.2.9.

**Exemple 1.2.20.** [KMT24] On pose  $\mathfrak{sl}_{n+1} = \{X \in \mathcal{M}_{n+1}(\mathbb{C}) \mid \text{tr}(X) = 0\}$ , et  $\mathfrak{so}_{2n+1} = \{X \in \mathcal{M}_{2n+1}(\mathbb{C}) \mid {}^t X + X = 0\}$ , où  $\text{Tr}$  désigne la trace, et  ${}^t X$  désigne la transposée d'une matrice. Pour ces algèbres de Lie, on peut associer des fonctions de zêta de Witten :

$$\begin{aligned} \zeta_{\mathfrak{sl}(2)}(s) &= \zeta(s), \\ \zeta_{\mathfrak{sl}(3)}(s) &= 2^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s}, \\ \zeta_{\mathfrak{so}(5)}(s) &= 6^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s}. \end{aligned}$$

Pour l'algèbre de Lie exceptionnelle  $\mathfrak{g}_2$  décrite dans [Bou81], on a

$$\zeta_{\mathfrak{g}_2}(s) = 120^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_2)^s (2n_1 + 3n_2)^s}.$$

Notons que dans la littérature, on trouve parfois des variantes multivariées de ces fonctions zêta dans les travaux de Matsumoto, Komori, Yasushi et Tsumura [KMT10c], [KMT10b], [KMT12], [KMT11b]. Ces auteurs donnent une expression de l'ensemble des singularités de ces fonctions multizêta de Witten. Cela nous permet en particulier d'obtenir des candidats pôles pour la fonction zêta de Witten univariée correspondante.

- De [KMT10d, Théorème 6.2], on en déduit que la fonction zêta  $\zeta_{\mathfrak{so}(5)}(s)$  admet des pôles de la forme

$$s = \frac{1}{2}, \quad s = \frac{1-k}{3} \quad (k \in \mathbb{N}). \quad (1.5)$$

- De [KMT11a, Théorème 3.1], on en déduit que la fonction zêta  $\zeta_{\mathfrak{g}_2}(s)$  admet des pôles de la forme

$$s = \frac{1}{3}, \quad s = \frac{1-k}{5} \quad (k \in \mathbb{N}). \quad (1.6)$$

On montrera via la Proposition 2.2.4 que les entiers négatifs ne sont pas des pôles pour  $\zeta_{\mathfrak{so}(5)}(s)$  et pour  $\zeta_{\mathfrak{g}_2}(s)$ .

Ces fonctions zêta sont très adaptées pour étudier les représentations des algèbres de Lie semi-simples, et on peut exprimer leurs valeurs spéciales afin d'obtenir des informations sur ses représentations. En particulier, des travaux de Romik [Rom17] permettent d'étudier le comportement asymptotique du nombre de représentations de degré  $n$  de  $\mathfrak{sl}(3)$  en fonctions des valeurs  $\zeta_{\mathfrak{sl}(3)}(0)$  et  $\zeta'_{\mathfrak{sl}(3)}(0)$ , ainsi que des résidus de  $\zeta_{\mathfrak{sl}(3)}(s)$  en des singularités de  $\zeta_{\mathfrak{sl}(3)}$ . Kurokawa et Ochiai dans [KO13] ont étudié les valeurs aux entiers négatifs de  $\zeta_{\mathfrak{sl}(3)}(s)$ , puis en ont déduit des résultats pour les valeurs de certaines fonctions zêta de Witten  $p$ -adiques.

Onodera a par la suite obtenu d'autres formules en étudiant une fonction zêta de type Mordell-Tornheim :

$$\zeta_P^{MT}(s) := \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{(n_1 + d_1)^s \dots (n_P + d_P)^s (n_1 + d_1 + \dots + n_P + d_P)^s},$$

avec  $P \geq 2$  un entier. Observons qu'en posant  $P = 2$ , on obtient  $2^{-s} \zeta_{\mathfrak{sl}(3)}(s)$ .

Onodera calcule les valeurs de cette fonction zêta en les entiers négatifs via une expansion de Crandall. Cette stratégie consiste à découper le domaine d'intégration d'une représentation intégrale de la fonction zêta qui nous intéresse, le domaine dépendant d'une variable libre. Dans les faits, Onodera obtient l'expression suivante :

$$\begin{aligned} \zeta_P^{MT}(s, \mathbf{d}) &= \frac{1}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{(n_1 + d_1)^s \dots (n_P + d_P)^s (n_1 + d_1 + \dots + n_P + d_P)^s} \int_{\theta(n_1 + d_1 + \dots + n_P + d_P)}^{+\infty} e^{-y} y^{s-1} dy \\ &\quad + \frac{1}{\Gamma(s)} \int_0^\theta \prod_{p=1}^P e^{-d_p y_p} \phi(e^{-y}, s, d_p) dy_1 \dots dy_P \end{aligned}$$

avec  $\phi$  la fonction zêta de Lerch décrite dans la Définition 1.2.7, et où  $\theta$  est une variable libre réelle suffisamment petite. Le premier terme à droite de l'égalité précédente aura le bon goût d'être holomorphe et de s'annuler en les entiers négatifs. Le second terme peut être réexprimé via la formule d'Erdélyi 2.6, et Onodera obtient finalement les formules suivantes :

**Théorème 1.2.21.** [Ono21, Théorème 3, Théorème 4] Soit  $\mathbf{d} \in \mathbb{R}_*^{+P}$  et  $t \in \llbracket 1, P-1 \rrbracket$  tel que  $t < |\mathbf{d}|$ . Pour tout entier  $N \in \mathbb{N}$ , on a

$$\begin{aligned} \zeta_P^{MT}(-N, \mathbf{d}) &= \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \frac{(-1)^{(N+1)(|\mathcal{P}|)} N!^{|\mathcal{P}|+1}}{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|_{|\mathcal{P}|} = (|\mathcal{P}|+1)N + |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!} \\ \zeta_P^{MT'}(-N, \mathbf{d}) &= \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}^c|(N+1)-1} N!^{|\mathcal{P}^c|} \left( \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ u \geq 0 \\ |\mathbf{k}|_{|\mathcal{P}|+u} = (|\mathcal{P}^c|+1)N + |\mathcal{P}^c|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!} \right) \\ &\quad \cdot \left( \frac{(-1)^N \zeta'(-N - u, |\mathbf{d}| - t)}{u!} + \frac{1}{|\mathcal{P}^c|} \sum_{p \in \mathcal{P}^c} \frac{(-1)^N \zeta'(-N - u, d_p)}{u!} \right) \end{aligned}$$

où l'on a noté  $\zeta_P^{MT'}(s, \mathbf{d}) := \partial_s \zeta_P^{MT}(s, \mathbf{d})$

Onodera obtient également des valeurs pour les dérivées secondes [Ono21, Théorème 5], que l'on ne détaillera pas ici. On cherchera dans cette thèse à généraliser le théorème ci-dessus pour une classe de fonctions zêta plus large. Cela nous permettra en particulier d'obtenir des valeurs aux entiers négatifs de  $\zeta_{\mathfrak{so}(5)}(s)$  et de  $\zeta_{\mathfrak{g}_2}(s)$ , ainsi que celles de leurs dérivées premières.

### 1.2.2 Quelques résultats sur certaines fonctions multizêta

On donnera ici quelques résultats sur les valeurs spéciales de certaines fonctions multizêta rattachées à des produits de formes linéaires. En particulier, on détaillera des expressions de ces valeurs pour les fonctions multizêta d'Euler Zagier :

**Définition 1.2.22.** Soit  $P \in \mathbb{N}^*$ , et  $\mathbf{c} = (c_1, \dots, c_P) \in H_0^P$ ,  $\mathbf{d} = (d_1, \dots, d_P) \in \mathbb{C}^P$  des complexes tels que  $\operatorname{Re}(d_i) > -\operatorname{Re}(c_i)$  pour tout  $p \in \llbracket 1, P \rrbracket$ . On appelle fonction multizêta d'Euler-Zagier généralisée la fonction

$$\zeta_P^{EZ}(\mathbf{s}, \mathbf{d}) = \sum_{\substack{n_1 \geq 1 \\ n_2, \dots, n_P \geq 0}} \prod_{p=1}^P (c_1 n_1 + \dots + c_p n_p + d_p)^{-s_p}.$$

On notera  $\zeta_P^{EZ}(\mathbf{s}) := \zeta(\mathbf{s}, \mathbf{1}, (0, 1, \dots, P-1))$  la fonction multizêta d'Euler Zagier.

Observons que, via un changement de variable, on peut réécrire la fonction zêta d'Euler-Zagier "classique" sous la forme suivante,

$$\zeta_P^{EZ}(\mathbf{s}) = \sum_{0 < n_1 < \dots < n_P} \frac{1}{n_1^{s_1} \dots n_P^{s_P}}.$$

**Proposition 1.2.23.** La fonction multizêta d'Euler-Zagier généralisée  $\zeta_P^{EZ}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{d})$  est holomorphe sur  $\{(s_1, \dots, s_P) \in \mathbb{C}^P \mid \sigma_k + \dots + \sigma_P > P + 1 - k, 1 \leq k \leq P\}$ , et est méromorphe sur  $\mathbb{C}^P$ , avec des pôles inclus dans les hyperplans de la forme

$$s_p + \dots + s_P = (P + 1 - p) - k_p, \quad (1 \leq p \leq P, k_1, \dots, k_P \in \mathbb{N}).$$

On voit de la proposition précédente que la quantité  $\zeta_P^{EZ}(-\mathbf{N}, \mathbf{d})$  n'a pas toujours un sens. De plus, un résultat de [AET01] précise même que la plupart des multi-entiers négatifs sont en fait des singularités pour  $\zeta_P^{EZ}(\mathbf{s})$ . Afin d'étudier tout de même les valeurs de cette fonction, Akiyama, Egami et Tanigawa étudient les valeurs "régulières" de cette fonction zêta en posant

$$\zeta_P^{EZ, \text{reg}}(-\mathbf{N}) := \lim_{s_1 \rightarrow -N_1} \dots \lim_{s_P \rightarrow -N_P} \zeta_P^{EZ}(\mathbf{s}).$$

Les auteurs introduisent également des valeurs "non régulières" en inversant les limites précédentes, en posant

$$\zeta_P^{EZ, \text{non-reg}}(-\mathbf{N}) := \lim_{s_P \rightarrow -N_P} \dots \lim_{s_1 \rightarrow -N_1} \zeta_P^{EZ}(\mathbf{s}).$$

Dans [AET01] et dans [AT01], les auteurs obtiennent une formule récursive entre les valeurs régulières  $\zeta_P^{EZ, \text{reg}}(-\mathbf{N})$  (respectivement  $\zeta_P^{EZ, \text{non-reg}}(-\mathbf{N})$  avec  $\mathbf{N} \in \mathbb{N}^P$ , et les valeurs régulières de la forme  $\zeta_{P-1}^{EZ, \text{reg}}(-\mathbf{N}')$  (respectivement  $\zeta_{P-1}^{EZ, \text{non-reg}}(-\mathbf{N}')$ ), avec  $\mathbf{N}' \in \mathbb{N}^{P-1}$  afin d'obtenir une expression explicite des valeurs régulières et non régulières aux entiers négatifs. Ces résultats ont par la suite été généralisés par Sasaki dans [Sas09a] et dans [Sas09b] à des ordres de limites différentes que celles des valeurs régulières et non-régulières aux multi-entiers négatif.

On peut également étudier les valeurs "directionnelles" de cette fonction multizêta. Dans [Kom10], Komori introduit la notion de valeur directionnelle pour de telles fonctions multizêta, et donne une expression de celles-ci pour des fonctions multi-zêta de type Hurwitz-Lerch. Komori en déduit alors des formules pour les valeurs aux entiers négatifs de la fonction zêta de Shintani et d'Euler-Zagier, en utilisant des nombres de Bernoulli généralisés (souvent non explicites).

**Définition 1.2.24.** Soit  $P, Q \in \mathbb{N}^*$  des entiers,  $\mathbf{d} = (d_1, \dots, d_Q)$ ,  $\mathbf{c} = (c_{q,p})_{(q,p) \in \llbracket 1, Q \rrbracket \times \llbracket 1, P \rrbracket}$  et  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_P)$  des complexes tels que  $c_{q,p} \in \overline{H}_0$ ,  $d_q \in H_0$  et  $\xi_q \in \mathbb{T} = \mathbb{C}/2i\pi\mathbb{Z}$ . On appelle fonction multizêta d'Hurwitz-Lerch la fonction multizêta

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s}) = \sum_{n_1=0}^{+\infty} \dots \sum_{n_P=0}^{+\infty} \frac{e^{\xi_1 n_1} \dots e^{\xi_P n_P}}{(c_{1,1} n_1 + \dots + c_{1,P} n_P + d_1)^{s_1} \dots (c_{Q,1} n_1 + \dots + c_{Q,P} n_P + d_Q)^{s_Q}}.$$

On pose également les valeurs directionnelles de  $\zeta^{HL}$ , suivant une direction  $\boldsymbol{\mu} \in \mathbb{C}^P$  :

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N}) := \lim_{s \rightarrow 0} \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N} + s\boldsymbol{\mu}).$$

Komori montre que cette fonction multizêta est holomorphe sur  $\{\mathbf{s} \in \mathbb{C} \mid \forall q \in \llbracket 1, Q \rrbracket, \operatorname{Re}(s_q) > \sigma_{0,q}\}$  avec  $\sigma_{0,q} > 0$  pour tout  $q \in \llbracket 1, Q \rrbracket$ . Il démontre également que cette fonction multizêta admet un prolongement méromorphe sur  $\mathbb{C}^Q$ , et que les valeurs directionnelles définies précédemment ont un sens. Avant de brièvement résumer ses résultats, on doit imposer les conditions suivantes sur les données de la fonction multizêta  $\zeta^{HL}$  :

On suppose qu'il existe un entier  $P_0 \in \llbracket 1, P \rrbracket$  et un ensemble  $\mathcal{B} \subset \llbracket 1, Q \rrbracket \times \llbracket 1, P_0 \rrbracket$  tel que

$$\begin{cases} c_{q,p} = 0 \text{ si et seulement si } (q,p) \in \mathcal{B}^c \\ \xi_p = 0 \text{ si et seulement si } p \in \llbracket 1, P_0 \rrbracket, \end{cases}$$

avec  $\mathcal{B}^c = \llbracket 1, Q \rrbracket \times \llbracket 1, P_0 \rrbracket \setminus \mathcal{B}$ . On pose  $\mathcal{B}_{\bullet p} := \{q \mid (q,p) \in \mathcal{B}\}$ . On suppose que  $\mathcal{B}_{\bullet p} \neq \emptyset$  pour tout  $p \in \llbracket 1, P_0 \rrbracket$ .

Komori donne dans [Kom10] une représentation intégrale surfacique (c'est-à-dire une représentation intégrale dont le domaine est une surface) de la fonction multizêta  $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s})$ . Cette représentation intégrale surfacique généralise en particulier la représentation intégrale usuelle de la fonction zêta de Riemann

$$\zeta(s) = \frac{1}{\Gamma(s)(e^{2i\pi s} - 1)} \int_C \frac{z^{s-1}}{e^z - 1} dz,$$

où  $C$  désigne le contour de Hankel [EMOT81, §1.10]. Au passage, il est bon de rappeler que cette représentation intégrale de  $\zeta(s)$  permet en particulier d'obtenir une formule pour les valeurs de la fonction zêta de Riemann aux entiers négatifs à l'aide des nombres de Bernoulli, en utilisant la relation

$$\frac{z}{e^z - 1} = \sum_{k=0}^{+\infty} B_k \frac{z^k}{k!}.$$

Dans la même veine, Komori utilise ensuite la représentation intégrale surfacique obtenue dans [Kom10, Théorème 3.14] pour obtenir une expression des valeurs directionnelles en fonction de nombres de Bernoulli généralisés  $B_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{k})$  avec  $\mathbf{k} \in \mathbb{N}^Q$ . Pour voir la construction de ces nombres, voir [Kom10, Définition 3.20].

**Théorème 1.2.25.** [Kom10, Théorème 3.14, Corollaire 3.16] *Sous les hypothèses décrites précédemment, la fonction multizêta  $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s})$  admet un prolongement méromorphe en  $\mathbf{s}$  sur tout  $\mathbb{C}^P$ . De plus, ses singularités sont incluses dans les hyperplans de la forme*

$$\begin{cases} \sum_{q \in \mathcal{Q}} s_q = Q - k & (\mathcal{Q} \subset \llbracket 1, Q \rrbracket, |\mathcal{Q}| \geq 2, k \in \mathbb{N}) \\ s_q = 1, \dots, P & (1 \leq q \leq Q). \end{cases}$$

De plus, Komori démontre dans [KMT10a, Théorème 3.22] que les valeurs directionnelles  $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}}$  ont un sens, et il en donne une expression non explicite en général, en imposant une condition de non-annulation sur la somme des directions. On obtient au passage que, si l'on arrive à réexprimer les nombres de Bernoulli généralisés dans la formule obtenue par Komori, alors cette formule permet de montrer si un multientier négatif est une singularité pour  $\zeta^{HL}$ . En effet, si l'on obtient deux valeurs distinctes de  $\zeta^{HL}$  en un multientier négatifs, selon deux directions différentes, cela implique en particulier que le multientier négatif est une singularité.

**Remarque 1.2.26.** *La formule directionnelle [KMT10a, Théorème 3.22] obtenue par Komori n'est pas très explicite, puisqu'il est parfois difficile d'obtenir une expression satisfaisante des nombres de Bernoulli généralisés qu'il introduit dans son article.*

Dans [EM20], Essouabri et Matsumoto ont étudié les valeurs directionnelles de la fonction multizêta d'Euler-Zagier généralisée. On donnera ici seulement une formule très simplifiée et beaucoup moins détaillée :

**Théorème 1.2.27.** [EM20, Théorème 1] Soit  $\mathbf{N} \in \mathbb{N}^P$ , et  $\mathbf{c} = (c_1, \dots, c_P) \in \mathbb{C}^P$  tel que  $\operatorname{Re}(\mu_p + \dots + \mu_P) \neq 0$  pour tout  $p \in \llbracket 1, P \rrbracket$ . Alors la quantité  $\zeta_{\mathbf{P}}^{\text{EZ}}(-\mathbf{N}, \mathbf{c}, \mathbf{d}) := \lim_{s \rightarrow 0} \zeta_{\mathbf{P}}^{\text{EZ}}(-\mathbf{N} + s\boldsymbol{\mu}, \mathbf{c}, \mathbf{d})$  existe, et on a :

$$\zeta_{\mathbf{P}}^{\text{EZ}}(-\mathbf{N}, \mathbf{c}, \mathbf{d}) = \sum_{\mathbf{k}=(k_1, \dots, k_P) \in \llbracket 0, |\mathbf{N}|+P+1 \rrbracket^P} P_{\mathbf{k}}(\mathbf{b}, \mathbf{c}, \boldsymbol{\mu}) \prod_{i=1}^P B_{k_i},$$

avec  $B_n$  qui désigne le  $n$ -ième nombre de Bernoulli, et  $P_{\mathbf{k}}(\mathbf{b}, \boldsymbol{\gamma}, \boldsymbol{\mu})$  qui est un polynôme explicite en les  $\gamma_p, d_p$  et  $\mu_p$  pour  $p \in \llbracket 1, P \rrbracket$ .

**Remarque 1.2.28.** 1) Dans [EM21], Essouabri et Matsumoto ont obtenu un résultat similaire pour les valeurs spéciales d'une classe de fonction multizêta plus générale de la forme

$$\zeta_m(\mathbf{s}, \mathbf{P}) = \sum_{n_1, \dots, n_m} \frac{1}{\prod_{j=1}^m P_j(n_1, \dots, n_j)},$$

avec  $P_j$  un polynôme en les  $X_1, \dots, X_j$  variables, pour  $1 \leq j \leq m$ , vérifiant la condition HOS (voir [Ess97]) tels que

$$\lim_{x_1 + \dots + x_j \rightarrow +\infty} P_j(x_1, \dots, x_j) = +\infty,$$

et en imposant que le polynôme  $P_m$  est homogène et elliptique.

2) Dans [MW02], Matsumoto et Weng ont étudié des fonctions zêta de la forme

$$\zeta^{\text{MW}}(s) = \sum_{n=1}^{+\infty} \frac{P(n)}{Q(n)^s},$$

avec  $P$  et  $Q$  deux polynômes non nuls tels que  $Q$  ne s'annule pas en les entiers strictement positifs. Matsumoto et Weng obtiennent une formule de la dérivée en  $s = 0$ . L'intérêt de ce genre de formule provient de l'étude du déterminant régularisé du Laplacien de la  $n$ -sphère (voir [Var88], [WY96], [QC96], [Kum99]).

### 1.2.3 Sur les formules asymptotiques de différents nombres de partitions

On détaillera dans cette sous-section quelques applications au calcul des valeurs spéciales de certaines fonctions zêta de Witten, et de leurs dérivées. On discutera également du théorème de Meinardus qui fournit une formule asymptotique de certains nombres de partition générés par des produits infinis. On mentionnera également des travaux récents de Bridges, Brindle, Bringmann, Craig, Franke, Males à propos de formule asymptotique de certains nombres de partition dans les préprints [BB23], [BBF23], [BCM23], [BBBBF23]. On donnera en particulier quelques résultats obtenus par Bridges, Bringmann, Brindle, Franke dans [BBBBF23], et on parlera de leur approche.

En combinatoire, il est classique d'utiliser une série génératrice de la forme

$$\omega(q) = \sum_{n \geq 0} r(n)q^n,$$

afin d'étudier le comportement d'une suite de réels  $r(n)$ . Si la fonction  $\omega$  est holomorphe sur  $|q| < 1$ , alors on a par théorème de Cauchy que

$$r(n) = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{\omega(q)}{q^{n+1}} dq,$$

avec  $\mathcal{C}$  un cercle centré en 0 parcouru dans le sens trigonométrique, inclus dans le disque unité. En général, il n'existe pas de résultat permettant d'obtenir une relation asymptotique pour une suite  $r(n)$ .



On se concentrera dans la suite de cette section sur une série génératrice s'écrivant sous la forme d'un produit infini.

On considère une fonction  $G_f(q)$  de la forme

$$G_f(q) = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f(n)}},$$

avec  $f(n)$  des réels. En distribuant le produit, on obtient

$$G_f(q) = \sum_{n=0}^{+\infty} p_f(n) q^n, \quad (1.7)$$

On considère la série de Dirichlet

$$L_f(s) = \sum_{n=1}^{+\infty} \frac{f(n)}{n^s} \quad (s = \sigma + i\tau, \sigma > \sigma_0),$$

avec  $\sigma_0$  son abscisse de convergence.

**Exemple 1.2.29.** [And76, 5.1.2] *Le plus simple produit infini auquel on puisse penser correspond à l'étude du nombre de partitions  $p(n)$  d'un entier  $n$  :*

$$\prod_{n=1}^{+\infty} \frac{1}{1 - q^n} = \sum_{n=1}^{+\infty} p(n) q^n.$$

*La série de Dirichlet correspondante est la fonction zêta de Riemann. Hardy et Ramanujan ont démontré que, lorsque  $n \rightarrow +\infty$ , on a*

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

On pose maintenant

$$g_f(z) := \sum_{n=1}^{+\infty} f(n) e^{-nz}.$$

On suppose les conditions suivantes :

- i)  $\sigma_0 > 0$ .
- ii)  $L_f$  admet un prolongement méromorphe sur  $H_{-C_0}$ , avec  $0 < C_0 < 1$  et tel que  $L_f$  n'ait qu'un seul pôle en  $s = \sigma_0$  sur ce domaine, que ce pôle soit d'ordre 1, et de résidu  $A$ .
- iii) On a la majoration suivante lorsque  $\tau \rightarrow +\infty$ , et uniforme en  $\sigma > -C_0$ ,

$$|L_f(s)| = O(|\tau|^{C_1}),$$

avec  $C_1 > 0$  et  $s = \tau + i\sigma$ .

- iv) Soit  $\epsilon > 0$ . Pour tout  $z = x + 2i\pi y$  tel que  $\arg(z) > \frac{\pi}{4}$ ,  $|x| \leq \frac{1}{2}$  et  $|y| \ll 1$ , on a

$$\operatorname{Re}(g_f(z)) - g_f(x) \leq -C_2 x^{-\epsilon},$$

avec  $C_2 > 0$  une constante dépendant de  $\epsilon$ .

Le théorème de Meinardus donne alors une relation asymptotique pour la suite  $p_f(n)$

**Théorème 1.2.30** (de Meinardus). [And76, Théorème 6.2] Pour  $n \rightarrow +\infty$ , on a

$$p_f(n) = Cn^\kappa \exp\left(n^{\frac{\alpha}{\alpha+1}} \left(1 + \frac{1}{\alpha}\right) (A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha}}\right) (1 + O(n^{-\kappa_1})),$$

avec  $p_f(n)$  définie via la formule (1.7), et  $C, \kappa, \kappa_1$  des réels dépendant de  $L_f(0), L'_f(0), C_0$  et  $\alpha$ .

La démonstration de ce théorème repose sur la méthode du point selle, dont on expliquera quelques points clés ici. Pour les détails de la méthode, on se réfère à [FS09, VIII.3]. La méthode exploite le fait que l'intégrale  $\frac{1}{2i\pi} \int_{\mathcal{C}} \frac{G_f(q)}{q^{n+1}} dq$  est indépendante du rayon du cercle  $\mathcal{C}$  inclus dans le disque unité. On peut alors utiliser cette indépendance en choisissant judicieusement un rayon  $\rho_0(n)$  adéquat. La méthode du point selle consiste à choisir ce rayon de tel sorte à minimiser  $\left| \frac{G_f(q)}{q^n} \right|$  sur un arc de cercle mineur du cercle  $\mathcal{C}_1(\theta_0(n))$ . Concrètement, on choisit un tel point selle de tel sorte que la dérivée de l'intégrande en  $q = \rho_0(n)$  soit une  $O(n^\delta)$ , avec  $\delta$  bien choisie. Alternativement, on peut chercher  $\rho_n$  en étudiant la dérivée de la fonction

$$h_f(z) := \ln(L_f(z)) - n \log(z).$$

On cherche alors  $\rho_n$  de tel sorte que  $h'_f(\rho_0(n)) = -n + O(n^\delta)$ .

On aura alors une approximation quadratique lorsque  $\theta \rightarrow 0$  :

$$h_f(\rho_0(n)e^{i\theta}) - h_f(\rho_0(n)) =: -\frac{1}{2}\beta(\rho_0(n))\theta^2 + o(\theta^3) + O(n^\delta),$$

où  $\beta(\rho(n))$  est une constante dépendant de  $\rho(n)$ , et du choix de  $\delta$ . On cherchera alors à déterminer un angle  $\theta_0(n)$  de tel sorte que

$$\beta(\rho_0(n))\theta_0(n)^2 \xrightarrow{\theta \rightarrow 0} +\infty, \quad h''(\rho_0(n))\theta_0(n)^3 \xrightarrow{\theta \rightarrow 0} 0.$$

Pour un tel choix de  $\theta_0(n)$ , on en déduit en particulier que le comportement de l'intégrale sur l'arc mineur  $\mathcal{C}_0$  se comporte comme une intégrale gaussienne, dont les bornes dépendent de  $n$ , que l'on sait approximer.

Debruyne et Tennenbaum ont obtenu dans [DT20] une version plus générale du théorème de Meinardus en raffinant la condition iii) sur la croissance de  $L_f$  lorsque la partie imaginaire  $\tau$  est grande. En étudiant les fonctions zêta de Witten, on s'apercevra que l'on ne peut pas appliquer simplement le théorème de Meinardus pour obtenir une expression asymptotique du nombre de représentations des algèbres de Lie  $r_{\mathfrak{g}}(n)$  qui nous intéressent. Cependant, en modifiant astucieusement la preuve du Théorème de Meinardus, puis en étudiant les résidus de  $\zeta_{\mathfrak{sl}(3)}(s)$  en  $s = \frac{1}{2}, \frac{2}{3}$ , et en calculant les valeurs spéciales  $\zeta_{\mathfrak{sl}(3)}(0)$  et  $\zeta'_{\mathfrak{sl}(3)}(0)$ , Romik a obtenu le Théorème suivant :

**Théorème 1.2.31.** [Rom17, Théorème 1.1] Au voisinage de  $+\infty$ , on a

$$r_{\mathfrak{sl}(3)}(n) = (1 + o(1)) \frac{K}{n^{3/5}} \exp\left(A_1 n^{2/5} - A_2 n^{3/10} - A_3 n^{1/5} - A_4 n^{1/10}\right),$$

avec les constantes  $A_1, A_2, A_3, A_4$  et  $K$  des constantes explicites : On pose tout d'abord

$$X = \left(\frac{1}{9}\Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{5}{3}\right)\right), \quad Y = -\sqrt{\pi}\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right).$$

Les constantes  $A_1, A_2, A_3, A_4, K$  sont alors définies par les relations suivantes :

$$A_1 = 5X^2 = 6.858260476\dots$$

$$A_2 = X^{-1}Y = 5.77360174\dots$$

$$A_3 = \frac{3}{80}X^{-4}Y^2 = 0.91134107\dots$$

$$A_4 = \frac{11}{3200}X^{-7}Y^3 = 0.35163754\dots$$

$$K = \frac{2\sqrt{3\pi}}{\sqrt{5}}X^{1/3} \exp\left(-\frac{1}{2560}X^{-10}Y^4\right) = 2.44629033486\dots$$

Notons que dans [Rom17], le calcul de la constante  $Y$  dépend en particulier de la valeur spéciale  $\zeta'_{\mathfrak{sl}(3)}(0) = -\ln(2\pi) + \ln(2)$  calculée dans [BD18]. De plus, toutes ces constantes dépendent également du calcul des résidus de  $\zeta_{\mathfrak{sl}(3)}(s)$  en  $s = \frac{1}{2}$  et en  $s = \frac{2}{3}$ .

Par la suite, Bridges, Bringmann et Franke ont obtenu dans [BBF23] une formule asymptotique pour le nombre de représentations irréductible de dimension  $n$  de  $\mathfrak{sl}(3)$  en étudiant la fonction zêta de Witten  $\zeta_{\mathfrak{sl}(3)}(s)$ , et en utilisant un théorème Taubérien.

Romik conjecture qu'il est possible d'obtenir des résultats similaires sur des nombres de représentations  $r_{\mathfrak{g}}(n)$  pour d'autres algèbres de Lie  $\mathfrak{g}$ . Pour étendre ce résultat, on a besoin d'établir un théorème de type Meinardus, et également d'obtenir des informations sur les pôles et les résidus de la fonction zêta de Witten correspondant dans la bande  $0 \leq \Re(s) \leq 1$  pour la fonction zêta de Witten  $\zeta_{\mathfrak{g}}(s)$ , puis d'obtenir une expression de  $\zeta'_{\mathfrak{g}}(0)$ .

Motivés par cette question, Bridges, Brindle, Bringmann et Franke ont établi une variante du Théorème de Meinardus plus général dans le preprint [BBBF23]. Soit  $f : \mathbb{N}^* \rightarrow \mathbb{N}$ . On pose pour tout  $q = e^{-z}$ ,  $z \in H_0$  les fonctions

$$G_f(z) = \sum_{n \geq 0} p_f(n) q^n = \prod_{n \geq 1} \frac{1}{(1 - q^n)^{f(n)}}, \quad L_f(s) = \sum_{n \geq 1} \frac{f(n)}{n^s}.$$

On pose  $\Lambda := \mathbb{N} \setminus f^{-1}(\{0\})$ . On suppose les conditions suivantes :

(P1) Soit  $\alpha > 0$  le pôle le plus large de  $L_f$ . Il existe un entier  $L \in \mathbb{N}$  tel que, pour tout nombre premier  $p$ ,

$$|\Lambda \setminus (p\mathbb{N} \cap \Lambda)| \geq L > \frac{\alpha}{2}.$$

(P2) Il existe un réel  $R \in \mathbb{R}_+$  tel que  $L_f$  soit méromorphe sur  $\overline{H}_{-R} = \{z \in \mathbb{C}, \Re(z) \geq -R\}$ , et tel que  $L_f$  soit holomorphe sur la droite ( $\Re(z) = -R$ ). De plus, on suppose que la fonction méromorphe  $L_f^*(s) := \Gamma(s)\zeta(s+1)L_f(s)$  possède uniquement des pôles réels  $\alpha := \gamma_1 > \dots$ , que ces pôles sont simples à part en  $s = 0$ , où le pôle peut être un pôle double.

(P3) Il existe un réel  $a < \frac{\pi}{2}$  tel que, sur chaque bande  $\sigma_1 \leq \sigma \leq \sigma_2$  dans le domaine d'holomorphie de  $L_f$ , on a

$$L_f(s) = O_{\sigma_1, \sigma_2} \left( e^{a|\tau|} \right) \quad (s = \sigma + i\tau),$$

lorsque  $|\tau| \rightarrow +\infty$ .

**Théorème 1.2.32.** [BBBF23, Théorème 1.4] *On suppose les conditions (P1), (P2) et (P3). Notons  $L$  le réel de la condition (P1) et  $R$  le réel de la condition (P2). Alors pour  $M, N \in \mathbb{N}$ , on a*

$$p_f(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp \left( A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^M A_j n^{\alpha_j} \right) \left( 1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R} \left( n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}} \right) \right),$$

avec  $0 \leq \alpha_M < \dots < \alpha_1 := \frac{\alpha}{\alpha+1}$ ,  $0 < \nu_2 < \dots < \nu_N$ ,  $A_1, \dots, A_M$  et  $B_2, \dots, B_N$  des réels explicites.

Lorsque la fonction méromorphe  $L_f$  ne possède que deux pôles strictement positifs, Bridges, Brindle, Bringmann et Franke obtiennent une formule plus précise. Introduisons tout d'abord les constantes suivantes : Posons  $\alpha > \beta > 0$  les deux seuls pôles positifs de  $L_f$ . On note  $\omega_\alpha = \text{Res}_{s=\alpha}(L_f(s))$  et  $\omega_\beta = \text{Res}_{s=\beta}(L_f(s))$  les résidus de  $L_f$  en ses pôles positifs, et on pose  $c_1 = \omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1)$ ,  $c_2 = \omega_\beta \Gamma(\beta+1)\zeta(\beta+1)$ ,  $c_3 = L_f(0)$ . On note

$$C = \frac{e^{L_f'(0)} (\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{\frac{1}{2}-L_f(0)}{\alpha+1}}}{\sqrt{2\pi(\alpha+1)}} \quad b = \frac{1 - L_f(0) + \frac{\alpha}{2}}{\alpha+1}.$$

On note  $\mathcal{P}_R$  l'union du singleton  $\{0\}$  et des pôles plus grand que  $-R$  de  $L_f^*$

$$\mathcal{L} = \frac{1}{\alpha+1} \mathcal{P}_R + \sum_{\mu \in \mathcal{P}_R} \left( \frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N},$$

$$\mathcal{M} := \frac{\alpha}{\alpha+1} \mathbb{N} + \left( - \sum_{\mu \in \mathcal{P}_R} \left( \frac{\mu+1}{\alpha-1} - 1 \right) \mathbb{N} \right) \cap \left[ 0, \frac{R+\alpha}{\alpha+1} \right].$$

**Théorème 1.2.33.** [BBBF23, Théorème 4.4] *On suppose les conditions (P1), (P2) et (P3). Notons  $L$  le réel de la condition (P1) et  $R$  le réel de la condition (P2). De plus, on suppose que  $L_f$  n'a que deux pôles  $\alpha > \beta > 0$  tels qu'il existe un entier  $l \in \mathbb{N}^*$  vérifiant l'inégalité  $\frac{l+1}{l}\beta < \alpha < \frac{l}{l-1}\beta$ . Alors on a*

$$p_f(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp \left( A_1 n^{\frac{\alpha}{\alpha+1}} + A_2 n^{\frac{\beta}{\alpha+1}} + \sum_{j=3}^{l+1} A_j n^{\frac{(j-1)\beta}{\alpha+1} + \frac{j-2}{\alpha+1} + 2-j} \right) \left( 1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R} \left( n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}} \right) \right),$$

avec  $A_1 := (\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{1}{\alpha+1}} \left( 1 + \frac{1}{\alpha} \right)$ , et  $A_2 := \frac{\omega_\beta \Gamma(\beta) \zeta(\beta+1)}{(\omega_\alpha \Gamma(\alpha) \zeta(\alpha+1))^{\frac{\beta}{\alpha+1}}}$ , et pour tout  $j \geq 3$ ,

$$A_j := K_j + \frac{c_1^{\frac{1}{\alpha+1}}}{\alpha} \sum_{m=1}^l \binom{-\alpha}{m} \sum_{\substack{0 \leq k_1 \leq \dots \leq k_l \leq m \\ |\mathbf{k}|=m \\ k_1+2k_2+\dots+l k_l=j-1}} \binom{m}{\mathbf{k}} \frac{K_2^{k_1} \dots K_{l+1}^{k_l}}{c_1^{\frac{m}{\alpha+1}}}$$

$$+ \frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}} \sum_{m=1}^l \binom{-\beta}{m} \sum_{\substack{0 \leq k_1 \leq \dots \leq k_l \leq m \\ |\mathbf{k}|=m \\ k_1+2k_2+\dots+l k_l=j-2}} \binom{m}{\mathbf{k}} \frac{K_2^{k_1} \dots K_{l+1}^{k_l}}{c_1^{\frac{m}{\alpha+1}}},$$

avec  $(K_j)_{j \geq 3}$  décrit dans [BBBF23, Lemme 4.3], et  $0 < \nu_2 < \dots$  les éléments strictement positifs de  $\mathcal{L} + \mathcal{M}$ .

**Remarque 1.2.34.** *Pour étudier le comportement asymptotique de  $r_{\mathfrak{g}_2}(n)$ , on aurait besoin de connaître  $K_3$ , et d'après [BBBF23, Lemme 4.3], on a*

$$K_3 = \frac{c_2^2 (\alpha - 2\beta)}{2(\alpha+1)^2 c_1^{\frac{2\beta+1}{\alpha+1}}},$$

avec  $c_1$  et  $c_2$  des constantes explicites telles que

$$c_1 = \omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1), \quad c_2 = \omega_\beta \Gamma(\beta+1) \zeta(\beta+1).$$

De plus, pour étudier le comportement asymptotique de  $r_{\mathfrak{g}_2}(n)$ , on verra que l'entier  $l = 2$  fonctionne car, on aurait alors  $L_f(s) = \zeta_{\mathfrak{g}_2}(s)$ , et les pôles positifs de cette fonction zêta sont  $s = \frac{1}{3}$  et  $s = \frac{1}{5}$ .

En appliquant ce théorème, les 4 auteurs ont obtenu des résultats similaires à Romik pour le nombre de représentations  $r_{\mathfrak{so}(5)}(n)$  :

**Théorème 1.2.35.** [BBBF23, Théorème 1.3] *Pour tout entier  $N \geq 1$ , on a*

$$r_{\mathfrak{so}(5)}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^{\frac{7}{12}}} \exp \left( A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4 \right) \left( 1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N \left( n^{-\frac{N+1}{9}} \right) \right),$$

avec

$$\begin{aligned}
b &= \frac{7}{12}, \quad C = \frac{e^{\zeta'_{\mathfrak{so}(5)}(0)} \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}}{2^{\frac{1}{3}} 3^{\frac{11}{24}} \sqrt{\pi}}, \quad A_1 = \frac{3^{\frac{4}{3}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{3}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{8}{3}}}, \\
A_2 &= \frac{2^{\frac{8}{9}} \left(2^{\frac{1}{3}} + 1\right) \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right)}{3^{\frac{7}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{9}}}, \quad A_3 = \frac{2^{\frac{40}{9}} \left(2^{\frac{1}{3}} + 1\right)^2 \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{1}{3}\right)^2 \zeta\left(\frac{4}{3}\right)^2}{3^{\frac{44}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{20}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{10}{9}}}, \\
A_4 &= \frac{2^8 \left(2^{\frac{1}{3}} + 1\right)^3 \Gamma\left(\frac{1}{3}\right)^3 \zeta\left(\frac{1}{3}\right)^3 \zeta\left(\frac{4}{3}\right)^3}{3^8 \Gamma\left(\frac{1}{4}\right)^4 \zeta\left(\frac{3}{2}\right)^2}.
\end{aligned}$$

En revanche, ces 4 auteurs ne disposaient pas d'une formule explicite de  $\zeta'_{\mathfrak{so}(5)}(0)$ . On donnera dans notre travail une expression des valeurs spéciales  $\zeta'_{\mathfrak{so}(5)}(0)$  et  $\zeta'_{\mathfrak{g}_2}(0)$ . On donnera également des formules pour les résidus de  $\zeta_{\mathfrak{so}(5)}(s)$  en  $s = \frac{1}{2}$  et en  $s = \frac{1}{3}$ , et des formules pour les résidus de  $\zeta_{\mathfrak{g}_2}(s)$  en  $s = \frac{1}{3}$  et en  $s = \frac{1}{5}$ . On pourra alors obtenir une formule asymptotique explicite pour le nombre de représentations  $r_{\mathfrak{so}(5)}(n)$ , en donnant une expression du coefficient  $C$  du théorème précédent.

Notons au passage que nous trouverons une valeur différente pour  $\zeta_{\mathfrak{so}(5)}(0)$  que dans [BBBF23, Proposition 5.15]. Nous interprétons cette différence de résultats en suggérant que les auteurs ont vraisemblablement commis une erreur en remplaçant  $\zeta(0)$  par  $\frac{1}{2}$ , à la place de  $-\frac{1}{2}$  dans la preuve de la Proposition 5.15.

On obtiendra également des formules explicites pour  $\zeta_{\mathfrak{g}_2}(0)$  et  $\zeta'_{\mathfrak{g}_2}(0)$ , et on pourra alors établir un théorème similaire au théorème précédent, pour le nombre de représentations  $r_{\mathfrak{g}_2}(n)$  en utilisant le Théorème 1.2.33.

**Remarque 1.2.36.** *Quand on appliquera le Théorème 1.2.32 à l'étude asymptotique du nombre de représentations  $r_{\mathfrak{g}}(n)$  d'une algèbre de Lie semi-simple, on verra que les coefficients  $f(n)$  correspondent au nombre de représentations irréductibles de dimension  $n$  de  $\mathfrak{g}$ .*

### 1.3 Premières définitions

On fixera dans tout ce manuscrit une famille  $(l_q)_{q \in \llbracket 1, Q \rrbracket}$  de formes linéaires de rang  $P$  à coefficients dans le demi-plan de Poincaré  $H_0 = \{s = \sigma + i\tau \in \mathbb{C} \mid \sigma > 0\}$  avec  $P \geq 1$  et  $Q \geq 1$  des entiers. On fixe également  $\mathbf{d} = (d_p)_{p \in \llbracket 1, P \rrbracket}$ ,  $\mathbf{d}' = (d'_q)_{q \in \llbracket 1, Q \rrbracket}$  des complexes dans  $H_0$  tels que  $d'_q := l_q(\mathbf{d})$  pour tout  $1 \leq q \leq Q$ .

**Notation.** Soit  $q \in \llbracket 1, Q \rrbracket$ , on note

$$l_q(\mathbf{n}) = \sum_{p=1}^P c_{q,p} n_p.$$

Par souci de clarté, on choisira tout le long de ce manuscrit d'utiliser l'indice  $p$  (ou l'indice  $i$  si  $p$  est déjà utilisée) lorsque l'on parcourra n'importe quel sous-ensemble  $\mathcal{P}$  de  $\llbracket 1, P \rrbracket$ , et on utilisera l'indice  $q$  (ou l'indice  $j$  si  $q$  est déjà utilisée) lorsque l'on parcourra n'importe quel sous-ensemble  $\mathcal{Q}$  de  $\llbracket 1, Q \rrbracket$ .

On définira tout d'abord les fonctions gamma incomplètes

**Définition 1.3.1.** Soit  $\nu \in H_0$ ,  $\theta > 0$ ,  $s \in \mathbb{C}$ . On pose alors

$$\begin{aligned} \Gamma(s, \theta, \nu) &= \int_{\theta}^{+\infty} e^{-\nu y} y^{s-1} dy, \\ \gamma(s, \theta, \nu) &= \int_0^{\theta} e^{-\nu y} y^{s-1} dy, \quad \text{lorsque } \operatorname{Re}(s) > 0. \end{aligned}$$

Ces fonctions gamma sont plus générales que celles que l'on peut trouver dans la littérature (par exemple dans [EMOT81, Chap.IX] ou dans [Cra12, §3]). On retrouve les fonctions gamma incomplètes classiques en posant  $\nu = 1$ . Pour nos usages, on aura besoin d'un  $\nu$  quelconque dans  $H_0$ .

On introduit maintenant des constantes qui apparaîtront dans le théorème D. Par une décomposition en éléments simples, on trouve aisément :

**Proposition 1.3.2.** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ . Notons  $\mathbb{K}$  le corps engendré par  $\mathbb{Q}$  et par les coefficients  $c_{q,p}$ , avec  $1 \leq q \leq Q$  et  $1 \leq p \leq P$ . Soit  $\mathcal{P} \subset \llbracket 1, P \rrbracket$ ,  $j, f \in \llbracket 1, Q \rrbracket$ ,  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}}$ ,  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c}$ . On note  $\mathcal{P}^c = \llbracket 1, P \rrbracket \setminus \mathcal{P}$ ,  $\mathbf{w}_p = (w_{p,q})_{q \in \llbracket 1, Q \rrbracket}$  et  $\mathbf{v}_p = (v_{p,q})_{q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}}$ . Il existe un polynôme  $\tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'} \in \mathbb{K}[x]$  et deux familles de constantes  $(C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}'))_{\lambda \in \mathbb{N}}$  et  $(D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}'))_{\lambda \in \mathbb{N}, p \in \mathcal{P}}$  dont chacun des termes appartient au corps  $\mathbb{K}$ , tels que

$$\begin{aligned} x^{-N'_j - 1 + \mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x)^{-N_p - 1 - |\mathbf{v}_p|} &= \tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}(x) + \sum_{\lambda=1}^{N'_j + 1 - \mathbf{w}(f)} \frac{C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')}{x^\lambda} \\ &+ \sum_{p \in \mathcal{P}} \sum_{\lambda=1}^{(|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|)} \frac{D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}')}{(c_{j,p} + c_{f,p} x)^\lambda} \end{aligned}$$

On note  $P_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}$  la primitive s'annulant en 1 de  $\tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}$ . On note  $E_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \in \mathbb{K}$  le terme constant de  $P_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}$ .

On note

$$\begin{aligned} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') &= -E_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_j + 1 - \mathbf{w}(f)} \frac{C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} \\ &- \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{(|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|)} \frac{D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}) \\ &+ \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (1, p)}(\mathbf{N}, \mathbf{N}') \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \right). \end{aligned}$$

**Remarque 1.3.3.** 1) Notons que si  $-N'_f - 1 + \mathbf{w}(f) \geq 0$ , alors  $C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda} = 0$  pour tout  $\lambda$ .

2) Dans le produit  $\prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p}x)^{(-N_p - 1 - |\mathbf{v}_p|)}$ , certains termes du produit peuvent apparaître plusieurs

fois. C'est la raison pour laquelle la somme  $\sum_{\lambda=1}^{(|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|)}$  porte jusqu'à  $|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|$ . Si cela se produit, on a alors plusieurs choix possibles pour les constantes  $D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}$ . Notons que ces différents choix n'auront aucun impact dans la définition de la constante  $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ .

3) La constante  $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$  apparaîtra dans le calcul d'une intégrale, dans le Lemme 3.1.4.

**Exemple 1.3.4.** Soit  $\mathcal{P} = \emptyset$ , on a pour tout  $\lambda \geq 1$ ,

$$\begin{aligned} C_{\emptyset,j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}') &= \delta_{\lambda, N'_f + 1 - \mathbf{w}(f)} \\ D_{\emptyset,j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}') &= 0 \\ E_{\emptyset,j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &= 0 \text{ si } \mathbf{w}(f) \leq N'_f \\ E_{\emptyset,j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &= -\frac{1}{\mathbf{w}(f) - N'_f} \text{ si } \mathbf{w}(f) > N'_f, \end{aligned}$$

où l'on a noté  $\delta$  le symbole de Kronecker.

On obtient en particulier que  $F_{\emptyset,j,f,\mathbf{0},\mathbf{0}}(\mathbf{0}, \mathbf{0}) = 0$ .

**Exemple 1.3.5.** Soit  $\mathcal{P} = \{p\}$ ,  $j \neq f \in \llbracket 1, Q \rrbracket$ ,  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ ,  $\mathbf{v} = \mathbf{0}$  et  $\mathbf{w} = \mathbf{e}_f$ , alors on a

$$x_f^{-N'_f - 1 + \mathbf{w}(f)} (c_{j,p} + c_{f,p}x_f)^{-N_p - 1 - |\mathbf{v}_p|} = (c_{j,p} + c_{f,p}x_f)^{-1}.$$

Ainsi, on a  $F_{\{p\},j,f,\mathbf{0},\mathbf{e}_f} = \frac{1}{c_{f,p}} \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \right)$ .

**Exemple 1.3.6.** Soit  $\mathcal{P} = \{p\}$ ,  $j \neq f \in \llbracket 1, Q \rrbracket$ ,  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ ,  $\mathbf{v} = \mathbf{0}$  et  $\mathbf{w} = \mathbf{e}_j$ , alors on a

$$x_f^{-N'_f - 1 + \mathbf{w}(f)} (c_{j,p} + c_{f,p}x_f)^{-N_p - 1 - |\mathbf{v}_p|} = x_f^{-1} (c_{j,p} + c_{f,p}x_f)^{-1} = \frac{1}{c_{j,p}} x^{-1} - \frac{c_{f,p}}{c_{j,p}} (c_{j,p} + c_{f,p}x_f)^{-1}.$$

Ainsi, on a  $F_{\{p\},j,f,\mathbf{0},\mathbf{e}_j} = -\frac{1}{c_{j,p}} \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \right)$ .

On étudiera dans ce manuscrit la fonction multizêta suivante :

**Définition 1.3.7.** Soit  $(\mathbf{s}, \mathbf{s}') \in H_1^P \times H_1^Q$ , on pose

$$Z(\mathbf{s}, \mathbf{s}') = Z(\mathbf{c}, \mathbf{d}, \mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}.$$

**Remarque 1.3.8.** Cette fonction multizêta est de la même forme d'une fonction multizêta de type Hurwitz-Lerch décrite dans 1.2.24, sans torsion :

$$Z(\mathbf{s}, \mathbf{s}') = \zeta^{HL}(\mathbf{0}, \tilde{\mathbf{d}}, \tilde{\mathbf{c}}, (s_1, \dots, s_P, s'_1, \dots, s'_Q)) = \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{\prod_{q=1}^{\tilde{Q}} \left( \tilde{d}_q + \sum_{p=1}^P \tilde{c}_{q,p} n_p \right)^{s'_q}},$$

avec  $P_0 = P$ ,  $\tilde{Q} := P + Q$ ,  $\mathcal{B}^c = \cup_{1 \leq p \leq P} \llbracket 1, P \rrbracket \setminus \{p\} \times \{p\} \subset \llbracket 1, Q + P \rrbracket \times \llbracket 1, P \rrbracket$ ,  $\tilde{\mathbf{d}} = (\mathbf{d}, \mathbf{d}')$ , et avec  $\tilde{\mathbf{c}} = (c_{(q,p)})_{(q,p) \in \llbracket 1, Q \rrbracket \times \llbracket 1, P \rrbracket}$  tels que

$$\begin{cases} \tilde{c}_{q,p} = 0 & \text{si } q \neq p \text{ et } q \leq P \\ \tilde{c}_{p,p} = 1 & \text{si } p \in \llbracket 1, P \rrbracket \\ \tilde{c}_{q,p} = c_{q-P,p} & \text{si } q > P. \end{cases}$$

En particulier, on pourra appliquer le Théorème 1.2.25 à  $Z(\mathbf{s}, \mathbf{s}')$ .

A l'aide de la remarque précédente, et du Théorème 1.2.25), on voit que la fonction multizêta  $Z(\mathbf{s}, \mathbf{s}')$  est méromorphe sur tout  $\mathbb{C}^P \times \mathbb{C}^Q$ , et ce même théorème fournit également des informations sur l'ensemble des singularités de  $Z(\mathbf{s}, \mathbf{s}')$ . On dispose également d'un domaine de convergence, cependant pour nos besoins, on a besoin d'un domaine de convergence plus large que ce que propose le Théorème 1.2.25. On montrera alors un domaine de convergence plus précis dans la proposition suivante :

**Proposition 1.3.9.** *On pose*

$$\mathcal{D} = \mathcal{D}_{P,Q} := \left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket \right\}.$$

La fonction multizêta  $Z(\mathbf{s}, \mathbf{s}')$  a pour domaine de convergence  $\mathcal{D}$ , et est méromorphe sur  $\mathbb{C}^P \times \mathbb{C}^Q$ , avec des singularités incluses dans l'union d'hyperplans :

$$\begin{cases} \sum_{p \in \mathcal{P}} s_p + \sum_{q \in \mathcal{Q}} s'_q = Q + P - n & (\mathcal{P} \subset \llbracket 1, P \rrbracket, \mathcal{Q} \subset \llbracket 1, Q \rrbracket, |\mathcal{P} \cup \mathcal{Q}| \geq 2, n \in \mathbb{N}) \\ s_p = 1, \dots, P & (1 \leq p \leq P) \\ s'_q = 1, \dots, P & (1 \leq q \leq Q). \end{cases}$$

On notera de la même manière  $Z(\mathbf{s}, \mathbf{s}')$  son prolongement méromorphe. En général, les valeurs aux multientiers négatifs n'ont pas de sens puisque certains multientiers négatifs sont des pôles pour  $Z$ . C'est pourquoi on fixera une direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}')$  et un multi-entier négatif  $-(\mathbf{N}, \mathbf{N}')$ , et l'on étudiera la fonction multizêta  $Z$  en ce multi-entier, et selon la direction choisie. Il découle de la Proposition 1.3.9 que  $s \mapsto Z(-\mathbf{N} + s\boldsymbol{\mu}, -\mathbf{N}' + s\boldsymbol{\mu}')$  est méromorphe sur  $\mathbb{C}$ , et est holomorphe pour  $s = \sigma + i\tau$ , avec  $\sigma$  suffisamment grand. On peut alors définir la fonction zêta suivante :

**Définition 1.3.10.** *Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in (\mathbb{R}_+)^P \times (\mathbb{R}_+^*)^Q$ . On pose alors pour tout  $s \in \mathbb{C}$ ,*

$$Z_{\mathbf{N}, \mathbf{N}'}(s) := Z(-\mathbf{N} + s\boldsymbol{\mu}, -\mathbf{N}' + s\boldsymbol{\mu}').$$

On considère également la fonction univariable suivante, qui consiste à évaluer la fonction décrite précédemment selon la direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ , et pour  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ .

**Définition 1.3.11.** *Soit  $s \in \mathbb{C}$  tel que  $\sigma \gg 1$ . On pose*

$$Z_{\Delta} : s \in \mathbb{C} \mapsto Z((s, \dots, s), (s, \dots, s)).$$

**Remarque 1.3.12.** *Si l'on pose  $\boldsymbol{\mu} = \mathbf{0}$ , et  $\boldsymbol{\mu}' = \mathbf{1}$ , alors  $Z_{\mathbf{0}, \mathbf{0}'}(s)$  correspond à une fonction zêta de type Shintani*

$$Z_{\mathbf{0}, \mathbf{0}'}(s) = \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s}.$$

Gardons tout de même en tête que les coefficients  $d'_q$  ne sont pas libres, et dépendent des choix de  $d_p$  et des choix des coefficients  $(c_{p,q})_{1 \leq p \leq P, 1 \leq q \leq Q}$ .

Par la Proposition 1.3.9, on sait que  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  est holomorphe sur  $\{s \in \mathbb{C} \mid (-\mathbf{N} + s\boldsymbol{\mu}, -\mathbf{N}' + s\boldsymbol{\mu}') \in \mathcal{D}\}$ . Ainsi, il existe un réel  $\sigma_{\mathbf{0}, \mathbf{N}, \mathbf{N}'}$  tel que l'application  $Z_{\mathbf{N}, \mathbf{N}'}$  soit holomorphe sur  $H_{\sigma_{\mathbf{0}, \mathbf{N}, \mathbf{N}'}}$ . On dira que  $Z_{\mathbf{N}, \mathbf{N}'}$  est holomorphe pour  $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$ , ou parfois seulement  $\sigma \gg 1$  lorsqu'il n'y aura pas d'ambiguïté.

Afin de donner un sens aux valeurs directionnelles de  $Z$  en les multi-entiers négatifs, on doit prouver que  $Z_{\mathbf{N}, \mathbf{N}'}$  est régulière en  $s = 0$ . Des travaux de Komori prouvent bien que cette valeur directionnelle a un sens. Dans ce manuscrit, on détaillera une méthode de prolongement différente dans la Proposition 2.2.4 pour prouver que ces valeurs directionnelles ont un sens, ce qui nous permettra **surtout** d'obtenir des formules explicites sur les valeurs spéciales directionnelles de  $Z$ , ainsi que celles de sa dérivée.



**Notation.** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in (\mathbb{R}_+)^P \times (\mathbb{R}_+^*)^Q$ . On pose alors la valeur en  $(-\mathbf{N}, -\mathbf{N}')$  de  $Z(\mathbf{s}, \mathbf{s}')$  selon la direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}')$ ,

$$Z(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} := Z_{\mathbf{N}, \mathbf{N}'}(0)_{\boldsymbol{\mu}, \boldsymbol{\mu}'},$$

et on pose la valeur de la dérivée directionnelle en  $(-\mathbf{N}, -\mathbf{N}')$  de  $Z(\mathbf{s}, \mathbf{s}')$  selon la direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}')$ ,

$$Z'(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} := \partial_s (Z_{\mathbf{N}, \mathbf{N}'}(s))_{\boldsymbol{\mu}, \boldsymbol{\mu}'}|_{s=0}.$$

On introduit désormais des fonctions auxiliaires (que l'on appellera des fonctions zêta de Barnes généralisés) dont le rôle sera crucial dans les formules de  $Z'(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ .

**Définition 1.3.13.** Soit  $\mathbf{R} \in \mathbb{N}^P$ , et  $j \in \llbracket 1, Q \rrbracket$ . On pose pour tout  $\sigma \gg 1$ ,

$$\varphi_{\mathbf{R}}^j(s) = \sum_{\mathbf{n} \in \mathbb{N}^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d'_j)^{-s}.$$

**Remarque 1.3.14.** Si  $\mathbf{R} = \mathbf{0}$ , on obtient une fonction zêta de Barnes :  $\varphi_{\mathbf{0}}^j(s) = \zeta_P^B(s, d'_j | c_{j,1}, \dots, c_{j,P})$ .

La fonction  $\varphi_{\mathbf{R}}^j(s)$  correspond à une fonction de type  $Z_{\mathbf{N}, \mathbf{N}'}(s)_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ , avec  $Q = 1$ ,  $\boldsymbol{\mu} = \mathbf{0}$ ,  $\boldsymbol{\mu}' = 1$ ,  $\mathbf{N} = \mathbf{R}$  et  $\mathbf{N}' = 0$ . On en déduit en particulier qu'elle admet un prolongement méromorphe, et qu'elle est régulière en les entiers négatifs. La dérivée première de ces fonctions auxiliaires interviendra dans l'expression de  $Z'(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ , il est donc très intéressant d'avoir une expression exploitable des valeurs spéciales en les entiers négatifs de ces fonctions auxiliaires.

## 1.4 Principaux résultats

Comme il est mentionné dans la section précédente, on démontrera par une autre méthode le résultat de Komori sur l'existence des valeurs directionnelles  $Z(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  dans la Proposition 2.2.4. Pour obtenir ce résultat, on utilise dans le chapitre 2 une stratégie appelée expansion de Crandall, développée dans [Cra12], qui était initialement utilisée pour obtenir des approximations numériques de certaines fonctions zêta et certaines fonctions  $L$ . Cette stratégie a par la suite été utilisée par Borwein et Dilcher dans [BD18] pour étudier les valeurs d'une fonction zêta de type Mordell-Torheim, et approximer sa dérivée en les entiers négatifs. Par la suite, Onodera a donné des formules exactes pour les valeurs aux entiers négatifs d'une fonction zêta de type Mordell-Torheim généralisée, ainsi que des valeurs de ses dérivées d'ordre 1 et 2 dans [Ono21]. L'astuce utilisée par Onodera pour calculer les valeurs des dérivées aux entiers négatifs repose en partie sur le fait d'introduire des fonctions auxiliaires ressemblant à des fonctions zêta de Barnes, avec un terme polynomial au numérateur du terme général de la série, puis de donner une formule des dérivées de ces fonctions auxiliaires.

Plus précisément, on décrira une formule de prolongement pour la fonction méromorphe  $Z_{\mathbf{N}, \mathbf{N}'}(s)_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  dans la Proposition 2.2.4 :

$$Z_{\mathbf{N}, \mathbf{N}'}(s)_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{\boldsymbol{\mu}, \boldsymbol{\mu}'} + J_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{\boldsymbol{\mu}, \boldsymbol{\mu}'},$$

où  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  sera holomorphe sur tout  $\mathbb{C}$ , et s'annulant en  $s = 0$ , et  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  une fonction méromorphe sur  $\mathbb{C}$ , avec des pôles qui sont tous simples, de la forme

$$\begin{aligned} s &= \frac{n}{\mu_p} & (1 \leq p \leq P \text{ tel que } \mu_p \neq 0, n \in \mathbb{Z}_{\geq N_p+1}) \\ s &= \frac{n}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} & (\emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket, n \in \mathbb{Z}_{\leq |\mathbf{N}|_{\mathcal{P}} + |\mathbf{N}'|_{\mathcal{P}}}) \end{aligned}$$

En particulier, on constatera que 0 n'est pas un pôle pour la fonction méromorphe  $Z_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}$ , et donc on retrouvera que les valeurs directionnelles ont un sens pour la fonction  $Z(\mathbf{s}, \mathbf{s}')$ .

Le fait que  $Z_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}$  admette un prolongement méromorphe sur  $\mathbb{C}$  découle directement de la Proposition 1.3.9, et ce même théorème donne également des candidats pour les pôles de cette fonction. Il suffit en effet de remplacer  $s_p$  par  $-N_p + \mu_p s$  et  $s'_q$  par  $-N'_q + \mu'_q s$  pour obtenir des candidats pôles. Cependant, l'ensemble de pôles que l'on donne ici est plus précis.

Remarquons au passage que le terme  $K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s)$  ne contribuera pas à la formule des valeurs spéciales directionnelles, puisqu'il s'annule en  $s = 0$ . En revanche, si on dérive ce terme, il n'est pas nul en général, et est particulièrement pathologique. On introduira alors des fonctions auxiliaires  $(\varphi_{\mathbf{R}}^j)_{j \in \llbracket 1, Q \rrbracket}$  qui aideront à récupérer l'information du terme  $\partial_s(K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s))$  en  $s = 0$ . On pourra alors évaluer la formule de prolongement en  $s = 0$  pour obtenir une formule explicite sur les valeurs directionnelles des dérivées.

**Théorème A.** Soit  $(\mu, \mu') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ . On a

$$Z(-(\mathbf{N}, \mathbf{N}'))_{\mu, \mu'} = (-1)^{|\mathbf{N}'|} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|} \left( \prod_{p \in \mathcal{P}} N_p! \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} \sum_{j=1}^Q \frac{(-1)^{N'_j} \mu'_j N'_j!}{|\mu'| + |\mu|_{|\mathcal{P}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$$

$$\prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!},$$

avec

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, \mathcal{P}}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, \mathcal{P}}^{v_{p,q}} \right),$$

où l'on rappelle les notations

$$\forall q \in \llbracket 1, Q \rrbracket, \quad \mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p,q},$$

et

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad \mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p,q}.$$

**Remarque 1.4.1.** 1) L'expression décrivant  $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$  est une somme finie. En effet, on voit que

$$\sum_{\substack{q=1 \\ q \neq j}}^Q (N'_q - \mathbf{w}(q)) = \sum_{\substack{q=1 \\ q \neq j}}^Q \mathbf{v}(q) = \sum_{(p,q) \in \mathcal{P} \times (\llbracket 1, Q \rrbracket \setminus \{j\})} v_{p,q}. \text{ Comme } \mathbf{w}(q) \text{ est positif pour tout } q \in \llbracket 1, Q \rrbracket, \text{ on en}$$

déduit la majoration suivante :

$$\sum_{(p,q) \in \mathcal{P} \times (\llbracket 1, Q \rrbracket \setminus \{j\})} v_{p,q} \leq |\mathbf{N}'|.$$

Ainsi, pour tout  $p \in \mathcal{P}$ , et tout  $q \in \llbracket 1, Q \rrbracket \setminus \{j\}$ , on a que  $0 \leq v_{p,q} \leq |\mathbf{N}'|$ .

2) L'expression des valeurs de  $Z_{\mu, \mu'}(\mathbf{N}, \mathbf{N}')$  est décrite par une somme finie. En effet, pour tout  $\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}$  tel que  $|\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$  on a  $0 \leq k_p \leq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$  pour tout  $p \in \mathcal{P}^c$ .

3) Les termes de la forme  $\zeta(-n, d)$  avec  $n \in \mathbb{N}$  et  $d \in H_0$  s'expriment à l'aide des polynômes de Bernoulli :

$$\zeta(-n, d) = -\frac{B_{n+1}(d)}{n+1}.$$

De plus, en  $d = 1$ , on a  $\zeta(-n, 1) = \zeta(-n)$ , et

$$\zeta(-n) = (-1)^n \frac{b_{n+1}}{n+1},$$

où  $b_{n+1}$  est le  $n+1$ -ième nombre de Bernoulli.

4) Si tous les coefficients  $d_p$  valent 1, alors le produit  $\prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}$  sera fréquemment nul. En effet, s'il existe  $p \in \mathcal{P}^c$  tel que  $N_p + k_p$  soit pair et plus grand que 2, alors le produit est nul.

En  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ , on obtient les expressions simplifiées suivantes :

**Corollaire A1.** Soit  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_*^{+Q}$ . On a

$$Z(-(\mathbf{0}, \mathbf{0})) = \sum_{\substack{\boldsymbol{\mu}, \boldsymbol{\mu}' \\ \mathcal{P} \not\subseteq \llbracket 1, P \rrbracket}} (-1)^{|\mathcal{P}|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \sum_{j=1}^Q \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p, d_p)}{k_p!}. \quad (1.9)$$

**Remarque 1.4.2.** On obtient au passage la même formule que [Ono21] en  $N = 0$ , une fois avoir remplacé toutes les directions par 1.

Via le Théorème A, on a également une formule pour les valeurs sur la diagonale de  $Z(\mathbf{s}, \mathbf{s}')$  en considérant la direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$  :

**Corollaire A2.** Soit  $N \in \mathbb{N}$ . On a

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \not\subseteq \llbracket 1, P \rrbracket} \frac{(-1)^{(|\mathcal{P}|+1+Q)N+|\mathcal{P}|}}{Q+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=(Q+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!},$$

avec

$$Q_N^0(\mathcal{P}, \mathbf{k}) = (-1)^{N(Q-1)} N^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right),$$

où l'on rappelle les notations

$$\forall q \in \llbracket 1, Q \rrbracket, \quad \mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p,q},$$

et

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad \mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p,q}.$$

**Remarque 1.4.3.** Observons à l'aide du corollaire précédent que, si l'on pose  $Q = 1$ ,  $P \geq 1$  un entier quelconque, et  $l_1(\mathbf{n}) = n_1 + \dots + n_P$ , on trouve via les formules précédentes que  $Q_N^0(\mathcal{P}, \mathbf{k}) = 1$ , et que

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subsetneq [1, P]} \frac{(-1)^{|\mathcal{P}|N+|\mathcal{P}|}}{1+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=(1+|\mathcal{P}|)N+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!}.$$

Cette formule coïncide avec celle obtenue par Onodera dans [Ono21, Théorème 3].

On peut également fournir une variante qualitative du Théorème A sur les valeurs directionnelles aux entiers négatifs de  $Z(\mathbf{s}, \mathbf{s}')$  :

**Corollaire A3.** Soit  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ , et  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ . On note  $\mathbb{K}$  le corps engendré par  $\mathbb{Q}$ , par les coefficients complexes  $(d_p)_{p \in [1, P]}$  et  $(c_{q,p})_{p \in [1, P], q \in [1, Q]}$ , et par les directions  $\mu_p$  pour tout  $p \in [1, P]$ , et  $\mu'_q$  pour tout  $q \in [1, Q]$ . Alors  $Z(-(\mathbf{N}, \mathbf{N}')) \in \mathbb{K}$ .

On a également une expression de la dérivée directionnelle en les entiers négatifs de  $Z$ . Cette expression fait intervenir des fonctions "auxiliaires", que l'on appellera fonction zêta de type Barnes généralisée.

Si l'on suppose qu'une forme linéaire  $l_j$  est à coefficients rationnels, on dispose d'une relation entre la fonction auxiliaire  $\varphi_{\mathbf{R}}^j(s)$  et la fonction zêta de Hurwitz :

**Proposition B.** Soit  $j \in [1, Q]$  un entier. On suppose que  $l_j$  est une forme linéaire à coefficients rationnels pour  $1 \leq j \leq Q$ . Pour tout  $1 \leq p \leq P$ , on pose  $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$  la décomposition irréductible de  $c_{j,p}$  avec  $a_{j,p} > 0$  et  $b_{j,p} > 0$ . On pose alors  $x_j(\mathbf{c}) = \frac{\text{ppcm}(a_{j,p})}{\text{pgcd}(b_{j,p})} \in \mathbb{Q}_+^*$ , et  $\beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}^*$  pour tout  $1 \leq p \leq P$ . Alors pour tout  $\mathbf{R} \in \mathbb{N}^P$ , la fonction  $\varphi_{\mathbf{R}}^j(s)$  admet la relation suivante pour tout  $s \in \mathbb{C} \setminus \mathbb{N}^*$ ,

$$\begin{aligned} \varphi_{\mathbf{R}}^j(s) = & x_j(\mathbf{c})^{-s} \prod_{p=1}^P \beta_{j,p}^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ \dots \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^{\mathcal{P}}, \mathbf{k}' \in \mathbb{N} \\ |\mathbf{k}| + \mathbf{k}' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!} \\ & \cdot \frac{\zeta(s - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} \end{aligned}$$

En dérivant en  $s$  la fonction auxiliaire  $\varphi_{\mathbf{R}}^j(s)$ , puis en évaluant la dérivée en  $s = -N$  avec  $N$  un entier positif, on trouve aisément le corollaire suivant via la proposition précédente.

**Corollaire B1.** Avec les mêmes notations que la Proposition B, on a pour tout  $\mathbf{R} \in \mathbb{N}^P$ ,

$$\begin{aligned} \varphi_{\mathbf{R}}^j(-N) = & x_j(\mathbf{c})^N \prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ \dots \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq [1, P]} \left( \prod_{p \in \mathcal{P}^c} R_p! \right) \\ & \cdot \sum_{\substack{\mathbf{k} \in \mathbb{N}^{\mathcal{P}}, \mathbf{k}' \in \mathbb{N} \\ |\mathbf{k}| + \mathbf{k}' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!} \\ & \cdot \left( \frac{\zeta'(-N - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} \right). \end{aligned}$$

On commence par énoncer une version qualitative sur les valeurs spéciales de la dérivée directionnelle de  $Z(\mathbf{s}, \mathbf{s}')$  :

**Théorème C.** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ . On note  $\mathbb{K}$  le corps engendré par  $\mathbb{Q}$ , par les coefficients complexes  $(d_p)_{p \in \llbracket 1, P \rrbracket}$  et  $(c_{q,p})_{p \in \llbracket 1, P \rrbracket, q \in \llbracket 1, Q \rrbracket}$ , et par les directions  $(\mu_p)_{p \in \llbracket 1, P \rrbracket}$ ,  $(\mu'_q)_{q \in \llbracket 1, Q \rrbracket}$ . Alors,

$$Z'_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(-(\mathbf{N}, \mathbf{N}')) \in \text{Vect}_{\mathbb{K}} \left[ \gamma, \ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right)_{1 \leq q, j \leq Q, 1 \leq p \leq P}, (\zeta'(-n, d_p))_{n \in \mathfrak{D}, |\mathbf{N}| + |\mathbf{N}'| + P}, \right. \\ \left. (\varphi_{\mathbf{n}}^{j'}(-N'_j))_{\mathbf{n} \in [0, |\mathbf{N}| + |\mathbf{N}'|]^P, j \in \llbracket 1, Q \rrbracket} \right],$$

où  $\gamma$  désigne la constante d'Euler, et les  $\varphi_j$  sont les fonctions auxiliaires, dites de Barnes généralisées, décrites dans la Définition 1.3.13.

Si les formes linéaires  $(l_j)_{j \in \llbracket 1, Q \rrbracket}$  sont à coefficients rationnels, alors on peut utiliser le Corollaire B1 pour obtenir un résultat plus précis :

**Corollaire C1.** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ . On suppose que les formes linéaires  $(l_j)_{j \in \llbracket 1, Q \rrbracket}$  et les coefficients  $(d_p)_{p \in \llbracket 1, P \rrbracket}$  sont rationnels. On note  $\mathbb{K}$  le corps engendré par  $\mathbb{Q}$ , et par les directions  $(\mu_p)_{p \in \llbracket 1, P \rrbracket}$ ,  $(\mu'_q)_{q \in \llbracket 1, Q \rrbracket}$ . Pour tout  $1 \leq p \leq P$ , on pose  $c_{q,p} = \frac{a_{q,p}}{b_{q,p}} \in \mathbb{Q}$  la décomposition irréductible de  $c_{q,p}$  avec  $a_{q,p} > 0$  et  $b_{q,p} > 0$ . On pose alors  $x_q(\mathbf{c}) = \frac{\text{ppcm}(a_{q,p})}{\text{pgcd}(b_{q,p})} \in \mathbb{Q}_+$ , et  $\beta_{q,p}(\mathbf{c}) := \frac{x_q(\mathbf{c})}{c_{q,p}} \in \mathbb{N}^*$  pour tout  $1 \leq p \leq P$ . Posons également l'ensemble

$$A := \{l_j(x_{\mathbf{c}}^{-1}(\mathbf{d} + \mathbf{v})) \mid j \in \llbracket 1, Q \rrbracket, \mathbf{v} \in [0, \beta_{j,1} - 1] \times \dots \times [0, \beta_{j,P} - 1]\}.$$

On a alors

$$Z'_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(-(\mathbf{N}, \mathbf{N}')) \in \text{Vect}_{\mathbb{K}} \left[ \gamma, \ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right)_{q, j \in \llbracket 1, Q \rrbracket}, (\ln(x_j(\mathbf{c})))_{1 \leq j \leq Q}, (\zeta'(-n, y))_{0 \leq n \leq P|\mathbf{N}| + |\mathbf{N}'| + P, y \in A} \right],$$

où  $\gamma$  désigne la constante d'Euler

La formule pour  $Z'_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(-(\mathbf{N}, \mathbf{N}'))$  est en fait explicite. Elle admet une formule faisant intervenir les valeurs spéciales des dérivées des fonctions zêta de Barnes généralisées  $\varphi_j^{\mathbf{R}}$ , des valeurs spéciales des dérivées en  $s$  de la fonctions zêta de Hurwitz, et des valeurs spéciales de la fonction zêta de Hurwitz. On détaille cette formule dans le théorème ci-dessous.

**Théorème D.** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ , alors

$$\begin{aligned}
Z'(-(\mathbf{N}, \mathbf{N}')) &= \tag{1.10} \\
&\sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q (-1)^{|\mathbf{N}|+|\mathbf{N}'|+|\mathcal{P}|+N'_j} \left( \prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\
&\cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \sum_{p \in \mathcal{P}} \mu_p(\gamma - h_{N_p}) + \mu'_j(\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
&+ \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|+N'_j} } \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
&\cdot \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
&+ \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{s=-N'_j} \\
&- \sum_{j=1}^Q (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\
&\cdot \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}|+|\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}|+|\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \Bigg),
\end{aligned}$$

où l'on a posé

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket},$$

pour tout  $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$ , et tout  $j \in \llbracket 1, Q \rrbracket$ ,

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) &= \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, N'_q = \mathbf{v}(q) + \mathbf{w}(q)}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{q,p}} \right) \\
&\cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{q,p}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!.
\end{aligned}$$

et

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k})_{\mu, \mu'} &= (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\
&\quad \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
&+ (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \\
&\quad \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')
\end{aligned}$$

avec  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  la constante posée dans la Proposition 1.3.2.

**Remarque 1.4.4.** Ce théorème ramène ainsi le calcul des dérivées directionnelles en les multientiers négatifs  $Z'(-(\mathbf{N}, \mathbf{N}'))_{\mu, \mu'}$  aux calculs des valeurs aux entiers négatifs de la dérivée des fonctions auxiliaires

$$\varphi_{\mathbf{R}}^j(s) = \sum_{\mathbf{n} \in \mathbb{N}^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d'_j)^{-s} \quad (j \in \llbracket 1, Q \rrbracket, \mathbf{R} \in \mathbb{N}^P).$$

Si les coefficients des formes linéaires  $l_j$  sont rationnels, alors on dispose d'une expression explicite de la valeur de  $\partial_s \varphi_{\mathbf{R}}^j(-N)$  en fonction des valeurs spéciales des dérivées de la fonction zêta d'Hurwitz via la Proposition B.

**Remarque 1.4.5.** On observe que la formule pour  $Z'(-(\mathbf{N}, \mathbf{N}'))_{\mu, \mu'}$  ne fait intervenir que des sommes finies.

En effet, il est clair que la somme  $\sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|}}$  est finie. De même, on a que la somme

$\sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}}$  présente dans le calcul de  $Q_{\mathbf{N}, \mathbf{N}'}^1$  et de  $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})_{\mu, \mu'}$  est finie puisque

$$\forall a \in \mathcal{P}, \forall b \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad 0 \leq v_{a,b} \leq \sum_{\substack{q=1 \\ q \neq j}}^Q \mathbf{v}(q) + \mathbf{w}(q) = |\mathbf{N}'|$$

$$\forall p \in \mathcal{P}^c, \forall q \in \llbracket 1, Q \rrbracket, \quad 0 \leq w_{p,q} \leq |\mathbf{w}_p| = k_p \leq |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}^c} + |\mathcal{P}|.$$

Pour les mêmes raisons, la somme  $\sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}}$  est également finie.

Si les coefficients  $\mathbf{c}$  sont rationnels, on peut remplacer les dérivées des fonctions auxiliaires aux entiers négatifs par la formule obtenue dans le Corollaire B1 :

**Corollaire D1.** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ . Pour tout  $1 \leq p \leq P$ , on pose  $c_{q,p} = \frac{a_{q,p}}{b_{q,p}} \in \mathbb{Q}$  la décomposition irréductible de  $c_{q,p}$  avec  $a_{q,p} > 0$  et  $b_{q,p} > 0$ . On pose alors  $x_q(\mathbf{c}) = \frac{\text{ppcm}(a_{q,p})}{\text{pgcd}(b_{q,p})} \in \mathbb{Q}_+^*$ , et  $\beta_{q,p}(\mathbf{c}) := \frac{x_q(\mathbf{c})}{c_{q,p}} \in \mathbb{N}^*$  pour tout  $1 \leq p \leq P$ . On a

$$\begin{aligned}
Z'(-(\mathbf{N}, \mathbf{N}')) &= \\
& \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q (-1)^{|\mathbf{N}| + |\mathbf{N}'| + |\mathcal{P}| + N'_j} \left( \prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\
& \cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
& + \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|} + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
& \cdot \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
& + \sum_{j=1}^Q \mu'_j x_j(\mathbf{c})^{N'_j} \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p} \\
& \cdot \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c}) - 1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c}) - 1}} \sum_{\mathcal{P} \subsetneq [1, P]} \left( \prod_{p \in \mathcal{P}^c} \mathbf{R}(\mathbf{N}, j, \mathbf{u})_p! \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}^c| + |\mathcal{P}|} - 1}} (-1)^{|\mathbf{k}|} \left( \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})})}{k_p!} \right) \\
& \cdot \left( \frac{\zeta'(-N'_j - k', l_j(x_j(\mathbf{c}))^{-1}(\mathbf{d} + \mathbf{v}))}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N'_j - k', l_j(x_j(\mathbf{c}))^{-1}(\mathbf{d} + \mathbf{v}))}{k'!} \right) \\
& - \sum_{j=1}^Q (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\
& \cdot \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}'|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}'|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!}
\end{aligned}$$

où l'on a posé

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + \mathbf{u}(p))_{p \in [1, P]},$$



pour tout  $\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}}$ , et tout  $j \in [1, Q]$ ,

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \in [1, Q] \setminus \{j\}, N'_q = \mathbf{v}(q) + \mathbf{w}(q)}}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ \cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!$$

et

$$Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\ + (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}}} \\ \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}').$$

avec  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  la constante posée dans la Proposition 1.3.2.

**Remarque 1.4.6.** Posons  $Q = 1$ , et  $P \geq 1$  un entier quelconque, on considère  $l_1(\mathbf{n}) = n_1 + \dots + n_P$  (et donc  $c_{1,1} = \dots = c_{1,P} = 1$ ),  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ , et  $(\mathbf{N}, \mathbf{N}') = (N_\Delta, N_\Delta)$ , en rappelant que  $N_\Delta = (N, \dots, N)$ . On trouve alors via le corollaire précédent que

- Pour tout  $\mathcal{P} \subsetneq [1, P]$ , et  $\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}$ , on a  $Q_{\mathbf{N}, \mathbf{N}'}^0(1, \mathcal{P}, \mathbf{k}) = 1$  et  $Q_{\mathbf{N}, \mathbf{N}'}^1(1, \mathcal{P}, \mathbf{k}) = 0$ .
- Avec les notations du Corollaire D1, on a  $\mathbf{x}_1(\mathbf{c}) = 1$ , et  $\beta_{1,p} = 1$  pour tout  $1 \leq p \leq P$ .
- On a  $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := \mathbf{N}_\Delta \in \mathbb{N}^P$ .
- Dans l'expression de  $Z'(-(N_\Delta, N_\Delta))$  du corollaire précédent, on trouve que la première somme de la formule s'annule avec la dernière somme de la formule.

Via les quatre points précédents, on trouve que

$$Z'(-(N_\Delta, N_\Delta)) = \sum_{\mathcal{P} \subsetneq [1, P]} \frac{N^{|\mathcal{P}|+1}}{|\mathcal{P}|+1} (-1)^{|\mathcal{P}|(N+1)} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|(N+1) + |\mathcal{P}|}} \\ \left( \sum_{i \in \mathcal{P}^c} \frac{\zeta'(-N - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N - k_p, d_p)}{k_p!} \right) \\ + \sum_{\mathcal{P} \subsetneq [1, P]} N^{|\mathcal{P}^c|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}} \in \mathbb{N}^{\mathcal{P}}, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathcal{P}^c|(N+1) + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \frac{\zeta'(-N - k', d'_1)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta(-N - k_p, d_p)}{k_p!}.$$

Cette expression correspond à une forme développée de la formule obtenue par Onodera dans [Ono21, Théorème 4] (avec le paramètre  $d_1 = 0$  dans le Théorème d'Onodera).

En posant  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$  dans le Théorème D, on obtient le corollaire suivant :

**Corollaire D2.** Soit  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ , alors

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = & \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \left( \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \gamma(\mu'_j + |\boldsymbol{\mu}|_{\mathcal{P}}) + Q^1_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p, d_p)}{k_p!} \right) \\ & + \sum_{j=1}^Q \mu'_j (\zeta^B)'(0, d'_j | \mathbf{c}_j) - \sum_{j=1}^Q \gamma \mu'_j \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p, d_p)}{k_p!}, \end{aligned}$$

avec

$$\begin{aligned} Q^1_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(j, \mathcal{P}, \mathbf{k}) = & \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} [0, k_p]} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{j,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ & + \gamma |\boldsymbol{\mu}'|_{[1, Q] \setminus \{j\}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}, \end{aligned}$$

et  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N})$  la constante définie dans la Proposition 1.3.2.

**Remarque 1.4.7.** Les termes de la forme  $F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0})$  dans le corollaire précédent découlent de l'étude de la fraction rationnelle

$$x^{-1+k_p} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x)^{-1}.$$

**Remarque 1.4.8.** Notons que Matsumoto et Weng ont obtenu dans [MW02] un résultat similaire pour la valeur de la dérivée en  $s = 0$  d'une fonction zêta définie par deux polynômes dans  $\mathbb{C}[X]$ .

En étudiant  $\boldsymbol{\mu} = \mathbf{0}$ , et  $\boldsymbol{\mu}' = \mathbf{1}$ , on obtient que  $Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{0}, \mathbf{1}} = \partial_s (\zeta^{Sh}(s, (\mathbf{c}_1, \dots, \mathbf{c}_Q), \mathbf{d}')|_{s=0})$ , c'est-à-dire la dérivée première par rapport à  $s$ , en  $s = 0$ , de la fonction zêta de Shintani définie dans 1.2.13. Rappelons que, avec la Définition 1.2.16 et la Définition 1.2.14, on a  $(\partial_s \zeta^B(s, d | c_1, \dots, c_P))|_{s=0} = \ln(\Gamma_P(d | c_1, \dots, c_P))$ , et  $\partial_s (\zeta^{Sh}(s, \mathbf{d}' | \mathbf{c}))|_{s=0} = \ln(\Gamma_P(\mathbf{d}' | \mathbf{c}))$ . On obtient alors via le corollaire précédent que le logarithme de la fonction multigamma de Shintani  $\Gamma_P^{Sh}(\mathbf{d}' | \mathbf{c}_1, \dots, \mathbf{c}_Q)$  s'exprime complètement à l'aide du logarithme de la fonction gamma de type Barnes.

**Corollaire D3.** On a

$$\begin{aligned} \ln(\Gamma_P(\mathbf{d}' | \mathbf{c})|_{s=0}) = & \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|}}{Q} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \left( \gamma \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) + Q^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{j=1}^Q \ln(\Gamma_P(d'_j | \mathbf{c}_j)) - \gamma \sum_{j=1}^Q \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p, d_p)}{k_p!}, \end{aligned}$$

avec

$$Q^1(j, \mathcal{P}, \mathbf{k}) = \sum_{\substack{f=1 \\ f \neq j}}^Q \sum_{\mathbf{w}' = (w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ + \gamma(Q-1) \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}.$$

En appliquant le Corollaire A1 et le Corollaire D2 aux deux fonctions zêta de Witten  $\zeta_{\mathfrak{g}_2}$  et  $\zeta_{\mathfrak{so}(5)}$ , on obtient les formules suivantes :

**Théorème E.** *On a les valeurs spéciales pour  $\zeta_{\mathfrak{g}_2}(s)$  et  $\zeta_{\mathfrak{so}(5)}(s)$  en  $s = 0$  :*

$$\zeta_{\mathfrak{g}_2}(0) = \frac{29}{60}, \\ \zeta_{\mathfrak{so}(5)}(0) = \frac{7}{18}.$$

De plus, on a les valeurs spéciales pour  $\zeta'_{\mathfrak{g}_2}(s)$  et  $\zeta'_{\mathfrak{so}(5)}(s)$  en  $s = 0$  :

$$\zeta'_{\mathfrak{g}_2}(0) = \frac{29}{60} \ln(120) + 2\gamma\zeta(0)^2 + \frac{1}{4}\zeta(0)\zeta'(0) - 4\gamma\zeta(-1) + \frac{1}{5}\zeta(-1)(\ln(3) + 2\ln(2)) - \frac{14}{5}\zeta'(-1) \\ - \frac{11}{9} \ln(2) - \frac{23}{24} \ln(3) - \frac{1}{2} \ln(\pi) - 5\zeta'(0) + 3\zeta'(-1) - \frac{4}{3}\zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3}\zeta'\left(0, \frac{4}{3}\right) \\ + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) - \frac{1}{6}\zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6}\zeta'\left(0, \frac{5}{6}\right), \\ \zeta'_{\mathfrak{so}(5)}(0) = \frac{7}{18} \ln(6) - \frac{11}{24} \ln(2) - \frac{1}{4} \ln(\pi) + \left(\frac{1}{2}\zeta(0)^2 - \frac{1}{2}\zeta(-1)\right) \gamma - \frac{11}{4}\zeta'(0) - \frac{13}{6}\zeta'(-1)$$

On peut simplifier la formule de  $\zeta'_{\mathfrak{so}(5)}(0)$  sous cette forme :

$$\zeta'_{\mathfrak{so}(5)}(0) = -\frac{13}{6} \left( \frac{1}{12} - \ln(A) \right) + \frac{1}{6}\gamma + \frac{47}{36} \ln(2) + \frac{7}{18} \ln(3) + \frac{9}{8} \ln(\pi) \simeq 2.94635\dots$$

avec  $A$  la constante de Gleicher-Kinkelin [OEI23b], et  $\gamma$  la constante d'Euler [OEI23a].

Via la formule précédente, on obtient une expression explicite de la constante  $C$  dans le Théorème 1.2.35 :

$$C = \frac{e^{\frac{1}{6}\gamma} 2^{\frac{35}{36}} \pi^{\frac{9}{8}} A^{\frac{13}{6}} \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}}{3^{\frac{5}{72}} e^{\frac{13}{72}} \sqrt{\pi}}.$$

On obtient alors une version explicite du Théorème 1.2.35 obtenu par Bridges, Brindle, Bringmann et Franke dans [BBBF23, Théorème 1.3] :

**Corollaire F.** *Pour tout entier  $N \geq 1$ , on a*

$$r_{\mathfrak{so}(5)}(n) \underset{n \rightarrow +\infty}{\sim} \frac{C}{n^{\frac{7}{12}}} \exp\left(A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4\right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N\left(n^{-\frac{N+1}{9}}\right)\right),$$

avec

$$b = \frac{7}{12}, \quad C = \frac{2^{\frac{35}{36}} \pi^{\frac{9}{8}} A^{\frac{13}{6}} \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}}{3^{\frac{5}{72}} e^{\frac{13}{72} + \frac{1}{9}\gamma} \sqrt{\pi}}, \quad A_1 = \frac{3^{\frac{4}{3}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{3}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{8}{3}}}, \\ A_2 = \frac{2^{\frac{8}{9}} \left(2^{\frac{1}{3}} + 1\right) \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right)}{3^{\frac{7}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{9}}}, \quad A_3 = \frac{2^{\frac{40}{9}} \left(2^{\frac{1}{3}} + 1\right)^2 \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{1}{3}\right)^2 \zeta\left(\frac{4}{3}\right)^2}{3^{\frac{44}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{20}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{10}{9}}}, \\ A_4 = \frac{2^8 \left(2^{\frac{1}{3}} + 1\right)^3 \Gamma\left(\frac{1}{3}\right)^3 \zeta\left(\frac{1}{3}\right)^3 \zeta\left(\frac{4}{3}\right)^3}{3^8 \Gamma\left(\frac{1}{4}\right)^4 \zeta\left(\frac{3}{2}\right)^2}.$$

On souhaite désormais appliquer les résultats obtenus dans le Théorème E afin d'obtenir une formule asymptotique du nombre de représentations  $r_{\mathfrak{g}_2}(n)$  de l'algèbre de Lie exceptionnelle  $\mathfrak{g}_2$ , formule que l'on obtient via le Théorème 1.2.33 obtenu par Bridges, Brindle, Bringmann et Franke dans [BBBF23, Théorème 4.4]. On pose tout d'abord

$$\begin{aligned} l_1^*(x_1, x_2) &= x_1 + x_2 + x_3 + 2x_4, \\ l_2^*(x_1, x_2) &= x_1 + 2x_2 + 3x_3 + 4x_4, \end{aligned}$$

et on pose

$$\begin{aligned} \omega_\beta := & 120^{\frac{1}{5}} \frac{\Gamma\left(\frac{4}{5}\right)}{\Gamma\left(\frac{1}{5}\right)^4} \zeta\left(\frac{1}{5}\right) \left( \int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{4}{5}} (l_1^*(1, x_2, x_3, x_4))^{-\frac{4}{5}} + l_2^*(1, x_2, x_3, x_4)^{-\frac{4}{5}} dx_2 dx_3 dx_4 \right. \\ & + \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{4}{5}} (l_1^*(x_1, 1, x_3, x_4))^{-\frac{4}{5}} + l_2^*(x_1, 1, x_3, x_4)^{-\frac{4}{5}} dx_1 dx_3 dx_4 \\ & + \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{4}{5}} (l_1^*(x_1, x_2, 1, x_4))^{-\frac{4}{5}} + l_2^*(x_1, x_2, 1, x_4)^{-\frac{4}{5}} dx_1 dx_2 dx_4 \\ & \left. + \int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{4}{5}} (l_1^*(x_1, x_2, x_3, 1))^{-\frac{4}{5}} + l_2^*(x_1, x_2, x_3, 1)^{-\frac{4}{5}} dx_1 dx_2 dx_3 \right) \end{aligned}$$

et

$$\begin{aligned} \omega_\alpha = & 120^{\frac{1}{3}} \frac{\Gamma\left(\frac{2}{3}\right)^2}{\Gamma\left(\frac{1}{3}\right)} \left( \int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, x_3, 1)^{-\frac{2}{3}} dx_1 dx_2 dx_3, \right. \\ & + \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, 1, x_4)^{-\frac{2}{3}} dx_1 dx_2 dx_4 \\ & + \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, 1, x_3, x_4)^{-\frac{2}{3}} dx_1 dx_3 dx_4 \\ & \left. + \int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(1, x_2, x_3, x_4)^{-\frac{2}{3}} dx_2 dx_3 dx_4 \right). \end{aligned}$$

On trouve alors une formule asymptotique de  $r_{\mathfrak{g}_2}(n)$ , dont certains de ces coefficients s'expriment en fonction des résidus  $\omega_\alpha$  et  $\omega_\beta$ , et des valeurs spéciales  $\zeta_{\mathfrak{g}_2}(0)$  et  $\zeta'_{\mathfrak{g}_2}(0)$ .

