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type Hurwitz généralisées et de leurs dérivées.
Applications à certaines fonctions zêta de Witten
rattachées à des algèbres de Lie semi-simples

Simon Rutard

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**THÈSE de DOCTORAT DE L'UNIVERSITÉ JEAN MONNET
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**Ecole Doctorale N°488
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**Valeurs aux entiers négatifs de fonctions multizêta de type
Hurwitz généralisées et de leurs dérivées. Applications à
certaines fonctions zêta de Witten rattachées à des algèbres de
Lie semi-simples**

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Valeurs aux entiers négatifs de fonctions multizêta de type Hurwitz généralisées et de leurs dérivées. Applications à certaines fonctions zêta de Witten rattachées à des algèbres de Lie semi-simples

Résumé

Le but de cette thèse est de fournir des formules explicites pour les valeurs directionnelles et les valeurs des dérivées directionnelles de fonctions multizêta de Hurwitz généralisées en des multi-entiers négatifs. On exprimera en particulier les valeurs directionnelles de ces fonctions multizêta en fonction de polynômes de Bernoulli. De plus, on exprimera les valeurs des dérivées directionnelles de ces fonctions multizêta en fonction de polynômes de Bernoulli, de valeurs aux entiers négatifs de la dérivée suivant la variable s de la fonction zêta d'Hurwitz, et de valeurs aux entiers négatifs de la dérivée de certaines fonctions zêta de type Barnes généralisé.

Comme première application, on déterminera des formules pour les valeurs en $s = 0$ de la fonction zêta de Witten $\zeta_{\mathfrak{g}_2}$ et de sa dérivée, ainsi que des valeurs en $s = 0$ de la fonction zêta de Witten $\zeta_{s_0(5)}$ et de sa dérivée. On trouvera en particulier une formule asymptotique explicite du nombre de représentations de l'algèbre de Lie \mathfrak{g}_2 en combinant nos résultats avec un théorème de type Meinardus généralisé obtenu par Bridges, Brindle, Bringmann et Franke.

Comme seconde application, on déterminera des formules explicites des fonctions multigamma de type Shintani via des fonctions multigamma de type Barnes.

Mots-clés : Fonctions multizêta de type Hurwitz généralisées, Fonctions zêta de Witten, Valeurs et valeurs des dérivées aux entiers négatifs des fonctions multizêta, Représentations des algèbres de Lie semi-simples

Special values of generalized multiple Hurwitz zeta functions and their derivatives. Applications towards some Witten zeta functions attached to semi-simple Lie algebras

Abstract

The goal of this thesis is to provide explicit formulas for directional values and directional derivative values of generalized Hurwitz multizeta functions at non positive integers. We will express the directional values of these multizeta functions in terms of Bernoulli polynomials. Furthermore, we will express the directional derivative values of these multizeta functions in terms of Bernoulli polynomials, of values at nonpositive integer values of the derivative with respect to the variable s of the Hurwitz zeta function, and of values at nonpositive integers of the derivative of some generalized Barnes zeta functions.

As a first application, we will determine formulas for the values at $s = 0$ of the Witten zeta function $\zeta_{\mathfrak{g}_2}$ and its derivative, as well as values at $s = 0$ of the Witten zeta function $\zeta_{s_0(5)}$ and its derivative. In particular, we will find an explicit asymptotic formula for the number of representations of the exceptional Lie algebra \mathfrak{g}_2 by combining our result with a generalized Meinardus-type theorem proved by Bridges, Brindle, Bringmann, and Franke.

As a second application, we will determine explicit formulas for Shintani's multigamma functions in terms of Barnes' multigamma functions.

Keywords: Generalized multiple zeta functions of Hurwitz type, Witten zeta functions, Values and derivative values at nonpositive integers of multiple zeta functions, Representations of semi-simple Lie algebra

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Thèse de doctorat en Mathématiques
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12 Janvier 2024

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Notations

Notation. Soit A, B des ensembles. On notera l'inclusion large $A \subset B$ (ce que certains auteurs notent $A \subseteq B$), et l'inclusion stricte $A \subsetneq B$.

Notation. Soit A un ensemble, et $B \subset A$ un sous-ensemble de A , on note $B^c = A \setminus B$ le complémentaire de B dans A , et on note $|A| \in \mathbb{N} \cup \{+\infty\}$ le cardinal de A .

Notation. Soit A, B des ensembles, et $(c_{a,b})_{a,b \in A \times B}$ des nombres complexes. On notera

$$\sum_{a \in A} \sum_{b \in B} c_{a,b} := \sum_{a \in A} \left(\sum_{b \in B} c_{a,b} \right).$$

Notation. Soit A un ensemble, on note δ_a^b le symbole de Kronecker, valant 1 si $a = b$, et 0 sinon. Soit $(c_a)_{a \in A}$ des nombres complexes, on considère $P : A \rightarrow \{\text{Vrai}, \text{Faux}\}$ une propriété de A . On note alors

$$\sum_{\substack{a \in A \\ P(a)}} c_a := \sum_{a \in A} c_a \delta_{\text{Vrai}}^{P(a)},$$

c'est-à-dire la somme des termes c_a avec $a \in A$ et tel que $P(a)$ soit vraie.

Notation. On utilisera les lettres en gras pour désigner les multi-indices de la forme $\mathbf{x} = (x_a)_{a \in A} \in \mathbb{C}^A$ avec A un ensemble fini. Soit $B \subset A$, on notera $|\mathbf{x}|_B := \sum_{b \in B} x_b$. Si $B = A$, on notera directement

$|\mathbf{x}| := |\mathbf{x}|_A$. Notons qu'il ne faut pas confondre cette notation avec celle du module d'un nombre complexe.

Notation. Soit $1 \leq k \leq n$ des entiers. On notera $\mathbf{e}_k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{C}^n$ le vecteur avec un 1 à la k -ième composante, et des 0 ailleurs.

Notation. Soit A et B des ensembles finis, et $\mathbf{x} = (\mathbf{x}_a)_{a \in A} \in (\mathbb{N}^B)^A$, avec $\mathbf{x}_a = (x_{a,b})_{b \in B}$ pour tout $a \in A$. On pose pour tout $a \in A$, $b \in B$,

$$|\mathbf{x}_{\bullet b}| := \sum_{a \in A} x_{a,b}, \quad |\mathbf{x}_a| := \sum_{b \in B} x_{a,b}.$$

Par convention, on dira qu'une somme sur l'ensemble vide vaut 0, et qu'un produit sur l'ensemble vide vaut 1.

Remarque. Soit A et B des ensembles finis, et $\mathbf{x} = (\mathbf{x}_a)_{a \in A} \in (\mathbb{N}^B)^A$. Le multi-indice \mathbf{x} correspond à une application $\mathbf{x} : A \times B \rightarrow \mathbb{N}, (a, b) \mapsto x_{a,b}$. En particulier, si A ou B est vide, \mathbf{x} correspond à l'application vide. Dans ce cas, on a alors pour tout sous-ensemble $\mathcal{E} \subset A \times B$ et pour tout $b \in B$,

$$|\mathbf{x}|_{\mathcal{E}} = 0, \quad |\mathbf{x}_{\bullet b}| = 0.$$

Notation. Soit $s \in \mathbb{C}$ un nombre complexe. On note $s = \sigma + i\tau$ avec σ et τ respectivement la partie réelle et la partie imaginaire de s .

Notation. Pour tout réel $x \in \mathbb{R}$ on note H_x le demi-plan complexe ouvert d'abscisse x , et \overline{H}_x son adhérence :

$$H_x := \{s \in \mathbb{C} \mid \sigma > x\}, \quad \overline{H}_x := \{s \in \mathbb{C} \mid \sigma \geq x\}.$$

Notation. Soit $a \in \mathbb{C}$, $r \in \mathbb{R}_+$. On note $\overline{D}_a(r)$ le disque fermé de centre a et de rayon r :

$$\overline{D}_a(r) := \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

Notation. Soit $s \in \mathbb{C}$ un nombre complexe, et $k \in \mathbb{N}$. On note $s_{\Delta, k} := (s, \dots, s) \in \mathbb{C}^k$. Lorsqu'il n'y a pas d'ambiguïté, on notera simplement s_{Δ} .

Notation. Soit $n \in \mathbb{N}$ un entier. On appelle n -ième nombre harmonique le nombre rationnel suivant

$$h_n := \sum_{k=1}^n \frac{1}{k},$$

avec la convention $h_0 = 0$.

Notation. On notera $\gamma = 0,5772156649\dots$ la constante gamma d'Euler [OEI23a], $A = 1,28242712\dots$ la constante de Glaisher-Kinkelin [OEI23c], et $e = 2,718281828459045\dots$ le nombre d'Euler [OEI23b].

Proposition-Définition ([EMOT81, Chap.1]). On note Γ le prolongement méromorphe sur \mathbb{C} de la fonction gamma d'Euler définie sur H_0 par l'intégrale

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx \quad (\operatorname{Re}(s) > 0).$$

La fonction Γ est holomorphe sur H_0 , et possède des pôles simples aux entiers négatifs. On note également ψ le prolongement méromorphe sur \mathbb{C} de la fonction digamma définie sur H_0 par la relation

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} \quad (\operatorname{Re}(s) > 0).$$

La fonction ψ est holomorphe sur H_0 , et possède des pôles simples aux entiers négatifs.

Rappelons que Γ admet l'équation fonctionnelle suivante

$$\Gamma(s+1) = s\Gamma(s) \quad (s \notin -\mathbb{N}),$$

et en appliquant cette équation fonctionnelle aux entiers positifs, on trouve que $\Gamma(N+1) = N!$. Rappelons aussi que Γ satisfait la formule des compléments

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)} \quad (s \notin \mathbb{Z}).$$

Au voisinage de $s = 0$, on a le développement asymptotique suivant

$$\Gamma(s) = \frac{1}{s} - \gamma + \frac{1}{2} \left(\gamma^2 + \frac{\pi^2}{6} \right) s + O(s^2).$$

Rappelons également que la fonction $\frac{1}{\Gamma}$ est holomorphe sur \mathbb{C} et qu'elle s'annule en les entiers négatifs.

Notation. Pour tout $k \in \mathbb{Z}$, et $s \in \mathbb{C}$, on pose le coefficient binomial

$$\binom{s}{k} = \begin{cases} \frac{s(s-1)\dots(s-k+1)}{k!} & \text{si } k \geq 0, \\ 0 & \text{sinon.} \end{cases}$$

Pour tous les entiers $\mathbf{k} = (k_1, \dots, k_P)$ tels que $n = k_1 + \dots + k_P$, on considère le coefficient multinomial :

$$\binom{n}{\mathbf{k}} = \begin{cases} \frac{n!}{k_1! \dots k_P!} & \text{si } k_p \geq 0 \text{ pour tout } p \in \llbracket 1, P \rrbracket, \\ 0 & \text{sinon.} \end{cases}$$

Dans ce manuscrit, on utilisera la détermination principale du logarithme, et on écrira pour tout complexe $z \in \mathbb{C}^*$, $z = |z|e^{i \arg(z)}$ avec $\arg(z) \in]-\pi, \pi]$.

Chapitre 1

Introduction

Dans [Kom08], Komori obtient une nouvelle représentation intégrale surfacique de la fonction double zêta de Mordell-Tornheim introduite par Matsumoto [Mat03] puis il déduit de cette représentation une formule des valeurs aux entiers négatifs de cette fonction zêta. Par la suite, via une méthode similaire et en utilisant des nombres de Bernoulli généralisés, il étendit ses résultats dans [Kom10] en établissant une formule pour les valeurs directionnelles aux entiers négatifs de fonctions multizêta de type Hurwitz-Lerch, cependant ces nombres de Bernoulli généralisés sont souvent non explicites. Des cas particuliers ont par la suite été étudiés. Pour des fonctions multizêta de type Euler-Zagier, des formules explicites ont été obtenus par Onozuka [Ono13], puis par Essouabri et Matsumoto [EM20] pour une classe plus générale de fonctions multizêta de type Euler-Zagier. Des travaux de Borwein et Dilcher [BD18] établissent des formules explicites des valeurs et des valeurs de la dérivée première de la fonction multizêta de Mordell-Tornheim en s'inspirant d'une technique due à Crandall [Cra12]. Par la suite Onodera [Ono21] généralisera cette technique et ces résultats en établissant des formules explicites pour les valeurs et les valeurs des dérivées d'ordre 1 et 2 de la fonction multizêta de Mordell-Tornheim de rang quelconque sur la diagonale.

Cette thèse s'inscrit dans la continuité des travaux ci-dessus. On s'intéressera à une classe de fonctions multizêta de type Hurwitz généralisées définie par la série de Dirichlet

$$Z(\mathbf{s}, \mathbf{s}') = \sum_{\mathbf{n}=(n_1, \dots, n_P) \in \mathbb{N}^P} \frac{1}{\prod_{p=1}^P (n_p + d_p)^{s_p} \prod_{q=1}^Q (l_q(\mathbf{n} + \mathbf{d}))^{s'_q}},$$

avec des entiers $P, Q \geq 1$, des nombres complexes $\mathbf{d} = (d_1, \dots, d_P) \in H_0^P$, et l_1, \dots, l_Q des formes linéaires de rang P dépendants de chaque variable n_1, \dots, n_P . Notons qu'une telle classe de fonctions multizêta contient les fonctions multizêta de type Shintani, les fonctions multizêta de type Mordell-Tornheim et les fonctions multizêta rattachées à des systèmes de racines de rang 2. Cette classe contient en particulier les fonctions zêta de Witten $\zeta_{\mathfrak{sl}(3)}$, $\zeta_{\mathfrak{so}(5)}$ et $\zeta_{\mathfrak{g}_2}$ à un facteur multiplicatif près. On étudiera tout d'abord le domaine de convergence et le prolongement méromorphe de ces fonctions multizêta de type Hurwitz généralisées dans le chapitre 2. Ensuite on appliquera dans ce même chapitre une technique de prolongement méromorphe due à Crandall et déjà utilisée par Borwein et Dilcher puis par Onodera afin d'étudier les fonctions multizêta de type Mordell-Tornheim. On établira à l'aide de ce prolongement des formules explicites pour les valeurs directionnelles aux entiers négatifs de cette classe de fonction multizêta dans le chapitre 4. On établira également à l'aide de ce même prolongement des formules pour les dérivées directionnelles pour cette même classe de fonctions multizêta dans le chapitre 5. Notons que ces deux formules directionnelles dépendent de coefficients issus de l'étude d'une intégrale que l'on effectuera dans le chapitre 3. On verra la construction des fonctions zêta de Witten rattachées à des algèbres de Lie semi-simple dans le chapitre 6, puis on établira également des formules pour les valeurs et les valeurs dérivées des fonctions $\zeta_{\mathfrak{so}(5)}$ et $\zeta_{\mathfrak{g}_2}$, et on étudiera également les résidus aux pôles positifs de ces deux fonctions zêta de Witten. Enfin, on utilisera ensuite ces résultats pour établir une relation asymptotique complètement explicite du nombre de représentations de dimension n des algèbres de Lie $\mathfrak{so}(5)$ et \mathfrak{g}_2 via un théorème de Meinardus [BBBF24], complétant ainsi certains travaux de Bridges, Brindle, Bringmann et Franke.

1.1 Tour d'horizon sur les fonctions zêta et multizêta

L'étude des valeurs de fonctions zêta commence avec l'étude des valeurs de la série

$$\zeta(k) = \sum_{n=1}^{+\infty} \frac{1}{n^k} \quad (k \geq 2),$$

qui correspondent aux valeurs de la fonction zêta de Riemann aux entiers positifs. Il est bien connu depuis Euler que pour $k = 2$ on a

$$\zeta(2) = \frac{\pi^2}{6}.$$

Lorsque k est pair on dispose également de formules explicites bien connues, ce qui n'est pas le cas lorsque k est impair. Un autre exemple historique tout aussi important et mystérieux est l'étude des valeurs zêta multiples, c'est-à-dire les valeurs des séries de la forme

$$\zeta(k_1, \dots, k_r) := \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}} \quad (k_1, \dots, k_{r-1} \geq 1, k_r \geq 2).$$

Dans certains cas il existe des relations entre ces valeurs zêta multiples et les valeurs de la fonction zêta de Riemann. Par exemple, une identité d'Euler fournit la relation

$$\zeta(3) = \zeta(1, 2).$$

Ces valeurs zêta multiples vérifient également des relations dites de quasi-battages, comme par exemple la relation

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2),$$

pour plus de détails sur ces relations on se réfère à [Zag94] ou encore à [LM95]. Une version de ces relations de quasi-battages existe également pour les fonctions multizêta de type Hurwitz définies par

$$\zeta(s_1, \dots, s_r, \mathbf{d}) := \sum_{0 < n_1 < \dots < n_r} \frac{1}{(n_1 + d_1)^{s_1} \dots (n_r + d_r)^{s_r}} \quad (\mathbf{d} = (d_1, \dots, d_r) \in \mathbb{R}_+^r, s_1, \dots, s_{r-1} \geq 1, s_r \geq 2)$$

via des travaux de Manchon et Paycha [MP10]. Notons que certaines des relations obtenues par ces deux auteurs impliquent des valeurs aux entiers négatifs, nécessitant alors une discussion sur le sens que l'on donne à ces valeurs qui sont situées en dehors du domaine de convergence de la série de Dirichlet. Ainsi l'article en question s'intéresse à une étude du prolongement méromorphe de la fonction analytique $(s_1, \dots, s_r) \mapsto \zeta(s_1, \dots, s_r, \mathbf{d})$. Notons également que d'autres relations de quasi-battages ont été obtenu pour des fonctions multizêta rattachées à des systèmes de racines par Matsumoto, Tsumura et Komori (voir [KMT23]).

1.1.1 Quelques résultats sur les fonctions zêta

On souhaite donner ici un bref récapitulatif des résultats sur les valeurs spéciales de la fonction zêta de Riemann, sur la fonction zêta de Hurwitz, sur les fonctions zêta de Barnes, sur les fonctions zêta d'Euler-Zagier, sur certaines fonctions zêta de Witten et sur les fonctions zêta de Shintani. On mentionnera également leur domaine d'holomorphie et de méromorphie. On donnera en particulier des résultats sur les valeurs aux entiers négatifs pour certaines de ces fonctions.

Toutes les fonctions zêta mentionnées dans cette sous-section ont un domaine de convergence de la forme $H_{\sigma_0} = \{s = \sigma + i\tau \in \mathbb{C} \mid \sigma > \sigma_0\}$ avec $\sigma_0 \in \mathbb{R}$, et elles sont holomorphes sur leur demi-plan de convergence. Pour donner un sens des valeurs aux entiers $s = N \in \mathbb{Z}$ ne se situant pas dans le demi-plan de convergence de ces fonctions, on doit au préalable discuter de leur prolongement méromorphe, et de montrer que ce prolongement est régulier en l'entier que l'on considère. Plusieurs de ces prolongements peuvent s'établir à l'aide de la formule d'Euler-Maclaurin.

Théorème 1.1.1 (Formule d'Euler-Maclaurin, [Ten15]). *Pour tout entier positif $k \geq 0$ et pour toute fonction f de classe C^{k+1} sur $[a, b]$, avec $a, b \in \mathbb{Z}$, on a*

$$\sum_{n=a}^{b-1} f(n) = \int_a^b f(t)dt + \sum_{n=0}^k \frac{(-1)^{n+1} B_{n+1}}{(n+1)!} (f^{(n)}(b) - f^{(n)}(a)) \\ + \frac{(-1)^k}{(k+1)!} \int_a^b b_{k+1}(x) f^{(k+1)}(x) dx,$$

où B_n désigne le n -ième nombre de Bernoulli et $b_n(x)$ désigne le n -ième polynôme de Bernoulli périodisé.

On rappelle également l'expression de la fonction zêta de Riemann.

Définition 1.1.2. *On appelle fonction zêta de Riemann la fonction définie par la série de Dirichlet suivante*

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (\operatorname{Re}(s) > 1).$$

Cette fonction se prolonge sur \mathbb{C} :

Proposition 1.1.3. *La fonction zêta de Riemann est holomorphe sur H_1 , et admet un prolongement méromorphe sur \mathbb{C} avec un seul pôle en $s = 1$, qui est simple, de résidu 1.*

On peut démontrer le prolongement méromorphe à l'aide de la formule d'Euler-Maclaurin (1.1.1) ou alors via la représentation intégrale de contour

$$\zeta(s) = \frac{e^{-i\pi s} \Gamma(1-s)}{2i\pi} \int_{\mathcal{C}} \frac{z^{s-1}}{e^z - 1} dz$$

où \mathcal{C} correspond au contour de Hankel partant de $+\infty + i\delta$, passant autour de l'origine dans le sens trigonométrique, puis rejoignant $+\infty - i\delta$. Rappelons qu'il est classique de définir les nombres de Bernoulli par la fonction génératrice suivante

$$\frac{z}{e^z - 1} = \sum_{k=0}^{+\infty} \frac{B_k}{k!} z^k.$$

En injectant cette expression dans la représentation intégrale de contour précédente, on obtient une relation entre les valeurs de la fonction zêta de Riemann aux entiers négatifs et les nombres de Bernoulli

$$\forall N \in \mathbb{N}, \quad \zeta(-N) = (-1)^N \frac{B_{N+1}}{N+1}.$$

En particulier, on en déduit que $\zeta(-2N) = 0$ pour tout entier $N \in \mathbb{N}^*$.

La fonction zêta de Riemann possède une certaine symétrie qui prend la forme d'une équation fonctionnelle. De cette symétrie provient plusieurs résultats sur les valeurs de la fonction zêta de Riemann, mais aussi quelques résultats sur l'étude de ces zéros.

Théorème 1.1.4. *La fonction zêta de Riemann vérifie l'équation fonctionnelle*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (s \in \mathbb{C} \setminus \{0, 1\}).$$

De cette équation fonctionnelle on retrouve directement que la fonction zêta de Riemann s'annule en les entiers pairs strictement négatifs. De plus, en dérivant l'équation fonctionnelle, on trouve que

$$\forall N \in \mathbb{N}^*, \quad \zeta'(-2N) = \frac{(-1)^N (2N)!}{2^{2N+1} \pi^{2N}} \zeta(2N+1).$$

Notons que les dérivées de certaines petites valeurs sont bien connues :

$$\zeta'(0) = -\frac{1}{2} \ln(2\pi), \\ \zeta'(-1) = \frac{1}{12} - \ln(A),$$

avec A la constante de Glaisher-Kinkelin.

On sait depuis Euler que les valeurs aux entiers pairs strictement positifs correspondent à un multiple rationnel d'une puissance de π ,

$$\zeta(2N) \in \mathbb{Q}\pi^{2N} \quad (N \geq 1).$$

En fait, par l'équation fonctionnelle de la fonction zêta de Riemann, on a même une expression de ces valeurs en fonction de π et des nombres de Bernoulli

$$\zeta(2N) = (-1)^{N+1} \frac{2^{2N-1} B_{2N}}{(2N)!} \pi^{2N}.$$

En revanche l'étude des valeurs positives impairs de la fonction zêta de Riemann demeure un problème très largement ouvert, et l'on dispose surtout de résultat qualitatif sur ces valeurs. En effet, on sait par un résultat d'Apéry [Apé79] que $\zeta(3)$ est irrationnel, et l'on sait également via un résultat de Rivoal [Riv00] que la suite $(\zeta(2N+1))_{N \geq 1}$ comporte une infinité d'irrationnels.

Définition 1.1.5. Soit $d \in H_0$ un complexe, on appelle fonction zêta d'Hurwitz la fonction zêta

$$\zeta(s, d) := \sum_{n=0}^{+\infty} \frac{1}{(n+d)^s} \quad (\operatorname{Re}(s) > 1).$$

On notera dans ce manuscrit $\zeta'(s, d) := \frac{\partial}{\partial s} \zeta(s, d)$ la dérivée de la fonction zêta d'Hurwitz selon la variable s . Observons qu'en $d = 1$, on retrouve la fonction zêta de Riemann. Il est bien connu que cette fonction zêta admet un prolongement méromorphe sur \mathbb{C} selon la variable s .

Proposition 1.1.6. La fonction zêta de Hurwitz se prolonge méromorphiquement sur \mathbb{C} selon la variable s , avec un unique pôle en $s = 1$ qui est un pôle simple. De plus, le résidu en ce pôle vaut 1.

Démonstration. Soit $n \in \mathbb{N}^*$ un entier. Via la formule d'Euler-Maclaurin, on obtient pour $\sigma > 1$ que

$$\begin{aligned} \zeta(s, d) &= \frac{d^{1-s}}{s-1} + \sum_{i=0}^n \binom{-s}{i} \frac{(-1)^i B_{i+1}}{i+1} d^{-s-i} \\ &\quad + (-1)^n \binom{-s}{n+1} \int_0^{+\infty} b_{n+1}(x) (x+d)^{-s-n-1} dx. \end{aligned}$$

L'intégrale présente dans la formule précédente converge absolument pour tout $\sigma > -n$. Ainsi, cette formule permet de prolonger méromorphiquement la fonction zêta de Hurwitz pour $\sigma > -n$, avec un pôle unique en $s = 1$, d'ordre 1, et de résidu 1. \square

En partant d'une représentation intégrale de contour de la fonction zêta d'Hurwitz similaire à celle mentionnée plus haut

$$\zeta(s, x) = \frac{e^{-i\pi s} \Gamma(1-s)}{2i\pi} \int_{\mathcal{C}} \frac{z^{s-1} e^{xz}}{e^z - 1} dz$$

avec \mathcal{C} le contour de Hankel, et en utilisant la fonction génératrice des polynômes de Bernoulli

$$\frac{ze^{zx}}{e^z - 1} = \sum_{k=0}^{+\infty} \frac{B_k(x)}{k!} z^k,$$

on obtient des relations bien connues entre les valeurs aux entiers négatifs de la fonction zêta de Hurwitz et les polynômes de Bernoulli (voir [Apo76, Théorème 12.13])

$$\forall N \in \mathbb{N}, \quad \zeta(-N, x) = -\frac{B_{N+1}(x)}{n+1}.$$

De plus, il était connu par Hurwitz (voir [Ber85, Formule (3)]) que

$$\zeta'(0, d) = \ln(\Gamma(d)) + \zeta'(0) = \ln(\Gamma(d)) - \frac{1}{2} \ln(2\pi).$$

Observons également que, pour tout $s \neq 1$, et tout $d \in H_0$, on a $\zeta(s, d) = d^{-s} + \zeta(s, d+1)$. En dérivant selon s , on trouve alors

$$\zeta'(-N, d+1) = d^N \ln(d) + \zeta'(-N, d).$$

Par un jeu de réécriture de la série de Dirichlet $\zeta\left(s, \frac{1}{2}\right)$, on obtient que $\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s)$. Ainsi, en dérivant des deux côtés par rapport à s , on trouve en particulier

$$\zeta'\left(-1, \frac{1}{2}\right) = -\frac{\ln(2)}{24} - \frac{1}{2}\zeta'(-1) = -\frac{\ln(2)}{24} - \frac{1}{24} + \frac{\ln(A)}{2}.$$

Des travaux de Miller et Adamchik (voir [MA98]) ont établi des relations explicites entre les valeurs des dérivées de la fonction zêta d'Hurwitz suivant s en les entiers négatifs avec un coefficient rationnel d , et des valeurs spéciales de logarithme, de fonctions polygamma, et de la fonction zêta de Riemann. Ces travaux utilisent en particulier la formule de multiplication pour la fonction zêta d'Hurwitz :

$$\zeta(s, kz) = \sum_{n=0}^{k-1} \zeta\left(s, z + \frac{n}{k}\right) \quad (k \in \mathbb{N}^*). \quad (1.1)$$

Remarquons qu'en évaluant la dérivée de cette formule selon la variable s en $s = 0$, et en utilisant la relation $\zeta'(0, d) = \ln(\Gamma(d)) - \frac{1}{2} \ln(2\pi)$, on retrouve la formule de multiplication pour la fonction gamma d'Euler

$$\prod_{n=0}^{k-1} \Gamma\left(z + \frac{n}{k}\right) = (2\pi)^{\frac{k-1}{2}} k^{\frac{1}{2} - kz} \Gamma(kz).$$

La fonction zêta d'Hurwitz admet également une relation fonctionnelle appelée formule d'Hurwitz. Avant de l'exprimer, on a besoin d'introduire la fonction zêta de Lerch :

Définition 1.1.7. Soit $s \in \mathbb{C}$ tel que $\sigma > 1$, $d \in H_0$, et $z \in \mathbb{C}$ tel que $|z| \leq 1$. On définit la fonction zêta de Lerch par la relation

$$\phi(z, s, d) := \sum_{n \geq 0} \frac{z^n}{(n+d)^s}.$$

La fonction $s \mapsto \phi(z, s, d)$ admet un prolongement méromorphe sur \mathbb{C} . Elle est même holomorphe sur \mathbb{C} lorsque $z \neq 1$, et elle admet un unique pôle, qui est simple, en $s = 1$ lorsque $z = 1$ (voir [EMOT81, §1.11]).

Théorème 1.1.8 (Formule d'Hurwitz [Apo76, §12.7]). On a

$$\zeta(1-s, d) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-i\pi s/2} \phi(e^{2i\pi d}, s, 1) + e^{i\pi s/2} \phi(e^{-2i\pi d}, s, 1) \right) \quad (d \in]0, 1[, s \in \mathbb{C} \setminus \{0, 1\}).$$

Remarquons au passage que le cas $d = 1$ de la formule d'Hurwitz précédente est également vraie puisque ce cas correspond à l'équation fonctionnelle de la fonction zêta de Riemann. Cette équation fonctionnelle est utilisée par Onodera (voir [Ono21, Lemme 2.1]) afin d'obtenir une majoration de la fonction zêta d'Hurwitz. Cette majoration est en particulier cruciale dans l'expansion de Crandall de l'article du même auteur, et nous utiliserons également cette majoration dans le chapitre 2.

Certains résultats de cette thèse s'exprimeront à l'aide de fonctions zêta de type Barnes généralisées. Avant de mentionner la généralisation en question, on se propose de définir et de mentionner quelques résultats classiques pour les fonctions zêta de Barnes. On mentionnera également quelques résultats sur les valeurs et les valeurs des dérivées de ces fonctions.

Définition 1.1.9. Soit $P \in \mathbb{N}$ un entier, (c_1, \dots, c_P) des nombres complexes dans H_0 , et $d \in H_0$. On appelle fonction zêta de Barnes la fonction zêta de la forme

$$\forall s \in H_P, \quad \zeta^B(s, d | c_1, \dots, c_P) := \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{(\sum_{p=1}^P c_p n_p + d)^s}.$$

Il était connu de Barnes que cette fonction zêta était méromorphe selon la variable s sur \mathbb{C} . Plus précisément :

Proposition 1.1.10 ([Bar04]). *La fonction zêta de Barnes $s \mapsto \zeta^B(s, d|c_1, \dots, c_P)$ admet un prolongement méromorphe sur \mathbb{C} , avec comme seules singularités des pôles simples en les $s = 1, \dots, P$.*

La valeur de la dérivée en $s = 0$ de ces fonctions a été étudiée par Barnes en développant la théorie des fonctions multigamma (voir [Bar01] et [Bar04]). Ce type de fonction possède des liens intéressants avec la géométrie. En effet, Vardi a obtenu une relation explicite entre la valeur en $\frac{1}{2}$ de la fonction multigamma de dimension n , et le déterminant régularisé du Laplacien de la n -sphère dans [Var88]. Il fournit également une formule explicite de la valeur en $\frac{1}{2}$ de la fonction multigamma de dimension n en fonction des valeurs aux entiers négatifs de la dérivée de la fonction zêta de Riemann.

Quelques-uns de nos résultats s'exprimeront en fonction de valeurs dérivées de ces fonctions de Barnes. On aura en particulier besoin d'un résultat démontré par Sakane et Aoki dans [SA22]. On pose tout d'abord

$$C_{P,x}(t) := (t - x + P - 1)(t - x + P - 2) \dots (t - x + 1) \in (\mathbb{Z}[x])[t] \quad (P \geq 2),$$

et $C_{1,x}(t) = 1$. Remarquons au passage que $C_{2,x}(t) = t - x + 1$.

Théorème 1.1.11 ([SA22, Théorème 4]). *Soit $d \in H_0$, et $\mathbf{c} = (c_1, \dots, c_P) \in \mathbb{Q}^P$ des rationnels. On écrit $c_1 = \frac{a_1}{b_1}, \dots, c_P = \frac{a_P}{b_P}$ avec $a_p, b_p \geq 1$ des entiers premiers entre eux pour tout p . On pose $x(\mathbf{c}) := \frac{\text{ppcm}(a_1, \dots, a_P)}{\text{pgcd}(b_1, \dots, b_P)}$, et $\beta_1 := \frac{x(\mathbf{c})}{c_1}, \dots, \beta_P := \frac{x(\mathbf{c})}{c_P}$, alors on a*

$$\zeta^B(s, d|c_1, \dots, c_P) = \frac{x(\mathbf{c})^{-s}}{(P-1)!} \sum_{v_1=0}^{\beta_1-1} \dots \sum_{v_{P-1}=0}^{\beta_{P-1}-1} \sum_{k=0}^{P-1} C_{P, \frac{d+c_1v_1+\dots+c_Pv_P}{x(\mathbf{c})}}^{(k)}(0) \zeta\left(s-k, \frac{d+c_1v_1+\dots+c_Pv_P}{x(\mathbf{c})}\right)$$

On démontrera une généralisation du théorème précédent dans le Théorème B, en utilisant une stratégie similaire à celle utilisée par Aoki et Sakane dans [SA22].

Exemple 1.1.12. *On trouve*

$$\begin{aligned} \zeta^B(s, d|1, 1) &= (1-d)\zeta(s, d) + \zeta(s-1, d), \\ \zeta^B(s, d|1, 2) &= 2^{-s} \left[\left(1 - \frac{d}{2}\right) \zeta\left(s, \frac{d}{2}\right) + \left(1 - \frac{d+1}{2}\right) \zeta\left(s, \frac{d+1}{2}\right) + \zeta\left(s-1, \frac{d}{2}\right) + \zeta\left(s-1, \frac{d+1}{2}\right) \right], \\ \zeta^B(s, d|1, 3) &= 3^{-s} \left[\left(1 - \frac{d+2}{3}\right) \zeta\left(s, \frac{d+2}{3}\right) + \left(1 - \frac{d+1}{3}\right) \zeta\left(s, \frac{d+1}{3}\right) + \left(1 - \frac{d}{3}\right) \zeta\left(s, \frac{d}{3}\right) \right. \\ &\quad \left. + \zeta\left(s-1, \frac{d+2}{3}\right) + \zeta\left(s-1, \frac{d+1}{3}\right) + \zeta\left(s-1, \frac{d}{3}\right) \right], \\ \zeta^B(s, d|2, 3) &= 6^{-s} \left[\left(1 - \frac{d+7}{6}\right) \zeta\left(s, \frac{d+7}{6}\right) + \left(1 - \frac{d+5}{6}\right) \zeta\left(s, \frac{d+5}{6}\right) + \left(1 - \frac{d+4}{6}\right) \zeta\left(s, \frac{d+4}{6}\right) \right. \\ &\quad \left. + \left(1 - \frac{d+3}{6}\right) \zeta\left(s, \frac{d+3}{6}\right) + \left(1 - \frac{d+2}{6}\right) \zeta\left(s, \frac{d+2}{6}\right) + \left(1 - \frac{d}{6}\right) \zeta\left(s, \frac{d}{6}\right) \right. \\ &\quad \left. + \zeta\left(s-1, \frac{d+7}{6}\right) + \zeta\left(s-1, \frac{d+5}{6}\right) + \zeta\left(s-1, \frac{d+4}{6}\right) + \zeta\left(s-1, \frac{d+3}{6}\right) \right. \\ &\quad \left. + \zeta\left(s-1, \frac{d+2}{6}\right) + \zeta\left(s-1, \frac{d}{6}\right) \right]. \end{aligned}$$

On obtient au passage les valeurs spéciales

$$\begin{aligned}\zeta^B(0, 2|1, 1) &= \frac{5}{12}, & \zeta^B(0, 3|1, 2) &= \frac{11}{24}, & \zeta^B(0, 4|1, 3) &= \frac{19}{36}, & \zeta^B(0, 5|2, 3) &= \frac{31}{72}, \\ (\zeta^B)'(0, 2|1, 1) &= \frac{1}{2} \ln(2\pi) - \ln(A) + \frac{1}{12}, & (\zeta^B)'(0, 3|1, 2) &= \frac{1}{4} \ln(2) + \frac{1}{2} \ln(\pi) - \frac{1}{2} \ln(A) + \frac{1}{24}, \\ (\zeta^B)'(0, 4|1, 3) &= -\frac{5}{9} \ln(3) + \ln(2\pi) - \frac{1}{3} \ln(A) - \frac{2}{3} \ln\left(\Gamma\left(\frac{2}{3}\right)\right) - \frac{1}{3} \ln\left(\Gamma\left(\frac{1}{3}\right)\right) + \frac{1}{36}, \\ (\zeta^B)'(0, 5|2, 3) &= -\frac{11}{18} \ln(3) + \frac{13}{12} \ln(2) + \frac{4}{3} \ln(\pi) - \frac{1}{6} \ln(A) - \frac{2}{3} \ln\left(\Gamma\left(\frac{2}{3}\right)\right) - \ln\left(\Gamma\left(\frac{1}{3}\right)\right) + \frac{1}{72},\end{aligned}$$

en notant $(\zeta^B)'(s, d|\mathbf{c}) = \partial_s \zeta^B(s, d|\mathbf{c})$ et $\zeta'(s, d) = \partial_s \zeta(s, d)$.

Notons que dans l'exemple précédent, nous avons simplifié les termes à l'aide de la formule de multiplication de la fonction zêta d'Hurwitz (1.1), et nous avons également utilisé le résultat de Borwein et Zucker [BZ92, Table 3] :

$$\Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)^2}{\sqrt[3]{2}\sqrt{\pi}},$$

afin d'obtenir la relation suivante à l'aide de la formule des compléments

$$\frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{5}{6}\right)} = \frac{3\Gamma\left(\frac{1}{3}\right)^4}{2^{5/3}\pi^2}.$$

Shintani a introduit des fonctions zêta lui permettant d'étudier des fonctions zêta de Dedekind rattachées à un corps de nombre totalement réel (voir [Shi76], [Shi77a], [Shi77b], [Shi77c], [Shi80]). Ces fonctions généralisent en particulier les fonctions zêta de Barnes :

Définition 1.1.13. Soit $P, Q \in \mathbb{N}$ des entiers, et $\mathbf{c} = (c_{q,p})_{1 \leq p \leq P, 1 \leq q \leq Q}$, $\mathbf{d} = (d_q)_{1 \leq q \leq Q}$ des nombres complexes dans H_0 . On appelle fonction zêta de Shintani la fonction zêta suivante

$$\forall s \in H_{P/Q}, \quad \zeta^{Sh}(s, \mathbf{d}|\mathbf{c}) := \sum_{n_1, \dots, n_P \geq 0} \prod_{q=1}^Q (c_{q,1}n_1 + \dots + c_{q,P}n_P + d_q)^{-s}.$$

Observons que, pour $Q = 1$, on retrouve la définition de la fonction zêta de Barnes. Cette fonction zêta est holomorphe en la variable s sur $H_{P/Q}$, et admet un prolongement méromorphe sur \mathbb{C} . Shintani a prouvé que cette fonction possède les mêmes pôles que ceux de la fonction $\frac{\Gamma(Ps - Q)}{\Gamma(s)}$, et en particulier cette fonction est régulière en les entiers négatifs. Il a également prouvé certaines formules aux entiers négatifs de ces fonctions. Une partie de notre travail permet en particulier d'étudier les valeurs de certaines de ces fonctions aux entiers négatifs, ainsi que les valeurs de leurs dérivées.

La valeur de leurs dérivées par rapport à la variable s en $s = 0$ permettent de définir des fonctions multigamma, introduites par Barnes dans [Bar04], puis généralisées par Friedmann et Ruijsenaars.

Définition 1.1.14. Soit $P \in \mathbb{N}$ un entier, (c_1, \dots, c_P) des nombres complexes dans H_0 , et $d \in H_0$. On appelle fonction multigamma de Barnes la fonction

$$\Gamma_P(d|c_1, \dots, c_P) := \exp\left(\partial_s(\zeta^B(s, d|c_1, \dots, c_P))\Big|_{s=0}\right).$$

Exemple 1.1.15. Soit $d \in H_0$, on a

$$\begin{aligned}\Gamma_0(d) &= \frac{1}{d} \\ \Gamma_1(d|1) &= \frac{1}{\sqrt{2\pi}}\Gamma(d),\end{aligned}$$

où Γ désigne la fonction gamma d'Euler.

Barnes obtient aussi une relation fonctionnelle

$$\Gamma_P(d|c_1, \dots, c_P) = \Gamma_{P-1}(d|c_1, \dots, c_{P-1})\Gamma_P(d + c_P|c_1, \dots, c_P),$$

qui généralise la relation fonctionnelle de la fonction gamma d'Euler. Via les fonctions zêta de type Shintani, Friedman et Ruijsenaars ont introduit dans [FR04] des fonctions multigamma de type Shintani, via une construction similaire à celle utilisée par Barnes.

Définition 1.1.16. Soit $P, Q \in \mathbb{N}$ des entiers, et $\mathbf{c} = (c_{q,p})_{\substack{1 \leq p \leq P \\ 1 \leq q \leq Q}}$, $\mathbf{d} = (d_q)_{1 \leq q \leq Q}$ des nombres complexes dans H_0 . On appelle fonction multigamma de Shintani la fonction :

$$\Gamma_P(\mathbf{d}|\mathbf{c}_1, \dots, \mathbf{c}_P) := \exp(\partial_s(\zeta^{Sh}(s, \mathbf{d}|\mathbf{c}_1, \dots, \mathbf{c}_P))|_{s=0}),$$

en notant $\mathbf{c}_q = (c_{q,p})_{p \in [1, P]}$.

Dans [FR04], Friedman et Ruijsenaars étudient ces fonctions multigamma de Shintani pour en déduire des formules de type Raabe, généralisant la formule de Raabe suivante :

$$\int_0^1 \ln(\Gamma(x))dx = \ln(\sqrt{2\pi}).$$

On déduira de nos travaux des formules explicites pour les valeurs et les valeurs des dérivées de certaines fonctions zêta de type Shintani, et on retrouvera également la même formule de type Raabe obtenue par Friedman et Ruijsenaars.

On discutera à présent d'une classe de fonctions zêta rattachées à des algèbres de Lie. Witten a étudié dans [Wit91] des espaces de modules provenant de la théorie conforme des champs, rattachées à un groupe de jauge G . Il établit alors une formule de volume pour ces espaces de modules faisant intervenir une série parcourant les représentations irréductibles du groupe de Lie G . Witten obtient alors au passage le résultat qualitatif

$$\sum_{\varphi} \frac{1}{\dim(\varphi)^{2N}} \in \pi^{2kN} \mathbb{Q} \quad (N \in \mathbb{N}^*),$$

où φ parcourt les classes d'isomorphismes des représentations irréductibles de dimension finie du groupe de Lie G , et où k désigne le nombre de racines positives du groupe de Lie G . Zagier et Garoufalidis redémontrent ce résultat qualitatif via des techniques plus classiques en utilisant au choix l'analyse de Fourier ou une décomposition en éléments simples (voir [Zag94]). Zagier introduit également les fonctions zêta de Witten :

Définition 1.1.17 ([Zag94]). Soit \mathfrak{g} une algèbre de Lie semi-simple, on appelle fonction zêta de Witten rattachée à \mathfrak{g} la fonction zêta

$$\zeta_{\mathfrak{g}}(s) := \sum_{\varphi} \dim(\varphi)^{-s},$$

où φ parcourt les classes d'isomorphismes des représentations irréductibles de dimension finie de \mathfrak{g} .

Un résultat de Larsen et Lubotzky dans [LL08] établit que l'abscisse de convergence de $\zeta_{\mathfrak{g}}(s)$ est $\frac{r}{|\Delta_+(\mathfrak{g})|}$, où r est le rang de \mathfrak{g} et où $\Delta_+(\mathfrak{g})$ désigne l'ensemble des racines positives de \mathfrak{g} . Notons également que ce résultat fut généralisé par Häsä et Stasinski dans [HS19] pour une classe de fonctions zêta plus large. On verra dans le chapitre 6 que les fonctions zêta de Witten correspondent à des fonctions zêta rattachées à des formes linéaires via une formule de Weyl. En particulier, elles vérifieront la condition $H0S$ de l'article d'Essouabri [Ess97], et elles admettent donc un prolongement méromorphe. De plus, ces fonctions zêta sont régulières aux entiers négatifs grâce à un résultat de Komori [Kom10, Théorème 3.22].

On peut également fournir une définition alternative des fonctions zêta de Witten pour des groupes de Lie. Considérons un groupe de Lie compact connexe et semi-simple. On peut lui associer la fonction zêta de Witten

$$\zeta_G(s) = \sum_{\varphi} \frac{1}{\dim(\varphi)^s},$$

où φ parcourt les représentations irréductibles de dimension finie de G à isomorphisme près. Par hypothèse, le groupe de Lie G est nécessairement réel car tout groupe de Lie complexe connexe et compact est abélien (c'est-à-dire un tore), et est donc en particulier non semi-simple (voir [FH91, Exercice 8.44]). En notant $\text{Lie}(G)$ l'algèbre de Lie correspondante au groupe de Lie G , on considère $\mathfrak{g} = \mathbb{C} \otimes \text{Lie}(G)$ l'algèbre de Lie complexifiée de $\text{Lie}(G)$. Si l'on suppose que G est simplement connexe, alors il y a une correspondance entre les représentations d'un groupe de Lie et les représentations de son algèbre de Lie correspondante, puisqu'on peut associer canoniquement à chaque représentation $\rho : G \mapsto GL(E)$ sa différentielle $d\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(E)$ (voir [FH91, §8.1]). Sous cette hypothèse, on obtient finalement que $\zeta_G(s) = \zeta_{\text{Lie}(G)}(s) = \zeta_{\mathfrak{g}}(s)$.

Si l'on considère maintenant un groupe fini G , on peut également définir une fonction zêta de Witten rattachée au groupe G qui correspond alors à une somme finie sur l'ensemble des caractères irréductibles de G . Comme les caractères irréductibles d'un groupe fini forment une base de l'espace vectoriel des fonctions centrales, on trouve que

$$\zeta_G(-2) = \sum_{\chi} \dim(\chi)^2 = |G|,$$

pour plus de détails on se réfère à [FH91, §2.2]. De ce fait et de l'étude des fonctions zêta de Witten $\zeta_{\mathfrak{sl}(2)}(s)$ et $\zeta_{\mathfrak{sl}(3)}(s)$ en $s = -2$, Kurokawa et Ochiai émettent la conjecture suivante :

Conjecture 1.1.18 (Kurokawa et Ochiai [KO13]). *Pour toute algèbre de Lie semi-simple \mathfrak{g} , on a*

$$\zeta_{\mathfrak{g}}(-2) = 0.$$

On sait seulement de cette conjecture qu'elle est valable dans quelques cas particuliers. Cette conjecture est vraie pour les algèbres de Lie $\mathfrak{sl}(2)$ et $\mathfrak{sl}(3)$, et on montrera dans cette thèse que la conjecture est également vraie pour les algèbres de Lie $\mathfrak{so}(5)$ et \mathfrak{g}_2 .

Définition 1.1.19. *Soit \mathfrak{g} une algèbre de Lie semi-simple. Notons $r_{\mathfrak{g}}(n)$ le nombre de représentations de dimension n de \mathfrak{g} , à isomorphisme près. Posons également $f_{\mathfrak{g}}(n)$ le nombre de représentations irréductibles de dimension n de \mathfrak{g} , à isomorphisme près.*

En réécrivant la série de Dirichlet d'une fonction zêta de Witten quelconque il est clair que

$$\zeta_{\mathfrak{g}}(s) = \sum_{n=1}^{+\infty} \frac{f_{\mathfrak{g}}(n)}{n^s}.$$

Il est bien connu qu'une représentation d'une algèbre de Lie semi-simple se décompose de manière unique en somme directe de représentations irréductibles. Ainsi pour une représentation ρ de \mathfrak{g} , on a en particulier qu'il existe des entiers k_1, \dots, k_m et des représentations irréductibles $\varphi_1, \dots, \varphi_m$ tels que

$$\dim(\rho) = \sum_{i=1}^m k_i \dim(\varphi_i).$$

En utilisant la correspondance entre les représentations d'une algèbre de Lie semi-simple \mathfrak{g} et son système de poids on trouve aisément que

$$1 + \sum_{n=1}^{+\infty} r_{\mathfrak{g}}(n)q^n = \prod_{q=1}^{+\infty} \frac{1}{(1 - q^n)^{f_{\mathfrak{g}}(n)}}. \quad (1.2)$$

Ainsi le nombre de représentations de dimension n de \mathfrak{g} correspond à un nombre de partitions. En utilisant la théorie des systèmes de racines et la formule de Weyl on trouve une expression plus adaptée des nombres de représentations des algèbres de Lie $\mathfrak{sl}(3)$, $\mathfrak{so}(5)$ et de l'algèbre de Lie exceptionnelle \mathfrak{g}_2 (voir sa description dans [Bou81]).

Exemple 1.1.20 ([Hum72, §24.3]). *On pose*

$$\mathfrak{sl}(n) = \{X \in \mathcal{M}_n(\mathbb{C}) \mid \text{Tr}(X) = 0\}, \quad \mathfrak{so}(n) = \{X \in \mathcal{M}_n(\mathbb{C}) \mid {}^tX + X = 0\},$$

où Tr désigne la trace, et tX désigne la transposée d'une matrice. Via la théorie des systèmes de racines, on a les résultats suivants :

- 1) *L'ensemble des classes d'isomorphismes des représentations irréductibles de $\mathfrak{sl}(3)$ est de la forme $(\varphi_{i,j})_{i,j \geq 1}$, avec*

$$\dim(\varphi_{i,j}) = \frac{ij(i+j)}{2}.$$

On obtient que le nombre de représentations irréductibles de dimension n de $\mathfrak{sl}(3)$ est

$$f_{\mathfrak{sl}(3)}(n) = \left| \left\{ (i,j) \in \mathbb{N}^{*2} \mid \frac{ij(i+j)}{2} = n \right\} \right|,$$

et on obtient une expression du nombre de représentations $r_{\mathfrak{sl}(3)}(n)$ de $\mathfrak{sl}(3)$:

$$\begin{aligned} r_{\mathfrak{sl}(3)}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\varphi_{i,j}) = n \right\} \right|, \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)}{2} = n \right\} \right|. \end{aligned}$$

- 2) *L'ensemble des classes d'isomorphismes des représentations irréductibles de $\mathfrak{so}(5)$ est de la forme $(\varphi_{i,j})_{i,j \geq 1}$, avec*

$$\dim(\varphi_{i,j}) = \frac{ij(i+j)(i+2j)}{3!}.$$

On obtient que le nombre de représentations irréductibles de dimension n de $\mathfrak{so}(5)$ est

$$f_{\mathfrak{so}(5)}(n) = \left| \left\{ (i,j) \in \mathbb{N}^{*2} \mid \frac{ij(i+j)(i+2j)}{3!} = n \right\} \right|,$$

et on obtient une expression du nombre de représentations $r_{\mathfrak{so}(5)}(n)$ de $\mathfrak{so}(5)$:

$$\begin{aligned} r_{\mathfrak{so}(5)}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\varphi_{i,j}) = n \right\} \right| \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)}{3!} = n \right\} \right|. \end{aligned}$$

- 3) *L'ensemble des classes d'isomorphismes des représentations irréductibles de \mathfrak{g}_2 est de la forme $(\varphi_{i,j})_{i,j \geq 1}$, avec*

$$\dim(\varphi_{i,j}) = \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!}.$$

On obtient que le nombre de représentations irréductibles de dimension n de \mathfrak{g}_2 est

$$f_{\mathfrak{g}_2}(n) = \left| \left\{ (i,j) \in \mathbb{N}^{*2} \mid \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|,$$

et on obtient une expression du nombre de représentations $r_{\mathfrak{g}_2}(n)$ de \mathfrak{g}_2 :

$$\begin{aligned} r_{\mathfrak{g}_2}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\varphi_{i,j}) = n \right\} \right| \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^{*2}} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|. \end{aligned}$$

En utilisant les expressions du nombre de représentations des algèbres de Lie précédentes on peut préciser les fonctions zêta de Witten correspondantes.

Exemple 1.1.21 ([KMT23]). *Via la formule de Weyl on trouve*

$$\begin{aligned}\zeta_{\mathfrak{sl}(2)}(s) &= \zeta(s), \\ \zeta_{\mathfrak{sl}(3)}(s) &= 2^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s}, \\ \zeta_{\mathfrak{so}(5)}(s) &= 6^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s}, \\ \zeta_{\mathfrak{g}_2}(s) &= 120^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_2)^s (2n_1 + 3n_2)^s}.\end{aligned}$$

On trouve parfois dans la littérature des variantes multivariées de ces fonctions zêta dans les travaux de Matsumoto, Komori, Yasushi et Tsumura [KMT10c], [KMT10a], [KMT10b], [KMT12], [KMT11b], ainsi que dans les travaux de Zhaou et Zhou [ZZ11]. Ces fonctions multizêta sont en fait rattachées à des systèmes de racines, et ont autant de variables que le nombre de racines positives du système de racines correspondant. Ces auteurs donnent une expression de l'ensemble des singularités de ces fonctions multizêta, et obtiennent également des relations entre les valeurs positives. Ces résultats permettent en particulier d'obtenir des candidats pôles pour la fonction zêta de Witten univariée correspondante.

- De [KMT10d, Théorème 6.2], on en déduit que la fonction zêta $\zeta_{\mathfrak{so}(5)}(s)$ admet des pôles de la forme

$$s = \frac{1}{2}, \quad s = \frac{1-k}{3} \quad (k \in \mathbb{N}). \quad (1.3)$$

- De [KMT11a, Théorème 3.1], on en déduit que la fonction zêta $\zeta_{\mathfrak{g}_2}(s)$ admet des pôles de la forme

$$s = \frac{1}{3}, \quad s = \frac{1-k}{5} \quad (k \in \mathbb{N}). \quad (1.4)$$

Pour la fonction zêta de Witten $\zeta_{\mathfrak{sl}(3)}(s)$, on connaît plusieurs résultats sur les valeurs et les valeurs dérivées puisque cette fonction zêta a été amplement étudiée dans la littérature. La fonction $\zeta_{\mathfrak{sl}(3)}(s)$ correspond en effet à une fonction zêta classique étudiée indépendamment par Tornheim et Mordell, puis généralisée par Matsumoto.

Définition 1.1.22 ([Mat03]). *Soit $P \geq 1$ un entier, on appelle fonction multizêta de Mordell-Tornheim de rang P le prolongement méromorphe sur \mathbb{C}^{P+1} en les variables $\mathbf{s} = (s_1, \dots, s_{P+1})$ de la série de Dirichlet suivante*

$$\zeta_P^{MT}(\mathbf{s}, \mathbf{d}) := \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{(n_1 + d_1)^{s_1} \dots (n_P + d_P)^{s_P} (n_1 + \dots + n_P + |\mathbf{d}|)^{s_{P+1}}} \quad (\operatorname{Re}(d_1), \dots, \operatorname{Re}(d_P) > 0).$$

Le cas $P = 2$ et $d_1 = d_2 = 1$ évalué en $\mathbf{s} = (s, s, s)$ correspond à la fonction zêta de Tornheim. À un facteur 2^s près, elle correspond également à la zêta de Witten $\zeta_{\mathfrak{sl}(3)}$

$$\zeta_{\mathfrak{sl}(3)}(s) = 2^s \zeta_2^{MT}((s, s, s), (1, 1)).$$

Dans [BD18], via une expansion de Crandall, Borwein et Dilcher étudient les valeurs de la fonction multizêta $\zeta_2^{MT}((s_1, s_2, s_3), (1, 1))$ en obtenant des formules explicites et des approximations. En particulier ils obtiennent

$$\zeta_{\mathfrak{sl}(3)}(0) = \frac{1}{3}, \quad \zeta'_{\mathfrak{sl}(3)}(0) = \frac{\log(2)}{3} + \log(2\pi).$$

Des travaux de Romik [Rom17] ont par la suite utilisé ce résultat précédent afin d'étudier le comportement asymptotique du nombre de représentations de degré n de $\mathfrak{sl}(3)$ en fonctions des valeurs $\zeta_{\mathfrak{sl}(3)}(0)$ et

$\zeta'_{\mathfrak{sl}(3)}(0)$, ainsi que des résidus de $\zeta_{\mathfrak{sl}(3)}(s)$ en les pôles $s = 2/3$ et $s = 1/2$. Notons que Kurokawa et Ochiai [KO13] ont également étudié les valeurs aux entiers négatifs de $\zeta_{\mathfrak{sl}(3)}(s)$, puis en ont déduit des résultats pour les valeurs de certaines fonctions zêta de Witten p -adiques.

Par la suite, Onodera a étudié le cas général de ces fonctions multizêta de Mordell-Tornheim dans [Ono21] le long de la diagonale (c'est-à-dire les fonctions zêta de la forme $\zeta_P^{MT}((s, \dots, s), \mathbf{d})$ pour P quelconque). Via une expansion de Crandall, la même stratégie utilisée par Borwein et Dilcher, il calcule leurs valeurs et leurs valeurs dérivées d'ordre 1 et 2 en les entiers négatifs. Cette stratégie consiste à découper le domaine d'intégration d'une représentation intégrale de la fonction zêta qui nous intéresse, le domaine dépendant d'une variable libre. Dans les faits, Onodera obtient l'expression suivante :

$$\begin{aligned} \zeta_P^{MT}((s, \dots, s), \mathbf{d}) &= \frac{1}{\Gamma(s)} \int_0^\theta \prod_{p=1}^P e^{-d_p y} \phi(e^{-y}, s, d_p) dy \\ &+ \frac{1}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{(n_1 + d_1)^s \dots (n_P + d_P)^s (n_1 + d_1 + \dots + n_P + d_P)} \int_{\theta(n_1 + d_1 + \dots + n_P + d_P)}^{+\infty} e^{-y} y^{s-1} dy \end{aligned}$$

avec ϕ la fonction zêta de Lerch décrite dans la Définition 1.1.7, et où θ est une variable libre réelle strictement positive. Le second terme à droite de l'égalité précédente a le bon goût d'être holomorphe sur \mathbb{C} selon la variable s , et il s'annule en les entiers négatifs. Le premier terme peut être réexprimé via la formule d'Erdélyi (2.3), et Onodera obtient finalement les formules suivantes :

Théorème 1.1.23 ([Ono21, Théorème 3, Théorème 4]). *Soit $\mathbf{d} \in \mathbb{R}_*^+{}^P$ et $t \in \llbracket 1, P-1 \rrbracket$ tel que $t < |\mathbf{d}|$. Pour tout entier $N \in \mathbb{N}$, on a*

$$\begin{aligned} \zeta_P^{MT}(-N, \dots, N), \mathbf{d} &= \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} \frac{(-1)^{(N+1)(|\mathcal{P}|)} N!^{|\mathcal{P}|+1}}{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=(|\mathcal{P}|+1)N+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!} \\ (\zeta_P^{MT})'(-N, \mathbf{d}) &= \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}^c|(N+1)-1} N!^{|\mathcal{P}^c|} \left(\sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ u \geq 0 \\ |\mathbf{k}|+u=(|\mathcal{P}^c|+1)N+|\mathcal{P}^c|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!} \right) \\ &\cdot \left(\frac{(-1)^N \zeta'(-N - u, |\mathbf{d}| - t)}{u!} + \frac{1}{|\mathcal{P}^c|} \sum_{p \in \mathcal{P}^c} \frac{(-1)^N \zeta'(-N - u, d_p)}{u!} \right) \end{aligned}$$

où l'on a noté $(\zeta_P^{MT})'(s, \mathbf{d}) := \partial_s \zeta_P^{MT}((s, \dots, s), \mathbf{d})$.

On généralisera dans cette thèse la stratégie d'Onodera, et on déduira de nos travaux des formules pour les valeurs spéciales des fonctions zêta de Witten $\zeta_{\mathfrak{so}(5)}(s)$ et $\zeta_{\mathfrak{g}_2}(s)$ ainsi que de leurs dérivées. On montrera via la Proposition 2.2.4 que les entiers négatifs ne sont pas des pôles pour $\zeta_{\mathfrak{so}(5)}(s)$ et pour $\zeta_{\mathfrak{g}_2}(s)$. L'étude de ces fonctions zêta permettra en particulier d'obtenir une formule asymptotique du nombre de représentations de dimension n des algèbres de Lie $\mathfrak{so}(5)$ et \mathfrak{g}_2 via un théorème de type Meinardus, complétant ainsi des travaux de Bridges, Bringmann, Brindle et Franke.

1.1.2 Quelques résultats sur les fonctions multizêta

On donnera ici quelques résultats sur les valeurs spéciales de certaines fonctions multizêta rattachées à des produits de formes linéaires. En particulier, on détaillera des expressions de ces valeurs pour les fonctions multizêta d'Euler-Zagier :

Définition 1.1.24. *Soit $P \in \mathbb{N}^*$, et $\mathbf{c} = (c_1, \dots, c_P) \in H_0^P$, $\mathbf{d} = (d_1, \dots, d_P) \in \mathbb{C}^P$ des nombres complexes tels que $\operatorname{Re}(d_i) > -\operatorname{Re}(c_1)$ pour tout $p \in \llbracket 1, P \rrbracket$. On appelle fonction multizêta d'Euler-Zagier généralisée*

la fonction

$$\zeta_P^{EZ}(\mathbf{s}, \mathbf{c}, \mathbf{d}) := \sum_{\substack{n_1 \geq 1 \\ n_2, \dots, n_P \geq 0}} \prod_{p=1}^P (c_p n_p + d_p)^{-s_p}.$$

On notera $\zeta_P^{EZ}(\mathbf{s}) := \zeta_P^{EZ}(\mathbf{s}, \mathbf{1}, (0, 1, \dots, P-1))$ la fonction multizêta d'Euler-Zagier.

Observons que, via un changement de variable, on peut réécrire la fonction multizêta d'Euler-Zagier "classique" sous la forme suivante,

$$\zeta_P^{EZ}(\mathbf{s}) = \sum_{0 < n_1 < \dots < n_P} \frac{1}{n_1^{s_1} \dots n_P^{s_P}}.$$

Proposition 1.1.25 ([EM20]). *La fonction multizêta d'Euler-Zagier généralisée $\mathbf{s} \mapsto \zeta_P^{EZ}(\mathbf{s}, \mathbf{c}, \mathbf{d})$ est holomorphe sur $\{(s_1, \dots, s_P) \in \mathbb{C}^P \mid \sigma_k + \dots + \sigma_P > P + 1 - k, 1 \leq k \leq P\}$, et est méromorphe sur \mathbb{C}^P , avec des singularités situées dans des hyperplans de la forme*

$$s_p + \dots + s_P = (P + 1 - p) - k_p \quad (1 \leq p \leq P, k_1, \dots, k_P \in \mathbb{N}).$$

On voit de la proposition précédente que la valeur spéciale $\zeta_P^{EZ}(-\mathbf{N}, \mathbf{c}, \mathbf{d})$ n'a pas toujours un sens puisque $-\mathbf{N}$ est possiblement une singularité pour la fonction multizêta d'Euler-Zagier. De plus, un résultat de [AET01] précise même que la plupart des multi-entiers négatifs sont en fait des singularités pour $\zeta_P^{EZ}(\mathbf{s})$. Afin d'étudier tout de même les valeurs de cette fonction, Akiyama, Egami et Tanigawa étudient les valeurs "régulières" de cette fonction multizêta en posant

$$\zeta_P^{EZ}(-\mathbf{N})^{\text{reg}} := \lim_{s_1 \rightarrow -N_1} \dots \lim_{s_P \rightarrow -N_P} \zeta_P^{EZ}(\mathbf{s}).$$

Les auteurs introduisent également des valeurs "non régulières" en inversant les limites précédentes, en posant

$$\zeta_P^{EZ}(-\mathbf{N})^{\text{non-reg}} := \lim_{s_P \rightarrow -N_P} \dots \lim_{s_1 \rightarrow -N_1} \zeta_P^{EZ}(\mathbf{s}).$$

Dans [AET01] et dans [AT01], les auteurs obtiennent une formule récursive entre les valeurs régulières $\zeta_P^{EZ}(-\mathbf{N})^{\text{reg}}$ (respectivement non régulières $\zeta_P^{EZ}(-\mathbf{N})^{\text{non-reg}}$) avec $\mathbf{N} \in \mathbb{N}^P$, et les valeurs régulières de la forme $\zeta_{P-1}^{EZ}(-\mathbf{N}')^{\text{reg}}$ (respectivement non régulières $\zeta_{P-1}^{EZ}(-\mathbf{N}')^{\text{non-reg}}$) avec $\mathbf{N}' \in \mathbb{N}^{P-1}$. Ils déduisent alors une expression explicite des valeurs régulières et non régulières aux entiers négatifs. Ces résultats ont par la suite été généralisés par Sasaki dans [Sas09a] et dans [Sas09b] à des ordres de limites différentes que celles des valeurs régulières et non-régulières.

On peut également étudier les valeurs "directionnelles" de cette fonction multizêta. Dans [Kom10], Komori introduit la notion de valeur directionnelle pour de telles fonctions multizêta, et donne une expression de celles-ci pour des fonctions multizêta de type Hurwitz-Lerch. Komori en déduit alors des formules pour les valeurs aux entiers négatifs des fonctions zêta de Shintani et d'Euler-Zagier, en utilisant des nombres de Bernoulli généralisés (souvent non explicites).

Définition 1.1.26. *Soit $P, Q \in \mathbb{N}^*$ des entiers, $\mathbf{d} = (d_1, \dots, d_Q)$, $\mathbf{c} = (c_{q,p})_{(q,p) \in \llbracket 1, Q \rrbracket \times \llbracket 1, P \rrbracket}$ et $\boldsymbol{\xi} = (\xi_1, \dots, \xi_P)$ des nombres complexes tels que $c_{q,p} \in \overline{H}_0$, $d_q \in H_0$ et $\xi_q \in \mathbb{T} = \mathbb{C}/2i\pi\mathbb{Z}$. On appelle fonction multizêta d'Hurwitz-Lerch la fonction multizêta*

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s}) := \sum_{n_1=0}^{+\infty} \dots \sum_{n_P=0}^{+\infty} \frac{e^{\xi_1 n_1} \dots e^{\xi_P n_P}}{(c_{1,1} n_1 + \dots + c_{1,P} n_P + d_1)^{s_1} \dots (c_{Q,1} n_1 + \dots + c_{Q,P} n_P + d_Q)^{s_Q}}.$$

On pose également les valeurs directionnelles de ζ^{HL} en $-\mathbf{N} \in -\mathbb{N}^Q$ suivant une direction $\boldsymbol{\mu} \in \mathbb{C}^P$:

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} := \lim_{s \rightarrow 0} \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N} + s\boldsymbol{\mu}).$$

Komori montre que cette fonction multizêta est holomorphe sur $\{\mathbf{s} \in \mathbb{C} \mid \forall q \in \llbracket 1, Q \rrbracket, \operatorname{Re}(s_q) > \sigma_{0,q}\}$ avec $\sigma_{0,q} > 0$ pour tout $q \in \llbracket 1, Q \rrbracket$. Il démontre également que cette fonction multizêta admet un prolongement méromorphe sur \mathbb{C}^Q selon les variables \mathbf{s} , et que les valeurs directionnelles définies précédemment ont un sens. Avant de brièvement résumer ses résultats, on doit imposer les conditions suivantes sur les données de la fonction multizêta ζ^{HL} :

On suppose qu'il existe un entier $P_0 \in \llbracket 1, P \rrbracket$ et un ensemble $\mathcal{B} \subset \llbracket 1, Q \rrbracket \times \llbracket 1, P_0 \rrbracket$ tels que

$$\begin{cases} c_{q,p} = 0 \text{ si et seulement si } (q,p) \in \mathcal{B}^c \\ \xi_p = 0 \text{ si et seulement si } p \in \llbracket 1, P_0 \rrbracket, \end{cases}$$

avec $\mathcal{B}^c = \llbracket 1, Q \rrbracket \times \llbracket 1, P_0 \rrbracket \setminus \mathcal{B}$. En posant $\mathcal{B}_{\bullet p} := \{q \mid (q,p) \in \mathcal{B}\}$, on suppose que $\mathcal{B}_{\bullet p} \neq \emptyset$ pour tout $p \in \llbracket 1, P_0 \rrbracket$. Komori obtient alors les propriétés analytiques suivante sur cette fonction multizêta.

Théorème 1.1.27 ([Kom10, Corollaire 3.16]). *Sous les hypothèses précédentes, la fonction multizêta $\mathbf{s} \mapsto \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s})$ admet un prolongement méromorphe sur tout \mathbb{C}^P . De plus, ses singularités sont situées dans les hyperplans*

$$\begin{cases} \sum_{q \in \mathcal{Q}} s_q = Q - k \quad (\mathcal{Q} \subset \llbracket 1, Q \rrbracket, |\mathcal{Q}| \geq 2, k \in \mathbb{N}) \\ s_q = 1, \dots, P \quad (1 \leq q \leq Q). \end{cases}$$

Plus précisément, Komori démontre le prolongement dans [Kom10] en passant par une représentation intégrale surfacique (c'est-à-dire une représentation intégrale dont le domaine est une surface) de la fonction multizêta $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s})$. Posons tout d'abord

$$G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}, \mathbf{z}) := \frac{e^{(c_{1,1} + \dots + c_{1,P-d_1})y_{w,1}(\mathbf{z})} \dots e^{(c_{Q,1} + \dots + c_{Q,P-d_Q})y_{w,Q}(\mathbf{z})} z_1^{t_1-1} \dots z_Q^{t_Q-1}}{(e^{y_{w,1}(\mathbf{z})c_{1,1} + \dots + y_{w,Q}(\mathbf{z})c_{Q,1}} - e^{\xi_1}) \dots (e^{y_{w,1}(\mathbf{z})c_{1,P} + \dots + y_{w,Q}(\mathbf{z})c_{Q,P}} - e^{\xi_P})},$$

où $y_{w,q}(\mathbf{z})$ correspond à un certain produit explicite des nombres complexes $z_1, \dots, z_Q \in \mathbb{C}$, et où $t_{w,q} \in \mathbb{C}$.

Théorème 1.1.28 ([Kom10, Théorème 3.14]). *Sous les mêmes hypothèses que le théorème précédent, il existe des surfaces lisses par morceaux $\Sigma_w \subset (\mathbb{C}^*)^Q$ telles que*

$$\begin{aligned} \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s}) &= \frac{1}{\Gamma(s_1) \dots \Gamma(s_Q)} \sum_{w \in S_Q} \frac{1}{(e^{2i\pi t_{w,1}(\mathbf{s})} - 1) \dots (e^{2i\pi t_{w,Q}(\mathbf{s})} - 1)} \\ &\quad \cdot \int_{\Sigma_w} G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}_w(\mathbf{s}), \mathbf{z}) dz_1 \wedge \dots \wedge dz_P \end{aligned}$$

où l'on a posé $t_{w,q}(\mathbf{s}) := \sum_{p \in \mathcal{P}_{w,q}} s_p$ une forme linéaire avec $\mathcal{P}_{w,q} \subset \llbracket 1, Q \rrbracket$ pour tout $w \in S_Q \subset \mathfrak{S}_Q$ et tout $1 \leq q \leq Q$.

Notons que toutes les surfaces Σ_w définies dans [Kom10, formule (41)] évitent les singularités de la fonction $\mathbf{z} \mapsto G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}, \mathbf{z})$ correspondante. Komori montre même dans [Kom10, Proposition 6.7] que la fonction

$$I_w : (\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}) \mapsto \int_{\Sigma_w} G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}, \mathbf{z}) dz_1 \wedge \dots \wedge dz_P \quad (1.5)$$

est holomorphe en les variables \mathbf{t} sur \mathbb{C}^Q . Ainsi en composant la fonction holomorphe précédente par $\mathbf{s} \mapsto \mathbf{t}_w(\mathbf{s})$ on trouve que l'intégrale paramétrée dans la représentation intégrale surfacique précédente est holomorphe sur \mathbb{C}^Q . Komori obtient en particulier que la fonction

$$\begin{aligned} \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N} + s\boldsymbol{\mu}) &= \sum_{w \in S_Q} \frac{1}{\Gamma(-N_1 + \mu_1 s) \dots \Gamma(-N_Q + \mu_Q s) (e^{2i\pi s t_{w,1}(\boldsymbol{\mu})} - 1) \dots (e^{2i\pi s t_{w,Q}(\boldsymbol{\mu})} - 1)} \\ &\quad \cdot \int_{\Sigma_w} G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}_w(-\mathbf{N} + \boldsymbol{\mu}s), \mathbf{z}) dz_1 \wedge \dots \wedge dz_P \end{aligned} \quad (1.6)$$

est régulière en $s = 0$ pour toutes les directions en dehors de celles appartenant à une union d'hyperplans décrits par les relations $t_{w,q}(\boldsymbol{\mu}) = 0$. Ainsi les valeurs directionnelles ont bien un sens pour toutes les directions situées en dehors de ces hyperplans.

Notons que cette représentation intégrale surfacique généralise la représentation intégrale usuelle de la fonction zêta de Riemann

$$\zeta(s) = \frac{1}{\Gamma(s)(e^{2i\pi s} - 1)} \int_{\mathcal{C}} \frac{z^{s-1}}{e^z - 1} dz,$$

où \mathcal{C} désigne le contour de Hankel (voir [EMOT81, §1.10]). Au passage, il est bon de rappeler que cette représentation intégrale de $\zeta(s)$ permet en particulier d'obtenir la relation classique entre les nombres de Bernoulli et les valeurs de la fonction zêta de Riemann aux entiers négatifs. Dans la même veine, Komori utilise dans [Kom10, Théorème 3.22] sa représentation intégrale surfacique afin d'obtenir une expression des valeurs directionnelles en fonction de nombres de Bernoulli généralisés $B_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{k})$ avec $\mathbf{k} \in \mathbb{N}^Q$ définies par les fonctions génératrices $G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}, \mathbf{z})$ (voir [Kom10, Définition 3.20]). Remarquons que l'existence de ces valeurs directionnelles présuppose une condition de non-annulation sur la somme des composantes de la direction considérée.

Théorème 1.1.29 ([Kom10, Théorème 3.22]). *Pour toute direction $\boldsymbol{\mu} \in \mathbb{C}^Q$ telle que pour tout $w \in S_Q$, $t_{w,1}(\boldsymbol{\mu}), \dots, t_{w,Q}(\boldsymbol{\mu}) \neq 0$, on a*

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} = (-1)^{|\mathbf{N}|} N_1! \dots N_Q! \sum_{w \in S_Q} \frac{B_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{k}_w)}{k_{w,1}! \dots k_{w,Q}!} \prod_{q=1}^Q \frac{\mu_q}{t_{w,q}(\boldsymbol{\mu})}$$

avec $\mathbf{k}_w = (k_{w,1}, \dots, k_{w,Q}) \in \mathbb{N}^Q$ des entiers explicites dépendants de $\mathbf{N} = (N_1, \dots, N_Q) \in \mathbb{N}^Q$.

On obtient au passage que, si l'on arrive à réexprimer les nombres de Bernoulli généralisés dans la formule obtenue par Komori, alors cette formule permet de fournir un critère suffisant (mais pas nécessaire) pour prouver qu'un multi-entier négatif est une singularité pour ζ^{HL} . En effet, si l'on obtient deux valeurs distinctes $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} \neq \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}'}$ selon deux directions différentes $\boldsymbol{\mu} \neq \boldsymbol{\mu}'$ alors le multi-entier \mathbf{N} est une singularité.

Remarque 1.1.30. *La formule pour les valeurs directionnelles obtenue par Komori n'est pas très explicite puisqu'il est souvent difficile d'obtenir une expression satisfaisante des nombres de Bernoulli généralisés qu'il introduit dans son article.*

Komori ne s'est pas intéressé aux valeurs dérivées directionnelles des fonctions multizêta de type Hurwitz-Lerch définies par

$$(\zeta^{HL})'(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} := \partial_s (\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N} + s\boldsymbol{\mu}))_{|s=0}.$$

On ne peut pas déduire de la représentation intégrale surfacique du Théorème 1.1.28 une expression sympathique de ce genre de valeurs. Cependant on peut tout de même déduire une propriété analytique de ces valeurs, et l'on utilisera cette propriété plus tard dans le manuscrit.

Corollaire 1.1.31. *Pour tout multi-entier $\mathbf{N} \in \mathbb{N}^Q$, les deux fonctions*

$$\begin{aligned} \boldsymbol{\mu} &\mapsto \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} \\ \boldsymbol{\mu} &\mapsto (\zeta^{HL})'(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} \end{aligned}$$

sont méromorphes sur \mathbb{C}^Q avec des singularités situées dans des hyperplans de la forme $t_{w,q}(\boldsymbol{\mu}) = 0$ avec $w \in S_Q$ et $q \in [1, Q]$.

Démonstration. Du Théorème 1.1.29 on obtient que la valeur directionnelle $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}}$ est une fraction rationnelle en les variables μ_1, \dots, μ_Q avec des dénominateurs de la forme $t_{w,q}(\boldsymbol{\mu})$. On trouve alors que la fonction

$$\boldsymbol{\mu} \mapsto \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}}$$

est clairement méromorphe sur \mathbb{C}^Q .

Pour tout $w \in S_Q$ on pose

$$F_w(\mathbf{N}, \boldsymbol{\mu}) := \partial_s \left(\frac{1}{\Gamma(-N_1 + \mu_1 s) \dots \Gamma(-N_Q + \mu_Q s)} \frac{1}{(e^{2i\pi s t_{w,1}(\boldsymbol{\mu})} - 1) \dots (e^{2i\pi s t_{w,Q}(\boldsymbol{\mu})} - 1)} \right)_{|s=0},$$

et l'on observe par un calcul élémentaire que F_w est une fraction rationnelle en les variables μ_1, \dots, μ_Q avec des dénominateurs constituées de produits de certaines formes linéaires $t_{w,q}(\boldsymbol{\mu})$ avec $q \in \llbracket 1, Q \rrbracket$. En évaluant en $s = 0$ la dérivée de l'expression (1.6) on trouve alors

$$\begin{aligned} (\zeta^{HL})'(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N}) &= \sum_{w \in S_Q} F_w(\mathbf{N}, \boldsymbol{\mu}) \frac{B_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{k}_w)}{k_{w,1}! \dots k_{w,Q}!} \\ &+ (-1)^{|\mathbf{N}|} N_1! \dots N_Q! \sum_{w \in S_Q} \sum_{j=1}^Q \frac{\partial I_w}{\partial t_j}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N}) \prod_{\substack{q=1 \\ q \neq j}}^Q \frac{\mu_q}{t_{w,q}(\boldsymbol{\mu})} \end{aligned}$$

où $\mathbf{k}_w \in \mathbb{N}^Q$ sont des entiers explicites (voir [Kom10, Théorème 3.22]) pour chaque $w \in S_Q$, et où $\frac{\partial I_w}{\partial t_j}$ désigne la dérivée d'ordre 1 selon la variable t_j de l'intégrale paramétrique $I_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t})$ définie dans l'expression (1.5). On voit alors que l'expression obtenue pour $(\zeta^{HL})'(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})$ est méromorphe en la direction $\boldsymbol{\mu}$ sur \mathbb{C}^Q ce qui conclut. \square

Dans [EM20], Essouabri et Matsumoto ont étudié les valeurs directionnelles de la fonction multizêta d'Euler-Zagier généralisée. On donnera ici seulement une formule très simplifiée et beaucoup moins détaillée :

Théorème 1.1.32 ([EM20, Théorème 1]). *Considérons des nombres complexes $\mathbf{c} = (c_1, \dots, c_P) \in H_0^P$ et $\mathbf{d} = (d_1, \dots, d_P) \in \overline{H}_0^P$, un multi-entier $\mathbf{N} \in \mathbb{N}^P$ et une direction $\boldsymbol{\mu} \in \mathbb{C}^P$ telle que $\operatorname{Re}(\mu_p + \dots + \mu_P) \neq 0$ pour tout $p \in \llbracket 1, P \rrbracket$. Alors la valeur directionnelle*

$$\zeta_P^{EZ}(-\mathbf{N}, \mathbf{c}, \mathbf{d}) := \lim_{s \rightarrow 0} \zeta_P^{EZ}(-\mathbf{N} + s\boldsymbol{\mu}, \mathbf{c}, \mathbf{d})$$

existe, et on a :

$$\zeta_P^{EZ}(-\mathbf{N}, \mathbf{c}, \mathbf{d}) = \sum_{\mathbf{k}=(k_1, \dots, k_P) \in \llbracket 0, |\mathbf{N}|+P+1 \rrbracket^P} f_{\mathbf{k}}(\mathbf{c}, \mathbf{d}, \boldsymbol{\mu}) \prod_{i=1}^P B_{k_i},$$

avec B_n qui désigne le n -ième nombre de Bernoulli, et $f_{\mathbf{k}}(\mathbf{c}, \mathbf{d}, \boldsymbol{\mu})$ une fraction rationnelle explicite en les variables c_p et μ_p pour $p \in \llbracket 1, P \rrbracket$.

Remarque 1.1.33. 1) Dans [EM21], Essouabri et Matsumoto ont obtenu un résultat similaire pour les valeurs spéciales d'une classe de fonction multizêta plus générale de la forme

$$\zeta_m(\mathbf{s}, \mathbf{P}) := \sum_{n_1, \dots, n_m \geq 1} \frac{1}{\prod_{j=1}^m P_j(n_1, \dots, n_j)},$$

avec P_j un polynôme en les X_1, \dots, X_j variables vérifiant la condition H0S (voir [Ess97]) et tel que

$$\lim_{x_1 + \dots + x_j \rightarrow +\infty} P_j(x_1, \dots, x_j) = +\infty,$$

et en imposant que le polynôme P_m est homogène et elliptique.

2) Dans [MW02], Matsumoto et Weng ont étudié des fonctions zêta de la forme

$$\zeta^{MW}(s) := \sum_{n=1}^{+\infty} \frac{P(n)}{Q(n)^s},$$

avec P et Q deux polynômes non nuls tels que Q ne s'annule pas en les entiers strictement positifs. Matsumoto et Weng obtiennent une formule de la dérivée en $s = 0$. L'intérêt de ce genre de formule provient de l'étude du déterminant régularisé du Laplacien de la n -sphère (voir [Var88], [WY96], [QC96], [Kum99]).

1.1.3 Sur les formules asymptotiques de différents nombres de partitions

On détaillera dans cette sous-section quelques applications du calcul des valeurs spéciales de certaines fonctions zêta de Witten et de leurs dérivées. On discutera également du théorème de Meinardus qui fournit une formule asymptotique de certains nombres de partitions générés par des produits infinis. On mentionnera également des travaux récents de Bridges, Bringmann, Craig, Franke, Males, à propos de formules asymptotiques de certains nombres de partitions dans les articles [BB24], [BBF24], [BCM23], [BBBF24]. On donnera en particulier quelques résultats obtenus par Bridges, Bringmann, Brindle et Franke dans [BBBF24], et on parlera de leur approche.

En combinatoire, il est classique d'utiliser une série génératrice

$$\omega(q) := \sum_{n \geq 0} r(n)q^n$$

afin d'étudier le comportement d'une suite de nombres complexes $r(n)$. Si la fonction ω est holomorphe sur $|q| < 1$, alors on a par le théorème de Cauchy que

$$r(n) = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{\omega(q)}{q^{n+1}} dq,$$

avec \mathcal{C} un cercle centré en 0 parcouru dans le sens trigonométrique, inclus dans le disque unité. En général, il n'existe pas de résultat permettant d'obtenir une relation asymptotique pour une suite $r(n)$ quelconque. On se concentrera dans la suite de cette section sur une série génératrice s'écrivant sous la forme d'un produit infini.

On considère une fonction $G_f(q)$ définie par

$$G_f(q) := \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f(n)}}$$

avec $f(n) \in \mathbb{N}$ des entiers positifs. En distribuant le produit infini ci-dessus, on peut définir une suite $p_f(n)$ par la relation

$$G_f(q) =: \sum_{n=0}^{+\infty} p_f(n)q^n. \quad (1.7)$$

On considère la série de Dirichlet

$$L_f(s) := \sum_{n=1}^{+\infty} \frac{f(n)}{n^s} \quad (s = \sigma + i\tau, \sigma > \sigma_0),$$

où on a noté σ_0 son abscisse de convergence lorsqu'elle existe.

Exemple 1.1.34 ([HR17]). *Considérons $p(n)$ le nombre de partitions d'un entier n quelconque. Il est classique de définir ce nombre de partitions par la relation suivante*

$$1 + \sum_{n=1}^{+\infty} p(n)q^n := \prod_{n=1}^{+\infty} \frac{1}{1 - q^n}.$$

La série de Dirichlet correspondante est la fonction zêta de Riemann. Hardy et Ramanujan ont établi la formule asymptotique suivante

$$p(n) \underset{n \rightarrow +\infty}{\sim} \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

On pourra consulter une démonstration de cette formule asymptotique dans le livre d'Andrews [And76, 5.1.2].

On pose maintenant

$$g_f(z) := \sum_{n=1}^{+\infty} f(n)e^{-nz}.$$

On suppose les conditions suivantes :

- i) L'abscisse de convergence σ_0 de L_f est strictement positive.
- ii) La fonction L_f admet un prolongement méromorphe sur H_{-C_0} avec $0 < C_0 < 1$, et tel que L_f ne possède qu'un seul pôle en $s = \sigma_0$ sur ce domaine, que ce pôle soit d'ordre 1. Notons A le résidu en ce pôle.
- iii) On a la majoration suivante, uniforme sur $\sigma > -C_0$, lorsque $\tau \rightarrow +\infty$

$$|L_f(s)| = O(|\tau|^{C_1}) \quad (s = \sigma + i\tau)$$

avec $C_1 > 0$.

- iv) Soit $\epsilon > 0$. Pour tout $z = x + 2i\pi y$ tel que $\arg(z) > \frac{\pi}{4}$, $|x| \leq \frac{1}{2}$ et $|y| \ll 1$, on a

$$\operatorname{Re}(g_f(z)) - g_f(x) \leq -C_2 x^{-\epsilon},$$

avec $C_2 > 0$ une constante dépendant de ϵ .

Le théorème de Meinardus donne alors une relation asymptotique pour la suite $p_f(n)$.

Théorème 1.1.35 (de Meinardus, [And76, Théorème 6.2]). *Pour $n \rightarrow +\infty$, on a*

$$p_f(n) = Cn^\kappa \exp\left(n^{\frac{\alpha}{\alpha+1}} \left(1 + \frac{1}{\alpha}\right) (A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha}}\right) (1 + O(n^{-\kappa_1})),$$

avec $p_f(n)$ définie via la formule (1.7), et C, κ, κ_1 des réels dépendant de $L_f(0), L'_f(0), C_0$ et α .

La démonstration de ce théorème repose sur la méthode du point selle, dont on expliquera quelques points clés ici. Pour les détails de la méthode, on se réfère à [FS09, VIII.3]. La méthode exploite le fait que l'intégrale $\frac{1}{2i\pi} \int_{\mathcal{C}} \frac{G_f(q)}{q^{n+1}} dq$ est indépendante du rayon du cercle \mathcal{C} inclus dans le disque unité. On peut alors utiliser cette indépendance en choisissant judicieusement un rayon $\rho_0(n)$ adéquat. La méthode du point selle consiste à choisir un rayon proche de 1 de tel sorte à minimiser $\left| \frac{G_f(q)}{q^n} \right|$ sur un arc de cercle mineur du cercle $\mathcal{C}_1(\theta_0(n))$. Concrètement, on choisit ce rayon de tel sorte que la dérivée de l'intégrande en $q = \rho_0(n)$ soit un $O(n^\delta)$, avec δ bien choisie. Alternativement, on peut chercher $\rho_0(n)$ en étudiant la dérivée de la fonction

$$h_f(z) := \ln(L_f(z)) - n \log(z).$$

On cherche alors $\rho_0(n)$ de tel sorte que

$$h'_f(\rho_0(n)) = -n + O(n^\delta).$$

On aura ainsi une approximation quadratique lorsque $\theta \rightarrow 0$:

$$h_f(\rho_0(n)e^{i\theta}) - h_f(\rho_0(n)) =: -\frac{1}{2}\beta(\rho_0(n))\theta^2 + o(\theta^3) + O(n^\delta)$$

où $\beta(\rho_0(n))$ est une constante dépendant de $\rho_0(n)$, et du choix de δ . On cherchera alors à déterminer un angle $\theta_0(n)$ de tel sorte que

$$\beta(\rho_0(n))\theta_0(n)^2 \xrightarrow{\theta \rightarrow 0} +\infty, \quad h''(\rho_0(n))\theta_0(n)^3 \xrightarrow{\theta \rightarrow 0} 0.$$

Pour un tel choix de $\theta_0(n)$, on peut finalement déduire que le comportement de l'intégrale sur l'arc mineur C_0 se comporte comme une intégrale gaussienne, dont les bornes dépendent de n , que l'on sait approximer.

Debruyne et Tennenbaum ont obtenu dans [DT20] une version plus générale du théorème de Meinardus en enlevant la condition iv) sur la croissance de la fonction g_f . Observons aussi que certains nombres de partitions issus de la physique ont été étudié par Granovsky et Stark dans [GSE08]. Ceux-ci ont alors établi une version multipolaire du théorème de Meinardus, avec toutefois des conditions sur la répartition de la suite $(f(n))_{n \in \mathbb{N}^*}$ et des conditions de croissance sur la fonction analytique $L_f(s)$.

À l'aide de ces théorèmes généraux on peut espérer étudier des nombres de partitions exotiques comme par exemple le nombre de représentations de dimension n d'une algèbre de Lie semi-simple \mathfrak{g} . En effet, on a déjà vu via la formule (1.2) que ce nombre de représentations correspond à un nombre de partitions généré par un produit infini associé à la suite $(f_{\mathfrak{g}}(n))_{n \geq 1}$, et donc la fonction $L_{f_{\mathfrak{g}}}(s)$ correspondante est la fonction zêta de Witten $\zeta_{\mathfrak{g}}(s)$. On s'aperçoit alors que l'on ne peut pas appliquer simplement le théorème de Meinardus pour obtenir une expression asymptotique du nombre de représentations $r_{\mathfrak{g}}(n)$ de n'importe quelle algèbre de Lie puisque les fonctions zêta de Witten ont en général plus d'un seul pôle positif. Cependant, en modifiant astucieusement la démonstration du théorème de Meinardus, en étudiant les résidus de $\zeta_{\mathfrak{sl}(3)}(s)$ en $s = \frac{1}{2}$ et en $s = \frac{2}{3}$, puis en calculant les valeurs spéciales $\zeta_{\mathfrak{sl}(3)}(0)$ et $\zeta'_{\mathfrak{sl}(3)}(0)$, Romik a obtenu le théorème suivant :

Théorème 1.1.36 ([Rom17, Théorème 1.1]). *Lorsque $n \rightarrow +\infty$, on a*

$$r_{\mathfrak{sl}(3)}(n) \underset{n \rightarrow +\infty}{\sim} \frac{K}{n^{3/5}} \exp\left(A_1 n^{2/5} - A_2 n^{3/10} - A_3 n^{1/5} - A_4 n^{1/10}\right),$$

avec A_1, A_2, A_3, A_4 et K des constantes explicites. On pose tout d'abord

$$X := \left(\frac{1}{9} \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{5}{3}\right)\right), \quad Y := -\sqrt{\pi} \zeta\left(\frac{1}{2}\right) \zeta\left(\frac{3}{2}\right).$$

Les constantes A_1, A_2, A_3, A_4, K sont alors définies par les relations suivantes :

$$A_1 := 5X^2 = 6.858260476\dots$$

$$A_2 := X^{-1}Y = 5.77360174\dots$$

$$A_3 := \frac{3}{80} X^{-4} Y^2 = 0.91134107\dots$$

$$A_4 := \frac{11}{3200} X^{-7} Y^3 = 0.35163754\dots$$

$$K := \frac{2\sqrt{3\pi}}{\sqrt{5}} X^{1/3} \exp\left(-\frac{1}{2560} X^{-10} Y^4\right) = 2.44629033486\dots$$

Notons que dans [Rom17], le calcul de la constante Y dépend en particulier de la valeur spéciale $\zeta'_{\mathfrak{sl}(3)}(0) = -\ln(2\pi) + \ln(2)$ calculée dans [BD18]. De plus, toutes ces constantes dépendent également du calcul des résidus de $\zeta_{\mathfrak{sl}(3)}(s)$ en $s = \frac{1}{2}$ et en $s = \frac{2}{3}$.

Par la suite, Bridges, Bringmann et Franke ont obtenu dans [BBF24] une formule asymptotique pour le nombre de représentations irréductibles de dimension n de $\mathfrak{sl}(3)$ en étudiant la fonction zêta de Witten $\zeta_{\mathfrak{sl}(3)}(s)$, et en utilisant un théorème taubérien.

Romik conjecture qu'il est possible d'obtenir des résultats similaires sur des nombres de représentations $r_{\mathfrak{g}}(n)$ pour d'autres algèbres de Lie \mathfrak{g} . Pour étendre ce résultat, on a besoin d'établir un théorème de type Meinardus, et également d'obtenir des informations sur les pôles positifs et les résidus de la fonction zêta de Witten correspondante, puis d'obtenir une expression pour la valeur de la dérivée $\zeta'_{\mathfrak{g}}(0)$.

Motivés par cette question, Bridges, Brindle, Bringmann et Franke ont établi une variante plus générale du théorème de Meinardus dans l'article [BBBF24]. Soit $f : \mathbb{N}^* \rightarrow \mathbb{N}$ une fonction arithmétique. On pose pour tout $q = e^{-z}$, $z \in H_0$ les fonctions

$$G_f(z) := \sum_{n=0}^{+\infty} p_f(n) q^n = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f(n)}}, \quad L_f(s) := \sum_{n=1}^{+\infty} \frac{f(n)}{n^s}.$$

On pose $\Lambda := \mathbb{N} \setminus f^{-1}(\{0\})$. On suppose les conditions suivantes :

(P1) Soit $\alpha > 0$ le plus grand pôle de la fonction L_f . Il existe un entier $L \in \mathbb{N}$ tel que, pour tout nombre premier p ,

$$|\Lambda \setminus (p\mathbb{N} \cap \Lambda)| \geq L > \frac{\alpha}{2}.$$

(P2) Il existe un réel $R \in \mathbb{R}_+$ tel que la fonction L_f soit méromorphe sur $\overline{H}_{-R} = \{z \in \mathbb{C}, \operatorname{Re}(z) \geq -R\}$, et tel que la fonction L_f soit holomorphe sur la droite ($\operatorname{Re}(z) = -R$). De plus, on suppose que la fonction méromorphe $L_f^*(s) := \Gamma(s)\zeta(s+1)L_f(s)$ possède uniquement des pôles réels $\alpha := \gamma_1 > \dots$, que ces pôles sont simples, à part en $s = 0$ où le pôle peut être un pôle double.

(P3) Il existe un réel $a < \frac{\pi}{2}$ tel que, sur chaque bande $\sigma_1 \leq \sigma \leq \sigma_2$ dans le domaine d'holomorphie de L_f , on a

$$L_f(s) = O_{\sigma_1, \sigma_2} \left(e^{a|\tau|} \right) \quad (s = \sigma + i\tau)$$

lorsque $|\tau| \rightarrow +\infty$.

Théorème 1.1.37 ([BBBF24, Théorème 1.4]). *On suppose les conditions (P1), (P2) et (P3). Notons L le réel de la condition (P1) et R le réel de la condition (P2). Alors pour $N \in \mathbb{N}$, on a*

$$p_f(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp \left(A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^M A_j n^{\alpha_j} \right) \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R} \left(n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}} \right) \right)$$

avec $M \in \mathbb{N}^*$, $0 \leq \alpha_M < \dots < \alpha_1 := \frac{\alpha}{\alpha+1}$, $0 < \nu_2 < \dots < \nu_N$, A_1, \dots, A_M et B_2, \dots, B_N des constantes explicites.

Lorsque la fonction méromorphe L_f ne possède que deux pôles strictement positifs, Bridges, Brindle, Bringmann et Franke obtiennent une formule plus précise. Introduisons tout d'abord les constantes suivantes : Posons $\alpha > \beta > 0$ les deux seuls pôles positifs de L_f . On note $\omega_\alpha := \operatorname{Res}_{s=\alpha}(L_f(s))$ et $\omega_\beta := \operatorname{Res}_{s=\beta}(L_f(s))$ les résidus de L_f en ses pôles positifs, et on pose $c_1 := \omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1)$, $c_2 := \omega_\beta \Gamma(\beta+1)\zeta(\beta+1)$, $c_3 := L_f(0)$. On note

$$C := \frac{e^{L_f'(0)} (\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{\frac{1}{2}-L_f(0)}{\alpha+1}}}{\sqrt{2\pi(\alpha+1)}} \quad b := \frac{1 - L_f(0) + \frac{\alpha}{2}}{\alpha+1}.$$

On note \mathcal{P}_R l'union du singleton $\{0\}$ et des pôles plus grand que $-R$ de L_f^* . Posons également

$$\mathcal{L} := \frac{1}{\alpha+1} \mathcal{P}_R + \sum_{\mu \in \mathcal{P}_R} \left(\frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N},$$

$$\mathcal{M} := \frac{\alpha}{\alpha+1} \mathbb{N} + \left(- \sum_{\mu \in \mathcal{P}_R} \left(\frac{\mu+1}{\alpha-1} - 1 \right) \mathbb{N} \right) \cap \left[0, \frac{R+\alpha}{\alpha+1} \right[.$$

Théorème 1.1.38 ([BBBF24, Théorème 4.4]). *On suppose les conditions (P1), (P2) et (P3). Notons L le réel de la condition (P1) et R le réel de la condition (P2). De plus, on suppose que L_f n'a que deux pôles $\alpha > \beta > 0$ tels qu'il existe un entier $l \in \mathbb{N}^*$ vérifiant l'inégalité $\frac{l+1}{l}\beta < \alpha < \frac{l}{l-1}\beta$. Alors on a*

$$p_f(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp \left(A_1 n^{\frac{\alpha}{\alpha+1}} + A_2 n^{\frac{\beta}{\alpha+1}} + \sum_{j=3}^{l+1} A_j n^{\frac{(j-1)\beta}{\alpha+1} + \frac{j-2}{\alpha+1} + 2-j} \right) \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R} \left(n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}} \right) \right),$$

avec B_2, \dots, B_N des constantes explicites, et

$$\begin{aligned}
A_1 &:= (\omega_\alpha \Gamma(\alpha + 1) \zeta(\alpha + 1))^{\frac{1}{\alpha+1}} \left(1 + \frac{1}{\alpha}\right), & A_2 &:= \frac{\omega_\beta \Gamma(\beta) \zeta(\beta + 1)}{(w_\alpha \Gamma(\alpha + 1) \zeta(\alpha + 1))^{\frac{\beta}{\alpha+1}}}, \\
A_j &:= K_j + \frac{c_1^{\frac{1}{\alpha+1}}}{\alpha} \sum_{m=1}^l \binom{-\alpha}{m} \sum_{\substack{0 \leq k_1 \leq \dots \leq k_l \leq m \\ |\mathbf{k}|=m \\ k_1+2k_2+\dots+l k_l=j-1}} \binom{m}{\mathbf{k}} \frac{K_2^{k_1} \dots K_{l+1}^{k_l}}{c_1^{\frac{m}{\alpha+1}}} \\
&\quad + \frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}} \sum_{m=1}^l \binom{-\beta}{m} \sum_{\substack{0 \leq k_1 \leq \dots \leq k_l \leq m \\ |\mathbf{k}|=m \\ k_1+2k_2+\dots+l k_l=j-2}} \binom{m}{\mathbf{k}} \frac{K_2^{k_1} \dots K_{l+1}^{k_l}}{c_1^{\frac{m}{\alpha+1}}} \quad (j \geq 3),
\end{aligned}$$

avec $(K_j)_{j \geq 3}$ décrit dans [BBBF24, Lemme 4.3], et $0 < \nu_2 < \dots$ les éléments strictement positifs de $\mathcal{L} + \mathcal{M}$.

Remarque 1.1.39. Étant donné une algèbre de Lie semi-simple \mathfrak{g} , le nombre de représentations $r_{\mathfrak{g}}(n)$ correspond à un nombre de partitions associé à un produit infini

$$1 + \sum_{n=1}^{+\infty} r_{\mathfrak{g}}(n) q^n = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f_{\mathfrak{g}}(n)}}.$$

Lorsqu'on appliquera le Théorème de type Meinardus 1.1.37 à l'étude asymptotique de la suite des $r_{\mathfrak{g}}(n)$, on voit alors que les entiers $f(n)$ correspondent aux nombres de représentations irréductibles (à isomorphisme près) de dimension n de \mathfrak{g} , que l'on a noté $f_{\mathfrak{g}}(n)$. Ainsi la fonction $L_{f_{\mathfrak{g}}}(s)$ correspondante est en réalité la fonction zêta de Witten $\zeta_{\mathfrak{g}}(s)$. Comme la fonction $\zeta_{\mathfrak{g}}(s)$ vérifie la condition HOS d'Essouabri (voir [Ess97]), on trouve que la fonction $L_{f_{\mathfrak{g}}}(s)$ possède un prolongement méromorphe sur \mathbb{C} , et que la condition (P3) est vraie. Il reste alors à regarder si ce prolongement méromorphe ne possède que des pôles positifs qui sont simples afin de vérifier si la condition (P2) est vraie ou non.

Remarque 1.1.40. Pour étudier le comportement asymptotique de $r_{\mathfrak{g}_2}(n)$, on aurait besoin de connaître K_3 , et d'après [BBBF24, Lemme 4.3], on a

$$K_3 := \frac{c_2^2(\alpha - 2\beta)}{2(\alpha + 1)^2 c_1^{\frac{2\beta+1}{\alpha+1}}},$$

avec c_1 et c_2 les constantes précédemment décrites par les relations

$$c_1 := \omega_\alpha \Gamma(\alpha + 1) \zeta(\alpha + 1), \quad c_2 := \omega_\beta \Gamma(\beta + 1) \zeta(\beta + 1).$$

De plus, pour étudier le comportement asymptotique de $r_{\mathfrak{g}_2}(n)$, on verra que l'entier $l = 2$ fonctionne car les pôles positifs de la fonction $L_{f_{\mathfrak{g}_2}}(s) = \zeta_{\mathfrak{g}_2}(s)$ sont $s = \frac{1}{3}$ et $s = \frac{1}{5}$.

En appliquant ce théorème, les 4 auteurs ont obtenu des résultats similaires à Romik pour le nombre de représentations $r_{\mathfrak{so}(5)}(n)$:

Théorème 1.1.41 ([BBBF24, Théorème 1.3]). *Pour tout entier $N \geq 1$, on a*

$$r_{\mathfrak{so}(5)}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^{\frac{7}{12}}} \exp \left(A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4 \right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N \left(n^{-\frac{N+1}{9}} \right) \right),$$

avec B_2, \dots, B_{N+1} des constantes explicites, et

$$C = \frac{e^{\zeta'_{s_0(5)}(0)} \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}}{2^{\frac{1}{3}} 3^{\frac{11}{24}} \sqrt{\pi}}, \quad A_1 = \frac{3^{\frac{4}{3}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{3}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{8}{3}}},$$

$$A_2 = \frac{2^{\frac{8}{9}} \left(2^{\frac{1}{3}} + 1\right) \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right)}{3^{\frac{7}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{9}}}, \quad A_3 = \frac{2^{\frac{40}{9}} \left(2^{\frac{1}{3}} + 1\right)^2 \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{1}{3}\right)^2 \zeta\left(\frac{4}{3}\right)^2}{3^{\frac{44}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{20}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{10}{9}}},$$

$$A_4 = \frac{2^8 \left(2^{\frac{1}{3}} + 1\right)^3 \Gamma\left(\frac{1}{3}\right)^3 \zeta\left(\frac{1}{3}\right)^3 \zeta\left(\frac{4}{3}\right)^3}{3^8 \Gamma\left(\frac{1}{4}\right)^4 \zeta\left(\frac{3}{2}\right)^2}.$$

Les 4 auteurs de ce résultat ont calculé la valeur $\zeta_{s_0(5)}(0)$ et les résidus de la fonction zêta de Witten $\zeta_{s_0(5)}(s)$ en $s = \frac{1}{2}$ et en $s = \frac{1}{3}$, cependant ils ne disposaient pas d'une formule explicite de la valeur de la dérivée $\zeta'_{s_0(5)}(0)$. Dans le chapitre 6 on donnera d'autres expressions de ces même résidus, et on retrouvera la même valeur de $\zeta_{s_0(5)}(0)$ via une autre méthode que celle utilisée par les 4 auteurs. On donnera également une expression simple de la valeur de la dérivée $\zeta'_{s_0(5)}(0)$ qui fournira alors une expression explicite du coefficient C , permettant alors de préciser le théorème précédent en une formule asymptotique explicite pour le nombre de représentations $r_{s_0(5)}(n)$.

On obtiendra également dans le chapitre 6 des formules pour les résidus de $\zeta_{g_2}(s)$ en $s = \frac{1}{3}$ et en $s = \frac{1}{5}$, et des formules explicites pour la valeur spéciale $\zeta_{g_2}(0)$ et de la valeur de la dérivée $\zeta'_{g_2}(0)$. On pourra alors établir un théorème similaire au théorème précédent pour le nombre de représentations $r_{g_2}(n)$ en utilisant le Théorème de type Meinardus 1.1.38.

1.2 Premières définitions

On fixera dans tout ce manuscrit une famille $(l_q)_{q \in \llbracket 1, Q \rrbracket}$ de formes linéaires de rang P à coefficients dans le demi-plan de Poincaré $H_0 = \{s = \sigma + i\tau \in \mathbb{C} \mid \sigma > 0\}$ avec $P \geq 1$ et $Q \geq 1$ des entiers. On fixe également $\mathbf{d} = (d_p)_{p \in \llbracket 1, P \rrbracket}$, $\mathbf{d}' = (d'_q)_{q \in \llbracket 1, Q \rrbracket}$ des nombres complexes dans H_0 tels que $d'_q := l_q(\mathbf{d})$ pour tout $1 \leq q \leq Q$.

Notation. Pour tout entier $q \in \llbracket 1, Q \rrbracket$ on note

$$l_q(\mathbf{n}) = \sum_{p=1}^P c_{q,p} n_p.$$

Par souci de clarté, on choisira tout le long de ce manuscrit d'utiliser l'indice p (ou l'indice i si p est déjà utilisé) lorsque l'on parcourra n'importe quel sous-ensemble \mathcal{P} de $\llbracket 1, P \rrbracket$, et on utilisera l'indice q (ou l'indice j si q est déjà utilisé) lorsque l'on parcourra n'importe quel sous-ensemble \mathcal{Q} de $\llbracket 1, Q \rrbracket$.

On définit tout d'abord les fonctions gamma incomplètes

Définition 1.2.1. Soit $\nu \in H_0$, $\theta > 0$, $s \in \mathbb{C}$. On pose alors

$$\Gamma(s, \theta, \nu) = \int_{\theta}^{+\infty} e^{-\nu y} y^{s-1} dy,$$

$$\gamma(s, \theta, \nu) = \int_0^{\theta} e^{-\nu y} y^{s-1} dy \quad \text{lorsque } \operatorname{Re}(s) > 0.$$

On appelle $\Gamma(s, \theta, \nu)$ la fonction gamma incomplète supérieure, et $\gamma(s, \theta, \nu)$ la fonction gamma incomplète inférieure.

Ces fonctions gamma incomplètes sont plus générales que celles trouvées dans la littérature (par exemple dans [EMOT81, Chap.IX] ou dans [Cra12, §3]). On retrouve les fonctions gamma incomplètes classiques en posant $\nu = 1$. Pour nos usages, on aura besoin d'un ν quelconque dans H_0 .

On introduit maintenant des constantes qui apparaîtront dans le théorème D. Par une décomposition en éléments simples, on trouve aisément :

Proposition 1.2.2. *Soit $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ un multi-entier. Notons \mathbb{K} le corps engendré par \mathbb{Q} et par les coefficients $c_{q,p}$, avec $1 \leq q \leq Q$ et $1 \leq p \leq P$. Soit $\mathcal{P} \subset \llbracket 1, P \rrbracket$ un ensemble, $j, f \in \llbracket 1, Q \rrbracket$ des entiers, et on considère $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}}$ et $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c}$ des multi-indices. On note $\mathcal{P}^c = \llbracket 1, P \rrbracket \setminus \mathcal{P}$ le complémentaire de \mathcal{P} , et on note $\mathbf{w}_p = (w_{p,q})_{q \in \llbracket 1, Q \rrbracket}$ et $\mathbf{v}_p = (v_{p,q})_{q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}}$. Alors il existe un polynôme $\tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'} \in \mathbb{K}[x]$ et deux familles de constantes $(C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}'))_{\lambda \in \mathbb{N}}$ et $(D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}'))_{\lambda \in \mathbb{N}, p \in \mathcal{P}}$ dont chacun des termes appartient au corps \mathbb{K} , tels que*

$$x^{-N'_f - 1 + |\mathbf{w}_{\bullet f}|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p}x)^{-N_p - 1 - |\mathbf{v}_p|} = \tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}(x) + \sum_{\lambda=1}^{N'_f + 1 - |\mathbf{w}_{\bullet f}|} \frac{C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')}{x^\lambda} \\ + \sum_{p \in \mathcal{P}} \sum_{\lambda=1}^{|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|} \frac{D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}')}{(c_{j,p} + c_{f,p}x)^\lambda}$$

On note $P_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}$ la primitive s'annulant en $x = 1$ de $\tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}$. On note $E_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \in \mathbb{K}$ le terme constant de $P_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}$. On pose

$$F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') := -E_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_f + 1 - |\mathbf{w}_{\bullet f}|} \frac{C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} \\ - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|} \frac{D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}) \\ + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (1, p)}(\mathbf{N}, \mathbf{N}') \ln \left(1 + \frac{c_{f,p}}{c_{j,p}} \right).$$

Remarque 1.2.3. 1) Notons que si $-N'_f - 1 + |\mathbf{w}_{\bullet f}| \geq 0$, alors $C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda} = 0$ pour tout λ .

2) Dans le produit $\prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p}x)^{-N_p - 1 - |\mathbf{v}_p|}$, certains termes peuvent apparaître plusieurs fois.

C'est la raison pour laquelle la somme $\sum_{\lambda=1}^{(|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|)}$ porte jusqu'à $|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|$. Si cela se produit, on a alors plusieurs choix possibles pour les constantes $D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}$. Notons que la constante $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ est indépendante de ces choix.

3) La constante $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ apparaîtra dans le calcul d'une intégrale dans le Lemme 3.1.4.

Exemple 1.2.4. Soit $\mathcal{P} = \emptyset$, pour tout $\lambda \geq 1$ on a

$$C_{\emptyset, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}') = \delta_{N'_f + 1 - |\mathbf{w}_{\bullet f}|}^\lambda \\ D_{\emptyset, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}') = 0 \\ E_{\emptyset, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') = 0 \quad (\text{si } |\mathbf{w}_{\bullet f}| \leq N'_f) \\ E_{\emptyset, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') = -\frac{1}{|\mathbf{w}_{\bullet f}| - N'_f} \quad (\text{si } |\mathbf{w}_{\bullet f}| > N'_f),$$

où δ désigne le symbole de Kronecker. On obtient en particulier que $F_{\emptyset, j, f, \mathbf{0}, \mathbf{0}}(\mathbf{0}, \mathbf{0}) = 0$.

Exemple 1.2.5. Soit $\mathcal{P} = \{p\}$, $j \neq f \in \llbracket 1, Q \rrbracket$, $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$, $\mathbf{v} = \mathbf{0}$ et $\mathbf{w} = \mathbf{e}_f$, alors on a

$$x^{-N'_f - 1 + |\mathbf{w}_{\bullet f}|} (c_{j,p} + c_{f,p}x)^{-N_p - 1 - |\mathbf{v}_p|} = (c_{j,p} + c_{f,p}x)^{-1}.$$

Ainsi on a $F_{\{p\}, j, f, \mathbf{0}, \mathbf{e}_f} = \frac{1}{c_{f,p}} \ln \left(1 + \frac{c_{f,p}}{c_{j,p}} \right)$.

Exemple 1.2.6. Soit $\mathcal{P} = \{p\}$, $j \neq f \in \llbracket 1, Q \rrbracket$, $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$, $\mathbf{v} = \mathbf{0}$ et $\mathbf{w} = \mathbf{e}_j$, alors on a

$$x^{-N'_f-1+|\mathbf{w} \cdot \mathbf{f}|} (c_{j,p} + c_{f,p}x)^{-N_p-1-|\mathbf{v}_p|} = x_f^{-1} (c_{j,p} + c_{f,p}x)^{-1} = \frac{1}{c_{j,p}} x^{-1} - \frac{c_{f,p}}{c_{j,p}} (c_{j,p} + c_{f,p}x)^{-1}.$$

Ainsi on a $F_{\{p\}, j, f, \mathbf{0}, \mathbf{e}_j} = -\frac{1}{c_{j,p}} \ln \left(1 + \frac{c_{f,p}}{c_{j,p}} \right)$.

On étudiera dans ce manuscrit la fonction multizêta suivante :

Définition 1.2.7. Pour tout $(\mathbf{s}, \mathbf{s}') \in H_1^P \times H_1^Q$ on pose

$$Z(\mathbf{s}, \mathbf{s}') = Z(\mathbf{c}, \mathbf{d}, \mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}.$$

Remarque 1.2.8. Cette classe de fonctions multizêta est de type Hurwitz-Lerch sans torsion, au sens de la Définition 1.1.26 :

$$Z(\mathbf{s}, \mathbf{s}') = \zeta^{HL}(\mathbf{0}, \tilde{\mathbf{d}}, \tilde{\mathbf{c}}, (s_1, \dots, s_P, s'_1, \dots, s'_Q)) = \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{\prod_{q=1}^{\tilde{Q}} \left(\tilde{d}_q + \sum_{p=1}^P \tilde{c}_{q,p} n_p \right)^{s_q}},$$

avec $P_0 = P$, $\tilde{Q} := P + Q$, $\mathcal{B}^c = \bigcup_{p=1}^P \llbracket 1, P \rrbracket \setminus \{p\} \times \{p\} \subset \llbracket 1, Q + P \rrbracket \times \llbracket 1, P \rrbracket$, $\tilde{\mathbf{d}} = (\mathbf{d}, \mathbf{d}')$, et avec $\tilde{\mathbf{c}} = (c_{(q,p)})_{(q,p) \in \llbracket 1, Q' \rrbracket \times \llbracket 1, P \rrbracket}$ tels que

$$\begin{cases} \tilde{c}_{q,p} = 0 & \text{si } q \neq p \text{ et } q \leq P \\ \tilde{c}_{p,p} = 1 & \text{si } p \in \llbracket 1, P \rrbracket \\ \tilde{c}_{q,p} = c_{q-P,p} & \text{si } q > P. \end{cases}$$

En particulier, on pourra appliquer le Théorème 1.1.27 et ses corollaires à la fonction multizêta $Z(\mathbf{s}, \mathbf{s}')$. On a alors que la fonction

$$s \mapsto Z(-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N} + \boldsymbol{\mu}'s)$$

est méromorphe sur \mathbb{C} et régulière en $s = 0$, pour toutes directions à part celles appartenant à une union d'hyperplans explicites.

À l'aide de la remarque précédente et du Théorème 1.1.27, on voit que la fonction multizêta $Z(\mathbf{s}, \mathbf{s}')$ est méromorphe sur tout $\mathbb{C}^P \times \mathbb{C}^Q$, et ce même théorème fournit également des informations sur l'ensemble des singularités de $Z(\mathbf{s}, \mathbf{s}')$. On dispose également d'un domaine de convergence, cependant pour nos besoins on a besoin d'un domaine de convergence plus large que ce que propose le Théorème 1.1.27. On montrera alors un domaine de convergence plus précis dans la proposition suivante :

Proposition 1.2.9. On pose

$$\mathcal{D} = \mathcal{D}_{P,Q} := \left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket \right\}.$$

La fonction multizêta $Z(\mathbf{s}, \mathbf{s}')$ a pour domaine de convergence \mathcal{D} , et admet un prolongement méromorphe sur $\mathbb{C}^P \times \mathbb{C}^Q$, avec des singularités situées dans l'union d'hyperplans \mathcal{H} définie par :

$$\begin{cases} \sum_{p \in \mathcal{P}} s_p + \sum_{q \in \mathcal{Q}} s'_q = Q + P - n & (\mathcal{P} \subset \llbracket 1, P \rrbracket, \mathcal{Q} \subset \llbracket 1, Q \rrbracket, |\mathcal{P} \cup \mathcal{Q}| \geq 2, n \in \mathbb{N}) \\ s_p = 1, \dots, P & (1 \leq p \leq P) \\ s'_q = 1, \dots, P & (1 \leq q \leq Q). \end{cases}$$

En général, les valeurs aux multi-entiers négatifs n'ont pas de sens puisque certains multi-entiers négatifs sont des pôles pour $Z(\mathbf{s}, \mathbf{s}')$. C'est pourquoi on fixera une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}')$ et un multi-entier négatif $-(\mathbf{N}, \mathbf{N}')$. On étudiera alors la fonction multizêta $Z(\mathbf{s}, \mathbf{s}')$ en ce multi-entier, et selon la direction choisie.

Définition 1.2.10. Soit $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, et $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$. On pose

$$Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(s) := Z(-\mathbf{N} + s\boldsymbol{\mu}, -\mathbf{N}' + s\boldsymbol{\mu}').$$

Remarque 1.2.11. Si l'on pose $\boldsymbol{\mu} = \mathbf{0}$, et $\boldsymbol{\mu}' = \mathbf{1}$, alors $Z_{\mathbf{0}, \mathbf{0}}^{\mathbf{0}, \mathbf{1}}(s)$ correspond à une fonction zêta de type Shintani

$$Z_{\mathbf{0}, \mathbf{0}}^{\mathbf{0}, \mathbf{1}}(s) = \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s}.$$

Gardons tout de même en tête que les coefficients d'_q ne sont pas libres, et dépendent des choix de d_p et des choix des coefficients $(c_{p,q})_{\substack{1 \leq p \leq P \\ 1 \leq q \leq Q}}$.

On considère maintenant la fonction univariante suivante, qui consiste à évaluer la fonction décrite précédemment selon la direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$, et pour $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$.

Définition 1.2.12. Soit $s \in \mathbb{C}$ tel que $\sigma \gg 1$. On pose

$$Z_{\Delta} : s \in \mathbb{C} \mapsto Z((s, \dots, s), (s, \dots, s)).$$

Par la Proposition 1.2.9, on sait que la fonction $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ est holomorphe sur $\{s \in \mathbb{C} | (-\mathbf{N} + s\boldsymbol{\mu}, -\mathbf{N}' + s\boldsymbol{\mu}') \in \mathcal{D}\}$, pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. Ainsi, il existe un réel $\sigma_{\mathbf{0}, \mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ tel que la fonction $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ soit holomorphe sur $H_{\sigma_{\mathbf{0}, \mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}}$. Pour ces directions là, on dira que la fonction $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ est holomorphe pour $\sigma \gg_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'} 1$, ou parfois seulement $\sigma \gg 1$ lorsqu'il n'y aura pas d'ambiguïté.

Il découle également de la Proposition 1.2.9 (ou du Théorème 1.1.27) que la fonction $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ est méromorphe sur \mathbb{C} , pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ en dehors d'une union d'hyperplans. Par un résultat de Komori [Kom10, Théorème 3.17] on a que la fonction $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ est régulière en $s = 0$ pour ces mêmes directions. On détaillera dans la Proposition 2.2.4 une méthode de prolongement différente de l'approche de Komori afin de reprouver la régularité en $s = 0$ de la fonction $Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$. Ceci nous permettra surtout d'obtenir des formules explicites sur les valeurs spéciales directionnelles de la fonction $Z(\mathbf{s}, \mathbf{s}')$, ainsi que celles de sa dérivée.

Notation. Soit $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ un multi-entier, et $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ une direction telle que

$$\forall \mathcal{P} \subsetneq [1, P], \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0.$$

On pose alors la valeur directionnelle

$$Z(-(\mathbf{N}, \mathbf{N}'))^{\boldsymbol{\mu}, \boldsymbol{\mu}'} := Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(0),$$

et la valeur de la dérivée directionnelle

$$Z'(-(\mathbf{N}, \mathbf{N}'))^{\boldsymbol{\mu}, \boldsymbol{\mu}'} := \partial_s (Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(s))|_{s=0}.$$

Remarque 1.2.13. Par le Corollaire 1.1.31, on a que les deux fonctions

$$(\boldsymbol{\mu}, \boldsymbol{\mu}') \mapsto Z(-(\mathbf{N}, \mathbf{N}'))^{\boldsymbol{\mu}, \boldsymbol{\mu}'}, \quad (\boldsymbol{\mu}, \boldsymbol{\mu}') \mapsto Z'(-(\mathbf{N}, \mathbf{N}'))^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$$

sont méromorphes sur \mathbb{C}^{P+Q} avec des singularités situées dans des hyperplans explicites.

On introduit désormais des fonctions auxiliaires (que l'on appellera des fonctions zêta de Barnes généralisées) dont le rôle sera crucial dans les formules de $Z'(-(\mathbf{N}, \mathbf{N}'))$.

Définition 1.2.14. Soit $\mathbf{R} \in \mathbb{N}^P$, et $j \in \llbracket 1, Q \rrbracket$. On appelle fonction zêta de Barnes généralisée (ou fonction auxiliaire) la fonction

$$\varphi_{\mathbf{R}}^j(s) := \sum_{\mathbf{n} \in \mathbb{N}^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d_j)^{-s}.$$

Remarque 1.2.15. Si $\mathbf{R} = \mathbf{0}$, on obtient une fonction zêta de Barnes : $\varphi_{\mathbf{0}}^j(s) = \zeta_P^B(s, d_j | c_{j,1}, \dots, c_{j,P})$.

La fonction $\varphi_{\mathbf{R}}^j(s)$ correspond à une fonction du type $Z_{\mathbf{N}, \mathbf{N}'}(s)$ avec $Q = 1$, $\boldsymbol{\mu} = \mathbf{0}$, $\boldsymbol{\mu}' = 1$, $\mathbf{N} = \mathbf{R}$ et $\mathbf{N}' = \mathbf{0}$. On en déduit en particulier qu'elle admet un prolongement méromorphe, et qu'elle est régulière en les entiers négatifs. La dérivée première de ces fonctions auxiliaires interviendra dans l'expression de $Z'(-(\mathbf{N}, \mathbf{N}'))$, il est donc très intéressant d'avoir une expression exploitable des valeurs spéciales en les entiers négatifs de la dérivée de ces fonctions auxiliaires.

1.3 Principaux résultats

Comme il est mentionné dans la section précédente, on démontrera par une autre méthode le résultat de Komori sur l'existence des valeurs directionnelles $Z(-(\mathbf{N}, \mathbf{N}'))$ dans la Proposition 2.2.4. Pour obtenir ce résultat, on utilise dans le chapitre 2 une stratégie appelée expansion de Crandall, développée dans [Cra12], qui était initialement utilisée pour obtenir des approximations numériques de certaines fonctions zêta et certaines fonctions L . Cette stratégie a également été utilisée par Borwein et Dilcher dans [BD18] pour étudier les valeurs d'une fonction zêta de type Mordell-Torheim, et approximer sa dérivée en les entiers négatifs. À la suite de ces travaux, Onodera a donné des formules exactes pour les valeurs aux entiers négatifs d'une fonction zêta de type Mordell-Torheim généralisée, ainsi que des valeurs de ses dérivées d'ordre 1 et 2 dans [Ono21]. L'astuce utilisée par Onodera pour calculer ces valeurs repose en partie sur le fait d'introduire des fonctions auxiliaires ressemblant à des fonctions zêta de Barnes avec un terme polynomial au numérateur du terme général de la série de Dirichlet, puis de donner une formule des dérivées de ces fonctions auxiliaires.

Plus précisément, on décrira une formule de prolongement pour la fonction méromorphe $Z_{\mathbf{N}, \mathbf{N}'}(s)$ dans la Proposition 2.2.4 :

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = K_{\mathbf{N}, \mathbf{N}'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}(\theta, s),$$

où la fonction $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ est holomorphe sur tout \mathbb{C} en s'annulant en $s = 0$, et où la fonction $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ est méromorphe sur \mathbb{C} , avec des candidats pôles de la forme

$$\begin{aligned} s &= \frac{n}{\mu_p} && (1 \leq p \leq P \text{ tel que } \mu_p \neq 0, n \in \mathbb{Z}_{\geq N_p+1}) \\ s &= \frac{n}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}} && (\emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket, n \in \mathbb{Z}_{\leq |\mathbf{N}|_{|\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}|}^*) \end{aligned}$$

où l'on rappelle la notation $|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} = \mu'_1 + \dots + \mu'_Q + \sum_{p \in \mathcal{P}} \mu_p$. On constate en particulier que 0 n'est pas un pôle pour la fonction méromorphe $Z_{\mathbf{N}, \mathbf{N}'}(s)$, et donc on reprochera que les valeurs directionnelles ont un sens pour la fonction $Z(\mathbf{s}, \mathbf{s}')$.

Le fait que la fonction $s \mapsto Z_{\mathbf{N}, \mathbf{N}'}(s)$ admette un prolongement méromorphe sur \mathbb{C} découle directement de la Proposition 1.2.9, et cette même proposition donne également des candidats pour les pôles de cette

fonction. Il suffit en effet de remplacer s_p par $-N_p + \mu_p s$ et s'_q par $-N'_q + \mu'_q s$ pour obtenir des candidats pôles. Cependant, l'ensemble de candidats pôles que l'on donne ici est plus précis.

Remarquons au passage que le terme $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ ne contribuera pas à la formule des valeurs spéciales directionnelles, puisqu'il s'annule en $s = 0$. En revanche, si on dérive ce terme par rapport à la variable s , alors il n'est pas nul en général en $s = 0$ et est particulièrement pathologique. On utilisera alors des fonctions auxiliaires $(\varphi_{\mathbf{R}}^j)_{j \in \llbracket 1, Q \rrbracket}$ qui récupéreront l'information de la valeur de la dérivée $\partial_s(K_{\mathbf{N}, \mathbf{N}'}(\theta, s))$ en $s = 0$. On pourra alors évaluer en $s = 0$ la dérivée de la formule de prolongement de $Z_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(s)$ pour obtenir une formule explicite sur les valeurs directionnelles des dérivées.

Théorème A. *Pour tout multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ et pour toute direction $(\mu, \mu') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que*

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\mu'| + |\mu|_{|\mathcal{P}} \neq 0,$$

on a

$$Z_{\mu, \mu'}(-(\mathbf{N}, \mathbf{N}')) = (-1)^{|\mathbf{N}'|} \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \frac{(-1)^{N'_j + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|} \mu'_j N'_j!}{|\mu'| + |\mu|_{|\mathcal{P}}} \left(\prod_{p \in \mathcal{P}} N_p! \right) \cdot \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}, \quad (1.8)$$

avec

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) := (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet, q}| + |\mathbf{w}_{\bullet, q}| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j, p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, p}^{v_{p, q}} \right), \quad (1.9)$$

où l'on rappelle les notations

$$\forall q \in \llbracket 1, Q \rrbracket, \quad |\mathbf{w}_{\bullet, q}| = \sum_{p \in \mathcal{P}^c} w_{p, q},$$

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad |\mathbf{v}_{\bullet, q}| = \sum_{p \in \mathcal{P}} v_{p, q}.$$

Remarque 1.3.1. 1) L'expression décrivant $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ est une somme finie. En effet, on voit que

$$\sum_{\substack{q=1 \\ q \neq j}}^Q (N'_q - |\mathbf{w}_{\bullet, q}|) = \sum_{\substack{q=1 \\ q \neq j}}^Q |\mathbf{v}_{\bullet, q}| = \sum_{(p, q) \in \mathcal{P} \times (\llbracket 1, Q \rrbracket \setminus \{j\})} v_{p, q}. \text{ Comme } |\mathbf{w}_{\bullet, q}| \text{ est positif pour tout } q \in \llbracket 1, Q \rrbracket, \text{ on en}$$

déduit la majoration suivante :

$$\sum_{(p, q) \in \mathcal{P} \times (\llbracket 1, Q \rrbracket \setminus \{j\})} v_{p, q} \leq |\mathbf{N}'|.$$

Ainsi, pour tout $p \in \mathcal{P}$, et tout $q \in \llbracket 1, Q \rrbracket \setminus \{j\}$, on a que $0 \leq v_{p, q} \leq |\mathbf{N}'|$.

2) La formule pour les valeurs $Z_{\mu, \mu'}(-(\mathbf{N}, \mathbf{N}'))$ est décrite par une somme finie. En effet, pour tout

$\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}$ tel que $|\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ on a $0 \leq k_p \leq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ pour tout $p \in \mathcal{P}^c$.

3) Les valeurs $\zeta(-n, d)$ avec $n \in \mathbb{N}$ et $d \in H_0$ s'expriment à l'aide des polynômes de Bernoulli :

$$\zeta(-n, d) = -\frac{B_{n+1}(d)}{n+1}.$$

De plus, en $d = 1$, on a $\zeta(-n, 1) = \zeta(-n)$, et

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

où B_{n+1} est le $n+1$ -ième nombre de Bernoulli.

4) Si tous les coefficients d_p valent 1, alors le produit $\prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}$ sera fréquemment nul.

En effet, s'il existe $p \in \mathcal{P}^c$ tel que $N_p + k_p$ soit pair et plus grand que 2, alors le produit est nul.

En $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$, on obtient les expressions simplifiées suivantes :

Corollaire A1. Pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0,$$

on a

$$Z(-(\mathbf{0}, \mathbf{0})) = \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \frac{(-1)^{|\mathcal{P}|} \mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!}.$$

Via le Théorème A, on a également une formule pour les valeurs sur la diagonale de la fonction $Z(\mathbf{s}, \mathbf{s}')$ en les entiers négatifs en considérant la direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ et le multi-entier $(\mathbf{N}, \mathbf{N}') = (N, \dots, N)$ avec $N \in \mathbb{N}$.

Corollaire A2. Pour tout entier $N \in \mathbb{N}$ on a

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \frac{(-1)^{(|\mathcal{P}|+1+Q)N+|\mathcal{P}|}}{Q+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = (Q+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!},$$

avec

$$Q_N^0(\mathcal{P}, \mathbf{k}) = (-1)^{N(Q-1)} N^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right),$$

où l'on rappelle les notations

$$\forall q \in \llbracket 1, Q \rrbracket, \quad |\mathbf{w}_{\bullet q}| = \sum_{p \in \mathcal{P}^c} w_{p,q},$$

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad |\mathbf{v}_{\bullet q}| = \sum_{p \in \mathcal{P}} v_{p,q}.$$

On verra dans le Corollaire 6.3.2 une version simplifiée de cette formule dans le cas où $P = 2$ et où $\mathbf{d} = \mathbf{1}$.

Remarque 1.3.2. Observons à l'aide du corollaire précédent que si l'on pose $Q = 1$, $P \geq 1$ un entier quelconque, et $l_1(\mathbf{n}) = n_1 + \dots + n_P$, on trouve via les formules précédentes que $Q_N^0(\mathcal{P}, \mathbf{k}) = 1$ pour tout ensemble $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, et que

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \frac{(-1)^{|\mathcal{P}|N+|\mathcal{P}|}}{1+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=(1+|\mathcal{P}|)N+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!}.$$

Cette formule coïncide avec celle obtenue par Onodera dans [Ono21, Théorème 3].

On peut également fournir une variante qualitative du Théorème A :

Corollaire A3. Considérons un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ et une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

On note \mathbb{K} le corps engendré par \mathbb{Q} , par les coefficients complexes $(d_p)_{p \in \llbracket 1, P \rrbracket}$ et $(c_{q,p})_{p \in \llbracket 1, P \rrbracket, q \in \llbracket 1, Q \rrbracket}$, et par les composantes μ_p et μ'_q pour tout $p \in \llbracket 1, P \rrbracket$, $q \in \llbracket 1, Q \rrbracket$. Alors $Z(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} \in \mathbb{K}$.

On a également une expression de la dérivée directionnelle en les entiers négatifs de la fonction $Z(\mathbf{s}, \mathbf{s}')$. Cette expression fait intervenir les fonctions auxiliaires, aussi appelée fonctions zêta de type Barnes généralisées, posées dans la Définition 1.2.14. Si l'on suppose qu'une forme linéaire l_j est à coefficients rationnels, on dispose même d'une relation entre la fonction auxiliaire $\varphi_{\mathbf{R}}^j(s)$ et la fonction zêta de Hurwitz :

Proposition B. Soit $j \in \llbracket 1, Q \rrbracket$ un entier, on suppose que l_j est une forme linéaire à coefficients rationnels. Pour tout $1 \leq p \leq P$, on considère $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$ la décomposition irréductible de $c_{j,p}$ avec $a_{j,p} > 0$ et $b_{j,p} > 0$. On pose

$$x_j(\mathbf{c}) := \frac{\text{ppcm}(a_{j,p})}{\text{pgcd}(b_{j,p})} \in \mathbb{Q}_+^*, \quad \beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}$$

pour tout $1 \leq p \leq P$. Alors pour tout $\mathbf{R} \in \mathbb{N}^P$ et pour tout $s \in \mathbb{C} \setminus \mathbb{N}^*$, on a

$$\begin{aligned} \varphi_{\mathbf{R}}^j(s) &= x_j(\mathbf{c})^{-s} \left(\prod_{p=1}^P \beta_{j,p}^{R_p} \right) \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \left(\prod_{p \in \mathcal{P}^c} R_p! \right) \\ &\quad \cdot \sum_{\substack{\mathbf{k} \in \mathbb{N}^{\mathcal{P}}, k' \in \mathbb{N} \\ |\mathbf{k}|+k' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \frac{\zeta\left(s - k', \frac{l_j(\mathbf{d}+\mathbf{v})}{x_j(\mathbf{c})}\right)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!}. \end{aligned}$$

En dérivant en $s = -N$ la fonction auxiliaire $\varphi_{\mathbf{R}}^j(s)$ on obtient une expression explicite de la dérivée de cette fonction zêta de type Barnes généralisée en terme de valeurs spéciales et de valeurs de la dérivée de la fonction zêta d'Hurwitz.

Corollaire B1. Avec les mêmes hypothèses et notations que la Proposition B, on a pour tout $\mathbf{R} \in \mathbb{N}^P$,

$$\begin{aligned} (\varphi_{\mathbf{R}}^j)'(-N) &= x_j(\mathbf{c})^N \left(\prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{R_p} \right) \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \left(\prod_{p \in \mathcal{P}^c} R_p! \right) \sum_{\substack{\mathbf{k} \in \mathbb{N}^{\mathcal{P}}, k' \in \mathbb{N} \\ |\mathbf{k}|+k' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \\ &\quad \cdot \left(\frac{\zeta\left(-N - k', \frac{l_j(\mathbf{d}+\mathbf{v})}{x_j(\mathbf{c})}\right)}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta\left(-N - k', \frac{l_j(\mathbf{d}+\mathbf{v})}{x_j(\mathbf{c})}\right)}{k'!} \right) \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!}. \end{aligned}$$

On peut maintenant énoncer une version qualitative sur les valeurs spéciales de la dérivée directionnelle de $Z(\mathbf{s}, \mathbf{s}')$:

Théorème C. *Considérons un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ et une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que*

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0.$$

On note \mathbb{K} le corps engendré par \mathbb{Q} , par les coefficients complexes $(d_p)_{p \in \llbracket 1, P \rrbracket}$ et $(c_{q,p})_{p \in \llbracket 1, P \rrbracket, q \in \llbracket 1, Q \rrbracket}$, et par les composantes μ_p et μ'_q pour tout $p \in \llbracket 1, P \rrbracket$, $q \in \llbracket 1, Q \rrbracket$. Alors

$$Z'(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} \in \text{Vect}_{\mathbb{K}} \left[\gamma, (\ln(c_{q,p}), \ln(c_{j,p} + c_{q,p}))_{\substack{1 \leq q, j \leq Q \\ 1 \leq p \leq P}}, (\zeta'(-n, d_p)_{\substack{0 \leq n \leq |\mathbf{N}| + |\mathbf{N}'| + P \\ p \in \llbracket 1, P \rrbracket}}, \right. \\ \left. ((\varphi_{\mathbf{n}}^j)'(-N'_j))_{\substack{\mathbf{n} \in \llbracket 0, |\mathbf{N}| + |\mathbf{N}'| \rrbracket^P \\ 1 \leq j \leq Q}} \right],$$

où γ désigne la constante d'Euler, et les φ_j sont les fonctions auxiliaires, dites de Barnes généralisées, décrites dans la Définition 1.2.14.

Si les formes linéaires $(l_j)_{j \in \llbracket 1, Q \rrbracket}$ sont à coefficients rationnels, alors on peut utiliser le Corollaire B1 pour remplacer dans le corollaire précédent la valeur des dérivées $\varphi_{\mathbf{n}}^j(-N'_j)$ par des sommes finies de valeurs et de valeurs dérivées de la fonction zêta d'Hurwitz. On obtient alors le corollaire suivant :

Corollaire C1. *Considérons un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ et une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que*

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0.$$

On suppose que les formes linéaires l_1, \dots, l_Q sont à coefficients rationnels, et que les coefficients d_1, \dots, d_P sont des nombres rationnels. On note \mathbb{K} le corps engendré par \mathbb{Q} et par les composantes μ_p et μ'_q pour tout $p \in \llbracket 1, P \rrbracket$, $q \in \llbracket 1, Q \rrbracket$. Pour tout $1 \leq p \leq P$, on considère $c_{q,p} = \frac{a_{q,p}}{b_{q,p}} \in \mathbb{Q}$ la décomposition irréductible de $c_{q,p}$ avec $a_{q,p} > 0$ et $b_{q,p} > 0$. On pose alors

$$x_q(\mathbf{c}) := \frac{\text{ppcm}(a_{q,p})}{\text{pgcd}(b_{q,p})} \in \mathbb{Q}_+^*, \quad \beta_{q,p}(\mathbf{c}) := \frac{x_q(\mathbf{c})}{c_{q,p}} \in \mathbb{N}^*$$

pour tout $1 \leq p \leq P$. Posons également l'ensemble

$$A := \left\{ \frac{l_j(\mathbf{d} + \mathbf{v})}{x_{\mathbf{c}}} \mid j \in \llbracket 1, Q \rrbracket, \mathbf{v} \in \llbracket 0, \beta_{j,1} - 1 \rrbracket \times \dots \times \llbracket 0, \beta_{j,P} - 1 \rrbracket \right\}.$$

On a alors que

$$Z'(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} \in \text{Vect}_{\mathbb{K}} \left[\gamma, (\ln(c_{q,p}), \ln(c_{j,p} + c_{q,p}))_{\substack{1 \leq q, j \leq Q \\ 1 \leq p \leq P}}, (\ln(x_j(\mathbf{c})))_{1 \leq j \leq Q}, \right. \\ \left. (\zeta'(-n, y))_{\substack{0 \leq n \leq P(|\mathbf{N}|+1) + |\mathbf{N}'| \\ y \in A}} \right],$$

où γ désigne la constante d'Euler.

La formule pour $Z'(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ est en fait explicite. Elle admet une formule faisant intervenir les valeurs spéciales des dérivées des fonctions zêta de Barnes généralisées $\varphi_j^{\mathbf{R}}$, des valeurs spéciales de la fonction zêta d'Hurwitz et de sa dérivée selon la variable s . La formule fait également intervenir des logarithmes de certains entiers ainsi que la constante d'Euler γ . On détaille cette formule dans le théorème ci-dessous.

Théorème D. Pour tout multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ et pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0,$$

on a

$$\begin{aligned} Z'(-(\mathbf{N}, \mathbf{N}')) &= \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} (-1)^{|\mathbf{N}| + |\mathbf{N}'| + |\mathcal{P}| + N'_j} \left(\prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ &\quad \cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ &+ \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}| + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\ &\quad \cdot \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\ &+ \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) (\varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j)^{(-N'_j)} \\ &- \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}} + |\mathcal{P}|} \\ &\quad \cdot \prod_{p \in \mathcal{P}} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}} + |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}^{N_p + |\mathbf{u}_{\bullet p}| + 1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!}, \end{aligned} \tag{1.10}$$

où l'on a posé

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + |\mathbf{u}_{\bullet p}|)_{p \in \llbracket 1, P \rrbracket}, \text{ pour tout } \mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}},$$

et

$$\begin{aligned} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) &= (-1)^{|\mathbf{N}'|_{|\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, N'_q = |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ &\quad \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right), \end{aligned}$$

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) := & (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \quad (1.11) \\
& \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \\
& + (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\
& \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \frac{\prod_{q=1}^Q c_{q,p}^{v_{p,q}}}{\frac{q \neq j}{c_{j,p}^{N_p+1+|\mathbf{v}_p|}}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p+|\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right)
\end{aligned}$$

avec $j \in \llbracket 1, Q \rrbracket$, $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$, et où $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ désigne la constante posée dans la Proposition 1.2.2.

Remarque 1.3.3. Ce théorème ramène le calcul des valeurs des dérivées directionnelles $Z'(-(\mathbf{N}, \mathbf{N}'))_{\mu, \mu'}$ aux calculs des valeurs aux entiers négatifs de la dérivée des fonctions auxiliaires

$$\varphi_{\mathbf{R}}^j(s) = \sum_{\mathbf{n} \in \mathbb{N}^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d'_j)^{-s} \quad (j \in \llbracket 1, Q \rrbracket, \mathbf{R} \in \mathbb{N}^P).$$

Si les coefficients des formes linéaires l_j sont rationnels, alors on dispose d'une expression explicite de la valeur de $(\varphi_{\mathbf{R}}^j)'(-N)$ en fonction des valeurs spéciales des dérivées de la fonction zêta d'Hurwitz via le Corollaire B1.

Remarque 1.3.4. On observe que la formule pour $Z'(-(\mathbf{N}, \mathbf{N}'))_{\mu, \mu'}$ ne fait intervenir que des sommes finies.

En effet, il est clair que la somme $\sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|}} \mu, \mu'$ est finie. De même, on a que la somme

$$\sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} = \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}}$$

présente dans le calcul de $Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k})$ et de $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ est finie puisque

$$\begin{aligned}
\forall a \in \mathcal{P}, \forall b \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad 0 \leq v_{a,b} \leq \sum_{\substack{q=1 \\ q \neq j}}^Q |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = |\mathbf{N}'|, \\
\forall p \in \mathcal{P}^c, \forall q \in \llbracket 1, Q \rrbracket, \quad 0 \leq w_{p,q} \leq |\mathbf{w}_p| = k_p \leq |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}^c} + |\mathcal{P}|.
\end{aligned}$$

Pour les mêmes raisons, la somme

$$\sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} = \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}}$$

est également finie.

Si les coefficients \mathbf{c} sont rationnels, on peut remplacer les dérivées des fonctions auxiliaires aux entiers négatifs par la formule obtenue dans le Corollaire B1 :

Corollaire D1. *On suppose que les formes linéaires l_1, \dots, l_Q sont à coefficients rationnels. Considérons un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ et une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que*

$$\forall \mathcal{P} \subsetneq [1, P], \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0.$$

Pour tout $1 \leq p \leq P$, on considère $c_{q,p} = \frac{a_{q,p}}{b_{q,p}} \in \mathbb{Q}$ la décomposition irréductible de $c_{q,p}$ avec $a_{q,p} > 0$ et $b_{q,p} > 0$. On pose alors

$$x_q(\mathbf{c}) := \frac{\text{ppcm}(a_{q,p})}{\text{pgcd}(b_{q,p})} \in \mathbb{Q}_+^*, \quad \beta_{q,p}(\mathbf{c}) := \frac{x_q(\mathbf{c})}{c_{q,p}} \in \mathbb{N}^*$$

pour tout $1 \leq p \leq P$. On a

$$\begin{aligned} Z'(-(\mathbf{N}, \mathbf{N}')) &= \\ &= \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} (-1)^{|\mathbf{N}| + |\mathbf{N}'| + |\mathcal{P}| + N'_j} \left(\prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ &\cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ &+ \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\ &\cdot \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\ &+ \sum_{j=1}^Q \mu'_j x_j(\mathbf{c})^{N'_j} \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 0 \leq v_1 \leq \beta_{j,1}(\mathbf{c}) - 1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c}) - 1}} \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\ &\cdot \left(\prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p} \right) \left(\prod_{p \in \mathcal{P}^c} \mathbf{R}(\mathbf{N}, j, \mathbf{u})_p! \right) \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}} \in \mathbb{N}^{\mathcal{P}}, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \\ &\cdot \left(\frac{\zeta'(-N'_j - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})})}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N'_j - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})})}{k'!} \right) \prod_{p \in \mathcal{P}} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})})}{k_p!} \end{aligned}$$

$$\begin{aligned}
& - \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left((-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P C_{q,p}^{u_{q,p}} \right) \right. \\
& \cdot \left. \left(\prod_{p \in \mathcal{P}} R(\mathbf{N}, j, \mathbf{u})_p \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}}} \frac{\prod_{p \in \mathcal{P}^c} C_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} R(\mathbf{N}, j, \mathbf{u})_{p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \right),
\end{aligned}$$

où l'on a posé

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + |\mathbf{u}_{\bullet p}|)_{p \in \llbracket 1, P \rrbracket} \quad \text{pour } \mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}, \quad j \in \llbracket 1, Q \rrbracket,$$

et

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) & := (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, N'_q = |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q,p}^{w_{p,q}} \right) \\
& \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} C_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q C_{q,p}^{v_{p,q}} \right), \\
Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) & := (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \\
& \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q C_{q,p}^{v_{p,q}} \right) \\
& + (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q,p}^{w_{p,q}} \right) \\
& \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \frac{\prod_{q=1}^Q C_{q,p}^{v_{p,q}}}{C_{j,p}^{N_p + 1 + |\mathbf{v}_p|}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right)
\end{aligned}$$

avec $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ la constante posée dans la Proposition 1.2.2.

Remarque 1.3.5. Posons $Q = 1$, et $P \geq 1$ un entier quelconque, on considère $l_1(\mathbf{n}) = n_1 + \dots + n_P$ (et donc $c_{1,1} = \dots = c_{1,P} = 1$), $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$, et $(\mathbf{N}, \mathbf{N}') = (N, \dots, N)$ avec $N \in \mathbb{N}$. On trouve alors via le corollaire précédent que

- Pour tout $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, et $\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}$, on a $Q_{\mathbf{N}, \mathbf{N}'}^0(1, \mathcal{P}, \mathbf{k}) = 1$ et $Q_{\mathbf{N}, \mathbf{N}'}^1(1, \mathcal{P}, \mathbf{k}) = 0$.

- Avec les notations du Corollaire D1, on a $\mathbf{x}_1(\mathbf{c}) = 1$, et $\beta_{1,p} = 1$ pour tout $1 \leq p \leq P$.
- On a $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) = (N, \dots, N) \in \mathbb{N}^P$.
- Dans l'expression de $Z'(-(\mathbf{N}, \mathbf{N}'))$ du corollaire précédent, on trouve que la première somme de la formule s'annule avec la dernière somme de la formule.

Via les quatre points précédents, on trouve que

$$\begin{aligned}
Z'(-(\mathbf{N}, \mathbf{N}'))_{1,1} &= \sum_{\mathcal{P} \subsetneq [1, P]} \frac{N!^{|\mathcal{P}|+1}}{|\mathcal{P}|+1} (-1)^{|\mathcal{P}|(N+1)} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|(N+1)+|\mathcal{P}|}} \\
&\left(\sum_{i \in \mathcal{P}^c} \frac{\zeta'(-N - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N - k_p, d_p)}{k_p!} \right) \\
&+ \sum_{\mathcal{P} \subsetneq [1, P]} N!^{|\mathcal{P}^c|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}} \in \mathbb{N}^{\mathcal{P}}, k' \in \mathbb{N} \\ |\mathbf{k}|+k'=|\mathcal{P}^c|(N+1)+|\mathcal{P}^c|-1}} (-1)^{|\mathbf{k}|} \frac{\zeta'(-N - k', d_1)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta(-N - k_p, d_p)}{k_p!}.
\end{aligned}$$

Cette expression correspond à une forme développée de la formule obtenue par Onodera dans [Ono21, Théorème 4] (avec le paramètre $d_1 = 0$ dans le théorème d'Onodera).

En posant $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ dans le Théorème D, on obtient le corollaire suivant :

Corollaire D2. Pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que

$$\forall \mathcal{P} \subsetneq [1, P], \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0,$$

on a

$$\begin{aligned}
Z'(-(\mathbf{0}, \mathbf{0}))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} &= \sum_{j=1}^Q \mu'_j (\zeta^B)'(0, d'_j | \mathbf{c}_j) \\
&+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|} \mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \left(\sum_{p \in \mathcal{P}} \mu_p \ln(c_{j,p}) \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \\
&+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{1 \leq j \neq f \leq Q} \frac{(-1)^{|\mathcal{P}|} \mu'_j \mu'_f}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \\
&\cdot \sum_{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} [0, k_p]} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\
&+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|} \mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p, d_p)}{k_p!} \right)
\end{aligned}$$

avec $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{0}, \mathbf{0})$ la constante définie dans la Proposition 1.2.2, et avec la notation $(\zeta^B)'(s, d | \mathbf{c}) = \partial_s \zeta^B(s, d | \mathbf{c})$.

Remarque 1.3.6. Les termes de la forme $F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0})$ dans le corollaire précédent découlent de l'étude de la fraction rationnelle

$$x^{-1+k_p} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x)^{-1}.$$

Remarque 1.3.7. Notons que Matsumoto et Weng ont obtenu dans [MW02] un résultat ressemblant pour la valeur de la dérivée en $s = 0$ d'une fonction zêta définie par deux polynômes dans $\mathbb{C}[X]$.

En considérant $\boldsymbol{\mu} = \mathbf{0}$ et $\boldsymbol{\mu}' = \mathbf{1}$, on obtient que $Z'(-(\mathbf{0}, \mathbf{0})) = \partial_s(\zeta^{Sh}(s, (\mathbf{c}_1, \dots, \mathbf{c}_Q), \mathbf{d}'))|_{s=0}$, c'est-à-dire la dérivée première par rapport à s évaluée en $s = 0$ de la fonction zêta de Shintani définie dans la Définition 1.1.13. Rappelons que, avec la Définition 1.1.16 et la Définition 1.1.14, on a

$$\partial_s(\zeta^B(s, d'_j | \mathbf{c}_j))|_{s=0} = \ln(\Gamma_P(d'_j | \mathbf{c}_j)), \quad \partial_s(\zeta^{Sh}(s, \mathbf{d}' | \mathbf{c}))|_{s=0} = \ln(\Gamma_P(\mathbf{d}' | \mathbf{c})).$$

On obtient alors via le corollaire précédent que le logarithme de la fonction multigamma de Shintani $\Gamma_P(\mathbf{d}' | \mathbf{c}_1, \dots, \mathbf{c}_Q) = \Gamma_P((l_1(\mathbf{d}), \dots, l_Q(\mathbf{d})) | \mathbf{c}_1, \dots, \mathbf{c}_Q)$ s'exprime complètement à l'aide du logarithme de la fonction multigamma de type Barnes.

Corollaire D3. On a

$$\begin{aligned} \ln(\Gamma_P(\mathbf{d}' | \mathbf{c})) &= \sum_{j=1}^Q \ln(\Gamma_P(d'_j | \mathbf{c}_j)) \\ &+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{1 \leq j \neq f \leq Q} \frac{(-1)^{|\mathcal{P}|}}{Q} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \\ &\cdot \sum_{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} [0, k_p]} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right). \end{aligned}$$

Remarquons que ce corollaire implique en particulier une formule de type Raabe. En effet, il est connu depuis Barnes que

$$\int_{[0,1]^P} \ln(\Gamma_P(l_j(\mathbf{x}) | \mathbf{c}_j)) dx_1 \dots dx_P = 0.$$

De plus, par propriété des polynômes de Bernoulli, il est clair que

$$\forall n \in \mathbb{N}, \quad \int_0^1 \zeta(-n, x) dx = 0.$$

On retrouve alors via une stratégie différente la formule de type Raabe obtenue initialement par Friedmann et Ruijsenaars [FR04] :

$$\int_{[0,1]^P} \ln(\Gamma_P((l_1(\mathbf{x}), \dots, l_Q(\mathbf{x})) | \mathbf{c}_1, \dots, \mathbf{c}_Q)) dx_1 \dots dx_P = 0.$$

En appliquant le Corollaire A1 et le Corollaire D2 aux deux fonctions zêta de Witten $\zeta_{\mathfrak{g}_2}$ et $\zeta_{\mathfrak{so}(5)}$, on obtient les formules suivantes :

Théorème E. On a les valeurs spéciales pour $\zeta_{\mathfrak{g}_2}(s)$ et $\zeta_{\mathfrak{so}(5)}(s)$:

$-N$	0	-1	-2	-3	-4
$\zeta_{\mathfrak{so}(5)}(-N)$	$\frac{3}{8}$	$-\frac{11}{26880}$	0	$-\frac{509}{37847040}$	0
$\zeta_{\mathfrak{g}_2}(-N)$	$\frac{5}{12}$	$\frac{6641}{62705664}$	0	$\frac{12522872818983257}{109242202556140093440}$	0

De plus on a les valeurs suivantes des dérivées $\zeta'_{\mathfrak{g}_2}(s)$ et $\zeta'_{\mathfrak{so}(5)}(s)$ en $s = 0$:

$$\begin{aligned} \zeta'_{\mathfrak{g}_2}(0) &= \frac{5}{12} \ln(5) - \frac{1}{12} \ln(3) + \frac{13}{4} \ln(2) + \frac{5}{2} \ln(\pi) \approx 5,69360115756852203943195205888908833111\dots \\ \zeta'_{\mathfrak{so}(5)}(0) &= \frac{3}{8} \ln(3) + \frac{13}{8} \ln(2) + \frac{3}{2} \ln(\pi) \approx 3,2554386054345525232913601882450721298364388\dots \end{aligned}$$

Remarquons que ces deux fonctions zêta de Witten s'annulent en $s = -2$. Ce résultat fait écho à la conjecture de Kurokawa et Ochiai [KO13] qui prédit que les fonctions zêta de Witten s'annulent en $s = -2$. Notons que nous retrouvons la même valeur pour $\zeta_{\mathfrak{so}(5)}(0)$ que dans [BBBF24, Proposition 5.15]. Notons également que ces deux fonctions zêta de Witten s'annulent pour beaucoup d'entiers pairs strictement négatifs.

Via la formule précédente, on obtient une expression explicite de la constante C dans le Théorème 1.1.41 :

$$C = 3^{-\frac{1}{12}} 2^{\frac{31}{24}} \pi \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}.$$

On obtient alors une version explicite du Théorème 1.1.41 obtenu par Bridges, Brindle, Bringmann et Franke dans [BBBF24, Théorème 1.3] :

Corollaire F. *Pour tout entier $N \geq 1$ on a*

$$r_{\mathfrak{so}(5)}(n) \underset{n \rightarrow +\infty}{\sim} \frac{C}{n^{\frac{7}{12}}} \exp\left(A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4\right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N\left(n^{-\frac{N+1}{9}}\right)\right),$$

avec B_2, \dots, B_{N+1} des constantes explicites, et

$$\begin{aligned} C &= 3^{-\frac{1}{12}} 2^{\frac{31}{24}} \pi \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}, & A_1 &= \frac{3^{\frac{4}{3}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{3}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{8}{3}}}, \\ A_2 &= \frac{2^{\frac{8}{9}} \left(2^{\frac{1}{3}} + 1\right) \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right)}{3^{\frac{7}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{9}}}, & A_3 &= \frac{2^{\frac{40}{9}} \left(2^{\frac{1}{3}} + 1\right)^2 \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{1}{3}\right)^2 \zeta\left(\frac{4}{3}\right)^2}{3^{\frac{44}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{20}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{10}{9}}}, \\ A_4 &= \frac{2^8 \left(2^{\frac{1}{3}} + 1\right)^3 \Gamma\left(\frac{1}{3}\right)^3 \zeta\left(\frac{1}{3}\right)^3 \zeta\left(\frac{4}{3}\right)^3}{3^8 \Gamma\left(\frac{1}{4}\right)^4 \zeta\left(\frac{3}{2}\right)^2}. \end{aligned}$$

On souhaite désormais appliquer les résultats obtenus dans le Théorème E afin d'obtenir une formule asymptotique du nombre de représentations $r_{\mathfrak{g}_2}(n)$ de l'algèbre de Lie exceptionnelle \mathfrak{g}_2 , formule que l'on obtient via le Théorème 1.1.38 obtenu par Bridges, Brindle, Bringmann et Franke dans [BBBF24, Théorème 4.4]. On pose tout d'abord

$$\begin{aligned} l_1^*(x_1, x_2, x_3, x_4) &= x_1 + x_2 + x_3 + 2x_4, \\ l_2^*(x_1, x_2, x_3, x_4) &= x_1 + 2x_2 + 3x_3 + 3x_4, \end{aligned}$$

et on pose

$$\begin{aligned} \omega_\beta &:= \frac{120^{\frac{1}{5}} \Gamma\left(\frac{4}{5}\right)}{5 \Gamma\left(\frac{1}{5}\right)^4} \zeta\left(\frac{1}{5}\right) \int_{[0,1]^3} (xyz)^{-\frac{4}{5}} \left(3l_1^*(1, x, y, z)^{-\frac{4}{5}} + l_2^*(1, x, y, z)^{-\frac{4}{5}} \right. \\ &\quad \left. + l_2^*(x, 1, y, z)^{-\frac{4}{5}} + l_2^*(x, y, 1, z)^{-\frac{4}{5}} + l_1^*(x, y, z, 1)^{-\frac{4}{5}} + l_2^*(x, y, z, 1)^{-\frac{4}{5}}\right) dx dy dz, \\ \omega_\alpha &:= \frac{120^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)^2}{6 \Gamma\left(\frac{1}{3}\right)^4} \int_{[0,1]^3} (xyz)^{-\frac{2}{3}} \left(\prod_{p=1}^2 l_p^*(x, y, z, 1)^{-\frac{2}{3}} + \prod_{p=1}^2 l_p^*(x, y, 1, z)^{-\frac{2}{3}} \right. \\ &\quad \left. + \prod_{p=1}^2 l_p^*(x, 1, y, z)^{-\frac{2}{3}} + \prod_{p=1}^2 l_p^*(1, x, y, z)^{-\frac{2}{3}}\right) dx dy dz. \end{aligned}$$

Les constantes ω_α et ω_β correspondent en fait à des résidus de $\zeta_{\mathfrak{g}_2}$, résidus que l'on calculera dans la sous-section 6.3.1. On trouve alors une formule asymptotique de $r_{\mathfrak{g}_2}(n)$, dont certains de ces coefficients s'expriment en fonction des résidus ω_α et ω_β , et des valeurs spéciales $\zeta_{\mathfrak{g}_2}(0)$ et $\zeta'_{\mathfrak{g}_2}(0)$.

Théorème G. *Pour tout entier $N \geq 1$ on a*

$$r_{\mathfrak{g}_2}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^{\frac{9}{16}}} \exp\left(A_1 n^{\frac{1}{4}} + A_2 n^{\frac{3}{20}} + A_3 n^{\frac{1}{20}}\right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{20}}} + O\left(n^{-\frac{N+1}{20}}\right)\right),$$

avec B_2, \dots, B_{N+1} des constantes explicites, et

$$\begin{aligned} C &= 2^{\frac{7}{4}} 15^{\frac{5}{12}} \pi^2 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{-3}{16}}, \\ K_2 &= \frac{3}{4} \cdot \frac{\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{20}}}, \quad K_3 = -\frac{3}{160} \cdot \frac{\left(\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)\right)^2}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{21}{20}}}, \\ A_1 &:= 4 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{4}}, \quad A_2 := \frac{\omega_\beta \Gamma\left(\frac{1}{5}\right) \zeta\left(\frac{6}{5}\right)}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{20}}}, \\ A_3 &:= \frac{2K_2^2}{3 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{4}}} - \frac{\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{9}{10}}} K_2. \end{aligned}$$

Remarque 1.3.8. *Les B_j peuvent être calculés d'après les coefficients b_j obtenus via [BBBF24, Lemme 3.6].*

Chapitre 2

Expansion de Crandall de $Z(\mathbf{s}, \mathbf{s}')$

On fixera dans ce chapitre des coefficients $\mathbf{c} = (c_{q,p})_{\substack{1 \leq q \leq Q \\ 1 \leq p \leq P}}$ pour les formes linéaires l_1, \dots, l_Q , des nombres complexes $\mathbf{d} = (d_p)_{1 \leq p \leq P}$, $\mathbf{d}' = (d'_q)_{1 \leq q \leq Q}$ tels que $d_p \in H_0$ pour tout $1 \leq p \leq P$, et vérifiant pour tout $1 \leq q \leq Q$,

$$d'_q = l_q(\mathbf{d}) = \sum_{p=1}^P c_{q,p} d_p.$$

De plus, on fixera dans ce chapitre un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, et une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. Toutes les majorations de ce chapitre dépendront potentiellement de ces constantes, mais l'on choisira de ne pas spécifier cette dépendance pour des soucis de clarté dans la rédaction.

On étudiera tout d'abord le prolongement méromorphe de la fonction $Z(\mathbf{s}, \mathbf{s}')$ et on déterminera des hyperplans contenant toutes ses singularités dans la Proposition 1.2.9. Ensuite on décomposera la fonction méromorphe $Z(\mathbf{s}, \mathbf{s}')$ à l'aide d'une expansion de Crandall, et on obtiendra que une formule de prolongement pour la fonction univariable $s \mapsto Z(-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s)$. Plus précisément, on obtiendra décomposition de cette fonction méromorphe suivant une somme entre une fonction holomorphe s'annulant en $s = 0$, et une fonction méromorphe correspondant à une série de type Erdélyi. Cette formule de prolongement sera cruciale dans la démonstration des Théorèmes A et D.

2.1 Démonstration de la Proposition 1.2.9

Le prolongement méromorphe de $Z(\mathbf{s}, \mathbf{s}')$ et la forme des singularités découlent du Théorème 1.1.27. On cherche ici à montrer que $Z(\mathbf{s}, \mathbf{s}')$ est absolument convergente sur le domaine \mathcal{D} que l'on introduira dans la Proposition 2.1.1. Il suffira de montrer que la série décrivant $Z(\mathbf{s}, \mathbf{s}')$ est normalement convergente sur tout compact de \mathcal{D} . On se propose de démontrer la proposition suivante, plus générale que ce que l'on veut prouver dans la Proposition 1.2.9 :

Proposition 2.1.1. *Soit $P, Q \geq 1$ deux entiers, et $\boldsymbol{\gamma} = (\gamma_{q,p})_{\substack{1 \leq q \leq Q \\ 1 \leq p \leq P}} \in H_0^{Q \times P}$, $\boldsymbol{\alpha} = (\alpha_p)_{1 \leq p \leq P} \in H_0^P$, $\boldsymbol{\beta} = (\beta_q)_{1 \leq q \leq Q} \in H_0^Q$. On pose*

$$\mathcal{D} = \mathcal{D}_{P,Q} := \left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket \right\}.$$

La fonction

$$Z_{\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\beta}}^{P,Q}(\mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + \alpha_p)^{-s_p} \prod_{q=1}^Q \left(\sum_{p=1}^P \gamma_{q,p} n_p + \beta_q \right)^{-s'_q}$$

converge normalement sur tout compact de \mathcal{D} .

Démonstration. Toutes les constantes intervenantes dans la démonstration de cette proposition vont dépendre implicitement du choix des coefficients γ , α et β . On ne marquera pas cette dépendance pour améliorer la clarté de la rédaction. On démontrera cette proposition par principe de récurrence sur P , le cas $P = 1$ étant trivial.

Supposons que le résultat est vrai pour tout $k < P$. Soit $(\mathbf{s}, \mathbf{s}') \in H_0^{P+Q}$, et $\mathbf{n} \in \mathbb{N}^P$. Il est clair qu'on a les inégalités suivantes

$$(n_p + 1) \ll n_p + \operatorname{Re}(\alpha_p) \leq |n_p + \alpha_p| \leq n_p + |\alpha_p| \ll (n_p + 1),$$

De même, on obtient que

$$\sum_{p=1}^P n_p + P \ll \sum_{p=1}^P \operatorname{Re}(\gamma_{q,p})n_p + \operatorname{Re}(\beta_q) \leq \left| \sum_{p=1}^P \gamma_{q,p}n_p + \beta_q \right| \leq \sum_{p=1}^P |\gamma_{q,p}|n_p + |\beta_q| \ll \sum_{p=1}^P n_p + P.$$

En prenant le produit sur $\llbracket 1, P \rrbracket$ et $\llbracket 1, Q \rrbracket$, on trouve que

$$\left| \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \left(\sum_{p=1}^P \gamma_{q,p}n_p + \beta_q \right)^{-s'_q} \right| \ll e^{(\sum_{p=1}^P |\tau_p| + \sum_{q=1}^Q |\tau'_q|) \frac{\pi}{2}} \left(P + \sum_{p=1}^P n_p \right)^{-\sigma'_1 - \dots - \sigma'_Q} \prod_{p=1}^P (n_p + 1)^{-\sigma_p}.$$

On souhaite désormais montrer que la série

$$\sum_{\mathbf{n} \in \mathbb{N}^P} \left(P + \sum_{p=1}^P n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)} \prod_{p=1}^P (n_p + 1)^{-\sigma_p}$$

est normalement convergente sur tout compact de \mathcal{D} . On peut découper cette série en somme finie de séries de la forme

$$S_f(\boldsymbol{\sigma}, \boldsymbol{\sigma}') := \sum_{0 \leq n_{f(1)} \leq \dots \leq n_{f(P)}} \left(P + \sum_{p=1}^P n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)} \prod_{p=1}^P (n_p + 1)^{-\sigma_p}$$

avec $f \in \mathfrak{S}_P$, et en termes de la forme $Z_{1,1,P}^{k,1}((\sigma_1 + \dots + \sigma_{i_1}, \dots, \sigma_{i_{k-1}+1} + \dots + \sigma_{i_k}), \sigma'_1 + \dots + \sigma'_Q)$ avec $1 = i_1 < i_2 < \dots < i_k = P$ et $(i_1, \dots, i_k) \neq (1, \dots, P)$ (ce qui implique en particulier $k < P$). Par principe de récurrence, la série de Dirichlet $Z_{1,1,P}^{k,1}((\sigma_1 + \dots + \sigma_{i_1}, \dots, \sigma_{i_{k-1}+1} + \dots + \sigma_{i_k}), \sigma'_1 + \dots + \sigma'_Q)$ converge normalement sur tout compact de

$$\left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} (\sigma_{i_p} + \dots + \sigma_{i_{p+1}-1}) + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \mathcal{P} \subset \{i_1, \dots, i_{k-1}\} \right\},$$

et on observe que cet ensemble ci-dessus contient bien l'ensemble \mathcal{D} .

Fixons la permutation triviale $f = \operatorname{Id} \in \mathfrak{S}_P$. On trouve alors par changement de variable que

$$S_{\operatorname{Id}}(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \sum_{\mathbf{n} \in \mathbb{N}^P} \left(P + \sum_{p=1}^P pn_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)} \prod_{p=1}^P \left(1 + \sum_{i=1}^p n_i \right)^{-\sigma_p}.$$

Par une inégalité précédente, on trouve que :

$$S_{\operatorname{Id}}(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \ll \zeta_P^{EZ}(\sigma_1, \dots, \sigma_{P-1}, \sigma'_1 + \dots + \sigma'_Q + \sigma_P).$$

Par le Lemme 1.1.25, on en déduit que ce terme là converge normalement sur tout compact de l'ensemble

$$\{(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sigma_p + \dots + \sigma_P + \sigma'_1 + \dots + \sigma'_Q > P + 1 - p, 1 \leq p \leq P\}.$$

Mutatis mutandis, on trouve que pour tout $f \in \mathfrak{S}_P$, la série

$$\sum_{0 \leq n_{f(1)} \leq \dots \leq n_{f(P)}} \left(P + \sum_{p=1}^P n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)} \prod_{p=1}^P (n_p + 1)^{-\sigma_p}$$

converge normalement sur tout compact de

$$\{(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sigma_{f(p)} + \dots + \sigma_{f(P)} + \sigma'_1 + \dots + \sigma'_Q > P + 1 - p, 1 \leq p \leq P\}.$$

On trouve finalement que $Z_{\gamma, \alpha, \beta}^{P, Q}$ converge normalement sur tout compact de

$$\left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset [1, P] \right\}.$$

□

Pour obtenir le prolongement méromorphe de la fonction multizêta $Z(\mathbf{s}, \mathbf{s}')$ ainsi que les singularités de cette fonction multizêta, il suffit d'utiliser le Théorème 1.1.27. En effet, via la Remarque 1.2.8, la fonction multizêta $Z(\mathbf{s}, \mathbf{s}')$ est de la forme $\zeta^{HL}(\mathbf{0}, \tilde{\mathbf{d}}, \tilde{\mathbf{c}}, (s_1, \dots, s_P, s'_1, \dots, s'_Q))$ introduite dans la Définition 1.1.26. On peut donc appliquer le Théorème 1.1.27 pour cette fonction multizêta.

2.2 Enoncé de la formule de prolongement de $Z_{\mathbf{N}, \mathbf{N}'}(s)$

On souhaite ici établir une formule de prolongement de $Z_{\mathbf{N}, \mathbf{N}'}(s)$. Avant d'exprimer résultat, on se doit d'introduire des fonctions définies par des intégrales paramétriques qui apparaîtront dans la formule de prolongement.

Définition 2.2.1. *Pour tout ensemble $\mathcal{P} \subsetneq [1, P]$, tout entier $j \in \llbracket 1, Q \rrbracket$, et tout multi-index $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$ on pose $l_p^*(\mathbf{x}) = c_{1,p}x_1 + \dots + c_{Q,p}x_Q$ pour tout $1 \leq p \leq P$, et on considère les fonctions*

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx_j} \dots dx_Q,$$

et

$$h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx_j} \dots dx_Q,$$

où $s = \sigma + i\tau \in \mathbb{C}$ est un nombre complexe avec $\sigma \gg 1$ suffisamment grand, et où l'on a noté

$$\begin{cases} dx_1 \dots \widehat{dx_j} \dots dx_Q = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_Q \\ \widehat{\mathbf{x}}^j = (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_Q). \end{cases}$$

On dispose de la proposition suivante qui fournit une majoration de ces fonctions ainsi qu'une formule explicite que l'on utilisera dans le chapitre 3 pour calculer la valeur en $s = 0$ de la fonction $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ et de sa dérivée $h'_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$.

Proposition 2.2.2. *Soit $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ une direction et $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ un multi-entier. Pour tout sous-ensemble $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, tout entier $j \in \llbracket 1, Q \rrbracket$, et tout multi-indice $\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}$ on a :*

i) *Les fonctions $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}$ et $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}$ sont holomorphes sur \mathbb{C} .*

ii) *Pour tout entier $r \in \mathbb{N}^*$ on a la majoration suivante, uniformément pour $s \in \overline{D}_0(r)$:*

$$|h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)| \ll \frac{(QM)^{|\mathbf{k}|}}{r},$$

$$\text{avec } M := \max_{\substack{1 \leq q \leq Q \\ 1 \leq p \leq P}} (|c_{q,p}|).$$

iii) *Soit $\epsilon > 0$ suffisamment petit, et $s \in \mathbb{C}$, on a*

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) &= \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q,p}^{w_{p,q}} \right) \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)}}{\left(\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s) \right) \left(\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|) \right)} \\ &\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q, \end{aligned}$$

avec les notations $|\mathbf{v}_{\bullet q}| = \sum_{p \in \mathcal{P}} v_{p,q}$ pour tout $q \in \mathcal{Q}$, et $|\mathbf{w}_{\bullet q}| = \sum_{p \in \mathcal{P}^c} w_{p,q}$ pour tout $q \in \llbracket 1, Q \rrbracket$. De plus, on peut dériver le terme général de la série précédente en s pour obtenir une expression de la dérivée $(h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}})'(s)$.

Remarque 2.2.3. 1) *Le point iii) de cette proposition sera critique pour étudier les valeurs de la fonction $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ et de sa dérivée $(h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}})'(s)$ en $s = 0$ dans les sous-sections 3.2.2 et 3.2.3. Ces deux valeurs apparaîtront dans les calculs de la valeur directionnelle $Z(-(\mathbf{N}, \mathbf{N}'))$, et la valeur de la dérivée directionnelle $Z'(-(\mathbf{N}, \mathbf{N}'))$.*

2) *On peut obtenir des résultats similaires au points ii) et iii) de la proposition précédente pour les fonctions de la forme $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$. Cependant nous n'auront pas besoin d'étudier en détail la valeur et la valeur de la dérivée en $s = 0$ de ce type de fonctions puisque ces valeurs n'apparaîtront pas dans les calculs de la valeur directionnelle $Z(-(\mathbf{N}, \mathbf{N}'))$ et de la valeur de la dérivée directionnelle $Z'(-(\mathbf{N}, \mathbf{N}'))$.*

On peut désormais énoncer la formule de prolongement de $Z_{\mathbf{N}, \mathbf{N}'}(s)$. On découpera cette fonction en deux parties qui dépendront de s et d'une variable libre $\theta > 0$ suffisamment petite. La première, que l'on notera $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$, sera holomorphe sur \mathbb{C} en la variable s et s'annulera en $s = 0$. L'autre terme sera une série de type Erdélyi (dans le genre de la série présente dans la formule d'Erdélyi (2.3)), que l'on

notera $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$. Ce second terme contiendra les hypothétiques singularités de $Z_{\mathbf{N}, \mathbf{N}'}(s)$, et l'on notera cet ensemble de singularités $\mathcal{S}(\mathbf{N}, \mathbf{N}')$, dont l'expression est

$$\mathcal{S}(\mathbf{N}, \mathbf{N}') = \left[\bigcup_{\substack{1 \leq p \leq P \\ \mu_p \neq 0}} \frac{1}{\mu_p} \mathbb{Z}_{\geq N_p + 1} \right] \cup \left[\bigcup_{\mathcal{P} \subset [1, P]} \frac{1}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}} \mathbb{Z}_{\leq |\mathbf{N}|_{|\mathcal{P}}| + |\mathbf{N}'| + |\mathcal{P}|}^* \right].$$

Proposition 2.2.4. Soit $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ un multi-entier, et $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ une direction. Alors la fonction $Z_{\mathbf{N}, \mathbf{N}'}(s)$ admet un prolongement méromorphe sur \mathbb{C} , et ses pôles sont contenus dans l'ensemble $\mathcal{S}(\mathbf{N}, \mathbf{N}')$. Plus précisément, pour tout $\theta > 0$ suffisamment petit et tout $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$, on a

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}(\theta, s),$$

avec $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ et $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ deux fonctions méromorphes sur \mathbb{C} telles que :

1) Pour tout $s \in \mathbb{C}$, on pose

$$K_{\mathbf{N}, \mathbf{N}'}(\theta, s) := \sum_{\emptyset \neq \mathcal{Q} \subset [1, Q]} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \left(\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right). \quad (2.1)$$

2) Pour tout $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$, on pose

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) := \sum_{\mathcal{P} \subset [1, P]} \left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}'; \mathcal{P}, j, \mathbf{k}}(s) \cdot \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}|)s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}|)s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} + \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}'; [1, P], j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right).$$

De plus, on a que :

i) La fonction $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ est holomorphe sur \mathbb{C} , et s'annule en $s = 0$.

ii) La fonction $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ est méromorphe sur \mathbb{C} , avec des pôles inclus dans $\mathcal{S}(\mathbf{N}, \mathbf{N}')$. De plus, on peut dériver terme à terme par rapport à la variable s la série présente dans la formule (2.2).

Remarque 2.2.5. 1) Le terme $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ évalué en $s = 0$ sera l'unique contributeur lors du calcul de la valeur directionnelle $Z(-(\mathbf{N}, \mathbf{N}'))$, puisque la fonction $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ s'annule en $s = 0$. En revanche, lorsque l'on étudiera la valeur de la dérivée $Z'(-(\mathbf{N}, \mathbf{N}'))$, il faudra tenir compte de la dérivée selon s de la fonction $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$.

2) On verra dans la démonstration de la Proposition 2.2.4 que la série

$$s \mapsto \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \prod_{q \in A_Q} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_Q} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q),$$

décrit une fonction holomorphe sur \mathbb{C} (la série est en fait absolument convergente sur \mathbb{C}). En particulier, on pourra considérer la dérivée de cette série suivant s , et l'évaluer en $s = 0$ dans la démonstration du Théorème D. Notons que l'on ne cherchera jamais à calculer directement les valeurs de cette série, ou les valeurs de sa dérivée suivant s .

À l'aide de cette proposition, on obtiendra au passage que la fonction $s \mapsto Z_{\mathbf{N}, \mathbf{N}'}(s)$ est régulière en $s = 0$, et on aura alors démontré par une autre méthode que Komori dans [Kom10] l'existence de valeurs directionnelles pour la classe de fonction zêta qui nous intéresse. On obtient également une information sur les pôles de $Z_{\mathbf{N}, \mathbf{N}'}(s)$:

Corollaire 2.2.6. Soit $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, et $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. Alors la fonction $s \mapsto Z_{\mathbf{N}, \mathbf{N}'}(s)$ a des candidats pôles de la forme

$$\begin{aligned} s &= \frac{n}{\mu_p} & (1 \leq p \leq P, n \geq N_p + 1, \text{ si } \mu_p \neq 0) \\ s &= \frac{n}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}} & (\mathcal{P} \subset \llbracket 1, P \rrbracket, n \in \mathbb{Z} - \infty, |\mathbf{N}|_{|\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}| \setminus \{0\}). \end{aligned}$$

De plus, si $s \in \mathbb{C}$ est un pôle ne pouvant pas s'écrire sous la forme $s = \frac{n}{\mu_p}$ avec $1 \leq p \leq P$ et $n \geq N_p + 1$, alors s est nécessairement un pôle simple pour la fonction $s \mapsto Z_{\mathbf{N}, \mathbf{N}'}(s)$.

Remarque 2.2.7. On peut obtenir un ensemble de candidats pôles encore plus précis pour la fonction $Z_{\mathbf{N}, \mathbf{N}'}(s)$ en utilisant le Corollaire précédent et le Théorème de Komori 1.1.27.

Pour démontrer la Proposition 2.2.4, on multipliera la série de Dirichlet $Z(\mathbf{s}, \mathbf{s}')$ par le produit $\Gamma(s'_1) \dots \Gamma(s'_Q)$. Cette stratégie est classique pour prolonger et étudier les valeurs de la fonction zêta de Riemann (voir [Hid93, Chap. 2]), pour obtenir une représentation intégrale d'une fonction zêta de Shintani (voir [Hid93, Chap. 2]), et pour calculer les valeurs de la fonction zêta de Hurwitz multiple de type Mordell-Tornheim (voir [Ono21]).

Ensuite, on procédera à un découpage du domaine d'intégration en introduisant une variable libre θ . Ce faisant, on pourra isoler un voisinage de l'origine du reste du domaine d'intégration. Enfin, on remarquera que l'intégrale au voisinage de l'origine peut être réexprimée à l'aide de la formule d'Erdélyi (2.3), puis on démontrera également que le reste des intégrales définissent en fait une fonction holomorphe en $(\mathbf{s}, \mathbf{s}')$. Pour découper les intégrales apparaissant dans nos calculs, on utilisera les fonctions gamma incomplètes $\Gamma(s, \theta, \nu)$ et $\gamma(s, \theta, \nu)$ définies dans la Définition 1.2.1.

2.3 Expansion de Crandall directionnelle pour $Z(\mathbf{s}, \mathbf{s}')$

2.3.1 Lemmata

On commence par énoncer une identité classique qui découle d'un développement de Taylor :

Lemme 2.3.1. Soit X_1, \dots, X_d des nombres complexes tels que $|X_1 + \dots + X_d| < 1$. On a alors

$$(1 + X_1 + \dots + X_d)^{-s} = \sum_{\mathbf{k}=(k_p)_{p \in \llbracket 1, d \rrbracket}} \binom{-s}{|\mathbf{k}|} \binom{|\mathbf{k}|}{\mathbf{k}} X_1^{k_1} \dots X_d^{k_d}.$$

Les expressions présentes dans la preuve de la Proposition 2.2.4 feront intervenir une série dont le terme général contiendra $\zeta(s - k, d)$ avec $k \in \mathbb{N}$, et $d \in H_0$. Dans [Ono21], Onodera dispose également de ces termes dans une de ses expressions, et il contrôle ces termes à l'aide du lemme suivant, dans le cas où $d \in \mathbb{R}_+^*$:

Lemme 2.3.2 ([Ono21, Lemme 2.1]). *Soit $\alpha > 0$. On pose pour tout $\delta \in \mathbb{R}_*^+$ et $r \in \mathbb{N}^*$,*

$$D(\delta, r) := \left\{ s \in \mathbb{C} \mid \min_{n \in \mathbb{N}^*} |s - n| \geq \delta, |s| \leq r \right\}.$$

Alors, pour tout $s \in D(\delta, r)$, $k \in \mathbb{N}$, $d > 0$, $m \in \mathbb{N}$, on a

$$|\partial_s^m \zeta(s - k, d)| \ll_{m, r, \delta, \alpha} k!(k + 1)^{r + \alpha}.$$

Ce lemme n'est cependant pas suffisant ici, puisque l'on souhaite majorer $\zeta(s - k, d)$ avec $d \in H_0$. La démonstration du lemme précédent utilise l'équation fonctionnelle vérifiée par la fonction zêta d'Hurwitz $\zeta(s, d)$ pour $d \in]0, 1]$. A priori, on ne peut pas obtenir de la même manière une majoration similaire pour $d \in H_0$. On utilisera donc la formule d'Euler-Maclaurin sur la fonction zêta d'Hurwitz afin d'obtenir une inégalité similaire au lemme précédent.

Lemme 2.3.3. *Soit $\delta \in \mathbb{R}_+^*$, $r \in \mathbb{N}^*$, et $r' \in]1, +\infty[$. Alors on a*

$$\forall s \in D(\delta, r), \forall d \in \overline{H}_{1/r'} \cap \overline{D}_0(r'), \forall k \in \mathbb{N}, \quad |\zeta(s - k, d)| \ll_{r, r', \delta} \left(\frac{e}{\pi}\right)^k k!(k + 1)^{r + 1} (1 + r'^{2k}).$$

Démonstration. Soit $r \in \mathbb{N}^*$, on pose $n = r + 1$. On souhaite majorer $|\zeta(s - k, d)|$ uniformément en $s \in D(\delta, r)$ et en $d \in \overline{H}_{1/r'} \cap \overline{D}_0(r')$. Soit $s \in D(\delta, r)$ et $d \in \overline{H}_{1/r'} \cap \overline{D}_0(r')$. On remarque que l'on a $n > -\sigma$. Soit $k \in \mathbb{N}$, comme on a que $n + k > -\sigma + k$, on peut reprendre l'expression de la fonction zêta de Hurwitz par la formule d'Euler-Maclaurin (1.1.6) à l'ordre $n + k$, et l'évaluer en $s - k$

$$\begin{aligned} \zeta(s - k, d) &= \frac{d^{1-s+k}}{1-s+k} + \sum_{i=0}^{n+k} \binom{-s+k}{i} \frac{(-1)^i B_{i+1}}{i+1} d^{-s-i+k} \\ &\quad + (-1)^{n+k} \binom{-s+k}{n+1+k} \int_0^{+\infty} b_{n+k+1}(x) (x+d)^{-s-n-1} dx, \end{aligned}$$

avec $b_i(x)$ le i -ième polynôme de Bernoulli périodisé, et B_i le i -ième nombre de Bernoulli. En majorant chaque terme de l'égalité, on trouve

$$\begin{aligned} |\zeta(s - k, d)| &\leq \frac{|d|^{1-\sigma+k}}{|1-s+k|} e^{|\tau|\pi/2} + \sum_{i=0}^{n+k} \left| \binom{-s+k}{i} \right| \frac{|B_{i+1}|}{i+1} |d|^{-\sigma-i+k} e^{|\tau|\pi/2} \\ &\quad + \left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |b_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx. \end{aligned}$$

On cherche désormais à majorer chacun de ces termes.

On a clairement que $e^{|\tau|\pi/2} \leq e^{r\pi/2}$. Par continuité de la fonction $(x, y) \in [1/r', r'] \times [-r, r] \mapsto x^y$, on obtient que

$$|d|^{-\sigma+k} \ll_{r, r'} r'^k.$$

Par construction de $D(\delta, r)$, on a

$$\begin{aligned} \frac{|d|^{1-\sigma+k}}{|1-s+k|} &\ll_{r, r'} \frac{r'^k}{\delta} \\ &\ll_{r, r', \delta} r'^k. \end{aligned}$$

Pour la majoration des coefficients binomiaux, on observe tout d'abord que

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad \left| \binom{-s+k}{i} \right| \leq \binom{r+k+i-1}{i} \leq \binom{2n+2k}{i}.$$

Soit $i \in \mathbb{N}$. Par [Leh40], on sait que si $i \not\equiv 2 \pmod{4}$, on a

$$\forall x \in \mathbb{R}, \quad |b_i(x)| < \frac{2i!}{(2\pi)^i},$$

et lorsque $i \equiv 2 \pmod{4}$, on obtient que

$$\forall x \in \mathbb{R}, \quad |b_i(x)| < \frac{2i!\zeta(i)}{(2\pi)^i} \leq \frac{2i!\zeta(2)}{(2\pi)^i}.$$

Notons ψ la fonction digamma d'Euler. On sait que $\lim_{x \rightarrow +\infty} \psi(x) = +\infty$. Ainsi, il existe un réel x_0 tel que la fonction $\frac{\Gamma(x)}{(2\pi)^x}(\psi(x) - \ln(2\pi))$ est positive pour tout $x \geq x_0$. On trouve alors que la fonction

$$x \in \mathbb{R}_+^* \mapsto \frac{\Gamma(x)}{(2\pi)^x},$$

est croissante à partir de ce même réel x_0 . On en déduit donc que, pour n grand,

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad \frac{i!}{(2\pi)^i} \ll_n \frac{(n+k)!}{(2\pi)^{n+k}} \ll_n \frac{1}{(2\pi)^k} k!(k+1)\dots(k+n).$$

En majorant chaque terme du produit à droite de la dernière inégalité, on trouve aisément $(k+1)\dots(k+n) \ll_n (k+1)^n$, et donc comme $n = r+1$:

$$\forall i \in \llbracket 0, n+k \rrbracket, \forall x \in \mathbb{R}, \quad |b_i(x)| \ll_r \frac{k!}{(2\pi)^k} (k+1)^{r+1}.$$

En évaluant l'inégalité précédente en $x = 0$, on obtient que

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad |B_i| \ll_r \frac{k!}{(2\pi)^k} (k+1)^{r+1}.$$

On obtient alors l'inégalité suivante

$$\begin{aligned} \sum_{i=0}^{n+k} \left| \binom{-s+k}{i} \right| \frac{|B_{i+1}|}{i+1} |d|^{-\sigma-i+k} &\ll_r |d|^{-\sigma+k} \frac{k!}{(2\pi)^k} (k+1)^n \sum_{i=0}^{2n+2k} \binom{2n+2k}{i} |d|^{-i} \\ &\ll_{r,r'} \frac{k!}{(2\pi)^k} r'^k (k+1)^{r+1} (1+|d|^{-1})^{2n+2k} \\ &\ll_{r,r'} \frac{k!}{(2\pi)^k} (k+1)^{r+1} (1+r')^{2k}. \end{aligned}$$

Comme $-\sigma - n \leq -1$ il est clair que

$$\forall x \in \mathbb{R}_+, \quad |x+d|^{-\sigma-n-1} \leq (x+\operatorname{Re}(d))^{-\sigma-n-1}.$$

Comme cette dernière quantité est intégrable sur \mathbb{R}_+ , on obtient donc

$$\left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |b_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx \leq \binom{n+2k+1}{n+k+1} \frac{2(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} (\sigma+n)}.$$

Par l'inégalité classique

$$\binom{m}{i} \leq e^i \left(\frac{m}{i}\right)^i \quad (0 \leq i \leq m),$$

on obtient :

$$\begin{aligned} & \left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |b_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx \\ & \leq 2e^k \left(\frac{n+2k+1}{k} \right)^k \frac{(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} \sigma+n} \\ & \leq 2(2e)^k \left(1 + \frac{n+1}{2k} \right)^k \frac{(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} \sigma+n}. \end{aligned}$$

Par la majoration

$$\forall x \in]0, k[, \quad \left(1 + \frac{x}{k} \right)^k \leq e^x,$$

on en déduit que

$$\forall k > (n+1)/2, \quad \left(1 + \frac{n+1}{2k} \right)^k \leq e^{(n+1)/2}.$$

Comme $n = r + 1$, on obtient

$$\forall k \in \mathbb{N}, \quad \left(1 + \frac{n+1}{2k} \right)^k \ll_r 1.$$

On trouve finalement que

$$\begin{aligned} |\zeta(s-k, d)| & \ll_{r, r', \delta} r'^k + \frac{k!}{(2\pi)^k} (k+1)^n (1+r'^{2k}) + \frac{(2e)^k k!}{(2\pi)^k} (k+1)^{n+1} \\ & \ll_{r, r', \delta} \left(\frac{e}{\pi} \right)^k k! (k+1)^{r+1} (1+r'^{2k}). \end{aligned}$$

□

Pour tout entier naturel $a \in \mathbb{N}$, on observe que la fonction $s \mapsto \frac{1}{\Gamma(s)(s+a)}$ admet un prolongement holomorphe sur \mathbb{C} . On observe alors le lemme suivant :

Lemme 2.3.4. *Pour tout entier $r \in \mathbb{N}^*$ on a*

$$\begin{aligned} \forall s \in \overline{D}_0(r), \quad & \left| \frac{1}{\Gamma(s)} \right| \ll_r 1 \\ \forall s \in \overline{D}_0(r), \forall a \in \mathbb{N}, \quad & \left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r 1 \end{aligned}$$

Démonstration. Comme les fonctions $s \mapsto \frac{1}{\Gamma(s)}$ et $s \mapsto \frac{1}{\Gamma(s)s}$ sont holomorphes sur \mathbb{C} , on a uniformément sur $\overline{D}_0(r)$:

$$\begin{aligned} \left| \frac{1}{\Gamma(s)} \right| & \ll_r 1 \\ \left| \frac{1}{\Gamma(s)s} \right| & \ll_r 1 \end{aligned}$$

Enfin, il est clair que

$$\forall |s| > \frac{1}{2}, \quad \left| \frac{1}{s} \right| < 2.$$

On peut alors montrer l'inégalité souhaitée. Soit $s \in \mathbb{C}$ tel que $\sigma \in [-r, r]$.

Si $|s + a| > \frac{1}{2}$, alors

$$\left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r 1.$$

Si $|s + a| \leq \frac{1}{2}$ alors par l'inégalité triangulaire, on obtient que $a \leq |s| + 1/2 \leq r + 1/2$. Par l'équation fonctionnelle de Γ , on a que

$$\frac{1}{\Gamma(s)(s+a)} = \frac{(s+a-1)\dots s}{\Gamma(s+a)(s+a)},$$

on en déduit l'inégalité suivante :

$$\left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r r(r+1)\dots(2r+1).$$

Comme $a \leq 1/2 + r$, on obtient la majoration suivante pour le produit au numérateur

$$|(s+a-1)\dots s| \leq (2r+1)!.$$

On obtient alors que

$$\forall s \in \overline{D}_0(r), \quad \left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r 1.$$

□

La formule d'Erdélyi [EMOT81, §1.11] décrite dans la proposition suivante est cruciale pour réexprimer des intégrandes faisant intervenir la fonction zêta de Lerch que l'on a défini dans la Définition 1.1.7 :

Proposition 2.3.5 ([Ono21, Lemme 2.2]). *Soit $u \in \mathbb{C} \setminus]-\infty, 0]$ tel que $|u| < 2\pi$, $s \in \mathbb{C} \setminus \mathbb{N}^*$, et $d \in \mathbb{R}_+^*$. Alors*

$$e^{-du} \phi(e^{-u}, s, d) = \Gamma(1-s)u^{s-1} + \sum_{k=0}^{+\infty} \frac{(-u)^k}{k!} \zeta(s-k, d).$$

On souhaite obtenir une formule d'Erdélyi pour $d \in H_0$. La majoration que l'on a du terme $\zeta(s-k, d)$ n'est pas aussi bonne lorsque d est complexe que lorsque d est réel, ainsi on doit imposer une condition supplémentaire sur le nombre complexe u :

Proposition 2.3.6 (Formule d'Erdélyi). *Soit $d \in H_0$, $s \in \mathbb{C} \setminus \mathbb{N}^*$ et $u \in \mathbb{C} \setminus]-\infty, 0]$ avec $|u| \ll_d 1$. Alors*

$$e^{-du} \phi(e^{-u}, s, d) = \Gamma(1-s)u^{s-1} + \sum_{k=0}^{+\infty} \frac{(-u)^k}{k!} \zeta(s-k, d). \quad (2.3)$$

Démonstration. Soit $r' \in [1, +\infty[$ un réel, on cherchera à montrer le résultat pour $d \in \overline{H}_{1/r'} \cap \overline{D}_0(r')$. Soit $u \in \mathbb{C} \setminus]-\infty, 0]$ tel que $|u| < \frac{\pi}{e(1+r')^2}$. On fixe un $s \in \mathbb{C} \setminus \mathbb{N}^*$, et on pose $r := \lceil |s| \rceil$. On sait déjà que la formule que l'on souhaite démontrer est valable lorsque $d \in \mathbb{R}_+^*$. On cherche alors à démontrer cette formule pour tout $d \in H_0$ par prolongement analytique. Pour cela, on doit d'abord montrer que les différents termes dans l'expression de la formule voulue sont holomorphes en la variable d sur H_0 .

On sait que la fonction $d \in H_0 \mapsto e^{-du} \phi(e^{-u}, s, d)$ est holomorphe (voir [EMOT81, §1.11]), il ne reste qu'à montrer que la série présente dans la formule l'est aussi. On a déjà que chaque terme $d \in H_0 \mapsto \zeta(s-k, d)$ est holomorphe sur H_0 pour tout entier $k \in \mathbb{N}$. Pour démontrer cela, il suffit de reprendre le prolongement analytique de la fonction zêta d'Hurwitz via la formule d'Euler-Maclaurin, obtenu dans la démonstration de la Proposition 1.1.6. Il ne nous reste plus qu'à montrer que la série présente dans la formule d'Erdélyi (2.3) est normalement convergente en la variable d sur $\overline{H}_{1/r'} \cap \overline{D}_0(r')$. On a la majoration suivante

$$\forall k \in \mathbb{N}, \quad |\zeta(s-k, d)| \ll_{r, r', \delta} \left(\frac{e}{\pi}\right)^k k!(k+1)^{r+1}(1+r')^{2k}.$$

Ainsi on obtient que le terme général de la série présente dans la formule d'Erdélyi (2.3) admet la majoration suivante

$$\frac{|\zeta(s-k, d)|}{k!} |u|^k \ll_{r, r', \delta} \left(u \frac{e}{\pi} (1+r')^2 \right)^k (k+1)^{r+1},$$

or la série entière

$$\sum_{k=0}^{+\infty} \left(\frac{e}{\pi} (1+r')^2 \right)^k k^{r+1} z^k$$

est absolument convergente dans le disque ouvert $\left\{ z \in \mathbb{C}, |z| < \frac{\pi}{e(1+r')^2} \right\}$. On obtient alors que la série

$\sum_{k=0}^{+\infty} \frac{\zeta(s-k, d)}{k!} (-u)^k$ est normalement convergente en d sur $\overline{H}_{1/r'} \cap \overline{D}_0(r')$. Ainsi la formule d'Erdélyi (2.3) est valable pour tout $d \in H_0$ par prolongement analytique. \square

2.3.2 Expansion de Crandall

On souhaite établir dans cette section une expansion de Crandall pour la fonction multizêta

$$Z(\mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}.$$

Rappelons la Définition 1.2.1 : Soit $\nu \in H_0$, $\theta > 0$, $s \in \mathbb{C}$. On a posé

$$\begin{aligned} \Gamma(s, \theta, \nu) &= \int_{\theta}^{+\infty} e^{-\nu y} y^{s-1} dy, \\ \gamma(s, \theta, \nu) &= \int_0^{\theta} e^{-\nu y} y^{s-1} dy \quad (\text{lorsque } \operatorname{Re}(s) > 0). \end{aligned}$$

On notera que l'intégrale décrivant $\Gamma(s, \theta, \nu)$ est absolument convergente pour tout $s \in \mathbb{C}$, mais que celle décrivant $\gamma(s, \theta, \nu)$ n'est absolument convergente que pour $s \in H_0$. Plus précisément, on a :

Proposition 2.3.7. *Soit $\nu \in H_0$ et $\theta > 0$. Alors les fonctions $(s, \nu) \in \mathbb{C} \times H_0 \mapsto \Gamma(s, \theta, \nu)$ et $(s, \nu) \in H_0 \times H_0 \mapsto \gamma(s, \theta, \nu)$ sont holomorphes respectivement sur $\mathbb{C} \times H_0$ et sur $H_0 \times H_0$.*

Démonstration. La proposition découle du théorème d'holomorphic sous le signe intégral. \square

Proposition 2.3.8. *Soit $\nu \in H_0$, $\theta > 0$. On a*

$$\forall s \in H_0, \quad \Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) = \nu^{-s} \Gamma(s).$$

Démonstration. Supposons que $\nu \in \mathbb{R}_+^*$. On a alors par changement de variable :

$$\begin{aligned} \Gamma(s, \theta, \nu) &= \nu^{-s} \int_{\nu\theta}^{+\infty} e^{-x} x^{s-1} dx, \\ \gamma(s, \theta, \nu) &= \nu^{-s} \int_0^{\nu\theta} e^{-x} x^{s-1} dx \quad (\text{lorsque } \operatorname{Re}(s) > 0). \end{aligned}$$

En additionnant les deux termes, on obtient

$$\Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) = \nu^{-s} \Gamma(s).$$

Pour tout nombre complexe $s \in H_0$, la fonction $\nu \in H_0 \mapsto \Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) - \nu^{-s} \Gamma(s)$ est holomorphe, et elle s'annule sur tout \mathbb{R}_+^* . Ainsi par principe des zéros isolés on obtient le résultat voulu. \square

Corollaire 2.3.9. *La fonction $s \mapsto \gamma(s, \theta, \nu)$ admet un prolongement méromorphe sur \mathbb{C} avec des pôles en les entiers strictement négatifs, qui sont des pôles simples, et de résidu $\text{Res}_{s=-n} \gamma(s, \theta, \nu) = \frac{(-1)^n \nu^n}{n!}$.*

Lemme 2.3.10. *Pour tout réel positif $\theta > 0$, et tout nombres complexes $\nu \in H_0$ et $s \in \mathbb{C}$, on a la majoration*

$$|\Gamma(s, \theta, \nu)| \leq 2^\sigma e^{-\text{Re}(\nu)\theta/2} \Gamma\left(\sigma, \frac{\theta}{2}, \text{Re}(\nu)\right).$$

Démonstration. Après le changement de variable $x = \frac{x'}{2}$, on obtient

$$\Gamma(s, \theta, \nu) = 2^s \int_{\theta/2}^{+\infty} e^{-2\nu x} x^{s-1} dx.$$

Comme $|e^{-\nu x}| = e^{-\text{Re}(\nu)x}$ pour tout $x \in \mathbb{R}_+$ et tout complexe $\nu \in H_0$, on obtient la majoration suivante

$$\begin{aligned} |\Gamma(s, \theta, \nu)| &\leq |2^s| \int_{\theta/2}^{+\infty} e^{-\text{Re}(\nu)x} e^{-\text{Re}(\nu)x} x^{\sigma-1} dx \\ &\leq 2^\sigma \int_{\theta/2}^{+\infty} e^{-\text{Re}(\nu)\theta/2} e^{-\text{Re}(\nu)x} x^{\sigma-1} dx, \end{aligned}$$

ce qui conclut la preuve. \square

On étudie maintenant le produit $\prod_{q=1}^Q \Gamma(s'_q) Z(\mathbf{s}, \mathbf{s}')$. Soit $q \in \llbracket 1, Q \rrbracket$, $\theta > 0$, et $s'_q \in H_0$. Par la Proposition 2.3.8, on obtient

$$\Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} = \Gamma(s'_q, \theta, (l_q(\mathbf{n}) + d'_q)) + \gamma(s'_q, \theta, (l_q(\mathbf{n}) + d'_q)).$$

On se donne un multi-entier $\mathbf{n} \in \mathbb{N}^P$, et des nombres complexes $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times H_1^Q$. On obtient ainsi l'égalité suivante en distribuant la somme ci-dessus sur le produit suivant :

$$\begin{aligned} &\prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} \\ &= \prod_{q=1}^Q \left[\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) + \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right] \\ &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in \mathcal{Q}^c} \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) + \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in \mathcal{Q}^c} \left((l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) - \Gamma(s'_q, \theta, l'_q(\mathbf{n}) + d'_q) \right) \\ &\quad + \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned}$$

En distribuant les termes de la forme $(l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) - \Gamma(s'_q, \theta, l'_q(\mathbf{n}) + d'_q)$, on obtient

$$\begin{aligned} &\prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} = \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &+ \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in A_{\mathcal{Q}}} \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \mathcal{Q}^c \setminus A_{\mathcal{Q}}} (-\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)). \end{aligned}$$

En multipliant l'égalité précédente par le produit $\prod_{p=1}^P (n_p + d_p)^{-s_p}$, et en remarquant que $\mathcal{Q} \cup (\mathcal{Q}^c \setminus A_{\mathcal{Q}}) = \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}$, on trouve

$$\begin{aligned} \prod_{q=1}^Q \Gamma(s'_q) (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{p=1}^P (n_p + d_p)^{-s_p} &= \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &+ \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) \end{aligned} \quad (2.4)$$

et en sommant cette égalité sur $n_p \in \mathbb{N}$ pour tout $p \in \llbracket 1, P \rrbracket$, on obtient formellement la relation suivante

$$\begin{aligned} Z(\mathbf{s}, \mathbf{s}') &= \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &+ \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \frac{(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|}}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q)} \\ &\quad \cdot \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned}$$

On cherchera par la suite à donner un sens analytique à la relation précédente. On montrera notamment qu'elle est vraie pour tout $(\mathbf{s}, \mathbf{s}') \in \mathcal{D}$ (avec \mathcal{D} l'ensemble défini dans la Proposition 2.1.1), et on verra que les deux séries précédentes convergent absolument dans \mathcal{D} . Via la formule (2.4) et en utilisant le fait que la série de Dirichlet $Z(\mathbf{s}, \mathbf{s}')$ est absolument convergente sur \mathcal{D} , il suffira de montrer que l'une des deux séries précédentes est absolument convergente sur \mathcal{D} . En fait, nous montrerons dans la Proposition précédente que la seconde série est normalement convergente sur tout compact de $\mathbb{C}^P \times \mathbb{C}^Q$.

Proposition 2.3.11. *Pour tout réel $\theta > 0$ et tout $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q$, on pose*

$$\begin{aligned} K(\theta, \mathbf{s}, \mathbf{s}') &:= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \frac{(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|}}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q)} \\ &\quad \cdot \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned} \quad (2.5)$$

Alors la série précédente est normalement convergente sur tout compact de $\mathbb{C}^P \times \mathbb{C}^Q$. De plus, la fonction $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$ est holomorphe sur $\mathbb{C}^P \times \mathbb{C}^Q$, et $K(\theta, -\mathbf{N}, -\mathbf{N}') = 0$ pour tout multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$.

Démonstration. Soit $\theta > 0$ un réel. Pour démontrer l'holomorphie de la fonction $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$ sur $\mathbb{C}^P \times \mathbb{C}^Q$, il suffit de montrer que la série décrivant $K(\theta, \mathbf{s}, \mathbf{s}')$ est normalement convergente sur tout compact. On cherchera ainsi à majorer le terme général de la série définissant $K(\theta, \mathbf{s}, \mathbf{s}')$. On sait que la fonction $K(\theta, \mathbf{s}, \mathbf{s}')$ est une somme finie de termes de la forme

$$\frac{(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|}}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q), \quad (2.6)$$

avec $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$, et $A_{\mathcal{Q}} \subset \mathcal{Q}^c$. On observe que les fonctions $s'_q \mapsto \frac{1}{\Gamma(s'_q)}$ et $s'_q \mapsto \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)$ sont bien définies et holomorphes sur \mathbb{C} . Pour tout multi-entier $\mathbf{n} \in \mathbb{N}^P$ on note $T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')$ le terme général de la série présente dans la formule (2.6),

$$T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}') := \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \quad (2.7)$$

En utilisant le Lemme 2.3.10 on obtient que pour tout $q \in \llbracket 1, Q \rrbracket$ et pour tout $\mathbf{n} \in \mathbb{N}^P$, on a l'inégalité suivante

$$|\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)| \leq 2^{\sigma'_q} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \frac{\theta}{2}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(l_q(\mathbf{n}) + d'_q)).$$

A l'aide de l'inégalité précédente, et par décroissance de la fonction $x \in \mathbb{R}_*^+ \mapsto \Gamma(\sigma, \theta, x)$ pour $\theta > 0$ quelconque, on obtient finalement

$$|\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)| \leq 2^{\sigma'_q} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \frac{\theta}{2}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(d'_q)).$$

Enfin il est clair que, pour tout complexe $z \in H_0$, $s = \sigma + i\tau \in \mathbb{C}$, on a $|z^s| = |z|^\sigma e^{-\arg(z)\tau}$. Comme $\operatorname{Re}(z) > 0$ on en déduit que $\arg(z) \in]-\pi/2, \pi/2[$, et donc

$$|z^s| \leq |z|^\sigma e^{|\tau|\pi/2}.$$

On trouve alors l'inégalité suivante

$$\begin{aligned} & |T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')| \\ & \leq \left| \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s_q} \right| \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} 2^{\sigma'_q} e^{-\operatorname{Re}(l_q(\mathbf{n}) + d'_q) \frac{\theta}{2}} \Gamma\left(\sigma'_q, \frac{\theta}{2}, d'_q\right) \\ & \leq \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} e^{-\operatorname{Re}(l_q(\mathbf{n}) + d'_q) \frac{\theta}{2}} 2^{\sigma'_q} \Gamma\left(\sigma'_q, \frac{\theta}{2}, d'_q\right) \prod_{p=1}^P e^{|\tau_p| \frac{\pi}{2}} |n_p + d_p|^{-\sigma_p} \prod_{q \in A_{\mathcal{Q}}} e^{|\tau'_q| \frac{\pi}{2}} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}. \quad (2.8) \end{aligned}$$

Notons au passage que, si les coefficients $c_{p,q}$, d_p et d'_q sont tous réels pour tout $1 \leq p \leq P$ et tout $1 \leq q \leq Q$, on peut améliorer l'inégalité précédente en remplaçant les termes de la forme $e^{|\tau_p| \frac{\pi}{2}}$ et $e^{|\tau'_q| \frac{\pi}{2}}$ par 1.

Comme l'ensemble \mathcal{Q} est non vide, et que $A_{\mathcal{Q}} \subset \mathcal{Q}^c$, alors il contient au moins un entier $j \in A_{\mathcal{Q}}$, et par hypothèse, $l_j(\mathbf{n}) = \sum_{p=1}^P c_{j,p} n_p$, avec $c_{j,p} \in H_0$ pour tout $p \in \llbracket 1, P \rrbracket$. On a aussi $d'_j \in H_0$ pour tout $q \in \llbracket 1, Q \rrbracket$. on obtient alors

$$\prod_{q \in \mathcal{Q}} e^{-\operatorname{Re}(l_q(\mathbf{n}) + d'_q) \theta/2} \leq e^{-\operatorname{Re}(l_j(\mathbf{n})) \theta/2}.$$

Soit $r > 0$ un réel, et $(\mathbf{s}, \mathbf{s}') \in \overline{D}_0(r)^{P+Q}$, alors on a en particulier $(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in [-r, r]^{P+Q}$ et $(\boldsymbol{\tau}, \boldsymbol{\tau}') \in [-r, r]^{P+Q}$. Comme la fonction $x \in [-r, r] \mapsto \Gamma(x, \theta, \nu)$ est continue pour tout $\theta > 0$, et $\nu \in H_0$, on en déduit la majoration suivante, uniforme en $(\mathbf{s}, \mathbf{s}') \in \overline{D}_0(r)^P \times \overline{D}_0(r)^Q$

$$\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma\left(\sigma'_q, \frac{\theta}{2}, \operatorname{Re}(d'_q)\right) \ll \frac{1}{r}.$$

De l'inégalité (2.8), on en déduit alors que, pour tout $(\mathbf{s}, \mathbf{s}') \in \overline{D}_0(r)^P \times \overline{D}_0(r)^Q$

$$|T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')| \ll \frac{1}{r} \prod_{p=1}^P e^{-\operatorname{Re}(c_{j,p}) n_p \frac{\theta}{2}} |n_p + d_p|^{-\sigma_p} \prod_{q \in A_{\mathcal{Q}}} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}. \quad (2.9)$$

Comme $\theta > 0$ et que $\operatorname{Re}(c_{j,p}) > 0$ pour tout $p \in \llbracket 1, P \rrbracket$, on voit que la série

$$\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P e^{-\operatorname{Re}(c_{j,p}) n_p \theta/2} |n_p + d_p|^{-\sigma_p} \prod_{q \in A_{\mathcal{Q}}} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}$$

est normalement convergente sur tout compact de $\mathbb{C}^P \times \mathbb{C}^Q$. En particulier, on en déduit que la série $\sum_{\mathbf{n} \in \mathbb{N}^P} T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')$ est normalement convergente sur $\overline{D}_0(r)^{P+Q}$. On conclut alors que $K(\theta, \mathbf{s}, \mathbf{s}')$ est holomorphe sur $\mathbb{C}^P \times \mathbb{C}^Q$.