**Théorème G.** *Soit  $N \in \mathbb{N}^*$ . Lorsque  $n$  tend vers  $+\infty$ , on a*

$$r_{\mathfrak{g}_2}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp\left(A_1 n^{\frac{1}{4}} + A_2 n^{\frac{3}{20}} + A_3 n^{\frac{1}{20}}\right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{20}}} + O\left(n^{-\frac{N+1}{20}}\right)\right),$$

avec  $B_j$  des réels explicites, et avec  $C, b, A_1, A_2, A_3$  tels que :

$$\begin{aligned}
C &= \frac{e^{\zeta'_{\mathfrak{g}_2}(0)} (\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{1-6\zeta_{\mathfrak{g}_2}(0)}{8}} \sqrt{3}}{\sqrt{8\pi}}, \quad b = \frac{6-6\zeta_{\mathfrak{g}_2}(0)+1}{8}, \\
K_2 &= \frac{3}{4} \cdot \frac{\omega_\beta \Gamma(\frac{6}{5}) \zeta(\frac{6}{5})}{(\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{3}{20}}}, \quad K_3 = -\frac{3}{160} \cdot \frac{(\omega_\beta \Gamma(\frac{6}{5}) \zeta(\frac{6}{5}))^2}{(\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{21}{20}}}, \\
A_1 &:= 4 \left( \omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{3}{4}}, \quad A_2 := \frac{\omega_\beta \Gamma(\frac{1}{5}) \zeta(\frac{6}{5})}{(\omega_\alpha \Gamma(\frac{1}{3}) \zeta(\frac{4}{3}))^{\frac{3}{20}}}, \\
A_3 &:= K_3 + 3 \left( \binom{-1/3}{1} K_3 + \binom{-1/3}{2} \frac{K_2^2}{(\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{3}{4}}} \right) + \frac{5 (\omega_\beta \Gamma(\frac{6}{5}) \zeta(\frac{6}{5}))}{(\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{18}{20}}} \binom{-1/5}{1} K_2 \\
\zeta'_{\mathfrak{g}_2}(0) &= \frac{29}{60} \ln(120) + 2\gamma \zeta(0)^2 + \frac{1}{4} \zeta(0) \zeta'(0) - 4\gamma \zeta(-1) + \frac{1}{5} \zeta(-1) (\ln(3) + 2 \ln(2)) - \frac{14}{5} \zeta'(-1) \\
&\quad - \frac{11}{9} \ln(2) - \frac{23}{24} \ln(3) - \frac{1}{2} \ln(\pi) - 5\zeta'(0) + 3\zeta'(-1) - \frac{4}{3} \zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3} \zeta'\left(0, \frac{4}{3}\right) \\
&\quad + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) - \frac{1}{6} \zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6} \zeta'\left(0, \frac{5}{6}\right), \\
\zeta_{\mathfrak{g}_2}(0) &= \frac{29}{60}.
\end{aligned}$$

**Remarque 1.4.9.** Les  $B_j$  peuvent être calculés d'après les coefficients  $b_j$  obtenus via [BBBF23, Lemme 3.6].



## Chapitre 2

# Expansion de Crandall de $Z(\mathbf{s}, \mathbf{s}')$

On fixera dans ce chapitre des coefficients  $\mathbf{c} = (c_{q,p})_{q \in \llbracket 1, Q \rrbracket, p \in \llbracket 1, P \rrbracket}$ , des complexes  $\mathbf{d} = (d_p)_{p \in \llbracket 1, P \rrbracket}$ ,  $\mathbf{d}' = (d'_q)_{q \in \llbracket 1, Q \rrbracket}$  tels que  $d_p \in H_0$  pour tout  $1 \leq p \leq P$ , et vérifiant pour tout  $1 \leq q \leq Q$ ,

$$\forall q \in \llbracket 1, Q \rrbracket, \quad d'_q = \mathbf{c}_q \cdot \mathbf{d} = \sum_{p=1}^P c_{q,p} d_p.$$

De plus, on fixera dans ce chapitre un multi-entier  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et une direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^+ \times \mathbb{R}_*^+$ . Toutes les majorations de ce chapitre dépendront potentiellement de ces constantes, mais l'on choisira de ne pas spécifier cette dépendance pour des soucis de clarté dans la rédaction.

On étudiera tout d'abord le prolongement méromorphe de  $Z(\mathbf{s}, \mathbf{s}')$  et on étudiera ses singularités dans la démonstration de la Proposition 1.3.9. Ensuite on décomposera la fonction méromorphe  $Z(\mathbf{s}, \mathbf{s}')$  à l'aide d'une expansion de Crandall, et on obtiendra que une formule de prolongement pour la fonction univariante  $Z(-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N} + \boldsymbol{\mu}'s)$ . Plus précisément, on obtiendra décomposition de cette fonction méromorphe suivant une somme entre une fonction holomorphe s'annulant en  $s = 0$ , et une fonction méromorphe correspondant à une série de type Erdélyi. Cette formule de prolongement sera cruciale dans la démonstration des Théorèmes A et D.

### 2.1 Démonstration de la Proposition 1.3.9

Le prolongement méromorphe de  $Z(\mathbf{s}, \mathbf{s}')$  et la forme des singularités découlent du Théorème 1.2.25. On cherche ici à montrer que  $Z(\mathbf{s}, \mathbf{s}')$  est absolument convergente sur le domaine  $\mathcal{D}$  introduit dans la Proposition 2.1.1. Il suffira de montrer que la série décrivant  $Z(\mathbf{s}, \mathbf{s}')$  est normalement convergente sur tout compact de  $\mathcal{D}$ . On se propose de démontrer la proposition suivante, plus générale que ce que l'on veut prouver dans la Proposition 1.3.9 :

**Proposition 2.1.1.** *Soit  $P, Q \geq 1$  deux entiers, et  $\boldsymbol{\gamma} = (\gamma_q)_{q \in \llbracket 1, Q \rrbracket} \in (H_0^P)^Q$ ,  $\boldsymbol{\alpha} = (\alpha_p) \in H_0^P$ ,  $\boldsymbol{\beta} = (\beta_q) \in H_0^Q$ . On pose*

$$\mathcal{D} = \mathcal{D}_{P,Q} := \left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket \right\}.$$

La fonction  $Z_{\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\beta}}^{P,Q}(\mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + \alpha_p)^{-s_p} \prod_{q=1}^Q \left( \sum_{p=1}^P \gamma_{q,p} n_p + \beta_q \right)^{-s'_q}$  converge normalement sur tout compact de  $\mathcal{D}$ .

*Démonstration.* Toutes les constantes intervenantes dans la preuve de cette proposition dépendront implicitement du choix des coefficients  $(\gamma_q)_{q \in \llbracket 1, Q \rrbracket}$ , de  $\boldsymbol{\alpha}$  et de  $\boldsymbol{\beta}$ . On ne marquera pas cette dépendance

pour améliorer la clarté de la rédaction. On démontrera ce résultat par principe de récurrence. Le cas  $P = 1$  est trivial.

Supposons que le résultat est vrai pour tout  $k < P$ . Soit  $(\mathbf{s}, \mathbf{s}') \in H_0^{P+Q}$ , et  $\mathbf{n} \in \mathbb{N}^P$ . Alors on a les inégalités suivantes

$$(n_p + 1) \ll n_p + \operatorname{Re}(\alpha_p) \leq |n_p + \alpha_p| \leq n_p + |\alpha_p| \ll (n_p + 1),$$

De même, on obtient que

$$\sum_{p=1}^P n_p + P \ll \sum_{p=1}^P \operatorname{Re}(\gamma_{q,p}) n_p + \operatorname{Re}(\beta_q) \leq \left| \sum_{p=1}^P \gamma_{q,p} n_p + \beta_q \right| \leq \sum_{p=1}^P |\gamma_{q,p}| n_p + |\beta_q| \ll \sum_{p=1}^P n_p + P.$$

En prenant le produit sur  $\llbracket 1, P \rrbracket$  et  $\llbracket 1, Q \rrbracket$ , on trouve que

$$\left| \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q} \right| \ll e^{(\sum_{p=1}^P |\tau_p| + \sum_{q=1}^Q |\tau'_q|)\pi/2} \prod_{p=1}^P (n_p + 1)^{-\sigma_p} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)}.$$

On souhaite désormais montrer que la série  $\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + 1)^{-\sigma_p} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)}$  est normalement convergente sur tout compact de  $\mathcal{D}$ . On peut découper cette série en somme finie de séries de la forme  $\sum_{0 \leq n_{f(1)} \leq \dots \leq n_{f(P)}} \prod_{p=1}^P (n_p + 1)^{-\sigma_p} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)}$ , avec  $f \in \mathfrak{S}_P$ , et en termes de la forme  $Z_{\mathbf{1}, \mathbf{1}, P}^{k, \mathbf{1}}((\sigma_1 + \dots + \sigma_{i_1}, \dots, \sigma_{i_{k-1}+1} + \dots + \sigma_{i_k}), \sigma'_1 + \dots + \sigma'_Q)$  avec  $1 = i_1 < i_2 < \dots < i_k = P$  et  $(i_1, \dots, i_k) \neq (1, \dots, P)$  (ce qui implique en particulier  $k < P$ ). Par principe de récurrence, la série de Dirichlet  $Z_{\mathbf{1}, \mathbf{1}, P}^{k, \mathbf{1}}((\sigma_1 + \dots + \sigma_{i_1}, \dots, \sigma_{i_{k-1}+1} + \dots + \sigma_{i_k}), \sigma'_1 + \dots + \sigma'_Q)$  converge normalement sur tout compact de

$$\left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \left[ \sum_{p \in \mathcal{P}} (\sigma_{i_p} + \dots + \sigma_{i_{p+1}-1}) \right] + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \mathcal{P} \subset \{i_1, \dots, i_{k-1}\} \right\},$$

et on observe que cet ensemble ci-dessus contient bien l'ensemble  $\mathcal{D}_{P, Q}$ .

Fixons la permutation triviale  $f = \operatorname{Id} \in \mathfrak{S}_P$ . On trouve alors par changement de variable que

$$\sum_{0 \leq n_{f(1)} \leq \dots \leq n_{f(P)}} \prod_{p=1}^P (n_p + 1)^{-\sigma_p} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)} = \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P \left( 1 + \sum_{i=1}^p n_i \right)^{-\sigma_p} \left( P + \sum_{p=1}^P p n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)}.$$

Comme précédemment, on trouve la majoration suivante :

$$\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P \left( 1 + \sum_{i=1}^p n_i \right)^{-\sigma_p} \left( P + \sum_{p=1}^P p n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)} \ll \zeta_P^{EZ}(\sigma_1, \dots, \sigma_{P-1}, \sigma'_1 + \dots + \sigma'_Q + \sigma_P),$$

Par le Lemme 1.2.23, on en déduit que ce terme là converge normalement sur tout compact de l'ensemble

$$\{(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sigma_p + \dots + \sigma_P + \sigma'_1 + \dots + \sigma'_Q > P + 1 - p, 1 \leq p \leq P\}.$$

On obtient alors que la série  $\sum_{0 \leq n_{f(1)} \leq \dots \leq n_{f(P)}} \prod_{p=1}^P (n_p + 1)^{-\sigma_p} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)}$ , avec  $f \in \mathfrak{S}_P$  converge normalement sur tout compact de

$$\{(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sigma_{f(p)} + \dots + \sigma_{f(P)} + \sigma'_1 + \dots + \sigma'_Q > P + 1 - p, 1 \leq p \leq P\}.$$

On trouve finalement que  $Z_{\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\beta}}^{P, Q}$  converge normalement sur tout compact de

$$\left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset [1, P] \right\}.$$

□



Pour obtenir le prolongement méromorphe de la fonction multizêta  $Z(\mathbf{s}, \mathbf{s}')$ , et les singularités de cette fonction multizêta, il suffit d'utiliser le Théorème 1.2.25. En effet, via la Remarque 1.3.8, la fonction multizêta  $Z(\mathbf{s}, \mathbf{s}')$  est de la forme  $\zeta^{HL}(\mathbf{0}, \widetilde{\mathbf{d}}, \widetilde{\mathbf{c}}, (s_1, \dots, s_P, s'_1, \dots, s'_Q))$  introduite dans la Définition 1.2.24. On peut donc appliquer le Théorème 1.2.25 pour cette fonction multizêta.

## 2.2 Enoncé de la formule de prolongement de $Z_{\mathbf{N}, \mathbf{N}'}(s)$

On souhaite ici établir une formule de prolongement de  $Z_{\mathbf{N}, \mathbf{N}'}(s)$ . Avant d'exprimer cette formule, on se doit d'introduire des fonctions de la forme  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ , qui apparaitront dans la formule de prolongement.

**Définition 2.2.1.** Soit  $\mathcal{P} \subsetneq [1, P]$ ,  $j \in [1, Q]$ , et  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$ . On pose  $l_p^*(\mathbf{x}) = c_{1,p}x_1 + \dots + c_{Q,p}x_Q$  pour tout  $1 \leq p \leq P$ . Pour tout  $s = \sigma + i\tau \in \mathbb{C}$ , avec  $\sigma \gg 1$ , on considère les fonctions

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx_j} \dots dx_Q,$$

et

$$h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx_j} \dots dx_Q,$$

où l'on a posé  $dx_1 \dots \widehat{dx_j} \dots dx_Q = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_Q$ , et où l'on a posé  $\widehat{\mathbf{x}}^j = (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_Q)$ , avec  $1 \leq j \leq Q$ .

On dispose de la proposition suivante pour majorer ces fonctions, et pour calculer plus tard la valeur en  $s = 0$  de  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  et de  $h'_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ .

**Proposition 2.2.2.** Soit  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ , et  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ . Soit  $\mathcal{P} \subsetneq [1, P]$ ,  $j \in [1, Q]$  et  $\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}$ . Alors

i)  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}$  et  $h_{\mathbf{N}, \mathbf{N}', [1, P], j}$  sont holomorphes sur  $\mathbb{C}$ .

ii) Pour tout entier  $r \in \mathbb{N}^*$ , on a la majoration suivante, uniformément pour  $s \in D_0(r)$  :

$$|h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)| \ll \frac{(QM)^{|\mathbf{k}|}}{r}, \quad (2.1)$$

avec  $M := \max_{1 \leq q \leq Q, 1 \leq p \leq P} (|c_{q,p}|)$ .

iii) Soit  $\epsilon > 0$  suffisamment petit, et  $s \in \mathbb{C}$ , on a

$$\begin{aligned}
h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) &= \\
& \sum_{\mathcal{Q} \subset \llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}} \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, \mathcal{Q} \rrbracket} c_{q, p}^{w_{p, q}} \right) \\
& \cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q, p}^{v_{p, q}} \right) \frac{1}{\prod_{q \in \llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \\
& \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j, p} + \sum_{f \in \mathcal{Q}^c} c_{f, p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q, \quad (2.2)
\end{aligned}$$

avec les notations  $\mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p, q}$  pour tout  $q \in \mathcal{Q}$ , et  $\mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p, q}$  pour tout  $q \in \llbracket 1, \mathcal{Q} \rrbracket$ . De plus, on peut dériver le terme général de la série précédente en  $s$  pour obtenir une expression de  $\partial_s h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ .

**Remarque 2.2.3.** 1) Le point iii) de cette proposition sera critique pour évaluer  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  et  $\partial_s h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  en  $s = 0$  dans les sous-sections 3.2.2 et 3.2.3.

2) On peut obtenir des résultats similaires au points ii) et iii) de la proposition précédente pour les fonctions de la forme  $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$ , cependant nous n'auront pas besoin d'étudier en détail ce type de fonction.

On peut désormais énoncer la formule de prolongement de  $Z_{\mathbf{N}, \mathbf{N}'}(s)$ . On découpera cette fonction analytique en deux parties qui dépendront de  $s$  et d'une variable libre  $\theta > 0$  suffisamment petite. La première, que l'on notera  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ , sera holomorphe en  $s$  et s'annulera en  $s = 0$ . L'autre terme sera une série de type Erdélyi (dans le genre de la série présente dans la formule d'Erdélyi, c.f. 2.6), que l'on notera  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ . Ce second terme contiendra les hypothétiques singularités de  $Z_{\mathbf{N}, \mathbf{N}'}(s)$ , et l'on notera cet ensemble de singularités  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ , dont l'expression est

$$\mathcal{S}(\mathbf{N}, \mathbf{N}') = \left[ \bigcup_{\substack{1 \leq p \leq P \\ \mu_p \neq 0}} \frac{1}{\mu_p} \mathbb{Z}_{\geq N_p + 1} \right] \cup \left[ \bigcup_{\mathcal{P} \subset \llbracket 1, P \rrbracket} \frac{1}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|^{\mathcal{P}}} \mathbb{Z}_{\leq |\mathbf{N}|_{\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}|} \right].$$

**Proposition 2.2.4.** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{C}^P \times \mathbb{C}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ . Alors la fonction univariante  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  est méromorphe sur  $\mathbb{C}$ , et ses pôles sont contenus dans  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ . Plus précisément, on a la formule de prolongement suivante :

Pour tout  $\theta > 0$  suffisamment petit, et tout  $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ , on a

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}(\theta, s), \quad (2.3)$$

avec  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  et  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  tels que :

1) Pour tout  $s \in \mathbb{C}$ , on pose

$$K_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \sum_{\substack{\emptyset \neq \mathcal{Q} \subset \llbracket 1, \mathcal{Q} \rrbracket \\ A_{\mathcal{Q}} \subset \mathcal{Q}^c}} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in \llbracket 1, \mathcal{Q} \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\ \left. \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in \llbracket 1, \mathcal{Q} \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right). \quad (2.4)$$

2) Pour tout  $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ , on pose

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \right. \\ \left. \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}{\Gamma(-N'_j + \mu'_j s)(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\ + \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right). \quad (2.5)$$

De plus, on a que :

i)  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  est une fonction holomorphe en  $s$  sur tout  $\mathbb{C}$ , et s'annule en  $s = 0$ .

ii)  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  est une fonction méromorphe en  $s$  sur  $\mathbb{C}$ , avec des pôles inclus dans  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ . De plus, on peut dériver suivant  $s$  la série décrite dans l'expression de  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ , terme à terme.

**Remarque 2.2.5.** 1)  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  sera l'unique contributeur lors du calcul de la valeur directionnelle  $Z(-(\mathbf{N}, \mathbf{N}'))$ , puisque la fonction  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  s'annule en  $s = 0$ . En revanche, lorsque l'on étudiera la valeur de la dérivée  $Z'(-(\mathbf{N}, \mathbf{N}'))$ , il faudra tenir compte du terme  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ .

2) On verra dans la preuve de la Proposition 2.2.4 que la série

$$\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in \llbracket 1, \mathcal{Q} \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q),$$

décrit une fonction holomorphe en  $s$  sur  $\mathbb{C}$  (la série est en fait absolument convergente sur  $\mathbb{C}$ ). En particulier, on pourra considérer la dérivée de cette série en  $s$ , et l'évaluer en  $s = 0$  dans la démonstration du Théorème D. Notons que l'on ne cherchera jamais à calculer directement les valeurs de cette série, ou les valeurs de sa dérivée en  $s$ .

A l'aide de cette proposition, on obtiendra au passage que la fonction  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  est régulière en 0, et on aura alors démontré par une autre méthode que Komori dans [KMT10a] l'existence de valeurs directionnelles pour la classe de fonction zêta qui nous intéresse. On obtient également une information sur les pôles de  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  :

**Corollaire 2.2.6.** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in (\mathbb{R}_+)^P \times (\mathbb{R}_+^*)^Q$ . Alors la fonction  $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  a des pôles, qui sont tous simples, de la forme

$$s = \frac{n}{\mu_p} \quad (1 \leq p \leq P, n \in \llbracket 1, P \rrbracket)$$

$$s = \frac{n}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}} \quad (\emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket, n \in \llbracket -\infty, |\mathbf{N}'|_{|\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}| \rrbracket \setminus \{0\})$$

Pour montrer cette proposition, on multipliera la série de Dirichlet  $Z(\mathbf{s}, \mathbf{s}')$  par  $\Gamma(s'_1) \dots \Gamma(s'_Q)$ . Cela revient à considérer une transformation de Mellin multivariable en un certain sens. Cette stratégie est classique pour prolonger et étudier les valeurs de la fonction zêta de Riemann [Hid93, Chap. 2], pour obtenir une représentation intégrale d'une fonction zêta de Shintani dans [Hid93, Chap. 2], pour calculer les valeurs de la fonction zêta de Hurwitz multiple de type Mordell-Tornheim [Ono21].

Ensuite, on procédera à un découpage du domaine d'intégration en introduisant une variable libre  $\theta$ . Ce faisant, on pourra isoler un voisinage de l'origine du reste du domaine d'intégration. Enfin, on remarquera que l'intégrale au voisinage de l'origine peut être réexprimée à l'aide d'une formule d'Erdélyi (c.f. 2.6), puis on démontrera également que le reste des intégrales définissent en fait une fonction holomorphe en  $(\mathbf{s}, \mathbf{s}')$ . Pour découper les intégrales apparaissant dans nos calculs, on utilisera les fonctions gammas incomplètes  $\Gamma(s, \theta, \nu)$  et  $\gamma(s, \theta, \nu)$  définies dans la Définition 1.3.1.

## 2.3 Expansion de Crandall directionnelle pour $Z(\mathbf{s}, \mathbf{s}')$

### 2.3.1 Lemmes

On commence par énoncer une identité classique qui découle d'un développement de Taylor :

**Lemme 2.3.1.** Soit  $X_1, \dots, X_d$  des complexes tels que  $|X_1 + \dots + X_d| < 1$ . On a alors

$$(1 + X_1 + \dots + X_d)^{-s} = \sum_{\mathbf{k}=(k_p)_{p \in \llbracket 1, d \rrbracket}} \binom{-s}{|\mathbf{k}|} \binom{|\mathbf{k}|}{\mathbf{k}} X_1^{k_1} \dots X_d^{k_d}.$$

Les expressions présentes dans ce chapitre feront intervenir une série dont le terme général contiendra  $\zeta(s - k, d)$  avec  $k \in \mathbb{N}$ , et  $d \in H_0$ . Dans [Ono21], Onoeda dispose également de ces termes dans une de ses expressions, et il contrôle ces termes à l'aide du lemme suivant, dans le cas où  $d \in \mathbb{R}_+^*$  :

**Lemme 2.3.2.** [Ono21, Lemme 2.1] Soit  $\alpha > 0$ . On pose pour tout  $\delta \in \mathbb{R}_+^*$  et  $r \in \mathbb{N}^*$ ,

$$D(\delta, r) := \left\{ s \in \mathbb{C} \mid \min_{n \in \mathbb{N}^*} |s - n| \geq \delta, |s| \leq r \right\}.$$

Alors, pour tout  $s \in D(\delta, r)$ ,  $k \in \mathbb{N}$ ,  $x > 0$ ,  $m \in \mathbb{N}$ , on a

$$|\partial_s^m \zeta(s - k, d)| \ll_{m, r, \delta, \alpha} k!(k+1)^{r+\alpha}.$$

Ce lemme n'est cependant pas suffisant ici, puisque l'on souhaite majorer  $\zeta(s - k, d)$  avec  $d \in H_0$ . La démonstration du lemme précédent utilise une équation fonctionnelle vérifiée par la fonction zêta d'Hurwitz  $\zeta(s, d)$  pour  $d \in ]0, 1]$ . A priori, on ne peut pas obtenir de la même manière une majoration similaire pour  $d \in H_0$ . On utilisera donc la formule d'Euler-Maclaurin sur la fonction zêta d'Hurwitz afin d'obtenir une inégalité similaire au lemme précédent.

**Lemme 2.3.3.** Soit  $\delta \in \mathbb{R}_+^*$ ,  $r \in \mathbb{N}^*$ , et  $r' \in ]1, +\infty[$ . Alors on a

$$\forall s \in D(\delta, r), \forall d \in \overline{H}_{1/r'} \cap D_0(r'), \forall k \in \mathbb{N}, \quad |\zeta(s - k, d)| \ll_{r, r', \delta} \left(\frac{e}{\pi}\right)^k k!(k+1)^{r+1} (1 + r'^{2k}).$$

*Démonstration.* Soit  $r \in \mathbb{N}^*$ , on pose  $n = r + 1$ . On souhaite majorer  $|\zeta(s - k, d)|$  uniformément en  $s \in D(\delta, r)$  et en  $d \in \overline{H}_{1/r'} \cap D_0(r')$ . Soit  $s \in D(\delta, r)$  et  $d \in H_{1/r'} \cap D_0(r')$ . On remarque que l'on a  $n > -\sigma$ . Soit  $k \in \mathbb{N}$ , comme on a que  $n + k > -\sigma + k$ , on peut reprendre l'expression de la fonction zêta de Hurwitz par la formule d'Euler-Maclaurin 1.2.6 à l'ordre  $n + k$ , et l'évaluer en  $s - k$

$$\begin{aligned} \zeta(s - k, d) &= \frac{d^{1-s+k}}{1-s+k} + \sum_{i=0}^{n+k} \binom{-s+k}{i} \frac{(-1)^i B_{i+1}}{i+1} d^{-s-i+k} \\ &\quad + (-1)^{n+k} \binom{-s+k}{n+1+k} \int_0^{+\infty} B_{n+k+1}(x) (x+d)^{-s-n-1} dx, \end{aligned}$$

avec  $B_i(x)$  le  $i$ -ième polynôme de Bernoulli périodisé, et  $B_i$  le  $i$ -ième nombre de Bernoulli. En majorant chaque terme de l'égalité, on trouve

$$\begin{aligned} |\zeta(s - k, d)| &\leq \frac{|d|^{1-\sigma+k}}{|1-s+k|} e^{|\tau|\pi/2} + \sum_{i=0}^{n+k} \left| \binom{-s+k}{i} \right| \frac{|B_{i+1}|}{i+1} |d|^{-\sigma-i+k} e^{|\tau|\pi/2} \\ &\quad + \left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |B_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx. \end{aligned}$$

On cherche désormais à majorer chacun de ces termes. On a clairement que  $e^{|\tau|\pi/2} \leq e^{r\pi/2}$ . Par continuité de l'application  $(x, y) \in [1/r', r'] \times [-r, r] \mapsto x^y$ , on obtient que

$$|d|^{-\sigma+k} \ll_{r, r'} r'^k.$$

Par construction de  $D(\delta, r)$ , on a

$$\begin{aligned} \frac{|d|^{1-\sigma+k}}{|1-s+k|} &\ll_{r, r'} \frac{r'^k}{\delta} \\ &\ll_{r, r', \delta} r'^k. \end{aligned}$$

Pour la majoration des coefficients binomiaux, on observe tout d'abord que

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad \left| \binom{-s+k}{i} \right| \leq \binom{r+k+i-1}{i} \leq \binom{2n+2k}{i}.$$

Soit  $i \in \mathbb{N}$ . Par [Leh40], on sait que si  $i \not\equiv 2 \pmod{4}$ , on a

$$\forall i \in \mathbb{N}, \forall x \in \mathbb{R}, \quad |B_i(x)| < \frac{2i!}{(2\pi)^i},$$

et lorsque  $i \equiv 2 \pmod{4}$ , on obtient que

$$\forall x \in \mathbb{R}, \quad |B_i(x)| < \frac{2i!\zeta(i)}{(2\pi)^i} \leq \frac{2i!\zeta(2)}{(2\pi)^i}.$$

Notons  $\psi$  la fonction digamma d'Euler. On sait que  $\lim_{x \rightarrow +\infty} \psi(x) = +\infty$ . Ainsi, il existe un rang  $x_0$  tel que pour tout  $x \geq x_0$ , la fraction  $\frac{\Gamma(x)}{(2\pi)^x} (\psi(x) - \ln(2\pi))$  est positive. On trouve alors que la fonction

$$x \in \mathbb{R}_+^* \mapsto \frac{\Gamma(x)}{(2\pi)^x},$$

est croissante à partir de ce même  $x_0$ . On en déduit donc que, pour  $n$  grand,

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad \frac{i!}{(2\pi)^i} \ll_n \frac{(n+k)!}{(2\pi)^{n+k}} \ll_n \frac{1}{(2\pi)^k} k!(k+1)\dots(k+n).$$

En majorant chaque terme du produit à droite de la dernière inégalité, on trouve aisément  $(k+1)\dots(k+n) \ll_n (k+1)^n$ , et donc comme  $n = r+1$  :

$$\forall i \in \llbracket 0, n+k \rrbracket, \forall x \in \mathbb{R}, \quad |B_i(x)| \ll_r \frac{k!}{(2\pi)^k} (k+1)^{r+1}.$$

En évaluant l'inégalité précédente en  $x = 0$ , on obtient que

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad |B_i| \ll_r \frac{k!}{(2\pi)^k} (k+1)^{r+1}.$$

On obtient alors l'inégalité suivante

$$\begin{aligned} \sum_{i=0}^{n+k} \left| \binom{-s+k}{i} \right| \frac{|B_{i+1}|}{i+1} |d|^{-\sigma-i+k} &\ll_r |d|^{-\sigma+k} \frac{k!}{(2\pi)^k} (k+1)^n \sum_{i=0}^{2n+2k} \binom{2n+2k}{i} |d|^{-i} \\ &\ll_{r,r'} \frac{k!}{(2\pi)^k} r'^k (k+1)^{r+1} (1+|d|^{-1})^{2n+2k} \\ &\ll_{r,r'} \frac{k!}{(2\pi)^k} (k+1)^{r+1} (1+r')^{2k}. \end{aligned}$$

Comme  $-\sigma - n \leq -1$  il est clair que

$$\forall x \in \mathbb{R}_+, \quad |x+d|^{-\sigma-n-1} \leq (x + \operatorname{Re}(d))^{-\sigma-n-1}.$$

et cette dernière quantité est intégrable sur  $\mathbb{R}_+$ . On obtient donc

$$\left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |B_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx \leq \binom{n+2k+1}{n+k+1} \frac{2(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} (\sigma+n)}.$$

Par l'inégalité classique

$$\binom{m}{i} \leq e^i \left(\frac{m}{i}\right)^i \quad (0 \leq i \leq m),$$

on obtient :

$$\begin{aligned} \left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |B_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx \\ \leq 2e^k \left(\frac{n+2k+1}{k}\right)^k \frac{(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} (\sigma+n)} \\ \leq 2(2e)^k \left(1 + \frac{n+1}{2k}\right)^k \frac{(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} (\sigma+n)}. \end{aligned}$$

Comme on a la majoration suivante

$$\forall 0 \leq x < k, \quad \left(1 + \frac{x}{k}\right)^k \leq e^x,$$

on en déduit que

$$\forall k > (n+1)/2, \quad \left(1 + \frac{n+1}{2k}\right)^k \leq e^{(n+1)/2}.$$

Comme  $n = r+1$ , on obtient

$$\forall k \in \mathbb{N}, \quad \left(1 + \frac{n+1}{2k}\right)^k \ll_r 1.$$

On obtient finalement que

$$\begin{aligned} |\zeta(s-k, d)| &\ll_{r, r', \delta} r'^k + \frac{k!}{(2\pi)^k} (k+1)^n (1+r'^{2k}) + \frac{(2e)^k k!}{(2\pi)^k} (k+1)^{n+1} \\ &\ll_{r, r', \delta} \left(\frac{e}{\pi}\right)^k k! (k+1)^{r+1} (1+r'^{2k}). \end{aligned}$$

□

Pour tout entier naturel  $a \in \mathbb{N}$ , on observe que la fonction  $s \mapsto \frac{1}{\Gamma(s)(s+a)}$  admet un prolongement holomorphe sur  $\mathbb{C}$ . On observe alors le lemme suivant :

**Lemme 2.3.4.** *Soit  $r \in \mathbb{N}^*$ . Alors on a*

$$\begin{aligned} \forall s \in D_0(r), \quad \left| \frac{1}{\Gamma(s)} \right| &\ll_r 1 \\ \forall s \in D_0(r), \forall a \in \mathbb{N}, \quad \left| \frac{1}{\Gamma(s)(s+a)} \right| &\ll_r 1 \end{aligned}$$

*Démonstration.* Comme  $\frac{1}{\Gamma}$  et  $s \mapsto \frac{1}{\Gamma(s)s}$  sont holomorphes sur  $\mathbb{C}$ , on a uniformément sur  $D_0(r)$  :

$$\begin{aligned} \left| \frac{1}{\Gamma(s)} \right| &\ll_r 1 \\ \left| \frac{1}{\Gamma(s)s} \right| &\ll_r 1 \end{aligned}$$

Enfin, il est clair que

$$\forall |s| > \frac{1}{2}, \quad \left| \frac{1}{s} \right| < 2.$$

On peut alors montrer l'inégalité souhaitée. Soit  $s \in \mathbb{C}$  tel que  $\sigma \in [-r, r]$ .

Si  $|s+a| > \frac{1}{2}$ , alors

$$\left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r 1.$$

Si  $|s+a| \leq \frac{1}{2}$  alors par l'inégalité triangulaire, on obtient que  $a \leq |s| + 1/2 \leq r + 1/2$ . Par l'équation fonctionnelle de  $\Gamma$ , on a que

$$\frac{1}{\Gamma(s)(s+a)} = \frac{(s+a-1)\dots s}{\Gamma(s+a)(s+a)},$$

on en déduit l'inégalité suivante :

$$\left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r (2r+1)\dots r.$$

Comme  $a \leq 1/2 + r$ , on obtient la majoration suivante pour le produit au numérateur

$$|(s+a-1)\dots s| \leq (2r+1)!.$$

On obtient alors que

$$\forall s \in \bar{D}_0(r), \quad \left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r 1.$$

□

La formule d'Erdélyi [EMOT81, §1.11] décrite dans la proposition suivante est cruciale pour réexprimer des intégrandes faisant intervenir la fonction zêta de Lerch que l'on a défini dans la Définition 1.2.7 :

**Proposition 2.3.5.** [Ono21, Lemme 2.2]

Soit  $u \in \mathbb{C} \setminus ]-\infty, 0]$  tel que  $|u| < 2\pi$ ,  $s \in \mathbb{C} \setminus \mathbb{N}^*$ , et  $d \in \mathbb{R}_+^*$ . Alors

$$e^{-du} \phi(e^{-u}, s, d) = \Gamma(1-s) u^{s-1} + \sum_{k=0}^{+\infty} \frac{(-u)^k}{k!} \zeta(s-k, d).$$

On souhaite obtenir une formule d'Erdélyi pour  $d \in H_0$ . La majoration que l'on a du terme  $\zeta(s-k, d)$  n'est pas aussi bonne lorsque  $d$  est complexe que lorsque  $d$  est réel, ainsi on doit imposer une condition supplémentaire sur le complexe  $u$  :

**Proposition 2.3.6.** Soit  $d \in H_0$ ,  $s \in \mathbb{C} \setminus \mathbb{N}^*$  et  $u \in \mathbb{C} \setminus ]-\infty, 0]$  avec  $|u| \ll_{\frac{1}{d}} 1$ . Alors

$$e^{-du} \phi(e^{-u}, s, d) = \Gamma(1-s) u^{s-1} + \sum_{k=0}^{+\infty} \frac{(-u)^k}{k!} \zeta(s-k, d). \quad (2.6)$$

*Démonstration.* Soit  $r' \in [1, +\infty[$  un réel, on cherchera à montrer le résultat pour  $d \in H_{1/r'} \cap D_0(r')$ . Soit  $u \in \mathbb{C} \setminus ]-\infty, 0]$  tel que  $|u| < \frac{\pi}{e(1+r')^2}$ . On fixe un  $s \in \mathbb{C} \setminus \mathbb{N}^*$ , et on pose  $r := \lceil |s| \rceil$ . On sait déjà que la formule que l'on souhaite démontrer est valable lorsque  $d \in \mathbb{R}_+^*$ . On cherche alors à démontrer cette formule pour tout  $d \in H_0$  par prolongement analytique. Pour cela, on doit d'abord montrer que les différents termes dans l'expression de la formule voulue sont holomorphes en la variable  $d$  sur  $H_0$ .

On sait que l'application  $d \in H_0 \mapsto e^{-du} \phi(e^{-u}, s, d)$  est holomorphe [EMOT81, §1.11], il ne reste qu'à montrer que la série présente dans la formule l'est aussi. On a déjà que chaque terme  $d \in H_0 \mapsto \zeta(s-k, d)$  est holomorphe sur  $H_0$  pour tout entier  $k \in \mathbb{N}$ . Pour démontrer cela, il suffit de reprendre la formule obtenue par le développement d'Euler-Maclaurin de la fonction zêta d'Hurwitz obtenue dans la preuve de la Proposition 1.2.6. Il ne nous reste plus qu'à montrer que la série présente dans (2.6) est normalement convergente en  $d$  sur  $\overline{H_{1/r'}} \cap D_0(r')$ . On a la majoration suivante

$$\forall k \in \mathbb{N}, \quad |\zeta(s-k, d)| \ll_{r, r', \delta} \left(\frac{e}{\pi}\right)^k k!(k+1)^{r+1} (1+r')^{2k},$$

Ainsi, on obtient que le terme général de la série 2.6 admet la majoration suivante

$$\frac{|\zeta(s-k, d)|}{k!} |u|^k \ll_{r, r', \delta} \left(u \frac{e}{\pi} (1+r')^2\right)^k (k+1)^{r+1},$$

or la série entière  $\sum_{k=0}^{+\infty} \left(\frac{e}{\pi} (1+r')^2\right)^k k^{r+1} z^k$  est absolument convergente dans le disque ouvert

$\left\{ z \in \mathbb{C}, |z| < \frac{\pi}{e(1+r')^2} \right\}$ . On obtient alors que la série  $\sum_{k=0}^{+\infty} \frac{\zeta(s-k, d)}{k!} (-u)^k$  est normalement convergente en  $d$  sur  $H_{1/r'} \cap D_0(r')$ . Ainsi la formule (2.6) est vraie pour tout  $d \in H_0$  par prolongement analytique, via la Proposition 2.3.5.  $\square$

### 2.3.2 Résultats préliminaires

On souhaite établir dans cette section une expansion de Crandall pour la fonction multizêta

$$Z(\mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}.$$



Rappelons la Définition 1.3.1 : Soit  $\nu \in H_0$ ,  $\theta > 0$ ,  $s \in \mathbb{C}$ . On a posé

$$\begin{aligned}\Gamma(s, \theta, \nu) &= \int_{\theta}^{+\infty} e^{-\nu y} y^{s-1} dy, \\ \gamma(s, \theta, \nu) &= \int_0^{\theta} e^{-\nu y} y^{s-1} dy, \quad \text{lorsque } \operatorname{Re}(s) > 0.\end{aligned}$$

On notera que l'intégrale décrivant  $\Gamma(s, \theta, \nu)$  est absolument convergente pour tout  $s \in \mathbb{C}$ , mais que celle décrivant  $\gamma(s, \theta, \nu)$  n'est absolument convergente que pour  $s \in H_0$ . Plus précisément, on a :

**Proposition 2.3.7.** *Soit  $\nu \in H_0$  et  $\theta > 0$ . Alors les fonctions  $(s, \nu) \in \mathbb{C} \times H_0 \mapsto \Gamma(s, \theta, \nu)$  et  $(s, \nu) \in H_0 \times H_0 \mapsto \gamma(s, \theta, \nu)$  sont holomorphes respectivement sur  $\mathbb{C} \times H_0$  et sur  $H_0 \times H_0$ .*

*Démonstration.* La proposition découle du théorème d'holomorphicité sous le signe intégral.  $\square$

**Proposition 2.3.8.** *Soit  $\nu \in H_0$ ,  $\theta > 0$ . On a*

$$\forall s \in H_0, \quad \Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) = \nu^{-s} \Gamma(s).$$

*Démonstration.* Supposons que  $\nu \in \mathbb{R}_+^*$ . On a alors par changement de variable :

$$\begin{aligned}\Gamma(s, \theta, \nu) &= \nu^{-s} \int_{\nu\theta}^{+\infty} e^{-x} x^{s-1} dx, \\ \gamma(s, \theta, \nu) &= \nu^{-s} \int_0^{\nu\theta} e^{-x} x^{s-1} dx, \quad \text{lorsque } \operatorname{Re}(s) > 0.\end{aligned}$$

En additionnant les deux termes, on obtient

$$\Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) = \nu^{-s} \Gamma(s).$$

Soit  $s \in H_0$ . La fonction  $\nu \in H_0 \mapsto \Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) - \nu^{-s} \Gamma(s)$  est holomorphe, et s'annule sur tout  $\mathbb{R}_+^*$ , donc par principe des zéros isolés, on obtient le résultat voulu.  $\square$

**Corollaire 2.3.9.** *La fonction  $s \mapsto \gamma(s, \theta, \nu)$  admet un prolongement méromorphe sur  $\mathbb{C}$ , ayant pour seuls pôles des pôles simples en les entiers strictement négatifs, et le résidu en  $s = -n$  est  $\frac{(-1)^n \nu^n}{n!}$ .*

**Lemme 2.3.10.** *Soit  $\theta > 0$  un réel positif, et  $\nu \in H_0$ . Soit  $s \in \mathbb{C}$ . On a la majoration suivante :*

$$|\Gamma(s, \theta, \nu)| \leq 2^\sigma e^{-\operatorname{Re}(\nu)\theta/2} \Gamma\left(\sigma, \frac{\theta}{2}, \operatorname{Re}(\nu)\right).$$

*Démonstration.* Après le changement de variable  $x = \frac{x'}{2}$ , on obtient

$$\Gamma(s, \theta, \nu) = 2^s \int_{\theta/2}^{+\infty} e^{-2\nu x} x^{s-1} dx.$$

Comme  $|e^{-\nu x}| = e^{-\operatorname{Re}(\nu)x}$  pour tout  $x \in \mathbb{R}_+$  et tout complexe  $\nu \in H_0$ , on obtient la majoration suivante :

$$\begin{aligned}|\Gamma(s, \theta, \nu)| &\leq |2^s| \int_{\theta/2}^{+\infty} e^{-\operatorname{Re}(\nu)x} e^{-\operatorname{Re}(\nu)x} x^{\sigma-1} dx \\ &\leq 2^\sigma \int_{\theta/2}^{+\infty} e^{-\operatorname{Re}(\nu)\theta/2} e^{-\operatorname{Re}(\nu)x} x^{\sigma-1} dx,\end{aligned}$$

ce qui conclut la preuve.  $\square$

On étudie maintenant le produit  $\prod_{q=1}^Q \Gamma(s'_q) Z(\mathbf{s}, \mathbf{s}')$ . Soit  $q \in \llbracket 1, Q \rrbracket$ ,  $\theta > 0$ , et  $s'_q \in H_0$ . Par la Proposition 2.3.8, on obtient

$$\Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} = \Gamma(s'_q, \theta, (l_q(\mathbf{n}) + d'_q)) + \gamma(s'_q, \theta, (l_q(\mathbf{n}) + d'_q)).$$

On se donne un multi-entier  $\mathbf{n} \in \mathbb{N}^P$ , et des complexes  $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times H_1^Q$ . On obtient ainsi l'égalité suivante en distribuant la somme ci-dessus sur le produit suivant :

$$\begin{aligned} \prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} &= \prod_{q=1}^Q \left[ \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) + \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right] \\ &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in \mathcal{Q}^c} \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &\quad + \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in \mathcal{Q}^c} \left( (l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) - \Gamma(s'_q, \theta, l'_q(\mathbf{n}) + d'_q) \right) \\ &\quad + \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned}$$

En distribuant les termes de la forme  $(l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) - \Gamma(s'_q, \theta, l'_q(\mathbf{n}) + d'_q)$ , on obtient

$$\begin{aligned} \prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \\ &\quad \prod_{q \in A_{\mathcal{Q}}} \Gamma(s'_q) \prod_{q \in \mathcal{Q}^c \setminus A_{\mathcal{Q}}} (-1) \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &\quad + \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \end{aligned}$$

Soit  $(x_p)_{p \in \llbracket 1, P \rrbracket} \in H_0^P$ , alors par l'égalité précédente on a

$$\begin{aligned} \prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{p=1}^P x_p^{-s_p} &= \\ &\sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{p=1}^P x_p^{-s_p} \prod_{q \in \mathcal{Q} \cup (\mathcal{Q}^c \setminus A_{\mathcal{Q}})} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \left( \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) \right) \\ &\quad + \prod_{p=1}^P x_p^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \end{aligned}$$

Comme  $\mathcal{Q} \cup (\mathcal{Q}^c \setminus A_{\mathcal{Q}}) = \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}$ , en remplaçant  $x_p$  par  $n_p + d_p$  :

$$\begin{aligned} \prod_{q=1}^Q \Gamma(s'_q) (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{p=1}^P (n_p + d_p)^{-s_p} = \\ \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \left( \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) \right) \\ + \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned} \quad (2.7)$$

En sommant ces égalités sur  $n_p \in \mathbb{N}$  pour tout  $p \in \llbracket 1, P \rrbracket$ , on cherchera par la suite à prouver que, pour  $(\mathbf{s}, \mathbf{s}') \in \mathcal{D}$  (avec  $\mathcal{D}$  l'ensemble défini dans la Proposition 2.1.1), on a

$$\begin{aligned} Z(\mathbf{s}, \mathbf{s}') = \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(s'_q)} \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \right. \\ \left. \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right) \\ + \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \end{aligned}$$

Par la formule 2.7, il suffira de prouver que les séries présentes dans la série précédente converge absolument. On étudiera ces séries dans les deux propositions suivantes :

**Proposition 2.3.11.** *Soit  $\theta > 0$ , et  $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q$ . On pose :*

$$\begin{aligned} K(\theta, \mathbf{s}, \mathbf{s}') = \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(s'_q)} \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \right. \\ \left. \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right). \end{aligned} \quad (2.8)$$

Alors la série précédente est normalement convergente sur tout compact de  $\mathbb{C}^P \times \mathbb{C}^Q$ . De plus, la fonction  $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$  est holomorphe sur  $\mathbb{C}^P \times \mathbb{C}^Q$ , et  $K(\theta, -\mathbf{N}, -\mathbf{N}') = 0$  pour tout multi-entiers  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ .

*Démonstration.* Soit  $\theta > 0$  un réel. Pour démontrer l'holomorphie de  $K$  en les variables  $(\mathbf{s}, \mathbf{s}')$ , il suffit de montrer que la série décrivant  $K(\theta, \mathbf{s}, \mathbf{s}')$  est normalement convergente sur tout compact.

La fonction  $K(\theta, \mathbf{s}, \mathbf{s}')$  est une somme finie de termes de la forme

$$(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(s'_q)} \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right). \quad (2.9)$$

avec  $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$ , et  $A_{\mathcal{Q}} \subset \mathcal{Q}^c$ . On observe que les fonctions  $\frac{1}{\Gamma(s'_q)}$ , et  $\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)$  sont bien définis, et holomorphe, sur  $\mathbb{C}$ . On note

$$T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}') = \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q), \quad (\mathbf{n} \in \mathbb{N}^P) \quad (2.10)$$

le terme général de la série présente dans (2.9).

En utilisant le Lemme 2.3.10, on obtient, que pour tout  $q \in \llbracket 1, Q \rrbracket$ , et pour tout  $\mathbf{n} \in \mathbb{N}^P$ ,

$$|\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)| \leq 2^{\sigma'_q} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \frac{\theta}{2}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(l_q(\mathbf{n}) + d'_q)).$$

A l'aide de l'inégalité précédente, et par décroissance de  $x \in \mathbb{R}_*^+ \mapsto \Gamma(\sigma, \theta, x)$  pour  $\theta > 0$ , on obtient finalement

$$|\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)| \leq 2^{\sigma'_q} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \frac{\theta}{2}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(d'_q)).$$

Enfin, comme pour tout complexe  $z \in H_0$ ,  $s = \sigma + i\tau \in \mathbb{C}$ , on a  $|z^s| = |z|^\sigma e^{-\arg(z)\tau}$ . Comme  $z \in H_0$ , on en déduit que  $\arg(z) \in ]-\pi/2, \pi/2[$ , et donc

$$|z^s| \leq |z|^\sigma e^{|\tau|\pi/2}.$$

On en déduit alors l'inégalité suivante

$$\begin{aligned} |T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')| &\leq \left| \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \right| \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} 2^{\sigma'_q} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \frac{\theta}{2}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(d'_q)) \\ &\leq \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \theta/2} 2^{\sigma'_q} \Gamma(\sigma'_q, \theta/2, d'_q) \\ &\quad \cdot \prod_{p=1}^P e^{|\tau_p|\pi/2} |n_p + d_p|^{-\sigma_p} \prod_{q \in A_{\mathcal{Q}}} e^{|\tau'_q|\pi/2} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}. \end{aligned} \quad (2.11)$$

Notons au passage que, si les coefficients  $c_{p,q}$ ,  $d_p$  et  $d'_q$  sont tous réels pour tout  $1 \leq p \leq P$  et tout  $1 \leq q \leq Q$ , on peut remplacer les termes de la forme  $e^{|\tau_p|\pi/2}$  et  $e^{|\tau'_q|\pi/2}$  par 1.

Comme l'ensemble  $\mathcal{Q}$  est non vide, et que  $A_{\mathcal{Q}} \subset \mathcal{Q}^c$ , alors il contient au moins un entier  $j \in A_{\mathcal{Q}}$ , et par hypothèse,  $l_j(\mathbf{n}) = \sum_{p=1}^P c_{j,p} n_p$ , avec  $c_{j,p} \in H_0$  pour tout  $p \in \llbracket 1, P \rrbracket$ . On a aussi  $d'_q \in H_0$  pour tout  $q \in \llbracket 1, Q \rrbracket$ . on obtient alors

$$\prod_{q \in \mathcal{Q}} e^{-\operatorname{Re}(l_q(\mathbf{n}) + d'_q) \theta/2} \leq e^{-\operatorname{Re}(l_j(\mathbf{n})) \theta/2}.$$

Soit  $R > 0$  un réel, et  $(\mathbf{s}, \mathbf{s}') \in D_0(R)^{P+Q}$ , alors on a en particulier  $(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in [-R, R]^{P+Q}$  et  $(\boldsymbol{\tau}, \boldsymbol{\tau}') \in [-R, R]^{P+Q}$ . Comme l'application  $x \in [-R, R] \mapsto \Gamma(x, \theta, \nu)$  est continue pour tout  $\theta > 0$ , et  $\nu \in H_0$ , on en déduit la majoration suivante, uniforme en  $(\mathbf{s}, \mathbf{s}') \in D_0(R)^P \times D_0(R)^Q$

$$\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(d'_q)) \ll_R 1.$$

De l'inégalité 2.11, on en déduit alors que, pour tout  $(\mathbf{s}, \mathbf{s}') \in D_0(R)^P \times D_0(R)^Q$

$$|T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')| \ll_R 2^{(Q - |A_{\mathcal{Q}}|)R} \prod_{p=1}^P e^{\pi/2R} e^{-\operatorname{Re}(c_{j,p}) n_p \theta/2} |n_p + d_p|^{-\sigma_p} \cdot \prod_{q \in A_{\mathcal{Q}}} e^{\pi/2R} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}. \quad (2.12)$$

Comme  $\theta > 0$ , et que  $\operatorname{Re}(c_{j,p}) > 0$  pour tout  $p \in \llbracket 1, P \rrbracket$ , on sait que la série

$$\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P e^{-\operatorname{Re}(c_{j,p}) n_p \theta/2} |n_p + d_p|^{-\sigma_p} \prod_{q \in A_{\mathcal{Q}}} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q},$$

est normalement convergente sur tout compact de  $\mathbb{C}^P \times \mathbb{C}^Q$ . En particulier, on en déduit que la série  $\sum_{\mathbf{n} \in \mathbb{N}^P} T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')$  est normalement convergente sur  $D_0(R)^{P+Q}$ . On conclut alors que  $K$  est holomorphe sur  $\mathbb{C}^P \times \mathbb{C}^Q$ .

On observe enfin que pour tout  $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$  et pour tout  $A_{\mathcal{Q}} \subset \mathcal{Q}^c$ , on a que  $\llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}} \neq \emptyset$ . Ainsi, le produit  $\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(s'_q)}$  est non vide. En particulier, lorsque l'on évalue  $K(\theta, \mathbf{s}, \mathbf{s}')$  en  $\mathbf{s}' = -\mathbf{N}'$ , on obtient 0.  $\square$

**Remarque 2.3.12.** En évaluant la formule 2.8 en remplaçant  $\mathbf{s}$  par  $(-N_p + \mu_p s)_{p \in \llbracket 1, P \rrbracket}$ , et  $\mathbf{s}'$  par  $(-N'_q + \mu'_q s)_{q \in \llbracket 1, Q \rrbracket}$ , on retrouve l'expression 2.4 dans la Proposition 2.2.4.

**Proposition 2.3.13.** Soit  $\theta > 0$ , et  $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ , avec  $\mathcal{D}$  l'ensemble défini dans la Proposition 2.1.1. On pose

$$J(\theta, \mathbf{s}, \mathbf{s}') = \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \quad (2.13)$$

La série précédente est normalement convergente sur tout compact de  $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ . En particulier, la fonction  $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$  est holomorphe sur  $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ .

*Démonstration.* Soit  $\theta > 0$ , et  $R > 0$  des réels positifs. On reprend les notations de la proposition précédente, en particulier la formule 2.10 pour décrire le terme général des séries présentes dans  $K(\theta, \mathbf{s}, \mathbf{s}')$ . Par l'égalité 2.7, on sait que, pour tout  $\mathbf{n} \in \mathbb{N}^P$  et  $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times H_1^Q$  :

$$\begin{aligned} \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) = \\ - \left( \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \frac{(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|}}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q)} T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}') \right) + \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}. \end{aligned}$$

Dans la démonstration de la Proposition 2.3.11, on a montré à l'aide de l'inégalité 2.12 que la série de terme général  $T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')$  était normalement convergente sur tout compact de  $\mathbb{C}^P \times \mathbb{C}^Q$ . On a montré dans la Proposition 2.1.1 que la série de fonction en  $(\mathbf{s}, \mathbf{s}')$

$$\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q},$$

est normalement convergente sur tout compact de  $\mathcal{D}$ . Ainsi, on obtient que la série

$$\sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q),$$

est normalement convergente sur tout compact de  $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ .  $\square$

**Proposition 2.3.14.** Soit  $\theta > 0$ , et  $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ . On a :

$$Z(\mathbf{s}, \mathbf{s}') = K(\theta, \mathbf{s}, \mathbf{s}') + J(\theta, \mathbf{s}, \mathbf{s}'), \quad (2.14)$$

De plus, la fonction  $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$  admet un prolongement méromorphe sur  $\mathbb{C}^P \times \mathbb{C}^Q$ .

*Démonstration.* La première égalité découle des deux dernières propositions. Comme  $(\mathbf{s}, \mathbf{s}') \mapsto Z(\mathbf{s}, \mathbf{s}')$  est méromorphe sur  $\mathbb{C}^P \times \mathbb{C}^Q$ , et que  $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$  est holomorphe sur ce même espace, on en déduit que  $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$  est méromorphe sur  $\mathbb{C}^P \times \mathbb{C}^Q$ .  $\square$

**Corollaire 2.3.15.** La fonction méromorphe  $(\mathbf{s}, \mathbf{s}') \mapsto Z(\mathbf{s}, \mathbf{s}')$  et la fonction méromorphe  $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$  partagent exactement les mêmes singularités, et la multiplicité de leur diviseur polaire est identique en une même singularité. De plus,  $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$  est holomorphe sur le domaine  $\mathcal{D}$ , avec  $\mathcal{D}$  l'ensemble défini dans la Proposition 2.1.1.

Afin de calculer les valeurs de  $Z$  en les multi-entiers négatifs, il ne reste plus qu'à étudier  $J(\theta, \mathbf{s}, \mathbf{s}')$ . Pour ce faire, on réexprimera ce terme en tant qu'intégrale d'une fonction zêta de Lerch, puis on simplifiera l'intégrande à l'aide d'une formule de Erdélyi sur chaque terme du produit.

Rappelons les notations suivantes pour les fonctions  $l_p^*$  : Soit  $p \in \llbracket 1, P \rrbracket$ . On considère  $l_p^* : \mathbb{R}^Q \rightarrow \mathbb{C}$  la fonction linéaire suivante

$$l_p^*(x) := \sum_{q=1}^Q c_{q,p} x_q.$$

En reprenant l'expression précédente de  $J(\theta, \mathbf{s}, \mathbf{s}')$  pour  $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$  on obtient que

$$\begin{aligned} J(\theta, \mathbf{s}, \mathbf{s}') &= \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q e^{-(l_q(\mathbf{n}) + d'_q)x_q} x_q^{s'_q - 1} dx_q \\ &= \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(x)n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{s'_q - 1} dx_q. \end{aligned} \quad (2.15)$$

**Définition 2.3.16.** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . On pose pour tout  $s \in \mathbb{C}$ ,

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}(\theta, s) &:= J(\theta, -\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s) \\ K_{\mathbf{N}, \mathbf{N}'}(\theta, s) &:= K(\theta, -\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s) \end{aligned}$$

Pour  $\theta$  suffisamment petit, on a par (2.14) que, pour tout  $s \in \mathbb{C}$  tel que  $(-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s) \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ ,

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}(\theta, s).$$

Or par construction du domaine d'holomorphic  $\mathcal{D}$  dans la Proposition 1.3.9, on en déduit que la condition sur  $s$  précédente est vérifiée si

$$\begin{aligned} \forall \mathcal{P} \subset \llbracket 1, P \rrbracket, \quad -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| + (|\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)\sigma &> 1 + |\mathcal{P}| \\ \forall q \in \llbracket 1, Q \rrbracket, \quad -N'_q + \mu'_q \sigma &> 0 \end{aligned}$$

On pose  $\sigma_0(\mathbf{N}, \mathbf{N}') := \max_{\boldsymbol{\mu}, \boldsymbol{\mu}'} \left( \frac{N'_1}{\mu'_1}, \dots, \frac{N'_Q}{\mu'_Q}, \max_{\mathcal{P} \subset \llbracket 1, P \rrbracket} \left( \frac{1 + |\mathcal{P}| + |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}}}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}} \right) \right)$ .

On en déduit alors la proposition suivante :

**Proposition 2.3.17.** Pour tout  $s \in H_{\sigma_0(\mathbf{N}, \mathbf{N}')}_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ , on a

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}(\theta, s).$$

**Remarque 2.3.18.** 1) Plutôt que d'utiliser la notation  $s = \sigma + i\tau \in H_{\sigma_0(\mathbf{N}, \mathbf{N}')}_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ , on préférera noter  $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$ , voir même parfois  $\sigma \gg 1$ .

2) On utilise ici le fait que  $\mu'_q > 0$  pour tout  $1 \leq q \leq Q$ . On pourrait également généraliser cette approche en considérant des directions  $\mu'_q \in H_0$ , en revanche cela alourdirait les raisonnements pour ce chapitre.

On sait déjà par la Proposition 2.3.11 que  $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  est holomorphe et s'annule en  $s = 0$ . Il reste à étudier le terme  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ . Le point clé de l'étude de cette fonction est la formule d'Erdélyi (2.6).

**Proposition 2.3.19.** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . Soit  $s \in \mathbb{C}$  tel que  $\mu_p s \notin \mathbb{N}^*$  pour tout  $p \in \llbracket 1, P \rrbracket$ , et  $\sigma \gg 1$ . Alors,

$$J_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s) = \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-d_p l_p^*(x)} \phi(e^{-l_p^*(x)}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_q. \quad (2.16)$$

*Démonstration.* Soit  $s = \sigma + i\tau \in \mathbb{C}$  tel que  $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$ . Par la formule 2.15, on en déduit que

$$J_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s) = \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-(\sum_{q=1}^Q c_{q,p} x_q) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_q$$

On souhaite désormais effectuer une permutation série intégrale. On cherchera donc à majorer le terme général de la série précédente. On pose pour tout  $q \in \llbracket 1, Q \rrbracket$ ,

$$v_q(\mathbf{n}) := \sum_{p=1}^P \operatorname{Re}(c_{q,p}) n_p.$$

On a la majoration suivante :

$$\forall \mathbf{n} \in \mathbb{N}^P, \quad \left| \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-(\sum_{q=1}^Q c_{q,p} x_q) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q \right| \\ \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \mu_p \sigma} e^{\mu_p |\tau| \pi / 2} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q e^{-(v_q(\mathbf{n}) + \operatorname{Re}(d'_q)) x_q} x_q^{-N'_q + \mu'_q \sigma - 1} dx_1 \dots dx_Q$$

Par un changement de variable de la forme  $y_q := (v_q(\mathbf{n}) + \operatorname{Re}(d'_q)) x_q$  pour tout  $q \in \llbracket 1, Q \rrbracket$ , on obtient finalement :

$$\left| \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-(\sum_{q=1}^Q c_{q,p} x_q) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q \right| \\ \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \mu_p \sigma} e^{\mu_p |\tau| \pi / 2} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \mu'_q \sigma} \\ \cdot \int_0^{(v_1(\mathbf{n}) + \operatorname{Re}(d'_1)) \theta} \dots \int_0^{(v_Q(\mathbf{n}) + \operatorname{Re}(d'_Q)) \theta} \prod_{p=1}^P e^{-y_p} \prod_{q=1}^Q y_q^{-N'_q + \mu'_q \sigma - 1} dy_1 \dots dy_Q \\ \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \mu_p \sigma} e^{\mu_p |\tau| \pi / 2} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \mu'_q \sigma} \Gamma(-N'_q + \mu'_q \sigma)$$

Or la série  $\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P |n_p + d_p|^{N_p - \mu_p \sigma} e^{\mu_p |\tau| \pi / 2} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \mu'_q \sigma}$  est convergente pour  $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$  par la Proposition 2.1.1. Par le théorème de convergence dominée, on obtient alors pour tout  $s = \sigma + i\tau \in \mathbb{C}$  tel que  $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$ ,

$$J_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s) = \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(x) d_p} \left( \sum_{n_p=1}^{+\infty} (n_p + d_p)^{N_p - \mu_p s} e^{-l_p^*(x) n_p} \right) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q,$$

ce qui conclut la preuve.  $\square$

Une fois cette formule intégrale obtenue pour  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ , on peut utiliser la Formule d'Erdélyi sur chaque fonction zêta de Lerch dans le produit de l'intégrande, puis distribuer l'expression pour obtenir la proposition suivante :

**Proposition 2.3.20.** *Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . Soit  $s \in \mathbb{C}$  tel que  $\mu_p s \notin \mathbb{N}^*$  pour tout  $p \in \llbracket 1, P \rrbracket$ , et  $\sigma \gg 1$ . Alors, pour tout  $\theta > 0$  suffisamment petit, on a*

$$\begin{aligned}
J_{\mathbf{N}, \mathbf{N}'}(\theta, s) &= \\
& \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\
& \cdot \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(x)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(x)^{k_p} dx_1 \dots dx_Q \\
& + \frac{\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s)}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(x)^{-N_p + \mu_p s - 1} dx_1 \dots dx_Q. \tag{2.17}
\end{aligned}$$

*Démonstration.* Soit  $M = \max_{1 \leq p \leq P, 1 \leq q \leq Q} (|c_{q,p}|)$ , et  $\theta \in \left[0, \frac{1}{QM}\right]$ . Soit  $s \in \mathbb{C}$  tel que  $\mu_p s \notin \mathbb{N}^*$  pour tout  $p \in \llbracket 1, P \rrbracket$ , et  $\sigma \gg 1$ . Par la relation 2.16, on a que

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \int_0^\theta \dots \int_0^\theta \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P e^{-d_p l_p^*(x)} \phi(e^{-l_p^*(x)}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_q.$$

Par la Proposition 2.6, on a que, pour tout  $(x_q)_{q \in \llbracket 1, Q \rrbracket} \in ]0, \theta]^Q$ ,

$$\begin{aligned}
& \prod_{p=1}^P e^{-d_p l_p^*(x)} \phi(e^{-l_p^*(x)}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \\
& = \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P \left( \Gamma(1 + N_p - \mu_p s) l_p^*(x)^{-N_p + \mu_p s - 1} + \sum_{k_p \geq 0} (-1)^{k_p} \frac{l_p^*(x)^{k_p}}{k_p!} \zeta(-N_p + \mu_p s - k_p, d_p) \right)
\end{aligned}$$

En distribuant le produit précédent, on obtient pour tout  $\mathbf{x} = (x_q)_{q \in \llbracket 1, Q \rrbracket} \in [0, \theta]^Q$ ,

$$\begin{aligned}
& \prod_{p=1}^P e^{-d_p l_p^*(x)} \phi(e^{-l_p^*(x)}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \\
& = \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \\
& \cdot \prod_{p \in \mathcal{P}} l_p^*(x)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(x)^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\
& + \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \prod_{p=1}^P l_p^*(x)^{-N_p + \mu_p s - 1}. \tag{2.18}
\end{aligned}$$



On obtient alors que

$$\begin{aligned}
J_{\mathbf{N}, \mathbf{N}'}(\theta, s) &= \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \int_0^\theta \dots \int_0^\theta \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \quad (2.19) \\
&\prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \cdot \prod_{p \in \mathcal{P}} l_p^*(x)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(x)^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} dx_1 \dots dx_Q \\
&+ \frac{\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s)}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(x)^{-N_p + \mu_p s - 1} dx_1 \dots dx_Q.
\end{aligned}$$

On cherche maintenant à majorer le terme général de la série présente dans l'intégrande. Par compacité, on obtient uniformément sur  $\mathbf{x} = (x_1, \dots, x_Q) \in \left[0, \frac{1}{QM}\right]^Q$  que

$$\begin{aligned}
\left| \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \right| &\ll_s 1 \\
|l_p^*(x)^{-N_p + \mu_p s - 1}| &\ll_s 1.
\end{aligned}$$

Comme  $l_p^*$  est une forme linéaire pour tout  $p \in \llbracket 1, P \rrbracket$ , on a que

$$\forall x \in [0, \theta]^Q, \quad |l_p^*(x)| \leq QM\theta.$$

Enfin, par le Lemme 2.3.2, on a la majoration suivante :

$$\forall p \in \llbracket 1, P \rrbracket, \quad \left| \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \ll_s \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil \mu_p |s| \rceil}.$$

On obtient alors la majoration suivante : pour tout  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ , et pour tout  $\mathbf{x} = (x_q)_{q \in \llbracket 1, Q \rrbracket} \in [0, \theta]^Q$ , on a

$$\begin{aligned}
&\left| \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(x)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(x)^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \\
&\ll_s \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil \mu_p |s| \rceil} (QM\theta)^{k_p},
\end{aligned}$$

Or la série suivante converge absolument sur tout  $\theta \in \left[0, \frac{1}{QM}\right]$  :

$$\sum_{\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil \mu_p |s| \rceil} (QM\theta)^{k_p},$$

on peut alors permuter la série et l'intégrale pour obtenir le résultat demandé.  $\square$

**Proposition 2.3.21.** *Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . Soit  $s \in \mathbb{C}$  tel que  $\mu_{ps} \notin \mathbb{N}^*$*

pour tout  $p \in \llbracket 1, P \rrbracket$ , et  $\sigma \gg 1$ . Alors

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \frac{\sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)}{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!},} \\ + \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right),$$

où l'on rappelle les notations de la Proposition 2.2.4 : Pour  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ ,  $j \in \llbracket 1, Q \rrbracket$ , et  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$ ,

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \\ \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx}_j \dots dx_Q$$

et

$$h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \\ \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx}_j \dots dx_Q,$$

avec les notations suivantes :

- 1)  $\widehat{\mathbf{x}}^j = (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_Q)$ ,
- 2)  $dx_1 \dots \widehat{dx}_j \dots dx_Q = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_Q$ .

*Démonstration.* Soit  $s \in \mathbb{C}$  tel que  $\mu_p s \notin \mathbb{N}^*$  pour tout  $p \in \llbracket 1, P \rrbracket$ , et  $\sigma \gg 1$ . Pour prouver cette formule, il suffit de reprendre la formule obtenue dans la Proposition précédente, et de réexprimer l'intégrale présente dans cette formule. On simplifiera cette intégrale à l'aide d'un éclatement.

On remarque d'abord que  $[0, \theta]^Q = \bigcup_{j=1}^Q V_j$ , avec  $V_j := \{\mathbf{x} \in [0, \theta]^Q \mid \forall q \neq j, x_j \geq x_q\}$ , et on observe également que l'intersection deux à deux de ces ensembles est de mesure nulle. Par Chasles, on obtient

$$\int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q \\ = \sum_{j=1}^Q \int_{V_j} \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q$$

On considère le changement de variable suivant :

$$f_j : [0, 1]^{j-1} \times [0, \theta] \times [0, 1]^{Q-j} \rightarrow V_j \\ \mathbf{x} = (x_1, \dots, x_Q) \mapsto (x_1 x_j, \dots, x_j, \dots, x_Q x_j),$$

dont le jacobien est  $Jac(f_j) = x_j^{Q-1}$ .

Quitte à effectuer une permutation des coordonnées, il suffit d'étudier l'intégrale dont le domaine est  $V_Q$ . Tout d'abord, on remarque que pour tout  $\mathbf{x} = (x_1, \dots, x_Q) \in [0, 1]^{Q-1} \times [0, \theta]$ , on a  $l_p^*(f_Q(\mathbf{x})) = x_Q l_p^*(\widehat{\mathbf{x}}^Q)$ . Ainsi, on trouve que

$$\begin{aligned} & \int_{V_Q} \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q \\ &= \int_0^1 \dots \int_0^1 \int_0^\theta x_Q^{-|\mathbf{N}'| + |\boldsymbol{\mu}'|s - Q - |\mathbf{N}|_{|\mathcal{P}} + |\boldsymbol{\mu}|_{|\mathcal{P}^c} s - |\mathcal{P}| + |\mathbf{k}|} \prod_{q=1}^{Q-1} x_q^{-N'_q + \mu'_q s - 1} \\ & \quad \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^Q)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^Q)^{k_p} x_Q^{Q-1} dx_1 \dots dx_{Q-1} \\ &= \int_0^1 \dots \int_0^1 \prod_{q=1}^{Q-1} x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^Q)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} u_p(\widehat{\mathbf{x}}^Q)^{k_p} dx_1 \dots dx_{Q-1} \\ & \quad \cdot \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}, \end{aligned}$$

ce qui conclut la preuve.  $\square$

La proposition précédente est cruciale pour démontrer la Proposition 2.2.4. On montrera en effet la régularité de  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  en  $s = 0$  via la formule de la proposition précédente. On utilisera en particulier la Proposition 2.2.2, qui fournit une majoration de  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  uniforme en  $s$  sur tout compact. On démontrera la Proposition 2.2.2 dans la prochaine sous-section.

### 2.3.3 Démonstration de la Proposition 2.2.2

On démontrera dans cette sous-section la Proposition 2.2.2. Rappelons la notation  $l_p^*(x) = \sum_{q=1}^Q c_{q,p} x_q$  pour tout  $p \in [1, P]$  et tout  $x \in \mathbb{R}^Q$ . On établira un lemme avant de démontrer cette proposition. Pour tout  $\epsilon \in \left] 0, \frac{1}{2MQ} \right]$ , pour chaque sous-ensemble  $\mathcal{Q} \subset [1, Q] \setminus \{j\}$ , et pour tout  $q \in [1, Q] \setminus \{j\}$ , on pose

$$\Delta_{\mathcal{Q}, j}(\epsilon, q) = \begin{cases} [0, \epsilon] & \text{si } q \in \mathcal{Q} \\ [\epsilon, 1] & \text{si } q \in \mathcal{Q}^c \end{cases}$$

et on note ensuite  $F_{\mathcal{Q}, j}(\epsilon) := \prod_{\substack{q=1 \\ q \neq j}}^Q \Delta_{\mathcal{Q}, j}(\epsilon, q)$ . On peut alors découper le domaine d'intégration suivant :

$$\{(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_Q) \mid \forall q \neq j, x_q \in [0, 1]\} = \bigcup_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} F_{\mathcal{Q}, j}(\epsilon).$$

Via la relation de Chasles, on peut ainsi réécrire les termes  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  et  $h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s)$  sous forme d'une somme d'intégrales :

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\boldsymbol{\mu}, \boldsymbol{\mu}', \epsilon, s) \quad (2.20)$$

$$h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\boldsymbol{\mu}, \boldsymbol{\mu}', \epsilon, s), \quad (2.21)$$

en ayant noté pour tout sous-ensemble  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ ,

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx_j} \dots dx_Q,$$

et

$$I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s) = \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx_j} \dots dx_Q,$$

pour tout  $s \in \mathbb{C}$  tel que  $\sigma \gg 1$ .

**Lemme 2.3.22.** Soit  $(\mu, \mu') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ ,  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ ,  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ ,  $j \in \llbracket 1, Q \rrbracket$ ,  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ , et  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$ . Alors on a

i) Pour tout  $\epsilon > 0$  suffisamment petit, la fonction  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  est holomorphe en  $s$  sur  $\mathbb{C}$ .

ii) Pour tout  $\epsilon > 0$  suffisamment petit et pour tout  $r \in \mathbb{N}^*$ , on a la majoration suivante, uniforme pour  $s \in D_0(r)$  :

$$\left| I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right|_{r, \epsilon} \ll (QM)^{|\mathbf{k}|},$$

$$\text{avec } M = \max_{1 \leq q \leq Q, 1 \leq p \leq P} (|c_{q,p}|).$$

iii) Pour tout  $s \in \mathbb{C}$  et  $0 < \epsilon \ll 1$ , on a

$$\begin{aligned} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^Q)^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\ &\cdot \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))}}{\prod_{q \in \mathcal{Q}} \Gamma(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \\ &\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q, \quad (2.22) \end{aligned}$$

avec les notations  $\mathcal{Q}^c = (\llbracket 1, Q \rrbracket \setminus \{j\}) \setminus \mathcal{Q}$ ,  $\mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p,q}$  pour tout  $q \in \mathcal{Q}$ , et  $\mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p,q}$  pour tout  $q \in \llbracket 1, Q \rrbracket$ . En notant  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{w}, \mathbf{v}}(\epsilon, s)$  le terme général de la série précédente, on a également

$$\partial_s I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^Q)^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \partial_s T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{w}, \mathbf{v}}(\epsilon, s). \quad (2.23)$$

iv) Pour  $\epsilon > 0$  suffisamment petit, et pour tout  $s \in \mathbb{C}$ , on a

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s). \quad (2.24)$$

*Démonstration.* On cherchera ici à développer l'intégrande via des développements en séries entières au voisinage de 0 pour les éléments de  $F_{\mathcal{Q},j}(\epsilon)$ . Notons que ces séries entières convergeront uniformément sur le disque  $D_0(\eta)$ , pour  $\eta$  suffisamment petit. On pourra donc intervertir la série et l'intégrale.

Soit  $s \in \mathbb{C}$  tel que  $\sigma \gg 1$ . Soit  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ . Soit  $\epsilon \in \left] 0, \frac{1}{2MQ} \right]$ , avec  $M = \max_{1 \leq q \leq Q, 1 \leq p \leq P} (|c_{q,p}|)$ , on peut développer en série entière en  $(x_q)_{q \in \mathcal{Q}}$  les fonctions de la forme  $l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1}$ , avec  $x \in F_{\mathcal{Q},j}$  et  $p \in \mathcal{P}$ . Par le Lemme 2.3.1, on obtient uniformément en  $\widehat{x}^j$  :

$$\begin{aligned} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} &= \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q + \sum_{q \in \mathcal{Q}} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1} \\ &= \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1} \left( 1 + \sum_{f \in \mathcal{Q}} \frac{c_{f,p}}{c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q} x_f \right)^{-N_p + \mu_p s - 1} \\ &= \sum_{\mathbf{v}_p = (v_{p,q})_{q \in \llbracket 1, Q \rrbracket} \in \mathbb{N}^{\mathcal{Q}}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} x_q^{v_{p,q}}. \end{aligned} \quad (2.25)$$

Pour  $p \in \mathcal{P}^c$ , et  $k_p \in \mathbb{N}$ , on a par le multinôme de Newton :

$$l_p^*(\widehat{\mathbf{x}}^j)^{k_p} = \sum_{\substack{\mathbf{w}_p = (w_{p,q})_{q \in \llbracket 1, Q \rrbracket} \in \mathbb{N}^{\mathcal{Q}} \\ |\mathbf{w}_p| = k_p}} \binom{k_p}{\mathbf{w}_p} c_{j,p}^{w_{p,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{w_{p,q}} x_q^{w_{p,q}}.$$

La série présente dans (2.25) converge uniformément pour  $(x_e)_{e \in \mathcal{Q}} \in [0, \epsilon]^{\mathcal{Q}}$ . Par permutation série intégrale, on peut réexprimer  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  de la manière suivante :

$$\begin{aligned} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \\ &= \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} c_{j,p}^{w_{p,j}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \right) \\ &\cdot \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q},j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \left( \prod_{p \in \mathcal{P}^c} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{w_{p,q}} x_q^{w_{p,q}} \right) \\ &\cdot \prod_{p \in \mathcal{P}} \left( \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} x_q^{v_{p,q}} \right) \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_1 \dots \widehat{dx}_j \dots dx_Q. \end{aligned}$$

En remarquant que  $[1, Q] \setminus \{j\} = \mathcal{Q} \cup \mathcal{Q}^c$ , on peut séparer le produit  $\prod_{q \in [1, Q] \setminus \{j\}}$  en deux parties :

$$\begin{aligned}
& I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \\
& \quad \mu, \mu' \\
& = \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
& \cdot \frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{p \in \mathcal{P}^c} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{w_{p,q}} \right) \prod_{p \in \mathcal{P}} \left( \prod_{q \in \mathcal{Q}} x_q^{v_{p,q}} \right) \\
& \cdot \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots \widehat{dx}_j \dots dx_Q \\
& = \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
& \cdot \frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q}, j}} \prod_{q \in \mathcal{Q}} x_q^{-N'_q + \mu'_q s - 1 + \sum_{p \in \mathcal{P}} v_{p,q} + \sum_{p \in \mathcal{P}^c} w_{p,q}} \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \sum_{p \in \mathcal{P}^c} w_{p,q}} \\
& \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_1 \dots \widehat{dx}_j \dots dx_Q.
\end{aligned}$$

Soit  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}}$  avec  $\mathbf{v}_p \in \mathbb{N}^{\mathcal{Q}}$  pour tout  $p \in \mathcal{P}$ , et  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c}$  avec  $\mathbf{w}_p \in \mathbb{N}^{\mathcal{Q}}$  pour tout  $p \in \mathcal{P}^c$ . On rappelle la notation  $\mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p,q}$  tout  $q \in \mathcal{Q}$ , et  $\mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p,q}$  pour tout  $q \in [1, Q]$ . Via le théorème de Fubini, on obtient l'expression suivante pour  $\sigma \gg 1$  :

$$\begin{aligned}
& I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \\
& \quad \mu, \mu' \\
& = \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
& \cdot \frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \left( \prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q)}}{-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q)} \right) \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \\
& \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q.
\end{aligned}$$

On obtient donc que la formule 2.22 est vraie pour tout  $s \in \mathbb{C}$  tel que  $\sigma \gg 1$ .

On cherche maintenant à prolonger  $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  sur tout  $\mathbb{C}$  via l'expression précédente. On

a clairement que, pour tout  $\sigma \gg 1$ ,

$$\begin{aligned}
I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \\
& \frac{1}{\prod_{q \in \mathcal{Q}^c} \Gamma(-N'_q + \mu'_q s)} \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q,p}^{w_{p,q}} \right) \\
& \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} C_{q,p}^{v_{p,q}} \right) \prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q)}}{\Gamma(-N'_q + \mu'_q s) (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \\
& \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \left( \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \right) \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q. \quad (2.26)
\end{aligned}$$

Notons  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  le terme général de la série précédente. On a alors

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s).$$

Tous les termes présents dans l'expression de  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  sont holomorphes en  $s$  sur tout  $\mathbf{C}$ . Pour montrer que  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  est holomorphe en  $s$ , il suffit de montrer que la série précédente converge normalement en la variable  $s$  sur tout compact. On cherche alors à majorer le terme général sur un disque  $\overline{D}_0(r)$ , avec  $r \in \mathbb{N}^*$  un entier. Les majorations que l'on effectuera dépendront toutes de  $\mathbf{N}, \mathbf{N}', \mu, \mu', \mathcal{P}, j, \mathcal{Q}$ , mais on omettra cette dépendance afin de simplifier la rédaction.

On sait que la fonction

$$s \mapsto \frac{1}{\Gamma(-N'_q + \mu'_q s) (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))},$$

est holomorphe sur  $\mathbf{C}$ , pour tout  $q \in \mathcal{Q}$ . On majore maintenant le terme général de la série 2.26 à l'aide du Lemme 2.3.4. On a

$$\begin{aligned}
\forall q \in \mathcal{Q}, \forall s \in D_0(r), \forall a \in \mathbb{N}, & \quad \left| \frac{1}{\Gamma(-N'_q + \mu'_q s) (-N'_q + \mu'_q s + a)} \right| \ll_r 1 \\
\forall q \in \llbracket 1, Q \rrbracket, & \quad \left| \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right| \ll_r 1.
\end{aligned}$$

On sait que la fonction

$$s \in \mathbf{C} \mapsto \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q,$$

est holomorphe car l'intégrande est régulière.

Soit  $s \in D_0(r)$ ,  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}}$  avec  $\mathbf{v}_p \in \mathbb{N}^{\mathcal{Q}}$ , et  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c}$  avec  $\mathbf{w}_p \in \mathbb{N}^{\mathcal{Q}}$  tel que  $|\mathbf{w}_p| = k_p$  pour tout  $p \in \mathcal{P}^c$ . On observe alors que  $\sum_{\substack{q=1 \\ q \neq j}}^Q \mathbf{w}(q) = |\mathbf{k}| - \mathbf{w}(j)$ . De plus, par compacité, on obtient la majoration suivante uniformément en  $q \in \mathcal{Q}^c$  et en  $x_q \in [\epsilon, 1]$ ,

$$\left| x_q^{-N'_q + \mu'_q s - 1} \right| \ll_r 1.$$

Comme  $\mathbf{w}(q) \geq 0$ , on obtient que  $|x_q^{\mathbf{w}(q)}| \leq 1$ . De plus, comme  $\arg \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right) \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$ , on a pour tout  $s \in D_0(r)$ , et tout  $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$ ,

$$\forall s \in D_0(r), \quad \left| \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \right| \leq \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right|^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} e^{\mu_p r \pi / 2}.$$

On remarque que l'on a la majoration suivante pour tout  $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$

$$K + K\epsilon \leq \left( \operatorname{Re} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} \operatorname{Re}(c_{q,p} \epsilon) \right) \right) \leq \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \leq \sum_{q=1}^Q |c_{q,p}| \leq QM,$$

avec  $M = \max_{1 \leq p \leq P, 1 \leq q \leq Q} (|c_{q,p}|)$  et  $K := \min_{1 \leq q \leq Q, 1 \leq p \leq P} \operatorname{Re}(c_{q,p})$ . Pour  $p \in \mathcal{P}$ , on trouve alors

$$\left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right|^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \leq \begin{cases} (QM)^{\mu_p \sigma - N_p - |\mathbf{v}_p| - 1} & \text{si } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \geq 1 \text{ et } \mu_p \sigma - N_p - |\mathbf{v}_p| - 1 \geq 0 \\ 1 & \text{si } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \geq 1 \text{ et } \mu_p \sigma - N_p - |\mathbf{v}_p| - 1 < 0 \\ 1 & \text{si } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| < 1 \text{ et } \mu_p \sigma - N_p - |\mathbf{v}_p| - 1 \geq 0 \\ (K(\epsilon + 1))^{\mu_p \sigma - N_p - |\mathbf{v}_p| - 1} & \text{sinon} \end{cases}$$

On a uniformément pour  $s \in D_0(r)$  :

$$\forall p \in \mathcal{P}, \quad (QM)^{\mu_p \sigma - N_p - |\mathbf{v}_p| - 1} \ll_{r,\epsilon} (QM)^{-|\mathbf{v}_p|}.$$

Soit  $p \in \llbracket 1, P \rrbracket$ . Par compacité, on a uniformément pour  $s \in D_0(r)$  :

$$(K\epsilon)^{\mu_p \sigma} \ll_{\epsilon,r} 1.$$

On a également

$$(K(\epsilon + 1))^{-N_p - |\mathbf{v}_p| - 1} \ll_{\epsilon,r} (K(\epsilon + 1))^{-|\mathbf{v}_p|} \ll_{r,\epsilon} K^{-|\mathbf{v}_p|}.$$

On obtient finalement la majoration suivante uniformément en  $s \in D_0(r)$  et en  $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$ ,

$$\left| \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \right| \ll_{r,\epsilon} \max(K^{-|\mathbf{v}_p|}, (QM)^{-|\mathbf{v}_p|}, 1) \prod_{p=1}^P e^{\mu_p r \pi / 2} \ll_{r,\epsilon} \min(K, QM, 1)^{-|\mathbf{v}_p|}.$$

Maintenant que l'on a majoré l'intégrande, on obtient finalement que l'intégrale est bornée de la façon suivante :

$$\left| \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q \right| \ll_{r,\epsilon} \prod_{p \in \mathcal{P}} \min(K, QM, 1)^{-|\mathbf{v}_p|} \ll_{r,\epsilon} \min(K, QM, 1)^{-\sum_{p \in \mathcal{P}} |\mathbf{v}_p|}$$



On a la majoration suivante uniforme en  $s \in D_0(r)$  :

$$\left| \epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \right| \ll_{r, \epsilon} \epsilon^{\sum_{q \in \mathcal{Q}} \mathbf{v}(q)}$$

De plus, on a que

$$\forall n \in \mathbb{N}, \forall a \in \mathbb{N}, \forall b \in \mathbb{R}_+, \forall s \in D_0(r), \quad \left| \binom{-n + bs - 1}{a} \right| \leq \binom{n + \lceil br \rceil + a}{a}.$$

On trouve alors que

$$\begin{aligned} & \left| \frac{1}{\prod_{q \in \mathcal{Q}^c} \Gamma(-N'_q + \mu'_q s)} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \right. \\ & \cdot \frac{\epsilon^{-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q)}}{\Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \\ & \cdot \left. \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 + |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q \right| \\ & \ll_{r, \epsilon} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} M^{\sum_{q \in [1, Q]} w_{p,q}} \right) \prod_{p \in \mathcal{P}} \binom{N_p + \lceil \mu_p r \rceil + |\mathbf{v}_p|}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} (M\epsilon)^{\sum_{q \in \mathcal{Q}} \mathbf{v}(q)} \min(K, QM, 1)^{-\sum_{q \in \mathcal{Q}} \mathbf{v}(q)} \\ & \ll_{r, \epsilon} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} M^{k_p} \right) \prod_{p \in \mathcal{P}} \binom{N_p + \lceil \mu_p r \rceil + |\mathbf{v}_p|}{\mathbf{v}_p} \left( \frac{M\epsilon}{\min(K, QM, 1)} \right)^{\sum_{q \in \mathcal{Q}} \mathbf{v}(q)}. \end{aligned} \quad (2.27)$$

En sommant le terme de droite dans l'expression 2.27 sur  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}}$  avec  $\mathbf{v}_p \in \mathbb{N}^{\mathcal{Q}}$ , et sur  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c}$  avec  $\mathbf{w}_p \in \mathbb{N}^{\mathcal{Q}}$  tel que  $|\mathbf{w}_p| = k_p$ , pour tout  $p \in \mathcal{P}^c$ . On obtient alors la série

$$\begin{aligned} & M^{|\mathbf{k}|} \left( \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \right) \left( \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \prod_{p \in \mathcal{P}} \binom{N_p + \lceil \mu_p r \rceil + |\mathbf{v}_p|}{\mathbf{v}_p} \right) \left( \frac{M\epsilon}{\min(K, QM, 1)} \right)^{\sum_{q \in \mathcal{Q}} \mathbf{v}(q)} \\ & = (QM)^{|\mathbf{k}|} \left( \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \prod_{p \in \mathcal{P}} \binom{N_p + \lceil \mu_p r \rceil + |\mathbf{v}_p|}{\mathbf{v}_p} \right) \left( \frac{M\epsilon}{\min(K, QM, 1)} \right)^{\sum_{q \in \mathcal{Q}} \mathbf{v}(q)} \end{aligned}$$

La série entière ci-dessus précédemment converge absolument pour tout  $0 < \epsilon < \frac{\min(K, QM, 1)}{M}$ . En fixant un  $\epsilon$  suffisamment petit, la série en  $(v_{p,q})_{p \in \mathcal{P}, q \in \mathcal{Q}}$  est une constante ne dépendant que de  $r, \epsilon$  et de  $\mathbf{N}$ . On obtient alors l'inégalité suivante :

$$\left| I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right| \ll_{r, \epsilon} (QM)^{|\mathbf{k}|}.$$

On a donc obtenu le point i) et iii) par théorème d'holomorphic sous le signe somme, et par prolongement analytique. Le point ii) est vrai par la majoration précédente. Comme  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  est holomorphe, et que pour tout  $s = \sigma + i\tau \in \mathbb{C}$  tel que  $\sigma \gg 1$ , on a

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s),$$

et par 2.20, on peut prolonger l'égalité précédente sur tout  $\mathbb{C}$  par prolongement analytique, d'où le point iv).  $\square$

En modifiant légèrement la démonstration du lemme précédent, on obtient le lemme suivant :

**Lemme 2.3.23.** Soit  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ , et  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ ,  $j \in \llbracket 1, Q \rrbracket$ ,  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ . Pour tout  $s \in \mathbb{C}$  et  $0 < \epsilon \ll 1$ , on a :

$$I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s) = \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \llbracket 1, P \rrbracket} \in (\mathbb{N}^Q)^P} \left( \prod_{q \in \llbracket 1, Q \rrbracket} c_{q,p}^{w_{p,q}} \right) \left( \prod_{p=1}^P \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \cdot \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q))}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q))} \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1} \cdot \prod_{p=1}^P \left( c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q,$$

avec les notations  $\mathcal{Q}^c = (\llbracket 1, Q \rrbracket \setminus \{j\}) \setminus \mathcal{Q}$ ,  $\mathbf{v}(q) = \sum_{p=1}^P v_{p,q}$  pour tout  $q \in \mathcal{Q}$ .

De plus,  $s \in \mathbb{C} \mapsto I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s)$  admet un prolongement holomorphe sur  $\mathbb{C}$ .

**Remarque 2.3.24.** Contrairement au Lemme 2.3.22, on n'a pas besoin d'obtenir une majoration dans le lemme 2.3.23, puisque le terme  $\mathbf{k}$  y est absent.

On dispose de tous les ingrédients nécessaires à la démonstration de la Proposition 2.2.2 :

*Démonstration.* Rappelons que  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  et  $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$  s'expriment de la façon suivante, pour tout  $s \in \mathbb{C}$  tel que  $\sigma \gg 1$  :

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s),$$

$$h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s).$$

Comme on sait que les fonctions de la forme  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  et  $I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s)$  sont holomorphes en  $s$  sur  $\mathbb{C}$ , on peut prolonger holomorphiquement  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  et  $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$  sur tout  $s \in \mathbb{C}$ . On obtient alors le point i) de la Proposition 2.2.2. Le point ii) et iii) de la Proposition 2.2.2 découle du point ii) et iii) du Lemme 2.3.22.  $\square$

### 2.3.4 Démonstration de la Proposition 2.2.4

Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ . On exploite l'expression  $Z(\mathbf{s}, \mathbf{s}') = K(\theta, \mathbf{s}, \mathbf{s}') + J(\theta, \mathbf{s}, \mathbf{s}')$  pour obtenir la preuve du théorème de prolongement méromorphe selon la direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}')$ . On sait déjà que  $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$  est holomorphe sur  $\mathbf{C}^P \times \mathbf{C}^Q$ . Il reste donc à prouver que l'application  $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  est méromorphe et régulière en  $s = 0$ . Afin de montrer la Proposition 2.2.4, il ne nous reste plus qu'à établir que  $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  est méromorphe sur  $\mathbb{C}$  avec des pôles inclus dans

$$\mathcal{S}(\mathbf{N}, \mathbf{N}') = \left[ \bigcup_{\substack{1 \leq p \leq P \\ \mu_p \neq 0}} \frac{1}{\mu_p} \mathbb{Z}_{\geq N_p + 1} \right] \cup \left[ \bigcup_{\mathcal{P} \subset \llbracket 1, P \rrbracket} \frac{1}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \mathbb{Z}_{\leq |\mathbf{N}|_{\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}|} \right].$$

On sait que la formule 2.5 est valable sur le complémentaire de cet ensemble, pour  $\sigma \gg 1$ . Pour tout  $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$  tel que  $\sigma \gg 1$ , on a alors

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\ + \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)$$

. A l'aide de ces rappels, on peut démontrer la Proposition 2.2.4 :

*Démonstration.* Soit  $0 < \theta \ll 1$ . On sait par la Proposition 2.3.21 que la formule 2.5 est valable pour tout  $s = \sigma + i\tau \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ , tel que  $\sigma \gg 1$ .

Il est clair que le terme

$$\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)},$$

correspond à une fonction méromorphe en  $s$  sur  $\mathbb{C}$ , régulière en  $s = 0$ , et avec des pôles de la forme  $s = \frac{|\mathbf{N}'| + |\mathbf{N}| + P}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|}$ . Il reste alors à démontrer que la série de fonctions en  $s$

$$\sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \quad (2.28)$$

est méromorphe sur  $\mathbb{C}$ , régulier en  $s = 0$ , et avec des pôles inclus dans  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ .

On pose

$$\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r) := \left\{ s \in \overline{D}_0(r) \mid \left( \forall p \in \llbracket 1, P \rrbracket, \min_{\substack{n \in \mathbb{N} \\ n > N_p}} |\mu_p s - n| \geq \delta \right) \wedge \left( \forall \mathcal{P} \subset \llbracket 1, P \rrbracket, \min_{\substack{n \in \mathbb{Z}^* \\ n \geq -|\mathbf{N}|_{\mathcal{P}} - |\mathbf{N}'| - |\mathcal{P}|}} |(|\boldsymbol{\mu}|_{\mathcal{P}} + |\boldsymbol{\mu}'|)s + n| \geq \delta \right) \right\}.$$

Notons que :

1) On a clairement que  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$  contient un voisinage ouvert de 0.

2) La condition

$$\min_{\substack{n \in \mathbb{N} \\ n > N_p}} |\mu_p s - n| \geq \delta$$

apparaît pour éviter les singularités de  $\zeta(-N_p + \mu_p s - k, d_p)$ . On observe qu'au voisinage de  $s = 0$ , ce terme est parfaitement défini.

3) La condition

$$\min_{\substack{n \in \mathbb{Z}^* \\ n \geq -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| - |\mathcal{P}|}} |(|\boldsymbol{\mu}'|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)s + n| \geq \delta$$

apparaît pour éviter les singularités de  $\frac{1}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}$  qui ne sont pas nécessairement compensées par les zéros de  $\frac{1}{\Gamma(-N'_j + \mu'_j s)}$ . On remarque tout de même qu'en  $s = 0$ , lorsque  $-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}| = 0$ , la singularité de  $\frac{1}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s}$  est compensée par le zéro de  $\frac{1}{\Gamma(-N'_j + \mu'_j s)}$ .

4) L'ensemble  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$  est inclus dans le complémentaire de  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ .

Fixons un sous-ensemble  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ . On sait par la Proposition 2.3.21 que la formule 2.5 est valable pour tout  $s = \sigma + i\tau \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ , tel que  $\sigma \gg 1$ . On souhaite prouver que la série en  $\mathbf{k}$  présente dans la formule (2.28) est normalement convergente sur  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ .

Soit  $r' > 1$  un réel tel que, pour tout  $p \in \llbracket 1, P \rrbracket$ ,  $\operatorname{Re}(d_p) \geq 1/r'$ , et  $|d_p| < r'$ . Soit  $r > 0$ ,  $1 > \delta > 0$ . Par le Lemme 2.3.3, on a

$$\forall p \in \llbracket 1, P \rrbracket, \forall s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \forall k_p \in \mathbb{N}, \quad \zeta(-N_p + \mu_p s - k_p, d_p) \ll_{r, r', \delta} (k_p + N_p)! (k_p + N_p + 1)^{r+1} (1 + r'^{2k_p}).$$

Par le Lemme 2.2.2, on a

$$\forall s \in D_0(r), \forall \mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}, \quad |h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)| \ll_r (QM)^{|\mathbf{k}|}.$$

Comme l'application  $s \mapsto \Gamma(1 + N_p - \mu_p s)$  est holomorphe sur  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$  pour tout  $1 \leq p \leq P$ , on en déduit la majoration suivante

$$\forall p \in \llbracket 1, P \rrbracket, \forall s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \quad |\Gamma(1 + N_p - \mu_p s)| \ll_{r, \delta} 1.$$

Enfin, par le Lemme 2.3.4 on a la majoration suivante

$$\forall s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \quad \left| \frac{1}{\Gamma(-N'_j + \mu'_j s) (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \right| \ll_{r, \delta} 1.$$

Par compacité, de  $\bar{D}_0(r)$  on obtient

$$\forall s \in \bar{D}_0(r), |\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}|}| \ll_r 1.$$

Ainsi, pour tout  $s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ , et pour  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$ , on a

$$\begin{aligned} & \left| (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right. \\ & \left. \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}{\Gamma(-N'_j + \mu'_j s) (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \\ & \ll_{r, r', \delta} (QM\theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1} (1 + r'^{2k_p}) \\ & \ll_{r, r', \delta} (2r'^2 QM\theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1}. \end{aligned}$$

Or la série

$$\sum_{\mathbf{N} \in \mathbb{N}^{\mathcal{P}^c}} (2r'^2 Q M \theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1}$$

converge absolument pour tout  $\theta \in \mathbb{C}$  tel que  $|\theta| < \frac{1}{(2r'^2 Q M)}$ . On en déduit en particulier que la formule 2.5 est valable pour tout  $s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ . Par théorème d'holomorphic sous le signe somme, on obtient que la fonction  $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  est holomorphe sur  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ . Comme  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$  contient un voisinage ouvert de 0, on obtient par le théorème d'holomorphic sous le signe somme que l'on peut dériver en  $s$  sur ce même voisinage la fonction  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ ,

$$\partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \quad (2.29)$$

$$\begin{aligned} & \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} \partial_s \left( \left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \right. \\ & \left. \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right), \\ & + \partial_s \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right). \end{aligned}$$

Comme  $1 > \delta > 0$  est arbitraire dans la définition de  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ , et que  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$  contient un voisinage ouvert de 0, on obtient en particulier que  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  est régulière en  $s = 0$ , et que ses pôles sont inclus dans l'ensemble  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$  décrit dans la Proposition 2.2.4.  $\square$

**Remarque 2.3.25.** *Via la formule 2.5, on peut en déduire une expression des résidus de  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  en ses pôles. En effet, on sait que cette fonction se réexprime sous la forme d'une somme entre une fonction holomorphe et une fonction méromorphe en  $s$*

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = K_{\mathbf{N}, \mathbf{N}'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}(\theta, s).$$

Comme  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  est holomorphe en  $s$ , cette fonction ne contient aucune singularité. En revanche, dans la formule 2.5, on peut lire les résidus en étudiant finement les deux termes suivants, pour tout  $j \in \llbracket 1, Q \rrbracket$

- $\frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \quad (\mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \mathbf{k} \in \mathbb{N}^{\mathcal{P}^c})$
- $\frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)}$

Notons que les pôles en  $s$  de ces termes là peuvent être parfois compensés par des zéros des fonctions de la forme  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  pour le premier terme, et par des zéros de fonctions de la forme  $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$  pour le second terme.



## Chapitre 3

# Calculs des coefficients $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ $\mu, \mu'$ et $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ $\mu, \mu'$

Par la Proposition 2.2.2, on sait que la fonction  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}$  est holomorphe, en particulier, on peut considérer ses coefficients d'ordre 0 et 1 au voisinage de 0 dans son développement de Taylor :

**Notation.** Soit  $(\mu, \mu') \in \mathbb{R}^P \times \mathbb{R}_*^Q$ , et  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ . Soit  $\mathcal{P} \subset \llbracket 1, P \rrbracket$  et  $\mathbf{k} \in \mathbb{N}^{P^c}$ . Comme  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}$  est holomorphe, on note au voisinage de 0

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k}) + Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})s + O(s^2).$$

Les coefficients  $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ , pour tout  $j \in \llbracket 1, Q \rrbracket$ ,  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ , s'avéreront cruciaux dans l'expression des formules de  $Z(-(\mathbf{N}, \mathbf{N}'))$ . De plus, les coefficients  $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$  et  $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ , pour tout  $j \in \llbracket 1, Q \rrbracket$ ,  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ , seront présents dans la formule de la dérivée directionnelle en les multi-entiers négatifs  $Z'(-(\mathbf{N}, \mathbf{N}'))$ . Observons également que l'on a pas besoin de calculer  $Q_{\mathbf{N}, \mathbf{N}'}^0(\llbracket 1, P \rrbracket, j)$  et  $Q_{\mathbf{N}, \mathbf{N}'}^1(\llbracket 1, P \rrbracket, j)$ , puisqu'ils n'apparaissent pas dans les expressions contenues dans le Théorème A et dans le Théorème D. On montrera dans nos calculs que  $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$  ne dépend pas des directions  $(\mu, \mu')$ . On notera donc simplement ce coefficient  $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ .

### 3.1 Lemmes

**Lemme 3.1.1.** Soit  $a, c \in H_0$ ,  $b \in \mathbb{N}$  et  $d \in \mathbb{Z}$ . On pose  $g : \mathbb{C} \rightarrow \mathbb{C}$  la fonction méromorphe décrite par l'expression

$$g(s) := \frac{1}{\Gamma(as - b)(cs + d)}.$$

Alors  $g$  est régulière en 0. De plus

$$g(0) = \begin{cases} \frac{a}{c}(-1)^b b! & \text{si } d = 0 \\ 0 & \text{si } d \neq 0. \end{cases},$$

et

$$\partial_s (g(s))_{s=0} = \begin{cases} \frac{a^2}{c} (-1)^b b! (\gamma - h_b) & \text{si } d = 0 \\ \frac{a}{d} (-1)^b b! & \text{si } d \neq 0. \end{cases}$$

où  $h_b$  désigne le nombre harmonique  $h_b := \sum_{i=1}^b \frac{1}{i}$ . Enfin, on a

$$\partial_s \left( \frac{1}{\Gamma(s)} \right)_{|s=-N} = (-1)^N N!.$$

*Démonstration.* Au voisinage de  $s = 0$ , on a

$$\frac{1}{\Gamma(s-N)} = (-1)^N N! s + O(s^2),$$

ce qui prouve la formule pour  $\partial_s \left( \frac{1}{\Gamma(s)} \right)_{|s=-N}$ .

Pour démontrer les autres formules, on étudiera le développement limité de  $g$  au voisinage de  $s = 0$ . Par l'équation fonctionnelle de la fonction gamma, on obtient

$$g(s) = \frac{(as-1)\dots(as-b)}{\Gamma(as)(cs+d)}.$$

On a au voisinage de 0 que

$$\begin{aligned} (as-1)\dots(as-b) &= (-1)^b b! + a(-1)^{b-1} b! h_b s + O(s^2), \\ \frac{1}{\Gamma(s)} &= s + \gamma s^2 + O(s^3). \end{aligned}$$

Supposons que  $d = 0$ . Alors on a au voisinage de 0 :

$$\begin{aligned} g(s) &= \frac{a}{c} \frac{(as-1)\dots(as-b)}{\Gamma(as)as} \\ &= \frac{a}{c} ((-1)^b b! + as(-1)^{b-1} b! h_b)(1 + \gamma as) + O(s^2) \\ &= \frac{a}{c} (-1)^b b! + \frac{a^2}{c} (-1)^{b-1} b! h_b s + \frac{a^2}{c} \gamma (-1)^b b! s + O(s^2) \end{aligned}$$

Supposons maintenant que  $d \neq 0$ , alors on a au voisinage de 0 :

$$\begin{aligned} g(s) &= \frac{(as-1)\dots(as-b)}{\Gamma(as)(cs+d)} \\ &= \frac{(as-1)\dots(as-b)}{cs+d} (as) + O(s^2) \\ &= \frac{a}{d} (-1)^b b! s + O(s^2). \end{aligned}$$

□

**Lemme 3.1.2.** Soit  $\mathcal{P} \subset \llbracket 1, P \rrbracket$ ,  $(\mu_p)_{p \in \mathcal{P}} \in \mathbb{R}^{\mathcal{P}}$ ,  $(N_p)_{p \in \mathcal{P}} \in \mathbb{N}^{\mathcal{P}}$ ,  $(n_p)_{p \in \mathcal{P}} \in \mathbb{N}^{\mathcal{P}}$ , on a

$$\partial_s \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} \right)_{|s=0} = \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p} \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + n_p}) \right),$$

où  $h_n$  désigne le  $n$ -ième nombre harmonique.



*Démonstration.* Rappelons que l'on a

$$\prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} = \prod_{p \in \mathcal{P}} \frac{(-N_p - 1 + \mu_p s) \dots (-N_p + \mu_p s - n_p)}{n_p!}.$$

En développant le produit  $\frac{(-N_p - 1 + \mu_p s) \dots (-N_p - n_p + \mu_p s)}{n_p!}$  avec  $s$  au voisinage de 0, on trouve que

$$\frac{(-N_p - 1 + \mu_p s) \dots (-N_p - n_p + \mu_p s)}{n_p!} = \binom{-N_p - 1}{n_p} + \mu_p \left( \sum_{i=0}^{n_p-1} \frac{(-N_p - 1) \dots \widehat{(-N_p - 1 - i)} \dots (-N_p - n_p)}{n_p!} \right) s + O(s^2),$$

où  $\widehat{(-N_p - 1 - i)}$  désigne l'élément que l'on oublie dans le produit. On peut simplifier l'égalité précédente sous la forme

$$\begin{aligned} \frac{(-N_p - 1 + \mu_p s) \dots (-N_p - n_p + \mu_p s)}{n_p!} &= \binom{-N_p - 1}{n_p} + \mu_p \binom{-N_p - 1}{n_p} \left( \sum_{i=0}^{n_p-1} \frac{1}{-N_p - 1 - i} \right) s \\ &= \binom{-N_p - 1}{n_p} + \mu_p \binom{-N_p - 1}{n_p} (h_{N_p} - h_{N_p+n_p}) s + O(s^2). \end{aligned}$$

Au voisinage de  $s = 0$ , on obtient alors :

$$\prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} = \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p} + \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p} \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p+n_p}) \right) s + O(s^2).$$

□

**Lemme 3.1.3.** Soit  $M \in \mathbb{Z}_{\leq 0}$  un entier, et  $(a_k)_{k \in [M, +\infty[}$  et  $(b_k)_{k \in [M, +\infty[}$  deux suites de nombres complexes tels que la série entière  $\sum_{k=0}^{+\infty} a_k x^k$  et la série entière  $\sum_{k=0}^{+\infty} b_k x^k$  aient deux rayons de convergence  $R_1$  et  $R_2$  non nuls. Supposons de plus qu'il existe un réel  $0 < R < \min(R_1, R_2)$  tel que

$$\forall x \in ]0, R[, \quad \sum_{k=M}^{+\infty} a_k x^k + \sum_{k=M}^{+\infty} b_k x^k \ln(x) = 0,$$

alors, pour tout  $k \in [0, +\infty[$ ,  $a_k = 0$  et  $b_k = 0$ .

*Démonstration.* Si tous les  $b_k$  sont nuls (respectivement tous les  $a_k$  sont nuls) pour tout  $k \geq M$ , le résultat est trivial. Sinon, on suppose par l'absurde qu'il existe  $i, j \geq M$  tels que  $a_i \neq 0$  et  $b_j \neq 0$ . Supposons que  $i = \min\{k | a_k \neq 0\}$  et  $j = \min\{k | b_k \neq 0\}$ , on a alors au voisinage de 0 que

$$a_i x^i + b_j x^j \ln(x) + o(x^j \ln(x)) + o(x^i) = 0,$$

et donc  $b_j x^{j-i} \ln(x) \sim a_i$  au voisinage de 0, contradiction. □

Le lemme précédent bien qu'évident, jouera un rôle crucial dans la détermination des formules des valeurs spéciales dans la démonstration du Théorème A et du Théorème D. En effet, dans le découpage de  $Z_{(\mathbf{N}, \mathbf{N}')}(\theta, s)$  obtenue dans la Proposition 2.2.4, on a introduit artificiellement une variable libre  $\theta$ . Il est évident que  $K_{(\mathbf{N}, \mathbf{N}')}(\theta, s)$  s'annule en  $s = 0$ , la seule difficulté réside alors à évaluer  $J_{(\mathbf{N}, \mathbf{N}')}(\theta, 0)$  via la formule 2.5 en  $s = 0$ , on obtiendra alors une expression faisant intervenir une série de Laurent en les  $\theta$ , or la quantité  $Z_{(\mathbf{N}, \mathbf{N}')}(\theta, 0) = Z_{(\mathbf{N}, \mathbf{N}')}(\theta, 0)$  ne dépendant pas de  $\theta$ , on ne gardera de la formule obtenue que les termes constants de la série de Laurent en les  $\theta$ . Pour le calcul de  $Z'_{(\mathbf{N}, \mathbf{N}')}(\theta, 0) = Z'_{(\mathbf{N}, \mathbf{N}')}(\theta, 0)$ , on utilisera également ce lemme pour effectuer la même simplification, sauf que cette fois ci, en dérivant la formule 2.5 par rapport à  $s$ , on fera apparaître un terme en  $\ln(\theta)$ . De la même manière, on utilisera ce lemme afin de calculer les coefficients  $Q^0$  et  $Q^1$ , afin d'éliminer les séries en  $\epsilon$  et les termes en  $\ln(\epsilon)$ .

**Lemme 3.1.4.** *Il existe une fonction définie pour tout  $0 < \epsilon \ll 1$  tel que*

$$\alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) = a \ln(\epsilon) + \sum_{n \geq m} b_n \epsilon^n,$$

avec  $m \in \mathbb{Z}$ , avec un terme constant nul, (i.e.  $b_0 = 0$ ), et tel que

$$\int_{\epsilon}^1 x_f^{-N'_f-1+\mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) + F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}'),$$

avec  $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ , la constante définie dans la Proposition 1.3.2 :

$$\begin{aligned} F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &= -E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_f+1-\mathbf{w}(f)} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} \\ &\quad - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{(|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|)} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda-1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}) \\ &\quad + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \right). \end{aligned}$$

*Démonstration.* Afin de calculer l'intégrale, on utilise la Proposition 1.3.2. Avec les notations de cette proposition, on a

$$\begin{aligned} x_f^{-N'_f-1+\mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{(-N_p-1-|\mathbf{v}_p|)} &= \tilde{P}_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f) + \sum_{\lambda=1}^{N'_f+1-\mathbf{w}(f)} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{x_f^\lambda} \\ &\quad + \sum_{p \in \mathcal{P}} \sum_{\lambda=1}^{(|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|)} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{(c_{j,p} + c_{f,p} x_f)^\lambda}. \end{aligned}$$

On a noté  $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$  la primitive du polynôme  $\tilde{P}_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$  s'annulant en 1, et on note  $E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$  le terme constant de  $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$ . On note  $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(x_f) := P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f) - E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ . On trouve alors

$$\begin{aligned} \int_{\epsilon}^1 x_f^{-N'_f-1+\mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_f &= \\ &= -E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) - C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},1}(\mathbf{N}, \mathbf{N}') \ln(\epsilon) - \sum_{\lambda=2}^{N'_f+1-\mathbf{w}(f)} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} (1 - \epsilon^{1-\lambda}) \\ &\quad + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \left( \ln(c_{j,p} + c_{f,p}) - \ln(c_{j,p}) - \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \epsilon \right) \right) \\ &\quad - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{(|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|)} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda-1} ((c_{j,p} + c_{f,p})^{1-\lambda} - (c_{j,p} + c_{f,p} \epsilon)^{1-\lambda}) \end{aligned}$$

En développant au voisinage de  $\epsilon = 0$  les termes  $(c_{j,p} + c_{f,p} \epsilon)^{1-\lambda}$  et  $\ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \epsilon \right)$ , on peut découper la formule obtenue précédemment en un terme constant, et en une somme entre un terme en  $\ln(\epsilon)$  et une série de Laurent sans terme constant en  $\epsilon$ , pour  $0 < \epsilon \ll 1$ . Notons le terme non constant  $\alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon)$ .

On a alors

$$\begin{aligned} & \int_{\epsilon}^1 x_f^{-N'_j-1+\mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) \\ & - E_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_j+1-\mathbf{w}(f)} \frac{C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (1,p)}(\mathbf{N}, \mathbf{N}') \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \right) \\ & - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{(|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|)} \frac{D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda-1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}) \end{aligned}$$

□

Remarquons que l'on peut obtenir une expression plus simple de l'intégrale précédente, moins explicite, mais plus pratique à calculer avec un logiciel de calcul formel :

Considérons la primitive  $G_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}(x_f)$  de  $x_f^{-N'_j-1+\mathbf{w}(f)} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|}$  s'annulant en  $x_f = 1$ . Cette primitive s'exprime sous la forme d'une somme entre un multiple de  $\ln(x_f)$  et d'une série de Laurent en  $x_f$ , pour  $0 < x_f \ll 1$ . On remarque alors que le terme constant de cette série de Laurent est  $-F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ .

**Remarque 3.1.5.** *Il paraît plus pratique avec un logiciel de calcul formel de calculer directement le coefficient  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  en déterminant une primitive de  $x_f^{-N'_j-1+\mathbf{w}(f)} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|}$  plutôt que de calculer les coefficients  $C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')$ ,  $D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda,p)}(\mathbf{N}, \mathbf{N}')$ , et  $E_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  en utilisant la décomposition en éléments simples de la Proposition 1.3.2.*

## 3.2 Valeurs et dérivées de $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$

Soit  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ ,  $j \in \llbracket 1, Q \rrbracket$ , et  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$ . On cherchera dans cette section à calculer explicitement les coefficients  $Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^0$  et  $Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^1$ , dont on rappelle les définitions :

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) &= Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^0, \\ \partial_s h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) &= Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^1. \end{aligned}$$

Rappelons que l'on a montré dans la Proposition 2.2.2 que la fonction

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) &= \\ & \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx_j} \dots dx_Q, \end{aligned}$$

était holomorphe.

On avait également obtenu dans cette proposition une expression de la dérivée de  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  en  $s = 0$ .

### 3.2.1 Préliminaires

Soit  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ ,  $j \in \llbracket 1, Q \rrbracket$ , et  $0 < \epsilon \ll 1$ . Comme on a pour tout  $s \in \mathbb{C}$ ,

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s),$$

on en déduit qu'il suffit d'étudier le développement de Taylor de  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  au voisinage de 0 pour  $0 < \epsilon \ll 1$  afin d'obtenir celui de  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ . On rappelle le point iii) obtenu dans la

Proposition 2.2.2:

Soit  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$ . Pour tout  $s \in \mathbb{C}$ , et pour tout  $\epsilon > 0$  suffisamment petit, on a :

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) =$$

$$\begin{aligned} & \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in [1, Q]} c_{q, p}^{w_{p, q}} \right) \\ & \cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q, p}^{v_{p, q}} \right) \frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \\ & \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j, p} + \sum_{f \in \mathcal{Q}^c} c_{f, p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q. \end{aligned}$$

Notons  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  le terme général de la série précédente :

$$\begin{aligned} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) &= \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q, p}^{v_{p, q}} \right) \\ & \cdot \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \\ & \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j, p} + \sum_{f \in \mathcal{Q}^c} c_{f, p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q. \end{aligned} \quad (3.1)$$

Rappelons également que, avec cette définition de  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  on a par le point iii) du Lemme 2.3.22 :

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0) = \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0), \quad (3.2)$$

et par ce même point iii), on a également une expression de sa dérivée selon  $s$  :

$$\left( \partial_s I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right)_{|s=0} = \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left( \partial_s T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0}. \quad (3.3)$$

Grâce à l'égalité 2.24, on a

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\mathbf{v}, \mathbf{w}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0), \quad (3.4)$$

$$\left( \partial_s h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \right)_{|s=0} = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \left( \partial_s I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right)_{|s=0}. \quad (3.5)$$

Ainsi, calculer les coefficients  $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$  et  $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$  revient à étudier le terme général  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  au voisinage de  $s = 0$ .

**Lemme 3.2.1.**

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) = \begin{cases} O(1) & \text{si } |\mathcal{Q}| = Q - 1, \text{ et } \forall q \in \mathcal{Q}, & \mathbf{v}(q) + \mathbf{w}(q) = N'_q \\ O(s) & \text{si } |\mathcal{Q}| = Q - 1 \text{ et } \exists f \in \mathcal{Q} \setminus \{j\}, \forall q \in \mathcal{Q} \setminus \{f\}, & \mathbf{v}(q) + \mathbf{w}(q) = N'_q \\ & & \mathbf{v}(f) + \mathbf{w}(f) \neq N'_f \\ O(s) & \text{si } |\mathcal{Q}| = Q - 2 \text{ et } \forall q \in \mathcal{Q}, & \mathbf{v}(q) + \mathbf{w}(q) = N'_q \\ O(s^2) & \text{sinon.} \end{cases}$$

*Démonstration.* On sait que la fonction méromorphe  $s \mapsto \prod_{q \in \mathcal{Q}} \frac{1}{-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q)}$  admet un pôle d'ordre  $|\{q \in \mathcal{Q} | N'_q = \mathbf{v}(q) + \mathbf{w}(q)\}|$  en  $s = 0$ . En particulier, l'ordre est nécessairement plus petit que  $|\mathcal{Q}|$ . Par propriété de la fonction  $\Gamma$ , on sait que la fonction  $s \mapsto \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + s\mu'_q)}$  admet un zéro d'ordre  $Q - 1$  en  $s = 0$ . On obtient alors que  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  est d'ordre  $O(s^n)$  avec

$$n = Q - 1 - |\{q \in \mathcal{Q} | N'_q = \mathbf{v}(q) + \mathbf{w}(q)\}|.$$

□

### 3.2.2 Calcul de $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$

**Proposition 3.2.2.** Soit  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ .

- Si  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$  et  $\forall q \in \mathcal{Q}, \mathbf{v}(q) + \mathbf{w}(q) = N'_q$ , alors

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0) = \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!$$

- Sinon, on a

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0) = 0.$$

*Démonstration.* Dans le premier cas, par le lemme 3.1.1, on a pour tout  $q \neq j$  :

$$\left( \frac{1}{\Gamma(-N'_q + \mu'_q s)} \frac{1}{\mu'_q s} \right)_{s=0} = (-1)^{N'_q} N'_q! \frac{\mu'_q}{\mu'_q}.$$

En évaluant en  $s = 0$  l'expression de  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$ , on trouve la première formule de la proposition. La seconde égalité découle du Lemme 3.2.1. □

Par la formule 3.2 et par la proposition précédente, on obtient :

**Corollaire 3.2.3.** Soit  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ , et  $0 < \epsilon \ll 1$ .

- Si  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ , alors on a

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

- Sinon, on a

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0) = 0.$$

Par la formule 2.24, on a que

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0),$$

on trouve alors par le corollaire précédent que

$$Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathcal{Q}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right). \quad (3.6)$$

### 3.2.3 Calcul de $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$

En évaluant en  $s = 0$  la dérivée en  $s$  du terme général

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) = \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \cdot \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q,$$

on obtient :

**Proposition 3.2.4.** Soit  $\mathcal{Q} \subset \mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ ,

- Si  $\mathcal{Q} = [1, Q] \setminus \{j\}$  et  $\forall q \in \mathcal{Q}$ ,  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$ , alors

$$\begin{aligned}
 \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \\
 & (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, p}^{v_{p, q}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \\
 & \cdot \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
 & + (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}} + |\boldsymbol{\mu}'|_{\mathcal{Q}}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, p}^{v_{p, q}} \right) \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \ln(\epsilon).
 \end{aligned} \tag{3.7}$$

- Si  $\mathcal{Q} = [1, Q] \setminus \{j\}$  et s'il existe  $f \in \mathcal{Q} \setminus \{j\}$  tel que  $\mathbf{v}(f) + \mathbf{w}(f) \neq N'_f$  et tel que, pour tout  $q \in \mathcal{Q} \setminus \{f\}$ ,  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$ , alors

$$\begin{aligned}
 \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \\
 & (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, p}^{v_{p, q}} \right) \\
 & \cdot \frac{\mu'_f}{-N'_f + \mathbf{v}(f) + \mathbf{w}(f)} \epsilon^{-N'_f + \mathbf{v}(f) + \mathbf{w}(f)}.
 \end{aligned} \tag{3.8}$$

- Si  $\mathcal{Q} = [1, Q] \setminus \{j, f\}$  tel que  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$  pour tout  $q \in \mathcal{Q}$ , on a qu'il existe un terme  $\beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon)$  correspondant à la somme entre un terme en  $\ln(\epsilon)$  et d'une série de Laurent en  $\epsilon$  définie pour  $0 < \epsilon \ll 1$  sans coefficient constant, et tel que

$$\begin{aligned}
 \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \\
 & \mu'_f (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q, p}^{v_{p, q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \\
 & + \beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon),
 \end{aligned} \tag{3.9}$$

avec  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  définie dans la Proposition 1.3.2.

- Sinon, on a

$$\partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} = 0.$$

Démonstration. • Si  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ , on a alors

$$\begin{aligned}
& \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} = \\
& \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \partial_s \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right)_{|s=0} \\
& \cdot \left( \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \right)_{|s=0} \sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + \mathbf{v}(q) + \mathbf{w}(q)) \epsilon \\
& + \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
& \cdot \partial_s \left( \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \right)_{|s=0} \sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + \mathbf{v}(q) + \mathbf{w}(q)) \epsilon \\
& + |\mu'|_{|\mathcal{Q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
& \cdot \left( \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \right)_{|s=0} \sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + \mathbf{v}(q) + \mathbf{w}(q)) \ln(\epsilon) \epsilon
\end{aligned}$$

Par le Lemme 3.1.2, on a

$$\partial_s \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \right)_{|s=0} = \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) \right) \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|},$$

ce qui conclut la première formule.

• Si  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$  et  $\forall q \in \mathcal{Q}$ ,  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$ , on a par le Lemme 3.1.1,

$$\begin{aligned}
& \partial_s \left( \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right)_{|s=0} = \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f (-1)^{N'_f} N'_f! (\gamma - h_{N'_f}) \prod_{\substack{q=1 \\ q \neq j, f}}^Q (-1)^{N'_q} N'_q! \\
& = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left( \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
& \left( \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right)_{|s=0} = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q!
\end{aligned}$$



Ainsi, on trouve

$$\begin{aligned}
 \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0} &= \\
 & \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & \cdot (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \\
 & + \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & \cdot (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left( \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
 & + |\mu'|_{\mathcal{Q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & \cdot (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \ln(\epsilon)
 \end{aligned}$$

En factorisant la première et la seconde expression de cette somme, on obtient la formule 3.7.

• Si  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$  et s'il existe  $f \in \mathcal{Q} \setminus \{j\}$  tel que  $\mathbf{v}(f) + \mathbf{w}(f) \neq N'_f$  et tel que, pour tout  $q \in \mathcal{Q} \setminus \{f\}$ , on a  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$ , on a par le Lemme 3.1.1,

$$\begin{aligned}
 & \left( \frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + \mathbf{v}(f) + \mathbf{w}(f)) \prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = 0, \\
 \partial_s \left( \frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + \mathbf{v}(f) + \mathbf{w}(f)) \prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} &= \\
 & (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \frac{\mu'_f}{-N'_f + \mathbf{v}(f) + \mathbf{w}(f)}.
 \end{aligned}$$

On trouve alors

$$\begin{aligned} \partial_s \left( T_{\mathbf{N}, \mathbf{N}'}^{\mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \\ \epsilon \sum_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} (-N'_q + \mathbf{v}(q) + \mathbf{w}(q)) &\left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\ \cdot \partial_s \left( \frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + \mathbf{v}(f) + \mathbf{w}(f)) \prod_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} \Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \right)_{|s=0}. \end{aligned}$$

En observant que  $\sum_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} (-N'_q + \mathbf{v}(q) + \mathbf{w}(q)) = -N'_f + \mathbf{v}(f) + \mathbf{w}(f) \neq 0$ , on obtient la formule 3.8.

- Si  $\mathcal{Q} = \llbracket 1, \mathcal{Q} \rrbracket \setminus \{j, f\}$  tel que  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$  pour tout  $q \in \mathcal{Q}$ , on a par le Lemme 3.1.1

$$\begin{aligned} &\left( \frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right)_{|s=0} = 0 \\ \partial_s \left( \frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right)_{|s=0} &= \mu'_f \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} (-1)^{N'_q} N'_q!, \end{aligned}$$

En dérivant le terme général  $T_{\mathbf{N}, \mathbf{N}'}^{\mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  suivant  $s$ , puis en l'évaluant en  $s = 0$  on trouve pour tout  $0 < \epsilon \ll 1$ ,

$$\begin{aligned} \partial_s \left( T_{\mathbf{N}, \mathbf{N}'}^{\mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \quad (3.10) \\ \cdot \partial_s \left( \frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right)_{|s=0} &\epsilon \sum_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} (-N'_q + \mathbf{v}(q) + \mathbf{w}(q)) \\ \cdot \int_{\epsilon}^1 x_f^{-N'_f - 1 + \mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p - 1 - |\mathbf{v}_p|} dx_f. \end{aligned}$$

Avec les conditions que l'on a imposé sur les multi-indices  $\mathbf{v}$  et  $\mathbf{w}$ , on trouve que  $\sum_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} (-N'_q + \mathbf{v}(q) + \mathbf{w}(q)) = 0$ .

Par le Lemme 3.1.4, on a

$$\int_{\epsilon}^1 x_f^{-N'_f - 1 + \mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p - 1 - |\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) + F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$$

avec  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  définie dans la Proposition 1.3.2, et  $\alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon)$  une fonction en epsilon de la forme  $a \ln(\epsilon) + \sum_{n \geq m} b_n \epsilon^n$ ,  $m \in \mathbb{Z}$  et  $a, b_n$  des complexes tels que  $b_0 = 0$ .

En injectant cette expression dans la formule 3.10, on trouve

$$\begin{aligned} \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \\ \mu'_f \prod_{\substack{q=1 \\ q \neq j}}^{N'_q} (-1)^{N'_q} N'_q! &\left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \\ &\cdot \left( \alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) + F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \right). \end{aligned}$$

En notant

$$\beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) = \mu'_f \prod_{\substack{q=1 \\ q \neq j}}^{N'_q} (-1)^{N'_q} N'_q! \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon),$$

on obtient la formule 3.9.

• Si l'on n'est pas dans un des trois cas ci-dessus, alors on a par le Lemme 3.2.1 au voisinage de  $s = 0$ , pour tout  $0 < \epsilon \ll 1$ ,

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) = O(s^2),$$

ce qui conclut. □

Par la formule 3.3 et par la proposition précédente, on trouve le corollaire suivant :

**Corollaire 3.2.5.** *Soit  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ .*

- Si  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ , alors il existe un terme  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon)$  correspondant à la somme entre un terme en  $\ln(\epsilon)$  et d'une série de Laurent définie pour  $0 < \epsilon \ll 1$  sans terme constant, et tel que

$$\begin{aligned} \partial_s \left( I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \llbracket \mathcal{Q} \rrbracket \setminus \{j\}, \mathbf{k}}(\epsilon, s) \right)_{|s=0} &= \tag{3.11} \\ (-1)^{|\mathbf{N}'|_{\llbracket \llbracket 1, Q \rrbracket \setminus \{j\} \rrbracket}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! &\sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ &\left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\ &+ I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon). \end{aligned}$$

- Si  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j, f\}$ , on a qu'il existe un terme  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon)$  correspondant à la somme entre un

terme en  $\ln(\epsilon)$  et d'une série de Laurent définie pour  $0 < \epsilon \ll 1$  sans terme constant, et tel que

$$\partial_s \left( I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, s) \right)_{|s=0} = \quad (3.12)$$

$$I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k}}^*(\epsilon) + (-1)^{|\mathbf{N}'|_{[1,\mathcal{Q}] \setminus \{j\}}} \mu'_f \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1,\mathcal{Q}] \setminus \{j,f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}'),$$

avec  $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$  définie dans la Proposition 1.3.2.

- Sinon, on a

$$\partial_s \left( T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s) \right)_{|s=0} = 0.$$

Par la formule 3.5, on a que

$$\partial_s \left( h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s) \right)_{|s=0} = \sum_{\mathcal{Q} \subset [1,\mathcal{Q}] \setminus \{j\}} \partial_s \left( I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, s) \right)_{|s=0},$$

on trouve alors grâce au corollaire précédent que

$$\begin{aligned} Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k}) &= \partial_s \left( I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,[1,\mathcal{Q}] \setminus \{j\},\mathbf{k}}(\epsilon, s) \right)_{|s=0} + \sum_{\substack{f=1 \\ f \neq j}}^Q \partial_s \left( I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,[1,\mathcal{Q}] \setminus \{j,f\},\mathbf{k}}(\epsilon, s) \right)_{|s=0} \\ &= (-1)^{|\mathbf{N}'|_{[1,\mathcal{Q}] \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1,\mathcal{Q}] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ &\quad \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\ &+ (-1)^{|\mathbf{N}'|_{[1,\mathcal{Q}] \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1,\mathcal{Q}] \setminus \{j,f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \\ &\quad \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') \\ &+ I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,[1,\mathcal{Q}] \setminus \{j\},\mathbf{k}}^*(\epsilon) + \sum_{\substack{f=1 \\ f \neq j}}^Q I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,[1,\mathcal{Q}] \setminus \{j,f\},\mathbf{k}}^*(\epsilon). \end{aligned}$$

On conclut via le Lemme 3.1.3 en remarquant que  $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$  ne dépend pas de  $\epsilon$ . On trouve alors que le terme constant en  $\epsilon$  dans la formule précédente vaut nécessairement  $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ .

## Chapitre 4

# Démonstration du Théorème A et de ses corollaires

Le but de ce chapitre est d'étudier les valeurs directionnelles de  $Z(\mathbf{s}, \mathbf{s}')$  aux multi-entiers négatifs. En montrant que la fonction  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  est régulière en  $s = 0$ , on a montré que les quantités  $Z(-(\mathbf{N}, \mathbf{N}'))$  et  $Z'(-(\mathbf{N}, \mathbf{N}'))$  ont bien un sens. Afin de calculer la première quantité, on exploitera la Proposition 2.2.4 afin d'évaluer  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  en  $s = 0$ .

### 4.1 Enoncé du Théorème A

Rappelons tout d'abord l'énoncé du théorème :

**Théorème (A).** Soit  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_*^{+Q}$ . Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ . On a

$$Z(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = (-1)^{|\mathbf{N}'|} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|} \left( \prod_{p \in \mathcal{P}} N_p! \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}| = |\mathbf{k}|}} \sum_{j=1}^Q \frac{(-1)^{N'_j} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$$

$$\prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!},$$

avec

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \\ \forall p \in \mathcal{P}, \mathbf{v}_p \in \mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, N'_q = \mathbf{v}(q) + \mathbf{w}(q)}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q, p}^{w_{p, q}} \right)$$

$$\cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q C_{q, p}^{v_{p, q}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!.$$

où l'on rappelle les notations

$$\forall q \in \llbracket 1, Q \rrbracket, \quad \mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p, q},$$

et

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad \mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p, q}.$$

## 4.2 Démonstration du Théorème A

*Démonstration.* De la Proposition 2.2.4, on obtient pour tout  $0 < \theta \ll 1$ ,

$$Z_{\mathbf{N}, \mathbf{N}'}(0) = K_{\mathbf{N}, \mathbf{N}'}(\theta, 0) + J_{\mathbf{N}, \mathbf{N}'}(\theta, 0),$$

Or on sait que  $K_{\mathbf{N}, \mathbf{N}'}(\theta, 0) = 0$  par la Proposition 2.3.11. Il ne reste qu'à étudier la valeur de  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  en  $s = 0$ . Par la formule 2.5, on a

$$\begin{aligned} Z_{\mathbf{N}, \mathbf{N}'}(0) = & \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} N_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \sum_{j=1}^P h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) \\ & \left( \frac{1}{\Gamma(-N'_j + \mu'_j s) - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\mu'| + |\mu|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}} \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\mu'| + |\mu|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}}} \right) \prod_{s=0} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p=1}^P \Gamma(1 + N_p) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(0) \left( \frac{\theta^{(|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \right)_{|s=0} \end{aligned}$$

Observons tout d'abord que l'expression

$$\left( \prod_{p=1}^P \Gamma(1 + N_p) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(0) \left( \frac{\theta^{(|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \right)_{|s=0},$$

est nulle. En effet, comme on a  $-|\mathbf{N}'| - |\mathbf{N}| - P \neq 0$ , alors le zéro de  $\frac{1}{\Gamma(-N'_j + \mu'_j s)}$  en  $s = 0$  n'est compensé par aucun pôle venant de la fraction  $\frac{1}{((|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P)}$  en  $s = 0$ .

Étudions plus en détail le premier terme de la formule de  $Z_{\mathbf{N}, \mathbf{N}'}(0)$ . On sait que  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) = Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ . Soit  $0 < \theta \ll 1$  et  $\mathcal{P} \subsetneq [1, P]$ .

- Si  $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} \neq -|\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}$ , alors on sait par le Lemme 3.1.1 que le terme

$$\frac{1}{\Gamma(-N'_j + \mu'_j s) - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\mu'| + |\mu|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}}$$

s'annule en  $s = 0$ .

- Si  $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} = -|\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}$ , alors par le Lemme 3.1.1 on a :

$$\left( \frac{1}{\Gamma(-N'_j + \mu'_j s) - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\mu'| + |\mu|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}} \right)_{|s=0} = \frac{\mu'_j}{|\mu'| + |\mu|_{|\mathcal{P}}} (-1)^{N'_j} N'_j!$$

On en déduit alors le Théorème A. L'expression de  $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$  dans le Théorème A découle de l'expression obtenue dans l'égalité (3.6).  $\square$

## 4.3 Sur les corollaires du Théorème A

Une fois le Théorème A démontré, les corollaires sont rapides à démontrer.

### 4.3.1 Démonstration du Corollaire A1

Pour prouver le Corollaire A1, on applique le Théorème A, en posant  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$  :

$$Z(-(\mathbf{0}, \mathbf{0}))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \sum_{j=1}^Q \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!},$$

avec

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = 0}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

On peut simplifier le coefficient  $Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k})$  en remarquant que la condition  $\mathbf{v}(q) + \mathbf{w}(q) = 0$  pour tout  $q \neq j$  implique que

$$\begin{aligned} \forall p \in \mathcal{P}, \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad w_{p,q} &= 0 \\ \forall p \in \mathcal{P}^c, \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad w_{p,q} &= 0. \end{aligned}$$

On obtient alors que  $k_p = |\mathbf{w}_p| = w_{p,j}$  pour tout  $p \in \mathcal{P}^c$ . Ainsi, on peut simplifier l'expression de  $Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k})$  :

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}.$$

### 4.3.2 Démonstration du Corollaire A2

Sur le Corollaire A2, en posant  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$  et  $(\mathbf{N}, \mathbf{N}') = (N, \dots, N)$ , on trouve via le Théorème A que

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} (-1)^{N(|\mathcal{P}| + Q + 1) + |\mathcal{P}|} N!^{|\mathcal{P}| + 1} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N(Q + |\mathcal{P}|) + |\mathcal{P}|}} \sum_{j=1}^Q \frac{1}{Q + |\mathcal{P}|} Q_N^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!},$$

avec

$$Q_N^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

En notant

$$Q_N^0(\mathcal{P}, \mathbf{k}) = \sum_{j=1}^Q Q_N^0(j, \mathcal{P}, \mathbf{k}),$$

on trouve la formule annoncée dans le corollaire.

### 4.3.3 Démonstration du Corollaire A3

Le Corollaire A3 correspond à une version qualitative, non explicite du Théorème A. Soit  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_*^{+Q}$ . Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ . On note  $\mathbb{K}$  le corps engendré par  $\mathbb{Q}$ , par les coefficients complexes  $(d_p)_{p \in \llbracket 1, P \rrbracket}$  et  $(c_{q,p})_{p \in \llbracket 1, P \rrbracket, q \in \llbracket 1, Q \rrbracket}$ , et par les directions  $\mu_p$  pour tout  $p \in \llbracket 1, P \rrbracket$ , et  $\mu'_q$  pour tout  $q \in \llbracket 1, Q \rrbracket$ . Par le Théorème A, on a que

$$Z(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = (-1)^{|\mathbf{N}'|} \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|} \left( \prod_{p \in \mathcal{P}} N_p! \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} \sum_{j=1}^Q \frac{(-1)^{N'_j} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$$

$$\prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!},$$

avec

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

Observons tout d'abord que les coefficients  $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$  est dans le corps  $\mathbb{K}$ , en tant que somme rationnels de produit de puissance des coefficients  $c_{q,p} \in \mathbb{K}$  pour  $1 \leq p \leq P$  et  $1 \leq q \leq Q$ . Ensuite, par propriété de la fonction zêta de Hurwitz, on sait que ses valeurs aux entiers négatifs s'expriment en fonction de polynômes de Bernoulli, et ceux-ci sont à coefficients rationnels. Ainsi, on obtient que  $\frac{\zeta(-N_p - k_p, d_p)}{k_p!} \in \mathbb{Q}[d_1, \dots, d_P] \subset \mathbb{K}$ , ce qui prouve le corollaire.



## Chapitre 5

# Démonstration de la Proposition B et du Théorème D

### 5.1 Enoncé de la Proposition B et du Théorème D

On cherchera dans ce chapitre à démontrer la Proposition B et le Théorème D, dont on rappelle les énoncés ci-dessous.

**Proposition (B).** Soit  $j \in \llbracket 1, Q \rrbracket$  un entier. On suppose que  $l_j$  est une forme linéaire à coefficients rationnels pour  $1 \leq j \leq Q$ . Pour tout  $1 \leq p \leq P$ , on pose  $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$  la décomposition irréductible de  $c_{j,p}$  avec  $a_{j,p} > 0$  et  $b_{j,p} > 0$ . On pose alors  $x_j(\mathbf{c}) = \frac{\text{ppcm}(a_{j,p})}{\text{pgcd}(b_{j,p})} \in \mathbb{Q}_+$ , et  $\beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}^*$  pour tout  $1 \leq p \leq P$ . Alors pour tout  $\mathbf{R} \in \mathbb{N}^P$ , la fonction  $\varphi_{\mathbf{R}}^j(s)$  admet la relation suivante pour tout  $s \in \mathbb{C} \setminus \mathbb{N}^*$ ,

$$\varphi_{\mathbf{R}}^j(s) = x_j(\mathbf{c})^{-s} \prod_{p=1}^P \beta_{j,p}^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq [1,P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!} \cdot \frac{\zeta(s - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!}$$

**Théorème (D).** Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ , alors

$$\begin{aligned}
Z'(-(\mathbf{N}, \mathbf{N}')) &= \tag{5.1} \\
&\sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q (-1)^{|\mathbf{N}|+|\mathbf{N}'|+|\mathcal{P}|+N'_j} \left( \prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\
&\cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \sum_{p \in \mathcal{P}} \mu_p(\gamma - h_{N_p}) + \mu'_j(\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
&+ \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|+N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
&\cdot \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
&+ \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{|s=-N'_j} \\
&- \sum_{j=1}^Q (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\
&\cdot \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}|+|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=N'_j+|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}|+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \Bigg),
\end{aligned}$$

où l'on a posé

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket},$$

pour tout  $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$ , et tout  $j \in \llbracket 1, Q \rrbracket$ , avec  $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$  et  $Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k})$  déterminés dans le Chapitre 3.

## 5.2 Fonctions auxiliaires

On appliquera dans cette section la Proposition 2.2.4 aux fonctions auxiliaires  $\varphi_{\mathbf{R}}^j(s)$ , afin de faire intervenir les valeurs spéciales de ces fonctions par la suite dans le calcul des valeurs de la dérivée directionnelle de  $Z$ .

On rappelle que, pour  $\mathbf{R} = (R_1, \dots, R_P) \in \mathbb{N}^P$ , et  $j \in \llbracket 1, Q \rrbracket$ , on a posé

$$\varphi_{\mathbf{R}}^j(s) = \sum_{\mathbf{n} \in \mathbb{N}^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d'_j)^{-s}.$$

Cette série de Dirichlet se prolonge méromorphiquement sur  $\mathbb{C}$ , et on sait qu'elle est régulière en les entiers négatifs par le Corollaire 2.2.6.

**Remarque 5.2.1.** Pour  $\mathbf{R} = \mathbf{0}$ , on retrouve la fonction zêta de Barnes définie par la série

$$\zeta^B(s, d'_j | c_{j,1}, \dots, c_{j,P}) = \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{(l_j(\mathbf{n}) + d'_j)^s}.$$

On applique la Proposition 2.2.4 à la fonction auxiliaire  $\varphi_{\mathbf{R}}^j(s)$ , en posant  $\boldsymbol{\mu} = \mathbf{0}$ ,  $\mathbf{N} = \mathbf{R}$ ,  $\mathbf{N}' = \mathbf{0}$ , et  $\boldsymbol{\mu}' = 1$ . On sait que cette fonction auxiliaire est régulière en les entiers négatifs, et par cette même proposition, on obtient que l'ensemble des pôles est inclus dans l'ensemble

$$\mathcal{S}(\mathbf{R}) := \mathbb{Z}_{\geq 1}.$$

On obtient que la fonction auxiliaire s'exprime sous la forme suivante, pour  $0 < \theta \ll 1$ , et pour tout  $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{R})$ ,

$$\varphi_{\mathbf{R}}^j(s) = J_{\mathbf{R}}^j(\theta, s) + K_{\mathbf{R}}^j(\theta, s), \quad (5.2)$$

avec, pour tout  $s \in \mathbb{C}$ ,

$$K_{\mathbf{R}}^j(\theta, s) = \frac{1}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p},$$

et pour tout  $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{R})$ , on a

$$\begin{aligned} J_{\mathbf{R}}^j(\theta, s) = & \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{R}, \mathcal{P}, \mathbf{k}}(s) \frac{\theta^{s-|\mathbf{R}|_{|\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(s)(s-|\mathbf{R}|_{|\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p=1}^P R_p! \right) h_{\mathbf{R}, \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'|+|\boldsymbol{\mu}|)s-|\mathbf{N}'|-|\mathbf{N}|-P}}{\Gamma(s)(s-|\mathbf{R}|-P)}. \end{aligned}$$

avec

$$h_{\mathbf{R}, \mathcal{P}, \mathbf{k}}(s) = \prod_{p \in \mathcal{P}} c_{j,p}^{-N_p-1} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p},$$

et

$$h_{\mathbf{R}, \llbracket 1, P \rrbracket, j}(s) = \prod_{p=1}^P c_{j,p}^{-N_p-1}.$$

On trouve alors

$$\begin{aligned} J_{\mathbf{R}}^j(\theta, s) = & \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \frac{\theta^{s-|\mathbf{R}|_{|\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(s)(s-|\mathbf{R}|_{|\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p=1}^P R_p! \right) \left( \prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \frac{\theta^{s-|\mathbf{R}|-P}}{\Gamma(s)(s-|\mathbf{R}|-P)}. \end{aligned} \quad (5.3)$$

**Proposition 5.2.2.** Soit  $\mathbf{R} \in \mathbb{N}^P$ , et  $N \in \mathbb{N}$ . Soit  $0 < \theta \ll 1$  et  $j \in \llbracket 1, Q \rrbracket$ , alors  $s \mapsto \varphi_{\mathbf{R}}^j(s)$  admet un prolongement méromorphe sur  $\mathbb{C}$ , et est régulière en les entiers négatifs. De plus,

$$\partial_s K_{\mathbf{R}}^j(\theta, -N) = (-1)^N N! \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-N, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \quad (5.4)$$

$$\partial_s J_{\mathbf{R}}^j(\theta, -N) = \quad (5.5)$$

$$J_{\mathbf{R}}^{*j}(\theta, -N) + N!(\gamma - h_N) \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{R}|_{|\mathcal{P}}+|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=N+|\mathbf{R}|_{|\mathcal{P}}+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!},$$

avec  $J_{\mathbf{R}}^{*j}(\theta, -N)$  de la forme  $J_{\mathbf{R}}^{*j}(\theta, -N) = \sum_{n \geq -N - |\mathbf{R}| - P} \alpha_n \theta^n + \beta \ln(\theta)$  pour tout  $0 < \theta \ll 1$ , et avec un terme constant nul, i.e.  $\alpha_0 = 0$ .

*Démonstration.* Par le Théorème 2.2.6, on sait que  $\varphi_{\mathbf{R}}^j$  admet un prolongement méromorphe régulier en les entiers négatifs. Par la Proposition 2.3.11, on sait que l'application  $s \in \mathbb{C} \mapsto \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p}$  est holomorphe sur  $\mathbb{C}$ . En dérivant l'expression de  $K_{\mathbf{R}}^j$ , on obtient

$$\begin{aligned} \partial_s K_{\mathbf{R}}^j(\theta, s) &= -\frac{\psi(s)}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \\ &\quad + \frac{1}{\Gamma(s)} \partial_s \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \right), \end{aligned}$$

où  $\psi$  désigne la fonction digamma.

On peut alors évaluer cette somme de deux termes en  $s = -N$ . Comme  $\frac{1}{\Gamma(s)}$  s'annule en  $s = -N$ , et que par le Lemme 3.1.1,  $\left( \frac{\psi(s)}{\Gamma(s)} \right)_{s=-N} = (-1)^{N+1} N!$ , on obtient bien la formule voulue.

De la formule 5.3, on en déduit

$$\begin{aligned} \partial_s J_{\mathbf{R}}^j(\theta, s) &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &\quad \cdot \partial_s \left( \frac{1}{\Gamma(s)(s - |\mathbf{R}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \theta^{s - |\mathbf{R}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ &\quad + \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &\quad \cdot \frac{1}{\Gamma(s)(s - |\mathbf{R}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \theta^{s - |\mathbf{R}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \ln(\theta) \\ &\quad - \left( \prod_{p=1}^P R_p! \right) \left( \prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \theta^{s - |\mathbf{R}| - P} \left( \frac{\ln(\theta)}{\Gamma(s)(s - |\mathbf{R}| - P)} + \partial_s \left( \frac{1}{\Gamma(s)(s - |\mathbf{R}| - P)} \right) \right) \end{aligned}$$

Pour obtenir la formule pour  $(\partial_s J_{\mathbf{R}}^j(\theta, s))_{s=-N}$ , on doit évaluer les termes de la forme  $\frac{1}{(s - |\mathbf{R}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)}$ , et ceux de la forme  $\partial_s \frac{1}{(s - |\mathbf{R}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)}$ . On évalue ces expressions en  $s = -N$  à l'aide du Lemme 3.1.1 :

$$\left( \frac{1}{(s - |\mathbf{R}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)} \right)_{s=-N} = \begin{cases} (-1)^N N! & \text{si } |\mathbf{k}| = N + |\mathbf{R}|_{\mathcal{P}} + |\mathcal{P}| \\ 0 & \text{si } |\mathbf{k}| \neq N + |\mathbf{R}|_{\mathcal{P}} + |\mathcal{P}|. \end{cases}$$

et

$$\partial_s \left( \frac{1}{\Gamma(s)(s - |\mathbf{R}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{s=-N} = \begin{cases} (-1)^N (\gamma - h_N) N! & \text{si } |\mathbf{k}| = N + |\mathbf{R}|_{\mathcal{P}} + |\mathcal{P}| \\ \frac{(-1)^N N!}{-N - |\mathbf{R}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} & \text{si } |\mathbf{k}| \neq N + |\mathbf{R}|_{\mathcal{P}} + |\mathcal{P}|. \end{cases}$$

On trouve alors

$$\begin{aligned}
\partial_s J_{\mathbf{R}}^j(\theta, -N) = & (\gamma - h_N) N! \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p - 1} R_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\
& + (-1)^N N! \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p - 1} R_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|}} (-1)^{|\mathbf{k}|} \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \frac{\theta^{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\
& + N! \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p - 1} R_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|}} \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \ln(\theta) \\
& + (-1)^N N! \left( \prod_{p=1}^P R_p! \right) \left( \prod_{p=1}^P c_{j,p}^{-R_p - 1} \right) \frac{\theta^{-N - |\mathbf{R}| - P}}{-N - |\mathbf{R}| - P}
\end{aligned}$$

On peut alors poser  $J_{\mathbf{R}}^{*j}(\theta, -N)$  telle que

$$\begin{aligned}
J_{\mathbf{R}}^{*j}(\theta, -N) = & (-1)^N N! \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p - 1} R_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|}} (-1)^{|\mathbf{k}|} \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \frac{\theta^{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\
& + N! \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p - 1} R_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|}} \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \ln(\theta) \\
& + (-1)^N N! \left( \prod_{p=1}^P R_p! \right) \left( \prod_{p=1}^P c_{j,p}^{-R_p - 1} \right) \frac{\theta^{-N - |\mathbf{R}| - P}}{-N - |\mathbf{R}| - P}
\end{aligned}$$

□

**Remarque 5.2.3.** Le terme  $J_{\mathbf{R}}^{*j}(\theta, -N)$  correspond à la somme entre un multiple de  $\ln(\theta)$  et une série de Laurent à terme constant non nul.

Observons que l'on ne peut pas obtenir une expression explicite des valeurs de  $\partial_s \varphi_{\mathbf{R}}^j$  en les entiers négatifs à l'aide de (5.4) et de (5.5), puisqu'il n'est pas vrai que le terme  $\partial_s K_{\mathbf{R}}^j(\theta, -N)$  s'annule. On a ainsi besoin d'une autre stratégie pour calculer les valeurs de  $\partial_s \varphi_{\mathbf{R}}^j$  en les entiers négatifs.

### 5.2.1 Démonstration de la Proposition B

On souhaite dans cette section fournir des valeurs explicites des dérivées de ces fonctions auxiliaires dans le cas où les coefficients  $(c_{q,p})_{q \in [1, Q] \times [1, P]}$  sont rationnels.

Soit  $P \in \mathbb{N}^*$ , et  $\mathbf{R} \in \mathbb{N}^P$ . Pour simplifier les écritures dans cette sous-section, on écrira pour tout complexe  $s$  tel que  $\sigma \gg_{\mathbf{R}} 1$

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) := \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{(n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P}}{\left( \sum_{p=1}^P w_p (n_p + d_p) \right)^s},$$

avec  $\mathbf{d} = (d_p)_{p \in [1, P]}$  et  $w_p, d_p \in H_0$  pour tout entier  $p$ . Cette fonction admet un prolongement méromorphe sur  $\mathbb{C}$  en la variable  $s$ . On souhaite dans cette sous-section établir une relation explicite

entre  $\varphi_{\mathbf{R}}^j(s)$  et la fonction zêta d'Hurwitz  $\zeta(s, x)$ , avec  $x \in H_0$ . On suit pour cela une stratégie développée par Aoki et Sakane dans [SA22], qui a été utilisée initialement pour expliciter les valeurs spéciales des dérivées supérieures des fonctions zêta de Barnes à coefficients rationnels. Ici, les fonctions auxiliaires  $\varphi_{\mathbf{R}}(s, x|w_1, \dots, w_P)$  ne sont pas exactement des fonctions zêta de Barnes à cause des termes au numérateur dans la série, cependant les techniques utilisées par Aoki et Sakane fonctionnent de la même manière pour étudier cette fonction.

**Lemme 5.2.4.** *Soit  $w_1, \dots, w_P \in \mathbb{Q}_*^+$  tels que  $w_p = \frac{a_p}{b_p}$  avec  $a_p$  et  $b_p$  des entiers strictement positifs. Pour tout multiple commun  $\alpha$  à  $b_1, \dots, b_P$ , et tout multiple commun  $\beta$  à  $\alpha w_1, \dots, \alpha w_P$ , on a*

$$\varphi_{\mathbf{R}}(s, \mathbf{d}|w_1, \dots, w_P) = \left(\frac{\alpha}{\beta}\right)^s \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq k_1 \leq \beta_1 - 1 \\ \vdots \\ 0 \leq k_P \leq \beta_P - 1}} \varphi_{\mathbf{R}}\left(s, \left(\frac{d_p + k_p}{\beta_p}\right)_{p \in \llbracket 1, P \rrbracket} \middle| \mathbf{1}\right), \quad (5.6)$$

$$\text{avec } \beta_1 = \frac{\beta}{\alpha w_1}, \dots, \beta_P = \frac{\beta}{\alpha w_P}.$$

*Démonstration.* Soit  $s \in \mathbb{C}$  tel que  $\sigma \gg 1$ . Pour obtenir le résultat souhaité, on découpera chaque somme

$\sum_{n_p=0}^{+\infty}$  par  $\sum_{\substack{n_p=0 \\ n_p = k_p \text{ mod } \beta_p}}^{+\infty}$ , avec  $k_p$  un entier variant dans l'ensemble  $\llbracket 0, \beta_p - 1 \rrbracket$ . On obtient alors que

$$\varphi_{\mathbf{R}}(s, \mathbf{d}|w_1, \dots, w_P) = \left(\frac{\alpha}{\beta}\right)^s \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\substack{\mathbf{n} \in \mathbb{N}^P \\ \forall p, n_p = k_p \text{ mod } \beta_p}} \frac{\prod_{p=1}^P (n_p + d_p)^{R_p}}{\left(\frac{\alpha}{\beta} (\sum_{p=1}^P (w_p d_p + w_p k_p + w_p (n_p - k_p)))\right)^s}.$$

En effectuant un changement de variable de la forme  $n'_p = \beta_p n_p + k_p$  pour tout entier  $p$ , on obtient

$$\varphi_{\mathbf{R}}(s, \mathbf{d}|w_1, \dots, w_P) = \left(\frac{\alpha}{\beta}\right)^s \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{\prod_{p=1}^P (\beta_p n_p + k_p + d_p)^{R_p}}{\left(\frac{\alpha}{\beta} \sum_{p=1}^P w_p d_p + w_p k_p + w_p \beta_p n_p\right)^s}.$$

En factorisant par  $\beta_p^{R_p}$  au numérateur, et en remarquant que  $\frac{\alpha w_p \beta_p}{\beta} = 1$ , on a

$$\varphi_{\mathbf{R}}(s, \mathbf{d}|w_1, \dots, w_P) = \left(\frac{\alpha}{\beta}\right)^s \prod_{p=1}^P \beta_p^{R_p} \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{\prod_{p=1}^P \left(n_p + \frac{k_p + d_p}{\beta_p}\right)^{R_p}}{\left(\sum_{p=1}^P n_p + \frac{\alpha}{\beta} (w_p d_p + w_p k_p)\right)^s}.$$

Il ne reste qu'à observer que

$$\frac{k_p + d_p}{\beta_p} = \frac{\alpha}{\beta} (w_p d_p + w_p k_p),$$

pour tout  $1 \leq p \leq P$ . On obtient donc la formule 5.6 pour tout  $s \in \mathbb{C}$  tel que  $\sigma \gg 1$ . On peut alors étendre l'égalité sur tout  $\mathbb{C}$  par prolongement analytique.  $\square$

On peut préciser la formule 5.6 en choisissant un coefficient rationnel  $\frac{\alpha}{\beta}$  plus agréable. On peut en effet considérer directement un tel coefficient rationnel qui dépendra directement des coefficients  $w_1, \dots, w_P$  ce qui facilitera grandement nos calculs explicites par la suite.

**Proposition 5.2.5.** *Soit  $\mathbf{d} \in H_0$   $w_1 = \frac{a_1}{b_1}, \dots, w_P = \frac{a_P}{b_P}$  des rationnels avec  $a_p, b_p > 0$  des entiers premiers entre eux pour tout  $p \in \llbracket 1, P \rrbracket$ . On pose  $w := \frac{\text{ppcm}(a_1, \dots, a_P)}{\text{pgcd}(b_1, \dots, b_P)}$ , et pour tout  $p \in \llbracket 1, P \rrbracket$ , on*

pose  $\beta_1 := \frac{w}{w_1}, \dots, \beta_P := \frac{w}{w_P}$ . Alors

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = w^{-s} \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq k_1 \leq \beta_1 - 1 \\ \vdots \\ 0 \leq k_P \leq \beta_P - 1}} \varphi_{\mathbf{R}} \left( s, \left( \frac{d_p + k_p}{\beta_p} \right)_{p \in [1, P]} \mid \mathbf{1} \right).$$

*Démonstration.* On pose  $\alpha = \text{ppcm}(b_1, \dots, b_P)$  et  $\beta = \text{ppcm}(\alpha w_1, \dots, \alpha w_P)$ . Par [SA22, p.3], on sait que  $\frac{\alpha}{\beta} = w^{-1}$ . On a alors qu'à utiliser la formule 5.6 avec le  $\alpha$  et le  $\beta$  fixés ci-dessus.  $\square$

Finalement, on obtient une formule similaire à celle démontrée par [Ono21, Proposition 4.1] pour réexprimer  $\varphi_{\mathbf{R}}(s, \mathbf{d} | 1, \dots, 1)$ , en modifiant très légèrement la preuve d'Onodera.

**Proposition 5.2.6.** *Soit  $\mathbf{d} \in H_0^P$ ,  $\mathbf{R} \in \mathbb{N}^P$ , et  $s \in \mathbb{C} \setminus \mathbb{N}^*$ . Alors on a*

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | \mathbf{1}) = \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k'_i \geq 0 \\ |\mathbf{k}|_{\mathcal{P}} + k'_i = |\mathbf{R}|_{\mathcal{P}} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \frac{\zeta(s - k', |\mathbf{d}|)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}.$$

Pour démontrer cette formule, on utilisera un lemme d'Onodera [Ono21, Lemme 4.2] qui est une généralisation de la formule de Faulhaber suivante

$$\sum_{n=1}^y n^r = \frac{1}{r+1} \sum_{k=0}^r \binom{r+1}{k} (-1)^k B_k y^{r-k+1}.$$

**Lemme 5.2.7.** [Ono21] *Soit  $\mathbf{R} = (R_1, \dots, R_P) \in \mathbb{N}^P$  des entiers positifs, et  $y \in \mathbb{Z}$  un entier relatif. Alors on a*

$$\begin{aligned} & \sum_{\substack{\mathbf{n} \in \mathbb{N}^P \\ |\mathbf{n}|=y}} \prod_{p=1}^P (n_p + d_p)^{R_p} + (-1)^{P-1} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{\leq -1}^P \\ |\mathbf{n}|=y}} \prod_{p=1}^P (n_p + d_p)^{R_p} = \\ & \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k'_i \in \mathbb{N} \\ |\mathbf{k}| + k'_i = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} \frac{(|\mathbf{d}| + y)^{k'_i}}{k'_i!} \prod_{p \in \mathcal{P}} (-1)^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \end{aligned}$$

On remarque au passage que si  $y \in \mathbb{N}$ , alors le second terme à gauche de l'égalité du terme précédent est nul. On peut maintenant prouver la Proposition 5.2.6 :

*Démonstration.* On commence par remarquer que, pour tout  $s \in \mathbb{C}$  tel que  $\sigma \gg_{\mathbf{R}} 1$ , on a

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | \mathbf{1}) = \sum_{n'=0}^{+\infty} \frac{1}{(n' + |\mathbf{d}|)^s} \sum_{\substack{\mathbf{n} \in \mathbb{N}^P \\ |\mathbf{n}|=n'}} \prod_{p=1}^P (n_p + d_p)^{R_p}.$$

Par le Lemme 5.2.7, on obtient alors

$$\begin{aligned} \varphi_{\mathbf{R}}(s, \mathbf{d} | \mathbf{1}) &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k'_i \in \mathbb{N} \\ |\mathbf{k}| + k'_i = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} \prod_{p \in \mathcal{P}} (-1)^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \sum_{n'=0}^{+\infty} \frac{(n' + |\mathbf{d}|)^{k'_i}}{k'_i!} \frac{1}{(n' + |\mathbf{d}|)^s} \\ &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k'_i \in \mathbb{N} \\ |\mathbf{k}| + k'_i = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \left( \prod_{p \in \mathcal{P}} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \frac{\zeta(s - k', |\mathbf{d}|)}{k'!} \end{aligned}$$

$\square$

En appliquant la Proposition 5.2.6 à la formule obtenue dans la Proposition 5.2.5, on obtient la proposition suivante :

**Proposition 5.2.8.** *Soit  $\mathbf{d} \in H_0^P$ ,  $\mathbf{R} \in \mathbb{N}^P$ , et  $s \in \mathbb{C} \setminus \mathbb{N}^*$ , et  $w_1 = \frac{a_1}{b_1}, \dots, w_P = \frac{a_P}{b_P}$  des rationnels avec  $a_p, b_p > 0$  des entiers premiers entre eux pour tout  $p \in \llbracket 1, P \rrbracket$ . On pose  $w := \frac{\text{ppcm}(a_1, \dots, a_P)}{\text{pgcd}(b_1, \dots, b_P)}$ , et pour tout  $p \in \llbracket 1, P \rrbracket$ , on pose  $\beta_1 := \frac{w}{w_1}, \dots, \beta_P := \frac{w}{w_P}$ . Alors on a*

$$\begin{aligned} \varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = & \\ & w^{-s} \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq u_1 \leq \beta_1 - 1 \\ 0 \leq u_P \leq \beta_P - 1}} \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + u_p}{\beta_p}\right)}{k_p!} \\ & \cdot \frac{\zeta(s - k', w^{-1}(w_1(d_1 + u_1) + \dots + w_P(d_P + u_P)))}{k'!} \end{aligned}$$

### 5.3 Démonstration du Théorème D

Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ ,  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . Par la Proposition 2.2.4, on a pour tout complexe  $s$  au voisinage de 0, et pour tout  $0 < \theta \ll 1$ ,

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = K_{\mathbf{N}, \mathbf{N}'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}(\theta, s),$$

avec

$$\begin{aligned} K_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \sum_{\substack{\mathcal{Q} \subseteq \llbracket 1, Q \rrbracket \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subseteq \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s)} \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\ & \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right) \end{aligned}$$

et

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \\ & \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \\ & \cdot \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|_{\mathcal{P}^c}}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|_{\mathcal{P}^c})} \\ & \cdot \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right). \end{aligned}$$

**Proposition 5.3.1.** *Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ ,  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . Il existe une fonction  $K_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  définie pour  $0 < \theta \ll 1$ , tel que*

$$\forall 0 < \theta \ll 1, \quad K_{\mathbf{N}, \mathbf{N}'}^*(\theta) = \beta \ln(\theta) + \sum_{n=m}^{+\infty} \alpha_n \theta^n,$$



avec  $m \in \mathbb{Z}$  un entier, un complexe  $\beta \in \mathbb{C}$  et une suite de complexes  $(\alpha_n)_{n \in [m, +\infty[}$  vérifiant  $\alpha_0 = 0$ , et vérifiant l'égalité suivante :

$$\begin{aligned}
\partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} = & \tag{5.7} \\
& \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{\mathbf{u}(p)} \partial_s \varphi_{(N_p + \mathbf{u}(p))_{p \in [1, P]}}^j(s)_{s=-N'_j} \\
& - \sum_{j=1}^Q (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{\mathbf{u}(p)} \right) \\
& \cdot \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \\
& + K_{\mathbf{N}, \mathbf{N}'}^*(\theta)_{\mu, \mu'}
\end{aligned}$$

*Démonstration.* En dérivant le terme  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  par rapport à  $s$  dans la formule 2.4, on obtient

$$\begin{aligned}
\partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \tag{5.8} \\
& \sum_{\substack{\mathcal{Q} \subset [1, Q] \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \partial_s \left( \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right) \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\
& \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right) \\
& + \sum_{\substack{\mathcal{Q} \subset [1, Q] \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \partial_s \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\
& \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right).
\end{aligned}$$

A l'aide de l'inégalité 2.12, on a vu dans la preuve de la Proposition 2.3.11 que la fonction

$$s \in \mathbb{C} \mapsto \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right)$$

était holomorphe sur  $\mathbb{C}$ . Comme  $s \mapsto \frac{1}{\Gamma(s)}$  s'annule en les entiers négatifs, on en déduit que le second terme de 5.8 s'annule en  $s = 0$ . De plus, on sait que, au voisinage de 0,

$$\frac{1}{\Gamma(-N'_q + \mu'_q s)} = O(s),$$

On en déduit que, pour tout  $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$  et pour tout  $A_{\mathcal{Q}} \subset \mathcal{Q}^c$ , on a

$$\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} = \begin{cases} O(s^2) & \text{si } |\mathcal{Q}| \geq 2, \\ O(s) & \text{si } \mathcal{Q} = \{j\}, A_{\mathcal{Q}} = \llbracket 1, Q \rrbracket \setminus \{j\} \text{ avec } j \in \llbracket 1, Q \rrbracket, \\ O(s^2) & \text{sinon.} \end{cases}$$

Ainsi, on en déduit que si  $|\mathcal{Q}| \geq 2$ , alors

$$\partial_s \left( \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right)_{s=0} = 0.$$

On trouve alors

$$\begin{aligned} \partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} &= \sum_{j=1}^Q \partial_s \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)} \right)_{s=0} \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p} \right. \\ &\quad \left. \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} \right). \end{aligned}$$

Par le Lemme 3.1.1, on a que

$$\partial_s \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)} \right)_{s=0} = (-1)^{N'_j} N'_j! \mu'_j.$$

On obtient alors

$$\begin{aligned} \partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} &= \sum_{j=1}^Q \mu'_j (-1)^{N'_j} N'_j! \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p} \right. \\ &\quad \left. \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} \right) \end{aligned}$$

Pour tout  $q \in \llbracket 1, Q \rrbracket$ ,  $\mathbf{n} \in \mathbb{N}^P$ , on a  $l_q(\mathbf{n}) + d'_q = \sum_{p=1}^P c_{q,p}(n_p + d_p)$ . Ainsi par le multinôme de Newton, on obtient

$$(l_q(\mathbf{n}) + d'_q)^{N'_q} = \sum_{\substack{\mathbf{u}_q \in \mathbb{N}^P \\ |\mathbf{u}_q| = N'_q}} \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} (n_p + d_p)^{u_{q,p}}.$$

En effectuant le produit sur  $q \in \llbracket 1, Q \rrbracket \setminus \{j\}$ , on trouve

$$\begin{aligned} \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} &= \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \left( \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} (n_p + d_p)^{u_{q,p}} \right) \\ &= \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \prod_{p=1}^P (n_p + d_p)^{\mathbf{u}^{(p)}}, \end{aligned}$$

où l'on a noté

$$\mathbf{u}(p) = \sum_{\substack{q=1 \\ q \neq j}}^Q u_{q,p}.$$

On obtient finalement

$$\begin{aligned} \partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} &= \sum_{j \in \llbracket 1, Q \rrbracket} \mu'_j (-1)^{N'_j} N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \\ &\quad \left( \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p + \mathbf{u}(p)} \right). \end{aligned} \quad (5.9)$$

On sait par la formule (5.2) que, pour tout  $m \in \mathbb{N}$ ,

$$(-1)^m m! \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-m, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} = \partial_s \varphi_{\mathbf{R}}^j(s)_{s=-m} - \partial_s J_{\mathbf{R}}^j(\theta, s)_{s=-m}.$$

En substituant  $(-1)^m m! \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-m, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p}$  dans 5.9 par la formule obtenue précédemment, on obtient

$$\begin{aligned} \partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} &= \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \\ &\quad \left( \partial_s \varphi_{(N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket}}^j(s)_{s=-N'_j} - \partial_s J_{(N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket}}^j(\theta, s)_{s=-N'_j} \right). \end{aligned} \quad (5.10)$$

On rappelle que l'on a posé  $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) = (N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket}$  pour tout  $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$  dans les notations du Théorème D. Par la formule (5.5), on a

$$\begin{aligned} \partial_s J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(\theta, s)_{s=-N'_j} &= (\gamma - h_{N'_j}) N'_j! \sum_{\mathcal{P} \subset \llbracket 1, P \rrbracket} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \\ &\quad \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \\ &\quad + J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j). \end{aligned} \quad (5.11)$$

On pose pour  $0 < \theta \ll 1$ ,

$$K_{\mathbf{N}, \mathbf{N}'}^*(\theta) := - \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j) \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}}.$$

Par construction des  $J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j)$ , observons que  $K_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  est de la forme  $\sum_{n \geq m} \alpha_n \theta^n + \beta \ln(\theta)$  avec  $\alpha_n$  et  $\beta$  des complexes pour tout  $n \geq m$ , et  $\alpha_0 = 0$ . En injectant 5.11 dans 5.10, on trouve alors la

formule désirée :

$$\begin{aligned}
\partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} = & \quad (5.12) \\
& \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \partial_s \varphi_{(N_p + \mathbf{u}(p))_{p \in [1, P]}}^j(s)_{s=-N'_j} \\
& - \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \right) \prod_{p=1}^P c_{q,p}^{u_{q,p}} (-1)^{N'_j} N'_j! (\gamma - h_{N'_j}) \\
& \cdot \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \\
& + K_{\mathbf{N}, \mathbf{N}'}^*(\theta).
\end{aligned}$$

□

Il ne reste qu'à calculer les valeurs de  $\partial_s(J_{\mathbf{N}, \mathbf{N}'}(\theta, s))$  en  $s = 0$  afin de prouver le Théorème D.

**Proposition 5.3.2.** *Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ . Il existe une fonction  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  définie pour  $0 < \theta \ll 1$ , tel que*

$$\forall 0 < \theta \ll 1, \quad J_{\mathbf{N}, \mathbf{N}'}^*(\theta) = \beta \ln(\theta) + \sum_{n=m}^{+\infty} \alpha_n \theta^n,$$

avec  $m \in \mathbb{Z}$  un entier, un complexe  $\beta \in \mathbb{R}$  et une suite de complexes  $(\alpha_n)_{n \in [m, +\infty[}$  tel que  $\alpha_0 = 0$ , et vérifiant l'égalité suivante pour tout  $0 < \theta \ll 1$  :

$$\begin{aligned}
\partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \sum_{\mu, \mu'} \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|}}} \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}| + N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\
& \cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
& + \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|} + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
& \cdot \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
& + J_{\mathbf{N}, \mathbf{N}'}^*(\theta).
\end{aligned} \tag{5.13}$$

*Démonstration.* Rappelons les notations suivantes :

$$Q_{\mathbf{N},\mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) := h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(0)$$

$$Q_{\mathbf{N},\mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) := \partial_s (h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s))|_{s=0},$$

pour  $j \in \llbracket 1, Q \rrbracket$ ,  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ ,  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$ .

Notons  $T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(\theta, s)$  le terme général de la série présente dans 2.5 :

$$T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(\theta, s) = \frac{\left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s)}{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!},$$

pour tout  $j \in \llbracket 1, Q \rrbracket$ ,  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ , et  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$ . On évalue la formule 2.29 en  $s = 0$  pour réexprimer  $\partial_s (J_{\mathbf{N},\mathbf{N}'}(\theta, s))_{s=0}$ . On obtient

$$\begin{aligned} \partial_s (J_{\mathbf{N},\mathbf{N}'}(\theta, s))_{s=0} &= \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} \sum_{\mathcal{P} \subset \llbracket 1, P \rrbracket} \sum_{j=1}^Q \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} \partial_s T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(\theta, s)_{s=0} \\ &+ \partial_s \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} h_{\mathbf{N},\mathbf{N}',\llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{s=0}. \end{aligned} \quad (5.14)$$

Étudions tout d'abord le second terme de cette somme. Remarquons que, en  $s = 0$ , la fonction  $\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)}$  s'annule. On obtient alors

$$\begin{aligned} \partial_s \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} h_{\mathbf{N},\mathbf{N}',\llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{s=0} &= \\ \prod_{p=1}^P N_p! \sum_{j=1}^Q \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} h_{\mathbf{N},\mathbf{N}',\llbracket 1, P \rrbracket, j}(0) \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P} \partial_s \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{s=0}. \end{aligned}$$

Par le lemme 3.1.1, on a

$$\partial_s \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{s=0} = (-1)^{N'_j+1} N'_j! \frac{\mu'_j}{|\mathbf{N}'| + |\mathbf{N}| + P}.$$

On trouve alors

$$\begin{aligned} \partial_s \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} h_{\mathbf{N},\mathbf{N}',\llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{s=0} &= \\ \left( \prod_{p=1}^P N_p! \sum_{j=1}^Q \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} h_{\mathbf{N},\mathbf{N}',\llbracket 1, P \rrbracket, j}(0) (-1)^{N'_j+1} N'_j! \frac{\mu'_j}{|\mathbf{N}'| + |\mathbf{N}| + P} \right) \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P}. \end{aligned}$$

On inclura le terme précédent à l'expression de  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ , puisque  $-|\mathbf{N}'| - |\mathbf{N}| - P \neq 0$ .

Étudions à présent la dérivée du terme général  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$  en  $s = 0$ .

- Si  $|\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ , alors la fonction  $\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)}$  s'annule en  $s = 0$ . On trouve alors

$$\begin{aligned} \partial_s (T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s))_{s=0} &= \\ & \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta) \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ & \partial_s \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{s=0} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}. \end{aligned}$$

Via le Lemme 3.1.1, on obtient

$$\begin{aligned} \partial_s (T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s))_{s=0} &= \\ & \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta) \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} (-1)^{N'_j} N'_j! \frac{\mu'_j}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ & \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}. \end{aligned}$$

On remarque que la puissance de  $\theta$  dans le terme précédent est non nul. On inclura donc tous les termes de la forme  $\partial_s (T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s))_{s=0}$  tel que  $|\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$  dans l'expression de  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ .

- Si  $|\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ , alors par le Lemme 3.1.1, on a

$$\begin{aligned} & \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{s=0} = (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \\ \partial_s & \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{s=0} = (-1)^{N'_j} \frac{\mu_j'^2}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! (\gamma - h_{N'_j}). \end{aligned}$$

En dérivant par rapport à  $s$  le terme général  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$ , on a

$$\begin{aligned} \partial_s (T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s))_{s=0} = & \\ & - \left( \sum_{i \in \mathcal{P}} \mu_i \Gamma'(1 + N_i) \prod_{p \in \mathcal{P} \setminus \{i\}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}, \\ & + \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}, \\ & + \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}) \ln(\theta) (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}, \\ & + \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} \frac{\mu_j^2}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! (\gamma - h_{N'_j}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right), \end{aligned}$$

On sait que  $\Gamma'(1 + N_p) = N'_p!(-\gamma + h_{N'_p})$ . On peut réduire l'expression précédente sous une forme plus pratique à manipuler.

$$\begin{aligned} \partial_s (T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s))_{s=0} = & \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}| + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & \cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \mu'_j (\gamma - h_{N'_j}) - \sum_{p \in \mathcal{P}} \frac{\mu_p \Gamma'(1 + N_p)}{N_p!} \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right), \\ & + \left( \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}| + N'_j} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \mu'_j N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \ln(\theta). \end{aligned}$$

On inclura le terme en  $\ln(\theta)$  dans l'expression de  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ .

On pose  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  tel que

$$\begin{aligned}
J_{\mathbf{N}, \mathbf{N}'}^*(\theta) = & \left( \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} \left( \prod_{p \in \mathcal{P}} N_p! \right) \sum_{j=1}^Q \mu'_j N'_j! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}}+|\mathcal{P}|}} (-1)^{|\mathbf{k}|+N'_j} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \ln(\theta) \\
& + \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| \neq |\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}}+|\mathcal{P}|}} \left( \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} N'_j! \frac{\mu'_j}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \right. \\
& \cdot \left. \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right) \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\
& + \left( \prod_{p=1}^P N_p! \sum_{j=1}^Q Q_{\mathbf{N}, \mathbf{N}'}^0(j, \llbracket 1, P \rrbracket) (-1)^{N'_j+1} N'_j! \frac{\mu'_j}{|\mathbf{N}'| + |\mathbf{N}| + P} \right) \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P}.
\end{aligned}$$

On découpe la série présente dans la formule 5.14 suivant que  $|\mathbf{k}|$  soit égal ou non à  $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ . Avec le terme  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  que l'on vient de poser, on obtient alors la formule 5.13.  $\square$

On observe que les séries  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  et  $K_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  ont un terme constant nul, et que  $\partial_s Z_{\mathbf{N}, \mathbf{N}'}(s) = \partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + \partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ . Comme  $\partial_s (Z_{\mathbf{N}, \mathbf{N}'}(s))_{s=0}$  ne dépend pas de  $\theta$ , on a par le lemme 3.1.3 que

$$\partial_s (Z_{\mathbf{N}, \mathbf{N}'}(s))_{s=0} = \partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} + \partial_s (K_{\mathbf{N}, \mathbf{N}'}(\theta, s))_{s=0} - K_{\mathbf{N}, \mathbf{N}'}^*(\theta) - J_{\mathbf{N}, \mathbf{N}'}^*(\theta),$$

pour tout  $0 < \theta \ll 1$ . On en déduit alors le Théorème D.

On obtient également dans le même temps le Théorème C, qui correspond à une reformulation qualitative du Théorème D.

**Remarque 5.3.3.** De l'égalité  $K_{\mathbf{N}, \mathbf{N}'}^*(\theta) + J_{\mathbf{N}, \mathbf{N}'}^*(\theta) = 0$  pour tout  $0 < \theta \ll 1$ , et du lemme 3.1.3, on en déduit que chaque coefficient dans la série de Laurent de  $K_{\mathbf{N}, \mathbf{N}'}^*(\theta) + J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  s'annule, et le coefficient devant  $\ln(\theta)$  s'annule aussi. On pourrait donc en déduire des relations entre les valeurs de  $\zeta(-k, d_p)$ , avec  $k \in \mathbb{N}$  et  $1 \leq p \leq P$ .

## 5.4 Sur les corollaires du Théorème D

Pour démontrer les corollaires du Théorème D, il suffit d'utiliser la formule donnée par le théorème et de remplacer la direction ou les valeurs  $-(\mathbf{N}, \mathbf{N}')$  que l'on souhaite étudier.

### 5.4.1 Démonstration du Théorème C

Soit  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ , et  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^+{}^P \times \mathbb{R}^+{}^Q$ . Il s'agit ici de reprendre la formule obtenue pour  $Z'(-(\mathbf{N}, \mathbf{N}'))$ , et d'y étudier les termes qualitativement. On note  $\mathbb{K}$  le corps engendré par  $\mathbb{Q}$ , par les coefficients complexes  $(d_p)_{p \in \llbracket 1, P \rrbracket}$  et  $(c_{q,p})_{p \in \llbracket 1, P \rrbracket, q \in \llbracket 1, Q \rrbracket}$ , et par les directions  $(\mu_p)_{p \in \llbracket 1, P \rrbracket}$ ,  $(\mu'_q)_{q \in \llbracket 1, Q \rrbracket}$ .



Par le Théorème D, on a

$$\begin{aligned}
Z'(-(\mathbf{N}, \mathbf{N}')) &= \\
&\sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q (-1)^{|\mathbf{N}|+|\mathbf{N}'|+|\mathcal{P}|+N'_j} \left( \prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\
&\cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \sum_{p \in \mathcal{P}} \mu_p(\gamma - h_{N_p}) + \mu'_j(\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
&+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|+N'_j} } \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
&\cdot \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
&+ \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{s=-N'_j} \\
&- \sum_{j=1}^Q (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\
&\cdot \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}|+|\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}|+|\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \Bigg),
\end{aligned}$$

où l'on a posé

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + \mathbf{u}(p))_{p \in [1, P]},$$

pour tout  $\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}}$ , et tout  $j \in [1, Q]$ ,

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) &= \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \in [1, Q] \setminus \{j\}, N'_q = \mathbf{v}(q) + \mathbf{w}(q)}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{q,p}} \right) \\
&\cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{q,p}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!,
\end{aligned}$$

et

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k})_{\mu, \mu'} &= (-1)^{|\mathbf{N}'|_{[\mathbb{1}, Q] \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[\mathbb{1}, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\
&\quad \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
&+ (-1)^{|\mathbf{N}'|_{[\mathbb{1}, Q] \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[\mathbb{1}, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \\
&\quad \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}'),
\end{aligned}$$

avec  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  la constante posée dans la Proposition 1.3.2.

Observons tout d'abord que  $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \in \mathbb{K}$ . De plus, les coefficients de la forme  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  appartiennent à l'espace vectoriel sur  $\mathbb{K}$  engendré par les coefficients de la forme  $\ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right)$  pour  $1 \leq q, j \leq Q$  et  $1 \leq p \leq P$ , donc  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \in \text{Vect}_{\mathbb{K}} \left[ \left( \ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right) \right)_{\substack{a \leq j, q \leq Q \\ 1 \leq p \leq P}} \right]$ . On trouve alors que

$$Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k})_{\mu, \mu'} \in \text{Vect}_{\mathbb{K}} \left[ \gamma, \left( \ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right) \right)_{\substack{a \leq j, q \leq Q \\ 1 \leq p \leq P}} \right].$$

Comme les valeurs spéciales de la forme  $\zeta(-N, d_p)$  sont dans le corps engendré  $\mathbb{Q}$  et par les coefficients  $d_1, \dots, d_P$ , on a que ces valeurs de la fonction zêta de Hurwitz sont dans  $\mathbb{K}$ , on obtient que  $Z'(-(\mathbf{N}, \mathbf{N}'))_{\mu, \mu'}$  appartient bien à l'espace vectoriel sur  $\mathbb{K}$  engendré par  $\gamma$ , par les coefficients de la forme  $\ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right)$ , par les valeurs spéciales aux entiers négatifs des dérivées des fonctions auxiliaires  $\partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{|s=-N'_j}$  et par des valeurs spéciales aux entiers négatifs de la dérivée selon  $s$  de la fonction zêta d'Hurwitz.

## 5.4.2 Démonstration du Corollaire C1

On utilise le fait que, lorsque les formes linéaires sont rationnels, on a par le Corollaire B1 une expression de  $\partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{|s=-N'_j}$  en fonction de la dérivée de la fonction zêta de Hurwitz. On remplace alors les termes de la forme  $\partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{|s=-N'_j}$  dans le Théorème C, par des termes de la forme  $\zeta'(-n, y)$ .

## 5.4.3 Démonstration du Corollaire D1

On obtient le Corollaire D1 en appliquant la formule obtenue pour  $Z'(-(\mathbf{N}, \mathbf{N}'))_{\mu, \mu'}$  dans le Théorème A, en remplaçant les termes de la forme  $\varphi_{\mathbf{R}}^j(-N)$  par la formule donnée par le Corollaire B1.

Comme les formes linéaires  $l_j$  sont à coefficients rationnels, on peut poser  $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$  pour tout  $1 \leq p \leq P$ , en supposant que cette fraction est irréductible. On pose alors  $x_j(\mathbf{c}) = \frac{\text{ppcm}(a_{j,p})}{\text{pgcd}(b_{j,p})} \in \mathbb{Q}_+^*$ , et

$\beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}^*$  pour tout  $1 \leq p \leq P$ .

Par le Corollaire B1, pour tout  $\mathbf{R} \in \mathbb{N}^P$ , on a

$$\begin{aligned} \varphi_{\mathbf{R}}^{j'}(-N'_j) &= \\ & x_j(\mathbf{c})^N \prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq [1,P]} \left( \prod_{p \in \mathcal{P}^c} R_p! \right) \sum_{\substack{\mathbf{k} \in \mathbb{N}^{\mathcal{P}}, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \\ & \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!} \left( \frac{\zeta'(-N - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} \right). \\ & = x_j(\mathbf{c})^N \prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq [1,P]} \left( \prod_{p \in \mathcal{P}^c} \mathbf{R}(\mathbf{N}, j, \mathbf{u})_p! \right) \sum_{\substack{\mathbf{k} \in \mathbb{N}^{\mathcal{P}}, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \\ & \cdot \left( \prod_{p \in \mathcal{P}} \frac{\zeta\left(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!} \right) \\ & \cdot \left( \frac{\zeta'(-N'_j - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N'_j - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} \right). \end{aligned}$$

En injectant cette formule dans celle obtenue pour  $Z'(-(\mathbf{N}, \mathbf{N}'))$  dans le Théorème D, on trouve le résultat.