On observe enfin que pour tout $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$ et pour tout $A_{\mathcal{Q}} \subset \mathcal{Q}^c$, on a que $\llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}} \neq \emptyset$. Ainsi, le produit $\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(s'_q)}$ est non vide. En particulier, lorsque l'on évalue $K(\theta, \mathbf{s}, \mathbf{s}')$ en $\mathbf{s}' = -\mathbf{N}'$, on obtient 0. \square

Remarque 2.3.12. En évaluant la formule (2.5) en remplaçant \mathbf{s} par $(-N_p + \mu_p s)_{p \in \llbracket 1, P \rrbracket}$, et \mathbf{s}' par $(-N'_q + \mu'_q s)_{q \in \llbracket 1, Q \rrbracket}$, on retrouve l'expression (2.1) dans la Proposition 2.2.4.

Proposition 2.3.13. Soit $\theta > 0$ et $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$, avec \mathcal{D} l'ensemble défini dans la Proposition 2.1.1. On pose

$$J(\theta, \mathbf{s}, \mathbf{s}') := \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q).$$

La série précédente est normalement convergente sur tout compact de $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$. En particulier, la fonction $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$ est holomorphe sur $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$.

Démonstration. Soit $\theta > 0$ un réel positif. On reprend les notations de la proposition précédente, en particulier la formule (2.7) pour décrire le terme général des séries présentes dans la définition de $K(\theta, \mathbf{s}, \mathbf{s}')$. Par la formule (2.4), on sait que, pour tout $\mathbf{n} \in \mathbb{N}^P$ et tout $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times H_0^Q$,

$$\begin{aligned} \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) &= \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q} \\ &\quad - \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \frac{(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|}}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q)} T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}'). \end{aligned}$$

Dans la démonstration de la Proposition 2.3.11, on a montré à l'aide de l'inégalité (2.9) que la série de terme général $T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')$ était normalement convergente sur tout compact de $\mathbb{C}^P \times \mathbb{C}^Q$. On a montré dans la Proposition 2.1.1 que la série de fonction en $(\mathbf{s}, \mathbf{s}')$

$$\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q},$$

est normalement convergente sur tout compact de \mathcal{D} . Ainsi, on obtient que la série

$$\sum_{\mathbf{n} \in \mathbb{N}^P} \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q),$$

est normalement convergente sur tout compact de $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$. \square

Proposition 2.3.14. Soit $\theta > 0$, et $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$. On a :

$$Z(\mathbf{s}, \mathbf{s}') = K(\theta, \mathbf{s}, \mathbf{s}') + J(\theta, \mathbf{s}, \mathbf{s}'). \quad (2.10)$$

De plus, la fonction $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$ admet un prolongement méromorphe sur $\mathbb{C}^P \times \mathbb{C}^Q$.

Démonstration. La première égalité découle des deux dernières propositions. Comme la fonction $(\mathbf{s}, \mathbf{s}') \mapsto Z(\mathbf{s}, \mathbf{s}')$ est méromorphe sur $\mathbb{C}^P \times \mathbb{C}^Q$, et que la fonction $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$ est holomorphe sur ce même espace, on en déduit que la fonction $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$ est méromorphe sur $\mathbb{C}^P \times \mathbb{C}^Q$. \square

Corollaire 2.3.15. Pour tout réel $\theta > 0$, les fonctions méromorphes $(\mathbf{s}, \mathbf{s}') \mapsto Z(\mathbf{s}, \mathbf{s}')$ et $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$ partagent exactement les mêmes singularités, et les multiplicités de leurs diviseurs polaires respectifs sont identiques. De plus, la fonction $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$ est holomorphe sur \mathcal{D} , avec \mathcal{D} l'ensemble défini dans la Proposition 2.1.1.

Afin de calculer les valeurs de la fonction $Z(\mathbf{s}, \mathbf{s}')$ en les multi-entiers négatifs, il ne reste plus qu'à étudier $J(\theta, \mathbf{s}, \mathbf{s}')$. Pour ce faire, on réexprimera ce terme en tant qu'intégrale d'un produit de fonctions zêta de Lerch, puis on simplifiera l'intégrande à l'aide d'une formule de Erdélyi sur chaque terme du produit.

Rappelons les notations suivantes pour les fonctions l_p^* : Pour tout $p \in \llbracket 1, P \rrbracket$, on considère $l_p^* : \mathbb{C}^Q \rightarrow \mathbb{C}$ la fonction linéaire suivante

$$l_p^*(\mathbf{x}) := \sum_{q=1}^Q c_{q,p} x_q.$$

En reprenant l'expression précédente de $J(\theta, \mathbf{s}, \mathbf{s}')$ pour $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$ on obtient que

$$\begin{aligned} J(\theta, \mathbf{s}, \mathbf{s}') &= \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q e^{-(l_q(\mathbf{n}) + d'_q) x_q} x_q^{s'_q - 1} dx_q \\ &= \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(\mathbf{x}) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{s'_q - 1} dx_q. \end{aligned} \quad (2.11)$$

On considère désormais des fonctions directionnelles définies à partir des fonctions $J(\theta, \mathbf{s}, \mathbf{s}')$ et $K(\theta, \mathbf{s}, \mathbf{s}')$.

Définition 2.3.16. Soit un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, et une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. On pose pour $s \in \mathbb{C}$,

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) := J(\theta, -\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s)$$

$$K_{\mathbf{N}, \mathbf{N}'}(\theta, s) := K(\theta, -\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s).$$

Pour θ suffisamment petit, on a par la formule (2.10) que, pour tout $s \in \mathbb{C}$ tel que $(-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s) \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$,

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}(\theta, s).$$

Or par construction du domaine d'holomorphic \mathcal{D} dans la Proposition 1.2.9, on en déduit que la condition sur s précédente est vérifiée si

$$\begin{aligned} \forall \mathcal{P} \subset \llbracket 1, P \rrbracket, & \quad -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| + (|\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)\sigma > 1 + |\mathcal{P}| \\ \forall q \in \llbracket 1, Q \rrbracket, & \quad -N'_q + \mu'_q \sigma > 0 \end{aligned}$$

Posons $\sigma_0(\mathbf{N}, \mathbf{N}') := \max\left(\frac{N'_1}{\mu'_1}, \dots, \frac{N'_Q}{\mu'_Q}, \max_{\mathcal{P} \subset \llbracket 1, P \rrbracket} \left(\frac{1 + |\mathcal{P}| + |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}}}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}\right)\right)$. On trouve alors la proposition suivante :

Proposition 2.3.17. Pour tout $s \in H_{\sigma_0(\mathbf{N}, \mathbf{N}')}_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$, on a

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}(\theta, s).$$

Remarque 2.3.18. Plutôt que d'utiliser la notation $s = \sigma + i\tau \in H_{\sigma_0(\mathbf{N}, \mathbf{N}')}_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$, on préférera noter $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$, voir même parfois $\sigma \gg 1$.

On sait déjà par la Proposition 2.3.11 que la fonction $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ est holomorphe et s'annule en $s = 0$. Il reste à étudier le prolongement méromorphe et la régularité en $s = 0$ de $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$. On se propose dans la proposition précédente d'établir une représentation intégrale de $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ à l'aide de la fonction zêta de Lerch.

Proposition 2.3.19. *Soit un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$, et un nombre complexe $s \in \mathbb{C}$ tel que $\mu_p s \notin \mathbb{N}^*$ pour tout $p \in \llbracket 1, P \rrbracket$, et tel que $\sigma \gg 1$. Alors on a*

$$J_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s) = \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-d_p l_p^*(\mathbf{x})} \phi(e^{-l_p^*(\mathbf{x})}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_q. \quad (2.12)$$

Démonstration. Soit $s = \sigma + i\tau \in \mathbb{C}$ tel que $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$. Par la formule (2.11), on en déduit que

$$J_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s) = \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(\mathbf{x}) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_q,$$

où l'on rappelle la définition des formes linéaires $l_p^*(\mathbf{x}) := \sum_{q=1}^Q c_{q,p} x_q$ avec $1 \leq p \leq P$. On souhaite désormais effectuer une permutation série intégrale, on cherchera donc à majorer le terme général de la série précédente. On pose pour tout $q \in \llbracket 1, Q \rrbracket$,

$$v_q(\mathbf{n}) := \sum_{p=1}^P \operatorname{Re}(c_{q,p}) n_p.$$

Pour tout $\mathbf{n} \in \mathbb{N}^P$ on a la majoration suivante :

$$\begin{aligned} & \left| \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(\mathbf{x}) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q \right| \\ & \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \operatorname{Re}(\mu_p s)} e^{\operatorname{Im}(\mu_p s) \frac{\pi}{2}} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q e^{-(v_q(\mathbf{n}) + \operatorname{Re}(d'_q)) x_q} x_q^{-N'_q + \operatorname{Re}(\mu'_q s) - 1} dx_1 \dots dx_Q. \end{aligned}$$

Par un changement de variable de la forme $y_q := (v_q(\mathbf{n}) + \operatorname{Re}(d'_q)) x_q$ pour tout $q \in \llbracket 1, Q \rrbracket$, on obtient finalement l'inégalité suivante

$$\begin{aligned} & \left| \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(\mathbf{x}) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q \right| \\ & \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \operatorname{Re}(\mu_p s)} e^{\operatorname{Im}(\mu_p s) \frac{\pi}{2}} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \operatorname{Re}(\mu'_q s)} \\ & \quad \cdot \int_0^{(v_1(\mathbf{n}) + \operatorname{Re}(d'_1)) \theta} \dots \int_0^{(v_Q(\mathbf{n}) + \operatorname{Re}(d'_Q)) \theta} \prod_{p=1}^P e^{-y_p} \prod_{q=1}^Q y_q^{-N'_q + \operatorname{Re}(\mu'_q s) - 1} dy_1 \dots dy_Q \\ & \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \operatorname{Re}(\mu_p s)} e^{\operatorname{Im}(\mu_p s) \pi/2} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \operatorname{Re}(\mu'_q s)} \Gamma(-N'_q + \operatorname{Re}(\mu'_q s)) \end{aligned}$$

Or la série

$$\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P |n_p + d_p|^{N_p - \operatorname{Re}(\mu_p s)} e^{\operatorname{Im}(\mu_p s) \pi/2} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \operatorname{Re}(\mu'_q s)}$$

est convergente pour tout $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$ par la Proposition 2.1.1. Par le théorème de convergence dominée,

on obtient alors pour tout $s = \sigma + i\tau \in \mathbb{C}$ tel que $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$,

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \cdot \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(\mathbf{x}) d_p} \left(\sum_{n_p=1}^{+\infty} (n_p + d_p)^{N_p - \mu_p s} e^{-l_p^*(\mathbf{x}) n_p} \right) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q,$$

ce qui conclut la preuve. \square

Une fois cette représentation intégrale obtenue pour $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$, on peut utiliser la formule d'Erdélyi sur chaque fonction zêta de Lerch dans le produit de l'intégrande, puis distribuer l'expression afin d'obtenir la proposition suivante :

Proposition 2.3.20. *Soit un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ et $s \in \mathbb{C}$ un nombre complexe tel que $\mu_p s \notin \mathbb{N}^*$ pour tout $p \in \llbracket 1, P \rrbracket$, et tel que $\sigma \gg 1$. Alors, pour tout $\theta > 0$ suffisamment petit, on a*

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \frac{\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s)}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} dx_1 \dots dx_Q$$

$$+ \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}$$

$$\cdot \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q.$$

Démonstration. Soit $M = \max_{1 \leq p \leq P, 1 \leq q \leq Q} (|c_{q,p}|)$, et $\theta \in \left[0, \frac{1}{QM}\right]$. Soit $s \in \mathbb{C}$ tel que $\mu_p s \notin \mathbb{N}^*$ pour tout $p \in \llbracket 1, P \rrbracket$, et $\sigma \gg 1$. Par la relation (2.12), on a que

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-d_p l_p^*(\mathbf{x})} \phi(e^{-l_p^*(\mathbf{x})}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_q.$$

Par la formule d'Erdélyi (2.3), on a que, pour tout $(x_q)_{q \in \llbracket 1, Q \rrbracket} \in]0, \theta]^Q$,

$$\prod_{p=1}^P e^{-d_p l_p^*(\mathbf{x})} \phi(e^{-l_p^*(\mathbf{x})}, -N_p + \mu_p s, d_p)$$

$$= \prod_{p=1}^P \left(\Gamma(1 + N_p - \mu_p s) l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} + \sum_{k_p \geq 0} (-1)^{k_p} \frac{l_p^*(\mathbf{x})^{k_p}}{k_p!} \zeta(-N_p + \mu_p s - k_p, d_p) \right).$$

En distribuant le produit précédent, on obtient pour tout $\mathbf{x} = (x_q)_{q \in \llbracket 1, Q \rrbracket} \in [0, \theta]^Q$,

$$\prod_{p=1}^P e^{-d_p l_p^*(\mathbf{x})} \phi(e^{-l_p^*(\mathbf{x})}, -N_p + \mu_p s, d_p) = \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \prod_{p=1}^P l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1}$$

$$+ \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}.$$

On obtient alors que

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}(\theta, s) &= \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \int_0^\theta \dots \int_0^\theta \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \\ &\prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \cdot \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} dx_1 \dots dx_Q \\ &+ \frac{\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s)}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} dx_1 \dots dx_Q. \end{aligned}$$

On cherche maintenant à majorer le terme général de la série présente dans l'intégrande. Par compacité, on obtient uniformément sur $\mathbf{x} = (x_1, \dots, x_Q) \in \left[0, \frac{1}{QM}\right]^Q$ que

$$\begin{aligned} \left| \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \right| &\ll_s 1 \\ |l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1}| &\ll_s 1. \end{aligned}$$

Comme l_p^* est une forme linéaire pour tout $p \in \llbracket 1, P \rrbracket$, on a que

$$\forall x \in [0, \theta]^Q, \quad |l_p^*(\mathbf{x})| \leq QM\theta.$$

Enfin, par le Lemme 2.3.2, on a la majoration suivante :

$$\forall p \in \llbracket 1, P \rrbracket, \quad \left| \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \ll_s \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil |\mu_p s| \rceil}.$$

On obtient finalement la majoration suivante : pour tout $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, et pour tout $\mathbf{x} = (x_q)_{q \in \llbracket 1, Q \rrbracket} \in [0, \theta]^Q$, on a

$$\begin{aligned} \left| \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \\ \ll_s \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil |\mu_p s| \rceil} (QM\theta)^{k_p}, \end{aligned}$$

Or la série suivante converge absolument sur tout $\theta \in \left[0, \frac{1}{QM}\right]$:

$$\sum_{\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil |\mu_p s| \rceil} (QM\theta)^{k_p},$$

on peut alors permuter la série et l'intégrale pour obtenir le résultat souhaité. \square

Par un éclatement en l'origine on peut simplifier les intégrales dans la proposition précédente pour obtenir une expression de $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ où l'on distingue plus clairement la contribution de la variable libre θ .

Proposition 2.3.21. Soit un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ et $s \in \mathbb{C}$ un nombre complexe tel que $\mu_p s \notin \mathbb{N}^*$ pour tout $p \in \llbracket 1, P \rrbracket$, et tel que $\sigma \gg 1$. Alors

$$J_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s) = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(s) \\ \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\ + \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right),$$

où l'on rappelle les notations de la Proposition 2.2.4 : Pour $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, $j \in \llbracket 1, Q \rrbracket$, et $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$,

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \\ \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx}_j \dots dx_Q \\ h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \\ \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx}_j \dots dx_Q,$$

avec les notations suivantes

$$\begin{cases} dx_1 \dots \widehat{dx}_j \dots dx_Q = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_Q \\ \widehat{\mathbf{x}}^j = (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_Q). \end{cases}$$

Démonstration. Soit $s \in \mathbb{C}$ tel que $\mu_p s \notin \mathbb{N}^*$ pour tout $p \in \llbracket 1, P \rrbracket$, et $\sigma \gg 1$. Pour prouver cette formule, il suffit de reprendre la formule obtenue dans la proposition précédente, et de réexprimer l'intégrale présente dans cette formule. On simplifiera l'intégrale à l'aide d'un éclatement.

On remarque d'abord que $[0, \theta]^Q = \bigcup_{j=1}^Q V_j$, avec $V_j := \{\mathbf{x} \in [0, \theta]^Q \mid \forall q \neq j, x_j \geq x_q\}$, et on observe également que l'intersection deux à deux de ces ensembles est de mesure de Lebesgue nulle. Par Chasles, on obtient

$$\int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q \\ = \sum_{j=1}^Q \int_{V_j} \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q.$$

On considère le changement de variable suivant :

$$f_j : [0, 1]^{j-1} \times [0, \theta] \times [0, 1]^{Q-j} \rightarrow V_j \\ \mathbf{x} = (x_1, \dots, x_Q) \mapsto (x_1 x_j, \dots, x_j, \dots, x_Q x_j),$$

dont le jacobien est $\text{Jac}(f_j) = x_j^{Q-1}$. Quitte à effectuer une permutation des coordonnées, il suffit d'étudier l'intégrale dont le domaine est V_Q . Tout d'abord, on remarque que pour tout $\mathbf{x} = (x_1, \dots, x_Q) \in [0, 1]^{Q-1} \times [0, \theta]$, on a $l_p^*(f_Q(\mathbf{x})) = x_Q l_p^*(\widehat{\mathbf{x}}^Q)$. Ainsi, on trouve que

$$\begin{aligned} & \int_{V_Q} \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q \\ &= \int_0^1 \dots \int_0^1 \int_0^\theta x_Q^{-|\mathbf{N}'| + |\boldsymbol{\mu}'|s - Q - |\mathbf{N}|_{|\mathcal{P}} + |\boldsymbol{\mu}|_{|\mathcal{P}^c} s - |\mathcal{P}| + |\mathbf{k}|} \prod_{q=1}^{Q-1} x_q^{-N'_q + \mu'_q s - 1} \\ & \quad \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^Q)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^Q)^{k_p} x_Q^{Q-1} dx_1 \dots dx_Q \\ &= \int_0^1 \dots \int_0^1 \prod_{q=1}^{Q-1} x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^Q)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} u_p(\widehat{\mathbf{x}}^Q)^{k_p} dx_1 \dots dx_{Q-1} \\ & \quad \cdot \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}^c})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}^c})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \end{aligned}$$

ce qui conclut la preuve. \square

La proposition précédente est cruciale pour démontrer la Proposition 2.2.4, puisque l'on montrera la régularité de $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ en $s = 0$ via la formule de la proposition précédente. On utilisera en particulier la Proposition 2.2.2, qui fournit une majoration uniforme sur tout compact de la fonction $s \mapsto h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$. On démontrera cette proposition dans la prochaine sous-section.

2.3.3 Démonstration de la Proposition 2.2.2

Soit un ensemble $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ et un entier $j \in \llbracket 1, Q \rrbracket$. On démontrera dans cette sous-section la Proposition 2.2.2. Rappelons la notation $l_p^*(\mathbf{x}) = \sum_{q=1}^Q c_{q,p} x_q$ pour tout $p \in \llbracket 1, P \rrbracket$ et tout $x \in \mathbb{C}^Q$. On établira un lemme avant de démontrer cette proposition. Pour tout $\epsilon \in \left] 0, \frac{1}{2MQ} \right]$, pour chaque sous-ensemble $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$, on pose

$$F_{\mathcal{Q}, j}(\epsilon) := \{(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_Q) \mid \forall q \in \mathcal{Q}, x_q \in [0, \epsilon], \forall q' \in \mathcal{Q}^c, x_{q'} \in [\epsilon, 1]\}$$

en ayant noté $\mathcal{Q}^c = \llbracket 1, Q \rrbracket \setminus (\mathcal{Q} \cup \{j\})$. On peut alors découper le domaine d'intégration suivant :

$$\{(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_Q) \mid \forall q \neq j, x_q \in [0, 1]\} = \bigcup_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} F_{\mathcal{Q}, j}(\epsilon).$$

Observons que l'intersection deux à deux des ensembles dans l'union ci-dessus est de mesure de Lebesgue nulle. Via la relation de Chasles, on peut ainsi réécrire les fonctions $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ et $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$ sous forme d'une somme d'intégrales :

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) &= \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \\ h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) &= \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s), \end{aligned} \tag{2.13}$$

en ayant noté pour tout sous-ensemble $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$,

$$I_{\mu, \mu'}^{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \cdot \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx}_j \dots dx_Q,$$

$$I_{\mu, \mu'}^{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s) = \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \cdot \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx}_j \dots dx_Q,$$

pour tout $s \in \mathbb{C}$ tel que $\sigma \gg 1$.

Lemme 2.3.22. *Soit une direction $(\mu, \mu') \in \overline{H}_0^P \times H_0^Q$, un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, un entier $j \in \llbracket 1, Q \rrbracket$, deux ensembles $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ et $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$, et un multi-indice $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$. On a que :*

i) *Pour tout $\epsilon > 0$ suffisamment petit, la fonction $s \mapsto I_{\mu, \mu'}^{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ est holomorphe sur \mathbb{C} .*

ii) *Pour tout $\epsilon > 0$ suffisamment petit et pour tout $r \in \mathbb{N}^*$, on a la majoration suivante, uniforme pour $s \in \overline{D}_0(r)$:*

$$\left| I_{\mu, \mu'}^{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right| \underset{r, \epsilon}{\ll} (QM)^{|\mathbf{k}|},$$

$$\text{avec } M = \max_{1 \leq q \leq Q, 1 \leq p \leq P} (|c_{q,p}|).$$

iii) *Pour tout $s \in \mathbb{C}$ et $0 < \epsilon \ll 1$, on a*

$$I_{\mu, \mu'}^{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \quad (2.14)$$

$$\cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q,p}^{w_{p,q}} \right) \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)}$$

$$\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q,$$

avec les notations $\mathcal{Q}^c = \llbracket 1, Q \rrbracket \setminus (\mathcal{Q} \cup \{j\})$, $|\mathbf{v}_{\bullet q}| = \sum_{p \in \mathcal{P}} v_{p,q}$ pour tout $q \in \mathcal{Q}$, et $|\mathbf{w}_{\bullet q}| = \sum_{p \in \mathcal{P}^c} w_{p,q}$ pour tout $q \in \llbracket 1, Q \rrbracket$. En notant $T_{\mu, \mu'}^{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{w}, \mathbf{v}}(\epsilon, s)$ le terme général de la série précédente, on a également

$$\partial_s I_{\mu, \mu'}^{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \partial_s T_{\mu, \mu'}^{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{w}, \mathbf{v}}(\epsilon, s).$$

iv) *Pour tout $0 < \epsilon \ll 1$ suffisamment petit et pour tout $s \in \mathbb{C}$, on a*

$$h_{\mu, \mu'}^{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\mu, \mu'} I_{\mu, \mu'}^{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s). \quad (2.15)$$

Démonstration. On cherchera ici à développer l'intégrande via des développements en séries entières au voisinage de 0 pour les éléments $\mathbf{x} \in F_{\mathcal{Q},j}(\epsilon)$ avec $\epsilon > 0$ suffisamment petit. Notons que ces séries entières convergeront uniformément sur le disque $D_0(\eta)$, pour η suffisamment petit. On pourra donc intervertir la série et l'intégrale.

Soit $s \in \mathbb{C}$ un nombre complexe tel que $\sigma \gg 1$, et $\epsilon \in \left]0, \frac{1}{2MQ}\right]$ avec $M := \max_{1 \leq q \leq Q, 1 \leq p \leq P} (|c_{q,p}|)$. Fixons un ensemble $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$. On peut développer en série entière en les $(x_q)_{q \in \mathcal{Q}}$ les fonctions de la forme $l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1}$, avec $\mathbf{x} \in F_{\mathcal{Q},j}(\epsilon)$ et $p \in \mathcal{P}$. Par le Lemme 2.3.1, on obtient uniformément en $\widehat{\mathbf{x}}^j = (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_Q)$:

$$\begin{aligned} & l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \\ &= \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q + \sum_{q \in \mathcal{Q}} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1} \\ &= \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1} \left(1 + \sum_{f \in \mathcal{Q}} \frac{c_{f,p}}{c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q} x_f \right)^{-N_p + \mu_p s - 1} \\ &= \sum_{\mathbf{v}_p = (v_{p,q})_{q \in \llbracket 1, Q \rrbracket} \in \mathbb{N}^{\mathcal{Q}}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} x_q^{v_{p,q}}. \quad (2.16) \end{aligned}$$

Pour $p \in \mathcal{P}^c$, et $k_p \in \mathbb{N}$, on a par le multinôme de Newton :

$$l_p^*(\widehat{\mathbf{x}}^j)^{k_p} = \sum_{\substack{\mathbf{w}_p = (w_{p,q})_{q \in \llbracket 1, Q \rrbracket} \in \mathbb{N}^{\mathcal{Q}} \\ |\mathbf{w}_p| = k_p}} \binom{k_p}{\mathbf{w}_p} c_{j,p}^{w_{p,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{w_{p,q}} x_q^{w_{p,q}}.$$

Il est clair que la série présente dans l'expression (2.16) converge uniformément pour $(x_q)_{q \in \mathcal{Q}} \in [0, \epsilon]^{\mathcal{Q}}$. Par permutation série intégrale, on peut réexprimer $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ de la manière suivante :

$$\begin{aligned} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} c_{j,p}^{w_{p,j}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \right) \\ &\cdot \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q},j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \left(\prod_{p \in \mathcal{P}^c} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{w_{p,q}} x_q^{w_{p,q}} \right) \\ &\cdot \prod_{p \in \mathcal{P}} \left(\prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} x_q^{v_{p,q}} \right) \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_1 \dots \widehat{dx}_j \dots dx_Q. \end{aligned}$$

En remarquant que $\llbracket 1, Q \rrbracket \setminus \{j\} = \mathcal{Q} \cup \mathcal{Q}^c$, on peut séparer le produit $\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$ en deux parties :

$$\begin{aligned} & I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \\ &= \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q},j}(\epsilon)} \prod_{p \in \mathcal{P}^c} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{w_{p,q}} \right) \prod_{p \in \mathcal{P}} \left(\prod_{q \in \mathcal{Q}} x_q^{v_{p,q}} \right) \\
& \cdot \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots \widehat{dx}_j \dots dx_Q \\
& = \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
& \cdot \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q},j}} \prod_{q \in \mathcal{Q}} x_q^{-N'_q + \mu'_q s - 1 + \sum_{p \in \mathcal{P}} v_{p,q} + \sum_{p \in \mathcal{P}^c} w_{p,q}} \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \sum_{p \in \mathcal{P}^c} w_{p,q}} \\
& \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_1 \dots \widehat{dx}_j \dots dx_Q.
\end{aligned}$$

Soit $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}}$ avec $\mathbf{v}_p \in \mathbb{N}^{\mathcal{Q}}$ pour tout $p \in \mathcal{P}$, et $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c}$ avec $\mathbf{w}_p \in \mathbb{N}^{\mathcal{Q}}$ pour tout $p \in \mathcal{P}^c$. On rappelle la notation $|\mathbf{v}_{\bullet,q}| = \sum_{p \in \mathcal{P}} v_{p,q}$ tout $q \in \mathcal{Q}$, et $|\mathbf{w}_{\bullet,q}| = \sum_{p \in \mathcal{P}^c} w_{p,q}$ pour tout $q \in \llbracket 1, Q \rrbracket$. Via le théorème de Fubini, on obtient l'expression suivante pour $\sigma \gg 1$:

$$\begin{aligned}
I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
& \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \left(\prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|}}{-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|} \right) \\
& \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \left(\prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet,q}|} \right) \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q.
\end{aligned}$$

On obtient donc que la formule (2.14) est vraie pour tout $s \in \mathbb{C}$ tel que $\sigma \gg 1$.

On cherche maintenant à prolonger la fonction $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ sur tout \mathbb{C} via l'expression précédente. On a clairement que, pour tout $\sigma \gg 1$,

$$\begin{aligned}
I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \frac{1}{\prod_{q \in \mathcal{Q}^c} \Gamma(-N'_q + \mu'_q s)} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \quad (2.17) \\
& \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|}}{\Gamma(-N'_q + \mu'_q s) (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|)} \\
& \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \left(\prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet,q}|} \right) \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q.
\end{aligned}$$

Notons $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$ le terme général de la série précédente. On a alors

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s).$$

Tous les termes présents dans l'expression de $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$ sont holomorphes sur \mathbb{C} suivant la variable s . Pour montrer que la fonction $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ est holomorphe sur \mathbb{C} , il suffit de montrer que la série précédente converge normalement selon la variable s sur tout compact de \mathbb{C} . On cherche alors à majorer le terme général sur un disque $\overline{D}_0(r)$, avec $r \in \mathbb{N}^*$ un entier. Les majorations que l'on effectuera dépendront toutes de $\mathbf{N}, \mathbf{N}', \mu, \mu', \mathcal{P}, j, \mathcal{Q}$, mais on omettra cette dépendance afin de simplifier la rédaction.

On sait que la fonction

$$s \mapsto \frac{1}{\Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)}$$

est holomorphe sur \mathbb{C} pour tout $q \in \mathcal{Q}$. On majore maintenant le terme général de la série présente dans la formule (2.17) à l'aide du Lemme 2.3.4. On a

$$\begin{aligned} \forall q \in \mathcal{Q}, \forall s \in \overline{D}_0(r), \forall a \in \mathbb{N}, & \quad \left| \frac{1}{\Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + a)} \right| \ll_r 1 \\ \forall q \in \llbracket 1, Q \rrbracket, & \quad \left| \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right| \ll_r 1. \end{aligned}$$

On sait que la fonction

$$s \in \mathbb{C} \mapsto \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q$$

est holomorphe car l'intégrande est régulière.

Soit $s \in \overline{D}_0(r)$, $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}}$ avec $\mathbf{v}_p \in \mathbb{N}^{\mathcal{Q}}$, et $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c}$ avec $\mathbf{w}_p \in \mathbb{N}^{\mathcal{Q}}$ tel que $|\mathbf{w}_p| = k_p$ pour tout $p \in \mathcal{P}^c$. On observe alors que $\sum_{\substack{q=1 \\ q \neq j}}^Q |\mathbf{w}_{\bullet q}| = |\mathbf{k}| - |\mathbf{w}_{\bullet j}|$. De plus, par compacité, on obtient la majoration suivante uniformément en $q \in \mathcal{Q}^c$ et en $x_q \in [\epsilon, 1]$,

$$\left| x_q^{-N'_q + \mu'_q s - 1} \right| \ll_r 1.$$

Comme $|\mathbf{w}_{\bullet q}| \geq 0$, on obtient que $|x_q^{|\mathbf{w}_{\bullet q}|}| \leq 1$. De plus, comme $\arg \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right) \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$, on a pour tout $s \in \overline{D}_0(r)$, et tout $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$,

$$\left| \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \right| \leq \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right|^{-N_p + \operatorname{Re}(\mu_p s) - 1 - |\mathbf{v}_p|} e^{|\mu_p| r \pi / 2}.$$

On remarque que l'on a la majoration suivante pour tout $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$

$$K + K\epsilon \leq \left(\operatorname{Re} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} \operatorname{Re}(c_{q,p} \epsilon) \right) \right) \leq \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \leq \sum_{q=1}^Q |c_{q,p}| \leq QM$$

avec $M = \max_{1 \leq p \leq P, 1 \leq q \leq Q} (|c_{q,p}|)$ et $K := \min_{1 \leq q \leq Q, 1 \leq p \leq P} \operatorname{Re}(c_{q,p})$. Pour $p \in \mathcal{P}$, on trouve alors

$$\left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right|^{-N_p + \operatorname{Re}(\mu_p s) - 1 - |\mathbf{v}_p|} \leq \begin{cases} (QM)^{\operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1} & \text{si } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \geq 1 \\ & \text{et } \operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1 \geq 0 \\ 1 & \text{si } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \geq 1 \text{ et } \operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1 < 0 \\ 1 & \text{si } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| < 1 \text{ et } \operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1 \geq 0 \\ (K(\epsilon + 1))^{\operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1} & \text{sinon.} \end{cases}$$

On a uniformément pour $s \in \overline{D}_0(r)$:

$$\forall p \in \mathcal{P}, \quad (QM)^{\operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1} \ll_r (QM)^{-|\mathbf{v}_p|}.$$

Soit $p \in \llbracket 1, P \rrbracket$. Par compacité, on a uniformément pour $s \in \overline{D}_0(r)$:

$$(K\epsilon)^{\mu_p \sigma} \ll_{\epsilon, r} 1.$$

On a également

$$(K(\epsilon + 1))^{\operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1} \ll_{\epsilon, r} (K(\epsilon + 1))^{-|\mathbf{v}_p|} \ll_{r, \epsilon} K^{-|\mathbf{v}_p|}.$$

On obtient finalement la majoration suivante uniformément en $s \in \overline{D}_0(r)$ et en $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$,

$$\left| \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p \sigma - 1 - |\mathbf{v}_p|} \right| \ll_{r, \epsilon} \max(K^{-|\mathbf{v}_p|}, (QM)^{-|\mathbf{v}_p|}, 1) \prod_{p=1}^P e^{|\mu_p| r \pi / 2} \ll_{r, \epsilon} \min(K, QM, 1)^{-|\mathbf{v}_p|}.$$

Maintenant que l'on a majoré l'intégrande, on obtient finalement que l'intégrale est bornée de la façon suivante :

$$\left| \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q \right| \ll_{r, \epsilon} \min(K, QM, 1)^{-\sum_{p \in \mathcal{P}} |\mathbf{v}_p|}.$$

On a la majoration suivante uniforme en $s \in \overline{D}_0(r)$:

$$\left| \epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)} \right| \ll_{r, \epsilon} \sum_{q \in \mathcal{Q}} |\mathbf{v}_{\bullet q}|.$$

De plus, on a que

$$\forall n \in \mathbb{N}, \forall a \in \mathbb{N}, \forall b \in \overline{H}_0, \forall s \in \overline{D}_0(r), \quad \left| \binom{-n + bs - 1}{a} \right| \leq \binom{n + \lceil |b|R \rceil + a}{a}.$$

On trouve alors que

$$\begin{aligned}
& \left| \frac{1}{\prod_{q \in \mathcal{Q}^c} \Gamma(-N'_q + \mu'_q s)} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \right. \\
& \cdot \prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|}}{\Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)} \\
& \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 + |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q \left. \right| \\
& \ll_{r,\epsilon} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} M^{|\mathbf{w}_p|} \right) \prod_{p \in \mathcal{P}} \binom{N_p + \lceil |\mu_p| R \rceil + |\mathbf{v}_p|}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} (M\epsilon)^{\sum_{q \in \mathcal{Q}} |\mathbf{v}_{\bullet q}|} \min(K, QM, 1)^{-\sum_{q \in \mathcal{Q}} |\mathbf{v}_{\bullet q}|} \\
& \ll_{r,c} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} M^{k_p} \right) \prod_{p \in \mathcal{P}} \binom{N_p + \lceil |\mu_p| R \rceil + |\mathbf{v}_p|}{(\mathbf{v}_p, \lceil |\mu_p| R \rceil)} \left(\frac{M\epsilon}{\min(K, QM, 1)} \right)^{\sum_{q \in \mathcal{Q}} |\mathbf{v}_{\bullet q}|}. \tag{2.18}
\end{aligned}$$

On souhaite maintenant montrer que le majorant obtenu précédemment est le terme général d'une série absolument convergente en la variable $\epsilon > 0$ lorsque l'on considère la somme $\sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}}$.

Remarquons d'abord que par le multinôme de Newton, on trouve

$$\sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} M^{k_p} = (QM)^{|\mathbf{k}|}.$$

De plus, la série

$$\sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \prod_{p \in \mathcal{P}} \binom{N_p + \lceil |\mu_p| R \rceil + |\mathbf{v}_p|}{(\mathbf{v}_p, \lceil |\mu_p| R \rceil)} \left(\frac{M\epsilon}{\min(K, QM, 1)} \right)^{\sum_{q \in \mathcal{Q}} |\mathbf{v}_{\bullet q}|}$$

converge absolument pour tout $0 < \epsilon < \frac{\min(K, QM, 1)}{M}$. En fixant un ϵ suffisamment petit, la série en $(v_{p,q})_{p \in \mathcal{P}, q \in \mathcal{Q}}$ est une constante ne dépendant que de r, ϵ et de \mathbf{N} . On obtient alors l'inégalité suivante :

$$\left| I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right|_{\mu, \mu'} \leq (QM)^{|\mathbf{k}|}.$$

Les points i) et iii) se déduisent donc du théorème d'holomorphic sous le signe somme, et par prolongement analytique. Le point ii) est vrai par la majoration précédente. Comme la fonction $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ est holomorphe sur \mathbb{C} , et que par la formule (2.13) on a

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \quad (\sigma \gg 1),$$

alors on peut prolonger l'égalité précédente sur tout \mathbb{C} par prolongement analytique, d'où le point iv). \square

En modifiant légèrement la démonstration du lemme précédent, on obtient le lemme suivant :

Lemme 2.3.23. Soit une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$, un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, un entier $j \in \llbracket 1, Q \rrbracket$, et un ensemble $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$. Pour tout $s \in \mathbb{C}$ et $0 < \epsilon \ll 1$, on a que :

$$I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s) = \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \llbracket 1, P \rrbracket} \in (\mathbb{N}^Q)^P} \left(\prod_{q \in \llbracket 1, Q \rrbracket} c_{q, p, q}^{w_{p, q}} \right) \left(\prod_{p=1}^P \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q, p}^{v_{p, q}} \right) \\ \cdot \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s) \prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}|)} \\ \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1} \cdot \prod_{p=1}^P \left(c_{j, p} + \sum_{f \in \mathcal{Q}^c} c_{f, p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q,$$

avec les notations $\mathcal{Q}^c = (\llbracket 1, Q \rrbracket \setminus \{j\}) \setminus \mathcal{Q}$, $|\mathbf{v}_{\bullet q}| = \sum_{p=1}^P v_{p, q}$ pour tout $q \in \mathcal{Q}$. De plus, la fonction $s \in \mathbb{C} \mapsto I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s)$ admet un prolongement holomorphe sur \mathbb{C} .

Remarque 2.3.24. Contrairement au Lemme 2.3.22, on n'a pas besoin d'obtenir une majoration dans le lemme 2.3.23, puisque le terme \mathbf{k} y est absent.

On dispose de tous les ingrédients nécessaires à la démonstration de la Proposition 2.2.2 :

Démonstration. Rappelons que $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ et $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$ s'expriment de la façon suivante, pour tout $s \in \mathbb{C}$ tel que $\sigma \gg 1$:

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\substack{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\} \\ \boldsymbol{\mu}, \boldsymbol{\mu}'}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s), \\ h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) = \sum_{\substack{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\} \\ \boldsymbol{\mu}, \boldsymbol{\mu}'}} I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s).$$

Comme on sait que les fonctions $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ et $s \mapsto I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s)$ sont holomorphes sur \mathbb{C} , on peut prolonger holomorphiquement les fonctions $s \mapsto h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ et $s \mapsto h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$ sur \mathbb{C} . On obtient alors le point i) de la Proposition 2.2.2. Le point ii) et iii) de la Proposition 2.2.2 découle du point ii) et iii) du Lemme 2.3.22. \square

2.3.4 Démonstration de la Proposition 2.2.4

Soit un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, et une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. On exploite l'expression $Z(\mathbf{s}, \mathbf{s}') = K(\theta, \mathbf{s}, \mathbf{s}') + J(\theta, \mathbf{s}, \mathbf{s}')$ pour obtenir la preuve du théorème de prolongement méromorphe selon la direction $(\boldsymbol{\mu}, \boldsymbol{\mu}')$. On sait déjà que la fonction $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$ est holomorphe sur $\mathbb{C}^P \times \mathbb{C}^Q$. Il reste donc à prouver que la fonction $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ est méromorphe et régulière en $s = 0$. Afin de montrer la Proposition 2.2.4, il ne nous reste plus qu'à établir que la fonction $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ est méromorphe sur \mathbb{C} avec des pôles inclus dans

$$\mathcal{S}(\mathbf{N}, \mathbf{N}') = \left[\bigcup_{\substack{1 \leq p \leq P \\ \mu_p \neq 0}} \frac{1}{\mu_p} \mathbb{Z}_{\geq N_p + 1} \right] \cup \left[\bigcup_{\mathcal{P} \subset \llbracket 1, P \rrbracket} \frac{1}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|^{\mathcal{P}}} \mathbb{Z}_{\leq |\mathbf{N}|_{\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}|} \right].$$

On sait que la formule (2.2) est valable sur le complémentaire de cet ensemble, pour $\sigma \gg 1$. Pour

tout $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ tel que $\sigma \gg 1$, on a alors

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} \left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} + \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right).$$

À l'aide de ces rappels on peut démontrer la Proposition 2.2.4 :

Démonstration. Soit $0 < \theta \ll 1$. On sait par la Proposition 2.3.21 que la formule (2.2) est valable pour tout nombre complexe $s = \sigma + i\tau \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ tel que $\sigma \gg 1$. Il est clair que la fonction

$$s \mapsto \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)},$$

est méromorphe sur \mathbb{C} , régulière en $s = 0$, et avec des pôles de la forme $s = \frac{|\mathbf{N}'| + |\mathbf{N}| + P}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|}$. Il reste alors à démontrer que la série de fonctions en s

$$s \mapsto \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} \left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \quad (2.19)$$

est méromorphe sur \mathbb{C} , régulier en $s = 0$, et avec des pôles inclus dans $\mathcal{S}(\mathbf{N}, \mathbf{N}')$. On pose

$$\tilde{D}_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\mathbf{N}, \mathbf{N}', \delta, r) := \left\{ s \in \overline{D}_0(r) \left| \begin{array}{l} \forall p \in \llbracket 1, P \rrbracket, \min_{\substack{n \in \mathbb{N} \\ n > N_p}} |\mu_p s - n| \geq \delta \\ \forall \mathcal{P} \subset \llbracket 1, P \rrbracket, \min_{\substack{n \in \mathbb{Z}^* \\ n \geq -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| - |\mathcal{P}|}} |(|\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)s + n| \geq \delta \end{array} \right. \right\}.$$

Notons que :

1) On a clairement que $\tilde{D}_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\mathbf{N}, \mathbf{N}', \delta, r)$ contient un voisinage ouvert de 0.

2) La condition

$$\min_{\substack{n \in \mathbb{N} \\ n > N_p}} |\mu_p s - n| \geq \delta$$

apparaît pour éviter les singularités de $\zeta(-N_p + \mu_p s - k, d_p)$. On observe qu'au voisinage de $s = 0$, ce terme est parfaitement défini.

3) La condition

$$\min_{\substack{n \in \mathbb{Z}^* \\ n \geq -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| - |\mathcal{P}|}} |(|\boldsymbol{\mu}'|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)s + n| \geq \delta$$

apparaît pour éviter les singularités de $\frac{1}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}$ qui ne sont pas nécessairement compensées par les zéros de $\frac{1}{\Gamma(-N'_j + \mu'_j s)}$. On remarque tout de même qu'en $s = 0$, lorsque $-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}| = 0$, la singularité de $\frac{1}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s}$ est compensée par le zéro de $\frac{1}{\Gamma(-N'_j + \mu'_j s)}$.

4) L'ensemble $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ est inclus dans le complémentaire de $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$.

Fixons un sous-ensemble $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$. On sait par la Proposition 2.3.21 que la formule (2.2) est valable pour tout $s = \sigma + i\tau \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$, tel que $\sigma \gg 1$. On souhaite prouver que la série en \mathbf{k} présente dans la formule (2.19) est normalement convergente sur $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$.

Soit $r' > 1$ un réel tel que, pour tout $p \in \llbracket 1, P \rrbracket$, $\operatorname{Re}(d_p) \geq 1/r'$, et $|d_p| < r'$. Soit $r > 0$, $1 > \delta > 0$. Par le Lemme 2.3.3, on a

$$\forall p \in \llbracket 1, P \rrbracket, \forall s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \forall k_p \in \mathbb{N}, \quad \zeta(-N_p + \mu_p s - k_p, d_p) \ll_{r, r', \delta} (k_p + N_p)! (k_p + N_p + 1)^{r+1} (1 + r'^{2k_p}).$$

Par le Lemme 2.2.2, on a

$$\forall s \in \overline{D}_0(r), \forall \mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}, \quad |h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)| \ll_r (QM)^{|\mathbf{k}|}.$$

Comme la fonction $s \mapsto \Gamma(1 + N_p - \mu_p s)$ est holomorphe sur $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ pour tout $1 \leq p \leq P$, on en déduit la majoration suivante

$$\forall p \in \llbracket 1, P \rrbracket, \forall s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \quad |\Gamma(1 + N_p - \mu_p s)| \ll_{r, \delta} 1.$$

Enfin, par le Lemme 2.3.4 on a la majoration suivante

$$\forall s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \quad \left| \frac{1}{\Gamma(-N'_j + \mu'_j s) (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \right| \ll_{r, \delta} 1.$$

Par compacité de $\overline{D}_0(r)$ on obtient

$$\forall s \in \overline{D}_0(r), \quad |\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}|}| \ll_r 1.$$

Ainsi, pour tout $s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$, et pour $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$, on a

$$\begin{aligned} & \left| (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right. \\ & \left. \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s) (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \\ & \ll_{r, r', \delta} (QM\theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1} (1 + r'^{2k_p}) \\ & \ll_{r, r', \delta} (2r'^2 QM\theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1}. \end{aligned}$$

Or la série

$$\sum_{\mathbf{N} \in \mathbb{N}^{\mathcal{P}^c}} (2r'^2 Q M \theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1}$$

converge absolument pour tout $\theta \in \mathbb{C}$ tel que $|\theta| < \frac{1}{(2r'^2 Q M)}$. On en déduit en particulier que la formule (2.2) est valable pour tout $s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$. Par théorème d'holomorphic sous le signe somme, on obtient que la fonction $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ est holomorphe sur $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$. Comme $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ contient un voisinage ouvert de 0, on obtient par le théorème d'holomorphic sous le signe somme que l'on peut dériver suivant s sur ce même voisinage la fonction $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$,

$$\partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \tag{2.20}$$

$$\begin{aligned} & \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} \partial_s \left(\left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \right. \\ & \left. \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}) s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}) s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right), \\ & + \partial_s \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|) s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|) s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right). \end{aligned}$$

Comme $1 > \delta > 0$ est arbitraire dans la définition de $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$, et que $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ contient un voisinage ouvert de 0, on obtient en particulier que $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ est régulière en $s = 0$, et que ses pôles sont inclus dans l'ensemble $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ décrit dans la Proposition 2.2.4. \square

Remarque 2.3.25. *Via la formule (2.2), on peut en déduire une expression des résidus de $Z_{\mathbf{N}, \mathbf{N}'}(s)$ en ses pôles. En effet, on sait que cette fonction se réexprime sous la forme d'une somme entre une fonction holomorphe et une fonction méromorphe sur \mathbb{C} :*

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = K_{\mathbf{N}, \mathbf{N}'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}(\theta, s).$$

Comme la fonction $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ est holomorphe sur \mathbb{C} , cette fonction ne contient aucune singularité. En revanche, dans la formule (2.2), on peut lire les résidus en étudiant finement les deux termes suivants :

- $\frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}) s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}) s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}$
- $\frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|) s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|) s - |\mathbf{N}'| - |\mathbf{N}| - P)}$

Notons que les pôles en la variable s de ces termes là peuvent être parfois compensés par des zéros des fonctions de la forme $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ pour le premier terme, et par des zéros de fonctions de la forme $h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s)$ pour le second terme.

Chapitre 3

Calculs des coefficients $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ et $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ μ, μ'

On calculera dans ce chapitre les coefficients $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ et $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ apparaissant dans les Théorèmes A et D. Pour des raisons que l'on expliquera en détail dans les deux chapitres suivant, on se propose de ne calculer ces coefficients que pour des directions $(\mu, \mu') \in \overline{H}_0^P \times H_0^Q$. En effet, dans les deux prochains chapitres on démontrera les Théorèmes A et D pour ces directions-là, puis on étendra les deux théorèmes à un ensemble de directions plus large via un prolongement analytique.

Par la Proposition 2.2.2, on sait que la fonction $s \mapsto h_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\mathcal{P}, j, \mathbf{k})(s)$ est holomorphe sur \mathbb{C} . En particulier, on peut considérer ses coefficients d'ordre 0 et 1 au voisinage de $s = 0$ dans son développement de Taylor.

Notation. On considère une direction $(\mu, \mu') \in \overline{H}_0^P \times H_0^Q$ et un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$. Soit $\mathcal{P} \subset \llbracket 1, P \rrbracket$ un ensemble et $\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}$ un multi-indice. Comme la fonction $h_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\mathcal{P}, j, \mathbf{k})(s)$ est holomorphe, on note au voisinage de $s = 0$

$$h_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\mathcal{P}, j, \mathbf{k})(s) = Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k}) + Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})s + O(s^2).$$

On fixe à partir de maintenant pour tout ce chapitre une direction $(\mu, \mu') \in \overline{H}_0^P \times H_0^Q$ et un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$. Les coefficients $Q_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\mathcal{P}, j, \mathbf{k})$, pour tout entier $j \in \llbracket 1, Q \rrbracket$ et tout ensemble $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ s'avéreront cruciaux dans l'expression des formules de $Z(-(\mathbf{N}, \mathbf{N}'))$. De plus, les coefficients $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ et $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ seront présents dans la formule de la dérivée directionnelle en les multi-entiers négatifs $Z'(-(\mathbf{N}, \mathbf{N}'))$. Observons également que l'on a pas besoin de calculer $Q_{\mathbf{N}, \mathbf{N}'}^0(\llbracket 1, P \rrbracket, j)$ et $Q_{\mathbf{N}, \mathbf{N}'}^1(\llbracket 1, P \rrbracket, j)$, puisqu'ils n'apparaissent ni dans les expressions contenues dans les Théorèmes A et D, ni dans leurs preuves. On montrera dans nos calculs que $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ ne dépend pas des directions (μ, μ') . On notera donc simplement ce coefficient $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$.

3.1 Lemmata

Lemme 3.1.1. Soit $a, c \in H_0$, $b \in \mathbb{N}$ et $d \in \mathbb{Z}$. On pose $g : \mathbb{C} \rightarrow \mathbb{C}$ la fonction méromorphe décrite par l'expression

$$g(s) := \frac{1}{\Gamma(as - b)(cs + d)}.$$

Alors g est régulière en 0. De plus

$$g(0) = \begin{cases} \frac{a}{c}(-1)^b b! & \text{si } d = 0 \\ 0 & \text{si } d \neq 0, \end{cases}$$

et

$$g'(0) = \begin{cases} \frac{a^2}{c}(-1)^b b! (\gamma - h_b) & \text{si } d = 0 \\ \frac{a}{d}(-1)^b b! & \text{si } d \neq 0, \end{cases}$$

où h_b désigne le nombre harmonique $h_b := \sum_{i=1}^b \frac{1}{i}$. On a également que

$$\left(\frac{1}{\Gamma}\right)'(-N) = (-1)^N N!.$$

Démonstration. Au voisinage de $s = 0$, on a

$$\frac{1}{\Gamma(s - N)} = (-1)^N N! s + O(s^2),$$

ce qui prouve la formule pour $\left(\frac{1}{\Gamma}\right)'(-N)$.

Pour démontrer les autres formules, on étudiera le développement limité de g au voisinage de $s = 0$. Par l'équation fonctionnelle de la fonction gamma, on obtient

$$g(s) = \frac{(as - 1) \dots (as - b)}{\Gamma(as)(cs + d)}.$$

On a au voisinage de 0 que

$$\begin{aligned} (as - 1) \dots (as - b) &= (-1)^b b! + a(-1)^{b-1} b! h_b s + O(s^2), \\ \frac{1}{\Gamma(s)} &= s + \gamma s^2 + O(s^3). \end{aligned}$$

Supposons que $d = 0$. Alors on a au voisinage de 0 :

$$\begin{aligned} g(s) &= \frac{a}{c} \frac{(as - 1) \dots (as - b)}{\Gamma(as)as} \\ &= \frac{a}{c} ((-1)^b b! + as(-1)^{b-1} b! h_b)(1 + \gamma as) + O(s^2) \\ &= \frac{a}{c} (-1)^b b! + \frac{a^2}{c} (-1)^{b-1} b! h_b s + \frac{a^2}{c} \gamma (-1)^b b! s + O(s^2). \end{aligned}$$

Supposons maintenant que $d \neq 0$, alors on a au voisinage de 0 :

$$\begin{aligned} g(s) &= \frac{(as - 1) \dots (as - b)}{\Gamma(as)(cs + d)} \\ &= \frac{(as - 1) \dots (as - b)}{cs + d} (as) + O(s^2) \\ &= \frac{a}{d} (-1)^b b! s + O(s^2). \end{aligned}$$

□

Lemme 3.1.2. *Pour tout $\mathcal{P} \subset \llbracket 1, P \rrbracket$, $(\mu_p)_{p \in \mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$, $(N_p)_{p \in \mathcal{P}} \in \mathbb{N}^{\mathcal{P}}$, $(n_p)_{p \in \mathcal{P}} \in \mathbb{N}^{\mathcal{P}}$, on a*

$$\partial_s \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} \right) \Big|_{s=0} = \left(\sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + n_p}) \right) \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p},$$

où h_n désigne le n -ième nombre harmonique.

Démonstration. Rappelons que l'on a

$$\prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} = \prod_{p \in \mathcal{P}} \frac{(-N_p - 1 + \mu_p s) \dots (-N_p + \mu_p s - n_p)}{n_p!}.$$

En développant le produit $\frac{(-N_p - 1 + \mu_p s) \dots (-N_p - n_p + \mu_p s)}{n_p!}$ avec s au voisinage de 0, on trouve que

$$\frac{(-N_p - 1 + \mu_p s) \dots (-N_p + \mu_p s - n_p)}{n_p!} = \binom{-N_p - 1}{n_p} + \mu_p s \sum_{i=0}^{n_p-1} \frac{(-N_p - 1) \dots \widehat{(-N_p - 1 - i)} \dots (-N_p - n_p)}{n_p!} + O(s^2),$$

où $\widehat{(-N_p - 1 - i)}$ désigne l'élément que l'on oublie dans le produit. On peut simplifier l'égalité précédente à l'aide des nombres harmoniques

$$\begin{aligned} \frac{(-N_p - 1 + \mu_p s) \dots (-N_p + \mu_p s - n_p)}{n_p!} &= \binom{-N_p - 1}{n_p} + \mu_p \binom{-N_p - 1}{n_p} \left(\sum_{i=0}^{n_p-1} \frac{1}{-N_p - 1 - i} \right) s + O(s^2) \\ &= \binom{-N_p - 1}{n_p} + \mu_p \binom{-N_p - 1}{n_p} (h_{N_p} - h_{N_p + n_p}) s + O(s^2). \end{aligned}$$

Au voisinage de $s = 0$, on obtient alors :

$$\prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} = \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p} + \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p} \left(\sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + n_p}) \right) s + O(s^2).$$

□

Lemme 3.1.3. *Soit $M \in \mathbb{Z}_{\leq 0}$ un entier négatif, et $(a_k)_{k \in \llbracket M, +\infty \rrbracket}$, $(b_k)_{k \in \llbracket M, +\infty \rrbracket}$ deux suites de nombres complexes telles que les séries entières $\sum_{k=0}^{+\infty} a_k x^k$ et $\sum_{k=0}^{+\infty} b_k x^k$ aient deux rayons de convergence R_1 et R_2 non nuls. Supposons de plus qu'il existe un réel $0 < R < \min(R_1, R_2)$ tel que*

$$\forall x \in]0, R[, \quad \sum_{k=M}^{+\infty} a_k x^k + \sum_{k=M}^{+\infty} b_k x^k \ln(x) = 0,$$

alors pour tout $k \in \llbracket M, +\infty \rrbracket$ on a que $a_k = b_k = 0$.

Démonstration. Si tous les b_k sont nuls (respectivement tous les a_k sont nuls) pour tout $k \geq M$, le résultat est trivial. Sinon, on suppose par l'absurde qu'il existe $i, j \geq M$ tels que $a_i \neq 0$ et $b_j \neq 0$. Supposons que $i = \min\{k | a_k \neq 0\}$ et $j = \min\{k | b_k \neq 0\}$, on a alors au voisinage de 0 que

$$a_i x^i + b_j x^j \ln(x) + o(x^j \ln(x)) + o(x^i) = 0,$$

et donc $b_j x^{j-i} \ln(x) \sim -a_i$ au voisinage de 0, contradiction. □

Le lemme précédent jouera un rôle crucial dans la détermination des formules des valeurs spéciales dans la démonstration du Théorème D dans le chapitre 5, et du calcul des coefficients $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ dans la fin du chapitre actuel. En effet, dans le prolongement analytique de la fonction $Z_{\mathbf{N},\mathbf{N}'}(s)$ obtenue dans la Proposition 2.2.4, on a introduit artificiellement une variable libre θ , et deux fonctions $K_{\mathbf{N},\mathbf{N}'}(\theta, s)$ et $J_{\mathbf{N},\mathbf{N}'}(\theta, s)$. En dérivant cette formule de prolongement, on obtient $Z'(-(\mathbf{N}, \mathbf{N}')) = \partial_s(K_{\mathbf{N},\mathbf{N}'}(\theta, s))|_{s=0} + \partial_s(J_{\mathbf{N},\mathbf{N}'}(\theta, s))|_{s=0}$. En dérivant la fonction $J_{\mathbf{N},\mathbf{N}'}(\theta, s)$ par rapport à s et en l'évaluant en $s = 0$ via la formule (2.2), on fera apparaître une série de Laurent en la variable θ et un multiple de $\ln(\theta)$. De même, en dérivant la fonction $K_{\mathbf{N},\mathbf{N}'}(\theta, s)$ par rapport à s et en l'évaluant en $s = 0$ via la formule (2.1), on fera apparaître une série de Laurent en la variable θ et un multiple de $\ln(\theta)$. Il est clair que la valeur de la dérivée directionnelle $Z'(-(\mathbf{N}, \mathbf{N}'))$ est indépendante de θ . On obtient alors par le lemme précédent que le terme constant en θ dans la somme des deux séries de Laurent issus du calcul de $\partial_s(J_{\mathbf{N},\mathbf{N}'}(\theta, s))|_{s=0}$ et de $\partial_s(K_{\mathbf{N},\mathbf{N}'}(\theta, s))|_{s=0}$ correspond à la valeur de la dérivée directionnelle $Z'(-(\mathbf{N}, \mathbf{N}'))$. De la même manière, on utilisera ce même lemme lors du calcul des coefficients $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ afin d'éliminer les séries en la variable ϵ et les multiples de $\ln(\epsilon)$ dans nos calculs.

Lemme 3.1.4. *Il existe une fonction $\alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) = a \ln(\epsilon) + \sum_{n \geq m} b_n \epsilon^n$ définie pour tout $0 < \epsilon \ll 1$, avec $m \in \mathbb{Z}$ et un terme constant nul (i.e. $b_0 = 0$), et telle que*

$$\int_{\epsilon}^1 x_f^{-N'_f-1+|\mathbf{w}_{\bullet f}|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) + F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}'),$$

avec $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$, la constante définie dans la Proposition 1.2.2 :

$$\begin{aligned} F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &= -E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_f+1-|\mathbf{w}_{\bullet f}|} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} \\ &\quad - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|+|\mathbf{v}|}} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda-1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}) \\ &\quad + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \ln \left(1 + \frac{c_{f,p}}{c_{j,p}} \right). \end{aligned}$$

Démonstration. Afin de calculer l'intégrale, on utilise la Proposition 1.2.2. Avec les notations de cette proposition, on a

$$\begin{aligned} x_f^{-N'_f-1+|\mathbf{w}_{\bullet f}|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|} &= \tilde{P}_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f) + \sum_{\lambda=1}^{N'_f+1-|\mathbf{w}_{\bullet f}|} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{x_f^\lambda} \\ &\quad + \sum_{p \in \mathcal{P}} \sum_{\lambda=1}^{(|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|+|\mathbf{v}|})} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{(c_{j,p} + c_{f,p} x_f)^\lambda}. \end{aligned}$$

On rappelle que $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$ désigne la primitive du polynôme $\tilde{P}_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$ s'annulant en 1, et on note $E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ le terme constant de $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$. En notant le polynôme

$$P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(x_f) := P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f) - E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$$

on trouve

$$\begin{aligned} & \int_{\epsilon}^1 x_f^{-N'_f-1+|\mathbf{w}\cdot f|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|} dx_f = -E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) \\ & - C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},1}(\mathbf{N}, \mathbf{N}') \ln(\epsilon) - \sum_{\lambda=2}^{N'_f+1-|\mathbf{w}\cdot f|} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} (1 - \epsilon^{1-\lambda}) \\ & + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \left(\ln(c_{j,p} + c_{f,p}) - \ln(c_{j,p}) - \ln\left(1 + \frac{c_{f,p}}{c_{j,p}} \epsilon\right) \right) \\ & - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda-1} ((c_{j,p} + c_{f,p})^{1-\lambda} - (c_{j,p} + c_{f,p}\epsilon)^{1-\lambda}). \end{aligned}$$

En développant au voisinage de $\epsilon = 0$ les termes $(c_{j,p} + c_{f,p}\epsilon)^{1-\lambda}$ et $\ln\left(1 + \frac{c_{f,p}}{c_{j,p}} \epsilon\right)$, on peut découper la formule obtenue précédemment en un terme constant, et en une somme entre un multiple de $\ln(\epsilon)$ et une série de Laurent sans terme constant en la variable ϵ , pour $0 < \epsilon \ll 1$. Notons $\alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon)$ la somme entre le multiple de $\ln(\epsilon)$ et la série sans terme constant. On a alors

$$\begin{aligned} & \int_{\epsilon}^1 x_f^{-N'_f-1+|\mathbf{w}\cdot f|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) \\ & - E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_f+1-|\mathbf{w}\cdot f|} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \ln\left(1 + \frac{c_{f,p}}{c_{j,p}} \epsilon\right) \\ & - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda-1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}). \end{aligned}$$

□

Remarquons que l'on peut obtenir une expression plus simple de l'intégrale précédente, moins explicite, mais plus pratique à calculer avec un logiciel de calcul formel. Considérons la primitive $G_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$ de la fonction $x_f \mapsto x_f^{-N'_f-1+|\mathbf{w}\cdot f|} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|}$ s'annulant en $x_f = 1$. Cette primitive s'exprime sous la forme d'une somme entre un multiple de $\ln(x_f)$ et d'une série de Laurent en x_f , pour $0 < x_f \ll 1$. On remarque alors que le terme constant de cette série de Laurent est $-F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$.

Remarque 3.1.5. *Il paraît plus pratique avec un logiciel de calcul formel de calculer directement le coefficient $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ en déterminant une primitive de la fonction*

$$x_f \mapsto x_f^{-N'_f-1+|\mathbf{w}\cdot f|} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|}$$

plutôt que de calculer les coefficients $C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')$, $D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')$, et $E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ en utilisant la décomposition en éléments simples de la Proposition 1.2.2.

3.2 Valeurs et dérivées de $h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s)$ μ, μ'

Soit un ensemble $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, un entier $j \in \llbracket 1, Q \rrbracket$, et un multi-indice $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$. On cherchera dans cette section à calculer explicitement les coefficients $Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ et $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$,
 μ, μ'

dont on rappelle les définitions :

$$\begin{aligned} h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(0) &= Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k}), \\ (h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}})'(0) &= Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k}). \end{aligned}$$

Rappelons que l'on a montré dans la Proposition 2.2.2 que la fonction

$$\begin{aligned} h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s) &= \\ &= \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx}_j \dots dx_Q, \end{aligned}$$

était holomorphe sur \mathbb{C} . On avait également obtenu dans cette proposition une expression de la dérivée de $h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s)$ en $s = 0$.

3.2.1 Préliminaires

Soit un ensemble $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, un entier $j \in \llbracket 1, Q \rrbracket$, et un réel $0 < \epsilon \ll 1$ suffisamment petit. Comme on a pour tout $s \in \mathbb{C}$,

$$h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s) = \sum_{\substack{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\} \\ \mu, \mu'}} I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, s),$$

on en déduit qu'il suffit d'étudier le développement de Taylor de $I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, s)$ au voisinage de $s = 0$ pour $0 < \epsilon \ll 1$ suffisamment petit afin d'obtenir celui de $h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s)$. On rappelle tout d'abord le point iii) obtenu dans la Proposition 2.2.2. Soit $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$, alors pour tout $s \in \mathbb{C}$ et pour tout réel $0 < \epsilon \ll 1$ suffisamment petit, on a :

$$\begin{aligned} h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s) &= \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_p,q} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q,p}^{w_p,q} \right) \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|)}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|)} \\ &\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet,q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q. \end{aligned}$$

Notons $T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s)$ le terme général de la série précédente :

$$\begin{aligned} T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s) &= \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_p,q} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q,p}^{w_p,q} \right) \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|)}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|)} \\ &\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet,q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q. \end{aligned}$$

Rappelons également que, avec cette définition de $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$, on a par le point iii) du Lemme 2.3.22 :

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0) = \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0), \quad (3.1)$$

et par ce même point iii), on sait que l'on peut dériver terme à terme la série précédente pour obtenir

$$\partial_s \left(I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right) \Big|_{s=0} = \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0}. \quad (3.2)$$

Grâce à la formule (2.15), on a

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) &= \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0), \\ (h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}})'(0) &= \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \partial_s \left(I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right) \Big|_{s=0}. \end{aligned} \quad (3.3)$$

Ainsi le calcul des coefficients $Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^0$ et $Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^1$ se ramène à l'étude au voisinage de $s = 0$ du terme général $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$.

Lemme 3.2.1. *Pour tout $\epsilon > 0$, au voisinage de $s = 0$ on a*

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) = \begin{cases} O(1) & \text{si } |\mathcal{Q}| = Q - 1 \text{ et } \forall q \in \mathcal{Q}, & |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q \\ O(s) & \text{si } |\mathcal{Q}| = Q - 1 \text{ et } \exists f \in \mathcal{Q}, \forall q \in \mathcal{Q} \setminus \{f\}, & |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q \\ & & |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}| \neq N'_f \\ O(s) & \text{si } |\mathcal{Q}| = Q - 2 \text{ et } \forall q \in \mathcal{Q}, & |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q \\ O(s^2) & \text{sinon.} \end{cases}$$

Démonstration. On sait que la fonction méromorphe $s \mapsto \prod_{q \in \mathcal{Q}} \frac{1}{-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|}$ admet un pôle d'ordre $|\{q \in \mathcal{Q} | N'_q = |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|\}|$ en $s = 0$. En particulier, l'ordre est nécessairement plus petit que $|\mathcal{Q}|$. Comme la fonction gamma d'Euler possède un pôle d'ordre 1 en chaque entier négatif, on trouve que la fonction $s \mapsto \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + s\mu'_q)}$ admet un zéro d'ordre $Q - 1$ en $s = 0$. On obtient alors que

$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) = \tilde{O}(s^n)$ au voisinage de $s = 0$, avec

$$n := Q - 1 - |\{q \in \mathcal{Q} | N'_q = |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|\}|.$$

□

3.2.2 Calcul de $Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^0$

Proposition 3.2.2. *Pour tout réel $\epsilon > 0$ et tout ensemble $\mathcal{Q} \subset [1, Q] \setminus \{j\}$ on a que :*

- Si $\mathcal{Q} = [1, Q] \setminus \{j\}$ et $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ pour tout $q \in \mathcal{Q}$, alors

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0) = \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in [1, Q]} c_{q, p}^{w_{p, q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j, p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}} c_{q, p}^{v_{p, q}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!.$$

- Sinon, on a

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0) = 0.$$

Démonstration. Dans le premier cas, par le lemme 3.1.1, on a pour tout $q \neq j$:

$$\left(\frac{1}{\Gamma(-N'_q + \mu'_q s)} \frac{1}{\mu'_q s} \right)_{|s=0} = (-1)^{N'_q} N'_q! \frac{\mu'_q}{\mu'_q}.$$

En évaluant en $s = 0$ l'expression du terme général $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$, on trouve le premier point de la proposition. Le second point découle du Lemme 3.2.1. \square

Par la formule (3.1) et par la proposition précédente, on obtient :

Corollaire 3.2.3. *Pour tout ensemble $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ et tout réel $0 < \epsilon \ll 1$ suffisamment petit, on a que :*

- Si $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$, alors on a

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0)_{\mu, \mu'} = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \cdot \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_q| + |\mathbf{w}_q| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

- Sinon, on a

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0)_{\mu, \mu'} = 0.$$

Par la formule (2.15), on a que

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0)_{\mu, \mu'} = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0)_{\mu, \mu'}$$

on trouve alors par le corollaire précédent que

$$Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathcal{Q}, \mathbf{k})_{\mu, \mu'} = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_q| + |\mathbf{w}_q| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

Remarquons que ce coefficient ne dépend pas des directions, ce qui justifie le choix de notation de $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathcal{Q}, \mathbf{k})$ dans les Théorèmes A et D.

3.2.3 Calcul de $Q_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}^1$

En évaluant en $s = 0$ la dérivée suivant s du terme général

$$\begin{aligned}
 T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) &= \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
 &\cdot \frac{1}{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)}} \\
 &\cdot \frac{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)}{\prod_{q \in \mathcal{Q}} \Gamma(-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)} \\
 &\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q,
 \end{aligned}$$

on obtient alors la proposition suivante.

Proposition 3.2.4. *Pour tout réel $\epsilon > 0$ et tout ensemble $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ on a que :*

- Si $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ et $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ pour tout $q \in \mathcal{Q}$, alors

$$\begin{aligned}
 \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \tag{3.4} \\
 &(-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} |\boldsymbol{\mu}'|_{\mathcal{Q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \ln(\epsilon) \\
 &+ (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 &\cdot \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right).
 \end{aligned}$$

- Si $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ et s'il existe $f \in \mathcal{Q}$ tel que $|\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}| \neq N'_f$ et tel que $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ pour tout $q \in \mathcal{Q} \setminus \{f\}$, alors

$$\begin{aligned}
 \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \tag{3.5} \\
 &\cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \frac{\mu'_f}{-N'_f + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|} \epsilon^{-N'_f + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|}.
 \end{aligned}$$

- Si $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j, f\}$ tel que $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ pour tout $q \in \mathcal{Q}$, alors il existe une fonction à valeurs complexes $\beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) = \beta \ln(\epsilon) + \sum_{n=m}^{+\infty} \alpha_n \epsilon^n$ définie pour tout $0 < \epsilon \ll 1$ avec $m \in \mathbb{Z}$

et $\alpha_0 = 0$, telle que

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) \\ &+ \mu'_f (-1)^{|\mathbf{N}'|_{\llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}} N'_q! \right) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\ &\cdot F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \end{aligned} \quad (3.6)$$

avec $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ définie dans la Proposition 1.2.2.

• Sinon, on a

$$\partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} = 0.$$

Démonstration. • Si $\mathcal{Q} = \llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}$ et $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ pour tout $q \in \mathcal{Q}$, on a par le Lemme 3.1.1,

$$\begin{aligned} \partial_s \left(\frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right)_{|s=0} &= \sum_{\substack{f=1 \\ f \neq j}}^{\mathcal{Q}} \mu'_f (-1)^{N'_f} N'_f! (\gamma - h_{N'_f}) \prod_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} (-1)^{N'_q} N'_q! \\ &= (-1)^{|\mathbf{N}'|_{\llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} N'_q! \right) \left(\sum_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} \mu'_q (\gamma - h_{N'_q}) \right) \\ &\left(\frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right)_{|s=0} = (-1)^{|\mathbf{N}'|_{\llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} N'_q!. \end{aligned}$$

Par le Lemme 3.1.2, on a

$$\partial_s \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \right)_{|s=0} = \left(\sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) \right) \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|}.$$

Enfin, il est clair que

$$\partial_s \left(\prod_{p \in \mathcal{P}} c_{j,p}^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \right)_{|s=0} = \left(\sum_{p \in \mathcal{P}} \mu_p \ln(c_{j,p}) \right) \prod_{p \in \mathcal{P}} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|}.$$

Ainsi, on trouve

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= |\mu'|_{\mathcal{Q}} (-1)^{|\mathbf{N}'|_{\llbracket 1, \mathcal{Q} \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} N'_q! \right) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \ln(\epsilon) \end{aligned}$$

$$\begin{aligned}
 & + (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p + |\mathbf{v}_p|}) \right) \\
 & \quad \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & + \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
 & \quad \cdot (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left(\sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right).
 \end{aligned}$$

En factorisant la première et la seconde expression de cette somme, on obtient la formule (3.4).

• Si $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ et s'il existe $f \in \mathcal{Q} \setminus \{j\}$ tel que $|\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}| \neq N'_f$ et tel que $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ pour tout $q \in \mathcal{Q} \setminus \{f\}$, on a par le Lemme 3.1.1

$$\begin{aligned}
 & \left(\frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|) \prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = 0, \\
 & \partial_s \left(\frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|) \prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = \\
 & \quad (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \frac{\mu'_f}{-N'_f + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|}.
 \end{aligned}$$

On trouve alors

$$\begin{aligned}
 & \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0} = \\
 & \partial_s \left(\frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)} \right) \Big|_{s=0} \\
 & \quad \cdot \epsilon \sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).
 \end{aligned}$$

En observant que $\sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|) = -N'_f + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}| \neq 0$, on obtient la formule (3.5).

• Si $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j, f\}$ tel que $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ pour tout $q \in \mathcal{Q}$, on a par le Lemme 3.1.1

$$\left(\frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = 0$$

$$\partial_s \left(\frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = \mu'_f \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!.$$

En dérivant le terme général $T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s)_{\mu, \mu'}$ suivant s , puis en l'évaluant en $s = 0$ on trouve pour tout $0 < \epsilon \ll 1$,

$$\begin{aligned} \partial_s \left(T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s)_{\mu, \mu'} \right) \Big|_{s=0} &= \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \quad (3.7) \\ &\cdot \partial_s \left(\frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} \sum_{\substack{q=1 \\ q \neq j, f}}^Q (-N'_q + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|) \\ &\cdot \int_{\epsilon}^1 x_f^{-N'_f - 1 + |\mathbf{w}_{\bullet f}|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p - 1 - |\mathbf{v}_p|} dx_f. \end{aligned}$$

Avec les conditions que l'on a imposé sur les multi-indices \mathbf{v} et \mathbf{w} on peut simplifier l'expression ci-dessus en remarquant que $\sum_{\substack{q=1 \\ q \neq j, f}}^Q (-N'_q + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|) = 0$. Par le Lemme 3.1.4, on a

$$\int_{\epsilon}^1 x_f^{-N'_f - 1 + |\mathbf{w}_{\bullet f}|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p - 1 - |\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) + F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$$

avec $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ définie dans la Proposition 1.2.2, et avec $\alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon)$ une fonction en la variable ϵ de la forme $a \ln(\epsilon) + \sum_{n \geq m} b_n \epsilon^n$, avec $m \in \mathbb{Z}$, et $a, (b_n)_{n \geq m}$ des nombres complexes tels que $b_0 = 0$.

En injectant cette expression dans la formule (3.7), on trouve

$$\begin{aligned} \partial_s \left(T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s)_{\mu, \mu'} \right) \Big|_{s=0} &= \\ &\mu'_f \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q! \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \\ &\cdot (\alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) + F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')). \end{aligned}$$

En notant

$$\begin{aligned} \beta_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) &= \\ &\mu'_f \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q! \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon), \end{aligned}$$

on obtient la formule (3.6).

• Si l'on n'est pas dans l'un des trois cas ci-dessus, alors par le Lemme 3.2.1 on a au voisinage de $s = 0$,

$$T_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s)_{\mu, \mu'} = O(s^2),$$

ce qui conclut. \square

Par la formule (3.2) et par la proposition précédente, on trouve le corollaire suivant :

Corollaire 3.2.5. *Pour tout ensemble $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ et tout réel $0 < \epsilon \ll 1$ suffisamment petit, on a que :*

- Si $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$, alors il existe une fonction à valeurs complexes $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon) = \beta \ln(\epsilon) + \sum_{n=m}^{+\infty} \alpha_n \epsilon^n$ définie pour tout $0 < \epsilon \ll 1$, avec $m \in \mathbb{Z}$ et $\alpha_0 = 0$, telle que

$$\begin{aligned} \partial_s \left(I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right) \Big|_{s=0} &= \\ & (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\ & \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\ & + I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon). \end{aligned}$$

- Si $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j, f\}$, alors il existe une fonction à valeurs complexes $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon) = \beta \ln(\epsilon) + \sum_{n=m}^{+\infty} \alpha_n \epsilon^n$ définie pour tout $0 < \epsilon \ll 1$ avec $m \in \mathbb{Z}$ et $\alpha_0 = 0$, telle que

$$\begin{aligned} \partial_s \left(I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right) \Big|_{s=0} &= I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon) \\ & + (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \mu'_f \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \\ & \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right), \end{aligned}$$

avec $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ définie dans la Proposition 1.2.2.

- Sinon, on a

$$\partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0} = 0.$$

Par la formule (3.3), on a que

$$(h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}})_{\mu, \mu'}'(0) = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \partial_s \left(I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right) \Big|_{s=0},$$

on trouve alors grâce au corollaire précédent que

$$\begin{aligned}
& Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})_{\mu, \mu'} \\
&= \partial_s \left(I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,[1,Q]\setminus\{j\},\mathbf{k}}(\epsilon, s)_{\mu, \mu'} \right)_{|s=0} + \sum_{\substack{f=1 \\ f \neq j}}^Q \partial_s \left(I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,[1,Q]\setminus\{j,f\},\mathbf{k}}(\epsilon, s)_{\mu, \mu'} \right)_{|s=0} \\
&= (-1)^{|\mathbf{N}'|_{[1,Q]\setminus\{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1,Q]\setminus\{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, |\mathbf{v}_q|+|\mathbf{w}_{\bullet q}|=N'_q}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
&\quad \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p+|\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
&+ (-1)^{|\mathbf{N}'|_{[1,Q]\setminus\{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1,Q]\setminus\{j,f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, f, |\mathbf{v}_q|+|\mathbf{w}_{\bullet q}|=N'_q}} F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') \\
&\quad \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \\
&+ I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,[1,Q]\setminus\{j\},\mathbf{k}}^*_{\mu, \mu'}(\epsilon) + \sum_{\substack{f=1 \\ f \neq j}}^Q I_{\mathbf{N},\mathbf{N}',\mathcal{P},j,[1,Q]\setminus\{j,f\},\mathbf{k}}^*_{\mu, \mu'}(\epsilon).
\end{aligned}$$

On conclut via le Lemme 3.1.3 en remarquant que $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})_{\mu, \mu'}$ ne dépend pas de ϵ . On trouve alors que le terme constant en ϵ dans la formule précédente vaut nécessairement $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})_{\mu, \mu'}$.

Chapitre 4

Démonstration du Théorème A et de ses corollaires

Le but de ce chapitre est d'étudier les valeurs directionnelles de $Z(\mathbf{s}, \mathbf{s}')$ aux multi-entiers négatifs. En montrant que la fonction $Z_{\mathbf{N}, \mathbf{N}'}(s)$ est régulière en $s = 0$, on a montré que les valeurs $Z(-(\mathbf{N}, \mathbf{N}'))$ et $Z'(-(\mathbf{N}, \mathbf{N}'))$ ont bien un sens. Afin de calculer la première valeur, on exploitera la Proposition 2.2.4 afin d'évaluer $Z_{\mathbf{N}, \mathbf{N}'}(s)$ en $s = 0$.

4.1 Enoncé du Théorème A

Rappelons tout d'abord l'énoncé du théorème :

Théorème (A). *Pour tout multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ et pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que*

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0,$$

on a

$$Z_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(-(\mathbf{N}, \mathbf{N}')) = (-1)^{|\mathbf{N}'|} \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \frac{(-1)^{N'_j + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \left(\prod_{p \in \mathcal{P}} N_p! \right) \\ \cdot \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!},$$

avec

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, \mathcal{P}}^{w_{p, q}} \right) \\ \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j, \mathcal{P}}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, \mathcal{P}}^{v_{p, q}} \right),$$

où l'on rappelle les notations

$$\begin{aligned} \forall q \in \llbracket 1, Q \rrbracket, \quad |\mathbf{w}_{\bullet q}| &= \sum_{p \in \mathcal{P}^c} w_{p,q}, \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad |\mathbf{v}_{\bullet q}| &= \sum_{p \in \mathcal{P}} v_{p,q}. \end{aligned}$$

4.2 Démonstration du Théorème A

Considérons tout d'abord une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$, et fixons un réel $0 < \theta \ll 1$ suffisamment petit. De la Proposition 2.2.4, on obtient

$$Z_{\mathbf{N}, \mathbf{N}'}(0) = K_{\mathbf{N}, \mathbf{N}'}(\theta, 0) + J_{\mathbf{N}, \mathbf{N}'}(\theta, 0),$$

or on sait que $K_{\mathbf{N}, \mathbf{N}'}(\theta, 0) = 0$ par la Proposition 2.3.11. Il ne reste qu'à évaluer la fonction $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ en $s = 0$ pour obtenir le Théorème A. Par la formule (2.2), on a

$$\begin{aligned} Z_{\mathbf{N}, \mathbf{N}'}(0) &= \\ & \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} N_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \sum_{j=1}^P h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) \\ & \left(\frac{1}{\Gamma(-N'_j + \mu'_j s) - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & + \left(\prod_{p=1}^P \Gamma(1 + N_p) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(0) \left(\frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \right)_{|s=0}. \end{aligned}$$

Observons d'abord que l'expression

$$\left(\prod_{p=1}^P \Gamma(1 + N_p) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(0) \left(\frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \right)_{|s=0}$$

est nulle. En effet, comme on a $-|\mathbf{N}'| - |\mathbf{N}| - P \neq 0$, alors la fonction $s \mapsto \frac{1}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}$ est régulière en $s = 0$, et la fonction $s \mapsto \frac{1}{\Gamma(-N'_j + \mu'_j s)}$ s'annule en $s = 0$. Étudions plus en détail le premier terme de la formule de $Z_{\mathbf{N}, \mathbf{N}'}(0)$. Pour tout ensemble $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, on sait que $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) = Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$. De plus par le Lemme 3.1.1 on a que :

- Si $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} \neq -|\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}$, alors la fonction

$$s \mapsto \frac{1}{\Gamma(-N'_j + \mu'_j s) - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}}$$

s'annule en $s = 0$.

- Si $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} = -|\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}$, alors

$$\left(\frac{1}{\Gamma(-N'_j + \mu'_j s) - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}} \right)_{|s=0} = \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} (-1)^{N'_j} N'_j!.$$

On en déduit alors le Théorème A pour n'importe quelle direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$, et l'expression de $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ dans le Théorème A découle des calculs effectués dans le chapitre 3.

On souhaite maintenant montrer le Théorème A reste vrai pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

Par le Corollaire 1.1.31, on sait que la fonction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \mapsto Z(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$ est méromorphe sur $\mathbb{C}^P \times \mathbb{C}^Q$. Le terme à droite de la formule (1.8) est une fraction rationnelle en les variables $\mu_1, \dots, \mu_P, \mu'_1, \dots, \mu'_Q$ avec des dénominateurs de la forme $|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}$, et l'on vient de prouver cette formule pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. Ainsi, par principe de prolongement analytique, il est clair que la formule (1.8) est vraie pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

□

4.3 Sur les corollaires du Théorème A

Une fois le Théorème A démontré, les corollaires s'en déduisent très rapidement.

4.3.1 Démonstration du Corollaire A1

Pour prouver le Corollaire A1, on applique le Théorème A, en posant $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$:

$$Z(-(\mathbf{0}, \mathbf{0}))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \sum_{j=1}^Q \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!}$$

avec

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = 0}} \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

On peut simplifier le coefficient $Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k})$ en remarquant que la condition $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = 0$ pour tout $q \neq j$ implique que

$$\begin{aligned} \forall p \in \mathcal{P}, \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad & v_{p,q} = 0 \\ \forall p \in \mathcal{P}^c, \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad & w_{p,q} = 0. \end{aligned}$$

On obtient alors que $k_p = |\mathbf{w}_p| = w_{p,j}$ pour tout $p \in \mathcal{P}^c$. Ainsi, on peut simplifier l'expression de $Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k})$:

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}}.$$

□

4.3.2 Démonstration du Corollaire A2

Sur le Corollaire A2, en posant $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ et $(\mathbf{N}, \mathbf{N}') = (N, \dots, N)$, on trouve via le Théorème A que

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{N(|\mathcal{P}|+Q+1)+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=N(Q+|\mathcal{P}|)+|\mathcal{P}|}} \sum_{j=1}^Q \frac{1}{Q+|\mathcal{P}|} Q_N^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N-k_p, d_p)}{k_p!},$$

avec

$$Q_N^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_{\bullet, q}| + |\mathbf{w}_{\bullet, q}| = N}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

En notant

$$Q_N^0(\mathcal{P}, \mathbf{k}) := \sum_{j=1}^Q Q_N^0(j, \mathcal{P}, \mathbf{k}),$$

on trouve la formule annoncée dans le corollaire. \square

4.3.3 Démonstration du Corollaire A3

Le Corollaire A3 correspond à une version qualitative et non explicite du Théorème A. Soit $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ un multi-entier et $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ une direction telle que

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

On note \mathbb{K} le corps engendré par \mathbb{Q} , par les coefficients complexes $(d_p)_{p \in \llbracket 1, P \rrbracket}$ et $(c_{q,p})_{p \in \llbracket 1, P \rrbracket, q \in \llbracket 1, Q \rrbracket}$, et par les composantes μ_p pour tout $p \in \llbracket 1, P \rrbracket$, et μ'_q pour tout $q \in \llbracket 1, Q \rrbracket$. Par le Théorème A, on a que

$$Z_{(-\mathbf{N}, \mathbf{N}')}_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = (-1)^{|\mathbf{N}'|} \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \frac{(-1)^{N'_j + |\mathbf{N}'|_{|\mathcal{P}| + |\mathcal{P}'|} \mu'_j N'_j!}}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \left(\prod_{p \in \mathcal{P}} N_p! \right) \cdot \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}'|_{|\mathcal{P}|+|\mathcal{P}'|}}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!},$$

avec

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_{\bullet, q}| + |\mathbf{w}_{\bullet, q}| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

Observons tout d'abord que les coefficients $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ sont dans le corps \mathbb{K} , en tant que somme de multiples rationnels de produits des coefficients $c_{q,p} \in \mathbb{K}$ pour $1 \leq p \leq P$ et $1 \leq q \leq Q$. Ensuite,

on sait que ses valeurs aux entiers négatifs de la fonction zêta d'Hurwitz s'expriment en fonction de polynômes de Bernoulli. En notant que ces polynômes sont à coefficients rationnels, on obtient que $\frac{\zeta(-N_p - k_p, d_p)}{k_p!} \in \mathbb{Q}[d_1, \dots, d_P] \subset \mathbb{K}$, ce qui démontre le corollaire. \square

Chapitre 5

Démonstration de la Proposition B et du Théorème D

5.1 Enoncé de la Proposition B et du Théorème D

On cherchera dans ce chapitre à démontrer la Proposition B et le Théorème D, dont on rappelle les énoncés ci-dessous.

Proposition (B). Soit $j \in \llbracket 1, Q \rrbracket$ un entier, on suppose que l_j est une forme linéaire à coefficients rationnels. Pour tout $1 \leq p \leq P$, on pose $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$ la décomposition irréductible de $c_{j,p}$ avec $a_{j,p} > 0$ et $b_{j,p} > 0$. On pose

$$x_j(\mathbf{c}) := \frac{\text{ppcm}(a_{j,p})}{\text{pgcd}(b_{j,p})} \in \mathbb{Q}_+^*, \quad \beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}$$

pour tout $1 \leq p \leq P$. Alors pour tout $\mathbf{R} \in \mathbb{N}^P$, la fonction $\varphi_{\mathbf{R}}^j(s)$ admet la relation suivante pour tout $s \in \mathbb{C} \setminus \mathbb{N}^*$,

$$\begin{aligned} \varphi_{\mathbf{R}}^j(s) = & x_j(\mathbf{c})^{-s} \prod_{p=1}^P \beta_{j,p}^{R_p} \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \left(\prod_{p \in \mathcal{P}^c} R_p! \right) \\ & \cdot \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \frac{\zeta\left(s - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})}\right)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!}. \end{aligned}$$

Théorème (D). Pour tout multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ et pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0,$$

on a

$$\begin{aligned} Z'(-(\mathbf{N}, \mathbf{N}')) = & \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} (-1)^{|\mathbf{N}| + |\mathbf{N}'| + |\mathcal{P}| + N'_j} \left(\prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \end{aligned}$$

$$\begin{aligned}
& \cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\
& + \sum_{\substack{\mathcal{P} \subseteq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
& \cdot \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
& + \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) (\varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j)^{(-N'_j)} \\
& - \sum_{\substack{\mathcal{P} \subseteq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|} \\
& \cdot \prod_{p \in \mathcal{P}} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}^{N_p + |\mathbf{u}_{\bullet p}| + 1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!},
\end{aligned}$$

où l'on a posé

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + |\mathbf{u}_{\bullet p}|)_{p \in \llbracket 1, P \rrbracket},$$

pour tout $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$, et tout $j \in \llbracket 1, Q \rrbracket$, avec $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ et $Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k})$ définies dans les formules (1.9) et (1.11).

5.2 Fonctions auxiliaires

On appliquera dans cette section la Proposition 2.2.4 aux fonctions auxiliaires $\varphi_{\mathbf{R}}^j(s)$, afin de faire intervenir les valeurs spéciales de ces fonctions par la suite dans le calcul des valeurs de la dérivée directionnelle de la fonction $Z(\mathbf{s}, \mathbf{s}')$. On rappelle que, pour $\mathbf{R} = (R_1, \dots, R_P) \in \mathbb{N}^P$, et $j \in \llbracket 1, Q \rrbracket$, on a posé

$$\varphi_{\mathbf{R}}^j(s) = \sum_{\mathbf{n} \in \mathbb{N}^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d'_j)^{-s}.$$

On applique la Proposition 2.2.4 à la fonction auxiliaire $\varphi_{\mathbf{R}}^j(s)$, en posant $\boldsymbol{\mu} = \mathbf{0}$, $\mathbf{N} = \mathbf{R}$, $\mathbf{N}' = \mathbf{0}$, et $\boldsymbol{\mu}' = 1$. On sait que cette fonction auxiliaire est régulière en les entiers négatifs, et par cette même proposition, on obtient que l'ensemble des pôles est inclus dans l'ensemble

$$\mathcal{S}(\mathbf{R}) := \mathbb{N}^*.$$

On obtient que la fonction auxiliaire s'exprime sous la forme suivante, pour tout $0 < \theta \ll 1$

$$\varphi_{\mathbf{R}}^j(s) = J_{\mathbf{R}}^j(\theta, s) + K_{\mathbf{R}}^j(\theta, s) \quad (s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{R})) \quad (5.1)$$

avec

$$\begin{aligned}
K_{\mathbf{R}}^j(\theta, s) &= \frac{1}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \quad (s \in \mathbb{C}), \\
J_{\mathbf{R}}^j(\theta, s) &= \\
&\sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{R}, \mathcal{P}, \mathbf{k}}(s) \frac{\theta^{s-|\mathbf{R}|_{\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(s)(s-|\mathbf{R}|_{\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\
&+ \left(\prod_{p=1}^P R_p! \right) h_{\mathbf{R}, \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\mu'|+|\mu|)s-|\mathbf{N}'|-|\mathbf{N}|-P}}{\Gamma(s)(s-|\mathbf{R}|-P)} \quad (s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{R})),
\end{aligned}$$

et où l'on a posé

$$\begin{aligned}
h_{\mathbf{R}, \mathcal{P}, \mathbf{k}}(s) &= \prod_{p \in \mathcal{P}} c_{j,p}^{-N_p-1} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}, \\
h_{\mathbf{R}, \llbracket 1, P \rrbracket, j}(s) &= \prod_{p=1}^P c_{j,p}^{-N_p-1}.
\end{aligned}$$

On trouve alors

$$\begin{aligned}
J_{\mathbf{R}}^j(\theta, s) &= \left(\prod_{p=1}^P R_p! \right) \left(\prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \frac{\theta^{s-|\mathbf{R}|-P}}{\Gamma(s)(s-|\mathbf{R}|-P)} \quad (5.2) \\
&+ \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \left(\prod_{p \in \mathcal{P}} R_p! \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R_p+1}} \frac{(-1)^{|\mathbf{k}|} \theta^{s-|\mathbf{R}|_{\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(s)(s-|\mathbf{R}|_{\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}.
\end{aligned}$$

Proposition 5.2.1. *Soit un multi-entier $\mathbf{R} \in \mathbb{N}^P$ et des entiers $N \in \mathbb{N}$, $j \in \llbracket 1, Q \rrbracket$. La fonction $s \mapsto \varphi_{\mathbf{R}}^j(s)$ admet un prolongement méromorphe sur \mathbb{C} , et est régulière en les entiers négatifs. De plus, pour tout réel $0 < \theta \ll 1$ suffisamment petit, on a*

$$\partial_s \left(K_{\mathbf{R}}^j(\theta, s) \right) \Big|_{s=-N} = (-1)^N N! \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-N, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \quad (5.3)$$

$$\partial_s \left(J_{\mathbf{R}}^j(\theta, s) \right) \Big|_{s=-N} = J_{\mathbf{R}}^{*j}(\theta, -N) \quad (5.4)$$

$$+ N!(\gamma - h_N) \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{R}|_{\mathcal{P}}+|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=N+|\mathbf{R}|_{\mathcal{P}}+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!},$$

avec $J_{\mathbf{R}}^{*j}(\theta, -N)$ de la forme $J_{\mathbf{R}}^{*j}(\theta, -N) = \sum_{n \geq -N-|\mathbf{R}|-P} \alpha_n \theta^n + \beta \ln(\theta)$ sans terme constant (i.e. $\alpha_0 = 0$).

Démonstration. Par le Corollaire 2.2.6, on sait que $\varphi_{\mathbf{R}}^j$ admet un prolongement méromorphe régulier en les entiers négatifs. Par la Proposition 2.3.11, on sait que la fonction

$$s \in \mathbb{C} \mapsto \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p}$$

est holomorphe sur \mathbb{C} . En dérivant la fonction $K_{\mathbf{R}}^j(\theta, s)$ selon la variable s , on obtient

$$\begin{aligned} \partial_s K_{\mathbf{R}}^j(\theta, s) &= -\frac{\psi(s)}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \\ &\quad + \frac{1}{\Gamma(s)} \partial_s \left(\sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \right), \end{aligned}$$

où ψ désigne la fonction digamma. On peut alors évaluer cette somme de deux termes en $s = -N$. Comme $\frac{1}{\Gamma(s)}$ s'annule en $s = -N$, et que par le Lemme 3.1.1, $\left(\frac{\psi(s)}{\Gamma(s)}\right)_{|s=-N} = (-1)^{N+1} N!$, on obtient bien la formule voulue.

De la formule (5.2), on en déduit

$$\begin{aligned} \partial_s J_{\mathbf{R}}^j(\theta, s) &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &\quad \cdot \partial_s \left(\frac{1}{\Gamma(s)(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \theta^{s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ &\quad + \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &\quad \cdot \frac{1}{\Gamma(s)(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \theta^{s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \ln(\theta) \\ &\quad - \left(\prod_{p=1}^P R_p! \right) \left(\prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \theta^{s - |\mathbf{R}| - P} \left(\frac{\ln(\theta)}{\Gamma(s)(s - |\mathbf{R}| - P)} + \partial_s \left(\frac{1}{\Gamma(s)(s - |\mathbf{R}| - P)} \right) \right) \end{aligned}$$

Afin d'évaluer en $s = -N$ la dérivée $\partial_s (J_{\mathbf{R}}^j(\theta, s))$, il suffit d'évaluer les termes $\frac{1}{(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)}$ et $\partial_s \left(\frac{1}{(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)} \right)$ en $s = -N$. On évalue ces expressions à l'aide du Lemme 3.1.1 :

$$\left(\frac{1}{(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)} \right)_{|s=-N} = \begin{cases} (-1)^N N! & \text{si } |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}| \\ 0 & \text{si } |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}| \end{cases}$$

et

$$\partial_s \left(\frac{1}{\Gamma(s)(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{|s=-N} = \begin{cases} (-1)^N (\gamma - h_N) N! & \text{si } |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}| \\ \frac{(-1)^N N!}{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} & \text{si } |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|. \end{cases}$$

On trouve alors

$$\begin{aligned} \partial_s \left(J_{\mathbf{R}}^j(\theta, s) \right)_{|s=-N} &= (\gamma - h_N) N! \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|} \left(\prod_{p \in \mathcal{P}} R_p! \right) \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=N+|\mathbf{R}|_{|\mathcal{P}}+|\mathcal{P}|}} (-1)^{|\mathbf{k}|} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}^{R_p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &\quad + (-1)^N N! \sum_{\mathcal{P} \subsetneq [1, P]} \left(\prod_{p \in \mathcal{P}} R_p! \right) \end{aligned}$$

$$\begin{aligned}
& \cdot \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|}} (-1)^{|\mathbf{k}|} \frac{\theta^{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R_p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\
& + N! \ln(\theta) \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|} \left(\prod_{p \in \mathcal{P}} R_p! \right) \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R_p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\
& + (-1)^N N! \left(\prod_{p=1}^P R_p! \right) \left(\prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \frac{\theta^{-N - |\mathbf{R}| - P}}{-N - |\mathbf{R}| - P}.
\end{aligned}$$

On peut alors poser $J_{\mathbf{R}}^j(\theta, -N)$ telle que

$$\begin{aligned}
J_{\mathbf{R}}^j(\theta, -N) &= (-1)^N N! \left(\prod_{p=1}^P R_p! \right) \left(\prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \frac{\theta^{-N - |\mathbf{R}| - P}}{-N - |\mathbf{R}| - P} \\
& + (-1)^N N! \sum_{\mathcal{P} \subsetneq [1, P]} \left(\prod_{p \in \mathcal{P}} R_p! \right) \\
& \cdot \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|}} (-1)^{|\mathbf{k}|} \frac{\theta^{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R_p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\
& + N! \ln(\theta) \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|} \left(\prod_{p \in \mathcal{P}} R_p! \right) \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R_p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}.
\end{aligned}$$

□

Observons que l'on ne peut pas obtenir une expression explicite des valeurs de la dérivée $(\varphi_{\mathbf{R}}^j)'(s)$ en les entiers négatifs à l'aide des formules (5.3) et (5.4), puisqu'il n'est pas vrai en général que la fonction $s \mapsto \partial_s K_{\mathbf{R}}^j(\theta, s)$ s'annule en les entiers négatifs. On a ainsi besoin d'une autre stratégie pour calculer les valeurs de la dérivée des fonctions auxiliaires en les entiers négatifs.

5.2.1 Démonstration de la Proposition B

On souhaite dans cette section fournir des valeurs explicites des dérivées de ces fonctions auxiliaires dans le cas où les coefficients $(c_{q,p})_{q \in [1, Q] \times [1, P]}$ sont rationnels. Soit $P \in \mathbb{N}^*$ un entier, et $\mathbf{R} \in \mathbb{N}^P$ un multi-entier. Pour simplifier les écritures dans cette sous-section, on pose pour tout complexe $s \in \mathbb{C}$ tel que $\sigma \gg_{\mathbf{R}} 1$ la fonction zêta de Barnes généralisée suivante,

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) := \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{(n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P}}{\left(\sum_{p=1}^P w_p (n_p + d_p) \right)^s},$$

avec $\mathbf{d} = (d_p)_{p \in [1, P]}$ et $w_1, \dots, w_P, d_1, \dots, d_P \in H_0$. Cette fonction admet un prolongement méromorphe sur \mathbb{C} en la variable s . On souhaite dans cette sous-section établir une relation explicite entre $\varphi_{\mathbf{R}}^j(s)$ et la fonction zêta d'Hurwitz $\zeta(s, x)$, avec $x \in H_0$. On suit pour cela une stratégie développée par Aoki et Sakane dans [SA22], qui a été utilisée initialement pour expliciter les valeurs spéciales des dérivées supérieures des fonctions zêta de Barnes à coefficients rationnels. Ici, les fonctions auxiliaires $\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P)$ ne sont pas exactement des fonctions zêta de Barnes à cause des termes au numérateur dans la série, cependant les techniques utilisées par Aoki et Sakane fonctionnent de la même manière.

Lemme 5.2.2. Soit $w_1, \dots, w_P \in \mathbb{Q}_*^+$ tels que $w_p = \frac{a_p}{b_p}$ avec a_p et b_p des entiers strictement positifs. Pour tout multiple commun α à b_1, \dots, b_P , et tout multiple commun β à $\alpha w_1, \dots, \alpha w_P$, on a

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta} \right)^s \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq k_1 \leq \beta_1 - 1 \\ \vdots \\ 0 \leq k_P \leq \beta_P - 1}} \varphi_{\mathbf{R}} \left(s, \left(\frac{d_p + k_p}{\beta_p} \right)_{p \in \llbracket 1, P \rrbracket} \middle| \mathbf{1} \right), \quad (5.5)$$

avec $\beta_1 = \frac{\beta}{\alpha w_1}, \dots, \beta_P = \frac{\beta}{\alpha w_P}$.

Démonstration. Soit $s \in \mathbb{C}$ un nombre complexe tel que $\sigma \gg_{\mathbf{R}} 1$. Pour obtenir le résultat souhaité, on découpera chaque somme présente dans la définition de $\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P)$ via des congruences :

$$\sum_{n_p=0}^{+\infty} = \sum_{k_p=0}^{\beta_p-1} \sum_{\substack{n_p \geq 0 \\ n_p = k_p \pmod{\beta_p}}}.$$

On obtient alors que

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta} \right)^s \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\substack{\mathbf{n} \in \mathbb{N}^P \\ \forall p, n_p = k_p \pmod{\beta_p}}} \frac{\prod_{p=1}^P (n_p + d_p)^{R_p}}{\left(\frac{\alpha}{\beta} (\sum_{p=1}^P (w_p d_p + w_p k_p + w_p (n_p - k_p))) \right)^s}.$$

En effectuant un changement de variable de la forme $n'_p = \beta_p n_p + k_p$ pour tout entier p , on obtient

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta} \right)^s \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{\prod_{p=1}^P (\beta_p n_p + k_p + d_p)^{R_p}}{\left(\frac{\alpha}{\beta} \sum_{p=1}^P w_p d_p + w_p k_p + w_p \beta_p n_p \right)^s}.$$

En factorisant par $\beta_p^{R_p}$ au numérateur, et en remarquant que $\frac{\alpha w_p \beta_p}{\beta} = 1$, on a

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta} \right)^s \prod_{p=1}^P \beta_p^{R_p} \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\mathbf{n} \in \mathbb{N}^P} \frac{\prod_{p=1}^P \left(n_p + \frac{k_p + d_p}{\beta_p} \right)^{R_p}}{\left(\sum_{p=1}^P n_p + \frac{\alpha}{\beta} (w_p d_p + w_p k_p) \right)^s}.$$

Il ne reste qu'à observer que

$$\frac{k_p + d_p}{\beta_p} = \frac{\alpha}{\beta} (w_p d_p + w_p k_p)$$

pour tout $1 \leq p \leq P$. On obtient donc la formule (5.5) pour tout $s \in \mathbb{C}$ tel que $\sigma \gg_{\mathbf{R}} 1$. On peut alors étendre l'égalité pour tout nombre complexe $s \in \mathbb{C}$ en dehors des singularités par prolongement analytique. \square

On peut préciser la formule (5.5) en choisissant un coefficient rationnel $\frac{\alpha}{\beta}$ plus agréable. On peut en effet poser un tel coefficient rationnel ne dépendant que des pgcd et des ppcm des numérateurs et dénominateurs des coefficients w_1, \dots, w_P , ce qui simplifiera grandement nos calculs par la suite.

Proposition 5.2.3. Soit $\mathbf{d} \in H_0^P$, et $w_1 = \frac{a_1}{b_1}, \dots, w_P = \frac{a_P}{b_P}$ des rationnels avec $a_p, b_p > 0$ des entiers copremiers pour tout $p \in \llbracket 1, P \rrbracket$. On pose $w := \frac{\text{ppcm}(a_1, \dots, a_P)}{\text{pgcd}(b_1, \dots, b_P)}$ et $\beta_1 := \frac{w}{w_1}, \dots, \beta_P := \frac{w}{w_P}$. On a alors

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = w^{-s} \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq k_1 \leq \beta_1 - 1 \\ \vdots \\ 0 \leq k_P \leq \beta_P - 1}} \varphi_{\mathbf{R}} \left(s, \left(\frac{d_p + k_p}{\beta_p} \right)_{p \in \llbracket 1, P \rrbracket} \middle| \mathbf{1} \right).$$

Démonstration. On pose $\alpha = \text{ppcm}(b_1, \dots, b_P)$ et $\beta = \text{ppcm}(\alpha w_1, \dots, \alpha w_P)$. Par [SA22, p.3], on sait que $\frac{\alpha}{\beta} = w^{-1}$. Il suffit alors d'utiliser la formule (5.5) avec le α et le β fixés ci-dessus. \square

Finalement, on obtient une formule similaire à celle obtenue par Onodera dans [Ono21, Proposition 4.1] pour réexprimer $\varphi_{\mathbf{R}}(s, \mathbf{d}|1, \dots, 1)$, en modifiant très légèrement sa démonstration.