On obtient encore une fois le Corollaire C1, qui correspond à une reformulation qualitative du Corollaire D1.

#### 5.4.4 Démonstration du Corollaire D2

En posant  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$  dans le théorème D, on peut simplifier plusieurs sommes en remarquant les faits suivants :

- 1) Soit  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}}$  et  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c}$  tels que

$$\begin{aligned} |\mathbf{w}_p| &= k_p & (p \in \mathcal{P}^c) \\ \mathbf{v}(q) + \mathbf{w}(q) &= 0 & (q \in \llbracket 1, Q \rrbracket \setminus \{j\}). \end{aligned}$$

Alors on a nécessairement que ces indices ont plusieurs composantes nulles :

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \forall p \in \mathcal{P}, \quad v_{p,q} = 0,$$

et

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \forall p \in \mathcal{P}^c, \quad w_{p,q} = 0.$$

Ainsi, comme  $w_{p,j} = |w_p| = k_p$  pour tout  $p \in \mathcal{P}^c$ , on obtient que  $\mathbf{w} = (k_p \mathbf{e}_j)_{p \in \mathcal{P}^c}$  avec  $\mathbf{e}_j \in \mathbb{N}^Q$  le vecteur valant 1 en la  $j$ -ième coordonnée, et 0 ailleurs.

- 2) Soit  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}}$  et  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c}$  tels que

$$\begin{aligned} |\mathbf{w}_p| &= k_p & (p \in \mathcal{P}^c) \\ \mathbf{v}(q) + \mathbf{w}(q) &= 0 & (q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}). \end{aligned}$$

On obtient alors que

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}, \forall p \in \mathcal{P}, \quad v_{p,q} = 0,$$

et

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}, \forall p \in \mathcal{P}^c, \quad w_{p,q} = 0.$$

Ainsi, on trouve que l'indice  $\mathbf{w}$  est de la forme  $(w_{p,j}\mathbf{e}_j + w_{p,f}\mathbf{e}_f)_{p \in \mathcal{P}^c}$ , avec  $w_{p,j} + w_{p,f} = k_p$  pour tout  $p \in \mathcal{P}^c$ . On trouve ainsi que le coefficient multinomial présent dans l'expression de  $Q^1$  vaut :

$$\binom{k_p}{\mathbf{w}_p} = \frac{k_p!}{w_{p,j}!w_{p,f}!}.$$

Pour simplifier cette expression, on parcourra  $w'_p \in \llbracket 0, k_p \rrbracket$ , et on posera  $w_{p,f} = w'_p$  et  $w_{p,j} = k_p - w'_p$

3) Soit  $j \in \llbracket 1, Q \rrbracket$ , et  $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$  tel que

$$|\mathbf{u}_q| = N'_q = 0 \quad (q \in \llbracket 1, Q \rrbracket \setminus \{j\}).$$

On obtient trivialement que  $\mathbf{u} = \mathbf{0}$ , et donc  $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket} = \mathbf{0}$ . En particulier, les fonctions auxiliaires  $\varphi_{(N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket}}^j(s)$  sont en fait des fonctions zêta de Barnes de la forme

$$\varphi_{\mathbf{0}}^j(s) = \zeta^B(s, d'_j | c_{j,1}, \dots, c_{j,P}).$$

On rappelle au passage que la dérivée de cette fonction zêta de Barnes en  $s = 0$  correspond à  $\ln(\Gamma_P(d'_j | c_{j,1}, \dots, c_{j,P}))$ , avec  $\Gamma_P$  correspondant aux fonctions multigamma de Barnes introduites dans la Définition 1.2.14.

On a déjà vu dans la preuve du Corollaire A1 que

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}.$$

A l'aide de ces trois points précédents, et du fait que le nombre harmonique  $h_0$  vaut 0, on obtient :

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} &= \\ & \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \left( \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \gamma(\mu'_j + |\boldsymbol{\mu}|_{\mathcal{P}}) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k})_{\boldsymbol{\mu}, \boldsymbol{\mu}'} \right) \\ & + \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p, d_p)}{k_p!} \right) \\ & + \sum_{j=1}^Q \mu'_j \partial_s \zeta^B(s, d'_j | \mathbf{c}_j)_{|s=0} - \sum_{j=1}^Q \gamma \mu'_j \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p, d_p)}{k_p!}. \end{aligned}$$

Par les points 1) et 2) précédent, on obtient :

$$\begin{aligned} Q_{\boldsymbol{\mu}, \boldsymbol{\mu}'}^1(j, \mathcal{P}, \mathbf{k}) &= \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket} E_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ & + \gamma |\boldsymbol{\mu}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}. \end{aligned}$$

### 5.4.5 Démonstration du Corollaire D3

En appliquant le Corollaire D2, en remplaçant  $\boldsymbol{\mu}$  par  $(0, \dots, 0)$ , et en remplaçant  $\boldsymbol{\mu}'$  par  $(1, \dots, 1)$ , on obtient

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{0}, \mathbf{1}} &= \\ & \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|}}{Q} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \left( \gamma \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) + Q^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{j=1}^Q \ln(\Gamma_P(d'_j | \mathbf{c}_j)) - \gamma \sum_{j=1}^Q \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p, d_p)}{k_p!}, \end{aligned}$$

avec

$$\begin{aligned} Q^1(j, \mathcal{P}, \mathbf{k}) &= \sum_{\substack{f=1 \\ f \neq j}}^Q \sum_{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ & + \gamma(Q-1) \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}. \end{aligned}$$

Or par la Définition 1.2.16  $Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{0}, \mathbf{1}} = \ln(\partial_s(\zeta^{Sh}(s, (\mathbf{c}_1, \dots, \mathbf{c}_Q), \mathbf{d}')|_{s=0}))$ , et par la Définition 1.2.14, on a  $(\partial_s \zeta^B(s, d|c_1, \dots, c_P))|_{s=0} = \ln(\Gamma_P(d|c_1, \dots, c_P))$ , et  $\partial_s(\zeta^{Sh}(s, \mathbf{d}'|\mathbf{c}))|_{s=0} = \ln(\Gamma_P(\mathbf{d}'|\mathbf{c}))$ . On trouve alors le Corollaire D3.



# Chapitre 6

## Fonctions zêta de Witten

On rappelle ici la théorie des fonctions zêta de Witten sous l'angle de Komori, Matsumoto et de Tsumura. Les détails des preuves sont disponibles dans le livre [KMT24] qui paraîtra en novembre 2023. Alternativement, on peut aussi voir le détail dans les articles [KMT10b], [KMT10c], [KMT11b], [KMT12]. Pour les détails de la théorie de Lie, on se réfère à [Hum72], [Bou06], [Bou81]. On appliquera ensuite certains de nos résultats à l'étude des fonctions zêta de Witten  $\zeta_{\mathfrak{g}_2}(s)$ , et  $\zeta_{\mathfrak{so}(5)}(s)$ , pour en déduire l'expression explicite de la constante  $C$  dans le Théorème 1.2.35, et une formule asymptotique pour le nombre de représentations  $r_{\mathfrak{g}_2}(n)$  à l'aide du Théorème 1.2.33.

### 6.1 Généralités sur les algèbres de Lie

On appelle  $\mathfrak{g}$  une algèbre de Lie un  $\mathbb{C}$ -espace vectoriel de dimension finie muni d'une forme bilinéaire  $[\cdot, \cdot]$  vérifiant les conditions suivantes

1.  $\forall x \in \mathfrak{g}, [x, x] = 0$
2.  $\forall x, y, z \in \mathfrak{g}, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ .

On appellera  $[\cdot, \cdot]$  le crochet de Lie associé à l'algèbre de Lie  $\mathfrak{g}$ .

Soit  $\mathfrak{g}_1, \mathfrak{g}_2$  des algèbres de Lie munies respectivement des crochets de Lie  $[\cdot, \cdot]_1$  et  $[\cdot, \cdot]_2$ . Soit  $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  un morphisme d'espace vectoriel. On dira que  $f$  est un morphisme d'algèbre de Lie si  $f$  respecte le crochet de Lie, c'est-à-dire que

$$\forall x, y \in \mathfrak{g}_1, [f(x), f(y)]_2 = f([x, y]_1).$$

Si de plus  $f$  est un isomorphisme d'espaces vectoriels, on dira que  $f$  est un isomorphisme d'algèbres de Lie.

Pour  $x \in \mathfrak{g}$ , on notera  $ad_x$  la dérivation suivante :

$$ad_x : \mathfrak{g} \rightarrow \mathfrak{g}, y \mapsto [x, y].$$

**Définition 6.1.1.** Soit  $\mathfrak{g}$  une algèbre de Lie. Soit  $\mathfrak{a} \subset \mathfrak{g}$  un sous-espace vectoriel de  $\mathfrak{g}$ . On dit que  $\mathfrak{a}$  est une sous-algèbre de  $\mathfrak{g}$  si  $\mathfrak{a}$  est stable par le crochet de Lie, c'est-à-dire lorsque  $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$ .

Soit  $\mathfrak{a}_1$  et  $\mathfrak{a}_2$  des sous-algèbres d'une algèbre de Lie  $\mathfrak{g}$ , on définit  $[\mathfrak{a}_1, \mathfrak{a}_2]$  comme étant la sous-algèbre de Lie de  $\mathfrak{g}$  engendrée par les éléments de la forme  $[x, y]$ , avec  $x \in \mathfrak{a}_1$ , et  $y \in \mathfrak{a}_2$ .

Soit  $\mathfrak{a}$  une sous-algèbre d'une algèbre de Lie  $\mathfrak{g}$ . On note  $N(\mathfrak{a})$  la sous-algèbre de  $\mathfrak{g}$  constituée d'éléments  $x$  vérifiant  $ad_x(\mathfrak{a}) \subset \mathfrak{a}$ . Si  $N(\mathfrak{a}) = \mathfrak{g}$ , alors on dira que  $\mathfrak{a}$  est un idéal de  $\mathfrak{g}$ .

**Définition 6.1.2.** On construit la suite  $(\mathfrak{g}^{(n)})_{n \in \mathbb{N}}$  définie par la relation

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \forall n \in \mathbb{N}, \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}].$$

On dira que  $\mathfrak{g}$  est résoluble s'il existe un entier  $n \in \mathbb{N}$  tel que  $\mathfrak{g}^{(n)} = 0$ . On notera  $rad(\mathfrak{g})$  (l'unique) idéal maximal résoluble de  $\mathfrak{g}$ .

**Définition 6.1.3.** Soit  $\mathfrak{g}$  une algèbre de Lie, on construit la suite  $(\mathfrak{g}^n)_{n \in \mathbb{N}}$

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \forall n \in \mathbb{N}, \quad \mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n].$$

On dira que  $\mathfrak{g}$  est nilpotente si il existe un entier  $n \in \mathbb{N}$  tel que  $\mathfrak{g}^n = 0$ .

**Définition 6.1.4.** Soit  $\mathfrak{g}$  une algèbre de Lie. On dit de  $\mathfrak{g}$  est semi-simple si le seul idéal résoluble de  $\mathfrak{g}$  est  $\{0\}$ . De plus, si  $\mathfrak{g}$  ne contient qu'un seul idéal, on dit que  $\mathfrak{g}$  est simple.

On donne maintenant quelques exemples fondamentaux d'algèbre de Lie semi-simple :

**Exemple 6.1.5.** Soit  $r \in \mathbb{N}^*$ ,

Le type  $A_r$  :  $A_r := \mathfrak{sl}_{r+1} = \{X \in \mathcal{M}_{r+1}(\mathbb{C}) \mid \text{tr}(X) = 0\}$ .

Le type  $B_r$  :  $B_r := \mathfrak{so}_{2r+1} = \{X \in \mathcal{M}_{2r+1}(\mathbb{C}) \mid {}^t X + X = 0\}$ .

Le type  $C_r$  :  $C_r := \mathfrak{sp}_r = \{X \in \mathcal{M}_{2r}(\mathbb{C}) \mid {}^t X J_r + J_r X = 0\}$ , avec

$$J_r := \begin{pmatrix} 0 & Id_r \\ -Id_r & 0 \end{pmatrix}.$$

Le type  $D_r$  :  $D_r := \mathfrak{so}_{2r} = \{X \in \mathcal{M}_{2r}(\mathbb{C}) \mid {}^t X + X = 0\}$ , pour  $r \geq 2$ .

Dans tous ces exemples, le crochet de Lie correspond à  $[x, y] := x \cdot y - y \cdot x$ , avec  $x, y \in \mathcal{M}_r$  et où  $\cdot$  correspond à la multiplication matricielle.

Dans l'exemple précédent, toutes ces algèbres de Lie sont simples, à part pour  $D_2$ . On a de plus la décomposition  $D_2 = A_1 \oplus A_1$ .

**Proposition 6.1.6.** Une algèbre de Lie semi-simple est une somme directe d'algèbres de Lie simples.

Ainsi, pour classifier les algèbres de Lie semi-simples, il suffit de classifier les algèbres de Lie simples. On verra via le Théorème de Cartan-Killing que les 4 types d'algèbres que l'on a décrites dans l'exemple précédent classifient presque totalement l'ensemble des algèbres de Lie semi-simples.

**Définition 6.1.7.** Une représentation d'une algèbre de Lie  $\mathfrak{g}$  sur un  $\mathbb{C}$ -espace vectoriel  $E$  correspond à un morphisme d'espace-vectoriel  $\varphi : \mathfrak{g} \rightarrow GL(E)$ . On appellera  $\dim(\varphi)$  la dimension de cette représentation. On dira que la représentation  $\varphi$  est irréductible si elle n'admet pas de sous-espace vectoriel non trivial stable par l'application linéaire  $\varphi$ .

**Exemple 6.1.8.** L'application adjointe induit une représentation d'une algèbre de Lie  $ad(\mathfrak{g})$  sur l'espace vectoriel  $\mathfrak{g}$  :

$$ad(\mathfrak{g}) : \mathfrak{g} \rightarrow GL(\mathfrak{g}), \quad x \mapsto ad_x.$$

On peut alors définir la forme bilinéaire de Killing sur  $\mathfrak{g} \times \mathfrak{g}$ :

$$\langle x, y \rangle := \text{Tr}(Ad_x \circ Ad_y).$$

**Théorème 6.1.9.** Soit  $\mathfrak{g}$  une algèbre de Lie. Les conditions suivantes sont équivalentes :

- $\mathfrak{g}$  est semi-simple.
- $rad(\mathfrak{g}) = 0$
- La forme de Killing  $\langle, \rangle$  est non dégénérée.

Le théorème précédent permet en particulier d'identifier une algèbre de Lie semi-simple avec son dual. Ainsi, à un élément  $x \in \mathfrak{g}$  on lui associera canoniquement un élément  $x^* \in \mathfrak{g}^*$ .

## 6.2 Système de racine d'une algèbre de Lie

Soit  $\mathfrak{g}$  une algèbre de Lie. On appelle sous-algèbre de Cartan une sous-algèbre de Lie nilpotente de  $\mathfrak{g}$  telle que  $N(\mathfrak{h}) = \mathfrak{h}$ . Les sous-algèbres de Cartan ne sont pas uniques, mais elles partagent la même dimension sur  $\mathbb{C}$ . On appelle alors cette dimension le rang de  $\mathfrak{g}$ , noté  $rg(\mathfrak{g})$ .

On se fixe pour tout cette section une algèbre de Lie semi-simple  $\mathfrak{g}$ , et une sous-algèbre de Cartan  $\mathfrak{h}$  de  $\mathfrak{g}$ . La forme de Killing s'étend à la sous-algèbre de Cartan  $\mathfrak{h}$ , et cette dernière est non dégénérée sur  $\mathfrak{h}$ . Via l'identification de  $\mathfrak{h}$  avec son dual, on peut définir la forme de Killing sur  $\mathfrak{h}^*$ .



**Définition 6.2.1.** Soit  $\alpha \in \mathfrak{h}^*$  un élément du dual de  $\mathfrak{h}$ . On dit que  $\alpha$  est une racine de  $\mathfrak{g}$  relativement à  $\mathfrak{h}$  lorsqu'il existe un élément non nul  $x \in \mathfrak{g} \setminus \{0\}$  tel que

$$\forall h \in \mathfrak{h}, \quad ad_h(x) = \alpha(h)x. \quad (6.1)$$

On notera  $g_\alpha$  l'ensemble des  $x \in \mathfrak{g}$  vérifiant 6.1 On notera  $\Delta(\mathfrak{g})$  l'ensemble des racines de  $\mathfrak{g}$  relativement à  $\mathfrak{h}$ . On appellera  $\Delta(\mathfrak{g})$  le système de racines de l'algèbre de Lie semi-simple  $\mathfrak{g}$ .

Le système de racines de  $\mathfrak{g}$  est un ensemble fini. De plus, les algèbres de Lie  $\mathfrak{g}_\alpha$  sont de dimension 1, et elles donnent une décomposition de  $\mathfrak{g}$  :

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} g_\alpha.$$

Par l'identification entre  $\mathfrak{h}$  et son dual, on considère pour toute racine  $\alpha \in \mathfrak{h}^*$  l'élément  $\alpha' \in \mathfrak{h}$  lui correspondant. On pose  $\mathfrak{h}_0 := \text{Vect}_{\mathbb{R}}((\alpha')_{\alpha \in \Delta(\mathfrak{g})})$  le  $\mathbb{R}$ -espace vectoriel engendré par les éléments  $\alpha'$ . On note alors

$$\alpha^\vee := \frac{2}{\langle \alpha', \alpha' \rangle} \alpha'.$$

Pour tout  $\alpha, \beta \in \Delta(\mathfrak{g})$ , on observe que

$$\beta(\alpha^\vee) = \frac{2 \langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

On note  $a(\beta, \alpha)$  la quantité précédente.

**Notation.** On note  $\mathfrak{h}_0$  le  $\mathbb{R}$ -espace vectoriel engendré par les racines  $\Delta(\mathfrak{g})$ , et on note  $\mathfrak{h}_0^*$  son dual.

**Fait 6.2.2.** Soit  $\alpha, \beta \in \Delta(\mathfrak{g})$ . On a que:

- 1)  $\beta(\alpha^\vee) \in \mathbb{Z}$ .
- 2)  $\beta - a(\beta, \alpha)\alpha \in \Delta(\mathfrak{g})$ .
- 3) Si il existe  $c \in \mathbb{C}$  tel que  $\beta = c\alpha$ , alors  $c = \pm 1$ .

Ces trois propriétés caractérisent un système de racines "abstrait" au sens de [Ser66] et de [Bou81]. Étant donné deux algèbres de Lie semi-simples, on sait qu'il existe une isométrie  $(\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_1)), \|\cdot\|_1) \rightarrow (\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_2)), \|\cdot\|_2)$  envoyant  $\Delta(\mathfrak{g}_1)$  sur  $\Delta(\mathfrak{g}_2)$  si et seulement si  $\mathfrak{g}_1$  est isomorphe à  $\mathfrak{g}_2$  en tant qu'algèbre de Lie, en notant  $\|\cdot\|_1$  et  $\|\cdot\|_2$  les normes euclidiennes des deux espaces vectoriels  $\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_1))$  et  $\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_2))$ . Cette propriété permet en particulier de ramener l'étude des algèbres de Lie simple à l'étude de système de racines. Il découle de cette correspondance le théorème suivant, permettant de classer les algèbres de Lie simples.

**Théorème 6.2.3.** [Bou81] [Théorème de Cartan-Killing]

Une algèbre de Lie simple quelconque est isomorphe soit à  $A_r$ ,  $B_r$ ,  $C_r$ , ou  $D_r$ , ou à une algèbre de Lie exceptionnelle  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$  ou  $E_8$ .

Pour voir la construction des algèbres de Lie exceptionnelles  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$  ou  $E_8$ , voir [Bou81].

Notons  $H_{\alpha^\vee} := \{\beta \in \mathfrak{h}_0^* \mid \langle \beta, \alpha^\vee \rangle = 0\}$

On remarque que, l'ensemble des racines étant fini, on a que l'ensemble

$$\mathfrak{h}_0^* \setminus \bigcup_{\alpha \in \Delta(\mathfrak{g})} H_{\alpha^\vee}$$

est non vide. Soit  $\gamma$  un élément de cet ensemble. On a alors la décomposition suivante :

$$\Delta(\mathfrak{g}) = \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle > 0\} \cup \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle < 0\}.$$

On notera  $\Delta_+(\mathfrak{g}, \gamma) = \Delta_+(\mathfrak{g}) := \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle > 0\}$  l'ensemble des racines positives.

**Définition 6.2.4.** Soit  $\alpha$  une racine positive. On dira que  $\alpha$  est une racine fondamentale (ou simple) s'il n'existe aucun couple  $(\beta_1, \beta_2) \in \Delta_+(\mathfrak{g})$  tel que  $\alpha = \beta_1 + \beta_2$ . On notera  $\psi(\mathfrak{g})$  l'ensemble des racines fondamentales.

**Fait 6.2.5.** L'ensemble des racines fondamentales forme une base du  $\mathbb{R}$ -espace vectoriel  $\mathfrak{h}_0^* = \text{Vect}_{\mathbb{R}}((\alpha')_{\alpha \in \Delta(\mathfrak{g})})$ . On obtient au passage qu'il y a  $r = \dim_{\mathbb{R}}(\mathfrak{h}_0)$  racines fondamentales.

On se propose maintenant d'introduire la notion de poids :

**Définition 6.2.6.** Soit  $\varphi : \mathfrak{g} \rightarrow GL(E)$  une représentation de l'algèbre de Lie semi-simple  $\mathfrak{g}$ , avec  $E$  de dimension finie. Soit  $u \in E$  un vecteur. On dira que  $u$  est un vecteur propre s'il existe une forme linéaire  $\lambda \in \mathfrak{h}^*$  sur  $\mathfrak{h}$  tel que

$$\forall H \in \mathfrak{h}, \quad \varphi(H)u = \lambda(H)u.$$

On appellera  $\lambda$  le poids de  $u$ .

Si de plus, on a  $\lambda(\alpha^\vee) \geq 0$  pour toute racine simple  $\alpha$  de  $\mathfrak{g}$ , alors on dira que  $\lambda$  est un poids dominant.

Notons  $\psi(\mathfrak{g}) = \{\alpha_1, \dots, \alpha_r\}$  l'ensemble des racines simples de  $\mathfrak{g}$ . On pose alors  $\lambda_j(\alpha_i^\vee) := \delta_{i,j}$  pour tout  $1 \leq i, j \leq r$ . Les  $\lambda_j$  sont en particulier des poids dominants. On les appelle des poids fondamentaux (ou simples).

**Fait 6.2.7.** Chaque poids dominant  $\lambda$  admet une décomposition unique à l'aide des poids fondamentaux

$$\lambda = n_1 \lambda_1 + \dots + n_r \lambda_r \quad (n_1, \dots, n_r \in \mathbb{N}).$$

**Fait 6.2.8.** Il y a une correspondance bijective entre l'ensemble des représentations irréductibles de dimension finie (à isomorphisme près), et l'ensemble des poids dominants.

**Théorème 6.2.9.** [de Weyl]

Soit  $\varphi$  une représentation irréductible de  $\mathfrak{g}$ . Alors

$$\dim(\varphi) = \prod_{\alpha \in \Delta^+(\mathfrak{g})} \frac{\langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle}{\langle \alpha^\vee, \lambda_1 + \dots + \lambda_r \rangle}.$$

**Remarque 6.2.10.** A priori, les résultats énoncés dépendent du choix de la sous-algèbre de Cartan  $\mathfrak{h}$ , et du choix de  $\gamma$ . Cependant, d'après [KMT24, §2.3], ceux-ci sont indépendants de ces choix. La raison étant que, d'après [Hum72, §16.2], deux sous-algèbres de Cartan sont nécessairement conjuguées.

Notons au passage que le Fait 6.2.8 et le Théorème 6.2.9 permettent en particulier de déterminer une expression de  $r_{\mathfrak{g}}$ , le nombre de représentations irréductibles (à isomorphisme près) de dimension  $n$  d'une algèbre de Lie semi-simple  $\mathfrak{g}$ .

**Définition 6.2.11.** [Zag94] On associe à une algèbre de Lie semi-simple  $\mathfrak{g}$  une fonction zêta

$$\zeta_{\mathfrak{g}}(s) := \sum_{\varphi} \frac{1}{\dim(\varphi)^s},$$

où l'on somme sur les représentations irréductibles de dimension finie  $\varphi$  de  $\mathfrak{g}$  (à isomorphisme près).

Cette fonction zêta généralise les séries qui interviennent dans les formules de volume de Witten dans [Wit91].

Du Fait 6.2.8 et du Théorème de Weyl 6.2.9, on en déduit que les fonctions zêta de Witten s'écrivent sous la forme d'une série de produits de formes linéaires en  $r$  variables,  $r$  étant le nombre de racines fondamentales. Plus précisément, on a l'écriture

$$\zeta_{\mathfrak{g}}(s) := \prod_{\alpha \in \Delta^+(\mathfrak{g})} \langle \alpha^\vee, \lambda_1 + \dots + \lambda_r \rangle^s \sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{\alpha \in \Delta^+(\mathfrak{g})} \langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle^s}.$$

Matsumoto, Tsumura et Komori ont introduit dans [KMT10b] des analogues multizêta des fonctions zêta de Witten, en considérant des séries de Dirichlet de la forme

$$\sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{\alpha \in \Delta^+(\mathfrak{g})} \langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle^{s_\alpha}}.$$

Cela permet d'obtenir des formules récursives pour ces fonctions multizêta (voir [KMT10b, Théorème 3.1]), et d'obtenir plusieurs formules à certains multi-entiers positifs.

### 6.3 Quelques résultats sur $\zeta_{\mathfrak{g}_2}$ et $\zeta_{\mathfrak{so}(5)}$

On souhaite ici détailler les formules obtenues dans le Théorème A et dans le Théorème D pour les deux fonctions zêta de Witten  $\zeta_{\mathfrak{g}_2}$  et  $\zeta_{\mathfrak{so}(5)}$ , en étudiant principalement les valeurs en  $s = 0$ . Les valeurs spéciales  $\zeta_{\mathfrak{g}_2}(0)$  et  $\zeta_{\mathfrak{so}(5)}(0)$  sont très simples à calculer à l'aide du Corollaire A1. En revanche, les calculs des valeurs  $\zeta'_{\mathfrak{g}_2}(0)$  et  $\zeta'_{\mathfrak{so}(5)}(0)$  s'avèreront plus ardues. On se propose d'établir une formule en utilisant le Corollaire D2 lorsque  $P = 2$ , c'est-à-dire lorsque la série de Dirichlet décrivant  $Z(\mathbf{s}, \mathbf{s}')$  n'est qu'à deux variables  $n_1, n_2$ . On fixe de plus la direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ , et on pose  $d_1 = d_2 = 1$ . Commençons par étudier les valeurs de  $Z_\Delta(s)$  en les entiers négatifs par le Corollaire A2 :

**Lemme 6.3.1.** *Soit  $N \in \mathbb{N}$ . On a*

$$\begin{aligned} Z_\Delta(-N) &= \frac{(-1)^{(1+Q)N}}{Q} N! \sum_{k=0}^{QN} Q_N^0(\emptyset, (k, QN - k)) \frac{\zeta(-N - k)}{k!} \frac{\zeta(-(Q+1)N + k)}{(QN - k)!} \\ &\quad + \frac{(-1)^{QN+1}}{Q+1} N!^2 (Q_N^0(\{1\}, (Q+1)N + 1) + Q_N^0(\{2\}, (Q+1)N + 1)) \frac{\zeta(-(Q+2)N - 1)}{((Q+1)N + 1)!}, \end{aligned}$$

avec

$$\begin{aligned} Q_N^0(\emptyset, (k_1, k_2)) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q, w_{1,q} + w_{2,q} = N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}}, \quad (k_1 + k_2 = QN) \\ Q_N^0(\{1\}, (Q+1)N + 1) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}^Q \\ |\mathbf{w}_2|=(Q+1)N+1 \\ w_{2,j} \geq 2N+1}} \frac{(-N-1)\dots(N+1-w_{2,j})}{\prod_{q \neq j} w_{2,q}!} c_{j,2}^{w_{2,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_{2,q}} c_{q,2}^{w_{2,q}}, \\ Q_N^0(\{2\}, (Q+1)N + 1) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_1 \in \mathbb{N}^Q \\ |\mathbf{w}_1|=(Q+1)N+1 \\ w_{1,j} \geq 2N+1}} \frac{(-N-1)\dots(N+1-w_{1,j})}{\prod_{q \neq j} w_{1,q}!} c_{j,1}^{w_{1,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,2}^{N-w_{1,q}} c_{q,1}^{w_{1,q}}. \end{aligned}$$

*Démonstration.* Par le Corollaire A2, on a

$$\begin{aligned} Z_\Delta(-N) &= \sum_{\mathcal{P} \subseteq [1,2]} \frac{(-1)^{(|\mathcal{P}|+1+Q)N+|\mathcal{P}|}}{Q+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=(Q+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p)}{k!}, \\ &= \frac{(-1)^{(1+Q)N}}{Q} N! \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 = QN}} Q_N^0(\emptyset, (k_1, k_2)) \prod_{p=1}^2 \frac{\zeta(-N - k_p)}{k_p!} \\ &\quad + \frac{(-1)^{QN+1}}{Q+1} N!^2 (Q_N^0(\{1\}, (Q+1)N + 1) + Q_N^0(\{2\}, (Q+1)N + 1)) \frac{\zeta(-(Q+2)N - 1)}{((Q+1)N + 1)!}, \end{aligned}$$

avec

$$Q_N^0(\mathcal{P}, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathbb{I}^1, Q \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

- Si  $\mathcal{P} = \emptyset$  et  $k_1 + k_2 = QN$ , alors on a

$$Q_N^0(\emptyset, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q \neq j, w_{1,q} + w_{2,q} = N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}}.$$

Observons que

$$\begin{aligned} w_{1,j} + w_{2,j} &= \sum_{q=1}^Q w_{1,q} + w_{2,q} - \sum_{\substack{q=1 \\ q \neq j}}^Q w_{1,q} + w_{2,q}, \\ &= k_1 + k_2 - (Q-1)N, \\ &= N. \end{aligned}$$

Soit  $\mathbf{w} = (\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}^Q)^2$ . La condition  $|\mathbf{w}_1| = k_1, |\mathbf{w}_2| = k_2$ , et  $w_{1,q} + w_{2,q} = N$  pour tout  $q \neq j$ , est équivalente à la condition  $w_{1,q} + w_{2,q} = N$  pour tout  $q$  et  $|\mathbf{w}_1| = k_1, |\mathbf{w}_2| = k_2$ . On trouve alors

$$Q_N^0(\emptyset, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q, w_{1,q} + w_{2,q} = N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}}.$$

- Si  $\mathcal{P} = \{1\}$  et  $k_2 = (Q+1)N + 1$ , alors on a

$$Q_N^0(\{1\}, (Q+1)N + 1) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}^Q \\ |\mathbf{w}_2| = (Q+1)N + 1 \\ \forall q \neq j, w_{2,q} \leq N}} \left( \binom{k_1}{(w_{2,1}, \dots, w_{2,Q})} \prod_{q=1}^Q c_{q,2}^{w_{2,q}} \right) \left( \binom{-N-1}{\sum_{q \neq j} (N - w_{2,q})} \binom{\sum_{q \neq j} (N - w_{2,q})}{(N - w_{2,q})_{q \neq j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N - w_{2,q}} \right).$$

On remarque que

$$\begin{aligned} \sum_{\substack{q=1 \\ q \neq j}}^Q (N - w_{2,q}) &= (Q-1)N - |\mathbf{w}_2| + w_{2,j} \\ &= -2N - 1 + w_{2,j}. \end{aligned}$$

Si ce terme est strictement négatif, alors le coefficient binomial  $\binom{-N-1}{\sum_{q \neq j} (N - w_{2,q})}$  est nul. Ainsi, on peut imposer la condition  $w_{2,j} \geq 2N + 1$  :

$$Q_N^0(\{1\}, k_2) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}^Q \\ |\mathbf{w}_2| = (Q+1)N+1 \\ w_{2,j} \geq 2N+1}} \frac{(-N-1) \dots (N+1 - w_{2,j})}{\prod_{q \neq j} w_{2,q}!} c_{j,2}^{w_{2,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_{2,q}} c_{q,2}^{w_{2,q}}.$$

- Si  $\mathcal{P} = \{2\}$  et  $k_1 = (Q+1)N + 1$ , alors on a

$$Q_N^0(\{2\},) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_1 \in \mathbb{N}^Q \\ |\mathbf{w}_1| = (Q+1)N+1 \\ w_{1,j} \geq 2N+1}} \frac{(-N-1) \dots (N+1 - w_{1,j})}{\prod_{q \neq j} w_{1,q}!} c_{j,1}^{w_{1,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,2}^{N-w_{1,q}} c_{q,1}^{w_{1,q}}.$$

□

Via le lemme précédent et en utilisant le fait que  $\zeta(-n) = (-1)^{n+1} \frac{B_{n+1}}{n+1}$ , on peut simplifier les expressions pour qu'elles soient plus pratiques à calculer explicitement :

$$Z_{\Delta}(-N) = \frac{(-1)^{QN}}{Q} N!^Q \sum_{k=0}^{QN} \alpha_k(N) \frac{B_{N+k+1}}{(N+k+1)k!} \frac{B_{(Q+1)N-k+1}}{((Q+1)N-k+1)(QN-k)!} \quad (6.2)$$

avec

$$\alpha_k(N) = \sum_{j=1}^Q \sum_{\substack{(w_q)_{q \in [1, Q]} \in [0, N]^Q \\ w_1 + \dots + w_Q = k}} \binom{k}{(w_1, \dots, w_Q)} \binom{QN-k}{(N-w_1, \dots, N-w_Q)} \prod_{q=1}^Q c_{q,1}^{w_q} c_{q,2}^{N-w_q},$$

$$\beta(N) = \sum_{j=1}^Q \sum_{\substack{(w_q)_{q \in [1, Q]} \in [0, (Q+1)N+1]^Q \\ w_1 + \dots + w_Q = (Q+1)N+1 \\ w_j \geq 2N+1}} \frac{(-N-1) \dots (N+1 - w_j)}{\prod_{q \neq j} w_q!} \left( c_{j,2}^{w_j} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_q} c_{q,2}^{w_q} + c_{j,1}^{w_j} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,2}^{N-w_q} c_{q,1}^{w_q} \right).$$

### Lemme 6.3.2.

$$\begin{aligned} Z'_{\Delta}(0) = & Q\zeta(0)^2 \gamma - \zeta(-1) \left( \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \right) \gamma \\ & - \frac{1}{Q+1} \zeta(-1) \sum_{j=1}^Q \sum_{\substack{f=1 \\ f \neq j}}^Q \left( \frac{1}{c_{j,2}} (c_{f,1} - c_{j,1}) \ln \left( 1 + \frac{c_{f,2}}{c_{j,2}} \right) + \frac{1}{c_{j,1}} (c_{f,2} - c_{j,2}) \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right) \right) \\ & + \frac{\zeta'(0)\zeta(0)}{Q} - \frac{\zeta'(-1)}{Q+1} \sum_{j=1}^Q (c_{j,1} + c_{j,2}) - \gamma \zeta(0)^2 \sum_{j=1}^Q (c_{j,1} c_{j,2})^{-1} + \gamma \zeta(-1) \sum_{j=1}^Q (c_{j,1}^{-1} c_{j,2} + c_{j,2}^{-1} c_{j,1}) \\ & + \sum_{j=1}^Q \partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})|_{s=0}. \end{aligned}$$

*Démonstration.* Rappelons que  $Z'_{\Delta}(0) = Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{1}, \mathbf{1}}$ . Comme on a posé  $P = 2$  et que l'on a fixé les directions  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ , on a par le Corollaire D2 :

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{1}, \mathbf{1}} &= \\ & \sum_{\mathcal{P} \subsetneq [1, 2]} (-1)^{|\mathcal{P}|} \frac{1}{Q + |\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p)}{k_p!} \sum_{j=1}^Q \left( \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \gamma(1 + |\mathcal{P}|) + Q^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{\mathcal{P} \subsetneq [1, 2]} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{1}{Q + |\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \left( \sum_{i \in \mathcal{P}^c} \frac{\zeta'(-k_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p)}{k_p!} \right) \\ & + \sum_{j=1}^Q \partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})|_{s=0} - \sum_{j=1}^Q \gamma \sum_{\mathcal{P} \subsetneq [1, 2]} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-1} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p)}{k_p!}, \end{aligned}$$

avec

$$\begin{aligned} Q^1(j, \mathcal{P}, \mathbf{k}) &= \sum_{\substack{f=1 \\ f \neq j}}^Q \sum_{\mathbf{w}' = (w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} [0, k_p]} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} \frac{c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p}}{c_{j,p}^{k_p}} \right) \\ & + (Q - 1) \gamma \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}. \end{aligned}$$

en rappelant la notation  $\mathbf{e}_n = (0, \dots, 1, \dots, 0) \in \mathbb{R}^Q$  avec un 1 sur la  $n$ -ième composante et des 0 ailleurs.

En développant la somme présente dans l'expression de  $Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{1}, \mathbf{1}}$ , on obtient

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{1}, \mathbf{1}} &= \\ & \zeta(0)^2 \gamma - \frac{2\zeta(-1)}{Q+1} \gamma \left( \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \right) + \frac{1}{Q} \zeta(0)^2 \sum_{j=1}^Q Q^1(j, \emptyset, (0, 0)) \\ & - \frac{1}{Q+1} \zeta(-1) \sum_{j=1}^Q (Q^1(j, \{1\}, 1) + Q^1(j, \{2\}, 1)) + \frac{\zeta'(0)\zeta(0)}{Q} - \frac{\zeta'(-1)}{Q+1} \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \\ & + \sum_{j=1}^Q \partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})|_{s=0} - \gamma \zeta(0)^2 \sum_{j=1}^Q (c_{j,1} c_{j,2})^{-1} + \gamma \zeta(-1) \sum_{j=1}^Q (c_{j,1}^{-1} c_{j,2} + c_{j,2}^{-1} c_{j,1}). \end{aligned}$$

On étudie maintenant les valeurs de  $Q^1(j, \mathcal{P}, \mathbf{k})$  :

- Si  $\mathcal{P} = \emptyset$  et  $\mathbf{k} = (0, 0)$ , on a

$$Q^1(j, \emptyset, (0, 0)) = \sum_{\substack{f=1 \\ f \neq j}}^Q F_{\emptyset, j, f, \mathbf{0}}(\mathbf{0}, \mathbf{0}) + (Q - 1) \gamma.$$

D'après l'Exemple 1.3.4, on a que  $F_{\emptyset, j, f, \mathbf{0}}(\mathbf{0}, \mathbf{0}) = 0$  donc  $Q^1(j, \emptyset, (0, 0)) = (Q - 1) \gamma$ .

- Si  $\mathcal{P} = \{1\}$  et  $k_2 = 1$ , on a

$$\begin{aligned} Q^1(j, \{1\}, 1) &= (Q-1)\gamma c_{j,2} + \sum_{\substack{f=1 \\ f \neq j}}^Q \sum_{w'=0}^1 F_{\{1\},j,f,\mathbf{0},w'\mathbf{e}_f+(1-w')\mathbf{e}_j}(\mathbf{0},\mathbf{0}) \left( \binom{1}{w'} c_{f,2}^{w'} c_{j,2}^{1-w'} \right) \\ &= (Q-1)\gamma c_{j,2} + \sum_{\substack{f=1 \\ f \neq j}}^Q (F_{\{1\},j,f,\mathbf{0},\mathbf{e}_f}(\mathbf{0},\mathbf{0}) c_{f,2} + F_{\{1\},j,f,\mathbf{0},\mathbf{e}_j}(\mathbf{0},\mathbf{0}) c_{j,2}). \end{aligned}$$

On trouve à l'aide de l'Exemple 1.3.6 que

$$F_{\{1\},j,f,\mathbf{0},\mathbf{e}_j}(\mathbf{0},\mathbf{0}) = -\frac{1}{c_{j,1}} \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right),$$

et à l'aide de l'Exemple 1.3.5, on a que

$$F_{\{1\},j,f,\mathbf{0},\mathbf{e}_f}(\mathbf{0},\mathbf{0}) = \frac{1}{c_{f,1}} \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right),$$

On obtient alors

$$Q^1(j, \{1\}, 1) = (Q-1)\gamma c_{j,2} + \sum_{\substack{f=1 \\ f \neq j}}^Q \left( \frac{c_{f,2}}{c_{f,1}} - \frac{c_{j,2}}{c_{j,1}} \right) \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right).$$

- Si  $\mathcal{P} = \{1\}$  et  $k_1 = 1$ , mutatis mutandis, on trouve

$$Q^1(j, \{2\}, 1) = (Q-1)\gamma c_{j,1} + \sum_{\substack{f=1 \\ f \neq j}}^Q \left( \frac{c_{f,1}}{c_{f,2}} - \frac{c_{j,1}}{c_{j,2}} \right) \ln \left( 1 + \frac{c_{f,2}}{c_{j,2}} \right).$$

Des trois points précédents, on obtient

$$\begin{aligned} Z'(-(\mathbf{0},\mathbf{0}))_{1,1} &= \\ &= Q\zeta(0)^2\gamma - \frac{2}{Q+1}\zeta(-1) \left( \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \right) \gamma - \frac{Q-1}{Q+1}\zeta(-1) \left( \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \right) \gamma \\ &\quad - \frac{1}{Q+1}\zeta(-1) \sum_{j=1}^Q \sum_{\substack{f=1 \\ f \neq j}}^Q \left( \left( \frac{c_{f,1}}{c_{f,2}} - \frac{c_{j,1}}{c_{j,2}} \right) \ln \left( 1 + \frac{c_{f,2}}{c_{j,2}} \right) + \left( \frac{c_{f,2}}{c_{f,1}} - \frac{c_{j,2}}{c_{j,1}} \right) \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right) \right) \\ &\quad + \frac{\zeta'(0)\zeta(0)}{Q} - \frac{\zeta'(-1)}{Q+1} \sum_{j=1}^Q (c_{j,1} + c_{j,2}) - \gamma\zeta(0)^2 \sum_{j=1}^Q (c_{j,1}c_{j,2})^{-1} + \gamma\zeta(-1) \sum_{j=1}^Q (c_{j,1}^{-1}c_{j,2} + c_{j,2}^{-1}c_{j,1}) \\ &\quad + \sum_{j=1}^Q \partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})|_{s=0}. \end{aligned}$$

□

**Remarque 6.3.3.** Avec les hypothèses du lemme précédent, et en fournissant l'entier  $Q \geq 1$  et les

coefficients  $\mathbf{c} = (c_{q,p})_{q \in [1,Q], p \in [1,2]}$ , le code SAGE 7.1 permet de simplifier l'expression

$$\begin{aligned} & Q\zeta(0)^2\gamma - \frac{Q-1}{Q+1}\zeta(-1) \left( \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \right) \gamma \\ & - \frac{1}{Q+1}\zeta(-1) \sum_{j=1}^Q \sum_{\substack{f=1 \\ f \neq j}}^Q \left( \left( \frac{c_{f,1}}{c_{f,2}} - \frac{c_{j,1}}{c_{j,2}} \right) \ln \left( 1 + \frac{c_{f,2}}{c_{j,2}} \right) + \left( \frac{c_{f,2}}{c_{f,1}} - \frac{c_{j,2}}{c_{j,1}} \right) \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right) \right) \\ & + \frac{\zeta'(0)\zeta(0)}{Q} - \frac{\zeta'(-1)}{Q+1} \sum_{j=1}^Q (c_{j,1} + c_{j,2}) - \gamma\zeta(0)^2 \sum_{j=1}^Q (c_{j,1}c_{j,2})^{-1} + \gamma\zeta(-1) \sum_{j=1}^Q (c_{j,1}^{-1}c_{j,2} + c_{j,2}^{-1}c_{j,1}). \end{aligned}$$

Notons au passage que le résultat du code n'est pas une approximation, mais s'exprime en fonction de la constante d'Euler  $\gamma$ ,  $\zeta(0)$ ,  $\zeta(-1)$ ,  $\zeta'(0)$ ,  $\zeta'(-1)$  et de certains logarithmes d'entiers.

Même si on dispose d'une formule pour calculer les valeurs aux entiers négatifs de la dérivée suivant  $s$  de la fonction zêta de Barnes  $\partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})|_{s=0}$  via le Corollaire B1, en posant  $N = 0$  et  $\mathbf{R} = \mathbf{0}$ , on utilisera plutôt le Théorème 1.2.11 démontré par Aoki et Sakane. La formule de ce théorème donne des formules plus facilement exploitable. En particulier, on utilisera les formules obtenues dans l'Exemple 1.2.12 pour simplifier les expressions de  $\zeta'_{g_2}(0)$  et de  $\zeta'_{s\sigma(5)}(0)$ . On simplifiera des termes de cet exemple via les formules (1.4), (1.3) et (1.2).

$$\begin{aligned} \zeta^{B'}(0, 2|1, 1) &= -\zeta'(0) + \zeta'(-1), \\ &= \frac{1}{2} \ln(2\pi) + \frac{1}{12} - \ln(A) \end{aligned} \tag{6.3}$$

$$\begin{aligned} \zeta^{B'}(0, 3|1, 2) &= -\frac{5}{8} \ln(2) - \frac{1}{4} \ln(\pi) - \frac{3}{2} \zeta'(0) + \frac{1}{2} \zeta'(-1), \\ &= \frac{1}{8} \ln(2) + \frac{1}{2} \ln(\pi) + \frac{1}{24} - \frac{\ln(A)}{2} \end{aligned} \tag{6.4}$$

$$\begin{aligned} \zeta^{B'}(0, 4|1, 3) &= -\frac{19 \ln(3)}{36} - \zeta'(0) - \frac{2}{3} \zeta' \left( 0, \frac{5}{3} \right) - \frac{1}{3} \zeta' \left( 0, \frac{4}{3} \right) + \zeta'(-1) + \zeta' \left( -1, \frac{5}{3} \right) + \zeta' \left( -1, \frac{4}{3} \right), \end{aligned} \tag{6.5}$$

$$\begin{aligned} \zeta^{B'}(0, 5|2, 3) &= -\frac{31 \ln(6)}{72} - \frac{\ln(2)}{6} - \frac{1}{4} \ln(\pi) - \frac{3}{2} \zeta'(0) - \frac{2}{3} \zeta' \left( 0, \frac{5}{3} \right) - \frac{1}{6} \zeta' \left( 0, \frac{7}{6} \right) + \frac{1}{6} \zeta' \left( 0, \frac{5}{6} \right) \\ &+ \frac{1}{2} \zeta'(-1) + \zeta' \left( -1, \frac{5}{3} \right) + \zeta' \left( -1, \frac{4}{3} \right) + \zeta' \left( -1, \frac{7}{6} \right) + \zeta' \left( -1, \frac{5}{6} \right), \end{aligned} \tag{6.6}$$

on a notamment utilisé le fait que

$$\begin{aligned} \zeta' \left( 0, \frac{3}{2} \right) &= \zeta'(0) - \ln(2) + \ln \Gamma \left( \frac{1}{2} \right), \\ &= \zeta'(0) + \ln(\sqrt{\pi}) - \ln(2), \end{aligned}$$

et que

$$\begin{aligned} \zeta' \left( -1, \frac{3}{2} \right) &= \zeta' \left( -1, \frac{1}{2} \right) - \frac{1}{2} \ln(2) \\ &= -\frac{\ln(2)}{6} - \frac{1}{2} \zeta'(-1) - \frac{1}{2} \ln(2) \\ &= -\frac{2}{3} \ln(2) - \frac{1}{2} \zeta'(-1). \end{aligned}$$



### 6.3.1 Pôles et résidus de $\zeta_{\mathfrak{g}_2}$

Via [KMT11a], on a vu en 1.6 que  $\zeta_{\mathfrak{g}_2}$  a des pôles de la forme  $s = \frac{1}{3}$  et en  $s = \frac{1-k}{5}$ , avec  $k \geq 0$  et  $k \neq 1 \pmod{5}$ . On souhaite ici donner une expression des résidus de  $\zeta_{\mathfrak{g}_2}$  en  $s = \frac{1}{5}$  et en  $s = \frac{1}{3}$ . On pose

$$l_1(x_1, x_2) = x_1 + x_2, \quad l_2(x_1, x_2) = x_1 + 2x_2, \quad l_3(x_1, x_2) = x_1 + 3x_2, \quad l_4(x_1, x_2) = 2x_1 + 3x_2$$

$$d_1 = 1, \quad d_2 = 1, \quad d'_1 = 2, \quad d'_2 = 3, \quad d'_3 = 4, \quad d'_4 = 5.$$

Via la Proposition 2.2.4, on trouve que

$$\zeta_{\mathfrak{g}_2}(s) = K(\theta, s) + J(\theta, s),$$

avec

$$K(\theta, s) = 120^{-s} \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1,4 \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \frac{1}{\Gamma(s)^{4 - |A_{\mathcal{Q}}|}} \left( \sum_{\mathbf{n} \in \mathbb{N}^2} (n_1 + 1)^{-s} (n_2 + 1)^{-s} \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n} + 1))^{-s} \prod_{q \in \llbracket 1,4 \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s, \theta, l_q(\mathbf{n} + 1)) \right),$$

et

$$J(\theta, s) = 120^s \sum_{\mathcal{P} \subsetneq \llbracket 1,2 \rrbracket} \Gamma(1-s)^{|\mathcal{P}|} \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^4 h_{\mathcal{P}, j, \mathbf{k}}(s) \right) \frac{\theta^{(4+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(s)((4+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(s - k_p)}{k_p!}$$

$$+ 120^s \Gamma(1-s)^2 \left( \sum_{j=1}^4 h_{\llbracket 1,2 \rrbracket, j}(s) \right) \frac{\theta^{6s-2}}{\Gamma(s)(6s-2)}, \quad (6.7)$$

avec

$$h_{\mathcal{P}, 1, \mathbf{k}}(s) = \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_2 x_3 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(1, x_2, x_3, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(1, x_2, x_3, x_4)^{k_p} dx_2 dx_3 dx_4,$$

$$h_{\mathcal{P}, 2, \mathbf{k}}(s) = \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_3 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, 1, x_3, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, 1, x_3, x_4)^{k_p} dx_1 dx_3 dx_4,$$

$$h_{\mathcal{P}, 3, \mathbf{k}}(s) = \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, x_2, 1, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, x_2, 1, x_4)^{k_p} dx_1 dx_2 dx_4,$$

$$h_{\mathcal{P}, 4, \mathbf{k}}(s) = \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_3)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, x_2, x_3, 1)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, x_2, x_3, 1)^{k_p} dx_1 dx_2 dx_3,$$

$$l_1^*(x_1, x_2) = x_1 + x_2 + x_3 + 2x_4,$$

$$l_2^*(x_1, x_2) = x_1 + 2x_2 + 3x_3 + 4x_4,$$

et où on a également

$$\begin{aligned} \sum_{j=1}^4 h_{[1,2],j}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_3)^{s-1} \prod_{p=1}^2 l_p^*(x_1, x_2, x_3, 1)^{s-1} dx_1 dx_2 dx_3, \\ &+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_4)^{s-1} \prod_{p=1}^2 l_p^*(x_1, x_2, 1, x_4)^{s-1} dx_1 dx_2 dx_4 \\ &+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_3 x_4)^{s-1} \prod_{p=1}^2 l_p^*(x_1, 1, x_3, x_4)^{s-1} dx_1 dx_3 dx_4 \\ &+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_2 x_3 x_4)^{s-1} \prod_{p=1}^2 l_p^*(1, x_2, x_3, x_4)^{s-1} dx_2 dx_3 dx_4. \end{aligned}$$

On observe que  $K(\theta, s)$  est holomorphe sur  $\mathbb{C}$ , donc ne contribue pas aux résidus. Il ne reste qu'à étudier les dénominateurs des termes présents dans la formule 6.7.

- Soit  $\mathcal{P} \subsetneq [1, 2]$ , et  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$ . Pour  $s = \frac{1}{5}$ , on a

$$(4 + \mathcal{P})s - |\mathcal{P}| + |\mathbf{k}| = 0 \text{ si et seulement si } |\mathcal{P}| = 1, \mathbf{k} = 0.$$

On trouve alors

$$\begin{aligned} \text{Res}_{s=\frac{1}{5}}(\zeta_{\mathfrak{g}_2}(s)) &= 120^{\frac{1}{5}} \frac{\Gamma(\frac{4}{5})}{\Gamma(\frac{1}{5})} \left( \sum_{j=1}^4 h_{\{1\},j,0}\left(\frac{1}{5}\right) + h_{\{2\},j,0}\left(\frac{1}{5}\right) \right) \zeta\left(\frac{1}{5}\right), \\ &= 120^{\frac{1}{5}} \frac{\Gamma(\frac{4}{5})}{\Gamma(\frac{1}{5})^4} \zeta\left(\frac{1}{5}\right) \left( \int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{4}{5}} (l_1^*(1, x_2, x_3, x_4)^{-\frac{4}{5}} + l_2^*(1, x_2, x_3, x_4)^{-\frac{4}{5}}) dx_2 dx_3 dx_4 \right. \\ &+ \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{4}{5}} (l_1^*(x_1, 1, x_3, x_4)^{-\frac{4}{5}} + l_2^*(x_1, 1, x_3, x_4)^{-\frac{4}{5}}) dx_1 dx_3 dx_4 \\ &+ \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{4}{5}} (l_1^*(x_1, x_2, 1, x_4)^{-\frac{4}{5}} + l_2^*(x_1, x_2, 1, x_4)^{-\frac{4}{5}}) dx_1 dx_2 dx_4 \\ &\left. + \int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{4}{5}} (l_1^*(x_1, x_2, x_3, 1)^{-\frac{4}{5}} + l_2^*(x_1, x_2, x_3, 1)^{-\frac{4}{5}}) dx_1 dx_2 dx_3 \right). \end{aligned}$$

- Via un raisonnement similaire, on trouve

$$\begin{aligned} \text{Res}_{s=\frac{1}{3}}(\zeta_{\mathfrak{g}_2}(s)) &= 120^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})^4} \left( \sum_{j=1}^4 h_{\{1,2\},j}\left(\frac{1}{3}\right) + h_{\{1,2\},j}\left(\frac{1}{5}\right) \right) \zeta\left(\frac{1}{3}\right), \\ &= 120^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})} \left( \int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, x_3, 1)^{-\frac{2}{3}} dx_1 dx_2 dx_3, \right. \\ &+ \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, 1, x_4)^{-\frac{2}{3}} dx_1 dx_2 dx_4 \\ &+ \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, 1, x_3, x_4)^{-\frac{2}{3}} dx_1 dx_3 dx_4 \\ &\left. + \int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(1, x_2, x_3, x_4)^{-\frac{2}{3}} dx_2 dx_3 dx_4 \right). \end{aligned}$$

### 6.3.2 Valeurs aux entiers négatifs de $\zeta_{\mathfrak{g}_2}$ et de $\zeta'_{\mathfrak{g}_2}$

On reprend les notations de la sous-section précédente :

$$l_1(x_1, x_2) = x_1 + x_2, \quad l_2(x_1, x_2) = x_1 + 2x_2, \quad l_3(x_1, x_2) = x_1 + 3x_2, \quad l_4(x_1, x_2) = 2x_1 + 3x_2$$

$$d_1 = 1, \quad d_2 = 1, \quad d'_1 = 2, \quad d'_2 = 3, \quad d'_3 = 4, \quad d'_4 = 5.$$

La fonction zêta  $\zeta_{\mathfrak{g}_2}$  est de la forme

$$\zeta_{\mathfrak{g}_2}(s) = 120^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_2)^s (2n_1 + 3n_2)^s} = 120^s Z_{\Delta}(s).$$

On a en particulier que

$$\zeta_{\mathfrak{g}_2}(-N) = 120^{-N} Z(-((N, N), (N, N, N, N)), \quad \zeta'_{\mathfrak{g}_2}(-N) = 120^{-N} Z'(-((N, N), (N, N, N, N)) + \ln(120)\zeta_{\mathfrak{g}_2}(-N)).$$

(1,1),(1,1,1,1) \qquad \qquad \qquad (1,1),(1,1,1,1)

Via le Corollaire A1, on obtient alors

$$\zeta_{\mathfrak{g}_2}(0) = \sum_{\mathcal{P} \subsetneq [1,2]} \frac{(-1)^{|\mathcal{P}|}}{4 + |\mathcal{P}|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \left( \sum_{j=1}^4 \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \frac{\zeta(-k_p)}{k_p!}.$$

1) Si  $\mathcal{P} = \emptyset$  et  $|\mathbf{k}| = 0$ , alors  $\mathbf{k} = (0, 0)$ , et

$$\sum_{j=1}^4 \prod_{p=1}^2 c_{j,p}^0 = 4.$$

2) Si  $\mathcal{P} = \{1\}$  et  $|\mathbf{k}| = 1$ , alors  $\mathbf{k} = k_2 = 1$ , et

$$\sum_{j=1}^4 c_{j,2}^1 = 9.$$

3) Si  $\mathcal{P} = \{2\}$  et  $|\mathbf{k}| = 1$ , alors  $\mathbf{k} = k_1 = 1$ , et

$$\sum_{j=1}^4 c_{j,1}^1 = 5.$$

En utilisant le fait que  $\zeta(0) = -\frac{1}{2}$ , et que  $\zeta(-1) = -\frac{1}{12}$ , on obtient alors la valeur de  $\zeta_{\mathfrak{g}_2}$  en  $s = 0$  :

$$\zeta_{\mathfrak{g}_2}(0) = \zeta(0)^2 - \frac{14}{5}\zeta(-1)$$

$$= \frac{29}{60}.$$

On peut également fournir une expression des valeurs aux entiers négatifs en général. Par le Corollaire A2, on a

$$\zeta_{\mathfrak{g}_2}(-N) = 120^{-N} \sum_{\mathcal{P} \subsetneq [1,2]} \frac{(-1)^{(|\mathcal{P}|+5)N+|\mathcal{P}|}}{4 + |\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=(4+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p)}{k_p!},$$

avec

$$Q_N^0(\mathcal{P}, \mathbf{k}) = (-1)^{3N} N!^3 \sum_{j=1}^4 \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^4)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\{1,4\} \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^4 c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^4 c_{q,p}^{v_{p,q}} \right).$$

Pour calculer  $\zeta'_{\mathfrak{g}_2}(0)$ , on remarque que

$$\zeta'_{\mathfrak{g}_2}(0) = \ln(120)\zeta_{\mathfrak{g}_2}(0) + Z'_{\Delta}(0).$$

En appliquant le Lemme 6.3.2, et en utilisant le code SAGE 7.1 on obtient

$$\begin{aligned} \zeta'_{\mathfrak{g}_2}(0) &= \ln(120)\zeta_{\mathfrak{g}_2}(0) + 2\gamma\zeta(0)^2 + \frac{1}{4}\zeta(0)\zeta'(0) - 4\gamma\zeta(-1) + \frac{1}{5}\zeta(-1)(\ln(3) + 2\ln(2)) - \frac{14}{5}\zeta'(-1) \\ &\quad + \zeta^{B'}(0, 2|1, 1) + \zeta^{B'}(0, 3|1, 2) + \zeta^{B'}(0, 4|1, 3) + \zeta^{B'}(0, 5|2, 3). \end{aligned}$$

Par les formules 6.3, 6.4, 6.5 et 6.6, on trouve que

$$\begin{aligned} &\zeta^{B'}(0, 2|1, 1) + \zeta^{B'}(0, 3|1, 2) + \zeta^{B'}(0, 4|1, 3) + \zeta^{B'}(0, 5|2, 3) = \\ &\quad - \zeta'(0) + \zeta'(-1) - \frac{5}{8}\ln(2) - \frac{1}{4}\ln(\pi) - \frac{3}{2}\zeta'(0) + \frac{1}{2}\zeta'(-1) \\ &\quad - \frac{19\ln(3)}{36} - \zeta'(0) - \frac{2}{3}\zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3}\zeta'\left(0, \frac{4}{3}\right) + \zeta'(-1) + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) \\ &\quad - \frac{31\ln(6)}{72} - \frac{\ln(2)}{6} - \frac{1}{4}\ln(\pi) - \frac{3}{2}\zeta'(0) - \frac{2}{3}\zeta'\left(0, \frac{5}{3}\right) - \frac{1}{6}\zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6}\zeta'\left(0, \frac{5}{6}\right) \\ &\quad + \frac{1}{2}\zeta'(-1) + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) + \zeta'\left(-1, \frac{7}{6}\right) + \zeta'\left(-1, \frac{5}{6}\right). \end{aligned}$$

En simplifiant ces termes, on trouve :

$$\begin{aligned} \zeta'_{\mathfrak{g}_2}(0) &= \ln(120)\zeta_{\mathfrak{g}_2}(0) + 2\gamma\zeta(0)^2 + \frac{1}{4}\zeta(0)\zeta'(0) - 4\gamma\zeta(-1) + \frac{1}{5}\zeta(-1)(\ln(3) + 2\ln(2)) - \frac{14}{5}\zeta'(-1) \\ &\quad - \frac{11}{9}\ln(2) - \frac{23}{24}\ln(3) - \frac{1}{2}\ln(\pi) - 5\zeta'(0) + 3\zeta'(-1) - \frac{4}{3}\zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3}\zeta'\left(0, \frac{4}{3}\right) \\ &\quad + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) - \frac{1}{6}\zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6}\zeta'\left(0, \frac{5}{6}\right). \end{aligned}$$

### 6.3.3 Pôles et résidus de $\zeta_{\mathfrak{so}(5)}$

Via [KMT10d], on a vu en 1.5 que  $\zeta_{\mathfrak{so}(5)}$  a des pôles de la forme  $s = \frac{1}{2}$  et en  $s = \frac{1-k}{3}$ , pour  $k \geq 0$  un entier quelconque tel que  $k \not\equiv 1 \pmod{3}$ . On se propose ici d'obtenir des expressions des résidus en  $s = \frac{1}{3}$  et en  $s = \frac{1}{2}$  pour  $\zeta_{\mathfrak{so}(5)}(s)$ . Notons que Bridges, Brindle, Bringmann, et Franke ont déjà obtenu une expression très simple de ces deux résidus dans [BBBF23, Proposition 5.16] via des techniques différentes.

On sait que, pour  $\theta > 0$  suffisamment petit, la fonction zêta  $\zeta_{\mathfrak{so}(5)}$  admet la décomposition  $\zeta_{\mathfrak{so}(5)}(s) = K(\theta, s) + J(\theta, s)$ , avec  $K(\theta, s)$  une fonction holomorphe en  $s$  sur tout  $\mathbb{C}$ , et  $J(\theta, s)$  une fonction méromorphe

en  $s$  sur tout  $\mathbb{C}$ , ayant les mêmes pôles que  $\zeta_{\mathfrak{so}(5)}(s)$ . Par la Proposition 2.2.4, on a

$$\begin{aligned} J(\theta, s) = & 6^s \sum_{\mathcal{P} \subsetneq [1, 2]} \Gamma(1-s)^{|\mathcal{P}|} \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^2 h_{\mathcal{P}, j, \mathbf{k}}(s) \right) \frac{\theta^{(2+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(s)((2+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(s - k_p)}{k_p!} \\ & + 6^s \Gamma(1-s)^2 \left( \sum_{j=1}^2 h_{[1, 2], j, \mathbf{0}}(s) \right) \frac{\theta^{4s-2}}{\Gamma(s)(4s-2)}, \end{aligned} \quad (6.8)$$

avec

$$\begin{aligned} h_{\mathcal{P}, 1, \mathbf{k}}(s) &= \frac{1}{\Gamma(s)} \int_0^1 x_2^{s-1} \prod_{p \in \mathcal{P}} l_p^*(1, x_2)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(1, x_2)^{k_p} dx_2, \\ h_{\mathcal{P}, 2, \mathbf{k}}(s) &= \frac{1}{\Gamma(s)} \int_0^1 x_1^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, 1)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, 1)^{k_p} dx_1, \\ l_1^*(x_1, x_2) &= x_1 + x_2, \\ l_2^*(x_1, x_2) &= x_1 + 2x_2. \end{aligned}$$

Comme  $K(\theta, s)$  est holomorphe en  $s$  sur  $\mathbb{C}$ , on devrait pouvoir lire les résidus de  $\zeta_{\mathfrak{so}(5)}(s)$  sur l'expression de  $J(\theta, s)$  ci-dessus. En  $s = 1/2$ , le seul terme contribuant au résidu de  $\zeta_{\mathfrak{so}(5)}(s)$  est

$$\Gamma(1-s)^2 \left( \sum_{j=1}^2 h_{[1, 2], j, \mathbf{0}}(s) \right) \frac{\theta^{4s-2}}{\Gamma(s)(4s-2)}. \text{ On obtient alors une expression du résidu en } s = 1/2 :$$

$$\begin{aligned} \text{Res}_{s=1/2}(\zeta_{\mathfrak{so}(5)}(s)) &= \sqrt{6} \frac{\Gamma(3/2)^2}{4\Gamma(1/2)} \left( \sum_{j=1}^2 h_{[1, 2], j, \mathbf{0}}(1/2) \right), \\ &= \sqrt{6} \frac{\Gamma(3/2)^2}{4\Gamma(1/2)} \frac{1}{\Gamma(1/2)} \left( \int_0^1 x^{-1/2}(x+1)^{-1/2}(x+2)^{-1/2} dx + \int_0^1 x^{-1/2}(1+x)^{-1/2}(1+2x)^{-1/2} dx \right) \\ &= \sqrt{6} \frac{\Gamma(3/2)^2}{4\Gamma(1/2)} \frac{1}{\Gamma(1/2)} \left( \int_0^1 x^{-1/2}(x+1)^{-1/2}(x+2)^{-1/2} dx + \int_0^1 x^{-1/2}(1+x)^{-1/2}(1+2x)^{-1/2} dx \right). \end{aligned}$$

Pour les pôles de la forme  $\frac{n}{3}$ , avec  $n \leq 1$  un entier tel que  $n \neq 0 \pmod{3}$ , les seuls termes contribuant au résidu en  $s = \frac{n}{3}$  correspondent aux termes de la forme

$$-\Gamma(1-n/3)^{|\mathcal{P}|} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^2 h_{\mathcal{P}, j, \mathbf{k}}(1-n/3) \right) \frac{1}{\Gamma(n/3)((2+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(n/3 - k_p)}{k_p!},$$

avec  $\mathcal{P} \subsetneq [1, 2]$ ,  $\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}$  vérifiant la condition  $(2+|\mathcal{P}|)\frac{n}{3} - |\mathcal{P}| + |\mathbf{k}| = 0$ . Cette condition implique nécessairement que  $\mathcal{P} = \{i\}$  est un singleton, et en notant  $\mathcal{P}^c = \{p\}$ , on a  $|k| = k_p = 1 - n \in \mathbb{N}$ . On trouve alors

$$\text{Res}_{s=n/3}(\zeta_{\mathfrak{so}(5)}(s)) = (-1)^{1-n} \Gamma\left(\frac{3-n}{3}\right) \left( \sum_{i=1}^2 \sum_{j=1}^2 h_{\{i\}, j, \mathbf{k}}\left(1 - \frac{n}{3}\right) \right) \frac{1}{3\Gamma\left(\frac{n}{3}\right)} \frac{\zeta\left(\frac{4n}{3} - 1\right)}{(1-n)!},$$

Notons que, en considérant les termes généraux de la série présente dans 6.8 avec  $\mathcal{P} = \emptyset$ , on remarque des termes de la forme  $\frac{\theta^{2s+|\mathbf{k}|}}{\Gamma(s)(2s+|\mathbf{k}|)}$ , avec  $\mathbf{k} = (k_1, k_2) \in \mathbb{N}$ . Les zéros de  $2s + |\mathbf{k}|$  ne constituent pas des pôles puisque ces pôles sont annulés par les zéros des termes de la forme  $\sum_{j=1}^2 h_{\emptyset, j, \mathbf{k}}$ . Par exemple,

en  $s = -\frac{1}{2}$ , on a que le terme  $2s + |\mathbf{k}|$  est nul si et seulement si  $|\mathbf{k}| = k_1 + k_2 = 1$ , c'est-à-dire lorsque  $\mathbf{k} \in \{(1, 0), (0, 1)\}$ . Or on remarque que

$$\begin{aligned} h_{1,\emptyset,(1,0)}(s) + h_{2,\emptyset,(1,0)}(s) + h_{1,\emptyset,(0,1)}(s) + h_{2,\emptyset,(0,1)}(s) &= \frac{1}{\Gamma(s)} \int_0^1 (x^{s-1}(1+2x) + 2x^{s-1}(1+x) + x^{s-1}(2+x)) dx \\ &= \frac{5}{\Gamma(s)} \left( \frac{1}{s+1} + \frac{1}{s} \right), \end{aligned}$$

et en  $s = -\frac{1}{2}$ , ce terme s'annule. Cela est cohérent avec la liste des pôles possibles donnée par Komori, Matsumoto et Tsumura dans [KMT10d, Théorème 6.2].

### 6.3.4 Valeurs aux entiers négatifs de $\zeta_{\mathfrak{so}(5)}$ et $\zeta'_{\mathfrak{so}(5)}$

On pose

$$\begin{aligned} l_1(x_1, x_2) &= x_1 + x_2, \quad l_2(x_1, x_2) = x_1 + 2x_2, \\ d_1 &= 1, \quad d_2 = 1, \quad d'_1 = 2, \quad d'_2 = 3, \end{aligned}$$

alors  $\zeta_{\mathfrak{so}(5)}(s)$  est de la forme

$$\zeta_{\mathfrak{so}(5)}(s) = 6^s \sum_{\mathbf{n} \in \mathbb{N}^2} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s} = 6^s Z_{\Delta}(s).$$

On a en particulier que

$$\zeta_{\mathfrak{so}(5)}(-N) = 6^{-N} Z(-((N, N), (N, N))), \quad \zeta'_{\mathfrak{g}_2}(-N) = 120^{-N} Z'(-((N, N), (N, N)) + \ln(6)\zeta_{\mathfrak{g}_2}(-N)).$$

(1,1),(1,1) \qquad\qquad\qquad (1,1),(1,1)

En  $N = 0$ , on trouve par le Corollaire A1 :

$$\begin{aligned} \zeta_{\mathfrak{so}(5)}(0) &= \sum_{\mathcal{P} \subseteq [1,2]} \frac{(-1)^{|\mathcal{P}|}}{2 + |\mathcal{P}|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathcal{P}|=|\mathbf{k}|}} \left( \sum_{j=1}^2 \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p)}{k_p!} \\ &= \zeta(0)^2 - \frac{5}{3} \zeta(-1) \\ &= \frac{7}{18}. \end{aligned}$$

On peut également fournir une expression des valeurs aux entiers négatifs en général. Par le Corollaire A2, on a

$$\zeta_{\mathfrak{so}(5)}(-N) = 6^{-N} \sum_{\mathcal{P} \subseteq [1,2]} \frac{(-1)^{(|\mathcal{P}|+3)N+|\mathcal{P}|}}{3 + |\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=(3+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p)}{k_p!},$$

avec

$$\begin{aligned} Q_N^0(\mathcal{P}, \mathbf{k}) &= N! \sum_{j=1}^2 \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^2)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1,2] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N}} \\ &\left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^2 c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^2 c_{q,p}^{v_{p,q}} \right). \end{aligned}$$

Enfin, par le Lemme 6.3.2, en s'aidant du Code SAGE 7.1 on a l'expression suivante de sa dérivée en 0 :

$$\begin{aligned} \zeta'_{\mathfrak{so}(5)}(0) &= \ln(6)\zeta_{\mathfrak{so}(5)}(0) + \frac{1}{2}\gamma\zeta(0)^2 + \frac{1}{2}\zeta(0)\zeta'(0) - \frac{1}{2}\gamma\zeta(-1) - \frac{5}{3}\zeta'(-1) + \frac{1}{6}\ln(2) \\ &\quad + \partial_s \zeta^B(s, 2|1, 1)|_{s=0} + \partial_s \zeta^B(s, 3|1, 2)|_{s=0}, \end{aligned}$$

et avec les formules (6.3) et (6.4), on trouve :

$$\begin{aligned} \zeta'_{\mathfrak{so}(5)}(0) &= \ln(6)\zeta_{\mathfrak{so}(5)}(0) + \frac{1}{2}\gamma\zeta(0)^2 + \frac{1}{2}\zeta(0)\zeta'(0) - \frac{1}{2}\gamma\zeta(-1) - \frac{5}{3}\zeta'(-1) + \frac{1}{6}\ln(2) \\ &\quad - \zeta'(0) + \zeta'(-1) - \frac{5}{8}\ln(2) - \frac{1}{4}\ln(\pi) - \frac{3}{2}\zeta'(0) + \frac{1}{2}\zeta'(-1). \end{aligned}$$

En simplifiant ces termes, on trouve

$$\zeta'_{\mathfrak{so}(5)}(0) = \frac{7}{18}\ln(6) - \frac{11}{24}\ln(2) - \frac{1}{4}\ln(\pi) + \left(\frac{1}{2}\zeta(0)^2 - \frac{1}{2}\zeta(-1)\right)\gamma - \frac{11}{4}\zeta'(0) - \frac{13}{6}\zeta'(-1).$$

L'expression ci-dessus nous permet d'obtenir une expression explicite du coefficient  $C$  dans le Théorème 1.2.35.

## 6.4 Application à l'étude du comportement asymptotique de $r_{\mathfrak{g}_2}(n)$

Rappelons que

$$r_{\mathfrak{g}_2}(n) = \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|.$$

Posons  $P(i, j) := \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!}$ . On observe que l'on a

$$\prod_{i,j \geq 1} \frac{1}{1 - q^{P(i,j)}} = \prod_{i,j \geq 1} \sum_{k_{i,j} \geq 0} q^{k_{i,j} P(i,j)},$$

en développant ce produit infini, et en utilisant l'expression de l'ensemble  $r_{\mathfrak{g}_2}(n)$ , on trouve que :

$$\sum_{n=1}^{+\infty} r_{\mathfrak{g}_2}(n)q^n = \prod_{i,j \geq 1} \frac{1}{1 - q^{P(i,j)}}.$$

En posant  $f(n) = |\{(i, j) \in \mathbb{N}^{*2} \mid P(i, j) = n\}|$ , on obtient que

$$\sum_{n=1}^{+\infty} r_{\mathfrak{g}_2}(n)q^n = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f(n)}}.$$

On note désormais  $L_f(s) = \zeta_{\mathfrak{g}_2}(s)$ , et  $L_f^*(s) = \zeta_{\mathfrak{g}_2}(s)\Gamma(s)\zeta(s+1)$ . Observons que  $L_f^*(s)$  ne possède que deux pôles dans  $H_0$ , en  $s = \frac{1}{3}$  et en  $s = \frac{1}{5}$ . De plus,  $L_f^*(s)$  possède un pôle double en  $s = 0$ . Remarquons également que, grâce à la liste des pôles obtenues en 1.6, on a que  $L_f^*$  possède des pôles simples en  $s = \frac{1}{3}$  et en  $s = \frac{k}{5}$ ,  $k \in \mathbb{Z}_{\leq 1}$ , avec  $k \not\equiv 0 \pmod{5}$ .

On souhaite appliquer le Théorème 1.2.33. Vérifions tout d'abord si, avec ces données, les conditions du théorème sont valides. Posons  $\Lambda := \mathbb{N} \setminus f^{-1}(\{0\})$ . Remarquons que  $\frac{3}{2} \cdot \frac{1}{5} < \frac{1}{3} < \frac{2}{1} \cdot \frac{1}{5}$ . Ainsi, l'entier  $l = 2$  vérifie l'hypothèse du Théorème 1.2.33. Vérifions maintenant si les hypothèses (P1), (P2) et (P3) sont vérifiées :

(P1) Soit  $p \geq 2$  un nombre premier. On a que  $|\Lambda \setminus (p\mathbb{N} \cap \Lambda)| = +\infty$ . En effet, les suites

$$\begin{aligned} u(2) &:= (P(8k+1, 1))_{k \in \mathbb{N}}, \\ u(3) &:= (P(9k+1, 1))_{k \in \mathbb{N}}, \\ u(5) &:= (P(25k+1, 1))_{k \in \mathbb{N}}, \\ u(p) &:= (P(kp+1, 1))_{k \geq 1} \quad \text{si } p \geq 7, \end{aligned}$$

sont strictement croissantes, et appartiennent à  $\Lambda$  par construction. De plus, on observe que la suite  $u(p)$  est incluse dans  $\Lambda \setminus (p\mathbb{N} \cap \Lambda)$ . On obtient alors que n'importe quel réel  $L \geq \frac{1}{6}$  vérifie la condition (P1).

(P2) On a déjà vu précédemment que  $L_f^*$  possède deux pôles (qui sont simples) en  $s = \frac{1}{3}$  et en  $s = \frac{1}{5}$ , un pôle double en  $s = 0$ , et des pôles simples en  $s = \frac{k}{5}$ ,  $k \in \mathbb{Z}_{\leq 1}$ , avec  $k \not\equiv 0 \pmod{5}$ . Ainsi, pour tout réel  $R \in \mathbb{R}_+^*$  tel que  $R \neq \frac{k}{5}$  avec  $k \not\equiv 0 \pmod{5}$ ,  $R$  vérifie la condition de la seconde hypothèse.

(P3) On remarque que le polynôme  $P$  vérifie la condition H0S de l'article [Ess97]. En particulier, on sait par [Ess97, Théorème 3] que la fonction zêta rattachée à ce polynôme, (i.e.  $\zeta_{\mathfrak{g}_2}$ ) admet une borne polynomiale en la partie imaginaire de  $s = \sigma + i\tau$  sur chaque bande verticale  $\sigma_1 \leq \sigma \leq \sigma_2$ . On en déduit alors que la condition (P3) est vérifiée.

**Remarque 6.4.1.** Dans la condition (P2),  $R$  est arbitrairement grand, et dans la condition (P1),  $L$  est arbitrairement grand.

Notons  $\alpha = \frac{1}{3}$ ,  $\beta = \frac{1}{5}$ ,  $\omega_\alpha := \text{Res}_{s=\frac{1}{3}} \zeta_{\mathfrak{g}_2}(s)$  et  $\omega_\beta := \text{Res}_{s=\frac{1}{5}} \zeta_{\mathfrak{g}_2}(s)$ . En appliquant le Théorème 1.2.33, on trouve alors

$$r_{\mathfrak{g}_2}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp\left(A_1 n^{\frac{1}{4}} + A_2 n^{\frac{3}{20}} + A_3 n^{\frac{1}{20}}\right) \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R}\left(n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}}\right)\right),$$

avec

$$C = \frac{e^{\zeta_{\mathfrak{g}_2}(0)} (\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{1-6\zeta_{\mathfrak{g}_2}(0)}{8}} \sqrt{3}}{\sqrt{8\pi}}, \quad b = \frac{6 - 6\zeta_{\mathfrak{g}_2}(0) + 1}{8},$$

et  $A_1 := 4 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{4}}$ ,  $A_2 := \frac{\omega_\beta \Gamma\left(\frac{1}{5}\right) \zeta\left(\frac{6}{5}\right)}{(\omega_\alpha \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}}$ , et

$$\begin{aligned} A_3 := & K_3 + 3 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{4}} \left( \binom{-1/3}{1} \frac{K_3}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{4}}} + \binom{-1/3}{2} \frac{K_2^2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{2}}} \right) \\ & + \frac{5 (\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right))}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}} \binom{-1/5}{1} \frac{K_2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{4}}}, \end{aligned}$$

avec  $0 < \nu_2 < \dots$  les éléments strictement positifs de  $\mathcal{N} + \mathcal{M}$ .



Par [BBBF23, Lemme 4.3], on a

$$K_2 = \frac{3}{4} \cdot \frac{\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{20}}}$$

$$K_3 = -\frac{3}{160} \cdot \frac{\left(\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)\right)^2}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{21}{20}}}.$$

Des ensembles décrits dans (1.8), (1.9), et (1.10) dans [BBBF23], on trouve que la suite  $\nu_j$  du Théorème 1.2.33 donne, avec nos hypothèses,  $\nu_2 = \frac{1}{20}$ ,  $\nu_3 = \frac{2}{20}$ ,  $\nu_4 = \frac{3}{20}$ , ...



# Chapitre 7

## Annexe

### 7.1 Codes SAGE

Listing 7.1: Code SAGE pour calculer  $Z'_{\Delta}(0)$  avec  $P = 2$  et  $d_1 = d_2 = 1$

```
from sage.all import *

c_g2 = [[1, 1], [1, 2], [1, 3], [2, 3]]
c_so5 = [[1, 1], [1, 2]]

gamma = euler_gamma
zeta0 = var('zeta0')
zeta_m1 = var('zeta_m1')
zeta0_prime = var('zeta0_prime')
zeta_m1_prime = var('zeta_m1_prime')

def Z_delta_prime_zero(c_matrix):
    Q=len(c_matrix)
    sum1 = sum(c[j][0] + c[j][1] for j in range(Q))
    sum2 = sum(sum((c[f][0]/c[f][1] - c[j][0]/c[j][1])
        * log(1 + c[f][1]/c[j][1])+(c[f][1]/c[f][0] - c[j][1]/c[j][0])
        * log(1 + c[f][0]/c[j][0])
    for f in range(Q) if f != j) for j in range(Q))
    sum3 = sum((c[j][0] * c[j][1])^-1 for j in range(Q))
    sum4 = sum(c[j][0]^(-1) * c[j][1] + c[j][1]^(-1) * c[j][0] for j in range(Q))
    result = Q*zeta0^2*gamma - zeta_m1*sum1*gamma - 1/(Q+1)*zeta_m1*sum2
        +zeta0_prime*zeta0/Q - zeta_m1_prime/(Q+1)*sum1 - gamma*zeta0^2*sum3
        + gamma*zeta_m1*sum4
    return(result)
print(result)
```

Listing 7.2: Code SAGE pour calculer  $Z_{\Delta}(-N)$  avec  $P = 2$

```
from math import comb
from itertools import product

def alpha_k_N(Q, N, k, c_matrix):
    alpha = 0
    for j in range(1, Q + 1):
        for w in product(range(N + 1), repeat=Q):
```

```

    if sum(w) == k:
        coef1 = multinomial(w)
        coef2 = multinomial([N - w_i for w_i in w])
        prod = 1
        for q in range(Q):
            prod *= (c_matrix[q][0] ** w[q]) * (c_matrix[q][1] ** (N - w[q]))
        alpha += coef1 * coef2 * prod
return alpha

def beta_N(Q, N, c_matrix):
    beta = 0
    for j in range(1, Q + 1):
        for w in product(range((Q + 1) * N + 2), repeat=Q):
            if sum(w) == (Q + 1) * N + 1 and w[j - 1] >= 2 * N + 1:
                numerator = 1
                for i in range(N + 1 - w[j - 1], -N - 1):
                    numerator *= i

                denominator = 1
                for q in range(Q):
                    if q != j - 1:
                        denominator *= factorial(w[q])

                term1 = c_matrix[j - 1][1] ** w[j - 1]
                term2 = c_matrix[j - 1][0] ** w[j - 1]

                for q in range(Q):
                    if q != j - 1:
                        term1 *= (c_matrix[q][0] ** (N - w[q]))
                        term1 *= (c_matrix[q][1] ** w[q])
                        term2 *= (c_matrix[q][1] ** (N - w[q]))
                        term2 *= (c_matrix[q][0] ** w[q])

                beta += (numerator / denominator) * (term1 + term2)
    return beta

def Z_Delta_minus_N(N, c_matrix):
    Q = len(c_matrix)
    sum1 = 0
    N_fact_Q = factorial(N) ** Q
    for k in range(Q * N + 1):
        alpha = alpha_k_N(Q, N, k, c_matrix)
        term1 = zeta(-N - k) / factorial(k)
        term2 = zeta(-(Q + 1) * N + k) / factorial(Q * N - k)
        sum1 += alpha * term1 * term2

    beta = beta_N(Q, N, c_matrix)
    sum2 = ((-1) ** (N + 1)) * (N_fact_Q * factorial(N)) * beta
        * zeta(-(Q + 2) * N - 1) / factorial((Q + 1) * N + 1)

    Z = (1 / Q) * N_fact_Q * sum1 + (1 / (Q + 1)) * sum2

    return Z

```

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**Special values of generalized multiple Hurwitz zeta  
functions and their derivatives. Applications towards  
some Witten zeta functions attached to semi-simple  
Lie algebras**

PhD in Mathematics  
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# Chapter 1

## Introduction

### 1.1 Notations

**Notation.** Let  $A, B$  be sets. We denote the non-strict inclusion as  $A \subset B$  (which some authors denote as  $A \subseteq B$ ), and the strict inclusion as  $A \subsetneq B$ .

**Notation.** Let  $A$  be a set, and  $B \subset A$  a subset of  $A$ , we denote  $B^c = A \setminus B$ . We denote  $|A| \in \mathbb{N}_0 \cup \{+\infty\}$  the cardinal of  $A$ .

**Notation.** We use a boldface letter to represent multi-index of the form  $\mathbf{x} = (x_a)_{a \in A} \in \mathbb{C}^A$  with  $A$  a finite set. If all  $x_a$  are integers and if  $|A| = n$ , we will sometimes call such index a  $n$ -tuple integer. Given  $B \subset A$ , we denote  $|\mathbf{x}|_B := \sum_{b \in B} x_b$ . If  $B = A$ , we directly denote  $|\mathbf{x}| := |\mathbf{x}|_A$ .

**Notation.** Let  $1 \leq k \leq n$  be positive integers. We denote  $\mathbf{e}_k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{C}^n$  the vector with a 1 at the  $k$ -th component, and 0's elsewhere.

**Notation.** Let  $A$  and  $B$  be finite sets, and  $\mathbf{x} = (\mathbf{x}_a)_{a \in A} \in (\mathbb{N}_0^B)^A$ , with  $\mathbf{x}_a = (x_{a,b})_{b \in B}$  for all  $a \in A$ . We define for every  $b \in B$ ,

$$\mathbf{x}(b) := \sum_{a \in A} x_{a,b}.$$

**Remark 1.1.1.** Let  $A$  and  $B$  be finite sets, and  $\mathbf{x} = (\mathbf{x}_a)_{a \in A} \in (\mathbb{N}_0^B)^A$ . The multi-index  $\mathbf{x}$  corresponds to a function  $\mathbf{x} : A \times B \rightarrow \mathbb{N}_0, (a, b) \mapsto x_{a,b}$ . In particular, if  $A$  or  $B$  is empty,  $\mathbf{x}$  corresponds to the empty function. In this case, for every subset  $\mathcal{A} \subset A$ ,

$$|\mathbf{x}|_{\mathcal{A}} = 0,$$

and for every  $b \in B$ ,

$$\mathbf{x}(b) = 0.$$

**Notation.** Let  $s \in \mathbb{C}$  be a complex number. We write  $s = \sigma + i\tau$  where  $\sigma$  and  $\tau$  are respectively the real and imaginary parts of  $s$ .

**Notation.** Let  $x \in \mathbb{R}$  be a real number. We denote  $H_x$  as the open complex half-plane with abscissa  $x$ , and  $\overline{H}_x$  as its closure:

$$\begin{aligned} H_x &:= \{s \in \mathbb{C} \mid \sigma > x\}, \\ \overline{H}_x &:= \{s \in \mathbb{C} \mid \sigma \geq x\}. \end{aligned}$$

**Notation.** Let  $a \in \mathbb{C}$ ,  $r \in \mathbb{R}_+$ . We denote  $\overline{D}_a(r)$  as the closed disc centered at  $a$  with radius  $r$ :

$$\overline{D}_a(r) := \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

**Notation.** Let  $s \in \mathbb{C}$  be a complex number, and  $k \in \mathbb{N}_0$ . We denote  $s_{\Delta, k} := (s, \dots, s) \in \mathbb{C}^k$ . When there is no ambiguity, we will simply denote  $s_{\Delta}$ .

**Notation.** Let  $n \in \mathbb{N}_0$  be an integer. We call the  $n$ -th harmonic number the following rational number

$$h_n := \sum_{k=1}^n \frac{1}{k},$$

with the convention  $h_0 = 0$ .

We will denote  $\gamma = 0.5772156649\dots$  as Euler's constant [OEI23a].

**Definition-Property 1.1.2.** [EMOT81, Chap.1] We denote by  $\Gamma$  the Euler's gamma function defined on  $H_0$  by the integral

$$\forall s \in H_0, \quad \Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx.$$

The function  $\Gamma$  is holomorphic on  $H_0$ , and has a meromorphic extension to  $\mathbb{C}$  with simple poles at the nonpositive integers. We denote this extension also by  $\Gamma$ . We also denote  $\psi$  the digamma function defined on  $H_0$  by the relation

$$\forall s \in H_0, \quad \psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}.$$

The function  $\psi$  is holomorphic on  $H_0$ , and has a meromorphic extension to  $\mathbb{C}$ , with simple poles at the nonpositive integers.

Recall that  $\Gamma$  satisfies the following functional equation:

$$\Gamma(s) = (s-1)\Gamma(s-1) \quad (s \notin -\mathbb{N}_0),$$

and in the neighborhood of  $s = 0$ , we have the following asymptotic expansion

$$\Gamma(s) = \frac{1}{s} - \gamma + \frac{1}{2} \left( \gamma^2 + \frac{\pi^2}{6} \right) s + O(s^2).$$

For any nonnegative integers  $N$ , we have

$$\Gamma(N+1) = N!.$$

We also recall that the function  $\frac{1}{\Gamma}$  is holomorphic on  $\mathbb{C}$ , and it vanishes at the nonpositive integers.

**Notation.** We define the binomial and multinomial coefficients as follows:  
For all  $k \in \mathbb{Z}$ , and  $s \in \mathbb{C}$ , we set

$$\binom{s}{k} = \begin{cases} \frac{s(s-1)\dots(s-k+1)}{k!} & \text{if } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For all integers  $\mathbf{k} = (k_1, \dots, k_P)$  such that  $n = k_1 + \dots + k_P$ , we consider the multinomial coefficient:

$$\binom{n}{\mathbf{k}} = \begin{cases} \frac{n!}{k_1! \dots k_P!} & \text{if } k_p \geq 0 \text{ for all } p \in \llbracket 1, P \rrbracket, \\ 0 & \text{otherwise.} \end{cases}$$

In this manuscript, we will use the principal determination of the logarithm, and we will write for any complex  $z \in \mathbb{C}$ ,  $z = |z|e^{i \arg(z)}$  with  $\arg(z) \in ]-\pi, \pi]$ .

By convention, we will say that a sum over an empty set is zero, and that a product over an empty set is 1.

## 1.2 Some history on zeta and multizeta functions

In this thesis, we will focus on a class of multizeta functions defined by the Dirichlet series

$$Z(\mathbf{s}, \mathbf{s}') = Z(\mathbf{c}, \mathbf{d}, \mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n} + \mathbf{d}))^{-s'_q},$$

with  $P \geq 1$ ,  $Q \geq 1$ ,  $\mathbf{d} \in H_0^P$ , and  $l_q$  being linear forms of rank  $P$ , dependent on each variable  $n_p$  for  $1 \leq p \leq P$ . We will study its domain of convergence, its meromorphic extension, and establish formulas for directional values at nonpositive integers of  $Z$ . We will also give a closed formula for the derivative values at nonpositive integers along the same direction.

In [Kom08], Komori gave a new integral representation formula of the Mordell-Tornheim double zeta function, and derived from this work a formula for the values at nonpositive integers of this zeta function. Further on, he provided an expression for directional values in terms of generalized Bernoulli numbers for a broader class of multizeta functions in [Kom10]. However, these generalized Bernoulli numbers are sometimes difficult to compute. For some generalized multiple zeta functions of Euler-Zagier type, explicit formulas at nonpositive integers along a directions have been computed in [Ono13], and more generally in [EM20], using only classical Bernoulli numbers. The work presented here provides explicit relations in terms of classical Bernoulli polynomials for directional values at nonpositive integers of  $Z(\mathbf{s}, \mathbf{s}')$ . Additionally, explicit values for the directional derivative at nonpositive integers of  $Z(\mathbf{s}, \mathbf{s}')$  are also obtained. These expressions are achieved using a Crandall expansion (c.f. [Cra12, §9.2]). In [BD18], this strategy has already yielded approximations for  $\zeta^{AV}(-N)$  and  $\partial_s \zeta^{AV}(-N)$ , with  $\zeta^{AV}$  being the Apostol-Vu type zeta function along the diagonal:

$$\zeta^{AV}(s) = \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s}.$$

In [BD18], Borwein and Tomkins studied a Tornheim-type zeta function:

$$\zeta_P^T(s) := \sum_{n_1, \dots, n_P \geq 1} \frac{1}{n_1^s \dots n_P^s (n_1 + \dots + n_P)^s},$$

and computed the values at nonpositive integers of  $\zeta_P^T(s)$ , as well as the derivative  $\partial_s \zeta_P^T(s)$ . In [Ono21], Onodera also studied this class of zeta functions and computed the values of the second derivative with respect to  $s$  of  $\zeta_P^T(s)$  at nonpositive integers.

From our work, we can derive formulas for the special values of the following two Witten zeta functions, as well as their derivatives:

$$\begin{aligned} \zeta_{\mathfrak{so}(5)}(s) &:= 6^s \sum_{n_1, n_2=1}^{+\infty} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s} \\ \zeta_{\mathfrak{g}_2}(s) &:= 120^s \sum_{n_1, n_2=1}^{+\infty} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_2)^s (2n_1 + 3n_2)^s}. \end{aligned}$$

Let's recall the Shintani zeta functions

$$\zeta^{Sh}(s, \mathbf{d} | \mathbf{c}_1, \dots, \mathbf{c}_Q) := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{q=1}^Q \left( d_q + \sum_{p=1}^P c_{q,p} n_p \right)^{-s},$$

with  $P, Q \geq 1$  being positive integers,  $d_q \in H_0$  and  $c_{q,p} \in H_0$  being complex number in  $H_0$  for all  $1 \leq p \leq P$  and  $1 \leq q \leq Q$ . From our work, we will also get formulas for special values of some Shintani zeta functions, and formulas for the special values of their derivatives as well.

### 1.2.1 Some results on special values of zeta functions

We aim to provide here a brief summary of results on the special values of the Riemann zeta function, the Hurwitz zeta function, Barnes zeta functions, Euler-Zagier zeta functions, some Witten zeta functions, and Shintani zeta functions. We will also discuss their domains of holomorphy and meromorphy. In particular, we will present results on values at nonpositive integers for some of these functions.

All the zeta functions mentioned in the previous paragraph have a domain of convergence of the form  $H_{\sigma_0} = \{s = \sigma + i\tau \in \mathbb{C} \mid \sigma > \sigma_0\}$  with  $\sigma_0 \in \mathbb{R}$ , and they are holomorphic on their half-plane of convergence. To give meaning to the value at  $s = N \in \mathbb{Z}$  when  $N$  is not in the half-plane of convergence for these functions, we must first discuss their meromorphic extension and prove that this extension is regular at the integer we considered.

We first recall the Euler-Maclaurin formula.

**Theorem 1.2.1** (Euler-Maclaurin formula). *[Ten15] Let  $k \in \mathbb{N}_0$  be a nonnegative integer, and  $f : [a, b + 1] \rightarrow \mathbb{C}$  a function of the class  $C^{k+1}$  on  $[a, b]$ . Then we have that*

$$\begin{aligned} \sum_{n=a}^{b-1} f(n) &= \int_a^b f(t) dt + \sum_{n=0}^k \frac{(-1)^{n+1} B_{n+1}}{(n+1)!} (f^{(n)}(b) - f^{(n)}(a)) \\ &\quad + \frac{(-1)^k}{(k+1)!} \int_a^b b_{k+1}(x) f^{(k+1)}(x) dx, \end{aligned}$$

where  $B_n$  is the  $n$ -th Bernoulli number, and  $b_n(t)$  is the  $n$ -th periodised Bernoulli polynomial.

We also recall the expression of the Riemann zeta function.

**Definition 1.2.2.** *Let  $s \in \mathbb{C}$ . The Riemann zeta function is defined as follows*

$$\forall s \in H_1, \quad \zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

We will see in the following proposition that this function can be extended over  $\mathbb{C}$ :

**Proposition 1.2.3.** *The Riemann zeta function is holomorphic on  $H_1$ , and it extends meromorphically over  $\mathbb{C}$  with a single pole at  $s = 1$ , which is simple, with a residue of 1.*

The meromorphic extension can be proven using the Euler-Maclaurin formula 1.2.1, and this formula also provides the values of the Riemann zeta function at nonpositive integers in terms of the Bernoulli numbers:

$$\forall N \in \mathbb{N}_0, \quad \zeta(-N) = (-1)^N \frac{B_{N+1}}{N+1}.$$

In particular, it follows that  $\zeta(-2N) = 0$  for all integers  $N \in \mathbb{N}$ .

Using the functional equation of the Riemann zeta function, we can also find relations between the values of  $\zeta$  at positive integers with the values of its derivative at nonpositive integers.

**Theorem 1.2.4.** *Let  $s \in \mathbb{C} \setminus \{0, 1\}$ , we have the functional relation*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s).$$

In particular, we deduce that, for all  $n \in \mathbb{N}$ ,

$$\zeta'(-2N) = \frac{(-1)^N (2N)!}{2^{2N+1} \pi^{2N}} \zeta(2N+1).$$

For small values, we also have these results:

$$\begin{aligned} \zeta'(0) &= -\frac{1}{2} \ln(2\pi), \\ \zeta'(-1) &= \frac{1}{12} - \ln(A), \end{aligned}$$

where  $A = 1.28242712\dots$  is the Glaisher-Kinkelin constant defined in [OEI23b].



**Definition 1.2.5.** Let  $d \in H_0$  be a complex number, the Hurwitz zeta function is defined by

$$\forall s \in H_0, \quad \zeta(s, d) := \sum_{n=0}^{+\infty} \frac{1}{(n+d)^s}.$$

Note that for  $d = 1$ , we recover the Riemann zeta function. This zeta function admits a meromorphic extension over  $\mathbb{C}$ .

**Proposition 1.2.6.** The Hurwitz zeta function is meromorphic on  $\mathbb{C}$ , with a single pole at  $s = 1$ , which is a simple pole. Moreover, the residue at that pole is 1.

*Proof.* Let  $n \in \mathbb{N}$  be a positive integer. Via Euler-Maclaurin formula, we get that for all  $\sigma > 1$ ,

$$\begin{aligned} \zeta(s, d) &= \frac{d^{1-s}}{1-s} + \sum_{i=0}^n \binom{-s}{i} \frac{(-1)^i B_{i+1}}{i+1} d^{-s-i} \\ &\quad + (-1)^n \binom{-s}{n+1} \int_0^{+\infty} b_{n+1}(x) (x+d)^{-s-n-1} dx. \end{aligned} \quad (1.1)$$

The integral in the previous formula converges absolutely for all  $\sigma > -n$ . Thus, this formula allows for a meromorphic extension of the Hurwitz zeta function for  $\sigma > -n$ , with a unique pole at  $s = 1$ , of order 1, and with residue 1.  $\square$

We also have results on the values at nonpositive integers of the Hurwitz zeta function (see [Apo76, Theorem 12.13]):

$$\forall N \in \mathbb{N}_0, \quad \zeta(-N, x) = -\frac{b_{N+1}(x)}{n+1},$$

where  $b_{N+1}(x)$  is the  $N + 1$ -th Bernoulli polynomial. Similarly, it was known by Hurwitz (see [Ber85, Formula (3)]) that

$$(\partial_s \zeta(s, d))|_{s=0} = \ln(\Gamma(d)) + \zeta'(0) = \ln(\Gamma(d)) - \frac{1}{2} \ln(2\pi) \quad (1.2)$$

Let us also observe that, for all  $s \neq 1$ , and for all  $d \in H_0$ , we have  $\zeta(s, d) = d^{-s} + \zeta(s, d+1)$ . Differentiating with respect to  $s$ , we then find

$$\partial_s \zeta(s, d)|_{s=-N} = -d^N \ln(d) + \partial_s \zeta(s, d+1)|_{s=-N}. \quad (1.3)$$

Rewriting the Dirichlet series  $\zeta\left(s, \frac{1}{2}\right)$ , we can prove that  $\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s)$ . Therefore, by differentiating both sides of the last equality with respect to  $s$ , we find that

$$\partial_s \zeta\left(s, \frac{1}{2}\right)|_{s=-1} = -\frac{\ln(2)}{6} - \frac{1}{2} \zeta'(-1). \quad (1.4)$$

Miller and Adamchik have established in [MA98] an explicit relation between the values of the derivatives of the Hurwitz zeta function with respect to  $s$  at nonpositive integers, with a rational coefficient  $d$ , and special values of logarithms, polygamma functions, and the Riemann zeta function.

The Hurwitz zeta function admits a functional relationship called Hurwitz's formula. Before stating it, we need to introduce the Lerch zeta function:

**Definition 1.2.7.** Let  $s \in \mathbb{C}$  such that  $\sigma > 1$ ,  $d \in H_0$ , and  $z \in \mathbb{C}$  such that  $|z| \leq 1$ . We define

$$\phi(z, s, d) := \sum_{n \geq 0} \frac{z^n}{(n+d)^s}.$$

This zeta function has a meromorphic extension on  $\mathbb{C}$ . It is even holomorphic for all  $z \neq 1$ , and it has a unique, simple pole at  $s = 1$  when  $z = 1$  [EMOT81, §1.11].

**Theorem 1.2.8** (Hurwitz Formula). [Apo76, §12.7] Let  $d \in ]0, 1]$  be a real number, and  $s \in \mathbb{C}$  be a complex number such that we have either  $d \in ]0, 1[$  and  $\operatorname{Re}(s) > 0$ , or  $d = 1$  and  $\operatorname{Re}(s) > 1$ . We then get that

$$\zeta(1-s, d) = \frac{\Gamma(s)}{(2\pi)^s} \left( e^{-i\pi s/2} \phi(e^{2i\pi d}, s, 1) + e^{i\pi s/2} \phi(e^{-2i\pi d}, s, 1) \right).$$

Let's note that the case  $d = 1$  in the previous formula corresponds to the functional equation of the Riemann zeta function.

We now mention some results about Barnes zeta functions. We will use some of these results later in the manuscript.

**Definition 1.2.9.** Let  $P \in \mathbb{N}_0$  be a nonnegative integer, and  $(c_1, \dots, c_P)$  a family of complex numbers in  $H_0$ , and  $d \in H_0$ . We call the Barnes zeta function the zeta function of the form

$$\forall s \in H_P, \quad \zeta^B(s, d|c_1, \dots, c_P) := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{\left( \sum_{p=1}^P c_p n_p + d \right)^s}.$$

Barnes proved that this zeta function has a meromorphic extension over  $\mathbb{C}$  with simple poles at some positive integers. More precisely:

**Proposition 1.2.10.** [Bar04] The Barnes zeta function  $s \mapsto \zeta^B(s, d|c_1, \dots, c_P)$  has a meromorphic extension over  $\mathbb{C}$ , with its only singularities being simple poles at  $s = 1, \dots, P$ .

The derivative value at  $s = 0$  of such zeta functions was initially studied by Barnes in the theory of multigamma functions (see [Bar01] and [Bar04]). Later on, Vardi found in [Var88] an explicit formula between the value at  $\frac{1}{2}$  of the  $n$ -th multigamma function and the regularized determinant of the Laplacian of the  $n$ -sphere. He also gave an explicit formula of the value at  $\frac{1}{2}$  of the  $n$ -th multigamma function using only values at nonpositive integers of the derivative of the Riemann zeta function.

Some of our results will be expressed in terms of derivative values at nonpositive integers of Barnes zeta functions. In particular, we will need a result proved by Sakane and Aoki in [SA22]. We first define

$$C_{P,x}(t) := (t-x+P-1)(t-x+P-2)\dots(t-x+1) \in (\mathbb{Z}[x])[t] \quad (P \geq 2),$$

and  $C_{1,x}(t) = 1$ . Let's note that  $C_{2,x}(t) = t-x+1$ .

**Theorem 1.2.11.** [SA22, Theorem 4] Let  $d \in H_0$ , and  $\mathbf{c} = (c_1, \dots, c_P) \in \mathbb{Q}^P$  be rational numbers. We write  $c_1 = \frac{a_1}{b_1}, \dots, c_P = \frac{a_P}{b_P}$  with  $a_p, b_p \geq 1$  being coprime positive integers for all  $1 \leq p \leq P$ . Let  $x(\mathbf{c}) := \frac{\operatorname{lcm}(a_1, \dots, a_P)}{\operatorname{gcd}(b_1, \dots, b_P)}$ , and  $\beta_1 := \frac{x(\mathbf{c})}{c_1}, \dots, \beta_P := \frac{x(\mathbf{c})}{c_P}$ . We then have that

$$\zeta^B(s, d|c_1, \dots, c_P) = \frac{x(\mathbf{c})^{-s}}{(P-1)!} \sum_{v_1=0}^{\beta_1-1} \dots \sum_{v_P=0}^{\beta_P-1} \sum_{k=0}^{P-1} C_{P, \frac{d+c_1 v_1 + \dots + c_P v_P}{x(\mathbf{c})}}^{(k)}(0) \zeta \left( s-k, \frac{d+c_1 v_1 + \dots + c_P v_P}{x(\mathbf{c})} \right).$$

In Theorem B, we shall prove a generalized version of the above theorem, using a similar strategy as the one used by Aoki and Sakane in [SA22].

**Example 1.2.12.** *We have*

$$\begin{aligned}\zeta^B(s, d|1, 1) &= (1-d)\zeta(s, d) + \zeta(s-1, d), \\ \zeta^B(s, d|1, 2) &= 2^{-s} \left[ \left(1 - \frac{d}{2}\right) \zeta\left(s, \frac{d}{2}\right) + \left(1 - \frac{d+1}{2}\right) \zeta\left(s, \frac{d+1}{2}\right) + \zeta\left(s-1, \frac{d}{2}\right) + \zeta\left(s-1, \frac{d+1}{2}\right) \right], \\ \zeta^B(s, d|1, 3) &= 3^{-s} \left[ \left(1 - \frac{d+2}{3}\right) \zeta\left(s, \frac{d+2}{3}\right) + \left(1 - \frac{d+1}{3}\right) \zeta\left(s, \frac{d+1}{3}\right) + \left(1 - \frac{d}{3}\right) \zeta\left(s, \frac{d}{3}\right) \right. \\ &\quad \left. + \zeta\left(s-1, \frac{d+2}{3}\right) + \zeta\left(s-1, \frac{d+1}{3}\right) + \zeta\left(s-1, \frac{d}{3}\right) \right], \\ \zeta^B(s, d|2, 3) &= 6^{-s} \left[ \left(1 - \frac{d+7}{6}\right) \zeta\left(s, \frac{d+7}{6}\right) + \left(1 - \frac{d+5}{6}\right) \zeta\left(s, \frac{d+5}{6}\right) + \left(1 - \frac{d+4}{6}\right) \zeta\left(s, \frac{d+4}{6}\right) \right. \\ &\quad \left. + \left(1 - \frac{d+3}{6}\right) \zeta\left(s, \frac{d+3}{6}\right) + \left(1 - \frac{d+2}{6}\right) \zeta\left(s, \frac{d+2}{6}\right) + \left(1 - \frac{d}{6}\right) \zeta\left(s, \frac{d}{6}\right) \right. \\ &\quad \left. + \zeta\left(s-1, \frac{d+7}{6}\right) + \zeta\left(s-1, \frac{d+5}{6}\right) + \zeta\left(s-1, \frac{d+4}{6}\right) + \zeta\left(s-1, \frac{d+3}{6}\right) + \zeta\left(s-1, \frac{d+2}{6}\right) \right. \\ &\quad \left. + \zeta\left(s-1, \frac{d}{6}\right) \right].\end{aligned}$$

We then get the following special values  $\zeta^B(0, 2|1, 1) = \frac{5}{12}$ ,  $\zeta^B(0, 3|1, 2) = \frac{11}{24}$ ,  $\zeta^B(0, 4|1, 3) = \frac{19}{36}$ , and  $\zeta^B(0, 5|2, 3) = \frac{31}{72}$ . We also can differentiate the previous formulas in respect to  $s$  for  $\zeta^B(s, d|c_1, c_2)$ . We then get

$$\begin{aligned}(\zeta^B)'(0, 2|1, 1) &= -\zeta'(0, 2) + \zeta'(-1, 2), \\ (\zeta^B)'(0, 3|1, 2) &= -\frac{11 \ln(2)}{24} - \frac{1}{2} \zeta'\left(0, \frac{3}{2}\right) - \zeta'(0, 2) + \zeta'\left(-1, \frac{3}{2}\right) + \zeta'(-1, 2), \\ (\zeta^B)'(0, 4|1, 3) &= -\frac{19 \ln(3)}{36} - \zeta'(0, 2) - \frac{2}{3} \zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3} \zeta'\left(0, \frac{4}{3}\right) + \zeta'(-1, 2) + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right), \\ (\zeta^B)'(0, 5|2, 3) &= -\frac{31 \ln(6)}{72} - \zeta'(0, 2) - \frac{2}{3} \zeta'\left(0, \frac{5}{3}\right) - \frac{1}{2} \zeta'\left(0, \frac{3}{2}\right) - \frac{1}{3} \zeta'\left(0, \frac{4}{3}\right) - \frac{1}{6} \zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6} \zeta'\left(0, \frac{5}{6}\right) \\ &\quad + \zeta'(-1, 2) + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{3}{2}\right) + \zeta'\left(-1, \frac{4}{3}\right) + \zeta'\left(-1, \frac{7}{6}\right) + \zeta'\left(-1, \frac{5}{6}\right),\end{aligned}$$

where we noted  $(\zeta^B)' = \partial_s \zeta^B$ , and  $\zeta'(s, d) = \partial_s \zeta(s, d)$ .

Shintani introduced zeta functions that allow him to study Dedekind zeta functions associated with a totally real number field (see [Shi76b], [Shi76a], [Shi77a], [Shi77b], [Shi77c], [Shi80]). These functions, in particular, generalize Barnes' zeta functions:

**Definition 1.2.13.** *Let  $P, Q \in \mathbb{N}_0$  be nonnegative integers,  $\mathbf{c} = (c_{q,p})_{1 \leq p \leq P, 1 \leq q \leq Q}$  and  $(d_q)_{1 \leq q \leq Q}$  be two families of complex numbers in  $H_0$ . We call the following function the Shintani zeta function:*

$$\zeta^{Sh}(s, \mathbf{d}|\mathbf{c}) = \sum_{n_1, \dots, n_P \geq 0} \prod_{q=1}^Q (c_{q,1}n_1 + \dots + c_{q,P}n_P + d_q)^{-s}.$$

Note that, for  $Q = 1$ , we recover the definition of Barnes' zeta function. This zeta function is holomorphic on  $H_{P/Q}$  and has a meromorphic extension over  $\mathbb{C}$ . Shintani proved that this function has the same poles as those of the function  $\frac{\Gamma(Ps - Q)}{\Gamma(s)}$ . He also proved certain formulas at the nonpositive integers of these functions. A part of our work particularly allows us to study the values of some of these functions at nonpositive integers, as well as the values of their derivatives.

We deduce from the above that the zeta functions of Barnes and Shintani are regular at nonpositive integers. The values of their derivatives with respect to the variable  $s$  at  $s = 0$  allow us to define multigamma functions, introduced by Barnes in [Bar04], and then generalized by Friedmann and Ruijsenaars.

**Definition 1.2.14.** *Let  $P \in \mathbb{N}_0$  be a nonnegative integer, and  $(c_1, \dots, c_P)$  be complex numbers in  $H_0$ , and  $d \in H_0$ . We call Barnes multigamma function the following*

$$\Gamma_P(d|c_1, \dots, c_P) := \exp((\partial_s \zeta^B(s, d|c_1, \dots, c_P))|_{s=0}).$$

**Example 1.2.15.** *Let  $d \in H_0$ , we have that*

$$\begin{aligned} \Gamma_0(d) &= \frac{1}{d} \\ \Gamma_1(d|1) &= \frac{1}{\sqrt{2\pi}} \Gamma(d), \end{aligned}$$

where  $\Gamma$  is Euler's gamma function.

Barnes also obtains the following functional relation

$$\Gamma_P(d|c_1, \dots, c_P) = \Gamma_{P-1}(d|c_1, \dots, c_{P-1}) \Gamma_P(d + c_P|c_1, \dots, c_P),$$

which generalizes the classical recurrence relation of Euler's gamma function.

Through Shintani-type zeta functions, Friedman and Ruijsenaars introduced in [FR04] Shintani-type multigamma functions, using a construction similar to the one used by Barnes.

**Definition 1.2.16.** *Let  $P, Q \in \mathbb{N}_0$  be nonnegative integers,  $\mathbf{c} = (c_{q,p})_{1 \leq p \leq P, 1 \leq q \leq Q}$  and  $(d_q)_{1 \leq q \leq Q}$  be two families of complex numbers in  $H_0$ . The Shintani multigamma function is defined as:*

$$\Gamma_P(\mathbf{d}|\mathbf{c}_1, \dots, \mathbf{c}_P) := \exp((\partial_s \zeta^{Sh}(s, \mathbf{d}|\mathbf{c}_1, \dots, \mathbf{c}_q))|_{s=0}),$$

where  $\mathbf{c}_q = (c_{q,p})_{p \in \llbracket 1, P \rrbracket}$ .

In [FR04], Friedman and Ruijsenaars studied these Shintani multigamma functions to deduce Raabe-type formulas, generalizing the following Raabe formula:

$$\int_0^1 \ln(\Gamma(x)) dx = \ln(\sqrt{2\pi}).$$

We will now discuss a class of zeta functions related to Lie algebras. Witten introduced in [Wit91] the series

$$\sum_{\rho} \frac{1}{\dim(\rho)^{2g}},$$

where  $g$  is a positive integer, and  $\rho$  runs over the irreducible finite representations, up to isomorphism, of a Lie algebra  $\mathfrak{g}$ , and  $g \in \mathbb{N}$ . Witten particularly proved that

$$\sum_{\rho} \frac{1}{\dim(\rho)^{2g}} \in \pi^{2g} \mathbb{Q}.$$

Zagier then introduced the following Witten zeta function :

**Definition 1.2.17.** [Zag94] *Let  $\mathfrak{g}$  be a semi-simple Lie algebra. We define a Witten zeta function attached to  $\mathfrak{g}$  the following zeta function:*

$$\zeta_{\mathfrak{g}}(s) := \sum_{\rho} \dim(\rho)^{-s},$$

where  $\rho$  runs over the finite irreducible representations, up to isomorphism, of the Lie algebra  $\mathfrak{g}$ .

We can introduce  $r_{\mathfrak{g}}(n)$  as the number of isomorphism classes among the representations of dimension  $n$ .

**Definition 1.2.18.** *Let  $\mathfrak{g}$  be a semi-simple Lie algebra. We set  $r_{\mathfrak{g}}(n)$  as the number of representations of  $\mathfrak{g}$  of dimension  $n$ , up to isomorphism.*

It is well-known that a representation of a semi-simple Lie algebra uniquely decomposes into a direct sum of irreducible representations. Thus, for a representation  $\rho$  of  $\mathfrak{g}$ , we have that there exist integers  $k_1, \dots, k_m$  and irreducible representations  $\phi_1, \dots, \phi_m$  such that

$$\dim(\rho) = \sum_{i=1}^m k_i \dim(\phi_i).$$

More specifically, for the Lie algebras  $\mathfrak{so}(5)$  and  $\mathfrak{g}_2$ , we have:

**Example 1.2.19.** *[Hum72, §24.3]*

1) *The set of isomorphism class of irreducible representations of  $\mathfrak{sl}(3)$  is of the form  $(\phi_{i,j})_{i,j \geq 1}$ , with*

$$\dim(\phi_{i,j}) = \frac{ij(i+j)}{2}.$$

*Then we get the following expression for the number of irreducible representations of dimension  $n$  of  $\mathfrak{sl}(3)$ :*

$$\left| \left\{ (i,j) \in \mathbb{N}^2 \mid \frac{ij(i+j)}{2} = n \right\} \right|,$$

*and we get the following expression of the number of representations  $r_{\mathfrak{sl}(3)}(n)$  of  $\mathfrak{sl}(3)$  is*

$$\begin{aligned} r_{\mathfrak{sl}(3)}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\phi_{i,j}) = n \right\} \right|, \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)}{2} = n \right\} \right|. \end{aligned}$$

2) *The set of isomorphism class of irreducible representations of  $\mathfrak{so}(5)$  is of the form  $(\phi_{i,j})_{i,j \geq 1}$ , with*

$$\dim(\phi_{i,j}) = \frac{ij(i+j)(i+2j)}{3!}.$$

*Then we get the following expression for the number of irreducible representations of dimension  $n$  of  $\mathfrak{so}(5)$ :*

$$\left| \left\{ (i,j) \in \mathbb{N}^2 \mid \frac{ij(i+j)(i+2j)}{3!} = n \right\} \right|,$$

*and we get the following expression of the number of representation  $r_{\mathfrak{so}(5)}(n)$  of  $\mathfrak{so}(5)$ :*

$$\begin{aligned} r_{\mathfrak{so}(5)}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\phi_{i,j}) = n \right\} \right| \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)}{3!} = n \right\} \right|. \end{aligned}$$

3) The set of isomorphism class of irreducible representations of  $\mathfrak{g}_2$  is of the form  $(\phi_{i,j})_{i,j \geq 1}$ , with

$$\dim(\phi_{i,j}) = \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!}.$$

Then we get the following expression for the number of irreducible representations of dimension  $n$  of  $\mathfrak{g}_2$ :

$$\left| \left\{ (i,j) \in \mathbb{N}^2 \mid \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|,$$

and we get the following expression of the number of representation  $r_{\mathfrak{g}_2}(n)$  of  $\mathfrak{g}_2$ :

$$\begin{aligned} r_{\mathfrak{g}_2}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\phi_{i,j}) = n \right\} \right| \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|. \end{aligned}$$

Let's write  $f_{\mathfrak{g}}(n)$  the number of irreducible representations of dimension  $n$  of a semi-simple Lie algebra  $\mathfrak{g}$ . Then by definition of the Witten zeta function, we get that

$$\zeta_{\mathfrak{g}}(s) = \sum_{n=1}^{+\infty} \frac{f_{\mathfrak{g}}(n)}{n^s}.$$

We will see in Chapter 6 that this zeta function can be simplified using a product of linear forms, thanks to Weyl's Theorem 6.2.9.

**Example 1.2.20.** [Zag94], [KMT24] Let  $\mathfrak{sl}_{n+1} = \{X \in \mathcal{M}_{n+1}(\mathbb{C}) \mid \text{tr}(X) = 0\}$ , and  $\mathfrak{so}_{2n+1} = \{X \in \mathcal{M}_{2n+1}(\mathbb{C}) \mid {}^tX + X = 0\}$ , where  $\text{Tr}$  denotes the trace, and  ${}^tX$  denotes the transpose of a matrix. For these Lie algebras, we can associate Witten's zeta functions:

$$\begin{aligned} \zeta_{\mathfrak{sl}(2)}(s) &= \zeta(s), \\ \zeta_{\mathfrak{sl}(3)}(s) &= 2^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s}, \\ \zeta_{\mathfrak{so}(5)}(s) &= 6^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s}. \end{aligned}$$

For the exceptional Lie algebra  $\mathfrak{g}_2$  described in [Bou81], we get that

$$\zeta_{\mathfrak{g}_2}(s) = 120^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_2)^s (2n_1 + 3n_2)^s}.$$

Note that in the literature, we sometimes find multivariable variants of these zeta functions in the works of Matsumoto, Komori, Yasushi, and Tsumura [KMT10c], [KMT10b], [KMT12], [KMT11b]. These authors provide an expression for the set of singularities of these Witten multizeta functions. This allows us in particular to obtain candidate poles for the corresponding univariable Witten zeta function.

- From [KMT10d, Theorem 6.2], we deduce that the zeta function  $\zeta_{\mathfrak{so}(5)}(s)$  has poles of the form

$$s = \frac{1}{2}, \quad s = \frac{1-k}{3} \quad (k \in \mathbb{N}_0). \quad (1.5)$$

- From [KMT11a, Theorem 3.1], we deduce that the zeta function  $\zeta_{\mathfrak{g}_2}(s)$  has poles of the form

$$s = \frac{1}{3}, \quad s = \frac{1-k}{5} \quad (k \in \mathbb{N}_0). \quad (1.6)$$

We will show via Proposition 2.2.4 that nonpositive integers are not poles for  $\zeta_{\mathfrak{so}(5)}(s)$  and for  $\zeta_{\mathfrak{g}_2}(s)$ .

These zeta functions are well-suited for studying the representations of semi-simple Lie algebras, and one can express their special values to obtain information about these representations. Indeed, recent work by Romik [Rom17] allows us to study the asymptotic behavior of the number representations of dimension  $n$  of  $\mathfrak{sl}(3)$  in terms of the values  $\zeta_{\mathfrak{sl}(3)}(0)$  and  $\zeta'_{\mathfrak{sl}(3)}(0)$ , as well as the residues of  $\zeta_{\mathfrak{sl}(3)}(s)$  at the singularities of  $\zeta_{\mathfrak{sl}(3)}$ . Kurokawa and Ochiai in [KO13] have studied the values at nonpositive integers of  $\zeta_{\mathfrak{sl}(3)}(s)$ , and have deduced results for the values of certain  $p$ -adic Witten zeta functions.

Onodera subsequently obtained other formulas by studying a Mordell-Tornheim-type zeta function:

$$\zeta_P^{MT}(s) := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{(n_1 + d_1)^s \dots (n_P + d_P)^s (n_1 + d_1 + \dots + n_P + d_P)^s},$$

with  $P \geq 2$  an integer. Note that by setting  $P = 2$ , we obtain  $2^{-s} \zeta_{\mathfrak{sl}(3)}(s)$ .

Onodera computed the values of this zeta function at nonpositive integers via a Crandall expansion. This strategy consists of partitioning the integration domain of an integral representation of the zeta function whose values one is looking to study, with the domain depending on a free variable. More precisely, Onodera obtained the following expression:

$$\begin{aligned} \zeta_P^{MT}(s, \mathbf{d}) &= \frac{1}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{(n_1 + d_1)^s \dots (n_P + d_P)^s (n_1 + d_1 + \dots + n_P + d_P)^s} \int_{\theta(n_1 + d_1 + \dots + n_P + d_P)}^{+\infty} e^{-y} y^{s-1} dy \\ &\quad + \frac{1}{\Gamma(s)} \int_0^\theta \prod_{p=1}^P e^{-d_p y_p} \phi(e^{-y}, s, d_p) dy_1 \dots dy_P, \end{aligned}$$

with  $\phi$  being the Lerch zeta function described in Definition 1.2.7, and where  $\theta$  is a sufficiently small real free variable. The first term on the right-hand side of the previous equation is holomorphic and vanish at nonpositive integers. The second term can be re-expressed via the Erdélyi formula 2.6, and Onodera ultimately obtains the following formulas:

**Theorem 1.2.21.** [Ono21, Theorem 3, Theorem 4] *Let  $\mathbf{d} \in \mathbb{R}_*^+{}^P$  and  $t \in \llbracket 1, P-1 \rrbracket$  such that  $t < |\mathbf{d}|$ . For all nonnegative integers  $N \in \mathbb{N}_0$ , we have that*

$$\begin{aligned} \zeta_P^{MT}(-N, \mathbf{d}) &= \sum_{\mathcal{P} \subsetneq [1, P]} \frac{(-1)^{(N+1)(|\mathcal{P}|)} N!^{|\mathcal{P}|+1}}{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|_{|\mathcal{P}^c|}=(|\mathcal{P}^c|+1)N+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!} \\ \zeta_P^{MT'}(-N, \mathbf{d}) &= \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathcal{P}^c|(N+1)-1} N!^{|\mathcal{P}^c|} \left( \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ u \geq 0 \\ |\mathbf{k}|_{|\mathcal{P}^c|}+u=(|\mathcal{P}^c|+1)N+|\mathcal{P}^c|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!} \right) \\ &\quad \cdot \left( \frac{(-1)^N \zeta'(-N - u, |\mathbf{d}| - t)}{u!} + \frac{1}{|\mathcal{P}^c|} \sum_{p \in \mathcal{P}^c} \frac{(-1)^N \zeta'(-N - u, d_p)}{u!} \right), \end{aligned}$$

where we noted  $\zeta_P^{MT'}(s, \mathbf{d}) := \partial_s \zeta_P^{MT}(s, \mathbf{d})$ .

Onodera also obtains values for the second derivatives [Ono21, Theorem 5], which we will not detail here. In this thesis, we will aim to generalize the above theorem for a broader class of zeta functions. This will specifically allow us to obtain values at nonpositive integers for  $\zeta_{\mathfrak{so}(5)}(s)$  and  $\zeta_{\mathfrak{g}_2}(s)$ , as well as those for their first derivatives.

## 1.2.2 Some results on special values of multizeta functions

We will provide some results here on the special values of certain multizeta functions related to products of linear forms. In particular, we will give expressions of these values for Euler-Zagier multizeta functions:

**Definition 1.2.22.** *Let  $P \in \mathbb{N}$ , and  $\mathbf{c} = (c_1, \dots, c_P) \in H_0^P$ ,  $\mathbf{d} = (d_1, \dots, d_P) \in \mathbb{C}^P$  be complex numbers such that  $\operatorname{Re}(d_i) > -\operatorname{Re}(c_i)$  for all  $p \in \llbracket 1, P \rrbracket$ . The generalized Euler-Zagier multizeta function is defined as*

$$\zeta_P^{EZ}(\mathbf{s}, \mathbf{d}) = \sum_{\substack{n_1 \geq 1 \\ n_2, \dots, n_P \geq 0}} \prod_{p=1}^P (c_p n_p + d_p)^{-s_p}.$$

We denote  $\zeta_P^{EZ}(\mathbf{s}) := \zeta(\mathbf{s}, \mathbf{1}, (0, 1, \dots, P-1))$  the Euler-Zagier multizeta function.

Note that, through a change of variables, we can rewrite the "classical" Euler-Zagier zeta function in the following form,

$$\zeta_P^{EZ}(\mathbf{s}) = \sum_{0 < n_1 < \dots < n_P} \frac{1}{n_1^{s_1} \dots n_P^{s_P}}.$$

**Proposition 1.2.23.** *The generalized Euler-Zagier multizeta function  $\zeta_P^{EZ}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{d})$  is holomorphic on  $\{(s_1, \dots, s_P) \in \mathbb{C}^P \mid \sigma_k + \dots + \sigma_P > P + 1 - k, 1 \leq k \leq P\}$ , and is meromorphic on  $\mathbb{C}^P$ , with poles included in hyperplanes of the form*

$$s_p + \dots + s_P = (P + 1 - p) - k_p, \quad (1 \leq p \leq P, k_1, \dots, k_P \in \mathbb{N}_0).$$

From the previous proposition, we see that the quantity  $\zeta_P^{EZ}(-\mathbf{N}, \mathbf{d})$  does not always make sense. Moreover, a result from [AET01] specifies that most of the nonpositive integers are actually singularities for  $\zeta_P^{EZ}(\mathbf{s})$ . To still study the values of this function, Akiyama, Egami, and Tanigawa investigated the "regular" values of this zeta function by defining

$$\zeta_P^{EZ, \text{reg}}(-\mathbf{N}) := \lim_{s_1 \rightarrow -N_1} \dots \lim_{s_P \rightarrow -N_P} \zeta_P^{EZ}(\mathbf{s}).$$

The authors also introduced "non-regular" values by reversing the previous limits, defining

$$\zeta_P^{EZ, \text{non-reg}}(-\mathbf{N}) := \lim_{s_P \rightarrow -N_P} \dots \lim_{s_1 \rightarrow -N_1} \zeta_P^{EZ}(\mathbf{s}).$$

In [AET01] and in [AT01], the authors obtained a recursive formula between the regular values  $\zeta_P^{EZ, \text{reg}}(-\mathbf{N})$  (respectively  $\zeta_P^{EZ, \text{non-reg}}(-\mathbf{N})$  with  $\mathbf{N} \in \mathbb{N}_0^P$ , and the regular values of the form  $\zeta_{P-1}^{EZ, \text{reg}}(-\mathbf{N}')$  (respectively  $\zeta_{P-1}^{EZ, \text{non-reg}}(-\mathbf{N}')$ , with  $\mathbf{N}' \in \mathbb{N}_0^{P-1}$  to obtain an explicit expression of regular and non-regular values at nonpositive integers. These results were later generalized by Sasaki in [Sas09a] and in [Sas09b] to different orders of limits than those of regular and non-regular values at nonpositive tuples.

We can also study the "directional" values of this multizeta function. In [Kom10], Komori introduces the notion of directional value for such multizeta functions and provides an expression for these in the case of Hurwitz-Lerch multizeta functions. Komori then deduces formulas for the values at nonpositive integers of the Shintani and Euler-Zagier zeta functions, using generalized Bernoulli numbers (often not explicit).

**Definition 1.2.24.** *Let  $P, Q \in \mathbb{N}$  be integers,  $\mathbf{d} = (d_1, \dots, d_Q)$ ,  $\mathbf{c} = (c_{q,p})_{(q,p) \in \llbracket 1, Q \rrbracket \times \llbracket 1, P \rrbracket}$  and  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_P)$  be complex numbers such that  $c_{q,p} \in \overline{H}_0$ ,  $d_q \in H_0$  and  $\xi_q \in \mathbb{T} = \mathbb{C}/2i\pi\mathbb{Z}$ . We call the function  $\zeta^{HL}$  the Hurwitz-Lerch multizeta function, defined as*

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s}) = \sum_{n_1=0}^{+\infty} \dots \sum_{n_P=0}^{+\infty} \frac{e^{\xi_1 n_1} \dots e^{\xi_P n_P}}{(c_{1,1} n_1 + \dots + c_{1,P} n_P + d_1)^{s_1} \dots (c_{Q,1} n_1 + \dots + c_{Q,P} n_P + d_Q)^{s_Q}}.$$

We also define the directional values of  $\zeta^{HL}$ , following a direction  $\boldsymbol{\mu} \in \mathbb{C}^P$ :

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N}) := \lim_{s \rightarrow 0} \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N} + s\boldsymbol{\mu}).$$



Komori shows that this multizeta function is holomorphic on  $\{\mathbf{s} \in \mathbb{C} \mid \forall q \in \llbracket 1, Q \rrbracket, \operatorname{Re}(s_q) > \sigma_{0,q}\}$ , where  $\sigma_{0,q} > 0$  for all  $q \in \llbracket 1, Q \rrbracket$ . He also proves that this multizeta function has a meromorphic extension on  $\mathbb{C}^Q$ , and that the previously defined directional values make sense. Before briefly summarizing his results, we must assume the following conditions on the data of the multizeta function  $\zeta^{HL}$ .

Let's suppose that there exist an integer  $P_0 \in \llbracket 1, P \rrbracket$  and a set  $\mathcal{B} \subset \llbracket 1, Q \rrbracket \times \llbracket 1, P_0 \rrbracket$  such that

$$\begin{cases} c_{q,p} = 0 \text{ if and only if } (q,p) \in \mathcal{B}^c \\ \xi_p = 0 \text{ if and only if } p \in \llbracket 1, P_0 \rrbracket, \end{cases}$$

with  $\mathcal{B}^c = \llbracket 1, Q \rrbracket \times \llbracket 1, P_0 \rrbracket \setminus \mathcal{B}$ . Let  $\mathcal{B}_{\bullet,p} := \{q \mid (q,p) \in \mathcal{B}\}$ . Let's suppose that  $\mathcal{B}_{\bullet,p} \neq \emptyset$  for all  $p \in \llbracket 1, P_0 \rrbracket$ .

Komori gave in [Kom10] a surface integral representation (i.e an integral representation on a surface) of the multizeta function  $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s})$ . This surface integral representation particularly generalizes the usual integral representation of Riemann's zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)(e^{2i\pi s} - 1)} \int_C \frac{z^{s-1}}{e^z - 1} dz,$$

where  $C$  denotes the Hankel's contour [EMOT81, §1.10]. It is good to note that this integral representation of  $\zeta(s)$  particularly allows for obtaining a formula for the values of the Riemann zeta function at nonpositive integers using Bernoulli numbers, by using

$$\frac{z}{e^z - 1} = \sum_{k=0}^{+\infty} B_k \frac{z^k}{k!}.$$

In the same vein, Komori subsequently used the surface integral representation obtained in [Kom10, Theorem 3.14] to get an expression for the directional values in terms of generalized Bernoulli numbers  $B_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{k})$  with  $\mathbf{k} \in \mathbb{N}_0^Q$ . To see the construction of these numbers, we refer to [Kom10, Definition 3.20].

**Theorem 1.2.25.** [Kom10, Theorem 3.14, Corollary 3.16] *Under the above assumptions, the multizeta function  $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s})$  is meromorphic in  $\mathbf{s}$  on all  $\mathbb{C}^P$ . Moreover, his singularities lies in hyperplanes of the form*

$$\begin{cases} \sum_{q \in \mathcal{Q}} s_q = Q - k & (\mathcal{Q} \subset \llbracket 1, Q \rrbracket, |\mathcal{Q}| \geq 2, k \in \mathbb{N}_0) \\ s_q = 1, \dots, P & (1 \leq q \leq Q). \end{cases}$$

Moreover, Komori shows in [Kom10, Theorem 3.22] that the directional values  $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})$  are well-defined, and he provides a non-explicit expression for them, imposing a non-vanishing condition on the sum of the directions. Incidentally, it turns out that if one manages to re-express the generalized Bernoulli numbers in the formula obtained by Komori, then this formula can be used to determine whether a nonpositive tuple is a singularity for  $\zeta^{HL}$ . Indeed, if one obtains two distinct values of  $\zeta^{HL}$  at a nonpositive tuple, along two different directions, this particularly implies that the nonpositive tuple is a singularity.

**Remark 1.2.26.** *The directional formula [Kom10, Theorem 3.22] obtained by Komori is not very explicit, as it is sometimes difficult to get at a satisfactory expression for the generalized Bernoulli numbers he introduces in his article.*

In [EM20], Essouabri and Matsumoto studied the directional values of the generalized Euler-Zagier multizeta function. Here, we will only provide a very simplified and much less detailed formula:

**Theorem 1.2.27.** [EM20, Theorem 1] *Let  $\mathbf{N} \in \mathbb{N}_0^P$ , and  $\mathbf{c} = (c_1, \dots, c_P) \in \mathbb{C}^P$  such that  $\operatorname{Re}(\mu_p + \dots + \mu_P) \neq 0$  for all  $p \in \llbracket 1, P \rrbracket$ . Then  $\zeta_{\boldsymbol{\mu}}^{EZ}(-\mathbf{N}, \mathbf{c}, \mathbf{d}) := \lim_{s \rightarrow 0} \zeta_{\boldsymbol{\mu}}^{EZ}(-\mathbf{N} + s\boldsymbol{\mu}, \mathbf{c}, \mathbf{d})$  exists, and we have that:*

$$\zeta_{\boldsymbol{\mu}}^{EZ}(-\mathbf{N}, \mathbf{c}, \mathbf{d}) = \sum_{\mathbf{k}=(k_1, \dots, k_P) \in \llbracket 0, |\mathbf{N}|+P+1 \rrbracket^P} P_{\mathbf{k}}(\mathbf{b}, \mathbf{c}, \boldsymbol{\mu}) \prod_{i=1}^P B_{k_i},$$

with  $B_n$  being the  $n$ -th Bernoulli number, and  $P_{\mathbf{k}}(\mathbf{b}, \mathbf{c}, \boldsymbol{\mu})$  an explicit polynomial in  $c_p, d_p$  and  $\mu_p$  for  $p \in \llbracket 1, P \rrbracket$ .

**Remark 1.2.28.** 1) In [EM21], Essouabri and Matsumoto generalized this last result for a larger class of multizeta functions, which are of the form

$$\zeta_m(\mathbf{s}, \mathbf{P}) = \sum_{n_1, \dots, n_m} \frac{1}{\prod_{j=1}^m P_j(n_1, \dots, n_j)^{s_j}},$$

where  $P_j$  is a polynomial in  $X_1, \dots, X_j$  for all  $1 \leq j \leq m$ , such that they respect the HOS condition (see [Ess97]), where  $P_m$  is homogeneous and elliptic and such that

$$\forall 1 \leq j \leq m, \lim_{x_1 + \dots + x_j \rightarrow +\infty} P_j(x_1, \dots, x_j) = +\infty.$$

2) In [MW02], Matsumoto and Weng studied zeta functions of the form

$$\zeta^{MW}(s) = \sum_{n=1}^{+\infty} \frac{P(n)}{Q(n)^s},$$

with  $P$  and  $Q$  two nonzero polynomials such that  $Q$  doesn't vanish at positive integers. They provided an explicit formula for its derivative value at  $s = 0$ . This formula is of particular interest when studying the regularized determinant of the Laplacian of the  $n$ -sphere (see [Var88], [WY96], [QC96], [Kum99]).

### 1.2.3 On asymptotic formulas for some partitions

We will explain in this subsection some arithmetic applications of computing special values of Witten's zeta functions, and their derivatives. We will also discuss Meinardus' theorem on the asymptotic behavior of some partition numbers generated by infinite product. We will also discuss some recent work from Bridges, Brindle, Bringmann, Craig, Franke, Males on asymptotic formulas for more general partition numbers in the preprints [BB23], [BBF23], [BCM23]. We will give in particular the statements of the results obtained by Bridges, Bringmann, Brindle, Franke in [BBBF23], and we will talk about their approach.

In combinatorics, it is standard to use a generating series of the form

$$\omega(q) = \sum_{n \geq 0} r(n)q^n$$

in order to study the behavior of a sequence of real numbers  $r(n)$ . If the function  $\omega$  is holomorphic for  $|q| < 1$ , then by Cauchy's theorem we have

$$r(n) = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{\omega(q)}{q^{n+1}} dq,$$

where  $\mathcal{C}$  is a circle centered at zero running counterclockwise, and included in the unit disk. In the general case, no general result exists to obtain an asymptotic formula for a sequence  $r(n)$ . In the rest of this section, we will focus on sequences obtained by considering a generating series written as an infinite product.

We consider a function  $G_f(q)$  of the form

$$G_f(q) = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f(n)}},$$

where  $f(n)$  are real numbers. By distributing the product, we obtain

$$G_f(q) = \sum_{n=0}^{+\infty} p_f(n)q^n. \tag{1.7}$$

We consider the Dirichlet series

$$L_f(s) = \sum_{n=1}^{+\infty} \frac{f(n)}{n^s} \quad (s = \sigma + i\tau, \sigma > \sigma_0),$$

with  $\sigma_0$  its abscissa of convergence.

**Example 1.2.29.** [And76, 5.1.2] *The simplest infinite product one can think of corresponds to studying the number of partitions  $p(n)$  of a positive integer  $n$ :*

$$\prod_{n=1}^{+\infty} \frac{1}{1-q^n} = \sum_{n=1}^{+\infty} p(n)q^n.$$

*The corresponding Dirichlet series is Riemann's zeta function. Hardy and Ramanujan showed that as  $n \rightarrow +\infty$ ,*

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

We now set

$$g_f(z) := \sum_{n=1}^{+\infty} f(n)e^{-nz}.$$

We assume the following conditions:

- i)  $\sigma_0 > 0$ .
- ii)  $L_f$  is meromorphic continuation to  $H_{-C_0}$ , with  $0 < C_0 < 1$  and such that  $L_f$  has only one pole at  $s = \sigma_0$  in this domain, and that this pole is of order 1 and has residue  $A$ .
- iii) We have the following upper bound as  $\tau \rightarrow +\infty$ , and uniform in  $\sigma > -C_0$ ,

$$|L_f(s)| = O(|\tau|^{C_1}),$$

with  $C_1 > 0$  and  $s = \sigma + i\tau$ .

- iv) Let  $\epsilon > 0$ . For all  $z = x + 2i\pi y$  such that  $\arg(z) > \frac{\pi}{4}$ ,  $|x| \leq \frac{1}{2}$  and  $|y| \ll 1$ , we have

$$\operatorname{Re}(g_f(z)) - g_f(x) \leq -C_2 x^{-\epsilon},$$

with  $C_2 > 0$  a constant depending on  $\epsilon$ .

Meinardus' theorem then provides an asymptotic relation for the sequence  $p_f(n)$ .

**Theorem 1.2.30** (of Meinardus). [And76, Theorem 6.2] *As  $n \rightarrow +\infty$ , we have*

$$p_f(n) = Cn^\kappa \exp\left(n^{\frac{\alpha}{\alpha+1}} \left(1 + \frac{1}{\alpha}\right) (A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha}}\right) (1 + O(n^{-\kappa_1})),$$

where  $p_f(n)$  are defined in formula 1.7, and  $C, \kappa, \kappa_1$  are real numbers depending on  $L_f(0), L'_f(0), C_0$ , and  $\alpha$ .

The proof of this theorem relies on the saddle point method, key aspects of which will be explained here. For details of the method, we refer to [FS09, VIII.3]. The method takes advantage of the fact that the integral  $\frac{1}{2i\pi} \int_{\mathcal{C}} \frac{G_f(q)}{q^{n+1}} dq$  is independent of the radius of circle  $\mathcal{C}$  within the unit disk. This independence can be exploited by judiciously choosing an appropriate radius  $\rho_0(n)$ . The saddle point

method involves selecting this radius in such a way as to minimize  $\left| \frac{G_f(q)}{q^n} \right|$  over a minor arc  $\mathcal{C}_1(\theta_0(n))$ . Specifically, we choose such a saddle point so that the derivative of the integrand at  $q = \rho_0(n)$  is  $O(n^\delta)$ , where  $\delta$  is well-chosen. Alternatively, we can find  $\rho_n$  by studying the derivative of the function

$$h_f(z) := \ln(L_f(z)) - n \log(z).$$

We then look for  $\rho_n$  such that  $h'_f(\rho_0(n)) = -n + O(n^\delta)$ .

We will then have a quadratic approximation when  $\theta \rightarrow 0$ :

$$h_f(\rho_0(n)e^{i\theta}) - h_f(\rho_0(n)) =: -\frac{1}{2}\beta(\rho_0(n))\theta^2 + o(\theta^3) + O(n^\delta),$$

where  $\beta(\rho(n))$  is a constant depending on  $\rho(n)$ , and the choice of  $\delta$ . We will then aim to determine an angle  $\theta_0(n)$  such that

$$\beta(\rho_0(n))\theta_0(n)^2 \xrightarrow{\theta \rightarrow 0} +\infty, \quad h''(\rho_0(n))\theta_0(n)^3 \xrightarrow{\theta \rightarrow 0} 0.$$

For such a choice of  $\theta_0(n)$ , we deduce in particular that the behavior of the integral over the minor arc  $\mathcal{C}_0$  behaves like a Gaussian integral, whose bounds depend on  $n$ , which can be approximated.

Debruyne and Tennenbaum obtained a more general version of Meinardus' theorem in [DT20] by weakening condition iii) on the growth of  $L_f$  when the imaginary part  $\tau$  is large. Upon studying Witten's zeta functions, we will realize that Meinardus' theorem cannot be straightforwardly applied to obtain an asymptotic expression for the number of representations of the Lie algebra  $r_{\mathfrak{g}}(n)$  that are of interest to us. However, by cleverly modifying the proof of Meinardus' Theorem, and then studying the residues of  $\zeta_{\mathfrak{sl}(3)}(s)$  at  $s = \frac{1}{2}, \frac{2}{3}$ , and by calculating the special values  $\zeta_{\mathfrak{sl}(3)}(0)$  and  $\zeta'_{\mathfrak{sl}(3)}(0)$ , Romik obtained the following theorem:

**Theorem 1.2.31.** [Rom17, Theorem 1.1] *We have*

$$r_{\mathfrak{sl}(3)}(n) \underset{n \rightarrow +\infty}{\sim} \frac{K}{n^{3/5}} \exp\left(A_1 n^{2/5} - A_2 n^{3/10} - A_3 n^{1/5} - A_4 n^{1/10}\right),$$

with the constant  $A_1, A_2, A_3, A_4$  and  $K$  being explicit:

Let

$$X = \left( \frac{1}{9} \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{5}{3}\right) \right), \quad Y = -\sqrt{\pi} \zeta\left(\frac{1}{2}\right) \zeta\left(\frac{3}{2}\right).$$

The constants  $A_1, A_2, A_3, A_4, K$  are such that:

$$A_1 = 5X^2 = 6.858260476\dots$$

$$A_2 = X^{-1}Y = 5.77360174\dots$$

$$A_3 = \frac{3}{80}X^{-4}Y^2 = 0.91134107\dots$$

$$A_4 = \frac{11}{3200}X^{-7}Y^3 = 0.35163754\dots$$

$$K = \frac{2\sqrt{3\pi}}{\sqrt{5}}X^{1/3} \exp\left(-\frac{1}{2560}X^{-10}Y^4\right) = 2.44629033486\dots$$

Note that in [Rom17], the computation of the constant  $Y$  particularly depends on the special value  $\zeta'_{\mathfrak{sl}(3)}(0) = -\ln(2\pi) + \ln(2)$  computed in [BD18]. Moreover, all these constants also depend on the calculation of the residues of  $\zeta_{\mathfrak{sl}(3)}(s)$  at  $s = \frac{1}{2}$  and  $s = \frac{2}{3}$ .

Later on, Bridges, Bringmann and Franke obtained in [BBF23] an asymptotic formula for the number of irreducible representations of dimension  $n$  of  $\mathfrak{sl}(3)$  by studying the Witten zeta function  $\zeta_{\mathfrak{sl}(3)}(s)$ , and using a Tauberian type theorem.

Romik conjectures that it is possible to obtain similar results on the representation numbers  $r_{\mathfrak{g}}(n)$  for other Lie algebras  $\mathfrak{g}$ . To extend this result, one needs to establish a Meinardus-type theorem, and also obtain information about the poles and residues of the corresponding Witten zeta function in the strip  $0 \leq \operatorname{Re}(s) \leq 1$  for the Witten zeta function  $\zeta_{\mathfrak{g}}(s)$ , and then obtain an expression for  $\zeta'_{\mathfrak{g}}(0)$ .

Motivated by this question, Bridges, Brindle, Bringmann, and Franke have established a more general variant of Meinardus' Theorem in the preprint [BBBF23]. Let  $f : \mathbb{N} \rightarrow \mathbb{N}_0$ . We define for all  $q = e^{-z}$ ,  $z \in H_0$ , the functions

$$G_f(z) = \sum_{n \geq 0} p_f(n) q^n = \prod_{n \geq 1} \frac{1}{(1 - q^n)^{f(n)}}, \quad L_f(s) = \sum_{n \geq 1} \frac{f(n)}{n^s}.$$

Let  $\Lambda := \mathbb{N}_0 \setminus f^{-1}(\{0\})$ . We assume that:

(P1) Let  $\alpha > 0$  be the largest poles of  $L_f$ . There exists an integer  $L \in \mathbb{N}_0$  such that, for all prime numbers  $p$ ,

$$|\Lambda \setminus (p\mathbb{N}_0 \cap \Lambda)| \geq L > \frac{\alpha}{2}.$$

(P2) There exists a real number  $R \in \mathbb{R}_+$  such that  $L_f$  is meromorphic on  $\overline{H}_{-R} = \{z \in \mathbb{C}, \operatorname{Re}(z) \geq -R\}$ , and such that  $L_f$  is holomorphic on the line  $(\operatorname{Re}(z) = -R)$ . Furthermore, we assume that the meromorphic function  $L_f^*(s) := \Gamma(s)\zeta(s+1)L_f(s)$  has only real poles  $\alpha := \gamma_1 > \dots$ , and that these poles are simple except at  $s = 0$ , where the pole may be a double pole.

(P3) There exists a real number  $a < \frac{\pi}{2}$  such that, on each strip  $\sigma_1 \leq \sigma \leq \sigma_2$  in the domain of holomorphy of  $L_f$ , we have

$$L_f(s) = O_{\sigma_1, \sigma_2} \left( e^{a|\tau|} \right) \quad (s = \sigma + i\tau),$$

as  $|\tau| \rightarrow +\infty$ .

**Theorem 1.2.32.** [BBBF23, Theorem 1.4] *Assume conditions (P1), (P2), and (P3). Let  $L$  be the real number from condition (P1) and  $R$  be the real number from condition (P2). Then for some  $M, N \in \mathbb{N}_0$ ,*

$$p_f(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp \left( A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^M A_j n^{\alpha_j} \right) \left( 1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R} \left( n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}} \right) \right),$$

with  $0 \leq \alpha_M < \dots < \alpha_1 := \frac{\alpha}{\alpha+1}$ ,  $0 < \nu_2 < \dots < \nu_N$ ,  $A_1, \dots, A_M$  and  $B_2, \dots, B_N$  being explicit constants.

When the meromorphic function  $L_f$  has only two strictly positive poles, Bridges, Brindle, Bringmann, and Franke obtained a more precise formula. Let us first introduce the following constants: Let  $\alpha > \beta > 0$  be the only two positive poles of  $L_f$ . We denote  $\omega_\alpha = \operatorname{Res}_{s=\alpha}(L_f(s))$  and  $\omega_\beta = \operatorname{Res}_{s=\beta}(L_f(s))$  as the residues of  $L_f$  at its positive poles, and we set  $c_1 = \omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1)$ ,  $c_2 = \omega_\beta \Gamma(\beta+1)\zeta(\beta+1)$ ,  $c_3 = L_f(0)$ . Let

$$C = \frac{e^{L'_f(0)(\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1)) \frac{\frac{1}{2} - L_f(0)}{\alpha+1}}}{\sqrt{2\pi(\alpha+1)}} \quad b = \frac{1 - L_f(0) + \frac{\alpha}{2}}{\alpha+1}.$$

Let  $\mathcal{P}_R$  be the union of  $\{0\}$  and of the poles greater than  $-R$  of  $L_f^*$

$$\mathcal{L} = \frac{1}{\alpha+1} \mathcal{P}_R + \sum_{\mu \in \mathcal{P}_R} \left( \frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N}_0,$$

$$\mathcal{M} := \frac{\alpha}{\alpha+1} \mathbb{N}_0 + \left( - \sum_{\mu \in \mathcal{P}_R} \left( \frac{\mu+1}{\alpha-1} - 1 \right) \mathbb{N}_0 \right) \cap \left[ 0, \frac{R+\alpha}{\alpha+1} \right].$$

**Theorem 1.2.33.** [BBBF23, Theorem 4.4] We assume conditions (P1), (P2), and (P3). Let  $L$  be the real number from condition (P1) and  $R$  be the real number from condition (P2). Moreover, we assume that  $L_f$  has only two poles  $\alpha > \beta > 0$  such that there exists an integer  $l \in \mathbb{N}$  satisfying the inequality  $\frac{l+1}{l}\beta < \alpha < \frac{l}{l-1}\beta$ . Then we have

$$p_f(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp \left( A_1 n^{\frac{\alpha}{\alpha+1}} + A_2 n^{\frac{\beta}{\alpha+1}} + \sum_{j=3}^{l+1} A_j n^{\frac{(j-1)\beta}{\alpha+1} + \frac{j-2}{\alpha+1} + 2-j} \right) \left( 1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R} \left( n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}} \right) \right),$$

with  $A_1 := (\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{1}{\alpha+1}} \left( 1 + \frac{1}{\alpha} \right)$ , and  $A_2 := \frac{\omega_\beta \Gamma(\beta) \zeta(\beta+1)}{(\omega_\alpha \Gamma(\alpha) \zeta(\alpha+1))^{\frac{\beta}{\alpha+1}}}$ , and for all  $j \geq 3$ ,

$$A_j := K_j + \frac{c_1^{\frac{1}{\alpha+1}}}{\alpha} \sum_{m=1}^l \binom{-\alpha}{m} \sum_{\substack{0 \leq k_1 \leq \dots \leq k_l \leq m \\ |\mathbf{k}|=m \\ k_1+2k_2+\dots+l k_l=j-1}} \binom{m}{\mathbf{k}} \frac{K_2^{k_1} \dots K_{l+1}^{k_l}}{c_1^{\frac{m}{\alpha+1}}} \\ + \frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}} \sum_{m=1}^l \binom{-\beta}{m} \sum_{\substack{0 \leq k_1 \leq \dots \leq k_l \leq m \\ |\mathbf{k}|=m \\ k_1+2k_2+\dots+l k_l=j-2}} \binom{m}{\mathbf{k}} \frac{K_2^{k_1} \dots K_{l+1}^{k_l}}{c_1^{\frac{m}{\alpha+1}}},$$

with  $(K_j)_{j \geq 3}$  described in [BBBF23, Lemma 4.3], and  $0 < \nu_2 < \dots$  running over the positive elements of  $\mathcal{L} + \mathcal{M}$ .

**Remark 1.2.34.** To study the asymptotic behavior of  $r_{\mathfrak{g}_2}(n)$ , we would need to know  $K_3$ , and according to [BBBF23, Lemma 4.3], we have

$$K_3 = \frac{c_2^2(\alpha - 2\beta)}{2(\alpha + 1)^2 c_1^{\frac{2\beta+1}{\alpha+1}}},$$

with  $c_1$  and  $c_2$  being explicit constants such that

$$c_1 = \omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1), \quad c_2 = \omega_\beta \Gamma(\beta+1) \zeta(\beta+1).$$

Moreover, to study the asymptotic behavior of  $r_{\mathfrak{g}_2}(n)$ , we will see that the integer  $l = 2$  works because, we would then have  $L_f(s) = \zeta_{\mathfrak{g}_2}(s)$ , and the positive poles of this zeta function are  $s = \frac{1}{3}$  and  $s = \frac{1}{5}$ .

By applying this theorem, the four authors obtained results similar to Romik for the number of representations  $r_{\mathfrak{so}(5)}(n)$ :

**Theorem 1.2.35.** [BBBF23, Theorem 1.3] For all positive integers  $N \geq 1$ , we have

$$r_{\mathfrak{so}(5)}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^{\frac{7}{12}}} \exp \left( A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4 \right) \left( 1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N \left( n^{-\frac{N+1}{9}} \right) \right),$$

with

$$b = \frac{7}{12}, \quad C = \frac{e^{\zeta'_{\mathfrak{so}(5)}(0)} \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}}{2^{\frac{1}{3}} 3^{\frac{11}{24}} \sqrt{\pi}}, \quad A_1 = \frac{3^{\frac{4}{3}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{3}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{8}{3}}}, \\ A_2 = \frac{2^{\frac{8}{9}} \left(2^{\frac{1}{3}} + 1\right) \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right)}{3^{\frac{7}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{9}}}, \quad A_3 = \frac{2^{\frac{40}{9}} \left(2^{\frac{1}{3}} + 1\right)^2 \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{1}{3}\right)^2 \zeta\left(\frac{4}{3}\right)^2}{3^{\frac{44}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{20}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{10}{9}}}, \\ A_4 = \frac{2^8 \left(2^{\frac{1}{3}} + 1\right)^3 \Gamma\left(\frac{1}{3}\right)^3 \zeta\left(\frac{1}{3}\right)^3 \zeta\left(\frac{4}{3}\right)^3}{3^8 \Gamma\left(\frac{1}{4}\right)^4 \zeta\left(\frac{3}{2}\right)^2}.$$

On the other hand, these four authors did not have an explicit formula for  $\zeta'_{\mathfrak{so}(5)}(0)$ . In our work, we will provide an expression for the special values  $\zeta'_{\mathfrak{so}(5)}(0)$  and  $\zeta'_{\mathfrak{g}_2}(0)$ . We will also provide formulas for the residues of  $\zeta_{\mathfrak{so}(5)}(s)$  at  $s = \frac{1}{2}$  and  $s = \frac{1}{3}$ , and formulas for the residues of  $\zeta_{\mathfrak{g}_2}(s)$  at  $s = \frac{1}{3}$  and  $s = \frac{1}{5}$ . We will then be able to obtain an explicit asymptotic formula for the number of representations  $r_{\mathfrak{so}(5)}(n)$  from the computation of  $\zeta'_{\mathfrak{so}(5)}(0)$ , allowing us to have an explicit expression for the coefficient  $C$  from the previous theorem.

We should note that we will find a different value for  $\zeta_{\mathfrak{so}(5)}(0)$  than in [BBBF23, Proposition 5.15]. We interpret this difference in results by suggesting that the authors may have made an error in replacing  $\zeta(0)$  by  $\frac{1}{2}$  instead of  $-\frac{1}{2}$  in the proof of Proposition 5.15.

We will also obtain explicit formulas for  $\zeta_{\mathfrak{g}_2}(0)$  and  $\zeta'_{\mathfrak{g}_2}(0)$ , and we will then be able to establish a theorem similar to the previous one, for the number of representations  $r_{\mathfrak{g}_2}(n)$  using Theorem 1.2.33.

**Remark 1.2.36.** *When we will apply the Theorem 1.2.32 to the number of representations  $r_{\mathfrak{g}}(n)$  of a semi-simple Lie algebra, we will see that the coefficients  $f(n)$  will correspond to the number of irreducible representations of dimension  $n$  of  $\mathfrak{g}$ . Therefore, the function  $L_f$  will correspond to the Witten zeta function  $\zeta_{\mathfrak{g}}$ .*

### 1.3 First definitions

In this manuscript, we shall consider a family  $(l_q)_{q \in \llbracket 1, Q \rrbracket}$  of linear forms of rank  $P$  with coefficients in the Poincaré half-plane  $H_0 = \{s = \sigma + i\tau \in \mathbb{C} \mid \sigma > 0\}$ , where  $P \geq 1$  and  $Q \geq 1$  are positive integers. We also fix  $\mathbf{d} = (d_p)_{p \in \llbracket 1, P \rrbracket}$ ,  $\mathbf{d}' = (d'_q)_{q \in \llbracket 1, Q \rrbracket}$  as complex numbers in  $H_0$  such that  $d'_q = l_q(\mathbf{d})$  for all  $1 \leq q \leq Q$ .

**Notation.** Let  $q \in \llbracket 1, Q \rrbracket$ , we denote

$$l_q(\mathbf{n}) = \sum_{p=1}^P c_{q,p} n_p.$$

For the sake of clarity, throughout this manuscript, we will choose to use the index  $p$  (or index  $i$  if  $p$  is already used) when summing or taking product over any subset  $\mathcal{P}$  of  $\llbracket 1, P \rrbracket$ , and we will use the index  $q$  (or the index  $j$  if  $q$  is already used) when summing or taking product over any subset  $\mathcal{Q}$  of  $\llbracket 1, Q \rrbracket$ .

We will first define the incomplete gamma functions.

**Definition 1.3.1.** Let  $\nu \in H_0$ ,  $\theta > 0$ ,  $s \in \mathbb{C}$ . We call incomplete gamma functions the following functions:

$$\begin{aligned} \Gamma(s, \theta, \nu) &= \int_{\theta}^{+\infty} e^{-\nu y} y^{s-1} dy, \\ \gamma(s, \theta, \nu) &= \int_0^{\theta} e^{-\nu y} y^{s-1} dy, \quad \text{when } \operatorname{Re}(s) > 0. \end{aligned}$$

These gamma functions are more general than those that can be found in the literature (for example, in [EMOT81, Chap.IX] or in [Cra12, §3]). The classical incomplete gamma functions can be recovered by setting  $\nu = 1$ . For our purposes, we will need an arbitrary  $\nu$  in  $H_0$ .

We now introduce constants that will appear in theorem D. Through partial fraction decomposition, we can easily find:

**Proposition 1.3.2.** Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ . Let's consider  $\mathbb{K}$  the field above  $\mathbb{Q}$ , generated by the coefficients  $c_{q,p}$ , with  $1 \leq q \leq Q$  and  $1 \leq p \leq P$ . Let  $\mathcal{P} \subset \llbracket 1, P \rrbracket$ ,  $j, f \in \llbracket 1, Q \rrbracket$ ,  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}}$ ,  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c}$ . Let's note  $\mathcal{P}^c = \llbracket 1, P \rrbracket \setminus \mathcal{P}$ ,  $\mathbf{w}_p = (w_{p,q})_{q \in \llbracket 1, Q \rrbracket}$  and  $\mathbf{v}_p = (v_{p,q})_{q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}}$ . There exist a polynomial  $\tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'} \in \mathbb{K}[x]$  and two sequences of constants  $(C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}'))_{\lambda \in \mathbb{N}_0}$  and  $(D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}'))_{\lambda \in \mathbb{N}_0, p \in \mathcal{P}}$  such that all their terms belong in the field  $\mathbb{K}$ , and such that

$$\begin{aligned} x^{-N'_f - 1 + \mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x)^{-N_p - 1 - |\mathbf{v}_p|} &= \tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}(x) + \sum_{\lambda=1}^{N'_f + 1 - \mathbf{w}(f)} \frac{C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')}{x^\lambda} \\ &+ \sum_{p \in \mathcal{P}} \sum_{\lambda=1}^{(|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|)} \frac{D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}')}{(c_{j,p} + c_{f,p} x)^\lambda}. \end{aligned}$$

We consider  $P_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}$  the primitive function which vanish at 1 of the function  $\tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}$ . Let's note  $E_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \in \mathbb{K}$  the constant term of  $P_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}$ .

We now set

$$\begin{aligned} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') &= -E_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_f + 1 - \mathbf{w}(f)} \frac{C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} \\ &- \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{(|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|)} \frac{D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}) \\ &+ \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (1, p)}(\mathbf{N}, \mathbf{N}') \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \right). \end{aligned}$$



**Remark 1.3.3.** 1) Note that if  $-N'_f - 1 + \mathbf{w}(f) \geq 0$ , then  $C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda} = 0$  for all  $\lambda$ .

2) In the product  $\prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p}x)^{(-N_p-1-|\mathbf{v}_p|)}$ , some terms of the product may appear multiple times.

This is why the sum  $\sum_{\lambda=1}^{(|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|)}$  goes up to  $|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|$ . If this occurs, then we have multiple possible choices for the constants  $D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}$ . Note that these different choices will have no impact on the definition of the constant  $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ .

3) The constant  $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$  will appear in the calculation of an integral, in Lemma 3.1.4.

**Example 1.3.4.** Let  $\mathcal{P} = \emptyset$ , we consider for all  $\lambda \geq 1$ ,

$$\begin{aligned} C_{\emptyset,j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}') &= \delta_{\lambda, N'_f+1-\mathbf{w}(f)} \\ D_{\emptyset,j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}') &= 0 \\ E_{\emptyset,j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &= 0 \text{ if } \mathbf{w}(f) \leq N'_f \\ E_{\emptyset,j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &= -\frac{1}{\mathbf{w}(f) - N'_f} \text{ if } \mathbf{w}(f) > N'_f, \end{aligned}$$

where we noted  $\delta_{\cdot, \cdot}$  the Kronecker symbol.

In particular, we get that  $F_{\emptyset,j,f,\mathbf{0},\mathbf{0}}(\mathbf{0}, \mathbf{0}) = 0$ .

**Example 1.3.5.** Let  $\mathcal{P} = \{p\}$ ,  $j \neq f \in \llbracket 1, Q \rrbracket$ ,  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ ,  $\mathbf{v} = \mathbf{0}$  and  $\mathbf{w} = \mathbf{e}_f$ , we then get that

$$x_f^{-N'_f-1+\mathbf{w}(f)} (c_{j,p} + c_{f,p}x_f)^{-N_p-1-|\mathbf{v}_p|} = (c_{j,p} + c_{f,p}x_f)^{-1}.$$

Therefore, we have  $F_{\{p\},j,f,\mathbf{0},\mathbf{e}_f} = \frac{1}{c_{f,p}} \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \right)$ .

**Example 1.3.6.** Let  $\mathcal{P} = \{p\}$ ,  $j \neq f \in \llbracket 1, Q \rrbracket$ ,  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ ,  $\mathbf{v} = \mathbf{0}$  and  $\mathbf{w} = \mathbf{e}_j$ , we then get that

$$x_f^{-N'_f-1+\mathbf{w}(f)} (c_{j,p} + c_{f,p}x_f)^{-N_p-1-|\mathbf{v}_p|} = x_f^{-1} (c_{j,p} + c_{f,p}x_f)^{-1} = \frac{1}{c_{j,p}} x^{-1} - \frac{c_{f,p}}{c_{j,p}} (c_{j,p} + c_{f,p}x_f)^{-1}.$$

Therefore, we have  $F_{\{p\},j,f,\mathbf{0},\mathbf{e}_j} = -\frac{1}{c_{j,p}} \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \right)$ .

We shall study in our thesis the following multizeta function:

**Definition 1.3.7.** Let  $(\mathbf{s}, \mathbf{s}') \in H_1^P \times H_1^Q$ , then we set

$$Z(\mathbf{s}, \mathbf{s}') = Z(\mathbf{c}, \mathbf{d}, \mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}.$$

**Remark 1.3.8.** This multizeta function is of the multiple Hurwitz-Lerch type described in Definition 1.2.24, without torsion:

$$Z(\mathbf{s}, \mathbf{s}') = \zeta^{HL}(\mathbf{0}, \tilde{\mathbf{d}}, \tilde{\mathbf{c}}, (s_1, \dots, s_P, s'_1, \dots, s'_Q)) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{\prod_{q=1}^{\tilde{Q}} \left( \tilde{d}_q + \sum_{p=1}^P \tilde{c}_{q,p} n_p \right)^{s'_q}},$$

with  $P_0 = P$ ,  $\tilde{Q} := P + Q$ ,  $\mathcal{B}^c = \cup_{1 \leq p \leq P} \llbracket 1, P \rrbracket \setminus \{p\} \times \{p\} \subset \llbracket 1, Q + P \rrbracket \times \llbracket 1, P \rrbracket$ ,  $\tilde{\mathbf{d}} = (\mathbf{d}, \mathbf{d}')$ , and with  $\tilde{\mathbf{c}} = (c_{(q,p)})_{(q,p) \in \llbracket 1, Q \rrbracket \times \llbracket 1, P \rrbracket}$  such that

$$\begin{cases} \tilde{c}_{q,p} = 0 & \text{if } q \neq p \text{ and } q \leq P \\ \tilde{c}_{p,p} = 1 & \text{if } p \in \llbracket 1, P \rrbracket \\ \tilde{c}_{q,p} = c_{q-P,p} & \text{if } q > P. \end{cases}$$

Therefore, we can apply the Theorem 1.2.25 for  $Z(\mathbf{s}, \mathbf{s}')$ .

With the help of the previous remark, and Theorem 1.2.25, we see that the multizeta function  $Z(\mathbf{s}, \mathbf{s}')$  is meromorphic over the entire  $\mathbb{C}^P \times \mathbb{C}^Q$ , and this theorem also provides information about the set of singularities of  $Z(\mathbf{s}, \mathbf{s}')$ . We also have a domain of holomorphy, however for our needs, we require a larger domain of convergence than what is provided by Theorem 1.2.25. We will then show a more precise domain of convergence in the following proposition:

**Proposition 1.3.9.** *We set*

$$\mathcal{D} = \mathcal{D}_{P,Q} := \left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket \right\}.$$

The multizeta function  $Z(\mathbf{s}, \mathbf{s}')$  has a domain of convergence included in  $\mathcal{D}$ , and is meromorphic over  $\mathbb{C}^P \times \mathbb{C}^Q$ , with singularities included in the following union of hyperplanes:

$$\begin{cases} \sum_{p \in \mathcal{P}} s_p + \sum_{q \in \mathcal{Q}} s'_q = Q + P - n & (\mathcal{P} \subset \llbracket 1, P \rrbracket, \mathcal{Q} \subset \llbracket 1, Q \rrbracket, |\mathcal{P} \cup \mathcal{Q}| \geq 2, n \in \mathbb{N}_0) \\ s_p = 1, \dots, P & (1 \leq p \leq P) \\ s'_q = 1, \dots, P & (1 \leq q \leq Q). \end{cases}$$

We will denote in the same way  $Z(\mathbf{s}, \mathbf{s}')$  its meromorphic extension. In general, the values at nonpositive tuple do not make sense since some nonpositive tuple are singularities for  $Z$ . That's why we will fix a direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}')$  and a  $P+Q$ -tuple  $-(\mathbf{N}, \mathbf{N}')$ , and study the multizeta function  $Z$  at this  $P+Q$ -tuple, and in the chosen direction. It follows from Proposition 1.3.9 that  $s \mapsto Z(-\mathbf{N} + s\boldsymbol{\mu}, -\mathbf{N}' + s\boldsymbol{\mu}')$  is meromorphic on  $\mathbb{C}$ , and is holomorphic for  $s = \sigma + i\tau$ , with  $\sigma$  sufficiently large. We can then define the following zeta function:

**Definition 1.3.10.** *Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in (\mathbb{R}_+)^P \times (\mathbb{R}_+^*)^Q$ . We then define for  $s \in \mathbb{C}$ ,*

$$Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(s) := Z(-\mathbf{N} + s\boldsymbol{\mu}, -\mathbf{N}' + s\boldsymbol{\mu}').$$

We also consider the following univariate function, which consists of evaluating the previously described function in the direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ , and for  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ .

**Definition 1.3.11.** *We also define for  $s \in \mathbb{C}$  the following function*

$$Z_{\Delta} : s \in \mathbb{C} \mapsto Z((s, \dots, s), (s, \dots, s)).$$

**Remark 1.3.12.** *If we set  $\boldsymbol{\mu} = \mathbf{0}$ , and  $\boldsymbol{\mu}' = \mathbf{1}$ , then  $Z_{\mathbf{0}, \mathbf{0}'}^{\mathbf{0}, \mathbf{1}}(s)$  corresponds to a Shintani-type zeta function*

$$Z_{\mathbf{0}, \mathbf{0}'}^{\mathbf{0}, \mathbf{1}}(s) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s}.$$

Keep in mind that the coefficients  $d'_q$  are not free, and depend on the choices of  $d_p$  and on the choices of the coefficients  $(c_{p,q})_{1 \leq p \leq P, 1 \leq q \leq Q}$ .

By Proposition 1.3.9, we know that  $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  is holomorphic on  $\{s \in \mathbb{C} \mid (-\mathbf{N} + s\boldsymbol{\mu}, -\mathbf{N}' + s\boldsymbol{\mu}') \in \mathcal{D}\}$ . Therefore, there is a real  $\sigma_{\mathbf{0}, \mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  such that the function  $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  is holomorphic on  $H_{\sigma_{\mathbf{0}, \mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}}$ . We shall say that  $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  is holomorphic on  $\sigma \gg_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'} 1$ , or sometimes only  $\sigma \gg 1$  when there is no ambiguity.

In order to give meaning to the directional values of  $Z$  at nonpositive tuple, we must prove that  $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  is regular at  $s = 0$ . Works by Komori indeed prove that this directional value has a meaning. In this manuscript, we will detail a different extension method in Proposition 2.2.4 to prove that these directional values have meaning. That proposition will **mainly** allow us to obtain explicit formulas for the special directional values of  $Z$ , as well as those of its derivative.

**Notation.** Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in (\mathbb{R}_+)^P \times (\mathbb{R}_+^*)^Q$ . We consider the values in  $(-\mathbf{N}, -\mathbf{N}')$  of  $Z(\mathbf{s}, \mathbf{s}')$  along the direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}')$ ,

$$Z(-(\mathbf{N}, \mathbf{N}')) := Z_{\mathbf{N}, \mathbf{N}'}(0),$$

and we consider the derivative value along a direction  $(-\mathbf{N}, -\mathbf{N}')$  of  $Z(\mathbf{s}, \mathbf{s}')$ , along the direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}')$ ,

$$Z'(-(\mathbf{N}, \mathbf{N}')) := \partial_s(Z_{\mathbf{N}, \mathbf{N}'}(s))|_{s=0}.$$

We will now introduce auxiliary functions (which we can call generalized Barnes zeta functions), for the purpose of studying the derivative values along a direction  $Z'(-(\mathbf{N}, \mathbf{N}'))$ . These functions will appear in the formulas of such values.

**Definition 1.3.13.** Let  $\mathbf{R} \in \mathbb{N}_0^P$ , and  $j \in \llbracket 1, Q \rrbracket$ . We consider for all  $\sigma \gg 1$ ,

$$\varphi_{\mathbf{R}}^j(s) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d_j')^{-s}.$$

**Remark 1.3.14.** If  $\mathbf{R} = \mathbf{0}$ , we obtain the classical Barnes zeta function  $\varphi_{\mathbf{0}}^j(s) = \zeta_P^B(s, d_j' | c_{j,1}, \dots, c_{j,P})$ .

The function  $\varphi_{\mathbf{R}}^j(s)$  corresponds to  $Z_{\mathbf{N}, \mathbf{N}'}(s)$ , with  $Q = 1$ ,  $\boldsymbol{\mu} = \mathbf{0}$ ,  $\boldsymbol{\mu}' = 1$ ,  $\mathbf{N} = \mathbf{R}$  and  $\mathbf{N}' = 0$ . Particularly, we know that it is meromorphic on  $\mathbb{C}$ , and is regular at nonpositive integers. The first derivative of these auxiliary functions will appear in the formulas of  $Z'(-(\mathbf{N}, \mathbf{N}'))$ . Therefore, it is crucial to have an explicit formula for the derivative of such functions.

## 1.4 Main results

As mentioned in the previous section, we will prove, using another method, the result of Komori on the existence of the directional values  $Z(-(\mathbf{N}, \mathbf{N}'))$  in Proposition 2.2.4. To obtain this result, we use in Chapter 2 a strategy called Crandall's expansion, developed in [Cra12], which was initially used to obtain numerical approximations of certain zeta functions and certain  $L$  functions. This strategy was subsequently used by Borwein and Dilcher in [BD18] to study the values of a Mordell-Tornheim type zeta function, and to approximate its derivative at nonpositive integers. Subsequently, Onodera provided exact formulas for the values at nonpositive integers of a generalized Mordell-Tornheim type zeta function, as well as the values of its first and second order derivatives in [Ono21]. The trick used by Onodera to compute the values of the derivatives at nonpositive integers partly relies on introducing auxiliary functions resembling Barnes' zeta functions, with a polynomial term in the numerator of the general term of the series, and then providing a formula for the derivatives of these auxiliary functions.

More precisely, we shall use a we shall describe an analytic continuation formula for the meromorphic function  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  in Proposition 2.2.4:

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = K_{\mathbf{N}, \mathbf{N}'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}(\theta, s),$$

where  $K_{\mathbf{N}, \mathbf{N}'}(s)$  is holomorphic on  $\mathbb{C}$ , and vanish at  $s = 0$ , and  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  is meromorphic on  $\mathbb{C}$ , with poles that are all simple, of the form

$$s = \frac{n}{\mu_p} \quad (1 \leq p \leq P \text{ such that } \mu_p \neq 0, n \in \mathbb{Z}_{\geq N_p+1})$$

$$s = \frac{n}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \quad (\emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket, n \in \mathbb{Z}_{\leq |\mathbf{N}|_{\mathcal{P}} + |\mathbf{N}'|_{\mathcal{P}}}).$$

In particular, we will see that 0 isn't a pole for the meromorphic function  $Z_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}$ , therefore obtaining that directional values of the function  $Z(\mathbf{s}, \mathbf{s}')$  have a meaning.

The fact that  $Z_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}$  is meromorphic extension over  $\mathbb{C}$  directly follows from Proposition 1.3.9, and this same theorem also provides candidates for the poles of this function. Indeed, it is sufficient to replace  $s_p$  by  $-N_p + \mu_p s$  and  $s'_q$  by  $-N'_q + \mu'_q s$  to obtain candidate poles. However, the set of poles provided here is more precise.

Let's note that the term  $K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s)$  will not contribute to the formula for the special directional values, as it vanishes at  $s = 0$ . On the other hand, if we differentiate this term, it is non-zero in general, and is pathological. We will therefore use the auxiliary functions  $(\varphi_{\mathbf{R}}^j)_{j \in \llbracket 1, Q \rrbracket}$  that will help retrieve the information of the term  $\partial_s(K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s))$  at  $s = 0$ . We will then be able to evaluate the extension formula at  $s = 0$  to obtain an explicit formula for the directional values of the derivatives.

**Theorem A.** *Let  $(\mu, \mu') \in \mathbb{R}_+^P \times \mathbb{R}_*^+{}^Q$ . Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ . We have*

$$Z_{\mu, \mu'}(-(\mathbf{N}, \mathbf{N}')) = (-1)^{|\mathbf{N}'|} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|} \left( \prod_{p \in \mathcal{P}} N_p! \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} \sum_{j=1}^Q \frac{(-1)^{N'_j} \mu'_j N'_j!}{|\mu'| + |\mu|_{\mathcal{P}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$$

$$\prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!},$$
(1.8)

with

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, p}^{v_{p, q}} \right),$$

where we noted

$$\forall q \in \llbracket 1, Q \rrbracket, \quad \mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p, q},$$

and

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad \mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p, q}.$$

**Remark 1.4.1.** 1) *The formula describing  $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$  is a finite sum. Indeed, we see that*

$$\sum_{\substack{q=1 \\ q \neq j}}^Q (N'_q - \mathbf{w}(q)) = \sum_{\substack{q=1 \\ q \neq j}}^Q \mathbf{v}(q) = \sum_{(p, q) \in \mathcal{P} \times (\llbracket 1, Q \rrbracket \setminus \{j\})} v_{p, q}.$$

Since  $\mathbf{w}(q)$  is nonnegative for all  $q \in \llbracket 1, Q \rrbracket$ , we deduce the following upper bound:

$$\sum_{(p, q) \in \mathcal{P} \times (\llbracket 1, Q \rrbracket \setminus \{j\})} v_{p, q} \leq |\mathbf{N}'|.$$

Thus, for all  $p \in \mathcal{P}$  and all  $q \in \llbracket 1, Q \rrbracket \setminus \{j\}$ , we have  $0 \leq v_{p, q} \leq |\mathbf{N}'|$ .

2) The expression for the values of  $Z_{\mu, \mu'}(\mathbf{N}, \mathbf{N}')$  is described by a finite sum. Indeed, for all  $\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}$  such that  $|\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}|} + |\mathcal{P}|$  we have  $0 \leq k_p \leq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}|} + |\mathcal{P}|$  for all  $p \in \mathcal{P}^c$ .

3) Terms of the form  $\zeta(-n, d)$  with  $n \in \mathbb{N}_0$  and  $d \in H_0$  are expressed using Bernoulli polynomials:

$$\zeta(-n, d) = -\frac{B_{n+1}(d)}{n+1}.$$

Moreover, at  $d = 1$ , we have  $\zeta(-n, 1) = \zeta(-n)$ , and

$$\zeta(-n) = (-1)^n \frac{b_{n+1}}{n+1},$$

where  $b_{n+1}$  is the  $n+1$ -th Bernoulli number.

4) If all the coefficients  $d_p$  are equal to 1, then the product  $\prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}$  will often be zero. Indeed, if there exists  $p \in \mathcal{P}^c$  such that  $N_p + k_p$  is even and greater than 2, then the product is zero.

Taking  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ , we obtain the following simplified expressions:

**Corollary A1.** Let  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . We have

$$\begin{aligned} Z(-(\mathbf{0}, \mathbf{0})) = \\ \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ \boldsymbol{\mu}, \boldsymbol{\mu}'}} (-1)^{|\mathcal{P}|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \sum_{j=1}^Q \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p, d_p)}{k_p!}. \end{aligned} \quad (1.9)$$

**Remark 1.4.2.** We get the same formula as [Ono21] in  $N = 0$ , after replacing the direction with  $\mathbf{1} = (1, \dots, 1)$ , and taking  $Q = 1$ .

Via Theorem A, we also get a formula for  $Z(\mathbf{s}, \mathbf{s}')$  along the diagonal by taking the direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ :

**Corollary A2.** Let  $N \in \mathbb{N}_0$ . We have

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subsetneq [1, P]} \frac{(-1)^{(|\mathcal{P}|+1+Q)N+|\mathcal{P}|}}{Q+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=(Q+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!},$$

with

$$\begin{aligned} Q_N^0(\mathcal{P}, \mathbf{k}) = & (-1)^{N(Q-1)} N^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N}} \\ & \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right), \end{aligned}$$

where we noted

$$\forall q \in [1, Q], \quad \mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p,q},$$

and

$$\forall q \in [1, Q] \setminus \{j\}, \quad \mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p,q}.$$

**Remark 1.4.3.** If we take  $Q = 1$ ,  $P \geq 1$  a positive integer, and  $l_1(\mathbf{n}) = n_1 + \dots + n_P$ , we find via the previous corollary that  $Q_N^0(\mathcal{P}, \mathbf{k}) = 1$ , and that

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subsetneq [1, P]} \frac{(-1)^{|\mathcal{P}|N+|\mathcal{P}|}}{1+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=(1+|\mathcal{P}|)N+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!}.$$

This formula matches the one obtained by Onodera in [Ono21, Theorem 3].

We also can formulate a qualitative version of the Theorem A:

**Corollary A3.** Let  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ , and  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ . We consider  $\mathbb{K}$  the field above  $\mathbb{Q}$  generated by the coefficients  $(d_p)_{p \in [1, P]}$  and  $(c_{q,p})_{p \in [1, P], q \in [1, Q]}$ , and by the directions  $\mu_p$  for  $p \in [1, P]$ , and by the coefficients  $\mu'_q$  for  $q \in [1, Q]$ . We then have  $Z(-\mathbf{N}, \mathbf{N}') \in \mathbb{K}$ .

We also have an expression for the directional derivative at nonpositive tuple of  $Z$ . This expression involves "auxiliary" functions, which we will call generalized Barnes-type zeta functions.

If we assume that a linear form  $l_j$  has rational coefficients, we have a relationship between the auxiliary function  $\varphi_{\mathbf{R}}^j(s)$  and the Hurwitz zeta function:

**Proposition B.** Let  $j \in [1, Q]$  an positive integer. We assume that  $l_j$  is a linear form with rational coefficients for all  $1 \leq j \leq Q$ . For all  $1 \leq p \leq P$ , we then write  $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$  the irreducible decomposition of  $c_{j,p}$  with  $a_{j,p} > 0$  and  $b_{j,p} > 0$ . Let  $x_j(\mathbf{c}) = \frac{\text{lcm}(a_{j,p})}{\text{gcd}(b_{j,p})} \in \mathbb{Q}_+^*$ , and  $\beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}$  for all  $1 \leq p \leq P$ . Then, for all  $\mathbf{R} \in \mathbb{N}_0^P$ , we have for all  $s \in \mathbb{C} \setminus \mathbb{N}$ ,

$$\begin{aligned} \varphi_{\mathbf{R}}^j(s) = & x_j(\mathbf{c})^{-s} \prod_{p=1}^P \beta_{j,p}^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ \dots \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}, k' \in \mathbb{N}_0 \\ |\mathbf{k}|+k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!} \\ & \cdot \frac{\zeta\left(s - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v}))\right)}{k'!}. \end{aligned}$$

Deriving with respect to  $s$  the previous equality, and evaluating that derivative at  $s = -N$  with  $N$  being a nonnegative integer, we then found the following corollary:

**Corollary B1.** With the same notations as in Proposition B, we then get for all  $\mathbf{R} \in \mathbb{N}_0^P$ ,

$$\begin{aligned} \varphi_{\mathbf{R}}^{j'}(-N) = & x_j(\mathbf{c})^N \prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ \dots \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq [1, P]} \left( \prod_{p \in \mathcal{P}^c} R_p! \right) \\ & \cdot \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}, k' \in \mathbb{N}_0 \\ |\mathbf{k}|+k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!} \\ & \cdot \left( \frac{\zeta'(-N - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v}))\right)}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v}))\right)}{k'!} \right). \end{aligned}$$

We now can give a qualitative version of our result on special values of the directional derivative of  $Z(\mathbf{s}, \mathbf{s}')$  at nonpositive integers:

**Theorem C.** Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ . We consider  $\mathbb{K}$  the field above  $\mathbb{Q}$  generated by the coefficients  $(d_p)_{p \in [1, P]}$  and  $(c_{q,p})_{p \in [1, P], q \in [1, Q]}$ , and by the directions  $(\mu_p)_{p \in [1, P]}$ ,  $(\mu'_q)_{q \in [1, Q]}$ . Then,

$$Z'_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(-(\mathbf{N}, \mathbf{N}')) \in \text{Vect}_{\mathbb{K}} \left[ \gamma, \ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right)_{1 \leq q, j \leq Q, 1 \leq p \leq P}, (\zeta'(-n, d_p))_{\substack{n \in \mathbb{N}, |\mathbf{N}| + |\mathbf{N}'| + P \\ p \in [1, P]}}, \right. \\ \left. (\varphi_{\mathbf{n}}^{j'}(-N'_j))_{\mathbf{n} \in [0, |\mathbf{N}| + |\mathbf{N}'|]^P, j \in [1, Q]} \right],$$

where  $\gamma$  is Euler's constant, and  $\varphi_j$  are the zeta function of generalized Barnes type described in Definition 1.3.13.