Proposition 5.2.4. *Soit $\mathbf{d} \in H_0^P$, un multi-entier $\mathbf{R} \in \mathbb{N}^P$, et $s \in \mathbb{C} \setminus \mathbb{N}^*$. On a alors*

$$\varphi_{\mathbf{R}}(s, \mathbf{d}|1) = \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k' \geq 0 \\ |\mathbf{k}|_{\mathcal{P}} + k' = |\mathbf{R}|_{\mathcal{P}} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \frac{\zeta(s - k', |\mathbf{d}|)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}.$$

Pour démontrer cette formule, on utilisera un lemme d'Onodera [Ono21, Lemme 4.2] qui est une généralisation de la formule de Faulhaber suivante

$$\sum_{n=1}^y n^r = \frac{1}{r+1} \sum_{k=0}^r \binom{r+1}{k} (-1)^k B_k y^{r-k+1}.$$

Lemme 5.2.5 ([Ono21]). *Soit $\mathbf{R} = (R_1, \dots, R_P) \in \mathbb{N}^P$ un multi-entier, et $y \in \mathbb{Z}$ un entier relatif. Alors on a*

$$\begin{aligned} \sum_{\substack{\mathbf{n} \in \mathbb{N}^P \\ |\mathbf{n}|=y}} \prod_{p=1}^P (n_p + d_p)^{R_p} + (-1)^{P-1} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{\leq -1}^P \\ |\mathbf{n}|=y}} \prod_{p=1}^P (n_p + d_p)^{R_p} = \\ \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} \frac{(|\mathbf{d}| + y)^{k'}}{k'!} \prod_{p \in \mathcal{P}} (-1)^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}. \end{aligned}$$

On remarque au passage que si $y \in \mathbb{N}$, alors le second terme à gauche de l'égalité du terme précédent est nul. On peut maintenant prouver la Proposition 5.2.4 :

Démonstration. On commence par remarquer que, pour tout $s \in \mathbb{C}$ tel que $\sigma \gg_{\mathbf{R}} 1$, on a

$$\varphi_{\mathbf{R}}(s, \mathbf{d}|1) = \sum_{n'=0}^{+\infty} \frac{1}{(n' + |\mathbf{d}|)^s} \sum_{\substack{\mathbf{n} \in \mathbb{N}^P \\ |\mathbf{n}|=n'}} \prod_{p=1}^P (n_p + d_p)^{R_p}.$$

Par le Lemme 5.2.5, on obtient alors

$$\begin{aligned} \varphi_{\mathbf{R}}(s, \mathbf{d}|1) &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} \prod_{p \in \mathcal{P}} (-1)^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \sum_{n'=0}^{+\infty} \frac{(n' + |\mathbf{d}|)^{k'}}{k'!} \frac{1}{(n' + |\mathbf{d}|)^s} \\ &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \left(\prod_{p \in \mathcal{P}} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \frac{\zeta(s - k', |\mathbf{d}|)}{k'!}. \end{aligned}$$

\square

En appliquant la Proposition 5.2.4 à la formule obtenue dans la Proposition 5.2.3, on obtient le résultat suivant qui correspond exactement à la Proposition B.

Proposition 5.2.6. Soit $\mathbf{d} \in H_0^P$, $\mathbf{R} \in \mathbb{N}^P$, et $s \in \mathbb{C} \setminus \mathbb{N}^*$, et $w_1 = \frac{a_1}{b_1}, \dots, w_P = \frac{a_P}{b_P}$ des rationnels avec $a_p, b_p > 0$ des entiers premiers entre eux pour tout $p \in \llbracket 1, P \rrbracket$. On pose $w := \frac{\text{ppcm}(a_1, \dots, a_P)}{\text{pgcd}(b_1, \dots, b_P)}$ et $\beta_1 := \frac{w}{w_1}, \dots, \beta_P := \frac{w}{w_P}$. Alors on a

$$\begin{aligned} \varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = & \\ & w^{-s} \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq u_1 \leq \beta_1 - 1 \\ \dots \\ 0 \leq u_P \leq \beta_P - 1}} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}^P, \mathbf{k}' \in \mathbb{N} \\ |\mathbf{k}| + \mathbf{k}' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + u_p}{\beta_p}\right)}{k_p!} \\ & \cdot \frac{\zeta(s - k', w^{-1}(w_1(d_1 + u_1) + \dots + w_P(d_P + u_P)))}{k'!} \end{aligned}$$

5.3 Démonstration du Théorème D

Soit $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ un multi-entier. Pour les mêmes raisons que dans la démonstration du Théorème A, il suffit de montrer que la formule (1.10) est valable pour tout $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ pour prouver le Théorème. En effet on sait par le Corollaire 1.1.31 que la fonction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \mapsto Z'(-(\mathbf{N}, \mathbf{N}'))$ est méromorphe sur \mathbb{C}^{P+Q} , et le terme de droite de la formule (1.10) est une fraction rationnelle en les variables $\mu_1, \dots, \mu_P, \mu'_1, \dots, \mu'_Q$ avec des dénominateurs de la forme $|\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|$. Ainsi, si l'on montre que la formule (1.10) est valable pour toute direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$, on pourra la prolonger par principe de prolongement analytique à n'importe quelle direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

Fixons une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. Par la Proposition 2.2.4, on a pour tout complexe s au voisinage de 0, et pour tout réel $0 < \theta \ll 1$ suffisamment petit,

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = K_{\mathbf{N}, \mathbf{N}'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}(\theta, s),$$

avec

$$\begin{aligned} K_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \sum_{\substack{\boldsymbol{\mu}, \boldsymbol{\mu}' \\ \mathcal{Q} \subset \llbracket 1, Q \rrbracket \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s)} \left(\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\ & \left. \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right) \end{aligned}$$

et

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{j=1}^Q \Gamma(1 + N_p - \mu_p s) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, \mathbf{k}}(\boldsymbol{\mu}, \boldsymbol{\mu}')(s) \\ & \cdot \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c})}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c})} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\ & + \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(\boldsymbol{\mu}, \boldsymbol{\mu}') \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right). \end{aligned}$$

Proposition 5.3.1. Soit $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ un multi-entier, et $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ une direction. Il existe une fonction à valeurs complexes $K_{\mathbf{N}, \mathbf{N}'}^*(\theta) = \beta \ln(\theta) + \sum_{n=m}^{+\infty} \alpha_n \theta^n$ définie pour tout réel $0 < \theta \ll 1$ suffisamment petit, avec $m \in \mathbb{Z}$ un entier et $\alpha_0 = 0$, et vérifiant l'égalité suivante :

$$\begin{aligned} \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}^*(\theta, s) \right)_{|s=0} &= K_{\mathbf{N}, \mathbf{N}'}^*(\theta) \\ &+ \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) (\varphi^j_{(N_p + |\mathbf{u}_{\bullet,p}|)_{p \in [1, P]}})^{(-N'_j)} \\ &- \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}} R(\mathbf{N}, j, \mathbf{u})_p! \right) \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}|}}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R(\mathbf{N}, j, \mathbf{u})_p + 1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \end{aligned}$$

Démonstration. En dérivant la fonction $K_{\mathbf{N}, \mathbf{N}'}^*(\theta, s)$ par rapport à la variable s dans la formule (2.1), on obtient

$$\begin{aligned} \partial_s K_{\mathbf{N}, \mathbf{N}'}^*(\theta, s) &= \tag{5.6} \\ &\sum_{\substack{\mathcal{Q} \subset [1, Q] \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \partial_s \left(\prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right) \left(\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\ &\quad \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right) \\ &+ \sum_{\substack{\mathcal{Q} \subset [1, Q] \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \left(\prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right) \partial_s \left(\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\ &\quad \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right). \end{aligned}$$

À l'aide de l'inégalité (2.9), on a vu dans la preuve de la Proposition 2.3.11 que la fonction

$$s \in \mathbb{C} \mapsto \left(\sum_{\mathbf{n} \in \mathbb{N}^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right)$$

était holomorphe sur \mathbb{C} . Comme la fonction $s \mapsto \frac{1}{\Gamma(s)}$ s'annule en les entiers négatifs, on en déduit que le second terme dans la formule (5.6) s'annule en $s = 0$. De plus on sait que, au voisinage de $s = 0$, on a

$$\frac{1}{\Gamma(-N'_q + \mu'_q s)} = O(s).$$

On en déduit que, pour tout $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$ et pour tout $A_{\mathcal{Q}} \subset \mathcal{Q}^c$, on a

$$\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} = \begin{cases} O(s) & \text{si } \mathcal{Q} = \{j\}, A_{\mathcal{Q}} = \llbracket 1, Q \rrbracket \setminus \{j\} \text{ avec } j \in \llbracket 1, Q \rrbracket, \\ O(s^2) & \text{sinon.} \end{cases}$$

Ainsi, on en déduit que si $|\mathcal{Q}| \geq 2$, alors

$$\partial_s \left(\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right) \Big|_{s=0} = 0.$$

On trouve alors

$$\begin{aligned} \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right) \Big|_{s=0} &= \sum_{j=1}^Q \partial_s \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)} \right) \Big|_{s=0} \left(\sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p} \right. \\ &\quad \left. \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} \right). \end{aligned}$$

Par le Lemme 3.1.1, on a que

$$\partial_s \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)} \right) \Big|_{s=0} = (-1)^{N'_j} N'_j! \mu'_j.$$

On obtient alors

$$\begin{aligned} \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right) \Big|_{s=0} &= \sum_{j=1}^Q \mu'_j (-1)^{N'_j} N'_j! \left(\sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p} \right. \\ &\quad \left. \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} \right). \end{aligned}$$

Pour tout $q \in \llbracket 1, Q \rrbracket$, $\mathbf{n} \in \mathbb{N}^P$, on a $l_q(\mathbf{n}) + d'_q = \sum_{p=1}^P c_{q,p} (n_p + d_p)$. Ainsi par le multinôme de Newton, on obtient

$$(l_q(\mathbf{n}) + d'_q)^{N'_q} = \sum_{\substack{\mathbf{u}_q \in \mathbb{N}^P \\ |\mathbf{u}_q| = N'_q}} \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} (n_p + d_p)^{u_{q,p}}.$$

En effectuant le produit sur $q \in \llbracket 1, Q \rrbracket \setminus \{j\}$, on trouve

$$\begin{aligned} \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} &= \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \left(\binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} (n_p + d_p)^{u_{q,p}} \right) \\ &= \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \prod_{p=1}^P (n_p + d_p)^{|\mathbf{u}_{\bullet,p}|}, \end{aligned}$$

où l'on a noté

$$|\mathbf{u}_{\bullet p}| = \sum_{\substack{q=1 \\ q \neq j}}^Q u_{q,p}.$$

On obtient finalement

$$\begin{aligned} \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s) \right) \Big|_{s=0} &= \sum_{j \in [1, Q]} \mu'_j (-1)^{N'_j} N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \\ &\quad \left(\sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p + |\mathbf{u}_{\bullet p}|} \right). \end{aligned} \quad (5.7)$$

On sait par la formule (5.1) que, pour tout $m \in \mathbb{N}$,

$$(-1)^m m! \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-m, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} = (\varphi_{\mathbf{R}}^j)'(-m) - \partial_s \left(J_{\mathbf{R}}^j(\theta, s) \right) \Big|_{s=-m}.$$

En substituant les termes de la forme $(-1)^m m! \sum_{\mathbf{n} \in \mathbb{N}^P} \Gamma(-m, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p}$ dans la formule (5.7) par l'expression obtenue juste au-dessus avec des données $\mathbf{R} = (R_1, \dots, R_P)$ et m adaptées, on obtient

$$\begin{aligned} \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s) \right) \Big|_{s=0} &= \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \\ &\quad \cdot \left((\varphi_{(N_p + |\mathbf{u}_{\bullet p}|)_{p \in [1, P]}}^j)'(-N'_j) - \partial_s \left(J_{(N_p + |\mathbf{u}_{\bullet p}|)_{p \in [1, P]}}^j(\theta, s) \right) \Big|_{s=-N'_j} \right). \end{aligned} \quad (5.8)$$

On rappelle que l'on a posé $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) = (N_p + |\mathbf{u}_{\bullet p}|)_{p \in [1, P]}$ pour tout $\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}}$ dans les notations du Théorème D. Par la formule (5.4), on a

$$\begin{aligned} \partial_s \left(J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(\theta, s) \right) \Big|_{s=-N'_j} &= (\gamma - h_{N'_j}) N'_j! \sum_{\mathcal{P} \subset [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \\ &\quad \cdot \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{c_{j,p}^{k_p} \zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \\ &\quad + J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j). \end{aligned} \quad (5.9)$$

On pose pour tout réel $0 < \theta \ll 1$ suffisamment petit,

$$K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta) := - \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j) \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}}.$$

Par construction des $J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j)$, observons que $K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta)$ est de la forme $\sum_{n \geq m} \alpha_n \theta^n + \beta \ln(\theta)$ avec $(\alpha_n)_{n \geq m}$ et β des nombres complexes, et $\alpha_0 = 0$. En injectant l'expression (5.9) dans l'expression (5.8), on trouve alors la formule désirée pour $\partial_s \left(K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s) \right) \Big|_{s=0}$. \square

Il ne reste qu'à calculer les valeurs de la dérivée $\partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ en $s = 0$ afin de prouver le Théorème D.

Proposition 5.3.2. *Soit un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$, et une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0 \times H_0^Q$. Il existe une fonction à valeurs complexes $J_{\mathbf{N}, \mathbf{N}'}^*(\theta) = \beta \ln(\theta) + \sum_{n=m}^{+\infty} \alpha_n \theta^n$ définie pour tout réel $0 < \theta \ll 1$ suffisamment petit, avec $m \in \mathbb{Z}$ un entier et $\alpha_0 = 0$, et vérifiant l'égalité suivante :*

$$\begin{aligned} \partial_s \left(J_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{|s=0} &= \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}|} + |\mathcal{P}|}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}| + N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}|}} N'_j! \\ &\cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ &+ \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}|} + |\mathcal{P}| + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}|}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}|} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\ &\cdot \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\ &+ J_{\mathbf{N}, \mathbf{N}'}^*(\theta). \end{aligned} \quad (5.10)$$

Démonstration. Rappelons les notations suivantes :

$$\begin{aligned} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) &:= h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\mu, \mu')(0) \\ Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) &:= (h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\mu, \mu'))'(0), \end{aligned}$$

pour $j \in \llbracket 1, Q \rrbracket$, $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$. Notons $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$ le terme général de la série présente dans la formule (2.2) :

$$\begin{aligned} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) &= \left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\mu, \mu')(s) \\ &\cdot \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}|})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}|} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s) (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}|})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}|} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}, \end{aligned}$$

pour tout $j \in \llbracket 1, Q \rrbracket$, $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, et $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$. On évalue la formule (2.20) en $s = 0$ pour réexprimer $\partial_s \left(J_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{|s=0}$. On obtient

$$\begin{aligned} \partial_s \left(J_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{|s=0} &= \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) \right)_{|s=0} \\ &+ \partial_s \left(\left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s) (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P} \right)_{|s=0}. \end{aligned} \quad (5.11)$$

Étudions tout d'abord le second terme de la somme précédente. Remarquons que la fonction $s \mapsto \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)}$ s'annule en $s = 0$. On obtient alors

$$\begin{aligned} \partial_s \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \Big|_{s=0} &= \\ \prod_{p=1}^P N_p! \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(0) \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P} \partial_s \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \Big|_{s=0}. & \end{aligned}$$

Par le lemme 3.1.1, on a

$$\partial_s \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \Big|_{s=0} = (-1)^{N'_j+1} N'_j! \frac{\mu'_j}{|\mathbf{N}'| + |\mathbf{N}| + P}.$$

On trouve alors

$$\begin{aligned} \partial_s \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \Big|_{s=0} &= \\ \left(\prod_{p=1}^P N_p! \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(0) (-1)^{N'_j+1} N'_j! \frac{\mu'_j}{|\mathbf{N}'| + |\mathbf{N}| + P} \right) \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P}. & \end{aligned}$$

On inclura le terme précédent à l'expression de $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$, puisque $-|\mathbf{N}'| - |\mathbf{N}| - P \neq 0$. Étudions à présent la dérivée du terme général $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$ en $s = 0$.

• Si $|\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$, alors $s \mapsto \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)}$ est une fonction s'annulant en $s = 0$. On trouve alors

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) \right) \Big|_{s=0} &= \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ &\cdot \partial_s \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \Big|_{s=0} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}. \end{aligned}$$

Via le Lemme 3.1.1, on obtient

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) \right) \Big|_{s=0} &= \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} (-1)^{N'_j} N'_j! \\ &\cdot \frac{\mu'_j}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}. \end{aligned}$$

On remarque que la puissance de θ dans le terme précédent est non nul. On inclura donc tous les termes de la forme $\partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) \right) \Big|_{s=0}$ tel que $|\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ dans l'expression de $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$.

- Si $|\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$, alors par le Lemme 3.1.1, on a

$$\begin{aligned} & \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{|s=0} = (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \\ \partial_s & \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{|s=0} = (-1)^{N'_j} \frac{\mu'^2_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! (\gamma - h_{N'_j}). \end{aligned}$$

En dérivant par rapport à s le terme général $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$, et en utilisant la relation $\Gamma'(1 + N_p) = N_p!(-\gamma + h_{N_p})$, on trouve

$$\begin{aligned} \partial_s & \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) \right)_{|s=0} = \left(\prod_{p \in \mathcal{P}} N_p! \right) \frac{(-1)^{|\mathbf{k}| + N'_j} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & \cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\mu'_j (\gamma - h_{N'_j}) - \sum_{p \in \mathcal{P}} \frac{\mu_p \Gamma'(1 + N_p)}{N_p!} \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \left(\prod_{p \in \mathcal{P}} N_p! \right) Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \frac{(-1)^{N'_j + |\mathbf{k}|} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\ & + \left(\left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}| + N'_j} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \mu'_j N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \ln(\theta). \end{aligned}$$

On définit maintenant la fonction $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ en incluant dans son expression tous les termes des expressions précédentes qui sont des multiples de $\ln(\theta)$ ou des multiples de θ^k avec $k \neq 0$:

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}^*(\theta) & = \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P} \left(\prod_{p=1}^P N_p! \right) \sum_{j=1}^Q Q_{\mathbf{N}, \mathbf{N}'}^0(j, [1, P]) \frac{(-1)^{N'_j + 1} \mu'_j N'_j!}{|\mathbf{N}'| + |\mathbf{N}| + P} \\ & + \ln(\theta) \sum_{\substack{\mathcal{P} \subseteq [1, P] \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) \mu'_j N'_j! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} (-1)^{|\mathbf{k}| + N'_j} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & + \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \sum_{\substack{\mathcal{P} \subseteq [1, P] \\ 1 \leq j \leq Q}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} \left(\prod_{p \in \mathcal{P}} N_p! \right) Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \frac{(-1)^{N'_j + |\mathbf{k}|} \mu'_j N'_j!}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ & \cdot \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}. \end{aligned}$$

On découpe la série présente dans la formule (5.11) suivant que $|\mathbf{k}|$ soit égal ou non à $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$. Avec le terme $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ que l'on vient de poser, on obtient alors la formule (5.10). \square

On observe que les séries $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ et $K_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ ont un terme constant nul, et que $\partial_s Z_{\mathbf{N}, \mathbf{N}'}(s) = \partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + \partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$. Comme la dérivée $\partial_s \left(Z_{\mathbf{N}, \mathbf{N}'}(s) \right)_{|s=0}$ est indépendant de la variable θ , on a par le Lemme 3.1.3 que

$$\partial_s \left(Z_{\mathbf{N}, \mathbf{N}'}(s) \right)_{|s=0} = \partial_s \left(J_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{|s=0} + \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{|s=0} - K_{\mathbf{N}, \mathbf{N}'}^*(\theta) - J_{\mathbf{N}, \mathbf{N}'}^*(\theta),$$

pour tout $0 < \theta \ll 1$. On en déduit alors le Théorème D. \square

Remarque 5.3.3. De l'égalité $K_{\mathbf{N}, \mathbf{N}'}^*(\theta) + J_{\mathbf{N}, \mathbf{N}'}^*(\theta) = 0$ pour tout $0 < \theta \ll 1$, et du lemme 3.1.3, on en déduit que chaque coefficient dans la série de Laurent de $K_{\mathbf{N}, \mathbf{N}'}^*(\theta) + J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ s'annule, et que le multiple de $\ln(\theta)$ s'annule aussi. On pourrait donc en déduire des relations entre les valeurs de $\zeta(-k, d_p)$, avec $k \in \mathbb{N}$ et $1 \leq p \leq P$.

5.4 Sur les corollaires du Théorème D

Pour démontrer les corollaires du Théorème D il suffit d'utiliser la formule donnée par ce théorème, et de remplacer par des valeurs particulière la direction $(\boldsymbol{\mu}, \boldsymbol{\mu}')$ ou le multi-entier $(\mathbf{N}, \mathbf{N}')$.

5.4.1 Démonstration du Théorème C

Soit un multi-entier $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}^P \times \mathbb{N}^Q$ et une direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ telle que

$$\forall \mathcal{P} \subsetneq [1, P], \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0.$$

Il s'agit ici de reprendre la formule obtenue pour $Z'(-(\mathbf{N}, \mathbf{N}'))$, et d'y étudier les termes qualitativement.

On note \mathbb{K} le corps engendré par \mathbb{Q} , par les coefficients complexes $(d_p)_{p \in [1, P]}$ et $(c_{q,p})_{p \in [1, P], q \in [1, Q]}$, et par les directions $(\mu_p)_{p \in [1, P]}$, $(\mu'_q)_{q \in [1, Q]}$. Par le Théorème D, on a

$$\begin{aligned} Z'(-(\mathbf{N}, \mathbf{N}')) = & \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} (-1)^{|\mathbf{N}| + |\mathbf{N}'| + |\mathcal{P}| + N'_j} \left(\prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & \cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\ & \cdot \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\ & + \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) (\varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j)'(-N'_j) \\ & - \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|} \\ & \cdot \prod_{p \in \mathcal{P}} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}^{N_p + |\mathbf{u}_p| + 1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!}, \end{aligned}$$

où l'on a posé

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + |\mathbf{u}_{\bullet,p}|)_{p \in [1, P]}, \text{ pour tout } \mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}},$$

et

$$\begin{aligned} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) &:= (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, N'_q = |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q,p}^{w_{p,q}} \right) \\ &\quad \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q C_{q,p}^{v_{p,q}} \right), \\ Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) &:= (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, f, |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}| = N'_q}} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \\ &\quad \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q C_{q,p}^{v_{p,q}} \right) \\ &+ (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q,p}^{w_{p,q}} \right) \\ &\quad \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \frac{\prod_{\substack{q=1 \\ q \neq j}}^Q C_{q,p}^{v_{p,q}}}{c_{j,p}^{N_p + 1 + |\mathbf{v}_p|}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \end{aligned}$$

avec $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ la constante posée dans la Proposition 1.2.2.

Observons tout d'abord que $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \in \mathbb{K}$. De plus, les coefficients de la forme $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ appartiennent à l'espace vectoriel sur \mathbb{K} engendré par les coefficients de la forme

$$\ln \left(1 + \frac{c_{q,p}}{c_{j,p}} \right) = \ln(c_{j,p} + c_{q,p}) - \ln(c_{j,p}) \quad (1 \leq q, j \leq Q, 1 \leq p \leq P),$$

donc $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ $\in \text{Vect}_{\mathbb{K}} \left[(\ln(c_{j,p}), \ln(c_{j,p} + c_{q,p}))_{\substack{1 \leq j, q \leq Q \\ 1 \leq p \leq P}} \right]$. On trouve alors que

$$Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \in \text{Vect}_{\mathbb{K}} \left[\gamma, (\ln(c_{j,p}), \ln(c_{j,p} + c_{q,p}))_{\substack{1 \leq j, q \leq Q \\ 1 \leq p \leq P}} \right].$$

Comme les valeurs spéciales de la forme $\zeta(-N, d_p)$ sont dans le corps engendré par \mathbb{Q} et par les coefficients d_1, \dots, d_P , on a que ces valeurs de la fonction zêta de Hurwitz sont dans \mathbb{K} , on obtient que $Z'(-(\mathbf{N}, \mathbf{N}'))$ appartient bien à l'espace vectoriel sur \mathbb{K} engendré par γ , par les coefficients de la forme $\ln(c_{j,p} + c_{q,p})$ et de la forme $\ln(c_{j,p})$, par les valeurs spéciales aux entiers négatifs des dérivées des fonctions auxiliaires $(\varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(-N'_j))'$, et par des valeurs spéciales aux entiers négatifs de la dérivée selon s de la fonction zêta d'Hurwitz. \square

5.4.2 Démonstration du Corollaire C1

On utilise le fait que, lorsque les formes linéaires sont rationnels, le Corollaire B1 fournit une expression de $(\varphi_{\mathbf{R}(\mathbf{N},j,\mathbf{u})}^j)'(-N'_j)$ en fonction de la dérivée de la fonction zêta de Hurwitz. On remplace alors les termes de la forme $(\varphi_{\mathbf{R}(\mathbf{N},j,\mathbf{u})}^j)'(-N'_j)$ dans le Théorème C, par des termes de la forme $\zeta'(-n, y)$. \square

5.4.3 Démonstration du Corollaire D1

On obtient le Corollaire D1 en appliquant la formule obtenue pour $Z'(-(\mathbf{N}, \mathbf{N}'))$ dans le Théorème A, en remplaçant les termes de la forme $(\varphi_{\mathbf{R}}^j)'(-N)$ par la formule donnée par le Corollaire B1.

Comme les formes linéaires l_j sont à coefficients rationnels, on considère $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$ la fraction irréductible, avec un numérateur et un dénominateur positif, pour tout $1 \leq p \leq P$. On pose alors $x_j(\mathbf{c}) := \frac{\text{ppcm}(a_{j,p})}{\text{pgcd}(b_{j,p})} \in \mathbb{Q}_+^*$, et $\beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}^*$ pour tout $1 \leq p \leq P$. Par le Corollaire B1, pour tout $\mathbf{R} \in \mathbb{N}^P$, on a

$$\begin{aligned} (\varphi_{\mathbf{R}}^j)'(-N) &= x_j(\mathbf{c})^N \left(\prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{R_p} \right) \sum_{\substack{\mathcal{P} \subseteq [1,P] \\ 0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \left(\prod_{p \in \mathcal{P}^c} R_p! \right) \sum_{\substack{\mathbf{k} \in \mathbb{N}^{\mathcal{P}}, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \\ &\cdot \left(\frac{\zeta'(-N - k', \frac{l_j(\mathbf{d}+\mathbf{v})}{x_j(\mathbf{c})})}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N - k', \frac{l_j(\mathbf{d}+\mathbf{v})}{x_j(\mathbf{c})})}{k'!} \right) \prod_{p \in \mathcal{P}} \frac{\zeta(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})})}{k_p!}. \end{aligned}$$

En injectant cette formule dans celle obtenue pour $Z'(-(\mathbf{N}, \mathbf{N}'))$ via le Théorème D on obtient le Corollaire D1. \square

Remarquons que l'on obtient au passage encore une fois le Corollaire C1, qui correspond à une reformulation qualitative du résultat que l'on vient de montrer.

5.4.4 Démonstration du Corollaire D2

En posant $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ dans le Théorème D, on peut simplifier plusieurs sommes en remarquant les faits suivants :

- 1) Soit $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}}$ et $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c}$ tels que

$$\begin{aligned} |\mathbf{w}_p| &= k_p & (p \in \mathcal{P}^c) \\ |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| &= 0 & (q \in \llbracket 1, Q \rrbracket \setminus \{j\}). \end{aligned}$$

Alors on a nécessairement que ces indices ont plusieurs composantes nulles :

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \forall p \in \mathcal{P}, \quad v_{p,q} = 0,$$

et

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \forall p \in \mathcal{P}^c, \quad w_{p,q} = 0.$$

Ainsi, comme $w_{p,j} = |w_p| = k_p$ pour tout $p \in \mathcal{P}^c$, on obtient que $\mathbf{w} = (k_p \mathbf{e}_j)_{p \in \mathcal{P}^c}$ avec $\mathbf{e}_j \in \mathbb{N}^Q$ le vecteur valant 1 en la j -ième coordonnée, et 0 ailleurs.

- 2) Soit $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}}$ et $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c}$ tels que

$$\begin{aligned} |\mathbf{w}_p| &= k_p & (p \in \mathcal{P}^c) \\ |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| &= 0 & (q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}). \end{aligned}$$

On obtient alors que

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}, \forall p \in \mathcal{P}, \quad v_{p,q} = 0,$$

et

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}, \forall p \in \mathcal{P}^c, \quad w_{p,q} = 0.$$

Ainsi, on trouve que l'indice \mathbf{w} est de la forme $(w_{p,j}\mathbf{e}_j + w_{p,f}\mathbf{e}_f)_{p \in \mathcal{P}^c}$, avec $w_{p,j} + w_{p,f} = k_p$ pour tout $p \in \mathcal{P}^c$. On trouve ainsi que le coefficient multinomial présent dans l'expression de Q^1 vaut :

$$\binom{k_p}{\mathbf{w}_p} = \frac{k_p!}{w_{p,j}!w_{p,f}!}.$$

Pour simplifier cette expression, on sommera sur l'indice $w'_p \in \llbracket 0, k_p \rrbracket$, et on posera $w_{p,f} = w'_p$ et $w_{p,j} = k_p - w'_p$.

3) Soit $j \in \llbracket 1, Q \rrbracket$, et $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$ tel que

$$|\mathbf{u}_q| = N'_q = 0, \quad q \in \llbracket 1, Q \rrbracket \setminus \{j\}.$$

On obtient trivialement que $\mathbf{u} = \mathbf{0}$, et donc $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) = (N_p + |\mathbf{u}_{\bullet,p}|)_{p \in \llbracket 1, P \rrbracket} = \mathbf{0}$. En particulier, les fonctions auxiliaires $\varphi_{(N_p + |\mathbf{u}_{\bullet,p}|)_{p \in \llbracket 1, P \rrbracket}}^j(s)$ sont en fait des fonctions zêta de Barnes de la forme

$$\varphi_{\mathbf{0}}^j(s) = \zeta^B(s, d'_j | c_{j,1}, \dots, c_{j,P}).$$

On rappelle au passage que la dérivée de cette fonction zêta de Barnes en $s = 0$ correspond à $\ln(\Gamma_P(d'_j | c_{j,1}, \dots, c_{j,P}))$, où Γ_P désigne une fonction multigamma de Barnes introduite dans la Définition 1.1.14.

On a déjà vu dans la preuve du Corollaire A1 que

$$Q_{\mathbf{0},\mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}}.$$

En utilisant les trois points précédents, et le fait que le nombre harmonique h_0 vaut 0, on obtient :

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0})) &= \\ & \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ \mu, \mu'}} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|} \mu'_j}{|\mu'| + |\mu|^{|\mathcal{P}|}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \left(\frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \gamma(\mu'_j + |\mu|^{|\mathcal{P}|}) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|} \mu'_j}{|\mu'| + |\mu|^{|\mathcal{P}|}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p, d_p)}{k_p!} \right) \\ & + \sum_{j=1}^Q \mu'_j (\zeta^B)'(0, d'_j | \mathbf{c}_j) - \sum_{j=1}^Q \gamma \mu'_j \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!}. \end{aligned}$$

Par les points 1) et 2) énoncés précédemment, on obtient :

$$\begin{aligned} Q_{\mu, \mu'}^1(j, \mathcal{P}, \mathbf{k}) &= \gamma \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} |\mu'|_{\llbracket 1, Q \rrbracket \setminus \{j\}} + \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \left(\sum_{p \in \mathcal{P}} \mu_p \ln(c_{j,p}) \right) \\ & + \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\mathbf{w}' = (w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right). \end{aligned}$$

En remarquant que

$$\gamma \frac{(-1)^{\mathcal{P}} \mu'_j}{|\mu'| + |\mu|_{|\mathcal{P}}}} (\mu'_j + |\mu|_{|\mathcal{P}} + |\mu'|_{|\llbracket 1, Q \rrbracket \setminus \{j\}}}) = \gamma(-1)^{\mathcal{P}} \mu'_j,$$

on peut simplifier le premier terme et le dernier terme de la formule obtenue pour $Z'(-(\mathbf{0}, \mathbf{0}))_{\mu, \mu'}$, ce qui conclut la preuve du Corollaire D2. \square

5.4.5 Démonstration du Corollaire D3

En appliquant le Corollaire D2, en remplaçant μ par $(0, \dots, 0)$, et en remplaçant μ' par $(1, \dots, 1)$, on obtient

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{0}, \mathbf{1}} &= \sum_{j=1}^Q (\zeta^B)'(0, d'_j | \mathbf{c}_j) \\ &+ \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{1 \leq j \neq f \leq Q} \frac{(-1)^{|\mathcal{P}|}}{Q} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \\ &\cdot \sum_{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right). \end{aligned}$$

Or par la Définition 1.1.16 on voit que $Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{0}, \mathbf{1}} = \partial_s(\zeta^{Sh}(s, (\mathbf{c}_1, \dots, \mathbf{c}_Q), \mathbf{d}'))_{s=0}$, et par la Définition 1.1.14, on a $\partial_s(\zeta^B(s, d | c_1, \dots, c_P))_{s=0} = \ln(\Gamma_P(d | c_1, \dots, c_P))$, et $\partial_s(\zeta^{Sh}(s, \mathbf{d}' | \mathbf{c}))_{s=0} = \ln(\Gamma_P(\mathbf{d}' | \mathbf{c}))$. On trouve alors le Corollaire D3. \square

Chapitre 6

Fonctions zêta de Witten

On rappelle dans les deux premières sections la construction de la théorie des fonctions zêta de Witten ainsi que quelques résultats classiques de théorie de Lie, et on s'inspire pour cela de l'approche de Komori, Matsumoto et de Tsumura. Le détail des preuves des deux premières sections sont disponibles dans le livre [KMT23]. Alternativement, on peut aussi voir le détail de cette construction dans les articles [KMT10b], [KMT10c], [KMT11b], [KMT12], et pour les détails de la théorie de Lie on se réfère à [Hum72], [Bou06], [Bou81], [FH91], et [KMT20]. On appliquera enfin dans la dernière section certains de nos résultats sur les valeurs directionnelles et les valeurs des dérivées directionnelles de fonctions multizêta aux fonctions zêta de Witten $\zeta_{\mathfrak{g}_2}(s)$ et $\zeta_{\mathfrak{so}(5)}(s)$, et on étudiera aussi leurs résidus. La formule obtenue pour $\zeta'_{\mathfrak{so}(5)}(0)$ fournit une expression explicite de la constante C dans le Théorème 1.1.41. Les formules obtenues pour $\zeta_{\mathfrak{g}_2}(0)$, $\zeta'_{\mathfrak{g}_2}(0)$, $\text{Res}_{s=\frac{1}{3}} \zeta_{\mathfrak{g}_2}(s)$, et $\text{Res}_{s=\frac{1}{5}} \zeta_{\mathfrak{g}_2}(s)$ permettent d'obtenir une formule asymptotique pour le nombre de représentations $r_{\mathfrak{g}_2}(n)$ via le Théorème 1.1.38.

6.1 Généralités sur les algèbres de Lie

Une algèbre de Lie \mathfrak{g} est un espace vectoriel réel ou complexe de dimension finie muni d'une forme bilinéaire alternée $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ qui satisfait la relation de Jacobi, c'est-à-dire que $[\cdot, \cdot]$ est une forme bilinéaire vérifiant les conditions suivantes :

1. $\forall x \in \mathfrak{g}, \quad [x, x] = 0$
2. $\forall x, y, z \in \mathfrak{g}, \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$

On appelle $[\cdot, \cdot]$ le crochet de Lie associé à l'algèbre de Lie \mathfrak{g} . Pour le reste du chapitre on ne considérera que des algèbres de Lie complexes. On peut construire un premier exemple d'algèbre de Lie en considérant l'espace vectoriel des endomorphismes d'un espace vectoriel complexe E muni du crochet de Lie $[f, g] := f \circ g - g \circ f$ avec f et g des endomorphismes de E . On notera $\mathfrak{gl}(E)$ cette algèbre de Lie, remarquons que son groupe de Lie associé correspond au groupe des automorphismes de E , noté $GL(E)$.

Soit $\mathfrak{g}_1, \mathfrak{g}_2$ des algèbres de Lie munies respectivement des crochets de Lie $[\cdot, \cdot]_1$ et $[\cdot, \cdot]_2$. Soit $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ un morphisme d'espace vectoriel. On dit que f est un morphisme d'algèbres de Lie si f respecte le crochet de Lie, c'est-à-dire que

$$\forall x, y \in \mathfrak{g}_1, \quad [f(x), f(y)]_2 = f([x, y]_1).$$

Si de plus f est un isomorphisme d'espaces vectoriels, on dit que f est un isomorphisme d'algèbres de Lie. Le morphisme de dérivation constitue un premier exemple de morphisme d'algèbres de Lie :

$$\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}, \quad y \mapsto [x, y]$$

avec $x \in \mathfrak{g}$ quelconque.

Définition 6.1.1. Soit \mathfrak{g} une algèbre de Lie. Soit $\mathfrak{a} \subset \mathfrak{g}$ un sous-espace vectoriel de \mathfrak{g} . On dit que \mathfrak{a} est une sous-algèbre de \mathfrak{g} si \mathfrak{a} est stable par le crochet de Lie de \mathfrak{g} , c'est-à-dire lorsque $[x, y] \in \mathfrak{a}$ pour tout $x, y \in \mathfrak{a}$.

Soit \mathfrak{a}_1 et \mathfrak{a}_2 des sous-algèbres d'une algèbre de Lie \mathfrak{g} , on définit $[\mathfrak{a}_1, \mathfrak{a}_2]$ comme étant la sous-algèbre de Lie de \mathfrak{g} engendrée par les éléments de la forme $[x, y]$, avec $x \in \mathfrak{a}_1$, et $y \in \mathfrak{a}_2$.

Soit \mathfrak{a} une sous-algèbre d'une algèbre de Lie \mathfrak{g} . On note $N(\mathfrak{a})$ la sous-algèbre de \mathfrak{g} constituée d'éléments x vérifiant $\text{ad}_x(\mathfrak{a}) \subset \mathfrak{a}$. Si $N(\mathfrak{a}) = \mathfrak{g}$, alors on dira que \mathfrak{a} est un idéal de \mathfrak{g} .

Définition 6.1.2. Soit \mathfrak{g} une algèbre de Lie, on construit la suite $(\mathfrak{g}^{(n)})_{n \in \mathbb{N}}$ par les relations suivantes :

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}] \quad (n \in \mathbb{N}).$$

On dit que \mathfrak{g} est résoluble s'il existe un entier $n \in \mathbb{N}$ tel que $\mathfrak{g}^{(n)} = 0$. On notera $\text{rad}(\mathfrak{g})$ l'unique idéal maximal résoluble de \mathfrak{g} .

Définition 6.1.3. Soit \mathfrak{g} une algèbre de Lie, on construit la suite $(\mathfrak{g}^n)_{n \in \mathbb{N}}$ par les relations suivantes :

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \quad (n \in \mathbb{N}).$$

On dira que \mathfrak{g} est nilpotente s'il existe un entier $n \in \mathbb{N}$ tel que $\mathfrak{g}^n = 0$.

Définition 6.1.4. Soit \mathfrak{g} une algèbre de Lie. On dit de \mathfrak{g} est semi-simple si le seul idéal résoluble de \mathfrak{g} est 0. De plus, si \mathfrak{g} ne contient qu'un seul idéal, on dit que \mathfrak{g} est simple.

On donne maintenant quelques exemples fondamentaux d'algèbres de Lie semi-simples :

Exemple 6.1.5. Pour tout entier $r \in \mathbb{N}^*$ on pose :

- Le type A_r : $A_r := \mathfrak{sl}(r+1) = \{X \in \mathcal{M}_{r+1}(\mathbb{C}) \mid \text{Tr}(X) = 0\}$.
- Le type B_r : $B_r := \mathfrak{so}(2r+1) = \{X \in \mathcal{M}_{2r+1}(\mathbb{C}) \mid {}^t X + X = 0\}$.
- Le type C_r : $C_r := \mathfrak{sp}(r) = \{X \in \mathcal{M}_{2r}(\mathbb{C}) \mid {}^t X J_r + J_r X = 0\}$, avec

$$J_r := \begin{pmatrix} 0 & \text{Id}_r \\ -\text{Id}_r & 0 \end{pmatrix}.$$

- Le type D_r : $D_r := \mathfrak{so}(2r) = \{X \in \mathcal{M}_{2r}(\mathbb{C}) \mid {}^t X + X = 0\}$, pour $r \geq 2$.

Dans tous ces exemples, le crochet de Lie correspond à $[X, Y] := X \cdot Y - Y \cdot X$ où \cdot correspond à la multiplication matricielle.

Toutes les algèbres de Lie de l'exemple précédent sont simples, à part pour D_2 car on a $D_2 = A_1 \oplus A_1$.

Proposition 6.1.6. Une algèbre de Lie semi-simple est une somme directe d'algèbres de Lie simples.

Ainsi, pour classifier les algèbres de Lie semi-simples, il suffit de classifier les algèbres Lie simples. On verra via le théorème de Cartan-Killing que les 4 types d'algèbres décrits dans l'exemple précédent classifient presque totalement l'ensemble des algèbres de Lie semi-simples.

Définition 6.1.7. Une représentation d'une algèbre de lie \mathfrak{g} sur un \mathbb{C} -espace vectoriel E correspond à une application linéaire (i.e. un morphisme d'espaces vectoriels) $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(E)$. On appellera $\dim(\varphi) := \dim(E)$ la dimension de cette représentation. On dira que la représentation φ est irréductible si elle n'admet pas de sous-espace vectoriel non trivial stable par l'application linéaire φ .

Exemple 6.1.8. L'application adjointe induit une représentation d'une algèbre de Lie $\text{ad}(\mathfrak{g})$ sur l'espace vectoriel \mathfrak{g} :

$$\text{ad}(\mathfrak{g}) : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), \quad x \mapsto \text{ad}_x.$$

On peut alors définir la forme bilinéaire de Killing sur $\mathfrak{g} \times \mathfrak{g}$:

$$\langle x, y \rangle := \text{Tr}(\text{ad}_x \circ \text{ad}_y).$$

Théorème 6.1.9. Soit \mathfrak{g} une algèbre de Lie. Les conditions suivantes sont équivalentes :

- a) L'algèbre de Lie \mathfrak{g} est semi-simple.
- b) Le radical de \mathfrak{g} est trivial (i.e. $\text{rad}(\mathfrak{g}) = 0$).
- c) La forme de Killing $\langle \cdot, \cdot \rangle$ est non dégénérée.

Le théorème précédent permet en particulier d'identifier une algèbre de Lie semi-simple avec son dual. Ainsi, à un élément $x \in \mathfrak{g}$ on lui associera canoniquement un élément $x^* \in \mathfrak{g}^*$.

6.2 Système de racine d'une algèbre de Lie

Soit \mathfrak{g} une algèbre de Lie. On appelle \mathfrak{h} une sous-algèbre de Cartan si \mathfrak{h} est une sous-algèbre de Lie nilpotente de \mathfrak{g} telle que $N(\mathfrak{h}) = \mathfrak{h}$. Les sous-algèbres de Cartan ne sont pas uniques, mais elles partagent la même dimension sur \mathbb{C} . On appelle alors cette dimension le rang de \mathfrak{g} , noté $\text{rg}(\mathfrak{g})$.

On se fixe pour tout cette section une algèbre de Lie semi-simple \mathfrak{g} , et une sous-algèbre de Cartan \mathfrak{h} de \mathfrak{g} . La forme de Killing sur \mathfrak{g} se restreint à la sous-algèbre de Cartan \mathfrak{h} , et cette forme bilinéaire est non dégénérée sur \mathfrak{h} . Via l'identification de \mathfrak{h} avec son dual, on peut définir la forme de Killing sur \mathfrak{h}^* .

Définition 6.2.1. Soit $\alpha \in \mathfrak{h}^*$ un élément du dual de \mathfrak{h} . On dit que α est une racine de \mathfrak{g} relativement à \mathfrak{h} lorsqu'il existe un élément non nul $x \in \mathfrak{g} \setminus \{0\}$ tel que

$$\forall h \in \mathfrak{h}, \quad \text{ad}_h(x) = \alpha(h)x. \quad (6.1)$$

On notera \mathfrak{g}_α l'ensemble des $x \in \mathfrak{g}$ vérifiant la condition (6.1). On notera $\Delta(\mathfrak{g})$ l'ensemble des racines de \mathfrak{g} relativement à \mathfrak{h} . On appellera $\Delta(\mathfrak{g})$ le système de racines de l'algèbre de Lie semi-simple \mathfrak{g} .

Le système de racines de \mathfrak{g} est un ensemble fini. De plus, les algèbres de Lie \mathfrak{g}_α sont toutes de dimension 1, et elles donnent une décomposition de \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Par l'identification entre \mathfrak{h} et son dual, on considère pour toute racine $\alpha \in \mathfrak{h}^*$ l'élément $\alpha' \in \mathfrak{h}$ lui correspondant. On pose $\mathfrak{h}_0 := \text{Vect}_{\mathbb{R}}((\alpha')_{\alpha \in \Delta(\mathfrak{g})})$ le \mathbb{R} -espace vectoriel engendré par les éléments α' . On note alors

$$\alpha^\vee := \frac{2}{\langle \alpha', \alpha' \rangle} \alpha'.$$

Pour tout $\alpha, \beta \in \Delta(\mathfrak{g})$, on observe que

$$\beta(\alpha^\vee) = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

On note $a(\beta, \alpha)$ la quantité précédente.

Notation. On note \mathfrak{h}_0 le \mathbb{R} -espace vectoriel engendré par les racines $\Delta(\mathfrak{g})$, et on note \mathfrak{h}_0^* son dual.

Fait 6.2.2. Soit $\alpha, \beta \in \Delta(\mathfrak{g})$. On a que:

- 1) $\beta(\alpha^\vee) \in \mathbb{Z}$.
- 2) $\beta - a(\beta, \alpha)\alpha \in \Delta(\mathfrak{g})$.
- 3) Si il existe $c \in \mathbb{C}$ tel que $\beta = c\alpha$, alors $c = \pm 1$.

Ces trois propriétés caractérisent un système de racines "abstrait" au sens de [Ser66] et de [Bou81]. Étant donné deux algèbres de Lie semi-simples \mathfrak{g}_1 et \mathfrak{g}_2 , on sait qu'il existe une isométrie

$$(\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_1)), \|\cdot\|_1) \rightarrow (\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_2)), \|\cdot\|_2)$$

envoyant $\Delta(\mathfrak{g}_1)$ sur $\Delta(\mathfrak{g}_2)$ si et seulement si \mathfrak{g}_1 est isomorphe à \mathfrak{g}_2 en tant qu'algèbre de Lie, en notant $\|\cdot\|_1$ et $\|\cdot\|_2$ les normes euclidiennes (après avoir fixé une base) des deux espaces vectoriels $\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_1))$ et $\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_2))$. Cette correspondance permet en particulier de ramener l'étude des algèbres de Lie simples à l'étude des systèmes de racines. Il découle de cette correspondance le théorème suivant, permettant de classer les algèbres de Lie simples. Pour voir la construction des systèmes de racines, on se réfère à [Bou81].

Théorème 6.2.3 (Théorème de Cartan-Killing, [Bou81]). Une algèbre de Lie simple quelconque est isomorphe soit à A_r , B_r , C_r , ou D_r , ou à une algèbre de Lie exceptionnelle \mathfrak{g}_2 , \mathfrak{f}_4 , \mathfrak{e}_6 , \mathfrak{e}_7 ou \mathfrak{e}_8 .

Notons $H_{\alpha^\vee} := \{\beta \in \mathfrak{h}_0^* \mid \langle \beta, \alpha^\vee \rangle = 0\}$. On remarque que, l'ensemble des racines étant fini, on a que l'ensemble

$$\mathfrak{h}_0^* \setminus \bigcup_{\alpha \in \Delta(\mathfrak{g})} H_{\alpha^\vee}$$

est non vide. Soit γ un élément de cet ensemble. On a alors la décomposition suivante :

$$\Delta(\mathfrak{g}) = \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle > 0\} \cup \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle < 0\}.$$

On notera $\Delta_+(\mathfrak{g}, \gamma) = \Delta_+(\mathfrak{g}) := \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle > 0\}$ l'ensemble des racines positives.

Définition 6.2.4. Soit α une racine positive. On dira que α est une racine fondamentale (ou simple) s'il n'existe aucun couple $(\beta_1, \beta_2) \in \Delta_+(\mathfrak{g})$ tel que $\alpha = \beta_1 + \beta_2$. On notera $\psi(\mathfrak{g})$ l'ensemble des racines fondamentales.

Fait 6.2.5. L'ensemble des racines fondamentales forme une base du \mathbb{R} -espace vectoriel \mathfrak{h}_0^* . On obtient au passage qu'il y a $r = \dim_{\mathbb{R}}(\mathfrak{h}_0)$ racines fondamentales.

On se propose maintenant d'introduire la notion de poids :

Définition 6.2.6. Soit $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(E)$ une représentation de l'algèbre de Lie semi-simple \mathfrak{g} , et $u \in E$ un vecteur. On dit que u est un vecteur propre s'il existe une forme linéaire $\lambda \in \mathfrak{h}^*$ sur \mathfrak{h} tel que

$$\forall H \in \mathfrak{h}, \quad \varphi(H)u = \lambda(H)u.$$

On appellera λ le poids de u . Si de plus, on a $\lambda(\alpha^\vee) \geq 0$ pour toute racine simple α de \mathfrak{g} , alors on dira que λ est un poids dominant.

Notons $\psi(\mathfrak{g}) = \{\alpha_1, \dots, \alpha_r\}$ l'ensemble des racines simples de \mathfrak{g} . On pose alors des formes linéaires $\lambda_1, \dots, \lambda_r \in \mathfrak{h}^*$ telles que $\lambda_j(\alpha_i^\vee) = \delta_i^j$ pour tout $1 \leq i, j \leq r$, où δ désigne le symbole de Kronecker. Les formes linéaires $\lambda_1, \dots, \lambda_r$ sont des poids dominants, et on les appelle des poids fondamentaux (ou simples).

Fait 6.2.7. Chaque poids dominant λ admet une décomposition unique à l'aide des poids fondamentaux

$$\lambda = n_1 \lambda_1 + \dots + n_r \lambda_r \quad (n_1, \dots, n_r \in \mathbb{N}).$$

Fait 6.2.8. Il y a une correspondance bijective entre l'ensemble des représentations irréductibles de dimension finie (à isomorphisme près), et l'ensemble des poids dominants.

Théorème 6.2.9 (de Weyl). Soit φ une représentation irréductible de \mathfrak{g} . Alors

$$\dim(\varphi) = \prod_{\alpha \in \Delta_+(\mathfrak{g})} \frac{\langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle}{\langle \alpha^\vee, \lambda_1 + \dots + \lambda_r \rangle}.$$

Remarque 6.2.10. A priori, les résultats énoncés dépendent du choix de la sous-algèbre de Cartan \mathfrak{h} , et du choix de γ . Cependant, d'après [KMT23, §2.3], ceux-ci sont indépendants de ces choix. La raison étant que deux sous-algèbres de Cartan sont nécessairement conjuguées (voir [Hum72, §16.2]).

Notons au passage que le Fait 6.2.8 et le Théorème 6.2.9 permettent en particulier de déterminer une expression de $f_{\mathfrak{g}}(n)$, le nombre de représentations irréductibles (à isomorphisme près) de dimension n d'une algèbre de Lie semi-simple \mathfrak{g} .

Définition 6.2.11 ([Zag94]). On associe à une algèbre de Lie semi-simple \mathfrak{g} une fonction zêta

$$\zeta_{\mathfrak{g}}(s) := \sum_{\varphi} \frac{1}{\dim(\varphi)^s},$$

où l'on somme sur les représentations irréductibles de dimension finie φ de \mathfrak{g} (à isomorphisme près).

Cette fonction zêta généralise les séries qui interviennent dans les formules de volume de Witten dans [Wit91]. Du Fait 6.2.8 et du Théorème de Weyl 6.2.9, on en déduit que les fonctions zêta de Witten s'écrivent sous la forme d'une série de Dirichlet rattachée à un produit de formes linéaires en r variables, r étant le nombre de racines fondamentales. Plus précisément, on a l'écriture

$$\zeta_{\mathfrak{g}}(s) = \prod_{\alpha \in \Delta_+(\mathfrak{g})} \langle \alpha^\vee, \lambda_1 + \dots + \lambda_r \rangle^s \sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{\alpha \in \Delta_+(\mathfrak{g})} \langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle^s}.$$

Étant donné un système de racines, Matsumoto, Tsumura et Komori ont introduit dans [KMT10b] des analogues multizêta de ces fonctions zêta de Witten en considérant des séries de Dirichlet de la forme

$$\sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{\alpha \in \Delta_+(\mathfrak{g})} \langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle^{s_\alpha}}.$$

Cela permet d'obtenir des formules récursives pour ces fonctions multizêta rattachées à des systèmes de racines (voir [KMT10b, Théorème 3.1]), et d'obtenir plusieurs formules à certains multi-entiers positifs faisant écho aux relations de quasi-battage.

6.3 Quelques résultats sur $\zeta_{\mathfrak{g}_2}$ et $\zeta_{\mathfrak{so}(5)}$

On souhaite ici détailler les formules obtenues dans les Théorèmes A et D pour les deux fonctions zêta de Witten $\zeta_{\mathfrak{g}_2}(s)$ et $\zeta_{\mathfrak{so}(5)}(s)$, en étudiant principalement les valeurs en $s = 0$. Les valeurs spéciales $\zeta_{\mathfrak{g}_2}(0)$ et $\zeta_{\mathfrak{so}(5)}(0)$ sont très simples à calculer à l'aide du Corollaire A1. En revanche, les calculs des valeurs $\zeta'_{\mathfrak{g}_2}(0)$ et $\zeta'_{\mathfrak{so}(5)}(0)$ s'avéreront plus ardues. On se propose de spécifier nos résultats dans le cas où $P = 2$, c'est-à-dire lorsque la série de Dirichlet décrivant $Z(\mathbf{s}, \mathbf{s}')$ n'est qu'à deux variables n_1, n_2 . On fixe de plus la direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$, et on pose $d_1 = d_2 = 1$. Commençons par étudier les valeurs de la fonction $Z(\mathbf{s}, \mathbf{s}')$ sur la diagonale

$$Z_{\Delta}(s) = Z((s, s), (s, \dots, s)) = \sum_{n_1, n_2=1}^{+\infty} \frac{1}{n_1^s n_2^s l_1(n_1, n_2)^s \dots l_Q(n_1, n_2)^s}$$

en les entiers négatifs par le Corollaire A2 :

Lemme 6.3.1. *Soit $N \in \mathbb{N}$. On a*

$$\begin{aligned} Z_{\Delta}(-N) &= \frac{(-1)^{(1+Q)N}}{Q} N! \sum_{k=0}^{QN} Q_N^0(\emptyset, (k, QN - k)) \frac{\zeta(-N - k)}{k!} \frac{\zeta(-(Q+1)N + k)}{(QN - k)!} \\ &\quad + \frac{(-1)^{QN+1}}{Q+1} N!^2 (Q_N^0(\{1\}, (Q+1)N + 1) + Q_N^0(\{2\}, (Q+1)N + 1)) \frac{\zeta(-(Q+2)N - 1)}{((Q+1)N + 1)!}, \end{aligned}$$

avec

$$Q_N^0(\emptyset, (k_1, k_2)) = (-1)^{N(Q-1)} N!^{Q-1} Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q, w_{1,q} + w_{2,q} = N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}} \quad (k_1 + k_2 = QN),$$

$$\begin{aligned} Q_N^0(\{1\}, (Q+1)N + 1) &= \\ (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}^Q \\ |\mathbf{w}_2|=(Q+1)N+1 \\ w_{2,j} \geq 2N+1 \\ \forall q \neq j, w_{2,q} \leq N}} \binom{-N-1}{-2N-1+w_{2,j}} \binom{-2N-1+w_{2,j}}{(N-w_{2,q})_{q \neq j}} \prod_{q=1}^Q c_{q,1}^{N-w_{2,q}} c_{q,2}^{w_{2,q}}, \end{aligned}$$

$$Q_N^0(\{2\}, (Q+1)N + 1) =$$

$$(-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_1 \in \mathbb{N}^Q \\ |\mathbf{w}_1| = (Q+1)N+1 \\ w_{1,j} \geq 2N+1 \\ \forall q \neq j, w_{1,q} \leq N}} \binom{-N-1}{-2N-1+w_{1,j}} \binom{-2N-1+w_{1,j}}{(N-w_{1,q})_{q \neq j}} \prod_{q=1}^Q c_{q,2}^{N-w_{1,q}} c_{q,1}^{w_{1,q}}.$$

Démonstration. Par le Corollaire A2, on a

$$\begin{aligned} Z_{\Delta}(-N) &= \sum_{\mathcal{P} \subsetneq [1,2]} \frac{(-1)^{(|\mathcal{P}|+1+Q)N+|\mathcal{P}|}}{Q+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}|=(Q+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N-k_p)}{k_p!} \\ &= \frac{(-1)^{(1+Q)N}}{Q} N! \sum_{\substack{k_1, k_2 \geq 0 \\ k_1+k_2=QN}} Q_N^0(\emptyset, (k_1, k_2)) \prod_{p=1}^2 \frac{\zeta(-N-k_p)}{k_p!} \\ &\quad + \frac{(-1)^{QN+1}}{Q+1} N!^2 (Q_N^0(\{1\}, (Q+1)N+1) + Q_N^0(\{2\}, (Q+1)N+1)) \frac{\zeta(-(Q+2)N-1)}{((Q+1)N+1)!}, \end{aligned}$$

avec

$$\begin{aligned} Q_N^0(\mathcal{P}, \mathbf{k}) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}^{[1,Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_{\bullet,q}|+|\mathbf{w}_{\bullet,q}|=N}} \\ &\quad \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right). \end{aligned}$$

- Si $\mathcal{P} = \emptyset$ et $k_1 + k_2 = QN$, alors on a

$$Q_N^0(\emptyset, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q \neq j, w_{1,q}+w_{2,q}=N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}}.$$

Observons que

$$\begin{aligned} w_{1,j} + w_{2,j} &= \sum_{q=1}^Q w_{1,q} + w_{2,q} - \sum_{\substack{q=1 \\ q \neq j}}^Q w_{1,q} + w_{2,q} \\ &= k_1 + k_2 - (Q-1)N \\ &= N. \end{aligned}$$

Soit $\mathbf{w} = (\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}^Q)^2$. La condition $|\mathbf{w}_1| = k_1, |\mathbf{w}_2| = k_2$, et $w_{1,q} + w_{2,q} = N$ pour tout $q \neq j$, est équivalente à la condition $w_{1,q} + w_{2,q} = N$ pour tout q , et $|\mathbf{w}_1| = k_1, |\mathbf{w}_2| = k_2$. On trouve alors

$$Q_N^0(\emptyset, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q, w_{1,q}+w_{2,q}=N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}}.$$

- Si $\mathcal{P} = \{1\}$ et $k_2 = (Q+1)N+1$, alors on a

$$Q_N^0(\{1\}, (Q+1)N+1) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}^Q \\ |\mathbf{w}_2| = (Q+1)N+1 \\ \forall q \neq j, w_{2,q} \leq N}} \left(\binom{k_1}{(w_{2,1}, \dots, w_{2,Q})} \prod_{q=1}^Q c_{q,2}^{w_{2,q}} \right) \\ \left(\binom{-N-1}{\sum_{q \neq j} (N-w_{2,q})} \binom{\sum_{q \neq j} (N-w_{2,q})}{(N-w_{2,q})_{q \neq j}} c_{j,p}^{-N-1 - \sum_{q=1}^Q (N-w_{2,q})} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_{2,q}} \right).$$

On remarque que

$$\sum_{\substack{q=1 \\ q \neq j}}^Q (N-w_{2,q}) = (Q-1)N - |\mathbf{w}_2| + w_{2,j} \\ = -2N - 1 + w_{2,j}.$$

Si ce terme est strictement négatif, alors le coefficient binomial $\binom{-N-1}{\sum_{q \neq j} (N-w_{2,q})}$ est nul. Ainsi, on peut imposer la condition $w_{2,j} \geq 2N+1$:

$$Q_N^0(\{1\}, k_2) \\ = (-1)^{N(Q-1)} N!^{Q-1} \\ \cdot \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}^Q \\ |\mathbf{w}_2| = (Q+1)N+1 \\ w_{2,j} \geq 2N+1 \\ \forall q \neq j, w_{2,q} \leq N}} \binom{-N-1}{-2N-1+w_{2,j}} \binom{-2N-1+w_{2,j}}{(N-w_{2,q})_{q \neq j}} c_{j,1}^{N-w_{2,j}} c_{j,2}^{w_{2,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_{2,q}} c_{q,2}^{w_{2,q}} \\ = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}^Q \\ |\mathbf{w}_2| = (Q+1)N+1 \\ w_{2,j} \geq 2N+1 \\ \forall q \neq j, w_{2,q} \leq N}} \binom{-N-1}{-2N-1+w_{2,j}} \binom{-2N-1+w_{2,j}}{(N-w_{2,q})_{q \neq j}} \prod_{q=1}^Q c_{q,1}^{N-w_{2,q}} c_{q,2}^{w_{2,q}}.$$

- Si $\mathcal{P} = \{2\}$ et $k_1 = (Q+1)N+1$, par un raisonnement similaire on obtient

$$Q_N^0(\{2\}, k_1) = \\ (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_1 \in \mathbb{N}^Q \\ |\mathbf{w}_1| = (Q+1)N+1 \\ w_{1,j} \geq 2N+1 \\ \forall q \neq j, w_{1,q} \leq N}} \binom{-N-1}{-2N-1+w_{1,j}} \binom{-2N-1+w_{1,j}}{(N-w_{1,q})_{q \neq j}} \prod_{q=1}^Q c_{q,2}^{N-w_{1,q}} c_{q,1}^{w_{1,q}}.$$

□

On peut simplifier les expressions du lemme précédent afin d'obtenir le corollaire suivant. On utilisera l'expression suivante afin de calculer simplement les termes de la forme $Z_{\Delta}(-N)$ lorsque $P = 2$ avec le code SAGE 7.2.

Corollaire 6.3.2.

$$Z_{\Delta}(-N) = N!^Q \sum_{k=0}^{QN} \alpha_k(N) \frac{\zeta(-N-k) \zeta(-(Q+1)N+k)}{k! (QN-k)!} + \frac{(-1)^{N+1}}{Q+1} N!^{Q+1} \beta(N) \frac{\zeta(-(Q+2)N-1)}{((Q+1)N+1)!},$$

avec

$$\begin{aligned} \alpha_k(N) &= \sum_{\substack{(w_q)_{q \in [1, Q]} \in \llbracket 0, N \rrbracket^Q \\ w_1 + \dots + w_Q = k}} \binom{k}{(w_1, \dots, w_Q)} \binom{QN - k}{(N - w_1, \dots, N - w_Q)} \prod_{q=1}^Q c_{q,1}^{w_q} c_{q,2}^{N-w_q}, \\ \beta(N) &= \sum_{j=1}^Q \sum_{\substack{(w_q)_{q \in [1, Q]} \in \llbracket 0, (Q+1)N+1 \rrbracket^Q \\ w_1 + \dots + w_Q = (Q+1)N+1 \\ w_j \geq 2N+1 \\ \forall q \neq j, w_q \leq N}} \binom{-N-1}{-2N-1+w_j} \binom{-2N-1+w_j}{(N-w_q)_{q \neq j}} \\ &\quad \cdot \left(\prod_{q=1}^Q c_{q,1}^{N-w_q} c_{q,2}^{w_q} + \prod_{q=1}^Q c_{q,2}^{N-w_q} c_{q,1}^{w_q} \right). \end{aligned}$$

On prouve désormais une conséquence du Corollaire D2 dans le cas particulier $P = 2$:

Lemme 6.3.3. *On a*

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0})_{\mathbf{1}, \mathbf{1}}) &= \sum_{j=1}^Q (\zeta^B)'(0, d'_j | c_{j,1}, c_{j,2}) + 2\zeta(0)\zeta'(0) - \frac{\zeta'(-1)}{Q+1} \left(\sum_{j=1}^Q \frac{c_{j,2}}{c_{j,1}} + \frac{c_{j,1}}{c_{j,2}} \right) \\ &\quad - \sum_{j=1}^Q \frac{\zeta(-1)}{Q+1} \left(\frac{c_{j,2}}{c_{j,1}} \ln(c_{j,1}) + \frac{c_{j,1}}{c_{j,2}} \ln(c_{j,2}) \right) \\ &\quad + \sum_{\substack{f=1 \\ f \neq j}}^Q \left(\left(\frac{c_{f,2}}{c_{f,1}} - \frac{c_{j,2}}{c_{j,1}} \right) \ln \left(1 + \frac{c_{f,1}}{c_{j,1}} \right) + \left(\frac{c_{f,1}}{c_{f,2}} - \frac{c_{j,1}}{c_{j,2}} \right) \ln \left(1 + \frac{c_{f,2}}{c_{j,2}} \right) \right), \end{aligned}$$

où l'on a noté $(\zeta^B)'(s, d'_j | c_{j,1}, c_{j,2}) = \partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})$.

Démonstration. Rappelons que $Z'_{\Delta}(0) = Z'(-(\mathbf{0}, \mathbf{0})_{\mathbf{1}, \mathbf{1}})$, où l'on a posé $P = 2$ et on a fixé la direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1}) = ((1, 1), (1, \dots, 1)) \in \mathbb{R}^2 \times \mathbb{R}^Q$. Ainsi, par le Corollaire D2, on obtient :

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0})_{\mathbf{1}, \mathbf{1}}) &= \sum_{j=1}^Q (\zeta^B)'(0, l_j(\mathbf{1}) | \mathbf{c}_j) \\ &\quad + \sum_{\mathcal{P} \subsetneq [1, 2]} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|}}{Q + |\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p)}{k_p!} \left(\frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \sum_{p \in \mathcal{P}} \ln(c_{j,p}) \right) \\ &\quad + \sum_{\substack{f=1 \\ f \neq j}}^Q \sum_{\mathbf{w}' = (w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ &\quad + \sum_{\mathcal{P} \subsetneq [1, 2]} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{1}{Q + |\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \left(\sum_{i \in \mathcal{P}^c} \frac{\zeta'(-k_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p)}{k_p!} \right), \end{aligned}$$

où l'on rappelle que $\mathbf{e}_n = (0, \dots, 1, \dots, 0) \in \mathbb{R}^Q$ est le vecteur avec un 1 en la n -ième composante, et des

0 ailleurs. En développant la somme dans la formule de $Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{1}, \mathbf{1}}$, on trouve

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{1}, \mathbf{1}} &= \sum_{j=1}^Q (\zeta^B)'(0, d'_j | c_{j,1}, c_{j,2}) + \sum_{1 \leq j \neq f \leq Q} \frac{1}{Q} \zeta(0)^2 F_{\emptyset, j, f, \mathbf{0}, (\mathbf{0}, \mathbf{0})}(\mathbf{0}, \mathbf{0}) \\ &- \sum_{j=1}^Q \frac{\zeta(-1)}{Q+1} \left(\frac{c_{j,2}}{c_{j,1}} \ln(c_{j,1}) + \frac{c_{j,1}}{c_{j,2}} \ln(c_{j,2}) \right. \\ &\quad \left. + \sum_{\substack{f=1 \\ f \neq j}}^Q (F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) c_{f,2} + F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) c_{j,2} + F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) c_{f,1} + F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) c_{j,1}) \right) \\ &+ 2\zeta(0)\zeta'(0) - \frac{1}{Q+1} \left(\sum_{j=1}^Q \frac{c_{j,2}}{c_{j,1}} + \frac{c_{j,1}}{c_{j,2}} \right) \zeta'(-1). \end{aligned}$$

On étudie désormais les coefficients $F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0})$ qui sont présents dans la formule précédente.

- Si $\mathcal{P} = \emptyset$ et $\mathbf{k} = (0, 0)$, à l'aide de l'Exemple 1.2.4 on trouve que $F_{\emptyset, j, f, \mathbf{0}, (\mathbf{0}, \mathbf{0})}(\mathbf{0}, \mathbf{0}) = 0$.
- Si $\mathcal{P} = \{1\}$ et $k_2 = 1$, à l'aide de l'Exemple 1.2.6 on trouve que

$$F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) = -\frac{1}{c_{j,1}} \ln \left(1 + \frac{c_{f,1}}{c_{j,1}} \right),$$

et par l'Exemple 1.2.5, on trouve que

$$F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) = \frac{1}{c_{f,1}} \ln \left(1 + \frac{c_{f,1}}{c_{j,1}} \right).$$

- Si $\mathcal{P} = \{2\}$ et $k_1 = 1$, mutatis mutandis on trouve que

$$\begin{aligned} F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) &= -\frac{1}{c_{j,2}} \ln \left(1 + \frac{c_{f,2}}{c_{j,2}} \right) \\ F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) &= \frac{1}{c_{f,2}} \ln \left(1 + \frac{c_{f,2}}{c_{j,2}} \right). \end{aligned}$$

Ainsi on obtient

$$\begin{aligned} &\sum_{\substack{f=1 \\ f \neq j}}^Q (F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) c_{f,2} + F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) c_{j,2} + F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) c_{f,1} + F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) c_{j,1}) \\ &= \sum_{\substack{f=1 \\ f \neq j}}^Q \left(\left(\frac{c_{f,2}}{c_{f,1}} - \frac{c_{j,2}}{c_{j,1}} \right) \ln \left(1 + \frac{c_{f,1}}{c_{j,1}} \right) + \left(\frac{c_{f,1}}{c_{f,2}} - \frac{c_{j,1}}{c_{j,2}} \right) \ln \left(1 + \frac{c_{f,2}}{c_{j,2}} \right) \right). \end{aligned}$$

En injectant cette égalité dans la formule obtenue pour $Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{1}, \mathbf{1}}$, on trouve bien le résultat demandé. \square

En appliquant le lemme précédent à la fonction zêta de Tornheim on obtient le résultat bien connu

$$(\zeta_2^{MT})'(0) = \ln(2\pi).$$

Remarque 6.3.4. *Sous les conditions du Lemme précédent, et en donnant un entier $Q \geq 1$ et des coefficients $\mathbf{c} = (c_{q,p})_{q \in [1,Q], p \in [1,2]}$, le code SAGE 7.1 calcule $Z'_\Delta(0)$ lorsque les coefficients $c_{q,p}$ sont des entiers entre 1 et 3. Notons que le résultat est exact et n'est pas une approximation, et son expression dépend de $\zeta(0)$, $\zeta(-1)$, $\zeta'(0)$, $\zeta'(-1)$, du logarithme de certains entiers, et de certaines valeurs des fonctions gamma et digamma. On pourrait décrire un algorithme décrivant la formule de $Z'_\Delta(0)$ lorsque les coefficients $c_{q,p}$ sont des entiers quelconques, cependant on ne dispose pas de formule complètement explicites et satisfaisantes pour les dérivées de fonctions zêta de Barnes en $s = 0$ pour des couples d'entiers quelconques. Dans le cas général, on ne peut espérer que des expressions de ces dérivées faisant intervenir des valeurs spéciales des dérivées de la fonction zêta de Hurwitz.*

Même si on dispose d'une formule pour calculer les valeurs aux entiers négatifs de la dérivée suivant s de la fonction zêta de Barnes $(\zeta^B)'(0, d'_j | c_{j,1}, c_{j,2})$ via le Corollaire B1, en posant $N = 0$ et $\mathbf{R} = \mathbf{0}$, on utilisera plutôt le Théorème 1.1.11 démontré par Aoki et Sakane. La formule de ce théorème donne des formules plus facilement exploitable. En particulier, on utilisera les formules obtenues dans l'Exemple 1.1.12 pour simplifier les expressions de $\zeta'_{\mathfrak{g}_2}(0)$ et de $\zeta'_{\mathfrak{so}(5)}(0)$. On a codé ces expressions dans le code SAGE 7.1 afin de calculer les valeurs et les valeurs de la dérivée en $s = 0$ de ces deux fonctions zêta de Witten.

6.3.1 Pôles et résidus de $\zeta_{\mathfrak{g}_2}$

Via [KMT11a], on a vu dans la liste de pôles 1.4 que $\zeta_{\mathfrak{g}_2}(s)$ a des pôles de la forme $s = \frac{1}{3}$ et en $s = \frac{1-k}{5}$, avec $k \geq 0$ et $k \not\equiv 1 \pmod{5}$. On souhaite ici donner une expression des résidus de $\zeta_{\mathfrak{g}_2}(s)$ en $s = \frac{1}{5}$ et en $s = \frac{1}{3}$. On pose

$$\begin{aligned} l_1(x_1, x_2) &= x_1 + x_2, & l_2(x_1, x_2) &= x_1 + 2x_2, & l_3(x_1, x_2) &= x_1 + 3x_2, & l_4(x_1, x_2) &= 2x_1 + 3x_2, \\ d_1 &= 1, & d_2 &= 1, & d'_1 &= 2, & d'_2 &= 3, & d'_3 &= 4, & d'_4 &= 5, \\ Z_\Delta(s) &:= Z(-((s, s), (s, s, s, s))) = \sum_{\substack{(1,1), (1,1,1,1) \\ n_1, n_2 \geq 1}} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_2)^s (2n_1 + 3n_2)^s}. \end{aligned}$$

La fonction zêta de Witten $\zeta_{\mathfrak{g}_2}(s)$ est alors de la forme

$$\zeta_{\mathfrak{g}_2}(s) = 120^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_2)^s (2n_1 + 3n_2)^s} = 120^s Z_\Delta(s).$$

Via la Proposition 2.2.4, on trouve que

$$\zeta_{\mathfrak{g}_2}(s) = K(\theta, s) + J(\theta, s),$$

avec

$$\begin{aligned} K(\theta, s) &= 120^s \sum_{\emptyset \neq \mathcal{Q} \subset [1,4]} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \frac{1}{\Gamma(s)^{4 - |A_{\mathcal{Q}}|}} \left(\sum_{\mathbf{n} \in \mathbb{N}^2} (n_1 + 1)^{-s} (n_2 + 1)^{-s} \right. \\ &\quad \left. \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n} + \mathbf{1}))^{-s} \prod_{q \in [1,4] \setminus A_{\mathcal{Q}}} \Gamma(s, \theta, l_q(\mathbf{n} + \mathbf{1})) \right), \end{aligned}$$

et

$$\begin{aligned} J(\theta, s) &= 120^s \Gamma(1-s)^2 \left(\sum_{j=1}^4 h_{[1,2],j}(s) \right) \frac{\theta^{6s-2}}{\Gamma(s)(6s-2)} \\ &+ 120^s \sum_{\mathcal{P} \subsetneq [1,2]} \Gamma(1-s)^{|\mathcal{P}|} \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} \left(\sum_{j=1}^4 h_{\mathcal{P},j,\mathbf{k}}(s) \right) \frac{(-1)^{|\mathbf{k}|} \theta^{(4+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(s)((4+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(s - k_p)}{k_p!} \end{aligned} \quad (6.2)$$

avec

$$h_{\mathcal{P},1,\mathbf{k}}(s) = \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_2 x_3 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(1, x_2, x_3, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(1, x_2, x_3, x_4)^{k_p} dx_2 dx_3 dx_4,$$

$$h_{\mathcal{P},2,\mathbf{k}}(s) = \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_3 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, 1, x_3, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, 1, x_3, x_4)^{k_p} dx_1 dx_3 dx_4,$$

$$h_{\mathcal{P},3,\mathbf{k}}(s) = \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, x_2, 1, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, x_2, 1, x_4)^{k_p} dx_1 dx_2 dx_4,$$

$$h_{\mathcal{P},4,\mathbf{k}}(s) = \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_3)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, x_2, x_3, 1)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, x_2, x_3, 1)^{k_p} dx_1 dx_2 dx_3,$$

$$l_1^*(x_1, x_2) = x_1 + x_2 + x_3 + 2x_4,$$

$$l_2^*(x_1, x_2) = x_1 + 2x_2 + 3x_3 + 3x_4,$$

et où on a également

$$\begin{aligned} \sum_{j=1}^4 h_{[[1,2],j]}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_3)^{s-1} \prod_{p=1}^2 l_p^*(x_1, x_2, x_3, 1)^{s-1} dx_1 dx_2 dx_3 \\ &+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_4)^{s-1} \prod_{p=1}^2 l_p^*(x_1, x_2, 1, x_4)^{s-1} dx_1 dx_2 dx_4 \\ &+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_3 x_4)^{s-1} \prod_{p=1}^2 l_p^*(x_1, 1, x_3, x_4)^{s-1} dx_1 dx_3 dx_4 \\ &+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_2 x_3 x_4)^{s-1} \prod_{p=1}^2 l_p^*(1, x_2, x_3, x_4)^{s-1} dx_2 dx_3 dx_4. \end{aligned}$$

On observe que la fonction $s \mapsto K(\theta, s)$ est holomorphe sur \mathbb{C} , et donc ne contribue pas aux résidus. Il ne reste qu'à étudier les dénominateurs des termes présents dans la formule (6.2).

- Soit $\mathcal{P} \subsetneq [[1, 2]]$, et $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}$. Pour $s = \frac{1}{5}$, on a

$$(4 + |\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}| = 0 \text{ si et seulement si } |\mathcal{P}| = 1, \mathbf{k} = 0.$$

On trouve alors

$$\text{Res}_{s=\frac{1}{5}} \zeta_{\mathfrak{g}_2}(s) = \frac{120^{\frac{1}{5}} \Gamma\left(\frac{4}{5}\right)}{5 \Gamma\left(\frac{1}{5}\right)} \zeta\left(\frac{1}{5}\right) \sum_{j=1}^4 \left(h_{\{1\},j,0}\left(\frac{1}{5}\right) + h_{\{2\},j,0}\left(\frac{1}{5}\right) \right).$$

En remplaçant les fonctions h par leurs expressions, on trouve

$$\begin{aligned} \text{Res}_{s=\frac{1}{5}} \zeta_{\mathfrak{g}_2}(s) &= \frac{120^{\frac{1}{5}} \Gamma\left(\frac{4}{5}\right)}{5 \Gamma\left(\frac{1}{5}\right)^4} \zeta\left(\frac{1}{5}\right) \\ &\cdot \left(\int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{4}{5}} (l_1^*(1, x_2, x_3, x_4)^{-\frac{4}{5}} + l_2^*(1, x_2, x_3, x_4)^{-\frac{4}{5}}) dx_2 dx_3 dx_4 \right. \\ &+ \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{4}{5}} (l_1^*(x_1, 1, x_3, x_4)^{-\frac{4}{5}} + l_2^*(x_1, 1, x_3, x_4)^{-\frac{4}{5}}) dx_1 dx_3 dx_4 \\ &+ \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{4}{5}} (l_1^*(x_1, x_2, 1, x_4)^{-\frac{4}{5}} + l_2^*(x_1, x_2, 1, x_4)^{-\frac{4}{5}}) dx_1 dx_2 dx_4 \\ &\left. + \int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{4}{5}} (l_1^*(x_1, x_2, x_3, 1)^{-\frac{4}{5}} + l_2^*(x_1, x_2, x_3, 1)^{-\frac{4}{5}}) dx_1 dx_2 dx_3 \right). \end{aligned}$$

- Via un raisonnement similaire, on trouve

$$\operatorname{Res}_{s=\frac{1}{3}} \zeta_{\mathfrak{g}_2}(s) = \frac{120^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)^2}{6\Gamma\left(\frac{1}{3}\right)} \left(\sum_{j=1}^4 h_{\{1,2\},j}\left(\frac{1}{3}\right) + h_{\{1,2\},j}\left(\frac{1}{3}\right) \right),$$

et en remplaçant les fonctions h par leurs expressions, on trouve

$$\begin{aligned} \operatorname{Res}_{s=\frac{1}{3}} \zeta_{\mathfrak{g}_2}(s) &= \frac{120^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)^2}{6\Gamma\left(\frac{1}{3}\right)^4} \\ &\cdot \left(\int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, x_3, 1)^{-\frac{2}{3}} dx_1 dx_2 dx_3, \right. \\ &+ \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, 1, x_4)^{-\frac{2}{3}} dx_1 dx_2 dx_4 \\ &+ \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, 1, x_3, x_4)^{-\frac{2}{3}} dx_1 dx_3 dx_4 \\ &\left. + \int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(1, x_2, x_3, x_4)^{-\frac{2}{3}} dx_2 dx_3 dx_4 \right). \end{aligned}$$

Ces calculs justifient alors les formules données pour les résidus ω_α et ω_β dans le Théorème G.

6.3.2 Valeurs aux entiers négatifs de $\zeta_{\mathfrak{g}_2}$ et de $\zeta'_{\mathfrak{g}_2}$

On reprend les notations de la sous-section précédente. On a en particulier que

$$\zeta_{\mathfrak{g}_2}(-N) = 120^{-N} Z_\Delta(-N), \quad \zeta'_{\mathfrak{g}_2}(-N) = 120^{-N} Z'_\Delta(-N) + \ln(120) \zeta_{\mathfrak{g}_2}(-N).$$

En appliquant le Corollaire A1 avec les coefficients \mathbf{c} et \mathbf{d} adaptées, et selon la direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') = ((1, 1), (1, 1, 1, 1))$, on trouve directement

$$\zeta_{\mathfrak{g}_2}(0) = \zeta(0)^2 - 2\zeta(-1) = \frac{5}{12}.$$

Via le Corollaire 6.3.2, on obtient une expression simple et explicite de $\zeta_{\mathfrak{g}_2}(-N)$:

$$\zeta_{\mathfrak{g}_2}(-N) = 120^{-N} N!^4 \sum_{k=0}^{4N} \alpha_k(N) \frac{\zeta(-N-k)\zeta(-5N+k)}{k!(4N-k)!} + \frac{120^{-N}(-1)^{N+1}}{5} N!^5 \beta(N) \frac{\zeta(-6N-1)}{(5N+1)!},$$

avec

$$\begin{aligned} \alpha_k(N) &= \sum_{\substack{(w_q)_{q \in [1,4]} \in [0, N]^4 \\ w_1 + \dots + w_4 = k}} \binom{k}{(w_1, \dots, w_4)} \binom{4N-k}{(N-w_1, \dots, N-w_4)} \prod_{q=1}^4 c_{q,1}^{w_q} c_{q,2}^{N-w_q}, \\ \beta(N) &= \sum_{j=1}^4 \sum_{\substack{(w_q)_{q \in [1,4]} \in [0, 5N+1]^4 \\ w_1 + \dots + w_4 = 5N+1 \\ w_j \geq 2N+1 \\ \forall q \neq j, w_q \leq N}} \binom{-N-1}{-2N-1+w_j} \binom{-2N-1+w_j}{(N-w_q)_{q \neq j}} \left(\prod_{q=1}^4 c_{q,1}^{N-w_q} c_{q,2}^{w_q} + \prod_{q=1}^4 c_{q,2}^{N-w_q} c_{q,1}^{w_q} \right). \end{aligned}$$

En utilisant le code SAGE 7.2, on trouve les valeurs de $\zeta_{\mathfrak{g}_2}(-N)$ présentes dans le tableau du Théorème E.

Afin de calculer $\zeta'_{\mathfrak{g}_2}(0)$, on remarque tout d'abord que

$$\zeta'_{\mathfrak{g}_2}(0) = \ln(120)\zeta_{\mathfrak{g}_2}(0) + Z'_{\Delta}(0).$$

En utilisant $\zeta_{\mathfrak{g}_2}(0) = \frac{5}{12}$, et en calculant $Z'_{\Delta}(0)$ à l'aide du code SAGE 7.1, on trouve

$$\zeta'_{\mathfrak{g}_2}(0) = \frac{5}{12} \ln(5) - \frac{7}{12} \ln(3) + \frac{155}{36} \ln(2) + \frac{7}{2} \ln(\pi) - \frac{1}{6} \ln\left(\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{5}{6})}\right) - \frac{4}{3} \ln\left(\Gamma\left(\frac{2}{3}\right)\right) - \frac{2}{3} \ln\left(\Gamma\left(\frac{1}{3}\right)\right).$$

Par [BZ92, Table 3], on a

$$\Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{3}\Gamma(\frac{1}{3})^2}{\sqrt[3]{2}\sqrt{\pi}},$$

et en utilisant la formule des compléments on trouve

$$\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{5}{6})} = \frac{3\Gamma(\frac{1}{3})^4}{2^{5/3}\pi^2}.$$

En injectant cette formule dans l'expression obtenue pour $\zeta'_{\mathfrak{g}_2}(0)$, on trouve

$$\zeta'_{\mathfrak{g}_2}(0) = \frac{5}{12} \ln(5) - \frac{3}{4} \ln(3) + \frac{55}{12} \ln(2) + \frac{23}{6} \ln(\pi) - \frac{4}{3} \ln\left(\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\right).$$

Par la formule des compléments on a

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}},$$

et ainsi on obtient

$$\zeta'_{\mathfrak{g}_2}(0) = \frac{5}{12} \ln(5) - \frac{1}{12} \ln(3) + \frac{13}{4} \ln(2) + \frac{5}{2} \ln(\pi).$$

6.3.3 Pôles et résidus de $\zeta_{\mathfrak{so}(5)}$

Via [KMT10d], on a vu en dans la liste de pôles (1.3) que $\zeta_{\mathfrak{so}(5)}(s)$ a des pôles de la forme $s = \frac{1}{2}$ et en $s = \frac{1-k}{3}$, pour $k \geq 0$ un entier quelconque tel que $k \not\equiv 1 \pmod{3}$. On se propose ici d'obtenir des expressions des résidus en $s = \frac{1}{3}$ et en $s = \frac{1}{2}$ pour $\zeta_{\mathfrak{so}(5)}(s)$. Notons que Bridges, Brindle, Bringmann, et Franke ont déjà obtenu une expression très simple de ces deux résidus dans [BBBF24, Proposition 5.16] via des techniques différentes.

On pose

$$\begin{aligned} l_1(x_1, x_2) &= x_1 + x_2, & l_2(x_1, x_2) &= x_1 + 2x_2, \\ d_1 &= 1, & d_2 &= 1, & d'_1 &= 2, & d'_2 &= 3, \\ Z_{\Delta}(s) &:= Z(-((s, s), (s, s, s, s))) = \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s}. \end{aligned}$$

On a alors que

$$\zeta_{\mathfrak{so}(5)}(s) = 6^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s} = 6^s Z_{\Delta}(s).$$

On sait que, pour $\theta > 0$ suffisamment petit, la fonction zêta $\zeta_{\mathfrak{so}(5)}$ admet la décomposition $\zeta_{\mathfrak{so}(5)}(s) = K(\theta, s) + J(\theta, s)$, avec $s \mapsto K(\theta, s)$ une fonction holomorphe sur tout \mathbb{C} , et $s \mapsto J(\theta, s)$ une fonction méromorphe sur \mathbb{C} ayant les mêmes pôles que $\zeta_{\mathfrak{so}(5)}(s)$. Par la Proposition 2.2.4, on a

$$J(\theta, s) = 6^s \Gamma(1-s)^2 \left(\sum_{j=1}^2 h_{[1,2],j,\mathbf{0}}(s) \right) \frac{\theta^{4s-2}}{\Gamma(s)(4s-2)} \quad (6.3)$$

$$+ 6^s \sum_{\mathcal{P} \subsetneq [1,2]} \Gamma(1-s)^{|\mathcal{P}|} \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}^{\mathcal{P}^c}} \left(\sum_{j=1}^2 h_{\mathcal{P},j,\mathbf{k}}(s) \right) \frac{(-1)^{|\mathbf{k}|} \theta^{(2+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(s)((2+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(s - k_p)}{k_p!}$$

avec

$$h_{\mathcal{P},1,\mathbf{k}}(s) = \frac{1}{\Gamma(s)} \int_0^1 x_2^{s-1} \prod_{p \in \mathcal{P}} l_p^*(1, x_2)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(1, x_2)^{k_p} dx_2,$$

$$h_{\mathcal{P},2,\mathbf{k}}(s) = \frac{1}{\Gamma(s)} \int_0^1 x_1^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, 1)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, 1)^{k_p} dx_1,$$

$$l_1^*(x_1, x_2) = x_1 + x_2,$$

$$l_2^*(x_1, x_2) = x_1 + 2x_2.$$

Comme la fonction $s \mapsto K(\theta, s)$ est holomorphe sur \mathbb{C} , on devrait pouvoir lire les résidus de $\zeta_{\mathfrak{so}(5)}(s)$ sur l'expression de $J(\theta, s)$ ci-dessus. En $s = 1/2$, le seul terme contribuant au résidu de $\zeta_{\mathfrak{so}(5)}(s)$ est

$$\Gamma(1-s)^2 \left(\sum_{j=1}^2 h_{[1,2],j,\mathbf{0}}(s) \right) \frac{\theta^{4s-2}}{\Gamma(s)(4s-2)}. \text{ On obtient alors une expression du résidu en } s = 1/2 :$$

$$\text{Res}_{s=\frac{1}{2}} \zeta_{\mathfrak{so}(5)}(s) = \frac{\sqrt{6}\Gamma(1/2)^2}{4\Gamma(1/2)} \left(\sum_{j=1}^2 h_{[1,2],j,\mathbf{0}}(1/2) \right).$$

En remplaçant les fonctions h par leurs expressions, on obtient :

$$\begin{aligned} \text{Res}_{s=\frac{1}{2}} \zeta_{\mathfrak{so}(5)}(s) &= \frac{\sqrt{6}}{4} \left(\int_0^1 x^{-1/2}(x+1)^{-1/2}(x+2)^{-1/2} dx + \int_0^1 x^{-1/2}(1+x)^{-1/2}(1+2x)^{-1/2} dx \right) \\ &= \frac{\sqrt{6}}{4} \int_0^1 x^{-1/2}(x+1)^{-1/2}((x+2)^{-1/2} + (1+2x)^{-1/2}) dx. \end{aligned}$$

Même si l'intégrale ci-dessus est difficile à réexprimer en des termes plus simples, il est bon de noter que l'on peut évaluer cette intégrale numériquement, et l'on obtient alors la même valeur que celle obtenue dans l'article [BBBF24].

Pour les pôles $s = \frac{n}{3}$, avec $n \leq 1$ un entier tel que $n \not\equiv 0 \pmod{3}$, les seuls termes contribuant au résidu correspondant sont des termes de la forme

$$-\Gamma(1-n/3)^{|\mathcal{P}|} (-1)^{|\mathbf{k}|} \left(\sum_{j=1}^2 h_{\mathcal{P},j,\mathbf{k}}(1-n/3) \right) \frac{1}{\Gamma(n/3)((2+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(n/3 - k_p)}{k_p!},$$

avec $\mathcal{P} \subsetneq [1,2]$, $\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}$ vérifiant la condition $(2+|\mathcal{P}|)\frac{n}{3} - |\mathcal{P}| + |\mathbf{k}| = 0$. Cette condition implique nécessairement que $\mathcal{P} = \{i\}$ est un singleton, et en notant $\mathcal{P}^c = \{p\}$, on a $|\mathbf{k}| = k_p = 1 - n \in \mathbb{N}$. On trouve alors

$$\text{Res}_{s=\frac{n}{3}} \zeta_{\mathfrak{so}(5)}(s) = (-1)^{1-n} \Gamma\left(\frac{3-n}{3}\right) \left(\sum_{i=1}^2 \sum_{j=1}^2 h_{\{i\},j,\mathbf{k}}\left(1-\frac{n}{3}\right) \right) \frac{1}{3\Gamma\left(\frac{n}{3}\right)} \frac{\zeta\left(\frac{4n}{3}-1\right)}{(1-n)!}.$$

Notons que, en considérant les termes généraux de la série présente dans la formule (6.3) avec $\mathcal{P} = \emptyset$, on remarque des termes de la forme $\frac{\theta^{2s+|\mathbf{k}|}}{\Gamma(s)(2s+|\mathbf{k}|)}$ avec $\mathbf{k} = (k_1, k_2) \in \mathbb{N}$. Les zéros de $2s + |\mathbf{k}|$ ne constituent pas des pôles puisque ces pôles sont annulés par les zéros des termes de la forme $\sum_{j=1}^2 h_{\emptyset, j, \mathbf{k}}$. Par exemple, en $s = -\frac{1}{2}$, on a que le terme $2s + |\mathbf{k}|$ est nul si et seulement si $|\mathbf{k}| = k_1 + k_2 = 1$, c'est-à-dire lorsque $\mathbf{k} \in \{(1, 0), (0, 1)\}$. Or on remarque que

$$\begin{aligned} & h_{1, \emptyset, (1, 0)}(s) + h_{2, \emptyset, (1, 0)}(s) + h_{1, \emptyset, (0, 1)}(s) + h_{1, \emptyset, (0, 1)}(s) \\ &= \frac{1}{\Gamma(s)} \int_0^1 (x^{s-1}(1+2x) + 2x^{s-1}(1+x) + x^{s-1}(2+x)) dx \\ &= \frac{5}{\Gamma(s)} \left(\frac{1}{s+1} + \frac{1}{s} \right), \end{aligned}$$

et en $s = -\frac{1}{2}$, ce dernier terme s'annule. Ce fait est cohérent avec la liste des candidats pôles (1.3).

6.3.4 Valeurs aux entiers négatifs de $\zeta_{\mathfrak{so}(5)}$ et $\zeta'_{\mathfrak{so}(5)}$

On reprend les notations de la sous-section précédente. On a en particulier que

$$\zeta_{\mathfrak{so}(5)}(-N) = 6^{-N} Z_{\Delta}(-N), \quad \zeta'_{\mathfrak{so}(5)}(-N) = 6^{-N} Z'_{\Delta}(-N) + \ln(6) \zeta_{\mathfrak{so}(5)}(-N).$$

En appliquant le Corollaire A1 on trouve directement

$$\zeta_{\mathfrak{so}(5)}(0) = \zeta(0)^2 - \frac{9}{6} \zeta(-1) = \frac{3}{8}.$$

Via le Corollaire 6.3.2, on a une expression simple et explicite de $Z_{\Delta}(-N)$. On trouve alors :

$$\zeta_{\mathfrak{so}(5)}(-N) = 6^{-N} N!^2 \sum_{k=0}^{2N} \alpha_k(N) \frac{\zeta(-N-k) \zeta(-3N+k)}{k!(2N-k)!} + \frac{6^{-N} (-1)^{N+1}}{3} N!^3 \beta(N) \frac{\zeta(-4N-1)}{(3N+1)!},$$

avec

$$\begin{aligned} \alpha_k(N) &= \sum_{\substack{(w_q)_{q \in [1, 2]} \in [0, N]^2 \\ w_1 + w_2 = k}} \binom{k}{(w_1, w_2)} \binom{2N-k}{(N-w_1, N-w_2)} \prod_{q=1}^2 c_{q,1}^{w_q} c_{q,2}^{N-w_q}, \\ \beta(N) &= \sum_{j=1}^2 \sum_{\substack{(w_q)_{q \in [1, 2]} \in [0, 3N+1]^2 \\ w_1 + w_2 = 3N+1 \\ w_j \geq 2N+1 \\ \forall q \neq j, w_q \leq N}} \binom{-N-1}{-2N-1+w_j} \binom{-2N-1+w_j}{(N-w_q)_{q \neq j}} \left(\prod_{q=1}^2 c_{q,1}^{N-w_q} c_{q,2}^{w_q} + \prod_{q=1}^2 c_{q,2}^{N-w_q} c_{q,1}^{w_q} \right). \end{aligned}$$

En utilisant le code SAGE 7.2, on trouve les valeurs de $\zeta_{\mathfrak{so}(5)}(-N)$ présentes dans le tableau du Théorème E.

Enfin, par le Lemme 6.3.3, en s'aidant du Code SAGE 7.1 on a l'expression suivante de sa dérivée en 0 :

$$\zeta'_{\mathfrak{so}(5)}(0) = \frac{3}{8} \ln(3) + \frac{13}{8} \ln(2) + \frac{3}{2} \ln(\pi).$$

L'expression de la dérivée en $s = 0$ de la fonction zêta de Witten $\zeta_{\mathfrak{so}(5)}$ permet d'obtenir une expression explicite du coefficient C dans le Théorème 1.1.41, répondant ainsi à une question ouverte de Bringmann, Bridges, Brindle et Franke [BBBBF24].

6.4 Application à l'étude du comportement asymptotique de $r_{\mathfrak{g}_2}(n)$

Rappelons que

$$r_{\mathfrak{g}_2}(n) = \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N})^{\mathbb{N}^*2} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|.$$

Posons $P(i,j) := \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!}$. On observe que l'on a

$$\prod_{i,j \geq 1} \frac{1}{1 - q^{P(i,j)}} = \prod_{i,j \geq 1} \sum_{k_{i,j} \geq 0} q^{k_{i,j} P(i,j)},$$

en développant ce produit infini, et en utilisant l'expression du nombre de représentations $r_{\mathfrak{g}_2}(n)$, on trouve que :

$$1 + \sum_{n=1}^{+\infty} r_{\mathfrak{g}_2}(n) q^n = \prod_{i,j \geq 1} \frac{1}{1 - q^{P(i,j)}}.$$

En posant $f_{\mathfrak{g}_2}(n) = |\{(i,j) \in \mathbb{N}^*2 \mid P(i,j) = n\}|$, on obtient que

$$1 + \sum_{n=1}^{+\infty} r_{\mathfrak{g}_2}(n) q^n = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f_{\mathfrak{g}_2}(n)}}.$$

On note désormais $L_{f_{\mathfrak{g}_2}}(s) = \zeta_{\mathfrak{g}_2}(s)$, et $L_{f_{\mathfrak{g}_2}}^*(s) = \zeta_{\mathfrak{g}_2}(s)\Gamma(s)\zeta(s+1)$. Par le Corollaire 2.2.6, on sait que tous les pôles de la fonction $\zeta_{\mathfrak{g}_2}(s)$ sont simples. Observons que la fonction $L_{f_{\mathfrak{g}_2}}^*(s)$ ne possède que deux pôles dans H_0 (qui sont des pôles simples) en $s = \frac{1}{3}$ et en $s = \frac{1}{5}$. De plus, la fonction $L_{f_{\mathfrak{g}_2}}^*(s)$ possède un pôle double en $s = 0$.

On souhaite appliquer le Théorème 1.1.38. Vérifions tout d'abord si, avec ces données, les conditions du théorème sont valides. Posons $\Lambda := \mathbb{N} \setminus f_{\mathfrak{g}_2}^{-1}(\{0\})$. Remarquons que $\frac{3}{2} \cdot \frac{1}{5} < \frac{1}{3} < \frac{2}{1} \cdot \frac{1}{5}$. Ainsi, l'entier $l = 2$ vérifie l'hypothèse du Théorème 1.1.38. Vérifions maintenant si les hypothèses (P1), (P2) et (P3) sont vérifiées :

(P1) Soit $p \geq 2$ un nombre premier. On a que $|\Lambda \setminus (p\mathbb{N} \cap \Lambda)| = +\infty$. En effet, les suites

$$\begin{aligned} u(2) &:= (P(8k+1, 1))_{k \in \mathbb{N}}, \\ u(3) &:= (P(9k+1, 1))_{k \in \mathbb{N}}, \\ u(5) &:= (P(25k+1, 1))_{k \in \mathbb{N}}, \\ u(p) &:= (P(kp+1, 1))_{k \in \mathbb{N}} \quad \text{si } p \geq 7 \end{aligned}$$

sont strictement croissantes, et appartiennent à Λ par construction. De plus, on observe que la suite $u(p)$ est incluse dans $\Lambda \setminus (p\mathbb{N} \cap \Lambda)$ pour tout nombre premier p . On obtient alors que n'importe quel réel $L \geq \frac{1}{6}$ vérifie la condition (P1).

(P2) On a déjà vu précédemment que $L_{f_{\mathfrak{g}_2}}^*$ possède deux pôles (qui sont simples) en $s = \frac{1}{3}$ et en $s = \frac{1}{5}$, un pôle double en $s = 0$, et des pôles simples de la forme $s = \frac{k}{5}$, avec $k \in \mathbb{Z}_{\leq 1}$ tel que $k \neq 0 \pmod{5}$. Ainsi, tout réel $R \in \mathbb{R}_+^* \setminus \frac{1}{5}\mathbb{N}^*$ vérifie la condition (P2).

(P3) On remarque que le polynôme P vérifie la condition H0S de l'article [Ess97]. En particulier, on sait par [Ess97, Théorème 3] que la fonction zêta rattachée à ce polynôme (i.e. ζ_{g_2}) admet une borne polynomiale en la partie imaginaire de $s = \sigma + i\tau$ sur chaque bande verticale $\sigma_1 \leq \sigma \leq \sigma_2$. On en déduit alors que la condition (P3) est vérifiée.

Remarque 6.4.1. Dans la condition (P2), on peut prendre R arbitrairement grand, et dans la condition (P1), on peut prendre L arbitrairement grand.

Notons $\alpha = \frac{1}{3}$, $\beta = \frac{1}{5}$, $\omega_\alpha := \text{Res}_{s=\frac{1}{3}} \zeta_{g_2}(s)$ et $\omega_\beta := \text{Res}_{s=\frac{1}{5}} \zeta_{g_2}(s)$. En appliquant le Théorème 1.1.38, on trouve alors

$$r_{g_2}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp\left(A_1 n^{\frac{1}{4}} + A_2 n^{\frac{3}{20}} + A_3 n^{\frac{1}{20}}\right) \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R}\left(n^{-\min\left(\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\right)}\right)\right),$$

avec

$$C = \frac{e^{\zeta'_{g_2}(0)} (\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{1-6\zeta_{g_2}(0)}{8}} \sqrt{3}}{\sqrt{8\pi}}, \quad b = \frac{6 - 6\zeta_{g_2}(0) + 1}{8},$$

et $A_1 := 4 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{4}}$, $A_2 := \frac{\omega_\beta \Gamma\left(\frac{1}{5}\right) \zeta\left(\frac{6}{5}\right)}{(\omega_\alpha \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}}$, et

$$A_3 := K_3 + 3 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{4}} \left(\binom{-1/3}{1} \frac{K_3}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{4}}} + \binom{-1/3}{2} \frac{K_2^2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{2}}} \right) \\ + \frac{5 (\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right))}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}} \binom{-1/5}{1} \frac{K_2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{4}}},$$

avec $0 < \nu_2 < \dots$ les éléments strictement positifs de $\mathcal{N} + \mathcal{M}$. Par la définition des ensembles (1.8), (1.9), et (1.10) décrits dans [BBBF24], on trouve que la suite ν_j du Théorème 1.1.38 donne, avec nos

hypothèses, $\nu_2 = \frac{1}{20}$, $\nu_3 = \frac{2}{20}$, $\nu_4 = \frac{3}{20}$, ...