If the linear forms  $(l_j)_{j \in [1, Q]}$  are all rationals, then we can use Corollary B1 in order to obtain a more precise result:

**Corollary C1.** Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ . We assume that the linear forms  $(l_j)_{j \in [1, Q]}$  and the coefficients  $(d_p)_{p \in [1, P]}$  are rationals. We consider the field  $\mathbb{K}$  above  $\mathbb{Q}$  generated by the directions  $(\mu_p)_{p \in [1, P]}$ ,  $(\mu'_q)_{q \in [1, Q]}$ . For all  $1 \leq p \leq P$ , we write  $c_{q,p} = \frac{a_{q,p}}{b_{q,p}} \in \mathbb{Q}$  the irreducible decomposition of  $c_{q,p}$  with  $a_{q,p} > 0$  and  $b_{q,p} > 0$ . We set  $x_q(\mathbf{c}) := \frac{\text{lcm}(a_{q,p})}{\text{gcd}(b_{q,p})} \in \mathbb{Q}_*^+$ , and  $\beta_{q,p}(\mathbf{c}) := \frac{x_q(\mathbf{c})}{c_{q,p}} \in \mathbb{N}$  for all  $1 \leq p \leq P$ . Let's also consider the following set

$$A := \{l_j(x_{\mathbf{c}}^{-1}(\mathbf{d} + \mathbf{v})) | j \in [1, Q], \mathbf{v} \in [0, \beta_{j,1} - 1] \times \dots \times [0, \beta_{j,P} - 1]\}.$$

We then have that

$$Z'_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(-(\mathbf{N}, \mathbf{N}')) \in \text{Vect}_{\mathbb{K}} \left[ \gamma, \ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right)_{q, j \in [1, Q]}, (\ln(x_j(\mathbf{c})))_{1 \leq j \leq Q}, (\zeta'(-n, y))_{\substack{0 \leq n \leq P|\mathbf{N}| + |\mathbf{N}'| + P \\ y \in A}} \right],$$

where  $\gamma$  is Euler's constant.

The formula for  $Z'_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(-(\mathbf{N}, \mathbf{N}'))$  is actually explicit. It has a formula involving the special values of the derivatives of the generalized Barnes zeta functions  $\varphi_j^{\mathbf{R}}$ , special values of the derivatives with respect to  $s$  of the Hurwitz zeta function, and special values of the Hurwitz zeta function itself. We detail this formula in the theorem below.

**Theorem D.** Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ , then

$$\begin{aligned}
Z'(-(\mathbf{N}, \mathbf{N}')) &= \tag{1.10} \\
&\sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q (-1)^{|\mathbf{N}|+|\mathbf{N}'|+|\mathcal{P}|+N'_j} \left( \prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\
&\cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
&+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|+N'_j} } \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
&\cdot \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
&+ \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{s=-N'_j} \\
&- \sum_{j=1}^Q (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\
&\cdot \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}|+|\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}|+|\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \Bigg),
\end{aligned}$$

with

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + \mathbf{u}(p))_{p \in [1, P]},$$

for all  $\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}}$ , and for all  $j \in [1, Q]$ ,

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) &= \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \in [1, Q] \setminus \{j\}, N'_q = \mathbf{v}(q) + \mathbf{w}(q)}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{q,p}} \right) \\
&\cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{q,p}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!,
\end{aligned}$$



and

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) &= (-1)^{|\mathbf{N}'|_{[[1, Q] \setminus \{j\}}]} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\
&\quad \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
&+ (-1)^{|\mathbf{N}'|_{[[1, Q] \setminus \{j\}}]} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \\
&\quad \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}'),
\end{aligned}$$

with  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  being the constant described in Proposition 1.3.2.

**Remark 1.4.4.** This theorem thus reduces the computation of directional derivatives at the nonpositive  $(Q+P)$ -tuple  $Z'(-(\mathbf{N}, \mathbf{N}'))$  to the computations of values at nonpositive integers of the derivative of the auxiliary functions  $\mu, \mu'$

$$\varphi_{\mathbf{R}}^j(s) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d'_j)^{-s} \quad (j \in [1, Q], \mathbf{R} \in \mathbb{N}_0^P).$$

If the coefficients of the linear forms  $l_j$  are rationals, then we have an explicit expression for the value of  $\partial_s \varphi_{\mathbf{R}}^j(-N)$  in terms of the special values of the derivatives of the Hurwitz zeta function via Proposition B.

**Remark 1.4.5.** We observe that the formula for  $Z'(-(\mathbf{N}, \mathbf{N}'))$  involves only finite sums. Indeed, it is clear that the sum  $\sum_{\mu, \mu'}$  is finite. On top of that the sum

$$\sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \quad \text{in the formula of } Q_{\mathbf{N}, \mathbf{N}'}^1 \text{ and of } Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \text{ is also finite}$$

because

$$\begin{aligned}
\forall a \in \mathcal{P}, \forall b \in [1, Q] \setminus \{j\}, \quad 0 \leq v_{a,b} \leq \sum_{\substack{q=1 \\ q \neq j}}^Q \mathbf{v}(q) + \mathbf{w}(q) = |\mathbf{N}'| \\
\forall p \in \mathcal{P}^c, \forall q \in [1, Q], \quad 0 \leq w_{p,q} \leq |\mathbf{w}_p| = k_p \leq |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}^c} + |\mathcal{P}|.
\end{aligned}$$

For the same reasons, the sum  $\sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}}$  is also finite.

If the coefficients  $\mathbf{c}$  are rationals, we can replace the derivatives of the auxiliary functions at nonpositive integers with the formula obtained in Corollary B1:

**Corollary D1.** Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ . For  $1 \leq p \leq P$ , we write  $c_{q,p} = \frac{a_{q,p}}{b_{q,p}} \in \mathbb{Q}$  the irreducible decomposition of  $c_{q,p}$  with  $a_{q,p} > 0$  and  $b_{q,p} > 0$ . We set  $x_q(\mathbf{c}) = \frac{\text{lcm}(a_{q,p})}{\text{gcd}(b_{q,p})} \in \mathbb{Q}_+^*$ , and  $\beta_{q,p}(\mathbf{c}) := \frac{x_q(\mathbf{c})}{c_{q,p}} \in \mathbb{N}$  for all  $1 \leq p \leq P$ . Then we have

$$\begin{aligned}
& Z'(-(\mathbf{N}, \mathbf{N}')) = \\
& \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q (-1)^{|\mathbf{N}| + |\mathbf{N}'| + |\mathcal{P}| + N'_j} \left( \prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\
& \cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
& + \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|} + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
& \cdot \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
& + \sum_{j=1}^Q \mu'_j x_j(\mathbf{c})^{N'_j} \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p} \\
& \cdot \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c}) - 1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c}) - 1}} \sum_{\mathcal{P} \subsetneq [1, P]} \left( \prod_{p \in \mathcal{P}^c} \mathbf{R}(\mathbf{N}, j, \mathbf{u})_p! \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c}, k' \in \mathbb{N}_0 \\ |\mathbf{k}| + k' = |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}^c| + |\mathcal{P}^c|} - 1}} (-1)^{|\mathbf{k}|} \left( \prod_{p \in \mathcal{P}} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})})}{k_p!} \right) \\
& \cdot \left( \frac{\zeta'(-N'_j - k', l_j(x_j(\mathbf{c}))^{-1}(\mathbf{d} + \mathbf{v}))}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N'_j - k', l_j(x_j(\mathbf{c}))^{-1}(\mathbf{d} + \mathbf{v}))}{k'!} \right) \\
& - \sum_{j=1}^Q (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\
& \cdot \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \Bigg),
\end{aligned}$$

where we set

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + \mathbf{u}(p))_{p \in [1, P]},$$

for all  $\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}}$ , and for all  $j \in [1, Q]$ ,

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \in [1, Q] \setminus \{j\}, N'_q = \mathbf{v}(q) + \mathbf{w}(q)}}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ \cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!$$

and

$$Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\ + (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}}} \\ \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}').$$

where  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  is the constant defined in Proposition 1.3.2.

**Remark 1.4.6.** Let  $Q = 1$ , and  $P \geq 1$  a positive integer, we set  $l_1(\mathbf{n}) = n_1 + \dots + n_P$  (and thus  $c_{1,1} = \dots = c_{1,P} = 1$ ),  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ , and  $(\mathbf{N}, \mathbf{N}') = (N_\Delta, N_\Delta)$  (we recall that  $N_\Delta = (N, \dots, N)$ ). Then we find by the previous corollary that

- For all  $\mathcal{P} \subsetneq [1, P]$ , and  $\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}$ , we have  $Q_{\mathbf{N}, \mathbf{N}'}^0(1, \mathcal{P}, \mathbf{k}) = 1$  and  $Q_{\mathbf{N}, \mathbf{N}'}^1(1, \mathcal{P}, \mathbf{k}) = 0$ .
- With the same notation as in Corollary D1, we have  $\mathbf{x}_1(\mathbf{c}) = 1$ , and  $\beta_{1,p} = 1$  for all  $1 \leq p \leq P$ .
- We have  $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := \mathbf{N}_\Delta \in \mathbb{N}_0^P$ .
- In the formula of  $Z'(-N_\Delta, N_\Delta)$  obtained by the previous corollary, we get that the first sum cancels the last sum.

Using the last 4 points, we get that

$$\begin{aligned} Z'(-(\mathbf{N}_\Delta, \mathbf{N}_\Delta)) &= \sum_{\mathcal{P} \subseteq [1, P]} \frac{N^{|\mathcal{P}|+1}}{|\mathcal{P}|+1} (-1)^{|\mathcal{P}|(N+1)} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|(N+1)+|\mathcal{P}|}} \\ &\left( \sum_{i \in \mathcal{P}^c} \frac{\zeta'(-N - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N - k_p, d_p)}{k_p!} \right) \\ &+ \sum_{\mathcal{P} \subseteq [1, P]} N^{|\mathcal{P}^c|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}} \in \mathbb{N}_0^{\mathcal{P}}, k' \in \mathbb{N}_0 \\ |\mathbf{k}|+k'=|\mathcal{P}^c|(N+1)+|\mathcal{P}^c|-1}} (-1)^{|\mathbf{k}|} \frac{\zeta'(-N - k', d'_1)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta(-N - k_p, d_p)}{k_p!}. \end{aligned}$$

This expression matches the one obtained by Onodera in [Ono21, Theorem 4] (with the parameter  $d_1 = 0$  in Onodera's theorem).

By considering  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$  in Theorem D, we get:

**Corollary D2.** Let  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ , then

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0})) &= \sum_{\mathcal{P} \subseteq [1, P]} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \left( \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \gamma(\mu'_j + |\boldsymbol{\mu}|_{\mathcal{P}}) + Q_{\boldsymbol{\mu}, \boldsymbol{\mu}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ &+ \sum_{\mathcal{P} \subseteq [1, P]} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p, d_p)}{k_p!} \right) \\ &+ \sum_{j=1}^Q \mu'_j (\zeta^B)'(0, d'_j | \mathbf{c}_j) - \sum_{j=1}^Q \gamma \mu'_j \sum_{\mathcal{P} \subseteq [1, P]} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p, d_p)}{k_p!}, \end{aligned}$$

with

$$\begin{aligned} Q_{\boldsymbol{\mu}, \boldsymbol{\mu}'}^1(j, \mathcal{P}, \mathbf{k}) &= \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} [0, k_p]} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ &+ \gamma |\boldsymbol{\mu}'|_{[1, Q] \setminus \{j\}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}, \end{aligned}$$

and  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N})$  being the constant defined in Proposition 1.3.2.

**Remark 1.4.7.** The constants  $F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0})$  in the previous corollary are linked with the partial fraction decomposition of

$$x^{-1+k_p} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x)^{-1}.$$

**Remark 1.4.8.** It is worth noting that Matsumoto and Weng obtained in [MW02, Theorem E] a similar expression of the derivative value at  $s = 0$  of a zeta function defined by two polynomials in  $\mathbb{C}[X]$ .

By taking  $\boldsymbol{\mu} = \mathbf{0}$ , and  $\boldsymbol{\mu}' = \mathbf{1}$ , we get  $Z'(-(\mathbf{0}, \mathbf{0})) = \partial_s (\zeta^{Sh}(s, (\mathbf{c}_1, \dots, \mathbf{c}_Q), \mathbf{d}'))|_{s=0}$ , which corresponds to the first derivative with respect to  $s$  of the Shintani zeta function at  $s = 0$ , defined in 1.2.13. Let's recall

that in Definition 1.2.16, and in Definition 1.2.14, we have defined multigamma function. Therefore, we have  $(\partial_s \zeta^B(s, d|c_1, \dots, c_P))|_{s=0} = \ln(\Gamma_P(d|c_1, \dots, c_P))$ , and  $\partial_s(\zeta^{Sh}(s, \mathbf{d}'|\mathbf{c}))|_{s=0} = \ln(\Gamma_P(\mathbf{d}'|\mathbf{c}))$ . From the previous corollary, and by definition of these multigamma functions, we get the logarithm of the Shintani multigamma function  $\Gamma_P^{Sh}(\mathbf{d}'|\mathbf{c}_1, \dots, \mathbf{c}_Q)$  has an explicit expression using logarithm of the Barnes zeta function.

**Corollary D3.** *We have*

$$\begin{aligned} \ln(\Gamma_P(\mathbf{d}'|\mathbf{c})|_{s=0}) = & \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|}}{Q} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \left( \gamma \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) + Q^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{j=1}^Q \ln(\Gamma_P(d'_j|\mathbf{c}_j)) - \gamma \sum_{j=1}^Q \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p, d_p)}{k_p!}, \end{aligned}$$

with

$$\begin{aligned} Q^1(j, \mathcal{P}, \mathbf{k}) = & \sum_{\substack{f=1 \\ f \neq j}}^Q \sum_{\substack{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} [0, k_p]}} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ & + \gamma(Q-1) \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}. \end{aligned}$$

Applying Corollary A1 and Corollary D2 to the two Witten zeta functions  $\zeta_{\mathfrak{g}_2}$  and  $\zeta_{\mathfrak{so}(5)}$ , we get the following formulas:

**Theorem E.** *We have the following formulas for the special values of  $\zeta_{\mathfrak{g}_2}(s)$  and  $\zeta_{\mathfrak{so}(5)}(s)$  at  $s = 0$ :*

$$\begin{aligned} \zeta_{\mathfrak{g}_2}(0) &= \frac{29}{60}, \\ \zeta_{\mathfrak{so}(5)}(0) &= \frac{7}{18}. \end{aligned}$$

Moreover, we have the following formulas for the special values of  $\zeta'_{\mathfrak{g}_2}(s)$  and  $\zeta'_{\mathfrak{so}(5)}(s)$  at  $s = 0$ :

$$\begin{aligned} \zeta'_{\mathfrak{g}_2}(0) &= \frac{29}{60} \ln(120) + 2\gamma\zeta(0)^2 + \frac{1}{4}\zeta(0)\zeta'(0) - 4\gamma\zeta(-1) + \frac{1}{5}\zeta(-1)(\ln(3) + 2\ln(2)) - \frac{14}{5}\zeta(-1) \\ &\quad - \frac{11}{9}\ln(2) - \frac{23}{24}\ln(3) - \frac{1}{2}\ln(\pi) - 5\zeta'(0) + 3\zeta'(-1) - \frac{4}{3}\zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3}\zeta'\left(0, \frac{4}{3}\right) \\ &\quad + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) - \frac{1}{6}\zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6}\zeta'\left(0, \frac{5}{6}\right), \\ \zeta'_{\mathfrak{so}(5)}(0) &= \frac{7}{18}\ln(6) - \frac{11}{24}\ln(2) - \frac{1}{4}\ln(\pi) + \left(\frac{1}{2}\zeta(0)^2 - \frac{1}{2}\zeta(-1)\right)\gamma - \frac{11}{4}\zeta'(0) - \frac{13}{6}\zeta'(-1). \end{aligned}$$

We can simplify the formula of  $\zeta'_{\mathfrak{so}(5)}(0)$ :

$$\zeta'_{\mathfrak{so}(5)}(0) = -\frac{13}{6} \left( \frac{1}{12} - \ln(A) \right) + \frac{1}{6}\gamma + \frac{47}{36}\ln(2) + \frac{7}{18}\ln(3) + \frac{9}{8}\ln(\pi) \simeq 2.94635\dots$$

with  $A$  being the Glaisher-Kinkelin constant [OEI23b], and  $\gamma$  being Euler's constant [OEI23a].

Using last formula, and by taking the expression of the constant  $C$  in Theorem 1.2.35, we get:

$$C = \frac{e^{\frac{1}{6}\gamma} 2^{\frac{35}{36}} \pi^{\frac{9}{8}} A^{\frac{13}{6}} \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}}{3^{\frac{5}{72}} e^{\frac{13}{72}} \sqrt{\pi}}.$$

We now obtain an explicit version of the Theorem 1.2.35 proved by Bridges, Brindle, Bringmann and Franke in [BBBF23, Theorem 1.3]:

**Corollary F.** *For all positive integers  $N \geq 1$ , we have*

$$r_{\mathfrak{so}(5)}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^{\frac{7}{12}}} \exp\left(A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4\right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N\left(n^{-\frac{N+1}{9}}\right)\right),$$

with

$$\begin{aligned} b &= \frac{7}{12}, \quad C = \frac{2^{\frac{35}{36}} \pi^{\frac{9}{8}} A^{\frac{13}{6}} \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}}{3^{\frac{5}{72}} e^{\frac{13}{72} + \frac{1}{9}\gamma} \sqrt{\pi}}, \quad A_1 = \frac{3^{\frac{4}{3}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{3}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{8}{3}}}, \\ A_2 &= \frac{2^{\frac{8}{9}} \left(2^{\frac{1}{3}} + 1\right) \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right)}{3^{\frac{7}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{9}}}, \quad A_3 = \frac{2^{\frac{40}{9}} \left(2^{\frac{1}{3}} + 1\right)^2 \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{1}{3}\right)^2 \zeta\left(\frac{4}{3}\right)^2}{3^{\frac{44}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{20}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{10}{9}}}, \\ A_4 &= \frac{2^8 \left(2^{\frac{1}{3}} + 1\right)^3 \Gamma\left(\frac{1}{3}\right)^3 \zeta\left(\frac{1}{3}\right)^3 \zeta\left(\frac{4}{3}\right)^3}{3^8 \Gamma\left(\frac{1}{4}\right)^4 \zeta\left(\frac{3}{2}\right)^2}. \end{aligned}$$

We now wish to apply the results obtained in Theorem E to derive an asymptotic formula for  $r_{\mathfrak{g}_2}(n)$ , which we get through Theorem 1.2.33 obtained by Bridges, Brindle, Bringmann, and Franke in [BBBF23, Theorem 4.4]. We first set

$$\begin{aligned} l_1^*(x_1, x_2) &= x_1 + x_2 + x_3 + 2x_4, \\ l_2^*(x_1, x_2) &= x_1 + 2x_2 + 3x_3 + 4x_4, \end{aligned}$$

where we set

$$\begin{aligned} \omega_\beta &:= 120^{\frac{1}{5}} \frac{\Gamma\left(\frac{4}{5}\right)}{\Gamma\left(\frac{1}{5}\right)^4} \zeta\left(\frac{1}{5}\right) \left( \int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{4}{5}} (l_1^*(1, x_2, x_3, x_4))^{-\frac{4}{5}} + l_2^*(1, x_2, x_3, x_4))^{-\frac{4}{5}} dx_2 dx_3 dx_4 \right. \\ &+ \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{4}{5}} (l_1^*(x_1, 1, x_3, x_4))^{-\frac{4}{5}} + l_2^*(x_1, 1, x_3, x_4))^{-\frac{4}{5}} dx_1 dx_3 dx_4 \\ &+ \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{4}{5}} (l_1^*(x_1, x_2, 1, x_4))^{-\frac{4}{5}} + l_2^*(x_1, x_2, 1, x_4))^{-\frac{4}{5}} dx_1 dx_2 dx_4 \\ &\left. + \int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{4}{5}} (l_1^*(x_1, x_2, x_3, 1))^{-\frac{4}{5}} + l_2^*(x_1, x_2, x_3, 1))^{-\frac{4}{5}} dx_1 dx_2 dx_3 \right), \end{aligned}$$

and

$$\begin{aligned} \omega_\alpha &= 120^{\frac{1}{3}} \frac{\Gamma\left(\frac{2}{3}\right)^2}{\Gamma\left(\frac{1}{3}\right)} \left( \int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, x_3, 1))^{-\frac{2}{3}} dx_1 dx_2 dx_3, \right. \\ &+ \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, 1, x_4))^{-\frac{2}{3}} dx_1 dx_2 dx_4 \\ &+ \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, 1, x_3, x_4))^{-\frac{2}{3}} dx_1 dx_3 dx_4 \\ &\left. + \int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(1, x_2, x_3, x_4))^{-\frac{2}{3}} dx_2 dx_3 dx_4 \right). \end{aligned}$$

We then state an asymptotic formula for the number of representation  $r_{\mathfrak{g}_2}(n)$  of the exceptional algebra  $\mathfrak{g}_2$ , some of the coefficients of the formula have an expression using the residues  $\omega_\alpha$  and  $\omega_\beta$ , and the special values  $\zeta_{\mathfrak{g}_2}(0)$  and  $\zeta'_{\mathfrak{g}_2}(0)$ .

**Theorem G.** Let  $N \in \mathbb{N}$ . When  $n$  tends to  $+\infty$ , we have

$$r_{\mathfrak{g}_2}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp\left(A_1 n^{\frac{1}{4}} + A_2 n^{\frac{3}{20}} + A_3 n^{\frac{1}{20}}\right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{20}}} + O\left(n^{-\frac{N+1}{20}}\right)\right),$$

with  $B_j$  being explicit constants, and  $C, b, A_1, A_2, A_3$  such that:

$$\begin{aligned} C &= \frac{e^{\zeta'_{\mathfrak{g}_2}(0)} (\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{1-6\zeta_{\mathfrak{g}_2}(0)}{8}} \sqrt{3}}{\sqrt{8\pi}}, \quad b = \frac{6 - 6\zeta_{\mathfrak{g}_2}(0) + 1}{8}, \\ K_2 &= \frac{3}{4} \cdot \frac{\omega_\beta \Gamma(\frac{6}{5}) \zeta(\frac{6}{5})}{(\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{3}{20}}}, \quad K_3 = -\frac{3}{160} \cdot \frac{(\omega_\beta \Gamma(\frac{6}{5}) \zeta(\frac{6}{5}))^2}{(\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{21}{20}}}, \\ A_1 &:= 4 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{4}}, \quad A_2 := \frac{\omega_\beta \Gamma\left(\frac{1}{5}\right) \zeta\left(\frac{6}{5}\right)}{(\omega_\alpha \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}}, \\ A_3 &:= K_3 + 3 \left( \binom{-1/3}{1} K_3 + \binom{-1/3}{2} \frac{K_2^2}{(\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{3}{4}}} \right) + \frac{5 (\omega_\beta \Gamma(\frac{6}{5}) \zeta(\frac{6}{5}))}{(\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{18}{20}}} \binom{-1/5}{1} K_2 \\ \zeta'_{\mathfrak{g}_2}(0) &= \frac{29}{60} \ln(120) + 2\gamma \zeta(0)^2 + \frac{1}{4} \zeta(0) \zeta'(0) - 4\gamma \zeta(-1) + \frac{1}{5} \zeta(-1) (\ln(3) + 2 \ln(2)) - \frac{14}{5} \zeta'(-1) \\ &\quad - \frac{11}{9} \ln(2) - \frac{23}{24} \ln(3) - \frac{1}{2} \ln(\pi) - 5\zeta'(0) + 3\zeta'(-1) - \frac{4}{3} \zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3} \zeta'\left(0, \frac{4}{3}\right) \\ &\quad + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) - \frac{1}{6} \zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6} \zeta'\left(0, \frac{5}{6}\right), \\ \zeta_{\mathfrak{g}_2}(0) &= \frac{29}{60}. \end{aligned}$$

**Remark 1.4.9.** The  $B_j$  can be computed through the coefficients  $b_j$  obtained by [BBBF23, Lemma 3.6].





## Chapter 2

# Crandall's Expansion of $Z(\mathbf{s}, \mathbf{s}')$

We shall fix in this chapter coefficients  $\mathbf{c} = (c_{q,p})_{q \in [1,Q], p \in [1,P]}$ ,  $\mathbf{d} = (d_p)_{p \in [1,P]}$ ,  $\mathbf{d}' = (d'_q)_{q \in [1,Q]}$  such that  $d_p \in H_0$  for all  $1 \leq p \leq P$ , and we assume that for all  $1 \leq q \leq Q$ ,

$$\forall q \in [1, Q], \quad d'_q = \mathbf{c}_q \cdot \mathbf{d} = \sum_{p=1}^P c_{q,p} d_p.$$

Moreover, we will consider a  $P+Q$ -tuple integer  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and a direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^+ \times \mathbb{R}_*^+$ . All of the bounds that we will state through this chapter will depend on the above constants. However, we will not note that dependence for the sake of clarity.

We will first study the meromorphic extension of  $Z(\mathbf{s}, \mathbf{s}')$  and examine its singularities in the proof of Proposition 1.3.9. Next, we will decompose the meromorphic function  $Z(\mathbf{s}, \mathbf{s}')$  using a Crandall expansion, and we will obtain an extension formula for the univariate function  $Z(-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N} + \boldsymbol{\mu}'s)$ . More specifically, we will obtain a decomposition of this meromorphic function as a sum between a holomorphic function that vanishes at  $s = 0$ , and a meromorphic function corresponding to an Erdélyi-type series. This extension formula will be crucial in the proofs of Theorems A and D.

### 2.1 Proof of Proposition 1.3.9

The meromorphic extension of  $Z(\mathbf{s}, \mathbf{s}')$  and the form of the singularities follow from Theorem 1.2.25. We aim to show here that  $Z(\mathbf{s}, \mathbf{s}')$  is absolutely convergent on the domain  $\mathcal{D}$  introduced in Proposition 2.1.1. It is enough to show that the series describing  $Z(\mathbf{s}, \mathbf{s}')$  is normally convergent on every compact subset of  $\mathcal{D}$ . We intend to prove the following proposition, which is more general than what we aim to prove in Proposition 1.3.9:

**Proposition 2.1.1.** *Let  $P, Q \geq 1$  be two positive integers, and  $\boldsymbol{\gamma} = (\gamma_q)_{q \in [1,Q]} \in (H_0^P)^Q$ ,  $\boldsymbol{\alpha} = (\alpha_p) \in H_0^P$ ,  $\boldsymbol{\beta} = (\beta_q) \in H_0^Q$ . We set*

$$\mathcal{D} = \mathcal{D}_{P,Q} := \left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset [1, P] \right\}.$$

*Then the function  $Z_{\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\beta}}^{P,Q}(\mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + \alpha_p)^{-s_p} \prod_{q=1}^Q \left( \sum_{p=1}^P \gamma_{q,p} n_p + \beta_q \right)^{-s'_q}$  normally converges on all compact sets of  $\mathcal{D}$ .*

*Proof.* All the constants involved in the proof of this proposition will implicitly depend on the choice of the coefficients  $(\gamma_q)_{q \in [1,Q]}$ ,  $\boldsymbol{\alpha}$ , and  $\boldsymbol{\beta}$ . We will not show this dependency to improve the clarity of the proof. We will prove this result by induction. The case  $P = 1$  is trivial.

Let's assume that the result is true for all  $k < P$ . Let  $(\mathbf{s}, \mathbf{s}') \in H_0^{P+Q}$ , and  $\mathbf{n} \in \mathbb{N}_0^P$ . We then have the following upper bound

$$(n_p + 1) \ll n_p + \operatorname{Re}(\alpha_p) \leq |n_p + \alpha_p| \leq n_p + |\alpha_p| \ll (n_p + 1).$$

Thus, we have that

$$\sum_{p=1}^P n_p + P \ll \sum_{p=1}^P \operatorname{Re}(\gamma_{q,p}) n_p + \operatorname{Re}(\beta_q) \leq \left| \sum_{p=1}^P \gamma_{q,p} n_p + \beta_q \right| \leq \sum_{p=1}^P |\gamma_{q,p}| n_p + |\beta_q| \ll \sum_{p=1}^P n_p + P.$$

Taking the product over  $[1, P]$  and  $[1, Q]$ , we find that

$$\left| \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q} \right| \ll e^{(\sum_{p=1}^P |\tau_p| + \sum_{q=1}^Q |\tau'_q|)\pi/2} \prod_{p=1}^P (n_p + 1)^{-\sigma_p} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)}.$$

We now aim to show that the series  $\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + 1)^{-\sigma_p} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)}$  is normally convergent on every compact sets of  $\mathcal{D}$ . This series can be split into a finite sum of series of the form

$$\sum_{0 \leq n_{f(1)} \leq \dots \leq n_{f(P)}} \prod_{p=1}^P (n_p + 1)^{-\sigma_p} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)},$$

with  $f \in \mathfrak{S}_P$ , and in terms of terms of the form  $Z_{\mathbf{1}, \mathbf{1}, P}^{k, 1}((\sigma_1 + \dots + \sigma_{i_1}, \dots, \sigma_{i_{k-1}+1} + \dots + \sigma_{i_k}), \sigma'_1 + \dots + \sigma'_Q)$  with  $1 = i_1 < i_2 < \dots < i_k = P$  and  $(i_1, \dots, i_k) \neq (1, \dots, P)$  (which in particular implies  $k < P$ ). By induction principle, the Dirichlet series  $Z_{\mathbf{1}, \mathbf{1}, P}^{k, 1}((\sigma_1 + \dots + \sigma_{i_1}, \dots, \sigma_{i_{k-1}+1} + \dots + \sigma_{i_k}), \sigma'_1 + \dots + \sigma'_Q)$  converges normally on every compact sets of

$$\left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \left[ \sum_{p \in \mathcal{P}} (\sigma_{i_p} + \dots + \sigma_{i_{p+1}-1}) \right] + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \mathcal{P} \subset \{i_1, \dots, i_{k-1}\} \right\},$$

and we see that the above set do contains the set  $\mathcal{D}_{P, Q}$ .

Let's consider the trivial permutation  $f = Id \in \mathfrak{S}_P$ . By a change of variables we see that

$$\sum_{0 \leq n_{f(1)} \leq \dots \leq n_{f(P)}} \prod_{p=1}^P (n_p + 1)^{-\sigma_p} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)} = \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P \left( 1 + \sum_{i=1}^p n_i \right)^{-\sigma_p} \left( P + \sum_{p=1}^P p n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)}.$$

By a previous argument, we get that:

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P \left( 1 + \sum_{i=1}^p n_i \right)^{-\sigma_p} \left( P + \sum_{p=1}^P p n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)} \ll \zeta_P^{EZ}(\sigma_1, \dots, \sigma_{P-1}, \sigma'_1 + \dots + \sigma'_Q + \sigma_P).$$

By Lemma 1.2.23, we find that the right hand term is normally convergent on all compact sets of

$$\{(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sigma_p + \dots + \sigma_P + \sigma'_1 + \dots + \sigma'_Q > P + 1 - p, 1 \leq p \leq P\}.$$

We then get that the series  $\sum_{0 \leq n_{f(1)} \leq \dots \leq n_{f(P)}} \prod_{p=1}^P (n_p + 1)^{-\sigma_p} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)}$ , with  $f \in \mathfrak{S}_P$  converges normally on all compact subset of

$$\{(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sigma_{f(p)} + \dots + \sigma_{f(P)} + \sigma'_1 + \dots + \sigma'_Q > P + 1 - p, 1 \leq p \leq P\}.$$

We then get that  $Z_{\gamma, \alpha, \beta}^{P, Q}$  converges normally on all compact sets of

$$\left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset [1, P] \right\}.$$

□

To obtain the meromorphic extension of the multizeta function  $Z(\mathbf{s}, \mathbf{s}')$ , and the singularities of this multizeta function, we can simply use Theorem 1.2.25. Indeed, via Remark 1.3.8, the multizeta function  $Z(\mathbf{s}, \mathbf{s}')$  is of the form  $\zeta^{HL}(\mathbf{0}, \widetilde{\mathbf{d}}, \widetilde{\mathbf{c}}, (s_1, \dots, s_P, s'_1, \dots, s'_Q))$  as introduced in Definition 1.2.24. Therefore, we can apply Theorem 1.2.25 to this multizeta function.

## 2.2 Statement of the extension formula for $Z_{\mathbf{N}, \mathbf{N}'}(s)$

We aim to establish an extension formula for  $Z_{\mathbf{N}, \mathbf{N}'}(s)$ . Before expressing this formula, we need to introduce the functions  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ , which will appear in the extension formula.

**Definition 2.2.1.** Let  $\mathcal{P} \subsetneq [1, P]$ ,  $j \in [1, Q]$ , and  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$ . We set  $l_p^*(\mathbf{x}) = c_{1,p}x_1 + \dots + c_{Q,p}x_Q$  for all  $1 \leq p \leq P$ . For all  $s = \sigma + i\tau \in \mathbb{C}$ , with  $\sigma \gg 1$ , we consider

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx_j} \dots dx_Q,$$

and

$$h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx_j} \dots dx_Q,$$

where we have set  $dx_1 \dots \widehat{dx_j} \dots dx_Q = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_Q$ , and  $\widehat{\mathbf{x}}^j = (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_Q)$ , with  $1 \leq j \leq Q$ .

We now state in the following proposition some results about this function. Particularly, we shall state an upper bound of these functions in order to control later a series, and we shall give an expression of these functions in order to compute the values at  $s = 0$  of  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  and of  $h'_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ .

**Proposition 2.2.2.** Let  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ , and  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ . Let  $\mathcal{P} \subsetneq [1, P]$ ,  $j \in [1, Q]$  and  $\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}$ . We have that

i)  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}$  and  $h_{\mathbf{N}, \mathbf{N}', [1, P], j}$  are holomorphic on  $\mathbb{C}$ .

ii) For all positive integer  $r \in \mathbb{N}$ , we have the following upper bound uniformly for  $s \in D_0(r)$ :

$$|h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)| \ll_r (QM)^{|\mathbf{k}|} \quad (2.1)$$

with  $M := \max_{1 \leq q \leq Q, 1 \leq p \leq P} (|c_{q,p}|)$ .

iii) Let  $\epsilon > 0$  small enough and  $s \in \mathbb{C}$ , we have

$$\begin{aligned}
h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) &= \\
& \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in [1, Q]} c_{q, p}^{w_{p, q}} \right) \\
& \cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q, p}^{v_{p, q}} \right) \frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \\
& \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j, p} + \sum_{f \in \mathcal{Q}^c} c_{f, p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q, \quad (2.2)
\end{aligned}$$

with the notations  $\mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p, q}$  for all  $q \in \mathcal{Q}$ , and  $\mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p, q}$  for all  $q \in [1, Q]$ . Furthermore, we can differentiate the general term of the previous series with respect to  $s$  to obtain an expression for  $\partial_s h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ .

**Remark 2.2.3.** 1) Point iii) of this proposition will be critical for evaluating  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  and  $\partial_s h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  at  $s = 0$  in the sub-sections 3.2.2 and 3.2.3.

2) We can obtain results similar to points ii) and iii) of the previous proposition for functions of the form  $h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s)$ , however we will not need to study this type of function in detail.

We can now state the extension formula for  $Z_{\mathbf{N}, \mathbf{N}'}(s)$ . We will decompose this analytic function into two parts that will depend on  $s$  and a sufficiently small free variable  $\theta > 0$ . The first, denoted as  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ , will be holomorphic in  $s$  and will vanish at  $s = 0$ . The other term will be an Erdélyi-type series (in the style of the series present in the Erdélyi formula, cf. 2.6), denoted as  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ . This second term will contain the hypothetical singularities of  $Z_{\mathbf{N}, \mathbf{N}'}(s)$ , and we will denote the set of these hypothetical singularities as  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ , the expression of which is

$$\mathcal{S}(\mathbf{N}, \mathbf{N}') = \left[ \bigcup_{\substack{1 \leq p \leq P \\ \mu_p \neq 0}} \frac{1}{\mu_p} \mathbb{Z}_{\geq N_p + 1} \right] \cup \left[ \bigcup_{\mathcal{P} \subset [1, P]} \frac{1}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|^{\mathcal{P}}} \mathbb{Z}_{\leq |\mathbf{N}|_{\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}|} \right].$$

**Proposition 2.2.4.** Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{C}^P \times \mathbb{C}^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ . Then the function  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  is meromorphic on  $\mathbb{C}$ , and his poles are included in  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ . Furthermore, we have the following analytical continuation formula:

For all  $\theta > 0$  small enough, and for all  $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ , we have

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}(\theta, s) \quad (2.3)$$

with  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  and  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  such that:

1) For all  $s \in \mathbb{C}$ , we set

$$K_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \sum_{\emptyset \neq \mathcal{Q} \subset [1, Q]} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right). \quad (2.4)$$

2) For all  $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ , we set

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \sum_{\mathcal{P} \subsetneq [1, P]} \left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right) + \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right). \quad (2.5)$$

Moreover, we have that:

- i)  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  is holomorphic in  $s$  on all  $\mathbb{C}$ , and vanish at  $s = 0$ .
- ii)  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  is meromorphic in  $s$  on all  $\mathbb{C}$ , with poles inside  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ . Furthermore, one can compute the derivative with respect to  $s$  of  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  by deriving the series in the expression of  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  termwise.

**Remark 2.2.5.** 1)  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  will be the sole contributor when computing the directional value  $Z(-(\mathbf{N}, \mathbf{N}'))$ , since the function  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  vanishes at  $s = 0$ . However, when studying the value of the derivative  $Z'(-(\mathbf{N}, \mathbf{N}'))$ , we will have to consider the derivative of the term  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ .

2) We will see in the proof of Proposition 2.2.4 that the series

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q),$$

describes a holomorphic function in  $s$  on all  $\mathbb{C}$  (the series is actually absolutely convergent on all  $\mathbb{C}$ ). In particular, we can consider the derivative of this series with respect to  $s$ , and evaluate it at  $s = 0$  in the proof of Theorem D. Note that we will never attempt to directly compute the values of this series, or the values of its derivative with respect to  $s$ .

With the help of this proposition, we will then establish that the function  $Z_{\mathbf{N}, \mathbf{N}'}$  is regular at 0, and will thus have proven by another method than Komori in [KMT10a] the existence of directional values for the class of zeta functions that interest us. We also retrieve some information on the poles of  $Z_{\mathbf{N}, \mathbf{N}'}$ :

**Corollary 2.2.6.** *Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in (\mathbb{R}_+)^P \times (\mathbb{R}_+^*)^Q$ . Then the function  $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  has poles, which are all simple, of the form*

$$\begin{aligned} s &= \frac{n}{\mu_p} & (1 \leq p \leq P, n \in \llbracket 1, P \rrbracket) \\ s &= \frac{n}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}} & (\emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket, n \in ]-\infty, |\mathbf{N}|_{|\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}|\llbracket \setminus \{0\} \rrbracket). \end{aligned}$$

To prove this proposition, we will multiply the Dirichlet series  $Z(\mathbf{s}, \mathbf{s}')$  by  $\Gamma(s'_1) \dots \Gamma(s'_Q)$ . In a way, this is equivalent to consider a multivariable Mellin transform. This strategy is classic for extending and studying the values of the Riemann zeta function [Hid93, Chap. 2], for obtaining an integral representation of a Shintani zeta function in [Hid93, Chap. 2], and for calculating the values of the multiple Hurwitz-type Mordell-Tornheim zeta function [Ono21].

Next, we will proceed to partition the integration domain by introducing a free variable  $\theta$ . By doing so, we will isolate a neighborhood of the origin from the rest of the integration domain. Finally, we will note that the integral around the neighborhood of the origin can be re-expressed using an Erdélyi formula (see 2.6), and we will also prove that the remaining integrals actually define a holomorphic function in  $(\mathbf{s}, \mathbf{s}')$ . To partition the integrals appearing in our calculations, we will use the incomplete gamma functions  $\Gamma(s, \theta, \nu)$  and  $\gamma(s, \theta, \nu)$  defined in the Definition 1.3.1.

## 2.3 Directional Crandall's expansion for $Z(\mathbf{s}, \mathbf{s}')$

### 2.3.1 Lemmata

We shall start by stating a few lemma. Firstly, let's state a classical identity obtained by a multivariable Taylor's theorem:

**Lemma 2.3.1.** *Let  $X_1, \dots, X_d$  be complex numbers such that  $|X_1 + \dots + X_d| < 1$ . We then have that*

$$(1 + X_1 + \dots + X_d)^{-s} = \sum_{\mathbf{k}=(k_p)_{p \in \llbracket 1, d \rrbracket}} \binom{-s}{|\mathbf{k}|} \binom{|\mathbf{k}|}{\mathbf{k}} X_1^{k_1} \dots X_d^{k_d}.$$

The expressions in this chapter will involve a series whose general term will contain  $\zeta(s - k, d)$  where  $k \in \mathbb{N}_0$ , and  $d \in H_0$ . In [Ono21], Onodera also had these terms in one of his expressions, and he controlled them using the following lemma, in the case where  $d \in \mathbb{R}_+^*$ :

**Lemma 2.3.2.** [Ono21, Lemma 2.1] *Let  $\alpha > 0$ . We set for all  $\delta \in \mathbb{R}_+^*$  and all  $r \in \mathbb{N}$ ,*

$$D(\delta, r) := \left\{ s \in \mathbb{C} \mid \min_{n \in \mathbb{N}} |s - n| \geq \delta, |s| \leq r \right\}.$$

*Then for all  $s \in D(\delta, r)$ ,  $k \in \mathbb{N}_0$ ,  $x > 0$ ,  $m \in \mathbb{N}_0$ , we have*

$$|\partial_s^m \zeta(s - k, d)| \ll_{m, r, \delta, \alpha} k!(k+1)^{r+\alpha}.$$

However, this lemma is not sufficient here, as we want to bound  $\zeta(s - k, d)$  with  $d \in H_0$ . The proof of the previous lemma uses a functional equation satisfied by the Hurwitz zeta function  $\zeta(s, d)$  for  $d \in ]0, 1]$ . We cannot prove a similar bound for  $d \in H_0$  in such a fashion. We will therefore use Euler-Maclaurin's formula on the Hurwitz zeta function to obtain an inequality similar to the previous lemma.

**Lemma 2.3.3.** *Let  $\delta \in \mathbb{R}_+^*$ ,  $r \in \mathbb{N}$ , and  $r' \in ]1, +\infty[$ . Then we have that*

$$\forall s \in D(\delta, r), \forall d \in \overline{H}_{1/r'} \cap D_0(r'), \forall k \in \mathbb{N}_0, \quad |\zeta(s - k, d)| \ll_{r, r', \delta} \left(\frac{e}{\pi}\right)^k k!(k+1)^{r+1} (1 + r'^{2k}).$$

*Proof.* Let  $r \in \mathbb{N}$ , we set  $n = r + 1$ . We aim to bound  $|\zeta(s - k, d)|$  uniformly in  $s \in D(\delta, r)$  and in  $d \in \overline{H}_{1/r'} \cap D_0(r')$ . Let  $s \in D(\delta, r)$  and  $d \in H_{1/r'} \cap D_0(r')$ . We note that  $n > -\sigma$ . Let  $k \in \mathbb{N}_0$ , as we have  $n + k > -\sigma + k$ , we can use the expression for the Hurwitz zeta function from Maclaurin's formula 1.2.6 up to order  $n + k$ , and evaluate it at  $s - k$ .

$$\begin{aligned} \zeta(s - k, d) &= \frac{d^{1-s+k}}{1-s+k} + \sum_{i=0}^{n+k} \binom{-s+k}{i} \frac{(-1)^i B_{i+1}}{i+1} d^{-s-i+k} \\ &\quad + (-1)^{n+k} \binom{-s+k}{n+1+k} \int_0^{+\infty} B_{n+k+1}(x) (x+d)^{-s-n-1} dx, \end{aligned}$$

with  $B_i(x)$  the  $i$ -th periodised Bernoulli polynomial Bernoulli, and  $B_i$  the  $i$ -th Bernoulli number. We now have the following inequality

$$\begin{aligned} |\zeta(s - k, d)| &\leq \frac{|d|^{1-\sigma+k}}{|1-s+k|} e^{|\tau|\pi/2} + \sum_{i=0}^{n+k} \left| \binom{-s+k}{i} \right| \frac{|B_{i+1}|}{i+1} |d|^{-\sigma-i+k} e^{|\tau|\pi/2} \\ &\quad + \left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |B_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx. \end{aligned}$$

Let's find an upper bound for each of these terms. We clearly have that  $e^{|\tau|\pi/2} \leq e^{r\pi/2}$ . By compactness, we get that the continuous function  $(x, y) \in [1/r', r'] \times [-r, r] \mapsto x^y$  verifies the following inequality

$$|d|^{-\sigma+k} \ll_{r, r'} r'^k.$$

By construction of the set  $D(\delta, r)$ , we have that

$$\begin{aligned} \frac{|d|^{1-\sigma+k}}{|1-s+k|} &\ll_{r, r'} \frac{r'^k}{\delta} \\ &\ll_{r, r', \delta} r'^k. \end{aligned}$$

We have the following inequality for the multinomial coefficients

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad \left| \binom{-s+k}{i} \right| \leq \binom{r+k+i-1}{i} \leq \binom{2n+2k}{i}.$$

Let  $i \in \mathbb{N}_0$ . By [Leh40], we know that if  $i \not\equiv 2 \pmod{4}$ , we have

$$\forall i \in \mathbb{N}_0, \forall x \in \mathbb{R}, \quad |B_i(x)| < \frac{2i!}{(2\pi)^i},$$

and if we have  $i \equiv 2 \pmod{4}$ , we get

$$\forall x \in \mathbb{R}, \quad |B_i(x)| < \frac{2i!\zeta(i)}{(2\pi)^i} \leq \frac{2i!\zeta(2)}{(2\pi)^i}.$$

Let's note  $\psi$  the digamma function. We know that  $\lim_{x \rightarrow +\infty} \psi(x) = +\infty$ . Therefore, there exist a real  $x_0$  such that for all  $x \geq x_0$ , the fraction  $\frac{\Gamma(x)}{(2\pi)^x} (\psi(x) - \ln(2\pi))$  is positive. We then find that the function

$$x \in \mathbb{R}_+^* \mapsto \frac{\Gamma(x)}{(2\pi)^x}$$

is increasing after  $x_0$ . Therefore, for  $n$  big enough, we have that

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad \frac{i!}{(2\pi)^i} \ll_n \frac{(n+k)!}{(2\pi)^{n+k}} \ll_n \frac{1}{(2\pi)^k} k!(k+1)\dots(k+n).$$

Taking an upper bound of each term inside the product in the right hand side have the last equality, we get that  $(k+1)\dots(k+n) \ll_n (k+1)^n$ . Because  $n = r+1$ , we then have

$$\forall i \in \llbracket 0, n+k \rrbracket, \forall x \in \mathbb{R}, \quad |B_i(x)| \ll_r \frac{k!}{(2\pi)^k} (k+1)^{r+1}.$$

By taking  $x = 0$  in the previous inequality, we have

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad |B_i| \ll_r \frac{k!}{(2\pi)^k} (k+1)^{r+1}.$$

By all the above, we have

$$\begin{aligned} \sum_{i=0}^{n+k} \left| \binom{-s+k}{i} \right| \frac{|B_{i+1}|}{i+1} |d|^{-\sigma-i+k} &\ll_r |d|^{-\sigma+k} \frac{k!}{(2\pi)^k} (k+1)^n \sum_{i=0}^{2n+2k} \binom{2n+2k}{i} |d|^{-i} \\ &\ll_{r,r'} \frac{k!}{(2\pi)^k} r'^k (k+1)^{r+1} (1+|d|^{-1})^{2n+2k} \\ &\ll_{r,r'} \frac{k!}{(2\pi)^k} (k+1)^{r+1} (1+r')^{2k}. \end{aligned}$$

As  $-\sigma - n \leq -1$  it is clear that

$$\forall x \in \mathbb{R}_+, \quad |x+d|^{-\sigma-n-1} \leq (x+\operatorname{Re}(d))^{-\sigma-n-1},$$

and the right hand side term is integrable over  $\mathbb{R}_+$ . Therefore, we obtain

$$\left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |B_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx \leq \binom{n+2k+1}{n+k+1} \frac{2(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} (\sigma+n)}.$$

By a classical inequality, we have

$$\binom{m}{i} \leq e^i \left(\frac{m}{i}\right)^i \quad (0 \leq i \leq m).$$

We then find that

$$\begin{aligned} \left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |B_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx \\ \leq 2e^k \left(\frac{n+2k+1}{k}\right)^k \frac{(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} (\sigma+n)} \\ \leq 2(2e)^k \left(1 + \frac{n+1}{2k}\right)^k \frac{(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} (\sigma+n)}. \end{aligned}$$

From the following inequality

$$\forall 0 \leq x < k, \quad \left(1 + \frac{x}{k}\right)^k \leq e^x,$$

we then find that

$$\forall k > (n+1)/2, \quad \left(1 + \frac{n+1}{2k}\right)^k \leq e^{(n+1)/2}.$$

As  $n = r+1$ , we obtain

$$\forall k \in \mathbb{N}_0, \quad \left(1 + \frac{n+1}{2k}\right)^k \ll_r 1.$$



Finally, we get that

$$\begin{aligned} |\zeta(s-k, d)| &\underset{r, r', \delta}{\ll} r'^k + \frac{k!}{(2\pi)^k} (k+1)^n (1+r'^{2k}) + \frac{(2e)^k k!}{(2\pi)^k} (k+1)^{n+1} \\ &\underset{r, r', \delta}{\ll} \left(\frac{e}{\pi}\right)^k k! (k+1)^{r+1} (1+r'^{2k}). \end{aligned}$$

□

For all nonnegative integers  $a \in \mathbb{N}_0$ , we observe that the function  $s \mapsto \frac{1}{\Gamma(s)(s+a)}$  is holomorphic on all  $\mathbb{C}$ . We get the following lemma:

**Lemma 2.3.4.** *Let  $r \in \mathbb{N}$ . We have*

$$\begin{aligned} \forall s \in D_0(r), \quad \left| \frac{1}{\Gamma(s)} \right| &\underset{r}{\ll} 1 \\ \forall s \in D_0(r), \forall a \in \mathbb{N}_0, \quad \left| \frac{1}{\Gamma(s)(s+a)} \right| &\underset{r}{\ll} 1. \end{aligned}$$

*Proof.* As  $\frac{1}{\Gamma}$  and  $s \mapsto \frac{1}{\Gamma(s)s}$  are holomorphic on  $\mathbb{C}$ , we have uniformly on  $D_0(r)$ :

$$\begin{aligned} \left| \frac{1}{\Gamma(s)} \right| &\underset{r}{\ll} 1 \\ \left| \frac{1}{\Gamma(s)s} \right| &\underset{r}{\ll} 1. \end{aligned}$$

Thus it is clear that

$$\forall |s| > \frac{1}{2}, \quad \left| \frac{1}{s} \right| < 2.$$

One can then show the inequality required. Let  $s \in \mathbb{C}$  such that  $\sigma \in [-r, r]$ .

If  $|s+a| > \frac{1}{2}$ , then

$$\left| \frac{1}{\Gamma(s)(s+a)} \right| \underset{r}{\ll} 1.$$

If  $|s+a| \leq \frac{1}{2}$  then by the triangular inequality, we get  $a \leq |s| + 1/2 \leq r + 1/2$ . Using the functional equation of  $\Gamma$  we get

$$\frac{1}{\Gamma(s)(s+a)} = \frac{(s+a-1)\dots s}{\Gamma(s+a)(s+a)},$$

thus we have the following inequality:

$$\left| \frac{1}{\Gamma(s)(s+a)} \right| \underset{r}{\ll} (2r+1)\dots r.$$

As  $a \leq 1/2 + r$ , we then get the following upper bound for the product in the numerator

$$|(s+a-1)\dots s| \leq (2r+1)!.$$

Therefore, we find that

$$\forall s \in \overline{D}_0(r), \quad \left| \frac{1}{\Gamma(s)(s+a)} \right| \underset{r}{\ll} 1.$$

□

Erdélyi's formula [EMOT81, §1.11] described in the following proposition is crucial for re-expressing integrands involving the Lerch zeta function that we defined in Definition 1.2.7:

**Proposition 2.3.5.** [Ono21, Lemma 2.2]

Let  $u \in \mathbb{C} \setminus ]-\infty, 0]$  such that  $|u| < 2\pi$ ,  $s \in \mathbb{C} \setminus \mathbb{N}$ , and  $d \in \mathbb{R}_+^*$ . Then

$$e^{-du} \phi(e^{-u}, s, d) = \Gamma(1-s)u^{s-1} + \sum_{k=0}^{+\infty} \frac{(-u)^k}{k!} \zeta(s-k, d).$$

We aim to obtain an Erdélyi formula for  $d \in H_0$ . The bound we have on the term  $\zeta(s-k, d)$  is not as good when  $d$  is a complex number as when  $d$  is real, thus we must assume an additional condition on the complex  $u$ :

**Proposition 2.3.6.** Let  $d \in H_0$ ,  $s \in \mathbb{C} \setminus \mathbb{N}$  and  $u \in \mathbb{C} \setminus ]-\infty, 0]$  such that  $|u| \ll_d 1$ . then

$$e^{-du} \phi(e^{-u}, s, d) = \Gamma(1-s)u^{s-1} + \sum_{k=0}^{+\infty} \frac{(-u)^k}{k!} \zeta(s-k, d) \quad (2.6)$$

*Proof.* Let  $r' \in [1, +\infty[$  be a real number, we will aim to prove the result for  $d \in H_{1/r'} \cap D_0(r')$ . Let  $u \in \mathbb{C} \setminus ]-\infty, 0]$  such that  $|u| < \frac{\pi}{e(1+r')^2}$ . We fix  $s \in \mathbb{C} \setminus \mathbb{N}$ , and set  $r := \lceil |s| \rceil$ . We already know that the formula we wish to prove holds when  $d \in \mathbb{R}_+^*$ . We then aim to prove this formula for all  $d \in H_0$  through analytic continuation. To do this, we first need to show that the different terms in the expression of the desired formula are holomorphic in the variable  $d$  over  $H_0$ .

We know that the map  $d \in H_0 \mapsto e^{-du} \phi(e^{-u}, s, d)$  is holomorphic [EMOT81, §1.11]; it remains to show that the series appearing in the formula is holomorphic as well. We already know that each terms  $d \in H_0 \mapsto \zeta(s-k, d)$  are holomorphic over  $H_0$  for every integer  $k \in \mathbb{N}_0$ . To prove this, it is enough to refer to the formula obtained by Maclaurin's expansion of the Hurwitz zeta function, found in the proof of Proposition 1.2.6. What remains is to show that the series appearing in (2.6) converges normally in  $d$  over  $\overline{H}_{1/r'} \cap D_0(r')$ . We have the following bound

$$\forall k \in \mathbb{N}_0, \quad |\zeta(s-k, d)| \ll_{r, r', \delta} \left(\frac{e}{\pi}\right)^k k!(k+1)^{r+1} (1+r')^{2k},$$

thus, we find that the general term of the series 2.6 has the following upper bound

$$\frac{|\zeta(s-k, d)|}{k!} |u|^k \ll_{r, r', \delta} \left(u \frac{e}{\pi} (1+r')^2\right)^k (k+1)^{r+1},$$

but the power series  $\sum_{k=0}^{+\infty} \left(\frac{e}{\pi} (1+r')^2\right)^k k^{r+1} z^k$  is absolutely convergent in the open disc  $\left\{z \in \mathbb{C}, |z| < \frac{\pi}{e(1+r')^2}\right\}$ .

We then find that the series  $\sum_{k=0}^{+\infty} \frac{\zeta(s-k, d)}{k!} (-u)^k$  is normally convergent in  $d$  over  $H_{1/r'} \cap D_0(r')$ .

Therefore, the formula (2.6) holds for all  $d \in H_0$  by analytic continuation, via Proposition 2.3.5.  $\square$

### 2.3.2 Preliminary Results

In this section, we wish to establish a Crandall expansion for the multizeta function.

$$Z(\mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}.$$

We recall Definition 1.3.1: Let  $\nu \in H_0$ ,  $\theta > 0$ ,  $s \in \mathbb{C}$ . We defined

$$\begin{aligned}\Gamma(s, \theta, \nu) &= \int_{\theta}^{+\infty} e^{-\nu y} y^{s-1} dy, \\ \gamma(s, \theta, \nu) &= \int_0^{\theta} e^{-\nu y} y^{s-1} dy, \quad \text{when } \operatorname{Re}(s) > 0.\end{aligned}$$

Let's note that the integral defining  $\Gamma(s, \theta, \nu)$  is absolutely convergent for all  $s \in \mathbb{C}$ , but the one defining  $\gamma(s, \theta, \nu)$  is absolutely convergent only for  $s \in H_0$ .

**Proposition 2.3.7.** *Let  $\theta > 0$ . Then the functions  $(s, \nu) \in \mathbb{C} \times H_0 \mapsto \Gamma(s, \theta, \nu)$  and  $(s, \nu) \in H_0 \times H_0 \mapsto \gamma(s, \theta, \nu)$  are holomorphic respectively on  $\mathbb{C} \times H_0$  and on  $H_0 \times H_0$ .*

*Proof.* The proposition follows from the theorem of holomorphy under the integral sign.  $\square$

**Proposition 2.3.8.** *Let  $\nu \in H_0$ ,  $\theta > 0$ . We have*

$$\forall s \in H_0, \quad \Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) = \nu^{-s} \Gamma(s).$$

*Proof.* Let's assume that  $\nu \in \mathbb{R}_+^*$ . By a change of variables we get:

$$\begin{aligned}\Gamma(s, \theta, \nu) &= \nu^{-s} \int_{\nu\theta}^{+\infty} e^{-x} x^{s-1} dx, \\ \gamma(s, \theta, \nu) &= \nu^{-s} \int_0^{\nu\theta} e^{-x} x^{s-1} dx, \quad \text{when } \operatorname{Re}(s) > 0.\end{aligned}$$

Summing up these 2 terms, we have

$$\Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) = \nu^{-s} \Gamma(s).$$

Let  $s \in H_0$ . The function  $\nu \in H_0 \mapsto \Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) - \nu^{-s} \Gamma(s)$  is holomorphic, and vanish in  $s \in \mathbb{R}_+^*$ . By the isolated zero theorem, we get the final result.  $\square$

**Corollary 2.3.9.** *The function  $s \mapsto \gamma(s, \theta, \nu)$  is meromorphic on all  $\mathbb{C}$ , having poles (which are simple) only at nonpositive integers, and the residue at  $s = -n$  is  $\frac{(-1)^n \nu^n}{n!}$ .*

**Lemma 2.3.10.** *Let  $\theta > 0$  be a positive number, and  $\nu \in H_0$ . Let  $s \in \mathbb{C}$ . We have the following bound:*

$$|\Gamma(s, \theta, \nu)| \leq 2^\sigma e^{-\operatorname{Re}(\nu)\theta/2} \Gamma\left(\sigma, \frac{\theta}{2}, \operatorname{Re}(\nu)\right).$$

*Proof.* By the change of variables  $x = \frac{x'}{2}$ , we found

$$\Gamma(s, \theta, \nu) = 2^s \int_{\theta/2}^{+\infty} e^{-2\nu x} x^{s-1} dx.$$

As  $|e^{-\nu x}| = e^{-\operatorname{Re}(\nu)x}$  for all  $x \in \mathbb{R}_+$  and for all complex numbers  $\nu \in H_0$  we find the following bound:

$$\begin{aligned}|\Gamma(s, \theta, \nu)| &\leq |2^s| \int_{\theta/2}^{+\infty} e^{-\operatorname{Re}(\nu)x} e^{-\operatorname{Re}(\nu)x} x^{\sigma-1} dx \\ &\leq 2^\sigma \int_{\theta/2}^{+\infty} e^{-\operatorname{Re}(\nu)\theta/2} e^{-\operatorname{Re}(\nu)x} x^{\sigma-1} dx,\end{aligned}$$

which concludes the proof.  $\square$

We now study the product  $\prod_{q=1}^Q \Gamma(s'_q)Z(\mathbf{s}, \mathbf{s}')$ . Let  $q \in \llbracket 1, Q \rrbracket$ ,  $\theta > 0$ , and  $s'_q \in H_0$ . By Proposition 2.3.8, we get

$$\Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} = \Gamma(s'_q, \theta, (l_q(\mathbf{n}) + d'_q)) + \gamma(s'_q, \theta, (l_q(\mathbf{n}) + d'_q)).$$

We fix a  $P$ -tuple  $\mathbf{n} \in \mathbb{N}_0^P$ , and complex numbers  $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times H_0^Q$ . We thus obtain the following equality by distributing the sum above over the subsequent product:

$$\begin{aligned} \prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} &= \prod_{q=1}^Q \left[ \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) + \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right] \\ &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in \mathcal{Q}^c} \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &\quad + \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in \mathcal{Q}^c} \left( (l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) - \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right) \\ &\quad + \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned}$$

By distributing the terms  $(l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) - \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)$ , we get

$$\begin{aligned} \prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \\ &\quad \prod_{q \in A_{\mathcal{Q}}} \Gamma(s'_q) \prod_{q \in \mathcal{Q}^c \setminus A_{\mathcal{Q}}} (-1) \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &\quad + \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned}$$

Let  $(x_p)_{p \in \llbracket 1, P \rrbracket} \in H_0^P$ , then by the previous equality we have that

$$\begin{aligned} \prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{p=1}^P x_p^{-s_p} &= \\ &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{p=1}^P x_p^{-s_p} \prod_{q \in \mathcal{Q} \cup (\mathcal{Q}^c \setminus A_{\mathcal{Q}})} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \left( \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) \right) \\ &\quad + \prod_{p=1}^P x_p^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned}$$

As  $\mathcal{Q} \cup (\mathcal{Q}^c \setminus A_{\mathcal{Q}}) = \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}$ , by replacing  $x_p$  by  $n_p + d_p$  we get:

$$\begin{aligned} \prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{p=1}^P (n_p + d_p)^{-s_p} &= \tag{2.7} \\ &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \left( \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) \right) \\ &\quad + \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned}$$

Summing the last equality over  $n_p \in \mathbb{N}_0$  for all  $p \in \llbracket 1, P \rrbracket$ , we shall prove later on that for  $(\mathbf{s}, \mathbf{s}') \in \mathcal{D}$  (with  $\mathcal{D}$  the set defined in Proposition 2.1.1), we get

$$\begin{aligned} Z(\mathbf{s}, \mathbf{s}') &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(s'_q)} \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \right. \\ &\quad \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right) \\ &\quad + \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned}$$

By formula 2.7, it is enough to show that the previous series converge absolutely. We shall study those series in the two following propositions:

**Proposition 2.3.11.** *Let  $\theta > 0$ , and  $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q$ . We set*

$$\begin{aligned} K(\theta, \mathbf{s}, \mathbf{s}') &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(s'_q)} \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \right. \\ &\quad \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right). \end{aligned} \quad (2.8)$$

Then the series above converges normally on all compact sets of  $\mathbb{C}^P \times \mathbb{C}^Q$ . Moreover, the function  $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$  is holomorphic on  $\mathbb{C}^P \times \mathbb{C}^Q$ , and  $K(\theta, -\mathbf{N}, -\mathbf{N}') = 0$  for all  $Q + T$ -tuple  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ .

*Proof.* Let  $\theta > 0$  be a real number. In order to prove that  $K$  is holomorphic in the variables  $(\mathbf{s}, \mathbf{s}')$  on all  $\mathbb{C}^P \times \mathbb{C}^Q$ , it is enough to show that the series defining  $K(\theta, \mathbf{s}, \mathbf{s}')$  converges normally on all compact sets.

The function  $K(\theta, \mathbf{s}, \mathbf{s}')$  is a finite sum of the following terms

$$(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(s'_q)} \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right), \quad (2.9)$$

with  $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$ , and  $A_{\mathcal{Q}} \subset \mathcal{Q}^c$ . We see that the functions  $\frac{1}{\Gamma(s'_q)}$ , and  $\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)$  are well defined, and holomorphic on  $\mathbb{C}$ . We define

$$T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}') = \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q), \quad (\mathbf{n} \in \mathbb{N}_0^P) \quad (2.10)$$

as the general term of the series in (2.9).

By Lemma 2.3.10, we get that, for all  $q \in \llbracket 1, Q \rrbracket$  and all  $\mathbf{n} \in \mathbb{N}_0^P$ ,

$$|\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)| \leq 2^{\sigma'_q} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \frac{\theta}{2}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(l_q(\mathbf{n}) + d'_q)).$$

By the previous bound, and using the fact that the following function  $x \in \mathbb{R}_*^+ \mapsto \Gamma(\sigma, \theta, x)$  is decreasing, for  $\theta > 0$ , we get

$$|\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)| \leq 2^{\sigma'_q} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \frac{\theta}{2}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(d'_q)).$$

Finally, for all complex numbers  $z \in H_0$ ,  $s = \sigma + i\tau \in \mathbb{C}$ , we get  $|z^s| = |z|^\sigma e^{-\arg(z)\tau}$ . As  $z \in H_0$ , we then find that  $\arg(z) \in ]-\pi/2, \pi/2[$ , and thus

$$|z^s| \leq |z|^\sigma e^{|\tau|\pi/2}.$$

From all the above, we get the following bound for the general term:

$$\begin{aligned} |T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')| &\leq \left| \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \right| \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} 2^{\sigma'_q} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \frac{\theta}{2}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(d'_q)) \\ &\leq \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \theta/2} 2^{\sigma'_q} \Gamma(\sigma'_q, \theta/2, d'_q) \\ &\quad \cdot \prod_{p=1}^P e^{|\tau_p|\pi/2} |n_p + d_p|^{-\sigma_p} \prod_{q \in A_{\mathcal{Q}}} e^{|\tau'_q|\pi/2} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}. \end{aligned} \quad (2.11)$$

Let's note that, if all the coefficients  $c_{p,q}$ ,  $d_p$  and  $d'_q$  are real numbers, then we can get rid of the  $e^{|\tau_p|\frac{\pi}{2}}$  and  $e^{|\tau'_q|\frac{\pi}{2}}$  terms in the last inequality.

Because the set  $\mathcal{Q}$  is non empty, and that  $A_{\mathcal{Q}} \subset \mathcal{Q}^c$ , then this last set contains at least an integer  $j \in A_{\mathcal{Q}}$ , and by our assumptions,  $l_j(\mathbf{n}) = \sum_{p=1}^P c_{j,p} n_p$ , with  $c_{j,p} \in H_0$  for all  $p \in \llbracket 1, P \rrbracket$ . We also have that  $d'_q \in H_0$  for all  $q \in \llbracket 1, Q \rrbracket$ . Therefore,

$$\prod_{q \in \mathcal{Q}} e^{-\operatorname{Re}(l_q(\mathbf{n}) + d'_q) \theta/2} \leq e^{-\operatorname{Re}(l_j(\mathbf{n})) \theta/2}.$$

Let  $R > 0$ , and  $(\mathbf{s}, \mathbf{s}') \in D_0(R)^{P+Q}$ . We then found that  $(\sigma, \sigma') \in [-R, R]^{P+Q}$  and  $(\tau, \tau') \in [-R, R]^{P+Q}$ . The map  $x \in [-R, R] \mapsto \Gamma(x, \theta, \nu)$  being a continuous function for all  $\theta > 0$ , and all  $\nu \in H_0$ , we found the following bound, uniform in  $(\mathbf{s}, \mathbf{s}') \in D_0(R)^P \times D_0(R)^Q$ ,

$$\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(d'_q)) \ll \frac{1}{R}.$$

By the bound 2.11, we found that for all  $(\mathbf{s}, \mathbf{s}') \in D_0(R)^P \times D_0(R)^Q$ ,

$$|T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')| \ll \frac{2^{(Q-|A_{\mathcal{Q}}|)R}}{R} \prod_{p=1}^P e^{\pi/2R} e^{-\operatorname{Re}(c_{j,p}) n_p \theta/2} |n_p + d_p|^{-\sigma_p} \cdot \prod_{q \in A_{\mathcal{Q}}} e^{\pi/2R} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}. \quad (2.12)$$

As  $\theta > 0$ , and  $\operatorname{Re}(c_{j,p}) > 0$  for all  $p \in \llbracket 1, P \rrbracket$ , we know that the series

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P e^{-\operatorname{Re}(c_{j,p}) n_p \theta/2} |n_p + d_p|^{-\sigma_p} \prod_{q \in A_{\mathcal{Q}}} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}$$

converges normally on all compact sets of  $\mathbb{C}^P \times \mathbb{C}^Q$ . Thus, we found that the series  $\sum_{\mathbf{n} \in \mathbb{N}_0^P} T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')$

converges normally on  $D_0(R)^{P+Q}$ . We then get the fact that  $K$  is holomorphic on  $\mathbb{C}^P \times \mathbb{C}^Q$ .

Finally, we see that, for all  $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$  and for all  $A_{\mathcal{Q}} \subset \mathcal{Q}^c$ , we have that  $\llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}} \neq \emptyset$ . Therefore, the product  $\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(s'_q)}$  is non-empty. In particular, when we take the values of  $K(\theta, \mathbf{s}, \mathbf{s}')$  at  $\mathbf{s}' = -\mathbf{N}'$ , we get 0.  $\square$

**Remark 2.3.12.** By evaluating formula 2.8 and replacing  $\mathbf{s}$  by  $(-N_p + \mu_p s)_{p \in \llbracket 1, P \rrbracket}$ , and  $\mathbf{s}'$  by  $(-N'_q + \mu'_q s)_{q \in \llbracket 1, Q \rrbracket}$ , we recover expression 2.4 in Proposition 2.2.4.

**Proposition 2.3.13.** *Let  $\theta > 0$ , and  $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ , with  $\mathcal{D}$  the set defined in Proposition 2.1.1. We define*

$$J(\theta, \mathbf{s}, \mathbf{s}') = \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \quad (2.13)$$

The preceding series is normally convergent on any compact subset of  $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ . In particular, the function  $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$  is holomorphic on  $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ .

*Proof.* Let  $\theta > 0$ , and  $R > 0$  be positive real numbers. We use the notations from the previous proposition, particularly formula 2.10 to describe the general term of the series present in  $K(\theta, \mathbf{s}, \mathbf{s}')$ . By equality 2.7, we know that, for all  $\mathbf{n} \in \mathbb{N}_0^P$  and  $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times H_1^Q$ :

$$\begin{aligned} & \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) = \\ & - \left( \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \frac{(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|}}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q)} T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}') \right) + \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}. \end{aligned}$$

In the proof of Proposition 2.3.11, we showed using inequality 2.12 that the series with general term  $T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')$  is normally convergent on any compact subset of  $\mathbb{C}^P \times \mathbb{C}^Q$ . We proved in Proposition 2.1.1 that the series of functions in  $(\mathbf{s}, \mathbf{s}')$

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}$$

is normally convergent on any compact subset of  $\mathcal{D}$ . Thus, we obtain that the series

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)$$

is normally convergent on any compact subset of  $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ .  $\square$

**Proposition 2.3.14.** *Let  $\theta > 0$ , and  $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ . We have:*

$$Z(\mathbf{s}, \mathbf{s}') = K(\theta, \mathbf{s}, \mathbf{s}') + J(\theta, \mathbf{s}, \mathbf{s}'), \quad (2.14)$$

Furthermore, the function  $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$  admits a meromorphic extension on  $\mathbb{C}^P \times \mathbb{C}^Q$ .

*Proof.* The first equality follows from the last two propositions. Since  $(\mathbf{s}, \mathbf{s}') \mapsto Z(\mathbf{s}, \mathbf{s}')$  is meromorphic on  $\mathbb{C}^P \times \mathbb{C}^Q$ , and  $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$  is holomorphic on the same space, it follows that  $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$  is meromorphic on  $\mathbb{C}^P \times \mathbb{C}^Q$ .  $\square$

**Corollary 2.3.15.** *The meromorphic function  $(\mathbf{s}, \mathbf{s}') \mapsto Z(\mathbf{s}, \mathbf{s}')$  and the meromorphic function  $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$  share exactly the same singularities, and the multiplicity of their polar divisor is identical at the same singularity. Moreover,  $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$  is holomorphic on the domain  $\mathcal{D}$ , where  $\mathcal{D}$  is the set defined in Proposition 2.1.1.*

To compute the values of  $Z$  at nonpositive integers, it remains to study  $J(\theta, \mathbf{s}, \mathbf{s}')$ . To do this, we will express this term as an integral of a Lerch zeta function, and then simplify the integrand using an Erdélyi formula for each term of the product.

Let's recall the following notations for the functions  $l_p^*$ : Let  $p \in \llbracket 1, P \rrbracket$ . We consider  $l_p^* : \mathbb{R}^Q \rightarrow \mathbb{C}$  the following linear form

$$l_p^*(x) := \sum_{q=1}^Q c_{q,p} x_q.$$

By taking the previous expression of  $J(\theta, \mathbf{s}, \mathbf{s}')$  for  $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ , we get that

$$\begin{aligned} J(\theta, \mathbf{s}, \mathbf{s}') &= \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q e^{-(l_q(\mathbf{n}) + d'_q)x_q} x_q^{s'_q - 1} dx_q \\ &= \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(x)n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{s'_q - 1} dx_q. \end{aligned} \quad (2.15)$$

**Definition 2.3.16.** Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ . We define for  $s \in \mathbb{C}$ ,

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}(\theta, s) &:= J(\theta, -\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s) \\ K_{\mathbf{N}, \mathbf{N}'}(\theta, s) &:= K(\theta, -\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s). \end{aligned}$$

By (2.14), for  $s \in \mathbb{C}$  such that  $(-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s) \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ , we have

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}(\theta, s).$$

By definition of the convergence domain  $\mathcal{D}$  in Proposition 1.3.9, we get that the previous assumption on  $s$  is verified if

$$\begin{aligned} \forall \mathcal{P} \subset \llbracket 1, P \rrbracket, \quad & -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| + (|\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)\sigma > |\mathcal{P}| \\ \forall q \in \llbracket 1, Q \rrbracket, \quad & -N'_q + \mu'_q \sigma > 0. \end{aligned}$$

We define  $\sigma_0(\mathbf{N}, \mathbf{N}') := \max\left(\frac{N'_1}{\mu'_1}, \dots, \frac{N'_Q}{\mu'_Q}, \max_{\mathcal{P} \subset \llbracket 1, P \rrbracket} \left(\frac{1 + |\mathcal{P}| + |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}}}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}\right)\right)$ . We then get the following proposition:

**Proposition 2.3.17.** For all  $s \in H_{\sigma_0(\mathbf{N}, \mathbf{N}')}_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ , we have

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}(\theta, s).$$

**Remark 2.3.18.** 1) Rather than using the notation  $s = \sigma + i\tau \in H_{\sigma_0(\mathbf{N}, \mathbf{N}')}_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ , we prefer to denote  $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$ , or even sometimes  $\sigma \gg 1$ .

2) Here we use the fact that  $\mu'_q > 0$  for all  $1 \leq q \leq Q$ . One should be able to generalize this approach by considering directions  $\mu'_q \in H_0$ , however, this would complicate the reasoning for this chapter.

We already know from Proposition 2.3.11 that  $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  is holomorphic and vanishes at  $s = 0$ .

What remains to be studied is the term  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ . The key point in studying this function is Erdélyi's formula.

**Proposition 2.3.19.** Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ . Let  $s \in \mathbb{C}$  such that  $\mu_p s \notin \mathbb{N}$  for all  $p \in \llbracket 1, P \rrbracket$ , and  $\sigma \gg 1$ . We then have

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-d_p l_p^*(x)} \phi(e^{-l_p^*(x)}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_q. \quad (2.16)$$

*Proof.* Let  $s = \sigma + i\tau \in \mathbb{C}$  such that  $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$ . By formula 2.15, we found that

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}(\theta, s) &= \\ &= \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-(\sum_{q=1}^Q c_{q,p} x_q)n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_q. \end{aligned}$$



We know wish to interchange the integral sign with the sum. Firstly, we will bound the general term in the previous series. For all  $q \in \llbracket 1, Q \rrbracket$ , we define

$$v_q(\mathbf{n}) := \sum_{p=1}^P \operatorname{Re}(c_{q,p}) n_p.$$

We then found the following bound:

$$\begin{aligned} \forall \mathbf{n} \in \mathbb{N}_0^P, \quad & \left| \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-(\sum_{q=1}^Q c_{q,p} x_q) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q \right| \\ & \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \mu_p \sigma} e^{\mu_p |\tau| \pi / 2} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q e^{-(v_q(\mathbf{n}) + \operatorname{Re}(d'_q)) x_q} x_q^{-N'_q + \mu'_q \sigma - 1} dx_1 \dots dx_Q. \end{aligned}$$

By a change of variables  $y_q := (v_q(\mathbf{n}) + \operatorname{Re}(d'_q)) x_q$  for all  $q \in \llbracket 1, Q \rrbracket$ , we get:

$$\begin{aligned} & \left| \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-(\sum_{q=1}^Q c_{q,p} x_q) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q \right| \\ & \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \mu_p \sigma} e^{\mu_p |\tau| \pi / 2} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \mu'_q \sigma} \\ & \quad \cdot \int_0^{(v_1(\mathbf{n}) + \operatorname{Re}(d'_1)) \theta} \dots \int_0^{(v_Q(\mathbf{n}) + \operatorname{Re}(d'_Q)) \theta} \prod_{p=1}^P e^{-y_p} \prod_{q=1}^Q y_q^{-N'_q + \mu'_q \sigma - 1} dy_1 \dots dy_Q \\ & \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \mu_p \sigma} e^{\mu_p |\tau| \pi / 2} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \mu'_q \sigma} \Gamma(-N'_q + \mu'_q \sigma). \end{aligned}$$

We can see that the series  $\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P |n_p + d_p|^{N_p - \mu_p \sigma} e^{\mu_p |\tau| \pi / 2} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \mu'_q \sigma}$  is convergent for  $\sigma \gg_{\substack{\mathbf{N}, \mathbf{N}' \\ \mu, \mu'}} 1$  by Proposition 2.1.1. By Lebesgue's Theorem, for all  $s = \sigma + i\tau \in \mathbb{C}$  such that  $\sigma \gg_{\substack{\mathbf{N}, \mathbf{N}' \\ \mu, \mu'}} 1$ , we get that

$$\begin{aligned} J_{\substack{\mathbf{N}, \mathbf{N}' \\ \mu, \mu'}}(\theta, s) = \\ \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(x) d_p} \left( \sum_{n_p=1}^{+\infty} (n_p + d_p)^{N_p - \mu_p s} e^{-l_p^*(x) n_p} \right) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q, \end{aligned}$$

which concludes the proof.  $\square$

Once this integral formula is obtained for  $J_{\substack{\mathbf{N}, \mathbf{N}' \\ \mu, \mu'}}(\theta, s)$ , we can use Erdélyi's Formula on each Lerch zeta function in the integrand's product, and then distribute the expression to get the following proposition:

**Proposition 2.3.20.** *Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . Let  $s \in \mathbb{C}$  such that  $\mu_{ps} \notin \mathbb{N}$*

for all  $p \in \llbracket 1, P \rrbracket$ , and  $\sigma \gg 1$ . Then, we find that for  $\theta > 0$  sufficiently small,

$$\begin{aligned}
J_{\mathbf{N}, \mathbf{N}'}(\theta, s) &= \\
& \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\
& \cdot \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(x)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(x)^{k_p} dx_1 \dots dx_Q \\
& + \frac{\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s)}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(x)^{-N_p + \mu_p s - 1} dx_1 \dots dx_Q. \tag{2.17}
\end{aligned}$$

*Proof.* Let  $M = \max_{1 \leq p \leq P, 1 \leq q \leq Q} (|c_{q,p}|)$ , and  $\theta \in \left[0, \frac{1}{QM}\right]$ . Let  $s \in \mathbb{C}$  such that  $\mu_p s \notin \mathbb{N}$  for all  $p \in \llbracket 1, P \rrbracket$ , and  $\sigma \gg 1$ . By the formula 2.16, we have that

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \int_0^\theta \dots \int_0^\theta \sum_{\mathbf{n} \in \mathbb{N}_0^{\mathcal{P}}} \prod_{p=1}^P e^{-d_p l_p^*(x)} \phi(e^{-l_p^*(x)}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_q.$$

By Proposition 2.6, for all  $(x_q)_{q \in \llbracket 1, Q \rrbracket} \in ]0, \theta]^Q$ , we have

$$\begin{aligned}
& \prod_{p=1}^P e^{-d_p l_p^*(x)} \phi(e^{-l_p^*(x)}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \\
& = \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P \left( \Gamma(1 + N_p - \mu_p s) l_p^*(x)^{-N_p + \mu_p s - 1} + \sum_{k_p \geq 0} (-1)^{k_p} \frac{l_p^*(x)^{k_p}}{k_p!} \zeta(-N_p + \mu_p s - k_p, d_p) \right).
\end{aligned}$$

Distributing the previous product, we get for all  $\mathbf{x} = (x_q)_{q \in \llbracket 1, Q \rrbracket} \in ]0, \theta]^Q$ ,

$$\begin{aligned}
& \prod_{p=1}^P e^{-d_p l_p^*(x)} \phi(e^{-l_p^*(x)}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \\
& = \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \\
& \cdot \prod_{p \in \mathcal{P}} l_p^*(x)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(x)^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\
& + \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \prod_{p=1}^P l_p^*(x)^{-N_p + \mu_p s - 1}. \tag{2.18}
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
J_{\mathbf{N}, \mathbf{N}'}(\theta, s) &= \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \int_0^\theta \dots \int_0^\theta \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \\
& \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \cdot \prod_{p \in \mathcal{P}} l_p^*(x)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(x)^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} dx_1 \dots dx_Q \\
& + \frac{\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s)}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(x)^{-N_p + \mu_p s - 1} dx_1 \dots dx_Q. \tag{2.19}
\end{aligned}$$

We shall now bound the general term in the series inside the integrand. By compactness, we have uniformly on  $\mathbf{x} = (x_1, \dots, x_Q) \in \left[0, \frac{1}{QM}\right]^Q$  that

$$\left| \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \right| \ll_s 1$$

$$|l_p^*(x)^{-N_p + \mu_p s - 1}| \ll_s 1.$$

Since  $l_p^*$  is a linear form for all  $p \in \llbracket 1, P \rrbracket$ , we get that

$$\forall x \in [0, \theta]^Q, \quad |l_p^*(x)| \leq QM\theta.$$

By Lemma 2.3.2, we have the following bound:

$$\forall p \in \llbracket 1, P \rrbracket, \quad \left| \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \ll_s \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil \mu_p |s| \rceil}.$$

For all  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ , and for all  $\mathbf{x} = (x_q)_{q \in \llbracket 1, Q \rrbracket} \in [0, \theta]^Q$ , we then get the following bound

$$\left| \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(x)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(x)^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right|$$

$$\ll_s \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil \mu_p |s| \rceil} (QM\theta)^{k_p}.$$

Since the following series

$$\sum_{\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil \mu_p |s| \rceil} (QM\theta)^{k_p},$$

is absolutely convergent on  $\theta \in \left[0, \frac{1}{QM}\right]$ , we then can interchange the series and the integral to prove the proposition.  $\square$

**Proposition 2.3.21.** *Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . Let  $s \in \mathbb{C}$  such that  $\mu_p s \notin \mathbb{N}$  for all  $p \in \llbracket 1, P \rrbracket$ , and  $\sigma \gg 1$ . We have*

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) =$$

$$\sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$$

$$\frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!},$$

$$+ \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right),$$

where we recall the notations of Proposition 2.2.4: For  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ ,  $j \in \llbracket 1, Q \rrbracket$ , and  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$ ,

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)}$$

$$\cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx_j} \dots dx_Q,$$

and

$$h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx_j} \dots dx_Q.$$

with:

- 1)  $\widehat{\mathbf{x}}^j = (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_Q)$ ,
- 2)  $dx_1 \dots \widehat{dx_j} \dots dx_Q = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_Q$ .

*Proof.* Let  $s \in \mathbb{C}$  such that  $\mu_p s \notin \mathbb{N}$  for all  $p \in \llbracket 1, P \rrbracket$ , and  $\sigma \gg 1$ . To prove the required formula, it's enough to revisit the formula obtained in the previous proposition, and to re-express the integral in this formula. This integral will be simplified using a blowing-up.

First, we note that  $[0, \theta]^Q = \bigcup_{j=1}^Q V_j$ , where  $V_j := \{\mathbf{x} \in [0, \theta]^Q \mid \forall q \neq j, x_j \geq x_q\}$ , and we also observe that the pairwise intersection of these sets has zero measure. By Chasles' relation, we obtain

$$\begin{aligned} & \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q \\ &= \sum_{j=1}^Q \int_{V_j} \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q. \end{aligned}$$

We now consider the following change of variables:

$$\begin{aligned} f_j : [0, 1]^{j-1} \times [0, \theta] \times [0, 1]^{Q-j} &\rightarrow V_j \\ \mathbf{x} = (x_1, \dots, x_Q) &\mapsto (x_1 x_j, \dots, x_j, \dots, x_Q x_j), \end{aligned}$$

with the corresponding jacobian determinant  $Jac(f_j) = x_j^{Q-1}$ .

We can just study the integral with the domain  $V_Q$ , and extrapolate the other integrals by a permutation of variables. Firstly, let's note that for all  $\mathbf{x} = (x_1, \dots, x_Q) \in [0, 1]^{Q-1} \times [0, \theta]$ , we have  $l_p^*(f_Q(\mathbf{x})) = x_Q l_p^*(\widehat{\mathbf{x}}^Q)$ . Therefore we get that

$$\begin{aligned} & \int_{V_Q} \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q \\ &= \int_0^1 \dots \int_0^1 \int_0^\theta x_Q^{-|\mathbf{N}'| + |\boldsymbol{\mu}'|s - Q - |\mathbf{N}|_{|\mathcal{P}} + |\boldsymbol{\mu}|_{|\mathcal{P}^c} s - |\mathcal{P}| + |\mathbf{k}|} \prod_{q=1}^{Q-1} x_q^{-N'_q + \mu'_q s - 1} \\ & \quad \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^Q)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^Q)^{k_p} x_Q^{Q-1} dx_1 \dots dx_Q \\ &= \int_0^1 \dots \int_0^1 \prod_{q=1}^{Q-1} x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^Q)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} u_p(\widehat{\mathbf{x}}^Q)^{k_p} dx_1 \dots dx_{Q-1} \\ & \quad \cdot \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}^c})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}^c})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}, \end{aligned}$$

which concludes the proof.  $\square$

The previous proposition is crucial to the proof of Proposition 2.2.4. We shall prove that  $J_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s)$  is regular at  $s = 0$  via the previous proposition. Particularly, we will use Proposition 2.2.2, which gives a bound for the functions  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^{\mu, \mu'}(s)$ , uniform in  $s$  on all compact sets of  $\mathbb{C}$ . We shall prove Proposition 2.2.2 in the next sub-section.

### 2.3.3 Proof of Proposition 2.2.2

In this subsection, we will prove Proposition 2.2.2. Let us recall the notation  $l_p^*(x) = \sum_{q=1}^Q c_{q,p} x_q$  for every  $p \in [1, P]$  and every  $x \in \mathbb{R}^Q$ . We will establish a lemma before proving this proposition. For every  $\epsilon \in \left] 0, \frac{1}{2MQ} \right]$ , for each subset  $\mathcal{Q} \subset [1, Q] \setminus \{j\}$ , and for all  $q \in [1, Q] \setminus \{j\}$ , we set

$$\Delta_{\mathcal{Q}, j}(\epsilon, q) = \begin{cases} [0, \epsilon] & \text{if } q \in \mathcal{Q} \\ [\epsilon, 1] & \text{if } q \in \mathcal{Q}^c \end{cases}$$

and we define  $F_{\mathcal{Q}, j}(\epsilon) := \prod_{\substack{q=1 \\ q \neq j}}^Q \Delta_{\mathcal{Q}, j}(\epsilon, q)$ . We then can partition the following integration domain:

$$\{(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_Q) \mid \forall q \neq j, x_q \in [0, 1]\} = \bigcup_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} F_{\mathcal{Q}, j}(\epsilon).$$

Via Chasles' formula, we can write  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^{\mu, \mu'}(s)$  and  $h_{\mathbf{N}, \mathbf{N}', [1, P], j}^{\mu, \mu'}$  as sums of the following integrals:

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^{\mu, \mu'}(s) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^{\mu, \mu'}(\epsilon, s) \quad (2.20)$$

$$h_{\mathbf{N}, \mathbf{N}', [1, P], j}^{\mu, \mu'}(s) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}^{\mu, \mu'}(\epsilon, s), \quad (2.21)$$

where we noted for all subset  $\mathcal{Q} \subset [1, Q] \setminus \{j\}$ ,

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^{\mu, \mu'}(\epsilon, s) =$$

$$\frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx}_j \dots dx_Q,$$

and

$$I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}^{\mu, \mu'}(\epsilon, s) = \frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx}_j \dots dx_Q,$$

for all  $s \in \mathbb{C}$  such that  $\sigma \gg 1$ .

**Lemma 2.3.22.** *Let  $(\mu, \mu') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ ,  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ ,  $\mathcal{P} \subsetneq [1, P]$ ,  $j \in [1, Q]$ ,  $\mathcal{Q} \subset [1, Q] \setminus \{j\}$ , and  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$ . Then we have that*

i) *For all  $\epsilon > 0$  small enough, the function  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^{\mu, \mu'}(\epsilon, s)$  is holomorphic in  $s$  on all  $\mathbb{C}$ .*

ii) *For  $\epsilon > 0$  small enough, and for every positive integer  $r \in \mathbb{N}$ , we have the following bound, uniformly in  $s \in D_0(r)$ :*

$$\left| I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^{\mu, \mu'}(\epsilon, s) \right| \ll_{r, \epsilon} (QM)^{|\mathbf{k}|},$$

$$\text{with } M = \max_{1 \leq q \leq Q, 1 \leq p \leq P} (|c_{q,p}|).$$

iii) For all  $s \in \mathbb{C}$  and  $0 < \epsilon \ll 1$ , we have

$$\begin{aligned}
I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \\
&\sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, \mathcal{Q} \rrbracket} c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
&\cdot \frac{1}{\prod_{q \in \llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s) \prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \\
&\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q, \quad (2.22)
\end{aligned}$$

where we set  $\mathcal{Q}^c = (\llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}) \setminus \mathcal{Q}$ ,  $\mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p,q}$  for all  $q \in \mathcal{Q}$ , and  $\mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p,q}$  for all  $q \in \llbracket 1, \mathcal{Q} \rrbracket$ . Noting  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{w}, \mathbf{v}}(\epsilon, s)$  as the general term in the previous series, we also have

$$\partial_s I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \sum_{\substack{\mu, \mu'}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \partial_s T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{w}, \mathbf{v}}(\epsilon, s) \quad (2.23)$$

iv) For  $\epsilon > 0$  small enough, and for  $s \in \mathbb{C}$ , we have

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\substack{\mu, \mu'}} \sum_{\mathcal{Q} \subset \llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \quad (2.24)$$

*Proof.* Here we aim to expand the integrand using power series expansions around 0, in  $F_{\mathcal{Q}, j}(\epsilon)$ . Note that these power series will converge uniformly on the disk  $D_0(\eta)$ , for  $\eta$  sufficiently small. We can therefore interchange the series and the integral.