Par [BBBF24, Lemme 4.3], on a

$$K_2 = \frac{3}{4} \cdot \frac{\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}}$$

$$K_3 = -\frac{3}{160} \cdot \frac{(\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right))^2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{21}{20}}}.$$

On obtient finalement le Théorème G. □

Chapitre 7

Annexe : Codes SAGE

Listing 7.1: Code SAGE pour calculer $Z'_\Delta(0)$ avec $P = 2$ et $d_1 = d_2 = 1$

```
from sage.all import *
A=var('A') #Constante de Glaisher-Kinkelin
zeta_der_1=1/12-log(A) #Valeur de  $\zeta'(-1)$ 
c_so5=[[1,1],[1,2]] #Data de  $\zeta_{s_0(5)}(s)$ 
c_sl3=[[1,1]] #Data de  $\zeta_{sl(3)}(s)$ 
c_g2=[[1,1],[1,2],[1,3],[2,3]] #Data de  $\zeta_{g_2}(s)$ 
def barnes_eval_deriv(N,d,c): #Calcul de quelques  $(\zeta^B)'(0,d|c)$ 
    if d==2 and c==[1,1]:
        return(1/2*log(2*pi)+1/12-log(A))
    elif d==3 and c==[1,2]:
        return(1/4*log(2)+1/2*log(pi)-1/2*log(A)+1/24)
    elif d==4 and c==[1,3]:
        return(-5/9*log(3)+log(2)+log(pi)-1/3*log(A)
            -2/3*log(gamma(2/3))-1/3*log(gamma(1/3))+1/36)
    elif d==5 and c==[2,3]:
        return(-11/18*log(3)+13/12*log(2)+4/3*log(pi)-1/6*log(A)
            -2/3*log(gamma(2/3))-log(gamma(1/3))+1/72)
    else:
        return('uncomputed_value')
def Z_delta_prime_zero(c): #Calcul de  $Z'_\Delta(0)$ .
    Q=len(c)
    sum2 = 0
    sum1 = sum(c[j][0]/c[j][1]+c[j][1]/c[j][0] for j in range(Q))
    sum3 = sum(barnes_eval_deriv(0,sum(c[j]),c[j]) for j in range(Q))
    sum2 = sum(c[j][1]/c[j][0]*log(c[j][0])+c[j][0]/c[j][1]*log(c[j][1])
        for j in range(Q))
    +sum(sum((c[f][0]/c[f][1]-c[j][0]/c[j][1])*log(1+c[f][1]/c[j][1])
        +(c[f][1]/c[f][0]-c[j][1]/c[j][0])*log(1+c[f][0]/c[j][0])
        for f in range(Q) if f != j) for j in range(Q))
    result = -zeta(0)*log(2*pi)-sum1/(Q+1)*zeta_der_1-sum2/(Q+1)*zeta(-1)
    +sum3
    if not(sage.rings.rational.is_Rational(result)):
        result=result.canonicalize_radical()
    return(result)
print(Z_delta_prime_zero(c_so5)+log(6)*3/8) #Calcul de  $\zeta'_{s_0(5)}(0)$ 
print(Z_delta_prime_zero(c_g2)+log(120)*5/12) #Calcul de  $\zeta'_{g_2}(0)$ 
```