Let  $s \in \mathbb{C}$  such that  $\sigma \gg 1$ . Let  $\mathcal{Q} \subset \llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}$ . Let  $\epsilon \in \left] 0, \frac{1}{2M\mathcal{Q}} \right]$ , with  $M = \max_{1 \leq q \leq \mathcal{Q}, 1 \leq p \leq \mathcal{P}} (|c_{q,p}|)$ , we can expand into a power series in  $(x_q)_{q \in \mathcal{Q}}$  the functions of the form  $l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1}$ , where  $x \in F_{\mathcal{Q}, j}$  and  $p \in \mathcal{P}$ . By Lemma 2.3.1, we obtain uniformly in  $\widehat{x}^j$ :

$$\begin{aligned}
l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} &= \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q + \sum_{q \in \mathcal{Q}} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1} \\
&= \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1} \left( 1 + \sum_{f \in \mathcal{Q}} \frac{c_{f,p}}{c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q} x_f \right)^{-N_p + \mu_p s - 1} \\
&= \sum_{\mathbf{v}_p=(v_{p,q})_{q \in \llbracket 1, \mathcal{Q} \rrbracket} \in \mathbb{N}_0^{\mathcal{Q}}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} x_q^{v_{p,q}}. \quad (2.25)
\end{aligned}$$

For  $p \in \mathcal{P}^c$ , and  $k_p \in \mathbb{N}_0$ , we have by Newton's multinomial:

$$l_p^*(\widehat{\mathbf{x}}^j)^{k_p} = \sum_{\substack{\mathbf{w}_p=(w_{p,q})_{q \in \llbracket 1, \mathcal{Q} \rrbracket} \in \mathbb{N}_0^{\mathcal{Q}} \\ |\mathbf{w}_p|=k_p}} \binom{k_p}{\mathbf{w}_p} c_{j,p}^{w_{p,j}} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} x_q^{w_{p,q}}.$$

The series in (2.25) converges uniformly in  $(x_e)_{e \in \mathcal{Q}} \in [0, \epsilon]^{\mathcal{Q}}$ . By interchanging the integral and the series, we get that  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  is of the following form:

$$\begin{aligned}
I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \\
&\sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} c_{j,p}^{w_{p,j}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \right) \\
&\cdot \frac{1}{\prod_{q \in [1, \mathcal{Q}] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} x_q^{-N'_q + \mu'_q s - 1} \left( \prod_{p \in \mathcal{P}^c} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} x_q^{w_{p,q}} \right) \\
&\cdot \prod_{p \in \mathcal{P}} \left( \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} x_q^{v_{p,q}} \right) \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_1 \dots \widehat{dx}_j \dots dx_{\mathcal{Q}}.
\end{aligned}$$

Let's note that  $[1, \mathcal{Q}] \setminus \{j\} = \mathcal{Q} \cup \mathcal{Q}^c$ . Then we can split the products  $\prod_{q \in [1, \mathcal{Q}] \setminus \{j\}}$  into 2 parts, and we get:

$$\begin{aligned}
I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \\
&\sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
&\cdot \frac{1}{\prod_{q \in [1, \mathcal{Q}] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{p \in \mathcal{P}^c} \left( \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} x_q^{w_{p,q}} \right) \prod_{p \in \mathcal{P}} \left( \prod_{q \in \mathcal{Q}} x_q^{v_{p,q}} \right) \\
&\cdot \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \cdot \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots \widehat{dx}_j \dots dx_{\mathcal{Q}} \\
&= \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
&\cdot \frac{1}{\prod_{q \in [1, \mathcal{Q}] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q}, j}} \prod_{q \in \mathcal{Q}} x_q^{-N'_q + \mu'_q s - 1 + \sum_{p \in \mathcal{P}} v_{p,q} + \sum_{p \in \mathcal{P}^c} w_{p,q}} \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \sum_{p \in \mathcal{P}^c} w_{p,q}} \\
&\cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_1 \dots \widehat{dx}_j \dots dx_{\mathcal{Q}}.
\end{aligned}$$

Let  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}}$  with  $\mathbf{v}_p \in \mathbb{N}_0^{\mathcal{Q}}$  for all  $p \in \mathcal{P}$ , and  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c}$  with  $\mathbf{w}_p \in \mathbb{N}_0^{\mathcal{Q}}$  for all  $p \in \mathcal{P}^c$ . We recall the notation  $\mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p,q}$  for all  $q \in \mathcal{Q}$ , and  $\mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p,q}$  for all  $q \in [1, \mathcal{Q}]$ . Via

Fubini's theorem, we get the following expression for all  $\sigma \gg 1$ :

$$\begin{aligned}
I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \\
& \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
& \cdot \frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \left( \prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q)}}{-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q)} \right) \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \\
& \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q.
\end{aligned}$$

Thus we obtain that 2.22 is true for all  $s \in \mathbb{C}$  such that  $\sigma \gg 1$ .

We're now looking to prove an analytic continuation of  $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  on all  $\mathbb{C}$  via the formula that we just obtained. We clearly have that, for all  $\sigma \gg 1$ ,

$$\begin{aligned}
I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \\
& \frac{1}{\prod_{q \in \mathcal{Q}^c} \Gamma(-N'_q + \mu'_q s)} \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\
& \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q)}}{\Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \\
& \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \left( \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \right) \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q. \quad (2.26)
\end{aligned}$$

Let's define  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  as the general term of the previous series. We then have that

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s).$$

All the terms inside the expression of  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  are holomorphic in  $s$  on all  $\mathbb{C}$ . In order to show that  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  is holomorphic in  $s$ , it's enough to show that the last series converges normally on all compact sets of  $\mathbb{C}$ . We're now looking to find a bound of that general term over a compact disk  $\overline{D}_0(r)$ , with  $r \in \mathbb{N}$  being a positive integer. Such bound will depend on  $\mathbf{N}, \mathbf{N}', \boldsymbol{\mu}, \boldsymbol{\mu}', \mathcal{P}, j, \mathcal{Q}$ , but we shall not specify this dependence to increase the clarity of the proof.

We know that the function

$$s \mapsto \frac{1}{\Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))}$$

is holomorphic on  $\mathbb{C}$ , for all  $q \in \mathcal{Q}$ . We now bound the general term of the series 2.26 thanks to Lemma 2.3.4. We have

$$\begin{aligned}
\forall q \in \mathcal{Q}, \forall s \in D_0(r), \forall a \in \mathbb{N}_0, & \quad \left| \frac{1}{\Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + a)} \right| \ll_r 1 \\
\forall q \in [1, Q], & \quad \left| \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right| \ll_r 1.
\end{aligned}$$



We know that the function

$$s \in \mathbb{C} \mapsto \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q,$$

is holomorphic because the integrand is regular.

Let  $s \in D_0(r)$ ,  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}}$  with  $\mathbf{v}_p \in \mathbf{N}^{\mathcal{Q}}$ , and  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c}$  with  $\mathbf{w}_p \in \mathbb{N}_0^{\mathcal{Q}}$  such that  $|\mathbf{w}_p| = k_p$  for all  $p \in \mathcal{P}^c$ . We see that  $\sum_{\substack{q=1 \\ q \neq j}}^Q \mathbf{w}(q) = |\mathbf{k}| - \mathbf{w}(j)$ . Moreover, by compactness, we find the following bound which is uniform in  $x_q \in [\epsilon, 1]$ ,

$$\left| x_q^{-N'_q + \mu'_q s - 1} \right| \ll_r 1.$$

As  $\mathbf{w}(q) \geq 0$ , we find that  $|x_q^{\mathbf{w}(q)}| \leq 1$ . Moreover, since  $\arg \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right) \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$ , we have for all  $s \in D_0(r)$ , and for all  $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$ ,

$$\forall s \in D_0(r), \quad \left| \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \right| \leq \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right|^{-N_p + \mu_p \sigma - 1 - |\mathbf{v}_p|} e^{\mu_p r \pi / 2}.$$

Let's note that we have the following upper bound for all  $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$ ,

$$K + K\epsilon \leq \left( \operatorname{Re} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} \operatorname{Re}(c_{q,p} \epsilon) \right) \right) \leq \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \leq \sum_{q=1}^Q |c_{q,p}| \leq QM,$$

with  $M = \max_{1 \leq p \leq P, 1 \leq q \leq Q} (|c_{q,p}|)$  and  $K := \min_{1 \leq q \leq Q, 1 \leq p \leq P} \operatorname{Re}(c_{q,p})$ . For  $p \in \mathcal{P}$ , we then find that

$$\left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right|^{-N_p + \mu_p \sigma - 1 - |\mathbf{v}_p|} \leq \begin{cases} (QM)^{\mu_p \sigma - N_p - |\mathbf{v}_p| - 1} & \text{if } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \geq 1 \text{ and } \mu_p \sigma - N_p - |\mathbf{v}_p| - 1 \geq 0 \\ 1 & \text{if } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \geq 1 \text{ and } \mu_p \sigma - N_p - |\mathbf{v}_p| - 1 < 0 \\ 1 & \text{if } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| < 1 \text{ and } \mu_p \sigma - N_p - |\mathbf{v}_p| - 1 \geq 0 \\ (K(\epsilon + 1))^{\mu_p \sigma - N_p - |\mathbf{v}_p| - 1} & \text{otherwise} \end{cases}$$

We have uniformly in  $s \in D_0(r)$ :

$$\forall p \in \mathcal{P}, \quad (QM)^{\mu_p \sigma - N_p - |\mathbf{v}_p| - 1} \ll_r (QM)^{-|\mathbf{v}_p|}.$$

Let  $p \in \llbracket 1, P \rrbracket$ . By compactness, we have uniformly in  $s \in D_0(r)$ :

$$(K\epsilon)^{\mu_p \sigma} \ll_{\epsilon, r} 1.$$

We also have

$$(K(\epsilon + 1))^{-N_p - |\mathbf{v}_p| - 1} \ll_{\epsilon, r} (K(\epsilon + 1))^{-|\mathbf{v}_p|} \ll_{r, \epsilon} K^{-|\mathbf{v}_p|}.$$

Finally, we have the following bound uniformly in  $s \in D_0(r)$  and in  $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$ ,

$$\left| \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p \sigma - 1 - |\mathbf{v}_p|} \right| \underset{r, \epsilon}{\ll} \max(K^{-|\mathbf{v}_p|}, (QM)^{-|\mathbf{v}_p|}, 1) \prod_{p=1}^P e^{\mu_p r \pi / 2} \underset{r, \epsilon}{\ll} \min(K, QM, 1)^{-|\mathbf{v}_p|}.$$

Now that we have a bound of the integrand, we can give a bound of the corresponding integral:

$$\left| \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q \right| \underset{r, \epsilon}{\ll} \prod_{p \in \mathcal{P}} \min(K, QM, 1)^{-|\mathbf{v}_p|} \underset{r, \epsilon}{\ll} \min(K, QM, 1)^{-\sum_{p \in \mathcal{P}} |\mathbf{v}_p|}.$$

We have the following bound, uniformly in  $s \in D_0(r)$ :

$$\left| \epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \right| \underset{r, \epsilon}{\ll} \epsilon^{\sum_{q \in \mathcal{Q}} \mathbf{v}(q)}.$$

Moreover, we have

$$\forall n \in \mathbb{N}_0, \forall a \in \mathbb{N}_0, \forall b \in \mathbb{R}_+, \forall s \in D_0(r), \quad \left| \binom{-n + bs - 1}{a} \right| \leq \binom{n + \lceil br \rceil + a}{a}.$$

We then find that

$$\begin{aligned} & \left| \frac{1}{\prod_{q \in \mathcal{Q}^c} \Gamma(-N'_q + \mu'_q s)} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \right. \\ & \cdot \prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q)}}{\Gamma(-N'_q + \mu'_q s) (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \\ & \cdot \left. \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q \right| \\ & \underset{r, \epsilon}{\ll} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} M^{\sum_{q \in [1, Q]} w_{p,q}} \right) \prod_{p \in \mathcal{P}} \binom{N_p + \lceil \mu_p r \rceil + |\mathbf{v}_p|}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} (M\epsilon)^{\sum_{q \in \mathcal{Q}} \mathbf{v}(q)} \min(K, QM, 1)^{-\sum_{q \in \mathcal{Q}} \mathbf{v}(q)} \\ & \underset{r, \epsilon}{\ll} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} M^{k_p} \right) \prod_{p \in \mathcal{P}} \binom{N_p + \lceil \mu_p r \rceil + |\mathbf{v}_p|}{\mathbf{v}_p} \left( \frac{M\epsilon}{\min(K, QM, 1)} \right)^{\sum_{q \in \mathcal{Q}} \mathbf{v}(q)}. \end{aligned} \quad (2.27)$$

Taking the sum of the right hand side of 2.27 over  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}}$  with  $\mathbf{v}_p \in \mathbf{N}^{\mathcal{Q}}$ , and over  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c}$  with  $\mathbf{w}_p \in \mathbb{N}_0^Q$  such that  $|\mathbf{w}_p| = k_p$  for all  $p \in \mathcal{P}^c$ , we get

$$\begin{aligned} & M^{|\mathbf{k}|} \left( \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \right) \left( \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \prod_{p \in \mathcal{P}} \binom{N_p + \lceil \mu_p r \rceil + |\mathbf{v}_p|}{\mathbf{v}_p} \left( \frac{M\epsilon}{\min(K, QM, 1)} \right)^{\sum_{q \in \mathcal{Q}} \mathbf{v}(q)} \right) \\ & = (QM)^{|\mathbf{k}|} \left( \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \prod_{p \in \mathcal{P}} \binom{N_p + \lceil \mu_p r \rceil + |\mathbf{v}_p|}{\mathbf{v}_p} \left( \frac{M\epsilon}{\min(K, QM, 1)} \right)^{\sum_{q \in \mathcal{Q}} \mathbf{v}(q)} \right). \end{aligned}$$

The previous power series converges absolutely on  $0 < \epsilon < \frac{\min(K, QM, 1)}{M}$ . By taking a  $\epsilon > 0$  small enough, we then find that the previous series over  $(v_{p,q})_{p \in \mathcal{P}, q \in \mathcal{Q}}$  is a constant depending only on  $r, \epsilon$  and on  $\mathbf{N}$ . We then get the following inequality:

$$\left| I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right|_{r, \epsilon} \leq (QM)^{|\mathbf{k}|}.$$

Thus we get i) and iii) by the Lebesgue's theorem, and by analytic continuation. ii) is true by the above upper bound. As  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  is holomorphic, and because for all  $s = \sigma + i\tau \in \mathbb{C}$  such that  $\sigma \gg 1$ , we have

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s),$$

by 2.20, we obtain that the previous equality stands on all  $\mathbb{C}$  by analytic continuation, thus proving iv).  $\square$

By a similar reasoning, we get the following lemma:

**Lemma 2.3.23.** *Let  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ , and  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ ,  $j \in [1, Q]$ ,  $\mathcal{Q} \subset [1, Q] \setminus \{j\}$ . For all  $s \in \mathbb{C}$  and  $0 < \epsilon \ll 1$ , we have:*

$$\begin{aligned} I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\epsilon, s) = & \sum_{\mathbf{v} = (v_p)_{p \in [1, P]} \in (\mathbb{N}_0^Q)^P} \left( \prod_{q \in [1, Q]} c_{q, p}^{w_{p, q}} \right) \left( \prod_{p=1}^P \binom{-N_p + \mu_p s - 1}{|v_p|} \binom{|v_p|}{v_p} \prod_{q \in \mathcal{Q}} c_{q, p}^{v_{p, q}} \right) \\ & \cdot \frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q))}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q))} \\ & \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1} \cdot \prod_{p=1}^P \left( c_{j, p} + \sum_{f \in \mathcal{Q}^c} c_{f, p} x_f \right)^{-N_p + \mu_p s - 1 - |v_p|} \prod_{q \in \mathcal{Q}^c} dx_q, \end{aligned}$$

with the notations  $\mathcal{Q}^c = ([1, Q] \setminus \{j\}) \setminus \mathcal{Q}$ ,  $\mathbf{v}(q) = \sum_{p=1}^P v_{p, q}$  for all  $q \in \mathcal{Q}$ .

Moreover,  $s \in \mathbb{C} \mapsto I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\epsilon, s)$  is holomorphic in  $\mathbb{C}$ .

**Remark 2.3.24.** *As opposed to Lemma 2.3.22, we don't need a bound in Lemma 2.3.23, because there is no  $\mathbf{k}$  inside the expression of  $I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\epsilon, s)$ .*

We know have all the ingredients required to prove Proposition 2.2.2:

*Proof.* Let's recall that  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  and  $h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s)$  have the following expression for all  $s \in \mathbb{C}$  such that  $\sigma \gg 1$ :

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) &= \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s), \\ h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) &= \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\epsilon, s). \end{aligned}$$

Since the functions  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$  and  $I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\epsilon, s)$  are holomorphic in  $s$  on  $\mathbb{C}$ , we have an analytic continuation of  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  and of  $h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s)$  on all  $s \in \mathbb{C}$ . We then obtain point i) of Proposition 2.2.2. Point ii) and iii) of Proposition 2.2.2 follows directly from point ii) and iii) of Lemma 2.3.22.  $\square$

### 2.3.4 Proof of Proposition 2.2.4

Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ . We exploit the expression  $Z(s, s') = K(\theta, s, s') + J(\theta, s, s')$  to obtain the proof of the meromorphic extension theorem in the direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}')$ . We already know that  $(s, s') \mapsto K(\theta, s, s')$  is holomorphic on  $\mathbf{C}^P \times \mathbf{C}^Q$ . What remains is to prove that the function  $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  is meromorphic and regular at  $s = 0$ . In order to show Proposition 2.2.4, all that remains is to establish that  $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  is meromorphic on  $\mathbb{C}$  with poles contained in

$$\mathcal{S}(\mathbf{N}, \mathbf{N}') = \left[ \bigcup_{\substack{1 \leq p \leq P \\ \mu_p \neq 0}} \frac{1}{\mu_p} \mathbb{Z}_{\geq N_p + 1} \right] \cup \left[ \bigcup_{\mathcal{P} \subset [1, P]} \frac{1}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \mathbb{Z}_{\leq |\mathbf{N}'|_{\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}|} \right].$$

We know that formula 2.5 is valid on the complement of this set, for  $\sigma \gg 1$ . For all  $s = \sigma + i\tau \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$  such that  $\sigma \gg 1$ , we have that

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \sum_{\mathcal{P} \subsetneq [1, P]} \left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \right. \\ & \left. \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right). \end{aligned}$$

Via all the above, we can now prove Proposition 2.2.4:

*Proof.* Let  $0 < \theta \ll 1$ . We know from Proposition 2.3.21 that the formula 2.5 is true for all  $s = \sigma + i\tau \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ , such that  $\sigma \gg 1$ .

It is clear that

$$\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)},$$

is meromorphic in  $s$  on all  $\mathbb{C}$ , and regular at  $s = 0$ , with poles of the form  $s = \frac{|\mathbf{N}'| + |\mathbf{N}| + P}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|}$ . Then, it remains to show that the series of functions in  $s$

$$\begin{aligned} & \sum_{\mathcal{P} \subsetneq [1, P]} \left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \right. \\ & \left. \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \end{aligned} \quad (2.28)$$

is meromorphic on  $\mathbb{C}$ , regular at  $s = 0$ , and with poles inside  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ .

We define

$$\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r) := \left\{ s \in \overline{D}_0(r) \mid \left( \forall p \in \llbracket 1, P \rrbracket, \min_{\substack{n \in \mathbb{N}_0 \\ n > N_p}} |\mu_p s - n| \geq \delta \right) \wedge \left( \forall \mathcal{P} \subset \llbracket 1, P \rrbracket, \min_{\substack{n \in \mathbb{Z}^* \\ n \geq -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| - |\mathcal{P}|}} (|\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)s + n \geq \delta \right) \right\}.$$

Let's note that

1) We clearly have that  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$  contains an open neighbourhood of 0.

2) The condition

$$\min_{\substack{n \in \mathbb{N}_0 \\ n > N_p}} |\mu_p s - n| \geq \delta$$

is here to avoid the singularities of  $\zeta(-N_p + \mu_p s - k, d_p)$ . We see that, around  $s = 0$ , this last term is regular.

3) The condition

$$\min_{\substack{n \in \mathbb{Z}^* \\ n \geq -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| - |\mathcal{P}|}} (|\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)s + n \geq \delta$$

is here to avoid the singularities of  $\frac{1}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}$  which are not necessarily compensated by the zeros of  $\frac{1}{\Gamma(-N'_j + \mu'_j s)}$ . We shall note that around  $s = 0$ , when  $-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}| = 0$ , the singularity of  $\frac{1}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s}$  is compensated by the zero of  $\frac{1}{\Gamma(-N'_j + \mu'_j s)}$ .

4) The set  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$  is inside the complement of  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$  in  $\mathbb{C}$ .

Let's fix a subset  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ . We see that by Proposition 2.3.21, the formula 2.5 is true for all  $s = \sigma + i\tau \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ , such that  $\sigma \gg 1$ . We wish to prove that the series in  $\mathbf{k}$  in formula (2.28) converges normally on  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ .

Let  $r' > 1$  be a real such that, for all  $p \in \llbracket 1, P \rrbracket$ , we have  $\operatorname{Re}(d_p) \geq 1/r'$ , and  $|d_p| < r'$ . Let  $r > 0$ ,  $1 > \delta > 0$ . By Lemma 2.3.3, we have

$$\forall p \in \llbracket 1, P \rrbracket, \forall s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \forall k_p \in \mathbb{N}_0, \quad \zeta(-N_p + \mu_p s - k_p, d_p) \underset{r, r', \delta}{\ll} (k_p + N_p)! (k_p + N_p + 1)^{r+1} (1 + r'^{2k_p}).$$

By Lemma 2.2.2, we have

$$\forall s \in D_0(r), \forall \mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}, \quad |h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)| \underset{r}{\ll} (QM)^{|\mathbf{k}|}.$$

Since the function  $s \mapsto \Gamma(1 + N_p - \mu_p s)$  is holomorphic on  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$  for all  $1 \leq p \leq P$ , we then find the following bound

$$\forall p \in \llbracket 1, P \rrbracket, \forall s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \quad |\Gamma(1 + N_p - \mu_p s)| \underset{r, \delta}{\ll} 1.$$

By Lemma 2.3.4 we find the following bound

$$\forall s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \quad \left| \frac{1}{\Gamma(-N'_j + \mu'_j s) (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right| \underset{r, \delta}{\ll} 1.$$

By compactness of  $\bar{D}_0(r)$  we get

$$\forall s \in \bar{D}_0(r), |\theta^{(|\boldsymbol{\mu}'|+|\boldsymbol{\mu}|_{|\mathcal{P}})s-|\mathbf{N}'|-|\mathbf{N}|_{|\mathcal{P}}-|\mathcal{P}|}| \ll_r 1.$$

Therefore, for all  $s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ , and for  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$ , we find

$$\begin{aligned} & \left| (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right. \\ & \left. \frac{\theta^{(|\boldsymbol{\mu}'|+|\boldsymbol{\mu}|_{|\mathcal{P}})s-|\mathbf{N}'|-|\mathbf{N}|_{|\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'|+|\boldsymbol{\mu}|_{|\mathcal{P}})s-|\mathbf{N}'|-|\mathbf{N}|_{|\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \\ & \ll_{r, r', \delta} (QM\theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1} (1 + r'^{2k_p}) \\ & \ll_{r, r', \delta} (2r'^2 QM\theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1}. \end{aligned}$$

Since the series

$$\sum_{\mathbf{N} \in \mathbb{N}_0^{\mathcal{P}^c}} (2r'^2 QM\theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1}$$

converges absolutely for all  $\theta \in \mathbb{C}$  such that  $|\theta| < \frac{1}{(2r'^2 QM)}$ , we find that formula 2.5 holds for all

$s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ . By holomorphy theorem under the sum sign, we get that the function  $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  is holomorphic on  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ . As  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$  contains an open neighbourhood of 0, we see that by the same holomorphy theorem under the sum sign, we can derive termwise in respect to  $s$  around  $s = 0$  the function  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ ,

$$\partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \tag{2.29}$$

$$\begin{aligned} & \sum_{\mathcal{P} \subseteq [1, P]} \sum_{j=1}^Q \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} \partial_s \left( \left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \right. \\ & \left. \frac{\theta^{(|\boldsymbol{\mu}'|+|\boldsymbol{\mu}|_{|\mathcal{P}})s-|\mathbf{N}'|-|\mathbf{N}|_{|\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'|+|\boldsymbol{\mu}|_{|\mathcal{P}})s-|\mathbf{N}'|-|\mathbf{N}|_{|\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right), \\ & + \partial_s \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta^{(|\boldsymbol{\mu}'|+|\boldsymbol{\mu}|)s-|\mathbf{N}'|-|\mathbf{N}|-P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'|+|\boldsymbol{\mu}|)s-|\mathbf{N}'|-|\mathbf{N}|-P)} \right). \end{aligned}$$

As  $1 > \delta > 0$  is arbitrary in the definition of  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ , and that  $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$  contains an open neighbourhood of  $s = 0$ , we then get that  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  is regular at  $s = 0$ , and that the poles are inside  $\mathcal{S}(\mathbf{N}, \mathbf{N}')$  described in Proposition 2.2.4.  $\square$

**Remark 2.3.25.** Via formula 2.5, one can find expressions of the residues of  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  at these poles. In fact, we know that this function can be decomposed into two sums, one one which is holomorphic on  $\mathbb{C}$ , and the other one is meromorphic:

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = K_{\mathbf{N}, \mathbf{N}'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}(\theta, s).$$

Since  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  is holomorphic in  $s$ , this function doesn't have any singularities. On the other hand, in formula 2.5, one can see that compute the residues by studying carefully the two following terms for  $j \in \llbracket 1, Q \rrbracket$ ,

$$\begin{aligned} & \bullet \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \quad (\mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}) \\ & \bullet \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)}. \end{aligned}$$

Let's note that these poles can sometimes be compensated by zeros of the functions  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  for the first term, and of zeros of the functions  $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$  for the second term.





## Chapter 3

# Computation of the coefficients

$$Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$$

$$\text{and } Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$$

By Proposition 2.2.2, we know that the function  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  is holomorphic on  $\mathbb{C}$ . Therefore, we can consider his Taylor's coefficients of order 0 and 1 near  $s = 0$ :

**Notation.** Let  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^P \times \mathbb{R}_*^Q$ , and  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ . Let  $\mathcal{P} \subset \llbracket 1, P \rrbracket$  and  $\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}$ . As  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  is holomorphic on  $\mathbb{C}$ , we consider the following coefficients:

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})_{\boldsymbol{\mu}, \boldsymbol{\mu}'} + Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})_{\boldsymbol{\mu}, \boldsymbol{\mu}'} s + O(s^2).$$

The coefficients  $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ , for all  $j \in \llbracket 1, Q \rrbracket$ ,  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ , will be crucial in the expression of the formulas for  $Z(-(\mathbf{N}, \mathbf{N}'))$ . Furthermore, the coefficients  $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  and  $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ , for all  $j \in \llbracket 1, Q \rrbracket$ ,  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ , will appear in the formula for the directional derivative at nonpositive integers  $Z'(-(\mathbf{N}, \mathbf{N}'))$ . It is also worth noting that we do not need to compute  $Q_{\mathbf{N}, \mathbf{N}'}^0(\llbracket 1, P \rrbracket, j)_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  and  $Q_{\mathbf{N}, \mathbf{N}'}^1(\llbracket 1, P \rrbracket, j)_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ , as they do not appear in the expressions contained in Theorem A and in Theorem D. We will show in our calculations that  $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$  is independent of the directions  $(\boldsymbol{\mu}, \boldsymbol{\mu}')$ . We will therefore simply denote this coefficient as  $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ .

### 3.1 Lemmata

**Lemma 3.1.1.** Let  $a, c \in H_0$ ,  $b \in \mathbb{N}_0$  and  $d \in \mathbb{Z}$ . We define  $g : \mathbb{C} \rightarrow \mathbb{C}$  the following meromorphic function:

$$g(s) := \frac{1}{\Gamma(as - b)(cs + d)}.$$

Then  $g$  is regular near  $s = 0$ . Moreover, we have:

$$g(0) = \begin{cases} \frac{a}{c}(-1)^b b! & \text{if } d = 0 \\ 0 & \text{if } d \neq 0, \end{cases},$$

and

$$\partial_s (g(s))_{s=0} = \begin{cases} \frac{a^2}{c} (-1)^b b! (\gamma - h_b) & \text{if } d = 0 \\ \frac{a}{d} (-1)^b b! & \text{if } d \neq 0, \end{cases}$$

where  $h_b$  is the harmonic number  $h_b := \sum_{i=1}^b \frac{1}{i}$ .

We also have

$$\partial_s \left( \frac{1}{\Gamma(s)} \right)_{|s=-N} = (-1)^N N!.$$

*Proof.* Around  $s = 0$ , we have

$$\frac{1}{\Gamma(s - N)} = (-1)^N N! s + O(s^2),$$

which proves the formula for  $\partial_s \left( \frac{1}{\Gamma(s)} \right)_{|s=-N}$ .

In order to prove the other formulas, we shall study the behavior of  $g$  near  $s = 0$ . By Gamma's functional equation, we get

$$g(s) = \frac{(as - 1) \dots (as - b)}{\Gamma(as)(cs + d)}.$$

Around  $s = 0$ , we have

$$\begin{aligned} (as - 1) \dots (as - b) &= (-1)^b b! + a(-1)^{b-1} b! h_b s + O(s^2), \\ \frac{1}{\Gamma(s)} &= s + \gamma s^2 + O(s^3). \end{aligned}$$

Let's assume that  $d = 0$ . Therefore, near  $s = 0$  we have:

$$\begin{aligned} g(s) &= \frac{a}{c} \frac{(as - 1) \dots (as - b)}{\Gamma(as)as} \\ &= \frac{a}{c} ((-1)^b b! + as(-1)^{b-1} b! h_b)(1 + \gamma as) + O(s^2) \\ &= \frac{a}{c} (-1)^b b! + \frac{a^2}{c} (-1)^{b-1} b! h_b s + \frac{a^2}{c} \gamma (-1)^b b! s + O(s^2). \end{aligned}$$

Let's assume that  $d \neq 0$ . Therefore, near  $s = 0$  we have:

$$\begin{aligned} g(s) &= \frac{(as - 1) \dots (as - b)}{\Gamma(as)(cs + d)} \\ &= \frac{(as - 1) \dots (as - b)}{cs + d} (as) + O(s^2) \\ &= \frac{a}{d} (-1)^b b! s + O(s^2). \end{aligned}$$

□

**Lemma 3.1.2.** Let  $\mathcal{P} \subset \llbracket 1, P \rrbracket$ ,  $(\mu_p)_{p \in \mathcal{P}} \in \mathbb{R}^{\mathcal{P}}$ ,  $(N_p)_{p \in \mathcal{P}} \in \mathbb{N}_0^{\mathcal{P}}$ ,  $(n_p)_{p \in \mathcal{P}} \in \mathbb{N}_0^{\mathcal{P}}$ , we have

$$\partial_s \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} \right)_{|s=0} = \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p} \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + n_p}) \right),$$

where  $h_n$  is the  $n$ -th harmonic number.

*Proof.* Let's recall that

$$\prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} = \prod_{p \in \mathcal{P}} \frac{(-N_p - 1 + \mu_p s) \dots (-N_p + \mu_p s - n_p)}{n_p!}.$$

By developing the product  $\frac{(-N_p - 1 + \mu_p s) \dots (-N_p - n_p + \mu_p s)}{n_p!}$  near  $s = 0$ , we get

$$\frac{(-N_p - 1 + \mu_p s) \dots (-N_p + \mu_p s - n_p)}{n_p!} = \binom{-N_p - 1}{n_p} + \mu_p \left( \sum_{i=0}^{n_p-1} \frac{(-N_p - 1) \dots \widehat{(-N_p - 1 - i)} \dots (-N_p - n_p)}{n_p!} \right) s + O(s^2),$$

where  $\widehat{(-N_p - 1 - i)}$  means that we forget that term in the product. We can simplify that last expression:

$$\begin{aligned} \frac{(-N_p - 1 + \mu_p s) \dots (-N_p + \mu_p s - n_p)}{n_p!} &= \binom{-N_p - 1}{n_p} + \mu_p \binom{-N_p - 1}{n_p} \left( \sum_{i=0}^{n_p-1} \frac{1}{-N_p - 1 - i} \right) s \\ &= \binom{-N_p - 1}{n_p} + \mu_p \binom{-N_p - 1}{n_p} (h_{N_p} - h_{N_p+n_p}) s + O(s^2). \end{aligned}$$

Near  $s = 0$ , we then get that:

$$\prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} = \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p} + \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p} \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p+n_p}) \right) s + O(s^2).$$

□

**Lemma 3.1.3.** *Let  $M \in \mathbb{Z}_{\leq 0}$  be a nonpositive integer, and  $(a_k)_{k \in \llbracket M, +\infty[}$  and  $(b_k)_{k \in \llbracket M, +\infty[}$  be two complex number sequences such that the power series  $\sum_{k=0}^{+\infty} a_k x^k$  and the power series  $\sum_{k=0}^{+\infty} b_k x^k$  have two positive radius of convergence  $R_1$  and  $R_2$ . Let's also assume that there exist a positive real number  $0 < R < \min(R_1, R_2)$  such that*

$$\forall x \in ]0, R[, \quad \sum_{k=M}^{+\infty} a_k x^k + \sum_{k=M}^{+\infty} b_k x^k \ln(x) = 0,$$

*Then for all  $k \in \llbracket 0, +\infty[$ , we have  $a_k = 0$  and  $b_k = 0$ .*

*Proof.* If all  $b_k$  are zeros (respectively all  $a_k$  are zeros) for all  $k \geq M$ , the result is trivial. Otherwise, we shall use a reductio ad absurdum argument. Let's assume that there exist  $i, j \geq M$  such that  $a_i \neq 0$  and  $b_j \neq 0$ . Suppose that  $i = \min\{k | a_k \neq 0\}$  and  $j = \min\{k | b_k \neq 0\}$ , then in the neighborhood of 0 we have

$$a_i x^i + b_j x^j \ln(x) + o(x^j \ln(x)) + o(x^i) = 0,$$

which implies  $b_j x^{j-i} \ln(x) \sim -a_i$  near  $x = 0$ , a contradiction. □

The preceding lemma, although trivial, will play a crucial role in determining formulas for special values in the proof of Theorem A and Theorem D. Indeed, in the decomposition of  $Z_{(\mathbf{N}, \mathbf{N}')} (s)$  obtained in Proposition 2.2.4, we have artificially introduced a free variable  $\theta$ . It is clear that  $K_{(\mathbf{N}, \mathbf{N}')}^{\mu, \mu'}(\theta, s)$  vanishes at  $s = 0$ , the only challenge then is to evaluate  $J_{(\mathbf{N}, \mathbf{N}')}^{\mu, \mu'}(\theta, 0)$  using formula 2.5 at  $s = 0$ , which will yield an expression involving a Laurent series in  $\theta$ . Since the quantity  $Z(-(\mathbf{N}, \mathbf{N}')) = Z_{(\mathbf{N}, \mathbf{N}')}^{\mu, \mu'}(0)$  does not depend on  $\theta$ , we will only retain the constant terms of the Laurent series in  $\theta$  from the obtained formula. For the

calculation of  $Z'(-(\mathbf{N}, \mathbf{N}')) = Z'_{\substack{(\mathbf{N}, \mathbf{N}') \\ \mu, \mu'}}(0)$ , we will also use this lemma to make the same simplification, except that this time, by differentiating formula 2.5 with respect to  $s$ , a term involving  $\ln(\theta)$  will appear. In the same way, we will use this lemma to compute the coefficients  $Q^0$  and  $Q^1$ , in order to eliminate the series in  $\epsilon$  and the multiples of  $\ln(\epsilon)$ .

**Lemma 3.1.4.** *There exist a function defined for all  $0 < \epsilon \ll 1$  such that*

$$\alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) = a \ln(\epsilon) + \sum_{n \geq m} b_n \epsilon^n,$$

with  $m \in \mathbb{Z}$ , without constant terms (i.e.  $b_0 = 0$ ), and such that

$$\int_{\epsilon}^1 x_f^{-N'_f-1+\mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) + F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}'),$$

with  $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$  being the constant defined in Proposition 1.3.2:

$$\begin{aligned} F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &= -E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_f+1-\mathbf{w}(f)} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} \\ &\quad - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{(|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|)} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda-1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}) \\ &\quad + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \right). \end{aligned}$$

*Proof.* In order to compute the integral above, we shall use Proposition 1.3.2. With the same notations as in that proposition, we get

$$\begin{aligned} x_f^{-N'_f-1+\mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{(-N_p-1-|\mathbf{v}_p|)} &= \tilde{P}_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f) + \sum_{\lambda=1}^{N'_f+1-\mathbf{w}(f)} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{x_f^{\lambda}} \\ &\quad + \sum_{p \in \mathcal{P}} \sum_{\lambda=1}^{(|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|)} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{(c_{j,p} + c_{f,p} x_f)^{\lambda}}. \end{aligned}$$

We noted  $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$  the primitive of the polynomial  $\tilde{P}_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$  which fades at  $x_f = 1$ , and we noted  $E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$  the constant term of  $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$ . We also noted  $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(x_f) := P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f) - E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ . Therefore we find that

$$\begin{aligned} \int_{\epsilon}^1 x_f^{-N'_f-1+\mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_f &= \\ &= -E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) - C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},1}(\mathbf{N}, \mathbf{N}') \ln(\epsilon) - \sum_{\lambda=2}^{N'_f+1-\mathbf{w}(f)} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} (1 - \epsilon^{1-\lambda}) \\ &\quad + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \left( \ln(c_{j,p} + c_{f,p}) - \ln(c_{j,p}) - \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \epsilon \right) \right) \\ &\quad - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{(|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|)} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda-1} ((c_{j,p} + c_{f,p})^{1-\lambda} - (c_{j,p} + c_{f,p} \epsilon)^{1-\lambda}). \end{aligned}$$

By expanding the terms  $(c_{j,p} + c_{f,p} \epsilon)^{1-\lambda}$  and  $\ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \epsilon \right)$  near  $\epsilon = 0$ , we can decompose the previously obtained formula into a constant term, and into a sum between a term in  $\ln(\epsilon)$  and a Laurent series

without a constant term in  $\epsilon$ , for  $0 < \epsilon \ll 1$ . Let's denote the non-constant term  $\alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon)$ . We then have

$$\begin{aligned} & \int_{\epsilon}^1 x_f^{-N'_j - 1 + \mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) \\ & - E_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_j + 1 - \mathbf{w}(f)} \frac{C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (1,p)}(\mathbf{N}, \mathbf{N}') \ln \left( 1 + \frac{c_{f,p}}{c_{j,p}} \right) \\ & - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{(|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|)} \frac{D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}). \end{aligned}$$

□

We note that a simpler, although less explicit, expression of the previous integral can be obtained, using a symbolic computation software:

Consider the primitive  $G_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}(x_f)$  of  $x_f^{-N'_j - 1 + \mathbf{w}(f)} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|}$  that vanishes at  $x_f = 1$ . This primitive is expressed as a sum of a multiple of  $\ln(x_f)$  and a Laurent series in  $x_f$ , for  $0 < x_f \ll 1$ . We then observe that the constant term of this Laurent series is  $-F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ .

**Remark 3.1.5.** *It seems more practical to directly compute the coefficient  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  using symbolic computation software by determining a primitive of  $x_f^{-N'_j - 1 + \mathbf{w}(f)} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|}$ , rather than computing the coefficients  $C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')$ ,  $D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda,p)}(\mathbf{N}, \mathbf{N}')$ , and  $E_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  using the partial fraction decomposition from Proposition 1.3.2.*

## 3.2 Value and derivative of $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ at $s = 0$

Let  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ ,  $j \in \llbracket 1, Q \rrbracket$ , and  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$ . In this section, we're now aiming to compute the coefficients  $Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^0$  and  $Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^1$ , which we recall the definition:

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) &= Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^0, \\ \partial_s h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) &= Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^1. \end{aligned}$$

Let's recall that we prove in Proposition 2.2.2 that the function

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) &= \\ & \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx_j} \dots dx_Q \end{aligned}$$

is holomorphic.

We also recall that we obtained from that proposition an expression of the derivative  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$  at  $s = 0$ .

### 3.2.1 Preliminaries

Let  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ ,  $j \in \llbracket 1, Q \rrbracket$ , and  $0 < \epsilon \ll 1$ . We know that for all  $s \in \mathbb{C}$ , we have

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s),$$

we then deduce the fact that it is enough to study the values and the derivative values of  $I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, s)$  at  $s = 0$  for  $0 < \epsilon \ll 1$  in order to get the values and the derivative values of  $h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(\epsilon, s)$  at  $s = 0$ . We now recall point iii) obtained in Proposition 2.2.2:

Let  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$ . For all  $s \in \mathbb{C}$ , and for  $\epsilon > 0$  small enough, we have:

$$h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(\epsilon, s) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in [1, Q]} c_{q,p}^{w_{p,q}} \right) \cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q.$$

Let's consider  $T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s)$  the general term of the above series:

$$T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s) = \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q. \quad (3.1)$$

Let's recall that, with that definition in mind, we have by point iii) of Lemma 2.3.22:

$$I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, 0) = \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \sum_{\mu, \mu'} T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, 0). \quad (3.2)$$

and by that same point iii), we also get an expression of the derivative at  $s = 0$ :

$$\left( \partial_s I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, s) \right)_{|s=0} = \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left( \partial_s T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s) \right)_{|s=0}. \quad (3.3)$$

Thanks to the formula (2.24), we get

$$h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(0) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, 0), \quad (3.4)$$

$$\left( \partial_s h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s) \right)_{|s=0} = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \left( I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, s) \right)_{|s=0}. \quad (3.5)$$

Therefore, we can compute the coefficients  $Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$  and  $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$  by computing the value and the derivative value of  $T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s)$  at  $s = 0$ .

**Lemma 3.2.1.**

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) = \begin{cases} O(1) & \text{if } |\mathcal{Q}| = Q - 1, \text{ and } \forall q \in \mathcal{Q}, & \mathbf{v}(q) + \mathbf{w}(q) = N'_q \\ O(s) & \text{if } |\mathcal{Q}| = Q - 1 \text{ and } \exists f \in \mathcal{Q} \setminus \{j\}, \forall q \in \mathcal{Q} \setminus \{f\}, & \mathbf{v}(q) + \mathbf{w}(q) = N'_q \\ & & \mathbf{v}(f) + \mathbf{w}(f) \neq N'_f \\ O(s) & \text{if } |\mathcal{Q}| = Q - 2 \text{ and } \forall q \in \mathcal{Q}, & \mathbf{v}(q) + \mathbf{w}(q) = N'_q \\ O(s^2) & \text{else.} \end{cases}$$

*Proof.* We know that the meromorphic function

$$s \mapsto \prod_{q \in \mathcal{Q}} \frac{1}{-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q)}$$

has a pole of order

$$|\{q \in \mathcal{Q} | N'_q = \mathbf{v}(q) + \mathbf{w}(q)\}|$$

at  $s = 0$ . In particular, the order is necessarily smaller than  $|\mathcal{Q}|$ . By the property of the  $\Gamma$  function, we know that the function  $s \mapsto \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + s\mu'_q)}$  has a zero of order  $Q - 1$  at  $s = 0$ . We then obtain that

$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  has a zero of order

$$n = Q - 1 - |\{q \in \mathcal{Q} | N'_q = \mathbf{v}(q) + \mathbf{w}(q)\}|.$$

□

### 3.2.2 Computation of $Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^0$

**Proposition 3.2.2.** Let  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ .

- If  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$  and  $\forall q \in \mathcal{Q}, \mathbf{v}(q) + \mathbf{w}(q) = N'_q$ , then

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0) = \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q, p}^{w_{p, q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q, p}^{v_{p, q}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!.$$

- Otherwise, we have

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0) = 0.$$

*Proof.* In the first case, by Lemma 3.1.1, we have that:

$$\left( \frac{1}{\Gamma(-N'_q + \mu'_q s)} \frac{1}{\mu'_q s} \right)_{s=0} = (-1)^{N'_q} N'_q! \frac{\mu'_q}{\mu'_q}.$$

By Taking  $s = 0$  in the expression of  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$ , we obtain the first formula from the proposition. The second formula follows from Lemma 3.2.1. □

By formula 3.2, and by the previous proposition, we get:

**Corollary 3.2.3.** Let  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ , and  $0 < \epsilon \ll 1$ .

- If  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ , then we have

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

- Otherwise we have

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0) = 0.$$

By formula 2.24, we recall that we have

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0),$$

by the previous corollary, we then obtain

$$Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathcal{Q}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right). \quad (3.6)$$

### 3.2.3 Computation of $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$

Taking  $s = 0$  in the derivative with respect to  $s$  of the general term

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) = \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \cdot \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \frac{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \mathbf{w}(q)} \cdot \prod_{p \in \mathcal{P}} \left( c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q,$$

we get:

**Proposition 3.2.4.** *Let  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ ,*



- If  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$  and  $\forall q \in \mathcal{Q}$ ,  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$ , we have

$$\begin{aligned}
 \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= & (3.7) \\
 & (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \\
 & \cdot \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
 & + (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} |\boldsymbol{\mu}'|_{\mathcal{Q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \ln(\epsilon).
 \end{aligned}$$

- If  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$  and if there exist an integer  $f \in \mathcal{Q} \setminus \{j\}$  such that  $\mathbf{v}(f) + \mathbf{w}(f) \neq N'_f$  and such that for all  $q \in \mathcal{Q} \setminus \{f\}$ ,  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$ , we then have

$$\begin{aligned}
 \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= & (3.8) \\
 & (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & \cdot \frac{\mu'_f}{-N'_f + \mathbf{v}(f) + \mathbf{w}(f)} \epsilon^{-N'_f + \mathbf{v}(f) + \mathbf{w}(f)}.
 \end{aligned}$$

- If  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j, f\}$  such that  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$  for all  $q \in \mathcal{Q}$ , then there exist a function  $\beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon)$  which has a decomposition as a sum between a multiple of  $\ln(\epsilon)$ , and a Laurent series in  $\epsilon$ , for  $0 < \epsilon$  small enough, without any constant term and such that

$$\begin{aligned}
 \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= & (3.9) \\
 & \mu'_f (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \\
 & + \beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon),
 \end{aligned}$$

with  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  being the constant defined in Proposition 1.3.2.

- Otherwise we have

$$\partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} = 0.$$

*Proof.* • If  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ , we then find that

$$\begin{aligned}
 & \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0} = \\
 & \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \partial_s \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \Big|_{s=0} \\
 & \cdot \left( \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \right) \Big|_{s=0} \epsilon^{\sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + \mathbf{v}(q) + \mathbf{w}(q))} \\
 & + \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & \cdot \partial_s \left( \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \right) \Big|_{s=0} \epsilon^{\sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + \mathbf{v}(q) + \mathbf{w}(q))} \\
 & + |\mu'|_{|\mathcal{Q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & \cdot \left( \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \right) \Big|_{s=0} \ln(\epsilon) \epsilon^{\sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + \mathbf{v}(q) + \mathbf{w}(q))}.
 \end{aligned}$$

By Lemma 3.1.2, we get

$$\partial_s \left( \prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \right) \Big|_{s=0} = \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) \right) \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|},$$

which concludes the first formula.

• If  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$  and  $\forall q \in \mathcal{Q}$ ,  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$ , we then get by Lemma 3.1.1,

$$\begin{aligned}
 \partial_s \left( \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} &= \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f (-1)^{N'_f} N'_f! (\gamma - h_{N'_f}) \prod_{\substack{q=1 \\ q \neq j, f}}^Q (-1)^{N'_q} N'_q! \\
 &= (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left( \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
 \left( \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} &= (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q!.
 \end{aligned}$$

Therefore we find that

$$\begin{aligned}
 \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0} &= \\
 & \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & \cdot (-1)^{|\mathbf{N}'|_{\llbracket [1, Q] \setminus \{j\} \rrbracket}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \\
 & + \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & \cdot (-1)^{|\mathbf{N}'|_{\llbracket [1, Q] \setminus \{j\} \rrbracket}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left( \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
 & + |\boldsymbol{\mu}'|_{\mathcal{Q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & \cdot (-1)^{|\mathbf{N}'|_{\llbracket [1, Q] \setminus \{j\} \rrbracket}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \ln(\epsilon).
 \end{aligned}$$

Factoring the first and second term of this last sum, we get 3.7.

• If  $\mathcal{Q} = \llbracket [1, Q] \setminus \{j\} \rrbracket$  and if there exist an integer  $f \in \mathcal{Q} \setminus \{j\}$  such that  $\mathbf{v}(f) + \mathbf{w}(f) \neq N'_f$  and such that for all  $q \in \mathcal{Q} \setminus \{f\}$ , we have  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$ , we then find by Lemma 3.1.1,

$$\begin{aligned}
 & \left( \frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + \mathbf{v}(f) + \mathbf{w}(f)) \prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = 0, \\
 \partial_s \left( \frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + \mathbf{v}(f) + \mathbf{w}(f)) \prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} &= \\
 & (-1)^{|\mathbf{N}'|_{\llbracket [1, Q] \setminus \{j\} \rrbracket}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \frac{\mu'_f}{-N'_f + \mathbf{v}(f) + \mathbf{w}(f)}.
 \end{aligned}$$

Therefore, we get

$$\begin{aligned} \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0} &= \\ & \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\ & \cdot \partial_s \left( \frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + \mathbf{v}(f) + \mathbf{w}(f)) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + \mathbf{v}(q) + \mathbf{w}(q))} \right) \Big|_{s=0} \\ & \epsilon^{\sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + \mathbf{v}(q) + \mathbf{w}(q))}. \end{aligned}$$

Noticing that  $\sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + \mathbf{v}(q) + \mathbf{w}(q)) = -N'_f + \mathbf{v}(f) + \mathbf{w}(f) \neq 0$ , we then obtain formula 3.8.

- If  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j, f\}$  such that  $\mathbf{v}(q) + \mathbf{w}(q) = N'_q$  for all  $q \in \mathcal{Q}$ , we then find by Lemma 3.1.1 that

$$\begin{aligned} & \left( \frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = 0 \\ \partial_s & \left( \frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = \mu'_f \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!. \end{aligned}$$

Deriving the general term  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$  with respect to  $s$ , and by taking  $s = 0$  we then find that

$$\begin{aligned} \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0} &= \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \quad (3.10) \\ & \cdot \partial_s \left( \frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} \epsilon^{\sum_{\substack{q=1 \\ q \neq j, f}}^Q (-N'_q + \mathbf{v}(q) + \mathbf{w}(q))} \\ & \cdot \int_{\epsilon}^1 x_f^{-N'_f - 1 + \mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p - 1 - |\mathbf{v}_p|} dx_f. \end{aligned}$$

With the conditions on the index  $\mathbf{v}$  and on the index  $\mathbf{w}$ , we find that  $\sum_{\substack{q=1 \\ q \neq j, f}}^Q (-N'_q + \mathbf{v}(q) + \mathbf{w}(q)) = 0$ . By Lemma 3.1.4, we get

$$\int_{\epsilon}^1 x_f^{-N'_f - 1 + \mathbf{w}(f)} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p - 1 - |\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) + F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}'),$$

with  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  defined in Proposition 1.3.2, and  $\alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon)$  being a function in  $\epsilon$  of the form  $a \ln(\epsilon) + \sum_{n \geq m} b_n \epsilon^n$ ,  $m \in \mathbb{Z}$  and  $a, b_n$  being complex numbers with  $b_0 = 0$ .

Injecting those last expressions into formula 3.10, we then get

$$\begin{aligned} \partial_s \left( T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \\ & \mu'_f \prod_{\substack{q=1 \\ q \neq j}}^{N'_q} (-1)^{N'_q} N'_q! \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \\ & \cdot (\alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) + F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')). \end{aligned}$$

Let's define

$$\beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) = \mu'_f \prod_{\substack{q=1 \\ q \neq j}}^{N'_q} (-1)^{N'_q} N'_q! \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon),$$

we then get the formula 3.9.

- If we're not in one of these three last cases above, therefore by Lemma 3.2.1, we find that

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) = O(s^2),$$

which concludes this particular point.  $\square$

By formula 3.3 and by the previous proposition, we then find the following corollary:

**Corollary 3.2.5.** *Let  $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ .*

- *If  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ , then there exist a function  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon)$  which corresponds to a sum of a multiple of  $\ln(\epsilon)$  and a Laurent series defined for all  $0 < \epsilon \ll 1$ , without any constant term, and such that*

$$\begin{aligned} \partial_s \left( I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \llbracket \mathcal{Q} \setminus \{j\} \rrbracket, \mathbf{k}}(\epsilon, s) \right)_{|s=0} &= \tag{3.11} \\ & (-1)^{|\mathbf{N}'|_{\llbracket \llbracket 1, Q \rrbracket \setminus \{j\} \rrbracket}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\} \rrbracket})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ & \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\ & + I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon). \end{aligned}$$

- *If  $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j, f\}$ , then we find that there is a function  $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon)$  which corresponds to a sum of a multiple of  $\ln(\epsilon)$  and a Laurent series defined for all  $0 < \epsilon \ll 1$  without any constant term,*

and such that

$$\begin{aligned} \partial_s \left( I_{\mathbf{N}, \mathbf{N}'}^* \right)_{\mu, \mu'} \Big|_{s=0} &= \\ & I_{\mathbf{N}, \mathbf{N}'}^* \Big|_{\mu, \mu'} + (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \mu'_f \left( \prod_{\substack{q=1 \\ q \neq j}} N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \\ & \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q, p}^{v_{p, q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}'), \end{aligned} \quad (3.12)$$

with  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  defined in Proposition 1.3.2.

- Otherwise, we find that

$$\partial_s \left( T_{\mathbf{N}, \mathbf{N}'} \right)_{\mu, \mu'} \Big|_{s=0} = 0.$$

By formula 3.5, we then get that

$$\partial_s \left( h_{\mathbf{N}, \mathbf{N}'} \right)_{\mu, \mu'} \Big|_{s=0} = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \partial_s \left( I_{\mathbf{N}, \mathbf{N}'} \right)_{\mu, \mu'} \Big|_{s=0},$$

therefore by the previous corollary we obtains the following expression for the coefficient  $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ :

$$\begin{aligned} Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k}) &= \partial_s \left( I_{\mathbf{N}, \mathbf{N}'} \right)_{\mu, \mu'} \Big|_{s=0} + \sum_{\substack{f=1 \\ f \neq j}}^Q \partial_s \left( I_{\mathbf{N}, \mathbf{N}'} \right)_{\mu, \mu'} \Big|_{s=0} \\ &= (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \\ & \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, p}^{v_{p, q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\ & + (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \\ & \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q, p}^{v_{p, q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')) \\ & + I_{\mathbf{N}, \mathbf{N}'}^* \Big|_{\mu, \mu'} + \sum_{\substack{f=1 \\ f \neq j}}^Q I_{\mathbf{N}, \mathbf{N}'}^* \Big|_{\mu, \mu'} \end{aligned}$$

We now conclude by using Lemma 3.1.3 by noticing the fact that  $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$  is independent of  $\epsilon$ . We then find that the constant term of the right hand term is equal to the coefficient  $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ .

## Chapter 4

# Proof of Theorem A, and of his corollaries

The aim of this chapter is to study the directional values of  $Z(\mathbf{s}, \mathbf{s}')$  at nonpositive integers. By showing that the function  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  is regular at  $s = 0$ , we have established that the quantities  $Z(-(\mathbf{N}, \mathbf{N}'))$  and  $Z'(-(\mathbf{N}, \mathbf{N}'))$  are well-defined. To compute the first quantity, we will use Proposition 2.2.4 to evaluate  $Z_{\mathbf{N}, \mathbf{N}'}(s)$  at  $s = 0$ .

### 4.1 Statement of Theorem A

Let's recall the statement of the theorem:

**Theorem (A).** *Let  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ . Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ . We have*

$$Z(-(\mathbf{N}, \mathbf{N}')) = (-1)^{|\mathbf{N}'|} \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|} \left( \prod_{p \in \mathcal{P}} N_p! \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} \sum_{j=1}^Q \frac{(-1)^{N'_j} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$$

$$\prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!},$$

with

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, \mathcal{P}}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, \mathcal{P}}^{v_{p,q}} \right),$$

where we noted

$$\forall q \in \llbracket 1, Q \rrbracket, \quad \mathbf{w}(q) = \sum_{p \in \mathcal{P}^c} w_{p,q},$$

and

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad \mathbf{v}(q) = \sum_{p \in \mathcal{P}} v_{p,q}.$$

## 4.2 Proof of Theorem A

*Proof.* By Proposition 2.2.4, we get for all  $0 < \theta \ll 1$  that

$$Z_{\mathbf{N}, \mathbf{N}'}(0) = K_{\mathbf{N}, \mathbf{N}'}(\theta, 0) + J_{\mathbf{N}, \mathbf{N}'}(\theta, 0).$$

Since  $K_{\mathbf{N}, \mathbf{N}'}(\theta, 0) = 0$  by Proposition 2.3.11, we only have to study the value of  $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  at  $s = 0$ . By formula 2.5, we have

$$\begin{aligned} Z_{\mathbf{N}, \mathbf{N}'}(0) = & \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} N_p! \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \sum_{j=1}^P h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) \\ & \left( \frac{1}{\Gamma(-N'_j + \mu'_j s) - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\mu'| + |\mu|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}} \right) \prod_{s=0} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p=1}^P \Gamma(1 + N_p) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(0) \left( \frac{\theta(|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \Big|_{s=0} \right). \end{aligned}$$

Let's first observe that the expression

$$\left( \prod_{p=1}^P \Gamma(1 + N_p) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(0) \left( \frac{\theta(|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \Big|_{s=0} \right),$$

vanish because  $-|\mathbf{N}'| - |\mathbf{N}| - P \neq 0$ . Therefore, the zero of  $\frac{1}{\Gamma(-N'_j + \mu'_j s)}$  at  $s = 0$  is not compensated by any poles coming from the fraction  $\frac{1}{(|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P}$  at  $s = 0$ .

By studying the first term of the formula of  $Z_{\mathbf{N}, \mathbf{N}'}(0)$ , we see that  $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) = Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ . Let  $0 < \theta \ll 1$  and  $\mathcal{P} \subsetneq [1, P]$ .

- If  $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} \neq -|\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}$ , then by Lemma 3.1.1 we find that

$$\frac{1}{\Gamma(-N'_j + \mu'_j s) - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\mu'| + |\mu|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}}$$

vanish at  $s = 0$ .

- If  $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} = -|\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}$ , then by Lemma 3.1.1 we find that:

$$\left( \frac{1}{\Gamma(-N'_j + \mu'_j s) - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\mu'| + |\mu|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}} \right) \Big|_{s=0} = \frac{\mu'_j}{|\mu'| + |\mu|_{|\mathcal{P}}} (-1)^{N'_j} N'_j!,$$

therefore we prove the Theorem A. The expression of  $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$  in Theorem A is easily deduced by the formula (3.6).  $\square$

## 4.3 On corollaries of Theorem A

We easily derive from Theorem A his corollaries.



### 4.3.1 Proof of Corollary A1

To prove Corollary A1, we apply Theorem A by setting  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ :

$$Z(-(\mathbf{0}, \mathbf{0})) = \sum_{\substack{\boldsymbol{\mu}, \boldsymbol{\mu}' \\ \mathcal{P} \subseteq \llbracket 1, P \rrbracket}} (-1)^{|\mathcal{P}|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \sum_{j=1}^Q \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!},$$

with

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = 0}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

We can simplify the coefficient  $Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k})$  by noticing that the condition  $\mathbf{v}(q) + \mathbf{w}(q) = 0$  for all  $q \neq j$  implies that

$$\begin{aligned} \forall p \in \mathcal{P}, \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad v_{p,q} &= 0 \\ \forall p \in \mathcal{P}^c, \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad w_{p,q} &= 0. \end{aligned}$$

We then obtain that  $k_p = |\mathbf{w}_p| = w_{p,j}$  for all  $p \in \mathcal{P}^c$ . Therefore we have the following expression for  $Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k})$ :

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}.$$

### 4.3.2 Proof of Corollary A2

On Corollary A2, we set  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$  and  $(\mathbf{N}, \mathbf{N}') = (N, \dots, N)$ , and we find by Theorem A that

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} (-1)^{N(|\mathcal{P}|+Q+1)+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=N(Q+|\mathcal{P}|)+|\mathcal{P}|}} \sum_{j=1}^Q \frac{1}{Q+|\mathcal{P}|} Q_N^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N-k_p, d_p)}{k_p!},$$

with

$$Q_N^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{N(Q-1)} N^{Q-1} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right),$$

where we noted

$$Q_N^0(\mathcal{P}, \mathbf{k}) = \sum_{j=1}^Q Q_N^0(j, \mathcal{P}, \mathbf{k}),$$

therefore proving the required result.

### 4.3.3 Proof of Corollary A3

The Corollary A3 corresponds to a qualitative and non explicit version of Theorem A. Let  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_*^{+Q}$ . Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ . We define  $\mathbb{K}$  the field above  $\mathbb{Q}$  generated by the complex numbers  $(d_p)_{p \in \llbracket 1, P \rrbracket}$  and  $(c_{q,p})_{p \in \llbracket 1, P \rrbracket, q \in \llbracket 1, Q \rrbracket}$ , and by the directions  $\mu_p$  for all  $p \in \llbracket 1, P \rrbracket$ , and  $\mu'_q$  for all  $q \in \llbracket 1, Q \rrbracket$ .

By Theorem A, we find that

$$Z(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = (-1)^{|\mathbf{N}'|} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|} \left( \prod_{p \in \mathcal{P}} N_p! \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} \sum_{j=1}^Q \frac{(-1)^{N'_j} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$$

$$\prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!},$$

with

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

Let's now observe that the coefficients  $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$  belong to the field  $\mathbb{K}$ . Moreover, by property of the special values of the Hurwitz zeta function, we know that the nonpositive values of this zeta function have an expression using only Bernoulli polynomials, which are all rational polynomials. Therefore we get that  $\frac{\zeta(-N_p - k_p, d_p)}{k_p!} \in \mathbb{Q}[d_1, \dots, d_P] \subset \mathbb{K}$ , which proves the corollary.

## Chapter 5

# Proof of Proposition B, Theorem D, and some corollaries

### 5.1 Statement of Proposition B and of Theorem D

Firstly we will prove in this chapter Proposition B, and secondly we shall prove Theorem D, which we recall the statements below:

**Proposition (B).** *Let  $j \in \llbracket 1, Q \rrbracket$  an positive integer. We assume that  $l_j$  is a linear form with rational coefficients for all  $1 \leq j \leq Q$ . For all  $1 \leq p \leq P$ , we then write  $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$  the irreducible decomposition of  $c_{j,p}$  with  $a_{j,p} > 0$  and  $b_{j,p} > 0$ . Let  $x_j(\mathbf{c}) = \frac{\text{lcm}(a_{j,p})}{\text{gcd}(b_{j,p})} \in \mathbb{Q}_+^*$ , and  $\beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}$  for all  $1 \leq p \leq P$ . Then, for all  $\mathbf{R} \in \mathbb{N}_0^P$ , we have for all  $s \in \mathbb{C} \setminus \mathbb{N}$ ,*

$$\begin{aligned} \varphi_{\mathbf{R}}^j(s) = & x_j(\mathbf{c})^{-s} \prod_{p=1}^P \beta_{j,p}^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq [1,P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, k' \in \mathbb{N}_0 \\ |\mathbf{k}| + k' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!} \\ & \cdot \frac{\zeta(s - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!}. \end{aligned}$$

**Theorem (D).** Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ , then

$$\begin{aligned}
Z'(-(\mathbf{N}, \mathbf{N}')) &= \tag{5.1} \\
&\sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q (-1)^{|\mathbf{N}|+|\mathbf{N}'|+|\mathcal{P}|+N'_j} \left( \prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\
&\cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \sum_{p \in \mathcal{P}} \mu_p(\gamma - h_{N_p}) + \mu'_j(\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
&+ \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|+N'_j} } \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
&\cdot \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
&+ \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{s=-N'_j} \\
&- \sum_{j=1}^Q (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\
&\cdot \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}|+|\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}|+|\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \Bigg),
\end{aligned}$$

with

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket},$$

for all  $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$ , and for all  $j \in \llbracket 1, Q \rrbracket$ ,

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) &= \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, N'_q = \mathbf{v}(q) + \mathbf{w}(q)}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{q,p}} \right) \\
&\cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{q,p}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!,
\end{aligned}$$

and

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) &= (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\
&\quad \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
&+ (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \\
&\quad \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}'),
\end{aligned}$$

with  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  being the constant described in Proposition 1.3.2.

## 5.2 Auxiliary functions

In this section, we shall apply to the auxiliary functions  $\varphi_{\mathbf{R}}^j(s)$  the Proposition 2.2.4 in order to use these formulas when we will compute the special values of the derivative  $Z'(-(\mathbf{N}, \mathbf{N}'))$ .

Let's recall that, for  $\mathbf{R} = (R_1, \dots, R_P) \in \mathbb{N}_0^P$  and  $j \in [1, Q]$ , we have set

$$\varphi_{\mathbf{R}}^j(s) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d'_j)^{-s}.$$

This Dirichlet series is meromorphic on  $\mathbb{C}$ , and we know that the auxiliary function is regular at nonpositive integers by Corollary 2.2.6.

**Remark 5.2.1.** For  $\mathbf{R} = \mathbf{0}$ , we retrieve the Barnes zeta function defined as

$$\zeta^B(s, d'_j | c_{j,1}, \dots, c_{j,P}) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{(l_j(\mathbf{n}) + d'_j)^s}.$$

We apply Proposition 2.2.4 to the auxiliary function  $\varphi_{\mathbf{R}}^j(s)$ , setting  $\boldsymbol{\mu} = \mathbf{0}$ ,  $\mathbf{N} = \mathbf{R}$ ,  $\mathbf{N}' = \mathbf{0}$ , and  $\boldsymbol{\mu}' = 1$ . We know that this auxiliary function is regular at nonpositive integers, and by this same proposition, we obtain that the set of poles is contained in the set

$$\mathcal{S}(\mathbf{R}) := \mathbb{Z}_{\geq 1}.$$

We find that the auxiliary function takes the following form, for  $0 < \theta \ll 1$ , and for all  $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{R})$ ,

$$\varphi_{\mathbf{R}}^j(s) = J_{\mathbf{R}}^j(\theta, s) + K_{\mathbf{R}}^j(\theta, s), \quad (5.2)$$

such that for all  $s \in \mathbb{C}$ ,

$$K_{\mathbf{R}}^j(\theta, s) = \frac{1}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p},$$

and for all  $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{R})$ , we have

$$\begin{aligned} J_{\mathbf{R}}^j(\theta, s) = & \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{R}, \mathcal{P}, \mathbf{k}}(s) \frac{\theta^{s-|\mathbf{R}|_{|\mathcal{P}|-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(s)(s-|\mathbf{R}|_{|\mathcal{P}|-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p=1}^P R_p! \right) h_{\mathbf{R}, [1, P], j}(s) \frac{\theta^{(|\boldsymbol{\mu}'|+|\boldsymbol{\mu}|)s-|\mathbf{N}'|-|\mathbf{N}|-P}}{\Gamma(s)(s-|\mathbf{R}|-P)}, \end{aligned}$$

with

$$h_{\mathbf{R}, \mathcal{P}, \mathbf{k}}(s) = \prod_{p \in \mathcal{P}} c_{j,p}^{-N_p-1} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p},$$

and

$$h_{\mathbf{R}, [1, P], j}(s) = \prod_{p=1}^P c_{j,p}^{-N_p-1}.$$

We then get

$$\begin{aligned} J_{\mathbf{R}}^j(\theta, s) = & \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \frac{\theta^{s-|\mathbf{R}|_{|\mathcal{P}|-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(s)(s-|\mathbf{R}|_{|\mathcal{P}|-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p=1}^P R_p! \right) \left( \prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \frac{\theta^{s-|\mathbf{R}|-P}}{\Gamma(s)(s-|\mathbf{R}|-P)}. \end{aligned} \tag{5.3}$$

**Proposition 5.2.2.** *Let  $\mathbf{R} \in \mathbb{N}_0^P$ , and  $N \in \mathbb{N}_0$ . Let  $0 < \theta \ll 1$  and  $j \in [1, Q]$ , then  $s \mapsto \varphi_{\mathbf{R}}^j(s)$  is meromorphic on  $\mathbb{C}$ , and is regular at nonpositive integers. Moreover, we have*

$$\partial_s K_{\mathbf{R}}^j(\theta, -N) = (-1)^N N! \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-N, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \tag{5.4}$$

$$\begin{aligned} \partial_s J_{\mathbf{R}}^j(\theta, -N) = & J_{\mathbf{R}}^{*j}(\theta, -N) + N!(\gamma - h_N) \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}+|\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=N+|\mathbf{R}|_{|\mathcal{P}+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}, \end{aligned} \tag{5.5}$$

with  $J_{\mathbf{R}}^{*j}(\theta, -N)$  being a function of the form  $J_{\mathbf{R}}^{*j}(\theta, -N) = \sum_{n \geq -N-|\mathbf{R}|-P} \alpha_n \theta^n + \beta \ln(\theta)$  for all  $0 < \theta \ll 1$ , and without any constant term (i.e.  $\alpha_0 = 0$ ).

*Proof.* By Theorem 2.2.6, we know that  $\varphi_{\mathbf{R}}^j$  is meromorphic at nonpositive integers. By Proposition 2.3.11, we know that the function  $s \in \mathbb{C} \mapsto \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p}$  is holomorphic on  $\mathbb{C}$ .

Deriving  $K_{\mathbf{R}}^j$  with respect to  $s$ , we get

$$\begin{aligned} \partial_s K_{\mathbf{R}}^j(\theta, s) = & -\frac{\psi(s)}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \\ & + \frac{1}{\Gamma(s)} \partial_s \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \right), \end{aligned}$$

where  $\psi$  is the digamma function.

Therefore we can evaluate the sum of these two terms at  $s = -N$ . Since  $\frac{1}{\Gamma(s)}$  vanish at  $s = -N$ , and by Lemma 3.1.1, we have  $\left(\frac{\psi(s)}{\Gamma(s)}\right)_{s=-N} = (-1)^{N+1}N!$ , we then obtain the required formula.

From formula 5.3, we get

$$\begin{aligned} \partial_s J_{\mathbf{R}}^j(\theta, s) &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &\cdot \partial_s \left( \frac{1}{\Gamma(s)(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \theta^{s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ &+ \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &\cdot \frac{1}{\Gamma(s)(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \theta^{s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \ln(\theta) \\ &- \left( \prod_{p=1}^P R_p! \right) \left( \prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \theta^{s - |\mathbf{R}| - P} \left( \frac{\ln(\theta)}{\Gamma(s)(s - |\mathbf{R}| - P)} + \partial_s \left( \frac{1}{\Gamma(s)(s - |\mathbf{R}| - P)} \right) \right). \end{aligned}$$

In order to obtain a formula for  $(\partial_s J_{\mathbf{R}}^j(\theta, s))_{s=-N}$ , we first need to evaluate the terms  $\frac{1}{(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)}$ , and the terms  $\partial_s \frac{1}{(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)}$ . That can easily be done using Lemma 3.1.1:

$$\left( \frac{1}{(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)} \right)_{s=-N} = \begin{cases} (-1)^N N! & \text{if } |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}| \\ 0 & \text{if } |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|. \end{cases}$$

and

$$\partial_s \left( \frac{1}{\Gamma(s)(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{s=-N} = \begin{cases} (-1)^N (\gamma - h_N) N! & \text{if } |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}| \\ \frac{(-1)^N N!}{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} & \text{if } |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|. \end{cases}$$

Therefore we find that

$$\begin{aligned} \partial_s J_{\mathbf{R}}^j(\theta, -N) &= (\gamma - h_N) N! \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=N+|\mathbf{R}|_{|\mathcal{P}}+|\mathcal{P}|}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &+ (-1)^N N! \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| \neq N+|\mathbf{R}|_{|\mathcal{P}}+|\mathcal{P}|}} (-1)^{|\mathbf{k}|} \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \frac{\theta^{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ &+ N! \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=N+|\mathbf{R}|_{|\mathcal{P}}+|\mathcal{P}|}} \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \ln(\theta) \\ &+ (-1)^N N! \left( \prod_{p=1}^P R_p! \right) \left( \prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \frac{\theta^{-N - |\mathbf{R}| - P}}{-N - |\mathbf{R}| - P}. \end{aligned}$$

We can set  $J_{\mathbf{R}}^{*j}(\theta, -N)$  such that

$$\begin{aligned}
J_{\mathbf{R}}^{*j}(\theta, -N) = & (-1)^N N! \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}|+|\mathcal{P}|}}} (-1)^{|\mathbf{k}|} \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \frac{\theta^{-N - |\mathbf{R}|_{|\mathcal{P}|+|\mathcal{P}|+|\mathbf{k}|}}}{-N - |\mathbf{R}|_{|\mathcal{P}|+|\mathcal{P}|+|\mathbf{k}|}} \\
& + N! \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}|+|\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}|+|\mathcal{P}|}}} \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \ln(\theta) \\
& + (-1)^N N! \left( \prod_{p=1}^P R_p! \right) \left( \prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \frac{\theta^{-N - |\mathbf{R}| - P}}{-N - |\mathbf{R}| - P}.
\end{aligned}$$

□

**Remark 5.2.3.** The term  $J_{\mathbf{R}}^{*j}(\theta, -N)$  corresponds to a sum of a multiple of  $\ln(\theta)$  and a Laurent series without any constant term.

Let us observe that we cannot obtain an explicit expression for the values of  $\partial_s \varphi_{\mathbf{R}}^j$  at nonpositive integers using (5.4) and (5.5), since it is not true that the term  $\partial_s K_{\mathbf{R}}^j(\theta, -N)$  vanishes. We thus need another strategy to compute the values of  $\partial_s \varphi_{\mathbf{R}}^j$  at nonpositive integers.

## 5.2.1 Proof of Proposition B

In this section, we aim to provide explicit values of the derivatives of these auxiliary functions assuming the coefficients  $(c_{q,p})_{q \in [1, Q] \times [1, P]}$  are rationals.

Let  $P \in \mathbb{N}$ , and  $\mathbf{R} \in \mathbb{N}_0^P$ . To simplify notations in this subsection, we will write for all complex  $s$  such that  $\sigma \gg_{\mathbf{R}} 1$

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{(n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P}}{\left( \sum_{p=1}^P w_p (n_p + d_p) \right)^s},$$

with  $\mathbf{d} = (d_p)_{p \in [1, P]}$  and  $w_p, d_p \in H_0$  for all integer  $p$ . This function admits a meromorphic extension to  $\mathbb{C}$  in the variable  $s$ . In this subsection, we wish to establish an explicit relation between  $\varphi_{\mathbf{R}}^j(s)$  and the Hurwitz zeta function  $\zeta(s, x)$ , with  $x \in H_0$ . To do this, we follow a strategy developed by Aoki and Sakane in [SA22], which was initially used to get formulas for special values of the higher derivatives of Barnes zeta functions with rational coefficients. Here, the auxiliary functions  $\varphi_{\mathbf{R}}(s, x | w_1, \dots, w_P)$  are not exactly Barnes zeta functions due to the terms in the numerator of the series, however, the strategy used by Aoki and Sakane works in the same way to study this function.

**Lemma 5.2.4.** Let  $w_1, \dots, w_P \in \mathbb{Q}_*^+$  such that  $w_p = \frac{a_p}{b_p}$  with  $a_p$  and  $b_p$  are positive integers. For any common multiple  $\alpha$  of  $b_1, \dots, b_P$ , and any common multiple  $\beta$  of  $\alpha w_1, \dots, \alpha w_P$ , we have

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left( \frac{\alpha}{\beta} \right)^s \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq k_1 \leq \beta_1 - 1 \\ 0 \leq k_P \leq \beta_P - 1}} \varphi_{\mathbf{R}} \left( s, \left( \frac{d_p + k_p}{\beta_p} \right)_{p \in [1, P]} \mid \mathbf{1} \right), \quad (5.6)$$

with  $\beta_1 = \frac{\beta}{\alpha w_1}, \dots, \beta_P = \frac{\beta}{\alpha w_P}$ .



*Proof.* Let  $s \in \mathbb{C}$  such that  $\sigma \gg_{\mathbf{R}} 1$ . In order to get the required result, we will decompose each sum  $\sum_{n_p=0}^{+\infty}$

using  $\sum_{\substack{n_p=0 \\ n_p \equiv k_p \pmod{\beta_p}}}^{+\infty}$ , with  $k_p$  running over the set  $\llbracket 0, \beta_p - 1 \rrbracket$ . We then get

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta}\right)^s \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\substack{\mathbf{n} \in \mathbb{N}_0^P \\ \forall p, n_p \equiv k_p \pmod{\beta_p}}} \frac{\prod_{p=1}^P (n_p + d_p)^{R_p}}{\left(\frac{\alpha}{\beta} \left(\sum_{p=1}^P (w_p d_p + w_p k_p + w_p (n_p - k_p))\right)\right)^s}.$$

By a change of variables  $n'_p = \beta_p n_p + k_p$  for each integer  $p$ , we get

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta}\right)^s \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{\prod_{p=1}^P (\beta_p n_p + k_p + d_p)^{R_p}}{\left(\frac{\alpha}{\beta} \sum_{p=1}^P w_p d_p + w_p k_p + w_p \beta_p n_p\right)^s}.$$

By factoring  $\beta_p^{R_p}$  in the numerator, and by noticing that  $\frac{\alpha w_p \beta_p}{\beta} = 1$ , we get

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta}\right)^s \prod_{p=1}^P \beta_p^{R_p} \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{\prod_{p=1}^P \left(n_p + \frac{k_p + d_p}{\beta_p}\right)^{R_p}}{\left(\sum_{p=1}^P n_p + \frac{\alpha}{\beta} (w_p d_p + w_p k_p)\right)^s}.$$

Finally, one can check that

$$\frac{k_p + d_p}{\beta_p} = \frac{\alpha}{\beta} (w_p d_p + w_p k_p),$$

for all  $1 \leq p \leq P$ . We then find that the formula 5.6 is true for all  $s \in \mathbb{C}$  such that  $\sigma \gg_{\mathbf{R}} 1$ . We can then verify that the formula is true on all  $\mathbb{C}$  by analytic continuation.  $\square$

We can specify the last formula 5.6 by taking an explicit rational coefficient  $\frac{\alpha}{\beta}$ . In fact, we can directly consider a coefficient computed through the lcm and the gcd of the numerator and denominator of  $w_1, \dots, w_P$ , which allows us for more explicit results later on.

**Proposition 5.2.5.** *Let  $\mathbf{d} \in H_0$   $w_1 = \frac{a_1}{b_1}, \dots, w_P = \frac{a_P}{b_P}$  be rational numbers with  $a_p, b_p > 0$  coprime integers for all  $p \in \llbracket 1, P \rrbracket$ . We set  $w := \frac{\text{lcm}(a_1, \dots, a_P)}{\text{gcd}(b_1, \dots, b_P)}$ , and for all  $p \in \llbracket 1, P \rrbracket$ , we define  $\beta_1 := \frac{w}{w_1}, \dots, \beta_P := \frac{w}{w_P}$ . We then have*

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = w^{-s} \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq k_1 \leq \beta_1 - 1 \\ \dots \\ 0 \leq k_P \leq \beta_P - 1}} \varphi_{\mathbf{R}}\left(s, \left(\frac{d_p + k_p}{\beta_p}\right)_{p \in \llbracket 1, P \rrbracket} \middle| \mathbf{1}\right).$$

*Proof.* Let  $\alpha = \text{lcm}(b_1, \dots, b_P)$  and  $\beta = \text{lcm}(\alpha w_1, \dots, \alpha w_P)$ . By [SA22, p.3], we know that  $\frac{\alpha}{\beta} = w^{-1}$ . Therefore, we can use formula 5.6 with such  $\alpha$  and  $\beta$ .  $\square$

Finally, we obtain a formula similar to the one proved in [Ono21, Proposition 4.1] to re-express  $\varphi_{\mathbf{R}}(s, \mathbf{d} | 1, \dots, 1)$ , by making a very slight modification to Onodera's proof.

**Proposition 5.2.6.** *Let  $\mathbf{d} \in H_0^P$ ,  $\mathbf{R} \in \mathbb{N}_0^P$ , and  $s \in \mathbb{C} \setminus \mathbb{N}$ . We then have that*

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | \mathbf{1}) = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, k'_i \geq 0 \\ |\mathbf{k}|_{\mathcal{P}} + k'_i = |\mathbf{R}|_{\mathcal{P}} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \frac{\zeta(s - k'_i, |\mathbf{d}|)}{k'_i!} \prod_{p \in \mathcal{P}} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}.$$

In order to prove such formula, we need to use a lemma proved by Onodera [Ono21, Lemma 4.2], which generalizes Faulhaber's formula:

$$\sum_{n=1}^y n^r = \frac{1}{r+1} \sum_{k=0}^r \binom{r+1}{k} (-1)^k B_k y^{r-k+1}.$$

**Lemma 5.2.7.** [Ono21] Let  $\mathbf{R} = (R_1, \dots, R_P) \in \mathbb{N}_0^P$  be nonnegative integers, and  $y \in \mathbb{Z}$  an integer. We have that

$$\begin{aligned} \sum_{\substack{\mathbf{n} \in \mathbb{N}_0^P \\ |\mathbf{n}|=y}} \prod_{p=1}^P (n_p + d_p)^{R_p} + (-1)^{P-1} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{\leq -1}^P \\ |\mathbf{n}|=y}} \prod_{p=1}^P (n_p + d_p)^{R_p} = \\ \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, \mathbf{k}' \in \mathbb{N}_0 \\ |\mathbf{k}| + \mathbf{k}' = |\mathbf{R}|_{|\mathcal{P}^c| + |\mathcal{P}^c| - 1}}} \frac{(|\mathbf{d}| + y)^{\mathbf{k}'}}{\mathbf{k}'!} \prod_{p \in \mathcal{P}} (-1)^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}. \end{aligned}$$

Let's note that if  $y \in \mathbb{N}_0$ , then the second sum in the last equality vanishes. We can now prove Proposition 5.2.6:

*Proof.* Let's first note that, for all  $s \in \mathbb{C}$  such that  $\sigma_{\mathbf{R}} \gg 1$ , we have

$$\varphi_{\mathbf{R}}(s, \mathbf{d}|\mathbf{1}) = \sum_{n'=0}^{+\infty} \frac{1}{(n' + |\mathbf{d}|)^s} \sum_{\substack{\mathbf{n} \in \mathbb{N}_0^P \\ |\mathbf{n}|=n'}} \prod_{p=1}^P (n_p + d_p)^{R_p}.$$

By Lemma 5.2.7, we then get that

$$\begin{aligned} \varphi_{\mathbf{R}}(s, \mathbf{d}|\mathbf{1}) &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, \mathbf{k}' \in \mathbb{N}_0 \\ |\mathbf{k}| + \mathbf{k}' = |\mathbf{R}|_{|\mathcal{P}^c| + |\mathcal{P}^c| - 1}}} \prod_{p \in \mathcal{P}} (-1)^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \sum_{n'=0}^{+\infty} \frac{(n' + |\mathbf{d}|)^{\mathbf{k}'}}{\mathbf{k}'!} \frac{1}{(n' + |\mathbf{d}|)^s} \\ &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, \mathbf{k}' \in \mathbb{N}_0 \\ |\mathbf{k}| + \mathbf{k}' = |\mathbf{R}|_{|\mathcal{P}^c| + |\mathcal{P}^c| - 1}}} (-1)^{|\mathbf{k}|} \left( \prod_{p \in \mathcal{P}} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \frac{\zeta(s - \mathbf{k}', |\mathbf{d}|)}{\mathbf{k}'!}. \end{aligned}$$

□

Applying Proposition 5.2.6 into the formula obtained in Proposition 5.2.5, we get the following proposition:

**Proposition 5.2.8.** Let  $\mathbf{d} \in H_0^P$ ,  $\mathbf{R} \in \mathbb{N}_0^P$ ,  $s \in \mathbb{C} \setminus \mathbb{N}$ , and  $w_1 = \frac{a_1}{b_1}, \dots, w_P = \frac{a_P}{b_P}$  be rational numbers with  $a_p, b_p > 0$  coprime integers for all  $p \in [1, P]$ . We define  $w := \frac{\text{lcm}(a_1, \dots, a_P)}{\text{gcd}(b_1, \dots, b_P)}$ , and for all  $p \in [1, P]$ , we define  $\beta_1 := \frac{w}{w_1}, \dots, \beta_P := \frac{w}{w_P}$ . Then we have

$$\begin{aligned} \varphi_{\mathbf{R}}(s, \mathbf{d}|w_1, \dots, w_P) &= \\ w^{-s} \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq u_1 \leq \beta_1 - 1 \\ 0 \leq u_P \leq \beta_P - 1}} \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, \mathbf{k}' \in \mathbb{N}_0 \\ |\mathbf{k}| + \mathbf{k}' = |\mathbf{R}|_{|\mathcal{P}^c| + |\mathcal{P}^c| - 1}}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + u_p}{\beta_p}\right)}{k_p!} \\ \cdot \frac{\zeta(s - \mathbf{k}', w^{-1}(w_1(d_1 + u_1) + \dots + w_P(d_P + u_P)))}{\mathbf{k}'!}. \end{aligned}$$

### 5.3 Proof of Theorem D

Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ ,  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ . By Proposition 2.2.4, for all complex  $s$  around 0, and for all  $0 < \theta \ll 1$  we have that

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = K_{\mathbf{N}, \mathbf{N}'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}(\theta, s),$$

with

$$\begin{aligned} K_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \sum_{\substack{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s)} \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\ & \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right), \end{aligned}$$

and

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \\ & \cdot \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c})}}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c})}} \\ & \cdot \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left( \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right). \end{aligned}$$

**Proposition 5.3.1.** *Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ ,  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}_+^P \times \mathbb{R}_+^{*Q}$ . There exist a function  $K_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  defined for all  $0 < \theta \ll 1$ , such that*

$$\forall 0 < \theta \ll 1, \quad K_{\mathbf{N}, \mathbf{N}'}^*(\theta) = \beta \ln(\theta) + \sum_{n=m}^{+\infty} \alpha_n \theta^n,$$

with  $m \in \mathbb{Z}$  and integer,  $\beta \in \mathbb{C}$  a complex number, and  $(\alpha_n)_{n \in [m, +\infty[}$  a complex sequence such that

$\alpha_0 = 0$ , and such that:

$$\begin{aligned}
\partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} = & \tag{5.7} \\
& \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{\mathbf{u}(p)} \partial_s \varphi_{(N_p + \mathbf{u}(p))_{p \in [1, P]}}^j(s)_{s=-N'_j} \\
& - \sum_{j=1}^Q (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{\mathbf{u}(p)} \right) \\
& \cdot \sum_{\mathcal{P} \subseteq [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \\
& + K_{\mathbf{N}, \mathbf{N}'}^*(\theta).
\end{aligned}$$

*Proof.* Taking the derivative of  $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$  with respect to  $s$  in formula 2.4, we get

$$\begin{aligned}
\partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \tag{5.8} \\
& \sum_{\substack{\mathcal{Q} \subseteq [1, Q] \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subseteq \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \partial_s \left( \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right) \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\
& \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right) \\
& + \sum_{\substack{\mathcal{Q} \subseteq [1, Q] \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subseteq \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \partial_s \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\
& \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right).
\end{aligned}$$

Thanks to the bound obtained in 2.12, we've already seen in the proof of Proposition 2.3.11 that the function

$$s \in \mathbb{C} \mapsto \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right)$$

was holomorphic on  $\mathbb{C}$ . As  $s \mapsto \frac{1}{\Gamma(s)}$  vanish at nonpositive integers, we find that the second term in 5.8 vanish at  $s = 0$ . Moreover, we know that around  $s = 0$ , we have

$$\frac{1}{\Gamma(-N'_q + \mu'_q s)} = O(s).$$

Therefore, we find that for all  $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$  and for all  $A_{\mathcal{Q}} \subset \mathcal{Q}^c = \llbracket 1, Q \rrbracket \setminus \mathcal{Q}$ , we have

$$\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} = \begin{cases} O(s^2) & \text{if } |\mathcal{Q}| \geq 2, \\ O(s) & \text{if } \mathcal{Q} = \{j\}, A_{\mathcal{Q}} = \llbracket 1, Q \rrbracket \setminus \{j\} \text{ with } j \in \llbracket 1, Q \rrbracket, \\ O(s^2) & \text{otherwise.} \end{cases}$$

Therefore, we obtain that if  $|\mathcal{Q}| \geq 2$ , we have

$$\partial_s \left( \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right)_{s=0} = 0.$$

We then get

$$\begin{aligned} \partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} &= \sum_{j=1}^Q \partial_s \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)} \right)_{s=0} \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p} \right. \\ &\quad \left. \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} \right). \end{aligned}$$

By Lemma 3.1.1, we obtain

$$\partial_s \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)} \right)_{s=0} = (-1)^{N'_j} N'_j! \mu'_j,$$

and therefore

$$\begin{aligned} \partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} &= \sum_{j=1}^Q \mu'_j (-1)^{N'_j} N'_j! \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p} \right. \\ &\quad \left. \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} \right). \end{aligned}$$

For all  $q \in \llbracket 1, Q \rrbracket$ ,  $\mathbf{n} \in \mathbb{N}_0^P$ , we have  $l_q(\mathbf{n}) + d'_q = \sum_{p=1}^P c_{q,p} (n_p + d_p)$ . By Newton's multinomial, we get

$$(l_q(\mathbf{n}) + d'_q)^{N'_q} = \sum_{\substack{\mathbf{u}_q \in \mathbb{N}_0^P \\ |\mathbf{u}_q| = N'_q}} \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} (n_p + d_p)^{u_{q,p}}.$$

By taking the product over  $q \in \llbracket 1, Q \rrbracket \setminus \{j\}$ , we find

$$\begin{aligned} \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} &= \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \left( \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} (n_p + d_p)^{u_{q,p}} \right) \\ &= \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \prod_{p=1}^P (n_p + d_p)^{\mathbf{u}^{(p)}}, \end{aligned}$$

where we noted

$$\mathbf{u}(p) = \sum_{\substack{q=1 \\ q \neq j}}^Q u_{q,p}.$$

We finally get

$$\begin{aligned} \partial_s K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s)_{s=0} &= \sum_{j \in \llbracket 1, Q \rrbracket} \mu'_j (-1)^{N'_j} N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \\ &\quad \left( \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p + \mathbf{u}(p)} \right). \end{aligned} \quad (5.9)$$

By formula (5.2) we know that for all  $m \in \mathbb{N}_0$ ,

$$(-1)^m m! \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-m, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} = \partial_s \varphi_{\mathbf{R}}^j(s)_{|s=-m} - \partial_s J_{\mathbf{R}}^j(\theta, s)_{|s=-m}.$$

Substituting  $(-1)^m m! \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-m, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p}$  in 5.9 by the above formula, we get

$$\begin{aligned} \partial_s K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s)_{s=0} &= \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \\ &\quad \left( \partial_s \varphi_{(N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket}}^j(s)_{|s=-N'_j} - \partial_s J_{(N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket}}^j(\theta, s)_{|s=-N'_j} \right). \end{aligned} \quad (5.10)$$

Let's note  $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) = (N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket}$  for all  $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$ . By formula (5.5), we get

$$\begin{aligned} \partial_s J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(\theta, s)_{s=-N'_j} &= (\gamma - h_{N'_j}) N'_j! \sum_{\mathcal{P} \subset \llbracket 1, P \rrbracket} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \\ &\quad \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \\ &\quad + J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j). \end{aligned} \quad (5.11)$$

For all  $0 < \theta \ll 1$ , we set

$$K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta) := - \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j) \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}}.$$

By definition of  $J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j)$ , we see that  $K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta)$  are of the forms  $\sum_{n \geq m} \alpha_n \theta^n + \beta \ln(\theta)$  with  $\alpha_n$  and  $\beta$  complex numbers for all  $n \geq m$ , and  $\alpha_0 = 0$ . Injecting formula 5.11 into formula 5.10, we find the

required formula:

$$\begin{aligned}
\partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)_{s=0} = & \quad (5.12) \\
& \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \partial_s \varphi_{(N_p + \mathbf{u}(p))_{p \in [1, P]}}^j(s)_{s=-N'_j} \\
& - \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \right) \prod_{p=1}^P c_{q,p}^{u_{q,p}} (-1)^{N'_j} N'_j! (\gamma - h_{N'_j}) \\
& \cdot \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \\
& + K_{\mathbf{N}, \mathbf{N}'}^*(\theta).
\end{aligned}$$

□

We then just need to compute  $\partial_s (J_{\mathbf{N}, \mathbf{N}'}(\theta, s))$  at  $s = 0$  in order to prove Theorem D.

**Proposition 5.3.2.** *Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ . There exist a function  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  defined for all  $0 < \theta \ll 1$ , such that*

$$\forall 0 < \theta \ll 1, \quad J_{\mathbf{N}, \mathbf{N}'}^*(\theta) = \beta \ln(\theta) + \sum_{n=m}^{+\infty} \alpha_n \theta^n,$$

with  $m \in \mathbb{Z}$  an integer,  $\beta \in \mathbb{R}$  a complex number and  $(\alpha_n)_{n \in [m, +\infty[}$  a complex number sequence such that  $\alpha_0 = 0$ , and such that for  $0 < \theta \ll 1$ :

$$\begin{aligned}
\partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \sum_{\mu, \mu'} \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|}}} \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}| + N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}|}} N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\
& \cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
& + \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|} + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}|}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
& \cdot \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
& + J_{\mathbf{N}, \mathbf{N}'}^*(\theta).
\end{aligned} \tag{5.13}$$

*Proof.* Let's recall that

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) := h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0)$$

$$Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) := \partial_s (h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s))|_{s=0},$$

for  $j \in \llbracket 1, Q \rrbracket$ ,  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ ,  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$ .

Let's note  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$  the general term in the series in formula 2.5:

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) =$$

$$\frac{\left( \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!},$$

for  $j \in \llbracket 1, Q \rrbracket$ ,  $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ , and  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$ . We're now going to evaluate formula 2.29 at  $s = 0$  in order to get the desired formula for  $\partial_s (J_{\mathbf{N}, \mathbf{N}'}(\theta, s))_{s=0}$ . We get

$$\partial_s (J_{\mathbf{N}, \mathbf{N}'}(\theta, s))_{s=0} = \sum_{\mu, \mu'} \sum_{\mathcal{P} \subset \llbracket 1, P \rrbracket} \sum_{j=1}^Q \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} \partial_s T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)_{s=0} \quad (5.14)$$

$$+ \partial_s \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{s=0}.$$

Firstly, let's study the second term of this sum. We see that, at  $s = 0$ , the function  $\frac{1}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)}$  vanish. Therefore, we get that

$$\partial_s \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{s=0} =$$

$$\prod_{p=1}^P N_p! \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(0) \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P} \partial_s \left( \frac{1}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{s=0}.$$

By Lemma 3.1.1, we have

$$\partial_s \left( \frac{1}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{s=0} = (-1)^{N'_j + 1} N'_j! \frac{\mu'_j}{|\mathbf{N}'| + |\mathbf{N}| + P}.$$

We then find that

$$\partial_s \left( \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{s=0} =$$

$$\left( \prod_{p=1}^P N_p! \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(0) (-1)^{N'_j + 1} N'_j! \frac{\mu'_j}{|\mathbf{N}'| + |\mathbf{N}| + P} \right) \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P}.$$

We shall include later that last term in the definition  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ , because  $-|\mathbf{N}'| - |\mathbf{N}| - P \neq 0$ .



Let's now study the derivative of the general term  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$  with respect to  $s$  at  $s = 0$ .

- If  $|\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ , then the function  $\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)}$  vanish at  $s = 0$ . Therefore we find that

$$\begin{aligned} \partial_s (T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s))_{s=0} &= \\ & \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ & \partial_s \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}. \end{aligned}$$

Via Lemma 3.1.1, we get

$$\begin{aligned} \partial_s (T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s))_{s=0} &= \\ & \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} (-1)^{N'_j} N'_j! \frac{\mu'_j}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ & \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}. \end{aligned}$$

We see that the power of  $\theta$  in the last term is non-zero. Therefore we shall later on include the terms  $\partial_s (T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s))_{s=0}$  such that  $|\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$  into the definition of  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ .

- If  $|\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ , then y Lemma 3.1.1, we have

$$\begin{aligned} & \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{s=0} = (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \\ \partial_s & \left( \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{s=0} = (-1)^{N'_j} \frac{\mu_j'^2}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! (\gamma - h_{N'_j}). \end{aligned}$$

Taking the derivative with respect to  $s$  of the general term  $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$ , we get

$$\begin{aligned} \partial_s (T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s))_{s=0} &= \\ & - \left( \sum_{i \in \mathcal{P}} \mu_i \Gamma'(1 + N_i) \prod_{p \in \mathcal{P} \setminus \{i\}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}, \\ & + \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}, \\ & + \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}) \ln(\theta) (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}, \\ & + \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} \frac{\mu_j'^2}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! (\gamma - h_{N'_j}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & + \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right), \end{aligned}$$

We know that  $\Gamma'(1 + N_p) = N_p!(-\gamma + h_{N'_p})$ . Therefore, we can simplify the last equality:

$$\begin{aligned} \partial_s(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s))_{s=0} &= \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}| + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ &\cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \mu'_j(\gamma - h_{N'_j}) - \sum_{p \in \mathcal{P}} \frac{\mu_p \Gamma'(1 + N_p)}{N_p!} \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ &+ \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right), \\ &+ \left( \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}| + N'_j} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \mu'_j N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \ln(\theta). \end{aligned}$$

We shall later on include the terms of the form  $\ln(\theta)$  into the definition of  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ .

We now define  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  such that

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}^*(\theta) &= \\ &\left( \sum_{\mathcal{P} \subsetneq [1, P]} \left( \prod_{p \in \mathcal{P}} N_p! \right) \sum_{j=1}^Q \mu'_j N'_j! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}}+|\mathcal{P}|}} (-1)^{|\mathbf{k}| + N'_j} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \ln(\theta) \\ &+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} \left( \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) (-1)^{N'_j} N'_j! \frac{\mu'_j}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \right. \\ &\cdot \left. \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right) \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ &+ \left( \prod_{p=1}^P N_p! \sum_{j=1}^Q Q_{\mathbf{N}, \mathbf{N}'}^0(j, [1, P]) (-1)^{N'_j + 1} N'_j! \frac{\mu'_j}{|\mathbf{N}'| + |\mathbf{N}| + P} \right) \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P}. \end{aligned}$$

By cutting off the series in formula 5.14 whether if  $|\mathbf{k}|$  is equal or not to  $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ , and by definition of the term  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  that we just defined, we get the required formula 5.13.  $\square$

We see that the series  $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  and  $K_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  have a zero constant term, and that  $\partial_s Z_{\mathbf{N}, \mathbf{N}'}(s) = \partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + \partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ . As  $\partial_s(Z_{\mathbf{N}, \mathbf{N}'}(s))_{s=0}$  does not depend on  $\theta$ , by Lemma 3.1.3 we have

$$\partial_s(Z_{\mathbf{N}, \mathbf{N}'}(s))_{s=0} = \partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s)|_{s=0} + \partial_s(K_{\mathbf{N}, \mathbf{N}'}(\theta, s))_{s=0} - K_{\mathbf{N}, \mathbf{N}'}^*(\theta) - J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$$

for every  $0 < \theta \ll 1$ . This leads us to Theorem D.

At the same time, we also obtain Theorem C, which corresponds to a qualitative reformulation of Theorem D.

**Remark 5.3.3.** From the equality  $K_{\mathbf{N}, \mathbf{N}'}^*(\theta) + J_{\mathbf{N}, \mathbf{N}'}^*(\theta) = 0$  for all  $0 < \theta \ll 1$ , and from Lemma 3.1.3, we get that each coefficient in the Laurent series in  $K_{\mathbf{N}, \mathbf{N}'}^*(\theta) + J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$  vanishes, and that the coefficient

in front of  $\ln(\theta)$  also vanishes. We can then get new relations between the special values  $\zeta(-k, d_p)$ , with  $k \in \mathbb{N}_0$  and  $1 \leq p \leq P$ .

## 5.4 On the corollaries of Theorem D

To prove the corollaries of Theorem D, we will use the formula given by the theorem, and substitute the direction or the values  $-(\mathbf{N}, \mathbf{N}')$  that one wishes to study.

### 5.4.1 Proof of Theorem C

Let  $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ , and  $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{R}^{+P} \times \mathbb{R}_*^{+Q}$ . The goal here is to revisit the formula obtained for  $Z'(-(\mathbf{N}, \mathbf{N}'))$  and to study its terms qualitatively. We denote  $\mathbb{K}$  as the field generated above  $\mathbb{Q}$  by the complex coefficients  $(d_p)_{p \in [1, P]}$  and  $(c_{q,p})_{p \in [1, P], q \in [1, Q]}$ , and the directions  $(\mu_p)_{p \in [1, P]}$ ,  $(\mu'_q)_{q \in [1, Q]}$ . By Theorem D, we have

$$\begin{aligned}
Z'(-(\mathbf{N}, \mathbf{N}')) = & \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q (-1)^{|\mathbf{N}| + |\mathbf{N}'| + |\mathcal{P}| + N'_j} \left( \prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\
& \cdot \left( Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left( \sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
& + \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \left( \prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|} + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}| + |\mathcal{P}|}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
& \cdot \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
& + \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s) |_{s = -N'_j} \\
& - \sum_{j=1}^Q (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\
& \cdot \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}}} \prod_{p \in \mathcal{P}^c} \frac{c_{j,p}^{k_p} \zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \Bigg),
\end{aligned}$$

where we have set

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + \mathbf{u}(p))_{p \in [1, P]},$$

for  $\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}}$ , and for  $j \in [1, Q]$ ,

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \in [1, Q] \setminus \{j\}, N'_q = \mathbf{v}(q) + \mathbf{w}(q)}}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ \cdot \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!$$

and

$$Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left( \sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\ + (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left( \prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, \mathbf{v}(q) + \mathbf{w}(q) = N'_q}} \\ \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}'),$$

with  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  the constant defined in Proposition 1.3.2.

Let's observe that  $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \in \mathbb{K}$ . Moreover, the coefficient  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$  belongs in the vector space on  $\mathbb{K}$  generated by coefficients  $\ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right)$  with  $1 \leq q, j \leq Q$  and  $1 \leq p \leq P$ . Therefore  $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \in \text{Vect}_{\mathbb{K}} \left[ \left( \ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right) \right)_{\substack{a \leq j, q \leq Q \\ 1 \leq p \leq P}} \right]$ . We then find that

$$Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \in \text{Vect}_{\mathbb{K}} \left[ \gamma, \left( \ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right) \right)_{\substack{a \leq j, q \leq Q \\ 1 \leq p \leq P}} \right].$$

As the special values  $\zeta(-N, d_p)$  belong in the field above  $\mathbb{Q}$  generated by the coefficients  $d_1, \dots, d_P$ , we find that these special values belong in  $\mathbb{K}$ . Therefore we get that  $Z'(-(\mathbf{N}, \mathbf{N}')) \in \text{Vect}_{\mathbb{K}} \left[ \zeta'(-n, y) \right]_{\mu, \mu'}$

on  $\mathbb{K}$  generated by  $\gamma$ , by coefficient of the form  $\ln \left( 1 + \frac{c_{q,p}}{c_{j,p}} \right)$ , by special values of the derivative of the auxiliary functions  $\partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{|s=-N'_j}$ , and by special values at nonpositive integers of the derivative with respect to  $s$  of the Hurwitz zeta functions.

## 5.4.2 Proof of Corollary C1

We use the fact that, when the linear forms are rational, Corollary B1 provides an expression for  $\partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{|s=-N'_j}$  in terms of the derivative of the Hurwitz zeta function. We then replace the terms of the form  $\partial_s \varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(s)_{|s=-N'_j}$  in Theorem C with terms of the form  $\zeta'(-n, y)$ .

### 5.4.3 Proof of Corollary D1

Corollary D1 is obtained by applying the formula found for  $Z'(-(\mathbf{N}, \mathbf{N}'))$  in Theorem A, and replacing the terms of the form  $\varphi_{\mathbf{R}}^{j'}(-N)$  with the formula provided by Corollary B1.

Since the linear forms  $l_j$  have rational coefficients, we can set  $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$  for all  $1 \leq p \leq P$ , assuming that this fraction is irreducible. We then define  $x_j(\mathbf{c}) = \frac{\text{lcm}(a_{j,p})}{\text{gcd}(b_{j,p})} \in \mathbb{Q}_+^*$ , and  $\beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}$  for all  $1 \leq p \leq P$ .

By Corollary B1, for  $\mathbf{R} \in \mathbb{N}_0^P$  we get that

$$\begin{aligned} \varphi_{\mathbf{R}}^{j'}(-N'_j) &= \\ & x_j(\mathbf{c})^N \prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ \dots \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq [1,P]} \left( \prod_{p \in \mathcal{P}^c} R_p! \right) \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, \mathbf{k}' \in \mathbb{N}_0 \\ |\mathbf{k}| + \mathbf{k}' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \\ & \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!} \left( \frac{\zeta'(-N - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} \right). \\ & = x_j(\mathbf{c})^N \prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ \dots \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq [1,P]} \left( \prod_{p \in \mathcal{P}^c} \mathbf{R}(\mathbf{N}, j, \mathbf{u})_p! \right) \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, \mathbf{k}' \in \mathbb{N}_0 \\ |\mathbf{k}| + \mathbf{k}' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \\ & \cdot \left( \prod_{p \in \mathcal{P}} \frac{\zeta\left(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!} \right) \\ & \cdot \left( \frac{\zeta'(-N'_j - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N'_j - k', l_j(x_j(\mathbf{c})^{-1}(\mathbf{d} + \mathbf{v})))}{k'!} \right). \end{aligned}$$

By injecting this formula in the one obtained for  $Z'(-(\mathbf{N}, \mathbf{N}'))$  in theorem D, we found the required result.

We obtain again Corollary C1, which corresponds to a qualitative reformulation of Corollary D1.

### 5.4.4 Proof of Corollary D2

By setting  $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$  in Theorem D, we can simplify several sums by noting the following facts:

- 1) Let  $\mathbf{v} = (v_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1,Q] \setminus \{j\}})^{\mathcal{P}}$  and  $\mathbf{w} = (w_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c}$  such that

$$\begin{aligned} |\mathbf{w}_p| &= k_p & (p \in \mathcal{P}^c) \\ \mathbf{v}(q) + \mathbf{w}(q) &= 0 & (q \in [1, Q] \setminus \{j\}). \end{aligned}$$

Then we found that the index have many zero components:

$$\forall q \in [1, Q] \setminus \{j\}, \forall p \in \mathcal{P}, \quad v_{p,q} = 0,$$

and

$$\forall q \in [1, Q] \setminus \{j\}, \forall p \in \mathcal{P}^c, \quad w_{p,q} = 0.$$

Therefore, as  $w_{p,j} = |w_p| = k_p$  for all  $p \in \mathcal{P}^c$ , we get that  $\mathbf{w} = (k_p \mathbf{e}_j)_{p \in \mathcal{P}^c}$  with  $\mathbf{e}_j \in \mathbb{N}_0^Q$  being the vector with a 1 in the  $j$ -th coordinate, and 0 elsewhere.

2) Let  $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}}$  and  $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c}$  such that

$$\begin{aligned} |\mathbf{w}_p| &= k_p & (p \in \mathcal{P}^c) \\ \mathbf{v}(q) + \mathbf{w}(q) &= 0 & (q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}). \end{aligned}$$

We found that

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}, \forall p \in \mathcal{P}, \quad v_{p,q} = 0,$$

and

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}, \forall p \in \mathcal{P}^c, \quad w_{p,q} = 0.$$

Thus, we find that the index  $\mathbf{w}$  takes the form  $(w_{p,j} \mathbf{e}_j + w_{p,f} \mathbf{e}_f)_{p \in \mathcal{P}^c}$ , with  $w_{p,j} + w_{p,f} = k_p$  for all  $p \in \mathcal{P}^c$ . We thus find that the multinomial coefficient present in the expression for  $Q^1$  is:

$$\binom{k_p}{\mathbf{w}_p} = \frac{k_p!}{w_{p,j}! w_{p,f}!}.$$

To simplify this expression, we will take the sum over  $w'_p \in \llbracket 0, k_p \rrbracket$ , and we will set  $w_{p,f} = w'_p$  and  $w_{p,j} = k_p - w'_p$ .

3) Let  $j \in \llbracket 1, Q \rrbracket$ , and  $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$  such that

$$|\mathbf{u}_q| = N'_q = 0 \quad (q \in \llbracket 1, Q \rrbracket \setminus \{j\}).$$

It trivially follows that  $\mathbf{u} = \mathbf{0}$ , and therefore  $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket} = \mathbf{0}$ . In particular, the auxiliary functions  $\varphi_{(N_p + \mathbf{u}(p))_{p \in \llbracket 1, P \rrbracket}}^j(s)$  are actually Barnes zeta functions of the form

$$\varphi_{\mathbf{0}}^j(s) = \zeta^B(s, d'_j | c_{j,1}, \dots, c_{j,P}).$$

We recall that the derivative of this Barnes zeta function at  $s = 0$  corresponds to  $\ln(\Gamma_P(d'_j | c_{j,1}, \dots, c_{j,P}))$ , where  $\Gamma_P$  corresponds to the Barnes multigamma functions introduced in Definition 1.2.14.

We have already seen in the proof of Corollary A1 that

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}.$$

Using these three preceding points, and the fact that the harmonic number  $h_0$  is 0, we obtain:

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0})) &= \\ &= \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \left( \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \gamma(\mu'_j + |\boldsymbol{\mu}|_{\mathcal{P}}) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ &+ \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \left( \sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p, d_p)}{k_p!} \right) \\ &+ \sum_{j=1}^Q \mu'_j \partial_s \zeta^B(s, d'_j | \mathbf{c}_j) |_{s=0} - \sum_{j=1}^Q \gamma \mu'_j \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-1} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{c_{j,p}^{k_p} \zeta(-k_p, d_p)}{k_p!}. \end{aligned}$$

By the point 1) and the point 2) previously stated, we get:

$$\begin{aligned} Q_{\boldsymbol{\mu}, \boldsymbol{\mu}'}^1(j, \mathcal{P}, \mathbf{k}) &= \sum_{\substack{j=1 \\ f \neq j}}^Q \mu'_f \sum_{\mathbf{w}' = (w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket} E_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ &+ \gamma |\boldsymbol{\mu}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}. \end{aligned}$$

### 5.4.5 Proof of Corollary D3

Applying Corollary D2, and by taking  $\boldsymbol{\mu} = (0, \dots, 0)$ , and  $\boldsymbol{\mu}' = (1, \dots, 1)$ , we get

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{0}, \mathbf{1}} &= \\ & \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|}}{Q} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \left( \gamma \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) + Q^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{j=1}^Q \ln(\Gamma_P(d'_j | \mathbf{c}_j)) - \gamma \sum_{j=1}^Q \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p, d_p)}{k_p!}, \end{aligned}$$

with

$$\begin{aligned} Q^1(j, \mathcal{P}, \mathbf{k}) &= \sum_{\substack{f=1 \\ f \neq j}}^Q \sum_{\substack{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket}} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ & + \gamma(Q-1) \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}. \end{aligned}$$

By Definition 1.2.16, we know that  $Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{0}, \mathbf{1}} = \ln(\partial_s(\zeta^{Sh}(s, (\mathbf{c}_1, \dots, \mathbf{c}_Q), \mathbf{d}')|_{s=0}))$ , and by Definition

1.2.14, we know that  $(\partial_s \zeta^B(s, d|c_1, \dots, c_P))|_{s=0} = \ln(\Gamma_P(d|c_1, \dots, c_P))$ , and  $\partial_s(\zeta^{Sh}(s, \mathbf{d}'|\mathbf{c}))|_{s=0} = \ln(\Gamma_P(\mathbf{d}'|\mathbf{c}))$ .

By all the above, we get Corollary D3.





# Chapter 6

## Witten zeta functions

We recall here the theory of Witten's zeta functions from the perspectives of Komori, Matsumoto, and Tsumura. Details of the proofs are available in the book [KMT24] which will be published in November 2023. Alternatively, one can also find details in the papers [KMT10b], [KMT10c], [KMT11b], [KMT12]. For details on Lie theory, we refer to [Hum72], [Bou06], [Bou81]. We will then apply some of our results to the study of Witten's zeta functions  $\zeta_{\mathfrak{g}_2}(s)$ , and  $\zeta_{\mathfrak{so}(5)}(s)$ , to deduce the explicit expression for the constant  $C$  in Theorem 1.2.35, and an asymptotic formula for the number of representations  $r_{\mathfrak{g}_2}(n)$  using Theorem 1.2.33.

### 6.1 Generalities on Lie algebras

We call  $\mathfrak{g}$  a Lie algebra, a  $\mathbb{C}$ -vector space of finite dimension equipped with a bilinear form  $[\cdot, \cdot]$  that satisfies the following conditions

1.  $\forall x \in \mathfrak{g}, [x, x] = 0$
2.  $\forall x, y, z \in \mathfrak{g}, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ .

We call  $[\cdot, \cdot]$  a Lie bracket associated with the Lie algebra  $\mathfrak{g}$ .

Let  $\mathfrak{g}_1, \mathfrak{g}_2$  be two Lie algebras with two Lie brackets  $[\cdot, \cdot]_1$  and  $[\cdot, \cdot]_2$  respectively. Let  $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  be a linear map between the two vector spaces. We say that  $f$  is a morphism of Lie algebra if  $f$  preserves the Lie brackets, i.e:

$$\forall x, y \in \mathfrak{g}_1, [f(x), f(y)]_2 = f([x, y]_1).$$

Moreover, if  $f$  is also a linear isomorphism, we say that  $f$  is an isomorphism of Lie algebras.

For  $x \in \mathfrak{g}$ , we note  $ad_x$  the following derivation:

$$ad_x : \mathfrak{g} \rightarrow \mathfrak{g}, y \mapsto [x, y].$$

**Definition 6.1.1.** Let  $\mathfrak{g}$  a Lie algebra. Let  $\mathfrak{a} \subset \mathfrak{g}$  be a subspace of  $\mathfrak{g}$ . We say that  $\mathfrak{a}$  is a Lie subalgebra of  $\mathfrak{g}$  if  $\mathfrak{a}$  is stable under the Lie bracket, i.e.,  $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$ .

Let  $\mathfrak{a}_1$  and  $\mathfrak{a}_2$  be subalgebras of a Lie algebra  $\mathfrak{g}$ , we define  $[\mathfrak{a}_1, \mathfrak{a}_2]$  as the Lie subalgebra of  $\mathfrak{g}$  generated by elements of the form  $[x, y]$ , where  $x \in \mathfrak{a}_1$ , and  $y \in \mathfrak{a}_2$ .

Let  $\mathfrak{a}$  be a subalgebra of a Lie algebra  $\mathfrak{g}$ . We denote  $N(\mathfrak{a})$  the subalgebra of  $\mathfrak{g}$  consisting of elements  $x$  such that  $ad_x(\mathfrak{a}) \subset \mathfrak{a}$ . If  $N(\mathfrak{a}) = \mathfrak{g}$ , then we say that  $\mathfrak{a}$  is an ideal of  $\mathfrak{g}$ .

**Definition 6.1.2.** We define a sequence  $(\mathfrak{g}^{(n)})_{n \in \mathbb{N}_0}$  such that

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \forall n \in \mathbb{N}_0, \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}].$$

We say that  $\mathfrak{g}$  is solvable if there is an integer  $n \in \mathbb{N}_0$  such that  $\mathfrak{g}^{(n)} = 0$ . We define  $rad(\mathfrak{g})$  as the (unique) maximal solvable ideal ideal of  $\mathfrak{g}$ .

**Definition 6.1.3.** Let  $\mathfrak{g}$  be a Lie algebra, we define the sequence  $(\mathfrak{g}^n)_{n \in \mathbb{N}_0}$ ,

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \forall n \in \mathbb{N}_0, \quad \mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n].$$

We say that  $\mathfrak{g}$  is nilpotent if there's an integer  $n \in \mathbb{N}_0$  such that  $\mathfrak{g}^n = 0$ .

**Definition 6.1.4.** Let  $\mathfrak{g}$  be a Lie algebra. We say that  $\mathfrak{g}$  is semi-simple if the only solvable ideal of  $\mathfrak{g}$  is  $\{0\}$ . Moreover, if  $\mathfrak{g}$  contains only a single ideal, we say that  $\mathfrak{g}$  is simple.

We now provide some fundamental examples of semi-simple Lie algebras:

**Example 6.1.5.** Let  $r \in \mathbb{N}$ ,

The type  $A_r$ :  $A_r := \mathfrak{sl}_{r+1} = \{X \in \mathcal{M}_{r+1}(\mathbb{C}) \mid \text{tr}(X) = 0\}$ .

The type  $B_r$ :  $B_r := \mathfrak{so}_{2r+1} = \{X \in \mathcal{M}_{2r+1}(\mathbb{C}) \mid {}^t X + X = 0\}$ .

The type  $C_r$ :  $C_r := \mathfrak{sp}_r = \{X \in \mathcal{M}_{2r}(\mathbb{C}) \mid {}^t X J_r + J_r X = 0\}$ , with

$$J_r := \begin{pmatrix} 0 & Id_r \\ -Id_r & 0 \end{pmatrix}.$$

The type  $D_r$ :  $D_r := \mathfrak{so}_{2r} = \{X \in \mathcal{M}_{2r}(\mathbb{C}) \mid {}^t X + X = 0\}$ , for  $r \geq 2$ .

In all these examples, the Lie bracket corresponds to  $[x, y] := x \cdot y - y \cdot x$ , with  $x, y \in \mathcal{M}_r$  and where  $\cdot$  means the usual matrix multiplication.

In the previous example, all these Lie algebras are simple, except for  $D_2$ . We also have the decomposition  $D_2 = A_1 \oplus A_1$ .

**Proposition 6.1.6.** A semi-simple Lie algebra is a direct sum of simple Lie algebras.

Thus, to classify semi-simple Lie algebras, it is sufficient to classify simple Lie algebras. We will see via the Cartan-Killing Theorem that the 4 types of algebras that we have described in the previous example almost completely classify the set of semi-simple Lie algebras.

**Definition 6.1.7.** A representation of a Lie algebra  $\mathfrak{g}$  over a  $\mathbb{C}$ -vector space  $E$  corresponds to a linear map  $\varphi : \mathfrak{g} \rightarrow GL(E)$ . We will call  $\dim(\varphi)$  the dimension of this representation. We say that the representation  $\varphi$  is irreducible if it does not admit any non-trivial vector subspace stable under the linear mapping  $\varphi$ .

**Example 6.1.8.** The derivation  $ad$  induces a representation of a Lie algebra  $ad(\mathfrak{g})$  on the vector space  $\mathfrak{g}$ :

$$Ad(\mathfrak{g}) : \mathfrak{g} \rightarrow GL(\mathfrak{g}), \quad x \mapsto ad_x.$$

We can then define the Killing bilinear form on  $\mathfrak{g} \times \mathfrak{g}$ :

$$\langle x, y \rangle := \text{Tr}(Ad_x \circ Ad_y).$$

**Theorem 6.1.9.** Let  $\mathfrak{g}$  be a Lie algebra. The following conditions are equivalent:

- $\mathfrak{g}$  is semi-simple.
- $\text{rad}(\mathfrak{g}) = 0$
- The Killing form  $\langle, \rangle$  is non-degenerate.

The previous theorem allows us in particular to identify a semi-simple Lie algebra with its dual. Thus, for an element  $x \in \mathfrak{g}$ , we can canonically associate an element  $x^* \in \mathfrak{g}^*$ .

## 6.2 Root System of a Lie Algebra

Let  $\mathfrak{g}$  be a Lie algebra. A Cartan subalgebra is defined as a nilpotent Lie subalgebra of  $\mathfrak{g}$  such that  $N(\mathfrak{h}) = \mathfrak{h}$ . Cartan subalgebras are not unique, but they share the same dimension over  $\mathbb{C}$ . This dimension is then called the rank of  $\mathfrak{g}$ , denoted  $\text{rg}(\mathfrak{g})$ .

Throughout this section, we fix a semi-simple Lie algebra  $\mathfrak{g}$  and a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . The Killing form extends to the Cartan subalgebra  $\mathfrak{h}$ , and it is non-degenerate on  $\mathfrak{h}$ . Through the identification of  $\mathfrak{h}$  with its dual, we can define the Killing form on  $\mathfrak{h}^*$ .

**Definition 6.2.1.** Let  $\alpha \in \mathfrak{h}^*$  be an element of the dual of  $\mathfrak{h}$ . We say that  $\alpha$  is a root of  $\mathfrak{g}$  relative to  $\mathfrak{h}$  when there exists a nonzero element  $x \in \mathfrak{g} \setminus \{0\}$  such that

$$\forall h \in \mathfrak{h}, \quad ad_h(x) = \alpha(h)x. \quad (6.1)$$

We will denote  $g_\alpha$  as the set of  $x \in \mathfrak{g}$  satisfying 6.1. We will denote  $\Delta(\mathfrak{g})$  as the set of roots of  $\mathfrak{g}$  relative to  $\mathfrak{h}$ . We will call  $\Delta(\mathfrak{g})$  the root system of the semi-simple Lie algebra  $\mathfrak{g}$ .

The root system of  $\mathfrak{g}$  is a finite set. Moreover, the Lie algebras  $\mathfrak{g}_\alpha$  have dimension 1, and they yield a decomposition of  $\mathfrak{g}$ :

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Through the identification between  $\mathfrak{h}$  and its dual, we consider for every root  $\alpha \in \mathfrak{h}^*$  the corresponding element  $\alpha' \in \mathfrak{h}$ . We set  $\mathfrak{h}_0 := \text{Vect}_{\mathbb{R}}((\alpha')_{\alpha \in \Delta(\mathfrak{g})})$  the  $\mathbb{R}$ -vector space generated by the elements  $\alpha'$ . We then define

$$\alpha^\vee := \frac{2}{\langle \alpha', \alpha' \rangle} \alpha'.$$

For every  $\alpha, \beta \in \Delta(\mathfrak{g})$ , we observe that

$$\beta(\alpha^\vee) = \frac{2 \langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

We denote  $a(\beta, \alpha)$  the above expression.

**Notation.** We denote  $\mathfrak{h}_0$  as the  $\mathbb{R}$ -vector space generated by the roots  $\Delta(\mathfrak{g})$ , and  $\mathfrak{h}_0^*$  as its dual.

**Fact 6.2.2.** Let  $\alpha, \beta \in \Delta(\mathfrak{g})$ . We have:

- 1)  $\beta(\alpha^\vee) \in \mathbb{Z}$ .
- 2)  $\beta - a(\beta, \alpha)\alpha \in \Delta(\mathfrak{g})$ .
- 3) If there exists  $c \in \mathbb{C}$  such that  $\beta = c\alpha$ , then  $c = \pm 1$ .

These three properties characterize an "abstract" root system in the sense of [Ser66] and [Bou81]. Given two semi-simple Lie algebras, we know that there exists an isometry  $(\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_1)), \|\cdot\|_1) \rightarrow (\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_2)), \|\cdot\|_2)$  mapping  $\Delta(\mathfrak{g}_1)$  to  $\Delta(\mathfrak{g}_2)$  if and only if  $\mathfrak{g}_1$  is isomorphic to  $\mathfrak{g}_2$  as a Lie algebra, denoting  $\|\cdot\|_1$  and  $\|\cdot\|_2$  as the Euclidean norms of the two vector spaces  $\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_1))$  and  $\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_2))$ . This property, in particular, reduces the study of simple Lie algebras to the study of root systems. From this correspondence follows the subsequent theorem, allowing the classification of simple Lie algebras.

**Theorem 6.2.3.** [Bou81] [Cartan-Killing Theorem]

Any simple Lie algebra is isomorphic either to  $A_r, B_r, C_r, D_r$ , or to an exceptional Lie algebra  $G_2, F_4, E_6, E_7, E_8$ .

To see the construction of the exceptional Lie algebras  $G_2, F_4, E_6, E_7$ , or  $E_8$ , refer to [Bou81].

Let us define  $H_{\alpha^\vee} := \{\beta \in \mathfrak{h}_0^* \mid \langle \beta, \alpha^\vee \rangle = 0\}$ .

We note that, since the set of roots is finite, the set

$$\mathfrak{h}_0^* \setminus \bigcup_{\alpha \in \Delta(\mathfrak{g})} H_{\alpha^\vee}$$

is non-empty. Let  $\gamma$  be an element of this set. We then have the following decomposition:

$$\Delta(\mathfrak{g}) = \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle > 0\} \cup \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle < 0\}.$$

We denote  $\Delta_+(\mathfrak{g}, \gamma) = \Delta_+(\mathfrak{g}) := \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle > 0\}$  as the set of positive roots. If a positive root  $\alpha$  cannot be decomposed as a sum of two positive roots, we call  $\alpha$  a fundamental (or simple) root.

**Definition 6.2.4.** Let  $\alpha$  be a positive root. We say that  $\alpha$  is a fundamental (or simple) root if  $\alpha$  cannot be decomposed as a sum of two positive roots. We shall note  $\psi(\mathfrak{g})$  as the set of fundamental roots.

**Fact 6.2.5.** *The set of fundamental roots forms a basis for the  $\mathbb{R}$ -vector space  $\mathfrak{h}_0^* = \text{Vect}_{\mathbb{R}}((\alpha')_{\alpha \in \Delta(\mathfrak{g})})$ . As a consequence, there are  $r = \dim_{\mathbb{R}}(\mathfrak{h}_0)$  fundamental roots.*

We now aim to introduce the concept of weight:

**Definition 6.2.6.** *Let  $\varphi : \mathfrak{g} \rightarrow GL(E)$  be a representation of the semi-simple Lie algebra  $\mathfrak{g}$ , with  $E$  being of finite dimension. Let  $u \in E$  be a vector. We say that  $u$  is an eigenvector if there exists a linear form  $\lambda \in \mathfrak{h}^*$  on  $\mathfrak{h}$  such that*

$$\forall H \in \mathfrak{h}, \quad \varphi(H)u = \lambda(H)u.$$

We call  $\lambda$  the weight of  $u$ .

If furthermore, we have  $\lambda(\alpha^\vee) \geq 0$  for every simple root  $\alpha$  of  $\mathfrak{g}$ , then we say that  $\lambda$  is a dominant weight.

Let  $\psi(\mathfrak{g}) = \{\alpha_1, \dots, \alpha_r\}$  be the set of simple roots of  $\mathfrak{g}$ . We then define  $\lambda_j(\alpha_i^\vee) := \delta_{i,j}$  for all  $1 \leq i, j \leq r$ . In particular, the  $\lambda_j$  are dominant weights. We call them fundamental weights (or simple weights).

**Fact 6.2.7.** *Each dominant weight  $\lambda$  has a unique decomposition in terms of the fundamental weights*

$$\lambda = n_1\lambda_1 + \dots + n_r\lambda_r \quad (n_1, \dots, n_r \in \mathbb{N}_0).$$

**Fact 6.2.8.** *There is a bijective correspondence between the set of irreducible finite-dimensional representations (up to isomorphism), and the set of dominant weights.*

**Theorem 6.2.9.** *[of Weyl]*

*Let  $\varphi$  be an irreducible representation of  $\mathfrak{g}$ . Then we have*

$$\dim(\varphi) = \prod_{\alpha \in \Delta^+(\mathfrak{g})} \frac{\langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle}{\langle \alpha^\vee, \lambda_1 + \dots + \lambda_r \rangle}.$$

**Remark 6.2.10.** *At first glance, the result above depends on the choice of the Cartan subalgebra  $\mathfrak{h}$ , and the choice of  $\gamma$ . However, according to [KMT24, §2.3], these are independent of these choices. The reason being that, according to [Hum72, §16.2], two Cartan subalgebras are necessarily conjugate.*

We note in passing that Fact 6.2.8 and Theorem 6.2.9 particularly allow us to determine an expression for  $r_{\mathfrak{g}}$ , the number of irreducible representations (up to isomorphism) of dimension  $n$  of a semi-simple Lie algebra  $\mathfrak{g}$ .

**Definition 6.2.11.** *[Zag94] We associate with a semi-simple Lie algebra  $\mathfrak{g}$  a zeta function*

$$\zeta_{\mathfrak{g}}(s) := \sum_{\varphi} \frac{1}{\dim(\varphi)^s},$$

where the sum is over the finite-dimensional irreducible representations  $\varphi$  of  $\mathfrak{g}$  (up to isomorphism).

This zeta function generalizes the series that appear in Witten's volume formulas in [Wit91].

From Fact 6.2.8 and Weyl's Theorem 6.2.9, it follows that Witten's zeta functions can be written as a series of products of linear forms in  $r$  variables,  $r$  being the number of fundamental roots. More precisely, we have the following expression:

$$\zeta_{\mathfrak{g}}(s) := \prod_{\alpha \in \Delta^+(\mathfrak{g})} \langle \alpha^\vee, \lambda_1 + \dots + \lambda_r \rangle^s \sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{\alpha \in \Delta^+(\mathfrak{g})} \langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle^s}.$$

Matsumoto, Tsumura, and Komori introduced in [KMT10b] multizeta analogues of Witten's zeta functions, by considering Dirichlet series of the form

$$\sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{\alpha \in \Delta^+(\mathfrak{g})} \langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle^{s_\alpha}}.$$

This allows for obtaining recursive formulas for these multizeta functions (see [KMT10b, Theorem 3.1]), and to derive several formulas for some positive integers

### 6.3 Some Results on $\zeta_{g_2}$ and $\zeta_{s_0(5)}$

Here, we aim to detail the formulas obtained in Theorem A and in Theorem D for the two Witten zeta functions  $\zeta_{g_2}$  and  $\zeta_{s_0(5)}$ , primarily focusing on the values at  $s = 0$ . The special values  $\zeta_{g_2}(0)$  and  $\zeta_{s_0(5)}(0)$  are very simple to compute using Corollary A1. On the other hand, the calculations for the values  $\zeta'_{g_2}(0)$  and  $\zeta'_{s_0(5)}(0)$  will prove to be more challenging. We propose to establish a formula using Corollary D2 when  $P = 2$ , that is, when the Dirichlet series describing  $Z(\mathbf{s}, \mathbf{s}')$  only involves two variables  $n_1, n_2$ . We furthermore fix the direction  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ , and set  $d_1 = d_2 = 1$ . Let's start by examining the values of  $Z_\Delta(s)$  at nonpositive integers using Corollary A2:

**Lemma 6.3.1.** *Let  $N \in \mathbb{N}_0$ . We have*

$$\begin{aligned} Z_\Delta(-N) &= \frac{(-1)^{(1+Q)N}}{Q} N! \sum_{k=0}^{QN} Q_N^0(\emptyset, (k, QN - k)) \frac{\zeta(-N - k)}{k!} \frac{\zeta(-(Q+1)N + k)}{(QN - k)!} \\ &\quad + \frac{(-1)^{QN+1}}{Q+1} N!^2 (Q_N^0(\{1\}, (Q+1)N + 1) + Q_N^0(\{2\}, (Q+1)N + 1)) \frac{\zeta(-(Q+2)N - 1)}{((Q+1)N + 1)!}, \end{aligned}$$

with

$$\begin{aligned} Q_N^0(\emptyset, (k_1, k_2)) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}_0^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q, w_{1,q} + w_{2,q} = N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}}, \quad (k_1 + k_2 = QN) \\ Q_N^0(\{1\}, (Q+1)N + 1) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}_0^Q \\ |\mathbf{w}_2|=(Q+1)N+1 \\ w_{2,j} \geq 2N+1}} \frac{(-N-1)\dots(N+1-w_{2,j})}{\prod_{q \neq j} w_{2,q}!} c_{j,2}^{w_{2,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_{2,q}} c_{q,2}^{w_{2,q}}, \\ Q_N^0(\{2\}, (Q+1)N + 1) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_1 \in \mathbb{N}_0^Q \\ |\mathbf{w}_1|=(Q+1)N+1 \\ w_{1,j} \geq 2N+1}} \frac{(-N-1)\dots(N+1-w_{1,j})}{\prod_{q \neq j} w_{1,q}!} c_{j,1}^{w_{1,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,2}^{N-w_{1,q}} c_{q,1}^{w_{1,q}}. \end{aligned}$$

*Proof.* By Corollary A2, we get

$$\begin{aligned} Z_\Delta(-N) &= \sum_{\mathcal{P} \subseteq [1,2]} \frac{(-1)^{(|\mathcal{P}|+1+Q)N+|\mathcal{P}|}}{Q+|\mathcal{P}|} N!^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=(Q+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p)}{k!}, \\ &= \frac{(-1)^{(1+Q)N}}{Q} N! \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 = QN}} Q_N^0(\emptyset, (k_1, k_2)) \prod_{p=1}^2 \frac{\zeta(-N - k_p)}{k_p!} \\ &\quad + \frac{(-1)^{QN+1}}{Q+1} N!^2 (Q_N^0(\{1\}, (Q+1)N + 1) + Q_N^0(\{2\}, (Q+1)N + 1)) \frac{\zeta(-(Q+2)N - 1)}{((Q+1)N + 1)!}, \end{aligned}$$

with

$$\begin{aligned} Q_N^0(\mathcal{P}, \mathbf{k}) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N}} \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right). \end{aligned}$$

- If  $\mathcal{P} = \emptyset$  and  $k_1 + k_2 = QN$ , then we have

$$Q_N^0(\emptyset, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}_0^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q \neq j, w_{1,q}+w_{2,q}=N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}}.$$

Let's observe that

$$\begin{aligned} w_{1,j} + w_{2,j} &= \sum_{q=1}^Q w_{1,q} + w_{2,q} - \sum_{\substack{q=1 \\ q \neq j}}^Q w_{1,q} + w_{2,q}, \\ &= k_1 + k_2 - (Q-1)N, \\ &= N. \end{aligned}$$

Let  $\mathbf{w} = (\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}_0^Q)^2$ . The condition  $|\mathbf{w}_1| = k_1$ ,  $|\mathbf{w}_2| = k_2$ , and  $w_{1,q} + w_{2,q} = N$  for all  $q \neq j$ , is equivalent to the condition  $w_{1,q} + w_{2,q} = N$  for all  $q$  and  $|\mathbf{w}_1| = k_1$ ,  $|\mathbf{w}_2| = k_2$ . We then find that

$$Q_N^0(\emptyset, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}_0^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q, w_{1,q}+w_{2,q}=N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}}.$$

- If  $\mathcal{P} = \{1\}$  and  $k_2 = (Q+1)N+1$ , then we have that

$$\begin{aligned} Q_N^0(\{1\}, (Q+1)N+1) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}_0^Q \\ |\mathbf{w}_2|=(Q+1)N+1 \\ \forall q \neq j, w_{2,q} \leq N}} \left( \binom{k_1}{(w_{2,1}, \dots, w_{2,Q})} \prod_{q=1}^Q c_{q,2}^{w_{2,q}} \right) \\ &\quad \left( \binom{-N-1}{\sum_{q \neq j} (N-w_{2,q})} \binom{\sum_{q \neq j} (N-w_{2,q})}{(N-w_{2,q})_{q \neq j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_{2,q}} \right). \end{aligned}$$

Let's note that

$$\begin{aligned} \sum_{\substack{q=1 \\ q \neq j}}^Q (N-w_{2,q}) &= (Q-1)N - |\mathbf{w}_2| + w_{2,j} \\ &= -2N - 1 + w_{2,j}. \end{aligned}$$

If this term is negative, then the binomial coefficient  $\binom{-N-1}{\sum_{q \neq j} (N-w_{2,q})}$  vanishes. Therefore we can assume the condition  $w_{2,j} \geq 2N+1$  instead:

$$Q_N^0(\{1\}, k_2) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}_0^Q \\ |\mathbf{w}_2|=(Q+1)N+1 \\ w_{2,j} \geq 2N+1}} \frac{(-N-1) \dots (N+1-w_{2,j})}{\prod_{q \neq j} w_{2,q}!} c_{j,2}^{w_{2,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_{2,q}} c_{q,2}^{w_{2,q}}.$$

- If  $\mathcal{P} = \{2\}$  and  $k_1 = (Q+1)N+1$ , therefore we get

$$Q_N^0(\{2\},) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_1 \in \mathbb{N}_0^Q \\ |\mathbf{w}_1|=(Q+1)N+1 \\ w_{1,j} \geq 2N+1}} \frac{(-N-1) \dots (N+1-w_{1,j})}{\prod_{q \neq j} w_{1,q}!} c_{j,1}^{w_{1,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,2}^{N-w_{1,q}} c_{q,1}^{w_{1,q}}.$$

□

Via the previous lemma, and by using the fact that  $\zeta(-n) = (-1)^{n+1} \frac{B_{n+1}}{n+1}$ , we have the simplified version of the Lemma, which is easier to compute:

$$Z_{\Delta}(-N) = \frac{(-1)^{QN}}{Q} N!^Q \sum_{k=0}^{QN} \alpha_k(N) \frac{B_{N+k+1}}{(N+k+1)k!} \frac{B_{(Q+1)N-k+1}}{((Q+1)N-k+1)(QN-k)!} \quad (6.2)$$

with

$$\alpha_k(N) = \sum_{j=1}^Q \sum_{\substack{(w_q)_{q \in [1, Q]} \in [0, N]^Q \\ w_1 + \dots + w_Q = k}} \binom{k}{(w_1, \dots, w_Q)} \binom{QN-k}{(N-w_1, \dots, N-w_Q)} \prod_{q=1}^Q c_{q,1}^{w_q} c_{q,2}^{N-w_q},$$

$$\beta(N) = \sum_{j=1}^Q \sum_{\substack{(w_q)_{q \in [1, Q]} \in [0, (Q+1)N+1]^Q \\ w_1 + \dots + w_Q = (Q+1)N+1 \\ w_j \geq 2N+1}} \frac{(-N-1) \dots (N+1-w_j)}{\prod_{q \neq j} w_q!} \left( c_{j,2}^{w_j} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_q} c_{q,2}^{w_q} + c_{j,1}^{w_j} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,2}^{N-w_q} c_{q,1}^{w_q} \right).$$

**Lemma 6.3.2.** *We have*

$$\begin{aligned} Z'_{\Delta}(0) = & \\ & Q\zeta(0)^2 \gamma - \zeta(-1) \left( \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \right) \gamma \\ & - \frac{1}{Q+1} \zeta(-1) \sum_{j=1}^Q \sum_{\substack{f=1 \\ f \neq j}}^Q \left( \frac{1}{c_{j,2}} (c_{f,1} - c_{j,1}) \ln \left( 1 + \frac{c_{f,2}}{c_{j,2}} \right) + \frac{1}{c_{j,1}} (c_{f,2} - c_{j,2}) \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right) \right) \\ & + \frac{\zeta'(0)\zeta(0)}{Q} - \frac{\zeta'(-1)}{Q+1} \sum_{j=1}^Q (c_{j,1} + c_{j,2}) - \gamma \zeta(0)^2 \sum_{j=1}^Q (c_{j,1} c_{j,2})^{-1} + \gamma \zeta(-1) \sum_{j=1}^Q (c_{j,1}^{-1} c_{j,2} + c_{j,2}^{-1} c_{j,1}) \\ & + \sum_{j=1}^Q \partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})|_{s=0}. \end{aligned}$$

*Proof.* Let's recall that  $Z'_{\Delta}(0) = Z'(-\mathbf{0}, \mathbf{0})_{\mathbf{1}, \mathbf{1}}$ . We set  $P = 2$  and we fixed the directions  $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ .

Therefore, by Corollary D2, we have:

$$\begin{aligned} Z'(-\mathbf{0}, \mathbf{0})_{\mathbf{1}, \mathbf{1}} = & \\ & \sum_{\mathcal{P} \subsetneq [1, 2]} (-1)^{|\mathcal{P}|} \frac{1}{Q + |\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p)}{k_p!} \sum_{j=1}^Q \left( \left( \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \gamma (1 + |\mathcal{P}|) + Q^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{\mathcal{P} \subsetneq [1, 2]} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{1}{Q + |\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \left( \sum_{i \in \mathcal{P}^c} \frac{\zeta'(-k_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p)}{k_p!} \right) \\ & + \sum_{j=1}^Q \partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})|_{s=0} - \sum_{j=1}^Q \gamma \sum_{\mathcal{P} \subsetneq [1, 2]} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-1} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-k_p)}{k_p!}, \end{aligned}$$

with

$$Q^1(j, \mathcal{P}, \mathbf{k}) = \sum_{\substack{f=1 \\ f \neq j}}^Q \sum_{\mathbf{w}' = (w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ + (Q-1)\gamma \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p},$$

where we recall that  $\mathbf{e}_n = (0, \dots, 1, \dots, 0) \in \mathbb{R}^Q$  is the vector with a 1 in the  $n$ -th component, and 0 elsewhere.

Developing the sum in the formula of  $Z'(-\mathbf{0}, \mathbf{0})$ , we get

$$Z'(-\mathbf{0}, \mathbf{0}) = \\ \zeta(0)^2 \gamma - \frac{2\zeta(-1)}{Q+1} \gamma \left( \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \right) + \frac{1}{Q} \zeta(0)^2 \sum_{j=1}^Q Q^1(j, \emptyset, (0, 0)) \\ - \frac{1}{Q+1} \zeta(-1) \sum_{j=1}^Q (Q^1(j, \{1\}, 1) + Q^1(j, \{2\}, 1)) + \frac{\zeta'(0)\zeta(0)}{Q} - \frac{\zeta'(-1)}{Q+1} \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \\ + \sum_{j=1}^Q \partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})|_{s=0} - \gamma \zeta(0)^2 \sum_{j=1}^Q (c_{j,1} c_{j,2})^{-1} + \gamma \zeta(-1) \sum_{j=1}^Q (c_{j,1}^{-1} c_{j,2} + c_{j,2}^{-1} c_{j,1}).$$

Let's now study the values of  $Q^1(j, \mathcal{P}, \mathbf{k})$ :

- If  $\mathcal{P} = \emptyset$  and  $\mathbf{k} = (0, 0)$ , we have

$$Q^1(j, \emptyset, (0, 0)) = \sum_{\substack{f=1 \\ f \neq j}}^Q F_{\emptyset, j, f, \mathbf{0}}(\mathbf{0}, \mathbf{0}) + (Q-1)\gamma.$$

By the Example 1.3.4, we have that  $F_{\emptyset, j, f, \mathbf{0}}(\mathbf{0}, \mathbf{0}) = 0$  therefore  $Q^1(j, \emptyset, (0, 0)) = (Q-1)\gamma$ .

- If  $\mathcal{P} = \{1\}$  and  $k_2 = 1$ , we have that

$$Q^1(j, \{1\}, 1) = (Q-1)\gamma c_{j,2} + \sum_{\substack{f=1 \\ f \neq j}}^Q \sum_{w'=0}^1 F_{\{1\}, j, f, \mathbf{0}, w' \mathbf{e}_f + (1-w') \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) \left( \binom{1}{w'} c_{f,2}^{w'} c_{j,2}^{1-w'} \right) \\ = (Q-1)\gamma c_{j,2} + \sum_{\substack{f=1 \\ f \neq j}}^Q (F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) c_{f,2} + F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) c_{j,2}).$$

We then find by Example 1.3.6 that

$$F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) = -\frac{1}{c_{j,1}} \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right),$$

and thanks to Example 1.3.5, we have that

$$F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) = \frac{1}{c_{f,1}} \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right),$$



therefore we obtain

$$Q^1(j, \{1\}, 1) = (Q-1)\gamma c_{j,2} + \sum_{\substack{f=1 \\ f \neq j}}^Q \left( \frac{c_{f,2}}{c_{f,1}} - \frac{c_{j,2}}{c_{j,1}} \right) \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right)$$

- If  $\mathcal{P} = \{1\}$  and  $k_1 = 1$ , mutatis mutandis, on get

$$Q^1(j, \{2\}, 1) = (Q-1)\gamma c_{j,1} + \sum_{\substack{f=1 \\ f \neq j}}^Q \left( \frac{c_{f,1}}{c_{f,2}} - \frac{c_{j,1}}{c_{j,2}} \right) \ln \left( 1 + \frac{c_{f,2}}{c_{j,2}} \right).$$

From the last three points, we get

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{1,1} = & \\ & Q\zeta(0)^2\gamma - \frac{2}{Q+1}\zeta(-1) \left( \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \right) \gamma - \frac{Q-1}{Q+1}\zeta(-1) \left( \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \right) \gamma \\ & - \frac{1}{Q+1}\zeta(-1) \sum_{j=1}^Q \sum_{\substack{f=1 \\ f \neq j}}^Q \left( \left( \frac{c_{f,1}}{c_{f,2}} - \frac{c_{j,1}}{c_{j,2}} \right) \ln \left( 1 + \frac{c_{f,2}}{c_{j,2}} \right) + \left( \frac{c_{f,2}}{c_{f,1}} - \frac{c_{j,2}}{c_{j,1}} \right) \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right) \right) \\ & + \frac{\zeta'(0)\zeta(0)}{Q} - \frac{\zeta'(-1)}{Q+1} \sum_{j=1}^Q (c_{j,1} + c_{j,2}) - \gamma\zeta(0)^2 \sum_{j=1}^Q (c_{j,1}c_{j,2})^{-1} + \gamma\zeta(-1) \sum_{j=1}^Q (c_{j,1}^{-1}c_{j,2} + c_{j,2}^{-1}c_{j,1}) \\ & + \sum_{j=1}^Q \partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})|_{s=0}. \end{aligned}$$

□

**Remark 6.3.3.** Under the assumptions of the previous lemma, by giving an integer  $Q \geq 1$  and the coefficients  $\mathbf{c} = (c_{q,p})_{q \in [1, Q], p \in [1, 2]}$ , the SAGE code 7.1 shall compute the following expression

$$\begin{aligned} & Q\zeta(0)^2\gamma - \frac{Q-1}{Q+1}\zeta(-1) \left( \sum_{j=1}^Q (c_{j,1} + c_{j,2}) \right) \gamma \\ & - \frac{1}{Q+1}\zeta(-1) \sum_{j=1}^Q \sum_{\substack{f=1 \\ f \neq j}}^Q \left( \left( \frac{c_{f,1}}{c_{f,2}} - \frac{c_{j,1}}{c_{j,2}} \right) \ln \left( 1 + \frac{c_{f,2}}{c_{j,2}} \right) + \left( \frac{c_{f,2}}{c_{f,1}} - \frac{c_{j,2}}{c_{j,1}} \right) \ln \left( 1 + \frac{c_{f,1}}{c_{j,1}} \right) \right) \\ & + \frac{\zeta'(0)\zeta(0)}{Q} - \frac{\zeta'(-1)}{Q+1} \sum_{j=1}^Q (c_{j,1} + c_{j,2}) - \gamma\zeta(0)^2 \sum_{j=1}^Q (c_{j,1}c_{j,2})^{-1} + \gamma\zeta(-1) \sum_{j=1}^Q (c_{j,1}^{-1}c_{j,2} + c_{j,2}^{-1}c_{j,1}). \end{aligned}$$

Let's note that the result of such computations are not rounded up, but give an expression using Euler's constant  $\gamma$ ,  $\zeta(0)$ ,  $\zeta(-1)$ ,  $\zeta'(0)$ ,  $\zeta'(-1)$  and some logarithm.

Even though we have a formula to compute at nonpositive integers of the derivative with respect to  $s$  of Barnes' zeta function  $\partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})|_{s=0}$  via Corollary B1, by setting  $N = 0$  and  $\mathbf{R} = \mathbf{0}$ , we will instead use Theorem 1.2.11 proved by Aoki and Sakane. The formula in this theorem provides more easily exploitable expressions. In particular, we will use the formulas obtained in Example 1.2.12 to simplify the expressions for  $\zeta'_{\mathfrak{g}_2}(0)$  and  $\zeta'_{\mathfrak{so}(5)}(0)$ . We will use the formulas (1.4), (1.3) and (1.2) in order to get

a more suitable expression of the derivative of the values at nonpositive integers of the derivative of the Barnes zeta function:

$$\begin{aligned}\zeta^{B'}(0, 2|1, 1) &= -\zeta'(0) + \zeta'(-1), \\ &= \frac{1}{2} \ln(2\pi) + \frac{1}{12} - \ln(A)\end{aligned}\tag{6.3}$$

$$\begin{aligned}\zeta^{B'}(0, 3|1, 2) &= -\frac{5}{8} \ln(2) - \frac{1}{4} \ln(\pi) - \frac{3}{2} \zeta'(0) + \frac{1}{2} \zeta'(-1), \\ &= \frac{1}{8} \ln(2) + \frac{1}{2} \ln(\pi) + \frac{1}{24} - \frac{\ln(A)}{2}\end{aligned}\tag{6.4}$$

$$\zeta^{B'}(0, 4|1, 3) = -\frac{19 \ln(3)}{36} - \zeta'(0) - \frac{2}{3} \zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3} \zeta'\left(0, \frac{4}{3}\right) + \zeta'(-1) + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right),\tag{6.5}$$

$$\begin{aligned}\zeta^{B'}(0, 5|2, 3) &= -\frac{31 \ln(6)}{72} - \frac{\ln(2)}{6} - \frac{1}{4} \ln(\pi) - \frac{3}{2} \zeta'(0) - \frac{2}{3} \zeta'\left(0, \frac{5}{3}\right) - \frac{1}{6} \zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6} \zeta'\left(0, \frac{5}{6}\right) \\ &\quad + \frac{1}{2} \zeta'(-1) + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) + \zeta'\left(-1, \frac{7}{6}\right) + \zeta'\left(-1, \frac{5}{6}\right).\end{aligned}\tag{6.6}$$

We hereby used the following facts:

$$\begin{aligned}\zeta'\left(0, \frac{3}{2}\right) &= \zeta'(0) - \ln(2) + \ln \Gamma\left(\frac{1}{2}\right), \\ &= \zeta'(0) + \ln(\sqrt{\pi}) - \ln(2),\end{aligned}$$

and

$$\begin{aligned}\zeta'\left(-1, \frac{3}{2}\right) &= \zeta'\left(-1, \frac{1}{2}\right) - \frac{1}{2} \ln(2) \\ &= -\frac{\ln(2)}{6} - \frac{1}{2} \zeta'(-1) - \frac{1}{2} \ln(2) \\ &= -\frac{2}{3} \ln(2) - \frac{1}{2} \zeta'(-1).\end{aligned}$$

### 6.3.1 Poles and residues of $\zeta_{\mathfrak{g}_2}$

Via [KMT11a], we've seen in 1.6 that  $\zeta_{\mathfrak{g}_2}$  has poles of the form  $s = \frac{1}{3}$  and  $s = \frac{1-k}{5}$ , with  $k \geq 0$  and  $k \not\equiv 1 \pmod{5}$ . We shall give an expression of the residues of  $\zeta_{\mathfrak{g}_2}$  at  $s = \frac{1}{5}$  and at  $s = \frac{1}{3}$ . We set

$$\begin{aligned}l_1(x_1, x_2) &= x_1 + x_2, \quad l_2(x_1, x_2) = x_1 + 2x_2, \quad l_3(x_1, x_2) = x_1 + 3x_2, \quad l_4(x_1, x_2) = 2x_1 + 3x_2 \\ d_1 &= 1, \quad d_2 = 1, \quad d'_1 = 2, \quad d'_2 = 3, \quad d'_3 = 4, \quad d'_4 = 5.\end{aligned}$$

Via Proposition 2.2.4, we find that

$$\zeta_{\mathfrak{g}_2}(s) = K(\theta, s) + J(\theta, s),$$

with

$$\begin{aligned}K(\theta, s) &= 120^{-s} \sum_{\emptyset \neq \mathcal{Q} \subset [1, 4]} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \frac{1}{\Gamma(s)^{4 - |A_{\mathcal{Q}}|}} \left( \sum_{\mathbf{n} \in \mathbb{N}_0^2} (n_1 + 1)^{-s} (n_2 + 1)^{-s} \right. \\ &\quad \left. \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n} + \mathbf{1}))^{-s} \prod_{q \in [1, 4] \setminus A_{\mathcal{Q}}} \Gamma(s, \theta, l_q(\mathbf{n} + \mathbf{1})) \right),\end{aligned}$$

and

$$\begin{aligned}
J(\theta, s) = & 120^s \sum_{\mathcal{P} \subsetneq [1,2]} \Gamma(1-s)^{|\mathcal{P}|} \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^4 h_{\mathcal{P},j,\mathbf{k}}(s) \right) \frac{\theta^{(4+|\mathcal{P}|)s-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(s)((4+|\mathcal{P}|)s-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(s-k_p)}{k_p!} \\
& + 120^s \Gamma(1-s)^2 \left( \sum_{j=1}^4 h_{[1,2],j}(s) \right) \frac{\theta^{6s-2}}{\Gamma(s)(6s-2)}, \tag{6.7}
\end{aligned}$$

with

$$\begin{aligned}
h_{\mathcal{P},1,\mathbf{k}}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_2 x_3 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(1, x_2, x_3, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(1, x_2, x_3, x_4)^{k_p} dx_2 dx_3 dx_4, \\
h_{\mathcal{P},2,\mathbf{k}}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_3 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, 1, x_3, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, 1, x_3, x_4)^{k_p} dx_1 dx_3 dx_4, \\
h_{\mathcal{P},3,\mathbf{k}}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, x_2, 1, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, x_2, 1, x_4)^{k_p} dx_1 dx_2 dx_4, \\
h_{\mathcal{P},4,\mathbf{k}}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_3)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, x_2, x_3, 1)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, x_2, x_3, 1)^{k_p} dx_1 dx_2 dx_3, \\
l_1^*(x_1, x_2) &= x_1 + x_2 + x_3 + 2x_4, \\
l_2^*(x_1, x_2) &= x_1 + 2x_2 + 3x_3 + 4x_4,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^4 h_{[1,2],j}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_3)^{s-1} \prod_{p=1}^2 l_p^*(x_1, x_2, x_3, 1)^{s-1} dx_1 dx_2 dx_3, \\
&+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_4)^{s-1} \prod_{p=1}^2 l_p^*(x_1, x_2, 1, x_4)^{s-1} dx_1 dx_2 dx_4 \\
&+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_3 x_4)^{s-1} \prod_{p=1}^2 l_p^*(x_1, 1, x_3, x_4)^{s-1} dx_1 dx_3 dx_4 \\
&+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_2 x_3 x_4)^{s-1} \prod_{p=1}^2 l_p^*(1, x_2, x_3, x_4)^{s-1} dx_2 dx_3 dx_4.
\end{aligned}$$

Let's recall that  $K(\theta, s)$  is holomorphic on  $\mathbb{C}$ , thus this term doesn't contribute to the residues. We just have to study the denominators of the terms inside of  $J(\theta, s)$ , in the formula 6.7.

- Let  $\mathcal{P} \subsetneq [1, 2]$ , and  $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$ . For  $s = \frac{1}{5}$ , we have

$$(4 + \mathcal{P})s - |\mathcal{P}| + |\mathbf{k}| = 0 \text{ if and only if } |\mathcal{P}| = 1, \mathbf{k} = 0.$$

We then find that

$$\begin{aligned}
\text{Res}_{s=\frac{1}{5}}(\zeta_{\mathfrak{g}_2}(s)) &= 120^{\frac{1}{5}} \frac{\Gamma(\frac{4}{5})}{\Gamma(\frac{1}{5})} \left( \sum_{j=1}^4 h_{\{1\},j,0} \left( \frac{1}{5} \right) + h_{\{2\},j,0} \left( \frac{1}{5} \right) \right) \zeta \left( \frac{1}{5} \right), \\
&= 120^{\frac{1}{5}} \frac{\Gamma(\frac{4}{5})}{\Gamma(\frac{1}{5})^4} \zeta \left( \frac{1}{5} \right) \left( \int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{4}{5}} (l_1^*(1, x_2, x_3, x_4)^{-\frac{4}{5}} + l_2^*(1, x_2, x_3, x_4)^{-\frac{4}{5}}) dx_2 dx_3 dx_4 \right. \\
&\quad + \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{4}{5}} (l_1^*(x_1, 1, x_3, x_4)^{-\frac{4}{5}} + l_2^*(x_1, 1, x_3, x_4)^{-\frac{4}{5}}) dx_1 dx_3 dx_4 \\
&\quad + \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{4}{5}} (l_1^*(x_1, x_2, 1, x_4)^{-\frac{4}{5}} + l_2^*(x_1, x_2, 1, x_4)^{-\frac{4}{5}}) dx_1 dx_2 dx_4 \\
&\quad \left. + \int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{4}{5}} (l_1^*(x_1, x_2, x_3, 1)^{-\frac{4}{5}} + l_2^*(x_1, x_2, x_3, 1)^{-\frac{4}{5}}) dx_1 dx_2 dx_3 \right).
\end{aligned}$$

- Via a similar argument, we get that

$$\begin{aligned}
\text{Res}_{s=\frac{1}{3}}(\zeta_{\mathfrak{g}_2}(s)) &= 120^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})^4} \left( \sum_{j=1}^4 h_{\{1,2\},j} \left( \frac{1}{3} \right) + h_{\{1,2\},j} \left( \frac{1}{5} \right) \right) \zeta \left( \frac{1}{3} \right), \\
&= 120^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})} \left( \int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, x_3, 1)^{-\frac{2}{3}} dx_1 dx_2 dx_3, \right. \\
&\quad + \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, 1, x_4)^{-\frac{2}{3}} dx_1 dx_2 dx_4 \\
&\quad + \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, 1, x_3, x_4)^{-\frac{2}{3}} dx_1 dx_3 dx_4 \\
&\quad \left. + \int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(1, x_2, x_3, x_4)^{-\frac{2}{3}} dx_2 dx_3 dx_4 \right).
\end{aligned}$$

### 6.3.2 Special values of $\zeta_{\mathfrak{g}_2}$ and of $\zeta'_{\mathfrak{g}_2}$

Let's take the same notations as in the previous sub-section:

$$\begin{aligned}
l_1(x_1, x_2) &= x_1 + x_2, \quad l_2(x_1, x_2) = x_1 + 2x_2, \quad l_3(x_1, x_2) = x_1 + 3x_2, \quad l_4(x_1, x_2) = 2x_1 + 3x_2 \\
d_1 &= 1, \quad d_2 = 1, \quad d'_1 = 2, \quad d'_2 = 3, \quad d'_3 = 4, \quad d'_4 = 5.
\end{aligned}$$

The zeta function  $\zeta_{\mathfrak{g}_2}$  is defined as

$$\zeta_{\mathfrak{g}_2}(s) = 120^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_2)^s (2n_1 + 3n_2)^s} = 120^s Z_{\Delta}(s)$$

Therefore, we have

$$\zeta_{\mathfrak{g}_2}(-N) = 120^{-N} Z(-((N, N), (N, N, N, N)), \quad \zeta'_{\mathfrak{g}_2}(-N) = 120^{-N} Z'(-((N, N), (N, N, N, N)) + \ln(120)\zeta_{\mathfrak{g}_2}(-N)).$$

(1,1),(1,1,1,1) (1,1),(1,1,1,1)

Via Corollary A1, we then get that

$$\zeta_{\mathfrak{g}_2}(0) = \sum_{\mathcal{P} \subseteq [1,2]} \frac{(-1)^{|\mathcal{P}|}}{4 + |\mathcal{P}|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \left( \sum_{j=1}^4 \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \frac{\zeta(-k_p)}{k_p!}.$$

1) If  $\mathcal{P} = \emptyset$  and  $|\mathbf{k}| = 0$ , then  $\mathbf{k} = (0, 0)$ , and

$$\sum_{j=1}^4 \prod_{p=1}^2 c_{j,p}^0 = 4.$$

2) If  $\mathcal{P} = \{1\}$  and  $|\mathbf{k}| = 1$ , then  $\mathbf{k} = k_2 = 1$ , and

$$\sum_{j=1}^4 c_{j,2}^1 = 9.$$

3) If  $\mathcal{P} = \{2\}$  and  $|\mathbf{k}| = 1$ , then  $\mathbf{k} = k_1 = 1$ , and

$$\sum_{j=1}^4 c_{j,1}^1 = 5.$$

Using the fact that  $\zeta(0) = -\frac{1}{2}$ , and that  $\zeta(-1) = -\frac{1}{12}$ , we get the value of  $\zeta_{\mathfrak{g}_2}$  at  $s = 0$ :

$$\begin{aligned} \zeta_{\mathfrak{g}_2}(0) &= \zeta(0)^2 - \frac{14}{5}\zeta(-1) \\ &= \frac{29}{60}. \end{aligned}$$

We also find thanks to Corollary A2 the following formula

$$\zeta_{\mathfrak{g}_2}(-N) = 120^{-N} \sum_{\mathcal{P} \subseteq [1,2]} \frac{(-1)^{(|\mathcal{P}|+5)N+|\mathcal{P}|}}{4+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=(4+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p)}{k_p!},$$

with

$$\begin{aligned} Q_N^0(\mathcal{P}, \mathbf{k}) &= (-1)^{3N} N!^3 \sum_{j=1}^4 \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^4)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1,4] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N}} \\ &\left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^4 c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^4 c_{q,p}^{v_{p,q}} \right). \end{aligned}$$

In order to compute  $\zeta'_{\mathfrak{g}_2}(0)$ , we see that

$$\zeta'_{\mathfrak{g}_2}(0) = \ln(120)\zeta_{\mathfrak{g}_2}(0) + Z'_{\Delta}(0).$$

By applying Lemma 6.3.2, we see by using the SAGE code 7.1 that

$$\begin{aligned} \zeta'_{\mathfrak{g}_2}(0) &= \ln(120)\zeta_{\mathfrak{g}_2}(0) + 2\gamma\zeta(0)^2 + \frac{1}{4}\zeta(0)\zeta'(0) - 4\gamma\zeta(-1) + \frac{1}{5}\zeta(-1)(\ln(3) + 2\ln(2)) - \frac{14}{5}\zeta'(-1) \\ &\quad + \zeta^{B'}(0, 2|1, 1) + \zeta^{B'}(0, 3|1, 2) + \zeta^{B'}(0, 4|1, 3) + \zeta^{B'}(0, 5|2, 3). \end{aligned}$$

By the formulas 6.3, 6.4, 6.5 and 6.6, we find that

$$\begin{aligned} & \zeta^{B'}(0, 2|1, 1) + \zeta^{B'}(0, 3|1, 2) + \zeta^{B'}(0, 4|1, 3) + \zeta^{B'}(0, 5|2, 3) = \\ & -\zeta'(0) + \zeta'(-1) - \frac{5}{8}\ln(2) - \frac{1}{4}\ln(\pi) - \frac{3}{2}\zeta'(0) + \frac{1}{2}\zeta'(-1) \\ & - \frac{19\ln(3)}{36} - \zeta'(0) - \frac{2}{3}\zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3}\zeta'\left(0, \frac{4}{3}\right) + \zeta'(-1) + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) \\ & - \frac{31\ln(6)}{72} - \frac{\ln(2)}{6} - \frac{1}{4}\ln(\pi) - \frac{3}{2}\zeta'(0) - \frac{2}{3}\zeta'\left(0, \frac{5}{3}\right) - \frac{1}{6}\zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6}\zeta'\left(0, \frac{5}{6}\right) \\ & + \frac{1}{2}\zeta'(-1) + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) + \zeta'\left(-1, \frac{7}{6}\right) + \zeta'\left(-1, \frac{5}{6}\right). \end{aligned}$$

Simplifying those terms, we get:

$$\begin{aligned} \zeta'_{\mathfrak{g}_2}(0) &= \ln(120)\zeta_{\mathfrak{g}_2}(0) + 2\gamma\zeta(0)^2 + \frac{1}{4}\zeta(0)\zeta'(0) - 4\gamma\zeta(-1) + \frac{1}{5}\zeta(-1)(\ln(3) + 2\ln(2)) - \frac{14}{5}\zeta'(-1) \\ & - \frac{11}{9}\ln(2) - \frac{23}{24}\ln(3) - \frac{1}{2}\ln(\pi) - 5\zeta'(0) + 3\zeta'(-1) - \frac{4}{3}\zeta'\left(0, \frac{5}{3}\right) - \frac{1}{3}\zeta'\left(0, \frac{4}{3}\right) \\ & + \zeta'\left(-1, \frac{5}{3}\right) + \zeta'\left(-1, \frac{4}{3}\right) - \frac{1}{6}\zeta'\left(0, \frac{7}{6}\right) + \frac{1}{6}\zeta'\left(0, \frac{5}{6}\right). \end{aligned}$$

### 6.3.3 Poles and residues of $\zeta_{\mathfrak{so}(5)}$

Via [KMT10d], we saw in 1.5 that  $\zeta_{\mathfrak{so}(5)}$  has poles of the form  $s = \frac{1}{2}$  and  $s = \frac{1-k}{3}$ , for  $k \geq 0$  an arbitrary integer such that  $k \not\equiv 1 \pmod{3}$ . We aim here to obtain expressions for the residues at  $s = \frac{1}{3}$  and  $s = \frac{1}{2}$  for  $\zeta_{\mathfrak{so}(5)}(s)$ . We note that Bridges, Brindle, Bringmann, and Franke have already obtained a very simple expression for these two residues in [BBBF23, Proposition 5.16] using different techniques.

We know that, for  $\theta > 0$  sufficiently small, the zeta function  $\zeta_{\mathfrak{so}(5)}$  admits the decomposition  $\zeta_{\mathfrak{so}(5)}(s) = K(\theta, s) + J(\theta, s)$ , where  $K(\theta, s)$  is a holomorphic function in  $s$  over all  $\mathbb{C}$ , and  $J(\theta, s)$  is a meromorphic function in  $s$  over all  $\mathbb{C}$ , having the same poles as  $\zeta_{\mathfrak{so}(5)}(s)$ . By Proposition 2.2.4, we have

$$\begin{aligned} J(\theta, s) &= 6^s \sum_{\mathcal{P} \subsetneq [1, 2]} \Gamma(1-s)^{|\mathcal{P}|} \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^2 h_{\mathcal{P}, j, \mathbf{k}}(s) \right) \frac{\theta^{(2+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(s)((2+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(s - k_p)}{k_p!} \\ &+ 6^s \Gamma(1-s)^2 \left( \sum_{j=1}^2 h_{[1, 2], j, \mathbf{0}}(s) \right) \frac{\theta^{4s-2}}{\Gamma(s)(4s-2)}, \end{aligned} \quad (6.8)$$

with

$$\begin{aligned} h_{\mathcal{P}, 1, \mathbf{k}}(s) &= \frac{1}{\Gamma(s)} \int_0^1 x_2^{s-1} \prod_{p \in \mathcal{P}} l_p^*(1, x_2)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(1, x_2)^{k_p} dx_2, \\ h_{\mathcal{P}, 2, \mathbf{k}}(s) &= \frac{1}{\Gamma(s)} \int_0^1 x_1^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, 1)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, 1)^{k_p} dx_1, \\ l_1^*(x_1, x_2) &= x_1 + x_2, \\ l_2^*(x_1, x_2) &= x_1 + 2x_2. \end{aligned}$$

As  $K(\theta, s)$  is holomorphic in  $s$  on all  $\mathbb{C}$ , we can get the residues of  $\zeta_{\mathfrak{so}(5)}(s)$  by studying the expression of  $J(\theta, s)$  above. At  $s = 1/2$ , the only contributing term of the residue of  $\zeta_{\mathfrak{so}(5)}(s)$  is

$$\Gamma(1-s)^2 \left( \sum_{j=1}^2 h_{[1, 2], j, \mathbf{0}}(s) \right) \frac{\theta^{4s-2}}{\Gamma(s)(4s-2)}.$$

We then get the following residue at  $s = 1/2$ :

$$\begin{aligned} \text{Res}_{s=1/2}(\zeta_{\mathfrak{so}(5)}(s)) &= \sqrt{6} \frac{\Gamma(3/2)^2}{4\Gamma(1/2)} \left( \sum_{j=1}^2 h_{[1,2],j,\mathbf{0}}(1/2) \right), \\ &= \sqrt{6} \frac{\Gamma(3/2)^2}{4\Gamma(1/2)} \frac{1}{\Gamma(1/2)} \left( \int_0^1 x^{-1/2}(x+1)^{-1/2}(x+2)^{-1/2} dx + \int_0^1 x^{-1/2}(1+x)^{-1/2}(1+2x)^{-1/2} dx \right) \\ &= \sqrt{6} \frac{\Gamma(3/2)^2}{4\Gamma(1/2)} \frac{1}{\Gamma(1/2)} \left( \int_0^1 x^{-1/2}(x+1)^{-1/2}(x+2)^{-1/2} dx + \int_0^1 x^{-1/2}(1+x)^{-1/2}(1+2x)^{-1/2} dx \right). \end{aligned}$$

For the poles of the form  $\frac{n}{3}$ , with  $n \leq 1$  being an integer such that  $n \not\equiv 0 \pmod{3}$ , the only contributing term to the residues at  $s = \frac{n}{3}$  are of the form

$$-\Gamma(1-n/3)^{|\mathcal{P}|} (-1)^{|\mathbf{k}|} \left( \sum_{j=1}^2 h_{\mathcal{P},j,\mathbf{k}}(1-n/3) \right) \frac{1}{\Gamma(n/3)((2+|\mathcal{P}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(n/3 - k_p)}{k_p!},$$

with  $\mathcal{P} \subsetneq [1, 2]$ ,  $\mathbf{k} \in \mathbf{N}^{\mathcal{P}^c}$  such that  $(2+|\mathcal{P}|)\frac{n}{3} - |\mathcal{P}| + |\mathbf{k}| = 0$ . This condition implies that  $\mathcal{P} = \{i\}$  is a singleton. We set  $\mathcal{P}^c = \{p\}$ , we then have  $|k| = k_p = 1 - n \in \mathbb{N}_0$ . We found

$$\text{Res}_{s=n/3}(\zeta_{\mathfrak{so}(5)}(s)) = (-1)^{1-n} \Gamma\left(\frac{3-n}{3}\right) \left( \sum_{i=1}^2 \sum_{j=1}^2 h_{\{i\},j,\mathbf{k}}\left(1-\frac{n}{3}\right) \right) \frac{1}{3\Gamma\left(\frac{n}{3}\right)} \frac{\zeta\left(\frac{4n}{3}-1\right)}{(1-n)!},$$

Let's note that, when considering the general terms of the series present in 6.8 with  $\mathcal{P} = \emptyset$ , we observe terms of the form  $\frac{\theta^{2s+|\mathbf{k}|}}{\Gamma(s)(2s+|\mathbf{k}|)}$ , where  $\mathbf{k} = (k_1, k_2) \in \mathbb{N}_0$ . The zeros of  $2s + |\mathbf{k}|$  do not constitute poles since these poles are canceled by the zeros of the terms of the form  $\sum_{j=1}^2 h_{\emptyset,j,\mathbf{k}}$ . For example, at  $s = -1/2$ , we find that the term  $2s + |\mathbf{k}|$  is zero if and only if  $|\mathbf{k}| = k_1 + k_2 = 1$ , that is, when  $\mathbf{k} \in \{(1, 0), (0, 1)\}$ . However, we note that

$$\begin{aligned} h_{1,\emptyset,(1,0)}(s) + h_{2,\emptyset,(1,0)}(s) + h_{1,\emptyset,(0,1)}(s) + h_{2,\emptyset,(0,1)}(s) &= \frac{1}{\Gamma(s)} \int_0^1 (x^{s-1}(1+2x) + 2x^{s-1}(1+x) + x^{s-1}(2+x)) dx \\ &= \frac{5}{\Gamma(s)} \left( \frac{1}{s+1} + \frac{1}{s} \right), \end{aligned}$$

and at  $s = -\frac{1}{2}$ , this term vanish. This matches the list of poles that we obtained from Komori, Matsumoto and Tsumura in [KMT10d, Theorem 6.2].

### 6.3.4 Special values of $\zeta_{\mathfrak{so}(5)}$ and $\zeta'_{\mathfrak{so}(5)}$

Let

$$\begin{aligned} l_1(x_1, x_2) &= x_1 + x_2, \quad l_2(x_1, x_2) = x_1 + 2x_2, \\ d_1 &= 1, \quad d_2 = 1, \quad d'_1 = 2, \quad d'_2 = 3, \end{aligned}$$

then  $\zeta_{\mathfrak{so}(5)}(s)$  is of the form

$$\zeta_{\mathfrak{so}(5)}(s) = 6^s \sum_{\mathbf{n} \in \mathbb{N}_0^2} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s} = 6^s Z_{\Delta}(s)$$

we then get that

$$\zeta_{\mathfrak{so}(5)}(-N) = 6^{-N} Z(-((N, N), (N, N))), \quad \zeta'_{\mathfrak{g}_2}(-N) = 120^{-N} Z'(-((N, N), (N, N)) + \ln(6)\zeta_{\mathfrak{g}_2}(-N)).$$

(1,1),(1,1) \qquad (1,1),(1,1)

At  $N = 0$ , we find by Corollary A1:

$$\begin{aligned}\zeta_{\mathfrak{so}(5)}(0) &= \sum_{\mathcal{P} \subsetneq [1,2]} \frac{(-1)^{|\mathcal{P}|}}{2 + |\mathcal{P}|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathcal{P}|=|\mathbf{k}|}} \left( \sum_{j=1}^2 \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p)}{k_p!} \\ &= \zeta(0)^2 - \frac{5}{3} \zeta(-1) \\ &= \frac{7}{18}.\end{aligned}$$

We can also have an expression of the values at nonpositive integers using Corollary A2,

$$\zeta_{\mathfrak{so}(5)}(-N) = 6^{-N} \sum_{\mathcal{P} \subsetneq [1,2]} \frac{(-1)^{(|\mathcal{P}|+3)N+|\mathcal{P}|}}{3 + |\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=(3+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p)}{k_p!},$$

with

$$\begin{aligned}Q_N^0(\mathcal{P}, \mathbf{k}) &= N! \sum_{j=1}^2 \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^2)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1,2] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, \mathbf{v}(q) + \mathbf{w}(q) = N}} \\ &\quad \left( \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^2 c_{q,p}^{w_{p,q}} \right) \left( \prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^2 c_{q,p}^{v_{p,q}} \right).\end{aligned}$$

Then by Lemma 6.3.2, using SAGE code 7.1 we have:

$$\begin{aligned}\zeta'_{\mathfrak{so}(5)}(0) &= \ln(6) \zeta_{\mathfrak{so}(5)}(0) + \frac{1}{2} \gamma \zeta(0)^2 + \frac{1}{2} \zeta(0) \zeta'(0) - \frac{1}{2} \gamma \zeta(-1) - \frac{5}{3} \zeta'(-1) + \frac{1}{6} \ln(2) \\ &\quad + \partial_s \zeta^B(s, 2|1, 1)|_{s=0} + \partial_s \zeta^B(s, 3|1, 2)|_{s=0},\end{aligned}$$

and with the formulas (6.3) and (6.4), we get:

$$\begin{aligned}\zeta'_{\mathfrak{so}(5)}(0) &= \ln(6) \zeta_{\mathfrak{so}(5)}(0) + \frac{1}{2} \gamma \zeta(0)^2 + \frac{1}{2} \zeta(0) \zeta'(0) - \frac{1}{2} \gamma \zeta(-1) - \frac{5}{3} \zeta'(-1) + \frac{1}{6} \ln(2) \\ &\quad - \zeta'(0) + \zeta'(-1) - \frac{5}{8} \ln(2) - \frac{1}{4} \ln(\pi) - \frac{3}{2} \zeta'(0) + \frac{1}{2} \zeta'(-1).\end{aligned}$$

By injecting these last two formulas into the one obtained for  $\zeta'_{\mathfrak{so}(5)}(0)$ , we get:

$$\zeta'_{\mathfrak{so}(5)}(0) = \frac{7}{18} \ln(6) - \frac{11}{24} \ln(2) - \frac{1}{4} \ln(\pi) + \left( \frac{1}{2} \zeta(0)^2 - \frac{1}{2} \zeta(-1) \right) \gamma - \frac{11}{4} \zeta'(0) - \frac{13}{6} \zeta'(-1).$$

This last expression allows us to have an explicit expression of the constant  $C$  in Theorem 1.2.35, which allow us to get an explicit asymptotic formula for the number of representation  $r_{\mathfrak{so}(5)}$ .

## 6.4 Application to the study of the asymptotic behavior of $r_{\mathfrak{g}_2}(n)$

Let's recall that the number of representations of dimension  $n$  of the exceptional Lie algebra  $\mathfrak{g}_2$  is

$$r_{\mathfrak{g}_2}(n) = \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|.$$



We set  $P(i, j) := \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!}$ . We have that

$$\prod_{i,j \geq 1} \frac{1}{1 - q^{P(i,j)}} = \prod_{i,j \geq 1} \sum_{k_{i,j} \geq 1} q^{k_{i,j} P(i,j)},$$

Developing the infinite product, and using the expression of the set  $r_{\mathfrak{g}_2}(n)$ , we get:

$$\sum_{n=1}^{+\infty} r_{\mathfrak{g}_2}(n) q^n = \prod_{i,j \geq 1} \frac{1}{1 - q^{P(i,j)}}.$$

We set  $f(n) = |\{(i, j) \in \mathbb{N}^2 | P(i, j) = n\}|$ , we then get that

$$\sum_{n=1}^{+\infty} r_{\mathfrak{g}_2}(n) q^n = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f(n)}}.$$

Let's define  $L_f(s) = \zeta_{\mathfrak{g}_2}(s)$ , and  $L_f^*(s) = \zeta_{\mathfrak{g}_2}(s) \Gamma(s) \zeta(s+1)$ . We see that  $L_f^*(s)$  has only two poles in  $H_0$ , at  $s = \frac{1}{3}$  and  $s = \frac{1}{5}$ . Moreover,  $L_f^*(s)$  has a double pole at  $s = 0$ . We also note that, thanks to the list of poles obtained in 1.6,  $L_f^*$  has simple poles at  $s = \frac{1}{3}$  and  $s = \frac{k}{5}$ ,  $k \in \mathbb{Z}_{\leq 1}$ , with  $k \not\equiv 0 \pmod{5}$ .

We want to apply Theorem 1.2.33. First, let's verify if, with the data above, the conditions of the theorem are valid. Let  $\Lambda := \mathbb{N}_0 \setminus f^{-1}(\{0\})$ . We note that  $\frac{3}{2} \cdot \frac{1}{5} < \frac{1}{3} < \frac{2}{1} \cdot \frac{1}{5}$ . Thus, the integer  $l = 2$  satisfies the hypothesis of Theorem 1.2.33. Now, let's check whether the conditions (P1), (P2), and (P3) are met:

(P1) Let  $p \geq 2$  be a prime number. We have that  $|\Lambda \setminus (p\mathbb{N}_0 \cap \Lambda)| = +\infty$ . Indeed, the sequences

$$\begin{aligned} u(2) &:= (P(8k+1, 1))_{k \in \mathbb{N}_0}, \\ u(3) &:= (P(9k+1, 1))_{k \in \mathbb{N}_0}, \\ u(5) &:= (P(25k+1, 1))_{k \in \mathbb{N}_0}, \\ u(p) &:= (P(kp+1, 1))_{k \geq 1} \quad \text{if } p \geq 7. \end{aligned}$$

are strictly increasing, and belong to  $\Lambda$  by definition. Moreover, we observe that the sequence  $u(p)$  belongs in  $\Lambda \setminus (p\mathbb{N}_0 \cap \Lambda)$ . We then conclude that any real number  $L \geq \frac{1}{6}$  satisfies condition (P1).

(P2) We have already seen previously that  $L_f^*$  has two poles (who are simple) at  $s = \frac{1}{3}$  and  $s = \frac{1}{5}$ , a double pole at  $s = 0$ , and simple poles at  $s = \frac{k}{5}$ ,  $k \in \mathbb{Z}_{\leq 1}$ , with  $k \not\equiv 0 \pmod{5}$ . Therefore, for any real  $R \in \mathbb{R}_+^*$  such that  $R \neq \frac{k}{5}$  with  $k \not\equiv 0 \pmod{5}$ ,  $R$  satisfies the condition of the second hypothesis.

(P3) We note that the polynomial  $P$  satisfies the HOS condition of the article [Ess97]. In particular, we know from [Ess97, Theorem 3] that the zeta function associated with this polynomial, (i.e.,  $\zeta_{\mathfrak{g}_2}$ ) admits a polynomial bound in the imaginary part of  $s = \sigma + i\tau$  on each vertical strip  $\sigma_1 \leq \sigma \leq \sigma_2$ . We then conclude that condition (P3) is met.

**Remark 6.4.1.** *In condition (P2),  $R$  is arbitrarily large, and in condition (P1),  $L$  is also arbitrarily large.*

Let's note  $\alpha = \frac{1}{3}$ ,  $\beta = \frac{1}{5}$ ,  $\omega_\alpha := \text{Res}_{s=\frac{1}{3}} \zeta_{\mathfrak{g}_2}(s)$  and  $\omega_\beta := \text{Res}_{s=\frac{1}{5}} \zeta_{\mathfrak{g}_2}(s)$ . Applying Theorem 1.2.33, we get that

$$r_{\mathfrak{g}_2}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp\left(A_1 n^{\frac{1}{4}} + A_2 n^{\frac{3}{20}} + A_3 n^{\frac{1}{20}}\right) \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R}\left(n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}}\right)\right),$$

with

$$C = \frac{e^{\zeta'_{\mathfrak{g}_2}(0)} (\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{1-6\zeta_{\mathfrak{g}_2}(0)}{8}} \sqrt{3}}{\sqrt{8\pi}}, \quad b = \frac{6 - 6\zeta_{\mathfrak{g}_2}(0) + 1}{8},$$

and  $A_1 := 4 \left( \omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{3}{4}}$ ,  $A_2 := \frac{\omega_\beta \Gamma\left(\frac{1}{5}\right) \zeta\left(\frac{6}{5}\right)}{(\omega_\alpha \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}}$ , and

$$A_3 := K_3 + 3 \left( \omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{3}{4}} \left( \binom{-1/3}{1} \frac{K_3}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{4}}} + \binom{-1/3}{2} \frac{K_2^2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{2}}} \right) \\ + \frac{5 (\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right))}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}} \binom{-1/5}{1} \frac{K_2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{4}}},$$

with  $0 < \nu_2 < \dots$  being the positive elements of the set  $\mathcal{N} + \mathcal{M}$ .

By [BBBF23, Lemma 4.3], we get

$$K_2 = \frac{3}{4} \cdot \frac{\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}}$$

$$K_3 = -\frac{3}{160} \cdot \frac{(\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right))^2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{21}{20}}}.$$

From the definition of the sets (1.8), (1.9), and (1.10) in [BBBF23], we found that the sequence  $\nu_j$  in Theorem 1.2.33 give, with our assumptions,  $\nu_2 = \frac{1}{20}, \nu_3 = \frac{2}{20}, \nu_4 = \frac{3}{20}, \dots$

# Chapter 7

## Appendix

### 7.1 SAGE codes

Listing 7.1: SAGE Code to compute  $Z'_{\Delta}(0)$  with  $P = 2$  and  $d_1 = d_2 = 1$

```
from sage.all import *

c_g2 = [[1,1],[1,2],[1,3],[2,3]]
c_so5 = [[1,1],[1,2]]

gamma = euler_gamma
zeta0 = var('zeta0')
zeta_m1 = var('zeta_m1')
zeta0_prime = var('zeta0_prime')
zeta_m1_prime = var('zeta_m1_prime')

def Z_delta_prime_zero(c_matrix):
    Q=len(c_matrix)
    sum1 = sum(c[j][0] + c[j][1] for j in range(Q))
    sum2 = sum(sum((c[f][0]/c[f][1] - c[j][0]/c[j][1])
        * log(1 + c[f][1]/c[j][1])+(c[f][1]/c[f][0] - c[j][1]/c[j][0])
        * log(1 + c[f][0]/c[j][0])
    for f in range(Q) if f != j) for j in range(Q))
    sum3 = sum((c[j][0] * c[j][1])^-1 for j in range(Q))
    sum4 = sum(c[j][0]^-1 * c[j][1] + c[j][1]^-1 * c[j][0] for j in range(Q))
    result = Q*zeta0^2*gamma - zeta_m1*sum1*gamma - 1/(Q+1)*zeta_m1*sum2
        +zeta0_prime*zeta0/Q - zeta_m1_prime/(Q+1)*sum1 - gamma*zeta0^2*sum3
        + gamma*zeta_m1*sum4
    return(result)
print(result)
```

Listing 7.2: SAGE Code to compute  $Z_{\Delta}(-N)$  with  $P = 2$

```
from math import comb
from itertools import product

def alpha_k_N(Q, N, k, c_matrix):
    alpha = 0
    for j in range(1, Q + 1):
        for w in product(range(N + 1), repeat=Q):
```

```

    if sum(w) == k:
        coef1 = multinomial(w)
        coef2 = multinomial([N - w_i for w_i in w])
        prod = 1
        for q in range(Q):
            prod *= (c_matrix[q][0] ** w[q]) * (c_matrix[q][1] ** (N - w[q]))
        alpha += coef1 * coef2 * prod
return alpha

def beta_N(Q, N, c_matrix):
    beta = 0
    for j in range(1, Q + 1):
        for w in product(range((Q + 1) * N + 2), repeat=Q):
            if sum(w) == (Q + 1) * N + 1 and w[j - 1] >= 2 * N + 1:
                numerator = 1
                for i in range(N + 1 - w[j - 1], -N - 1):
                    numerator *= i

                denominator = 1
                for q in range(Q):
                    if q != j - 1:
                        denominator *= factorial(w[q])

                term1 = c_matrix[j - 1][1] ** w[j - 1]
                term2 = c_matrix[j - 1][0] ** w[j - 1]

                for q in range(Q):
                    if q != j - 1:
                        term1 *= (c_matrix[q][0] ** (N - w[q]))
                        term1 *= (c_matrix[q][1] ** w[q])
                        term2 *= (c_matrix[q][1] ** (N - w[q]))
                        term2 *= (c_matrix[q][0] ** w[q])

                beta += (numerator / denominator) * (term1 + term2)
    return beta

def Z_Delta_minus_N(N, c_matrix):
    Q = len(c_matrix)
    sum1 = 0
    N_fact_Q = factorial(N) ** Q
    for k in range(Q * N + 1):
        alpha = alpha_k_N(Q, N, k, c_matrix)
        term1 = zeta(-N - k) / factorial(k)
        term2 = zeta(-(Q + 1) * N + k) / factorial(Q * N - k)
        sum1 += alpha * term1 * term2

    beta = beta_N(Q, N, c_matrix)
    sum2 = ((-1) ** (N + 1)) * (N_fact_Q * factorial(N)) * beta
        * zeta(-(Q + 2) * N - 1) / factorial((Q + 1) * N + 1)

    Z = (1 / Q) * N_fact_Q * sum1 + (1 / (Q + 1)) * sum2

    return Z

```

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