Listing 7.2: Code SAGE pour calculer $Z_{\Delta}(-N)$ avec $P = 2$

```

from math import comb
from itertools import product
c_g2 = [[1,1],[1,2],[1,3],[2,3]] #Data de  $\zeta_{g_2}(s)$ 
c_so5 = [[1,1],[1,2]] #Data de  $\zeta_{so(5)}(s)$ 

def alpha_k_N(Q,N,k, c_matrix):
    alpha=0
    for w in product(range(N+1),repeat=Q):
        if sum(w)==k:
            coef1=multinomial(w)
            coef2=multinomial([N-w_i for w_i in w])
            prod=1
            for q in range(Q):
                prod*=(c_matrix[q][0]**w[q])*(c_matrix[q][1]**(N-w[q]))
            alpha+=coef1*coef2*prod
    return alpha

def beta_N(Q,N, c_matrix):
    beta=0
    for j in range(Q):
        for w in product(range((Q+1)*N+2),repeat=Q):
            if sum(w)==(Q+1)*N+1 and w[j]>=2*N+1 and
            max([w[q] for q in range(Q) if q!=j])<=N:
                coeff=multinomial(w)*binomial(-N-1,-2*N-1+w[j])
                    *multinomial([N-w[q] for q in range(Q) if q!=j])
                term1=1
                term2=1
                for q in range(Q):
                    term1*=(c_matrix[q][0]**(N-w[q]))
                        *(c_matrix[q][1]**w[q])
                    term2*=(c_matrix[q][1]**(N-w[q]))
                        *(c_matrix[q][0]**w[q])
                beta+=coeff*(term1 + term2)
    return beta

def Z_Delta_minus_N(N, c_matrix): #Calcul de  $Z_{\Delta}(-N)$ 
    Q=len(c_matrix)
    sum1=0
    sum2=0
    N_fact_Q=factorial(N)**Q
    for k in range(Q*N+1):
        alpha=alpha_k_N(Q,N,k, c_matrix)
        term1=zeta(-N-k) / factorial(k)
        term2=zeta(-(Q+1)*N+k)/factorial(Q*N-k)
        sum1+=alpha*term1*term2
    beta=beta_N(Q,N, c_matrix)
    sum2=(N_fact_Q*factorial(N))*beta*zeta(-(Q+2)*N-1)/factorial((Q+1)*N+1)
    Z=N_fact_Q*sum1+(-1)**(N+1)*(1/(Q+1))*sum2
    return Z

print(6**(-N)*Z_Delta_minus_N(N, c_so5)) #Calcul de  $\zeta_{so(5)}(-N)$ 
print(120**(-N)*Z_Delta_minus_N(N, c_g2)) #Calcul de  $\zeta_{g_2}(-N)$ 

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**Special values of generalized multiple Hurwitz zeta
functions and their derivatives. Applications towards
some Witten zeta functions attached to semi-simple
Lie algebras**

PhD in Mathematics
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Notations

Notation. Let A, B be sets. We denote the non-strict inclusion as $A \subset B$ (which some authors denote as $A \subseteq B$), and the strict inclusion as $A \subsetneq B$.

Notation. Let A be a set, and $B \subset A$ a subset of A , we denote $B^c = A \setminus B$ and we denote $|A| \in \mathbb{N}_0 \cup \{+\infty\}$ the cardinal of A .

Notation. Considering two sets A, B and complex numbers $(c_{a,b})_{a,b \in A \times B}$, we note

$$\sum_{a \in A} \sum_{b \in B} c_{a,b} := \sum_{a \in A} \left(\sum_{b \in B} c_{a,b} \right).$$

Notation. We note δ_a^b the Kronecker symbol which is equal to 1 when $a = b$, and 0 otherwise.

Notation. We consider a set A , complex numbers $(c_a)_{a \in A}$, and $P : A \rightarrow \{\text{True}, \text{False}\}$ a property of A . We then set

$$\sum_{\substack{a \in A \\ P(a)}} c_a := \sum_{a \in A} c_a \delta_{\text{True}}^{P(a)},$$

which corresponds to the sum over all $a \in A$ of the terms c_a such that $P(a)$ is true.

By convention, we will say that a sum over an empty set is 0, and that a product over an empty set is 1.

Notation. We use a boldface letter to represent multi-index of the form $\mathbf{x} = (x_a)_{a \in A} \in \mathbb{C}^A$ with A a finite set. If all x_a are integers and if $|A| = n$, we will sometimes call such index a n -tuple integer. Given $B \subset A$, we denote $|\mathbf{x}|_B := \sum_{b \in B} x_b$. If $B = A$, we directly denote $|\mathbf{x}| := |\mathbf{x}|_A$. One should be mindful not to mix this notation with the module of a complex number.

Notation. Let $1 \leq k \leq n$ be positive integers. We denote $\mathbf{e}_k = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{C}^n$ the vector with a 1 at the k -th component, and 0's elsewhere.

Notation. Let A and B be finite sets, and $\mathbf{x} = (\mathbf{x}_a)_{a \in A} \in (\mathbb{N}_0^B)^A$, with $\mathbf{x}_a = (x_{a,b})_{b \in B}$ for all $a \in A$. We define for every $a \in A$, $b \in B$,

$$|\mathbf{x}_{\bullet b}| := \sum_{a \in A} x_{a,b}, \quad |\mathbf{x}_a| := \sum_{b \in B} x_{a,b}.$$

Remark. Let A and B be finite sets, and $\mathbf{x} = (\mathbf{x}_a)_{a \in A} \in (\mathbb{N}_0^B)^A$. The multi-index \mathbf{x} corresponds to a function $\mathbf{x} : A \times B \rightarrow \mathbb{N}_0, (a, b) \mapsto x_{a,b}$. In particular, if A or B is empty, \mathbf{x} corresponds to the empty function. In this case, for every subset $\mathcal{E} \subset A \times B$ and for every $b \in B$,

$$|\mathbf{x}|_{\mathcal{E}} = 0, \quad |\mathbf{x}_{\bullet b}| = 0.$$

Notation. Let $s \in \mathbb{C}$ be a complex number. We write $s = \sigma + i\tau$ where σ and τ are respectively the real and imaginary parts of s .

Notation. Let $x \in \mathbb{R}$ be a real number. We denote H_x as the open complex half-plane with abscissa x , and \overline{H}_x as its closure:

$$\begin{aligned} H_x &:= \{s \in \mathbb{C} \mid \sigma > x\}, \\ \overline{H}_x &:= \{s \in \mathbb{C} \mid \sigma \geq x\}. \end{aligned}$$

Notation. Let $a \in \mathbb{C}$, $r \in \mathbb{R}_+$. We denote $\overline{D}_a(r)$ as the closed disc centered at a with radius r :

$$\overline{D}_a(r) := \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

Notation. Let $s \in \mathbb{C}$ be a complex number, and $k \in \mathbb{N}_0$. We denote $\mathbf{s}_{\Delta, k} := (s, \dots, s) \in \mathbb{C}^k$. When there is no ambiguity, we will simply denote it as \mathbf{s}_{Δ} .

Notation. Let $n \in \mathbb{N}_0$ be an integer. We call the n -th harmonic number the following rational number

$$h_n := \sum_{k=1}^n \frac{1}{k},$$

with the convention $h_0 = 0$.

Notation. We will denote $\gamma = 0.5772156649\dots$ as Euler's constant [OEI23a], $A = 1.28242712\dots$ as the Glaisher-Kinkelin constant [OEI23c], and $e = 2.718281828459\dots$ as Euler's number [OEI23b].

Proposition-Definition ([EMOT81, Chap.1]). We denote by Γ the meromorphic continuation on \mathbb{C} of Euler's gamma function defined on H_0 by the integral

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx \quad (\operatorname{Re}(s) > 0).$$

The function Γ is holomorphic on H_0 , and has simple poles at the nonpositive integers. We also note ψ the meromorphic continuation on \mathbb{C} of the digamma function defined on H_0 by the relation

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} \quad (\operatorname{Re}(s) > 0).$$

The function ψ is holomorphic on H_0 , and has simple poles at the nonpositive integers.

We recall that Γ satisfies the following functional equation:

$$\Gamma(s) = (s-1)\Gamma(s-1) \quad (s \notin -\mathbb{N}_0),$$

and using this functional equation at positive integers, we find that $\Gamma(N+1) = N!$. Let's also recall Euler's reflection formula

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)} \quad (s \notin \mathbb{Z}).$$

In the neighborhood of $s = 0$, we have the following asymptotic expansion

$$\Gamma(s) = \frac{1}{s} - \gamma + \frac{1}{2} \left(\gamma^2 + \frac{\pi^2}{6} \right) s + O(s^2).$$

We also recall that the function $\frac{1}{\Gamma}$ is holomorphic on \mathbb{C} and it vanishes at nonpositive integers.

Notation. For all $k \in \mathbb{Z}$, and $s \in \mathbb{C}$, we set the binomial coefficient

$$\binom{s}{k} = \begin{cases} \frac{s(s-1)\dots(s-k+1)}{k!} & \text{if } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For all integers $\mathbf{k} = (k_1, \dots, k_P)$ such that $n = k_1 + \dots + k_P$, we consider the multinomial coefficient:

$$\binom{n}{\mathbf{k}} = \begin{cases} \frac{n!}{k_1! \dots k_P!} & \text{if } k_p \geq 0 \text{ for all } p \in \llbracket 1, P \rrbracket, \\ 0 & \text{otherwise.} \end{cases}$$

In this manuscript, we will use the principal determination of the logarithm, and we will write for all complex $z \in \mathbb{C}^*$, $z = |z|e^{i \arg(z)}$ with $\arg(z) \in]-\pi, \pi]$.

Chapter 1

Introduction

In [Kom08], Komori obtained a surface integral representation of the double zeta Mordell-Tornheim zeta function introduced by Matsumoto [Mat03], he then obtained explicit formulas for the values at nonpositive integers of this function. Further on, by introducing generalized Bernoulli numbers and via a similar method, he generalized those results in [Kom10] by establishing a formula for the directional values at nonpositive integers of multiple zeta functions of generalized Hurwitz-Lerch type, however this formula is not quite explicit because the generalized Bernoulli numbers are often hard to compute. Later on some special cases have been studied. For multiple zeta functions of Euler-Zagier type, explicit formulas have been proved by Onozuka [Ono13], and later on by Essouabri and Matsumoto [EM20] for an even larger class. Some work from Borwein and Dilcher [BD18] provided explicit formulas and approximate for the values and derivative values of multiple zeta functions of Mordell-Tornheim of rank 2 by using a technique from Crandall [Cra12]. Then an article from Onodera [Ono21] generalized this approach and computed explicit results for the values and derivative values of order 1 and 2 for multiple zeta functions of Mordell-Tornheim of any rank, along the diagonal.

This thesis is a continuation of the work above. We will study a class of multiple zeta functions of generalized Hurwitz type defined by the Dirichlet series

$$Z(\mathbf{s}, \mathbf{s}') = \sum_{\mathbf{n}=(n_1, \dots, n_P) \in \mathbb{N}_0^P} \frac{1}{\prod_{p=1}^P (n_p + d_p)^{s_p} \prod_{q=1}^Q (l_q(\mathbf{n} + \mathbf{d}))^{s'_q}},$$

with $P \geq 1$, $Q \geq 1$, $\mathbf{d} = (d_1, \dots, d_P) \in H_0^P$, and with l_q being linear forms of rank P , depending on each variables n_1, \dots, n_P . Let's note that such a function class contains the multiple zeta function of Mordell-Tornheim, some multiple zeta functions of Shintani class, and it also contains multiple zeta functions associated with any root system of rank 2. In particular, this class contains the Witten zeta functions $\zeta_{\mathfrak{sl}(3)}$, $\zeta_{\mathfrak{so}(5)}$ and $\zeta_{\mathfrak{g}_2}$ up to a multiplicative factor. We will study the domain of convergence and the meromorphic continuation of the multiple zeta function of generalized Hurwitz type in chapter 2. Then we will apply in this chapter some techniques used by Onodera which originated from Crandall in order to prove a directional meromorphic continuation of the same multiple zeta functions. We will then evaluate the meromorphic continuation formula in order to find explicit directional values at nonpositive integers for the multiple zeta functions of generalized Hurwitz type in chapter 4. Thanks to the same meromorphic continuation formula, we will also establish an explicit formula for the derivative of the same class of multiple zeta function along a direction in chapter 5. Let's note that those two formulas rely on two constants that come from some integrals, and the computations of those constants are done in chapter 3. We will also see the construction of Witten zeta functions attached to semi-simple Lie algebras in chapter 6, then we will establish some formulas for the values and derivative values of the functions $\zeta_{\mathfrak{so}(5)}$ and $\zeta_{\mathfrak{g}_2}$ and we will study the residues at positive poles of those two Witten zeta functions. Finally we will use those values and residues in order to obtain an explicit asymptotic formula for the number of representations of dimension n of the Lie algebras $\mathfrak{so}(5)$ and \mathfrak{g}_2 via a Meinardus theorem [BBBF24], thus completing the work of Bridges, Brindle, Bringmann and Franke.

1.1 Some history on zeta and multizeta functions

The study of values of zeta functions starts with the values of the series

$$\zeta(k) = \sum_{n=1}^{+\infty} \frac{1}{n^k} \quad (k \geq 2),$$

which corresponds to special values of the Riemann zeta function at positive integers. It was known by Euler that for $k = 2$ we have

$$\zeta(2) = \frac{\pi^2}{6}.$$

When k is even we also have well known formulas, which is not the case when k is odd. Another important and mysterious series is featured in the study of multiple zeta values (MZV), which corresponds to values of the series

$$\zeta(k_1, \dots, k_r) := \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}} \quad (k_1, \dots, k_{r-1} \geq 1, k_r \geq 2).$$

In certain cases some relations between the multiple zeta values and special values of the Riemann zeta function exist. A first well known relation from Euler states that

$$\zeta(3) = \zeta(1, 2).$$

Such values also verify quasi-shuffle relations, an example of which is

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2),$$

and one can read more details about those relations in [Zag94] or in [LM95]. Those quasi-shuffle relations also exist for multiple zeta functions of Hurwitz type defined by

$$\zeta(s_1, \dots, s_r, \mathbf{d}) := \sum_{0 < n_1 < \dots < n_r} \frac{1}{(n_1 + d_1)^{s_1} \dots (n_r + d_r)^{s_r}} \quad (\mathbf{d} = (d_1, \dots, d_r) \in \mathbb{R}_+^r, s_1, \dots, s_{r-1} \geq 1, s_r \geq 2)$$

in recent works from Manchon and Paycha [MP10]. Let's note that some of the relations obtained by the two authors involve nonpositive integers, thus one needs to specify what it means to consider such values outside of the convergence domain of the Dirichlet series. In that regard, the article of Manchon and Paycha takes an interest in the meromorphic continuation of the function $\zeta(s_1, \dots, s_r, \mathbf{d})$. Let's also note that the multiple zeta functions associated with a root system introduced by Matsumoto, Tsumura and Komori in [KMT23] also verifies some quasi-shuffle relations.

1.1.1 Some results on special values of zeta functions

We aim to provide here a brief summary of results on the special values of the Riemann zeta function, the Hurwitz zeta function, Barnes zeta functions, Euler-Zagier zeta functions, some Witten zeta functions, and Shintani zeta functions. We will also discuss their domains of holomorphy and meromorphy. In particular, we will present results on values at nonpositive integers for some of these functions.

All the zeta functions mentioned in this subsection have a domain of convergence of the form $H_{\sigma_0} = \{s = \sigma + i\tau \in \mathbb{C} \mid \sigma > \sigma_0\}$ with $\sigma_0 \in \mathbb{R}$, and they are holomorphic on their half-plane of convergence. To give meaning to the value at $s = N \in \mathbb{Z}$ when N is not in the half-plane of convergence for these functions, we must first discuss their meromorphic continuation and prove that each continuation is regular at nonpositive integers. Some of those analytic continuations are established using the Euler-Maclaurin formula.

Theorem 1.1.1 (Euler-Maclaurin formula, [Ten15]). *For all integer $k \geq 0$ and any given function f of class C^{k+1} on $[a, b]$, $a, b \in \mathbb{Z}$, we have*

$$\begin{aligned} \sum_{n=a}^{b-1} f(n) &= \int_a^b f(t) dt + \sum_{n=0}^k \frac{(-1)^{n+1} B_{n+1}}{(n+1)!} (f^{(n)}(b) - f^{(n)}(a)) \\ &\quad + \frac{(-1)^k}{(k+1)!} \int_a^b b_{k+1}(x) f^{(k+1)}(x) dx, \end{aligned}$$

where B_n is the n -th Bernoulli number and $b_n(x)$ is the n -th periodised Bernoulli polynomial.

We also recall the expression of the Riemann zeta function.

Definition 1.1.2. *The Riemann zeta function corresponds to the following Dirichlet series*

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s} \quad (\operatorname{Re}(s) > 1).$$

This function has a continuation on \mathbb{C} :

Proposition 1.1.3. *The Riemann zeta function is holomorphic on H_1 , and it has a meromorphic continuation on \mathbb{C} with a single pole at $s = 1$, which is simple, with a residue of 1.*

The meromorphic continuation can be obtained using the Euler-Maclaurin formula (1.1.1), or via the contour integral representation formula

$$\zeta(s) = \frac{e^{-i\pi s} \Gamma(1-s)}{2i\pi} \int_{\mathcal{C}} \frac{z^{s-1}}{e^z - 1} dz,$$

where \mathcal{C} denotes Hankel's contour who starts from $+\infty + i\delta$, then pass around the origin counter clockwise, and then back to $+\infty - i\delta$. It is classical to define Bernoulli numbers using the following generating function

$$\frac{z}{e^z - 1} = \sum_{k=0}^{+\infty} \frac{B_k}{k!} z^k,$$

and by injecting this expression into the contour integral representation of the Riemann zeta function, we get a formula for the values of the Riemann zeta function at nonpositive integers in terms of the Bernoulli numbers

$$\forall N \in \mathbb{N}_0, \quad \zeta(-N) = (-1)^N \frac{B_{N+1}}{N+1}.$$

In particular, it follows that $\zeta(-2N) = 0$ for all integers $N \in \mathbb{N}$. Using the functional equation, we can also find relations between the values of the Riemann zeta function at positive integers with the values of its derivative at nonpositive integers.

Theorem 1.1.4. *The Riemann zeta function satisfies the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (s \in \mathbb{C} \setminus \{0, 1\}).$$

Let's note that one can easily see from this functional equation alone that the negative even integers are zeros of the Riemann zeta function. Moreover, by taking the derivative of the functional equation we get that

$$\forall N \in \mathbb{N}, \quad \zeta'(-2N) = \frac{(-1)^N (2N)!}{2^{2N+1} \pi^{2N}} \zeta(2N+1).$$

For small values, we also have these results:

$$\begin{aligned} \zeta'(0) &= -\frac{1}{2} \ln(2\pi), \\ \zeta'(-1) &= \frac{1}{12} - \ln(A), \end{aligned}$$

where A is the Glaisher-Kinkelin constant.

We know since Euler that the values at positive even integers corresponds to a rational multiple of a power of π ,

$$\zeta(2N) \in \mathbb{Q} \pi^{2N} \quad (N \geq 1).$$

In fact, using the functional equation of the Riemann zeta function we even have an explicit expression of those values using Bernoulli numbers

$$\forall N \in \mathbb{N}, \quad \zeta(2N) = (-1)^{N+1} \frac{2^{2N-1} B_{2N}}{(2N)!} \pi^{2N}.$$

Nevertheless the study of the values at positive odd numbers is still pretty much an open problem, but we do have qualitative results about those values. Indeed, a result from Apéry [Apé79] showed that $\zeta(3)$ is an irrational number, and by a result of Rivoal [Riv00] we know that the sequence $(\zeta(2N+1))_{N \geq 1}$ does contain an infinite number of irrationals.

Definition 1.1.5. Let $d \in H_0$ be a complex number, the Hurwitz zeta function is defined by

$$\zeta(s, d) := \sum_{n=0}^{+\infty} \frac{1}{(n+d)^s} \quad (\operatorname{Re}(s) > 1).$$

We will note in this manuscript $\zeta'(s, d) := \frac{\partial}{\partial s} \zeta(s, d)$. Note that for $d = 1$, we recover the Riemann zeta function. It is well known that this zeta function admits a meromorphic continuation on \mathbb{C} in the s variable.

Proposition 1.1.6. The Hurwitz zeta function is meromorphic on \mathbb{C} with respect to the s variable, with a single pole at $s = 1$, which is a simple pole. Moreover, the residue at that pole is 1.

Proof. Let $n \in \mathbb{N}$ be a positive integer. Via the Euler-Maclaurin formula, we get that for all $\sigma > 1$,

$$\begin{aligned} \zeta(s, d) &= \frac{d^{1-s}}{s-1} + \sum_{i=0}^n \binom{-s}{i} \frac{(-1)^i B_{i+1}}{i+1} d^{-s-i} \\ &\quad + (-1)^n \binom{-s}{n+1} \int_0^{+\infty} b_{n+1}(x) (x+d)^{-s-n-1} dx. \end{aligned}$$

The integral in the previous formula converges absolutely for all $\sigma > -n$. Thus, this formula allows for a meromorphic continuation of the Hurwitz zeta function for $\sigma > -n$, with a unique pole at $s = 1$, of order 1, and with residue 1. \square

Starting from a contour integral representation of the Hurwitz zeta function similar to the one mentioned above

$$\zeta(s, x) = \frac{e^{-i\pi s} \Gamma(1-s)}{2i\pi} \int_{\mathcal{C}} \frac{z^{s-1} e^{xz}}{e^z - 1} dz$$

where \mathcal{C} is Hankel's contour, and using the generating function for the Bernoulli polynomials

$$\frac{ze^{zx}}{e^z - 1} = \sum_{k=0}^{+\infty} \frac{B_k(x)}{k!} z^k,$$

one easily finds the following relations between the values at nonpositive integers of the Hurwitz zeta function and Bernoulli polynomials (see [Apo76, Theorem 12.13])

$$\forall N \in \mathbb{N}_0, \quad \zeta(-N, x) = -\frac{B_{N+1}(x)}{n+1}.$$

Also, it was known by Hurwitz (see [Ber85, Formula (3)]) that

$$\zeta'(0, d) = \ln(\Gamma(d)) + \zeta'(0) = \ln(\Gamma(d)) - \frac{1}{2} \ln(2\pi).$$

Let us also observe that, for all $s \neq 1$, and for all $d \in H_0$, we have $\zeta(s, d) = d^{-s} + \zeta(s, d+1)$. Differentiating with respect to s , we then find

$$\zeta'(-N, d+1) = d^N \ln(d) + \zeta'(-N, d).$$

Rewriting the Dirichlet series $\zeta\left(s, \frac{1}{2}\right)$, we can prove that $\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s)$. Therefore, by differentiating both sides of the last equality with respect to s , we find that

$$\zeta'\left(-1, \frac{1}{2}\right) = -\frac{\ln(2)}{24} - \frac{1}{2}\zeta'(-1).$$

Miller and Adamchik have established in [MA98] an explicit relation between the values of the derivatives of the Hurwitz zeta function with respect to s at nonpositive integers with a rational coefficient d , and special values of logarithms, polygamma functions, and the Riemann zeta function. This work involves the multiplication formula for the Hurwitz zeta function:

$$\zeta(s, kz) = \sum_{n=0}^{k-1} \zeta\left(s, z + \frac{n}{k}\right) \quad (k \in \mathbb{N}^*). \quad (1.1)$$

Let's note that by evaluating the derivative with respect to s of this function at $s = 0$, and using $\zeta'(0, d) = \ln(\Gamma(d)) - \frac{1}{2}\ln(2\pi)$, we find the well known multiplication formula for Euler's gamma function

$$\prod_{n=0}^{k-1} \Gamma\left(z + \frac{n}{k}\right) = (2\pi)^{\frac{k-1}{2}} k^{\frac{1}{2}-kz} \Gamma(kz).$$

The Hurwitz zeta function also satisfies a functional equation called Hurwitz's formula. Before stating it, we need to introduce the Lerch zeta function.

Definition 1.1.7. *Let $s \in \mathbb{C}$ such that $\sigma > 1$, $d \in H_0$, and $z \in \mathbb{C}$ such that $|z| \leq 1$. We define the Lerch zeta function as*

$$\phi(z, s, d) := \sum_{n \geq 0} \frac{z^n}{(n+d)^s}.$$

The function $s \mapsto \phi(z, s, d)$ has a meromorphic continuation on \mathbb{C} . It is even holomorphic on \mathbb{C} for all $z \neq 1$ and it has a unique pole, which is simple, at $s = 1$ when $z = 1$ (see [EMOT81, §1.11]).

Theorem 1.1.8 (Hurwitz Formula, [Apo76, §12.7]). *We have*

$$\zeta(1-s, d) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-i\pi s/2} \phi(e^{2i\pi d}, s, 1) + e^{i\pi s/2} \phi(e^{-2i\pi d}, s, 1) \right) \quad (d \in]0, 1[, s \in \mathbb{C} \setminus \{0, 1\}).$$

Let's note that the case $d = 1$ also holds in the previous formula, because it corresponds to the functional equation of the Riemann zeta function. That functional equation can be used in order to find an estimate of the Hurwitz zeta function (see [Ono21, Lemma 2.1]), and this upper bound obtained by Onodera is actually key in the Crandall expression that we will do in Chapter 2.

Some of the results of the thesis will feature a large class of zeta functions called generalized Barnes zeta functions. Before mentioning the generalization, we will first give a brief summary of the results known for the classical Barnes zeta functions, and we will also mention some known values and derivative values at nonpositive integers of such functions.

Definition 1.1.9. *Let $P \in \mathbb{N}_0$ be a nonnegative integer, $d \in H_0$, and (c_1, \dots, c_P) some complex numbers in H_0 . We call the Barnes zeta function the zeta function of the form*

$$\forall s \in H_P, \quad \zeta^B(s, d|c_1, \dots, c_P) := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{\left(\sum_{p=1}^P c_p n_p + d\right)^s}.$$

Barnes proved that this zeta function has a meromorphic continuation on \mathbb{C} with simple poles at some positive integers. More precisely:

Proposition 1.1.10 ([Bar04]). *The Barnes zeta function $s \mapsto \zeta^B(s, d|c_1, \dots, c_P)$ has a meromorphic continuation on \mathbb{C} , with its only singularities being simple poles at $s = 1, \dots, P$.*

The derivative value at $s = 0$ of such zeta functions was initially studied by Barnes in the theory of multigamma functions (see [Bar01] and [Bar04]). Such a function does have links with geometry. Indeed, Vardi found in [Var88] an explicit formula between the value at $\frac{1}{2}$ of the n -th multigamma function and the regularized determinant of the Laplacian of the n -sphere. He also gave an explicit formula of the value at $\frac{1}{2}$ of the n -th multigamma function using only values at nonpositive integers of the derivative of the Riemann zeta function.

Some of our results will be expressed in terms of derivative values at nonpositive integers of Barnes zeta functions. In particular, we will need a result proved by Sakane and Aoki in [SA22]. We first define

$$C_{P,x}(t) := (t - x + P - 1)(t - x + P - 2) \dots (t - x + 1) \in (\mathbb{Z}[x])[t] \quad (P \geq 2)$$

and $C_{1,x}(t) = 1$. Let's note that $C_{2,x}(t) = t - x + 1$.

Theorem 1.1.11 ([SA22, Theorem 4]). *Let $d \in H_0$, and $\mathbf{c} = (c_1, \dots, c_P) \in \mathbb{Q}^P$ be rational numbers. We write $c_1 = \frac{a_1}{b_1}, \dots, c_P = \frac{a_P}{b_P}$ with $a_p, b_p \geq 1$ being coprime positive integers for all $1 \leq p \leq P$. Let $x(\mathbf{c}) := \frac{\text{lcm}(a_1, \dots, a_P)}{\text{gcd}(b_1, \dots, b_P)}$, and $\beta_1 := \frac{x(\mathbf{c})}{c_1}, \dots, \beta_P := \frac{x(\mathbf{c})}{c_P}$. We then have that*

$$\zeta^B(s, d | c_1, \dots, c_P) = \frac{x(\mathbf{c})^{-s}}{(P-1)!} \sum_{v_1=0}^{\beta_1-1} \dots \sum_{v_P=0}^{\beta_P-1} \sum_{k=0}^{P-1} C_{P, \frac{d+c_1v_1+\dots+c_Pv_P}{x(\mathbf{c})}}^{(k)}(0) \zeta\left(s - k, \frac{d + c_1v_1 + \dots + c_Pv_P}{x(\mathbf{c})}\right).$$

In Theorem B, we shall prove a generalized version of the above theorem, using a similar strategy as the one used by Aoki and Sakane in [SA22].

Example 1.1.12. *We have*

$$\begin{aligned} \zeta^B(s, d | 1, 1) &= (1-d)\zeta(s, d) + \zeta(s-1, d), \\ \zeta^B(s, d | 1, 2) &= 2^{-s} \left[\left(1 - \frac{d}{2}\right) \zeta\left(s, \frac{d}{2}\right) + \left(1 - \frac{d+1}{2}\right) \zeta\left(s, \frac{d+1}{2}\right) + \zeta\left(s-1, \frac{d}{2}\right) + \zeta\left(s-1, \frac{d+1}{2}\right) \right], \\ \zeta^B(s, d | 1, 3) &= 3^{-s} \left[\left(1 - \frac{d+2}{3}\right) \zeta\left(s, \frac{d+2}{3}\right) + \left(1 - \frac{d+1}{3}\right) \zeta\left(s, \frac{d+1}{3}\right) + \left(1 - \frac{d}{3}\right) \zeta\left(s, \frac{d}{3}\right) \right. \\ &\quad \left. + \zeta\left(s-1, \frac{d+2}{3}\right) + \zeta\left(s-1, \frac{d+1}{3}\right) + \zeta\left(s-1, \frac{d}{3}\right) \right], \\ \zeta^B(s, d | 2, 3) &= 6^{-s} \left[\left(1 - \frac{d+7}{6}\right) \zeta\left(s, \frac{d+7}{6}\right) + \left(1 - \frac{d+5}{6}\right) \zeta\left(s, \frac{d+5}{6}\right) + \left(1 - \frac{d+4}{6}\right) \zeta\left(s, \frac{d+4}{6}\right) \right. \\ &\quad \left. + \left(1 - \frac{d+3}{6}\right) \zeta\left(s, \frac{d+3}{6}\right) + \left(1 - \frac{d+2}{6}\right) \zeta\left(s, \frac{d+2}{6}\right) + \left(1 - \frac{d}{6}\right) \zeta\left(s, \frac{d}{6}\right) \right. \\ &\quad \left. + \zeta\left(s-1, \frac{d+7}{6}\right) + \zeta\left(s-1, \frac{d+5}{6}\right) + \zeta\left(s-1, \frac{d+4}{6}\right) + \zeta\left(s-1, \frac{d+3}{6}\right) \right. \\ &\quad \left. + \zeta\left(s-1, \frac{d+2}{6}\right) + \zeta\left(s-1, \frac{d}{6}\right) \right]. \end{aligned}$$

We then get the following special values

$$\begin{aligned} \zeta^B(0, 2 | 1, 1) &= \frac{5}{12}, & \zeta^B(0, 3 | 1, 2) &= \frac{11}{24}, & \zeta^B(0, 4 | 1, 3) &= \frac{19}{36}, & \zeta^B(0, 5 | 2, 3) &= \frac{31}{72}, \\ (\zeta^B)'(0, 2 | 1, 1) &= \frac{1}{2} \ln(2\pi) - \ln(A) + \frac{1}{12}, & (\zeta^B)'(0, 3 | 1, 2) &= \frac{1}{4} \ln(2) + \frac{1}{2} \ln(\pi) - \frac{1}{2} \ln(A) + \frac{1}{24}, \\ (\zeta^B)'(0, 4 | 1, 3) &= -\frac{5}{9} \ln(3) + \ln(2\pi) - \frac{1}{3} \ln(A) - \frac{2}{3} \ln\left(\Gamma\left(\frac{2}{3}\right)\right) - \frac{1}{3} \ln\left(\Gamma\left(\frac{1}{3}\right)\right) + \frac{1}{36}, \\ (\zeta^B)'(0, 5 | 2, 3) &= -\frac{11}{18} \ln(3) + \frac{13}{12} \ln(2) + \frac{4}{3} \ln(\pi) - \frac{1}{6} \ln(A) - \frac{2}{3} \ln\left(\Gamma\left(\frac{2}{3}\right)\right) - \ln\left(\Gamma\left(\frac{1}{3}\right)\right) + \frac{1}{72}, \end{aligned}$$

where we noted $(\zeta^B)'(s, d|\mathbf{c}) = \partial_s \zeta^B(s, d|\mathbf{c})$ and $\zeta'(s, d) = \partial_s \zeta(s, d)$.

Let's note that in the previous example, we've simplified the terms thanks to the multiplicative formula for the Hurwitz zeta function (1.1), and we've also used a result from Borwein and Zucker [BZ92, Table 3]:

$$\Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{3}\Gamma(\frac{1}{3})^2}{\sqrt[3]{2}\sqrt{\pi}},$$

in order to obtain the following result thanks to Euler's reflection formula

$$\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{5}{6})} = \frac{3\Gamma(\frac{1}{3})^4}{2^{5/3}\pi^2}.$$

Shintani introduced zeta functions that allowed him to study Dedekind zeta functions associated with a totally real number field (see [Shi76], [Shi77a], [Shi77b], [Shi77c], [Shi80]). In particular, these functions generalize Barnes' zeta functions:

Definition 1.1.13. Let $P, Q \in \mathbb{N}_0$ be nonnegative integers, and $\mathbf{c} = (c_{q,p})_{\substack{1 \leq p \leq P \\ 1 \leq q \leq Q}}$, $\mathbf{d} = (d_q)_{1 \leq q \leq Q}$ be complex numbers in H_0 . We call the following function the Shintani zeta function:

$$\forall s \in H_{P/Q}, \quad \zeta^{Sh}(s, \mathbf{d}|\mathbf{c}) := \sum_{n_1, \dots, n_P \geq 0} \prod_{q=1}^Q (c_{q,1}n_1 + \dots + c_{q,P}n_P + d_q)^{-s}.$$

Note that, for $Q = 1$, we recover the definition of Barnes' zeta function. This zeta function is holomorphic in the s variable on $H_{P/Q}$ and has a meromorphic continuation on \mathbb{C} . Shintani proved that this function has the same poles as those of the function $\frac{\Gamma(Ps - Q)}{\Gamma(s)}$, thus this function is regular at nonpositive integers. He also proved certain formulas at the nonpositive integers of these functions. A part of our work allows us to study the values of some of these functions at nonpositive integers, as well as the values of their derivatives.

The values of their derivatives with respect to the variable s at $s = 0$ allows us to define multigamma functions, introduced by Barnes in [Bar04], and then generalized in [FR04] by Friedmann and Ruijsenaars.

Definition 1.1.14. Let $P \in \mathbb{N}_0$ be a nonnegative integer, and (c_1, \dots, c_P) be complex numbers in H_0 , and $d \in H_0$. We call Barnes multigamma function the following function

$$\Gamma_P(d|c_1, \dots, c_P) := \exp(\partial_s(\zeta^B(s, d|c_1, \dots, c_P))|_{s=0}).$$

Example 1.1.15. Let $d \in H_0$, we have that

$$\begin{aligned} \Gamma_0(d) &= \frac{1}{d} \\ \Gamma_1(d|1) &= \frac{1}{\sqrt{2\pi}}\Gamma(d), \end{aligned}$$

where Γ is Euler's gamma function.

Barnes also obtains the following functional relation

$$\Gamma_P(d|c_1, \dots, c_P) = \Gamma_{P-1}(d|c_1, \dots, c_{P-1})\Gamma_P(d + c_P|c_1, \dots, c_P),$$

which generalizes the classical functional equation of the Euler gamma function. Through Shintani-type zeta functions, Friedman and Ruijsenaars introduced in [FR04] some Shintani-type multigamma functions, using a construction similar to the one used by Barnes.

Definition 1.1.16. Let $P, Q \in \mathbb{N}_0$ be nonnegative integers, $\mathbf{c} = (c_{q,p})_{1 \leq p \leq P, 1 \leq q \leq Q}$ and $(d_q)_{1 \leq q \leq Q}$ be two families of complex numbers in H_0 . The Shintani multigamma function is defined as:

$$\Gamma_P(\mathbf{d}|\mathbf{c}_1, \dots, \mathbf{c}_P) := \exp(\partial_s(\zeta^{Sh}(s, \mathbf{d}|\mathbf{c}_1, \dots, \mathbf{c}_P))|_{s=0})$$

where $\mathbf{c}_q = (c_{q,p})_{p \in [1, P]}$.

In [FR04], Friedman and Ruijsenaars studied these Shintani multigamma functions to deduce Raabe-type formulas, generalizing the following Raabe formula:

$$\int_0^1 \ln(\Gamma(x)) dx = \ln(\sqrt{2\pi}).$$

In our work we will obtain explicit formulas for the values and derivative values at nonpositive integers of some zeta functions of Shintani type. In particular we will also prove the same Raabe type formula as the one obtained by Friedman and Ruijsenaars.

We will now discuss a class of zeta functions related to Lie algebras. Witten studied in [Wit91] some moduli spaces that comes from conformal field theory, attached to a gauge group G . He then established a volume formula that contains a series running over all the irreducible representations of the Lie group G . Witten then obtains the qualitative result

$$\sum_{\varphi} \frac{1}{\dim(\varphi)^{2N}} \in \pi^{2kN} \mathbb{Q} \quad (N \in \mathbb{N}),$$

where φ runs over the finite-dimensional irreducible representations, up to isomorphism, of the Lie group G , and where k is the number of positive roots of the Lie group G . Zagier and Garoufalidis showed the same qualitative result using classical tools, either by using Fourier analysis or partial fraction decomposition (see [Zag94]). Zagier also introduced the following Witten zeta function:

Definition 1.1.17 ([Zag94]). *Let \mathfrak{g} a semi-simple Lie algebra, we call Witten zeta function attached to \mathfrak{g} the function*

$$\zeta_{\mathfrak{g}}(s) := \sum_{\varphi} \dim(\varphi)^{-s},$$

where φ runs over the finite-dimensional irreducible representations, up to isomorphism, of the Lie algebra \mathfrak{g} .

A result from Larsen and Lubotzky in [LL08] establish that the abscissa of convergence of $\zeta_{\mathfrak{g}}(s)$ is $\frac{r}{|\Delta_+(\mathfrak{g})|}$, where r is the rank of \mathfrak{g} and $\Delta_+(\mathfrak{g})$ is the set of positive roots of \mathfrak{g} . Let's also note that this result has been generalized to a broader class of zeta functions by Häsä and Stasinski in [HS19]. We will see in Chapter 6 that Witten zeta functions corresponds to zeta functions associated with a product of linear forms thanks to Weyl's formula. In particular, all Witten zeta functions verify the $H0S$ condition from Essouabri's article [Ess97], and so they all have a meromorphic continuation. Moreover, those zeta functions are regular at nonpositive integers thanks to a result from Komori [Kom10, Theorem 3.22].

We can give another description of the Witten zeta functions via the Lie groups. We can associate to any compact connected semi-simple Lie group G the Witten zeta function

$$\zeta_G(s) = \sum_{\varphi} \frac{1}{\dim(\varphi)^s},$$

where φ runs over all the finite-dimensional irreducible representations of G , up to isomorphism. By our assumptions the Lie group G is necessarily real, because any compact connected complex Lie group is abelian (meaning a torus), and therefore cannot be semi-simple (see [FH91, Exercise 8.44]). By noting $\text{Lie}(G)$ the real Lie algebra associated with the Lie group G , we consider its complexification $\mathfrak{g} = \mathbb{C} \otimes \text{Lie}(G)$. Assuming that G is simply connected, then there is a correspondence between the representation of G and those of its Lie algebra $\text{Lie}(G)$, because we can canonically associate to any representation $\rho : G \mapsto GL(E)$ its differential $d\rho : \text{Lie}(G) \rightarrow \mathfrak{gl}(E)$ (see [FH91, §8.1]). Under this assumption, we then get that $\zeta_G(s) = \zeta_{\text{Lie}(G)}(s) = \zeta_{\mathfrak{g}}(s)$.

If we now take a finite group G , we can still define a Witten zeta function attached to G in the same fashion as we previously did, and it corresponds to a finite sum taken over all the irreducible characters of G . Since the irreducible characters form a basis for the vector space of central functions, we see that

$$\zeta_G(-2) = \sum_{\chi} \dim(\chi)^2 = |G|,$$

for more details we refer to [FH91, §2.2]. By that fact, and by studying the two Witten zeta functions $\zeta_{\mathfrak{sl}(2)}(s)$ and $\zeta_{\mathfrak{sl}(3)}(s)$ at $s = -2$, Kurokawa and Ochiai predicted the following fact.

Conjecture 1.1.18 (Kurokawa and Ochiai [KO13]). *For all semi-simple Lie algebra \mathfrak{g} , we have*

$$\zeta_{\mathfrak{g}}(-2) = 0.$$

We only know this conjecture to be true in a few cases. This conjecture is true for the Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$, and we will prove that the conjecture is also true for the Lie algebras $\mathfrak{so}(5)$ and \mathfrak{g}_2 .

Definition 1.1.19. *Let \mathfrak{g} be a semi-simple Lie algebra. We set $r_{\mathfrak{g}}(n)$ as the number of representations of \mathfrak{g} of dimension n , up to isomorphism, and $f_{\mathfrak{g}}(n)$ the number of irreducible representations of \mathfrak{g} of dimension n , up to isomorphism.*

By rewriting the Dirichlet sum of a Witten zeta function, we get that

$$\zeta_{\mathfrak{g}}(s) = \sum_{n=1}^{+\infty} \frac{f_{\mathfrak{g}}(n)}{n^s}.$$

It is well-known that a representation of a semi-simple Lie algebra uniquely decomposes into a direct sum of irreducible representations. Thus, for a representation ρ of \mathfrak{g} , we have that there exist integers k_1, \dots, k_m and irreducible representations $\varphi_1, \dots, \varphi_m$ such that

$$\dim(\rho) = \sum_{i=1}^m k_i \dim(\varphi_i).$$

Using the correspondence between the representations of a semi-simple Lie algebra \mathfrak{g} and its corresponding root system we find that

$$1 + \sum_{n=1}^{+\infty} r_{\mathfrak{g}}(n)q^n = \prod_{q=1}^{+\infty} \frac{1}{(1 - q^n)^{f_{\mathfrak{g}}(n)}}. \quad (1.2)$$

Therefore the number of representations of dimension n of the Lie algebra \mathfrak{g} corresponds to a partition number. Using root systems and Weyl formula we get a better expression of the number of representations of dimension n of the Lie algebras $\mathfrak{sl}(3)$, $\mathfrak{so}(5)$ and \mathfrak{g}_2 (see its definition in [Bou81]).

Example 1.1.20 ([Hum72, §24.3]). *Let $\mathfrak{sl}(n) = \{X \in \mathcal{M}_n(\mathbb{C}) \mid \text{Tr}(X) = 0\}$, and $\mathfrak{so}(n) = \{X \in \mathcal{M}_n(\mathbb{C}) \mid {}^t X + X = 0\}$, where Tr denotes the trace, and ${}^t X$ denotes the transpose of a matrix. Thanks to the theory of root system, we have the following results:*

- 1) *The set of isomorphism class of irreducible representations of $\mathfrak{sl}(3)$ is of the form $(\varphi_{i,j})_{i,j \geq 1}$, with*

$$\dim(\varphi_{i,j}) = \frac{ij(i+j)}{2}.$$

Then we get the following expression for the number of irreducible representations of dimension n of $\mathfrak{sl}(3)$:

$$f_{\mathfrak{sl}(3)}(n) = \left| \left\{ (i, j) \in \mathbb{N}^2 \mid \frac{ij(i+j)}{2} = n \right\} \right|,$$

and we get the following expression of the number of representations $r_{\mathfrak{sl}(3)}(n)$ of $\mathfrak{sl}(3)$ is

$$\begin{aligned} r_{\mathfrak{sl}(3)}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\varphi_{i,j}) = n \right\} \right|, \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)}{2} = n \right\} \right|. \end{aligned}$$

2) The set of isomorphism class of irreducible representations of $\mathfrak{so}(5)$ is of the form $(\varphi_{i,j})_{i,j \geq 1}$, with

$$\dim(\varphi_{i,j}) = \frac{ij(i+j)(i+2j)}{3!}.$$

Then we get the following expression for the number of irreducible representations of dimension n of $\mathfrak{so}(5)$:

$$f_{\mathfrak{so}(5)}(n) = \left| \left\{ (i,j) \in \mathbb{N}^2 \mid \frac{ij(i+j)(i+2j)}{3!} = n \right\} \right|,$$

and we get the following expression of the number of representations $r_{\mathfrak{so}(5)}(n)$ of $\mathfrak{so}(5)$:

$$\begin{aligned} r_{\mathfrak{so}(5)}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\varphi_{i,j}) = n \right\} \right| \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)}{3!} = n \right\} \right|. \end{aligned}$$

3) The set of isomorphism class of irreducible representations of \mathfrak{g}_2 is of the form $(\varphi_{i,j})_{i,j \geq 1}$, with

$$\dim(\varphi_{i,j}) = \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!}.$$

Then we get the following expression for the number of irreducible representations of dimension n of \mathfrak{g}_2 :

$$f_{\mathfrak{g}_2}(n) = \left| \left\{ (i,j) \in \mathbb{N}^2 \mid \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|,$$

and we get the following expression of the number of representations $r_{\mathfrak{g}_2}(n)$ of \mathfrak{g}_2 :

$$\begin{aligned} r_{\mathfrak{g}_2}(n) &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \dim(\varphi_{i,j}) = n \right\} \right| \\ &= \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|. \end{aligned}$$

Using the expressions for the number of representations of the previous Lie algebras we can give a more precise description of the corresponding Witten zeta functions.

Example 1.1.21 ([Zag94], [KMT23]). *Via Weyl's formula we find*

$$\begin{aligned} \zeta_{\mathfrak{sl}(2)}(s) &= \zeta(s), \\ \zeta_{\mathfrak{sl}(3)}(s) &= 2^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s}, \\ \zeta_{\mathfrak{so}(5)}(s) &= 6^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s}, \\ \zeta_{\mathfrak{g}_2}(s) &= 120^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_2)^s (2n_1 + 3n_2)^s}. \end{aligned}$$

In the literature we sometimes find multivariable variants of these zeta functions in the works of Matsumoto, Komori, Yasushi, and Tsumura [KMT10c], [KMT10a], [KMT10b], [KMT12], [KMT11b], and in the works of Zhao and Zhou [ZZ11]. These multizeta functions are attached to root systems, and the number of variables corresponds to the number of positive roots in the corresponding root system. These authors provide an expression for the set of singularities of these Witten multizeta functions, and also obtained relations between their positive values. In particular these results allow us to obtain candidate poles for the corresponding univariate Witten zeta function.

- From [KMT10d, Theorem 6.2] we get that the zeta function $\zeta_{\mathfrak{so}(5)}(s)$ has poles of the form

$$s = \frac{1}{2}, \quad s = \frac{1-k}{3} \quad (k \in \mathbb{N}_0). \quad (1.3)$$

- From [KMT11a, Theorem 3.1] we get that the zeta function $\zeta_{\mathfrak{g}_2}(s)$ has poles of the form

$$s = \frac{1}{3}, \quad s = \frac{1-k}{5} \quad (k \in \mathbb{N}_0). \quad (1.4)$$

For the Witten zeta function $\zeta_{\mathfrak{sl}(3)}(s)$, we already know several results about its singularities, its values and derivative values because this zeta function relates to another zeta function deeply studied in recent and old literature. Indeed, the function $\zeta_{\mathfrak{sl}(3)}(s)$ corresponds to a classical zeta function studied independently by Tornheim and Mordell, and later on generalized by Matsumoto.

Definition 1.1.22 ([Mat03]). *Let $P \geq 1$, we call multiple zeta function of Mordell-Tornheim of rank P the meromorphic continuation on \mathbb{C}^{P+1} in the variables $\mathbf{s} = (s_1, \dots, s_{P+1})$ of the following Dirichlet series*

$$\zeta_P^{MT}(\mathbf{s}, \mathbf{d}) := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{(n_1 + d_1)^{s_1} \dots (n_P + d_P)^{s_P} (n_1 + \dots + n_P + |\mathbf{d}|)^{s_{P+1}}} \quad (\operatorname{Re}(d_1), \dots, \operatorname{Re}(d_P) > 0).$$

The special case $P = 2$ and $d_1 = d_2 = 1$ at $\mathbf{s} = (s, s, s)$ corresponds to the Tornheim zeta function. Up to 2^s , it also corresponds to the Witten zeta function $\zeta_{\mathfrak{sl}(3)}$

$$\zeta_{\mathfrak{sl}(3)}(s) = 2^s \zeta_2^{MT}((s, s, s), (1, 1)).$$

In [BD18], using Crandall's expansion, Borwein and Dilcher studied the values of the multiple zeta function $\zeta_2^{MT}((s_1, s_2, s_3), (1, 1))$ and obtained explicit formulas and approximate for such function. In particular they obtained

$$\zeta_{\mathfrak{sl}(3)}(0) = \frac{1}{3}, \quad \zeta'_{\mathfrak{sl}(3)}(0) = \frac{\log(2)}{3} + \log(2\pi).$$

A recent work from Romik [Rom17] used that last result in order to get an asymptotic formula for the number of representations of dimension n of $\mathfrak{sl}(3)$ in terms of the values $\zeta_{\mathfrak{sl}(3)}(0)$ and $\zeta'_{\mathfrak{sl}(3)}(0)$, and in terms of the residues of $\zeta_{\mathfrak{sl}(3)}(s)$ at the poles $s = 2/3$ and $s = 1/2$. Let's also note that Kurokawa and Ochiai [KO13] studied the values at nonpositive integers of $\zeta_{\mathfrak{sl}(3)}(s)$, and then obtained some results on the values of p -adic variant of the Witten zeta function $\zeta_{\mathfrak{sl}(3)}(s)$.

Later on Onodera studied the general case in [Ono21] along the diagonal (meaning that he studied the zeta function $\zeta_P^{MT}((s, \dots, s), \mathbf{d})$ with $P \geq 1$). Via a Crandall's expansion, which is the same strategy used by Borwein and Dilcher, he then obtained explicit formulas for the values and derivative values of order 1 and 2 at nonpositive integers. This strategy consists of partitioning the integration domain of an integral representation of the zeta function whose values one is looking to study, with the domain depending on a free variable. More precisely, Onodera obtained the following expression:

$$\begin{aligned} \zeta_P^{MT}((s, \dots, s), \mathbf{d}) &= \frac{1}{\Gamma(s)} \int_0^\theta \prod_{p=1}^P e^{-d_p y} \phi(e^{-y}, s, d_p) dy \\ &+ \frac{1}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{(n_1 + d_1)^s \dots (n_P + d_P)^s (n_1 + d_1 + \dots + n_P + d_P)} \int_{\theta(n_1 + d_1 + \dots + n_P + d_P)}^{+\infty} e^{-y} y^{s-1} dy \end{aligned}$$

with ϕ being the Lerch zeta function described in Definition 1.1.7, and where θ is a free positive real number. The second term on the right-hand side of the previous equation is holomorphic on \mathbb{C} in the variable s , and vanishes at nonpositive integers. The first term can be re-expressed via the Erdélyi formula (2.3), and Onodera ultimately obtains the following formulas:

Theorem 1.1.23 ([Ono21, Theorem 3, Theorem 4]). *Let $\mathbf{d} \in \mathbb{R}_*^{+P}$ and $t \in \llbracket 1, P-1 \rrbracket$ such that $t < |\mathbf{d}|$. For all nonnegative integers $N \in \mathbb{N}_0$, we have that*

$$\begin{aligned} \zeta_P^{MT}(-N, \dots, N), \mathbf{d} &= \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \frac{(-1)^{(N+1)(|\mathcal{P}|)} N^{|\mathcal{P}|+1}}{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=(|\mathcal{P}|+1)N+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!} \\ (\zeta_P^{MT})'(-N, \mathbf{d}) &= \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}^c|(N+1)-1} N^{|\mathcal{P}^c|} \left(\sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ u \geq 0 \\ |\mathbf{k}|+u=(|\mathcal{P}^c|+1)N+|\mathcal{P}^c|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!} \right) \\ &\quad \cdot \left(\frac{(-1)^N \zeta'(-N - u, |\mathbf{d}| - t)}{u!} + \frac{1}{|\mathcal{P}^c|} \sum_{p \in \mathcal{P}^c} \frac{(-1)^N \zeta'(-N - u, d_p)}{u!} \right), \end{aligned}$$

where we noted $(\zeta_P^{MT})'(s, \mathbf{d}) := \partial_s \zeta_P^{MT}(s, \dots, s), \mathbf{d}$.

In this thesis, we will aim to generalize the strategy used by Onodera in order to obtain explicit formulas for the values and derivative values at nonpositive integers of a broader class of zeta functions. This will specifically allow us to obtain values and derivative values at nonpositive integers for $\zeta_{\mathfrak{so}(5)}(s)$ and $\zeta_{\mathfrak{g}_2}(s)$. We will show via Proposition 2.2.4 that nonpositive integers are not poles for $\zeta_{\mathfrak{so}(5)}(s)$ and for $\zeta_{\mathfrak{g}_2}(s)$. The study of such zeta functions will allow us to prove asymptotic formulas for the number of representations of dimension n of the Lie algebras $\mathfrak{so}(5)$ and \mathfrak{g}_2 via a Meinardus type theorem, thus completing some recent work from Bridges, Bringmann, Brindle and Franke.

1.1.2 Some results on special values of multizeta functions

We will provide some classical results on the special values of certain multizeta functions related to products of linear forms. In particular, we will give expressions of these values for Euler-Zagier multizeta functions:

Definition 1.1.24. *Let $P \in \mathbb{N}$, and $\mathbf{c} = (c_1, \dots, c_P) \in H_0^P$, $\mathbf{d} = (d_1, \dots, d_P) \in \mathbb{C}^P$ be complex numbers such that $\operatorname{Re}(d_i) > -\operatorname{Re}(c_1)$ for all $p \in \llbracket 1, P \rrbracket$. The generalized Euler-Zagier multizeta function is defined as*

$$\zeta_P^{EZ}(\mathbf{s}, \mathbf{c}, \mathbf{d}) := \sum_{\substack{n_1 \geq 1 \\ n_2, \dots, n_P \geq 0}} \prod_{p=1}^P (c_p n_p + d_p)^{-s_p}.$$

We denote $\zeta_P^{EZ}(\mathbf{s}) := \zeta_P^{EZ}(\mathbf{s}, \mathbf{1}, (0, 1, \dots, P-1))$ the Euler-Zagier multizeta function.

Note that, through a change of variables, we can rewrite the "classical" Euler-Zagier multizeta function in the following form,

$$\zeta_P^{EZ}(\mathbf{s}) = \sum_{0 < n_1 < \dots < n_P} \frac{1}{n_1^{s_1} \dots n_P^{s_P}}.$$

Proposition 1.1.25 ([EM20]). *The generalized Euler-Zagier multizeta function $\mathbf{s} \mapsto \zeta_P^{EZ}(\mathbf{s}, \mathbf{c}, \mathbf{d})$ is holomorphic on $\{(s_1, \dots, s_P) \in \mathbb{C}^P \mid \sigma_k + \dots + \sigma_P > P + 1 - k, 1 \leq k \leq P\}$, and is meromorphic on \mathbb{C}^P , with singularities located in hyperplanes of the form*

$$s_p + \dots + s_P = (P + 1 - p) - k_p \quad (1 \leq p \leq P, k_1, \dots, k_P \in \mathbb{N}_0).$$

From the previous proposition, we see that the special value $\zeta_P^{EZ}(-\mathbf{N}, \mathbf{c}, \mathbf{d})$ does not always make sense. Moreover, a result from [AET01] specifies that most of the nonpositive integers are actually

singularities for $\zeta_P^{EZ}(\mathbf{s})$. In order to study the values of this function, Akiyama, Egami, and Tanigawa investigated the "regular" values of this multizeta function by setting

$$\zeta_P^{EZ}(-\mathbf{N})^{reg} := \lim_{s_1 \rightarrow -N_1} \dots \lim_{s_P \rightarrow -N_P} \zeta_P^{EZ}(\mathbf{s}).$$

The authors also introduced "non-regular" values by reversing the previous limits, defining

$$\zeta_P^{EZ}(-\mathbf{N})^{non-reg} := \lim_{s_P \rightarrow -N_P} \dots \lim_{s_1 \rightarrow -N_1} \zeta_P^{EZ}(\mathbf{s}).$$

In [AET01] and in [AT01], Akiyama, Egami and Tanigawa obtained recursive formulas between the regular values $\zeta_P^{EZ}(-\mathbf{N})^{reg}$ (respectively the non regular values $\zeta_P^{EZ}(-\mathbf{N})^{non-reg}$) with $\mathbf{N} \in \mathbb{N}_0^P$, and the regular values of the form $\zeta_{P-1}^{EZ}(-\mathbf{N}')^{reg}$ (respectively the non regular values $\zeta_{P-1}^{EZ}(-\mathbf{N}')^{non-reg}$) with $\mathbf{N}' \in \mathbb{N}_0^{P-1}$. Thus they obtained some explicit expressions for the regular and non-regular values at nonpositive integers. These results were later generalized by Sasaki in [Sas09a] and in [Sas09b] to different orders of limits than those of regular and non-regular values.

We can also study the "directional" values of this multizeta function. In [Kom10], Komori introduces the notion of directional value for such multizeta functions and provides an expression for these in the case of Hurwitz-Lerch multizeta functions. Komori then deduces formulas for the values at nonpositive integers of the Shintani and Euler-Zagier zeta functions, using generalized Bernoulli numbers (often not explicit).

Definition 1.1.26. *Let $P, Q \in \mathbb{N}$ be integers, $\mathbf{d} = (d_1, \dots, d_Q)$, $\mathbf{c} = (c_{q,p})_{(q,p) \in \llbracket 1, Q \rrbracket \times \llbracket 1, P \rrbracket}$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_P)$ be complex numbers such that $c_{q,p} \in \overline{H}_0$, $d_q \in H_0$ and $\xi_q \in \mathbb{T} = \mathbb{C}/2i\pi\mathbb{Z}$. We call the function ζ^{HL} the Hurwitz-Lerch multiple zeta function, defined as*

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s}) := \sum_{n_1=0}^{+\infty} \dots \sum_{n_P=0}^{+\infty} \frac{e^{\xi_1 n_1} \dots e^{\xi_P n_P}}{(c_{1,1} n_1 + \dots + c_{1,P} n_P + d_1)^{s_1} \dots (c_{Q,1} n_1 + \dots + c_{Q,P} n_P + d_Q)^{s_Q}}.$$

We also define the directional values of ζ^{HL} at $-\mathbf{N} \in -\mathbb{N}_0^Q$ along a direction $\boldsymbol{\mu} \in \mathbb{C}^P$:

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} := \lim_{s \rightarrow 0} \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N} + s\boldsymbol{\mu}).$$

Komori shows that this multiple zeta function is holomorphic on $\{\mathbf{s} \in \mathbb{C} \mid \forall q \in \llbracket 1, Q \rrbracket, \operatorname{Re}(s_q) > \sigma_{0,q}\}$, where $\sigma_{0,q} > 0$ for all $q \in \llbracket 1, Q \rrbracket$. He also proves that this multiple zeta function has a meromorphic continuation on \mathbb{C}^Q in the variables \mathbf{s} , and that the previously defined directional values make sense. Before briefly summing up his results, we must assume the following conditions on the data of the multiple zeta function ζ^{HL} .

Let's assume that there exist an integer $P_0 \in \llbracket 1, P \rrbracket$ and a set $\mathcal{B} \subset \llbracket 1, Q \rrbracket \times \llbracket 1, P_0 \rrbracket$ such that

$$\begin{cases} c_{q,p} = 0 \text{ if and only if } (q,p) \in \mathcal{B}^c \\ \xi_p = 0 \text{ if and only if } p \in \llbracket 1, P_0 \rrbracket, \end{cases}$$

with $\mathcal{B}^c = \llbracket 1, Q \rrbracket \times \llbracket 1, P_0 \rrbracket \setminus \mathcal{B}$. Let $\mathcal{B}_{\bullet p} := \{q \mid (q,p) \in \mathcal{B}\}$. We assume that $\mathcal{B}_{\bullet p} \neq \emptyset$ for all $p \in \llbracket 1, P_0 \rrbracket$. Komori then obtains the following analytic properties for this multiple zeta function.

Theorem 1.1.27 ([Kom10, Corollary 3.16]). *Assuming all the conditions above, the function $\mathbf{s} \mapsto \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s})$ has a meromorphic continuation on \mathbb{C}^P . Moreover, the singularities lies on the hyperplanes*

$$\begin{cases} \sum_{q \in \mathcal{Q}} s_q = Q - k & (\mathcal{Q} \subset \llbracket 1, Q \rrbracket, |\mathcal{Q}| \geq 2, k \in \mathbb{N}_0) \\ s_q = 1, \dots, P & (1 \leq q \leq Q). \end{cases}$$

More precisely Komori shows the analytic continuation by proving a surface integral representation (meaning an integral representation over a surface) of the multiple zeta function $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s})$. First let's set

$$G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}, \mathbf{z}) := \frac{e^{(c_{1,1}+\dots+c_{1,P-d_1})y_{w,1}(\mathbf{z})} \dots e^{(c_{Q,1}+\dots+c_{Q,P-d_Q})y_{w,Q}(\mathbf{z})} z_1^{t_1-1} \dots z_Q^{t_Q-1}}{(e^{y_{w,1}(\mathbf{z})c_{1,1}+\dots+y_{w,Q}(\mathbf{z})c_{Q,1}} - e^{\xi_1}) \dots (e^{y_{w,1}(\mathbf{z})c_{1,P}+\dots+y_{w,Q}(\mathbf{z})c_{Q,P}} - e^{\xi_P})},$$

where $y_{w,q}(\mathbf{z})$ corresponds to a certain explicit product of the complex numbers $z_1, \dots, z_Q \in \mathbb{C}$, and where $t_{w,q} \in \mathbb{C}$.

Theorem 1.1.28 ([Kom10, Theorem 3.14]). *Assuming the same conditions as in the previous theorem, there exist piecewise smooth surfaces $\Sigma_w \subset (\mathbb{C}^*)^Q$ such that*

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{s}) = \frac{1}{\Gamma(s_1) \dots \Gamma(s_Q)} \sum_{w \in S_Q} \frac{1}{(e^{2i\pi t_{w,1}(s)} - 1) \dots (e^{2i\pi t_{w,Q}(s)} - 1)} \cdot \int_{\Sigma_w} G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}_w(s), \mathbf{z}) dz_1 \wedge \dots \wedge dz_P$$

where $t_{w,q}(s) := \sum_{p \in \mathcal{P}_{w,q}} s_p$ is a linear form with $\mathcal{P}_{w,q} \subset [1, Q]$ for all $w \in S_Q \subset \mathfrak{S}_Q$ and all $1 \leq q \leq Q$.

Note that the surfaces Σ_w as defined in [Kom10, Formula (41)] avoid all the singularities of the corresponding function $\mathbf{z} \mapsto G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}, \mathbf{z})$. Komori even shows in [Kom10, Proposition 6.7] that the function

$$I_w : (\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}) \mapsto \int_{\Sigma_w} G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}, \mathbf{z}) dz_1 \wedge \dots \wedge dz_P \quad (1.5)$$

is holomorphic in the variables \mathbf{t} on \mathbb{C}^Q . Komori then finds that the function

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N} + s\boldsymbol{\mu}) = \sum_{w \in S_Q} \frac{1}{\Gamma(-N_1 + \mu_1 s) \dots \Gamma(-N_Q + \mu_Q s) (e^{2i\pi s t_{w,1}(\boldsymbol{\mu})} - 1) \dots (e^{2i\pi s t_{w,Q}(\boldsymbol{\mu})} - 1)} \cdot \int_{\Sigma_w} G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}_w(-\mathbf{N} + \boldsymbol{\mu}s), \mathbf{z}) dz_1 \wedge \dots \wedge dz_P \quad (1.6)$$

is regular at $s = 0$ for all directions except for those who belong in a union of hyperplanes described by the relations $t_{w,q}(\boldsymbol{\mu}) = 0$. Therefore the directional values make sense for all the directions that are located outside of those hyperplanes.

Also note that this surface integral representation generalizes the usual integral representation of Riemann's zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)(e^{2i\pi s} - 1)} \int_C \frac{z^{s-1}}{e^z - 1} dz,$$

where C is Hankel's contour (see [EMOT81, §1.10]). It is good to recall that this integral representation of $\zeta(s)$ yields well known relations between Bernoulli numbers and the values of the Riemann zeta function at nonpositive integers. On the same trend, Komori used his surface integral representation in [Kom10, Theorem 3.22] in order to get formulas of the directional values at nonpositive integers in terms of generalized Bernoulli numbers $B_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{k}_w)$ with $\mathbf{k}_w \in \mathbb{N}_0^Q$ defined by the generating functions $G_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t}, \mathbf{z})$ (see [Kom10, Définition 3.20]). Note that the existence of such directional values assume a non-vanishing condition on the sum of the components of a given direction.

Theorem 1.1.29 ([Kom10, Theorem 3.22]). *For all direction $\boldsymbol{\mu} \in \mathbb{C}^Q$ such that $t_{w,1}(\boldsymbol{\mu}), \dots, t_{w,Q}(\boldsymbol{\mu}) \neq 0$ for any $w \in S_Q$, we have*

$$\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} = (-1)^{|\mathbf{N}|} N_1! \dots N_Q! \sum_{w \in S_Q} \frac{B_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{k}_w)}{k_{w,1}! \dots k_{w,Q}!} \prod_{q=1}^Q \frac{\mu_q}{t_{w,q}(\boldsymbol{\mu})}$$

with $\mathbf{k}_w = (k_{w,1}, \dots, k_{w,Q}) \in \mathbb{N}_0^Q$ being explicit integers depending on $\mathbf{N} = (N_1, \dots, N_Q) \in \mathbb{N}_0^Q$.

Incidentally, it turns out that if one manages to re-express the generalized Bernoulli numbers in the formula obtained by Komori, then this formula can be used in order to determine whether a nonpositive tuple is a singularity for the function ζ^{HL} . Indeed, if one obtains two distinct values $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N}) \neq \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}}$ along two different directions $\boldsymbol{\mu} \neq \boldsymbol{\mu}'$, then the nonpositive Q -tuple \mathbf{N} is a singularity.

Remark 1.1.30. *The directional value formula obtained by Komori is not very explicit, as it is often difficult to get at a satisfactory expression for the generalized Bernoulli numbers he introduces in his article.*

Komori didn't considered derivative values along a direction of the Hurwitz Lerch multiple zeta functions defined by

$$(\zeta^{HL})'(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N}) := \partial_s (\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N} + s\boldsymbol{\mu}))_{|s=0}.$$

From Komori's surface integral representation in Theorem 1.1.28, we won't be able to provide a nice formula for those values. Yet we still can state an analytical property of those values which will turn out useful in our later work.

Corollary 1.1.31. *For all Q -tuple $\mathbf{N} \in \mathbb{N}_0^Q$, the two functions*

$$\begin{aligned} \boldsymbol{\mu} &\mapsto \zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} \\ \boldsymbol{\mu} &\mapsto (\zeta^{HL})'(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} \end{aligned}$$

are meromorphic on \mathbb{C}^Q with singularities located in the hyperplanes $t_{w,q}(\boldsymbol{\mu}) = 0$ with $w \in S_Q$ and $q \in \llbracket 1, Q \rrbracket$.

Proof. From Theorem 1.1.29 we get that the directional value $\zeta^{HL}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}}$ is a rational fraction in the variables μ_1, \dots, μ_Q , with denominators of the form $t_{w,q}(\boldsymbol{\mu})$. The first function of the corollary is then clearly meromorphic on \mathbb{C}^Q .

For all $w \in S_Q$ we set

$$F_w(\mathbf{N}, \boldsymbol{\mu}) := \partial_s \left(\frac{1}{\Gamma(-N_1 + \mu_1 s) \dots \Gamma(-N_Q + \mu_Q s)} \frac{1}{(e^{2i\pi s t_{w,1}(\boldsymbol{\mu})} - 1) \dots (e^{2i\pi s t_{w,Q}(\boldsymbol{\mu})} - 1)} \right)_{|s=0},$$

and by easy computations we observe that F_w is a rational fraction in the variables μ_1, \dots, μ_Q , with the denominators being products of some of the linear forms $t_{w,q}(\boldsymbol{\mu})$ with $q \in \llbracket 1, Q \rrbracket$. By evaluating at $s = 0$ the derivative of the expression (1.6) we then get

$$\begin{aligned} (\zeta^{HL})'(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N})_{\boldsymbol{\mu}} &= \sum_{w \in S_Q} F_w(\mathbf{N}, \boldsymbol{\mu}) \frac{B_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{k}_w)}{k_{w,1}! \dots k_{w,Q}!} \\ &\quad + (-1)^{|\mathbf{N}|} N_1! \dots N_Q! \sum_{w \in S_Q} \sum_{j=1}^Q \frac{\partial I_w}{\partial t_j}(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, -\mathbf{N}) \prod_{\substack{q=1 \\ q \neq j}}^Q \frac{\mu_q}{t_{w,q}(\boldsymbol{\mu})} \end{aligned}$$

where $\mathbf{k}_w \in \mathbb{N}_0^Q$ are explicit integers (see [Kom10, Theorem 3.22]) for all $w \in S_Q$, and where $\frac{\partial I_w}{\partial t_j}$ is the first derivative with respect to the variable t_j of the function $I_w(\boldsymbol{\xi}, \mathbf{d}, \mathbf{c}, \mathbf{t})$ defined in the expression (1.5). We then find that the second function of the corollary is meromorphic on \mathbb{C}^Q . \square

In [EM20], Essouabri and Matsumoto studied the directional values of the generalized Euler-Zagier multizeta function. Here, we will only provide a very simplified and much less detailed formula:

Theorem 1.1.32 ([EM20, Theorem 1]). *Let's consider complex numbers $\mathbf{c} = (c_1, \dots, c_P) \in H_0^P$ and $\mathbf{d} = (d_1, \dots, d_P) \in \overline{H}_0^P$, a P -tuple $\mathbf{N} \in \mathbb{N}_0^P$, and a direction $\boldsymbol{\mu} \in \mathbb{C}^P$ such that $\operatorname{Re}(\mu_p + \dots + \mu_P) \neq 0$ for all $p \in \llbracket 1, P \rrbracket$. Then the directional value*

$$\zeta_P^{EZ}(-\mathbf{N}, \mathbf{c}, \mathbf{d}) := \lim_{s \rightarrow 0} \zeta_P^{EZ}(-\mathbf{N} + s\boldsymbol{\mu}, \mathbf{c}, \mathbf{d})$$

exists, and we have that:

$$\zeta_P^{EZ}(-\mathbf{N}, \mathbf{c}, \mathbf{d}) = \sum_{\mathbf{k}=(k_1, \dots, k_P) \in \llbracket 0, |\mathbf{N}|+P+1 \rrbracket^P} f_{\mathbf{k}}(\mathbf{c}, \mathbf{d}, \boldsymbol{\mu}) \prod_{i=1}^P B_{k_i},$$

with B_n being the n -th Bernoulli number, and $f_{\mathbf{k}}(\mathbf{c}, \mathbf{d}, \boldsymbol{\mu})$ an explicit rational fraction in the variables c_p and μ_p for $p \in \llbracket 1, P \rrbracket$.

Remark 1.1.33. 1) In [EM21], Essouabri and Matsumoto generalized this last result for a larger class of multizeta functions, which are of the form

$$\zeta_m(\mathbf{s}, \mathbf{P}) := \sum_{n_1, \dots, n_m \geq 1} \frac{1}{\prod_{j=1}^m P_j(n_1, \dots, n_j)^{s_j}},$$

where P_j is a polynomial in X_1, \dots, X_j such that it respects the HOS condition (see [Ess97]), where P_m is homogeneous and elliptic and such that

$$\forall 1 \leq j \leq m, \lim_{x_1 + \dots + x_j \rightarrow +\infty} P_j(x_1, \dots, x_j) = +\infty.$$

2) In [MW02], Matsumoto and Weng studied zeta functions of the form

$$\zeta^{MW}(s) := \sum_{n=1}^{+\infty} \frac{P(n)}{Q(n)^s},$$

with P and Q two nonzero polynomials such that Q doesn't vanishes at positive integers. They provided an explicit formula for its derivative value at $s = 0$. This formula is of particular interest when studying the regularized determinant of the Laplacian of the n -sphere (see [Var88], [WY96], [QC96], [Kum99]).

1.1.3 On asymptotic formulas for some partitions

We will explain in this subsection some arithmetic applications of computing special values of Witten's zeta functions, and their derivatives. We will also discuss Meinardus' theorem on the asymptotic behavior of some partition numbers generated by infinite product. We will also discuss some recent work from Bridges, Bringmann, Craig, Franke, Males on asymptotic formulas for more general partition numbers in the articles [BB24], [BBF24], [BCM23]. We will also state the results obtained by Bridges, Bringmann, Brindle and Franke in [BBBF24], and we will talk about their approach.

In combinatorics, it is standard to use a generating series

$$\omega(q) := \sum_{n \geq 0} r(n)q^n$$

in order to study the behavior of a sequence of complex numbers $r(n)$. If the function ω is holomorphic for $|q| < 1$, then by Cauchy's theorem we have

$$r(n) = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{\omega(q)}{q^{n+1}} dq,$$

where \mathcal{C} is a circle centered at zero running counterclockwise, and included in the unit disk. In the general case, no result exists to obtain an asymptotic formula for all sequence $r(n)$. In the rest of this section,

we will focus on sequences obtained by considering a generating series that corresponds to an infinite product.

We consider a function $G_f(q)$ defined by

$$G_f(q) := \prod_{n=1}^{+\infty} \frac{1}{(1-q^n)^{f(n)}},$$

where $f(n) \in \mathbb{N}_0$ are nonnegative integers. By distributing the product above, one can consider a sequence $p_f(n)$ by the relation

$$G_f(q) =: \sum_{n=0}^{+\infty} p_f(n)q^n. \quad (1.7)$$

We consider the Dirichlet series

$$L_f(s) := \sum_{n=1}^{+\infty} \frac{f(n)}{n^s} \quad (s = \sigma + i\tau, \sigma > \sigma_0),$$

with σ_0 its abscissa of convergence, when it exists.

Example 1.1.34 ([HR17]). *Let's consider the number of partitions $p(n)$ of a positive integer n . A classical way of defining such a sequence is by considering an infinite product*

$$1 + \sum_{n=1}^{+\infty} p(n)q^n := \prod_{n=1}^{+\infty} \frac{1}{1-q^n}.$$

The corresponding Dirichlet series is Riemann's zeta function. For such sequence, Hardy and Ramanujan showed the following asymptotic formula

$$p(n) \underset{n \rightarrow +\infty}{\sim} \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

One can check for the proof in Andrew's book [And76, 5.1.2].

We now set

$$g_f(z) := \sum_{n=1}^{+\infty} f(n)e^{-nz}.$$

We assume the following conditions:

- i) The abscissa of convergence σ_0 of L_f is positive.
- ii) The function L_f has a meromorphic continuation to H_{-C_0} with $0 < C_0 < 1$, and such that L_f has only one pole at $s = \sigma_0$ in this domain, and that this pole is of order 1. We note A the residue at that pole.
- iii) We have the following upper bound, which is uniform on $\sigma > -C_0$, as $\tau \rightarrow +\infty$

$$|L_f(s)| = O(|\tau|^{C_1}) \quad (s = \sigma + i\tau)$$

with $C_1 > 0$.

- iv) Let $\epsilon > 0$. For all $z = x + 2i\pi y$ such that $\arg(z) > \frac{\pi}{4}$, $|x| \leq \frac{1}{2}$ and $|y| \ll 1$, we have

$$\operatorname{Re}(g_f(z)) - g_f(x) \leq -C_2 x^{-\epsilon},$$

with $C_2 > 0$ a constant depending on ϵ .

Meinardus' theorem then provides an asymptotic relation for the sequence $p_f(n)$.

Theorem 1.1.35 (of Meinardus, [And76, Theorem 6.2]). *As $n \rightarrow +\infty$, we have*

$$p_f(n) = Cn^\kappa \exp\left(n^{\frac{\alpha}{\alpha+1}} \left(1 + \frac{1}{\alpha}\right) (A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha}}\right) (1 + O(n^{-\kappa_1})),$$

where $p_f(n)$ are defined in formula (1.7), and C, κ, κ_1 are real numbers depending on $L_f(0), L'_f(0), C_0$, and α .

The proof of this theorem relies on the saddle point method, key aspects of which will be explained here. For details of the method, we refer to [FS09, VIII.3]. The method takes advantage of the fact that the integral $\frac{1}{2i\pi} \int_{\mathcal{C}} \frac{G_f(q)}{q^{n+1}} dq$ is independent of the radius of the circle \mathcal{C} within the unit disk. This independence can be exploited by judiciously choosing an appropriate radius $\rho_0(n)$. The saddle point method involves selecting this radius near 1 in such a way as to minimize $\left| \frac{G_f(q)}{q^n} \right|$ over a minor arc $\mathcal{C}_1(\theta_0(n))$. Specifically, we can choose such a radius so that the derivative of the integrand at $q = \rho_0(n)$ is $O(n^\delta)$, where δ is well-chosen. Alternatively, we can find $\rho_0(n)$ by studying the derivative of the function

$$h_f(z) := \ln(L_f(z)) - n \log(z).$$

We then look for $\rho_0(n)$ such that

$$h'_f(\rho_0(n)) = -n + O(n^\delta).$$

We will then have a quadratic approximation when $\theta \rightarrow 0$:

$$h_f(\rho_0(n)e^{i\theta}) - h_f(\rho_0(n)) =: -\frac{1}{2}\beta(\rho_0(n))\theta^2 + o(\theta^3) + O(n^\delta)$$

where $\beta(\rho_0(n))$ is a constant depending on $\rho_0(n)$, and on the choice of δ . We will then aim to determine an angle $\theta_0(n)$ such that

$$\beta(\rho_0(n))\theta_0(n)^2 \xrightarrow{\theta \rightarrow 0} +\infty, \quad h''(\rho_0(n))\theta_0(n)^3 \xrightarrow{\theta \rightarrow 0} 0.$$

For such a choice of $\theta_0(n)$, one can get in particular that the behavior of the integral over the minor arc \mathcal{C}_0 behaves like a Gaussian integral, whose bounds depend on n , which can be approximated.

Debruyne and Tennenbaum obtained a more general version of Meinardus' theorem in [DT20] by weakening condition iv) on the growth of the function g_f . Let's also observe that some partition numbers coming from physics have been studied by Granovsky and Stark in [GSE08]. The two authors established a multipolar version of the Meinardus theorem assuming a nice repartition of the sequence $(f(n))_{n \in \mathbb{N}}$ and a growth condition on the analytic function $L_f(s)$.

Thanks to those general theorems one can hope to study more exotic number of partitions such as the number of representations of a semi-simple Lie algebra \mathfrak{g} . Indeed, we saw via formula (1.2) that this number of representations corresponds to a number of partitions generated by an infinite product associated to the sequence $(f_{\mathfrak{g}}(n))_{n \geq 1}$, and then the function $L_{f_{\mathfrak{g}}}(s)$ corresponds to the Witten zeta function $\zeta_{\mathfrak{g}}(s)$. One then realizes that Meinardus' theorem cannot be directly applied to obtain an asymptotic expression for the representation number $r_{\mathfrak{g}}(n)$ of any Lie algebra because the Witten zeta functions have more than one positive pole in general. However, by cleverly modifying the proof of Meinardus' theorem, and then studying the residues of $\zeta_{\mathfrak{sl}(3)}(s)$ at $s = \frac{1}{2}$ and at $s = \frac{2}{3}$, and by calculating the special values $\zeta_{\mathfrak{sl}(3)}(0)$ and $\zeta'_{\mathfrak{sl}(3)}(0)$, Romik obtained the following theorem:

Theorem 1.1.36 ([Rom17, Theorem 1.1]). *As $n \rightarrow +\infty$ we have*

$$r_{\mathfrak{sl}(3)}(n) \underset{n \rightarrow +\infty}{\sim} \frac{K}{n^{3/5}} \exp\left(A_1 n^{2/5} - A_2 n^{3/10} - A_3 n^{1/5} - A_4 n^{1/10}\right),$$

with the constant A_1, A_2, A_3, A_4 and K being explicit. Let

$$X := \left(\frac{1}{9} \Gamma \left(\frac{1}{3} \right)^2 \zeta \left(\frac{5}{3} \right) \right), \quad Y := -\sqrt{\pi} \zeta \left(\frac{1}{2} \right) \zeta \left(\frac{3}{2} \right).$$

The constants A_1, A_2, A_3, A_4, K are such that:

$$\begin{aligned} A_1 &:= 5X^2 = 6.858260476\dots \\ A_2 &:= X^{-1}Y = 5.77360174\dots \\ A_3 &:= \frac{3}{80}X^{-4}Y^2 = 0.91134107\dots \\ A_4 &:= \frac{11}{3200}X^{-7}Y^3 = 0.35163754\dots \\ K &:= \frac{2\sqrt{3\pi}}{\sqrt{5}}X^{1/3} \exp \left(-\frac{1}{2560}X^{-10}Y^4 \right) = 2.44629033486\dots \end{aligned}$$

Note that in [Rom17], the computation of the constant Y depends on the special value $\zeta'_{\mathfrak{sl}(3)}(0) = -\ln(2\pi) + \ln(2)$ computed in [BD18]. Moreover, all these constants also depend on the calculation of the residues of $\zeta_{\mathfrak{sl}(3)}(s)$ at $s = \frac{1}{2}$ and $s = \frac{2}{3}$.

Later on, Bridges, Bringmann and Franke obtained in [BBF24] an asymptotic formula for the number of irreducible representations of dimension n of $\mathfrak{sl}(3)$ by studying the Witten zeta function $\zeta_{\mathfrak{sl}(3)}(s)$, and using a Tauberian type theorem.

Romik conjectured that it is possible to obtain similar results on the representation numbers $r_{\mathfrak{g}}(n)$ for other Lie algebras \mathfrak{g} . To extend this result, one needs to establish a Meinardus-type theorem, and also obtain information about the positive poles and residues of the corresponding Witten zeta function, and then obtain an expression for the derivative value $\zeta'_{\mathfrak{g}}(0)$.

Motivated by this question, Bridges, Brindle, Bringmann, and Franke have established a more general variant of Meinardus' theorem in the article [BBBF24]. Let $f : \mathbb{N} \rightarrow \mathbb{N}_0$ an arithmetic function. We define for all $q = e^{-z}$, $z \in H_0$, the functions

$$G_f(z) := \sum_{n=0}^{+\infty} p_f(n)q^n = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f(n)}}, \quad L_f(s) := \sum_{n=1}^{+\infty} \frac{f(n)}{n^s}.$$

Let $\Lambda := \mathbb{N}_0 \setminus f^{-1}(\{0\})$. We assume that:

(P1) Let $\alpha > 0$ be the biggest pole of L_f . There exists an integer $L \in \mathbb{N}_0$ such that, for all prime numbers p ,

$$|\Lambda \setminus (p\mathbb{N}_0 \cap \Lambda)| \geq L > \frac{\alpha}{2}.$$

(P2) There exists a real number $R \in \mathbb{R}_+$ such that L_f is meromorphic on $\overline{H}_{-R} = \{z \in \mathbb{C}, \operatorname{Re}(z) \geq -R\}$, and such that L_f is holomorphic on the line $(\operatorname{Re}(z) = -R)$. Furthermore, we assume that the meromorphic function $L_f^*(s) := \Gamma(s)\zeta(s+1)L_f(s)$ has only real poles $\alpha := \gamma_1 > \dots$, and that these poles are all simple, except at $s = 0$ where it may be a double pole.

(P3) There exists a real number $a < \frac{\pi}{2}$ such that, on each strip $\sigma_1 \leq \sigma \leq \sigma_2$ in the domain of holomorphy of L_f , we have

$$L_f(s) = O_{\sigma_1, \sigma_2} \left(e^{a|\tau|} \right) \quad (s = \sigma + i\tau),$$

as $|\tau| \rightarrow +\infty$.

Theorem 1.1.37 ([BBBF24, Theorem 1.4]). *We assume the conditions (P1), (P2), and (P3). Let L be the real number from condition (P1) and R be the real number from condition (P2). Then for some $N \in \mathbb{N}_0$,*

$$p_f(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp \left(A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^M A_j n^{\alpha_j} \right) \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R} \left(n^{-\min(\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1})} \right) \right),$$

with $M \in \mathbb{N}$, $0 \leq \alpha_M < \dots < \alpha_1 := \frac{\alpha}{\alpha+1}$, $0 < \nu_2 < \dots < \nu_N$, A_1, \dots, A_M and B_2, \dots, B_N being explicit constants.

When the meromorphic function L_f has only two strictly positive poles, Bridges, Brindle, Bringmann, and Franke obtained a more precise formula. Let us first introduce the following constants: Let $\alpha > \beta > 0$ be the only two positive poles of L_f . We denote $\omega_\alpha := \text{Res}_{s=\alpha}(L_f(s))$ and $\omega_\beta := \text{Res}_{s=\beta}(L_f(s))$ as the residues of L_f at its positive poles, and we set $c_1 := \omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1)$, $c_2 := \omega_\beta \Gamma(\beta+1) \zeta(\beta+1)$, $c_3 := L_f(0)$. Let

$$C := \frac{e^{L_f(0)} (\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{\frac{1}{2}-L_f(0)}{\alpha+1}}}{\sqrt{2\pi(\alpha+1)}}, \quad b := \frac{1-L_f(0) + \frac{\alpha}{2}}{\alpha+1}.$$

Let \mathcal{P}_R be the union of $\{0\}$ and of the poles greater than $-R$ of L_f^* . We set

$$\mathcal{L} := \frac{1}{\alpha+1} \mathcal{P}_R + \sum_{\mu \in \mathcal{P}_R} \left(\frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N}_0,$$

$$\mathcal{M} := \frac{\alpha}{\alpha+1} \mathbb{N}_0 + \left(- \sum_{\mu \in \mathcal{P}_R} \left(\frac{\mu+1}{\alpha-1} - 1 \right) \mathbb{N}_0 \right) \cap \left[0, \frac{R+\alpha}{\alpha+1} \right].$$

Theorem 1.1.38 ([BBBF24, Theorem 4.4]). *We assume conditions (P1), (P2), and (P3). Let L be the real number from condition (P1) and R be the real number from condition (P2). Moreover, we assume that L_f has only two poles $\alpha > \beta > 0$ such that there exists an integer $l \in \mathbb{N}$ satisfying the inequality $\frac{l+1}{l} \beta < \alpha < \frac{l}{l-1} \beta$. Then we have*

$$p_f(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp \left(A_1 n^{\frac{\alpha}{\alpha+1}} + A_2 n^{\frac{\beta}{\alpha+1}} + \sum_{j=3}^{l+1} A_j n^{\frac{(j-1)\beta}{\alpha+1} + \frac{j-2}{\alpha+1} + 2-j} \right) \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R} \left(n^{-\min(\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1})} \right) \right),$$

with B_2, \dots, B_N being explicit constants, and

$$A_1 := (\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{1}{\alpha+1}} \left(1 + \frac{1}{\alpha} \right), \quad A_2 := \frac{\omega_\beta \Gamma(\beta) \zeta(\beta+1)}{(\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{\beta}{\alpha+1}}},$$

$$A_j := K_j + \frac{c_1^{\frac{1}{\alpha+1}}}{\alpha} \sum_{m=1}^l \binom{-\alpha}{m} \sum_{\substack{0 \leq k_1 \leq \dots \leq k_l \leq m \\ |\mathbf{k}|=m \\ k_1+2k_2+\dots+l k_l=j-1}} \binom{m}{\mathbf{k}} \frac{K_2^{k_1} \dots K_{l+1}^{k_l}}{c_1^{\frac{m}{\alpha+1}}}$$

$$+ \frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}} \sum_{m=1}^l \binom{-\beta}{m} \sum_{\substack{0 \leq k_1 \leq \dots \leq k_l \leq m \\ |\mathbf{k}|=m \\ k_1+2k_2+\dots+l k_l=j-2}} \binom{m}{\mathbf{k}} \frac{K_2^{k_1} \dots K_{l+1}^{k_l}}{c_1^{\frac{m}{\alpha+1}}} \quad (j \geq 3),$$

with $(K_j)_{j \geq 3}$ described in [BBBF24, Lemma 4.3], and $0 < \nu_2 < \dots$ running over the positive elements of $\mathcal{L} + \mathcal{M}$.

Remark 1.1.39. For a given semi-simple Lie algebra \mathfrak{g} , the number of representations $r_{\mathfrak{g}}(n)$ can be seen as a partition number associated with an infinite product

$$1 + \sum_{n=1}^{+\infty} r_{\mathfrak{g}}(n)q^n = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f_{\mathfrak{g}}(n)}}.$$

Upon applying the Meinardus type Theorem 1.1.37 to the study of the asymptotic behavior of the sequence $r_{\mathfrak{g}}(n)$, we then see that the integers $f(n)$ correspond to the sequence number of irreducible representations (up to isomorphism) of dimension n of the Lie algebra \mathfrak{g} , noted $f_{\mathfrak{g}}(n)$. Therefore, the function $L_{f_{\mathfrak{g}}}(s)$ corresponds to the Witten zeta function $\zeta_{\mathfrak{g}}(s)$. Since the function $\zeta_{\mathfrak{g}}(s)$ verifies the HOS condition of Essouabri (see [Ess97]), it follows that the function $L_{f_{\mathfrak{g}}}(s)$ has a meromorphic continuation on \mathbb{C} , and that the condition (P3) holds. It remains to check if all the positive poles are simple poles in order to prove that (P2) holds or not.

Remark 1.1.40. To study the asymptotic behavior of $r_{\mathfrak{g}_2}(n)$, we would need to know K_3 , and according to [BBBF24, Lemma 4.3], we have

$$K_3 = \frac{c_2^2(\alpha - 2\beta)}{2(\alpha + 1)^2 c_1^{\frac{2\beta+1}{\alpha+1}}},$$

with c_1 and c_2 being the constants already defined by

$$c_1 = \omega_{\alpha}\Gamma(\alpha + 1)\zeta(\alpha + 1), \quad c_2 = \omega_{\beta}\Gamma(\beta + 1)\zeta(\beta + 1).$$

Moreover, to study the asymptotic behavior of $r_{\mathfrak{g}_2}(n)$, we will see that the integer $l = 2$ works because we would then have $L_{f_{\mathfrak{g}_2}}(s) = \zeta_{\mathfrak{g}_2}(s)$, and the positive poles of the function $L_{f_{\mathfrak{g}_2}}(s) = \zeta_{\mathfrak{g}_2}(s)$ are $s = \frac{1}{3}$ and $s = \frac{1}{5}$.

By applying this theorem, the four authors obtained results similar to Romik for the number of representations $r_{\mathfrak{so}(5)}(n)$:

Theorem 1.1.41 ([BBBF24, Theorem 1.3]). For all positive integers $N \geq 1$, we have

$$r_{\mathfrak{so}(5)}(n) \underset{n \rightarrow +\infty}{\sim} \frac{C}{n^{\frac{7}{12}}} \exp\left(A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4\right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N\left(n^{-\frac{N+1}{9}}\right)\right),$$

with B_2, \dots, B_{N+1} being explicit constants, and

$$\begin{aligned} C &= \frac{e^{\zeta'_{\mathfrak{so}(5)}(0)} \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}}{2^{\frac{1}{3}} 3^{\frac{11}{24}} \sqrt{\pi}}, & A_1 &= \frac{3^{\frac{4}{3}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{3}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{8}{3}}}, \\ A_2 &= \frac{2^{\frac{8}{9}} \left(2^{\frac{1}{3}} + 1\right) \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right)}{3^{\frac{7}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{9}}}, & A_3 &= \frac{2^{\frac{40}{9}} \left(2^{\frac{1}{3}} + 1\right)^2 \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{1}{3}\right)^2 \zeta\left(\frac{4}{3}\right)^2}{3^{\frac{44}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{20}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{10}{9}}}, \\ A_4 &= \frac{2^8 \left(2^{\frac{1}{3}} + 1\right)^3 \Gamma\left(\frac{1}{3}\right)^3 \zeta\left(\frac{1}{3}\right)^3 \zeta\left(\frac{4}{3}\right)^3}{3^8 \Gamma\left(\frac{1}{4}\right)^4 \zeta\left(\frac{3}{2}\right)^2}. \end{aligned}$$

Those 4 authors computed the special value $\zeta_{\mathfrak{so}(5)}(0)$ and the residues of the Witten zeta function $\zeta_{\mathfrak{so}(5)}(s)$ at $s = \frac{1}{2}$ and at $s = \frac{1}{3}$, however they hadn't a nice formula for the derivative value $\zeta'_{\mathfrak{so}(5)}(0)$. In chapter 6 we will give other expressions of those residues, and we will find the same value for $\zeta_{\mathfrak{so}(5)}(0)$ via another method as the one used by the 4 authors. We will also give a simple expression for the derivative value $\zeta'_{\mathfrak{so}(5)}(0)$ which will allow us to compute explicitly the coefficient C , therefore giving an explicit asymptotic formula for the number of representations $r_{\mathfrak{so}(5)}(n)$.

We will also obtain in chapter 6 formulas for the residues of the Witten zeta function $\zeta_{\mathfrak{g}_2}(s)$ at $s = \frac{1}{3}$ and at $s = \frac{1}{5}$, and also provide explicit formulas of the value $\zeta_{\mathfrak{g}_2}(0)$ and of the derivative value $\zeta'_{\mathfrak{g}_2}(0)$. We will then establish a similar result as the theorem previously stated for the number of representations $r_{\mathfrak{g}_2}(n)$ by using the Meinardus type Theorem 1.1.38.

1.2 First definitions

In this manuscript, we shall consider a family $(l_q)_{q \in \llbracket 1, Q \rrbracket}$ of linear forms of rank P with coefficients in the Poincaré half-plane $H_0 = \{s = \sigma + i\tau \in \mathbb{C} \mid \sigma > 0\}$, where $P \geq 1$ and $Q \geq 1$ are positive integers. We also fix $\mathbf{d} = (d_p)_{p \in \llbracket 1, P \rrbracket}$, $\mathbf{d}' = (d'_q)_{q \in \llbracket 1, Q \rrbracket}$ as complex numbers in H_0 such that $d'_q = l_q(\mathbf{d})$ for all $1 \leq q \leq Q$.

Notation. Let $q \in \llbracket 1, Q \rrbracket$, we denote

$$l_q(\mathbf{n}) = \sum_{p=1}^P c_{q,p} n_p.$$

For the sake of clarity, throughout this manuscript, we will choose to use the index p (or index i if p is already used) when summing or taking product over any subset \mathcal{P} of $\llbracket 1, P \rrbracket$, and we will use the index q (or the index j if q is already used) when summing or taking product over any subset \mathcal{Q} of $\llbracket 1, Q \rrbracket$.

We will first define the incomplete gamma functions.

Definition 1.2.1. Let $\nu \in H_0$, $\theta > 0$, $s \in \mathbb{C}$. We call incomplete gamma functions the following functions:

$$\begin{aligned} \Gamma(s, \theta, \nu) &= \int_{\theta}^{+\infty} e^{-\nu y} y^{s-1} dy, \\ \gamma(s, \theta, \nu) &= \int_0^{\theta} e^{-\nu y} y^{s-1} dy \quad (\text{when } \operatorname{Re}(s) > 0). \end{aligned}$$

We call $\Gamma(s, \theta, \nu)$ the upper incomplete gamma function and $\gamma(s, \theta, \nu)$ the lower incomplete gamma function.

These incomplete gamma functions are more general than those that can be found in the literature (for example, in [EMOT81, Chap.IX] or in [Cra12, §3]). The classical incomplete gamma functions can be recovered by setting $\nu = 1$. For our purposes, we will need an arbitrary ν in H_0 .

We now introduce constants that will appear in Theorem D. Through partial fraction decomposition, we can easily find:

Proposition 1.2.2. Let a $P+Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$. Let's consider \mathbb{K} the field above \mathbb{Q} generated by the coefficients $c_{q,p}$, with $1 \leq q \leq Q$ and $1 \leq p \leq P$. Let $\mathcal{P} \subset \llbracket 1, P \rrbracket$ a set, $j, f \in \llbracket 1, Q \rrbracket$ some integers, and we consider $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}}$ and $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c}$ some tuple. Let's note $\mathcal{P}^c = \llbracket 1, P \rrbracket \setminus \mathcal{P}$ the complement of the set \mathcal{P} , $\mathbf{w}_p = (w_{p,q})_{q \in \llbracket 1, Q \rrbracket}$ and $\mathbf{v}_p = (v_{p,q})_{q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}}$. Then there exist a polynomial $\tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'} \in \mathbb{K}[x]$ and two sequences of constants $(C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}'))_{\lambda \in \mathbb{N}_0}$ and $(D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}'))_{\lambda \in \mathbb{N}_0, p \in \mathcal{P}}$ such that all their terms belong to the field \mathbb{K} , and such that

$$\begin{aligned} x^{-N'_j - 1 + |\mathbf{w}_{\bullet, f}|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x)^{-N_p - 1 - |\mathbf{v}_p|} &= \tilde{P}_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}(x) + \sum_{\lambda=1}^{N'_j + 1 - |\mathbf{w}_{\bullet, f}|} \frac{C_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \lambda}(\mathbf{N}, \mathbf{N}')}{x^\lambda} \\ &+ \sum_{p \in \mathcal{P}} \sum_{\lambda=1}^{|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|} \frac{D_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, (\lambda, p)}(\mathbf{N}, \mathbf{N}')}{(c_{j,p} + c_{f,p} x)^\lambda}. \end{aligned}$$

We consider $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}$ the antiderivative which vanishes at $x = 1$ of the function $\tilde{P}_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}$. Let's note $E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') \in \mathbb{K}$ the constant term of $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}$. We now set

$$\begin{aligned} F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &:= -E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_f+1-|\mathbf{w}_{\bullet f}|} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} \\ &- \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|+|\mathbf{v}|}} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda-1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}) \\ &+ \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \ln \left(1 + \frac{c_{f,p}}{c_{j,p}} \right). \end{aligned}$$

Remark 1.2.3. 1) Note that if $-N'_f - 1 + |\mathbf{w}_{\bullet f}| \geq 0$, then $C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda} = 0$ for all λ .

2) In the product $\prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p}x)^{(-N_p-1-|\mathbf{v}_p|)}$, some terms may appear multiple times. This is why the sum $\sum_{\lambda=1}^{(|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|+|\mathbf{v}|})}$ goes up to $|\mathbf{N}|_{|\mathcal{P}|+|\mathcal{P}|+|\mathbf{v}|}$. If this occurs, then we have many choices for the constants $D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}$. Note that the constant $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ is independent of those choices.

3) The constant $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ will appear in the calculation of an integral in Lemma 3.1.4.

Example 1.2.4. Let $\mathcal{P} = \emptyset$, for all $\lambda \geq 1$ we have

$$\begin{aligned} C_{\emptyset,j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}') &= \delta_{N'_f+1-|\mathbf{w}_{\bullet f}|}^\lambda \\ D_{\emptyset,j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}') &= 0 \\ E_{\emptyset,j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &= 0 \quad (\text{if } |\mathbf{w}_{\bullet f}| \leq N'_f) \\ E_{\emptyset,j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &= -\frac{1}{|\mathbf{w}_{\bullet f}| - N'_f} \quad (\text{if } |\mathbf{w}_{\bullet f}| > N'_f), \end{aligned}$$

where δ is the Kronecker symbol. In particular, we get that $F_{\emptyset,j,f,\mathbf{0},\mathbf{0}}(\mathbf{0}, \mathbf{0}) = 0$.

Example 1.2.5. Let $\mathcal{P} = \{p\}$, $j \neq f \in \llbracket 1, Q \rrbracket$, $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$, $\mathbf{v} = \mathbf{0}$ and $\mathbf{w} = \mathbf{e}_f$, we then get that

$$x^{-N'_f-1+|\mathbf{w}_{\bullet f}|} (c_{j,p} + c_{f,p}x)^{-N_p-1-|\mathbf{v}_p|} = (c_{j,p} + c_{f,p}x)^{-1}.$$

Therefore, we have $F_{\{p\},j,f,\mathbf{0},\mathbf{e}_f} = \frac{1}{c_{f,p}} \ln \left(1 + \frac{c_{f,p}}{c_{j,p}} \right)$.

Example 1.2.6. Let $\mathcal{P} = \{p\}$, $j \neq f \in \llbracket 1, Q \rrbracket$, $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$, $\mathbf{v} = \mathbf{0}$ and $\mathbf{w} = \mathbf{e}_j$, we then get that

$$x^{-N'_f-1+|\mathbf{w}_{\bullet f}|} (c_{j,p} + c_{f,p}x)^{-N_p-1-|\mathbf{v}_p|} = x^{-1} (c_{j,p} + c_{f,p}x)^{-1} = \frac{1}{c_{j,p}} x^{-1} - \frac{c_{f,p}}{c_{j,p}} (c_{j,p} + c_{f,p}x)^{-1}.$$

Therefore, we have $F_{\{p\},j,f,\mathbf{0},\mathbf{e}_j} = -\frac{1}{c_{j,p}} \ln \left(1 + \frac{c_{f,p}}{c_{j,p}} \right)$.

We shall study in our thesis the following multizeta function:

Definition 1.2.7. For all $(\mathbf{s}, \mathbf{s}') \in H_1^P \times H_1^Q$ we set

$$Z(\mathbf{s}, \mathbf{s}') = Z(\mathbf{c}, \mathbf{d}, \mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}.$$

Remark 1.2.8. This class of multizeta function is of the Hurwitz-Lerch type without any torsion, as described in Definition 1.1.26:

$$Z(\mathbf{s}, \mathbf{s}') = \zeta^{HL}(\mathbf{0}, \tilde{\mathbf{d}}, \tilde{\mathbf{c}}, (s_1, \dots, s_P, s'_1, \dots, s'_Q)) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{\prod_{q=1}^Q \left(\tilde{d}_q + \sum_{p=1}^P \tilde{c}_{q,p} n_p \right)^{s'_q}},$$

with $P_0 = P$, $\tilde{Q} := P + Q$, $\mathcal{B}^c = \bigcup_{1 \leq p \leq P} \llbracket 1, P \rrbracket \setminus \{p\} \times \{p\} \subset \llbracket 1, Q + P \rrbracket \times \llbracket 1, P \rrbracket$, $\tilde{\mathbf{d}} = (\mathbf{d}, \mathbf{d}')$, and with $\tilde{\mathbf{c}} = (c_{(q,p)})_{(q,p) \in \llbracket 1, Q' \rrbracket \times \llbracket 1, P \rrbracket}$ such that

$$\begin{cases} \tilde{c}_{q,p} = 0 & \text{if } q \neq p \text{ and } q \leq P \\ \tilde{c}_{p,p} = 1 & \text{if } p \in \llbracket 1, P \rrbracket \\ \tilde{c}_{q,p} = c_{q-P,p} & \text{if } q > P. \end{cases}$$

Therefore, we can apply the Theorem 1.1.27 and its corollaries for the multizeta function $Z(\mathbf{s}, \mathbf{s}')$. We then get that the function

$$s \mapsto Z(-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N} + \boldsymbol{\mu}'s)$$

is meromorphic on \mathbb{C} and regular at $s = 0$ for all directions except for those inside an explicit union of hyperplanes.

Thanks to the previous remark and Theorem 1.1.27, we see that the multizeta function $Z(\mathbf{s}, \mathbf{s}')$ is meromorphic on $\mathbb{C}^P \times \mathbb{C}^Q$, and this theorem also provides information about the set of singularities of $Z(\mathbf{s}, \mathbf{s}')$. We also have a domain of holomorphy, however for our needs, we require a larger domain of convergence than what is provided by Theorem 1.1.27. We will then show a more precise domain of convergence in the following proposition:

Proposition 1.2.9. *We set*

$$\mathcal{D} = \mathcal{D}_{P,Q} := \left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket \right\}.$$

The multizeta function $Z(\mathbf{s}, \mathbf{s}')$ has a domain of convergence inside of the set \mathcal{D} , and has a meromorphic continuation on $\mathbb{C}^P \times \mathbb{C}^Q$, with singularities inside in the following union of hyperplanes:

$$\begin{cases} \sum_{p \in \mathcal{P}} s_p + \sum_{q \in \mathcal{Q}} s'_q = Q + P - n & (\mathcal{P} \subset \llbracket 1, P \rrbracket, \mathcal{Q} \subset \llbracket 1, Q \rrbracket, |\mathcal{P} \cup \mathcal{Q}| \geq 2, n \in \mathbb{N}_0) \\ s_p = 1, \dots, P & (1 \leq p \leq P) \\ s'_q = 1, \dots, P & (1 \leq q \leq Q). \end{cases}$$

In general the values at nonpositive integers don't make any sense because they are poles for the function $Z(\mathbf{s}, \mathbf{s}')$. This is why we will consider the values at a $P + Q$ -tuple $-(\mathbf{N}, \mathbf{N}')$ along a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}')$. Thus we need to study the function $Z(\mathbf{s}, \mathbf{s}')$ at this $P + Q$ -tuple along such a direction:

Definition 1.2.10. *Let $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, and $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$. We set*

$$Z_{\mathbf{N}, \mathbf{N}'}(s) := Z(-\mathbf{N} + s\boldsymbol{\mu}, -\mathbf{N}' + s\boldsymbol{\mu}').$$

Remark 1.2.11. *If we set $\boldsymbol{\mu} = \mathbf{0}$, and $\boldsymbol{\mu}' = \mathbf{1}$, then $Z_{\mathbf{0}, \mathbf{0}}(s)$ corresponds to a Shintani-type zeta function*

$$Z_{\mathbf{0}, \mathbf{0}}(s) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s}.$$

Keep in mind that the coefficients d'_q are not free, and depend on the choices of d_p and on the choices of the coefficients $(c_{p,q})_{\substack{1 \leq p \leq P \\ 1 \leq q \leq Q}}$.

We also consider the following zeta function, which consists of evaluating the multizeta function $Z(\mathbf{s}, \mathbf{s}')$ along the diagonal.

Definition 1.2.12. We set for $s \in \mathbb{C}$ the zeta function associated with the data set \mathbf{c}, \mathbf{d}

$$Z_{\Delta} : s \in \mathbb{C} \mapsto Z((s, \dots, s), (s, \dots, s)).$$

By Proposition 1.2.9 we know that the function $Z_{\mathbf{N}, \mathbf{N}'}$ is holomorphic on $\{s \in \mathbb{C} | (-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s) \in \mathcal{D}\}$, for all directions $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. Therefore, there exists a real number $\sigma_{0, \mathbf{N}, \mathbf{N}'}$ such that the function $Z_{\mathbf{N}, \mathbf{N}'}$ is holomorphic on $H_{\sigma_{0, \mathbf{N}, \mathbf{N}'}}$ for such directions. We shall say that the function $Z_{\mathbf{N}, \mathbf{N}'}$ is holomorphic on $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$, or sometimes only $\sigma \gg 1$ when there is no ambiguity.

It also follows from Proposition 1.2.9 (or from Theorem 1.1.27) that the function $Z_{\mathbf{N}, \mathbf{N}'}$ is meromorphic on \mathbb{C} , for all directions $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ except for those who belong in an explicit union of hyperplanes. By Komori's result [Kom10, Theorem 3.17] we know that the function $Z_{\mathbf{N}, \mathbf{N}'}$ is regular at $s = 0$ for those same directions. In Proposition 2.2.4 we will detail an analytic continuation method different as Komori's approach in order to prove the regularity at $s = 0$ of the function $Z_{\mathbf{N}, \mathbf{N}'}$. This will mainly allow us to prove explicit formulas for the values and derivative values along a direction of the function $Z(\mathbf{s}, \mathbf{s}')$.

Notation. Let a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0.$$

We consider the directional value,

$$Z(-(\mathbf{N}, \mathbf{N}')) := Z_{\mathbf{N}, \mathbf{N}'}(0),$$

and we consider the directional derivative value

$$Z'(-(\mathbf{N}, \mathbf{N}')) := \partial_s(Z_{\mathbf{N}, \mathbf{N}'}(s))|_{s=0}.$$

Remark 1.2.13. By the Corollary 1.1.31, we have that the two functions

$$(\boldsymbol{\mu}, \boldsymbol{\mu}') \mapsto Z(-(\mathbf{N}, \mathbf{N}')), \quad (\boldsymbol{\mu}, \boldsymbol{\mu}') \mapsto Z'(-(\mathbf{N}, \mathbf{N}'))$$

are meromorphic on \mathbb{C}^{P+Q} with singularities located inside an explicit union of hyperplanes.

We will now introduce auxiliary functions (which we sometime call generalized Barnes zeta functions) for the purpose of studying the derivative values along a direction $Z'(-(\mathbf{N}, \mathbf{N}'))$.

Definition 1.2.14. Let $\mathbf{R} \in \mathbb{N}_0^P$, and $j \in \llbracket 1, Q \rrbracket$. We call auxiliary function (or generalized Barnes zeta function) the function

$$\varphi_{\mathbf{R}}^j(s) := \sum_{\mathbf{n} \in \mathbb{N}_0^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d'_j)^{-s}.$$

Remark 1.2.15. If $\mathbf{R} = \mathbf{0}$, we obtain the classical Barnes zeta function $\varphi_{\mathbf{0}}^j(s) = \zeta_P^B(s, d'_j | c_{j,1}, \dots, c_{j,P})$.

The function $\varphi_{\mathbf{R}}^j(s)$ corresponds to $Z_{\mathbf{N}, \mathbf{N}'}$ with $Q = 1$, $\boldsymbol{\mu} = \mathbf{0}$, $\boldsymbol{\mu}' = 1$, $\mathbf{N} = \mathbf{R}$ and $\mathbf{N}' = 0$. In particular, we know that this function is meromorphic on \mathbb{C} , and is regular at nonpositive integers. The first derivative of these auxiliary functions will appear in the formula for $Z'(-(\mathbf{N}, \mathbf{N}'))$. Therefore, it is crucial to have an explicit formula for the derivative of such auxiliary functions.

1.3 Main results

As mentioned in the previous section, we will prove, using another method, the result of Komori on the existence of the directional values $Z(-(\mathbf{N}, \mathbf{N}'))$ in Proposition 2.2.4. To obtain this result, we use in

Chapter 2 a strategy called Crandall's expansion, developed in [Cra12], which was initially used to obtain numerical approximations of certain zeta functions and certain L functions. This strategy was also used by Borwein and Dilcher in [BD18] to study the values of a Mordell-Tornheim type zeta function, and to approximate its derivative at nonpositive integers. Subsequently, Onodera provided exact formulas for the values at nonpositive integers of a generalized Mordell-Tornheim type zeta function, as well as the values of its first and second order derivatives in [Ono21]. The trick used by Onodera to compute those values partly relies on introducing auxiliary functions resembling Barnes' zeta functions with a polynomial term in the numerator of the general term of the Dirichlet series, and then providing a formula for the derivatives of these auxiliary functions.

More precisely, we shall state an analytic continuation formula for the meromorphic function $Z_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}$ in Proposition 2.2.4:

$$Z_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(s) = K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s),$$

where the function $s \mapsto K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s)$ is holomorphic on \mathbb{C} , and vanishes at $s = 0$, and the function $s \mapsto J_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s)$ is meromorphic on \mathbb{C} , with candidate poles that are of the form

$$\begin{aligned} s &= \frac{n}{\mu_p} && (1 \leq p \leq P \text{ such that } \mu_p \neq 0, n \in \mathbb{Z}_{\geq N_p+1}) \\ s &= \frac{n}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}} && (\emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket, n \in \mathbb{Z}_{\leq |\mathbf{N}|_{|\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}|}). \end{aligned}$$

where we recall the notation $|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} = \mu'_1 + \dots + \mu'_Q + \sum_{p \in \mathcal{P}} \mu_p$. We then see in particular that 0 isn't a pole for the meromorphic function $Z_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}$, therefore obtaining that directional values of the function $Z(\mathbf{s}, \mathbf{s}')$ are well defined.

The fact that the function $s \mapsto Z_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(s)$ is meromorphic on \mathbb{C} directly follows from Proposition 1.2.9, and this same proposition also provides candidates for the poles of this function. Indeed, it is sufficient to replace s_p by $-N_p + \mu_p s$ and s'_q by $-N'_q + \mu'_q s$ to obtain candidate poles. However, the set of candidate poles provided here is more precise.

Let's note that the term $K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s)$ will not contribute to the formula for the directional values, as it vanishes at $s = 0$. On the other hand, if we differentiate this term with respect to s , then it doesn't vanish at $s = 0$ in general. We will therefore use the auxiliary functions $(\varphi_{\mathbf{R}}^j(s))_{j \in \llbracket 1, Q \rrbracket}$ that will help retrieve the information of the derivative $\partial_s K_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(\theta, s)$ at $s = 0$. We will then be able to evaluate at $s = 0$ the derivative of the continuation formula of $Z_{\mathbf{N}, \mathbf{N}'}^{\mu, \mu'}(s)$ in order to obtain an explicit formula for the directional derivative values.

Theorem A. *For all $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and all direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that*

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0,$$

we have

$$\begin{aligned} Z(-(\mathbf{N}, \mathbf{N}'))^{\mu, \mu'} &= (-1)^{|\mathbf{N}'|} \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \frac{(-1)^{N'_j + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \left(\prod_{p \in \mathcal{P}} N_p! \right) \\ &\quad \cdot \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}, \end{aligned} \tag{1.8}$$

with

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) := (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, \mathcal{P}}^{w_{p,q}} \right) \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j, \mathcal{P}}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, \mathcal{P}}^{v_{p,q}} \right), \quad (1.9)$$

where we noted

$$\begin{aligned} \forall q \in \llbracket 1, Q \rrbracket, \quad |\mathbf{w}_{\bullet q}| &= \sum_{p \in \mathcal{P}^c} w_{p,q}, \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad |\mathbf{v}_{\bullet q}| &= \sum_{p \in \mathcal{P}} v_{p,q}. \end{aligned}$$

Remark 1.3.1. 1) The formula describing $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ is a finite sum. Indeed, we see that

$$\sum_{\substack{q=1 \\ q \neq j}}^Q (N'_q - |\mathbf{w}_{\bullet q}|) = \sum_{\substack{q=1 \\ q \neq j}}^Q |\mathbf{v}_{\bullet q}| = \sum_{(p,q) \in \mathcal{P} \times (\llbracket 1, Q \rrbracket \setminus \{j\})} v_{p,q}.$$

Since $|\mathbf{w}_{\bullet q}|$ is nonnegative for all $q \in \llbracket 1, Q \rrbracket$, we deduce the following upper bound:

$$\sum_{(p,q) \in \mathcal{P} \times (\llbracket 1, Q \rrbracket \setminus \{j\})} v_{p,q} \leq |\mathbf{N}'|.$$

Thus, for all $p \in \mathcal{P}$ and all $q \in \llbracket 1, Q \rrbracket \setminus \{j\}$, we have $0 \leq v_{p,q} \leq |\mathbf{N}'|$.

2) The formula for the values $Z(-(\mathbf{N}, \mathbf{N}'))$ is described by a finite sum. Indeed, for all $\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}$ such that $|\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ we have $0 \leq k_p \leq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ for all $p \in \mathcal{P}^c$.

3) The values $\zeta(-n, d)$ with $n \in \mathbb{N}_0$ and $d \in H_0$ are expressed using Bernoulli polynomials:

$$\zeta(-n, d) = -\frac{B_{n+1}(d)}{n+1}.$$

Moreover, at $d = 1$, we have $\zeta(-n, 1) = \zeta(-n)$, and

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

where B_{n+1} is the $n+1$ -th Bernoulli number.

4) If all the coefficients d_p are equal to 1, then the product $\prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!}$ will often be zero.

Indeed, if there exists $p \in \mathcal{P}^c$ such that $N_p + k_p$ is even and greater than 2, then the product is zero.

Taking $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$, we obtain the following simplified expressions:

Corollary A1. For all direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^{\mathcal{P}} \times \mathbb{C}^{\mathcal{Q}}$ such that

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0,$$

we have

$$Z(-(\mathbf{0}, \mathbf{0})) = \sum_{\substack{\boldsymbol{\mu}, \boldsymbol{\mu}' \\ \mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \frac{(-1)^{|\mathcal{P}|} \mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j, \mathcal{P}}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j, \mathcal{P}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!}.$$

Via Theorem A, we also get a formula for the function $Z(\mathbf{s}, \mathbf{s}')$ along the diagonal at non positive integers by taking the direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ and a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') = (N, \dots, N)$ with $N \in \mathbb{N}_0$.

Corollary A2. *For all integer $N \in \mathbb{N}_0$ we have*

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \frac{(-1)^{(|\mathcal{P}|+1+Q)N+|\mathcal{P}|}}{Q+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=(Q+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!},$$

with

$$Q_N^0(\mathcal{P}, \mathbf{k}) = (-1)^{N(Q-1)} N^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|=N}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right),$$

where we noted

$$\begin{aligned} \forall q \in \llbracket 1, Q \rrbracket, \quad |\mathbf{w}_{\bullet q}| &= \sum_{p \in \mathcal{P}^c} w_{p,q}, \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad |\mathbf{v}_{\bullet q}| &= \sum_{p \in \mathcal{P}} v_{p,q}. \end{aligned}$$

Remark 1.3.2. *If we take $Q = 1$, $P \geq 1$ a positive integer, and $l_1(\mathbf{n}) = n_1 + \dots + n_P$, we find via the previous corollary that $Q_N^0(\mathcal{P}, \mathbf{k}) = 1$ for all set $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, and that*

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \frac{(-1)^{|\mathcal{P}|N+|\mathcal{P}|}}{1+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=(1+|\mathcal{P}|)N+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N - k_p, d_p)}{k_p!}.$$

This formula matches the one obtained by Onodera in [Ono21, Theorem 3].

We also can formulate a qualitative version of the Theorem A:

Corollary A3. *Let's consider a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that*

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

We consider \mathbb{K} the field above \mathbb{Q} generated by the coefficients $(d_p)_{p \in \llbracket 1, P \rrbracket}$ and $(c_{q,p})_{p \in \llbracket 1, P \rrbracket, q \in \llbracket 1, Q \rrbracket}$, and by the coefficients μ_p for $p \in \llbracket 1, P \rrbracket$, and μ'_q for $q \in \llbracket 1, Q \rrbracket$. We then have $Z(-(\mathbf{N}, \mathbf{N}')) \in \mathbb{K}$.

We also have an expression for the directional derivative at nonpositive tuple of the function $Z(\mathbf{s}, \mathbf{s}')$. This expression involves the auxiliary functions, also called generalized Barnes zeta functions, that we set in Definition 1.2.14. If we assume that a linear form l_j has rational coefficients, we have a relation between the auxiliary function $\varphi_{\mathbf{R}}^j(s)$ and the Hurwitz zeta function:

Proposition B. *Let $j \in \llbracket 1, Q \rrbracket$ a positive integer, we assume that l_j is a linear form with rational coefficients. For all $1 \leq p \leq P$, we then write $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$ the irreducible fraction decomposition of $c_{j,p}$ with $a_{j,p} > 0$ and $b_{j,p} > 0$. Let*

$$x_j(\mathbf{c}) := \frac{\text{lcm}(a_{j,p})}{\text{gcd}(b_{j,p})} \in \mathbb{Q}_+^*, \quad \beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}$$

for all $1 \leq p \leq P$. Then for all $\mathbf{R} \in \mathbb{N}_0^P$ and all $s \in \mathbb{C} \setminus \mathbb{N}$, we have

$$\begin{aligned} \varphi_{\mathbf{R}}^j(s) &= x_j(\mathbf{c})^{-s} \prod_{p=1}^P \beta_{j,p}^{R_p} \sum_{\substack{\mathcal{P} \subsetneq [1,P] \\ 0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \left(\prod_{p \in \mathcal{P}^c} R_p! \right) \\ &\quad \cdot \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}}, k' \in \mathbb{N}_0 \\ |\mathbf{k}| + k' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \frac{\zeta\left(s - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})}\right)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!}. \end{aligned}$$

Taking the derivative at $s = -N$ of the generalized Barnes zeta function, we find an expression of this derivative using only special values and derivative values of the Hurwitz zeta function.

Corollary B1. *With the same notations as in Proposition B, we then get for all $\mathbf{R} \in \mathbb{N}_0^P$,*

$$\begin{aligned} (\varphi_{\mathbf{R}}^j)'(-N) &= x_j(\mathbf{c})^N \prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{R_p} \sum_{\substack{\mathcal{P} \subsetneq [1,P] \\ 0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \left(\prod_{p \in \mathcal{P}^c} R_p! \right) \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}}, k' \in \mathbb{N}_0 \\ |\mathbf{k}| + k' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \\ &\quad \cdot \left(\frac{\zeta'\left(-N - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})}\right)}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta\left(-N - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})}\right)}{k'!} \right) \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!}. \end{aligned}$$

We now can give a qualitative version of our result on special values of the directional derivative of $Z(\mathbf{s}, \mathbf{s}')$ at nonpositive integers:

Theorem C. *Let's consider a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that*

$$\forall \mathcal{P} \subsetneq [1, P], \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

We consider \mathbb{K} the field above \mathbb{Q} generated by the coefficients $(d_p)_{p \in [1, P]}$ and $(c_{q,p})_{p \in [1, P], q \in [1, Q]}$, and by the coefficients μ_p for $p \in [1, P]$, and μ'_q for $q \in [1, Q]$. Then,

$$\begin{aligned} Z'_{\boldsymbol{\mu}, \boldsymbol{\mu}'}(-\mathbf{N}, \mathbf{N}') &\in \text{Vect}_{\mathbb{K}} \left[\gamma, (\ln(c_{q,p}))_{\substack{1 \leq q \leq Q \\ 1 \leq p \leq P}}, (\ln(c_{j,p} + c_{q,p}))_{\substack{1 \leq q, j \leq Q \\ 1 \leq p \leq P}}, (\zeta'(-n, d_p))_{\substack{0 \leq n \leq |\mathbf{N}| + |\mathbf{N}'| + P \\ p \in [1, P]}}, \right. \\ &\quad \left. (\varphi_{\mathbf{n}}^j)'(-N'_j)_{\substack{\mathbf{n} \in [0, |\mathbf{N}| + |\mathbf{N}'|]^P \\ 1 \leq j \leq Q}} \right], \end{aligned}$$

where γ is Euler's constant, and φ_j are the zeta functions of generalized Barnes type described in Definition 1.2.14.

If the linear forms $(l_j)_{j \in [1, Q]}$ are all rationals, then we can use Corollary B1 in order to replace the derivative $\varphi_{\mathbf{n}}^j'(-N'_j)$ with an finite sum of special values at nonpositive integers of the derivative with respect to s of the Hurwitz zeta function. Thus, we get the following corollary:

Corollary C1. *Let's consider a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that*

$$\forall \mathcal{P} \subsetneq [1, P], \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

We assume that the linear forms l_1, \dots, l_Q have rational coefficients and that the coefficients d_1, \dots, d_P are rational numbers. We consider the field \mathbb{K} above \mathbb{Q} generated by the coefficients μ_p for $p \in [1, P]$, and μ'_q

for $q \in \llbracket 1, Q \rrbracket$. For all $1 \leq p \leq P$, we write $c_{q,p} = \frac{a_{q,p}}{b_{q,p}} \in \mathbb{Q}$ the irreducible fraction decomposition of $c_{q,p}$ with $a_{q,p} > 0$ and $b_{q,p} > 0$. We set

$$x_q(\mathbf{c}) := \frac{\text{lcm}(a_{q,p})}{\text{gcd}(b_{q,p})} \in \mathbb{Q}_+^*, \quad \beta_{q,p}(\mathbf{c}) := \frac{x_q(\mathbf{c})}{c_{q,p}} \in \mathbb{N}$$

for all $1 \leq p \leq P$, $1 \leq q \leq Q$. Let's also consider the following set

$$A := \left\{ \frac{l_j(\mathbf{d} + \mathbf{v})}{x_{\mathbf{c}}} \mid j \in \llbracket 1, Q \rrbracket, \mathbf{v} \in \llbracket 0, \beta_{j,1} - 1 \rrbracket \times \dots \times \llbracket 0, \beta_{j,P} - 1 \rrbracket \right\}.$$

We then have that

$$Z'(-(\mathbf{N}, \mathbf{N}')) \in \text{Vect}_{\mathbb{K}} \left[\gamma, (\ln(c_{q,p}))_{\substack{1 \leq q \leq Q \\ 1 \leq p \leq P}}, (\ln(c_{j,p} + c_{q,p}))_{\substack{1 \leq q, j \leq Q \\ 1 \leq p \leq P}}, (\ln(x_j(\mathbf{c})))_{1 \leq j \leq Q}, \right. \\ \left. (\zeta'(-n, y))_{\substack{0 \leq n \leq P(|\mathbf{N}|+1)+|\mathbf{N}'| \\ y \in A}} \right],$$

where γ is Euler's constant.

The formula for $Z'(-(\mathbf{N}, \mathbf{N}'))$ is actually explicit. It has a formula involving the special values of the derivatives of the generalized Barnes zeta functions $\varphi_j^{\mathbf{R}}$, special values of the Hurwitz zeta function and of its derivative with respect to s . The formula also features the logarithm of some integers, and Euler's gamma constant. We detail this formula in the theorem below.

Theorem D. For all $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and all direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0,$$

we have

$$\begin{aligned} Z'(-(\mathbf{N}, \mathbf{N}')) = & \tag{1.10} \\ & \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} (-1)^{|\mathbf{N}|+|\mathbf{N}'|+|\mathcal{P}|+N'_j} \left(\prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & \cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'|+|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\ & \cdot \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\ & + \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q|=N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) (\varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j)^{(-N'_j)} \\ & - \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q|=N'_q}} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}}+|\mathcal{P}|} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \end{aligned}$$

$$\cdot \prod_{p \in \mathcal{P}} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=N'_j+|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}+|\mathcal{P}|}}} \frac{\prod_{p \in \mathcal{P}^c} C_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} C_{j,p}^{N_p+|\mathbf{u}_{\bullet p}|+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!},$$

with

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + |\mathbf{u}_{\bullet p}|)_{p \in [1, P]}, \text{ for all } \mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}}$$

and with

$$\begin{aligned} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) &= (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, N'_q=|\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q,p}^{w_{p,q}} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} C_{j,p}^{-N_p-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q C_{q,p}^{v_{p,q}} \right), \\ Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) &:= (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|=N'_q}} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \quad (1.11) \\ &\cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q C_{q,p}^{v_{p,q}} \right) \\ &+ (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|=N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q C_{q,p}^{w_{p,q}} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \frac{\prod_{\substack{q=1 \\ q \neq j}}^Q C_{q,p}^{v_{p,q}}}{C_{j,p}^{N_p+1+|\mathbf{v}_p|}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p+|\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \end{aligned}$$

where $j \in [1, Q]$, $\mathcal{P} \subsetneq [1, P]$, $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$, and where $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ is the constant defined in Proposition 1.2.2.

Remark 1.3.3. This theorem reduces the computation of the directional derivative value $Z'(-(\mathbf{N}, \mathbf{N}'))$ to the computations of values at nonpositive integers of the derivative of the auxiliary functions $\varphi_{\mathbf{R}}^j$.

$$\varphi_{\mathbf{R}}^j(s) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d'_j)^{-s} \quad (j \in [1, Q], \mathbf{R} \in \mathbb{N}_0^P).$$

If the coefficients of the linear forms l_j are rationals, then we have an explicit expression for the value $(\varphi_{\mathbf{R}}^j)'(-N)$ in terms of the special values of the derivative of the Hurwitz zeta function via Corollary B1.

Remark 1.3.4. We observe that the formula for $Z'(-(\mathbf{N}, \mathbf{N}'))$ involves only finite sums. Indeed, it is

clear that the sum
$$\sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}}+|\mathcal{P}|}} \quad \text{is finite. On top of that the sum}$$

$$\sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|=N'_q}} = \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|=N'_q}}$$

in the formula of $Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k})$ and of $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ is also finite because

$$\forall a \in \mathcal{P}, \forall b \in [1, Q] \setminus \{j\}, \quad 0 \leq v_{a,b} \leq \sum_{\substack{q=1 \\ q \neq j}}^Q |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = |\mathbf{N}'|,$$

$$\forall p \in \mathcal{P}^c, \forall q \in [1, Q], \quad 0 \leq w_{p,q} \leq |\mathbf{w}_p| = k_p \leq |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}^c} + |\mathcal{P}|.$$

For the same reasons, the sum

$$\sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|=N'_q}} = \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|=N'_q}}$$

is also finite.

If the coefficients \mathbf{c} are rationals, we can replace the derivatives of the auxiliary functions at nonpositive integers with the formula obtained in Corollary B1:

Corollary D1. We assume that the linear forms l_1, \dots, l_Q have rational coefficients. Let's consider a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that

$$\forall \mathcal{P} \subsetneq [1, P], \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

For all $1 \leq p \leq P$, we write $c_{q,p} = \frac{a_{q,p}}{b_{q,p}} \in \mathbb{Q}$ the irreducible fraction decomposition of $c_{q,p}$ with $a_{q,p} > 0$ and $b_{q,p} > 0$. We set

$$x_q(\mathbf{c}) := \frac{\text{lcm}(a_{q,p})}{\text{gcd}(b_{q,p})} \in \mathbb{Q}_+^*, \quad \beta_{q,p}(\mathbf{c}) := \frac{x_q(\mathbf{c})}{c_{q,p}} \in \mathbb{N}$$

for all $1 \leq p \leq P$. Then we have

$$\begin{aligned} Z'(-(\mathbf{N}, \mathbf{N}')) &= \\ & \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} (-1)^{|\mathbf{N}|+|\mathbf{N}'|+|\mathcal{P}|+N'_j} \left(\prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}}+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & \cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\sum_{p \in \mathcal{P}} \mu_p(\gamma - h_{N_p}) + \mu'_j(\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}}+|\mathcal{P}|+N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{|\mathcal{P}}+|\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
& + \sum_{j=1}^Q \mu'_j x_j(\mathbf{c})^{N'_j} \sum_{\substack{\mathcal{P} \subseteq [1, P] \\ 0 \leq v_1 \leq \beta_{j,1}(\mathbf{c}) - 1 \\ \dots \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c}) - 1}} \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\
& \cdot \left(\prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p} \right) \left(\prod_{p \in \mathcal{P}^c} \mathbf{R}(\mathbf{N}, j, \mathbf{u})_p! \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}} \in \mathbb{N}_0^{\mathcal{P}}, k' \in \mathbb{N}_0 \\ |\mathbf{k}| + k' = |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}^c| + |\mathcal{P}^c| - 1}}} (-1)^{|\mathbf{k}|} \\
& \cdot \left(\frac{\zeta'(-N'_j - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})})}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta(-N'_j - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})})}{k'!} \right) \prod_{p \in \mathcal{P}} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})})}{k_p!} \\
& - \sum_{\substack{\mathcal{P} \subseteq [1, P] \\ 1 \leq j \leq Q}} (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left((-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}^c|}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \right) \\
& \cdot \left(\prod_{p \in \mathcal{P}} R(\mathbf{N}, j, \mathbf{u})_p! \right) \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}| + |\mathcal{P}^c|}}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R(\mathbf{N}, j, \mathbf{u})_p + 1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \Bigg),
\end{aligned}$$

where we set

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + |\mathbf{u}_{\bullet p}|)_{p \in [1, P]}, \text{ for all } \mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}}$$

and

$$\begin{aligned}
Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) & := (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, N'_q = |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\
& \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right), \\
Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) & := (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \\
& \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right)
\end{aligned}$$

$$\begin{aligned}
& + (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, |\mathbf{v}_q|+|\mathbf{w}_q|=N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\
& \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \frac{\prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}}}{c_{j,p}^{N_p+1+|\mathbf{v}_p|}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p+|\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right)
\end{aligned}$$

where $j \in \llbracket 1, Q \rrbracket$, $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$, and where $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ is the constant defined in Proposition 1.2.2.

Remark 1.3.5. Let $Q = 1$, and $P \geq 1$ a positive integer, we set $l_1(\mathbf{n}) = n_1 + \dots + n_P$ (and thus $c_{1,1} = \dots = c_{1,P} = 1$), $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$, and $(\mathbf{N}, \mathbf{N}') = (N, \dots, N)$ with $N \in \mathbb{N}_0$. Then we find by the previous corollary that

- For all $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, and $\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}$, we have $Q_{\mathbf{N}, \mathbf{N}'}^0(1, \mathcal{P}, \mathbf{k}) = 1$ and $Q_{\mathbf{N}, \mathbf{N}'}^1(1, \mathcal{P}, \mathbf{k}) = 0$.
- With the same notation as in Corollary D1, we have $\mathbf{x}_1(\mathbf{c}) = 1$, and $\beta_{1,p} = 1$ for all $1 \leq p \leq P$.
- We have $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) = (N, \dots, N) \in \mathbb{N}_0^P$.
- In the formula of $Z'(-(\mathbf{N}, \mathbf{N}'))_{1,1}$ obtained by the previous corollary, we get that the first sum cancels the last sum.

Using the last 4 points, we get that

$$\begin{aligned}
Z'(-(\mathbf{N}, \mathbf{N}'))_{1,1} &= \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \frac{N^{|\mathcal{P}|+1}}{|\mathcal{P}|+1} (-1)^{|\mathcal{P}|(N+1)} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|(N+1)+|\mathcal{P}|}} \\
& \left(\sum_{i \in \mathcal{P}^c} \frac{\zeta'(-N - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N - k_p, d_p)}{k_p!} \right) \\
& + \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} N^{|\mathcal{P}^c|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}} \in \mathbb{N}_0^{\mathcal{P}}, k' \in \mathbb{N}_0 \\ |\mathbf{k}|+k' = |\mathcal{P}^c|(N+1)+|\mathcal{P}^c|-1}} (-1)^{|\mathbf{k}|} \frac{\zeta'(-N - k', d_1)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta(-N - k_p, d_p)}{k_p!}.
\end{aligned}$$

This expression matches the one obtained by Onodera in [Ono21, Theorem 4] (with the parameter $d_1 = 0$ in Onodera's theorem).

By considering $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ in Theorem D, we get:

Corollary D2. For all direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0,$$

we have

$$\begin{aligned}
Z'(-(\mathbf{0}, \mathbf{0})) &= \sum_{\mu, \mu'}^Q \mu'_j (\zeta^B)'(0, d'_j | \mathbf{c}_j) \\
&+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|} \mu'_j}{|\mu'| + |\mu|^{|\mathcal{P}|}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}} \left(\sum_{p \in \mathcal{P}} \mu_p \ln(c_{j,p}) \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \\
&+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{1 \leq j \neq f \leq Q} \frac{(-1)^{|\mathcal{P}|} \mu'_j \mu'_f}{|\mu'| + |\mu|^{|\mathcal{P}|}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \\
&\cdot \sum_{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} [0, k_p]} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\
&+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{\mu'_j}{|\mu'| + |\mu|^{|\mathcal{P}|}} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}} \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p, d_p)}{k_p!} \right)
\end{aligned}$$

where $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}'}(\mathbf{0}, \mathbf{0})$ is the constant defined in Proposition 1.2.2, and with the notation $(\zeta^B)'(s, d | \mathbf{c}) = \partial_s \zeta^B(s, d | \mathbf{c})$.

Remark 1.3.6. The constants $F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0})$ in the previous corollary are linked with the partial fraction decomposition of

$$x^{-1+k_p} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x)^{-1}.$$

Remark 1.3.7. It is worth noting that Matsumoto and Weng obtained in [MW02, Theorem E] a similar-looking expression for the derivative value at $s = 0$ of a zeta function defined by two polynomials in $\mathbb{C}[X]$.

By taking $\mu = \mathbf{0}$, and $\mu' = \mathbf{1}$, we get $Z'(-(\mathbf{0}, \mathbf{0})) = \partial_s (\zeta^{Sh}(s, (\mathbf{c}_1, \dots, \mathbf{c}_Q), \mathbf{d}'))|_{s=0}$, which corresponds to the first derivative with respect to s of the Shintani zeta function at $s = 0$, as defined in Definition 1.1.13. Let's recall that in Definition 1.1.16, and in Definition 1.1.14, we have defined multigamma function. Therefore, we have

$$\partial_s (\zeta^B(s, d'_j | \mathbf{c}_j))|_{s=0} = \ln(\Gamma_P(d'_j | \mathbf{c}_j)), \quad \partial_s (\zeta^{Sh}(s, \mathbf{d}' | \mathbf{c}))|_{s=0} = \ln(\Gamma_P(\mathbf{d}' | \mathbf{c})).$$

From the previous corollary, and by definition of these multigamma functions, we get that the logarithm of the Shintani multigamma function $\Gamma_P(\mathbf{d}' | \mathbf{c}_1, \dots, \mathbf{c}_Q) = \Gamma_P((l_1(\mathbf{d}'), \dots, l_Q(\mathbf{d}')) | \mathbf{c}_1, \dots, \mathbf{c}_Q)$ has an explicit expression using the logarithm of Barnes multigamma function.

Corollary D3. We have

$$\begin{aligned}
\ln(\Gamma_P(\mathbf{d}' | \mathbf{c})) &= \sum_{j=1}^Q \ln(\Gamma_P(d'_j | \mathbf{c}_j)) \\
&+ \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{1 \leq j \neq f \leq Q} \frac{(-1)^{|\mathcal{P}|}}{Q} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \\
&\cdot \sum_{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} [0, k_p]} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right).
\end{aligned}$$

Note that through this corollary we obtain a Raabe type formula. Indeed, it is well known since Barnes that

$$\int_{[0,1]^P} \ln(\Gamma_P(l_j(\mathbf{x})|\mathbf{c}_j)) dx_1 \dots dx_P = 0.$$

Moreover, by property of Bernoulli polynomials, it is clear that

$$\forall n \in \mathbb{N}_0, \quad \int_0^1 \zeta(-n, x) dx = 0.$$

Thus, we obtain via a different strategy the following Raabe type formula first obtained by Friedmann and Ruijsenaars [FR04]:

$$\int_{[0,1]^P} \ln(\Gamma_P((l_1(\mathbf{x}), \dots, l_Q(\mathbf{x}))|\mathbf{c}_1, \dots, \mathbf{c}_Q)) dx_1 \dots dx_P = 0.$$

Applying Corollary A1 and Corollary D2 to the two Witten zeta functions $\zeta_{\mathfrak{g}_2}$ and $\zeta_{\mathfrak{so}(5)}$, we get the following formulas:

Theorem E. *We have the following formulas for the special values of $\zeta_{\mathfrak{g}_2}(s)$ and $\zeta_{\mathfrak{so}(5)}(s)$:*

$-N$	0	-1	-2	-3	-4
$\zeta_{\mathfrak{so}(5)}(-N)$	$\frac{3}{8}$	$-\frac{11}{26880}$	0	$-\frac{509}{37847040}$	0
$\zeta_{\mathfrak{g}_2}(-N)$	$\frac{5}{12}$	$\frac{6641}{62705664}$	0	$\frac{12522872818983257}{109242202556140093440}$	0

Moreover, we have the following formulas for the special values of $\zeta'_{\mathfrak{g}_2}(s)$ and $\zeta'_{\mathfrak{so}(5)}(s)$ at $s = 0$:

$$\begin{aligned} \zeta'_{\mathfrak{g}_2}(0) &= \frac{5}{12} \ln(5) - \frac{1}{12} \ln(3) + \frac{13}{4} \ln(2) + \frac{5}{2} \ln(\pi) \approx 5.693601157568522039431952058889088333111\dots \\ \zeta'_{\mathfrak{so}(5)}(0) &= \frac{3}{8} \ln(3) + \frac{13}{8} \ln(2) + \frac{3}{2} \ln(\pi) \approx 3.2554386054345525232913601882450721298364388\dots \end{aligned}$$

Note that these two Witten zeta functions vanish at $s = -2$. This result echoes Kurokawa and Ochiai's conjecture [KO13], which predicts that all Witten's zeta functions vanish at $s = -2$. Note also that we find the same value for $\zeta_{\mathfrak{so}(5)}(0)$ as in [BBBF24, Proposition 5.15].

Using last formula, and by taking the expression of the constant C in Theorem 1.1.41, we get:

$$C = 3^{-\frac{1}{12}} 2^{\frac{31}{24}} \pi \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}.$$

We now obtain an explicit version of the Theorem 1.1.41 proved by Bridges, Brindle, Bringmann and Franke in [BBBF24, Theorem 1.3]:

Corollary F. *For all integer $N \geq 1$ we have*

$$r_{\mathfrak{so}(5)}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^{\frac{7}{12}}} \exp\left(A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4\right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N\left(n^{-\frac{N+1}{9}}\right)\right),$$

with B_2, \dots, B_{N+1} being explicit constants, and

$$\begin{aligned} C &= 3^{-\frac{1}{12}} 2^{\frac{31}{24}} \pi \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}, & A_1 &= \frac{3^{\frac{4}{3}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{3}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{8}{3}}}, \\ A_2 &= \frac{2^{\frac{8}{9}} \left(2^{\frac{1}{3}} + 1\right) \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right)}{3^{\frac{7}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{9}}}, & A_3 &= \frac{2^{\frac{40}{9}} \left(2^{\frac{1}{3}} + 1\right)^2 \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{1}{3}\right)^2 \zeta\left(\frac{4}{3}\right)^2}{3^{\frac{44}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{20}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{10}{9}}}, \\ A_4 &= \frac{2^8 \left(2^{\frac{1}{3}} + 1\right)^3 \Gamma\left(\frac{1}{3}\right)^3 \zeta\left(\frac{1}{3}\right)^3 \zeta\left(\frac{4}{3}\right)^3}{3^8 \Gamma\left(\frac{1}{4}\right)^4 \zeta\left(\frac{3}{2}\right)^2}. \end{aligned}$$

We now wish to apply the results obtained in Theorem E in order to state an asymptotic formula for $r_{\mathfrak{g}_2}(n)$, which we get through Theorem 1.1.38 obtained by Bridges, Brindle, Bringmann, and Franke in [BBBF24, Theorem 4.4]. We first set

$$\begin{aligned} l_1^*(x_1, x_2, x_3, x_4) &= x_1 + x_2 + x_3 + 2x_4, \\ l_2^*(x_1, x_2, x_3, x_4) &= x_1 + 2x_2 + 3x_3 + 3x_4, \end{aligned}$$

and we set

$$\begin{aligned} \omega_\beta &:= \frac{120^{\frac{1}{5}} \Gamma\left(\frac{4}{5}\right) \zeta\left(\frac{1}{5}\right)}{5 \Gamma\left(\frac{1}{5}\right)^4} \int_{[0,1]^3} (xyz)^{-\frac{4}{5}} \left(3l_1^*(1, x, y, z)^{-\frac{4}{5}} + l_2^*(1, x, y, z)^{-\frac{4}{5}} \right. \\ &\quad \left. + l_2^*(x, 1, y, z)^{-\frac{4}{5}} + l_2^*(x, y, 1, z)^{-\frac{4}{5}} + l_1^*(x, y, z, 1)^{-\frac{4}{5}} + l_2^*(x, y, z, 1)^{-\frac{4}{5}} \right) dx dy dz, \\ \omega_\alpha &:= \frac{120^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)^2}{6 \Gamma\left(\frac{1}{3}\right)^4} \int_{[0,1]^3} (xyz)^{-\frac{2}{3}} \left(\prod_{p=1}^2 l_p^*(x, y, z, 1)^{-\frac{2}{3}} + \prod_{p=1}^2 l_p^*(x, y, 1, z)^{-\frac{2}{3}} \right. \\ &\quad \left. + \prod_{p=1}^2 l_p^*(x, 1, y, z)^{-\frac{2}{3}} + \prod_{p=1}^2 l_p^*(1, x, y, z)^{-\frac{2}{3}} \right) dx dy dz. \end{aligned}$$

We will see in subsection 6.3.1 that the coefficients ω_α and ω_β correspond to residues of the Witten zeta function $\zeta_{\mathfrak{g}_2}$. We now state an asymptotic formula for the number of representations $r_{\mathfrak{g}_2}(n)$ of the exceptional algebra \mathfrak{g}_2 , some of the coefficients of the formula have an expression using the residues ω_α and ω_β , and the special values $\zeta_{\mathfrak{g}_2}(0)$ and $\zeta'_{\mathfrak{g}_2}(0)$.

Theorem G. *For all integer $N \geq 1$ we have*

$$r_{\mathfrak{g}_2}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^{\frac{9}{16}}} \exp\left(A_1 n^{\frac{1}{4}} + A_2 n^{\frac{3}{20}} + A_3 n^{\frac{1}{20}}\right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{20}}} + O\left(n^{-\frac{N+1}{20}}\right)\right),$$

with B_2, \dots, B_{N+1} being explicit constants, and

$$\begin{aligned} C &= 2^{\frac{7}{4}} 15^{\frac{5}{12}} \pi^2 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{-3}{16}}, \\ K_2 &= \frac{3}{4} \cdot \frac{\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{20}}}, \quad K_3 = -\frac{3}{160} \cdot \frac{\left(\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)\right)^2}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{21}{20}}}, \\ A_1 &:= 4 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{3}{4}}, \quad A_2 := \frac{\omega_\beta \Gamma\left(\frac{1}{5}\right) \zeta\left(\frac{6}{5}\right)}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{20}}}, \\ A_3 &:= \frac{2K_2^2}{3 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{3}{4}}} - \frac{\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right)\right)^{\frac{9}{10}}} K_2. \end{aligned}$$

Remark 1.3.8. *The B_j can be computed through the coefficients b_j obtained by [BBBF24, Lemma 3.6].*

Chapter 2

Crandall's expansion of $Z(\mathbf{s}, \mathbf{s}')$

We shall fix in this chapter some coefficients $\mathbf{c} = (c_{q,p})_{\substack{1 \leq q \leq Q \\ 1 \leq p \leq P}}$ for the linear forms l_1, \dots, l_Q , and $\mathbf{d} = (d_p)_{1 \leq p \leq P}$, $\mathbf{d}' = (d'_q)_{1 \leq q \leq Q}$ some complex numbers such that $d_p \in H_0$ for all $1 \leq p \leq P$, and we assume that for all $1 \leq q \leq Q$,

$$d'_q = l_q(\mathbf{d}) = \sum_{p=1}^P c_{q,p} d_p.$$

Moreover, we will consider a $P+Q$ -tuple integer $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. All of the bounds that we will state through this chapter will depend on the above constants. However, we won't note that dependence for the sake of clarity.

Firstly we will establish a meromorphic continuation of $Z(\mathbf{s}, \mathbf{s}')$ and examine its singularities in Proposition 1.2.9. Next, we will decompose the meromorphic function $Z(\mathbf{s}, \mathbf{s}')$ using a Crandall expansion, and we will obtain a continuation formula for the univariate function $s \mapsto Z(-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s)$. More specifically, we will obtain a decomposition of this meromorphic function as a sum between a holomorphic function that vanishes at $s = 0$, and a meromorphic function corresponding to an Erdélyi-type series. This continuation formula will be crucial in the proofs of Theorems A and D.

2.1 Proof of Proposition 1.2.9

The meromorphic continuation of $Z(\mathbf{s}, \mathbf{s}')$ and the form of the singularities follow from Theorem 1.1.27. We aim to show here that $Z(\mathbf{s}, \mathbf{s}')$ is absolutely convergent on the domain \mathcal{D} that is yet to be introduced in Proposition 2.1.1. It is enough to show that the series describing $Z(\mathbf{s}, \mathbf{s}')$ is normally convergent on every compact subset of \mathcal{D} . We intend to prove the following proposition, which is more general than what we aim to prove in Proposition 1.2.9:

Proposition 2.1.1. *Let $P, Q \geq 1$ be two positive integers, and $\boldsymbol{\gamma} = (\gamma_{q,p})_{\substack{1 \leq q \leq Q \\ 1 \leq p \leq P}} \in H_0^{Q \times P}$, $\boldsymbol{\alpha} = (\alpha_p)_{1 \leq p \leq P} \in H_0^P$, $\boldsymbol{\beta} = (\beta_q)_{1 \leq q \leq Q} \in H_0^Q$. We set*

$$\mathcal{D} = \mathcal{D}_{P,Q} := \left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset \llbracket 1, P \rrbracket \right\}.$$

Then the function

$$Z_{\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\beta}}^{P,Q}(\mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + \alpha_p)^{-s_p} \prod_{q=1}^Q \left(\sum_{p=1}^P \gamma_{q,p} n_p + \beta_q \right)^{-s'_q}$$

normally converges on any compact subset of \mathcal{D} .

Proof. All the constants involved in the proof of this proposition will implicitly depend on the choice of the coefficients γ , α , and β . We will not show this dependency to improve the clarity of the proof. We will prove this proposition by induction on P . It is clear that the case $P = 1$ is trivial.

Let's assume that the result is true for all $k < P$. Let $(\mathbf{s}, \mathbf{s}') \in H_0^{P+Q}$, and $\mathbf{n} \in \mathbb{N}_0^P$. We then have the following upper bound

$$(n_p + 1) \ll n_p + \operatorname{Re}(\alpha_p) \leq |n_p + \alpha_p| \leq n_p + |\alpha_p| \ll (n_p + 1).$$

Thus, we have that

$$\sum_{p=1}^P n_p + P \ll \sum_{p=1}^P \operatorname{Re}(\gamma_{q,p}) n_p + \operatorname{Re}(\beta_q) \leq \left| \sum_{p=1}^P \gamma_{q,p} n_p + \beta_q \right| \leq \sum_{p=1}^P |\gamma_{q,p}| n_p + |\beta_q| \ll \sum_{p=1}^P n_p + P.$$

Taking the product over $\llbracket 1, P \rrbracket$ and $\llbracket 1, Q \rrbracket$, we find that

$$\left| \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \left(\sum_{p=1}^P \gamma_{q,p} n_p + \beta_q \right)^{-s'_q} \right| \ll e^{(\sum_{p=1}^P |\tau_p| + \sum_{q=1}^Q |\tau'_q|) \frac{\pi}{2}} \left(P + \sum_{p=1}^P n_p \right)^{-\sigma'_1 - \dots - \sigma'_Q} \prod_{p=1}^P (n_p + 1)^{-\sigma_p}.$$

We now aim to show that the series

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \left(P + \sum_{p=1}^P n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)} \prod_{p=1}^P (n_p + 1)^{-\sigma_p}$$

is normally convergent on any compact subset of \mathcal{D} . This series can be split into a finite sum of series of the form

$$S_f(\boldsymbol{\sigma}, \boldsymbol{\sigma}') := \sum_{0 \leq n_{f(1)} \leq \dots \leq n_{f(P)}} \left(P + \sum_{p=1}^P n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)} \prod_{p=1}^P (n_p + 1)^{-\sigma_p},$$

with $f \in \mathfrak{S}_P$, and in terms of the form $Z_{1,1,P}^{k,1}((\sigma_1 + \dots + \sigma_{i_1}, \dots, \sigma_{i_{k-1}+1} + \dots + \sigma_{i_k}), \sigma'_1 + \dots + \sigma'_Q)$ with $1 = i_1 < i_2 < \dots < i_k = P$ and $(i_1, \dots, i_k) \neq (1, \dots, P)$ (which implies $k < P$). By induction principle, the Dirichlet series $Z_{1,1,P}^{k,1}((\sigma_1 + \dots + \sigma_{i_1}, \dots, \sigma_{i_{k-1}+1} + \dots + \sigma_{i_k}), \sigma'_1 + \dots + \sigma'_Q)$ normally converges on any compact subset of

$$\left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} (\sigma_{i_p} + \dots + \sigma_{i_{p+1}-1}) + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \mathcal{P} \subset \{i_1, \dots, i_{k-1}\} \right\},$$

and we see that the above set contains the set \mathcal{D} .

Let's consider the trivial permutation $f = \operatorname{Id} \in \mathfrak{S}_P$. By a change of variables we see that

$$S_{\operatorname{Id}}(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \sum_{\mathbf{n} \in \mathbb{N}_0^P} \left(P + \sum_{p=1}^P p n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)} \prod_{p=1}^P \left(1 + \sum_{i=1}^p n_i \right)^{-\sigma_p}.$$

By a previous inequality we find that:

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \left(P + \sum_{p=1}^P p n_p \right)^{-(\sigma'_1 + \dots + \sigma'_Q)} \prod_{p=1}^P \left(1 + \sum_{i=1}^p n_i \right)^{-\sigma_p} \ll \zeta_P^{EZ}(\sigma_1, \dots, \sigma_{P-1}, \sigma'_1 + \dots + \sigma'_Q + \sigma_P).$$

By Lemma 1.1.25, we find that the right-hand side of the last inequality is normally convergent on any compact subset of

$$\{(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sigma_p + \dots + \sigma_P + \sigma'_1 + \dots + \sigma'_Q > P + 1 - p, 1 \leq p \leq P\}.$$

Mutatis mutandis we get that, for all $f \in \mathfrak{S}_P$, the series

$$\sum_{0 \leq n_{f(1)} \leq \dots \leq n_{f(P)}} (P + \sum_{p=1}^P n_p)^{-(\sigma'_1 + \dots + \sigma'_Q)} \prod_{p=1}^P (n_p + 1)^{-\sigma_p}$$

normally converges on any compact subset of

$$\{(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sigma_{f(p)} + \dots + \sigma_{f(P)} + \sigma'_1 + \dots + \sigma'_Q > P + 1 - p, 1 \leq p \leq P\}.$$

We then get that $Z_{\gamma, \alpha, \beta}^{P, Q}$ normally converges on any compact subset of

$$\left\{ (\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q \mid \sum_{p \in \mathcal{P}} \sigma_p + \sigma'_1 + \dots + \sigma'_Q > |\mathcal{P}|, \emptyset \neq \mathcal{P} \subset [1, P] \right\}.$$

□

To obtain the meromorphic continuation of the multizeta function $Z(\mathbf{s}, \mathbf{s}')$, and the singularities of this multizeta function, we can simply use Theorem 1.1.27. Indeed, via Remark 1.2.8, the multizeta function $Z(\mathbf{s}, \mathbf{s}')$ is of the form $\zeta^{HL}(\mathbf{0}, \tilde{\mathbf{d}}, \tilde{\mathbf{c}}, (s_1, \dots, s_P, s'_1, \dots, s'_Q))$ as introduced in Definition 1.1.26. Therefore, we can apply Theorem 1.1.27 to this multizeta function.

2.2 Statement of the continuation formula for $Z_{\mathbf{N}, \mathbf{N}'}(s)$

We aim to establish a continuation formula for $Z_{\mathbf{N}, \mathbf{N}'}(s)$. Before expressing this result, we need to introduce some functions defined by parametric integrals that will appear in the continuation formula.

Definition 2.2.1. For all set $\mathcal{P} \subsetneq [1, P]$, all integer $j \in \llbracket 1, Q \rrbracket$, and all multi-index $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$, we set $l_p^*(\mathbf{x}) = c_{1,p}x_1 + \dots + c_{Q,p}x_Q$ for all $1 \leq p \leq P$, and we also set the functions

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx}_j \dots dx_Q,$$

and

$$h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx}_j \dots dx_Q,$$

where $s = \sigma + i\tau \in \mathbb{C}$ is a complex number with $\sigma \gg 1$ big enough, and where we have set

$$\begin{cases} dx_1 \dots \widehat{dx}_j \dots dx_Q = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_Q \\ \widehat{\mathbf{x}}^j = (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_Q). \end{cases}$$

We now state in the following proposition some results about these functions. In particular, we shall state an upper bound of these functions in order to control later a series, and we shall give an expression of these functions in order to compute the values at $s = 0$ of the function $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ and of its derivative

$h'_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ in chapter 3.

Proposition 2.2.2. *Let $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ a direction, and $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ a $P + Q$ -tuple. Let $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, $j \in \llbracket 1, Q \rrbracket$ and $\mathbf{k} \in \mathbb{N}_0^{P^c}$. We have that*

i) *The functions $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}$ and $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}$ are holomorphic on \mathbb{C} .*

ii) *For all positive integer $r \in \mathbb{N}$, we have the following upper bound uniformly for $s \in \overline{D}_0(r)$:*

$$|h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)| \ll_r (QM)^{|\mathbf{k}|}$$

with $M := \max_{\substack{1 \leq q \leq Q \\ 1 \leq p \leq P}} (|c_{q,p}|)$.

iii) *Let $\epsilon > 0$ small enough and $s \in \mathbb{C}$, we have*

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) &= \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q,p}^{w_{p,q}} \right) \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|)}}{\left(\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s) \right) \left(\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|) \right)} \\ &\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet,q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q, \end{aligned}$$

with the notations $|\mathbf{v}_{\bullet,q}| = \sum_{p \in \mathcal{P}} v_{p,q}$ for all $q \in \mathcal{Q}$, and $|\mathbf{w}_{\bullet,q}| = \sum_{p \in \mathcal{P}^c} w_{p,q}$ for all $q \in \llbracket 1, Q \rrbracket$. Furthermore, we can differentiate the general term of the previous series with respect to s in order to obtain an expression for the derivative $(h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}})'(s)$.

Remark 2.2.3. 1) *Point iii) of this proposition will be critical for studying the values of the function $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ and of its derivative $(h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}})'(s)$ at $s = 0$ in the subsections 3.2.2 and 3.2.3. Both of these values will appear in the computation of the directional value $Z(-(\mathbf{N}, \mathbf{N}'))$ and the directional derivative value $Z'(-(\mathbf{N}, \mathbf{N}'))$.*

2) *One can obtain similar results to the points ii) and iii) of the previous proposition for the functions $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$. However we won't need to state such results because the value and derivative value at $s = 0$ of these type of functions won't appear in the computations of the directional value $Z'(-(\mathbf{N}, \mathbf{N}'))$ and the directional derivative value $Z'(-(\mathbf{N}, \mathbf{N}'))$.*

We can now state the continuation formula for $Z_{\mathbf{N}, \mathbf{N}'}(s)$. We will decompose this analytic function into two parts that will depend on s and a sufficiently small free variable $\theta > 0$. The first, denoted as $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$, will be holomorphic in s and will vanish at $s = 0$. The other term will be an Erdélyi-type series (in the same fashion as the series in Erdélyi formula (2.3)) denoted as $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$. This second term

will contain the hypothetical singularities of $Z_{\mathbf{N}, \mathbf{N}'}(s)$, and we will denote the set of these hypothetical singularities as $\mathcal{S}(\mathbf{N}, \mathbf{N}')$, the expression of which is

$$\mathcal{S}(\mathbf{N}, \mathbf{N}') = \left[\bigcup_{\substack{1 \leq p \leq P \\ \mu_p \neq 0}} \frac{1}{\mu_p} \mathbb{Z}_{\geq N_p + 1} \right] \cup \left[\bigcup_{\mathcal{P} \subset [1, P]} \frac{1}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \mathbb{Z}_{\leq |\mathbf{N}|_{\mathcal{P}} + |\mathbf{N}'|_{\mathcal{P}}} \right].$$

Proposition 2.2.4. *Let $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ a $P+Q$ -tuple and $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ a direction. Then the function $Z_{\mathbf{N}, \mathbf{N}'}(s)$ has a meromorphic continuation on \mathbb{C} , and its poles are included in the set $\mathcal{S}(\mathbf{N}, \mathbf{N}')$. Furthermore, for all $\theta > 0$ small enough, and for all $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$, we have*

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$$

with $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ and $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ being two meromorphic functions on \mathbb{C} such that:

1) For all $s \in \mathbb{C}$, we set

$$K_{\mathbf{N}, \mathbf{N}'}(\theta, s) := \sum_{\emptyset \neq \mathcal{Q} \subset [1, Q]} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \left(\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right). \quad (2.1)$$

2) For all $s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$, we set

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) := \sum_{\mathcal{P} \subsetneq [1, P]} \left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right) \prod_{p \in \mathcal{P}^c} \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right).$$

Moreover, we have that:

i) The function $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ is holomorphic on \mathbb{C} , and vanishes at $s = 0$.

ii) The function $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ is meromorphic on \mathbb{C} , with poles inside $\mathcal{S}(\mathbf{N}, \mathbf{N}')$. Furthermore, one can take the derivative with respect to s of the series in formula (2.2) termwise.

Remark 2.2.5. 1) The function $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ evaluated at $s = 0$ will be the sole contributor when computing the directional value $Z(-(\mathbf{N}, \mathbf{N}'))$, since the function $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ vanishes at $s = 0$. However, when studying the value of the derivative $Z'(-(\mathbf{N}, \mathbf{N}'))$, we will have to consider the derivative of the term $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$.

2) We will see in the proof of Proposition 2.2.4 that the series

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \prod_{q \in A_Q} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_Q} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q),$$

describes a holomorphic function in s on \mathbb{C} (the series is actually absolutely convergent on \mathbb{C}). In particular, we can consider the derivative of this series with respect to s , and evaluate it at $s = 0$ in the proof of Theorem D. Note that we will never attempt to directly compute the values of this series, or the values of its derivative with respect to s .

With the help of this proposition, we will then establish that the function $s \mapsto Z_{\mathbf{N}, \mathbf{N}'}(s)$ is regular at $s = 0$, and will thus have proven by another method than Komori in [Kom10] the existence of directional values for the class of zeta functions that we consider here. We also retrieve some information on the poles of $Z_{\mathbf{N}, \mathbf{N}'}(s)$:

Corollary 2.2.6. *Let a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. Then the function $s \mapsto Z_{\mathbf{N}, \mathbf{N}'}(s)$ has candidate poles of the form*

$$s = \frac{n}{\mu_p} \quad (1 \leq p \leq P, n \geq N_p + 1, \text{ if } \mu_p \neq 0)$$

$$s = \frac{n}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}} \quad (\mathcal{P} \subset \llbracket 1, P \rrbracket, n \in \mathbb{Z} - \infty, |\mathbf{N}|_{|\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}| \setminus \{0\}).$$

Moreover, if $s \in \mathbb{C}$ is a pole that cannot be written as $s = \frac{n}{\mu_p}$ with $1 \leq p \leq P$ and $n \geq N_p + 1$, then s is a simple pole of the function $s \mapsto Z_{\mathbf{N}, \mathbf{N}'}(s)$.

Remark 2.2.7. *One could get an even more precise set of candidate poles for the function $Z_{\mathbf{N}, \mathbf{N}'}(s)$ by taking into account the previous corollary and Komori's Theorem 1.1.27.*

In order to prove the Proposition 2.2.4 we will multiply the Dirichlet series $Z(\mathbf{s}, \mathbf{s}')$ by the product $\Gamma(s'_1) \dots \Gamma(s'_Q)$. This strategy is classic for studying the values of the Riemann zeta function (see [Hid93, Chap. 2]), for obtaining an integral representation of a Shintani zeta function (see [Hid93, Chap. 2]), and for computing the values of the multiple Hurwitz zeta function of Mordell-Tornheim type (see [Ono21]).

Next, we will proceed to partition the integration domain by introducing a free variable θ . By doing so, we will isolate a neighborhood of the origin from the rest of the integration domain. Finally, we will note that the integral around the neighborhood of the origin can be re-expressed using the Erdélyi formula (2.3), and we will also prove that the remaining integrals actually define a holomorphic function in $(\mathbf{s}, \mathbf{s}')$. To partition the integrals appearing in our calculations, we will use the incomplete gamma functions $\Gamma(s, \theta, \nu)$ and $\gamma(s, \theta, \nu)$ defined in the Definition 1.2.1.

2.3 Directional Crandall's expansion for $Z(\mathbf{s}, \mathbf{s}')$

2.3.1 Lemmata

We shall start by stating a few lemma. Firstly, let's state a classical identity obtained by Taylor's theorem:

Lemma 2.3.1. *Let X_1, \dots, X_d be complex numbers such that $|X_1 + \dots + X_d| < 1$. We then have that*

$$(1 + X_1 + \dots + X_d)^{-s} = \sum_{\mathbf{k} = (k_p)_{p \in \llbracket 1, d \rrbracket}} \binom{-s}{|\mathbf{k}|} \binom{|\mathbf{k}|}{\mathbf{k}} X_1^{k_1} \dots X_d^{k_d}.$$

The expressions in the proof of Proposition 2.2.4 will involve a series whose general term contains $\zeta(s-k, d)$ where $k \in \mathbb{N}_0$, and $d \in H_0$. In [Ono21], Onodera also had these terms in one of his expressions, and he controlled them using the following lemma, in the case where $d \in \mathbb{R}_+^*$:

Lemma 2.3.2 ([Ono21, Lemma 2.1]). *Let $\alpha > 0$. We set for all $\delta \in \mathbb{R}_+^*$ and all $r \in \mathbb{N}$,*

$$D(\delta, r) := \left\{ s \in \mathbb{C} \mid \min_{n \in \mathbb{N}} |s - n| \geq \delta, |s| \leq r \right\}.$$

Then for all $s \in D(\delta, r)$, $k \in \mathbb{N}_0$, $d > 0$, $m \in \mathbb{N}_0$, we have

$$|\partial_s^m \zeta(s - k, d)| \ll_{m, r, \delta, \alpha} k!(k+1)^{r+\alpha}.$$

However, this lemma is not sufficient here, as we want to bound $\zeta(s-k, d)$ with $d \in H_0$. The proof of the previous lemma uses the functional equation satisfied by the Hurwitz zeta function $\zeta(s, d)$ for $d \in]0, 1]$. We cannot prove a similar bound for $d \in H_0$ in such a fashion. We will therefore use Euler-Maclaurin's formula on the Hurwitz zeta function to obtain an inequality similar to the previous lemma.

Lemma 2.3.3. *Let $\delta \in \mathbb{R}_+^*$, $r \in \mathbb{N}$, and $r' \in]1, +\infty[$. Then we have that*

$$\forall s \in D(\delta, r), \forall d \in \overline{H}_{1/r'} \cap \overline{D}_0(r'), \forall k \in \mathbb{N}_0, \quad |\zeta(s - k, d)| \ll_{r, r', \delta} \left(\frac{e}{\pi}\right)^k k!(k+1)^{r+1} (1 + r'^{2k}).$$

Proof. Let $r \in \mathbb{N}$, we set $n = r + 1$. We aim to bound $|\zeta(s - k, d)|$ uniformly in $s \in D(\delta, r)$ and in $d \in \overline{H}_{1/r'} \cap \overline{D}_0(r')$. Let $s \in D(\delta, r)$ and $d \in \overline{H}_{1/r'} \cap \overline{D}_0(r')$. We note that $n > -\sigma$. Let $k \in \mathbb{N}_0$, as we have $n + k > -\sigma + k$, we can use the expression for the Hurwitz zeta function from Maclaurin's formula (1.1.6) up to order $n + k$, and evaluate it at $s - k$.

$$\begin{aligned} \zeta(s - k, d) &= \frac{d^{1-s+k}}{1-s+k} + \sum_{i=0}^{n+k} \binom{-s+k}{i} \frac{(-1)^i B_{i+1}}{i+1} d^{-s-i+k} \\ &\quad + (-1)^{n+k} \binom{-s+k}{n+1+k} \int_0^{+\infty} b_{n+k+1}(x) (x+d)^{-s-n-1} dx, \end{aligned}$$

with $b_i(x)$ the i -th periodised Bernoulli polynomial, and B_i the i -th Bernoulli number. We now have the following inequality

$$\begin{aligned} |\zeta(s - k, d)| &\leq \frac{|d|^{1-\sigma+k}}{|1-s+k|} e^{|\tau|\pi/2} + \sum_{i=0}^{n+k} \left| \binom{-s+k}{i} \right| \frac{|B_{i+1}|}{i+1} |d|^{-\sigma-i+k} e^{|\tau|\pi/2} \\ &\quad + \left| \binom{-s+k}{n+1+k} \right| \int_0^{+\infty} |b_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx. \end{aligned}$$

Let's find an upper bound for each of these terms.

We clearly have that $e^{|\tau|\pi/2} \leq e^{r\pi/2}$. By compactness, we get that the continuous function $(x, y) \in [1/r', r'] \times [-r, r] \mapsto x^y$ is bounded, thus we get

$$|d|^{-\sigma+k} \ll_{r, r'} r'^k.$$

By construction of the set $D(\delta, r)$, we have that

$$\begin{aligned} \frac{|d|^{1-\sigma+k}}{|1-s+k|} &\ll_{r, r'} \frac{r'^k}{\delta} \\ &\ll_{r, r', \delta} r'^k. \end{aligned}$$

We have the following inequality for the multinomial coefficients

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad \left| \binom{-s+k}{i} \right| \leq \binom{r+k+i-1}{i} \leq \binom{2n+2k}{i}.$$

Let $i \in \mathbb{N}_0$. By [Leh40], we know that if $i \not\equiv 2 \pmod{4}$, we have

$$\forall x \in \mathbb{R}, \quad |b_i(x)| < \frac{2i!}{(2\pi)^i},$$

and if we have $i \equiv 2 \pmod{4}$, we get

$$\forall x \in \mathbb{R}, \quad |b_i(x)| < \frac{2i!\zeta(i)}{(2\pi)^i} \leq \frac{2i!\zeta(2)}{(2\pi)^i}.$$

Let's note ψ the digamma function. We know that $\lim_{x \rightarrow +\infty} \psi(x) = +\infty$. Therefore, there exists a real number x_0 such that the function $\frac{\Gamma(x)}{(2\pi)^x}(\psi(x) - \ln(2\pi))$ is positive for all $x \geq x_0$. We then find that the function

$$x \in \mathbb{R}_+^* \mapsto \frac{\Gamma(x)}{(2\pi)^x}$$

is increasing after x_0 . Therefore, for n big enough, we have that

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad \frac{i!}{(2\pi)^i} \ll_n \frac{(n+k)!}{(2\pi)^{n+k}} \ll_n \frac{1}{(2\pi)^k} k!(k+1)\dots(k+n).$$

Taking an upper bound of each term inside the product in the right-hand side of the last equality, we get that $(k+1)\dots(k+n) \ll_n (k+1)^n$. Since $n = r+1$, we get

$$\forall i \in \llbracket 0, n+k \rrbracket, \forall x \in \mathbb{R}, \quad |b_i(x)| \ll_r \frac{k!}{(2\pi)^k} (k+1)^{r+1}.$$

By taking $x = 0$ in the previous inequality, we have

$$\forall i \in \llbracket 0, n+k \rrbracket, \quad |B_i| \ll_r \frac{k!}{(2\pi)^k} (k+1)^{r+1}.$$

By all the above, we have

$$\begin{aligned} \sum_{i=0}^{n+k} \left| \binom{-s+k}{i} \right| \frac{|B_{i+1}|}{i+1} |d|^{-\sigma-i+k} &\ll_r |d|^{-\sigma+k} \frac{k!}{(2\pi)^k} (k+1)^n \sum_{i=0}^{2n+2k} \binom{2n+2k}{i} |d|^{-i} \\ &\ll_{r,r'} \frac{k!}{(2\pi)^k} r'^k (k+1)^{r+1} (1+|d|^{-1})^{2n+2k} \\ &\ll_{r,r'} \frac{k!}{(2\pi)^k} (k+1)^{r+1} (1+r')^{2k}. \end{aligned}$$

As $-\sigma - n \leq -1$ it is clear that

$$\forall x \in \mathbb{R}_+, \quad |x+d|^{-\sigma-n-1} \leq (x + \operatorname{Re}(d))^{-\sigma-n-1}.$$

The right-hand side term is integrable over \mathbb{R}_+ , therefore we obtain

$$\left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |b_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx \leq \binom{n+2k+1}{n+k+1} \frac{2(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} (\sigma+n)}.$$

By a classical inequality, we have

$$\binom{m}{i} \leq e^i \left(\frac{m}{i}\right)^i \quad (0 \leq i \leq m).$$

We then find that

$$\begin{aligned} \left| \binom{-s+k}{n+k+1} \right| \int_0^{+\infty} |b_{n+k+1}(x)| |x+d|^{-\sigma-n-1} e^{|\tau|\pi/2} dx \\ \leq 2e^k \left(\frac{n+2k+1}{k} \right)^k \frac{(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} \sigma+n} \\ \leq 2(2e)^k \left(1 + \frac{n+1}{2k} \right)^k \frac{(n+k+1)! \operatorname{Re}(d)^{-\sigma-n}}{(2\pi)^{n+k+1} \sigma+n}. \end{aligned}$$

From the following inequality

$$\forall x \in]0, k[, \quad \left(1 + \frac{x}{k} \right)^k \leq e^x,$$

we then find that

$$\forall k > (n+1)/2, \quad \left(1 + \frac{n+1}{2k} \right)^k \leq e^{(n+1)/2}.$$

As $n = r + 1$, we obtain

$$\forall k \in \mathbb{N}_0, \quad \left(1 + \frac{n+1}{2k} \right)^k \ll_r 1.$$

Finally, we get that

$$\begin{aligned} |\zeta(s-k, d)| \ll_{r, r', \delta} r'^k + \frac{k!}{(2\pi)^k} (k+1)^n (1+r'^{2k}) + \frac{(2e)^k k!}{(2\pi)^k} (k+1)^{n+1} \\ \ll_{r, r', \delta} \left(\frac{e}{\pi} \right)^k k! (k+1)^{r+1} (1+r'^{2k}). \end{aligned}$$

□

For all nonnegative integers $a \in \mathbb{N}_0$, we observe that the function $s \mapsto \frac{1}{\Gamma(s)(s+a)}$ is holomorphic on \mathbb{C} . We get the following lemma:

Lemma 2.3.4. *For all integer $r \in \mathbb{N}$ we have*

$$\begin{aligned} \forall s \in \overline{D}_0(r), \quad \left| \frac{1}{\Gamma(s)} \right| \ll_r 1 \\ \forall s \in \overline{D}_0(r), \forall a \in \mathbb{N}_0, \quad \left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r 1. \end{aligned}$$

Proof. Since the functions $s \mapsto \frac{1}{\Gamma(s)}$ and $s \mapsto \frac{1}{\Gamma(s)s}$ are holomorphic on \mathbb{C} , we have uniformly on $\overline{D}_0(r)$:

$$\begin{aligned} \left| \frac{1}{\Gamma(s)} \right| \ll_r 1 \\ \left| \frac{1}{\Gamma(s)s} \right| \ll_r 1. \end{aligned}$$

Thus it is clear that

$$\forall |s| > \frac{1}{2}, \quad \left| \frac{1}{s} \right| < 2.$$

One can then show the inequality required. Let $s \in \mathbb{C}$ such that $\sigma \in [-r, r]$.

If $|s+a| > \frac{1}{2}$, then

$$\left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r 1.$$

If $|s + a| \leq \frac{1}{2}$ then by the triangular inequality, we get $a \leq |s| + 1/2 \leq r + 1/2$. Using the functional equation of Γ we get

$$\frac{1}{\Gamma(s)(s+a)} = \frac{(s+a-1)\dots s}{\Gamma(s+a)(s+a)},$$

thus we have the following inequality:

$$\left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r r(r+1)\dots(2r+1).$$

As $a \leq 1/2 + r$, we then get the following upper bound for the product in the numerator

$$|(s+a-1)\dots s| \leq (2r+1)!.$$

Therefore, we find that

$$\forall s \in \overline{D}_0(r), \quad \left| \frac{1}{\Gamma(s)(s+a)} \right| \ll_r 1.$$

□

Erdélyi's formula [EMOT81, §1.11] described in the following proposition is crucial for re-expressing integrands involving the Lerch zeta function that we defined in Definition 1.1.7:

Proposition 2.3.5 ([Ono21, Lemma 2.2]). *Let $u \in \mathbb{C} \setminus]-\infty, 0]$ such that $|u| < 2\pi$, $s \in \mathbb{C} \setminus \mathbb{N}$, and $d \in \mathbb{R}_+^*$. Then*

$$e^{-du} \phi(e^{-u}, s, d) = \Gamma(1-s)u^{s-1} + \sum_{k=0}^{+\infty} \frac{(-u)^k}{k!} \zeta(s-k, d).$$

We aim to obtain an Erdélyi formula for $d \in H_0$. The bound we have on the term $\zeta(s-k, d)$ is not as good when d is a complex number as when d is real, thus we must assume an additional condition on the complex u :

Proposition 2.3.6 (Erdélyi Formula). *Let $d \in H_0$, $s \in \mathbb{C} \setminus \mathbb{N}$ and $u \in \mathbb{C} \setminus]-\infty, 0]$ such that $|u| \ll_d 1$. then*

$$e^{-du} \phi(e^{-u}, s, d) = \Gamma(1-s)u^{s-1} + \sum_{k=0}^{+\infty} \frac{(-u)^k}{k!} \zeta(s-k, d). \quad (2.3)$$

Proof. Let $r' \in [1, +\infty[$ be a real number, we will aim to prove the result for $d \in \overline{H}_{1/r'} \cap \overline{D}_0(r')$. Let $u \in \mathbb{C} \setminus]-\infty, 0]$ such that $|u| < \frac{\pi}{e(1+r')^2}$. We fix $s \in \mathbb{C} \setminus \mathbb{N}$, and set $r := \lceil |s| \rceil$. We already know that the formula we wish to prove holds when $d \in \mathbb{R}_+^*$. We then aim to prove this formula for all $d \in H_0$ through analytic continuation. To do this, we first need to show that the different terms in the expression of the desired formula are holomorphic in the variable d on H_0 .

We know that the map $d \in H_0 \mapsto e^{-du} \phi(e^{-u}, s, d)$ is holomorphic (see [EMOT81, §1.11]), it remains to show that the series appearing in the formula is holomorphic as well. We already know that the function $d \in H_0 \mapsto \zeta(s-k, d)$ is holomorphic on H_0 for all integer $k \in \mathbb{N}_0$. To prove this, it is enough to refer to the analytical continuation of the Hurwitz zeta function via Euler Maclaurin's formula, as we did in the proof of Proposition 1.1.6. What remains is to show that the series appearing in Erdélyi's formula (2.3) normally converges in the variable d on the set $\overline{H}_{1/r'} \cap \overline{D}_0(r')$. We have the following bound

$$\forall k \in \mathbb{N}_0, \quad |\zeta(s-k, d)| \ll_{r, r', \delta} \left(\frac{e}{\pi}\right)^k k!(k+1)^{r+1}(1+r')^{2k},$$

thus, we find that the general term of the series in Erdélyi's formula (2.3) has the following upper bound

$$\frac{|\zeta(s-k, d)|}{k!} |u|^k \ll_{r, r', \delta} \left(u \frac{e}{\pi} (1+r')^2\right)^k (k+1)^{r+1},$$

but the power series

$$\sum_{k=0}^{+\infty} \left(\frac{e}{\pi} (1+r')^2 \right)^k k^{r+1} z^k$$

is absolutely convergent in the open disc $\left\{ z \in \mathbb{C}, |z| < \frac{\pi}{e(1+r')^2} \right\}$. We then find that the power series

$\sum_{k=0}^{+\infty} \frac{\zeta(s-k, d)}{k!} (-u)^k$ is normally convergent in the variable d on the set $\overline{H}_{1/r'} \cap \overline{D}_0(r')$. Therefore, Erdélyi's formula (2.3) holds for all $d \in H_0$ by the analytic continuation principle. \square

2.3.2 Crandall's expansion

In this section, we wish to establish a Crandall expansion for the multizeta function.

$$Z(\mathbf{s}, \mathbf{s}') := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}.$$

We recall Definition 1.2.1: Let $\nu \in H_0$, $\theta > 0$, $s \in \mathbb{C}$. We defined

$$\begin{aligned} \Gamma(s, \theta, \nu) &= \int_{\theta}^{+\infty} e^{-\nu y} y^{s-1} dy, \\ \gamma(s, \theta, \nu) &= \int_0^{\theta} e^{-\nu y} y^{s-1} dy \quad (\text{when } \operatorname{Re}(s) > 0). \end{aligned}$$

Let's note that the integral defining $\Gamma(s, \theta, \nu)$ is absolutely convergent for all $s \in \mathbb{C}$, but the one defining $\gamma(s, \theta, \nu)$ is absolutely convergent only for $s \in H_0$.

Proposition 2.3.7. *Let $\theta > 0$. Then the functions $(s, \nu) \in \mathbb{C} \times H_0 \mapsto \Gamma(s, \theta, \nu)$ and $(s, \nu) \in H_0 \times H_0 \mapsto \gamma(s, \theta, \nu)$ are holomorphic respectively on $\mathbb{C} \times H_0$ and on $H_0 \times H_0$.*

Proof. The proposition follows from the theorem of holomorphy under the integral sign. \square

Proposition 2.3.8. *Let $\nu \in H_0$, $\theta > 0$. We have*

$$\forall s \in H_0, \quad \Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) = \nu^{-s} \Gamma(s).$$

Proof. Let's assume that $\nu \in \mathbb{R}_+^*$. By a change of variables we get:

$$\begin{aligned} \Gamma(s, \theta, \nu) &= \nu^{-s} \int_{\nu\theta}^{+\infty} e^{-x} x^{s-1} dx, \\ \gamma(s, \theta, \nu) &= \nu^{-s} \int_0^{\nu\theta} e^{-x} x^{s-1} dx \quad (\text{when } \operatorname{Re}(s) > 0). \end{aligned}$$

Summing up these 2 terms, we have

$$\Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) = \nu^{-s} \Gamma(s).$$

For all $s \in H_0$, the function $\nu \in H_0 \mapsto \Gamma(s, \theta, \nu) + \gamma(s, \theta, \nu) - \nu^{-s} \Gamma(s)$ is holomorphic, and vanishes when $\nu \in \mathbb{R}_+^*$. By the isolated zeros principle, we get the final result. \square

Corollary 2.3.9. *The function $s \mapsto \gamma(s, \theta, \nu)$ is meromorphic on \mathbb{C} with poles at nonpositive integers, which are simple poles, and with residue $\operatorname{Res}_{s=-n} \gamma(s, \theta, \nu) = \frac{(-1)^n \nu^n}{n!}$.*

Lemma 2.3.10. *For all positive real number $\theta > 0$, and all complex numbers $\nu \in H_0$ and $s \in \mathbb{C}$ we have the following bound:*

$$|\Gamma(s, \theta, \nu)| \leq 2^\sigma e^{-\operatorname{Re}(\nu)\theta/2} \Gamma\left(\sigma, \frac{\theta}{2}, \operatorname{Re}(\nu)\right).$$

Proof. By the change of variable $x = \frac{x'}{2}$, we find

$$\Gamma(s, \theta, \nu) = 2^s \int_{\theta/2}^{+\infty} e^{-2\nu x} x^{s-1} dx.$$

As $|e^{-\nu x}| = e^{-\operatorname{Re}(\nu)x}$ for all $x \in \mathbb{R}_+$ and for all complex number $\nu \in H_0$, we find the following bound:

$$\begin{aligned} |\Gamma(s, \theta, \nu)| &\leq |2^s| \int_{\theta/2}^{+\infty} e^{-\operatorname{Re}(\nu)x} e^{-\operatorname{Re}(\nu)x} x^{s-1} dx \\ &\leq 2^\sigma \int_{\theta/2}^{+\infty} e^{-\operatorname{Re}(\nu)\theta/2} e^{-\operatorname{Re}(\nu)x} x^{\sigma-1} dx, \end{aligned}$$

which concludes the proof. \square

We now study the product $\prod_{q=1}^Q \Gamma(s'_q) Z(\mathbf{s}, \mathbf{s}')$. Let $q \in \llbracket 1, Q \rrbracket$, $\theta > 0$, and $s'_q \in H_0$. By Proposition 2.3.8, we get

$$\Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} = \Gamma(s'_q, \theta, (l_q(\mathbf{n}) + d'_q)) + \gamma(s'_q, \theta, (l_q(\mathbf{n}) + d'_q)).$$

We fix a P -tuple $\mathbf{n} \in \mathbb{N}_0^P$, and complex numbers $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times H_0^Q$. We thus obtain the following equality by distributing the sum above over the subsequent product:

$$\begin{aligned} &\prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} \\ &= \prod_{q=1}^Q \left[\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) + \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right] \\ &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in \mathcal{Q}^c} \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) + \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in \mathcal{Q}^c} \left((l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) - \Gamma(s'_q, \theta, l'_q(\mathbf{n}) + d'_q) \right) \\ &\quad + \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned}$$

By distributing the terms $(l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) - \Gamma(s'_q, \theta, l'_q(\mathbf{n}) + d'_q)$, we get

$$\begin{aligned} &\prod_{q=1}^Q \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} = \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &+ \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \prod_{q \in \mathcal{Q}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in A_{\mathcal{Q}}} \Gamma(s'_q)(l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \mathcal{Q}^c \setminus A_{\mathcal{Q}}} (-\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)). \end{aligned}$$

Multiplying the previous equality by $\prod_{p=1}^P (n_p + d_p)^{-s_p}$, and using the fact that $\mathcal{Q} \cup (\mathcal{Q}^c \setminus A_{\mathcal{Q}}) = \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}$, we have

$$\begin{aligned} \prod_{q=1}^Q \Gamma(s'_q) (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{p=1}^P (n_p + d_p)^{-s_p} &= \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ + \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} &\prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \Gamma(s'_q) \end{aligned} \quad (2.4)$$

and by summing the last equality over $n_p \in \mathbb{N}_0$ for all $p \in \llbracket 1, P \rrbracket$, we get at least formally that

$$\begin{aligned} Z(\mathbf{s}, \mathbf{s}') &= \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \\ &+ \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \frac{(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|}}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q)} \\ &\cdot \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned}$$

Further on we will give an analytical meaning of the above formula. More precisely, we will prove that it holds for all $(\mathbf{s}, \mathbf{s}') \in \mathcal{D}$ (with \mathcal{D} being the set defined in Proposition 2.1.1), and we will see that the two series in the above formula are absolutely convergent on \mathcal{D} . By formula (2.4) and using the fact that the Dirichlet series $Z(\mathbf{s}, \mathbf{s}')$ is absolutely convergent on \mathcal{D} , it is enough to show that one of the two series above converge absolutely on \mathcal{D} . In fact, we will show in the next proposition that the series in the second term normally converges on any compact subset of $\mathbb{C}^P \times \mathbb{C}^Q$.

Proposition 2.3.11. *For all $\theta > 0$ and all $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times \mathbb{C}^Q$, we set*

$$\begin{aligned} K(\theta, \mathbf{s}, \mathbf{s}') &:= \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \frac{(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|}}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q)} \\ &\cdot \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \end{aligned} \quad (2.5)$$

Then the series above normally converges on any compact subset of $\mathbb{C}^P \times \mathbb{C}^Q$. Moreover, the function $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$ is holomorphic on $\mathbb{C}^P \times \mathbb{C}^Q$, and $K(\theta, -\mathbf{N}, -\mathbf{N}') = 0$ for all $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$.

Proof. Let $\theta > 0$ be a real number. In order to prove that $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$ is holomorphic on $\mathbb{C}^P \times \mathbb{C}^Q$, it is enough to show that the series defining $K(\theta, \mathbf{s}, \mathbf{s}')$ normally converges on any compact subset of $\mathbb{C}^P \times \mathbb{C}^Q$. Therefore we will give an upper bound of the general term of the series that appear in the definition of $K(\theta, \mathbf{s}, \mathbf{s}')$. We know that the function $K(\theta, \mathbf{s}, \mathbf{s}')$ is a finite sum of the following terms

$$\frac{(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|}}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q)} \left(\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) \right), \quad (2.6)$$

with $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$, and $A_{\mathcal{Q}} \subset \mathcal{Q}^c$. We see that the functions $s'_q \mapsto \frac{1}{\Gamma(s'_q)}$ and $s'_q \mapsto \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)$ are well defined and holomorphic on \mathbb{C} . For all $\mathbf{n} \in \mathbb{N}_0$ we set $T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')$ as the general term of the

series in formula (2.6),

$$T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}') := \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q). \quad (2.7)$$

By Lemma 2.3.10 we get that for all $q \in [1, Q]$ and all $\mathbf{n} \in \mathbb{N}_0^P$, we have the following upper bound

$$|\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)| \leq 2^{\sigma'_q} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \frac{\theta}{2}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(l_q(\mathbf{n}) + d'_q)).$$

By the previous bound, and using the fact that the function $x \in \mathbb{R}_*^+ \mapsto \Gamma(\sigma, \theta, x)$ is decreasing for any given $\theta > 0$, we get

$$|\Gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)| \leq 2^{\sigma'_q} e^{-(\operatorname{Re}(l_q(\mathbf{n}) + d'_q)) \frac{\theta}{2}} \Gamma(\sigma'_q, \theta/2, \operatorname{Re}(d'_q)).$$

Finally, for all complex numbers $z \in H_0$, $s = \sigma + i\tau \in \mathbb{C}$, we get $|z^s| = |z|^\sigma e^{-\arg(z)\tau}$. As $\operatorname{Re}(z) > 0$, we then find that $\arg(z) \in]-\pi/2, \pi/2[$, and thus

$$|z^s| \leq |z|^\sigma e^{|\tau|\pi/2}.$$

From all the above, we get the following bound for the general term:

$$\begin{aligned} & |T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')| \\ & \leq \left| \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{-s'_q} \right| \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} 2^{\sigma'_q} e^{-\operatorname{Re}(l_q(\mathbf{n}) + d'_q) \frac{\theta}{2}} \Gamma\left(\sigma'_q, \frac{\theta}{2}, d'_q\right) \\ & \leq \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} e^{-\operatorname{Re}(l_q(\mathbf{n}) + d'_q) \frac{\theta}{2}} 2^{\sigma'_q} \Gamma\left(\sigma'_q, \frac{\theta}{2}, d'_q\right) \prod_{p=1}^P e^{|\tau_p| \frac{\pi}{2}} |n_p + d_p|^{-\sigma_p} \prod_{q \in A_{\mathcal{Q}}} e^{|\tau'_q| \frac{\pi}{2}} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}. \quad (2.8) \end{aligned}$$

Let's note that, if all the coefficients $c_{p,q}$, d_p and d'_q are real numbers, then we get an improved version of the previous inequality by replacing $e^{|\tau_p| \frac{\pi}{2}}$ and $e^{|\tau'_q| \frac{\pi}{2}}$ by 1.

Because the set \mathcal{Q} is non-empty, and that $A_{\mathcal{Q}} \subset \mathcal{Q}^c$, then this last set contains at least an integer $j \in A_{\mathcal{Q}}$, and by our assumptions, $l_j(\mathbf{n}) = \sum_{p=1}^P c_{j,p} n_p$, with $c_{j,p} \in H_0$ for all $p \in [1, P]$. We also have that $d'_q \in H_0$ for all $q \in [1, Q]$. Therefore,

$$\prod_{q \in \mathcal{Q}} e^{-\operatorname{Re}(l_q(\mathbf{n}) + d'_q) \theta/2} \leq e^{-\operatorname{Re}(l_j(\mathbf{n})) \theta/2}.$$

Let $r > 0$, and $(\mathbf{s}, \mathbf{s}') \in \overline{D}_0(r)^{P+Q}$. We then find that $(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \in [-r, r]^{P+Q}$ and $(\boldsymbol{\tau}, \boldsymbol{\tau}') \in [-r, r]^{P+Q}$. The map $x \in [-r, r] \mapsto \Gamma(x, \theta, \nu)$ being a continuous function for all $\theta > 0$, and all $\nu \in H_0$, we find the following bound, uniform in $(\mathbf{s}, \mathbf{s}') \in \overline{D}_0(r)^P \times \overline{D}_0(r)^Q$,

$$\prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma\left(\sigma'_q, \frac{\theta}{2}, \operatorname{Re}(d'_q)\right) \ll_r 1.$$

By the bound (2.8), we find that for all $(\mathbf{s}, \mathbf{s}') \in \overline{D}_0(r)^P \times \overline{D}_0(r)^Q$,

$$|T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')| \ll_r \prod_{p=1}^P e^{-\operatorname{Re}(c_{j,p}) n_p \frac{\theta}{2}} |n_p + d_p|^{-\sigma_p} \prod_{q \in A_{\mathcal{Q}}} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}. \quad (2.9)$$

As $\theta > 0$ and $\operatorname{Re}(c_{j,p}) > 0$ for all $p \in [1, P]$, we know that the series

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P e^{-\operatorname{Re}(c_{j,p}) n_p \theta/2} |n_p + d_p|^{-\sigma_p} \prod_{q \in A_{\mathcal{Q}}} |l_q(\mathbf{n}) + d'_q|^{-\sigma'_q}$$

normally converges on any compact subset of $\mathbb{C}^P \times \mathbb{C}^Q$. Thus, using the previous inequality, we find that the series $\sum_{\mathbf{n} \in \mathbb{N}_0^P} T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')$ normally converges on $\overline{D}_0(r)^{P+Q}$. We then get that $K(\theta, \mathbf{s}, \mathbf{s}')$ is holomorphic on $\mathbb{C}^P \times \mathbb{C}^Q$.

Finally, we see that, for all $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$ and for all $A_{\mathcal{Q}} \subset \mathcal{Q}^c$, we have that $\llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}} \neq \emptyset$. Therefore, the product $\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(s'_q)}$ is non-empty. In particular, when we take the values of $K(\theta, \mathbf{s}, \mathbf{s}')$ at $\mathbf{s}' = -\mathbf{N}'$, we get 0. \square

Remark 2.3.12. By evaluating formula (2.5) and replacing \mathbf{s} by $(-N_p + \mu_p s)_{p \in \llbracket 1, P \rrbracket}$, and \mathbf{s}' by $(-N'_q + \mu'_q s)_{q \in \llbracket 1, Q \rrbracket}$, we recover the expression (2.1) in Proposition 2.2.4.

Proposition 2.3.13. Let $\theta > 0$ and $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$, with \mathcal{D} being the set defined in Proposition 2.1.1. We set

$$J(\theta, \mathbf{s}, \mathbf{s}') := \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q).$$

The previous series is normally convergent on any compact subset of $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$. In particular, the function $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$ is holomorphic on $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$.

Proof. Let $\theta > 0$ be a positive real number. We use the notations from the previous proposition, specifically the expression (2.7) of the general term of the series in the definition of $K(\theta, \mathbf{s}, \mathbf{s}')$. By formula (2.4), we know that, for all $\mathbf{n} \in \mathbb{N}_0^P$ and all $(\mathbf{s}, \mathbf{s}') \in \mathbb{C}^P \times H_0^Q$,

$$\begin{aligned} \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q) &= \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q} \\ &\quad - \sum_{\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} \frac{(-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|}}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(s'_q)} T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}'). \end{aligned}$$

In the proof of Proposition 2.3.11, we showed using inequality (2.9) that the series with general term $T_{\mathcal{Q}, A_{\mathcal{Q}}}(\theta, \mathbf{n}, \mathbf{s}, \mathbf{s}')$ is normally convergent on any compact subset of $\mathbb{C}^P \times \mathbb{C}^Q$. We proved in Proposition 2.1.1 that the series of functions in $(\mathbf{s}, \mathbf{s}')$

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q (l_q(\mathbf{n}) + d'_q)^{-s'_q}$$

is normally convergent on any compact subset of \mathcal{D} . Thus, we obtain that the series

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \prod_{p=1}^P (n_p + d_p)^{-s_p} \prod_{q=1}^Q \gamma(s'_q, \theta, l_q(\mathbf{n}) + d'_q)$$

is normally convergent on any compact subset of $\mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$. \square

Proposition 2.3.14. Let $\theta > 0$, and $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$. We have:

$$Z(\mathbf{s}, \mathbf{s}') = K(\theta, \mathbf{s}, \mathbf{s}') + J(\theta, \mathbf{s}, \mathbf{s}'). \quad (2.10)$$

Furthermore, the function $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$ admits a meromorphic continuation on $\mathbb{C}^P \times \mathbb{C}^Q$.

Proof. The first equality follows from the last two propositions. Since $(\mathbf{s}, \mathbf{s}') \mapsto Z(\mathbf{s}, \mathbf{s}')$ is meromorphic on $\mathbb{C}^P \times \mathbb{C}^Q$, and $(\mathbf{s}, \mathbf{s}') \mapsto K(\theta, \mathbf{s}, \mathbf{s}')$ is holomorphic on the same space, it follows that $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$ is meromorphic on $\mathbb{C}^P \times \mathbb{C}^Q$. \square

Corollary 2.3.15. *For all $\theta > 0$, the meromorphic functions $(\mathbf{s}, \mathbf{s}') \mapsto Z(\mathbf{s}, \mathbf{s}')$ and $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$ share exactly the same singularities, and the multiplicities of their respective polar divisors are identical. Moreover, the function $(\mathbf{s}, \mathbf{s}') \mapsto J(\theta, \mathbf{s}, \mathbf{s}')$ is holomorphic on the domain \mathcal{D} , where \mathcal{D} is the set defined in Proposition 2.1.1.*

To compute the values of the function $Z(\mathbf{s}, \mathbf{s}')$ at nonpositive integers, it remains to study $J(\theta, \mathbf{s}, \mathbf{s}')$. To do this, we will express this term as an integral of a product of Lerch zeta functions, and then simplify the integrand using an Erdélyi formula for each term of the product.

Let's recall the following notations for the functions l_p^* : For all $p \in \llbracket 1, P \rrbracket$, we consider $l_p^* : \mathbb{C}^Q \rightarrow \mathbb{C}$ the following linear form

$$l_p^*(\mathbf{x}) := \sum_{q=1}^Q c_{q,p} x_q.$$

By taking the previous expression of $J(\theta, \mathbf{s}, \mathbf{s}')$ for $(\mathbf{s}, \mathbf{s}') \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$, we get that

$$\begin{aligned} J(\theta, \mathbf{s}, \mathbf{s}') &= \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q e^{-(l_q(\mathbf{n}) + d'_q) x_q} x_q^{s'_q - 1} dx_q \\ &= \frac{1}{\prod_{q=1}^Q \Gamma(s'_q)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{-s_p} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(\mathbf{x}) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{s'_q - 1} dx_q. \end{aligned} \quad (2.11)$$

We now set the directional functions corresponding to $J(\theta, \mathbf{s}, \mathbf{s}')$ and $K(\theta, \mathbf{s}, \mathbf{s}')$.

Definition 2.3.16. *Let a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. We define for $s \in \mathbb{C}$,*

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s) &:= J(\theta, -\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s) \\ K_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s) &:= K(\theta, -\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s). \end{aligned}$$

By formula (2.10), for $s \in \mathbb{C}$ such that $(-\mathbf{N} + \boldsymbol{\mu}s, -\mathbf{N}' + \boldsymbol{\mu}'s) \in \mathcal{D} \cap (\mathbb{C}^P \times H_0^Q)$, we have

$$Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(s) = J_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s).$$

By definition of the convergence domain \mathcal{D} in Proposition 1.2.9, we get that the previous assumption on s is verified if

$$\begin{aligned} \forall \mathcal{P} \subset \llbracket 1, P \rrbracket, \quad & -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| + (|\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)\sigma > |\mathcal{P}| \\ \forall q \in \llbracket 1, Q \rrbracket, \quad & -N'_q + \mu'_q \sigma > 0. \end{aligned}$$

We define $\sigma_0(\mathbf{N}, \mathbf{N}') := \max_{\boldsymbol{\mu}, \boldsymbol{\mu}'} \left(\frac{N'_1}{\mu'_1}, \dots, \frac{N'_Q}{\mu'_Q}, \max_{\mathcal{P} \subset \llbracket 1, P \rrbracket} \left(\frac{1 + |\mathcal{P}| + |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}}}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \right) \right)$. We then get the following proposition:

Proposition 2.3.17. *For all $s \in H_{\sigma_0(\mathbf{N}, \mathbf{N}')}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$, we have*

$$Z_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(s) = J_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s) + K_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}(\theta, s).$$

Remark 2.3.18. *Rather than using the notation $s = \sigma + i\tau \in H_{\sigma_0(\mathbf{N}, \mathbf{N}')}^{\boldsymbol{\mu}, \boldsymbol{\mu}'}$, we prefer to denote $\sigma \gg_{\mathbf{N}, \mathbf{N}'}^{\boldsymbol{\mu}, \boldsymbol{\mu}'} 1$, or even sometimes $\sigma \gg 1$.*

We already know from Proposition 2.3.11 that $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ is holomorphic and vanishes at $s = 0$. What remains to be studied is the meromorphic continuation and regularity at $s = 0$ of $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$. In the next proposition, we will give an integral representation formula of $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ which will feature the Lerch zeta function.

Proposition 2.3.19. *Let a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ and $s \in \mathbb{C}$ a complex number such that $\mu_p s \notin \mathbb{N}$ for all $p \in \llbracket 1, P \rrbracket$, and such that $\sigma \gg 1$. We then have*

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-d_p l_p^*(\mathbf{x})} \phi(e^{-l_p^*(\mathbf{x})}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_q. \quad (2.12)$$

Proof. Let $s = \sigma + i\tau \in \mathbb{C}$ such that $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$. By formula (2.11), we find that

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(\mathbf{x}) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_q,$$

where we recall that $l_p^*(\mathbf{x}) := \sum_{q=1}^Q c_{q,p} x_q$ with $1 \leq p \leq P$. We now wish to interchange the integral sign with the sum. Firstly, we will bound the general term in the previous series. For all $q \in \llbracket 1, Q \rrbracket$, we define

$$v_q(\mathbf{n}) := \sum_{p=1}^P \operatorname{Re}(c_{q,p}) n_p.$$

For all $\mathbf{n} \in \mathbb{N}_0^P$ we find the following bound:

$$\begin{aligned} & \left| \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(\mathbf{x}) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q \right| \\ & \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \operatorname{Re}(\mu_p s)} e^{\operatorname{Im}(\mu_p s) \frac{\pi}{2}} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q e^{-(v_q(\mathbf{n}) + \operatorname{Re}(d'_q)) x_q} x_q^{-N'_q + \operatorname{Re}(\mu'_q s) - 1} dx_1 \dots dx_Q. \end{aligned}$$

By a change of variables $y_q := (v_q(\mathbf{n}) + \operatorname{Re}(d'_q)) x_q$ for all $q \in \llbracket 1, Q \rrbracket$, we get:

$$\begin{aligned} & \left| \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(\mathbf{x}) n_p} \prod_{q=1}^Q e^{-d'_q x_q} x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q \right| \\ & \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \operatorname{Re}(\mu_p s)} e^{\operatorname{Im}(\mu_p s) \frac{\pi}{2}} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \operatorname{Re}(\mu'_q s)} \\ & \quad \cdot \int_0^{(v_1(\mathbf{n}) + \operatorname{Re}(d'_1))\theta} \dots \int_0^{(v_Q(\mathbf{n}) + \operatorname{Re}(d'_Q))\theta} \prod_{p=1}^P e^{-y_p} \prod_{q=1}^Q y_q^{-N'_q + \operatorname{Re}(\mu'_q s) - 1} dy_1 \dots dy_Q \\ & \leq \prod_{p=1}^P |n_p + d_p|^{N_p - \operatorname{Re}(\mu_p s)} e^{\operatorname{Im}(\mu_p s) \pi/2} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \operatorname{Re}(\mu'_q s)} \Gamma(-N'_q + \operatorname{Re}(\mu'_q s)) \end{aligned}$$

We can see that the series

$$\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P |n_p + d_p|^{N_p - \operatorname{Re}(\mu_p s)} e^{\operatorname{Im}(\mu_p s) \pi/2} \prod_{q=1}^Q (v_q(\mathbf{n}) + \operatorname{Re}(d'_q))^{N'_q - \operatorname{Re}(\mu'_q s)}$$

is convergent for all $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$ via Proposition 2.1.1. By Lebesgue's Theorem, for all $s = \sigma + i\tau \in \mathbb{C}$ such that $\sigma \gg_{\mathbf{N}, \mathbf{N}'} 1$, we get that

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \cdot \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-l_p^*(\mathbf{x}) d_p} \left(\sum_{n_p=1}^{+\infty} (n_p + d_p)^{N_p - \mu_p s} e^{-l_p^*(\mathbf{x}) n_p} \right) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots dx_Q,$$

which concludes the proof. \square

Once this integral representation formula is obtained for $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$, we can use the Erdélyi formula on each Lerch zeta function in the integrand, and then distribute the expression to get the following proposition:

Proposition 2.3.20. *Let a $P+Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ and $s \in \mathbb{C}$ a complex number such that $\mu_p s \notin \mathbb{N}$ for all $p \in \llbracket 1, P \rrbracket$, and such that $\sigma \gg 1$. Then, we find that for $\theta > 0$ sufficiently small,*

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \frac{\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s)}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} dx_1 \dots dx_Q$$

$$+ \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}$$

$$\cdot \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q.$$

Proof. Let $M = \max_{1 \leq p \leq P, 1 \leq q \leq Q} (|c_{q,p}|)$, and $\theta \in \left[0, \frac{1}{QM}\right]$. Let $s \in \mathbb{C}$ such that $\mu_p s \notin \mathbb{N}$ for all $p \in \llbracket 1, P \rrbracket$, and $\sigma \gg 1$. By formula (2.12), we have that

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \int_0^\theta \dots \int_0^\theta \prod_{p=1}^P e^{-d_p l_p^*(\mathbf{x})} \phi(e^{-l_p^*(\mathbf{x})}, -N_p + \mu_p s, d_p) \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} dx_q.$$

By Erdélyi's formula (2.3), for all $(x_q)_{q \in \llbracket 1, Q \rrbracket} \in]0, \theta]^Q$ we have that

$$\prod_{p=1}^P e^{-d_p l_p^*(\mathbf{x})} \phi(e^{-l_p^*(\mathbf{x})}, -N_p + \mu_p s, d_p)$$

$$= \prod_{p=1}^P \left(\Gamma(1 + N_p - \mu_p s) l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} + \sum_{k_p \geq 0} (-1)^{k_p} \frac{l_p^*(\mathbf{x})^{k_p}}{k_p!} \zeta(-N_p + \mu_p s - k_p, d_p) \right).$$

Distributing the previous product, we get for all $\mathbf{x} = (x_q)_{q \in \llbracket 1, Q \rrbracket} \in [0, \theta]^Q$,

$$\prod_{p=1}^P e^{-d_p l_p^*(\mathbf{x})} \phi(e^{-l_p^*(\mathbf{x})}, -N_p + \mu_p s, d_p) = \prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \prod_{p=1}^P l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1}$$

$$+ \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}.$$

Thus, we obtain

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}(\theta, s) &= \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \int_0^\theta \dots \int_0^\theta \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \\ &\quad \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \cdot \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} dx_1 \dots dx_Q \\ &\quad + \frac{\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s)}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} dx_1 \dots dx_Q. \end{aligned}$$

We shall now bound the general term in the series inside the integrand. By compactness, we have uniformly on $\mathbf{x} = (x_1, \dots, x_Q) \in \left[0, \frac{1}{QM}\right]^Q$ that

$$\begin{aligned} \left| \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \right| &\ll_s 1 \\ |l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1}| &\ll_s 1. \end{aligned}$$

Since l_p^* is a linear form for all $p \in \llbracket 1, P \rrbracket$, we get that

$$\forall x \in [0, \theta]^Q, \quad |l_p^*(\mathbf{x})| \leq QM\theta.$$

By Lemma 2.3.2, we have the following bound:

$$\forall p \in \llbracket 1, P \rrbracket, \quad \left| \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \ll_s \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil |\mu_p s| \rceil}.$$

For all $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, and for all $\mathbf{x} = (x_q)_{q \in \llbracket 1, Q \rrbracket} \in [0, \theta]^Q$, we then get the following bound

$$\begin{aligned} &\left| \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \\ &\ll_s \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil |\mu_p s| \rceil} (QM\theta)^{k_p}. \end{aligned}$$

Since the series

$$\sum_{\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}^c}} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{1 + \lceil |\mu_p s| \rceil} (QM\theta)^{k_p}$$

is absolutely convergent for all $\theta \in \left[0, \frac{1}{QM}\right]$, we then can interchange the series and the integral to prove the proposition. \square

By a blowup at the origin in the integrals of the previous proposition, we can get a nicer expression of $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ where we can see more clearly the contribution of the free variable θ .

Proposition 2.3.21. *Let a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ and $s \in \mathbb{C}$*

a complex number such that $\mu_p s \notin \mathbb{N}$ for all $p \in \llbracket 1, P \rrbracket$, and such that $\sigma \gg 1$. We have

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \sum_{\mu, \mu'} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \\ \cdot \frac{\theta^{(|\mu'| + |\mu|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\mu'| + |\mu|_{\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\ + \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\mu'| + |\mu|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right),$$

where we recall the notations of Proposition 2.2.4: For $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, $j \in \llbracket 1, Q \rrbracket$, and $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$,

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \\ \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx}_j \dots dx_Q \\ h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) = \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \\ \cdot \int_0^1 \dots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots \widehat{dx}_j \dots dx_Q,$$

with

$$\begin{cases} dx_1 \dots \widehat{dx}_j \dots dx_Q = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_Q \\ \widehat{\mathbf{x}}^j = (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_Q). \end{cases}$$

Proof. Let $s \in \mathbb{C}$ such that $\mu_p s \notin \mathbb{N}$ for all $p \in \llbracket 1, P \rrbracket$, and $\sigma \gg 1$. To prove the required formula, it's enough to revisit the formula obtained in the previous proposition, and to re-express the integral in this formula. The integral will be simplified using a blowing-up.

First, we note that $[0, \theta]^Q = \bigcup_{j=1}^Q V_j$, where $V_j := \{\mathbf{x} \in [0, \theta]^Q \mid \forall q \neq j, x_j \geq x_q\}$, and we also observe that each pairwise intersection of these sets has a zero Lebesgue measure. By Chasles' relation, we obtain

$$\int_0^\theta \dots \int_0^\theta \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q \\ = \sum_{j=1}^Q \int_{V_j} \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q.$$

We now consider the following change of variables:

$$f_j : [0, 1]^{j-1} \times [0, \theta] \times [0, 1]^{Q-j} \rightarrow V_j \\ \mathbf{x} = (x_1, \dots, x_Q) \mapsto (x_1 x_j, \dots, x_j, \dots, x_Q x_j),$$

with the corresponding Jacobian determinant $\text{Jac}(f_j) = x_j^{Q-1}$. We can just study the integral with the domain V_Q , and extrapolate the other integrals by doing a permutation of variables. Firstly, let's note

that for all $\mathbf{x} = (x_1, \dots, x_Q) \in [0, 1]^{Q-1} \times [0, \theta]$, we have $l_p^*(f_Q(\mathbf{x})) = x_Q l_p^*(\widehat{\mathbf{x}}^Q)$. Therefore we get that

$$\begin{aligned}
& \int_{V_Q} \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\mathbf{x})^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\mathbf{x})^{k_p} dx_1 \dots dx_Q \\
&= \int_0^1 \dots \int_0^1 \int_0^\theta x_Q^{-|\mathbf{N}'| + |\boldsymbol{\mu}'|s - Q - |\mathbf{N}|_{|\mathcal{P}} + |\boldsymbol{\mu}|_{|\mathcal{P}}s - |\mathcal{P}| + |\mathbf{k}|} \prod_{q=1}^{Q-1} x_q^{-N'_q + \mu'_q s - 1} \\
& \quad \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^Q)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^Q)^{k_p} x_Q^{Q-1} dx_1 \dots dx_Q \\
&= \int_0^1 \dots \int_0^1 \prod_{q=1}^{Q-1} x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^Q)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} u_p(\widehat{\mathbf{x}}^Q)^{k_p} dx_1 \dots dx_{Q-1} \\
& \quad \cdot \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|},
\end{aligned}$$

which concludes the proof. \square

The previous proposition is crucial to the proof of Proposition 2.2.4, because we shall prove that $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ is regular at $s = 0$ via the formula in the previous proposition. In particular, we will use Proposition 2.2.2, which gives a uniform bound on any compact subset of \mathbb{C} for the functions $s \mapsto h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$. We shall prove this proposition in the next subsection.

2.3.3 Proof of Proposition 2.2.2

We consider a set $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ and an integer $j \in \llbracket 1, Q \rrbracket$. In this subsection, we will prove Proposition 2.2.2. Let us recall the notation $l_p^*(\mathbf{x}) = \sum_{q=1}^Q c_{q,p} x_q$ for every $p \in \llbracket 1, P \rrbracket$ and every $x \in \mathbb{C}^Q$. We will establish a lemma before proving this proposition. For every $\epsilon \in \left] 0, \frac{1}{2MQ} \right]$, for each subset $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$, we set

$$F_{\mathcal{Q}, j}(\epsilon) := \{(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_Q) \mid \forall q \in \mathcal{Q}, x_q \in [0, \epsilon], \forall q' \in \mathcal{Q}^c, x_{q'} \in [\epsilon, 1]\}$$

where we noted $\mathcal{Q}^c = (\llbracket 1, Q \rrbracket \setminus (\mathcal{Q} \cup \{j\}))$. We then can partition the following integration domain:

$$\{(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_Q) \mid \forall q \neq j, x_q \in [0, 1]\} = \bigcup_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} F_{\mathcal{Q}, j}(\epsilon).$$

We see that each pairwise intersection of the sets in the union above has a zero Lebesgue measure. Via Chasles' formula, we can write the functions $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ and $h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s)$ as sums of the following integrals:

$$\begin{aligned}
h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) &= \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \\
h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) &= \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s),
\end{aligned} \tag{2.13}$$

where we noted for all subset $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$,

$$\begin{aligned} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \\ &\cdot \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots dx_j \dots dx_Q, \\ I_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j, \mathcal{Q}}(\epsilon, s) &= \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \\ &\cdot \int_{F_{\mathcal{Q}, j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p=1}^P l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} dx_1 \dots dx_j \dots dx_Q, \end{aligned}$$

for all $s \in \mathbb{C}$ such that $\sigma \gg 1$.

Lemma 2.3.22. *Let a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$, a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, an integer $j \in \llbracket 1, Q \rrbracket$, some sets $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ and $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$, and a multi-index $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$. Then we have that*

i) *For all $\epsilon > 0$ small enough, the function $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ is holomorphic on \mathbb{C} .*

ii) *For $\epsilon > 0$ small enough, and for every positive integer $r \in \mathbb{N}$, we have the following bound, uniformly in $s \in \overline{D}_0(r)$:*

$$\left| I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right|_{r, \epsilon} \ll (QM)^{|\mathbf{k}|},$$

$$\text{with } M = \max_{1 \leq q \leq Q, 1 \leq p \leq P} (|c_{q,p}|).$$

iii) *For all $s \in \mathbb{C}$ and $0 < \epsilon \ll 1$, we have*

$$\begin{aligned} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^Q)^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{\mathbf{v}_p, q} \right) \quad (2.14) \\ &\cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q,p}^{\mathbf{w}_p, q} \right) \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet, q}| + |\mathbf{w}_{\bullet, q}|)}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet, q}| + |\mathbf{w}_{\bullet, q}|)} \\ &\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet, q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q, \end{aligned}$$

where we set $\mathcal{Q}^c = \llbracket 1, Q \rrbracket \setminus (\mathcal{Q} \cup \{j\})$, $|\mathbf{v}_{\bullet, q}| = \sum_{p \in \mathcal{P}} v_{p,q}$ for all $q \in \mathcal{Q}$, and $|\mathbf{w}_{\bullet, q}| = \sum_{p \in \mathcal{P}^c} w_{p,q}$ for all $q \in \llbracket 1, Q \rrbracket$. Noting $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{w}, \mathbf{v}}(\epsilon, s)$ as the general term in the previous series, we also have

$$\partial_s I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^Q)^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \partial_s T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{w}, \mathbf{v}}(\epsilon, s).$$

iv) *For all $0 < \epsilon \ll 1$ small enough, and for all $s \in \mathbb{C}$, we have*

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s). \quad (2.15)$$

Proof. Here we aim to expand the integrand using power series expansions around 0 for each terms $\mathbf{x} \in F_{\mathcal{Q},j}(\epsilon)$ with $\epsilon > 0$ small enough. Note that these power series will converge uniformly on the disk $\overline{D}_0(\eta)$, for η sufficiently small. We will therefore interchange the series and the integral.

Let $s \in \mathbb{C}$ be a complex number such that $\sigma \gg 1$, and $\epsilon \in \left]0, \frac{1}{2MQ}\right]$ where we have set $M := \max_{1 \leq q \leq Q, 1 \leq p \leq P} (|c_{q,p}|)$. We consider a set $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$. We can expand into power series in the variables $(x_q)_{q \in \mathcal{Q}}$ the functions $l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1}$, where $\mathbf{x} \in F_{\mathcal{Q},j}(\epsilon)$ and $p \in \mathcal{P}$. By Lemma 2.3.1, we obtain uniformly in $\widehat{\mathbf{x}}^j = (x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_Q)$:

$$\begin{aligned} & l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \\ &= \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q + \sum_{q \in \mathcal{Q}} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1} \\ &= \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1} \left(1 + \sum_{f \in \mathcal{Q}} \frac{c_{f,p}}{c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q} x_f \right)^{-N_p + \mu_p s - 1} \\ &= \sum_{\mathbf{v}_p = (v_{p,q})_{q \in \llbracket 1, Q \rrbracket} \in \mathbb{N}_0^{\mathcal{Q}}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} x_q^{v_{p,q}}. \quad (2.16) \end{aligned}$$

For $p \in \mathcal{P}^c$, and $k_p \in \mathbb{N}_0$, we have by Newton's multinomial:

$$l_p^*(\widehat{\mathbf{x}}^j)^{k_p} = \sum_{\substack{\mathbf{w}_p = (w_{p,q})_{q \in \llbracket 1, Q \rrbracket} \in \mathbb{N}_0^{\mathcal{Q}} \\ |\mathbf{w}_p| = k_p}} \binom{k_p}{\mathbf{w}_p} c_{j,p}^{w_{p,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{w_{p,q}} x_q^{w_{p,q}}.$$

The series in formula (2.16) converges uniformly in $(x_q)_{q \in \mathcal{Q}} \in [0, \epsilon]^{\mathcal{Q}}$. By interchanging the integral and the series, we get that $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ is of the following form:

$$\begin{aligned} & I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \\ & \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} c_{j,p}^{w_{p,j}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \right) \\ & \cdot \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q},j}(\epsilon)} \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} \left(\prod_{p \in \mathcal{P}^c} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{w_{p,q}} x_q^{w_{p,q}} \right) \\ & \cdot \prod_{p \in \mathcal{P}} \left(\prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} x_q^{v_{p,q}} \right) \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_1 \dots \widehat{dx}_j \dots dx_Q. \end{aligned}$$

Let's note that $\llbracket 1, Q \rrbracket \setminus \{j\} = \mathcal{Q} \cup \mathcal{Q}^c$. Then we can split the products $\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$ into 2 parts, and we get:

$$\begin{aligned} & I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \\ &= \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q},j}(\epsilon)} \prod_{p \in \mathcal{P}^c} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{w_{p,q}} \right) \prod_{p \in \mathcal{P}} \left(\prod_{q \in \mathcal{Q}} x_q^{v_{p,q}} \right) \\
& \cdot \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q x_q^{-N'_q + \mu'_q s - 1} dx_1 \dots \widehat{dx}_j \dots dx_Q \\
& = \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
& \cdot \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s)} \int_{F_{\mathcal{Q},j}} \prod_{q \in \mathcal{Q}} x_q^{-N'_q + \mu'_q s - 1 + \sum_{p \in \mathcal{P}} v_{p,q} + \sum_{p \in \mathcal{P}^c} w_{p,q}} \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + \sum_{p \in \mathcal{P}^c} w_{p,q}} \\
& \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} dx_1 \dots \widehat{dx}_j \dots dx_Q.
\end{aligned}$$

Let $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}}$ with $\mathbf{v}_p \in \mathbb{N}_0^{\mathcal{Q}}$ for all $p \in \mathcal{P}$, and $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c}$ with $\mathbf{w}_p \in \mathbb{N}_0^{\mathcal{Q}}$ for all $p \in \mathcal{P}^c$. We recall the notation $|\mathbf{v}_{\bullet q}| = \sum_{p \in \mathcal{P}} v_{p,q}$ for all $q \in \mathcal{Q}$, and $|\mathbf{w}_{\bullet q}| = \sum_{p \in \mathcal{P}^c} w_{p,q}$ for all $q \in [1, Q]$. Via Fubini's theorem, we get the following expression for all $\sigma \gg 1$:

$$\begin{aligned}
I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\
& \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \frac{1}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \left(\prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|}}{-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|} \right) \\
& \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \left(\prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \right) \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q.
\end{aligned}$$

Thus we obtain that formula (2.14) is true for all $s \in \mathbb{C}$ such that $\sigma \gg 1$.

We're now looking to prove an analytic continuation of $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ on \mathbb{C} via the formula that we just obtained. We clearly have that, for all $\sigma \gg 1$,

$$\begin{aligned}
I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) &= \frac{1}{\prod_{q \in \mathcal{Q}^c} \Gamma(-N'_q + \mu'_q s)} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \quad (2.17) \\
& \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|}}{\Gamma(-N'_q + \mu'_q s) (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)} \\
& \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \left(\prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \right) \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q.
\end{aligned}$$

Let's define $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$ as the general term of the previous series. We then have that

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) = \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s).$$

All the terms inside the expression of $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$ are holomorphic in the variable s on \mathbb{C} . In order to show that the function $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ is holomorphic on \mathbb{C} , it's enough to show that the last series normally converges on any compact subset of \mathbb{C} . We're now looking to find a bound of that general term over a compact disk $\overline{D}_0(r)$, with $r \in \mathbb{N}$ being a positive integer. Such bound will depend on $\mathbf{N}, \mathbf{N}', \boldsymbol{\mu}, \boldsymbol{\mu}', \mathcal{P}, j, \mathcal{Q}$, but we shall not specify this dependence to increase the clarity of the proof.

We know that the function

$$s \mapsto \frac{1}{\Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)}$$

is holomorphic on \mathbb{C} , for all $q \in \mathcal{Q}$. We now bound the general term of the series in formula (2.17) thanks to Lemma 2.3.4. We have

$$\begin{aligned} \forall q \in \mathcal{Q}, \forall s \in \overline{D}_0(r), \forall a \in \mathbb{N}_0, & \quad \left| \frac{1}{\Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + a)} \right| \ll_r 1 \\ \forall q \in [1, Q], & \quad \left| \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right| \ll_r 1. \end{aligned}$$

We know that the function

$$s \in \mathbb{C} \mapsto \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q$$

is holomorphic because the integrand is regular.

Let $s \in \overline{D}_0(r)$, $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}}$ with $\mathbf{v}_p \in \mathbb{N}^{\mathcal{Q}}$, and $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c}$ with $\mathbf{w}_p \in \mathbb{N}_0^{\mathcal{Q}}$ such that $|\mathbf{w}_p| = k_p$ for all $p \in \mathcal{P}^c$. We see that $\sum_{\substack{q=1 \\ q \neq j}} |\mathbf{w}_{\bullet q}| = |\mathbf{k}| - |\mathbf{w}_{\bullet j}|$. Moreover, by compactness, we find the following bound which is uniform in $x_q \in [\epsilon, 1]$,

$$\left| x_q^{-N'_q + \mu'_q s - 1} \right| \ll_r 1.$$

As $|\mathbf{w}_{\bullet q}| \geq 0$, we find that $|x_q^{|\mathbf{w}_{\bullet q}|}| \leq 1$. Moreover, since $\arg \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right) \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right]$, we have for all $s \in \overline{D}_0(r)$, and for all $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$,

$$\left| \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \right| \leq \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right|^{-N_p + \operatorname{Re}(\mu_p s) - 1 - |\mathbf{v}_p|} e^{|\mu_p| r \pi / 2}.$$

Let's note that we have the following upper bound for all $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$,

$$K + K\epsilon \leq \left(\operatorname{Re} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} \operatorname{Re}(c_{q,p} \epsilon) \right) \right) \leq \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \leq \sum_{q=1}^Q |c_{q,p}| \leq QM,$$

with $M = \max_{1 \leq p \leq P, 1 \leq q \leq Q} (|c_{q,p}|)$ and $K := \min_{1 \leq q \leq Q, 1 \leq p \leq P} \operatorname{Re}(c_{q,p})$. For $p \in \mathcal{P}$, we then find that

$$\left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right|^{-N_p + \operatorname{Re}(\mu_p s) - 1 - |\mathbf{v}_p|} \leq \begin{cases} (QM)^{\operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1} & \text{if } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \geq 1 \\ & \text{and } \operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1 \geq 0 \\ 1 & \text{if } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| \geq 1 \text{ and } \operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1 < 0 \\ 1 & \text{if } \left| c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right| < 1 \text{ and } \operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1 \geq 0 \\ (K(\epsilon + 1))^{\operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1} & \text{otherwise.} \end{cases}$$

We also have uniformly in $s \in \overline{D}_0(r)$:

$$\forall p \in \mathcal{P}, \quad (QM)^{\operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1} \ll_r (QM)^{-|\mathbf{v}_p|}.$$

Let $p \in \llbracket 1, P \rrbracket$. By compactness, we have uniformly in $s \in \overline{D}_0(r)$:

$$(K\epsilon)^{\operatorname{Re}(\mu_p s)} \ll_{\epsilon, r} 1.$$

We also have

$$(K(\epsilon + 1))^{\operatorname{Re}(\mu_p s) - N_p - |\mathbf{v}_p| - 1} \ll_{\epsilon, r} (K(\epsilon + 1))^{-|\mathbf{v}_p|} \ll_{r, \epsilon} K^{-|\mathbf{v}_p|}.$$

Finally, we have the following uniform bound in the variables $s \in \overline{D}_0(r)$ and $(x_q)_{q \in \mathcal{Q}^c} \in [\epsilon, 1]^{\mathcal{Q}^c}$,

$$\left| \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p \sigma - 1 - |\mathbf{v}_p|} \right| \ll_{r, \epsilon} \max(K^{-|\mathbf{v}_p|}, (QM)^{-|\mathbf{v}_p|}, 1) \prod_{p=1}^P e^{|\mu_p| r \pi / 2} \ll_{r, \epsilon} \min(K, QM, 1)^{-|\mathbf{v}_p|}.$$

Now that we have a bound of the integrand, we can give a bound of the corresponding integral:

$$\left| \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q \right| \ll_{r, \epsilon} \min(K, QM, 1)^{-\sum_{p \in \mathcal{P}} |\mathbf{v}_p|}.$$

We have the following bound, uniformly in $s \in \overline{D}_0(r)$:

$$\left| \epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)} \right| \ll_{r, \epsilon} \epsilon^{\sum_{q \in \mathcal{Q}} |\mathbf{v}_{\bullet q}|}.$$

Moreover, we have

$$\forall n \in \mathbb{N}_0, \forall a \in \mathbb{N}_0, \forall b \in \overline{H}_0, \forall s \in \overline{D}_0(r), \quad \left| \binom{-n + bs - 1}{a} \right| \leq \binom{n + \lceil |b|r \rceil + a}{a}.$$

We then find that

$$\begin{aligned}
& \left| \frac{1}{\prod_{q \in \mathcal{Q}^c} \Gamma(-N'_q + \mu'_q s)} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \right. \\
& \cdot \prod_{q \in \mathcal{Q}} \frac{\epsilon^{-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|}}{\Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + |\mathbf{v}_{\bullet,q}| + |\mathbf{w}_{\bullet,q}|)} \\
& \cdot \left. \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet,q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{q \in \mathcal{Q}^c} c_{q,p} x_q \right)^{-N_p + \mu_p s - 1 + |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q \right| \\
& \ll_{r,\epsilon} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} M^{|\mathbf{w}_p|} \right) \prod_{p \in \mathcal{P}} \binom{N_p + \lceil |\mu_p| r \rceil + |\mathbf{v}_p|}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} (M\epsilon)^{\sum_{q \in \mathcal{Q}} |\mathbf{v}_{\bullet,q}|} \min(K, QM, 1)^{-\sum_{q \in \mathcal{Q}} |\mathbf{v}_{\bullet,q}|} \\
& \ll_{r,\epsilon} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} M^{k_p} \right) \prod_{p \in \mathcal{P}} \binom{N_p + \lceil |\mu_p| r \rceil + |\mathbf{v}_p|}{(\mathbf{v}_p, N_p + \lceil |\mu_p| r \rceil)} \left(\frac{M\epsilon}{\min(K, QM, 1)} \right)^{\sum_{q \in \mathcal{Q}} |\mathbf{v}_{\bullet,q}|}. \tag{2.18}
\end{aligned}$$

We now prove that the above upper bound is the general term of an absolutely convergent series in the variable ϵ when we take the sum $\sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^Q)^{\mathcal{P}} \\ \mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}}$. Let's note that by Newton's multinomial we have

$$\sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} M^{k_p} = (QM)^{|\mathbf{k}|}.$$

Moreover the series

$$\sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^Q)^{\mathcal{P}}} \prod_{p \in \mathcal{P}} \binom{N_p + \lceil |\mu_p| r \rceil + |\mathbf{v}_p|}{(\mathbf{v}_p, N_p + \lceil |\mu_p| r \rceil)} \left(\frac{M\epsilon}{\min(K, QM, 1)} \right)^{\sum_{q \in \mathcal{Q}} |\mathbf{v}_{\bullet,q}|}$$

converges absolutely for all $0 < \epsilon < \frac{\min(K, QM, 1)}{M}$. By taking a $\epsilon > 0$ small enough, we then find that the previous series over $(v_{p,q})_{p \in \mathcal{P}, q \in \mathcal{Q}}$ is a constant depending only on r, ϵ and on \mathbf{N} . We then get the following inequality:

$$\left| I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right|_{r,\epsilon} \leq (QM)^{|\mathbf{k}|}.$$

Thus we get i) and iii) by Lebesgue's theorem, and by analytic continuation. ii) is true by the above upper bound. As $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ is holomorphic on \mathbb{C} and by formula (2.13) we have

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\substack{\mathcal{Q} \subset [1, Q] \setminus \{j\} \\ \mu, \mu'}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \quad (\sigma \gg 1)$$

we obtain that the previous equality stands on \mathbb{C} by analytic continuation, thus proving iv). \square

By a similar reasoning, we get the following lemma:

Lemma 2.3.23. *Let a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$, a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, an integer $j \in [1, Q]$, and a set $\mathcal{Q} \subset [1, Q] \setminus \{j\}$. For all $s \in \mathbb{C}$ and $0 < \epsilon \ll 1$, we have:*

$$I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\epsilon, s) = \sum_{\mathbf{v}=(\mathbf{v}_p)_{p \in [1, P]} \in (\mathbb{N}_0^{\mathcal{Q}})^P} \left(\prod_{q \in [1, Q]} c_{q, p}^{w_{p, q}} \right) \left(\prod_{p=1}^P \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q, p}^{v_{p, q}} \right) \\ \cdot \frac{1}{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}|)}} \cdot \frac{\Gamma(-N'_q + \mu'_q s)}{\prod_{q \in [1, Q] \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{1}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}|)} \\ \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1} \cdot \prod_{p=1}^P \left(c_{j, p} + \sum_{f \in \mathcal{Q}^c} c_{f, p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q,$$

with the notations $\mathcal{Q}^c = ([1, Q] \setminus \{j\}) \setminus \mathcal{Q}$, $|\mathbf{v}_{\bullet q}| = \sum_{p=1}^P v_{p, q}$ for all $q \in \mathcal{Q}$. Moreover, the function $s \in \mathbb{C} \mapsto I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\epsilon, s)$ is holomorphic on \mathbb{C} .

Remark 2.3.24. *As opposed to Lemma 2.3.22, we don't need a bound in Lemma 2.3.23, because there is no \mathbf{k} inside the expression of $I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\epsilon, s)$.*

We now have all the ingredients required to prove Proposition 2.2.2:

Proof. Let's recall that $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ and $h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s)$ have the following expression for all $s \in \mathbb{C}$ such that $\sigma \gg 1$:

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s), \\ h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) = \sum_{\mathcal{Q} \subset [1, Q] \setminus \{j\}} \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\epsilon, s).$$

Since the functions $s \mapsto I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)$ and $s \mapsto I_{\mathbf{N}, \mathbf{N}', [1, P], j, \mathcal{Q}}(\epsilon, s)$ are holomorphic on \mathbb{C} , we have an analytic continuation of $s \mapsto h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ and of $s \mapsto h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s)$ on \mathbb{C} . We then obtain point i) of Proposition 2.2.2. Point ii) and iii) of Proposition 2.2.2 follows directly from point ii) and iii) of Lemma 2.3.22. \square

2.3.4 Proof of Proposition 2.2.4

Let a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. We exploit the expression $Z(\mathbf{s}, \mathbf{s}') = K(\boldsymbol{\theta}, \mathbf{s}, \mathbf{s}') + J(\boldsymbol{\theta}, \mathbf{s}, \mathbf{s}')$ to obtain the proof of the meromorphic continuation theorem in the direction $(\boldsymbol{\mu}, \boldsymbol{\mu}')$. We already know that the function $(\mathbf{s}, \mathbf{s}') \mapsto K(\boldsymbol{\theta}, \mathbf{s}, \mathbf{s}')$ is holomorphic on $\mathbb{C}^P \times \mathbb{C}^Q$. What remains is to prove that the function $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\boldsymbol{\theta}, s)$ is meromorphic and regular at $s = 0$. In order to show Proposition 2.2.4, all that remains is to establish that $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\boldsymbol{\theta}, s)$ is meromorphic on \mathbb{C} with poles contained in

$$\mathcal{S}(\mathbf{N}, \mathbf{N}') = \left[\bigcup_{\substack{1 \leq p \leq P \\ \mu_p \neq 0}} \frac{1}{\mu_p} \mathbb{Z}_{\geq N_p + 1} \right] \cup \left[\bigcup_{\mathcal{P} \subset [1, P]} \frac{1}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|^{\mathcal{P}}} \mathbb{Z}_{\leq |\mathbf{N}|^{\mathcal{P}} + |\mathbf{N}'| + |\mathcal{P}|}^* \right].$$

We know that formula (2.2) is valid on the complement of this set, for $\sigma \gg 1$. For all $s = \sigma + i\tau \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ such that $\sigma \gg 1$, we have that

$$J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = \sum_{\mathcal{P} \subsetneq [1, P]} \left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} + \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right).$$

Via all the above, we can now prove Proposition 2.2.4:

Proof. Let $0 < \theta \ll 1$. We know from Proposition 2.3.21 that the formula (2.2) is true for all $s = \sigma + i\tau \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$ such that $\sigma \gg 1$. It is clear that the function

$$s \mapsto \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)},$$

is meromorphic on \mathbb{C} and regular at $s = 0$, with poles of the form $s = \frac{|\mathbf{N}'| + |\mathbf{N}| + P}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|}$. Then, it remains to show that the function

$$s \mapsto \sum_{\mathcal{P} \subsetneq [1, P]} \left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \frac{\theta(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \quad (2.19)$$

is meromorphic on \mathbb{C} , regular at $s = 0$, and with poles inside $\mathcal{S}(\mathbf{N}, \mathbf{N}')$. We set

$$\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r) := \left\{ s \in \overline{D}_0(r) \left| \begin{array}{l} \forall p \in [1, P], \min_{\substack{n \in \mathbb{N}_0 \\ n > N_p}} |\mu_p s - n| \geq \delta \\ \forall \mathcal{P} \subsetneq [1, P], \min_{\substack{n \in \mathbb{Z}^* \\ n \geq -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| - |\mathcal{P}|}} |(|\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)s + n| \geq \delta \end{array} \right. \right\}.$$

Let's note that

1) We clearly have that $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ contains an open neighborhood of 0.

2) The condition

$$\min_{\substack{n \in \mathbb{N}_0 \\ n > N_p}} |\mu_p s - n| \geq \delta$$

is here to avoid the singularities of $\zeta(-N_p + \mu_p s - k, d_p)$. We see that, around $s = 0$, this last term is regular.

3) The condition

$$\min_{\substack{n \in \mathbb{Z}^* \\ n \geq -|\mathbf{N}|_{|\mathcal{P}} - |\mathbf{N}'| - |\mathcal{P}|}} |(|\boldsymbol{\mu}'|_{|\mathcal{P}} + |\boldsymbol{\mu}'|)s + n| \geq \delta$$

is here to avoid the singularities of $\frac{1}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}$ which are not always compensated by the zeros of $\frac{1}{\Gamma(-N'_j + \mu'_j s)}$. We shall note that around $s = 0$, when $-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}| = 0$, the singularity of $\frac{1}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s}$ is compensated by the zero of $\frac{1}{\Gamma(-N'_j + \mu'_j s)}$.

4) The set $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ is inside the complement of $\widetilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ in \mathbb{C} .

Let's fix a subset $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$. We see that by Proposition 2.3.21, the formula (2.2) is true for all $s = \sigma + i\tau \in \mathbb{C} \setminus \mathcal{S}(\mathbf{N}, \mathbf{N}')$, such that $\sigma \gg 1$. We wish to prove that the series in \mathbf{k} in formula (2.19) normally converges on $\widetilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$.

Let $r' > 1$ be a real number such that, for all $p \in \llbracket 1, P \rrbracket$, we have $\operatorname{Re}(d_p) \geq 1/r'$, and $|d_p| < r'$. Let $r > 0$, $1 > \delta > 0$. By Lemma 2.3.3, we have

$$\forall p \in \llbracket 1, P \rrbracket, \forall s \in \widetilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \forall k_p \in \mathbb{N}_0, \quad \zeta(-N_p + \mu_p s - k_p, d_p) \underset{r, r', \delta}{\ll} (k_p + N_p)! (k_p + N_p + 1)^{r+1} (1 + r'^{2k_p}).$$

By Lemma 2.2.2, we have

$$\forall s \in \overline{D}_0(r), \forall \mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}, \quad |h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)| \underset{r}{\ll} (QM)^{|\mathbf{k}|}.$$

Since the function $s \mapsto \Gamma(1 + N_p - \mu_p s)$ is holomorphic on $\widetilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ for all $1 \leq p \leq P$, we then find the following bound

$$\forall p \in \llbracket 1, P \rrbracket, \forall s \in \widetilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \quad |\Gamma(1 + N_p - \mu_p s)| \underset{r, \delta}{\ll} 1.$$

By Lemma 2.3.4 we find the following bound

$$\forall s \in \widetilde{D}(\mathbf{N}, \mathbf{N}', \delta, r), \quad \left| \frac{1}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right| \underset{r, \delta}{\ll} 1.$$

By compactness of $\overline{D}_0(r)$ we get

$$\forall s \in \overline{D}_0(r), \quad |\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}|}| \underset{r}{\ll} 1.$$

Therefore, for all $s \in \widetilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$, and for $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$, we find

$$\begin{aligned} & \left| (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right. \\ & \left. \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s) ((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right| \\ & \underset{r, r', \delta}{\ll} (QM\theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1} (1 + r'^{2k_p}) \\ & \underset{r, r', \delta}{\ll} (2r'^2 QM\theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1}. \end{aligned}$$

Since the series

$$\sum_{\mathbf{N} \in \mathbb{N}_0^c} (2r'^2 Q M \theta)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{(k_p + N_p)!}{k_p!} (k_p + N_p + 1)^{r+1}$$

converges absolutely for all $\theta \in \mathbb{C}$ such that $|\theta| < \frac{1}{(2r'^2 Q M)}$, we find that formula (2.2) holds for all $s \in \tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$. By the holomorphy theorem under the sum sign, we get that the function $s \mapsto J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ is holomorphic on $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$. As $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ contains an open neighborhood of 0, we see that by the same holomorphy theorem under the sum sign, we can take the derivative with respect to s termwise around $s = 0$ of the function $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$,

$$\begin{aligned} \partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \quad (2.20) \\ & \sum_{\mathcal{P} \subsetneq [1, P]} \sum_{j=1}^Q \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^c} \partial_s \left(\left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s) \right. \\ & \left. \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \right), \\ & + \partial_s \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right). \end{aligned}$$

As $1 > \delta > 0$ is arbitrary in the definition of $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$, and that $\tilde{D}(\mathbf{N}, \mathbf{N}', \delta, r)$ contains an open neighborhood of $s = 0$, we then get that $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ is regular at $s = 0$, and that the poles are inside $\mathcal{S}(\mathbf{N}, \mathbf{N}')$ described in Proposition 2.2.4. \square

Remark 2.3.25. Via formula (2.2), one can find expressions of the residues of $Z_{\mathbf{N}, \mathbf{N}'}(s)$ at these poles. In fact, we know that this function can be decomposed into two sums, one of which is holomorphic on \mathbb{C} , and the other one is meromorphic on \mathbb{C} :

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = K_{\mathbf{N}, \mathbf{N}'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}(\theta, s).$$

Since $s \mapsto K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ is holomorphic on \mathbb{C} , this function doesn't have any singularities. On the other hand, in formula (2.2), one can compute the residues by studying carefully the two following terms:

- $\frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}$
- $\frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)}.$

Let's note that these poles can sometimes be compensated by zeros of the functions $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)$ for the first term, and of zeros of the functions $h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s)$ for the second term.

Chapter 3

Computation of the coefficients

$$Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k}) \text{ and } Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$$

$$\mu, \mu'$$

We compute in this chapter the coefficients $Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ and $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ that appear in Theorems A and D. For reasons that we'll explain in the two following chapters, we will only compute such coefficients for any given direction $(\mu, \mu') \in \overline{H}_0^P \times H_0^Q$. Indeed, in the two following chapters we will prove that Theorem A and D holds for such directions, and we will extend those two results with more general directions using the analytic continuation principle.

By Proposition 2.2.2, we know that the function $s \mapsto h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s)$ is holomorphic on \mathbb{C} . Therefore, we can consider its Taylor coefficients of order 0 and 1 near $s = 0$.

Notation. Let's consider a direction $(\mu, \mu') \in \overline{H}_0^P \times H_0^Q$, and a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$. We consider a set $\mathcal{P} \subset \llbracket 1, P \rrbracket$ and a tuple $\mathbf{k} \in \mathbb{N}_0^{P+Q}$. Since the function $h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s)$ is holomorphic on \mathbb{C} , we consider the following coefficients:

$$h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s) = Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k}) + Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})s + O(s^2).$$

We shall fix for this chapter a direction $(\mu, \mu') \in \overline{H}_0^P \times H_0^Q$ and a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$. The coefficients $Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ for all integers $j \in \llbracket 1, Q \rrbracket$ and all sets $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$ will be crucial in the expression of the formulas for $Z(-(\mathbf{N}, \mathbf{N}'))$. Furthermore, the coefficients $Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ and $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ will appear in the formula for the directional derivative at nonpositive integers $Z'(-(\mathbf{N}, \mathbf{N}'))$. It is also worth noting that we do not need to compute $Q_{\mathbf{N},\mathbf{N}'}^0(\llbracket 1, P \rrbracket, j)$ and $Q_{\mathbf{N},\mathbf{N}'}^1(\llbracket 1, P \rrbracket, j)$, as they won't appear in the expressions and in the proofs of Theorems A and D. We will show in our calculations that $Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ is independent of the directions (μ, μ') . We will therefore simply denote this coefficient as $Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$.

3.1 Lemmata

Lemma 3.1.1. Let $a, c \in H_0$, $b \in \mathbb{N}_0$ and $d \in \mathbb{Z}$. We define $g : \mathbb{C} \rightarrow \mathbb{C}$ the following meromorphic function:

$$g(s) := \frac{1}{\Gamma(as - b)(cs + d)}.$$

Then g is regular near $s = 0$. Moreover, we have:

$$g(0) = \begin{cases} \frac{a}{c}(-1)^b b! & \text{if } d = 0 \\ 0 & \text{if } d \neq 0, \end{cases}$$

and

$$g'(0) = \begin{cases} \frac{a^2}{c}(-1)^b b! (\gamma - h_b) & \text{if } d = 0 \\ \frac{a}{d}(-1)^b b! & \text{if } d \neq 0, \end{cases}$$

where h_b is the harmonic number $h_b := \sum_{i=1}^b \frac{1}{i}$. We also have that

$$\left(\frac{1}{\Gamma}\right)'(-N) = (-1)^N N!.$$

Proof. Around $s = 0$, we have

$$\frac{1}{\Gamma(s-N)} = (-1)^N N! s + O(s^2),$$

which proves the formula for $\left(\frac{1}{\Gamma}\right)'(-N)$.

In order to prove the other formulas, we shall study the behavior of g near $s = 0$. By Gamma's functional equation, we get

$$g(s) = \frac{(as-1)\dots(as-b)}{\Gamma(as)(cs+d)}.$$

Around $s = 0$, we have

$$\begin{aligned} (as-1)\dots(as-b) &= (-1)^b b! + a(-1)^{b-1} b! h_b s + O(s^2), \\ \frac{1}{\Gamma(s)} &= s + \gamma s^2 + O(s^3). \end{aligned}$$

Let's assume that $d = 0$. Therefore, near $s = 0$ we have:

$$\begin{aligned} g(s) &= \frac{a}{c} \frac{(as-1)\dots(as-b)}{\Gamma(as)as} \\ &= \frac{a}{c} ((-1)^b b! + as(-1)^{b-1} b! h_b)(1 + \gamma as) + O(s^2) \\ &= \frac{a}{c} (-1)^b b! + \frac{a^2}{c} (-1)^{b-1} b! h_b s + \frac{a^2}{c} \gamma (-1)^b b! s + O(s^2). \end{aligned}$$

Let's assume that $d \neq 0$. Therefore, near $s = 0$ we have:

$$\begin{aligned} g(s) &= \frac{(as-1)\dots(as-b)}{\Gamma(as)(cs+d)} \\ &= \frac{(as-1)\dots(as-b)}{cs+d} (as) + O(s^2) \\ &= \frac{a}{d} (-1)^b b! s + O(s^2). \end{aligned}$$

□

Lemma 3.1.2. For all $\mathcal{P} \subset \llbracket 1, P \rrbracket$, $(\mu_p)_{p \in \mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$, $(N_p)_{p \in \mathcal{P}} \in \mathbb{N}_0^{\mathcal{P}}$, $(n_p)_{p \in \mathcal{P}} \in \mathbb{N}_0^{\mathcal{P}}$, we have

$$\partial_s \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} \right) \Big|_{s=0} = \left(\sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + n_p}) \right) \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p},$$

where h_n is the n -th harmonic number.

Proof. Let's recall that

$$\prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} = \prod_{p \in \mathcal{P}} \frac{(-N_p - 1 + \mu_p s) \dots (-N_p + \mu_p s - n_p)}{n_p!}.$$

By developing the product $\frac{(-N_p - 1 + \mu_p s) \dots (-N_p - n_p + \mu_p s)}{n_p!}$ near $s = 0$, we get

$$\frac{(-N_p - 1 + \mu_p s) \dots (-N_p + \mu_p s - n_p)}{n_p!} = \binom{-N_p - 1}{n_p} + \mu_p s \sum_{i=0}^{n_p-1} \frac{(-N_p - 1) \dots \widehat{(-N_p - 1 - i)} \dots (-N_p - n_p)}{n_p!} + O(s^2),$$

where $\widehat{(-N_p - 1 - i)}$ means that we forget that term in the product. We can simplify that last expression by using harmonic numbers:

$$\begin{aligned} \frac{(-N_p - 1 + \mu_p s) \dots (-N_p + \mu_p s - n_p)}{n_p!} &= \binom{-N_p - 1}{n_p} + \mu_p \binom{-N_p - 1}{n_p} \left(\sum_{i=0}^{n_p-1} \frac{1}{-N_p - 1 - i} \right) s + O(s^2) \\ &= \binom{-N_p - 1}{n_p} + \mu_p \binom{-N_p - 1}{n_p} (h_{N_p} - h_{N_p+n_p}) s + O(s^2). \end{aligned}$$

Near $s = 0$, we then get that:

$$\prod_{p \in \mathcal{P}} \binom{-N_p - 1 + \mu_p s}{n_p} = \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p} + \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{n_p} \left(\sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p+n_p}) \right) s + O(s^2).$$

□

Lemma 3.1.3. *Let $M \in \mathbb{Z}_{\leq 0}$ be a nonpositive integer, and $(a_k)_{k \in \llbracket M, +\infty[}$, $(b_k)_{k \in \llbracket M, +\infty[}$ be two complex number sequences such that the power series $\sum_{k=0}^{+\infty} a_k x^k$ and the power series $\sum_{k=0}^{+\infty} b_k x^k$ have two positive radius of convergence R_1 and R_2 . Let's also assume that there exists a positive real number $0 < R < \min(R_1, R_2)$ such that*

$$\forall x \in]0, R[, \quad \sum_{k=M}^{+\infty} a_k x^k + \sum_{k=M}^{+\infty} b_k x^k \ln(x) = 0,$$

then for all $k \in \llbracket M, +\infty[$ we have that $a_k = b_k = 0$.

Proof. If all b_k are zeros (respectively all a_k are zeros) for all $k \geq M$, the result is trivial. Otherwise, we shall use a reductio ad absurdum argument. Let's assume that there exist $i, j \geq M$ such that $a_i \neq 0$ and $b_j \neq 0$. Suppose that $i = \min\{k | a_k \neq 0\}$ and $j = \min\{k | b_k \neq 0\}$, then in the neighborhood of 0 we have

$$a_i x^i + b_j x^j \ln(x) + o(x^j \ln(x)) + o(x^i) = 0,$$

which implies $b_j x^{j-i} \ln(x) \sim -a_i$ near $x = 0$, a contradiction. □

The previous lemma will play a crucial role in determining formulas for the special values in the proof of Theorem D in chapter 5, and in the computations of the coefficients $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ in the end of this chapter. Indeed, in the meromorphic continuation of the function $Z_{\mathbf{N}, \mathbf{N}'}(s)$ obtained in Proposition 2.2.4, we've introduced a free variable θ and two functions $K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ and $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$. By taking the derivative of the analytical continuation formula, we get $Z'(-\mathbf{N}, \mathbf{N}') = \partial_s (K_{\mathbf{N}, \mathbf{N}'}(\theta, s))|_{s=0} + \partial_s (J_{\mathbf{N}, \mathbf{N}'}(\theta, s))|_{s=0}$.

By taking the derivative of the function $J_{\mathbf{N},\mathbf{N}'}(\theta, s)$ with respect to s at $s = 0$ via formula (2.2), we will get a sum between a Laurent series in θ and a multiple of $\ln(\theta)$. Similarly, by taking the derivative of the function $K_{\mathbf{N},\mathbf{N}'}(\theta, s)$ with respect to s at $s = 0$ via formula (2.1), we will get a sum between a Laurent series in θ and a multiple of $\ln(\theta)$. It is clear the directional derivative value $Z'(-(\mathbf{N}, \mathbf{N}'))$ is independent of θ . Hence we get from the previous lemma that the constant term in θ in the sum of the two Laurent series that come from the computations of $\partial_s(J_{\mathbf{N},\mathbf{N}'}(\theta, s))|_{s=0}$ and of $\partial_s(K_{\mathbf{N},\mathbf{N}'}(\theta, s))|_{s=0}$ corresponds to the directional derivative value $Z'(-(\mathbf{N}, \mathbf{N}'))$. In the same trend, we will use this same lemma in the computations of the coefficients $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ in order to eliminate in our computations the non-constant terms of the Laurent series in the variable ϵ , and also the multiple of $\ln(\epsilon)$.

Lemma 3.1.4. *There exists a function $\alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) = a \ln(\epsilon) + \sum_{n \geq m} b_n \epsilon^n$ defined for all $0 < \epsilon \ll 1$ with $m \in \mathbb{Z}$, without any constant term (i.e. $b_0 = 0$), and such that*

$$\int_{\epsilon}^1 x_f^{-N_f-1+|\mathbf{w}_{\bullet}f|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) + F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}'),$$

with $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ being the constant defined in Proposition 1.2.2:

$$\begin{aligned} F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') &= -E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N_f+1-|\mathbf{w}_{\bullet}f|} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} \\ &\quad - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{(|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|)} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda-1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}) \\ &\quad + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \ln \left(1 + \frac{c_{f,p}}{c_{j,p}} \right). \end{aligned}$$

Proof. In order to compute the integral above, we shall use Proposition 1.2.2. With the same notations as in that proposition, we get

$$\begin{aligned} x_f^{-N_f-1+|\mathbf{w}_{\bullet}f|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|} &= \tilde{P}_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f) + \sum_{\lambda=1}^{N_f+1-|\mathbf{w}_{\bullet}f|} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{x_f^{\lambda}} \\ &\quad + \sum_{p \in \mathcal{P}} \sum_{\lambda=1}^{(|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|+|\mathbf{v}|)} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{(c_{j,p} + c_{f,p} x_f)^{\lambda}}. \end{aligned}$$

We recall that $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$ is the antiderivative of the polynomial $\tilde{P}_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$ which fades at $x_f = 1$, and we noted $E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ the constant term of $P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$. Considering the polynomial

$$P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(x_f) := P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f) - E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$$

we find that

$$\begin{aligned} \int_{\epsilon}^1 x_f^{-N_f-1+|\mathbf{w}_{\bullet}f|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|} dx_f &= -E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - P_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) \\ &\quad - C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},1}(\mathbf{N}, \mathbf{N}') \ln(\epsilon) - \sum_{\lambda=2}^{N_f+1-|\mathbf{w}_{\bullet}f|} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda-1} (1 - \epsilon^{1-\lambda}) \\ &\quad + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \left(\ln(c_{j,p} + c_{f,p}) - \ln(c_{j,p}) - \ln \left(1 + \frac{c_{f,p}}{c_{j,p}} \epsilon \right) \right) \end{aligned}$$

$$- \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} ((c_{j,p} + c_{f,p})^{1-\lambda} - (c_{j,p} + c_{f,p}\epsilon)^{1-\lambda}).$$

By expanding the terms $(c_{j,p} + c_{f,p}\epsilon)^{1-\lambda}$ and $\ln\left(1 + \frac{c_{f,p}}{c_{j,p}}\epsilon\right)$ near $\epsilon = 0$, we can decompose the previously obtained formula into a constant term, and into a sum between a multiple of $\ln(\epsilon)$ and a Laurent series without any constant term in the variable ϵ , for $0 < \epsilon \ll 1$. Let's denote by $\alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon)$ the sum between the multiple of $\ln(\epsilon)$ and the series with a constant term. We then have

$$\begin{aligned} & \int_{\epsilon}^1 x_f^{-N'_f-1+|\mathbf{w}\bullet f|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p}x_f)^{-N_p-1-|\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}^*(\epsilon) \\ & - E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}') - \sum_{\lambda=2}^{N'_f+1-|\mathbf{w}\bullet f|} \frac{C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} + \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(1,p)}(\mathbf{N}, \mathbf{N}') \ln\left(1 + \frac{c_{f,p}}{c_{j,p}}\right) \\ & - \frac{1}{c_{f,p}} \sum_{p \in \mathcal{P}} \sum_{\lambda=2}^{|\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + |\mathbf{v}|} \frac{D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')}{\lambda - 1} ((c_{j,p} + c_{f,p})^{1-\lambda} - c_{j,p}^{1-\lambda}). \end{aligned}$$

□

We note that a simpler, although less explicit, expression of the previous integral can be obtained, using a symbolic computation software. Consider the antiderivative $G_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\mathbf{N},\mathbf{N}'}(x_f)$ of the function $x_f \mapsto x_f^{-N'_f-1+|\mathbf{w}\bullet f|} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p}x_f)^{-N_p-1-|\mathbf{v}_p|}$ that vanishes at $x_f = 1$. This primitive is expressed as a sum of a multiple of $\ln(x_f)$ and a Laurent series in x_f , for $0 < x_f \ll 1$. We then observe that the constant term of this Laurent series is $-F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$.

Remark 3.1.5. *It seems more practical to directly compute the coefficient $F_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ using a symbolic computation software by determining an antiderivative of the function*

$$x_f \mapsto x_f^{-N'_f-1+|\mathbf{w}\bullet f|} \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p}x_f)^{-N_p-1-|\mathbf{v}_p|}$$

rather than computing the coefficients $C_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},\lambda}(\mathbf{N}, \mathbf{N}')$, $D_{\mathcal{P},j,f,\mathbf{v},\mathbf{w},(\lambda,p)}(\mathbf{N}, \mathbf{N}')$, and $E_{\mathcal{P},j,f,\mathbf{v},\mathbf{w}}(\mathbf{N}, \mathbf{N}')$ using the partial fraction decomposition from Proposition 1.2.2.

3.2 Value and derivative of $h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s)$ at $s = 0$

We consider a set $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, an integer $j \in \llbracket 1, Q \rrbracket$, and a tuple $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$. In this section we're aiming to compute the coefficients $Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ and $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$, which we recall the definitions:

$$\begin{aligned} h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(0) &= Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k}), \\ (h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}})'(0) &= Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k}). \end{aligned}$$

Let's recall that we proved in Proposition 2.2.2 that the function

$$\begin{aligned} h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s) &= \\ & \frac{1}{\prod_{q=1}^Q \Gamma(-N'_q + \mu'_q s)} \int_0^1 \dots \int_0^1 \prod_{q=1}^Q x_q^{-N'_q + \mu'_q s - 1} \prod_{p \in \mathcal{P}} l_p^*(\widehat{\mathbf{x}}^j)^{-N_p + \mu_p s - 1} \prod_{p \in \mathcal{P}^c} l_p^*(\widehat{\mathbf{x}}^j)^{k_p} dx_1 \dots \widehat{dx_j} \dots dx_Q \end{aligned}$$

is holomorphic on \mathbb{C} . We also recall that we obtained from that proposition an expression of the derivative $h_{\mathbf{N},\mathbf{N}',\mathcal{P},j,\mathbf{k}}(s)$ at $s = 0$.

3.2.1 Preliminaries

We consider a set $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, an integer $j \in \llbracket 1, Q \rrbracket$, and a real number $0 < \epsilon \ll 1$ small enough. We know that for all $s \in \mathbb{C}$, we have

$$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)_{\mu, \mu'} = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)_{\mu, \mu'}$$

we then deduce the fact that it is enough to study the values and the derivative values of $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)_{\mu, \mu'}$ at $s = 0$ for $0 < \epsilon \ll 1$ small enough in order to get the values and the derivative values of $h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)_{\mu, \mu'}$ at $s = 0$. We now recall point iii) obtained in Proposition 2.2.2.

Let $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$, then for all $s \in \mathbb{C}$ and for all real number $0 < \epsilon \ll 1$ small enough, we have:

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(s)_{\mu, \mu'} &= \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\substack{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}} \\ \mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q, p}^{\mathbf{v}_p, q} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q, p}^{\mathbf{w}_p, q} \right) \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)} \\ &\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j, p} + \sum_{f \in \mathcal{Q}^c} c_{f, p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q. \end{aligned}$$

Let's consider $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)_{\mu, \mu'}$ the general term of the above series:

$$\begin{aligned} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)_{\mu, \mu'} &= \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q, p}^{\mathbf{v}_p, q} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in \llbracket 1, Q \rrbracket} c_{q, p}^{\mathbf{w}_p, q} \right) \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \Gamma(-N'_q + \mu'_q s)} \frac{\epsilon^{\sum_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)}}{\prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)} \\ &\cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j, p} + \sum_{f \in \mathcal{Q}^c} c_{f, p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q. \end{aligned}$$

Let's recall that, with that definition in mind, we have by point iii) of Lemma 2.3.22:

$$I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0)_{\mu, \mu'} = \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0)_{\mu, \mu'}. \quad (3.1)$$

and by that same point iii), we know that one can take the derivative termwise in order to get

$$\partial_s \left(I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s)_{\mu, \mu'} \right) \Big|_{s=0} = \sum_{\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}}} \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)_{\mu, \mu'} \right) \Big|_{s=0}. \quad (3.2)$$

Thanks to the formula (2.15), we get

$$\begin{aligned} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) &= \sum_{\substack{\mathcal{Q} \subset [1, Q] \setminus \{j\} \\ \mu, \mu'}} I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, 0), \\ (h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}})'(0) &= \sum_{\substack{\mathcal{Q} \subset [1, Q] \setminus \{j\} \\ \mu, \mu'}} \partial_s \left(I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right) \Big|_{s=0}. \end{aligned} \quad (3.3)$$

Therefore, we can compute the coefficients $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$ and $Q_{\mathbf{N}, \mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ by computing the value and the derivative value of $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$ at $s = 0$.

Lemma 3.2.1. *For all $\epsilon > 0$, we have at $s = 0$ that*

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) = \begin{cases} O(1) & \text{if } |\mathcal{Q}| = Q - 1 \text{ and } \forall q \in \mathcal{Q}, & |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q \\ O(s) & \text{if } |\mathcal{Q}| = Q - 1 \text{ and } \exists f \in \mathcal{Q} \setminus \{j\}, \forall q \in \mathcal{Q} \setminus \{f\}, & |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q \\ & & |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}| \neq N'_f \\ O(s) & \text{if } |\mathcal{Q}| = Q - 2 \text{ and } \forall q \in \mathcal{Q}, & |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q \\ O(s^2) & \text{else.} \end{cases}$$

Proof. We know that the meromorphic function $s \mapsto \prod_{q \in \mathcal{Q}} \frac{1}{-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|}$ has a pole of order $|\{q \in \mathcal{Q} | N'_q = |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|\}|$ at $s = 0$. In particular, the order is necessarily smaller than $|\mathcal{Q}|$. Since the Euler gamma function has a pole of order 1 at nonpositive integers, we get that the function $s \mapsto \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + s\mu'_q)}$ has a zero of order $Q - 1$ at $s = 0$. We then obtain that $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) = O(s^n)$ at $s = 0$, with

$$n := Q - 1 - |\{q \in \mathcal{Q} | N'_q = |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|\}|.$$

□

3.2.2 Computation of $Q_{\mathbf{N}, \mathbf{N}'}^0(\mathcal{P}, j, \mathbf{k})$

Proposition 3.2.2. *For all $\epsilon > 0$ and all set $\mathcal{Q} \subset [1, Q] \setminus \{j\}$ we have that:*

- If $\mathcal{Q} = [1, Q] \setminus \{j\}$ and $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ for all $q \in \mathcal{Q}$, then

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0) = \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q \in [1, Q]} c_{q, p}^{w_{p, q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j, p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}} c_{q, p}^{v_{p, q}} \right) \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N_q!.$$

- Otherwise, we have

$$T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, 0) = 0.$$

Proof. In the first case, by Lemma 3.1.1, we have that:

$$\left(\frac{1}{\Gamma(-N'_q + \mu'_q s)} \frac{1}{\mu'_q s} \right) \Big|_{s=0} = (-1)^{N'_q} N_q! \frac{\mu'_q}{\mu'_q}.$$

By Taking $s = 0$ in the expression of $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$, we obtain the first point of the proposition.

The second point follows from Lemma 3.2.1. □

By formula (3.1), and by the previous proposition, we get:

Corollary 3.2.3. *For all set $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$, and any given real number $0 < \epsilon \ll 1$ small enough, we have that:*

- If $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$, then we have

$$I_{\mathbf{N},\mathbf{N}';\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, 0)_{\mu, \mu'} = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \cdot \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_q| + |\mathbf{w}_q| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

- Otherwise we have

$$I_{\mathbf{N},\mathbf{N}';\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, 0)_{\mu, \mu'} = 0.$$

By formula (2.15), we recall that we have

$$h_{\mathbf{N},\mathbf{N}';\mathcal{P},j,\mathbf{k}}(0)_{\mu, \mu'} = \sum_{\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}} \sum_{\mu, \mu'} I_{\mathbf{N},\mathbf{N}';\mathcal{P},j,\mathcal{Q},\mathbf{k}}(\epsilon, 0),$$

and by the previous corollary, we then obtain

$$Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathcal{Q}, \mathbf{k})_{\mu, \mu'} = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_q| + |\mathbf{w}_q| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

Noting that the value doesn't depend on the direction, that justifies the previous notation used for $Q_{\mathbf{N},\mathbf{N}'}^0(\mathcal{P}, j, \mathcal{Q}, \mathbf{k})$ in Theorems A and D.

3.2.3 Computation of $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})_{\mu, \mu'}$

Taking $s = 0$ in the derivative with respect to s of the general term

$$T_{\mathbf{N},\mathbf{N}';\mathcal{P},j,\mathcal{Q},\mathbf{k},\mathbf{v},\mathbf{w}}(\epsilon, s)_{\mu, \mu'} = \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{q \in \mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \cdot \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \prod_{q \in \mathcal{Q}} (-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)} \cdot \int_{\epsilon}^1 \dots \int_{\epsilon}^1 \prod_{q \in \mathcal{Q}^c} x_q^{-N'_q + \mu'_q s - 1 + |\mathbf{w}_{\bullet q}|} \cdot \prod_{p \in \mathcal{P}} \left(c_{j,p} + \sum_{f \in \mathcal{Q}^c} c_{f,p} x_f \right)^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \prod_{q \in \mathcal{Q}^c} dx_q,$$

we then get the following proposition.

Proposition 3.2.4. *For all real number $\epsilon > 0$ and all set $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ we have that:*

- If $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ and $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ for all $q \in \mathcal{Q}$, we have

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \\ & (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} |\boldsymbol{\mu}'|_{|\mathcal{Q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \ln(\epsilon) \\ & + (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\ & \cdot \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right). \end{aligned} \quad (3.4)$$

- If $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ and if there exists an integer $f \in \mathcal{Q}$ such that $|\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}| \neq N'_f$ and such that $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ for all $q \in \mathcal{Q} \setminus \{f\}$, we then have

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \\ & \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \frac{\mu'_f}{-N'_f + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|} \epsilon^{-N'_f + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|}. \end{aligned} \quad (3.5)$$

- If $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j, f\}$ such that $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ for all $q \in \mathcal{Q}$, then there exists a complex valued function $\beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) = \beta \ln(\epsilon) + \sum_{n=m}^{+\infty} \alpha_n \epsilon^n$ defined for all real number $0 < \theta \ll 1$ small enough, with $m \in \mathbb{Z}$ and $\alpha_0 = 0$, and such that

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} &= \beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) \\ & + \mu'_f (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \\ & \cdot F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \end{aligned} \quad (3.6)$$

with $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ being the constant defined in Proposition 1.2.2.

- Otherwise we have

$$\partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} = 0.$$

Proof. • If $\mathcal{Q} = [\mathbb{1}, \mathcal{Q}] \setminus \{j\}$ and $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ for all $q \in \mathcal{Q}$, we then get by Lemma 3.1.1,

$$\begin{aligned} \partial_s \left(\frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} &= \sum_{\substack{f=1 \\ f \neq j}}^{\mathcal{Q}} \mu'_f (-1)^{N'_f} N'_f! (\gamma - h_{N'_f}) \prod_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} (-1)^{N'_q} N'_q! \\ &= (-1)^{|\mathbf{N}'|_{[\mathbb{1}, \mathcal{Q}] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} N'_q! \right) \left(\sum_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} \mu'_q (\gamma - h_{N'_q}) \right) \\ \left(\frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} &= (-1)^{|\mathbf{N}'|_{[\mathbb{1}, \mathcal{Q}] \setminus \{j\}}} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} N'_q!. \end{aligned}$$

By Lemma 3.1.2, we get that

$$\partial_s \left(\prod_{p \in \mathcal{P}} \binom{-N_p + \mu_p s - 1}{|\mathbf{v}_p|} \right) \Big|_{s=0} = \left(\sum_{p \in \mathcal{P}} \mu_p (h_{N_p} - h_{N_p + |\mathbf{v}_p|}) \right) \prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|}.$$

It is also clear that

$$\partial_s \left(\prod_{p \in \mathcal{P}} c_{j,p}^{-N_p + \mu_p s - 1 - |\mathbf{v}_p|} \right) \Big|_{s=0} = \left(\sum_{p \in \mathcal{P}} \mu_p \ln(c_{j,p}) \right) \prod_{p \in \mathcal{P}} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|}.$$

Therefore we find that

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0} &= |\boldsymbol{\mu}'|_{\mathcal{Q}} (-1)^{|\mathbf{N}'|_{[\mathbb{1}, \mathcal{Q}] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} N'_q! \right) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \\ &\quad \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \ln(\epsilon) \\ &+ (-1)^{|\mathbf{N}'|_{[\mathbb{1}, \mathcal{Q}] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} N'_q! \right) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p + |\mathbf{v}_p|}) \right) \\ &\quad \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\ &+ \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\ &\quad \cdot (-1)^{|\mathbf{N}'|_{[\mathbb{1}, \mathcal{Q}] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} N'_q! \right) \left(\sum_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} \mu'_q (\gamma - h_{N'_q}) \right). \end{aligned}$$

Factoring the first and second term of this last sum, we get formula (3.4).

• If $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$ and if there exist an integer $f \in \mathcal{Q} \setminus \{j\}$ such that $|\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}| \neq N'_f$ and such that $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ for all $q \in \mathcal{Q} \setminus \{f\}$, we then find by Lemma 3.1.1,

$$\begin{aligned} & \left(\frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|) \prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = 0, \\ \partial_s & \left(\frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|) \prod_{\substack{q=1 \\ q \neq j}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = \\ & (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \frac{\mu'_f}{-N'_f + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \partial_s & \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0} = \\ & \left(\frac{1}{\Gamma(-N'_f + \mu'_f s)(-N'_f + \mu'_f s + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}|) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s)(-N'_q + \mu'_q s + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|)} \right) \Big|_{s=0} \\ & \cdot \sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right). \end{aligned}$$

Noticing that $\sum_{\substack{q=1 \\ q \neq j}}^Q (-N'_q + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|) = -N'_f + |\mathbf{v}_{\bullet f}| + |\mathbf{w}_{\bullet f}| \neq 0$, we then obtain formula (3.5).

• If $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j, f\}$ such that $|\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q$ for all $q \in \mathcal{Q}$, we then find by Lemma 3.1.1 that

$$\begin{aligned} & \left(\frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = 0 \\ \partial_s & \left(\frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} = \mu'_f \prod_{\substack{q=1 \\ q \neq j}}^Q (-1)^{N'_q} N'_q!. \end{aligned}$$

Deriving the general term $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)$ with respect to s , and by taking $s = 0$ we then find that

$$\begin{aligned} \partial_s & \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right) \Big|_{s=0} = \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \quad (3.7) \\ & \cdot \partial_s \left(\frac{1}{\Gamma(-N'_f + \mu'_f s) \prod_{\substack{q=1 \\ q \neq j, f}}^Q \Gamma(-N'_q + \mu'_q s) \mu'_q s} \right) \Big|_{s=0} \sum_{\substack{q=1 \\ q \neq j, f}}^Q (-N'_q + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|) \epsilon \end{aligned}$$

$$\cdot \int_{\epsilon}^1 x_f^{-N'_f-1+|\mathbf{w}_{\bullet f}|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|} dx_f.$$

With the conditions on the index \mathbf{v} and on the index \mathbf{w} , we find that $\sum_{\substack{q=1 \\ q \neq j, f}}^Q (-N'_q + |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|) = 0$.

By Lemma 3.1.4, we get

$$\int_{\epsilon}^1 x_f^{-N'_f-1+|\mathbf{w}_{\bullet f}|} \cdot \prod_{p \in \mathcal{P}} (c_{j,p} + c_{f,p} x_f)^{-N_p-1-|\mathbf{v}_p|} dx_f = \alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) + F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}'),$$

with $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ defined in Proposition 1.2.2, and $\alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon)$ being a function in ϵ of the form $a \ln(\epsilon) + \sum_{n \geq m} b_n \epsilon^n$, with $m \in \mathbb{Z}$, and $a, (b_n)_{n \geq m}$ being complex numbers with $b_0 = 0$.

Injecting those last expressions into formula (3.7), we then get

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}'; \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{\mu, \mu'} \Big|_{s=0} = \\ \mu'_f \prod_{\substack{q=1 \\ q \neq j}}^{N'_q} (-1)^{N'_q} N'_q! \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \\ \cdot (\alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) + F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')). \end{aligned}$$

By setting

$$\beta_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon) = \mu'_f \prod_{\substack{q=1 \\ q \neq j}}^{N'_q} (-1)^{N'_q} N'_q! \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \alpha_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}, \mathbf{N}, \mathbf{N}'}^*(\epsilon),$$

we then get the formula (3.6).

- If we're not in one of these three last cases above, therefore by Lemma 3.2.1, we have that

$$T_{\mathbf{N}, \mathbf{N}'; \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s)_{\mu, \mu'} = O(s^2)$$

around $s = 0$, which concludes this particular point. \square

By formula (3.2) and by the previous proposition, we then find the following corollary:

Corollary 3.2.5. *For all set $\mathcal{Q} \subset \llbracket 1, Q \rrbracket \setminus \{j\}$ and any given real number $0 < \epsilon \ll 1$ small enough we have that:*

- If $\mathcal{Q} = \llbracket 1, Q \rrbracket \setminus \{j\}$, then there exists a complex valued function $I_{\mathbf{N}, \mathbf{N}'; \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon)_{\mu, \mu'} = \beta \ln(\epsilon) +$

$\sum_{n=m}^{+\infty} \alpha_n \epsilon^n$ defined for all $0 < \epsilon \ll 1$, with $m \in \mathbb{Z}$ and $\alpha_0 = 0$, such that

$$\begin{aligned} \partial_s \left(I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right)_{|s=0} &= \\ & (-1)^{|\mathbf{N}'|_{\llbracket [1, \mathcal{Q}] \setminus \{j\} \rrbracket}} \left(\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket [1, \mathcal{Q}] \setminus \{j\} \rrbracket})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} c_{q,p}^{v_{p,q}} \right) \\ & \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} \mu'_q (\gamma - h_{N'_q}) \right) \\ & + I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon). \end{aligned}$$

- If $\mathcal{Q} = \llbracket [1, \mathcal{Q}] \setminus \{j, f\} \rrbracket$, then there exists a function $I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon)$ which corresponds to a sum of a multiple of $\ln(\epsilon)$ and a Laurent series defined for all $0 < \epsilon \ll 1$ without any constant term, and such that

$$\begin{aligned} \partial_s \left(I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right)_{|s=0} &= I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}^*(\epsilon) \\ & + (-1)^{|\mathbf{N}'|_{\llbracket [1, \mathcal{Q}] \setminus \{j\} \rrbracket}} \mu'_f \left(\prod_{\substack{q=1 \\ q \neq j}}^{\mathcal{Q}} N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^{\mathcal{Q}})^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket [1, \mathcal{Q}] \setminus \{j, f\} \rrbracket})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \\ & \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^{\mathcal{Q}} c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^{\mathcal{Q}} c_{q,p}^{v_{p,q}} \right), \end{aligned}$$

with $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ defined in Proposition 1.2.2.

- Otherwise, we find that

$$\partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}, \mathbf{v}, \mathbf{w}}(\epsilon, s) \right)_{|s=0} = 0.$$

By formula (3.3), we then get that

$$(h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}})_{\mu, \mu'}'(0) = \sum_{\mathcal{Q} \subset \llbracket [1, \mathcal{Q}] \setminus \{j\} \rrbracket} \partial_s \left(I_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathcal{Q}, \mathbf{k}}(\epsilon, s) \right)_{|s=0},$$

therefore by the previous corollary we obtains the following expression for the coefficient $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$:

$$\begin{aligned}
& Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k}) \\
&= \partial_s \left(I_{\mathbf{N},\mathbf{N}'}^{\mu,\mu'}(\mathcal{P}, j, \llbracket 1, Q \rrbracket \setminus \{j\}, \mathbf{k}(\epsilon, s)) \right)_{|s=0} + \sum_{\substack{f=1 \\ f \neq j}}^Q \partial_s \left(I_{\mathbf{N},\mathbf{N}'}^{\mu,\mu'}(\mathcal{P}, j, \llbracket 1, Q \rrbracket \setminus \{j, f\}, \mathbf{k}(\epsilon, s)) \right)_{|s=0} \\
&= (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, |\mathbf{v}_q|+|\mathbf{w}_q|=N'_q}} \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right) \\
&\quad \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j,p}) + h_{N_p} - h_{N_p+|\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \\
&+ (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j, f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, f, |\mathbf{v}_q|+|\mathbf{w}_q|=N'_q}} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \\
&\quad \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q,p}^{v_{p,q}} \right) \\
&+ I_{\mathbf{N},\mathbf{N}'}^{\mu,\mu'}(\mathcal{P}, j, \llbracket 1, Q \rrbracket \setminus \{j\}, \mathbf{k}(\epsilon)) + \sum_{\substack{f=1 \\ f \neq j}}^Q I_{\mathbf{N},\mathbf{N}'}^{\mu,\mu'}(\mathcal{P}, j, \llbracket 1, Q \rrbracket \setminus \{j, f\}, \mathbf{k}(\epsilon)).
\end{aligned}$$

We now conclude by using Lemma 3.1.3 by noticing the fact that $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$ is independent of ϵ .

We then find that the constant term in ϵ of the right-hand side of the previous formula is equal to the coefficient $Q_{\mathbf{N},\mathbf{N}'}^1(\mathcal{P}, j, \mathbf{k})$.

Chapter 4

Proof of Theorem A and its corollaries

The aim of this chapter is to study the directional values of $Z(\mathbf{s}, \mathbf{s}')$ at nonpositive integers. By showing that the function $Z_{\mathbf{N}, \mathbf{N}'}(s)$ is regular at $s = 0$, we have established that the values $Z(-(\mathbf{N}, \mathbf{N}'))$ and $Z'(-(\mathbf{N}, \mathbf{N}'))$ are well-defined. To compute the first value, we will use Proposition 2.2.4 to evaluate $Z_{\mathbf{N}, \mathbf{N}'}(s)$ at $s = 0$.

4.1 Statement of Theorem A

Let's recall the statement of the theorem:

Theorem (A). For all $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and all direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that

$$\forall \mathcal{P} \subsetneq [1, P], \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0,$$

we have

$$Z(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = (-1)^{|\mathbf{N}'|} \sum_{\substack{\mathcal{P} \subseteq [1, P] \\ 1 \leq j \leq Q}} \frac{(-1)^{N'_j + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \left(\prod_{p \in \mathcal{P}} N_p! \right) \cdot \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!},$$

with

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right),$$

where we noted

$$\begin{aligned} \forall q \in \llbracket 1, Q \rrbracket, \quad |\mathbf{w}_{\bullet q}| &= \sum_{p \in \mathcal{P}^c} w_{p,q}, \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad |\mathbf{v}_{\bullet q}| &= \sum_{p \in \mathcal{P}} v_{p,q}. \end{aligned}$$

4.2 Proof of Theorem A

Let's first fix a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ and a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$, and we consider a real number $0 < \theta \ll 1$ small enough. By Proposition 2.2.4, we get

$$Z_{\mathbf{N}, \mathbf{N}'}(0) = K_{\mathbf{N}, \mathbf{N}'}(\theta, 0) + J_{\mathbf{N}, \mathbf{N}'}(\theta, 0).$$

Since $K_{\mathbf{N}, \mathbf{N}'}(\theta, 0) = 0$ by Proposition 2.3.11, we only have to evaluate the function $J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ at $s = 0$ in order to prove Theorem A. By formula (2.2), we have

$$\begin{aligned} Z_{\mathbf{N}, \mathbf{N}'}(0) &= \\ & \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} N_p! \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \sum_{j=1}^P h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) \\ & \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)} \frac{\theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}}}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}}} \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & + \left(\prod_{p=1}^P \Gamma(1 + N_p) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(0) \left(\frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \right)_{|s=0}. \end{aligned}$$

Let's first observe that the expression

$$\left(\prod_{p=1}^P \Gamma(1 + N_p) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(0) \left(\frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \right)_{|s=0}$$

vanishes. It follows from the fact that $-|\mathbf{N}'| - |\mathbf{N}| - P \neq 0$, thus $s \mapsto \frac{1}{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}$

is a function regular at $s = 0$, and the function $s \mapsto \frac{1}{\Gamma(-N'_j + \mu'_j s)}$ vanishes at $s = 0$. Let's now

study the first term of the formula obtained for $Z_{\mathbf{N}, \mathbf{N}'}(0)$. For all set $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, we first see that

$h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) = Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$. Moreover, by Lemma 3.1.1 we have:

- If $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} \neq -|\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}$, then the function

$$s \mapsto \frac{1}{\Gamma(-N'_j + \mu'_j s)} \frac{\theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}}}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}}$$

vanishes at $s = 0$.

- If $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} = -|\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}$, then

$$\left(\frac{1}{\Gamma(-N'_j + \mu'_j s)} \frac{\theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}}}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} + (|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c}}} \right)_{|s=0} = \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} (-1)^{N'_j} N'_j!.$$

Therefore we obtain that Theorem A holds for any given direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. The expression of $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ in Theorem A follows from the computations done in chapter 3.

We now wish to prove the theorem for all direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

Let's recall that by Corollary 1.1.31, we know that the function $(\boldsymbol{\mu}, \boldsymbol{\mu}') \mapsto Z(-(\mathbf{N}, \mathbf{N}'))$ is meromorphic on $\mathbb{C}^P \times \mathbb{C}^Q$. We first see that the right-hand side of formula (1.8) is a rational fraction in the variables $\mu_1, \dots, \mu_P, \mu'_1, \dots, \mu'_Q$ with denominators of the form $|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}$, and we've just proved that this formula holds for all direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. Thus, by the analytical continuation principle, it's clear that the formula (1.8) holds for any given direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

□

4.3 On corollaries of Theorem A

We easily derive from Theorem A its corollaries.

4.3.1 Proof of Corollary A1

To prove Corollary A1, we apply Theorem A with the data $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$:

$$Z(-(\mathbf{0}, \mathbf{0})) = \sum_{\substack{\boldsymbol{\mu}, \boldsymbol{\mu}' \\ \mathcal{P} \subsetneq \llbracket 1, P \rrbracket}} (-1)^{|\mathcal{P}|} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \sum_{j=1}^Q \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!}$$

with

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_{\bullet, q}| + |\mathbf{w}_{\bullet, q}| = 0}} \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j, p}^{-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, p}^{v_{p, q}} \right).$$

We can simplify the coefficient $Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k})$ by noticing that the condition $|\mathbf{v}_{\bullet, q}| + |\mathbf{w}_{\bullet, q}| = 0$ for all $q \neq j$ implies that

$$\begin{aligned} \forall p \in \mathcal{P}, \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad v_{p, q} &= 0 \\ \forall p \in \mathcal{P}^c, \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, \quad w_{p, q} &= 0. \end{aligned}$$

We then obtain that $k_p = |\mathbf{w}_p| = w_{p, j}$ for all $p \in \mathcal{P}^c$. Therefore we have the following expression for $Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k})$:

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \frac{\prod_{p \in \mathcal{P}^c} c_{j, p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j, p}}.$$

□

4.3.2 Proof of Corollary A2

In order to prove A2, we set $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$ and $(\mathbf{N}, \mathbf{N}') = (N, \dots, N)$, and we find by Theorem A that

$$Z_{\Delta}(-N) = \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{N(|\mathcal{P}|+Q+1)+|\mathcal{P}|} N^{|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=N(Q+|\mathcal{P}|)+|\mathcal{P}|}} \sum_{j=1}^Q \frac{1}{Q+|\mathcal{P}|} Q_N^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N-k_p, d_p)}{k_p!},$$

with

$$Q_N^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{N(Q-1)} N^{Q-1} \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|=N}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

By the two previous equalities, and by setting

$$Q_N^0(\mathcal{P}, \mathbf{k}) := \sum_{j=1}^Q Q_N^0(j, \mathcal{P}, \mathbf{k}),$$

we get the required result. \square

4.3.3 Proof of Corollary A3

The Corollary A3 corresponds to a qualitative and non explicit version of Theorem A. Let a $P+Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0.$$

We set \mathbb{K} the field above \mathbb{Q} generated by the complex numbers $(d_p)_{p \in \llbracket 1, P \rrbracket}$ and $(c_{q,p})_{p \in \llbracket 1, P \rrbracket, q \in \llbracket 1, Q \rrbracket}$, and by the complex numbers μ_p for all $p \in \llbracket 1, P \rrbracket$, and μ'_q for all $q \in \llbracket 1, Q \rrbracket$.

By Theorem A, we find that

$$Z(-(\mathbf{N}, \mathbf{N}')) = (-1)^{|\mathbf{N}'|} \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \frac{(-1)^{N'_j + |\mathbf{N}'|_{\mathcal{P}} + |\mathcal{P}|} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \left(\prod_{p \in \mathcal{P}} N_p! \right) \cdot \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathbf{N}'|+|\mathbf{N}|_{\mathcal{P}}+|\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!},$$

with

$$Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) = (-1)^{|\mathbf{N}'|_{\llbracket 1, Q \rrbracket \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{\llbracket 1, Q \rrbracket \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|=N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j,p}^{-N_p-1-|\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right).$$

Let's now observe that the coefficients $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ belong to the field \mathbb{K} . Moreover, we know that the nonpositive values of the Hurwitz zeta function have an expression using only Bernoulli polynomials, which are all rational polynomials. Therefore we get that $\frac{\zeta(-N_p - k_p, d_p)}{k_p!} \in \mathbb{Q}[d_1, \dots, d_P] \subset \mathbb{K}$, which proves the corollary. \square

Chapter 5

Proof of Proposition B, Theorem D, and its corollaries

5.1 Statement of Proposition B and of Theorem D

Firstly we will prove in this chapter Proposition B, and secondly we shall prove Theorem D, which we recall the statements below:

Proposition (B). *Let $j \in \llbracket 1, Q \rrbracket$ a positive integer, we assume that l_j is a linear form with rational coefficients. For all $1 \leq p \leq P$, we then write $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$ the irreducible fraction decomposition of $c_{j,p}$ with $a_{j,p} > 0$ and $b_{j,p} > 0$. Let*

$$x_j(\mathbf{c}) := \frac{\text{lcm}(a_{j,p})}{\text{gcd}(b_{j,p})} \in \mathbb{Q}_+, \quad \beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}$$

for all $1 \leq p \leq P$. Then, for all $\mathbf{R} \in \mathbb{N}_0^P$, we have for all $s \in \mathbb{C} \setminus \mathbb{N}$,

$$\begin{aligned} \varphi_{\mathbf{R}}^j(s) = & x_j(\mathbf{c})^{-s} \prod_{p=1}^P \beta_{j,p}^{R_p} \sum_{\substack{0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ \dots \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}^c} R_p! \\ & \cdot \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, k' \in \mathbb{N}_0 \\ |\mathbf{k}| + k' = |\mathbf{R}|_{|\mathcal{P}^c|} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \frac{\zeta\left(s - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})}\right)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})}\right)}{k_p!}. \end{aligned}$$

Theorem (D). *For all $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and all direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that*

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}|} \neq 0,$$

we have

$$\begin{aligned} Z'(-(\mathbf{N}, \mathbf{N}'))_{\boldsymbol{\mu}, \boldsymbol{\mu}'} = & \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} (-1)^{|\mathbf{N}| + |\mathbf{N}'| + |\mathcal{P}| + N'_j} \left(\prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}|}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{P^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}|} + |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & \cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k})_{\boldsymbol{\mu}, \boldsymbol{\mu}'} \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{\mathcal{P} \subseteq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\
& \cdot \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\
& + \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{\mathbf{u}_q, p} \right) (\varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j)^{(-N'_j)} \\
& - \sum_{\substack{\mathcal{P} \subseteq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|} \left(\prod_{q=1}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{\mathbf{u}_q, p} \right) \\
& \cdot \prod_{p \in \mathcal{P}} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}^{N_p + |\mathbf{u}_{\bullet p}| + 1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!},
\end{aligned}$$

with

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + |\mathbf{u}_{\bullet p}|)_{p \in \llbracket 1, P \rrbracket},$$

for all $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$, and for all $j \in \llbracket 1, Q \rrbracket$, and with $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k})$ and $Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k})$ as defined in formulas (1.9) and (1.11).

5.2 Auxiliary functions

In this section, we shall apply to the auxiliary functions $\varphi_{\mathbf{R}}^j(s)$ the Proposition 2.2.4 in order to use these formulas when we will compute the special values of the derivative $Z'(-(\mathbf{N}, \mathbf{N}'))$. Let's recall that, for $\boldsymbol{\mu}, \boldsymbol{\mu}'$

$\mathbf{R} = (R_1, \dots, R_P) \in \mathbb{N}_0^P$ and $j \in \llbracket 1, Q \rrbracket$, we have set

$$\varphi_{\mathbf{R}}^j(s) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} (n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P} (l_j(\mathbf{n}) + d'_j)^{-s}.$$

We apply Proposition 2.2.4 to the auxiliary function $\varphi_{\mathbf{R}}^j(s)$, setting $\boldsymbol{\mu} = \mathbf{0}$, $\mathbf{N} = \mathbf{R}$, $\mathbf{N}' = \mathbf{0}$, and $\boldsymbol{\mu}' = 1$. We know that this auxiliary function is regular at nonpositive integers, and by this same proposition, we obtain that the set of poles is contained in the set

$$\mathcal{S}(\mathbf{R}) := \mathbb{N}.$$

We find that the auxiliary function takes the following form, for all $0 < \theta \ll 1$:

$$\varphi_{\mathbf{R}}^j(s) = J_{\mathbf{R}}^j(\theta, s) + K_{\mathbf{R}}^j(\theta, s) \quad (s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{R})) \quad (5.1)$$

with

$$\begin{aligned}
K_{\mathbf{R}}^j(\theta, s) &= \frac{1}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \quad (s \in \mathbb{C}), \\
J_{\mathbf{R}}^j(\theta, s) &= \\
&\sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{R}, \mathcal{P}, \mathbf{k}}(s) \frac{\theta^{s-|\mathbf{R}|_{\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(s)(s-|\mathbf{R}|_{\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\
&+ \left(\prod_{p=1}^P R_p! \right) h_{\mathbf{R}, \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\mu'|+|\mu|)s-|\mathbf{N}'|-|\mathbf{N}|-P}}{\Gamma(s)(s-|\mathbf{R}|-P)} \quad (s \in \mathbb{C} \setminus \mathcal{S}(\mathbf{R})),
\end{aligned}$$

with

$$\begin{aligned}
h_{\mathbf{R}, \mathcal{P}, \mathbf{k}}(s) &= \prod_{p \in \mathcal{P}} c_{j,p}^{-N_p-1} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}, \\
h_{\mathbf{R}, \llbracket 1, P \rrbracket, j}(s) &= \prod_{p=1}^P c_{j,p}^{-N_p-1}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
J_{\mathbf{R}}^j(\theta, s) &= \left(\prod_{p=1}^P R_p! \right) \left(\prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \frac{\theta^{s-|\mathbf{R}|-P}}{\Gamma(s)(s-|\mathbf{R}|-P)} \quad (5.2) \\
&+ \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \left(\prod_{p \in \mathcal{P}} R_p! \right) \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}^{R_p+1}} \frac{(-1)^{|\mathbf{k}|} \theta^{s-|\mathbf{R}|_{\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|}}{\Gamma(s)(s-|\mathbf{R}|_{\mathcal{P}}-|\mathcal{P}|+|\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}.
\end{aligned}$$

Proposition 5.2.1. *We consider a P -tuple $\mathbf{R} \in \mathbb{N}_0^P$, and some integers $N \in \mathbb{N}_0$, $j \in \llbracket 1, Q \rrbracket$. Then the function $s \mapsto \varphi_{\mathbf{R}}^j(s)$ has a meromorphic continuation on \mathbb{C} , and is regular at nonpositive integers. Moreover, we have for all $0 < \theta \ll 1$,*

$$\partial_s \left(K_{\mathbf{R}}^j(\theta, s) \right) \Big|_{s=-N} = (-1)^N N! \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-N, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \quad (5.3)$$

$$\partial_s \left(J_{\mathbf{R}}^j(\theta, s) \right) \Big|_{s=-N} = J_{\mathbf{R}}^{*j}(\theta, -N) \quad (5.4)$$

$$+ N!(\gamma - h_N) \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathbf{R}|_{\mathcal{P}}+|\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=N+|\mathbf{R}|_{\mathcal{P}}+|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!},$$

with $J_{\mathbf{R}}^{*j}(\theta, -N)$ being a function of the form $J_{\mathbf{R}}^{*j}(\theta, -N) = \sum_{n \geq -N-|\mathbf{R}|-P} \alpha_n \theta^n + \beta \ln(\theta)$ without any constant term (i.e. $\alpha_0 = 0$).

Proof. By Corollary 2.2.6, we know that $\varphi_{\mathbf{R}}^j$ has a meromorphic continuation on \mathbb{C} , and is regular at nonpositive integers. By Proposition 2.3.11, we know that the function

$$s \in \mathbb{C} \mapsto \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p}$$

is holomorphic on \mathbb{C} . Taking the derivative of the function $K_{\mathbf{R}}^j(\theta, s)$ with respect to s , we get

$$\begin{aligned} \partial_s K_{\mathbf{R}}^j(\theta, s) &= -\frac{\psi(s)}{\Gamma(s)} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \\ &\quad + \frac{1}{\Gamma(s)} \partial_s \left(\sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(s, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} \right), \end{aligned}$$

where ψ is the digamma function. Therefore we can evaluate the sum of these two terms at $s = -N$. We know that the function $s \mapsto \frac{1}{\Gamma(s)}$ vanishes at $s = -N$, and by Lemma 3.1.1 we have

$$\left(\frac{\psi(s)}{\Gamma(s)} \right)_{|s=-N} = (-1)^{N+1} N!, \text{ thus we obtain the required formula.}$$

From formula (5.2), we get

$$\begin{aligned} \partial_s J_{\mathbf{R}}^j(\theta, s) &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &\quad \cdot \partial_s \left(\frac{1}{\Gamma(s)(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \theta^{s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ &+ \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}} c_{j,p}^{-R_p-1} R_p! \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &\quad \cdot \frac{1}{\Gamma(s)(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \theta^{s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \ln(\theta) \\ &- \left(\prod_{p=1}^P R_p! \right) \left(\prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \theta^{s - |\mathbf{R}| - P} \left(\frac{\ln(\theta)}{\Gamma(s)(s - |\mathbf{R}| - P)} + \partial_s \left(\frac{1}{\Gamma(s)(s - |\mathbf{R}| - P)} \right) \right). \end{aligned}$$

In order to evaluate at $s = -N$ the derivative $\partial_s (J_{\mathbf{R}}^j(\theta, s))$, it is enough to compute at $s = -N$ the terms $\frac{1}{(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)}$ and $\partial_s \left(\frac{1}{(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)} \right)$. That can easily be done using Lemma 3.1.1:

$$\left(\frac{1}{(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)\Gamma(s)} \right)_{|s=-N} = \begin{cases} (-1)^N N! & \text{if } |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}| \\ 0 & \text{if } |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}| \end{cases}$$

and

$$\partial_s \left(\frac{1}{\Gamma(s)(s - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{|s=-N} = \begin{cases} (-1)^N (\gamma - h_N) N! & \text{if } |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}| \\ \frac{(-1)^N N!}{-N - |\mathbf{R}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} & \text{if } |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|. \end{cases}$$

Therefore we find that

$$\begin{aligned} \partial_s (J_{\mathbf{R}}^j(\theta, s))_{|s=-N} &= (\gamma - h_N) N! \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}} + |\mathcal{P}|} \left(\prod_{p \in \mathcal{P}} R_p! \right) \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=N+|\mathbf{R}|_{|\mathcal{P}}+|\mathcal{P}|}} (-1)^{|\mathbf{k}|} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R_p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\ &+ (-1)^N N! \sum_{\mathcal{P} \subsetneq [1, P]} \left(\prod_{p \in \mathcal{P}} R_p! \right) \end{aligned}$$

$$\begin{aligned}
& \cdot \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}| + |\mathcal{P}|}}} (-1)^{|\mathbf{k}|} \frac{\theta^{-N - |\mathbf{R}|_{|\mathcal{P}| - |\mathcal{P}| + |\mathbf{k}|}}}{-N - |\mathbf{R}|_{|\mathcal{P}| - |\mathcal{P}| + |\mathbf{k}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R_p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\
& + N! \ln(\theta) \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}| + |\mathcal{P}|}} \left(\prod_{p \in \mathcal{P}} R_p! \right) \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}| + |\mathcal{P}|}}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R_p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\
& + (-1)^N N! \left(\prod_{p=1}^P R_p! \right) \left(\prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \frac{\theta^{-N - |\mathbf{R}| - P}}{-N - |\mathbf{R}| - P}.
\end{aligned}$$

We can set $J_{\mathbf{R}}^{*j}(\theta, -N)$ such that

$$\begin{aligned}
J_{\mathbf{R}}^{*j}(\theta, -N) &= (-1)^N N! \left(\prod_{p=1}^P R_p! \right) \left(\prod_{p=1}^P c_{j,p}^{-R_p-1} \right) \frac{\theta^{-N - |\mathbf{R}| - P}}{-N - |\mathbf{R}| - P} \\
& + (-1)^N N! \sum_{\mathcal{P} \subsetneq [1, P]} \left(\prod_{p \in \mathcal{P}} R_p! \right) \\
& \cdot \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| \neq N + |\mathbf{R}|_{|\mathcal{P}| + |\mathcal{P}|}}} (-1)^{|\mathbf{k}|} \frac{\theta^{-N - |\mathbf{R}|_{|\mathcal{P}| - |\mathcal{P}| + |\mathbf{k}|}}}{-N - |\mathbf{R}|_{|\mathcal{P}| - |\mathcal{P}| + |\mathbf{k}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R_p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \\
& + N! \ln(\theta) \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathbf{R}|_{|\mathcal{P}| + |\mathcal{P}|}} \left(\prod_{p \in \mathcal{P}} R_p! \right) \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N + |\mathbf{R}|_{|\mathcal{P}| + |\mathcal{P}|}}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R_p+1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}.
\end{aligned}$$

□

Let's observe that we cannot obtain an explicit expression for the values of the derivative $(\varphi_{\mathbf{R}}^j)'(s)$ at nonpositive integers using formulas (5.3) and (5.4), since it is not true in general that the function $s \mapsto \partial_s K_{\mathbf{R}}^j(\theta, s)$ vanishes at nonpositive integers. We thus need another strategy to compute the values of the auxiliary functions at nonpositive integers.

5.2.1 Proof of Proposition B

In this section, we aim to provide explicit values of the derivatives of these auxiliary functions assuming the coefficients $(c_{q,p})_{q \in [1, Q] \times [1, P]}$ are rationals. Let an integer $P \in \mathbb{N}$, and a P -tuple $\mathbf{R} \in \mathbb{N}_0^P$. To simplify notations in this subsection, we will set for all complex $s \in \mathbb{C}$ such that $\sigma \gg_{\mathbf{R}} 1$ the following generalized Barnes zeta function,

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) := \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{(n_1 + d_1)^{R_1} \dots (n_P + d_P)^{R_P}}{\left(\sum_{p=1}^P w_p (n_p + d_p) \right)^s},$$

with $\mathbf{d} = (d_p)_{p \in [1, P]}$ and $w_1, \dots, w_P, d_1, \dots, d_P \in H_0$. This function has a meromorphic continuation on \mathbb{C} in the variable s . In this subsection, we wish to establish an explicit relation between $\varphi_{\mathbf{R}}^j(s)$ and the Hurwitz zeta function $\zeta(s, x)$, with $x \in H_0$. To do this, we follow a strategy developed by Aoki and Sakane in [SA22], which was initially used to get formulas for special values of the higher derivatives of Barnes zeta functions with rational coefficients. Here, the auxiliary functions $\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P)$ are not exactly Barnes zeta functions due to the terms in the numerator of the series, however, the strategy used by Aoki and Sakane still works.

Lemma 5.2.2. Let $w_1, \dots, w_P \in \mathbb{Q}_*^+$ such that $w_p = \frac{a_p}{b_p}$ with a_p and b_p are positive integers. For all common multiple α of b_1, \dots, b_P , and all common multiple β of $\alpha w_1, \dots, \alpha w_P$, we have

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta} \right)^s \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq k_1 \leq \beta_1 - 1 \\ \vdots \\ 0 \leq k_P \leq \beta_P - 1}} \varphi_{\mathbf{R}} \left(s, \left(\frac{d_p + k_p}{\beta_p} \right)_{p \in \llbracket 1, P \rrbracket} \middle| \mathbf{1} \right), \quad (5.5)$$

with $\beta_1 = \frac{\beta}{\alpha w_1}, \dots, \beta_P = \frac{\beta}{\alpha w_P}$.

Proof. Let $s \in \mathbb{C}$ a complex number such that $\sigma \gg 1$. In order to get the required result, we will split each sums in the definition of $\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P)$ by using congruence:

$$\sum_{n_p=0}^{+\infty} = \sum_{k_p=0}^{\beta_p-1} \sum_{\substack{n_p \geq 0 \\ n_p = k_p \pmod{\beta_p}}}.$$

We then get

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta} \right)^s \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\substack{\mathbf{n} \in \mathbb{N}_0^P \\ \forall p, n_p = k_p \pmod{\beta_p}}} \frac{\prod_{p=1}^P (n_p + d_p)^{R_p}}{\left(\frac{\alpha}{\beta} (\sum_{p=1}^P (w_p d_p + w_p k_p + w_p (n_p - k_p))) \right)^s}.$$

By a change of variables $n'_p = \beta_p n_p + k_p$ for each integer p , we get

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta} \right)^s \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{\prod_{p=1}^P (\beta_p n_p + k_p + d_p)^{R_p}}{\left(\frac{\alpha}{\beta} \sum_{p=1}^P w_p d_p + w_p k_p + w_p \beta_p n_p \right)^s}.$$

By factoring $\beta_p^{R_p}$ in the numerator, and by noticing that $\frac{\alpha w_p \beta_p}{\beta} = 1$, we get

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = \left(\frac{\alpha}{\beta} \right)^s \prod_{p=1}^P \beta_p^{R_p} \sum_{k_1=0}^{\beta_1-1} \dots \sum_{k_P=0}^{\beta_P-1} \sum_{\mathbf{n} \in \mathbb{N}_0^P} \frac{\prod_{p=1}^P \left(n_p + \frac{k_p + d_p}{\beta_p} \right)^{R_p}}{\left(\sum_{p=1}^P n_p + \frac{\alpha}{\beta} (w_p d_p + w_p k_p) \right)^s}.$$

Finally, one can check that

$$\frac{k_p + d_p}{\beta_p} = \frac{\alpha}{\beta} (w_p d_p + w_p k_p)$$

for all $1 \leq p \leq P$. We then find that the formula (5.5) is true for all $s \in \mathbb{C}$ such that $\sigma \gg 1$. We can then verify that the formula is true for all complex number $s \in \mathbb{C}$ except for the singularities by the analytic continuation principle. \square

We can specify the last formula (5.5) by taking an explicit rational coefficient $\frac{\alpha}{\beta}$. In fact, we can set this coefficient such that it depends on the lcm and the gcd of the numerators and denominators of w_1, \dots, w_P , which gives us a more explicit result.

Proposition 5.2.3. Let $\mathbf{d} \in H_0^P$, $w_1 = \frac{a_1}{b_1}, \dots, w_P = \frac{a_P}{b_P}$ be rational numbers with $a_p, b_p > 0$ coprime integers for all $p \in \llbracket 1, P \rrbracket$. We set $w := \frac{\text{lcm}(a_1, \dots, a_P)}{\text{gcd}(b_1, \dots, b_P)}$ and $\beta_1 := \frac{w}{w_1}, \dots, \beta_P := \frac{w}{w_P}$. We then have

$$\varphi_{\mathbf{R}}(s, \mathbf{d} | w_1, \dots, w_P) = w^{-s} \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq k_1 \leq \beta_1 - 1 \\ \vdots \\ 0 \leq k_P \leq \beta_P - 1}} \varphi_{\mathbf{R}} \left(s, \left(\frac{d_p + k_p}{\beta_p} \right)_{p \in \llbracket 1, P \rrbracket} \middle| \mathbf{1} \right).$$

Proof. Let $\alpha = \text{lcm}(b_1, \dots, b_P)$ and $\beta = \text{lcm}(\alpha w_1, \dots, \alpha w_P)$. By [SA22, p.3], we know that $\frac{\alpha}{\beta} = w^{-1}$. Therefore, we can use formula (5.5) with such α and β . \square

Finally, we obtain a formula for $\varphi_{\mathbf{R}}(s, \mathbf{d}|\mathbf{1}, \dots, 1)$ similar to the one proved by Onodera in [Ono21, Proposition 4.1] by making a very slight modification to his proof.

Proposition 5.2.4. *Let $\mathbf{d} \in H_0^P$, $\mathbf{R} \in \mathbb{N}_0^P$, and $s \in \mathbb{C} \setminus \mathbb{N}$. We then have that*

$$\varphi_{\mathbf{R}}(s, \mathbf{d}|\mathbf{1}) = \sum_{\mathcal{P} \subsetneq [1, P]} (-1)^{|\mathcal{P}|} \prod_{p \in \mathcal{P}} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, k' \geq 0 \\ |\mathbf{k}|_{\mathcal{P}} + k' = |\mathbf{R}|_{\mathcal{P}} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \frac{\zeta(s - k', |\mathbf{d}|)}{k'!} \prod_{p \in \mathcal{P}} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}.$$

In order to prove such formula, we need to use a lemma proved by Onodera in [Ono21, Lemma 4.2], which generalizes Faulhaber's formula:

$$\sum_{n=1}^y n^r = \frac{1}{r+1} \sum_{k=0}^r \binom{r+1}{k} (-1)^k B_k y^{r-k+1}.$$

Lemma 5.2.5 ([Ono21]). *Let $\mathbf{R} = (R_1, \dots, R_P) \in \mathbb{N}_0^P$ a P -tuple, and $y \in \mathbb{Z}$ an integer. We have that*

$$\begin{aligned} \sum_{\substack{\mathbf{n} \in \mathbb{N}_0^P \\ |\mathbf{n}|=y}} \prod_{p=1}^P (n_p + d_p)^{R_p} + (-1)^{P-1} \sum_{\substack{\mathbf{n} \in \mathbb{Z}_{\leq -1}^P \\ |\mathbf{n}|=y}} \prod_{p=1}^P (n_p + d_p)^{R_p} = \\ \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, k' \in \mathbb{N}_0 \\ |\mathbf{k}| + k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} \frac{(|\mathbf{d}| + y)^{k'}}{k'!} \prod_{p \in \mathcal{P}} (-1)^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!}. \end{aligned}$$

Let's note that if $y \in \mathbb{N}_0$, then the second sum in the last equality vanishes. We can now prove Proposition 5.2.4:

Proof. Let's first note that, for all $s \in \mathbb{C}$ such that $\sigma_{\mathbf{R}} \gg 1$, we have

$$\varphi_{\mathbf{R}}(s, \mathbf{d}|\mathbf{1}) = \sum_{n'=0}^{+\infty} \frac{1}{(n' + |\mathbf{d}|)^s} \sum_{\substack{\mathbf{n} \in \mathbb{N}_0^P \\ |\mathbf{n}|=n'}} \prod_{p=1}^P (n_p + d_p)^{R_p}.$$

By Lemma 5.2.5, we then get that

$$\begin{aligned} \varphi_{\mathbf{R}}(s, \mathbf{d}|\mathbf{1}) &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, k' \in \mathbb{N}_0 \\ |\mathbf{k}| + k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} \prod_{p \in \mathcal{P}} (-1)^{k_p} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \sum_{n'=0}^{+\infty} \frac{(n' + |\mathbf{d}|)^{k'}}{k'!} \frac{1}{(n' + |\mathbf{d}|)^s} \\ &= \sum_{\mathcal{P} \subsetneq [1, P]} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, k' \in \mathbb{N}_0 \\ |\mathbf{k}| + k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \left(\prod_{p \in \mathcal{P}} \frac{\zeta(-R_p - k_p, d_p)}{k_p!} \right) \frac{\zeta(s - k', |\mathbf{d}|)}{k'!}. \end{aligned}$$

\square

Applying Proposition 5.2.4 into the formula obtained in Proposition 5.2.3, we get the following result which corresponds exactly to Proposition B.

Proposition 5.2.6. *Let $\mathbf{d} \in H_0^P$, $\mathbf{R} \in \mathbb{N}_0^P$, $s \in \mathbb{C} \setminus \mathbb{N}$, and $w_1 = \frac{a_1}{b_1}, \dots, w_P = \frac{a_P}{b_P}$ be rational numbers with $a_p, b_p > 0$ being coprime integers for all $p \in \llbracket 1, P \rrbracket$. We set $w := \frac{\text{lcm}(a_1, \dots, a_P)}{\text{gcd}(b_1, \dots, b_P)}$ and $\beta_1 := \frac{w}{w_1}, \dots, \beta_P := \frac{w}{w_P}$. Then we have*

$$\begin{aligned} \varphi_{\mathbf{R}}(s, \mathbf{d}|w_1, \dots, w_P) = & \\ & w^{-s} \prod_{p=1}^P \beta_p^{R_p} \sum_{\substack{0 \leq u_1 \leq \beta_1 - 1 \\ 0 \leq u_P \leq \beta_P - 1}} \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} \prod_{p \in \mathcal{P}^c} R_p! \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^P, \mathbf{k}' \in \mathbb{N}_0 \\ |\mathbf{k}| + \mathbf{k}' = |\mathbf{R}|_{|\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \prod_{p \in \mathcal{P}} \frac{\zeta\left(-R_p - k_p, \frac{d_p + u_p}{\beta_p}\right)}{k_p!} \\ & \cdot \frac{\zeta(s - k', w^{-1}(w_1(d_1 + u_1) + \dots + w_P(d_P + u_P)))}{k'!}. \end{aligned}$$

5.3 Proof of Theorem D

Let a $P+Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$. Following the same ideas as in the proof of Theorem A, it's enough to prove that formula (1.10) holds for all $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. Indeed we know by Corollary 1.1.31 that the function $(\boldsymbol{\mu}, \boldsymbol{\mu}') \mapsto Z'(-(\mathbf{N}, \mathbf{N}'))$ is meromorphic on \mathbb{C}^{P+Q} . We also see that the right-hand side of formula (1.10) is a rational fraction in the variables $\mu_1, \dots, \mu_P, \mu'_1, \dots, \mu'_Q$ with denominators of the form $|\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|$. Thus if we show that formula (1.10) holds for any given direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$, then we can extend by the analytical continuation principle this formula for all direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that

$$\forall \mathcal{P} \subsetneq \llbracket 1, P \rrbracket, \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}} \neq 0.$$

We consider a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. By Proposition 2.2.4, for all complex number s around 0, and for all real number $0 < \theta \ll 1$ small enough, we have that

$$Z_{\mathbf{N}, \mathbf{N}'}(s) = K_{\mathbf{N}, \mathbf{N}'}(\theta, s) + J_{\mathbf{N}, \mathbf{N}'}(\theta, s),$$

with

$$\begin{aligned} K_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \sum_{\substack{\boldsymbol{\mu}, \boldsymbol{\mu}' \\ \mathcal{Q} \subseteq \llbracket 1, Q \rrbracket \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \frac{1}{\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s)} \left(\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\ & \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right), \end{aligned}$$

and

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}(\theta, s) = & \sum_{\boldsymbol{\mu}, \boldsymbol{\mu}'} \sum_{\mathcal{P} \subseteq \llbracket 1, P \rrbracket} \prod_{j=1}^Q \Gamma(1 + N_p - \mu_p s) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, \mathbf{k}}(s) \\ & \cdot \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c})}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|_{|\mathcal{P}^c})} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!} \\ & + \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \left(\sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right). \end{aligned}$$

Proposition 5.3.1. *Let $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ a $P + Q$ -tuple, and $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$ a direction. There exists a complex valued function $K_{\mathbf{N}, \mathbf{N}'}^*(\theta) = \beta \ln(\theta) + \sum_{n=m}^{+\infty} \alpha_n \theta^n$ defined for all $0 < \theta \ll 1$, with $m \in \mathbb{Z}$ an integer and $\alpha_0 = 0$, and such that*

$$\begin{aligned} \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}^*(\theta, s) \right)_{|s=0} &= K_{\mathbf{N}, \mathbf{N}'}^*(\theta) \\ &+ \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) (\varphi_{(N_p + |\mathbf{u}_{\bullet p}|)_{p \in [1, P]}}^j)^{(-N'_j)} \\ &- \sum_{\substack{\mathcal{P} \subseteq [1, P] \\ 1 \leq j \leq Q}} (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u}=(\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}} + |\mathcal{P}|} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\ &\cdot \left(\prod_{p \in \mathcal{P}} R(\mathbf{N}, j, \mathbf{u})_p! \right) \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{|\mathcal{P}} + |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}^c} c_{j,p}^{R(\mathbf{N}, j, \mathbf{u})_p + 1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \end{aligned}$$

Proof. Taking the derivative with respect to s of the function $K_{\mathbf{N}, \mathbf{N}'}^*(\theta, s)$ in formula (2.1), we get

$$\partial_s K_{\mathbf{N}, \mathbf{N}'}^*(\theta, s) = \tag{5.6}$$

$$\begin{aligned} &\sum_{\substack{\mathcal{Q} \subset [1, Q] \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \partial_s \left(\prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right) \left(\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\ &\quad \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right) \\ &+ \sum_{\substack{\mathcal{Q} \subset [1, Q] \\ \mathcal{Q} \neq \emptyset}} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \left(\prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right) \partial_s \left(\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \right. \\ &\quad \cdot \left. \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right). \end{aligned}$$

Thanks to the inequality (2.9), we've already seen in the proof of Proposition 2.3.11 that the function

$$s \in \mathbb{C} \mapsto \left(\sum_{\mathbf{n} \in \mathbb{N}_0^P} \prod_{p=1}^P (n_p + d_p)^{N_p - \mu_p s} \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n}) + d'_q)^{N'_q - \mu'_q s} \prod_{q \in [1, Q] \setminus A_{\mathcal{Q}}} \Gamma(-N'_q + \mu'_q s, \theta, l_q(\mathbf{n}) + d'_q) \right)$$

was holomorphic on \mathbb{C} . Since the function $s \mapsto \frac{1}{\Gamma(s)}$ vanishes at nonpositive integers, we find that the second term in formula (5.6) vanishes at $s = 0$. Moreover we know that around $s = 0$, we have

$$\frac{1}{\Gamma(-N'_q + \mu'_q s)} = O(s).$$

Therefore, we find that for all $\emptyset \neq \mathcal{Q} \subset \llbracket 1, Q \rrbracket$ and for all $A_{\mathcal{Q}} \subset \mathcal{Q}^c = \llbracket 1, Q \rrbracket \setminus \mathcal{Q}$, we have

$$\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} = \begin{cases} O(s^2) & \text{if } |\mathcal{Q}| \geq 2, \\ O(s) & \text{if } \mathcal{Q} = \{j\}, A_{\mathcal{Q}} = \llbracket 1, Q \rrbracket \setminus \{j\} \text{ with } j \in \llbracket 1, Q \rrbracket, \\ O(s^2) & \text{otherwise.} \end{cases}$$

Therefore we obtain that if $|\mathcal{Q}| \geq 2$, we have

$$\partial_s \left(\prod_{q \in \llbracket 1, Q \rrbracket \setminus A_{\mathcal{Q}}} \frac{1}{\Gamma(-N'_q + \mu'_q s)} \right) \Big|_{s=0} = 0.$$

We then get

$$\begin{aligned} \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right) \Big|_{s=0} &= \sum_{j=1}^Q \partial_s \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)} \right) \Big|_{s=0} \left(\sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p} \right. \\ &\quad \left. \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} \right). \end{aligned}$$

By Lemma 3.1.1, we obtain

$$\partial_s \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)} \right) \Big|_{s=0} = (-1)^{N'_j} N'_j! \mu'_j,$$

and therefore

$$\begin{aligned} \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right) \Big|_{s=0} &= \sum_{j=1}^Q \mu'_j (-1)^{N'_j} N'_j! \left(\sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p} \right. \\ &\quad \left. \cdot \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} \right). \end{aligned}$$

For all $q \in \llbracket 1, Q \rrbracket$, $\mathbf{n} \in \mathbb{N}_0^P$, we have $l_q(\mathbf{n}) + d'_q = \sum_{p=1}^P c_{q,p} (n_p + d_p)$. By Newton's multinomial, we get

$$(l_q(\mathbf{n}) + d'_q)^{N'_q} = \sum_{\substack{\mathbf{u}_q \in \mathbb{N}_0^P \\ |\mathbf{u}_q| = N'_q}} \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} (n_p + d_p)^{u_{q,p}}.$$

By taking the product over $q \in \llbracket 1, Q \rrbracket \setminus \{j\}$, we find

$$\begin{aligned} \prod_{\substack{q=1 \\ q \neq j}}^Q (l_q(\mathbf{n}) + d'_q)^{N'_q} &= \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \left(\binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} (n_p + d_p)^{u_{q,p}} \right) \\ &= \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}} \in (\mathbb{N}_0^P)^{\llbracket 1, Q \rrbracket \setminus \{j\}} \\ \forall q \in \llbracket 1, Q \rrbracket \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \prod_{p=1}^P (n_p + d_p)^{|\mathbf{u}_{\bullet,p}|}, \end{aligned}$$

where we noted

$$|\mathbf{u}_{\bullet p}| = \sum_{\substack{q=1 \\ q \neq j}}^Q u_{q,p}.$$

We finally get

$$\begin{aligned} \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{\mu, \mu'} \Big|_{s=0} &= \sum_{j \in [1, Q]} \mu'_j (-1)^{N'_j} N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \\ &\cdot \left(\sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-N'_j, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{N_p + |\mathbf{u}_{\bullet p}|} \right). \end{aligned} \quad (5.7)$$

By formula (5.1) we know that for all $m \in \mathbb{N}_0$,

$$(-1)^m m! \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-m, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p} = (\varphi_{\mathbf{R}}^j)'(-m) - \partial_s \left(J_{\mathbf{R}}^j(\theta, s) \right)_{|s=-m}.$$

Substituting the terms of the form $(-1)^m m! \sum_{\mathbf{n} \in \mathbb{N}_0^P} \Gamma(-m, \theta, l_j(\mathbf{n}) + d'_j) \prod_{p=1}^P (n_p + d_p)^{R_p}$ in formula (5.7) by the above expression with the corresponding coefficients $\mathbf{R} = (R_1, \dots, R_P)$ and m , we get

$$\begin{aligned} \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{\mu, \mu'} \Big|_{s=0} &= \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \\ &\cdot \left((\varphi_{(N_p + |\mathbf{u}_{\bullet p}|)_{p \in [1, P]}}^j)'(-N'_j) - \partial_s \left(J_{(N_p + |\mathbf{u}_{\bullet p}|)_{p \in [1, P]}}^j(\theta, s) \right)_{|s=-N'_j} \right). \end{aligned} \quad (5.8)$$

Let's recall that we noted $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) = (N_p + |\mathbf{u}_{\bullet p}|)_{p \in [1, P]}$ for all $\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}}$ in the notations of Theorem D. By formula (5.4) we get

$$\begin{aligned} \partial_s \left(J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j(\theta, s) \right)_{|s=-N'_j} &= (\gamma - h_{N'_j}) N'_j! \sum_{\mathcal{P} \subset [1, P]} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|} \prod_{p \in \mathcal{P}} c_{j,p}^{-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - 1} R(\mathbf{N}, j, \mathbf{u})_p! \\ &\cdot \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{c_{j,p}^{k_p} \zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!} \\ &+ J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j). \end{aligned} \quad (5.9)$$

For all $0 < \theta \ll 1$, we set

$$K_{\mathbf{N}, \mathbf{N}'}^*(\theta) := - \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j) \prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}}.$$

By definition of $J_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^{*j}(\theta, -N'_j)$, we see that $K_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ is of the form $\sum_{n \geq m} \alpha_n \theta^n + \beta \ln(\theta)$ with $(\alpha_n)_{n \geq m}$ and β being complex numbers and $\alpha_0 = 0$. Injecting formula (5.9) into formula (5.8), we find the required formula for $\partial_s \left(K_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{\mu, \mu'} \Big|_{s=0}$. \square

We then just need to compute the derivative $\partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s)$ at $s = 0$ in order to prove Theorem D.

Proposition 5.3.2. *Let a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \overline{H}_0^P \times H_0^Q$. There exist a complex valued function $J_{\mathbf{N}, \mathbf{N}'}^*(\theta) = \beta \ln(\theta) + \sum_{n=m}^{+\infty} \alpha_n \theta^n$ defined for all $0 < \theta \ll 1$, with $m \in \mathbb{Z}$ an integer and $\alpha_0 = 0$, and such that*

$$\begin{aligned} \partial_s \left(J_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{|s=0} &= \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}| + N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \\ &\cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\sum_{p \in \mathcal{P}} \mu_p(\gamma - h_{N_p}) + \mu'_j(\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ &+ \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}| + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\ &\cdot \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\ &+ J_{\mathbf{N}, \mathbf{N}'}^*(\theta). \end{aligned} \quad (5.10)$$

Proof. Let's recall that

$$\begin{aligned} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) &:= h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\mu, \boldsymbol{\mu}') \\ Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) &:= (h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\mu, \boldsymbol{\mu}'))', \end{aligned}$$

for $j \in \llbracket 1, Q \rrbracket$, $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$. Let's note $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$ the general term of the series in formula (2.2):

$$\begin{aligned} T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) &= \left(\prod_{p \in \mathcal{P}} \Gamma(1 + N_p - \mu_p s) \right) \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\mu, \boldsymbol{\mu}') \\ &\cdot \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}, \end{aligned}$$

for $j \in \llbracket 1, Q \rrbracket$, $\mathcal{P} \subsetneq \llbracket 1, P \rrbracket$, and $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$. We're now going to evaluate formula (2.20) at $s = 0$ in order to get the desired formula for $\partial_s \left(J_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{|s=0}$. We get

$$\begin{aligned} \partial_s \left(J_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{|s=0} &= \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) \right)_{|s=0} \\ &+ \partial_s \left(\left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \right) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', \llbracket 1, P \rrbracket, j}(\mu, \boldsymbol{\mu}') \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right)_{|s=0}. \end{aligned} \quad (5.11)$$

At first we will study the second term of the hand-side term in the last formula. We see that the function $s \mapsto \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)}$ vanishes at $s = 0$. Therefore, we get that

$$\begin{aligned} \partial_s \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \Big|_{s=0} &= \\ \prod_{p=1}^P N_p! \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(0) \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P} \partial_s \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \Big|_{s=0}. & \end{aligned}$$

By Lemma 3.1.1, we have

$$\partial_s \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \Big|_{s=0} = (-1)^{N'_j+1} N'_j! \frac{\mu'_j}{|\mathbf{N}'| + |\mathbf{N}| + P}.$$

We then find that

$$\begin{aligned} \partial_s \left(\prod_{p=1}^P \Gamma(1 + N_p - \mu_p s) \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(s) \frac{\theta^{(|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P}}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|)s - |\mathbf{N}'| - |\mathbf{N}| - P)} \right) \Big|_{s=0} &= \\ \left(\prod_{p=1}^P N_p! \sum_{j=1}^Q h_{\mathbf{N}, \mathbf{N}', [1, P], j}(0) (-1)^{N'_j+1} N'_j! \frac{\mu'_j}{|\mathbf{N}'| + |\mathbf{N}| + P} \right) \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P}. & \end{aligned}$$

We shall include later on that last term in the definition $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$, because $-|\mathbf{N}'| - |\mathbf{N}| - P \neq 0$.

Let's now study the derivative of the general term $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$ with respect to s at $s = 0$.

• If $|\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ then $s \mapsto \frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)}$ is a function that vanishes at $s = 0$. Therefore we find that

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) \right) \Big|_{s=0} &= \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ &\cdot \partial_s \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right) \Big|_{s=0} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}. \end{aligned}$$

Via Lemma 3.1.1, we get

$$\begin{aligned} \partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) \right) \Big|_{s=0} &= \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}|} h_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(0) \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} (-1)^{N'_j} N'_j! \\ &\cdot \frac{\mu'_j}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}. \end{aligned}$$

We see that the power of θ in the last term is non-zero. Therefore we shall later on include the terms

$$\partial_s \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) \right) \Big|_{s=0} \text{ such that } |\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}| \text{ into the definition of } J_{\mathbf{N}, \mathbf{N}'}^*(\theta).$$

- If $|\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$ then by Lemma 3.1.1, we have

$$\begin{aligned} & \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{|s=0} = (-1)^{N'_j} \frac{\mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! \\ \partial_s & \left(\frac{1}{\Gamma(-N'_j + \mu'_j s)((|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}})s - |\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|)} \right)_{|s=0} = (-1)^{N'_j} \frac{\mu'^2_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} N'_j! (\gamma - h_{N'_j}). \end{aligned}$$

Taking the derivative with respect to s of the general term $T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s)$ and using $\Gamma'(1 + N_p) = N_p!(-\gamma + h_{N_p})$, we get

$$\begin{aligned} \partial_s & \left(T_{\mathbf{N}, \mathbf{N}', \mathcal{P}, j, \mathbf{k}}(\theta, s) \right)_{|s=0} = \left(\prod_{p \in \mathcal{P}} N_p! \right) \frac{(-1)^{|\mathbf{k}| + N'_j} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & \cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\mu'_j (\gamma - h_{N'_j}) - \sum_{p \in \mathcal{P}} \frac{\mu_p \Gamma'(1 + N_p)}{N_p!} \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \left(\prod_{p \in \mathcal{P}} N_p! \right) Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \frac{(-1)^{N'_j + |\mathbf{k}|} \mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}} \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\ & + \left(\left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{k}| + N'_j} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \mu'_j N'_j! \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \ln(\theta). \end{aligned}$$

We will now include all the terms that contain either a multiple of $\ln(\theta)$ or a multiple of θ^k with $k \neq 0$ into the definition of $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$:

$$\begin{aligned} J_{\mathbf{N}, \mathbf{N}'}^*(\theta) & = \theta^{-|\mathbf{N}'| - |\mathbf{N}| - P} \left(\prod_{p=1}^P N_p! \right) \sum_{j=1}^Q Q_{\mathbf{N}, \mathbf{N}'}^0(j, [1, P]) \frac{(-1)^{N'_j + 1} \mu'_j N'_j!}{|\mathbf{N}'| + |\mathbf{N}| + P} \\ & + \ln(\theta) \sum_{\substack{\mathcal{P} \subseteq [1, P] \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) \mu'_j N'_j! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} (-1)^{|\mathbf{k}| + N'_j} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & + \theta^{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \sum_{\substack{\mathcal{P} \subseteq [1, P] \\ 1 \leq j \leq Q}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| \neq |\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|}} \left(\prod_{p \in \mathcal{P}} N_p! \right) Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \frac{(-1)^{N'_j + |\mathbf{k}|} \mu'_j N'_j!}{-|\mathbf{N}'| - |\mathbf{N}|_{|\mathcal{P}} - |\mathcal{P}| + |\mathbf{k}|} \\ & \cdot \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p + \mu_p s - k_p, d_p)}{k_p!}. \end{aligned}$$

By cutting off the series in formula (5.11) whether if $|\mathbf{k}|$ is equal or not to $|\mathbf{N}'| + |\mathbf{N}|_{|\mathcal{P}} + |\mathcal{P}|$, and by definition of the term $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ that we just defined, we get the required formula (5.10). \square

We see that the series $J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ and $K_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ have a zero constant term, and that $\partial_s Z_{\mathbf{N}, \mathbf{N}'}(s) =$

$\partial_s J_{\mathbf{N}, \mathbf{N}'}(\theta, s) + \partial_s K_{\mathbf{N}, \mathbf{N}'}(\theta, s)$. Since $\partial_s \left(Z_{\mathbf{N}, \mathbf{N}'}(s) \right)_{|s=0}$ is independent of the variable θ , by Lemma 3.1.3 we have

$$\partial_s \left(Z_{\mathbf{N}, \mathbf{N}'}(s) \right)_{|s=0} = \partial_s \left(J_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{|s=0} + \partial_s \left(K_{\mathbf{N}, \mathbf{N}'}(\theta, s) \right)_{|s=0} - K_{\mathbf{N}, \mathbf{N}'}^*(\theta) - J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$$

for every $0 < \theta \ll 1$. This leads us to Theorem D. \square

Remark 5.3.3. From the equality $K_{\mathbf{N}, \mathbf{N}'}^*(\theta) + J_{\mathbf{N}, \mathbf{N}'}^*(\theta) = 0$ for all $0 < \theta \ll 1$, and from Lemma 3.1.3, we get that each coefficient in the Laurent series in $K_{\mathbf{N}, \mathbf{N}'}^*(\theta) + J_{\mathbf{N}, \mathbf{N}'}^*(\theta)$ vanishes, and that the multiple of $\ln(\theta)$ also vanishes. We can then get new relations between the special values $\zeta(-k, d_p)$, with $k \in \mathbb{N}_0$ and $1 \leq p \leq P$.

5.4 On the corollaries of Theorem D

To prove the corollaries of Theorem D we will use the formula given by that theorem, and consider a particular direction $(\boldsymbol{\mu}, \boldsymbol{\mu}')$ or a particular $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}')$.

5.4.1 Proof of Theorem C

Let a $P + Q$ -tuple $(\mathbf{N}, \mathbf{N}') \in \mathbb{N}_0^P \times \mathbb{N}_0^Q$ and a direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') \in \mathbb{C}^P \times \mathbb{C}^Q$ such that

$$\forall \mathcal{P} \subsetneq [1, P], \quad |\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}} \neq 0.$$

The goal here is to revisit the formula obtained for $Z'(-(\mathbf{N}, \mathbf{N}'))$ and to study its terms qualitatively. We denote \mathbb{K} as the field generated above \mathbb{Q} by the complex coefficients $(d_p)_{p \in [1, P]}$ and $(c_{q,p})_{p \in [1, P], q \in [1, Q]}$, and the directions $(\mu_p)_{p \in [1, P]}$, $(\mu'_q)_{q \in [1, Q]}$. By Theorem D, we have

$$\begin{aligned} Z'(-(\mathbf{N}, \mathbf{N}')) &= \\ & \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} (-1)^{|\mathbf{N}| + |\mathbf{N}'| + |\mathcal{P}| + N'_j} \left(\prod_{p \in \mathcal{P}} N_p! \right) \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \\ & \quad \cdot \left(Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \left(\sum_{p \in \mathcal{P}} \mu_p (\gamma - h_{N_p}) + \mu'_j (\gamma - h_{N'_j}) \right) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} \left(\prod_{p \in \mathcal{P}} N_p! \right) (-1)^{|\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}| + N'_j} \frac{\mu'_j N'_j!}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{\mathcal{P}}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathbf{N}'| + |\mathbf{N}|_{\mathcal{P}} + |\mathcal{P}|}} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \\ & \quad \cdot \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-N_i - k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-N_p - k_p, d_p)}{k_p!} \right) \\ & + \sum_{j=1}^Q \mu'_j \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) (\varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j)'(-N'_j) \\ & - \sum_{\substack{\mathcal{P} \subsetneq [1, P] \\ 1 \leq j \leq Q}} (\gamma - h_{N'_j}) \mu'_j N'_j! \sum_{\substack{\mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}} \in (\mathbb{N}_0^P)^{[1, Q] \setminus \{j\}} \\ \forall q \in [1, Q] \setminus \{j\}, |\mathbf{u}_q| = N'_q}} (-1)^{|\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q \binom{N'_q}{\mathbf{u}_q} \prod_{p=1}^P c_{q,p}^{u_{q,p}} \right) \\ & \quad \cdot \prod_{p \in \mathcal{P}} R(\mathbf{N}, j, \mathbf{u})_p! \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = N'_j + |\mathbf{R}(\mathbf{N}, j, \mathbf{u})|_{\mathcal{P}} + |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}^{N_p + |\mathbf{u}_p| + 1}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-\mathbf{R}(\mathbf{N}, j, \mathbf{u})_p - k_p, d_p)}{k_p!}, \end{aligned}$$

with

$$\mathbf{R}(\mathbf{N}, j, \mathbf{u}) := (N_p + |\mathbf{u}_{\bullet p}|)_{p \in [1, P]}, \text{ for all } \mathbf{u} = (\mathbf{u}_q)_{q \in [1, Q] \setminus \{j\}}$$

and with

$$\begin{aligned} Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) &= (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, N'_q = |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}|}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \\ &\quad \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} c_{j, p}^{-N_p - 1 - |\mathbf{v}_p|} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, p}^{v_{p, q}} \right), \\ Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) &= (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j, f\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, f, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \\ &\quad \cdot \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j, f}}^Q c_{q, p}^{v_{p, q}} \right) \\ &+ (-1)^{|\mathbf{N}'|_{[1, Q] \setminus \{j\}}} \left(\prod_{\substack{q=1 \\ q \neq j}}^Q N'_q! \right) \sum_{\substack{\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1, Q] \setminus \{j\}})^{\mathcal{P}} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p| = k_p \\ \forall q \neq j, |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| = N'_q}} \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q, p}^{w_{p, q}} \right) \\ &\quad \cdot \left(\prod_{p \in \mathcal{P}} \binom{-N_p - 1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \frac{\prod_{\substack{q=1 \\ q \neq j}}^Q c_{q, p}^{v_{p, q}}}{c_{j, p}^{N_p + 1 + |\mathbf{v}_p|}} \left(\sum_{p \in \mathcal{P}} \mu_p (\ln(c_{j, p}) + h_{N_p} - h_{N_p + |\mathbf{v}_p|}) + \sum_{\substack{q=1 \\ q \neq j}}^Q \mu'_q (\gamma - h_{N'_q}) \right) \right) \end{aligned}$$

with $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ the constant defined in Proposition 1.2.2.

Let's observe that $Q_{\mathbf{N}, \mathbf{N}'}^0(j, \mathcal{P}, \mathbf{k}) \in \mathbb{K}$. Moreover, the coefficient $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}')$ belongs in the vector space on \mathbb{K} generated by coefficients of the form

$$\ln \left(1 + \frac{c_{q, p}}{c_{j, p}} \right) = \ln(c_{j, p} + c_{q, p}) - \ln(c_{j, p}) \quad (1 \leq q, j \leq Q, 1 \leq p \leq P),$$

therefore $F_{\mathcal{P}, j, f, \mathbf{v}, \mathbf{w}}(\mathbf{N}, \mathbf{N}') \in \text{Vect}_{\mathbb{K}} \left[(\ln(c_{j, p}), \ln(c_{j, p} + c_{q, p}))_{\substack{1 \leq j, q \leq Q \\ 1 \leq p \leq P}} \right]$. We then find that

$$Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \in \text{Vect}_{\mathbb{K}} \left[\gamma, (\ln(c_{j, p}), \ln(c_{j, p} + c_{q, p}))_{\substack{1 \leq j, q \leq Q \\ 1 \leq p \leq P}} \right].$$

As the special values $\zeta(-N, d_p)$ belong in the field above \mathbb{Q} generated by the coefficients d_1, \dots, d_P , we find that these special values belong in \mathbb{K} . Therefore we get that $Z'(-(\mathbf{N}, \mathbf{N}'))$ belong in the vector space on \mathbb{K} generated by γ , by coefficient of the form $\ln(c_{j, p} + c_{q, p})$ and of the form $\ln(c_{j, p})$, by special values of the derivative of the auxiliary functions $(\varphi_{\mathbf{R}(\mathbf{N}, j, \mathbf{u})}^j)'(-N'_j)$, and by special values at nonpositive integers of the derivative with respect to s of the Hurwitz zeta function. \square

5.4.2 Proof of Corollary C1

We use the fact that, when the linear forms are rational, Corollary B1 provides an expression for the derivative $(\varphi_{\mathbf{R}(\mathbf{N},j,\mathbf{u})}^j)'(-N'_j)$ in terms of derivative values of the Hurwitz zeta function. We can then replace the derivative values $(\varphi_{\mathbf{R}(\mathbf{N},j,\mathbf{u})}^j)'(-N'_j)$ in Theorem C with terms of the form $\zeta'(-n, y)$. \square

5.4.3 Proof of Corollary D1

Corollary D1 is obtained by applying the formula found for $Z'(-(\mathbf{N}, \mathbf{N}'))$ in Theorem A, and replacing the terms of the form $(\varphi_{\mathbf{R}}^j)'(-N)$ with the formula provided by Corollary B1.

Since the linear forms l_j have rational coefficients, we can set $c_{j,p} = \frac{a_{j,p}}{b_{j,p}}$ for all $1 \leq p \leq P$, assuming that this fraction is irreducible. We then define $x_j(\mathbf{c}) = \frac{\text{lcm}(a_{j,p})}{\text{gcd}(b_{j,p})} \in \mathbb{Q}_+^*$, and $\beta_{j,p}(\mathbf{c}) := \frac{x_j(\mathbf{c})}{c_{j,p}} \in \mathbb{N}$ for all $1 \leq p \leq P$. By Corollary B1, for $\mathbf{R} \in \mathbb{N}_0^P$ we get that

$$\begin{aligned} (\varphi_{\mathbf{R}}^j)'(-N) &= x_j(\mathbf{c})^N \left(\prod_{p=1}^P \beta_{j,p}(\mathbf{c})^{R_p} \right) \sum_{\substack{\mathcal{P} \subsetneq [1,P] \\ 0 \leq v_1 \leq \beta_{j,1}(\mathbf{c})-1 \\ 0 \leq v_P \leq \beta_{j,P}(\mathbf{c})-1}} \left(\prod_{p \in \mathcal{P}^c} R_p! \right) \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{\mathcal{P}}, k' \in \mathbb{N} \\ |\mathbf{k}| + k' = |\mathbf{R}|_{\mathcal{P}^c} + |\mathcal{P}^c| - 1}} (-1)^{|\mathbf{k}|} \\ &\cdot \left(\frac{\zeta' \left(-N - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})} \right)}{k'!} - \ln(x_j(\mathbf{c})) \frac{\zeta \left(-N - k', \frac{l_j(\mathbf{d} + \mathbf{v})}{x_j(\mathbf{c})} \right)}{k'!} \right) \prod_{p \in \mathcal{P}} \frac{\zeta \left(-R_p - k_p, \frac{d_p + v_p}{\beta_{j,p}(\mathbf{c})} \right)}{k_p!}. \end{aligned}$$

By injecting this formula in the one obtained for $Z'(-(\mathbf{N}, \mathbf{N}'))$ in Theorem D, we obtain Corollary D1. \square

Note that we obtain once again Corollary C1, which corresponds to a qualitative reformulation of the result we've just obtained.

5.4.4 Proof of Corollary D2

We set $(\mathbf{N}, \mathbf{N}') = (\mathbf{0}, \mathbf{0})$ and we apply Theorem D in order to get a formula for $Z'(-(\mathbf{0}, \mathbf{0}))$. We first can simplify several sums in the formula already obtained by noting the following facts:

- 1) Let $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1,Q] \setminus \{j\}})^{\mathcal{P}}$ and $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c}$ such that

$$\begin{aligned} |\mathbf{w}_p| &= k_p & (p \in \mathcal{P}^c) \\ |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| &= 0 & (q \in [1, Q] \setminus \{j\}). \end{aligned}$$

Then we find that the index have many zero components:

$$\forall q \in [1, Q] \setminus \{j\}, \forall p \in \mathcal{P}, \quad v_{p,q} = 0,$$

and

$$\forall q \in [1, Q] \setminus \{j\}, \forall p \in \mathcal{P}^c, \quad w_{p,q} = 0.$$

Therefore, as $w_{p,j} = |w_p| = k_p$ for all $p \in \mathcal{P}^c$, we get that $\mathbf{w} = (k_p \mathbf{e}_j)_{p \in \mathcal{P}^c}$ with $\mathbf{e}_j \in \mathbb{N}_0^Q$ being the vector with a 1 in the j -th coordinate, and 0 elsewhere.

- 2) Let $\mathbf{v} = (\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{[1,Q] \setminus \{j,f\}})^{\mathcal{P}}$ and $\mathbf{w} = (\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c}$ such that

$$\begin{aligned} |\mathbf{w}_p| &= k_p & (p \in \mathcal{P}^c) \\ |\mathbf{v}_{\bullet q}| + |\mathbf{w}_{\bullet q}| &= 0 & (q \in [1, Q] \setminus \{j, f\}). \end{aligned}$$

We find that

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}, \forall p \in \mathcal{P}, \quad v_{p,q} = 0,$$

and

$$\forall q \in \llbracket 1, Q \rrbracket \setminus \{j, f\}, \forall p \in \mathcal{P}^c, \quad w_{p,q} = 0.$$

Thus, we find that the index \mathbf{w} takes the form $(w_{p,j}\mathbf{e}_j + w_{p,f}\mathbf{e}_f)_{p \in \mathcal{P}^c}$, with $w_{p,j} + w_{p,f} = k_p$ for all $p \in \mathcal{P}^c$. We thus find that the multinomial coefficient present in the expression for Q^1 is:

$$\binom{k_p}{\mathbf{w}_p} = \frac{k_p!}{w_{p,j}!w_{p,f}!}.$$

To simplify this expression, we will take the sum over $w'_p \in \llbracket 0, k_p \rrbracket$, and we will set $w_{p,f} = w'_p$ and $w_{p,j} = k_p - w'_p$.

3) Let $j \in \llbracket 1, Q \rrbracket$, and $\mathbf{u} = (\mathbf{u}_q)_{q \in \llbracket 1, Q \rrbracket \setminus \{j\}}$ such that

$$|\mathbf{u}_q| = N'_q = 0 \quad (q \in \llbracket 1, Q \rrbracket \setminus \{j\}).$$

It trivially follows that $\mathbf{u} = \mathbf{0}$, and therefore $\mathbf{R}(\mathbf{N}, j, \mathbf{u}) = (N_p + |\mathbf{u}_{\bullet p}|)_{p \in \llbracket 1, P \rrbracket} = \mathbf{0}$. In particular, the auxiliary functions $\varphi_{(N_p + |\mathbf{u}_{\bullet p}|)_{p \in \llbracket 1, P \rrbracket}}^j(s)$ are actually Barnes zeta functions of the form

$$\varphi_{\mathbf{0}}^j(s) = \zeta^B(s, d'_j | c_{j,1}, \dots, c_{j,P}).$$

We recall that the derivative of this Barnes zeta function at $s = 0$ is featured in the definition of the generalized multiple gamma functions, thus it is equal to $\ln(\Gamma_P(d'_j | c_{j,1}, \dots, c_{j,P}))$, where Γ_P corresponds to a Barnes multigamma function as introduced in Definition 1.1.14.

We have already seen in the proof of Corollary A1 that

$$Q_{\mathbf{0}, \mathbf{0}}^0(j, \mathcal{P}, \mathbf{k}) = \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}}.$$

Using these three preceding points, and the fact that the harmonic number h_0 is 0, we obtain:

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0})) &= \\ & \sum_{\substack{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket \\ \mu, \mu'}} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|} \mu'_j}{|\mu'| + |\mu|_{|\mathcal{P}|}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \left(\frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \gamma(\mu'_j + |\mu|_{|\mathcal{P}|}) + Q_{\mathbf{N}, \mathbf{N}'}^1(j, \mathcal{P}, \mathbf{k}) \right) \\ & + \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|} \mu'_j}{|\mu'| + |\mu|_{|\mathcal{P}|}} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \left(\sum_{i \in \mathcal{P}^c} \mu_i \frac{\zeta'(-k_i, d_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p, d_p)}{k_p!} \right) \\ & + \sum_{j=1}^Q \mu'_j (\zeta^B)'(0, d'_j | \mathbf{c}_j) - \sum_{j=1}^Q \gamma \mu'_j \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} (-1)^{|\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!}. \end{aligned}$$

By the point 1) and the point 2) previously stated, we get:

$$\begin{aligned} Q_{\mu, \mu'}^1(j, \mathcal{P}, \mathbf{k}) &= \gamma \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} |\mu'|_{\llbracket 1, Q \rrbracket \setminus \{j\}} + \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \left(\sum_{p \in \mathcal{P}} \mu_p \ln(c_{j,p}) \right) \\ & + \sum_{\substack{f=1 \\ f \neq j}}^Q \mu'_f \sum_{\mathbf{w}' = (w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right). \end{aligned}$$

Noticing that

$$\gamma \frac{(-1)^{\mathcal{P}} \mu'_j}{|\boldsymbol{\mu}'| + |\boldsymbol{\mu}|_{|\mathcal{P}}}} (\mu'_j + |\boldsymbol{\mu}|_{|\mathcal{P}} + |\boldsymbol{\mu}'|_{|\llbracket 1, Q \rrbracket \setminus \{j\}}}) = \gamma (-1)^{\mathcal{P}} \mu'_j,$$

we can simplify the first and last term of the formula obtained earlier for $Z'(-(\mathbf{0}, \mathbf{0}))_{\boldsymbol{\mu}, \boldsymbol{\mu}'}$, thus proving Corollary D2. \square

5.4.5 Proof of Corollary D3

Applying Corollary D2, and by taking $\boldsymbol{\mu} = (0, \dots, 0)$, and $\boldsymbol{\mu}' = (1, \dots, 1)$, we get

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{0}, \mathbf{1}} &= \sum_{j=1}^Q (\zeta^B)'(0, d'_j | \mathbf{c}_j) \\ &+ \sum_{\mathcal{P} \subsetneq \llbracket 1, P \rrbracket} \sum_{1 \leq j \neq f \leq Q} \frac{(-1)^{|\mathcal{P}|}}{Q} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=|\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p, d_p)}{k_p!} \\ &\cdot \sum_{\mathbf{w}'=(w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} \llbracket 0, k_p \rrbracket} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right). \end{aligned}$$

By Definition 1.1.16, we know that $Z'(-(\mathbf{0}, \mathbf{0}))_{\mathbf{0}, \mathbf{1}} = \partial_s(\zeta^{Sh}(s, (\mathbf{c}_1, \dots, \mathbf{c}_Q), \mathbf{d}'))_{s=0}$, and by Definition 1.1.14, we know that $\partial_s(\zeta^B(s, d|c_1, \dots, c_P))_{s=0} = \ln(\Gamma_P(d|c_1, \dots, c_P))$, and $\partial_s(\zeta^{Sh}(s, \mathbf{d}'|\mathbf{c}))_{s=0} = \ln(\Gamma_P(\mathbf{d}'|\mathbf{c}))$. By all the above, we get Corollary D3. \square

Chapter 6

Witten zeta functions

We recall in the first two sections the theory of Witten zeta functions and some well-known results on Lie theory from the perspectives of Komori, Matsumoto, and Tsumura. Details of the proofs are available in the book [KMT23]. Alternatively, one can also find details in the papers [KMT10b], [KMT10c], [KMT11b], [KMT12]. For details on Lie theory, we refer to [Hum72], [Bou06], [Bou81], [FH91], and [KMT20]. In the last section we apply our results on values and derivative values of multizeta functions to the Witten zeta functions $\zeta_{\mathfrak{g}_2}(s)$ and $\zeta_{\mathfrak{so}(5)}(s)$. The explicit formula of $\zeta'_{\mathfrak{so}(5)}(0)$ allows us to get an explicit expression for the constant C in Theorem 1.1.41. The formulas of $\zeta_{\mathfrak{g}_2}(0)$, $\zeta'_{\mathfrak{g}_2}(0)$, $\text{Res}_{s=\frac{1}{3}} \zeta_{\mathfrak{g}_2}(s)$, and $\text{Res}_{s=\frac{1}{5}} \zeta_{\mathfrak{g}_2}(s)$ allow us to give an asymptotic formula for the number of representations $r_{\mathfrak{g}_2}(n)$ using Theorem 1.1.38.

6.1 Generalities on Lie algebras

We call a Lie algebra \mathfrak{g} a complex or a real vector space of finite dimension equipped with an alternating bilinear form $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the Jacobi identity, meaning that $[\cdot, \cdot]$ is a bilinear form that satisfies the following conditions:

1. $\forall x \in \mathfrak{g}, \quad [x, x] = 0$
2. $\forall x, y, z \in \mathfrak{g}, \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$

We call $[\cdot, \cdot]$ a Lie bracket associated with the Lie algebra \mathfrak{g} . For the rest of the chapter we will just consider complex Lie algebras. A first example comes from taking the complex vector space of all the endomorphism of a complex vector space E with the Lie bracket $[f, g] := f \circ g - g \circ f$, with f and g being endomorphisms of E . We call $\mathfrak{gl}(E)$ such a Lie algebra, let's note that its corresponding Lie group is the group of automorphism of E , noted $GL(E)$.

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras with two Lie brackets $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$ respectively. Let $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ be a linear map between the two vector spaces. We say that f is a morphism of Lie algebras if f preserves the Lie brackets, i.e:

$$\forall x, y \in \mathfrak{g}_1, \quad [f(x), f(y)]_2 = f([x, y]_1).$$

Moreover, if f is also a linear isomorphism, we say that f is an isomorphism of Lie algebras. The derivation constitute a first example of morphism of Lie algebras:

$$\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}, \quad y \mapsto [x, y]$$

with $x \in \mathfrak{g}$.

Definition 6.1.1. Let \mathfrak{g} a Lie algebra. Let $\mathfrak{a} \subset \mathfrak{g}$ be a subspace of \mathfrak{g} . We say that \mathfrak{a} is a Lie subalgebra of \mathfrak{g} if \mathfrak{a} is stable under the Lie bracket of \mathfrak{g} , (i.e. if $[x, y] \in \mathfrak{a}$ for all $x, y \in \mathfrak{a}$).

Let \mathfrak{a}_1 and \mathfrak{a}_2 be subalgebras of a Lie algebra \mathfrak{g} , we define $[\mathfrak{a}_1, \mathfrak{a}_2]$ as the Lie subalgebra of \mathfrak{g} generated by elements of the form $[x, y]$, where $x \in \mathfrak{a}_1$, and $y \in \mathfrak{a}_2$.

Let \mathfrak{a} be a subalgebra of a Lie algebra \mathfrak{g} . We denote $N(\mathfrak{a})$ the subalgebra of \mathfrak{g} consisting of elements x such that $\text{ad}_x(\mathfrak{a}) \subset \mathfrak{a}$. If $N(\mathfrak{a}) = \mathfrak{g}$, then we say that \mathfrak{a} is an ideal of \mathfrak{g} .

Definition 6.1.2. Let \mathfrak{g} be a Lie algebra, we define a sequence $(\mathfrak{g}^{(n)})_{n \in \mathbb{N}_0}$ such that

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}] \quad (n \in \mathbb{N}_0).$$

We say that \mathfrak{g} is solvable if there exists an integer $n \in \mathbb{N}_0$ such that $\mathfrak{g}^{(n)} = 0$. We define $\text{rad}(\mathfrak{g})$ as the only maximal solvable ideal of \mathfrak{g} .

Definition 6.1.3. Let \mathfrak{g} be a Lie algebra, we define the sequence $(\mathfrak{g}^n)_{n \in \mathbb{N}_0}$ such that

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \quad (n \in \mathbb{N}_0).$$

We say that \mathfrak{g} is nilpotent if there exists an integer $n \in \mathbb{N}_0$ such that $\mathfrak{g}^n = 0$.

Definition 6.1.4. Let \mathfrak{g} be a Lie algebra. We say that \mathfrak{g} is semi-simple if the only solvable ideal of \mathfrak{g} is 0. Moreover, if \mathfrak{g} contains only one ideal, we say that \mathfrak{g} is simple.

We now provide some fundamental examples of semi-simple Lie algebras:

Example 6.1.5. For all integer $r \in \mathbb{N}$ we set:

- The type A_r : $A_r := \mathfrak{sl}(r+1) = \{X \in \mathcal{M}_{r+1}(\mathbb{C}) \mid \text{Tr}(X) = 0\}$.
- The type B_r : $B_r := \mathfrak{so}(2r+1) = \{X \in \mathcal{M}_{2r+1}(\mathbb{C}) \mid {}^t X + X = 0\}$.
- The type C_r : $C_r := \mathfrak{sp}(r) = \{X \in \mathcal{M}_{2r}(\mathbb{C}) \mid {}^t X J_r + J_r X = 0\}$, with

$$J_r := \begin{pmatrix} 0 & \text{Id}_r \\ -\text{Id}_r & 0 \end{pmatrix}.$$

- The type D_r : $D_r := \mathfrak{so}(2r) = \{X \in \mathcal{M}_{2r}(\mathbb{C}) \mid {}^t X + X = 0\}$, for $r \geq 2$.

In all these examples, the Lie bracket corresponds to $[X, Y] := X \cdot Y - Y \cdot X$ where \cdot means the usual matrix multiplication.

In the previous example, all these Lie algebras are simple, except for D_2 since we have $D_2 = A_1 \oplus A_1$.

Proposition 6.1.6. A semi-simple Lie algebra is a direct sum of simple Lie algebras.

Thus, to classify semi-simple Lie algebras, it is sufficient to classify simple Lie algebras. We will see via the Cartan-Killing Theorem that the 4 types of algebras that we have described in the previous example almost completely classify the set of semi-simple Lie algebras.

Definition 6.1.7. A representation of a Lie algebra \mathfrak{g} over a \mathbb{C} -vector space E corresponds to a linear map $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(E)$. We will call $\dim(\varphi) := \dim(E)$ the dimension of this representation. We say that the representation φ is irreducible if it does not admit any non-trivial vector subspace stable under the linear map φ .

Example 6.1.8. The derivation ad induces a representation of a Lie algebra $\text{ad}(\mathfrak{g})$ on the vector space \mathfrak{g} :

$$\text{ad}(\mathfrak{g}) : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), \quad x \mapsto \text{ad}_x.$$

We can then define the Killing bilinear form on $\mathfrak{g} \times \mathfrak{g}$:

$$\langle x, y \rangle := \text{Tr}(\text{ad}_x \circ \text{ad}_y).$$

Theorem 6.1.9. Let \mathfrak{g} be a Lie algebra. The following conditions are equivalent:

- a) The Lie algebra \mathfrak{g} is semi-simple.
- b) The radical of \mathfrak{g} is trivial (i.e. $\text{rad}(\mathfrak{g}) = 0$).
- c) The Killing form $\langle \cdot, \cdot \rangle$ is non-degenerate.

The previous theorem allows us in particular to identify a semi-simple Lie algebra with its dual. Thus, for an element $x \in \mathfrak{g}$, we can canonically associate an element $x^* \in \mathfrak{g}^*$.

6.2 Root System of a Lie Algebra

Let \mathfrak{g} be a Lie algebra. A Cartan subalgebra \mathfrak{h} is defined as a nilpotent Lie subalgebra of \mathfrak{g} such that $N(\mathfrak{h}) = \mathfrak{h}$. Cartan subalgebras are not unique, but they share the same dimension over \mathbb{C} . This dimension is then called the rank of \mathfrak{g} , denoted $\text{rk}(\mathfrak{g})$.

Throughout this section, we fix a semi-simple Lie algebra \mathfrak{g} and a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . The Killing form can be restricted to the Cartan subalgebra \mathfrak{h} , and it is a non-degenerate bilinear form on \mathfrak{h} . Through the identification of \mathfrak{h} with its dual, we can define the Killing form on \mathfrak{h}^* .

Definition 6.2.1. Let $\alpha \in \mathfrak{h}^*$ be an element of the dual of \mathfrak{h} . We say that α is a root of \mathfrak{g} relative to \mathfrak{h} when there exists a nonzero element $x \in \mathfrak{g} \setminus \{0\}$ such that

$$\forall h \in \mathfrak{h}, \quad \text{ad}_h(x) = \alpha(h)x. \quad (6.1)$$

We will denote g_α as the set of $x \in \mathfrak{g}$ satisfying the condition (6.1). We will denote $\Delta(\mathfrak{g})$ as the set of roots of \mathfrak{g} relative to \mathfrak{h} . We will call $\Delta(\mathfrak{g})$ the root system of the semi-simple Lie algebra \mathfrak{g} .

The root system of \mathfrak{g} is a finite set. Moreover, the Lie algebras \mathfrak{g}_α have dimension 1, and they yield a decomposition of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Through the identification between \mathfrak{h} and its dual, we consider for every root $\alpha \in \Delta(\mathfrak{g})$ the corresponding element $\alpha' \in \mathfrak{h}$. We set $\mathfrak{h}_0 := \text{Vect}_{\mathbb{R}}((\alpha')_{\alpha \in \Delta(\mathfrak{g})})$ the \mathbb{R} -vector space generated by the elements α' . We then define

$$\alpha^\vee := \frac{2}{\langle \alpha', \alpha' \rangle} \alpha'.$$

For every $\alpha, \beta \in \Delta(\mathfrak{g})$, we observe that

$$\beta(\alpha^\vee) = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

We denote $a(\beta, \alpha)$ the above expression.

Notation. We denote \mathfrak{h}_0 as the \mathbb{R} -vector space generated by the roots $\Delta(\mathfrak{g})$, and \mathfrak{h}_0^* as its dual.

Fact 6.2.2. Let $\alpha, \beta \in \Delta(\mathfrak{g})$. We have:

- 1) $\beta(\alpha^\vee) \in \mathbb{Z}$.
- 2) $\beta - a(\beta, \alpha)\alpha \in \Delta(\mathfrak{g})$.
- 3) If there exists $c \in \mathbb{C}$ such that $\beta = c\alpha$, then $c = \pm 1$.

These three properties characterize an "abstract" root system in the sense of [Ser66] and [Bou81]. Given two semi-simple Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , we know that there exists an isometry

$$(\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_1)), \|\cdot\|_1) \rightarrow (\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_2)), \|\cdot\|_2)$$

mapping $\Delta(\mathfrak{g}_1)$ to $\Delta(\mathfrak{g}_2)$ if and only if \mathfrak{g}_1 is isomorphic to \mathfrak{g}_2 as a Lie algebra, denoting $\|\cdot\|_1$ and $\|\cdot\|_2$ as the Euclidean norms (after fixing a basis) of the two vector spaces $\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_1))$ and $\text{Vect}_{\mathbb{R}}(\Delta(\mathfrak{g}_2))$. This property reduces the study of simple Lie algebras to the study of root systems. From this correspondence follows the subsequent theorem, allowing the classification of simple Lie algebras. To see the construction of the exceptional root systems we refer to [Bou81].

Theorem 6.2.3 (Cartan-Killing Theorem, [Bou81]). Any simple Lie algebra is isomorphic either to A_r , B_r , C_r , D_r , or to an exceptional Lie algebra \mathfrak{g}_2 , \mathfrak{f}_4 , \mathfrak{e}_6 , \mathfrak{e}_7 or \mathfrak{e}_8 .

Let us define $H_{\alpha^\vee} := \{\beta \in \mathfrak{h}_0^* \mid \langle \beta, \alpha^\vee \rangle = 0\}$. We note that, since the set of roots is finite, the set

$$\mathfrak{h}_0^* \setminus \bigcup_{\alpha \in \Delta(\mathfrak{g})} H_{\alpha^\vee}$$

is non-empty. Let γ be an element of this set. We then have the following decomposition:

$$\Delta(\mathfrak{g}) = \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle > 0\} \cup \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle < 0\}.$$

We denote $\Delta_+(\mathfrak{g}, \gamma) = \Delta_+(\mathfrak{g}) := \{\alpha \in \Delta(\mathfrak{g}) \mid \langle \gamma, \alpha^\vee \rangle > 0\}$ as the set of positive roots.

Definition 6.2.4. Let α be a positive root. We say that α is a fundamental (or simple) root if α cannot be decomposed as a sum of two positive roots. We shall note $\psi(\mathfrak{g})$ as the set of fundamental roots.

Fact 6.2.5. The set of fundamental roots forms a basis for the \mathbb{R} -vector space $\mathfrak{h}_0^* = \text{Vect}_{\mathbb{R}}((\alpha')_{\alpha \in \Delta(\mathfrak{g})})$. As a consequence, there are $r = \dim_{\mathbb{R}}(\mathfrak{h}_0)$ fundamental roots.

We now aim to introduce the concept of weight:

Definition 6.2.6. Let $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(E)$ be a representation of the semi-simple Lie algebra \mathfrak{g} , and $u \in E$ a vector. We say that u is an eigenvector if there exists a linear form $\lambda \in \mathfrak{h}^*$ on \mathfrak{h} such that

$$\forall H \in \mathfrak{h}, \quad \varphi(H)u = \lambda(H)u.$$

We call λ the weight of u . If furthermore, we have $\lambda(\alpha^\vee) \geq 0$ for every simple root α of \mathfrak{g} , then we say that λ is a dominant weight.

Let $\psi(\mathfrak{g}) = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots of \mathfrak{g} . We then define some linear forms $\lambda_1, \dots, \lambda_r \in \mathfrak{h}^*$ such that $\lambda_j(\alpha_i^\vee) = \delta_i^j$ for all $1 \leq i, j \leq r$, where δ is Kronecker's symbol. The linear forms $\lambda_1, \dots, \lambda_r$ are dominant weights, and we call them fundamental weights (or simple weights).

Fact 6.2.7. Each dominant weight λ has a unique decomposition in terms of the fundamental weights

$$\lambda = n_1 \lambda_1 + \dots + n_r \lambda_r \quad (n_1, \dots, n_r \in \mathbb{N}_0).$$

Fact 6.2.8. There is a bijective correspondence between the set of irreducible representations of finite dimension (up to isomorphism), and the set of dominant weights.

Theorem 6.2.9 (of Weyl). Let φ be an irreducible representation of \mathfrak{g} . Then we have

$$\dim(\varphi) = \prod_{\alpha \in \Delta_+(\mathfrak{g})} \frac{\langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle}{\langle \alpha^\vee, \lambda_1 + \dots + \lambda_r \rangle}.$$

Remark 6.2.10. At first glance the results stated above all depend on the choice of the Cartan subalgebra \mathfrak{h} , and the choice of γ . However, according to [KMT23, §2.3], they are actually independent of these choices. The reason being that two Cartan subalgebras are necessarily conjugate (see [Hum72, §16.2]).

We note that the Fact 6.2.8 and the Theorem 6.2.9 allow us to determine an expression for $f_{\mathfrak{g}}(n)$, the number of irreducible representations (up to isomorphism) of dimension n of a semi-simple Lie algebra \mathfrak{g} .

Definition 6.2.11 ([Zag94]). We associate with a semi-simple Lie algebra \mathfrak{g} a zeta function

$$\zeta_{\mathfrak{g}}(s) := \sum_{\varphi} \frac{1}{\dim(\varphi)^s},$$

where the sum is over the finite-dimensional irreducible representations φ of \mathfrak{g} (up to isomorphism).

This zeta function generalizes the series that appear in Witten's volume formulas in [Wit91]. From Fact 6.2.8 and Weyl's Theorem 6.2.9, it follows that Witten's zeta functions can be written as a Dirichlet series attached to a products of linear forms in r variables, r being the number of fundamental roots. More precisely, we have the following expression:

$$\zeta_{\mathfrak{g}}(s) := \prod_{\alpha \in \Delta_+(\mathfrak{g})} \langle \alpha^\vee, \lambda_1 + \dots + \lambda_r \rangle^s \sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{\alpha \in \Delta_+(\mathfrak{g})} \langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle^s}.$$

Considering a root system, Matsumoto, Tsumura, and Komori introduced in [KMT10b] some multizeta analogues of Witten's zeta functions, by considering Dirichlet series of the form

$$\sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{\alpha \in \Delta_+(\mathfrak{g})} \langle \alpha^\vee, (n_1 + 1)\lambda_1 + \dots + (n_r + 1)\lambda_r \rangle^{s_\alpha}}.$$

This allows for obtaining recursive formulas for these multizeta functions (see [KMT10b, Theorem 3.1]), and to obtain several formulas for some positive integers which could be considered as quasi-shuffle relations.

6.3 Some Results on $\zeta_{\mathfrak{g}_2}$ and $\zeta_{\mathfrak{so}(5)}$

Here, we aim to detail the formulas obtained in Theorems A and D for the two Witten zeta functions $\zeta_{\mathfrak{g}_2}$ and $\zeta_{\mathfrak{so}(5)}$, primarily focusing on the values at $s = 0$. The special values $\zeta_{\mathfrak{g}_2}(0)$ and $\zeta_{\mathfrak{so}(5)}(0)$ are very simple to compute using Corollary A1. On the other hand, the calculations for the values $\zeta'_{\mathfrak{g}_2}(0)$ and $\zeta'_{\mathfrak{so}(5)}(0)$ will prove to be more challenging. We propose to specify our result on values and derivative values when $P = 2$, that is, when the Dirichlet series describing $Z(\mathbf{s}, \mathbf{s}')$ only involves two variables n_1, n_2 . We furthermore fix the direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1})$, and set $d_1 = d_2 = 1$. Let's start by examining the values of $Z_\Delta(s)$ at nonpositive integers using Corollary A2:

Lemma 6.3.1. *Let $N \in \mathbb{N}_0$. We have*

$$\begin{aligned} Z_\Delta(-N) &= \frac{(-1)^{(1+Q)N}}{Q} N! \sum_{k=0}^{QN} Q_N^0(\emptyset, (k, QN - k)) \frac{\zeta(-N - k)}{k!} \frac{\zeta(-(Q+1)N + k)}{(QN - k)!} \\ &\quad + \frac{(-1)^{QN+1}}{Q+1} N!^2 (Q_N^0(\{1\}, (Q+1)N + 1) + Q_N^0(\{2\}, (Q+1)N + 1)) \frac{\zeta(-(Q+2)N - 1)}{((Q+1)N + 1)!}, \end{aligned}$$

with

$$Q_N^0(\emptyset, (k_1, k_2)) = (-1)^{N(Q-1)} N!^{Q-1} Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}_0^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q, w_{1,q}+w_{2,q}=N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}} \quad (k_1 + k_2 = QN),$$

$$\begin{aligned} Q_N^0(\{1\}, (Q+1)N + 1) &= \\ (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}_0^Q \\ |\mathbf{w}_2|=(Q+1)N+1 \\ w_{2,j} \geq 2N+1 \\ \forall q \neq j, w_{2,q} \leq N}} \binom{-N-1}{-2N-1+w_{2,j}} \binom{-2N-1+w_{2,j}}{(N-w_{2,q})_{q \neq j}} \prod_{q=1}^Q c_{q,1}^{N-w_{2,q}} c_{q,2}^{w_{2,q}}, \end{aligned}$$

$$\begin{aligned} Q_N^0(\{2\}, (Q+1)N + 1) &= \\ (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_1 \in \mathbb{N}_0^Q \\ |\mathbf{w}_1|=(Q+1)N+1 \\ w_{1,j} \geq 2N+1 \\ \forall q \neq j, w_{1,q} \leq N}} \binom{-N-1}{-2N-1+w_{1,j}} \binom{-2N-1+w_{1,j}}{(N-w_{1,q})_{q \neq j}} \prod_{q=1}^Q c_{q,2}^{N-w_{1,q}} c_{q,1}^{w_{1,q}}. \end{aligned}$$

Proof. By Corollary A2, we get

$$\begin{aligned} Z_{\Delta}(-N) &= \sum_{\mathcal{P} \subsetneq [1,2]} \frac{(-1)^{(|\mathcal{P}|+1+Q)N+|\mathcal{P}|}}{Q+|\mathcal{P}|} N^{!|\mathcal{P}|+1} \sum_{\substack{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}|=(Q+|\mathcal{P}|)N+|\mathcal{P}|}} Q_N^0(\mathcal{P}, \mathbf{k}) \prod_{p \in \mathcal{P}^c} \frac{\zeta(-N-k_p)}{k_p!} \\ &= \frac{(-1)^{(1+Q)N}}{Q} N! \sum_{\substack{k_1, k_2 \geq 0 \\ k_1+k_2=QN}} Q_N^0(\emptyset, (k_1, k_2)) \prod_{p=1}^2 \frac{\zeta(-N-k_p)}{k_p!} \\ &\quad + \frac{(-1)^{QN+1}}{Q+1} N!^2 (Q_N^0(\{1\}, (Q+1)N+1) + Q_N^0(\{2\}, (Q+1)N+1)) \frac{\zeta(-(Q+2)N-1)}{((Q+1)N+1)!}, \end{aligned}$$

with

$$\begin{aligned} Q_N^0(\mathcal{P}, \mathbf{k}) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p \in \mathcal{P}^c} \in (\mathbb{N}_0^Q)^{\mathcal{P}^c} \\ \forall p \in \mathcal{P}^c, |\mathbf{w}_p|=k_p}} \sum_{\substack{\mathbf{v}=(\mathbf{v}_p)_{p \in \mathcal{P}} \in (\mathbb{N}_0^{1, Q \setminus \{j\}})^{\mathcal{P}} \\ \forall q \neq j, |\mathbf{v}_{\bullet q}|+|\mathbf{w}_{\bullet q}|=N}} \\ &\quad \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{\mathbf{w}_p} \prod_{q=1}^Q c_{q,p}^{w_{p,q}} \right) \left(\prod_{p \in \mathcal{P}} \binom{-N-1}{|\mathbf{v}_p|} \binom{|\mathbf{v}_p|}{\mathbf{v}_p} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,p}^{v_{p,q}} \right). \end{aligned}$$

- If $\mathcal{P} = \emptyset$ and $k_1 + k_2 = QN$, then we have

$$Q_N^0(\emptyset, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}_0^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q \neq j, w_{1,q}+w_{2,q}=N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}}.$$

Let's observe that

$$\begin{aligned} w_{1,j} + w_{2,j} &= \sum_{q=1}^Q w_{1,q} + w_{2,q} - \sum_{\substack{q=1 \\ q \neq j}}^Q w_{1,q} + w_{2,q} \\ &= k_1 + k_2 - (Q-1)N \\ &= N. \end{aligned}$$

Let $\mathbf{w} = (\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}_0^Q)^2$. The condition $|\mathbf{w}_1| = k_1$, $|\mathbf{w}_2| = k_2$, and $w_{1,q} + w_{2,q} = N$ for all $q \neq j$, is equivalent to the condition $w_{1,q} + w_{2,q} = N$ for all q , and $|\mathbf{w}_1| = k_1$, $|\mathbf{w}_2| = k_2$. We then find that

$$Q_N^0(\emptyset, \mathbf{k}) = (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}=(\mathbf{w}_p)_{p=1,2} \in (\mathbb{N}_0^Q)^2 \\ |\mathbf{w}_1|=k_1, |\mathbf{w}_2|=k_2 \\ \forall q, w_{1,q}+w_{2,q}=N}} \binom{k_1}{\mathbf{w}_1} \binom{k_2}{\mathbf{w}_2} \prod_{q=1}^Q c_{q,1}^{w_{1,q}} c_{q,2}^{w_{2,q}}.$$

- If $\mathcal{P} = \{1\}$ and $k_2 = (Q+1)N+1$, then we have that

$$\begin{aligned} Q_N^0(\{1\}, (Q+1)N+1) &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}_0^Q \\ |\mathbf{w}_2|=(Q+1)N+1 \\ \forall q \neq j, w_{2,q} \leq N}} \left(\binom{k_1}{(w_{2,1}, \dots, w_{2,Q})} \prod_{q=1}^Q c_{q,2}^{w_{2,q}} \right) \\ &\quad \left(\binom{-N-1}{\sum_{q \neq j} (N-w_{2,q})} \binom{\sum_{q \neq j} (N-w_{2,q})}{(N-w_{2,q})_{q \neq j}} c_{j,p}^{-N-1-\sum_{\substack{q=1 \\ q \neq j}}^Q (N-w_{2,q})} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_{2,q}} \right). \end{aligned}$$

Let's note that

$$\begin{aligned} \sum_{\substack{q=1 \\ q \neq j}}^Q (N - w_{2,q}) &= (Q-1)N - |\mathbf{w}_2| + w_{2,j} \\ &= -2N - 1 + w_{2,j}. \end{aligned}$$

If this term is negative, then the binomial coefficient $\binom{-N-1}{\sum_{q \neq j} (N - w_{2,q})}$ vanishes. Therefore we can assume the condition $w_{2,j} \geq 2N + 1$ instead:

$$\begin{aligned} &Q_N^0(\{1\}, k_2) \\ &= (-1)^{N(Q-1)} N!^{Q-1} \\ &\quad \cdot \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}_0^Q \\ |\mathbf{w}_2| = (Q+1)N+1 \\ w_{2,j} \geq 2N+1 \\ \forall q \neq j, w_{2,q} \leq N}} \binom{-N-1}{-2N-1+w_{2,j}} \binom{-2N-1+w_{2,j}}{(N-w_{2,q})_{q \neq j}} c_{j,1}^{N-w_{2,j}} c_{j,2}^{w_{2,j}} \prod_{\substack{q=1 \\ q \neq j}}^Q c_{q,1}^{N-w_{2,q}} c_{q,2}^{w_{2,q}} \\ &= (-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_2 \in \mathbb{N}_0^Q \\ |\mathbf{w}_2| = (Q+1)N+1 \\ w_{2,j} \geq 2N+1 \\ \forall q \neq j, w_{2,q} \leq N}} \binom{-N-1}{-2N-1+w_{2,j}} \binom{-2N-1+w_{2,j}}{(N-w_{2,q})_{q \neq j}} \prod_{q=1}^Q c_{q,1}^{N-w_{2,q}} c_{q,2}^{w_{2,q}}. \end{aligned}$$

- If $\mathcal{P} = \{2\}$ and $k_1 = (Q+1)N + 1$, by the same reasoning we get

$$\begin{aligned} &Q_N^0(\{2\}, k_1) = \\ &(-1)^{N(Q-1)} N!^{Q-1} \sum_{j=1}^Q \sum_{\substack{\mathbf{w}_1 \in \mathbb{N}_0^Q \\ |\mathbf{w}_1| = (Q+1)N+1 \\ w_{1,j} \geq 2N+1 \\ \forall q \neq j, w_{1,q} \leq N}} \binom{-N-1}{-2N-1+w_{1,j}} \binom{-2N-1+w_{1,j}}{(N-w_{1,q})_{q \neq j}} \prod_{q=1}^Q c_{q,2}^{N-w_{1,q}} c_{q,1}^{w_{1,q}}. \end{aligned}$$

□

We can simplify the expressions from the previous lemma in order to get the following corollary. We shall use this next expression in order to compute the values of the zeta functions of the form $Z_{\Delta}(-N)$ when $P = 2$ with the SAGE code 7.2.

Corollary 6.3.2. *Let $N \in \mathbb{N}_0$, then*

$$Z_{\Delta}(-N) = N!^Q \sum_{k=0}^{QN} \alpha_k(N) \frac{\zeta(-N-k) \zeta(-(Q+1)N+k)}{k!(QN-k)!} + \frac{(-1)^{N+1}}{Q+1} N!^{Q+1} \beta(N) \frac{\zeta(-(Q+2)N-1)}{((Q+1)N+1)!},$$

with

$$\begin{aligned} \alpha_k(N) &= \sum_{\substack{(w_q)_{q \in [1, Q]} \in [0, N]^Q \\ w_1 + \dots + w_Q = k}} \binom{k}{(w_1, \dots, w_Q)} \binom{QN - k}{(N - w_1, \dots, N - w_Q)} \prod_{q=1}^Q c_{q,1}^{w_q} c_{q,2}^{N-w_q}, \\ \beta(N) &= \sum_{j=1}^Q \sum_{\substack{(w_q)_{q \in [1, Q]} \in [0, (Q+1)N+1]^Q \\ w_1 + \dots + w_Q = (Q+1)N+1 \\ w_j \geq 2N+1 \\ \forall q \neq j, w_q \leq N}} \binom{-N-1}{-2N-1+w_j} \binom{-2N-1+w_j}{(N-w_q)_{q \neq j}} \\ &\quad \cdot \left(\prod_{q=1}^Q c_{q,1}^{N-w_q} c_{q,2}^{w_q} + \prod_{q=1}^Q c_{q,2}^{N-w_q} c_{q,1}^{w_q} \right). \end{aligned}$$

By taking $P = 2$ and $\mathbf{d} = \mathbf{1}$ in Corollary D2, we get a more refined version of that corollary:

Lemma 6.3.3. *We have*

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0})_{\mathbf{1}, \mathbf{1}}) &= \sum_{j=1}^Q (\zeta^B)'(0, d'_j | c_{j,1}, c_{j,2}) + 2\zeta(0)\zeta'(0) - \frac{\zeta'(-1)}{Q+1} \left(\sum_{j=1}^Q \frac{c_{j,2}}{c_{j,1}} + \frac{c_{j,1}}{c_{j,2}} \right) \\ &\quad - \sum_{j=1}^Q \frac{\zeta(-1)}{Q+1} \left(\frac{c_{j,2}}{c_{j,1}} \ln(c_{j,1}) + \frac{c_{j,1}}{c_{j,2}} \ln(c_{j,2}) \right) \\ &\quad + \sum_{\substack{j=1 \\ f \neq j}}^Q \left(\left(\frac{c_{f,2}}{c_{f,1}} - \frac{c_{j,2}}{c_{j,1}} \right) \ln \left(1 + \frac{c_{f,1}}{c_{j,1}} \right) + \left(\frac{c_{f,1}}{c_{f,2}} - \frac{c_{j,1}}{c_{j,2}} \right) \ln \left(1 + \frac{c_{f,2}}{c_{j,2}} \right) \right), \end{aligned}$$

where $(\zeta^B)'(s, d'_j | c_{j,1}, c_{j,2}) = \partial_s \zeta^B(s, d'_j | c_{j,1}, c_{j,2})$.

Proof. Let's recall that $Z'_{\Delta}(0) = Z'(-(\mathbf{0}, \mathbf{0})_{\mathbf{1}, \mathbf{1}})$ where we have set $P = 2$ and we fixed the directions $(\boldsymbol{\mu}, \boldsymbol{\mu}') = (\mathbf{1}, \mathbf{1}) = ((1, 1), (1, \dots, 1)) \in \mathbb{R}^2 \times \mathbb{R}^Q$. By Corollary D2 we then have:

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0})_{\mathbf{1}, \mathbf{1}}) &= \sum_{j=1}^Q (\zeta^B)'(0, l_j(\mathbf{1}) | c_j) \\ &\quad + \sum_{\mathcal{P} \subsetneq [1, 2]} \sum_{j=1}^Q \frac{(-1)^{|\mathcal{P}|}}{Q + |\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \prod_{p \in \mathcal{P}^c} \frac{\zeta(-k_p)}{k_p!} \left(\frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \sum_{p \in \mathcal{P}} \ln(c_{j,p}) \right) \\ &\quad + \sum_{\substack{f=1 \\ f \neq j}}^Q \sum_{\mathbf{w}' = (w'_p)_{p \in \mathcal{P}^c} \in \prod_{p \in \mathcal{P}^c} [0, k_p]} F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0}) \left(\prod_{p \in \mathcal{P}^c} \binom{k_p}{w'_p} c_{f,p}^{w'_p} c_{j,p}^{k_p - w'_p} \right) \\ &\quad + \sum_{\mathcal{P} \subsetneq [1, 2]} \sum_{j=1}^Q (-1)^{|\mathcal{P}|} \frac{1}{Q + |\mathcal{P}|} \sum_{\substack{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c} \\ |\mathbf{k}| = |\mathcal{P}|}} \frac{\prod_{p \in \mathcal{P}^c} c_{j,p}^{k_p}}{\prod_{p \in \mathcal{P}} c_{j,p}} \left(\sum_{i \in \mathcal{P}^c} \frac{\zeta'(-k_i)}{k_i!} \prod_{p \in \mathcal{P}^c \setminus \{i\}} \frac{\zeta(-k_p)}{k_p!} \right), \end{aligned}$$

where we recall that $\mathbf{e}_n = (0, \dots, 1, \dots, 0) \in \mathbb{R}^Q$ is the vector with a 1 in the n -th component, and 0

elsewhere. Developing the sums in the formula of $Z'(-(\mathbf{0}, \mathbf{0}))_{1,1}$, we get

$$\begin{aligned} Z'(-(\mathbf{0}, \mathbf{0}))_{1,1} &= \sum_{j=1}^Q (\zeta^B)'(0, d'_j | c_{j,1}, c_{j,2}) + \sum_{1 \leq j \neq f \leq Q} \frac{1}{Q} \zeta(0)^2 F_{\emptyset, j, f, \mathbf{0}, (\mathbf{0}, \mathbf{0})}(\mathbf{0}, \mathbf{0}) \\ &- \sum_{j=1}^Q \frac{\zeta(-1)}{Q+1} \left(\frac{c_{j,2}}{c_{j,1}} \ln(c_{j,1}) + \frac{c_{j,1}}{c_{j,2}} \ln(c_{j,2}) \right. \\ &+ \sum_{\substack{f=1 \\ f \neq j}}^Q (F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) c_{f,2} + F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) c_{j,2} + F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) c_{f,1} + F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) c_{j,1}) \left. \right) \\ &+ 2\zeta(0)\zeta'(0) - \frac{1}{Q+1} \left(\sum_{j=1}^Q \frac{c_{j,2}}{c_{j,1}} + \frac{c_{j,1}}{c_{j,2}} \right) \zeta'(-1). \end{aligned}$$

Let's now study the values of the coefficients $F_{\mathcal{P}, j, f, \mathbf{0}, (w'_p \mathbf{e}_f + (k_p - w'_p) \mathbf{e}_j)_{p \in \mathcal{P}^c}}(\mathbf{0}, \mathbf{0})$ that are in the previous formula.

- If $\mathcal{P} = \emptyset$ and $\mathbf{k} = (0, 0)$, using the Example 1.2.4, we have that $F_{\emptyset, j, f, \mathbf{0}, (\mathbf{0}, \mathbf{0})}(\mathbf{0}, \mathbf{0}) = 0$.
- If $\mathcal{P} = \{1\}$ and $k_2 = 1$, we have : By Example 1.2.6 that

$$F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) = -\frac{1}{c_{j,1}} \ln \left(1 + \frac{c_{f,1}}{c_{j,1}} \right),$$

and thanks to Example 1.2.5, we have that

$$F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) = \frac{1}{c_{f,1}} \ln \left(1 + \frac{c_{f,1}}{c_{j,1}} \right).$$

- If $\mathcal{P} = \{2\}$ and $k_1 = 1$, mutatis mutandis we find that

$$\begin{aligned} F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) &= -\frac{1}{c_{j,2}} \ln \left(1 + \frac{c_{f,2}}{c_{j,2}} \right) \\ F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) &= \frac{1}{c_{f,2}} \ln \left(1 + \frac{c_{f,2}}{c_{j,2}} \right). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &\sum_{\substack{f=1 \\ f \neq j}}^Q (F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) c_{f,2} + F_{\{1\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) c_{j,2} + F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_f}(\mathbf{0}, \mathbf{0}) c_{f,1} + F_{\{2\}, j, f, \mathbf{0}, \mathbf{e}_j}(\mathbf{0}, \mathbf{0}) c_{j,1}) \\ &= \sum_{\substack{f=1 \\ f \neq j}}^Q \left(\left(\frac{c_{f,2}}{c_{f,1}} - \frac{c_{j,2}}{c_{j,1}} \right) \ln \left(1 + \frac{c_{f,1}}{c_{j,1}} \right) + \left(\frac{c_{f,1}}{c_{f,2}} - \frac{c_{j,1}}{c_{j,2}} \right) \ln \left(1 + \frac{c_{f,2}}{c_{j,2}} \right) \right). \end{aligned}$$

Injecting this equality into the formula obtained for $Z'(-(\mathbf{0}, \mathbf{0}))_{1,1}$, we get the required result. \square

Applying this lemma to the Tornheim zeta function we get the well-known result

$$(\zeta_2^{MT})'(0) = \ln(2\pi).$$

Remark 6.3.4. Under the assumptions of the previous lemma, by giving an integer $Q \geq 1$ and the coefficients $\mathbf{c} = (c_{q,p})_{q \in [1,Q], p \in [1,2]}$, the SAGE code 7.1 shall compute $Z'_\Delta(0)$ when the coefficients $c_{q,p}$ are integers between 1 and 3. Let's note that the result of such computations are not rounded up, but gives an expression using only $\zeta(0)$, $\zeta(-1)$, $\zeta'(0)$, $\zeta'(-1)$, the logarithms of some integers, and special values of the gamma and digamma functions. One could code an algorithm that can treat the case when the coefficients $c_{q,p}$ are any integers, but the caveat is that we don't have completely explicit and nice formulas for the derivative at $s = 0$ of the corresponding Barnes zeta functions. In the general case, one can only give an expression of such derivatives in terms of special values of the derivative of the Hurwitz zeta function.

Even though we have a formula to compute at nonpositive integers the derivative with respect to s of Barnes' zeta function $(\zeta^B)'(0, d'_j | c_{j,1}, c_{j,2})$ via Corollary B1, by setting $N = 0$ and $\mathbf{R} = \mathbf{0}$, we will instead use the computation already done in the Example 1.1.12 in order to simplify the derivative values of the Barnes zeta function that will arise in the computations of $\zeta'_{s\mathfrak{o}(5)}(0)$ and $\zeta'_{\mathfrak{g}_2}(0)$. We coded these expressions in SAGE code 7.1 in order to compute the values and derivative values of those two Witten zeta functions.

6.3.1 Poles and residues of $\zeta_{\mathfrak{g}_2}$

Via [KMT11a], we've seen in the list of poles (1.4) that $\zeta_{\mathfrak{g}_2}(s)$ has poles of the form $s = \frac{1}{3}$ and $s = \frac{1-k}{5}$, with $k \geq 0$ and $k \not\equiv 1 \pmod{5}$. We shall give an expression of the residues of $\zeta_{\mathfrak{g}_2}(s)$ at $s = \frac{1}{5}$ and at $s = \frac{1}{3}$. We set

$$\begin{aligned} l_1(x_1, x_2) &= x_1 + x_2, & l_2(x_1, x_2) &= x_1 + 2x_2, & l_3(x_1, x_2) &= x_1 + 3x_2, & l_4(x_1, x_2) &= 2x_1 + 3x_2, \\ d_1 &= 1, & d_2 &= 1, & d'_1 &= 2, & d'_2 &= 3, & d'_3 &= 4, & d'_4 &= 5, \\ Z_\Delta(s) &:= Z(-((s, s), (s, s, s, s))) = \sum_{\substack{(1,1), (1,1,1,1) \\ n_1, n_2 \geq 1}} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_3)^s (2n_1 + 3n_2)^s}. \end{aligned}$$

Let's recall that the Witten zeta function $\zeta_{\mathfrak{g}_2}$ is defined as

$$\zeta_{\mathfrak{g}_2}(s) = 120^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s (n_1 + 3n_3)^s (2n_1 + 3n_2)^s} = 120^s Z_\Delta(s)$$

Via Proposition 2.2.4, we find that

$$\zeta_{\mathfrak{g}_2}(s) = K(\theta, s) + J(\theta, s),$$

with

$$\begin{aligned} K(\theta, s) &= 120^s \sum_{\emptyset \neq \mathcal{Q} \subset [1,4]} \sum_{A_{\mathcal{Q}} \subset \mathcal{Q}^c} (-1)^{|\mathcal{Q}^c| - |A_{\mathcal{Q}}|} \frac{1}{\Gamma(s)^{4 - |A_{\mathcal{Q}}|}} \left(\sum_{\mathbf{n} \in \mathbb{N}_0^2} (n_1 + 1)^{-s} (n_2 + 1)^{-s} \right. \\ &\quad \left. \cdot \prod_{q \in A_{\mathcal{Q}}} (l_q(\mathbf{n} + \mathbf{1}))^{-s} \prod_{q \in [1,4] \setminus A_{\mathcal{Q}}} \Gamma(s, \theta, l_q(\mathbf{n} + \mathbf{1})) \right), \end{aligned}$$

and

$$\begin{aligned} J(\theta, s) &= 120^s \Gamma(1-s)^2 \left(\sum_{j=1}^4 h_{[1,2],j}(s) \right) \frac{\theta^{6s-2}}{\Gamma(s)(6s-2)} \\ &+ 120^s \sum_{\mathcal{P} \subsetneq [1,2]} \Gamma(1-s)^{|\mathcal{P}|} \sum_{\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} \left(\sum_{j=1}^4 h_{\mathcal{P},j,\mathbf{k}}(s) \right) \frac{(-1)^{|\mathbf{k}|} \theta^{(4+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(s)((4+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(s - k_p)}{k_p!} \end{aligned} \quad (6.2)$$

with

$$\begin{aligned} h_{\mathcal{P},1,\mathbf{k}}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_2 x_3 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(1, x_2, x_3, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(1, x_2, x_3, x_4)^{k_p} dx_2 dx_3 dx_4, \\ h_{\mathcal{P},2,\mathbf{k}}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_3 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, 1, x_3, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, 1, x_3, x_4)^{k_p} dx_1 dx_3 dx_4, \\ h_{\mathcal{P},3,\mathbf{k}}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_4)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, x_2, 1, x_4)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, x_2, 1, x_4)^{k_p} dx_1 dx_2 dx_4, \\ h_{\mathcal{P},4,\mathbf{k}}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_3)^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, x_2, x_3, 1)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, x_2, x_3, 1)^{k_p} dx_1 dx_2 dx_3, \\ l_1^*(x_1, x_2) &= x_1 + x_2 + x_3 + 2x_4, \\ l_2^*(x_1, x_2) &= x_1 + 2x_2 + 3x_3 + 3x_4, \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^4 h_{[1,2],j}(s) &= \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_3)^{s-1} \prod_{p=1}^2 l_p^*(x_1, x_2, x_3, 1)^{s-1} dx_1 dx_2 dx_3, \\ &+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_2 x_4)^{s-1} \prod_{p=1}^2 l_p^*(x_1, x_2, 1, x_4)^{s-1} dx_1 dx_2 dx_4 \\ &+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_1 x_3 x_4)^{s-1} \prod_{p=1}^2 l_p^*(x_1, 1, x_3, x_4)^{s-1} dx_1 dx_3 dx_4 \\ &+ \frac{1}{\Gamma(s)^3} \int_{[0,1]^3} (x_2 x_3 x_4)^{s-1} \prod_{p=1}^2 l_p^*(1, x_2, x_3, x_4)^{s-1} dx_2 dx_3 dx_4. \end{aligned}$$

Let's recall that the function $s \mapsto K(\theta, s)$ is holomorphic on \mathbb{C} , thus this term doesn't contribute to the residues. We just have to study the denominators of the terms inside of $J(\theta, s)$, in the formula (6.2).

- Let $\mathcal{P} \subsetneq [1, 2]$, and $\mathbf{k} = (k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}$. For $s = \frac{1}{5}$, we have

$$(4 + |\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}| = 0 \text{ if and only if } |\mathcal{P}| = 1, \mathbf{k} = 0.$$

We then find that

$$\text{Res}_{s=\frac{1}{5}} \zeta_{\mathfrak{g}_2}(s) = \frac{120^{\frac{1}{5}} \Gamma\left(\frac{4}{5}\right)}{5\Gamma\left(\frac{1}{5}\right)} \left(\sum_{j=1}^4 h_{\{1\},j,0}\left(\frac{1}{5}\right) + h_{\{2\},j,0}\left(\frac{1}{5}\right) \right) \zeta\left(\frac{1}{5}\right).$$

By using the expressions of the h functions we get

$$\begin{aligned} \text{Res}_{s=\frac{1}{5}} \zeta_{\mathfrak{g}_2}(s) &= \frac{120^{\frac{1}{5}} \Gamma\left(\frac{4}{5}\right)}{5\Gamma\left(\frac{1}{5}\right)^4} \zeta\left(\frac{1}{5}\right) \\ &\cdot \left(\int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{4}{5}} (l_1^*(1, x_2, x_3, x_4)^{-\frac{4}{5}} + l_2^*(1, x_2, x_3, x_4)^{-\frac{4}{5}}) dx_2 dx_3 dx_4 \right. \\ &+ \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{4}{5}} (l_1^*(x_1, 1, x_3, x_4)^{-\frac{4}{5}} + l_2^*(x_1, 1, x_3, x_4)^{-\frac{4}{5}}) dx_1 dx_3 dx_4 \\ &+ \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{4}{5}} (l_1^*(x_1, x_2, 1, x_4)^{-\frac{4}{5}} + l_2^*(x_1, x_2, 1, x_4)^{-\frac{4}{5}}) dx_1 dx_2 dx_4 \\ &\left. + \int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{4}{5}} (l_1^*(x_1, x_2, x_3, 1)^{-\frac{4}{5}} + l_2^*(x_1, x_2, x_3, 1)^{-\frac{4}{5}}) dx_1 dx_2 dx_3 \right). \end{aligned}$$

- Via a similar argument, we get that

$$\operatorname{Res}_{s=\frac{1}{3}} \zeta_{\mathfrak{g}_2}(s) = \frac{120^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)^2}{6\Gamma\left(\frac{1}{3}\right)} \left(\sum_{j=1}^4 h_{\{1,2\},j}\left(\frac{1}{3}\right) + h_{\{1,2\},j}\left(\frac{1}{3}\right) \right) \zeta\left(\frac{1}{3}\right),$$

and by using the expressions of the h functions we get

$$\begin{aligned} \operatorname{Res}_{s=\frac{1}{3}} \zeta_{\mathfrak{g}_2}(s) &= \frac{120^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)^2}{6\Gamma\left(\frac{1}{3}\right)^4} \\ &\cdot \left(\int_{[0,1]^3} (x_1 x_2 x_3)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, x_3, 1)^{-\frac{2}{3}} dx_1 dx_2 dx_3, \right. \\ &+ \int_{[0,1]^3} (x_1 x_2 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, x_2, 1, x_4)^{-\frac{2}{3}} dx_1 dx_2 dx_4 \\ &+ \int_{[0,1]^3} (x_1 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(x_1, 1, x_3, x_4)^{-\frac{2}{3}} dx_1 dx_3 dx_4 \\ &\left. + \int_{[0,1]^3} (x_2 x_3 x_4)^{-\frac{2}{3}} \prod_{p=1}^2 l_p^*(1, x_2, x_3, x_4)^{-\frac{2}{3}} dx_2 dx_3 dx_4 \right). \end{aligned}$$

The above computations justify the formulas given for ω_α and ω_β in Theorem G.

6.3.2 Special values of $\zeta_{\mathfrak{g}_2}$ and of $\zeta'_{\mathfrak{g}_2}$

Let's take the same notations as in the previous subsection. In particular we have that

$$\zeta_{\mathfrak{g}_2}(-N) = 120^{-N} Z_\Delta(-N), \quad \zeta'_{\mathfrak{g}_2}(-N) = 120^{-N} Z'_\Delta(-N) + \ln(120) \zeta_{\mathfrak{g}_2}(-N).$$

Via Corollary A1 with the corresponding coefficients \mathbf{c} and \mathbf{d} and with the direction $(\boldsymbol{\mu}, \boldsymbol{\mu}') = ((1, 1), (1, 1, 1, 1))$, we get that

$$\zeta_{\mathfrak{g}_2}(0) = \zeta(0)^2 - 2\zeta(-1) = \frac{5}{12}.$$

We also find thanks to Corollary 6.3.2 the following formula

$$\zeta_{\mathfrak{g}_2}(-N) = 120^{-N} N!^4 \sum_{k=0}^{4N} \alpha_k(N) \frac{\zeta(-N-k) \zeta(-5N+k)}{k!(4N-k)!} + \frac{120^{-N} (-1)^{N+1}}{5} N!^5 \beta(N) \frac{\zeta(-6N-1)}{(5N+1)!},$$

with

$$\begin{aligned} \alpha_k(N) &= \sum_{\substack{(w_q)_{q \in [1,4]} \in [0, N]^4 \\ w_1 + \dots + w_4 = k}} \binom{k}{(w_1, \dots, w_4)} \binom{4N-k}{(N-w_1, \dots, N-w_4)} \prod_{q=1}^4 c_{q,1}^{w_q} c_{q,2}^{N-w_q}, \\ \beta(N) &= \sum_{j=1}^4 \sum_{\substack{(w_q)_{q \in [1,4]} \in [0, 5N+1]^4 \\ w_1 + \dots + w_4 = 5N+1 \\ w_j \geq 2N+1 \\ \forall q \neq j, w_q \leq N}} \binom{-N-1}{-2N-1+w_j} \binom{-2N-1+w_j}{(N-w_q)_{q \neq j}} \left(\prod_{q=1}^4 c_{q,1}^{N-w_q} c_{q,2}^{w_q} + \prod_{q=1}^4 c_{q,2}^{N-w_q} c_{q,1}^{w_q} \right). \end{aligned}$$

Using the SAGE code 7.2, one can find the list of values for $\zeta_{\mathfrak{g}_2}$ in Theorem E.

In order to compute $\zeta'_{\mathfrak{g}_2}(0)$, we see that

$$\zeta'_{\mathfrak{g}_2}(0) = \ln(120) \zeta_{\mathfrak{g}_2}(0) + Z'_\Delta(0).$$

By using $\zeta_{\mathfrak{g}_2}(0) = \frac{5}{12}$ and by computing $Z'_\Delta(0)$ with the SAGE code 7.1, we find that

$$\zeta'_{\mathfrak{g}_2}(0) = \frac{5}{12} \ln(5) - \frac{7}{12} \ln(3) + \frac{155}{36} \ln(2) + \frac{7}{2} \ln(\pi) - \frac{1}{6} \ln\left(\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{5}{6})}\right) - \frac{4}{3} \ln\left(\Gamma\left(\frac{2}{3}\right)\right) - \frac{2}{3} \ln\left(\Gamma\left(\frac{1}{3}\right)\right).$$

By [BZ92, Table 3], we have

$$\Gamma\left(\frac{1}{6}\right) = \frac{\sqrt{3}\Gamma(\frac{1}{3})^2}{\sqrt[3]{2}\sqrt{\pi}},$$

and using Euler's reflection formula, we then get

$$\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{5}{6})} = \frac{3\Gamma(\frac{1}{3})^4}{2^{5/3}\pi^2}.$$

Using this formula inside the expression of $\zeta'_{\mathfrak{g}_2}(0)$, we find that

$$\zeta'_{\mathfrak{g}_2}(0) = \frac{5}{12} \ln(5) - \frac{3}{4} \ln(3) + \frac{55}{12} \ln(2) + \frac{23}{6} \ln(\pi) - \frac{4}{3} \ln\left(\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\right).$$

By Euler's reflection formula we have

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}},$$

thus we obtain

$$\zeta'_{\mathfrak{g}_2}(0) = \frac{5}{12} \ln(5) - \frac{1}{12} \ln(3) + \frac{13}{4} \ln(2) + \frac{5}{2} \ln(\pi).$$

6.3.3 Poles and residues of $\zeta_{\mathfrak{so}(5)}$

Via [KMT10d], we saw in the list of poles (1.3) that $\zeta_{\mathfrak{so}(5)}(s)$ has poles of the form $s = \frac{1}{2}$ and $s = \frac{1-k}{3}$, for $k \geq 0$ an arbitrary integer such that $k \not\equiv 1 \pmod{3}$. We aim here to obtain expressions for the residues at $s = \frac{1}{3}$ and $s = \frac{1}{2}$ for $\zeta_{\mathfrak{so}(5)}(s)$. We note that Bridges, Brindle, Bringmann, and Franke have already obtained a very simple expression for these two residues in [BBBF24, Proposition 5.16] using different techniques.

Let

$$\begin{aligned} l_1(x_1, x_2) &= x_1 + x_2, & l_2(x_1, x_2) &= x_1 + 2x_2, \\ d_1 &= 1, & d_2 &= 1, & d'_1 &= 2, & d'_2 &= 3, \\ Z_\Delta(s) &:= Z(-((s, s), (s, s, s, s))) = \sum_{(1,1), (1,1,1,1)} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s}, \end{aligned}$$

then $\zeta_{\mathfrak{so}(5)}(s)$ is of the form

$$\zeta_{\mathfrak{so}(5)}(s) = 6^s \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^s n_2^s (n_1 + n_2)^s (n_1 + 2n_2)^s} = 6^s Z_\Delta(s).$$

We know that, for $\theta > 0$ sufficiently small, the zeta function $\zeta_{\mathfrak{so}(5)}$ admits the decomposition $\zeta_{\mathfrak{so}(5)}(s) = K(\theta, s) + J(\theta, s)$, where $s \mapsto K(\theta, s)$ is a holomorphic function on \mathbb{C} , and $s \mapsto J(\theta, s)$ is a meromorphic function on \mathbb{C} having the same poles as $\zeta_{\mathfrak{so}(5)}(s)$. By Proposition 2.2.4, we have

$$\begin{aligned} J(\theta, s) &= 6^s \Gamma(1-s)^2 \left(\sum_{j=1}^2 h_{[1,2],j,\mathbf{0}}(s) \right) \frac{\theta^{4s-2}}{\Gamma(s)(4s-2)} \\ &+ 6^s \sum_{\mathcal{P} \subsetneq [1,2]} \Gamma(1-s)^{|\mathcal{P}|} \sum_{\mathbf{k}=(k_p)_{p \in \mathcal{P}^c} \in \mathbb{N}_0^{\mathcal{P}^c}} \left(\sum_{j=1}^2 h_{\mathcal{P},j,\mathbf{k}}(s) \right) \frac{(-1)^{|\mathbf{k}|} \theta^{(2+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|}}{\Gamma(s)((2+|\mathcal{P}|)s - |\mathcal{P}| + |\mathbf{k}|)} \prod_{p \in \mathcal{P}^c} \frac{\zeta(s - k_p)}{k_p!} \end{aligned} \quad (6.3)$$

with

$$\begin{aligned} h_{\mathcal{P},1,\mathbf{k}}(s) &= \frac{1}{\Gamma(s)} \int_0^1 x_2^{s-1} \prod_{p \in \mathcal{P}} l_p^*(1, x_2)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(1, x_2)^{k_p} dx_2, \\ h_{\mathcal{P},2,\mathbf{k}}(s) &= \frac{1}{\Gamma(s)} \int_0^1 x_1^{s-1} \prod_{p \in \mathcal{P}} l_p^*(x_1, 1)^{s-1} \prod_{p \in \mathcal{P}^c} l_p^*(x_1, 1)^{k_p} dx_1, \\ l_1^*(x_1, x_2) &= x_1 + x_2, \\ l_2^*(x_1, x_2) &= x_1 + 2x_2. \end{aligned}$$

As the function $s \mapsto K(\theta, s)$ is holomorphic on \mathbb{C} , we can get the residues of $\zeta_{s\mathfrak{o}(5)}(s)$ by studying the expression of $J(\theta, s)$ above. At $s = 1/2$, the only contributing term of the residue of $\zeta_{s\mathfrak{o}(5)}(s)$ is

$$\Gamma(1-s)^2 \left(\sum_{j=1}^2 h_{[1,2],j,\mathfrak{o}}(s) \right) \frac{\theta^{4s-2}}{\Gamma(s)(4s-2)}.$$

We then get the following residue at $s = 1/2$

$$\text{Res}_{s=\frac{1}{2}} \zeta_{s\mathfrak{o}(5)}(s) = \frac{\sqrt{6}\Gamma(1/2)^2}{4\Gamma(1/2)} \left(\sum_{j=1}^2 h_{[1,2],j,\mathfrak{o}}(1/2) \right).$$

Using the expressions of the h functions we finally get

$$\begin{aligned} \text{Res}_{s=\frac{1}{2}} \zeta_{s\mathfrak{o}(5)}(s) &= \frac{\sqrt{6}}{4} \left(\int_0^1 x^{-1/2}(x+1)^{-1/2}(x+2)^{-1/2} dx + \int_0^1 x^{-1/2}(1+x)^{-1/2}(1+2x)^{-1/2} dx \right) \\ &= \frac{\sqrt{6}}{4} \int_0^1 x^{-1/2}(x+1)^{-1/2}((x+2)^{-1/2} + (1+2x)^{-1/2}) dx. \end{aligned}$$

Although this last residue is hard to express in nicer terms, we still can compute it numerically, and we obtain exactly the same value as the results in [BBBF24].

For the poles $s = \frac{n}{3}$, with $n \leq 1$ being an integer such that $n \not\equiv 0 \pmod{3}$, the only contributing terms to the corresponding residues are of the form

$$-\Gamma(1-n/3)^{|\mathcal{P}|} (-1)^{|\mathbf{k}|} \left(\sum_{j=1}^2 h_{\mathcal{P},j,\mathbf{k}}(1-n/3) \right) \frac{1}{\Gamma(n/3)((2+|\mathcal{P}|))} \prod_{p \in \mathcal{P}^c} \frac{\zeta(n/3 - k_p)}{k_p!},$$

with $\mathcal{P} \subsetneq [1, 2]$, $\mathbf{k} \in \mathbb{N}^{\mathcal{P}^c}$ such that $(2+|\mathcal{P}|)\frac{n}{3} - |\mathcal{P}| + |\mathbf{k}| = 0$. This condition implies that $\mathcal{P} = \{i\}$ is a singleton. We set $\mathcal{P}^c = \{p\}$, we then have $|\mathbf{k}| = k_p = 1 - n \in \mathbb{N}_0$. We find

$$\text{Res}_{s=\frac{n}{3}} \zeta_{s\mathfrak{o}(5)}(s) = (-1)^{1-n} \Gamma\left(\frac{3-n}{3}\right) \left(\sum_{i=1}^2 \sum_{j=1}^2 h_{\{i\},j,\mathbf{k}}\left(1-\frac{n}{3}\right) \right) \frac{1}{3\Gamma\left(\frac{n}{3}\right)} \frac{\zeta\left(\frac{4n}{3}-1\right)}{(1-n)!}.$$

Let's note that, when considering the general terms of the series present in formula (6.3) with $\mathcal{P} = \emptyset$, we observe terms of the form $\frac{\theta^{2s+|\mathbf{k}|}}{\Gamma(s)(2s+|\mathbf{k}|)}$ where $\mathbf{k} = (k_1, k_2) \in \mathbb{N}_0$. The zeros of $2s + |\mathbf{k}|$ do not constitute poles since these poles are canceled by the zeros of the terms of the form $\sum_{j=1}^2 h_{\emptyset,j,\mathbf{k}}$. For example, at $s = -1/2$, we find that the term $2s + |\mathbf{k}|$ is zero if and only if $|\mathbf{k}| = k_1 + k_2 = 1$, that is,

when $\mathbf{k} \in \{(1, 0), (0, 1)\}$. However, we note that

$$\begin{aligned} & h_{1,\emptyset,(1,0)}(s) + h_{2,\emptyset,(1,0)}(s) + h_{1,\emptyset,(0,1)}(s) + h_{1,\emptyset,(0,1)}(s) \\ &= \frac{1}{\Gamma(s)} \int_0^1 (x^{s-1}(1+2x) + 2x^{s-1}(1+x) + x^{s-1}(2+x)) dx \\ &= \frac{5}{\Gamma(s)} \left(\frac{1}{s+1} + \frac{1}{s} \right), \end{aligned}$$

and at $s = -\frac{1}{2}$, this last term vanishes. This fact matches the list of poles (1.3).

6.3.4 Special values of $\zeta_{\mathfrak{so}(5)}$ and $\zeta'_{\mathfrak{so}(5)}$

Using the same notation as in the previous subsection we see that

$$\zeta_{\mathfrak{so}(5)}(-N) = 6^{-N} Z_{\Delta}(-N), \quad \zeta'_{\mathfrak{so}(5)}(-N) = 6^{-N} Z'_{\Delta}(-N) + \ln(6) \zeta_{\mathfrak{so}(5)}(-N).$$

At $N = 0$, we find by Corollary A1:

$$\zeta_{\mathfrak{so}(5)}(0) = \zeta(0)^2 - \frac{9}{6} \zeta(-1) = \frac{3}{8}.$$

We can also have an expression of the values at nonpositive integers using Corollary A2,

$$\zeta_{\mathfrak{so}(5)}(-N) = 6^{-N} N!^2 \sum_{k=0}^{2N} \alpha_k(N) \frac{\zeta(-N-k)\zeta(-3N+k)}{k!(2N-k)!} + \frac{6^{-N}(-1)^{N+1}}{3} N!^3 \beta(N) \frac{\zeta(-4N-1)}{(3N+1)!},$$

with

$$\begin{aligned} \alpha_k(N) &= \sum_{\substack{(w_q)_{q \in [1,2]} \in \llbracket 0, N \rrbracket^2 \\ w_1 + w_2 = k}} \binom{k}{(w_1, w_2)} \binom{2N-k}{(N-w_1, N-w_2)} \prod_{q=1}^2 c_{q,1}^{w_q} c_{q,2}^{N-w_q}, \\ \beta(N) &= \sum_{j=1}^2 \sum_{\substack{(w_q)_{q \in [1,2]} \in \llbracket 0, 3N+1 \rrbracket^2 \\ w_1 + w_2 = 3N+1 \\ w_j \geq 2N+1 \\ \forall q \neq j, w_q \leq N}} \binom{-N-1}{-2N-1+w_j} \binom{-2N-1+w_j}{(N-w_q)_{q \neq j}} \left(\prod_{q=1}^2 c_{q,1}^{N-w_q} c_{q,2}^{w_q} + \prod_{q=1}^2 c_{q,2}^{N-w_q} c_{q,1}^{w_q} \right). \end{aligned}$$

Using SAGE code 7.2, we then find the values of $\zeta_{\mathfrak{so}(5)}(-N)$ in the tabular of Theorem E.

By Lemma 6.3.3, using SAGE code 7.1 we have:

$$\zeta'_{\mathfrak{so}(5)}(0) = \frac{3}{8} \ln(3) + \frac{13}{8} \ln(2) + \frac{3}{2} \ln(\pi).$$

This expression allows us to have an explicit expression of the constant C in Theorem 1.1.41, which gives us a completely explicit asymptotic formula for the number of representations $r_{\mathfrak{so}(5)}(n)$.

6.4 Application to the study of the asymptotic behavior of $r_{\mathfrak{g}_2}(n)$

Let's recall that the number of representations of dimension n of the exceptional Lie algebra \mathfrak{g}_2 is

$$r_{\mathfrak{g}_2}(n) = \left| \left\{ (k_{i,j})_{i,j \geq 1} \in (\mathbb{N}_0)^{\mathbb{N}^2} \mid \sum_{i,j \geq 1} k_{i,j} \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!} = n \right\} \right|.$$

We set $P(i, j) := \frac{ij(i+j)(i+2j)(i+3j)(2i+3j)}{5!}$. We have that

$$\prod_{i,j \geq 1} \frac{1}{1 - q^{P(i,j)}} = \prod_{i,j \geq 1} \sum_{k_{i,j} \geq 0} q^{k_{i,j} P(i,j)},$$

Developing the infinite product, and using the expression of the number of representations $r_{\mathfrak{g}_2}(n)$, we get:

$$1 + \sum_{n=1}^{+\infty} r_{\mathfrak{g}_2}(n) q^n = \prod_{i,j \geq 1} \frac{1}{1 - q^{P(i,j)}}.$$

We set $f_{\mathfrak{g}_2}(n) = |\{(i, j) \in \mathbb{N}^2 | P(i, j) = n\}|$, we then get that

$$1 + \sum_{n=1}^{+\infty} r_{\mathfrak{g}_2}(n) q^n = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f_{\mathfrak{g}_2}(n)}}.$$

Let's define $L_{f_{\mathfrak{g}_2}}(s) = \zeta_{\mathfrak{g}_2}(s)$, and $L_{f_{\mathfrak{g}_2}}^*(s) = \zeta_{\mathfrak{g}_2}(s) \Gamma(s) \zeta(s+1)$. By Corollary 2.2.6, we know that all the poles of the function $\zeta_{\mathfrak{g}_2}(s)$ are simple. We see that the function $L_{f_{\mathfrak{g}_2}}^*(s)$ has only two poles in H_0 (which are simple poles) at $s = \frac{1}{3}$ and $s = \frac{1}{5}$. Moreover, the function $L_{f_{\mathfrak{g}_2}}^*(s)$ has a pole at $s = 0$ of order 2.

We want to apply Theorem 1.1.38. First, let's verify if, with the data above, the conditions of the theorem are valid. Let $\Lambda := \mathbb{N}_0 \setminus f_{\mathfrak{g}_2}^{-1}(\{0\})$. We note that $\frac{3}{2} \cdot \frac{1}{5} < \frac{1}{3} < \frac{2}{1} \cdot \frac{1}{5}$. Thus, the integer $l = 2$ satisfies the hypothesis of Theorem 1.1.38. Now, let's check whether the conditions (P1), (P2), and (P3) are met:

(P1) Let $p \geq 2$ be a prime number. We have that $|\Lambda \setminus (p\mathbb{N}_0 \cap \Lambda)| = +\infty$. Indeed, the sequences

$$\begin{aligned} u(2) &:= (P(8k+1, 1))_{k \in \mathbb{N}_0}, \\ u(3) &:= (P(9k+1, 1))_{k \in \mathbb{N}_0}, \\ u(5) &:= (P(25k+1, 1))_{k \in \mathbb{N}_0}, \\ u(p) &:= (P(kp+1, 1))_{k \in \mathbb{N}_0} \quad \text{if } p \geq 7 \end{aligned}$$

are strictly increasing, and belong to Λ by definition. Moreover, we observe that the sequence $u(p)$ belongs in $\Lambda \setminus (p\mathbb{N}_0 \cap \Lambda)$ for all prime number p . We then conclude that any real number $L \geq \frac{1}{6}$ satisfies the condition (P1).

(P2) We have already seen previously that $L_{f_{\mathfrak{g}_2}}^*$ has two poles (which are simple) at $s = \frac{1}{3}$ and $s = \frac{1}{5}$, a pole of order 2 at $s = 0$, and simple poles of the form $s = \frac{k}{5}$, with $k \in \mathbb{Z}_{\leq 1}$ such that $k \not\equiv 0 \pmod{5}$. Therefore, any given real number $R \in \mathbb{R}_+^* \setminus \frac{1}{5}\mathbb{N}$ satisfies the condition (P2).

(P3) We note that the polynomial P satisfies the *H0S* condition of the article [Ess97]. In particular, we know from [Ess97, Theorem 3] that the zeta function associated with this polynomial, (i.e., $\zeta_{\mathfrak{g}_2}$) admits a polynomial bound in the imaginary part of $s = \sigma + i\tau$ on each vertical strip $\sigma_1 \leq \sigma \leq \sigma_2$. We then conclude that the condition (P3) is met.

Remark 6.4.1. *In condition (P2), R is arbitrarily large, and in condition (P1), L is also arbitrarily large.*

Let's note $\alpha = \frac{1}{3}$, $\beta = \frac{1}{5}$, $\omega_\alpha := \text{Res}_{s=\frac{1}{3}} \zeta_{\mathfrak{g}_2}(s)$ and $\omega_\beta := \text{Res}_{s=\frac{1}{5}} \zeta_{\mathfrak{g}_2}(s)$. Applying Theorem 1.1.38, we get that

$$r_{\mathfrak{g}_2}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^b} \exp\left(A_1 n^{\frac{1}{4}} + A_2 n^{\frac{3}{20}} + A_3 n^{\frac{1}{20}}\right) \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R}\left(n^{-\min\left(\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\right)}\right)\right),$$

with

$$C = \frac{e^{\zeta'_{\mathfrak{g}_2}(0)} (\omega_\alpha \Gamma(\frac{4}{3}) \zeta(\frac{4}{3}))^{\frac{1-6\zeta_{\mathfrak{g}_2}(0)}{8}} \sqrt{3}}{\sqrt{8\pi}}, \quad b = \frac{6 - 6\zeta_{\mathfrak{g}_2}(0) + 1}{8},$$

and $A_1 := 4 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{3}{4}}$, $A_2 := \frac{\omega_\beta \Gamma\left(\frac{1}{5}\right) \zeta\left(\frac{6}{5}\right)}{(\omega_\alpha \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}}$, and

$$A_3 := K_3 + 3 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{3}{4}} \left(\binom{-1/3}{1} \frac{K_3}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{4}}} + \binom{-1/3}{2} \frac{K_2^2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{2}}} \right) \\ + \frac{5 (\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right))}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}} \binom{-1/5}{1} \frac{K_2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{4}}},$$

with $0 < \nu_2 < \dots$ being the positive elements of the set $\mathcal{N} + \mathcal{M}$. From the definition of the sets (1.8), (1.9), and (1.10) in [BBBF24], we find that the sequence ν_j in Theorem 1.1.38 corresponds to

$$\nu_2 = \frac{1}{20}, \nu_3 = \frac{2}{20}, \nu_4 = \frac{3}{20}, \dots$$

By [BBBF24, Lemma 4.3], we get

$$K_2 = \frac{3}{4} \cdot \frac{\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{3}{20}}}$$

$$K_3 = -\frac{3}{160} \cdot \frac{(\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right))^2}{(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right))^{\frac{21}{20}}}.$$

Thus we obtain Theorem G. □

Chapter 7

Appendix: SAGE codes

Listing 7.1: SAGE Code to compute $Z'_{\Delta}(0)$ with $P = 2$ and $d_1 = d_2 = 1$

```
from sage.all import *
A=var('A') #Glaisher-Kinkelin constant
zeta_der_1=1/12-log(A) #Value of  $\zeta'(-1)$ 
c_so5=[[1,1],[1,2]] #Data set of  $\zeta_{s_0(5)}(s)$ 
c_sl3=[[1,1]] #Data set of  $\zeta_{s_l(3)}(s)$ 
c_g2=[[1,1],[1,2],[1,3],[2,3]] #Data set of  $\zeta_{g_2}(s)$ 
def barnes_eval_deriv(N,d,c): #Compute some  $(\zeta^B)'(0,d|c)$ 
    if d==2 and c==[1,1]:
        return(1/2*log(2*pi)+1/12-log(A))
    elif d==3 and c==[1,2]:
        return(1/4*log(2)+1/2*log(pi)-1/2*log(A)+1/24)
    elif d==4 and c==[1,3]:
        return(-5/9*log(3)+log(2)+log(pi)-1/3*log(A)
                -2/3*log(gamma(2/3))-1/3*log(gamma(1/3))+1/36)
    elif d==5 and c==[2,3]:
        return(-11/18*log(3)+13/12*log(2)+4/3*log(pi)-1/6*log(A)
                -2/3*log(gamma(2/3))-log(gamma(1/3))+1/72)
    else:
        return('uncomputed_value')
def Z_delta_prime_zero(c): #Compute  $Z'_{\Delta}(0)$ .
    Q=len(c)
    sum2 = 0
    sum1 = sum(c[j][0]/c[j][1]+c[j][1]/c[j][0] for j in range(Q))
    sum3 = sum(barnes_eval_deriv(0,sum(c[j]),c[j]) for j in range(Q))
    sum2 = sum(c[j][1]/c[j][0]*log(c[j][0])+c[j][0]/c[j][1]*log(c[j][1])
                for j in range(Q))
    +sum(sum((c[f][0]/c[f][1]-c[j][0]/c[j][1])*log(1+c[f][1]/c[j][1])
                +(c[f][1]/c[f][0]-c[j][1]/c[j][0])*log(1+c[f][0]/c[j][0])
                for f in range(Q) if f != j) for j in range(Q))
    result = -zeta(0)*log(2*pi)-sum1/(Q+1)*zeta_der_1-sum2/(Q+1)*zeta(-1)
    +sum3
    if not(sage.rings.rational.is_Rational(result)):
        result=result.canonicalize_radical()
    return(result)
print(Z_delta_prime_zero(c_so5)+log(6)*3/8) #Compute  $\zeta'_{s_0(5)}(0)$ 
print(Z_delta_prime_zero(c_g2)+log(120)*5/12) #Compute  $\zeta'_{g_2}(0)$ 
```

Listing 7.2: SAGE Code to compute $Z_{\Delta}(-N)$ with $P = 2$

```

from math import comb
from itertools import product
c_g2=[[1,1],[1,2],[1,3],[2,3]] #Data set of  $\zeta_{g_2}(s)$ 
c_so5=[[1,1],[1,2]] #Data set of  $\zeta_{so(5)}(s)$ 
def alpha_k_N(Q, N,k, c_matrix):
    alpha=0
    for w in product(range(N+1),repeat=Q):
        if sum(w)==k:
            coef1=multinomial(w)
            coef2=multinomial([N-w_i for w_i in w])
            prod=1
            for q in range(Q):
                prod*=(c_matrix[q][0]**w[q])*(c_matrix[q][1]**(N-w[q]))
            alpha+=coef1*coef2*prod
    return alpha
def beta_N(Q,N, c_matrix):
    beta=0
    for j in range(Q):
        for w in product(range((Q+1)*N+2),repeat=Q):
            if sum(w)==(Q+1)*N+1 and w[j]>=2*N+1 and
            max([w[q] for q in range(Q) if q!=j])<=N:
                coeff=multinomial(w)*binomial(-N-1,-2*N-1+w[j])
                *multinomial([N-w[q] for q in range(Q) if q!=j])
                term1=1
                term2=1
                for q in range(Q):
                    term1*=(c_matrix[q][0]**(N-w[q]))
                    *(c_matrix[q][1]**w[q])
                    term2*=(c_matrix[q][1]**(N-w[q]))
                    *(c_matrix[q][0]**w[q])
                beta+=coeff*(term1 + term2)
    return beta
def Z_Delta_minus_N(N, c_matrix): #Compute  $Z_{\Delta}(-N)$ 
    Q=len(c_matrix)
    sum1=0
    sum2=0
    N_fact_Q=factorial(N)**Q
    for k in range(Q*N+1):
        alpha=alpha_k_N(Q,N,k, c_matrix)
        term1=zeta(-N-k) / factorial(k)
        term2=zeta(-(Q+1)*N+k)/factorial(Q*N-k)
        sum1+=alpha*term1*term2
    beta=beta_N(Q,N, c_matrix)
    sum2=(N_fact_Q*factorial(N))*beta*zeta(-(Q+2)*N-1)/factorial((Q+1)*N+1)
    Z=N_fact_Q*sum1+(-1)**(N+1)*(1/(Q+1))*sum2
    return Z
print (6**(-N)*Z_Delta_minus_N(N, c_so5)) #Compute  $\zeta_{so(5)}(-N)$ 
print (120**(-N)*Z_Delta_minus_N(N, c_g2)) #Compute  $\zeta_{g_2}(-N)$ 

```

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